Fundamentals

If you want a nice discussion of SPH, with some interesting background see Joe Monaghan's introduction on youtube (`https://youtu.be/tAXHCAEgSuE`). We wish to evaluate the equations of motion of a barotropic fluid (in Lagrangian form, i.e. dealing with material derivatives):

$$\begin{cases} \dfrac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u} \\ \dfrac{D\mathbf{u}}{Dt} = -\dfrac{1}{\rho}\nabla P + \dfrac{\nu}{\rho}\nabla^2\mathbf{u} + \mathbf{g} \\ \quad P = F(\rho) \end{cases} \tag{1}$$

These PDE's are continuum equations, we wish to discretise these onto a set of particles that we follow through time and space. Kernel interpolation theory begins by considering the integral representation of an arbitrary field function f through a volume Ω in the form

$$f(\mathbf{x}) = \int_\Omega f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')d\mathbf{x}', \tag{2}$$

where δ is the Dirac-delta function and $d\mathbf{x}$ a volume element. Equation 2 is an exact interpolation, but cannot be integrated, hence the delta function is replaced with a smoothing function $W(\mathbf{x}-\mathbf{x}', h)$ such that

$$< f(\mathbf{x}) >= \int_\Omega f(\mathbf{x}')W(\mathbf{x}-\mathbf{x}', h)d\mathbf{x}'. \tag{3}$$

The smoothing or kernel function W has a radius of influence proportional to a length scale h, termed the 'smoothing length'. As the kernel function is only an approximation of the delta function, equation 3 cannot be exact, but is second order accurate if W is symmetric and meets the normalisation condition $\int_\mathbf{x} W d\mathbf{x} = 1$ [1]. Hence, this step is defined by the kernel approximation operator $<>$. Similarly, the spatial derivative can be approximated by replacing $A(\mathbf{x})$ with $\nabla A(\mathbf{x})$,

$$< \nabla f(\mathbf{x}) >= \int_\Omega [\nabla f(\mathbf{x}')]W(\mathbf{x}-\mathbf{x}', h)d\mathbf{x}' \tag{4}$$

With the spatial derivative likely unknown, the identity

$$\nabla[f(\mathbf{x}')W(\mathbf{x}-\mathbf{x}', h)] = [\nabla f(\mathbf{x}')]W(\mathbf{x}-\mathbf{x}', h) + f(\mathbf{x}')[\nabla W(\mathbf{x}-\mathbf{x}', h)] \tag{5}$$

is applied. Resulting in

$$< \nabla f(\mathbf{x}) >= \int_\Omega \nabla[f(\mathbf{x}')W(\mathbf{x}-\mathbf{x}', h)]d\mathbf{x}' - \int_\Omega f(\mathbf{x}')[\nabla W(\mathbf{x}-\mathbf{x}', h)]d\mathbf{x}' \tag{6}$$

Through the divergence theorem, the first term of Equation 6 can be replaced with a surface integral

$$< \nabla f(\mathbf{x}) >= \int_S f(\mathbf{x}')W(\mathbf{x}-\mathbf{x}', h)\cdot\vec{\mathbf{n}}dS - \int_\Omega f(\mathbf{x}')[\nabla W(\mathbf{x}-\mathbf{x}', h)]d\mathbf{x}' \tag{7}$$

where $\vec{\mathbf{n}}$ is a unit normal to the surface S. With the surface integral being evaluated over the kernel support radius (κh), if the chosen kernel has compact support ($W(\kappa h, h)$=0) this term will disappear and equation 7 reduces to

$$< \nabla f(\mathbf{x}) >= -\int_\Omega f(\mathbf{x}')[\nabla W(\mathbf{x}-\mathbf{x}', h)]d\mathbf{x}', \tag{8}$$

With the continuous fluid being discretised by a finite number of particles, the integral in equations 3 and 8 can be approximated by a summation over N neighbouring particles, which are assigned mass and density, taking the form

$$< f(\mathbf{x}_i) >\approx \sum_j f(\mathbf{x}_j)W(\mathbf{x}_i-\mathbf{x}_j, h)\frac{m_j}{\rho_j}, \tag{9}$$

$$< \nabla f(\mathbf{x}_i) >\approx -\sum_j f(\mathbf{x}_j)\nabla W(\mathbf{x}_i-\mathbf{x}_j, h)\frac{m_j}{\rho_j}. \tag{10}$$

Equation 9 represents the SPH particle approximation of a variable or function. A fundamental application of SPH is to set $A(\mathbf{x})$ equal to $\rho(\mathbf{x})$, deriving the SPH density estimator

$$< \rho(\mathbf{x}_i) >\approx \sum_j m_j W(\mathbf{x}_i-\mathbf{x}_j, h). \tag{11}$$

With the particle approximation, the equations governing particle motion for fluid flow can now be formulated; initially consider the inviscid momentum equation from the Lagrangian perspective such that

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla P}{\rho} + g. \tag{12}$$

The kernel approximation operator is subsequently dropped for clarity. Equation 12 can be directly evaluated using the particle approximation, but results in a term that does not conserve linear nor angular momentum [2]. Thus, to symmetrise the pressure gradient the identity

$$\frac{\nabla P}{\rho} = \nabla\frac{P}{\rho} + \frac{P}{\rho^2}\nabla\rho \tag{13}$$

is applied, leading to

$$\frac{\nabla P}{\rho} = -\sum_j \frac{m_j}{\rho_j}\left[\frac{P_j}{\rho_j}\right]\nabla_i W_{ij} - \left[\frac{P_i}{\rho_i^2}\right]\sum_j \frac{m_j}{\rho_j}\rho_j \nabla_i W_{ij} \tag{14}$$

$$\frac{\nabla P}{\rho} = -\sum_j m_j\left[\frac{P_j}{\rho_j^2} + \frac{P_i}{\rho_i^2}\right]\nabla_i W_{ij}. \tag{15}$$

Momentum

$$\frac{D\mathbf{u}_i}{Dt} = -\sum_j m_j\left[\frac{P_j}{\rho_j^2} + \frac{P_i}{\rho_i^2} + \Pi_{ij}\right]\nabla W_{ij} + \sum_j m_j \nu \frac{\rho_i + \rho_j}{\rho_i\rho_j}\frac{\mathbf{x}_{ij}\cdot\nabla W_{ij}}{|\mathbf{x}_{ij}|^2 + 0.001h^2}\mathbf{u}_{ij} + \mathbf{a}_{ST,i} + \mathbf{g} \tag{16}$$

$$\Pi_{ij} = \left\{\begin{array}{ll} \frac{-\alpha\bar{c}_{ij}\mu_{ij} + \beta\mu_{nk}^2}{\rho_{ij}} & : \mathbf{u}_{ij}\cdot\mathbf{x}_{ij} < 0 \\ 0 & : \mathbf{u}_{ij}\cdot\mathbf{x}_{ij} > 0 \end{array}\right\}, \quad \mu_{ij} = \frac{h\mathbf{u}_{ij}\cdot\mathbf{x}_{ij}}{\mathbf{x}_{ij}^2 + 0.001h^2}, \quad \beta = 0. \tag{17}$$

Continuity

$$\frac{D\rho_i}{Dt} = \sum_j m_j \mathbf{u}_{ij}\cdot\nabla W_{ij} + \delta h c_0 \mathcal{D}_i \tag{18}$$

$$\mathcal{D}_i = 2\sum_j \psi_{ij}\frac{\mathbf{x}_{ij}\cdot\nabla W_{ij}}{|\mathbf{x}_{ij}|}V_j \tag{19}$$

$$\psi_{ij} = \left\{(\rho_i - \rho_j) - \frac{1}{2}\left(\langle\nabla\rho\rangle_i^L + \langle\nabla\rho\rangle_j^L\right)\cdot\mathbf{x}_{ij}\right\} \tag{20}$$

$$\langle\nabla\rho\rangle_i^L = \sum_j(\rho_i - \rho_j)\,\mathbf{L}_i\nabla W_{ij}V_j, \quad \mathbf{L}_i = \left[-\sum_j \mathbf{x}_{ij}\otimes\nabla W_{ij}V_j\right]^{-1} \tag{21}$$

Tait equation of state and C2 Kernel

$$P = \frac{\rho_0 c_0^2}{\gamma}\left[\left(\frac{\rho}{\rho_0}\right)^\gamma - 1\right] \tag{22}$$

$$W_{ij} = \alpha_d\left(1 - \frac{|\mathbf{x}_{ij}|}{2h}\right)_+^4\left(\frac{2|\mathbf{x}_{ij}|}{h} + 1\right) \tag{23}$$

$$\begin{aligned} \alpha_w &= \frac{7}{4\pi h^2}, \quad \text{for dim} = 2 \\ \alpha_w &= \frac{21}{16\pi h^3}, \quad \text{for dim} = 3 \end{aligned} \tag{24}$$

Surface tension

Renormalised surface normal:

$$\mathbf{n}_i = -\mathbf{L}_i\sum_j \nabla W_{ij}V_j \tag{25}$$

Shepard interpolant over normals:

$$\widetilde{\mathbf{n}}_i = \frac{\sum_j \mathbf{n}_i W_{ij} V_j}{\sum_j W_{ij} V_j} \tag{26}$$

Obtain boundary normal:

$$\mathbf{n}_i^{b\perp} = \sum_{j \in boundary} \nabla W_{ij} V_j \tag{27}$$

Giving boundary tangent (using unit vectors for these following rotations):

$$\hat{\mathbf{n}}_i^{b\|} = \hat{\widetilde{\mathbf{n}}}_i - \left(\hat{\mathbf{n}}_i^{b\perp} \cdot \hat{\widetilde{\mathbf{n}}}_i\right) \hat{\mathbf{n}}_i^{b\perp} \tag{28}$$

Prescribed normal's at wall, contact angle θ:

$$\mathbf{n}_i^{\theta} = \hat{\mathbf{n}}_i^{b\perp} cos(\theta) + \hat{\mathbf{n}}_i^{b\|} sin(\theta) \tag{29}$$

Smoothed normal prescription over distance y^+:

$$\mathbf{n}_i^{\alpha} = \alpha^+ \hat{\mathbf{n}}_i^{\theta} + (1 - \alpha^+)\widetilde{\mathbf{n}}_i \tag{30}$$

$$\alpha^+ = \begin{cases} 1 & \text{for } y^+ \leq \Delta r \\ 1 + (\Delta r - y^+)/2h & \text{for } \Delta r < y^+ \leq 2h + \Delta r \\ 0 & \text{for } y^+ > 2h + \Delta r \end{cases} \tag{31}$$

Final surface normal with smoothed contact angle prescription, length is set to original fluid magnitude:

$$\mathbf{n}_i^{mod} = \hat{\mathbf{n}}_i^{\alpha} |\widetilde{\mathbf{n}}_i| \tag{32}$$

Curvature calculations are only performed over particles (i) and their neighbours (j) that are identified to lie on the surface. For this we use the minimum eigenvalue (λ) of the renormalisation matrix ($\mathbf{L}_i$), if $\lambda_i \leq 0.75$, we say this particle lies on/near the surface and it is included in calculation and summations. A less binary surface identification would aid the stiffness of the ST method and improve convergence.

$$\kappa_i = \sum_j V_j \left(\hat{\mathbf{n}}_j^{mod} - \hat{\mathbf{n}}_i^{mod}\right) \cdot \nabla W_{ij} \tag{33}$$

Corrected curvature, $\kappa^* = \kappa/\mathcal{L}$, due to limited particle support from only using surface particles:

$$\mathcal{L}_i = \sum_j V_j W_{ij} \tag{34}$$

Surface tension body acceleration:

$$\mathbf{a}_{ST,i} = \frac{\sigma}{\rho_i} \kappa_i^* \mathbf{n}_i^{mod} \tag{35}$$

This first equation is incorrect. Modified normals should only influence effective curvature to set energy at boundaries, use smoothed normal as direction of surface tension acceleration:

$$\mathbf{a}_{ST,i} = \frac{\sigma}{\rho_i} \kappa_i^* \widetilde{\mathbf{n}}_i \tag{36}$$

Newmark integration

$$\dot{\mathbf{x}}^{n+1} = \dot{\mathbf{x}}^n + (1-\gamma)\Delta t \ddot{\mathbf{x}}^n + \gamma \Delta t \ddot{\mathbf{x}}^{n+1} \tag{37}$$

$$\mathbf{x}^{n+1} = \mathbf{x}^n + \Delta t \dot{\mathbf{x}}^n + (0.5-\beta)\Delta t^2 \ddot{\mathbf{x}}^n + \beta \Delta t^2 \ddot{\mathbf{x}}^{n+1} \tag{38}$$

References

[1] D. J. Price, "Smoothed particle hydrodynamics and magnetohydrodynamics," *Journal of Computational Physics*, vol. 231, no. 3, pp. 759 – 794, 2012, special Issue: Computational Plasma Physics.

[2] J. J. Monaghan, "Smoothed particle hydrodynamics," *Annual review of astronomy and astrophysics*, vol. 30, 1992.

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$$\frac{D\mathbf{u}_i}{Dt} = -\sum_j m_j \left[\frac{P_j}{\rho_j^2} + \frac{P_i}{\rho_i^2} + \Pi_{ij}\right] \nabla W_{ij}$$
$$+ \sum_j m_j \nu \frac{\rho_i + \rho_j}{\rho_i \rho_j} \frac{\mathbf{x}_{ij} \cdot \nabla W_{ij}}{|\mathbf{x}_{ij}|^2 + 0.001h^2} \mathbf{u}_{ij}$$
$$+ \mathbf{a}_{ST,i} + \mathbf{g}$$

$$\frac{D\rho_i}{Dt} = \sum_j m_j \mathbf{u}_{ij} \cdot \nabla W_{ij} + \delta h c_0 \mathcal{D}_i$$

$$P = \frac{\rho_0 c_0^2}{\gamma} \left[\left(\frac{\rho}{\rho_0}\right)^\gamma - 1\right]$$

$$M\ddot{x} + C_1 \dot{x} + C_2 |\dot{x}| \dot{x} + Kx = F_{sph}$$

m_{eq}	0.275	kg
k	1.1	N/mm
f	10.05	Hz
ζ	$0.23 - 0.34$	%

$$\mathbf{n}_i = -\mathbf{L}_i \sum_j \nabla W_{ij} V_j$$

$$\widetilde{\mathbf{n}}_i = \frac{\sum_j \mathbf{n}_i W_{ij} V_j}{\sum_j W_{ij} V_j}$$

$$\mathbf{n}_i^{mod} = f(\widetilde{\mathbf{n}}_i, \theta, y^+)$$

$$\mathbf{n}_i^\theta = \mathbf{n}_i^{b\perp} cos(\theta) + \mathbf{n}_i^{b\|} sin(\theta)$$

$$\mathbf{n}_i^\alpha = \alpha^+ \mathbf{n}_i^\theta + (1 - \alpha^+) \widetilde{\mathbf{n}}_i$$

$$\alpha^+ = \begin{cases} 1 & \text{for } y^+ \leq \Delta r \\ 1 + (\Delta r - y^+)/2h & \text{for } \Delta r < y^+ \leq 2h + \Delta r \\ 0 & \text{for } y^+ > 2h + \Delta r \end{cases}$$

$$\kappa_i = \sum_j V_j \left(\hat{\mathbf{n}}_j^{mod} - \hat{\mathbf{n}}_i^{mod}\right) \cdot \nabla W_{ij} \quad i, j : \lambda < 0.75$$

$$\mathcal{L}_i = \sum_j V_j W_{ij}$$

$$\kappa_i^* = \kappa_i / \mathcal{L}_i$$

$$\mathbf{a}_{ST,i} = \frac{\sigma}{\rho_i} \kappa_i^* \widetilde{\mathbf{n}}_i$$

$$\mathbf{F}_{ij} = s_{ij} \cos\left(\frac{1.5\pi}{3h} |\mathbf{x}_{ij}|\right) \frac{\mathbf{x}_{ij}}{|\mathbf{x}_{ij}|}, \quad |\mathbf{x}_{ij}| \leq h \tag{39}$$

$$(\mathbf{a}_{ST})_i = -\frac{\sigma}{\rho_i} (\nabla \cdot \hat{\mathbf{n}})_i \mathbf{n}_i \tag{40}$$