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Analysis on arithmetic schemes. II

by

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Как число в уме, на песке оставляя след,
океан громоздится во тьме, миллионы лет
мертвой зыбью баюкая щепку.
И если резко шагнуть с дебаркадера вбок, вовне,
будешь долго падать, руки по швам;
но не воспоследует всплеска.

J. Brodsky

Abstract

We construct adelic objects for rank two integral structures on arithmetic surfaces and develop measure and integration theory, as well as elements of harmonic analysis. Using the topological Milnor K_2-delic and $K_1 \times K_1$-delic objects associated to an arithmetic surface, an adelic zeta integral is defined. Its unramified version is closely related to the square of the zeta function of the surface. For a proper regular model of an elliptic curve over a global field, a two-dimensional version of the theory of Tate and Iwasawa is derived. Using adelic analytic duality and a two-dimensional theta formula, the study of the zeta integral is reduced to the study of a boundary integral term. The work includes first applications to three fundamental properties of the zeta function: its meromorphic continuation and functional equation and a hypothesis on its mean periodicity; the location of its poles and a hypothesis on the permanence of the sign of the fourth logarithmic derivative of a boundary function; and its pole at the central point where the boundary integral explicitly relates the analytic and arithmetic ranks.

Key Words: topological Milnor K-groups, explicit two-dimensional class field theory, adeles for arithmetic schemes, translation invariant integration and harmonic analysis, two-dimensional adelic analysis, zeta functions, zeta integral, analytic duality on arithmetic surfaces, boundary term, elliptic curves over global fields.

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Contents

Introduction

0. A conceptual way to understand the meromorphic continuation and functional equation of zeta functions of global fields and their twists by Dirichlet characters is to lift them to zeta integrals on appropriate adelic objects and then calculate the integrals in two different ways, using a powerful adelic duality. For number fields this first appeared in [24], [57], [25]; for the general case see e.g. [60]. For a continuous homomorphism χ from the idele class group to the multiplicative group of complex numbers, and an appropriate function f on the adeles, one derives the following formula concerning the zeta integral $\zeta(f,\chi)$:

$$\zeta(f,\chi) = \xi(f,\chi) + \xi(\widehat{f},\widehat{\chi}) + \omega(f,\chi), \qquad \Re(s(\chi)) > 1$$

in which the first two terms are absolutely convergent integrals on the plane. The boundary term $\omega(f,\chi)$ involves an integral of f and its transform $\widehat{f}$ over the set $\{0\}$, which is the (weak) boundary of the multiplicative group of the global field. This boundary term is a rational function either of s or of q^{-s}. The meromorphic continuation, functional equation, and the location of the poles of the zeta integral and the determination of their residues then follow from analytic properties of the boundary term.

This K_1-adelic method was extended to algebraic groups over global fields in [18], where one works with zeta integrals of an automorphic representation. The appropriate analogue of the boundary term has finitely many poles. In particular, in the cuspidal case the corresponding boundary term vanishes.

1. Instead of passing from the one-dimensional, commutative theory, which uses GL_1 or K_1 of one-dimensional local, global, and adelic objects, to the one-dimensional, noncommutative theory, i.e. using algebraic groups of the objects, this work goes in the direction of a commutative, two-dimensional theory in the following sense. The analogue of a global field is a two-dimensional global field, namely the field of rational functions of an arithmetic surface, which in this text will be a proper, regular model of a curve over a (one-dimensional) global field. Abelian extensions of a two-dimensional global field are described by higher class field theory, one exposition of which uses products, restricted in a two-dimensional sense, of topological Milnor K_2^t-groups of completed and semi-completed objects associated to points and curves on the surface. In this work, without using higher class field theory, but working with subobjects of the objects from explicit higher class field theory, we lift the zeta function of the arithmetic surface to a zeta integral on K_2^t-delic and $K_1\times K_1$-delic objects, and then develop a *two-dimensional, commutative generalization* of Tate and Iwasawa's one-dimensional theory.

Thus, unlike almost all works in arithmetic geometry, which study the zeta function of arithmetic schemes via the study of their L-function factors, the two-dimensional adelic analysis of this paper directly operates with the zeta function.

The key ingredient, which has not been available until recently, is the theory of translation invariant measure and integration on higher dimensional local fields, developed in [11], [12]. To introduce the zeta integral we first construct the theory of measure and integration on new adelic objects. Two-dimensional adelic analysis reveals various kinds of new adelic symmetries underlying the properties of the zeta function, which are not explicitly visible in the approaches which study the L-function factors. Milnor K-theory plays an important role too, both for the motivation of some of the main objects and for their use. There are numerous links between the commutative, two-dimensional adelic theory of this paper and the one-dimensional, noncommutative theories, including aspects of the arithmetic and geometric Langlands correspondences.

The idelès, introduced by Chevalley, simplified and clarified some of fundamental structures in algebraic number theory. Similarly, for a group G ($G = G_a$, K_n, or $K_1^{\oplus n}$), G-adeles in higher dimensions are expected to play a central role in arithmetic geometry. Deep theorems in the arithmetic of elliptic curves in characteristic zero have so far been proved only under special restrictions on the ground global field, using quite specific methods which unfortunately cannot be extended to an arbitrary global field. In contrast, adelic methods are universal and they work over an arbitrary global field without any restrictions. The adelic perspective in dimension two leads to a unified whole understanding of the connections between three fundamental aspects of zeta functions: the meromorphic continuation and functional equation, the location of poles, and the behaviour at the central point.

2. This text deals with the relative situation of a two-dimensional, arithmetic scheme $\mathcal{S}$ over a one-dimensional base B, the latter being the spectrum of the ring of integers of a global number field k in characteristic zero, or a proper, smooth, connected curve over a finite field. The central case on which we concentrate in this work is that of a proper, regular model of an elliptic curve over a global field, which is the first nontrivial case of an arithmetic surface. The zeta integral in this case converges in the best possible way. In the case of proper, regular models of hyperbolic curves over global fields one should renormalize the fibre products of local zeta integrals, using the projective space over B.

Prerequisites for this work are the one-dimensional theory [57], [24] and the higher dimensional, local theory [11], [12]. A closely related text which explains the main ideas, methods, constructions, and directions of applications and which is relatively free from technical details is [14]. The current text simplifies some of the objects in [14].

A brief review of [11] is included at the beginning of parts 1 and 3. The numeration of the sections continues that of [11], and its sections are referred to without explicit mentioning of [11]. The sections form rather natural divisions of the material of the paper and they are of varying length; such a simple marking of the text aims to simplify its reading. The two-dimensional theory of [11] was extended to higher dimensional local fields in [12], which also contains a discussion of the links between the integration on higher local fields and the Feynman path integral. For an alternative approach to the measure, integration and zeta integrals on valuation fields whose residue field is a local field see [39]; its method uses the lifting from the residue level in a systematic way and is related to section 13 of [11]. For the measure and integration theory on finite dimensional vector spaces over such fields see [40], and for some new, interesting behaviour of the Fubini property for invertible polynomial transformations on vector spaces see [41]. The study of new representation theoretical algebras associated to algebraic groups over two-dimensional local fields, related to [11], is contained in [32] and [36]. In the equal characteristic case, there is a geometric categorical approach to the study of representation theoretical aspects of finite and infinite dimensional groups over two-dimensional local fields; see e.g. [16], [6]. It has undoubtedly many links with the previously mentioned works. For a model theoretical approach to integration on henselian fields with residue field of characteristic zero, which unifies a finitely additive version of the measure and integration on higher local fields with the so called motivic integration, see [22], [23].

3. The paper consists of 4 parts. Part 1 develops the theory of adelic measure and integration on new, two-dimensional adelic objects associated to $\mathcal{S}$. Part 2 contains a short K_2-delic description of two-dimensional class field theory. Part 3 introduces and studies the zeta integral of $\mathcal{S}$, and, for a proper, regular model of an elliptic curve over a global field, it derives a two-dimensional analogue of the unramified part of the classical Tate–Iwasawa theory. Part 4 includes both a more explicit description of the zeta integral and the first applications of the two-dimensional adelic method to and its relations with several key issues in the arithmetic of elliptic curves over global fields.

Now we summarise the content of the four parts; see also their introductions and the introductions of their subparts.

3-1. In the theory of part 1 we do not strive for the maximal level of generality and place more emphasis on explicit constructions rather than developing a general approach. To an arithmetic scheme $\mathcal{S} \longrightarrow B$ corresponding to a proper, regular model of a smooth, projective curve over a global field (for the precise definition, see section 24), we can associate several adelic objects. Each of them is defined by two restricted product conditions.

There are two quite different adelic spaces on $\mathcal{S}$. Recall that every nonarchimedean, two-dimensional local field has two rings of integers: one is the ring of integers of a discrete valuation of rank one and the other is the ring of integers of a discrete valuation of rank two (for example, if the field is $\mathbb{Q}_p((t))$ then these two rings are $\mathbb{Q}_p[[t]]$ and $\mathbb{Z}_p + t\mathbb{Q}_p[[t]]$). The two adelic spaces on $\mathcal{S}$ will be adelic spaces associated to these two different integral structures.

Section 28 introduces a large adelic object $\mathbf{A}_{\mathcal{S}}$, for which one of the restricted product conditions is taken with respect to the rank one integral structure. In positive characteristic this adelic object was first defined by Parshin [44], [46]; see his subsequent works for their first geometric applications. This adelic space is suitable for geometric duality studies and for the study of 1-cocycles on the surface. It is not very useful for measure and integration, the zeta integral, and the study of 0-cycles. For all those purposes one uses the second adelic space $\mathbb{A}_{S_\prime}$, introduced in section 29. It is associated to a subset $S_\prime$ of the irreducible curves on $\mathcal{S}$ which contains only finitely many horizontal curves. This object satisfies two adelic conditions, one of which is taken with respect to the rank two integral structure. Using the two-dimensional local theory of [11], section 30 introduces an $\mathbb{R}((X))$-valued measure, a $\mathbb{C}((X))$-valued integration, and a transform for elements of certain functional spaces on the adelic object $\mathbb{A}_{S_\prime}$. If the vertical part of $S_\prime$ contains all vertical curves on $\mathcal{S}$ then the measure and integration theory on the object $\mathbb{A}_{S_\prime}$ works best when the genus of the generic fibre is 1. In the case of a general arithmetic surface, a renormalization is required.

The adelic object $\mathbb{A}_{S_\prime}$ contains a subspace $\mathbb{B}_{S_\prime}$ of local-global nature which, from the point of view of adelic dualities, plays the role of an analogue of the global elements in the one-dimensional adelic object (of course, there is also a purely discrete object, the two-dimensional global field K of functions on $\mathcal{S}$, but it is not really involved at this stage). Relations between various objects associated to $\mathbb{A}_{S_\prime}$ and the next level $\mathbb{B}_{S_\prime}$ are crucial for two-dimensional class field theory and zeta integral study. In section 32 we derive an analogue of the summation formula, whose left and right hand sides are integrals over finitely many curves, for elements in certain functional spaces on $\mathbb{A}$.

3-2. Part 2 introduces topological Milnor K_2-delic objects associated to $\mathcal{S}$, using the adelic objects of part 1 and local theory of [9]. It includes without proof the main theorem of two-dimensional class field theory for $\mathcal{S}$ in the language of K_2^t-delic objects. Abelian extensions of the field K of rational functions on $\mathcal{S}$ are described by open subgroups of $J_\mathcal{S}/P_\mathcal{S}$, where

$$J_\mathcal{S} = {\prod_y}' {\prod_{x\in y}}' K_2^t(K_{x,y}) \times {\prod_{\sigma,\omega\in S_0^\sigma}}' K_2^t(K_{\omega,\sigma}),$$

where $K_{x,y}$, $K_{\omega,\sigma}$ are two-dimensional local fields, or finite products of such fields, associated to points on curves and archimedean fibres, and

$$P_\mathcal{S} = \Delta {\prod_y}' K_2(K_y) + \Delta {\prod_x}' K_2(K_x) + \Delta \prod_\sigma K_2(K_\sigma),$$

where K_y, K_σ, and K_x are two-dimensional fields and rings associated to irreducible curves, archimedean fibres, and closed points, and Δ is the map induced by the diagonal map; see sections 24 and 33 for definitions. Keeping in mind the unramified theory and the following study of the zeta integral, we explain how to modify the class group $J_\mathcal{S}/P_\mathcal{S}$ to a more convenient J/P. The main theorem of part 2 is not used in the study of the zeta integrals. Analogously to the one-dimensional theories, the theory of zeta integrals employs some of adelic objects which naturally appear in class field theory, but none of the results.

3-3. For a scheme $\mathcal{S}$ of finite type its (arithmetic) zeta function is defined as the Euler product

$$\zeta_{\mathcal{S}}(s) = \prod_{x \in \mathcal{S}_0} (1 - |k(x)|^{-s})^{-1}$$

of factors corresponding to closed points x of $\mathcal{S}$, where $|k(x)|$ is the cardinality of the finite residue field at x.

Let $\mathcal{S} = \mathcal{E} \longrightarrow B$ be a two-dimensional, arithmetic scheme which is a regular model of an elliptic curve E over a global field k and which is proper over B. Up to a product $n_{\mathcal{E}}(s)$ of finitely many zeta functions of affine lines over finite fields associated to singular fibres of $\mathcal{E} \longrightarrow B$, $\zeta_{\mathcal{E}}(s)$ equals the Hasse–Weil zeta function

$$\zeta_E(s) = \frac{\zeta_B(s)\zeta_B(s-1)}{L_E(s)},$$

where $L_E(s)$ is the L-function of E and where ζ_B is the classical Dedekind zeta function ζ_{k} in characteristic zero and is the completed (at infinite valuations) zeta function of k, the function field of B, in positive characteristic. Traditionally, properties of the zeta function in dimension two are not studied directly, but via the study of the properties of the L-function, with the Galois group in the background being that generated over k by the torsion points of E. This text studies properties of the zeta function directly, using the commutative, two-dimensional adelic method, with the Galois group in the background now the maximal abelian extension of the field of rational functions on $\mathcal{E}$.

3-4. In the local theory of [11] we used a surjective homomorphism, natural from the point of view of class field theory, from $K_1 \times K_1$ of the ring of integers of rank one of a two-dimensional local field $K_{x,y}$ to the topological Milnor group $K_2^t(K_{x,y})$. We use it to integrate functions over $K_2^t(K_{x,y})$ via lifting them to functions on $K_1 \times K_1$ of the ring of integers. No information is lost as far as the unramified zeta integral is concerned. Using the local homomorphisms, we construct in part 3 an adelic homomorphism

$$\mathrm{t}\colon (\mathbb{A}_{S'} \times \mathbb{A}_{S'})^{\times} \longrightarrow J$$

which is canonical modulo units; see section 36. The homomorphism t is related to the symbol map $\mathbb{A}_{S,}^{\times} \otimes \mathbf{A}_{S,}^{\times} \longrightarrow J$ via a commutative diagramme in Lemma 36. This gives a certain relation between the multiplicative groups of two different integral structures on the surface.

Fix $S_\prime$ as the union of all fibres on $\mathcal{E}$ together with a finite set of horizontal curves, and work with $\mathbb{A} = \mathbb{A}_{S_\prime}$. We define certain subgroups T, T_1, T_0 of $(\mathbb{A} \times \mathbb{A})^\times$ with their measure and integration. In particular, the integration over T differs from the integration over $(\mathbb{A} \times \mathbb{A})^\times$; the measure on T_0 is the tensor product of lifts of rescaled discrete measures on the function fields of fibres and curves, and it differs from the lift of the discrete counting measure at the residue level. The objects T, T_1, T_0 are two-dimensional zeta integral analogues of the ideles, the ideles of module 1, and the multiplicative group of global elements in the one-dimensional case. The morphism $\mathfrak{t}$ sends T into the K_2^t-delic object J, and the local-global object T_0 to P modulo units. The kernel of $\mathfrak{t}$ is large, but it can be ignored in the unramified theory. We also define a subgroup $\mathfrak{T}$ of T, which differs at horizontal data.

For a function g in a two-dimensional Bruhat–Schwartz space and a continuous homomorphism $\chi\colon J/P \longrightarrow \mathbb{C}^\times$, we define an adelic zeta integral

$$\zeta(g,\chi) = \int_{\mathfrak{T}} g\,\chi_{\mathfrak{t}},$$

where $\chi_{\mathfrak{t}}$ is a certain version of $\chi \circ \mathfrak{t}$, modified at horizontal data; for the notation see sections 39 and 37.

The definition of the zeta integral entails that for reasonably nice functions g it factorizes into a product of zeta integrals on curves and fibres in $S_\prime$. The zeta integrals on fibres are different from the zeta integrals on horizontal curves. In general, the zeta integral takes values in $\mathbb{C}((X))$, but, for an unramified $\chi = |\,|_2^s$ and appropriate lift of functions at the residue level, its values are in $\mathbb{C}$.

3-5. Section 40 contains the first calculation of the zeta integral, under some restrictions on the types of singularities of the fibres. It shows that $\zeta(g, |\,|_2^s)$ on $\Re(s) > 2$ equals the product of an exponential factor $\mathfrak{c}_{\mathcal{E}}^{1-s}$, the square of the zeta function $\zeta_{\mathcal{E}}(s)$, and the product of the squares of the one-dimensional zeta integrals at $s/2$ of the horizontal curves in $S_\prime$, each of which satisfies a functional equation with respect to $s \to 2 - s$. In particular, through $\mathfrak{c}_{\mathcal{E}}$ we get an adelic interpretation of the non wild part of the conductors of the fibres. The reason why we add the horizontal data is to ensure that the image N of the module map $|\,|$ on T is a nice, locally compact group in characteristic zero, namely the multiplicative group of positive real numbers; the horizontal zeta integral contribution will then cancel out each other in the functional equation for $\mathcal{E}$. The set $S_\prime$ includes just finitely many horizontal curves to ensure that the product of the horizontal zeta integrals converges.

Using the summation formula from section 32 on infinitely many finite subsets of curves in $S_\prime$, we deduce in section 44 a two-dimensional theta formula

$$\int_{T_0} \big(f(\alpha\beta) - |\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta)\big)\, d\mu(\beta) = \int_{\partial T_0} \big(|\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta) - f(\alpha\beta)\big)\, d\mu(\beta).$$

Its shape reflects the difference between the additive structure and multiplicative structure (close to the unramified class field theory structure) in dimension two.

We perform the second calculation of the zeta integral $\zeta(f, |\,|_2^s)$ for a certain centrally normalized function f in section 45. This gives the decomposition

$$\zeta(f, |\,|_2^s) = \xi(|\,|_2^s) + \xi(|\,|_2^{2-s}) + \omega(|\,|_2^s)$$

on the half-plane $\Re(s) > 2$ (for notation see section 45). The function $\xi(|\,|_2^s)$ is an absolutely and uniformly convergent integral on the complex plane, and so it extends to an entire function on the complex plane. These two calculations of the zeta integral form a two-dimensional analogue for $\mathcal{E}$ of the (unramified) work of Tate and Iwasawa.

In section 46 we get an integral representation for the boundary term

$$\omega(|\,|_2^s) = \int_{N^-} h(n) n^{s-2}\, d\mu_{N^-}(n), \qquad h(n) = \int_{\mathfrak{T}_1} \big(n^2 f(\mathfrak{m}_n\gamma) - f(\mathfrak{m}_n^{-1}\gamma)\big)\, d\mu(\gamma),$$

where $h(n)$ is the boundary function, N^- is a subspace of the elements ≤ 1 of the measure space N such that N is the disjoint union of N^- and its inverse, and $\mathfrak{m}_n \in \mathfrak{T}$ are chosen such that $|\mathfrak{m}_n| = n^2$.

The integral representation of $\omega(|\,|_2^s)$ is actually a Laplace–Stieltjes transform: for example, in characteristic zero a logarithmic change of variable makes this integral the Laplace transform of the function $h(e^{-t})e^{2t}$. Using the relation between $\zeta(f, |\,|_2^s)$ and $\zeta_{\mathcal{E}}(s)$ we see that the zeros of the L-function of E and the poles of the zeta function essentially correspond to the poles of the boundary term. Compare this with the quite different one-dimensional situation, where finitely many poles of the corresponding boundary term match poles, and not zeros, of the twisted zeta function.

Using the two-dimensional theta formula of section 45 we obtain $h(n) = h_1(n) + h_2(n)$, where the integral $\int_{N^-} h_1(n) n^{s-2}\, d\mu_{N^-}(n)$ extends to an entire function on the complex plane and

$$h_2(n) = n^2 \int_{\mathfrak{T}_1/T_0} \int_{\partial T_0} \big(|\mathfrak{m}_n\gamma|^{-1} f(\mathfrak{m}_n^{-1}\nu^{-1}\gamma^{-1}\beta) - f(\mathfrak{m}_n\gamma\beta)\big)\, d\mu(\beta)\, d\mu(\gamma)$$

(see section 46 for the notation), so that the information about the analytic properties of the zeta integral is contained in $h_2(n)$. The latter involves an integral over the (weak) boundary ∂T_0 of the adelic object T_0, which explains the terminology 'boundary term' for $\omega(|\,|_2^s)$. All the information about the meromorphic continuation, functional equation, and poles of the zeta integral is contained in the boundary term $\omega(|\,|_2^s)$, which in turn is determined by the structure of the integral $h_2(n)$. Unlike the one-dimensional case, the boundary term in dimension two is highly nontrivial.

3-6. Part 4 contains first relations and applications of the previous theory to three fundamental aspects of the arithmetic of elliptic curves over global fields: the meromorphic continuation and functional equation of the zeta function, the location of its poles, and its behaviour at $s = 1$. For more on this see Analysis on arithmetic schemes. III.

We include several explicit descriptions of the zeta integral and function $h(n)$ in the 'classical' number theory language. In particular, we derive in 4.3 the following explicit formula for the zeta integral in characteristic zero when the set $S_{\prime}$ contains only one horizontal curve, the image of the zero section. Write a generalized Dirichlet series

$$\zeta_{\mathcal{E}}(s)^2 \mathfrak{c}_{\mathcal{E}}^{1-s} = \sum_{\mathfrak{n} \in \mathfrak{c}_{\mathcal{E}} \mathbb{N}} \frac{c(\mathfrak{n}^2)}{\mathfrak{n}^s}.$$

Then

$$\zeta_{\mathcal{E},S_{\prime}}(f, |\,|_2^s) = \mathfrak{e} \sum_{\mathfrak{n} \in \mathfrak{c}_{\mathcal{E}} \mathbb{N}} c(\mathfrak{n}^2) \int_0^\infty \int_0^\infty y_{a,\mathfrak{n}^2 a^{-1}}(n) n^s \, \frac{da}{a} \frac{dn}{n},$$

where

$$y_{a,b}(n) = \big(\Theta(n^2 a^2) - 1\big)\big(\Theta(n^2 b^2) - 1\big),$$

Θ is the theta-function of k, and $\mathfrak{e}$ is the square of the normalized classical measure on the idele class group of k. In this case,

$$h(n) = \mathfrak{e} \sum_{\mathfrak{n} \in \mathfrak{c}_{\mathcal{E}} \mathbb{N}} c(\mathfrak{n}^2) \int_0^\infty \big(n^2 y_{a,\mathfrak{n}^2 a^{-1}}(n) - y_{a,\mathfrak{n}^2 a^{-1}}(n^{-1})\big) \frac{da}{a}.$$

3-7. Aspects of the meromorphic continuation and functional equation of the zeta integral are discussed in sections 47 and 48. Since the zeta function of $\mathcal{E}$ does not remember much about automorphic properties of its denominator, the L-function of E, we need a replacement of automorphicity. We offer a new hypothesis of *mean-periodicity* in an appropriate space of complex valued functions on $\mathbb{R}$ or $\mathbb{Z}$ for the function $H(t)$, which is $h(e^{-t})$ in characteristic zero and $h(q^{-t})$, $t \in \mathbb{Z}$, in positive characteristic. Recall that a function l in a locally convex functional space X is called mean-periodic if its translations do not generate a dense subspace. The Hahn–Banach theorem on X shows that this is equivalent to l being a solution of a homogeneous convolution equation $l * \tau = 0$, where τ is a nonzero element of the dual space of X. In the presence of harmonic synthesis in X every mean-periodic function l in X is approximated by sums of exponential polynomials, each of which lies in the closure of the space generated by translations of l; this generalizes the classical Fourier series representation for periodic functions. See section 47 for more properties of mean-periodic functions. The Laplace–Stieltjes transform of a mean-periodic function of exponential growth has a meromorphic continuation to the complex plane, given by the Laplace–Carleman transform.

It is easy to see that $h(n^{-1})n = -h(n)n^{-1}$. In some sense the Laplace transform of h corresponds to the Mellin transform for modular L-functions, and the functional equation of h and the mean-periodicity of H are weak analogues of the modularity of L-functions.

In positive characteristic the function H is mean-periodic in the space of complex valued functions on $\mathbb{Z}$ of exponential growth. In characteristic zero H is expected to be mean-periodic in the space $\mathcal{C}^\infty_{\exp}(\mathbb{R})$ of smooth functions on $\mathbb{R}$ of exponential growth. The expected mean-periodicity of $H(t)$ would imply that the zeta integral extends to a meromorphic function and satisfies the functional equation. Hence we would get the same properties for the square of $\zeta_{\mathcal{E}}$ and of ζ_E; see section 48. One would also obtain a description of the poles of the zeta function on the critical line as the Carleman spectrum of the transform of a function related to h. The integral adelic representation for the function h and the two-dimensional theta formula will be of great help towards establishing the mean-periodic property of H.

In characteristic zero it is proved in [56] that if the zeta-function of E has a meromorphic continuation of the expected shape and satisfies the functional equation, then the corresponding function H is indeed mean-periodic in several functional spaces which include the space $\mathcal{C}^\infty_{\exp}(\mathbb{R})$. For modular curves the convolutor for H can be obtained using the theory of $GL(2)$ cuspidal automorphic representations, which in this sense is dual to the two-dimensional commutative theory; see [55].

More generally, to every zeta function of an arithmetic scheme in characteristic zero which extends to a meromorphic function on $\mathbb{C}$ of the expected shape and satisfies the functional equation, one can associate a mean-periodic function in $\mathcal{C}^\infty_{\exp}(\mathbb{R})$. For example, for the Riemann zeta function we get $x^{-1}(\vartheta(x^{-2}) - 1) - (\vartheta(x^2) - 1) = 1 - x^{-1}$ and the corresponding mean-periodic function is $1 - e^t$.

3-8. When one tries to meromorphically extend the zeta integral to the left, the issue of its poles come naturally into consideration. Relations of the study of the zeta integral to the (generalized) Riemann hypothesis for the zeta integral are discussed in sections 49–54 under the assumption that the set S_f contains exactly one horizontal curve, namely the image of the zero section. In sections 49–50 we describe the behaviour of the first three derivatives of $H(t)$ as $t \to \infty$. In section 51 we propose *hypothesis* $(*)$ which says that the fourth derivative of $H(t)$ keeps its sign for all sufficiently large t. This hypothesis is discussed for characteristic zero in section 52, and for positive characteristic in section 53. In section 54 we show that hypothesis $(*)$, together with the absence of noninteger, real poles for the zeta integral inside the critical strip (i.e. the real RH), imply the Riemann Hypothesis for the zeta integral. The real RH can be easily checked computationally for modular elliptic curves over the rationals ([48] includes the computational data for the curves of conductor < 8000). On the other hand, when $\mathrm{k} = \mathbb{Q}$ the analytic study of the function H in [54] shows that if $L_E(s)$ has a holomorphic continuation and satisfies the functional equation, the nonreal zeros of $L_E(s)$ on the critical line are of multiplicity not greater than the multiplicity at 1 (this is expected to be true) and some technical condition on the derivative of L_E holds, then the Riemann hypothesis for the zeta function implies hypothesis $(*)$.

In section 55 we discuss a relation between the Laplace–Carleman transform, the analytic continuations of the zeta integral and zeta function, and their functional equations. Without using mean-periodicity, but assuming hypothesis $(*)$ and non density of the Carleman spectrum of the Laplace–Stieltjes–Carleman transform of a function related to h, we describe another method to derive the analytic continuation and functional equation of the boundary term.

3-9. Another application of zeta integrals is to special values, as they give a new method of studying the local behaviour of the zeta function at $s = 1$. The point is that the analytic behaviour of the zeta integral at $s = 1$ is completely described by the behaviour of the boundary term at $s = 1$, and this latter term involves an integral over the boundary of T_0 whose structure incorporates the arithmetic rank. Assuming the meromorphic continuation and functional equation of the zeta function, a new method is sketched in section 58. It uses the relation between the multiplicative groups of the two adelic spaces indicated in *3-4*.

3-10. In section 57 we sketch modifications required to treat an arithmetic scheme corresponding to a curve of genus $g > 1$ over a global field. Since the zeta integral defined above diverges for such schemes, one has to implement a fibrewise renormalization, using the $(g-1)$st power of the zeta function of $\mathbb{P}^1(B)$.

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1. A-delic objects with respect to two-dimensional structure

Unlike the classical one-dimensional case where there is a unique adelic object $\mathbb{A}$, there are two quite different adelic objects associated to an arithmetic surface. The first two-dimensional adelic object $\mathbf{A}$, introduced by Parshin in positive characteristic, will be defined below in arbitrary characteristic; and we introduce and study a new two-dimensional adelic object $\mathbb{A}$. In the first approximation, the main difference between them is that one of the two adelic conditions for $\mathbf{A}$ is with respect to the integral structure of rank one, whereas a similar adelic condition for $\mathbb{A}$ is with respect to the integral structure of rank two. The bold adelic object is more suitable for geometric applications, including Poincare type dualities and relations with the Picard group at the multiplicative level, and is less suitable for arithmetic applications. In particular, one cannot integrate over $\mathbf{A}$, but one can integrate over $\mathbb{A}$. In 25 we define the adeles $\mathbb{A}_y$ associated to a curve y on $\mathcal{S}$ as a certain restricted product of local objects associated to points on the curve y. Its fraction version $\mathbf{A}_y$ is defined in 28. The adelic object $\mathbf{A}$ is a certain restricted product of all $\mathbf{A}_y$ with respect to the integral structures of rank one. Section 25 introduces the adeles $\mathbb{A}_{\mathcal{S}}(\{A\})$ associated to a choice of $O_{x,y}$-modules $A_{x,y}$ as a certain restricted product of $\mathbb{A}_y$. When each $A_{x,y}$ is chosen to be $O_{x,y}$ and when a set $S_\prime$ of curves on $\mathcal{S}$ includes only finitely many nonsingular horizontal curves, the corresponding object denoted by $\mathbb{A}_{S_\prime}$ or just $\mathbb{A}$ is a subspace of $\mathbf{A}$. We endow the adeles $\mathbb{A}_{S_\prime}$ with an appropriate topology. Using local characters, we get a pairing between $\mathbb{A}_{S_\prime}$ and a certain dual object $\mathbb{A}^{\circ}_{S_\prime}$; unlike the one-dimensional case the latter is not isomorphic to $\mathbb{A}_{S_\prime}$ when the set $S_\prime$ is infinite.

Sections 30, 31 and 32 contain a list of natural definitions of measures and integrals on the adelic objects. In 30, using the local theory of [11], we define a normalized measure on $\mathbb{A}$ and three functional spaces $R'_{\mathbb{A}}$, $R_{\mathbb{A}}$, $Q_{\mathbb{A}}$. Then we define and study integration on $\mathbb{A}$ and Fourier transform for functions on it. We get an analogue of the familiar formula for the Fourier transform of functions in $Q_{\mathbb{A}}$, just by reducing to the one-dimensional case.

Providing the most general constructions is not the aim of this work, since this would make the text too long. We choose those pieces of the general theory which are relevant for the study of the zeta integral. In 31 we extend the previous definitions to adelic groups $\mathbb{A}^{\times}$, $(\mathbb{A}\times\mathbb{A})^{\times}$. Measure and integration on $\mathbb{B}$, $\mathbb{B}\times\mathbb{B}$ are defined, and then we prove a summation formula in 32.

Now for the reader's convenience we briefly review parts of the local theory of [11].

For an introduction to higher dimensional local fields see [68] and other papers in [15]. Let F be a two-dimensional local field. Assume that the residue field E

of F is a nonarchimedean local field. Denote by t_2 a local parameter of F and by t_1 a lift of a local parameter of the residue field. Denote by $\mathcal{O}$ the ring of integers of F with respect to its discrete valuation of rank one. Denote by O the ring of integers of F with respect to any of its discrete valuations of rank two, for example corresponding to the choice of local parameters t_2, t_1; O does not depend on the choice of discrete valuation. When we pass to the adelic theory, the two different integral structures $\mathcal{O}$ and O give two different adelic objects $\mathbf{A}$ and $\mathbb{A}$ on $\mathcal{S}$.

Define a function μ on the ring $\mathcal{A}$ of sets generated by the closed balls $a + t_2^i t_1^j O$ with respect to the rank two integral structure (this ring of sets does not depend on the choice of discrete valuation of rank two)

$$\mu(a + t_2^i t_1^j O) := q^{-j} X^i,$$

where q is the cardinality of the residue field of E. The function μ is well defined, translation invariant and finitely additive; it depends on the choice of discrete valuation of rank two. Moreover, μ is countably additive in the following refined sense. Call a series $\sum \alpha_n$, $\alpha_n = \sum a_{i,n} X^i \in \mathbb{C}((X))$, absolutely convergent in the two-dimensional local field $\mathbb{C}((X))$ if there is i_0 such that $a_{i,n} = 0$ for all $i < i_0$ and all n and if for every i the series $\sum_n a_{i,n}$ absolutely converges in $\mathbb{C}$. Then the measure μ is countably additive in the following refined sense: for countably many disjoint sets A_n in $\mathcal{A}$ such that $\cup A_n \in \mathcal{A}$ and $\sum \mu(A_n)$ absolutely converges in $\mathbb{C}((X))$, we have $\mu(\cup A_n) = \sum \mu(A_n)$. See [11] and [12] for more details.

To define the space of integrable functions, we first consider the space R_F of all functions $f\colon F \longrightarrow \mathbb{C}((X))$ which can be written as a sum of a function which is zero outside finitely many points and of $\sum c_n char_{A_n}$ with countably many disjoint measurable sets A_n, $c_n \in \mathbb{C}((X))$, such that the series $\sum c_n \mu(A_n)$ absolutely converges in $\mathbb{C}((X))$ with respect to its two-dimensional topology. Then we define $\int f\, d\mu = \sum c_n \mu(A_n)$; this definition is consistent. For example, $\int char_O d\mu = 1$, $\int char_{\mathcal{O}} d\mu = 0$. In order to have an analogue of the Fourier transform, we have to extend this class of integrable functions to include functions of type $\alpha \mapsto \psi(\beta\alpha)$, where ψ is a continuous character of F with conductor O. On this larger space we then define the integral and check its consistency.

In particular, if a function $f_1\colon E \longrightarrow \mathbb{C}$ is (absolutely) integrable over E with respect to the normalized Haar measure μ_E, then the function $f_1 \circ p$ extended by zero outside the ring of integers $\mathcal{O}$ with respect to the discrete valuation of rank one on F is integrable and $\int_{\mathcal{O}} f_1 \circ p\, d\mu = \int_E f_1\, d\mu_E$. Denote by Q_F the subspace of integrable functions consisting of functions f with support in $\mathcal{O}$ and such that $f|_{\mathcal{O}} = g \circ p|_{\mathcal{O}}$ for a Bruhat–Schwartz complex valued function g on E.

Now for an integrable function f define its transform

$$\mathcal{F}(f)(\beta) = \int_F f(\alpha)\psi(\alpha\beta)\,d\mu(\alpha).$$

Given $f \in Q_F$, the function $\mathcal{F}(f)$ belongs to Q_F, and by reducing to the one-dimensional case one easily gets the familiar double transform formula $\mathcal{F}^2(f)(\alpha) = f(-\alpha)$.

For two-dimensional local fields whose residue field E is an archimedean local field, the ring of measurable sets is generated by $B = a + t^i D + t^{i+1}K[[t]]$ where D is an open ball in E and t is a local parameter. The measure is a translation invariant additive measure μ on this ring such that $\mu(B) = \mu_E(D)X^i$ where μ_E is the ordinary Lebesgue measure on E if E is real, and is twice the ordinary Lebesgue measure on E if E is complex. Define the character $\psi\colon E((t)) \longrightarrow \mathbb{C}^\times$ by $\sum a_i t^i \mapsto \exp(-2\pi i \operatorname{Tr}_{E/\mathbb{R}}(a_0))$. The minus sign is to make the theory compatible with that of Tate. The transform of an integrable function f is defined by the same formula as above.

See [11], [12], and [39] for a closely related but different approach which systematically uses lifting procedure from a locally compact residue level.

1.1. Basic objects

24. BASIC NOTATION. The following notation will be used throughout the text. The notation matches those in [11] and [12]. It slightly differs from some of the notation in [14] which has been partially simplified: most importantly, we do not use rescaling maps $\mathfrak{o}$ and $\mathfrak{o}'$ in the current text, see in particular Remark 3 in 39 and also Remark 2 in 36.

Let $\mathcal{S}$ be a scheme of finite type. Various data sets associated to the scheme $\mathcal{S}$ are denoted by S_*, S^*. In particular, for $i \geq 0$ the notation S_i stands for the set of all s in $\mathcal{S}$ whose closure $\overline{\{s\}}$ viewed as a closed integral subscheme has dimension i. *Closed points are often called just points. The closure of an element of S_1 is called a curve and often denoted by y. Thus every curve on $\mathcal{S}$ in this text is irreducible.*

Let B be an open subscheme of the following one-dimensional scheme $\mathcal{B}$: the spectrum of the ring of integers of a global number field k in characteristic zero or a proper smooth connected curve over a finite field with function field k in positive characteristic.

For a smooth projective geometrically irreducible curve over k, let $\mathcal{S}$ be an arithmetic scheme (surface) which is a proper regular model of the curve, i.e. a regular integral scheme which is proper and flat over B with fibre dimension 1 whose generic fibre is the curve. In this text *we assume that B is the complete one-dimensional scheme $\mathcal{B}$.*

Every curve on $\mathcal{S}$ is either a component of a fibre $\mathcal{S} \longrightarrow B$ or it projects onto the whole B. In the former case it is called a vertical curve, and in the latter a horizontal curve. Every horizontal curves is the closure on the surface of a closed point of the generic fibre. Objects of this paper are defined for the reduced structures of subschemes of $\mathcal{S}$. When we talk about a fibre $\star$ we always mean the *reduced scheme associated to the fibre*. To keep track of multiplicities of components we use the terminology 'special fibre $\mathcal{S}_b = \mathcal{S} \times_B k(b)$ over $b \in B_0$' and 'geometric fibre $\mathcal{S} \times_B k(b)^{\mathrm{sep}}$'. For a fibre $\star$, the notation $y \subset \star$ stands for an irreducible component $y \in S_1$ of the fibre.

We will work with a set $S_{\prime}$ of some of fibres and horizontal curves $\star$ on the surface, on which we impose more and more restrictions in 29, 35, 36, 40. Often we work with infinite products $\prod_{x\in\star}, \prod_{\star\in S_{\prime}}$ which are the limits (if they exist) $\lim_{X_0}\prod_{x\in X_o}, \lim_{S_o}\prod_{\star\in S_o}$ over the partially ordered set of finite subsets X_o of the set of closed points of $\star$ and finite subsets S_o of $S_{\prime}$.

Fields and rings associated to $\mathcal{S}$, y, x.

By K we denote the function field of $\mathcal{S}$. It is a two-dimensional global field. Its abelian extensions are described in two-dimensional class field theory, see Theorem 34 for an adelic statement.

For a curve y denote by K_y the field of fractions of the completion $\mathcal{O}_y$ of the local ring of $\mathcal{S}$ at y; we do not use the traditional hat notation for complete objects. Denote by $\mathcal{M}_y$ its maximal ideal. The field K_y is a complete discrete valuation field with residue field $k(y)$. A prime element of K_y (local parameter of y) will be denoted by t_y.

For every $x \in S_0$ denote $\mathcal{O}_x$ the completion of the local ring of $\mathcal{S}$ at x. Its residue field is denoted by $k(x)$. This is a finite field of cardinality q_x. Denote by K'_x the field of fractions of $\mathcal{O}_x$ and by K_x its subring generated by K and $\mathcal{O}_x$. In 36 we also define a subring Q_x of K'_x for a singular point x of a fibre $\star$.

For a place v of the one-dimensional global field k let k_v denote the completion of k at v. Let $\mathcal{S}^v = \mathcal{S} \times_B O_{\mathrm{k}_v}$ and denote its special fibre by $\mathcal{S}_v = \mathcal{S} \times_B k(v)$. The completion of the local ring of $\mathcal{S}^v$ at a closed point x of $\mathcal{S}_v$ is naturally isomorphic to $\mathcal{O}_x$. Denote by K_v the field of functions of the generic fibre $\mathcal{S}_{\mathrm{k}_v} = \mathcal{S} \times_B \mathrm{k}_v$ of $\mathcal{S}^v$. For an archimedean place σ of k let k_σ be the completion of k with respect to σ. Let $\mathcal{S}^\sigma = \mathcal{S} \times_B \mathrm{k}_\sigma$, $K_\sigma = k(\mathcal{S}^\sigma)$. For a closed point $\omega \in S_0^\sigma$ let $K_{\omega,\sigma}$ be the fraction field of the completion of the local ring of $\mathcal{S}^\sigma$ at ω; it is a two-dimensional local field. Denote by $\mathcal{O}_{\omega,\sigma}$ the ring of integers of $K_{\omega,\sigma}$, choose a local parameter t_ω with respect to the discrete valuation of rank one, and denote by $k_\sigma(\omega)$ its archimedean residue field. Choosing the coefficient subfield in $K_{\omega,\sigma}$ we get an isomorphism of the latter two-dimensional local field and $k_\sigma(\omega)((t_\omega))$.

Let $\star$ be a curve or a fibre on $\mathcal{S}$. For a closed point x in $\star$ denote by $\star(x) \subset (\operatorname{Spec}\mathcal{O}_x)_1$ the disjoint union of local branches $z \in y(x)$ of y at x, where y runs through all components of $\star$. If x is a nonsingular point of $\star$ then $\star(x)$ is a one element set. If x is a singular point of a curve $\star$ then its local branches z at x correspond to minimal prime ideals of the completion of the local ring of $\star$ at x and they also correspond to maximal ideals of its integral closure.

In part 1 and 2 of this text we use the following notation. For a map $\mathcal{K}$ from the set $\{(x,z)\}$, where x a closed point, $z \in \star(x)$ for a curve or fibre $\star$ on $\mathcal{S}$, to abelian groups or measures/functions, $\mathcal{K}_{x,\star}$ denotes $\prod_{z\in\star(x)}\mathcal{K}_{x,z}$ for groups and $\otimes_{z\in\star(x)}\mathcal{K}_{x,z}$ for measures and functions. There will be one exception from this rule in part 1: the object in roman font $\mathrm{O}_{x,\star}$ defined in 25 for a singular $x \in \star$. In part 3 and 4 of this work there will be several objects which do not follow the rule: e.g. the integral $\int_{T_{x,\star}}$ for singular $x \in \star$ differs from the product of (x,z)-objects and integrals, where z runs through $\star(x)$. In part 3, when all singular points are ordinary double points and $\mathcal{K}_{x,z}$ does not depend on the choice of a local branch z at x, we occasionally use the notation $\prod_{x\in\star}\mathcal{K}_{x,z}$ for the product of $\mathcal{K}_{x,z}$ where only one local branch z (it does not matter which one) of $\star$ at x is chosen for each $x \in \star$.

For a local branch z of $\star$ at x denote by $K_{x,z}$ the fraction field of the completion of the localization of $\mathcal{O}_x$ with respect to the corresponding prime ideal of $\mathcal{O}_x$. The field $K_{x,z}$ is a *two-dimensional local field*, see e.g. [15]. According to the notation of the previous paragraph we get the ring $K_{x,\star} = \prod_{z\in\star(x)} K_{x,z}$ which is associated to the data $x \in \star$. If x is a nonsingular point of $\star$ then $K_{x,\star} = K_{x,z}$ is a field. A local parameter t_y of K_y where y is a component of $\star$, $x \in y$, can be used as a main local parameter $t_{2\,x,z}$ of $K_{x,z}$.

Denote by $O_{x,z}$ be the ring of integers of $K_{x,z}$ with respect to the two-dimensional structure. Denote by $\mathcal{O}_{x,z}$ the ring of integers in $K_{x,z}$ with respect to the discrete valuation of rank one, and by $\mathcal{M}_{x,z}$ its maximal ideal. Denote by $E_{x,z} = \mathcal{O}_{x,z}/\mathcal{M}_{x,z}$ the first residue field of $K_{x,z}$. This is a locally compact field. The ring of integers of $E_{x,\star}$ is a semilocal ring isomorphic to the product of the ring of integers of $E_{x,y}$, y runs through components of $\star$. The ring of integers of $E_{x,\star}$ is the completion with respect to the intersection of its maximal ideals of the normalization of the local ring of y at x and it is isomorphic to the product of the completions of the local rings of the normalization $\hat{y}$ of y at preimages of x.

For a local branch z at x of a curve or a fibre $\star$ denote by $t_{1\,x,z}$ a lift in $\mathcal{O}_{x,z}$ of a local parameter (uniformizer) of $E_{x,z}$. Denote by $k_z(x)$ the quotient field of $O_{x,z}$ by its maximal ideal; it coincides with the residue field of $E_{x,z}$, let $q_{x,z}$ be the number of its elements. The finite field $k_z(x)$ is a finite extension of the field $k(x)$. We call a closed point x of $\star$ *split or totally rational* if the residue field $k(x)$ coincides with the residue field $k_z(x)$ for all local branches z of $\star$ at x. In particular, every regular point x of $\star$ is split. If x is a *split ordinary double point* of $\star$ then the completion of the local ring of $\star$ at x is isomorphic to the quotient of the integral formal power series over $k(x)$ in two variables by the ideal generated by their product.

Let y be a horizontal curve and let σ be an archimedean place of k. The ring $k(y) \otimes_k k_\sigma$ is the direct sum of the completions $k(y)_\Sigma$ of $k(y)$ with respect to all places Σ of $k(y)$ which extend the place σ, and is the direct sum of fields $k_\sigma(\omega)$ where ω runs through closed points of $\mathcal{S}^\sigma$ which are mapped to y under $\mathcal{S}^\sigma \longrightarrow \mathcal{S}$. We have natural identification of the two sets, $\Sigma \leftrightarrow \omega$, and without loss of generality we use the notation ω for Σ. Denote by $K_{\omega,y}$ the corresponding two-dimensional local field $k(y)_\omega((t_y))$ associated to the curve y and its archimedean place ω. The archimedean local field $k(y)_\omega$ coincides with $k_\sigma(\omega)$, and the field $K_{\omega,y}$ is isomorphic to $K_{\omega,\sigma}$. Denote by $\mathcal{O}_{\omega,y} = \mathcal{O}_{\omega,\sigma}$ its ring of integers. It will be convenient to put $O_{\omega,y} = \mathcal{O}_{\omega,y}$.

We have natural embeddings

$$K_x \longrightarrow K_{x,y}, \quad K_y \longrightarrow K_{x,y}, \quad K_y \longrightarrow K_{\omega,y}, \quad K_\sigma \longrightarrow K_{\omega,\sigma}.$$

For more properties of the local objects in relation to each other see e.g. §1 Ch. II of [29].

From now on for a horizontal curve y we include the objects associated to the archimedean points ω on y, like the field $K_{\omega,y}$, etc. In the list of data associated to y, we will often use the notation x, y for ω, y, and in particular this field will be among the fields $K_{x,y}$ associated to y.

When $x \in S_0, y \in S_1$, $x \in y$ we often call (x, y) a nonarchimedean datum, sometimes using abbreviation na *in displayed formulas; when the first residue of the corresponding $K_{x,y}$ is archimedean, abbreviation* a *is sometimes used. The notation* ns *is used for nonsingular and acrhimedean points of a curve or fibre, and* s *for singular points of a curve or fibre.*

Everywhere below in the text $\star$ stands for a fibre or a horizontal curve on $\mathcal{S} \longrightarrow B$.

25. We aim to define the adeles $\mathbb{A}$, $\mathbb{B}$, $\mathbf{A}$, $\mathbf{B}$ mentioned in the introduction. The first two will be defined in 1.1; the last two in 1.2.

The adelic object $\mathbb{A}_y^r$ for a curve y is informally

$$\mathbb{A}_y^r = \Big\{ \big(\sum_{i \ge r} a_{i,x} t_y^i\big)_{x \in y} = \sum_{i \ge r} a_i t_y^i : a_i = (a_{i,x})_{x \in y} \in (\mathcal{O}_{x,y})_{x \in y} \text{ are nice lifts of } \overline{a_i} \in \mathbb{A}_{k(y)} \Big\}.$$

As usual, there is some little extra work to be done in mixed characteristic.

DEFINITION OF $\mathbb{A}_y^r$. First let y be a nonsingular curve.

In equal characteristic let $l_y\colon k(y) \longrightarrow \mathcal{O}_y$ be the lifting which corresponds to a choice of a subfield of $\mathcal{O}_y$ which is projected isomorphically onto $k(y)$. Define

$$l_y^n\colon k(y)^{\oplus n} \longrightarrow \mathcal{O}_y, \quad (c_1,\ldots,c_n) \mapsto \sum_{i=1}^n l_y(c_i) t_y^{i-1}.$$

In mixed characteristic we follow a well known standard procedure. Fix a p-base $\overline{a_1},\ldots,\overline{a_r}$ of $k(y)$ and lifts a_i to $\mathcal{O}_y$, and define an additive map

$$k_n\colon k(y) \longrightarrow \mathcal{O}_y/p^n\mathcal{O}_y, \quad \sum \overline{a_1}^{i_1}\ldots\overline{a_r}^{i_r}\overline{b_{i_1,\ldots,i_r}}^{p^n} \mapsto \sum a_1^{i_1}\ldots a_r^{i_r} b_{i_1,\ldots,i_r}^{p^n} + p^n\mathcal{O}_y,$$

where b is any lift of $\overline{b}$. This map does not depend on the choice of lifts. Define an additive map

$$l_y^n\colon k(y)^{\oplus n} \longrightarrow \mathcal{O}_y/t_y^n\mathcal{O}_y, \quad (c_1,\ldots,c_n) \mapsto \sum_{i=1}^n k_n(c_i) t_y^{i-1} + t_y^n\mathcal{O}_y.$$

In arbitrary characteristic for every $x \in y$ define similarly the local maps

$$l_{x,y}^n\colon E_{x,y}^{\oplus n} \longrightarrow \mathcal{O}_{x,y}$$

(resp. $\longrightarrow \mathcal{O}_{x,y}/t_y^n\mathcal{O}_{x,y}$). We also get the adelic map

$$L_y^{n,r} = \oplus_{x\in y} t_y^r l_{x,y}^{n-r}\colon \mathbb{A}_{k(y)}^{\oplus n-r} \longrightarrow (K_{x,y})_{x\in y}$$

(resp. $\longrightarrow (t_y^r\mathcal{O}_{x,y}/t_y^n\mathcal{O}_{x,y})_{x\in y}$).

Now define

$$\mathbb{A}_y^r := \big\{(a_{x,y})_{x\in y} : a_{x,y} \in K_{x,y}, (a_{x,y}) + t_y^n\mathcal{O}_y \in \operatorname{im}(L_y^{n,r}) \text{ for every } n \geq 1\big\} \subset \prod_{x\in y} K_{x,y}.$$

Observe that the construction of the object $\mathbb{A}_y^r$ works for every complete discrete valuation field L whose residue field is a one-dimensional global field k and we similarly get the adelic object $\mathbb{A}_L^r$. Then $\mathbb{A}_{K_y}^r = \mathbb{A}_y^r$.

For a singular curve y define

$$\mathbb{A}_y^r := \{(a_{x,y})_{x\in y} : a_{x,y} = (a_{x,z})_{z\in y(x)} \in K_{x,y}, (a_{x,y}) + t_y^n\mathcal{O}_y \in \operatorname{im}(L_y^{n,r}) \text{ for every } n \geq 1\}.$$

This adelic object can also be denoted $\mathbb{A}_{K_y}^r$, it depends on K_y and r only.

If $k(y)$ is of positive characteristic, it is easy to see that $\mathbb{A}_y^r$ does not depend on the choice of t_y. The previous construction shows that $\mathbb{A}_y^r = \varprojlim_{m\ge 0} \mathbb{A}_y^r/\mathbb{A}_y^{r+m}$ and $\mathbb{A}_y^r/\mathbb{A}_y^{r+m}$ is isomorphic to $\mathbb{A}_{k(y)}^{\oplus m}$ as an additive group. In the positive characteristic case one can endow $\mathbb{A}_{k(y)}^{\oplus m}$ with a suitable structure of graded ring so that it is isomorphic to $\mathbb{A}_y^r/\mathbb{A}_y^{r+m}$.

DEFINITION OF $\mathbb{A}_y$, p_y. Put $\mathbb{A}_y := \mathbb{A}_y^0$. Denote by $p = p_y\colon \mathbb{A}_y \longrightarrow \mathbb{A}_{k(y)}$ the residue homomorphism $(a_{x,y}) \mapsto (\overline{a_{x,y}})$ induced by the residue maps $\mathcal{O}_{x,z} \longrightarrow E_{x,z}$, $\mathcal{O}_{\omega,y} \longrightarrow k(y)_\omega$.

DEFINITION OF $\mathbb{A}_\sigma$. For an archimedean place σ let $\mathbb{A}_\sigma$ be the restricted product of $K_{\omega,\sigma}$ with respect to $\mathcal{O}_{\omega,\sigma}$.

DEFINITION OF $\mathcal{O}_\star$, $K_\star$, $k(\star)$, $t_\star$, $\mathrm{O}_{x,\star}$. For a closed point x of $\star$ the objects $O_{x,\star}$, $\mathcal{O}_{x,\star}$, $K_{x,\star}$ are defined by the agreement in 24.

Denote by $\mathcal{O}_\star$ the completion of the product of the local rings of components y of $\star$ with respect to the $\mathfrak{r}$-adic topology, $\mathfrak{r}$ is the intersection of maximal ideals of the product. The ring $\mathcal{O}_\star$ can be identified with the product of $\mathcal{O}_y$ with y running through components of $\star$. Denote by $K_\star$ its fraction ring and by $k(\star)$ the ring of rational functions on $\star$.

Choose a local parameter $t_\star$ of $\star$, i.e an element of $\mathcal{O}_\star$ which serves as a local parameter in all $\mathcal{O}_y$, $y \subset \star$. We will use $t_\star$ as the t_2-parameter of all two-dimensional local fields $K_{x,z}$, $z \in \star(x)$.

If $\star$ is a fibre, denote by $\mathrm{O}_{x,\star}$ the sum of the diagonal image of $\mathcal{O}_x$ in $\mathcal{O}_{x,\star}$ and of $t_\star\mathcal{O}_{x,\star}$; this is a subring of $O_{x,\star}$. The ring $\mathrm{O}_{x,\star}$ coincides with the preimage with respect to the residue map $\mathcal{O}_{x,\star} \longrightarrow E_{x,\star}$ of the image of completion of the local ring of $\star$ at x in $E_{x,\star}$.

If $\star$ is a horizontal curve, denote by $\mathrm{O}_{x,\star}$ the preimage in $\mathcal{O}_{x,\star}$ with respect to the residue map of the image of the completion of the local ring of $\star$ at x in $E_{x,\star}$.

We have $\mathrm{O}_{x,\star} = O_{x,\star}$ if and only if x is a nonsingular point of $\star$.

Let x be an ordinary double point of a fibre $\star$. Then the local ring of $\star$ at x consists of rational functions which give two integral elements of $E_{x,z}$ and $E_{x,z'}$ with equal free terms. The subring $\mathrm{O}_{x,\star}$ of $O_{x,\star}$ is the preimage with respect to the residue map of the set of pairs of integral power series with equal free terms.

DEFINITION OF $\mathbb{A}_\star$, $\mathbb{A}_\star^r$ $p_\star$. Let $\star$ be a fibre. Denote by $\mathbb{A}_\star \subset \prod_{x\in\star} K_{x,\star}$ the adeles

$$\mathbb{A}_\star = \prod_{y\subset\star} \mathbb{A}_y,$$

where y runs through components of $\star$. Similarly to $\mathbb{A}_y$ above, the ring $\mathbb{A}_\star$ can be viewed as associated to $K_\star$, and can be denoted $\mathbb{A}_{K_\star}$.

Denote $\mathbb{A}^r_\star = \prod_{y\subset\star}\mathbb{A}^r_y$.

The maps $(p_y)_{y\subset\star}$ induce the surjective homomorphism

$$p = p_\star\colon \mathbb{A}_\star \longrightarrow \mathbb{A}_{k(\star)} := \prod_{y\subset\star}\mathbb{A}_{k(y)}.$$

DEFINITION OF $O\mathbb{A}_\star, \hat{O}\mathbb{A}_\star$. For a fibre $\star$ or a horizontal curve put

$$O\mathbb{A}_\star := \Big(\prod_{x\in\star,\mathrm{ns}} O_{x,\star} \times \prod_{x\in\star,\mathrm{s}} \mathrm{O}_{x,\star}\Big) \cap \mathbb{A}_\star,$$

see 24 for the notation ns and s; recall that we also agreed in 24 that $O_{\omega,y} = \mathcal{O}_{\omega,y}$.

Define

$$\hat{O}\mathbb{A}_\star := \prod_{x\in\star} O_{x,\star} \cap \mathbb{A}_\star.$$

Unlike $\mathbb{A}_\star$ and $\hat{O}\mathbb{A}_\star$, the object $O\mathbb{A}_\star$ keeps track of singular points of $\star$.

Recall that for a countable set of groups G_i, $i \in I$, and their subgroups H_i (defined for almost all i) the restricted product $G = \prod' G_i$ with respect to (H_i) is the subgroup of the product of G_i of elements (g_i) such that $g_i \in H_i$ for almost all i. For a finite subset J of I denote $G_J = \prod_{i\in J} G_i \times \prod_{i\notin J} H_i$. Then $G = \varinjlim G_J$.

DEFINITION OF $\mathbb{A}_\mathcal{S}(\{A\})$. For every curve y and nonarchimedean $x \in y$ fix $O_{x,z}$-modules $A_{x,z} \subset K_{x,z}$ of rank one such that they are commensurable with $O_{x,z}$ (two groups are called commensurable if their intersection is of finite index in each of them) and the module $A_{x,y}$ equals $O_{x,y}$ for almost all $x \in y$. For archimedean ω on horizontal y let $A_{\omega,y} = \mathcal{O}_{\omega,y}$.

So if $\star$ is a fibre or a horizontal curve, then $(A_{x,y})_{x\in\star}$ can be written as $\alpha_\star\,\hat{O}\mathbb{A}_\star$ for an appropriate $\alpha_\star \in \mathbb{A}^\times_\star$.

The following two-dimensional adelic object takes into account the integral structure of rank two on the surface and includes the data for 'archimedean fibres' $\mathcal{S}^\sigma$

$$\begin{aligned}\mathbb{A}_\mathcal{S}(\{A\}) &= \prod_\star{}' \mathbb{A}_\star \times \prod \mathbb{A}_\sigma \\ &= \{(a_{x,\star})_{x\in\star},\quad \text{where } \star \text{ runs through fibres or horizontal curves}\} \\ &\quad\times\{(b_{\omega,\sigma})_{\omega\in\mathcal{S}^\sigma_0},\quad \text{where } \sigma \text{ runs through archimedean places of k}\}\end{aligned}$$

such that

for all $\star$ the element $(a_{x,\star})_{x\in\star}$ belongs to $\mathbb{A}_\star$,
$(b_{\omega,\sigma})_{\omega\in S_0^\sigma} \in \mathbb{A}_\sigma$,
for almost all $(x,\star)$ the element $a_{x,\star}$ belongs to $A_{x,\star}$.
In particular for almost all $\star$ the element $(a_{x,\star})_{x\in\star}$ belongs to $\alpha_\star \hat{O}\mathbb{A}_\star$ with $\alpha_\star$ as above.

Thus, the part of $\mathbb{A}_{\mathcal{S}}(\{A\})$ outside 'archimedean fibres' is the restricted product of $\mathbb{A}_\star$ with respect to $\mathbb{A}_\star(\{A\}) = \mathbb{A}_\star \cap \prod A_{x,y}$.

DEFINITION OF $\mathbb{A}_{S_{\prime}}(\{A\})$. Choose a set $S_{\prime}$ of some fibres and some horizontal curves on $\mathcal{S}$.

Let $A_{x,y}$ be as in the previous definition. Define

$$\mathbb{A}_{S_{\prime}}(\{A\}) = \mathbb{A}_{\mathcal{S}}(\{A\}) \cap \prod_{\star\in S_{\prime}} \mathbb{A}_\star.$$

Note that $\mathbb{A}_{S_{\prime}}(\{A\})$ does not include any of the archimedean fibre data.

For a fixed $\star$ and $(a_{x,\star})_{x\in\star} \in \mathbb{A}_\star$ for almost all $x \in \star$ the element $a_{x,\star}$ belongs to $A_{x,\star}$. Hence, if $S_{\prime}$ is finite then $\mathbb{A}_{S_{\prime}}(\{A\}) = \prod_{\star\in S_{\prime}} \mathbb{A}_\star$; if $S_{\prime}$ contains infinitely many fibres, the equality does not hold.

We will fix the choice of modules $A_{x,y}$ in 27.

26. Suppose that H_i are open subgroups of G_i. Define the topology of the restricted product $\prod' G_i$ with respect to (H_i) as the sequential saturation topology [10] (i.e. the strongest topology with the same set of convergence sequences) of the translation invariant topology in which the fundamental system of neighbourhoods of the identity element is formed by subsets $\prod_{i\in J} V_i \times \prod_{i\notin J} H_i$ of G_J, where J runs through finite subset of I and V_i are neighbourhoods of the identity element in G_i.

DEFINITION OF THE TOPOLOGY OF $\mathbb{A}_{\mathcal{S}}(\{A\})$. For a fibre or a horizontal curve $\star$ define the topology on $\mathbb{A}_\star$ as the sequential saturation topology of the following translation invariant topology: it has $\prod_{x\in\star} W_{x,\star} \cap \mathbb{A}_\star$ as a fundamental system of neighbourhoods of zero, where $W_{x,\star}$ are open neighbourhoods of zero in $K_{x,\star}$ with respect to its topology (see e.g. [10]) and $W_{x,\star} = A_{x,\star}$ for almost all $x \in \star$. Then a subgroup of $\mathbb{A}_\star$ is open if and only if it is of the form $\prod_{x\in\star} W_{x,\star} \cap \mathbb{A}_\star$ where $W_{x,\star}$ are open subgroups of $K_{x,\star}$ and $W_{x,\star} = A_{x,\star}$ for almost all $x \in \star$.

The ring of integers $O_{x,z}$ with respect to the two-dimensional structure and the modules $A_{x,z}$ are open subsets in the ring of integers $\mathcal{O}_{x,z}$, and $O\mathbb{A}_\star$ and $\prod_{x\in\star} A_{x,\star} \cap \mathbb{A}_\star$ are open in $\mathbb{A}_\star$.

Endow $\mathbb{A}_S(\{A\})$ with the sequential saturation topology of the following translation invariant topology: it has $\left(\prod W_{x,y} \times \prod W_{\omega,\sigma}\right) \cap \mathbb{A}_S(\{A\})$ as a fundamental system of neighbourhoods of zero, where $W_{x,y}$ (resp. $W_{\omega,\sigma}$) are open neighbourhoods of zero in $K_{x,y}$, for almost all nonarchimedean $(x,\star)$ (resp. for almost all (ω,σ)) equal to $A_{x,\star}$ (resp. $A_{\omega,\sigma}$).

Thus, restricting to the vertical part of $\mathbb{A}_S(\{A\})$, its subgroup is open if and only if it is of the form $\prod_\star A_\star$ where $A_\star$ is an open subgroup of $\mathbb{A}_\star$ and $A_\star = \prod_{x\in\star} A_{x,\star} \cap \mathbb{A}_\star$ for almost all vertical $\star$.

One easily checks that every continuous homomorphism $\psi\colon \mathbb{A}_S(\{A\}) \longrightarrow \mathbb{C}^\times$ is trivial on almost all $A_{x,y}$, $\mathcal{O}_{\omega,\sigma}$, and is the product of local homomorphisms $\psi_{x,y} = \psi|_{\mathcal{O}_{x,y}}$, $\psi_{\omega,\sigma} = \psi|_{K_{\omega,\sigma}}$.

Definition of $\mathbb{B}_y$, $\mathbb{B}_\star$, $\mathbb{B}_S(\{A\})$, $\mathbb{B}_{S_\prime}(\{A\})$. If y is a horizontal curve or a nonsingular vertical curve, define $\mathbb{B}_y$ as the intersection of the diagonal image of K_y in $\prod K_{x,y}$ with $\mathbb{A}_y$, so it is equal to $\mathcal{O}_y$. For a singular fibre $\star$ define $\mathbb{B}_\star$ as the intersection of the diagonal image of $K_\star$ in $\prod K_{x,\star}$ with $\mathbb{A}_\star$. So $\mathbb{B}_\star$ can be identified with the product of $\mathbb{B}_y$ where y runs through components of $\star$.

Define $\mathbb{B}_\sigma$ as the intersection of the diagonal image of K_σ with $\mathbb{A}_\sigma$.

We get a homomorphism $\mathbb{B}_\star \longrightarrow k(\star)$ induced by $p_\star$.

Define

$$\mathbb{B}_S(\{A\}) = {\prod}' \mathbb{B}_\star \times \prod \mathbb{B}_\sigma$$

as the intersection of $\prod \mathbb{B}_\star \times \prod \mathbb{B}_\sigma$ with $\mathbb{A}_S(\{A\}) \subset \prod \mathbb{A}_\star \times \prod \mathbb{A}_\sigma$.

For a set $S_\prime$ of fibres and horizontal curves define $\mathbb{B}_{S_\prime}(\{A\}) = \mathbb{B}_S(\{A\}) \cap \mathbb{A}_{S_\prime}(\{A\})$.

1.2. Adelic dualities

In this section we study characters of adeles and fix adelic objects $\mathbb{A}_S$, $\mathbb{A}_S^\circ$, $\mathbb{B}_S$, $\mathbb{B}_S^\circ$ and then $\mathbb{A}$, $\mathbb{B}$.

27. For a nontrivial character ψ of a nonarchimedean two-dimensional $K_{x,\star}$ (resp. $K_{x,z}$) the conductor of ψ at $x,\star$ (resp. at x,z) is the orthogonal complement $A_{x,\star}$ of $O_{x,\star}$ (resp. $O_{x,z}$) with respect to ψ. Recall that in 24 we agreed to use the notation x,y for ω,y where ω is an archimedean place of $k(y)$ and y is a horizontal curve.

Proposition *Let $\star$ be a fibre or a horizontal curve.*

There are local characters $\psi_{x,\star}$, $x \in \star$, and $\psi_{\omega,\sigma}$, $\omega \in \mathcal{S}^\sigma$, such that their product

$$\psi_\star = \otimes_{x\in\star}\psi_{x,\star}, \quad \psi_{x,\star} = \otimes_{z\in\star(x)}\psi_{x,z}, \qquad (\textit{resp. } \psi_\sigma = \otimes_{\omega\in\mathcal{S}^\sigma}\psi_{\omega,\sigma})$$

is defined on $\mathbb{A}_\star$ *and is trivial on* $\mathbb{B}_\star$ (*resp. on* $\mathbb{A}_\sigma$ *and is trivial on* $\mathbb{B}_\sigma$).

Moreover, for a fibre or a horizontal curve $\star$ *one can choose the local characters in such a way that the conductors* $A_{x,\star}$ *of* $\psi_{x,\star}$ *at all nonarchimedean points* $x \in \star$ *are commensurable with* $O_{x,\star}$ *and coincide with* $O_{x,\star}$ *for almost all* $x \in \star$*; for every singular point* x *of* $\star$ *the* (x,z)*-conductor of* $\psi_\star$ *is* $O_{x,z}$ *for all* $z \in \star(x)$*; and there is a nontrivial character on* $\mathbb{A}_{k(\star)}$ *such that its composition with the residue map* $p_\star$ *gives* $\psi_\star$*.*

In addition, if $\star$ *is a fibre then the orthogonal complement* $\mathrm{O}_{x,\star}^{\perp}$ *of* $\mathrm{O}_{x,\star}$ *with respect to the* x*-part of* $\psi_\star$ *is of the form* $\alpha_{x,\star}\mathrm{O}_{x,\star}$ *where* $\alpha_{x,\star}$ *is the* $(x,\star)$*-part of an element* $\alpha_\star$ *of* $\mathbb{A}_\star^\times$*. One can also choose* $\psi_\star$ *such that at every split ordinary double point* x *of fibre* $\star$ *and* t_1*-local parameters* $t_{1x,z}, t_{1x,z'}$ *at two local branches* z, z' *of* $\star$ *at* x *the element* $\alpha_{x,\star}$ *is* $(t_{1x,z}^{-1}, t_{1x,z'}^{-1})$*.*

Proof: We will lift appropriate characters at the residue level to the level of the two-dimensional fields.

Denote by $\mathfrak{l}\colon \mathbb{F}_p \longrightarrow \mathbb{Z}$ any map which mod $p\mathbb{Z}$ coincides with the identity map of $\mathbb{F}_p$.

Consider several cases.

(a) First assume that $\star = y$ is nonsingular.

Choose characters $\overline{\psi}_{x,y}\colon E_{x,y} \longrightarrow \mathbb{C}_1^\times$, where $\mathbb{C}_1^\times$ is the group of complex numbers of module 1, such that $\overline{\psi}_y = \otimes_{x\in y}\overline{\psi}_{x,y} = 1$ on $k(y)$ and almost all conductors are the rings of integers of $E_{x,y}$. See e.g. [57] and [60]. For compatibility with [57] and [60], in characteristic zero at archimedean $x = \omega$ of y we choose $\overline{\psi}_{x,y}(\alpha) = \exp\big(-2\pi i \operatorname{Tr}_{k(y)_\omega/\mathbb{R}}(\alpha)\big)$, these conditions determine $\overline{\psi}_y$ uniquely in characteristic zero.

If $E_{x,y}$ is of positive characteristic write $\overline{\psi}_{x,y} = \exp(2\pi i\, \mathfrak{l} \circ \varphi_{x,y}/p)$ with a well defined map $\varphi_{x,y}\colon E_{x,y} \longrightarrow \mathbb{F}_p$ which we call the additive part of $\overline{\psi}_{x,y}$.

(a1) In equal characteristic define $\psi_{x,y} = \overline{\psi}_{x,y} \circ p_y$. Then the tensor product of $\psi_{x,y}$, $x \in y$, is trivial on $\mathbb{B}_y$.

(a2) For a nonarchimedean fibre choose a nonzero k_σ-differential form Ω_σ on K_σ and define

$$\psi_{\omega,\sigma}(\alpha) = \exp\big(-2\pi i \operatorname{Tr}_{k(y)_\omega/\mathbb{R}} \operatorname{res}_{t_\omega^{-1}}\big(\Omega_\sigma \alpha\big)\big)$$

(this does not depend on the choice of t_ω). Then $\otimes_{\omega\in S^\sigma}\psi_{\omega,\sigma}$ is trivial on $\mathbb{B}_\sigma$.

(a3) In mixed characteristic we consider first the unramified case and then the general case.

(a3.1) Assume first that K_y is absolutely unramified. Then we can use prime p as a local parameter t_y of y. Define $\psi_{x,y}\colon K_{x,y} \longrightarrow \mathbb{C}_1^\times$ by imposing condition

$\psi_{x,y}\big(p\mathcal{O}_{x,y}\big) = 1$ and for $-n < 0$ put

$$\psi_{x,y}\big(p^{-n+1} l^n_{x,y}(\overline{a_1},\dots,\overline{a_n}) + p\mathcal{O}_{x,y}\big) = \exp\big(2\pi i\, p^{-n} l^n(\varphi_{x,y}(\overline{a_1}),\dots,\varphi_{x,y}(\overline{a_n}))\big)$$

using the maps defined in 25, where $l^n\colon \mathbb{F}_p^{\oplus n} \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$ is the restriction of l^n_y defined in 25 with $t_y = p$. Since $\sum_{x\in y}\varphi_{x,y}(\overline{a}) = 0$ for $\overline{a} \in k(y)$, $\otimes_{x\in y}\psi_{x,y}$ equals 1 on $p^{-n+1} l^n_y(k(y)^{\oplus n}) + t_y\mathcal{O}_y$ for every n, and hence on $\mathbb{B}_y$.

(a3.2) Let L be a complete discrete valuation field with prime element p such that K_y/L is a totally ramified extension of complete discrete valuation fields of degree n.

The following argument is classical. Let $f(X)$ be the monic irreducible polynomial of t_y over L. Recall that $\mathrm{Tr}(t_y^i/f'(t_y)) = 0$ for $0 \le i \le n-2$ and $= 1$ for $i = n-1$, where Tr is the trace map $\mathrm{Tr}_{K_y/L}$. Denoting by $\mathcal{O}$ the integral structures with respect to rank one discrete valuation, let the integer m be such that $t_y^{-m}\mathcal{O}_{K_y} = f'(t_y)\mathcal{O}_{K_y}$. Then m is the minimal integer with the property $\mathrm{Tr}(t_y^m\mathcal{O}_{K_y}) \subset \mathcal{O}_L$.

For every place v of $k(y)$ there is a uniquely determined two-dimensional local field L_v which is an extension of L as a discrete valuation field, such that $\mathcal{O}_{L_v} \supset \mathcal{O}_L$, $\mathcal{M}_{L_v} = \mathcal{M}_L\mathcal{O}_{L_v}$ and $\mathcal{O}_{L_v}/\mathcal{M}_{L_v}$ is the completion of $k(y)$ with respect to v, see e.g. [27]. Then K_y/L and L_v/L are linearly disjoint and every two-dimensional local field$K_{x,y}$ associated to K_y is of the form K_yL_v. Let $\gamma \in t_y^m\mathcal{O}_{K_y} \setminus t_y^{m+1}\mathcal{O}_{K_y}$. Now it is easy to see that $\{\alpha \in K_yL_v : \mathrm{Tr}(\gamma\alpha O_{K_yL_v}) \subset O_{L_v}\} = O_{K_yL_v}$ for almost all places v, where Tr is the trace map $\mathrm{Tr}_{K_yL_v/L_v}$. The previous paragraph defines a character ψ_L of the adelic ring $\mathbb{A}_L$ associated to L and the trace map gives the trace map Tr from $\mathbb{A}_y$ to $\mathbb{A}_L$, which sends $\mathbb{B}_y$ to $\mathcal{O}_L$. Thus, put $\psi(\alpha) = \psi_L(\mathrm{Tr}(\gamma\alpha))$, $\alpha \in \mathbb{A}_y$.

(b) Now consider the case of a singular fibre or a horizontal curve $\star$. We describe the residue characters which are then lifted as above to the level of characters on two-dimensional objects.

At the residue level choose characters for the normalizations of components of $\star$, then their tensor product gives a nontrivial character $\overline{\psi}_\star$ on the one-dimensional adelic object $\mathbb{A}_{k(\star)}$. Using the approximation theorem we can assume that at the preimage of singular points of $\star$ the conductor of $\overline{\psi}_\star$ is trivial (i.e. it equals to $p_\star(O_{x,\star})$). Lift, as above, the residue character to the character $\psi_\star$ of $\mathbb{A}_\star$.

To explain the last paragraph of the theorem let $\star$ be a fibre on the regular surface $\mathcal{S}$, so $\star$ is of locally complete intersection. Denote by $O\mathbb{A}_\star^\perp$ the orthogonal complement of $O\mathbb{A}_\star$ defined in 25 with respect to $\psi_\star$. Denote $o = p_\star(O\mathbb{A}_\star)$, $o^\perp = p_\star(O\mathbb{A}_\star^\perp)$. If j is an open o-submodule of $\mathbb{A}_{k(\star)}$ such that its orthogonal complement with respect to $\overline{\psi}_\star$ is open, then it immediately follows that the double orthogonal complement of j with respect to $\overline{\psi}_\star$ is j. Since $j^\perp = (o^\perp : j)$, we get

$(o^\perp : (o^\perp : j)) = j$ for every such j, i.e. $o^\perp$ is a dualizing o-module by definition. The Gorenstein property of $\star$ is equivalent to o being a dualizing o-module and is equivalent to $o^\perp = \overline{\gamma}_\star o$ for some $\overline{\gamma}_\star \in \mathbb{A}^\times_{k(\star)}$. Denote by $\alpha_{x,\star}$ a lift of the x-components of $\overline{\gamma}_\star$. For a split ordinary double point x of $\star$ the element $p_\star(\alpha_{x,\star})$ can be chosen as $(bp_\star(t_{1\,x,z}^{-1}), cp_\star(t_{1\,x,z'}^{-1}))$ for some $b,c \in k(x)^\times$. Changing the character $\overline{\psi}_\star$ to $\overline{\psi}_\star(\delta\cdot)$ with appropriate $\delta \in k(\star)^\times$, we get a character for which $b = c = 1$. The additive part of $\overline{\psi}_\star$ can be written as $\overline{\alpha} \mapsto \sum_{x' \in \hat{\star}} \mathrm{Res}_{x'}(\overline{\beta}^{-1}\overline{\alpha})$ for some $\overline{\beta} \in \mathbb{A}^\times_{k(\star)}$, where x' runs through all points on the normalization $\hat{\star}$ of $\star$ (the disjoint union of the normalizations of components of $\star$) and $\mathrm{Res}_{x'} = \mathrm{Tr}_{k(x')/\mathbb{F}_p} \circ \mathrm{res}_{x'}$. $\square$

Remark Using the correspondence between characters on adeles and differential forms the previous results can be easily translated into the language of differential forms. A choice of a relative canonical divisor $\omega_{S/B}$ gives canonical divisors on fibres. In particular, the character $\overline{\psi}_\star(\gamma^{-1}\cdot)$ (for which $p_\star(\mathrm{O}_{x,\star})$ is its own complement) corresponds to a nonzero differential form $\Omega_\star$ on $\star$ which is regular at singular points of $\star$ (see [49], Ch. 4 for the notion of regularity in the case of singular curves and [37], Ch. 10 for the general case). The divisor of $\Omega_\star$ is the sum of canonical divisors on the components of the normalization $\hat{\star}$, whose support does not include preimages of singular points of $\star$ plus $\sum [x']$ for all preimages x' of singular points x.

CHOICE OF $\psi_\star$. From now on choose and fix characters $\psi_\star$, satisfying all the conditions of the previous Proposition. As usual, these characters are not uniquely determined, unless $\star$ is a horizontal curve of characteristic zero.

DEFINITION OF $d_{x,z}$, $d_{x,\star}$. Let $\star$ be a fibre or a horizontal curve. Define numbers $d_{x,z}$ and multi-indices $d_{x,\star}$ associated to the character $\psi_\star$ for nonarchimedean points $x \in \star$ in the following way.

If x is a nonsingular point of $\star$ and $\star(x) = \{z\}$, write the conductor $A_{x,z}$ of the nonarchimedean local character $\psi_{x,z}$, the (x,z)-component of ψ, as the fractional ideal $t_{1\,x,z}^{\,d_{x,z}} O_{x,z}$ with integer $d_{x,z}$.

Using the $\alpha_{x,\star}$ from the previous Proposition, for a singular point x of $\star$ and a local branch z of $\star$ at x let $d_{x,z}$ be the minimal integer such that the image $p(\alpha_{x,\star})$ in $E_{x,z}$ belongs to the $d_{x,z}$-power of the maximal ideal of its ring of integers. Let $d_{x,\star}$ be the multi-index whose component at z is $d_{x,z}$. In particular, if x is a split ordinary double point of a fibre $\star$ then $d_{x,z} = d_{x,z'} = -1$ for local branches z,z' of $\star$ at x and $d_{x,\star} = (-1,-1)$.

Using the module map $|\,|_{x,z}$ on $K_{x,z}$ (see 5) we get

$$|t_{1\,x,z}^{\,d_{x,z}}|_{x,z} = q_{x,z}^{-d_{x,z}},$$

$q_{x,z}$ is defined in 24.

DEFINITION OF $\mathbb{A}_{\mathcal{S}}$, $\mathbb{A}_{S_{\prime}}$, $\mathbb{B}_{\mathcal{S}}$, $\mathbb{B}_{S_{\prime}}$. Using the general definition in 25 and 26 denote by $\mathbb{A}_{\mathcal{S}}$, $\mathbb{A}_{S_{\prime}}$ and $\mathbb{B}_{\mathcal{S}}$, $\mathbb{B}_{S_{\prime}}$ the adelic objects with respect to $A_{x,y} = O_{x,y}$; recall that at the archimedean points $x = \omega$ of horizontal y we already fixed $A_{\omega,y} = \mathcal{O}_{\omega,y}$.

The general definition in 26 gives the topology on $\mathbb{A}_{\mathcal{S}}$.

Denote by $\mathbb{A}^{\circ}_{\mathcal{S}}$, $\mathbb{A}^{\circ}_{S_{\prime}}$ and $\mathbb{B}^{\circ}_{\mathcal{S}}$, $\mathbb{B}^{\circ}_{S_{\prime}}$ the adelic objects with respect to $A_{x,y}$ as in the previous Proposition, thus they depend on the choice of the characters.

DEFINITION OF S_{l}, S_h. Denote by S_{l}, S_h the sets of fibres (resp. horizontal curves) of $\mathcal{S} \longrightarrow B$. We have vertical and horizontal subspaces $\mathbb{A}_{S_{\mathrm{l}}}$, $\mathbb{A}_{S_h}$ of $\mathbb{A}_{\mathcal{S}}$.

28. Now we define the large adelic object $\mathbf{A}$ mentioned in the introduction of this part.

DEFINITION OF $\mathbf{A}_y$, $\mathbf{A}_\star$, $p_\star$. Put

$$\mathbf{A}_y = \cup_{r\in\mathbb{Z}} \mathbb{A}_y^r = \mathbb{A}_y[t_y^{-1}].$$

For a fibre $\star$ define

$$\mathbf{A}_\star = \prod_{y\subset\star} \mathbf{A}_y$$

where y runs through components of $\star$. Put

$$\mathbf{A}_\sigma = \mathbb{A}_\sigma.$$

Define the projection map $p_\star\colon \mathbf{A}_\star \longrightarrow \mathbb{A}_{k(\star)}$ as $p_\star\colon \mathbb{A}_\star \longrightarrow \mathbb{A}_{k(\star)}$ of 25, extended by zero outside $\mathbb{A}_\star$.

DEFINITION OF $\mathbf{A}_{\mathcal{S}}$, $\mathbf{A}_{S_{\prime}}$. Define

$$\begin{aligned}\mathbf{A}_{\mathcal{S}} &= {\prod_\star}' \mathbf{A}_\star \times \prod_\sigma \mathbf{A}_\sigma \\ &= \{(a_{x,\star})_{x\in\star}, \text{where } \star \text{ runs through fibres and horizontal curves}\} \\ &\quad \times \{(b_{\omega,\sigma})_{\omega\in\mathcal{S}^\sigma}, \text{where } \sigma \text{ runs through archimedean places of k}\},\end{aligned}$$

such that $(a_{x,\star})_{x\in\star} \in \mathbf{A}_\star$ for every fibre and horizontal curve $\star$, and

for almost all $\star$ the element $a_{x,\star}$ belongs to $\mathcal{O}_{x,\star}$ for all $x \in \star$, i.e. $(a_{x,\star})_{x\in\star} \in \mathbb{A}_\star$,

there is an integer r such that $(a_{x,\star})_{x\in\star}$ belongs to $\mathbb{A}_\star^r$ for every $\star$.

For a set $S_{\prime}$ of fibres and horizontal curves we similarly define

$$\mathbf{A}_{S_{\prime}} = \mathbf{A}_{\mathcal{S}} \cap \prod_{\star\in S_{\prime}} \mathbf{A}_\star.$$

Every element of $\mathbb{A}_{\mathcal{S}}$ is an element of $\mathbf{A}_{\mathcal{S}}$.

Remark 1 The object $\mathbf{A}_{\mathcal{S}}$ is the adelic object with respect to the integral structure of rank one: on its vertical part it is the restricted product of $\mathbf{A}_\star$, where $\star$ runs over all fibres, with respect to $\mathbb{A}_\star = \mathbf{A}_\star \cap \prod_{x\in\star} \mathcal{O}_{x,\star}$. In contrast, on its vertical part the object $\mathbb{A}_{\mathcal{S}}$ is the restricted product of $\mathbb{A}_\star$ with respect to $\hat{O}\mathbb{A}_\star = \mathbb{A}_\star \cap \prod_{x\in\star} O_{x,\star}$, and $\mathbb{A}_{\mathcal{S}}$ takes into account the more refined integral structures of rank two on $\mathcal{S}$.

The object $\mathbf{A}_{\mathcal{S}}$ has various relations with the geometry of $\mathcal{S}$ and its structure sheaf $\mathcal{O}_{\mathcal{S}}$. It would be very useful to find a refinement of the structure sheaf $\mathcal{O}_{\mathcal{S}}$ which takes into account the two-dimensional integral structures on $\mathcal{S}$, see also Remark 3 in 56.

Remark 2 As soon as the set $S_{\prime}$ contains infinitely many vertical curves, the condition '$a_{x,y}$ belongs to $\mathcal{O}_{x,y}$ for almost all $x \in y$' instead of '$a_{x,y}$ belongs to $O_{x,y}$ for almost all $x \in y$' makes it impossible to define a nontrivial translation invariant measure on $\mathbf{A}_{\mathcal{S}}$ as the tensor product of local translation invariant measures $\mu_{x,y}$ on two-dimensional local fields $K_{x,y}$, since in this case the product $\otimes_{\star\in S_{\prime}} \otimes_{x\in\star} \mu_{x,\star}$ diverges on $\mathbf{A}_{S_{\prime}} \cap \prod \mathcal{O}_{x,y}$.

DEFINITION OF $\mathbf{B}_{\mathcal{S}}$ AND $\mathbf{C}_{\mathcal{S}}$. Denote by $\mathbf{C}_{\mathcal{S}}$ the intersection of the diagonal image of $\prod K_x$ with $\mathbf{A}_{\mathcal{S}}$, similarly define $\mathbf{B}_{\mathcal{S}}$ as the intersection of the diagonal image of $\prod K_y$ with $\mathbf{A}_{\mathcal{S}}$.

For a set $S_{\prime}$ of some of horizontal curves and fibres define the adelic object $\mathbf{B}_{S_{\prime}}$ as the intersection of the diagonal image of $\prod_{y\in S_{\prime}} K_y$ with $\mathbf{A}_{S_{\prime}}$.

We get the following diagramme

$$\begin{tikzcd}[column sep=small]
 & \mathbf{A}_{\mathcal{S}} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbf{B}_{\mathcal{S}} \arrow[dr, dash] & & \mathbf{C}_{\mathcal{S}} \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

We briefly mention that one can appropriately choose local parameters t_y so that the tensor product of shifted characters $\tilde{\psi}_{x,y}(\alpha) = \psi_{x,y}(t_y\alpha)$, $\tilde{\psi}_{\omega,y}(\alpha) = \psi_{\omega,y}(t_y\alpha)$ is a nontrivial character $\tilde{\psi}$ of $\mathbf{A}_{\mathcal{S}}$, which vanishes on $\mathbf{B}_{\mathcal{S}}$ and $\mathbf{C}_{\mathcal{S}}$, and the adelic object $\mathbf{A}_{\mathcal{S}}$ is self-dual with respect to $\tilde{\psi}$.

Remark 3 In the case of smooth surfaces over finite fields the adelic object $\mathbf{A}_{\mathcal{S}}$ was first introduced in [44], [46]; for a generalization of the nonarchimedean part of adelic objects for noetherian schemes see [1]. In this case the pairing $\mathbf{A}_{\mathcal{S}} \times \mathbf{A}_{\mathcal{S}} \longrightarrow \mathbb{C}_1^\times$ associated to $\tilde{\psi}$ is related to the pairing introduced in [44], [46]: in the additive form it is the sum of local residues of a nonzero differential form in $\Omega^2_{\mathcal{S}}$ multiplied by the local components of α and β. Using this pairing one easily establishes additive duality on $\mathcal{S}$ see [46], [65], [66]. The object orthogonal to $\mathbf{B}_\mathcal{S}+\mathbf{C}_\mathcal{S}$ is $\mathbf{B}_\mathcal{S} \cap \mathbf{C}_\mathcal{S} = K$, and any continuous character trivial on $\mathbf{B}_\mathcal{S}+\mathbf{C}_\mathcal{S}$ is of the form $\alpha \mapsto \tilde{\psi}(a\alpha)$ for $a \in K$. All this is a simple two-dimensional version of the classical theory for curves over finite fields.

See the next section for a rational version of $\mathbb{A}$.

29. *From now on the notation $S_,$ stands for a fixed set of some fibres and some finitely many horizontal curves.* The set $S_,$ will further be specified in 35 and 36, and in the case of proper regular models of elliptic curves from 40 on.

DEFINITION OF $\mathbb{A}$, $\mathbb{B}$. Following the definition in 28 and using the set $S_,$ we will use the abbreviated notation

$$\mathbb{A} = \mathbb{A}_{S_,}, \qquad \mathbb{B} = \mathbb{B}_{S_,},$$

and similarly $\mathbb{A}^\circ$, $\mathbb{B}^\circ$. So the data associated with the archimedean fibres are not included in these objects.

Then $\mathbb{A}$ is the restricted product of $\mathbb{A}_\star$ with respect to $O\mathbb{A}_\star$, where $\star$ runs over horizontal curves in $S_,$ and fibres. Thus, every element of $\mathbb{A}$ has only finitely many (x,y)-components not in $O_{x,y}$.

The topology of $\mathbb{A}$ is the induced topology from $\mathbb{A}_\mathcal{S}$. Later in 44 we will define another weak topology on $\mathbb{A}$.

We have the following picture

$$\begin{array}{ccc}
\mathbb{A} & - & \mathbf{A}_\mathcal{S} \\
| & & | \\
\mathbb{B} & - & \mathbf{B}_\mathcal{S} \\
 & & | \\
 & & K.
\end{array}$$

Lemma *The map $\psi = \otimes_{\star\in S_,}\psi_\star$, where $\psi_\star$ are defined in 27, is a continuous character on $\mathbb{A}^\circ$ trivial on $\mathbb{B}^\circ$.*

Using it we get the continuous pairing

$$\mathbb{A} \times \mathbb{A}^\circ \longrightarrow \mathbb{C}_1^\times, \qquad (\alpha,\beta) \mapsto \psi(\alpha\beta).$$

With respect to this pairing the adelic objects $\mathbb{B}$ and $\mathbb{B}^\circ$ are orthogonal to each other, and the orthogonal complement of $\mathbb{B}^\circ$ is $\mathbb{B} + \mathbb{A} \cap \prod_{\star\in S_,}\mathbb{A}_\star^1$.

Choose an element $\rho \in \mathbb{A}^\circ$ such that $\rho_{x,\star} O_{x,\star} = t_{1\,x,\star}^{\,d_{x,\star}} O_{x,\star}$ for all nonsingular $x \in \star$ and $\rho_{x,\star} \in \mathcal{O}_{x,\star}^\times$ at singular $x \in \star$. Then

$$\mathbb{A}^\circ = \rho\mathbb{A}.$$

If $S_\prime$ is finite then $\mathbb{A} = \mathbb{A}^\circ$, $\mathbb{B} = \mathbb{B}^\circ$. If $S_\prime$ contains infinitely many vertical curves then $\mathbb{A} \neq \mathbb{A}^\circ$, since $d_{x,y} \neq 0$ for infinitely many $x \in y \in S_\prime$.

Proof: Use local duality from 3 and 20 and the construction of the local characters in 27, and argue entirely similar to the one-dimensional case [57]. □

DEFINITION OF A RATIONAL VERSION $\mathfrak{A}$ OF $\mathbb{A}$. Define another adelic object $\mathfrak{A}(\{A\})$ as the restricted product of $\mathbf{A}_\star$ with respect to $\mathbb{A}_\star^0 \cap \prod A_{x,y}$ in the following sense: $(a_{x,y}) \in \mathfrak{A}(\{A\})$ with $a_{x,y} \in K_{x,y}$ if for almost all y the element $a_{x,y}$ belongs to $A_{x,y}$ for all $x \in y$ and there is an integer r such that $(a_{x,y})_{x\in y}$ belongs to $\mathbb{A}_y^r$ for every y.

Define $\mathfrak{B}(\{A\})$ as the diagonal image of $\prod K_y$ intersected with $\mathfrak{A}(\{A\})$. Define, similarly to the above, objects $\mathfrak{A}$, $\mathfrak{A}^\circ$, $\mathfrak{B}$, $\mathfrak{B}^\circ$. These are 'rational' versions of the objects $\mathbb{A}$, $\mathbb{B}$, etc.

Using the character ψ we get a pairing $\mathfrak{A} \times \mathfrak{A}^\circ \longrightarrow \mathbb{C}_1^\times$. The orthogonal complement of each of $\mathfrak{B}, \mathfrak{B}^\circ$ is the other one.

We do not use the adelic object $\mathfrak{A}$ in this paper, since $\mathbb{A}$ and spaces related to it are sufficient for the zeta integrals.

1.3. Adelic measure and integration

Using the local theory of [11] we will introduce adelic measures, appropriate spaces of functions and integrals.

30 . Given an integral domain A with a principal prime ideal $P = tA$ and projection $A \longrightarrow A/P = B$ and given an $\mathbb{R}$-valued translation invariant countably additive measure μ_B on B, we described in 13 of [11] how to lift it to an $\mathbb{R}((X))$-valued translation invariant finitely additive measure (and countably additive in the refined sense) on A: the ring of measurable sets of A is generated by shifts of sets $t^i p^{-1}(L)$, with a measurable subset $L$of B, its measure is by definition $X^i \mu_B(L)$. This measure can be called the 'lifted measure' from the residue level. Every non empty measurable set can be written as a disjoint union of sets A_n each of which is in the form $A_n' \setminus A_n''$ with $A_n' \supset A_n''$, where A_n' is a finite disjoint union (over m) of sets $t^{i(n)} p^{-1}(B_m)$, and A_n'' is empty or a finite disjoint union (over l) of sets $t^{j(n)} p^{-1}(C_l)$ with some integer $j(n) > i(n)$, for the higher local field case see sect. 6 of [12]. One also easily extends the definitions to a measure on the field of fractions of A.

For a detailed presentation of the 'lifted measure' from locally compact fields to n-dimensional complete discrete valuation fields whose last residue field is the locally compact field see [39]. The measure and integration on $\mathbb{A}$ and $\mathbb{B}$ will be the 'lifted measure and integration'.

DEFINITION OF LOCAL MEASURES. Let $\star$ be a fibre or a horizontal curve. Using 4, 11, 12, 20 of [11] define measures

$$\mu_{x,\star} = \mu_{K_{x,\star}}$$

on $K_{x,\star}$ which are self-dual with respect to the local characters $\psi_{x,\star}$ defined and fixed in 27. (in accordance with 24 $\mu_{x,\star}$ is the tensor product of measures $\mu_{x,z}$ on the objects $K_{x,z}$ for local branches $z \in \star(x)$). Thus, these measures are normalized in the following way:
(a) for nonarchimedean nonsingular points $x \in \star$ we get $\mu_{x,\star}(O_{x,\star}) = q_{x,\star}^{d_{x,\star}/2}$, $d_{x,\star}$ are defined in 27.
(b) at each singular point x of a fibre $\star$ we get $\mu_{x,z}(O_{x,z}) = 1$ for each local branch z of $\star$ at x. So $\mu_{x,\star}(O_{x,\star}) = 1$ and for a split ordinary double point x of a fibre $\star$ we get $\mu_{x,\star}(\mathrm{O}_{x,\star}) = |k(x)|^{-1}$, $\mathrm{O}_{x,\star}$ was defined in 25.
(c) at each archimedean point x of a horizontal curve $\star$ the measure $\mu_{x,\star}$ is as it was defined as in [11] and reviewed at the end of the introduction of this part.

This normalization of the measures will result in the familiar property with respect to the chosen character ψ: the double Fourier transform with respect to $\mu_{x,\star}$ and $\psi_{x,\star}$ of a well chosen function f equals $f \circ \mathrm{j}$ where j is the involution $\alpha \mapsto -\alpha$, see Proposition below.

For standard facts about functions, measure and integrals on restricted products of locally compact groups, see e.g. [57], 3.1–3.3; there the restricted product is taken with respect to open compact subgroups whose measure is almost always 1. Those constructions can be extended in purely algebraic way to restricted products of groups arising in the two-dimensional adelic theory, groups which are not locally compact.

Let G_i, H_i be as in 26. Suppose that there are translation invariant measures (taking values in $\mathbb{R}((X))$) μ_i on G_i such that $\mu_i(H_i) = 1$ for almost all $i \in I$. Consider the ring of measurable subsets of $\prod' G_i$ generated by $A \subset G_J$, for finite $J \subset I$, where $A = \prod_{i \in J} A_i \times \prod_{i \notin J} H_i$ and A_i are measurable subsets of G_i. Then we get a translation invariant measure on G_J given by $\mu_J(A) = \prod_{i\in J} \mu_i(A_i)$. Define the measure μ on G as $\varinjlim \mu_J$, so $(G, \mu) = \varinjlim (G_J, \mu_J)$ as ($\mathbb{R}((X))$-valued) measure spaces. We will write $\mu = \otimes \mu_i$. Integrable functions on $\prod' G_i$ are those functions f on $\prod' G_i$ for which $f|_{H_i} = 1$ for almost all i and $\lim_J \int_{G^J} f_J$ exists where $G^J = \prod_{i\in J} G_i \times \prod_{i \notin J} e_i$ and $f_J = f|_{G^J}$, e_i is the identity element of G_i.

Using this general recipe, we now define $\mu_{\mathbb{A}_\star}$ and then $\mu_{\mathbb{A}}$.

DEFINITION OF $\mu_{\mathbb{A}_\star}$. Let $\star$ be a horizontal curve or a fibre. Using the local measures $\mu_{x,z}$ define an $\mathbb{R}((X))$-valued measure $\mu_{\mathbb{A}_\star}$ in the following way: the ring of measurable sets is generated by translations of sets $t_\star^i(\prod_{x\in\star} D_{x,\star})\cap\mathbb{A}_\star$, $i\geq 0$, where $D_{x,z}$ are measurable subsets of $K_{x,z}$, $D_{x,z}\subset\mathcal{O}_{x,z}$, $D_{x,z}+t_\star\mathcal{O}_{x,z}=D_{x,z}$ and where $D_{x,z}=O_{x,z}$ for almost all $x\in\star$.

The $\mu_{\mathbb{A}_\star}$-measure of this measurable set is equal by definition to $X^i\prod_{x\in\star}\mu_{x,\star}(D_{x,\star})\in\mathbb{R}((X))$. Note that $\mu_{x,\star}(D_{x,\star})=1$ for almost all $x\in\star$.

The function $\mu_{\mathbb{A}_\star}$ is well defined on the ring of measurable sets, translation invariant and finitely additive.

Allowing i above run through all integer values, one similarly defines a translation invariant $\mathbb{R}((X))$-valued measure on $\mathbf{A}_\star$.

In particular, for a nonsingular fibre y the measurable set $O\mathbb{A}_y$ is of $\mu_{\mathbb{A}_y}$-measure 1 if and only if $\prod_{x\in y} q_{x,y}^{d_{x,y}}=1$.

Using the definitions and 13 we obtain

Lemma 1 *Let $\star$ be a horizontal curve or a fibre. The $\mathbb{R}((X))$-valued translation invariant measure $\mu_{\mathbb{A}_\star}$ on $\mathbb{A}_\star$ lifts, in the sense of the beginning of this section, the appropriately normalized Haar measure $\mu_{\mathbb{A}_{k(\star)}}$ on $\mathbb{A}_{k(\star)}$, which is self-dual with respect to the character $\overline{\psi}_\star=\otimes_{x\in\star}\overline{\psi}_{x,\star}$ defined in* 27.

DEFINITION OF THE ADELIC MEASURE $\mu_{\mathbb{A}}$. Define the ring of measurable subsets of $\mathbb{A}$ as generated by translations of sets $C=\prod_{\star\in S_\prime} C_\star$ of $\mathbb{A}$, where $C_\star$ is a measurable subset of $\mathbb{A}_\star$, equal to $O\mathbb{A}_\star$ for almost all $\star\in S_\prime$, and $\mu_{\mathbb{A}_\star}(C_\star)=1$ for almost all $\star\in S_\prime$. Define

$$\mu_{\mathbb{A}}=\otimes_{\star\in S_\prime}\mu_{\mathbb{A}_\star}.$$

Thus, $\mu_{\mathbb{A}}(C)=\prod\mu_{\mathbb{A}_\star}(C_\star)$.

The measure $\mu_{\mathbb{A}}$, if it is defined, see Remark 1, is translation invariant and finitely additive.

Similar to the one-dimensional theory, this measure depends on the choice of the characters $\psi_\star$.

Similarly define the measure $\mu_{\mathfrak{A}}=\otimes_{\star\in S_\prime}\mu_{\mathbf{A}_\star}$ on the adelic object $\mathfrak{A}$ introduced in 29.

Remark 1 Measurable sets with respect to $\mu_{\mathbb{A}}$ have almost all their $\star$-components equal to $O\mathbb{A}_\star$ of $\mu_{\mathbb{A}_\star}$-measure 1. Hence we need the equality $\prod_{x\in\star} q_{x,\star}^{d_{x,\star}}=1$ to be satisfied for almost all $\star\in S_\prime$. For a nonsingular fibre y we get a classical formula $\prod_{x\in y} q_{x,y}^{d_{x,y}}=q_y^{2(1-g_y)}$ where q_y is the cardinality of the maximal finite subfield of $k(y)$ and g_y is the genus of y, see 40. Hence if $S_\prime$ contains infinitely many fibres then the measure $\mu_{\mathbb{A}}$ is defined on $\mathbb{A}_{S_\prime}$ if and only if the genus of the generic fibre of $\mathcal{S}$ is 1. In the general case we can renormalize the measure as in the following definition.

DEFINITION OF RENORMALIZED MEASURES $\tilde\mu_{\mathbb{A}_\star}$, $\tilde\mu_{\mathbb{A}^\circ_\star}$, $\tilde\mu_{\mathbb{A}}$, $\tilde\mu_{\mathbb{A}^\circ}$. For a nonsingular fibre y define

$$\tilde\mu_{\mathbb{A}_y} := q_y^{g_y-1}\mu_{\mathbb{A}_y}, \qquad \tilde\mu_{\mathbb{A}^\circ_y} := q_y^{1-g_y}\mu_{\mathbb{A}_y}$$

and for singular fibres and horizontal curves put $\tilde\mu_{\mathbb{A}_\star} = \tilde\mu_{\mathbb{A}^\circ_\star} = \mu_{\mathbb{A}_\star}$.

Then $\tilde\mu_{\mathbb{A}_\star}(O\mathbb{A}_\star) = 1$ for almost all $\star \in S_{\prime}$ and hence

$$\tilde\mu_{\mathbb{A}} = \otimes_{\star\in S_{\prime}}\tilde\mu_{\mathbb{A}_\star}, \qquad \tilde\mu_{\mathbb{A}^\circ} = \otimes_{\star\in S_{\prime}}\tilde\mu_{\mathbb{A}^\circ_\star}$$

are well defined additive translation invariant measures on $\mathbb{A}$ and $\mathbb{A}^\circ$.

DEFINITION OF MODULE $|\,|$. For a fibre or a horizontal curve $\star$ define

$$|\,|_\star = \prod_{x\in\star}|\,|_{x,\star}.$$

So $|\gamma|_\star = \frac{\mu_\star(\gamma_\star A_\star)}{\mu_\star(A_\star)}$ for every $\gamma_\star \in \mathbb{A}_\star$, where $A_\star$ is any measurable subset of $\mathbb{A}_\star$ of nonzero measure.

Define

$$|\,| = |\,|_{S_{\prime}} = \prod_{\star\in S_{\prime}}|\,|_\star.$$

So $|\gamma| = \dfrac{\mu(\gamma A)}{\mu(A)}$ for every $\gamma \in \mathbb{A}$, where A is any measurable subset of $\mathbb{A}$ of nonzero measure.

Now we define several functional spaces $R' \supset R \supset Q$ on $\mathbb{A}_\star$ and $\mathbb{A}$, using the local spaces of functions R, R', Q defined in [11], and then we define the integrals.

DEFINITION OF SPACE $R'_{\mathbb{A}_\star}$ AND INTEGRAL $\int f\,d\mu_{\mathbb{A}_\star}$. Let $\star$ be a horizontal curve or a fibre. Using the local spaces R' of functions defined in 7, 9, 11, let $R'_{\mathbb{A}_\star}$ be the linear space generated by $f_\star$, where $f_\star(\alpha) = g_\star(t_\star^{-i_\star}\alpha)$ extended by zero outside $t_\star^{i_\star}\mathbb{A}_\star$, $i_\star \geq 0$, and

$g_\star = \otimes_{x\in\star}g_{x,\star}$ with $g_{x,\star} \in R'_{K_{x,\star}} = \prod_{z\in\star(x)} R'_{K_{x,z}}$,

$g_\star = h_\star \circ p_\star$ for an integrable continuous function $h_\star$ on $\mathbb{A}_{k(\star)}$, where $p_\star$ was defined in 28.

For $f_\star \in R'_{\mathbb{A}_\star}$ as above define

$$\int f_\star d\mu_{\mathbb{A}_\star} := X^{i_\star}\int g_\star d\mu_{\mathbb{A}_\star}$$

and extend by linearity to all elements of $R'_{\mathbb{A}_\star}$. This is well defined, since the functions $f_\star$ with different $i_\star$ are linearly independent, the proof goes entirely similar to the proof of Prop. 1.5 in [39].

For $f_\star$ is as above we get $\int f_\star d\mu_{\mathbb{A}_\star} = X^{i_\star}\int_{\mathbb{A}_{k(\star)}} h_\star d\mu_{\mathbb{A}_{k(\star)}}$.

In particular, elements of $R'_{\mathbb{A}_\star}$ are continuous functions on $\mathbb{A}_\star$.

DEFINITION OF SPACE $R'_{\mathbb{A}}$ AND INTEGRAL $\int f\,d\mu_{\mathbb{A}}$. Let $R'_{\mathbb{A}}$ be the space generated by $f = \otimes_{\star\in S_\prime} f_\star$, where $f_\star \in R'_{\mathbb{A}_\star}$ and $\int f_\star\,d\mu_{\mathbb{A}_\star} = 1$ for almost all $\star \in S_\prime$.

The last condition implies that the indices $i_\star$ for $f \in R'_{\mathbb{A}}$ should be 0 for almost all $\star \in S_\prime$.

For $f = \otimes f_\star \in R'_{\mathbb{A}}$ put

$$\int f\,d\mu_{\mathbb{A}} = \prod_{\star\in S_\prime} \int f_\star\,d\mu_{\mathbb{A}_\star}$$

and extend by linearity to $R'_{\mathbb{A}}$.

Here and later extensions by linearity are well defined, since the adelic conditions reduce the verification to the case of functions on the product of finitely many spaces.

Similarly one can define a functional space $R'_{\mathfrak{A}}$ by letting $i_\star$ in the definition of $R'_{\mathfrak{A}_\star}$ run through all integers; similarly one defines the integral $\int f\,d\mu_{\mathfrak{A}}$ for $f \in R'_{\mathfrak{A}}$. We also mention that the previous definitions can be extended to let elements of $R'_{\mathfrak{A}}$ take values in $\mathbb{C}((X))$ rather than in $\mathbb{C}$.

Using the measure $\tilde{\mu}_{\mathbb{A}}$ one similarly defines the integral $\int f\,d\tilde{\mu}_{\mathbb{A}}$.

Example 1 Suppose that $S_\prime$ includes fibres only. Let

$$f = \otimes_{\star\in S_\prime} f_\star, \quad f_\star = \otimes_{x\in\star} f_{x,\star},$$

where

$$f_{x,z} = char_{t_\star^{i_\star} t_{1\,x,z}^{c_{x,z}} O_{x,z}}, \qquad i_\star \ge 0,$$

such that
(a) for every $\star \in S_\prime$ the integers $c_{x,z} = 0$ for almost all $x \in \star$, $z \in \star(x)$,
(b) the product $\prod_{x\in\star} q_{x,\star}^{d_{x,\star}/2 - c_{x,\star}} = 1$ for almost all $\star \in S_\prime$,
(c) $i_\star = 0$ for almost all $\star \in S_\prime$.

Then $f \in R'_{\mathbb{A}}$. Here, according to 24, $q_{x,\star}^{d_{x,\star}/2-c_{x,\star}} = \prod_{z\in\star(x)} q_{x,z}^{d_{x,z}/2-c_{x,z}}$. We get

$$\int f\,d\mu_{\mathbb{A}_{S_\prime}} = X^{\sum_{\star\in S_\prime} i_\star} \prod_{\star\in S_\prime} \Big(\prod_{x\in\star} q_{x,\star}^{e_{x,\star}/2-c_{x,\star}}\Big),$$

where $e_{x,z} = d_{x,z}$ at nonsingular $x \in \star$ and $e_{x,z} = 0$ at singular $x \in \star$.

If we integrate against the measure $\tilde{\mu}_{\mathbb{A}}$, we can replace (b) by $(\tilde{\text{b}})$ $\prod_{x\in\star} q_{x,\star}^{-c_{x,\star}} = 1$ for almost all $\star \in S_\prime$. Then for every f satisfying (a),$(\tilde{\text{b}})$,(c) we get

$$\int f\,d\tilde{\mu}_{\mathbb{A}_{S_\prime}} = X^{\sum_{\star\in S_\prime} i_\star} \prod_{\star\in S_\prime} \Big(\prod_{x\in\star} q_{x,\star}^{-c_{x,\star}}\Big).$$

Lemma 2 *For $f \in R'_{\mathbb{A}}$ and $\beta \in \mathbb{A}$, $\gamma \in \mathbb{A}^{\times}$ we have*

$$\int f(\alpha)d\mu_{\mathbb{A}}(\alpha) = \int f(\alpha+\beta)d\mu_{\mathbb{A}}(\alpha),$$
$$\int f(\alpha)d\mu_{\mathbb{A}}(\alpha) = |\gamma|_{S_\prime} \int f(\gamma\alpha)d\mu_{\mathbb{A}}(\alpha).$$

Now we define a subspace $R_{\mathbb{A}}$ of $R'_{\mathbb{A}}$.

DEFINITION OF SPACES $R_{\mathbb{A}}$, $R_{\mathbb{A}_\star}$. Using the local spaces R defined in 6 and 11, denote by $R_{\mathbb{A}}$ the subspace of $R'_{\mathbb{A}}$ generated by $f = \otimes_{\star\in S_\prime} \otimes_{x\in\star} f_{x,\star}$ with $f_{x,\star} \in R_{K_{x,\star}}$ and such that for almost all $(x,\star)$, where $x \in \star \in S_\prime$,

$$f_{x,z} = char_{t_\star^{i_\star} t_{1\,x,z}^{c_{x,z}} O_{x,z}}, \qquad z \in \star(x), \quad i_\star \geq 0.$$

Using $S_\prime = \{\star\}$ one gets the definition of space $R_{\mathbb{A}_\star}$.

Similarly one defines spaces $R_{\mathfrak{A}}$, $R_{\mathfrak{A}_\star}$ by letting $i_\star$ take arbitrary integer values.

For f as in the previous definition we deduce: for every $\star \in S_\prime$ for almost all $x \in \star$ the integer $c_{x,z}$ equals zero for $z \in \star(x)$; for almost all $\star \in S_\prime$ the product $\prod_{x\in\star} q_{x,\star}^{d_{x,\star}/2-c_{x,\star}} = 1$; $i_\star = 0$ for almost all $\star$.

The element f of Example 1 belongs to $R_{\mathbb{A}}$.

The group of invertible adeles $\mathbb{A}^{\times}$, defined in 29, acts on the spaces $R'_{\mathbb{A}}, R_{\mathbb{A}}$:

$$f \longrightarrow f_\alpha, \quad f_\alpha(\gamma) = f(\alpha\gamma).$$

To define the adelic transform we will use the local transforms $\mathcal{F}_{x,y}$,$\mathcal{F}_{\omega,y}$ defined in 9 and 11, and take them with respect to the local measures and characters introduced in 27.

The choice of the measure $\mu_{x,z}$ in 29 implies that the local transform of $char_{t_{1\,x,z}^{c_{x,z}} O_{x,z}}$ with respect to the $\psi_{x,z}$ and $\mu_{x,z}$ is

$$\mathcal{F}_{x,z}(char_{t_{1\,x,z}^{c_{x,z}} O_{x,z}}) = q_{x,z}^{d_{x,z}/2-c_{x,z}} char_{t_{1\,x,z}^{d_{x,z}-c_{x,z}} O_{x,z}}$$

at nonsingular $x \in \star$.

At a split ordinary double point $x \in \star$ for the transform $\mathcal{F}_{x,\star} = \mathcal{F}_{x,z} \otimes \mathcal{F}_{x,z'}$ we get

$$\mathcal{F}_{x,\star}(char_{\mathrm{O}_{x,\star}}) = q_x^{-1} char_{\mathrm{O}_{x,\star}^{\perp}}, \quad \mathcal{F}_{x,\star}(char_{\mathrm{O}_{x,\star}^{\perp}}) = q_x char_{\mathrm{O}_{x,\star}},$$

$\mathrm{O}_{x,\star}^{\perp}$ is defined in 27.

DEFINITION OF TRANSFORM $\mathcal{F}$ ON $R_{\mathbb{A}_\star}$. Let $f_\star \in R_{\mathbb{A}_\star}$, $f_\star(\alpha) = g_\star(t_\star^{-i_\star}\alpha)$ with $g_\star = \otimes_{x\in\star} g_{x,\star} \in R_{\mathbb{A}_\star}$ such that $g_{x,\star} = char_{t_{1\,x,\star}^{\;c_{x,\star}} O_{x,\star}}$ for almost all $x \in \star$. Using the local transforms in 9 and 11, introduce the adelic transform

$$\mathcal{F}(f_\star) = X^{i_\star} \otimes_{x\in\star} \mathcal{F}_{x,\star}(g_{x,\star})$$

and extend by linearity to $R_{\mathbb{A}_\star}$ and $R_{\mathfrak{A}_\star}$ (it is well defined).

Then

$$\mathcal{F}(f_\star)(\beta) = \int f_\star(\alpha)\psi_\star(\alpha\beta)\,d\mu_{\mathbb{A}_\star}(\alpha),$$

where $\psi_\star$ is the character on $\mathbb{A}_\star$ defined in 27.

DEFINITION OF TRANSFORM $\mathcal{F}$ ON $R_{\mathbb{A}}$. For $f \in R_{\mathbb{A}}$ introduce its transform

$$\mathcal{F}(f)(\gamma) = \int f(\alpha)\psi(\alpha\gamma)\,d\mu_{\mathbb{A}}(\alpha).$$

This is a $\mathbb{C}((X))$-valued function on $\mathbb{A}^\circ$.

We get

$$\mathcal{F}(f_\alpha)(\gamma) = |\alpha|_{S,}^{-1}\mathcal{F}(f)_{\alpha^{-1}}(\gamma).$$

Remark 2 Note that the function $\mathcal{F}(f_\star)$ is a $\mathbb{C}((X))$-valued function on $\mathbb{A}_\star^\circ = \mathbb{A}_\star$. Thus, for $\mathcal{F}$ to induce an endomorphism of spaces of functions, e.g. $R_{\mathbb{A}_\star}$, $R_{\mathbb{A}}$, so that its square is the involution it is natural to extend the spaces to include $\mathbb{C}((X))$-valued functions. This is all relatively straightforward, and we do not pursue this line in this text, since for the purposes of the zeta integral the space $Q_{\mathbb{A}}$ defined below and related to it spaces will be sufficient, and the transform $\mathcal{F}$ is an endomorphism of $Q_{\mathbb{A}}$.

Example 2 Suppose that $S,$ is the union of all the fibres of $\mathcal{S}$ and of finitely many nonsingular horizontal curves.

At nonsingular nonarchimedean x of $\star$ put $f_{x,\star} = char_{t_{1\,x,\star}^{\;c_{x,\star}} O_{x,\star}}$, such that for all $\star \in S,$ for almost all $x \in \star$ the integer $c_{x,\star}$ equals zero. At singular $x \in \star$ put $f_{x,\star} = char_{O_{x,\star}}$.

In characteristic zero, for a horizontal $\star = y$ and $\omega \in r_\sigma^{-1}(y)$ let

$$f_{\omega,y}(\alpha) = \exp\bigl(-e_\omega\pi\,|\mathrm{res}_{t_\omega^0}(\alpha)|^2\bigr)char_{\mathcal{O}_{\omega,y}}(\alpha),$$

where we use the notation of 24,

$$e_\omega = \mathrm{Tr}_{k(y)_\omega/\mathbb{R}}(1),$$

$||$ is the *usual absolute value*, both in real and complex places. So, in the notation of 11, $|| = ||_{k(y)_\omega}^{1/e_\omega}$. It is convenient to define $d_{\omega,y} = c_{\omega,y} = 0$.

Now suppose that $\prod_{x\in\star} q_{x,\star}^{d_{x,\star}/2-c_{x,\star}} = 1$ for almost all $\star \in S_{\prime}$. Then $f \in R_{\mathbb{A}}$.

The local transform of the function $f_{\omega,y}$ equals itself, and the transform of $f = \otimes_{\star\in S_\prime} \otimes_{x\in\star} f_{x,\star}$ is

$$\prod_{\star\in S_\prime} \Big(\prod_{x\in\star} q_{x,\star}^{d_{x,\star}/2-c_{x,\star}}\Big) f_1, \qquad f_1(\alpha) = f\big(\gamma\alpha\big),$$

where $\gamma_{x,z} = t_{1x,z}^{-d_{x,z}+c_{x,z}}$ at nonsingular $x \in \star$ and $\gamma_{x,\star} = (t_{1x,z}, t_{1x,z'})$ at split ordinary double $x \in \star$.

Remark 3 After appropriately defining local components of ρ introduced in 29 at finitely many singular points $x \in \star$ (for example $\rho_{x,\star} = (t_{1x,z}^{-1}, t_{1x,z'}^{-1})$ at a split ordinary double point) we get $\mathcal{F}(char_{O\mathbb{A}_y}) = |\rho|_y^{-1/2} char_{\rho O\mathbb{A}_y}$ and $|\rho|_y = q_y^{2(g-1)}$ for nonsingular fibres y where g is the genus of the generic fibre of S. Thus, the previous formula extends to the formula

$$\mathcal{F}(char_{O\mathbb{A}}) = |\rho|^{-1/2} char_{\rho O\mathbb{A}}$$

for infinite $S_\prime$ if and only if the genus of the generic fibre of $\mathcal{S}$ is 1.

For the zeta integral computation the following smaller spaces Q will be useful.

DEFINITION OF SPACES $Q_{\mathbb{A}}$, $Q_{\mathbb{A}_\star}$. Using the local spaces Q defined in 9 and 11, denote by $Q_{\mathbb{A}}$ the subspace of $R_{\mathbb{A}}$ generated by functions $\otimes_{\star\in S_\prime} \otimes_{x\in\star} f_{x,\star} \in R_{\mathbb{A}}$, where archimedean $f_{\omega,y}$ are pullbacks with respect to p_y of functions in the Schwartz space on $E_{\omega,y}$, extended by zero outside $\mathcal{O}_{\omega,y}$, and where nonarchimedean $f_{x,\star}$ are in the space $Q_{K_{x,\star}} = \otimes_{z\in\star(x)} Q_{K_{x,z}}$.

Restricting to one fibre or horizontal curve $\star$, we get the definition of the space $Q_{\mathbb{A}_\star}$. We have $Q_{\mathbb{A}_\star} = p_\star{}^*(Q_{\mathbb{A}_{k(\star)}})$ and the similar property for the integrals, where $Q_{\mathbb{A}_{k(\star)}}$ is the Bruhat–Schwartz space on $\mathbb{A}_{k(\star)}$.

In particular, for a function in $Q_{\mathbb{A}}$ the numbers $i_\star$ in the definition of $R'_{\mathbb{A}}$ are zero for all $\star$.

The function f in the previous example belongs to the space $Q_{\mathbb{A}}$.

The transform $\mathcal{F}$ induces an endomorphism of $Q_{\mathbb{A}}$.

Using 9, 11, 12 we deduce

Proposition *For $f \in Q_{\mathbb{A}}$ we have $\mathcal{F}(f) \in Q_{\mathbb{A}}$, $\mathcal{F}^2(f) \in Q_{\mathbb{A}}$ and*

$$\mathcal{F}^2(f)(\alpha) = f(-\alpha).$$

If we extend the space $R_{\mathfrak{A}}$ by allowing its elements take values in $\mathbb{C}((X))$, we then easily get a similar formula for the transform $\mathcal{F}^2$ on that space.

To complete this section we introduce the measure and integral on $\mathbb{B}$-spaces.

DEFINITION OF $\mu_{\mathbb{B}_\star}$, $\mu_{\mathbb{B}}$ AND $\int_{\mathbb{B}} d\mu_{\mathbb{B}}$. Let $\star$ be a fibre or a horizontal curve. Using 13 define an $\mathbb{R}((X))$-valued translation invariant measure $\mu_{\mathbb{B}_\star}$ on $\mathbb{B}_\star$ which lifts the discrete counting measure on $k(\star)$: the measurable sets are elements of the ring generated by the products of translations of sets of the type $t_\star^{i_\star} p_\star^{-1}(A)$, where $A \subset p_\star(\mathbb{B}_\star)$ is a measurable set with respect to the discrete counting measure $\mu_{k(\star)}$ on $k(\star)$. The measure of $t_\star^{i_\star} p_\star^{-1}(A)$ is $X^{i_\star}\mu_{k(\star)}(A)$.

Define the measure

$$\mu_{\mathbb{B}} = \otimes_{\star\in S_{\prime}} \mu_{\mathbb{B}_\star}.$$

Components of a measurable set with respect to this measure for almost all $\star \in S_{\prime}$ are sets $p_\star^{-1}(pt)$, the preimages of points, since the measure of almost all components should be 1.

For $f = \otimes f_\star \in Q_{\mathbb{A}}$, $f_\star = \otimes_{x\in\star} f_{x,y}$, $f_{x,y} \in Q_{K_{x,y}}$, $f_\star = g_\star \circ p_\star$, where $g_\star$ is an integrable function on $\mathbb{A}_{k(\star)}$, define

$$\int f(\beta)\, d\mu_{\mathbb{B}}(\beta) = \prod_{\star\in S_{\prime}} \int g_\star\, d\mu_{k(\star)}$$

and extend by linearity to the space $Q_{\mathbb{A}}$. The right hand side may diverge if the set $S_{\prime}$ is infinite.

For a subset $C = \prod'_{\star\in S_{\prime}} C_\star$ of $\mathbb{B}$, such that $C_\star = p_\star^{-1}(B_\star)$, $B_\star \subset p_\star(\mathbb{B}_\star)$ and for f as above define the integral

$$\int_C f(\beta)\, d\mu_{\mathbb{B}}(\beta) := \prod_\star \int_{p_\star(C_\star)} g_\star\, d\mu_{k(\star)}$$

and extend to $Q_{\mathbb{A}}$.

31. In this section we extend the previous definitions to some algebraic groups; this will be used in the following parts of the text. This is done in the same way how one defines measures on algebraic groups over adeles in dimension one. It is easy to develop a more general theory, for example for a quasi-projective variety V over K, a finite dimensional K-algebra A and an algebraic group G one can define the adelic objects $V(\mathbb{A})$, $A(\mathbb{A})$, $G(\mathbb{A})$ similarly to the one-dimensional case presented in [62]. In particular, for a group G the adelic object $G(\mathbb{A})$ is the restricted product of $G(\mathbb{A}_\star)$ with respect to $G(O\mathbb{A}_\star)$, the measure on it is the tensor product of the measures, etc.

DEFINITION OF $\mathbb{A}\times\mathbb{A}$, $\mu_{\mathbb{A}\times\mathbb{A}}$. Similarly to the definition of $\mathbb{A}_y=\mathbb{A}_y^0$ and $\mathbb{A}_\star$ in 25, define

$$\mathbb{A}_\star\times\mathbb{A}_\star=G_a^2(\mathbb{A}_\star)=\{(\alpha_{x,\star}^{(1)},\alpha_{x,\star}^{(2)})_{x\in\star}:(\alpha_{x,\star}^{(m)})_{x\in\star}\in\mathbb{A}_\star \text{ for } m=1,2\}.$$

Let $\mathbb{A}\times\mathbb{A}$ be the restricted product of $\mathbb{A}_\star\times\mathbb{A}_\star$ with respect to $O(\mathbb{A}_\star\times\mathbb{A}_\star)=O\mathbb{A}_\star\times O\mathbb{A}_\star=G_a^2(O\mathbb{A}_\star)$, $O\mathbb{A}_\star$ was defined in 25.

In particular,

$$\mathbb{A}\times\mathbb{A}=G_a^2(\mathbb{A})=\{(\alpha_{x,\star}^{(1)},\alpha_{x,\star}^{(2)})_{x\in\star\in S_\prime}:(\alpha_{x,\star}^{(m)})_{x\in\star\in S_\prime}\in\mathbb{A} \text{ for } m=1,2\}.$$

Endow $\mathbb{A}\times\mathbb{A}$ with the sequential saturation topology [10] of the following translation invariant product topology: it has $\prod W_{x,y}\times W_{x,y}\cap\mathbb{A}\times\mathbb{A}$ as the fundamental system of neighbourhoods of zero, where $W_{x,y}$ are open neighbourhoods of zero in $K_{x,y}$ with respect to its topology, almost all equal to $O_{x,y}$.

The adelic object $\mathbb{A}\times\mathbb{A}$ comes with its translation invariant measure $\mu_{\mathbb{A}\times\mathbb{A}}=\mu_{\mathbb{A}}\otimes\mu_{\mathbb{A}}$.

We will use the same notation $|\,|=|\,|_{S_\prime}$ for the module on $\mathbb{A}\times\mathbb{A}$, it is the product of the modules of the components.

Using the projection map $p_\star\colon \mathbf{A}_\star\longrightarrow\mathbb{A}_{k(\star)}$ defined in 28, we use the same notation for the map $\mathbf{A}_\star\times\mathbf{A}_\star\longrightarrow\mathbb{A}_{k(\star)}\times\mathbb{A}_{k(\star)}$.

DEFINITION OF SPACES $R'_{\mathbb{A}\times\mathbb{A}}, R_{\mathbb{A}\times\mathbb{A}}$ AND INTEGRAL $\int f\,d\mu_{\mathbb{A}\times\mathbb{A}}$. Define $R'_{\mathbb{A}_\star\times\mathbb{A}_\star}=R'_{\mathbb{A}_\star}\otimes R'_{\mathbb{A}_\star}$. For $f_\star=f_\star^{(1)}\otimes f_\star^{(2)}$, $f_\star^{(m)}\in R'_{\mathbb{A}_\star}$, define $\int f_\star\,d\mu_{\mathbb{A}_\star\times\mathbb{A}_\star}=\int f_\star^{(1)}\,d\mu_{\mathbb{A}_\star}\int f_\star^{(2)}\,d\mu_{\mathbb{A}_\star}$ and extend by linearity.

Define $R'_{\mathbb{A}\times\mathbb{A}}$ as the space generated by $f=\otimes_{\star\in S_\prime}f_\star$ with $f_\star\in R'_{\mathbb{A}_\star\times\mathbb{A}_\star}$ such that $\int f_\star\,d\mu_{\mathbb{A}_\star\times\mathbb{A}_\star}=1$ for almost all $\star\in S_\prime$. Put

$$\int f\,d\mu_{\mathbb{A}\times\mathbb{A}}=\prod_{\star\in S_\prime}\int f_\star\,d\mu_{\mathbb{A}_\star\times\mathbb{A}_\star}$$

and extend by linearity.

Define $R_{\mathbb{A}\times\mathbb{A}}$ as the subspace of $R'_{\mathbb{A}\times\mathbb{A}}$ generated by $f=\otimes_{\star\in S_\prime}\otimes_{x\in\star}f_{x,\star}$ where $f_{x,\star}\in R_{K_{x,\star}\times K_{x,\star}}$ are such that $f_{x,\star}=f_{x,\star}^{(1)}\otimes f_{x,\star}^{(2)}$ with $f_{x,z}^{(m)}=char_{t_\star^{i_{\star,m}}t_{1x,z}^{c_{x,z,m}}O_{x,z}}$, $z\in\star(x)$, for almost all $x\in\star\in S_\prime$.

For the f as in the previous paragraph the definitions imply: for all $\star$ for almost all $x\in\star$ the integer $c_{x,z,m}$ equals zero for $z\in\star(x)$; $\prod_{x\in\star}q_{x,\star}^{d_{x,\star}-c_{x,\star,1}-c_{x,\star,2}}=1$ for almost all $\star\in S_\prime$; $i_{\star,m}=0$ for almost all $\star$, $m=1,2$.

DEFINITION OF SPACE $Q_{\mathbb{A}\times\mathbb{A}}$ AND TRANSFORM ON $Q_{\mathbb{A}\times\mathbb{A}}$. Define the subspace $Q_{\mathbb{A}\times\mathbb{A}}$ as the subspace of $R_{\mathbb{A}\times\mathbb{A}}$ generated by $\otimes_{\star\in S,}\otimes_{x\in\star} f^{(1)}_{x,\star}\otimes f^{(2)}_{x,\star} \in Q_{\mathbb{A}}\otimes Q_{\mathbb{A}}$.

In particular, for an element of f of $Q_{\mathbb{A}\times\mathbb{A}}$ the numbers $i_{\star,m}$ are zero for all $\star$.

In the case of one fibre $\star$ or curve y this definition also gives the definition of the spaces $Q_{\mathbb{A}_\star\times\mathbb{A}_\star}$, $Q_{\mathbb{A}_y\times\mathbb{A}_y}$. In particular, the space $Q_{\mathbb{A}_\star\times\mathbb{A}_\star}$ coincides with $p_\star^*(Q_{\mathbb{A}_{k(\star)}}\otimes Q_{\mathbb{A}_{k(\star)}})$.

The group of invertible adeles $(\mathbb{A}\times\mathbb{A})^\times$ acts on the spaces $R'_{\mathbb{A}\times\mathbb{A}}$, $R_{\mathbb{A}\times\mathbb{A}}$, $Q_{\mathbb{A}\times\mathbb{A}}$: $f \longrightarrow f_\alpha, \quad f_\alpha(\gamma) = f(\alpha\gamma)$.

DEFINITION. Using the character $\otimes_\star\otimes_{x\in\star}(\psi_{x,\star}\otimes\psi_{x,\star})$ of $\mathbb{A}\times\mathbb{A}$ define the transform $\mathcal{F}$ on the space $Q_{\mathbb{A}\times\mathbb{A}}$.

Now we consider the multiplicative groups. The definitions imply that $\mathbb{A}^\times$ is the restricted product of $\mathbb{A}_\star^\times$ with respect to $(O\mathbb{A}_\star)^\times$.

DEFINITION OF THE TOPOLOGY ON $\mathbb{A}_S^\times$, $\mathbb{A}^\times$. Define the topology on $\mathbb{A}_S^\times$ (resp. $\mathbb{A}^\times$) as the induced topology from $\mathbb{A}_S\times\mathbb{A}_S$ (resp. $\mathbb{A}\times\mathbb{A}$) via

$$\mathbb{A}_S^\times \longrightarrow \mathbb{A}_S\times\mathbb{A}_S, \quad \mathbb{A}^\times \longrightarrow \mathbb{A}\times\mathbb{A}, \qquad \alpha\mapsto(\alpha,\alpha^{-1}).$$

DEFINITION OF $\mu_{\mathbb{A}_\star^\times}$. Similarly to 30 define a (multiplicative) translation invariant measure $\mu_{\mathbb{A}_\star^\times}$ using the normalized local measures $\mu_{K_{x,z}^\times}$, $z\in\star(x)$, from 14: $\mu_{K_{x,z}^\times} = (1-q_{x,z}^{-1})^{-1}\mu_{K_{x,z}}/|\,|_{x,z}$ for nonarchimedean $x\in\star$, $z\in\star(x)$; and $\mu_{K_{\omega,y}^\times} = \mu_{K_{\omega,y}}/|\,|_{\omega,y}$. The ring of measurable sets is generated by sets $D = \prod_{x\in\star} D_{x,\star}\cap\mathbb{A}_\star^\times$ where $D_{x,\star}\subset\mathcal{O}_{x,\star}^\times$ and $D_{x,\star} = O_{x,\star}^\times$ for almost all $x\in\star$; and then $\mu_{\mathbb{A}_\star^\times}(D) = \prod_{x\in\star}\mu_{K_{x,\star}^\times}(D_{x,\star})\in\mathbb{R}((X))$.

DEFINITION OF SPACE $R_{\mathbb{A}_\star^\times}$ AND INTEGRAL $\int f_\star d\mu_{\mathbb{A}_\star^\times}$. Define $R_{\mathbb{A}_\star^\times}$ as the linear space generated by $f_\star$ where $f_\star = \otimes_{x\in\star} f_{x,\star}$ is a continuous function on $\mathbb{A}_\star^\times$, with $f_{x,\star}\in R_{K_{x,\star}}$ for all $x\in\star$, such that $f_{x,\star}|_{O_{x,\star}^\times} = 1$ for almost all $x\in\star$. Define

$$\int f_\star d\mu_{\mathbb{A}_\star^\times} = \prod_{x\in\star}\int f_{x,\star}d\mu_{K_{x,\star}^\times}$$

and extend by linearity to $R_{\mathbb{A}_\star^\times}$.

If $f_\star = h_\star\circ p_\star$ for an integrable continuous function $h_\star$ on $\mathbb{A}_{k(\star)}^\times$ then $\int f_\star d\mu_{\mathbb{A}_\star^\times} = \int h_\star d\mu_{\mathbb{A}_{k(\star)}^\times}$ where $\mu_{\mathbb{A}_{k(\star)}^\times}$ is the multiplicative measure associated to the measure $\mu_{\mathbb{A}_{k(\star)}}$ defined in Lemma 1 of 30.

DEFINITION OF $\mu_{\mathbb{A}^\times}$, SPACE $R_{\mathbb{A}^\times}$ AND INTEGRAL $\int f\, d\mu_{\mathbb{A}^\times}$. Define $\mu_{\mathbb{A}^\times} = \otimes_{\star \in S_{\prime}} \mu_{\mathbb{A}_\star^\times}$. Define the space $R_{\mathbb{A}^\times}$ as the linear space generated by $\otimes_{\star \in S_{\prime}} f_\star$ with $f_\star \in R_{\mathbb{A}_\star^\times}$ for all $\star$.

For $f = \otimes f_\star \in R_{\mathbb{A}^\times}$ with $f_\star \in R_{\mathbb{A}_\star^\times}$ define

$$\int f\, d\mu_{\mathbb{A}^\times} = \prod_{\star \in S_{\prime}} \int f_\star\, d\mu_{\mathbb{A}_\star^\times}$$

(the product may diverge) and extend by linearity to $R_{\mathbb{A}^\times}$. Thus, the integral $\int f\, d\mu_{\mathbb{A}^\times}$ takes values in $\mathbb{C}((X))$ if converges.

The space $Q_{\mathbb{A}}$ is a subspace of $R_{\mathbb{A}^\times}$. Unlike the spaces $R'_{\mathbb{A}}$, $R_{\mathbb{A}}$, $Q_{\mathbb{A}}$ with respect to $\mu_{\mathbb{A}}$, the space $R_{\mathbb{A}^\times}$ includes nonintegrable functions with respect to $\mu_{\mathbb{A}^\times}$.

One can also define a multiplicative invariant measure on $\mathfrak{A}^\times$, the space $R_{\mathfrak{A}^\times}$ and the integral against this measure.

Example 1 Let $S_{\prime}$ consist of all fibres. Let $f = \otimes_\star f_\star$, $f_\star = \otimes_{x \in \star} f_{x,\star}$, where

$$f_{x,z} = |\,|_{x,z}^s\, char_{t_{1\,x,z}^{c_{x,z}} O_{x,z}}, \quad s \in \mathbb{C},$$

and where for every $\star \in S_{\prime}$ for almost all $x \in \star$ we have $c_{x,z} = 0$ for $z \in \star(x)$.

Then $f_\star \in R_{\mathbb{A}_\star^\times}$ and

$$\int f_\star\, d\mu_{\mathbb{A}_\star^\times} = \prod_{x \in \star} \prod_{z \in \star(x)} q_{x,z}^{e_{x,z}/2 - c_{x,z} s} \frac{1}{1 - q_{x,z}^{-s}},$$

$e_{x,z}$ is defined in Example 1 of 30.

We get

$$\int f\, d\mu_{\mathbb{A}^\times} = \prod_{\star \in S_{\prime}} \Bigg(\prod_{x \in \star} \prod_{z \in \star(x)} q_{x,z}^{e_{x,z}/2 - c_{x,z} s} \frac{1}{1 - q_{x,z}^{-s}} \Bigg).$$

For the product of the exponential factors to converge we need in addition the condition $\prod_{x \in \star} q_{x,\star}^{c_{x,\star}} = 1$ for almost all $\star \in S_{\prime}$. If so, then $\prod_{\star \in S_{\prime}} \int f_\star\, d\mu_{\mathbb{A}_\star^\times} \in \mathbb{C}$ absolutely converges for $\Re(s) > 2$, as follows from its comparison with the zeta function of $\mathcal{S}$ which is known to absolutely converge in that half-plane, see 38.

Note that according to 30 the measure μ is defined on $\mathbb{A}$ if and only if $\prod_{x \in \star} q_{x,\star}^{d_{x,\star}} = 1$ for almost all fibres $\star$. The function $g = \otimes_{\star \in S_{\prime}} \otimes_{x \in \star} char_{t_{1\,x,z}^{c_{x,z}} O_{x,z}}$ belongs to $Q_{\mathbb{A}}$ if and only if $\prod_{x \in \star} q_{x,\star}^{d_{x,\star}/2 - c_{x,\star}} = 1$ for almost all $\star \in S_{\prime}$.

Thus, if the measure is defined on $\mathbb{A}$, i.e. the generic fibre of $\mathcal{S}$ is of genus 1, then for every $g \in Q_{\mathbb{A}}$, $s \in \mathbb{C}$, we get $|\,|^s g \in R_{\mathbb{A}^\times}$ and $\int g |\,|^s\, d\mu_{\mathbb{A}^\times}$ absolutely converges if $\Re(s) > 2$.

When the generic fibre of $\mathcal{S}$ is of genus different from 1, one has to renormalize fibre integrals to ensure the convergence of their infinite product, see 57.

DEFINITION OF $(\mathbb{A}\times\mathbb{A})^\times$, $\mu_{(\mathbb{A}\times\mathbb{A})^\times}$, SPACE $R_{(\mathbb{A}\times\mathbb{A})^\times}$ AND INTEGRAL $\int f\,d\mu_{(\mathbb{A}\times\mathbb{A})^\times}$. The local normalized measures are $\mu_{(K_{x,z}\times K_{x,z})^\times} = (1-q_{x,z}^{-1})^{-2}\mu_{K_{x,z}\times K_{x,z}}/|\,|_{x,z}$ for nonarchimedean x,y, and $\mu_{(K_{\omega,y}\times K_{\omega,y})^\times} = \mu_{K_{\omega,y}\times K_{\omega,y}}/|\,|_{\omega,y}$. Here $|\,|_{x,z}$ is the module on the product of two copies of $K_{x,z}$.

Similar to the above, $(\mathbb{A}\times\mathbb{A})^\times = \mathbb{A}^\times\times\mathbb{A}^\times$ is the restricted product of $(\mathbb{A}_\star\times\mathbb{A}_\star)^\times$ with respect to $\big(O(\mathbb{A}_\star\times\mathbb{A}_\star)\big)^\times$.

Endow $G_a^4(\mathbb{A})$ with the topology similar to the case of $G_a^2(\mathbb{A})$. Define the topology on $(\mathbb{A}\times\mathbb{A})^\times$ as the induced from $G_a^4(\mathbb{A})$ via $(\mathbb{A}\times\mathbb{A})^\times \longrightarrow G_a^4(\mathbb{A})$, $(\alpha,\beta)\mapsto(\alpha,\alpha^{-1},\beta,\beta^{-1})$.

Define $\mu_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times} = \mu_{\mathbb{A}_\star^\times}\otimes\mu_{\mathbb{A}_\star^\times}$, $\mu_{(\mathbb{A}\times\mathbb{A})^\times} = \otimes_{\star\in S,}\mu_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$.

Define $R_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times} = R_{\mathbb{A}_\star^\times}\otimes R_{\mathbb{A}_\star^\times}$. For $f_\star = f_\star^{(1)}\otimes f_\star^{(2)}$ with $f_\star^{(m)}\in R_{\mathbb{A}_\star^\times}$ define the integral $\int f_\star\,d\mu_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times} = \int f_\star^{(1)}\,d\mu_{\mathbb{A}_\star^\times}\int f_\star^{(2)}\,d\mu_{\mathbb{A}_\star^\times}$ and extend by linearity to $R_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$.

If $f_\star = h_\star\circ p_\star$ for an integrable continuous function $h_\star$ on $(\mathbb{A}_{k(\star)}\times\mathbb{A}_{k(\star)})^\times$ then $\int f_\star\,d\mu_{\mathbb{A}_\star^\times} = \int h_\star\,d\mu_{(\mathbb{A}_{k(\star)}\times\mathbb{A}_{k(\star)})^\times}$ where $\mu_{(\mathbb{A}_{k(\star)}\times\mathbb{A}_{k(\star)})^\times}$ is the tensor product of $\mu_{\mathbb{A}_{k(\star)}^\times}$.

Define a functional space $R_{(\mathbb{A}\times\mathbb{A})^\times}$ as the linear space generated by $\otimes_{\star\in S,}f_\star$, $f_\star\in R_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$.

For $f=\otimes f_\star\in R_{(\mathbb{A}\times\mathbb{A})^\times}$ with $f_\star\in R_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$ define

$$\int f\,d\mu_{(\mathbb{A}\times\mathbb{A})^\times} = \prod_{\star\in S,}\int f_\star\,d\mu_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$$

(the product may diverge) and extend by linearity to $R_{(\mathbb{A}\times\mathbb{A})^\times}$.

In particular, $Q_{\mathbb{A}\times\mathbb{A}}$ is a subset of $R_{(\mathbb{A}\times\mathbb{A})^\times}$.

Similarly to the above we deduce that if the generic fibre of $\mathcal{S}$ is of genus 1, then for every $g\in Q_{\mathbb{A}\times\mathbb{A}}$, $s\in\mathbb{C}$, we get $|\,|^s g\in R_{(\mathbb{A}\times\mathbb{A})^\times}$ and the integral $\int g|\,|^s\,d\mu_{(\mathbb{A}\times\mathbb{A})^\times}$ absolutely converges if $\Re(s)>2$.

Remark 1 The theory of measure and integration on $\mathbb{A}^\times$ is in some sense simpler than the theory on $\mathbb{A}$, since $\mathbb{A}_\star^\times$ coincides with the preimage of its image with respect to $p_\star\colon\mathbb{A}_\star\longrightarrow\mathbb{A}_{k(\star)}$. Almost all functions we work with in the study of an *unramified* zeta integral are pullbacks of functions on $\mathbb{A}_{k(\star)}$. For the purposes of the application to the unramified zeta integral we can define the weak $\mathbb{R}$-valued measure $\mu_{\mathbb{A}_\star^\times}^w$ on $\mathbb{A}_\star^\times$ as the pullback of the measure on $\mathbb{A}_{k(\star)}^\times$, and the weak $\mathbb{R}$-valued measure $\mu_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}^w$ on $(\mathbb{A}_\star\times\mathbb{A}_\star)^\times$ as the pullback of the measure on $(\mathbb{A}_{k(\star)}\times\mathbb{A}_{k(\star)})^\times$. For a function $f=p_\star^*(h)\in Q$ with a function h at the residue level the integral $\int f\,d\mu^w$ coincides with the integral $\int f\,d\mu$.

Remark 2 In the study of the zeta integral we will mainly work with $(\mathbb{A}\times\mathbb{A})^\times$. We will introduce a subgroup $T < (\mathbb{A}\times\mathbb{A})^\times$ in 36 and an integral denoted $\int_T g$ in 37 which *is not* equal to $\int g\, char_T\, d\mu_{(\mathbb{A}\times\mathbb{A})^\times}$. It will differ from the latter at $(x,\star)$, x is a singular point of $\star$.

Finally, we extend the measure and integration from $\mathbb{B}$ to $\mathbb{B}\times\mathbb{B}$ in the most straightforward way.

DEFINITION OF $\mathbb{B}\times\mathbb{B}$, $\mu_{\mathbb{B}\times\mathbb{B}}$, AND INTEGRAL. Using $\mathbb{B}$ define the adelic object $\mathbb{B}\times\mathbb{B}$ in the similar way to the definition of $\mathbb{A}\times\mathbb{A}$ using $\mathbb{A}$.

Using 13 we get the $\mathbb{R}((X))$-valued translation invariant measure $\mu_{\mathbb{B}_\star\times\mathbb{B}_\star}$ on $\mathbb{B}_\star\times\mathbb{B}_\star$ which lifts the discrete counting measure on $k(\star)\times k(\star)$, similar to the definition of the measure on $\mu_{\mathbb{B}_\star}$.

The adelic object $\mathbb{B}\times\mathbb{B}$ comes with its measure

$$\mu_{\mathbb{B}\times\mathbb{B}} = \otimes_{\star\in S_\prime}\mu_{\mathbb{B}_\star\times\mathbb{B}_\star}.$$

For $f \in Q_{\mathbb{A}\times\mathbb{A}}$ define its integral against the measure $\mu_{\mathbb{B}\times\mathbb{B}}$ similarly to the definition of the integral against the measure $\mu_{\mathbb{B}}$.

DEFINITION OF $\mu_{\mathbb{B}^\times}, \mu_{(\mathbb{B}\times\mathbb{B})^\times}$, AND INTEGRAL. The measure on $k(\star)$ is counting discrete and the discrete measure on $k(\star)^\times$ is induced by it. Define the measure on $\mathbb{B}_\star^\times$ and on $(\mathbb{B}_\star\times\mathbb{B}_\star)^\times$ as induced from the measure on $\mathbb{B}_\star$ in 30 and on $\mathbb{B}_\star\times\mathbb{B}_\star$. So these measures are just lifts of the discrete measures on $k(\star)^\times$ and on $(k(\star)\times k(\star))^\times$.

Similarly define the measure on $\mathbb{B}^\times$ and $(\mathbb{B}\times\mathbb{B})^\times$ as the tensor product of the measures on the components. So these measures are just induced by the measures on $\mathbb{B}$ and $\mathbb{B}\times\mathbb{B}$. We get $\int_{\mathbb{B}^\times} f\, d\mu_{\mathbb{B}^\times} = \int_{\mathbb{B}} f\, char_{\mathbb{B}^\times}\, d\mu_{\mathbb{B}}$.

For $f = \otimes f_\star \in Q_{\mathbb{A}\times\mathbb{A}}$, $f_\star = g_\star\circ p_\star$ where $g_\star$ is an integrable function on $\mathbb{A}_{k(\star)}\times\mathbb{A}_{k(\star)}$ and a subset $C = \prod'_\star C_\star < (\mathbb{B}\times\mathbb{B})^\times$, $C_\star = p_\star^{-1}(B_\star)$ for a measurable subset $B_\star \subset p_\star(\mathbb{B}_\star\times\mathbb{B}_\star)$ define the integral $\int_C f\, d\mu_{(\mathbb{B}\times\mathbb{B})^\times}$ as $\prod_{\star\in S_\prime}\int_{p_\star(C_\star)} g_\star\, d\mu_{k(\star)\times k(\star)}$ and extend to $f\in Q_{\mathbb{A}\times\mathbb{A}}$.

32. In this section we get summation formulas on $\mathbb{B}$ and $\mathbb{B}\times\mathbb{B}$ in the case of finite $S_\prime$. These formulas lift the one-dimensional summation formulas.

Lemma *Let $f\in Q_{\mathbb{A}}$. Assume the set $S_\prime$ in* 29 *is finite. Then the integral $\int f(\beta)\,d\mu_{\mathbb{B}}(\beta)$ is finite and*

$$\int f(\beta)\,d\mu_{\mathbb{B}}(\beta) = \int \mathcal{F}(f)(\beta)\,d\mu_{\mathbb{B}}(\beta).$$

Proof: Use the definition of the Q-spaces and the calculation of the transform function in 9 (the transform of $g \circ p$ is $\mathcal{F}(g) \circ p$) and reduce to the one-dimensional case on every fibre and horizontal curve. $\square$

Remark 1 Define the measure on $\mathfrak{A} \cap \mathbf{A}_y$ as the lift of the measure on $\mathbb{A}_{k(y)}$ following 13, and then define the measure on $\mathfrak{A}$ and $\mathfrak{B}$, as above. The previous Lemma holds for these adelic objects and measures as well.

Remark 2 Using the measures $\mu_{\mathbb{A}_\star}$ and $\mu_{\mathbb{B}_\star}$, one can define the measure $\mu_{\mathbb{A}_\star/\mathbb{B}_\star}$ on $\mathbb{A}_\star/\mathbb{B}_\star$ which lifts the appropriate one-dimensional quotient measure $\mu_{\mathbb{A}_{k(\star)}/k(\star)}$ so that $\mu(\mathbb{A}_\star/\mathbb{B}_\star) = \mu(\mathbb{A}_{k(\star)}/k(\star))$. We have $\mu(\mathbb{A}_y/\mathbb{B}_y) = |\rho|_y^{-1/2} = q_y^{1-g_y}$ for a nonsingular fibre y, ρ is defined in 29, see also Remark 3 in 30. Hence, if the genus of the generic fibre of $\mathcal{S}$ is one, then we have the well defined number $\mu(\mathbb{A}/\mathbb{B})_{S_\prime} = \prod_{\star \in S_\prime} \mu(\mathbb{A}_\star/\mathbb{B}_\star)$ for infinite $S_\prime$.

The previous Lemma and Lemma 1 of 30 imply the following *summation formula on* $\mathbb{B}$: for $\alpha \in \mathbb{A}^\times$, $f \in Q_{\mathbb{A}}$ and finite $S_\prime$ we get

$$\int f(\alpha\beta)\, d\mu_{\mathbb{B}}(\beta) = \frac{1}{|\alpha|} \int \mathcal{F}(f)(\alpha^{-1}\beta)\, d\mu_{\mathbb{B}}(\beta),$$

where $\mathcal{F}(f)$ is the transform of f on $\mathbb{A}$, $|\,|$ is the module on $\mathbb{A}$ defined in 30.

If $f = char_{O_{\mathbb{A}_\star}}$ the previous formula and Remark in 27 immediately give the Riemann–Roch formula for the curve $\star$.

We can easily extend the previous constructions to the adelic object $\mathbb{A} \times \mathbb{A}$.

Proposition *Let $f \in Q_{\mathbb{A}\times\mathbb{A}}$. Let $S_\prime$ be finite. Then*

$$\int f(\beta)\, d\mu_{\mathbb{B}\times\mathbb{B}}(\beta) = \int \mathcal{F}(f)(\beta)\, d\mu_{\mathbb{B}\times\mathbb{B}}(\beta)$$

and

$$\int f(\alpha\beta)\, d\mu_{\mathbb{B}\times\mathbb{B}}(\beta) = \frac{1}{|\alpha|} \int \mathcal{F}(f)(\alpha^{-1}\beta)\, d\mu_{\mathbb{B}\times\mathbb{B}}(\beta)$$

for $\alpha \in (\mathbb{A} \times \mathbb{A})^\times$, $|\,|$ is the module on $\mathbb{A} \times \mathbb{A}$ defined in 31.

2. Two-dimensional K-delic objects

In this short part for an arithmetic surface $\mathcal{S}$ we introduce a K_2-delic object $C_{\mathcal{S}} = J_{\mathcal{S}}/P_{\mathcal{S}}$ which via two-dimensional class field theory describes abelian extensions of the field of rational functions of $\mathcal{S}$. In 35 we modify $J_{\mathcal{S}}/P_{\mathcal{S}}$ to another object J/P which will be used in the study of the zeta integrals. Similar to the one-dimensional case, in the study of the zeta integrals one does not use class field theory, but rather appropriate K-delic objects which naturally originate from it. We continue to follow the agreement on the notation in 24.

33. Definition of $I_{\mathcal{S}}$. The topological K_2^t-groups of two-dimensional local fields were defined in 14 and 22. For their basic properties see [10], [67]. Let

$$I_{\mathcal{S}} = \prod_y \prod_{x \in y} K_2^t(K_{x,y}) \times \prod_{\sigma,\omega} K_2^t(K_{\omega,\sigma}),$$

where y runs through curves on $\mathcal{S}$ and x runs through places of $k(y)$.

Using the connecting homomorphisms in K-theory (which are called border or boundary homomorphisms) we get the following their compositions

$$K_2^t(K_{x,y}) \longrightarrow K_1(E_{x,y}) \longrightarrow K_0(k_y(x)), \quad K_2^t(K_{x,y}) \longrightarrow K_1(E_{x,y}) \longrightarrow K_1(k_y(x)),$$
$$K_2^t(K_{x,y}) \longrightarrow K_2^t(E_{x,y}) \longrightarrow K_1(k_y(x)),\ K_2^t(K_{\omega,y}) \longrightarrow K_1(k(y)_\omega),\ K_2^t(K_{\omega,\sigma}) \longrightarrow K_1(k_\sigma(\omega)).$$

Since the border homomorphisms from K_n of a henselian field to $K_n \oplus K_{n-1}$ of its residue field are surjective, the previous compositions induce the following surjective homomorphism

$$\begin{aligned} I_{\mathcal{S}} \longrightarrow \prod_y \prod_{x \in y} K_0(k_y(x)) \oplus \prod_y \prod_{x \in y} K_1(k_y(x)) \oplus \prod_y \prod_{x \in y} K_1(k_y(x)) \\ \oplus \prod_{\omega,y} K_1(k(y)_\omega) \oplus \prod_{\sigma,\omega} K_1(k_\sigma(\omega)), \end{aligned}$$

whose kernel we denote by $I_{\mathcal{S}}^0$.

The choice of local parameters $t_{1x,y}, t_{2x,y}$, t_ω and of lifts from the residue field level gives the splitting

$$\begin{aligned} I_{\mathcal{S}} = I_{\mathcal{S}}^0 \oplus \prod_y \prod_{x \in y} K_0(k_y(x)) \oplus \prod_y \prod_{x \in y} K_1(k_y(x)) \\ \oplus \prod_y \prod_{x \in y} K_1(k_y(x)) \oplus \prod_{\omega,y} K_1(k(y)_\omega) \oplus \prod_{\sigma,\omega} K_1(k_\sigma(\omega)). \end{aligned}$$

Definition of $V\mathbb{A}_{\mathcal{S}}^\times$, $VJ_{\mathcal{S}}$. Introduce a subgroup $V\mathbb{A}_{\mathcal{S}}^\times$ of $\mathbb{A}_{\mathcal{S}}^\times$ which at the nonarchimedean data equals $\mathbb{A}_{\mathcal{S}}^\times \cap \prod O_{x,y}^\times$ and whose archimedean components are all equal 1. Endow it with the induced topology from the nonarchimedean part of $\mathbb{A}_{\mathcal{S}}^\times$, see 31.

Denote by $K_2^t(K_{\omega,y})_0$ (resp. $K_2^t(K_{\omega,\sigma})_0$) the kernel of the border map from $K_2^t(K_{\omega,y})$ (resp. $K_2^t(K_{\omega,\sigma})$) to K_1 of the residue field, i.e. the cyclic group generated by $\{-1,-1\}$.

The following local maps at the nonarchimedean data

$$\mathfrak{q}_{x,z} \colon (\alpha_{1x,z}, \alpha_{2x,z}) \mapsto \{t_{1x,z}, \alpha_{1x,z}\} + \{\alpha_{2x,z}, t_{2x,z}\}$$

and the zero map at the archimedean data induce

$$\mathfrak{q}\colon V\mathbb{A}_S^\times \times V\mathbb{A}_S^\times \longrightarrow \prod K_2^t(K_{x,y}).$$

Alternatively, at the nonarchimedean data one can use the homomorphism $\mathfrak{t}$ defined below in 36.

Denote by VJ_S the sum of the image of $\mathfrak{q}$ (which equals the image of $\mathfrak{t}$) and of $\oplus_{y,\omega} K_2^t(K_{\omega,y})_0$, and endow it with the product topology of the induced topology from $V\mathbb{A}_S^\times \times V\mathbb{A}_S^\times$ and the discrete topology on $\oplus_{y,\omega} K_2^t(K_{\omega,y})_0$.

DEFINITION OF J_S. Define a K-delic subgroup J_S of the group I_S:

$$J_S = VJ_S + \big(\oplus_{y,x} K_0(k_y(x)) \oplus 0 \oplus 0 \oplus \oplus_{\omega,y} K_1(k(y)_\omega) \oplus \oplus_{\sigma,\omega} K_1(k_\sigma(\omega))\big),$$

the sum is actually the inner direct sum; note the absence of $K_1(k_y(x))$-terms.

Define the topology on J_S as the sequential saturation of the product of the topology on VJ_S, the discrete topology on the K_0-term, and the lift of the discrete topology on $K_1(k(y)_\omega)/2K_1(k(y)_\omega)$ and on $K_1(k_\sigma(\omega))/2K_1(k_\sigma(\omega))$. So far all this depends on the choice of lifts and local parameters.

The group J_S can be viewed as the restricted product

$$\prod_y{}' \prod_{x\in y}{}' K_2^t(K_{x,y}) \times \prod_{\sigma,\omega}{}' K_2^t(K_{\omega,\sigma}), \qquad \prod_{\sigma,\omega}{}' = \prod_\sigma \prod_{\omega\in S_0^\sigma}{}' \quad .$$

Every element in the first restricted product has only finitely many (x,y)-components not lying in $\mathfrak{q}_{x,y}(O_{x,y}^\times \times O_{x,y}^\times) = K_2^t(O_{x,y})$.

Denote the nonarchimedean part of J_S by $J_{S,\mathrm{na}}$. Decompose $J_S = J_{S,\mathrm{na}} \times J_{S,\mathrm{a}}$ where $J_{S,\mathrm{a}}$ is the archimedean part of J_S sitting on the 'archimedean fibres' of S and on the 'archimedean points' of horizontal curves.

Denote by $\mathbb{A}_{S,\mathrm{na}}$ (resp. $\mathbf{A}_{S,\mathrm{na}}$) the nonarchimedean part of $\mathbb{A}_S$ (resp. of $\mathbf{A}_S$), i.e. the parts associated to all the closed points and curves passing through them on S.

It is easy to see that the local symbol maps $(\alpha,\beta) \mapsto \{\alpha,\beta\}$ induce the adelic symbol map whose image lies in J_S:

$$\mathbf{A}_{S,\mathrm{na}}^\times \times \mathbf{A}_{S,\mathrm{na}}^\times \longrightarrow J_S.$$

Using the surjectivity of the nonarchimedean local maps $\mathfrak{q}_{x,y}$ we get the following

Lemma *The nonarchimedean part $J_{S,\mathrm{na}}$ of J_S coincides with the subgroup generated by the image of $\mathbf{A}_{S,\mathrm{na}}^\times \times \mathbf{A}_{S,\mathrm{na}}^\times$ with respect to the symbol map.*

Proof: Use the definitions and the equality for $\{1-\alpha, 1-\beta\}$ in 16. $\square$

One corollary of this Lemma is that J_S/nJ_S for every $n \geq 1$ does not depend on the choice of local parameters and lifts from the residue level.

34. DEFINITION OF $P_{\mathcal{S}}$. Define

$$P_{\mathcal{S}} = \Delta\prod_{y}{}' K_2(K_y) + \Delta\prod_{x}{}' K_2(K_x) + \Delta\prod_{\sigma} K_2(K_\sigma),$$

where the restricted product signs mean the intersection of the product with $\Delta^{-1}(J_{\mathcal{S}})$ and Δ are obvious diagonal maps induced by field embeddings.

Endow $J_{\mathcal{S}}/P_{\mathcal{S}}$ with the induced topology from $J_{\mathcal{S}}$.

Using Kato–Saito's higher class field theory [29], [30] and working with the adelic version of explicit higher local class field theory [9] one can prove the following

Theorem *Let $\mathcal{S}$ be as in 24. Let K be its function field.*

The product $\Phi_{\mathcal{S}}$ of the local reciprocity maps $\Phi_{x,z}\colon K_2^t(K_{x,z}) \longrightarrow \mathrm{Gal}(K_{x,z}^{\mathrm{ab}}/K_{x,z})$ vanishes on $P_{\mathcal{S}}$.

Continuous characters of finite order of the Galois group of K are in one-to-one correspondence with continuous characters of finite order of the K-delic class group $C_{\mathcal{S}} = J_{\mathcal{S}}/P_{\mathcal{S}}$ via the reciprocity map

$$\Phi_{\mathcal{S}}\colon J_{\mathcal{S}}/P_{\mathcal{S}} \longrightarrow \mathrm{Gal}(K^{ab}/K).$$

For a finite Galois extension L/K the homomorphism $\Phi_{\mathcal{S}}$ induces

$$J_{\mathcal{S}}/(P_{\mathcal{S}} + N_{\mathcal{S}\times_K L/\mathcal{S}} J_{\mathcal{S}\times_K L}) \xrightarrow{\sim} \mathrm{Gal}(L\cap K^{ab}/K).$$

The reciprocity map restricted on $\prod_\sigma\prod_\omega' K_2^t(K_{\omega,\sigma})$ factorizes through the quotient of the group $\prod_{\sigma\ real}\prod_\omega' K_2^t(K_{\omega,\sigma})/2$.

This Theorem is not used in this text, and its proof is not included.

35. Using the first composed map among the five types in 33, consider the composite map

$$\mathfrak{i}_{x,z}\colon K_2^t(K_{x,z}) \longrightarrow K_1(E_{x,z}) \longrightarrow K_0(k_z(x)) \longrightarrow K_0(k(x)),$$

where the last map is multiplication by the degree of $k_z(x)/k(x)$. Denote by $UK_2^t(K_{x,z})$ the kernel of $\mathfrak{i}_{x,z}$.

Set $\mathfrak{i}_x = \sum_{y\ni x}\sum_{z\in y(x)}\mathfrak{i}_{x,z}$ and $\mathfrak{i} = \oplus_{x\in S_0} i_x$. Thus we get

$$\mathfrak{i}\colon J_{\mathcal{S},\mathrm{na}} \longrightarrow C_0(\mathcal{S}) = \oplus_{x\in S_0} K_0(k(x)).$$

The group $UJ_{\mathcal{S},\mathrm{na}}$ is defined as the intersection with $J_{\mathcal{S},\mathrm{na}}$ of the product of the local groups $UK_2^t(K_{x,y})$ over all nonarchimedean data (x,y). The kernel of the homomorphism $\mathfrak{i}$ contains $UJ_{\mathcal{S},\mathrm{na}}$.

If w is a local (i.e. formal or infinitesimal) curve passing through x, i.e. a prime ideal of height 1 of $\mathcal{O}_x$, not necessarily coming from a curve on the surface, we can work with a two-dimensional local field $K_{x,w}$ and similarly define the map $\mathfrak{i}_{x,w}$.

The K-localization theory, see e.g. [47], for $\operatorname{Spec}\mathcal{O}_x$ gives the exact sequence for regular points x of $\mathcal{S}$

$$K_2(\mathcal{O}_x) \longrightarrow K_2(K'_x) \longrightarrow \oplus_{w\ni x} K_1(E_{x,w}) \longrightarrow K_0(k(x)) \longrightarrow 0,$$

where w runs through all local curves passing through x (see 24 for the definition of K'_x). We get the induced exact sequence

$$K_2(K_x) \longrightarrow \oplus_{y\ni x} \oplus_{z\in y(x)} K_1(E_{x,z}) \longrightarrow K_0(k(x)) \longrightarrow 0.$$

DEFINITION OF S^* AND $S_\prime$. Choose a finite set S^* of horizontal curves. *In characteristic zero include in S^* at least one horizontal curve.*

Put

$$S_\prime = S_\shortmid \cup S^*.$$

The set $S_\prime$ will be further specified in 36 and then in 40.

DEFINITION OF J AND P. Put $UK_2^t(K_{\omega,y}) = K_2^t(K_{\omega,y})$. Denote by J the subgroup of $J_{\mathcal{S}}$ such that every its (x,y)-components for every horizontal curve $y \in S_h \setminus S^*$ lies in $UK_2^t(K_{x,y})$. We have

$$J = \prod_{y\in S_\prime}{}' \prod_{x\in y}{}' K_2^t(K_{x,y}) \oplus \prod_{y\notin S_\prime}{}' \prod_{x\in y}{}' UK_2^t(K_{x,y}) \oplus \prod_{\sigma} \prod_{\omega}{}' K_2^t(K_{\omega,\sigma}).$$

Denote by $J_{\mathrm{na}} := J \cap J_{\mathcal{S},\mathrm{na}}$ the nonarchimedean part of J, and by $J_{\mathrm{a}} := J \cap J_{\mathcal{S},\mathrm{a}}$ the archimedean part of J.

Denote

$$P = J \cap P_{\mathcal{S}}.$$

DEFINITION OF $|\,|_2$. Using the local modules $|\,|_{2,x,y}$ defined in 15 and the modules $|\,|_{2,\omega,\sigma}$ defined in 22, and the norm maps from $k_z(x)$ to $k(x)$, define the following K_2-module homomorphism

$$|\,|_2 \colon J_{\mathcal{S}} \longrightarrow \mathbb{R}^\times_{>0} = (0,+\infty)$$

as their product.

Denote the kernel of $|\,|_2$ by $J^1_{\mathcal{S}}$. Put $UJ_{\mathcal{S}} = J^1_{\mathcal{S}} \cap (J_{\mathcal{S},\mathrm{a}} + UJ_{\mathcal{S},\mathrm{na}})$. Then we have $VJ_{\mathcal{S}} < UJ_{\mathcal{S}} < J$ and $UJ_{\mathcal{S}} \cap J_{\mathcal{S},\mathrm{na}} = VJ_{\mathcal{S}} \cap J_{\mathcal{S},\mathrm{na}}$.

Denote $J^1 = J \cap J^1_{\mathcal{S}}$, $UJ = UJ_{\mathcal{S}}$.

Lemma *The homomorphism* $\mathfrak{i}$ *induces*

$$J^1_{\mathcal{S}}/(P_{\mathcal{S}}+UJ_{\mathcal{S}}) = J^1_{\mathcal{S},\mathrm{na}}/(P_{\mathcal{S}}\cap J_{\mathcal{S},\mathrm{na}}+UJ_{\mathcal{S},\mathrm{na}}) \simeq CH_0(\mathcal{S})^0,$$

where the right hand side is the (*degree zero in positive characteristic*) *part of the Chow group of zero cycles of* $\mathcal{S}$.

Suppose that all singular points x *of fibres* $\star$ *are split, i.e. there is* $z\in\star(x)$ *such that* $k_z(x)=k(x)$. *Then for* $x\in S_0$

$$\begin{aligned}
&\mathfrak{i}_x(J_{\mathrm{na}}+\Delta{\prod}' K_2(K_x)) = \mathfrak{i}_x(J_{\mathcal{S},\mathrm{na}}),\\
&J_{\mathcal{S},\mathrm{na}} = (J+P_{\mathcal{S}})\cap J_{\mathcal{S},\mathrm{na}}+UJ_{\mathcal{S},\mathrm{na}},\\
&J_{\mathcal{S}}/(P_{\mathcal{S}}+UJ_{\mathcal{S}}) \simeq J/(P+UJ),
\end{aligned}$$

so $J^1/(P+UJ)\simeq CH_0(\mathcal{S})^0$.

Proof: Follows immediately from the definitions, the localization sequence and the description of the kernel of $\mathfrak{i}$. □

DEFINITION OF N. Denote

$$N = |J_{\mathcal{S}}|_2 = |J|_2.$$

The $|\,|_2$-module value group N is a relatively nice group: the cyclic group $\{q^n, n\in\mathbb{Z}\}$ generated by the appropriate power q of p in positive characteristic p and $\mathbb{R}^\times_{>0}$ in characteristic zero. This is due to the assumption that there is at least one horizontal curve in $S_\prime$, i.e. $|S^*|\ge 1$.

We get

$$J/J^1 \simeq J_{\mathcal{S}}/J^1_{\mathcal{S}} \simeq N.$$

DEFINITION. Define

$$|\,|_0\colon C_0(\mathcal{S}) \longrightarrow \mathbb{R},\quad \sum a_x x \mapsto \prod |k(x)|^{-a_x}.$$

Then $|\,|_2 = |\,|_0\circ\mathfrak{i}$ on $J_{\mathcal{S},\mathrm{na}}$.

3. Two-dimensional zeta integral and its first properties

From now on we assume that every singular point of every fibre of $\mathcal{S}\longrightarrow B$ *is split ordinary double, if necessary by blowing up at singular points of fibres and passing to a finite extension of the ground field* k. *The set* $S_\prime$ *of curves and fibres on* $\mathcal{S}$ *will consist of all the fibres and of finitely many nonsingular horizontal curves.* If the genus of the generic fibre of $\mathcal{S}$ is 1, then ordinary double points are split ordinary double points and there is no need to pass to a finite extension of the ground field.

Keeping in mind extensions of the theory, several more general objects will be used than strictly necessary for the case of split ordinary double points.

This part consists of seven subparts. We sketch below the construction of the local zeta integral given in [11]. In 3.1 we introduce adelic versions of the local objects of [11]. The zeta integral will be a certain integral over a subgroup $\mathfrak{T}$ of a subgroup T of $(\mathbb{A}\times\mathbb{A})^\times$; T coincides with $(\mathbb{A}\times\mathbb{A})^\times$ if all singular points of fibres are split ordinary double. The subgroup $\mathfrak{T}$ is a twisted version of T, and we use a twisted module map $|| \, ||$ with different scaling for horizontal and vertical curves. In 37 we define integrals $\int_T g$ and $\int_{\mathfrak{T}} g$. These integrals differ from $\int char_T \, g \, d\mu_{(\mathbb{A}\times\mathbb{A})^\times}$, $\int char_{\mathfrak{T}} \, g \, d\mu_{(\mathbb{A}\times\mathbb{A})^\times}$ at the local data associated to singular $x \in \star$. It is natural and inevitable that at the local singular data we have to use new objects.

Subsection 3.3 introduces the two-dimensional zeta integral

$$\zeta(g,\chi) = \int_{\mathfrak{T}} g \, \chi_{\mathfrak{t}}$$

for a good function g on $\mathbb{A} \times \mathbb{A}$ and a character χ of the K_2-dele group J; the function $\chi_{\mathfrak{t}}$ on $\mathfrak{T}$ is the result of using the composite of χ with $\mathfrak{t}$ and some twisting.

In the rest of the text, except 57, we work with a scheme $\mathcal{E}$ corresponding to a proper regular model of an elliptic curve over a global field. In 3.4 we perform the first calculation of the zeta integral. Its horizontal components satisfy the functional equation $s \to 2 - s$. Each of its fibre components equals the square of the zeta function of the fibre times an exponential factor if the fibre contains singular points. Thus we get a comparison of the zeta integral and the square of the zeta function of $\mathcal{E}$, and in particular the convergence of the zeta integral on the half plane $\Re(s) > 2$. For arithmetic schemes which are proper regular models of curves of higher genus, the zeta integral diverges and has to be renormalized, see 57.

As a preparation for the second calculation of the zeta integral, 3.5 introduces a local-global subgroup T_0 of T. To ensure convergence of integrals over T_0 we have to use an appropriate rescaling of the lifting measure from the counting discrete measure at the residue level. Lemma 41 shows that the K_2-objects J and P are compatible modulo units via the map $\mathfrak{t}$ with the $K_1 \times K_1$-objects T and T_0. In 43 we define various integrals for the filtration $\mathfrak{T} > \mathfrak{T}_1 > \mathfrak{T}_0 = T_0$ and its quotient filtration, and Lemma 43 connects the integrals with each other. We work with the spaces $R_{(\mathbb{A}\times\mathbb{A})^\times} \supset Q_{\mathbb{A}\times\mathbb{A}}$ and spaces $R^0_{(\mathbb{A}\times\mathbb{A})^\times} \supset Q^0_{\mathbb{A}\times\mathbb{A}} \supset Q^*_{\mathbb{A}\times\mathbb{A}}$, the last three are defined in 37 and 43.

In 3.6 using the summation formulas from 32, we establish a two-dimensional theta formula for integrals over T_0. It is of independent interest and is supposed to play an important role in future developments. In particular, its structure takes into account both the adelic duality and the class field theory structures.

The second calculation of the zeta integral is performed in 3.7. The two calculations together imply that the analytic properties of the zeta function and the analytic properties of the zeta integral are now reduced to the analytic properties of a certain two-dimensional boundary term defined in 3.7. In the one-dimensional case the corresponding boundary term is of almost trivial structure. In higher dimensions the distance between the additive and K-delic structures is larger and the boundary term structure in dimension two is much more complicated.

Now we briefly review appropriate parts of the local theory of [11]. Assume that F is a nonarchimedean two-dimensional local field, and let $\mathcal{O}$, O be the rings of integers of F with respect to the discrete valuation of rank one and with respect to a discrete valuation of rank two; $O \subset \mathcal{O}$ are preimages with respect to the residue map of the ring of integers of the one-dimensional local field and of the field. As in [11], put $T = \mathcal{O}^\times \times \mathcal{O}^\times$, so choosing local parameters t_2, t_1 in the nonarchimedean residue field case we get $T = \{(t_1^j u_1, t_1^l u_2) : j,l \in \mathbb{Z}, u_1, u_2 \in O^\times\}$. Instead of integrating over the topological Milnor K_2-group whose structure is not completely known in mixed characteristic, we integrate over $K_1 \times K_1$-objects using the local surjective homomorphism defined in 16

$$\mathfrak{t}\colon T \longrightarrow K_2^t(F), \qquad (t_1^i u_1, t_1^j u_2) \mapsto (i+j)\{t_1,t_2\} + \{t_1,u_1\} + \{u_2,t_2\}.$$

The kernel of $\mathfrak{t}$ consists of units which can be ignored as far as the unramified theory is concerned. The homomorphism $\mathfrak{t}$ is related to the homomorphism $\mathfrak{q}$ defined and used in 33.

Using the translation invariant measure $\mu_{F\times F}$ on $F \times F$ and the associated module $|\,|$ we get the measure

$$\mu_{F^\times \times F^\times} = (1-q^{-1})^2 \mu_{F\times F}/|\,|$$

on $F^\times \times F^\times$, where q is the cardinality of the last finite residue field of F.

For $f \in R_{F\times F}$, the space defined in 10, and a continuous quasi-character χ of $K_2^t(F)$, the local zeta integral is

$$\zeta(f,\chi) = \int_T f\, \chi \circ \mathfrak{t}\, d\mu_{F^\times \times F^\times}.$$

This is equivalent to the several other definitions in 17. In general, the zeta integral takes values in $\mathbb{C}((X))$, if converges. If f belongs to $Q_{F\times F}$, the two-dimensional local Bruhat–Schwartz space defined in 10, then the integral takes values in $\mathbb{C}$, if converges. In the adelic theory this local zeta integral will appear on points of fibres.

For example, if $\chi = \chi_0 |\,|_2^s$, where χ_0 is of finite order and trivial on $\{t_1, t_2\}$, then

$$\zeta(f,\chi) = (1-q^{-1})^{-2} \sum_{j,l\in\mathbb{Z}} (q^{-s})^{j+l} \int_{O^\times \times O^\times} f(t_1^j u_1, t_1^l u_2)\, \chi_0\big(\mathfrak{t}(u_1,u_2)\big)\, d\mu_{F\times F}(u_1,u_2).$$

For more details see [11], [12]. See also [39] for the lifting approach to the local theory.

3.1. Group T and integrals over it

Under the assumption about the type of singular points of fibres stated at the beginning of this part 3, Lemma 35 shows that the quotient $J_\mathcal{E}/(P_\mathcal{E} + UJ_\mathcal{E})$, the analogue of the idele class group, is isomorphic to the quotient $J/(P + UJ)$. Working with all the vertical curves and finitely many horizontal curves is also compatible with unramified class field theory for curves over global fields contained in [28].

36. In this and the next section we define various adelic multiplicative analogues of the local objects from [11]; they will be useful for the two-dimensional adelic zeta integral.

DEFINITION OF ARCHIMEDEAN T. For a two-dimensional local field $L = E((t))$, where E is an archimedean local field, its ring of integers is $\mathcal{O}_L = E[[t]]$. Let $p\colon \mathcal{O}_L \longrightarrow E$, $p\colon \mathcal{O}_L^\times \longrightarrow E^\times$ be the residue maps. We slightly extend the definition in 23 and put

$$T_L := \mathcal{O}_L^\times \times \mathcal{O}_L^\times.$$

DEFINITION OF LOCAL $\mathfrak{v}, \mathfrak{u}$, Q_x, $T_{x,\star}$. For a nonsingular point $x \in \star$ we use the local K_1-objects $T_{x,\star} := (K_1 \times K_1)(\mathcal{O}_{x,\star}) = K_1(\mathcal{O}_{x,\star} \times \mathcal{O}_{x,\star}) = \mathcal{O}_{x,\star}^\times \times \mathcal{O}_{x,\star}^\times$. For a singular $x \in \star$ and $z \in \star(x)$ put $T_{x,z} := (K_1 \times K_1)(\mathcal{O}_{x,z})$.

At archimedean $\omega \in y$ we use the $T_{\omega,y} = T_{K_{\omega,y}}$ as in the previous definition.

Now we treat the case of a split ordinary double point x of a fibre $\star$. At the level of groups the object $T_{x,\star}$ will be $\mathcal{O}_{x,\star}^\times \times \mathcal{O}_{x,\star}^\times$, but integrals over these two groups will be different. We also introduce several related constructions, which are expected to be useful in the general case of singular points of fibres as well.

Denote the branches of $\star$ at x by z, z'. Choose an element $t = t_x \in \mathcal{O}_x$ such that its image in $\mathcal{O}_{x,z}$ is a $t_{1\,x,z}$-element there and its image in $\mathcal{O}_{x,z'}$ is a $t_{1\,x,z'}$-element there; in other words, the images with respect to p of the image of t are uniformizers of the local fields $E_{x,z}$ and $E_{x,z'}$. Denote by

$$Q_x := \mathcal{O}_x[t^{-1}]$$

the subring of K'_x. Even though it is defined non canonically, computations of the unramified zeta integral won't depend on the choice of t.

The ring $\mathcal{O}_x$ is isomorphic to the ring $O[[X,Y]]/(XY-\pi)$ where O is the ring of integers of the completion k_v with respect to the place v corresponding to the fibre, π is a prime element of O, and $z=(Y), z'=(X)$. Every element of $O[[X,Y]]/(XY-\pi)$ can be uniquely represented by a series in $O+XO[[X]]+YO[[Y]]$. The two-dimensional local ring of integers we get is $\mathcal{O}_{x,z}=O\{\{X\}\}$, the completion of $O((X))$ with respect to its prime ideal $\pi O((X))$, and similarly $\mathcal{O}_{x,z'}=O\{\{Y\}\}$. We have a natural embedding

$$\mathfrak{z}\colon \mathcal{O}_x \longrightarrow O\{\{X\}\}, \quad g(X,Y)\mapsto g(X,\pi X^{-1})$$

and similarly $\mathfrak{z}'\colon \mathcal{O}_x \longrightarrow O\{\{Y\}\}$ and using them both we get the 'diagonal map' (already used in the definition of $\mathrm{O}_{x,\star}$ in 25)

$$\mathfrak{x}=(\mathfrak{z},\mathfrak{z}')\colon \mathcal{O}_x \longrightarrow O\{\{X\}\}\times O\{\{Y\}\}.$$

It is easy to see $\mathfrak{x}(\alpha)\equiv 0 \mod \pi$ if and only if $\alpha\equiv 0 \mod \pi$.

Recall that $p^{-1}(p(\mathfrak{x}(\mathcal{O}_x)))=\mathrm{O}_{x,\star}$, see 25. Note that for a split ordinary double point $x\in\star$ the fraction ring of $p(\mathcal{O}_x)$ is generated by $p(\mathcal{O}_x)$ and the cyclic group generated by $p(t)$, and so $p^{-1}(p(\mathfrak{x}(Q_x)))=\mathcal{O}_{x,\star}$.

The subgring $\mathfrak{z}(Q_x)$ is a dense subset of $\mathcal{O}_{x,z}$, the subring $\mathfrak{x}(Q_x)$ is a dense subset of $\mathcal{O}_{x,\star}$. The ring Q_x is isomorphic to its image $\mathfrak{z}(Q_x)$ and is isomorphic to its image $\mathfrak{x}(Q_x)$. At the residue level $p(Q_x)$ is not isomorphic to a subring of $p(\mathcal{O}_{x,z})=k(x)((X))$, where $p\colon \mathcal{O}_{x,\star}\longrightarrow E_{x,\star}$ is the residue morphism. Denote the isomorphism

$$\mathfrak{v}\colon \mathfrak{z}(Q_x)\longrightarrow \mathfrak{x}(Q_x)$$

and denote the inverse isomorphism by $\mathfrak{u}$.

$$\begin{tikzcd}
\mathcal{O}_{x,z} & & \mathcal{O}_{x,\star} \\
\mathfrak{z}(Q_x) \arrow[u, hook] \arrow[rr, leftrightarrow, "\mathfrak{u}\qquad\qquad\mathfrak{v}"] & & \mathfrak{x}(Q_x) \arrow[u, hook] \\
& Q_x \arrow[ul, "\mathfrak{z}"] \arrow[ur, "\mathfrak{x}"'] &
\end{tikzcd}$$

The map $\mathfrak{v}$ is not continuous, e.g. $\mathfrak{z}(t^{-2i}Y^i)$ tends to 0 and $\mathfrak{x}(t^{-2i}Y^i)$ does not tend to 0 when i goes to infinity. If $\alpha\equiv\beta \mod \pi$ then $\mathfrak{u}(\alpha)\equiv\mathfrak{u}(\beta) \mod \pi$.

For example, one can take the image of $X+Y$ in $\mathcal{O}_x$ as t. The embeddings $\mathfrak{z},\mathfrak{x}$ extend to ring homomorphisms $\mathfrak{z}\colon Q_x \longrightarrow O\{\{X\}\}$, $\mathfrak{z}'\colon Q_x \longrightarrow O\{\{Y\}\}$, $\mathfrak{x}\colon Q_x \longrightarrow O\{\{X\}\}\times O\{\{Y\}\}$.

By abuse of notation use $\mathfrak{x}$ for $(\mathfrak{x},\mathfrak{x})$, similarly for $\mathfrak{z}$, $\mathfrak{u}$,$\mathfrak{v}$.

We could have defined $T_{x,\star}$ as $\mathfrak{x}(Q_x^\times \times Q_x^\times)$, but it is slightly more convenient to work with its saturated version with respect to p. We define

$$T_{x,\star} := \mathcal{O}_{x,\star}^\times \times \mathcal{O}_{x,\star}^\times.$$

We deduce that the subgroup $\mathfrak{z}(Q_x^\times \times Q_x^\times)$ of $T_{x,z}$ is isomorphic to the subgroup $\mathfrak{x}(Q_x^\times \times Q_x^\times)$ of $T_{x,\star}$, and at the residue level we get $p(\mathfrak{x}(Q_x^\times \times Q_x^\times)) = p(T_{x,\star}) \neq p(T_{x,z})$.

Thus, if all singular points of fibres are split ordinary double, we have the uniform description

$$T_{x,\star} = (K_1 \times K_1)\mathcal{O}_{x,\star}, \quad x \in \star \in S_\prime,$$

at the level of the group structure. However, at the level of the integration the integral over $T_{x,\star}$ will be quite different from the integral over $\mathcal{O}_{x,\star}^\times \times \mathcal{O}_{x,\star}^\times$ for singular $x\in\star$.

Remark 1 The reciprocity map $\Phi_S = \prod_x \prod_{y\ni x} \prod_{z\in y(x)} \Phi_{x,z}$ vanishes on the image of $K_2(K_x)$ in $\oplus_{z\in y(x), y\ni x} K_2^t(K_{x,z})$, see 34. In particular, if x is a singular point of $\star$ then the image of the element $\{X,Y\}\in K_2(K_x)$ is trivial in $K_2^t(K_{x,y})$ if $y\neq\star$ and is $(\{X,\pi X^{-1}\},\{\pi Y^{-1},Y\}) = (\{X,-\pi\},-\{Y,-\pi\}) \in K_2^t(K_{x,\star})$, which should then go to the identity Galois automorphism in K^{ab}/K via the reciprocity map Φ_S. Using local $\mathfrak{t}_{x,z}$ with $t_2 = t_\star = -\pi$ we can write $\{X,-\pi\} = \mathfrak{t}_{x,z}(X,1)$, $-\{Y,-\pi\} = \mathfrak{t}_{x,z'}(Y^{-1},1)$. Hence $(X^i,X^j,Y^k,Y^l)\in T_{x,z}\times T_{x,z'}$ should go to the identity Galois automorphism with respect to the composite of $(\mathfrak{t}_{x,z},\mathfrak{t}_{x,z'})$ and the global reciprocity map, if $i+j+k+l=0$. Factorizing $T_{x,z}\times T_{x,z'}$ by the subgroup $\mathrm{N}_{x,\star}$ generated by such elements and by $O_{x,\star}^\times \times O_{x,\star}^\times$, which do not matter for the unramified class field theory, we arrive at the object which is isomorphic to the quotient of $T_{x,z}$ modulo units. Thus, from the point of view of class field theory, at singular points the minimal $K_1\times K_1$-object which would cover the K_2^t-object and is compatible with the reciprocity map is not the full $(K_1\times K_1)(\mathcal{O}_{x,\star})$ and its image with respect to $(\mathfrak{t}_{x,z},\mathfrak{t}_{x,z'})$, but rather some of its subgroups which is isomorphic (perhaps modulo units) with $(K_1\times K_1)(\mathcal{O}_{x,z})$. It will be natural at the $(x,\star)$-components of the unramified zeta integral to use the integration over $T_{x,z}$, using the map $\mathfrak{u}$.

Remark 2 The definition of $T_{x,\star}$ at singular $x\in\star$ and the definitions of $\mathfrak{u},\mathfrak{v}$ above differ from the abbreviated description given in 4.3 of [14].

Remark 3 For simplicity, in this text we treat the case of split ordinary double points only. An appropriate definition of $T_{x,\star}$ and integration over it can be given for any singular point $x \in \star$, see also Remark 2 in 40.

DEFINITION OF $\mathfrak{t}_{x,\star}$ AND ASSOCIATED COMMUTATIVE DIAGRAMMES. For nonarchimedean (x,z), section 16 and the introduction of this part 3 contain the definition of the surjective homomorphism

$$\mathfrak{t}_{x,z}\colon T_{x,z} = \mathcal{O}_{x,z}^{\times} \times \mathcal{O}_{x,z}^{\times} \longrightarrow K_2^t(K_{x,z})$$

for $x \in \star$ and $z \in \star(x)$. We get the following commutative diagramme

$$\begin{array}{ccccc} & & \mathcal{O}_{x,z}^{\times} \otimes K_{x,z}^{\times}/\mathcal{O}_{x,z}^{\times} & & \\ & & \downarrow & \searrow & \\ T_{x,z} & \longrightarrow & \mathcal{O}_{x,z}^{\times} \times \mathcal{O}_{x,z}^{\times}/O_{x,z}^{\times} & \longrightarrow & K_2^t(K_{x,z})/UK_2^t(K_{x,z}), \end{array}$$

where the vertical map sends $\alpha \otimes t_\star^m$ to $(\alpha^m, 1)$, the horizontal map is induced by $\mathfrak{t}$ and the diagonal map is induced by the symbol map. This diagramme relates the symbol map with the map $\mathfrak{t}_{x,z}$ which we will use for the integration.

For a fibre $\star$ and singular $x \in \star$ define $\mathfrak{t}_{x,\star}$ as

$$(\mathfrak{t}_{x,z})_{z\in\star(x)}\colon \prod_{z\in\star(x)} T_{x,z} = T_{x,\star} \longrightarrow \prod_{z\in\star(x)} K_2^t(K_{x,z}) = K_2^t(K_{x,\star}).$$

Using (x,z)-diagrammes for all $z \in \star(x)$ we get the following commutative diagramme

$$\begin{array}{ccccc} & & \mathcal{O}_{x,\star}^{\times} \otimes K_{x,\star}^{\times}/\mathcal{O}_{x,\star}^{\times} & & \\ & & \downarrow & \searrow & \\ T_{x,\star} & \longrightarrow & \mathcal{O}_{x,\star}^{\times} \times \mathcal{O}_{x,\star}^{\times}/O_{x,\star}^{\times} & \longrightarrow & K_2^t(K_{x,\star})/UK_2^t(K_{x,\star}), \end{array}$$

where the left horizontal map is induced by the inclusion $T_{x,\star} = \mathcal{O}_{x,\star}^{\times} \times \mathcal{O}_{x,\star}^{\times}$.

Finally, for $K_{\omega,y} = E_{\omega,y}((t_\omega))$ with archimedean local field $E_{\omega,y}$ as in 24, we have a homomorphism

$$\mathfrak{t}_{\omega,y}\colon T_{\omega,y} = \mathcal{O}_{\omega,y}^{\times} \times \mathcal{O}_{\omega,y}^{\times} \longrightarrow E_{\omega,y}^{\times} \times E_{\omega,y}^{\times} \longrightarrow K_2^t(K_{\omega,y}),$$

where the first map is (p,p), p is the residue map, and the second map is $(\alpha,\beta) \mapsto \{\alpha\beta, t_\omega\}$, it was denoted by $\mathfrak{t}$ in 23. The following diagramme is commutative

$$\begin{array}{ccccc} & & \mathcal{O}_{\omega,y}^{\times} \otimes K_{\omega,y}^{\times}/\mathcal{O}_{\omega,y}^{\times} & & \\ & & \downarrow & \searrow & \\ T_{\omega,y} & \longrightarrow & \mathcal{O}_{\omega,y}^{\times} \times \mathcal{O}_{\omega,y}^{\times} & \longrightarrow & K_2^t(K_{\omega,y})/K_2^t(K_{\omega,y})_0, \end{array}$$

where $K_2^t(K_{\omega,y})_0$ is defined in 33, the vertical map sends $\alpha \otimes t_\omega^m$ to $(\alpha^m, 1)$, and the diagonal map is induced by the symbol map.

DEFINITION OF $T_\star$, $T_{S_\prime}$, T_{S_o}. In general, define

$$T_\star := \prod_{x\in\star} T_{x,\star} \cap \left(\mathbb{A}_\star \times \mathbb{A}_\star\right)^\times, \quad T_{S_\prime} := \prod_{\star\in S_\prime} T_\star \cap \left(\mathbb{A}_{S_\prime} \times \mathbb{A}_{S_\prime}\right)^\times,$$
$$T_{S_o} = \prod_{\star\in S_o} T_\star \cap \left(\mathbb{A}_{S_o} \times \mathbb{A}_{S_o}\right)^\times,$$

where S_o is a subset $S_\prime$.

If all singular points of fibres are split ordinary double, then

$$T_\star = (\mathbb{A}_\star \times \mathbb{A}_\star)^\times, \quad T_{S_\prime} = (\mathbb{A}_{S_\prime} \times \mathbb{A}_{S_\prime})^\times.$$

Define the topology on $T_{S_\prime}$ as the topology induced from $(\mathbb{A}_{S_\prime} \times \mathbb{A}_{S_\prime})^\times$.

Recall that, as agreed in 24, we include objects associated to archimedean points on horizontal curves in the list of data (x,y). No information associated to K_σ, i.e. 'archimedean fibres' over archimedean places, is contained in $T_{S_\prime}$.

DEFINITION OF $\mathbf{VA}_{\mathcal{S}}^\times$, $\mathbb{A}_{\mathcal{S}}^\times \times \mathbf{A}_{\mathcal{S}}^\times$, $\mathbb{A}_{\mathcal{S}}^\times \otimes \mathbf{A}_{\mathcal{S}}^\times/\mathbf{VA}_{\mathcal{S}}^\times$. Using the adelic object $\mathbf{A}_{\mathcal{S}}$ of 28, put $\mathbf{VA}_{\mathcal{S}}^\times = \mathbf{A}_{\mathcal{S}}^\times \cap \prod \mathcal{O}_{x,y}^\times$.

Define

$$\mathbb{A}_{\mathcal{S}}^\times \times \mathbf{A}_{\mathcal{S}}^\times = \{(\alpha_{x,\star}, \beta_{x,\star})_{x\in\star\in S_1}, \quad (\alpha_{x,\star}) \in \mathbb{A}_{\mathcal{S}}^\times, (\beta_{x,\star}) \in \mathbf{A}_{\mathcal{S}}^\times\}.$$

The quotient $\mathbf{A}_{\mathcal{S}}^\times/\mathbf{VA}_{\mathcal{S}}^\times$ is isomorphic to $\oplus_{x\in\star\in S_1} \oplus_{z\in\star(x)} \mathbb{Z}$. For an abelian group R the tensor product $R\otimes\mathbb{Z}$ consists of tensor elements. Hence define

$$\mathbb{A}_{\mathcal{S}}^\times \otimes \mathbf{A}_{\mathcal{S}}^\times/\mathbf{VA}_{\mathcal{S}}^\times = \{(\alpha_{x,\star} \otimes \gamma_{x,\star})_{x\in\star\in S_1}, \quad (\alpha_{x,\star}) \in \mathbb{A}_{\mathcal{S}}^\times, (\gamma_{x,\star}) \in \mathbf{A}_{\mathcal{S}}^\times/\mathbf{VA}_{\mathcal{S}}^\times\}.$$

Similarly define objects for $S_\prime$, e.g.

$$\mathbb{A}_{S_\prime}^\times \otimes \mathbf{A}_{S_\prime}^\times/\mathbf{VA}_{S_\prime}^\times = \{(\alpha_{x,\star} \otimes \gamma_{x,\star})_{x\in\star\in S_\prime}, \quad (\alpha_{x,\star}) \in \mathbb{A}_{S_\prime}^\times, (\gamma_{x,\star}) \in \mathbf{A}_{S_\prime}^\times/\mathbf{VA}_{S_\prime}^\times\}.$$

DEFINITION OF $\mathfrak{t}$. Using all $\mathfrak{t}_{x,z}$ we get the adelic morphism

$$\mathfrak{t}\colon T_{S_\prime} = (\mathbb{A}_{S_\prime} \times \mathbb{A}_{S_\prime})^\times \longrightarrow J \longrightarrow J_{\mathcal{S}}.$$

The object $VJ_{\mathcal{S}}$ was defined in 33 using the image of $\mathfrak{q}$, and it coincides with $\mathfrak{t}(V\mathbb{A}_{\mathcal{S}}^\times \times V\mathbb{A}_{\mathcal{S}}^\times)$.

The following diagramme is the adelic version of the previous diagrammes for $\mathfrak{t}_{x,\star}$

$$\begin{tikzcd}
 & \mathbb{A}_{\mathcal{S}}^{\times} \otimes \mathbf{A}_{\mathcal{S}}^{\times}/\mathbf{V}\mathbf{A}_{\mathcal{S}}^{\times} \arrow[d] \arrow[dr] & \\
T_{S_{/}} \arrow[r] & \mathbb{A}_{\mathcal{S}}^{\times} \times \mathbb{A}_{\mathcal{S}}^{\times}/V\mathbb{A}_{\mathcal{S}}^{\times} \arrow[r] & J_{\mathcal{S}}/VJ_{\mathcal{S}},
\end{tikzcd}$$

where we use $\mathbb{A}_{S_{/}}^{\times} \times \mathbb{A}_{S_{/}}^{\times} \longrightarrow \mathbb{A}_{\mathcal{S}}^{\times} \times \mathbb{A}_{\mathcal{S}}^{\times}$ which is the identity map there where it is defined with added 1 at the other components.

Definition of $S_{/}$. *From now on, the set $S_{/}$ of curves on $\mathcal{S}$, on which all the adelic objects are defined, contains all fibres and a positive number of nonsingular horizontal curves.* Set

$$S_{/} = S_{|} \cup S_{-}$$

where $S_{-} = S_h \cap S_{/}$ is the subset of horizontal curves in $S_{/}$.

We abbreviate $\mathbb{A} = \mathbb{A}_{S_{/}}$.

The set $S_{/}$ will be further specified in 40 for proper regular models of elliptic curves.

Definition of T. This $K_1 \times K_1$-object is just

$$T := T_{S_{/}}.$$

The restriction of $\mathfrak{t}$ gives

$$\mathfrak{t}\colon T \longrightarrow J_{S_{/}} := J \cap \prod_{\star\in S_{/}}{}' \prod_{x\in\star}{}' K_2^t(K_{x,\star}).$$

Lemma *The induced map $T \longrightarrow J_{S_{/}}/(J_{S_{/}} \cap (P_{\mathcal{S}} + VJ_{\mathcal{S}}))$ is surjective. The following diagramme, whose bottom line is induced by $\mathfrak{t}$, is commutative*

$$\begin{tikzcd}
 & \mathbb{A}^{\times} \otimes \mathbf{A}_{S_{/}}^{\times}/\mathbf{V}\mathbf{A}_{S_{/}}^{\times} \arrow[d] \arrow[dr] & \\
T \arrow[r] & \mathbb{A}^{\times} \times \mathbb{A}^{\times}/V\mathbb{A}^{\times} \arrow[r] & J_{S_{/}}/(J_{S_{/}} \cap VJ_{\mathcal{S}}).
\end{tikzcd}$$

Proof: The local $\mathfrak{t}_{x,\star}$ are surjective. $\square$

DEFINITION OF $|\,|, N_{S_o}, N, q, q_y$. Define

$$|\,|\colon (\mathbb{A}\times\mathbb{A})^\times \longrightarrow N, \qquad (\alpha_1,\alpha_2)\mapsto |\alpha_1||\alpha_2|,$$

N is defined in 35, $|\alpha|$ is defined in 30. In particular, $N = |T|$.

Similarly, for a set S_o of fibres and horizontal curves define $|\,|_{S_o}$ and put $N_{S_o} = |T_{S_o}|$.

If $k(y)$ is of positive characteristic then N_y is a cyclic group generated by $q_y > 1$.

In positive characteristic the group N is generated by $q > 1$, as in 35.

DEFINITION OF $T_{1\,x,z}$, $T_{1,\star}$, AND $M_\star$. Define a subgroup

$$T_{1\,x,z} = \{(\alpha_1,\alpha_2)\in T_{x,z} : |(\alpha_1,\alpha_2)|_{x,z} = |\alpha_1|_{x,z}|\alpha_2|_{x,z} = 1\}$$

of $T_{x,z}$.

For a fibre or a horizontal curve $\star$ denote by $T_{1,\star}$ the kernel of the restriction of $|\,|$ on $T_\star$.

Choose a subgroup $M_\star$ of $T_\star$ which is isomorphic to $N_\star = |T_\star|$; this gives the splitting $T_\star \simeq T_{1,\star}\times M_\star$. The choice of $T_\star$ is non canonical. Recall that in the one-dimensional theory [57] one makes a similar non canonical choice of representatives of $N_\star$.

Finally, we will need the following rescaled module and group.

DEFINITION OF $\|\,\|$, $\mathfrak{T}$. For a fibre $\star$ put $\|\,\|_\star = |\,|_\star$ and denote $\mathfrak{T}_\star = T_\star$.

For a horizontal curve $\star$ put $\|\,\|_\star = |\,|_\star^{1/2}$ and choose a maximal subgroup $\mathfrak{T}_\star$ of $T_\star$ such that $\|\mathfrak{T}_\star\| = |T_\star|$. In other words, $\mathfrak{T}_\star = T_\star$ for horizontal curves in characteristic zero and $\mathfrak{T}_\star = T_{1,\star}\times M_\star^2$ (of course, this depends on the choice of $M_\star$) for horizontal curves in positive characteristic.

Put

$$\mathfrak{T} = T\cap\prod_{\star\in S_\prime}\mathfrak{T}_\star = \prod_{\star\in S_\prime}{}^{\prime}\,\mathfrak{T}_\star,\quad \|\,\| = \prod_{\star\in S_\prime}\|\,\|_\star.$$

37. Now we define integrals $\int_T g$, $\int_{\mathfrak{T}} g$ for $g\in R^0_{(\mathbb{A}\times\mathbb{A})^\times}$. They *are not* $\int g\, char_T\, d\mu_{(\mathbb{A}\times\mathbb{A})^\times}$, etc., but differ at the $(x,\star)$-data for singular points $x\in\star$. It is important that we have to use a new integration at singular points, since the theory cannot come as the lift of a one-dimensional theory. This new integration will be used in the zeta integrals. In the first approximation, quite imprecisely, $\int_{T_\star} g$ is the integral of $g\circ\mathfrak{v}$ over $\prod'_{x\in\star,\mathrm{ns}} T_{x,\star}\times\prod_{x\in\star,\mathrm{s}} T_{x,z}$.

We define certain subspaces R^0 and certain diamond modifications of their elements, which we then use to define the new integration.

DEFINITION OF SPACE $Q^0_{K_{x,\star}\times K_{x,\star}}$ AND MODIFICATION $g\mapsto g^\diamond$. The local (x,z)-spaces Q were defined in 9 and 10.

If x is a nonsingular point of $\star$, put $Q^0_{K_{x,\star}\times K_{x,\star}} = Q_{K_{x,\star}\times K_{x,\star}}$ and let the diamond operator be the identity one.

For a singular $x\in\star$ choose a subspace $Q^0_{K_{x,\star}\times K_{x,\star}}$ of $Q_{K_{x,\star}\times K_{x,\star}}$, which includes the characteristic functions of $(\mathfrak{x}(t)^i \mathrm{O}_{x,\star}, \mathfrak{x}(t)^j \mathrm{O}_{x,\star})$, $i,j\in\mathbb{Z}$, and such that for each of its elements of the type $h_{x,\star}=\sum a_i char_{A_i}$ with $A_i=p^{-1}(p(A_i))$ the function $h^\diamond_{x,\star}=\sum a_i char_{B_i}$, where $B_i=p^{-1}(p(\mathfrak{u}(A_i\cap \mathrm{im}(\mathfrak{x}))))$, is a well defined element of $Q_{K_{x,z}\times K_{x,z}}$.

For example, one can take the subspace generated by the characteristic functions of all $\mathrm{O}_{x,\star}\times \mathrm{O}_{x,\star}$-submodules of $\mathcal{O}_{x,\star}\times\mathcal{O}_{x,\star}$ as $Q^0_{K_{x,\star}\times K_{x,\star}}$ (note that such characteristic functions are linearly independent). Since this space is enough for the purposes of this text, we fix it from now on.

Using the density of $\mathfrak{x}(Q_x)$ in $\mathcal{O}_{x,\star}$ we can choose generators of every $\mathrm{O}_{x,\star}\times \mathrm{O}_{x,\star}$-submodule to lie in $\mathfrak{x}(Q_x)\times\mathfrak{x}(Q_x)$. For every function $h_{x,\star}$ in the space of the previous paragraph we have the property $h^\diamond_{x,\star}=h_{x,\star}\circ\mathfrak{v}$ on $\mathfrak{z}(Q_x)$.

Note that $O_{x,\star}\cap\mathfrak{x}(Q_x)=\mathrm{O}_{x,\star}\cap\mathfrak{x}(Q_x)$, so for objects which do not take into account the singularity $x\in\star$ we loose some information when intersecting with $\mathfrak{x}(Q_x)$ and applying the diamond modification.

In particular, the function $char_{(\mathfrak{x}(t)^i\mathrm{O}_{x,\star},\mathfrak{x}(t)^j\mathrm{O}^\perp_{x,\star})}$ belongs to $Q^0_{K_{x,\star}\times K_{x,\star}}$ and its image with respect to the diamond operator is $char_{(\mathfrak{z}(t)^i O_{x,z},\mathfrak{z}(t)^{j-1}O_{x,z})}$ (recall that $\mathrm{O}^\perp_{x,\star}=\mathfrak{x}(t)^{-1}\mathrm{O}_{x,\star}$).

DEFINITION OF SPACE $R^0_{(K_{x,\star}\times K_{x,\star})^\times}$ AND MODIFICATIONS $g\mapsto g^\diamond$, $g\mapsto {}^\diamond g$. The local (x,z)spaces R were defined in 6 and 10.

If x is a nonsingular point of $\star$, denote $R^0_{(K_{x,\star}\times K_{x,\star})^\times}=R_{K_{x,\star}\times K_{x,\star}}$ and let the diamond modifications be the identity ones.

If x is a singular point of $\star$, denote by $R^0_{(K_{x,\star}\times K_{x,\star})^\times}$ the subspace of the space $R_{K_{x,\star}\times K_{x,\star}}=\otimes_{z\in\star(x)}R_{K_{x,z}\times K_{x,z}}$ generated by products $h_{x,\star}\pi_{x,\star}$, where $h_{x,\star}\in Q^0_{K_{x,\star}\times K_{x,\star}}$ and $\pi_{x,\star}$ is the extension by zero of a continuous quasi-character $(\mathcal{O}_{x,\star}\times\mathcal{O}_{x,\star})^\times\longrightarrow\mathbb{C}^\times$ which lifts a continuous quasi-character of $E^\times_{x,\star}\times E^\times_{x,\star}$ at the residue level.

Note that if $h^{(i)}_{x,\star}$ are linearly independent elements of $Q_{K_{x,\star}\times K_{x,\star}}$, then $h^{(i)}_{x,\star}\pi^{(i)}_{x,\star}$ are linearly independent elements of the space $R_{(K_{x,\star}\times K_{x,\star})^\times}$ for any $\pi^{(i)}_{x,\star}$ as above, since the same fact is true for at the residue level.

Introduce the linear operator $R^0_{(K_{x,\star}\times K_{x,\star})^\times}\longrightarrow R^0_{(K_{x,z}\times K_{x,z})^\times}$, $g\mapsto g^\diamond$: let first $g_{x,\star}=h_{x,\star}\pi_{x,\star}$ with $h_{x,\star}\in Q^0_{K_{x,\star}\times K_{x,\star}}$, then put $g^\diamond_{x,\star}:=h^\diamond_{x,\star}\pi_{x,z}$; extend by linearity to the space $R^0_{(K_{x,\star}\times K_{x,\star})^\times}$.

Introduce the linear operator $R^0_{(K_{x,\star}\times K_{x,\star})^\times} \longrightarrow R^0_{(K_{x,\star}\times K_{x,\star})^\times}, g \mapsto {}^\diamond g$: for a function $g_{x,\star} \in R^0_{(K_{x,\star}\times K_{x,\star})^\times}$ define ${}^\diamond g_{x,\star}$ as

$$ {}^\diamond g_{x,\star}(\alpha,\beta) = g^\diamond_{x,\star}(\alpha) char_{O^\times_{x,z'}\times O^\times_{x,z'}}(\beta), $$

where $\alpha \in K_{x,z} \times K_{x,z}, \beta \in K_{x,z'} \times K_{x,z'}$.

DEFINITION OF SPACES $R^0_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$, $R^0_{(\mathbb{A}\times\mathbb{A})^\times}$, $g^\diamond$, ${}^\diamond g$. Denote by $R^0_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$ the subspace of the space $R_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$ defined in 31, which is generated by $\otimes_{x\in\star} g_{x,\star}$ with $g_{x,\star} \in R^0_{(K_{x,\star}\times K_{x,\star})^\times}$.

For a function $g_\star = \otimes_{x\in\star} h_{x,\star} \in R^0_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$ define $g^\diamond_\star = \otimes_{x\in\star} g^\diamond_{x,\star}$, ${}^\diamond g_\star = \otimes_{x\in\star} {}^\diamond g_{x,\star}$ and extend by linearity.

Denote by $R^0_{(\mathbb{A}\times\mathbb{A})^\times}$ the subspace of $R_{(\mathbb{A}\times\mathbb{A})^\times}$ defined in 31, which is generated by $\otimes_\star g_\star$ with $g_\star \in R^0_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$.

For an element $g = \otimes_\star g_\star \in R^0_{(\mathbb{A}\times\mathbb{A})^\times}$ define $g^\diamond = \otimes_\star g^\diamond_\star$, ${}^\diamond g = \otimes_\star {}^\diamond g_\star$ and extend by linearity.

DEFINITION OF $Q^0_{\mathbb{A}\times\mathbb{A}}$, $Q^0_{\mathbb{A}_{S_o}\times\mathbb{A}_{S_o}}$. Recall that the space $Q_{\mathbb{A}\times\mathbb{A}}$ is a subspace of $R_{(\mathbb{A}\times\mathbb{A})^\times}$, see 31. Define $Q^0_{\mathbb{A}\times\mathbb{A}}$ as the subspace of $Q_{\mathbb{A}\times\mathbb{A}}$, whose local components belong to the local Q^0- local subspaces.

In particular, if $h \in Q^0_{\mathbb{A}\times\mathbb{A}}$ then $h|\,|^s \in R^0_{(\mathbb{A}\times\mathbb{A})^\times}$ and ${}^\diamond(h|\,|^s) = {}^\diamond h|\,|^s$.

Similarly define $Q^0_{\mathbb{A}_{S_o}\times\mathbb{A}_{S_o}}$ for a subset S_o of $S_\prime$.

The group T acts on the spaces $R^0_{(\mathbb{A}\times\mathbb{A})^\times}$, $Q^0_{\mathbb{A}\times\mathbb{A}}$: $f \longrightarrow f_\alpha$, $\quad f_\alpha(\gamma) = f(\alpha\gamma)$.

DEFINITION OF $\int_T g$. For $g_{x,\star} \in R^0_{(K_{x,\star}\times K_{x,\star})^\times}$ define

$$ \int_{T_{x,\star}} g_{x,\star} := \int g^\diamond_{x,\star}\, d\mu_{(K_{x,z}\times K_{x,z})^\times} = \int {}^\diamond g_{x,\star}\, d\mu_{(K_{x,\star}\times K_{x,\star})^\times}, $$

where $\mu_{(K_{x,z}\times K_{x,z})^\times}$ was defined in 31, $\mu_{(K_{x,\star}\times K_{x,\star})^\times} = \otimes_{z\in\star(x)} \mu_{(K_{x,z}\times K_{x,z})^\times}$.

At nonsingular $x \in \star$ this is just the integral $\int g_{x,\star}\, d\mu_{(K_{x,\star}\times K_{x,\star})^\times}$. At singular $x \in \star$ the integral of $g_{x,\star}$ is the integral of $g^\diamond_{x,\star}$ over $\mathcal{O}^\times_{x,z} \times \mathcal{O}^\times_{x,z}$.

For $g = \otimes_{x\in\star} g_{x,\star} \in R^0_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$ define

$$ \int_{T_\star} g_\star := \prod_{x\in\star} \int_{T_{x,\star}} g_{x,\star} $$

and extend the definition to linear combinations. For $g = \otimes_{\star\in S_\prime} g_\star \in R^0_{(\mathbb{A}\times\mathbb{A})^\times}$ define

$$ \int_T g := \prod_{\star\in S_\prime} \int_{T_\star} g_\star $$

and extend to linear combinations.

From the definitions we obtain

$$\int_T g = \int^{\diamond} g \, d\mu_{(\mathbb{A}\times\mathbb{A})^\times}.$$

In particular, if $h \in Q^0_{\mathbb{A}\times\mathbb{A}}$ then $\int_T h|\,|^s = \int^{\diamond} h|\,|^s d\mu_{(\mathbb{A}\times\mathbb{A})^\times}$.

DEFINITION OF $\int_{\mathfrak{T}} g$. If $g = g_{|} \otimes g_{-} \in R^0_{(\mathbb{A}\times\mathbb{A})^\times}$ with $g_{|} \in R^0_{(\mathbb{A}_{S_|}\times\mathbb{A}_{S_|})^\times}$, $g_- \in R^0_{(\mathbb{A}_{S_-}\times\mathbb{A}_{S_-})^\times}$, define

$$\int_{\mathfrak{T}} g := \int^{\diamond} g_{|} \, d\mu_{(\mathbb{A}_{S_|}\times\mathbb{A}_{S_|})^\times} \int g_- char_{\mathfrak{T}_{S_-}} \, d\mu_{(\mathbb{A}_{S_-}\times\mathbb{A}_{S_-})^\times}$$

and extend to linear combinations.

Thus, if $g = \otimes g_\star$ then $\int_{\mathfrak{T}} g = \prod \int_{\mathfrak{T}_\star} g_\star$ with the following factors. If $\star$ is a fibre or a horizontal curve in characteristic zero then $\mathfrak{T}_\star = T_\star$ and $\int_{\mathfrak{T}_\star} = \int_{T_\star}$. If $\star$ is a horizontal curve in positive characteristic and $g = u \circ p_\star \in R^0_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$ with a function u at the residue level then the integral $\int_{\mathfrak{T}_\star} g$ is equal to the one-dimensional integral $\int_{p_\star(\mathfrak{T}_\star)} u$. For $g \in R^0_{(\mathbb{A}\times\mathbb{A})^\times}$ we can also write

$$\int_{\mathfrak{T}} g = \int_T char_{\mathfrak{T}} g.$$

3.2. Zeta function

38. Recall that the (unramified) zeta function of a scheme $\mathcal{X}$ of finite type over $\mathbb{Z}$ is

$$\zeta_{\mathcal{X}}(s) = \prod_{x\in X_0} \left(1 - |k(x)|^{-s}\right)^{-1}.$$

If $\mathcal{X}$ is a scheme over an open subscheme C of B, with B as fixed in 24, then $\zeta_{\mathcal{X}}(s)$ is equal to the product $\prod_{b\in C_0} \zeta_{\mathcal{X}_b}(s)$, where $\mathcal{X}_b = \mathcal{X} \times_C b$. This arithmetic zeta function $\zeta_{\mathcal{X}}(s)$ was introduced by Hasse in the thirties, published in [21], and extended to schemes by Serre in [50].

DEFINITION OF χ AND ITS FINITE AND UNRAMIFIED PARTS. Let $\mathcal{S}$ be as in 33. For the K_2^t-delic group $J_{\mathcal{S}}$ defined in 33 consider those continuous homomorphisms $\chi\colon J_{\mathcal{S}} \longrightarrow \mathbb{C}^\times$, $\chi(P_{\mathcal{S}}) = 1$, such that

$$\chi = \chi_0 |\,|_2^s, \quad \chi_0\colon J_{\mathcal{S}} \longrightarrow \mathbb{C}^\times,$$

where the character χ_0 is of finite order and is a lift of a continuous character of $J^1_{\mathcal{S}}/P_{\mathcal{S}}$ of the same finite order, $|\,|_2$ was defined in 35. It is easy to see that for such χ both χ_0 and $|\,|_2^s$ are uniquely determined. We call $|\,|_2^s$ an unramified quasi-character of $J_{\mathcal{S}}$. Put $s = s(\chi) \in \mathbb{C}$. It is uniquely determined by χ if K is of characteristic zero, and uniquely up to $2\pi i/\log q$, q is defined in 36, if K is of positive characteristic. Set $\widehat{\chi} = \chi^{-1}|\,|_2^2$. For each (x,z) denote by $\chi_{x,z}$ the composite $K_2^t(K_{x,z}) \longrightarrow J_{\mathcal{S}} \longrightarrow \mathbb{C}^\times$, sometimes we denote it just by χ.

The following zeta function is a commutative two-dimensional analogue of the Dedekind zeta function twisted by a Dirichlet character. The definition below works for any two-dimensional (and with obvious extension for higher dimensional) regular integral projective arithmetic scheme $\mathcal{S}$. Define

$$\zeta_{\mathcal{S}}(\chi) = \zeta_{\mathcal{S}}(\chi_0,s) = \prod_{\star\in S_{\mid}}\prod_{x\in\star}\big(1-\chi_{x,\star}(\Pi_{x,\star})\big)^{-1}.$$

To explain the formula, let first $\star(x) = \{z\}$, i.e. x is a unibranch point of fibre $\star$. If $\chi_{x,z}$ is unramified with respect to the two-dimensional structure (i.e. it is trivial on the subgroup of $K_2^t(K_{x,z})$ generated by $K_1(O_{x,z})$), then by definition $\chi_{x,z}(\Pi_{x,z})$ is just the image of a "prime" element $\Pi_{x,z} \in K_2^t(K_{x,z})$, i.e. an element which is mapped to $1 \in K_0(k_z(x))$ with respect to the composite of two border maps defined in 33. If the quasi-character $\chi_{x,z}$ is not unramified, then by definition $\chi_{x,z}(\Pi_{x,z}) = 0$. Now let x be a singular point of $\star$. Then the $(x,\star)$-factor $1-\chi_{x,\star}(\Pi_{x,\star})$ is by definition $f_{x,\star}(q_x^{-s})$ where $f_{x,\star}(t)$ is the greatest common divisor of polynomials $f_{x,z}(t) \in \mathbb{C}[t]$ for all $z \in \star(x)$ and $f_{x,z}(q_x^{-s}) = 1-\chi_{x,z}(\Pi_{x,z})$ with the right hand side as above.

In particular, if $\mathcal{S}$ is as fixed in the beginning of this part then via the reciprocity map $\Phi_{\mathcal{S}}$ in 34 the character χ_0 corresponds to a continuous character of the absolute Galois group of K and we have $\zeta_{\mathcal{S}}(|\,|_2^s) = \zeta_{\mathcal{S}}(s)$, and $\zeta_{\mathcal{S}}(\chi)$ coincides with the twist of $\zeta_{\mathcal{S}}(s)$ by that character as defined in [50].

The function $\zeta_{\mathcal{S}}(\chi)$ absolutely converges on $\Re(s(\chi)) > 2$, this follows from the similar property of $\zeta_{\mathcal{S}}(s)$. Recall that we even know a stronger property of $\zeta_{\mathcal{S}}(s)$: it extends to a meromorphic function on $\Re(s) > 3/2$ with the only simple pole(s) at $s = 2$ in characteristic zero and $q^s = q^2$ in positive characteristic, see e.g. [50].

3.3. Definition of the zeta integral

In dimension one the zeta integral plays a major role in the study of the meromorphic continuation and functional equation of the twisted Dedekind zeta functions, and its calculation also gives a simple proof of the finiteness of class number and Dirichlet's theorem on units; however, it is not of much use for the study of the zeros of the zeta functions. In dimension two the zeta integral defined in this section and 57 is supposed to play a major role in the study of many open properties of the zeta functions of arithmetic surfaces which include the meromorphic continuation and functional equation, location of poles, behaviour at the central point and various finiteness results.

39. DEFINITION OF $\chi_{\mathfrak{t}}$. For a continuous homomorphism $\chi\colon J \longrightarrow \mathbb{C}^\times$ as in 38, write $\chi = \chi_0 |\,|_2^s$ as the product of the unramified quasi-character $|\,|_2^s$ and χ_0. Define

$$\chi_{\mathfrak{t}} := (\chi_0 \circ \mathfrak{t}) \|\,\|^s : \mathfrak{T} \longrightarrow \mathbb{C}^\times,$$

$\|\,\|_2$ and $\mathfrak{T}$ are defined in 36. By 38, χ uniquely determines $|\,|_2^s$ and χ_0, and hence $\|\,\|^s$ and $\chi_{\mathfrak{t}}$ as functions on $\mathfrak{T}$ are uniquely determined by χ.

On the vertical part of T we have $\chi_{\mathfrak{t}} = \chi \circ \mathfrak{t}$.

In particular, if $g \in Q^0_{\mathbb{A}\times\mathbb{A}}$ and $\chi\colon J \longrightarrow \mathbb{C}^\times$ is a continuous homomorphism as in 38, then $g\,\chi_{\mathfrak{t}} \in R^0_{(\mathbb{A}\times\mathbb{A})^\times}$ and the previous section gives the definition of the integral $\int_{\mathfrak{T}} g\,\chi_{\mathfrak{t}}$.

DEFINITION. Let $\mathcal{S}$ be as in 24. Let the set of curves $S_{\prime}$ be as in 36. For $g \in R^0_{(\mathbb{A}\times\mathbb{A})^\times}$ and a continuous homomorphism $\chi\colon J_{\mathcal{S}} \longrightarrow \mathbb{C}^\times$ as in 38, such that $\chi(P_{\mathcal{S}}) = 1$, the zeta integral is

$$\zeta_{\mathcal{S}}(g,\chi) = \int_{\mathfrak{T}} g\,\chi_{\mathfrak{t}}.$$

It takes values in $\mathbb{C}((X))$ where convergent. The integral $\zeta_{\mathcal{S}}(g, |\,|_2^s)$ for $g \in Q^0_{\mathbb{A}\times\mathbb{A}}$ takes values in $\mathbb{C}$.

For a subset S_o of $S_{\prime}$ similarly define $\zeta_{\mathcal{S},S_o}(g,\chi)$.

In particular, if $g \in Q^0_{\mathbb{A}\times\mathbb{A}}$ then

$$\zeta_{\mathcal{S}}(g, |\,|_2^s) = \int_{\mathfrak{T}} g \|\,\|^s.$$

It is equal to

$$\int^{\diamond} g_{\shortmid} |\,|^s \, d\mu_{(\mathbb{A}_{S_\shortmid}\times\mathbb{A}_{S_\shortmid})^\times} \int g_{-} char_{\mathfrak{T}_{S_-}} \|\,\|^s \, d\mu_{(\mathbb{A}_{S_-}\times\mathbb{A}_{S_-})^\times}$$

when $g = g_{\shortmid} \otimes g_{-}$ with $g_{-} \in Q^0_{\mathbb{A}_{S_-}\times\mathbb{A}_{S_-}}$, $g_{\shortmid} \in Q_{\mathbb{A}_{S_\shortmid}\times\mathbb{A}_{S_\shortmid}}$.

CURVE FACTOR PRODUCT REPRESENTATION OF THE ZETA INTEGRAL. If g is $\star$-decomposable, i.e. $g = \otimes_{\star\in S_\prime} g_\star$, where $\star$ runs through horizontal curves in $S_\prime$ and fibres, then

$$\zeta_{\mathcal{S}}(g,\chi) = \prod_{\star\in S_\prime} \zeta_{\mathcal{S},\star}(g_\star,\chi), \qquad \zeta_{\mathcal{S},\star}(g_\star,\chi) = \int_{\mathfrak{T}_\star} g_\star\,\chi_{\mathfrak{t}},$$

where the notation in the last integral means that one first restricts χ to the $\star$-part of $J_{\mathcal{S}}$.

Assume that $g_\star = \otimes_{x\in\star} g_{x,\star} \in Q^0_{\mathbb{A}_\star\times\mathbb{A}_\star}$.

(1) If $\star$ is a fibre then $\zeta_{\mathcal{S},\star}(g_\star,\chi) = \int_{T_\star} g_\star\,\chi_{\mathfrak{t}}$ and

$$\zeta_\star(g_\star,\chi) = \prod_{x\in\star} \zeta_{x,\star}(g_{x,\star},\chi),$$

where the local zeta integrals are

$$\zeta_{x,\star}(g_{x,\star},\chi) = \int_{T_{x,\star}} g_{x,\star}\,\chi_{\mathfrak{t}_{x,\star}}.$$

At nonsingular $x\in\star$ this coincides with $\int_{T_{x,z}} g_{x,z}\,\chi_{\mathfrak{t}_{x,z}}\,d\mu_{K^\times_{x,z}\times K^\times_{x,z}}$, which is reviewed at the beginning of this part. We have the following computational formula

$$\zeta_{x,\star}(g_{x,\star},\chi) = (1-q_{x,\star}^{-1})^{-2}\sum b_{n,r}\, r\, q_{x,\star}^{-n(s-1)},$$
$$b_{n,r} = \mu_{K_{x,\star}\times K_{x,\star}}\big(\{\alpha\in T_{x,\star} : |\alpha|_{x,\star} = q_{x,\star}^{-n},\ g_{x,\star}(\alpha)\,\chi_0(\mathfrak{t}_{x,\star}(\alpha)) = r\}\big).$$

At a split ordinary double point x of $\star$ we get

$$\zeta_{x,\star}(g_{x,\star},\chi) = \int g^\diamond_{x,\star}\,\chi_{\mathfrak{t}_{x,z}}\,d\mu_{K^\times_{x,z}\times K^\times_{x,z}}.$$

(2) If $\star = y$ is horizontal in characteristic zero then $\zeta_y(g_y,\chi) = \int_{T_y} g_y\,\chi_{\mathfrak{t}}$ and

$$\zeta_y(g_y,\chi) = \prod_{x\in y} \zeta_{x,y}(g_{x,y},\chi),$$

where the local zeta integral $\zeta_{x,y}(g_{x,y},\chi) = \int_{T_{x,y}} g_{x,y}\,\chi_{\mathfrak{t}_{x,y}}\,d\mu_{K^\times_{x,y}\times K^\times_{x,y}}$. In particular, we obtain

$$\zeta_y(g_y,|\,|_2^s) = \int_{T_y} g_y\,|\,|^{s/2}.$$

(3) If y is horizontal in positive characteristic then $\zeta_y(g_y,\chi) = \int_{\mathfrak{T}_y} g_y\,\chi_{\mathfrak{t}}$, hence $\zeta_y(g_y,|\,|_2^s) = \int_{\mathfrak{T}_y} g_y\,|\,|^{s/2}$.

It is convenient to introduce and use an auxiliary zeta integral

$$\zeta_y^a(g_y,\chi) = \int_{T_y} g_y\,\chi_{\mathfrak{t}}.$$

The latter is the product of the local integrals $\int_{T_{x,y}} g_y\,\chi_{\mathfrak{t}}\,d\mu_{K_{x,y}^\times \times K_{x,y}^\times}$ to calculate which we can use the formulas similar to those for horizontal curves in characteristic zero.

If $\zeta_y^a(g_y,|\,|_2^s) = \sum_{n\in N_y} c_n n^{-s/2}$ with $c_n = \int char_{T_{1,y}}\, g_y(\mathfrak{m}_n\gamma)\,d\mu(\gamma)$, where $\mathfrak{m}_n \in \mathfrak{T}_y$, $||\mathfrak{m}_n||_y = n$, then we immediately get $\zeta_y(g_y,|\,|_2^s) = \sum_{n\in N_y} c_{n^2} n^{-s}$ (see also the proof of Theorem 40 for more detail about this integral).

Remark 1 As in the local case [11], we integration over the $K_1 \times K_1$-objects, using the homomorphism $\mathfrak{t}$ which relates them to K_2^t-objects in class field theory.

Remark 2 The horizontal part of the zeta integral plays less significant role than its vertical part, the latter is closely related to the Euler factors for the zeta function. In the functional equation the horizontal zeta integrals contributions will cancel each other.

Remark 3 The zeta integral $\zeta(g,\chi)$ defined above is a simplified version of the zeta integral which was described in Remark 5.1 of [14]. Its vertical part coincides with the vertical part of the zeta integral defined in [14], but their horizontal parts are slightly different.

3.4. First calculation of the zeta integral

40. *From now on we assume that $\mathcal{S} = \mathcal{E} \longrightarrow B$ corresponds to a proper regular model of an elliptic curve E over a global field* k. *Even though it is not really necessary, for simplicity we further assume that $\mathcal{E} \longrightarrow B$ has normal crossings. Finally, we also impose the restriction that the reduction in residual characteristic 2 and 3 is good or multiplicative.*

Sections of the morphism $\mathcal{E} \longrightarrow B$ correspond to k-points of E. *We include in the set S_- the image of the zero section of $\mathcal{E} \longrightarrow B$.* Later on, in 50–55 we will assume that S_- consists of the image of the zero section.

For the description of special fibres of a minimal proper regular model and minimal proper regular model with normal crossings of E see section 10.2 of [37]. In particular, for a minimal proper regular model with normal crossings all singular points of fibres are split ordinary double.

The reason why $S_{\prime}$ includes horizontal curves is to ensure that the module value group $|T_{S_{\prime}}|$ is a complete group $\mathbb{R}_{>0}^\times$ in characteristic zero: notice that if S_- is empty then $|T_{S_{\prime}}|$ is generated by N_b, where N_b is a cyclic multiplicative group with generator $|k(b)|$ and b runs through nonarchimedean places of k, and so in characteristic zero it is not a complete group. The reason why S_- is finite is that to ensure the convergence of the horizontal part of the zeta integral: notice that in characteristic zero the zeta integral on each of horizontal curves contributes a gamma-factor.

A more general construction of the zeta integral in the case of hyperbolic curves over k is offered in 57.

DEFINITION OF $\mathfrak{c}_\star$, $\mathfrak{c}_{\mathcal{E}}$. For a horizontal curve in $S_\prime$ or a fibre $\star$ define

$$\mathfrak{c}_\star = \prod_{x\in\star,\mathrm{ns}} q_{x,\star}^{d_{x,\star}} \cdot \prod_{x\in\star,\mathrm{s}} \big(q_x \prod_{z\in\star(x)} q_{x,z}^{-1}\big) = \prod_{x\in\star,\mathrm{ns}} q_{x,\star}^{d_{x,\star}} \cdot \prod_{x\in\star} \big(q_x \prod_{z\in\star(x)} q_{x,z}^{-1}\big),$$

where the first product is taken with respect to nonsingular closed points $x \in \star$, $d_{x,\star}$ are defined in 27, and the second product is taken with respect to singular $x \in \star$. In the product $\prod_{x\in\star}\big(q_x \prod_{z\in\star(x)} q_{x,z}^{-1}\big)$ only finitely many factors, corresponding to singular points, are different from 1.

Following the proof of Proposition 27 and the definition of $d_{x,\star}$, we get the following formula for a fibre $\star$ over $b \in B_0$

$$\mathfrak{c}_\star = \prod_{y\subset\star} q_y^{2(1-g_y)} \prod_{x\in\star,\mathrm{s}} \big(q_x \prod_{z\in\star(x)} q_{x,z}^{-1}\big),$$

where y runs through components of $\star$, q_y was defined in 36, it is the cardinality of the maximal finite subfield of $k(y)$, g_y is the genus of the normalization $\hat{y}$ of y. Here we use the formula $\prod_{x\in y,\mathrm{ns}} q_{x,y}^{d_{x,y}} = q_y^{2(1-g_y)}$ which follows from the classical one-dimensional formula and part (c) of Proposition 27. Thus, $\mathfrak{c}_\star$ does not depend on the choice of character of $\mathbb{A}_\star$ and associated numbers $d_{x,\star}$.

When all singular points are ordinary double, we obtain

$$\mathfrak{c}_\star = \prod_{x\in\star} q_{x,z}^{d_{x,z}} = \prod_{y\subset\star} q_y^{2(1-g_y)} \prod_{x\in\star,\mathrm{s}} q_x^{-1},$$

where the product is taken in accordance with 24, i.e. for a singular point x of $\star$ only one branch is involved.

If q_y for all $y \subset \star$ and q_x for all singular points x of $\star$ are equal to $|k(b)|$, then $\mathfrak{c}_\star$ is just $|k(b)|$ raised to the Euler number of the dual graph of the fibre $\star$.

In particular, for all its nonsingular fibres of $\mathcal{E}$ we get $\mathfrak{c}_\star = 1$. Now define for $\mathcal{E}$

$$\mathfrak{c}_{\mathcal{E}} = \prod_{\star\in S_\shortmid} \mathfrak{c}_\star.$$

DEFINITION OF PRELIMINARY FUNCTION f^{pr}. As a preparation for the definition of a centrally normalized function $f \in Q^0_{\mathbb{A}\times\mathbb{A}}$ define a preliminary function $f^{pr} \in Q^0_{\mathbb{A}\times\mathbb{A}}$. Put

$$f^{pr} = \otimes_{\star\in S_{\prime}} \otimes_{x\in\star} f^{pr}_{x,\star} \colon \mathbb{A}\times\mathbb{A} \longrightarrow \mathbb{C}$$

with the following local components.

At nonarchimedean nonsingular $x \in \star$ the component $f^{pr}_{x,\star}$ is $char_{(O_{x,\star},O_{x,\star})}$.

Put

$$f^{pr}_{x,\star} = q_x^{-1} char_{(\mathrm{O}_{x,\star},\mathrm{O}^{\perp}_{x,\star})}$$

at split ordinary double $x \in \star$, where $\mathrm{O}^{\perp}_{x,\star}$ is the orthogonal complement of $\mathrm{O}_{x,\star}$ with respect to $\psi_\star$ as in 27, $q_x = |k(x)|$. So then $f^{pr\diamond}_{x,\star} = q_x^{-1} char_{(O_{x,z},t_1{}^{-1}_{x,z} O_{x,z})}$, where $t_{1x,z}$ is a local t_1-parameter in $K_{x,z}$.

Using the notation in Example 2 of 30, define the components of f^{pr} over archimedean places

$$f^{pr}_{\omega,y}(\alpha,\beta) = \exp\big(-e_\omega \pi \left(|p_y(\alpha)|^2 + |p_y(\beta)|^2\right)\big),$$

for $(\alpha,\beta) \in \mathcal{O}_{\omega,y}\times\mathcal{O}_{\omega,y}$, where p_y is the projection map.

Since $\mathfrak{c}_\star = 1$ for almost all fibres $\star$ in $S_{\prime}$, the function f^{pr} is in the space $Q^0_{\mathbb{A}\times\mathbb{A}}$.

By 30 at nonsingular x of $\star$ we get

$$\mathcal{F}_{x,z}\big(char_{(t_1^{c_{x,z}} O_{x,z},\, t_1^{c'_{x,z}} O_{x,z})}\big) = q_{x,z}^{d_{x,z}-c_{x,z}-c'_{x,z}} char_{(t_1^{d_{x,z}-c_{x,z}} O_{x,z},\, t_1^{d_{x,z}-c'_{x,z}} O_{x,z})}$$

and

$$\mathcal{F}_{x,\star}(char_{(\mathrm{O}_{x,\star},\mathrm{O}^{\perp}_{x,\star})}) = char_{(\mathrm{O}^{\perp}_{x,\star},\mathrm{O}_{x,\star})}$$

at singular x of $\star$.

Now we define a centrally normalized function $f \in Q^0_{\mathbb{A}\times\mathbb{A}}$ which is a two-dimensional analogue of the centrally normalized function in [25].

DEFINITION OF f AND ν. We consider several cases.

(a) For a nonsingular fibre $\star$ put $f_\star = \otimes_{x\in\star} f_{x,\star}$, $f_{x,\star} = f^{pr}_{x,\star}$. We have

$$\mathcal{F}(f_\star)(\alpha) = f_\star(\nu_\star^{-1}\alpha)$$

with $\nu_\star \in T_{1,\star}$. In fact, $\nu_\star = (t_1{}_{x,\star}^{d_{x,\star}}, t_1{}_{x,\star}^{d_{x,\star}}) = (\rho_\star,\rho_\star)$, ρ is as in Remark 3 of 30.

(b) Let $\star$ be a singular fibre. Using the formula for $\mathcal{F}$, choose a modification $f_{x,\star}$ of $f^{pr}_{x,\star}$ in finitely many nonsingular points $x \in \star$, such that for all $x \in \star$

$$f_{x,\star}(\alpha) = f^{pr}_{x,\star}(\varepsilon_{x,\star}\alpha)$$

with some $\varepsilon_\star = (\varepsilon_{x,\star})_{x\in\star} \in T_\star$ where $\varepsilon_{x,z} = (t_{1,x,z}^{-c_{x,z}}, t_{1,x,z}^{-c'_{x,z}})$, and such that for $f_\star = \otimes_{x\in\star} f_{x,\star}$ we have

$$\mathcal{F}(f_\star)(\alpha) = f_\star(\nu_\star^{-1}\alpha)$$

with $|\nu_\star| = 1$. Just choose finitely many nonzero $c_{x,\star}, c'_{x,\star}$ at nonsingular $x \in \star$ such that

$$\prod_{x\in\star} q_{x,z}^{d_{x,z}} = \prod_{x\in\star} q_{x,z}^{c_{x,z}+c'_{x,z}}.$$

Put $c_{x,z} = c_{x,z'} = c'_{x,z} = c'_{x,z'} = 0$ at singular $x \in \star$.

In particular, $\nu_{x,\star} = (\nu_{x,z}, \nu_{x,z'})$, $\nu_{x,z} = (t_{1\,x,z}^{-1}, t_{1\,x,z})$, $\nu_{x,z'} = (t_{1\,x,z'}^{-1}, t_{1\,x,z'})$ at singular $x \in \star$.

(c) Similarly, using the formula for $\mathcal{F}$, on a horizontal curve y in positive characteristic choose a modification of $f_{x,y}^{pr}$ in finitely many points x on y, such that for all $x \in \star$

$$f_{x,y}(\alpha) = f_{x,y}^{pr}(\varepsilon_{x,y}\alpha)$$

with $(\varepsilon_{x,y}) \in T_y$, such that for $f_y = \otimes_{x\in y} f_{x,y}$ we have

$$\mathcal{F}(f_y)(\alpha) = f_y(\nu_y^{-1}\alpha)$$

for an appropriate $\nu_y \in T_{1,y}$.

(d) On a horizontal curve y in characteristic zero put $\eta_y = \sqrt[2n]{\mathfrak{c}_y} \in \mathbb{R}_{>0}$ where $n = |k(y) : \mathbb{Q}|$, $\mathfrak{c}_y$ was defined above, in this case it is the inverse of the discriminant of $k(y)$. Define f_y as having the same components as f^{pr} at nonarchimedean data and as

$$f_{\omega,y}(\alpha) = f_{\omega,y}^{pr}((\eta_y, \eta_y)\alpha)$$

at archimedean (ω, y). Then on horizontal curves in characteristic zero we have

$$\mathcal{F}(f_y)(\alpha) = f_y(\nu_y^{-1}\alpha)$$

with appropriate $\nu_y \in T_{1,y}$.

Now put

$$f = \otimes_{\star\in S_/} f_\star, \quad f_- = \otimes_{y\in S_-} f_y, \quad f_| = \otimes_{\star\in S_|} f_\star.$$

Define

$$\nu = (\nu_\star)_{\star\in S_/} \in \mathfrak{T}.$$

For every $\star \in S_/$ we get $\nu_\star \in T_{1,\star}$.

The functions f and $f_{\nu^{-1}}$ belong to the space $Q^0_{\mathbb{A}\times\mathbb{A}}$ and we get

$$\mathcal{F}(f) = f_{\nu^{-1}}$$

and

$$\mathcal{F}(f_\alpha)(\gamma) = |\alpha|^{-1}\mathcal{F}(f)_{\alpha^{-1}}(\gamma) = |\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\gamma).$$

It is easy to see that for the transform $\mathcal{F}$ of any function $g \in Q_{\mathbb{A}\times\mathbb{A}}$ we get $\mathcal{F}(g)|_{x,\star} = g_{\nu^{-1}}|_{x,\star}$ at nonsingular $x \in \star$.

As in the one-dimensional case [57], [25] the zeta integral $\zeta(f,\chi)$ can be calculated in two ways.

The first calculation uses the fibre integrals to compare the zeta integrals $\zeta_\star(f,|\,|_2^s)$, $\zeta(f,|\,|_2^s)$ with the zeta functions $\zeta_\star(s)$ of the fibre $\star$ and $\zeta_{\mathcal{E}}(s)$.

Theorem *Let $\mathcal{S} = \mathcal{E}$ as in the beginning of this section. Let f be the centrally normalized function defined above.*

For every fibre $\star$ we have

$$\zeta_\star(f,|\,|_2^s) = \mathfrak{c}_\star^{1-s}\zeta_\star(s)^2 = \mathfrak{c}_\star^{1-s}\prod_{x\in\star}\left(\frac{1}{1-q_{x,z}^{-s}}\right)^2$$

and $\mathfrak{c}_\star = |k(b)|^{f_b+m_b-1}$ where m_b is the number of components of the geometric fibre, and f_b is the (exponent of the) conductor of E at b. Thus,

$$\zeta_\star(f,|\,|_2^s) = \zeta_\star(s)^2, \quad \textit{for a nonsingular fibre } \star \textit{ over } b \in B_0$$
$$\zeta_\star(f,|\,|_2^s) = |k(b)|^{(f_b+m_b-1)(1-s)}\zeta_\star(s)^2, \quad \textit{for a singular fibre } \star \textit{ over } b \in B_0.$$

For every nonsingular horizontal curve y the zeta integral $\zeta_y(f,|\,|_2^s)$ is a meromorphic function which satisfies the functional equation $\zeta_y(f,|\,|_2^s) = \zeta_y(f,|\,|_2^{2-s})$ and which is holomorphic outside its poles of multiplicity two at $s = 0,2$ in characteristic zero and at $q^s = 1,q^2$ in positive characteristic. For a horizontal curve y in characteristic zero the zeta integral $\zeta_y(f,|\,|_2^s)$ is the square of a certain one-dimensional Iwasawa–Tate zeta integral at $s/2$ on the adeles $\mathbb{A}_{k(y)}$.

Thus,

$$\zeta_{\mathcal{E},S_{\prime}}(f,|\,|_2^s) = c_{\mathcal{E},S_{\prime}}(|\,|_2^s)\zeta_{\mathcal{E}}(s)^2, \qquad \Re(s) > 2,$$

where

$$c_{\mathcal{E},S_{\prime}}(|\,|_2^s) = c_{\mathcal{E},S_{\shortmid}}(|\,|_2^s)c_{\mathcal{E},S_{-}}(|\,|_2^s).$$

The first factor

$$c_{\mathcal{E},S_{\shortmid}}(|\,|_2^s) = \mathfrak{c}_{\mathcal{E}}^{1-s}.$$

The second factor is has a meromorphic continuation to the complex plane and satisfies the functional equation

$$c_{\mathcal{E},S_{-}}(|\,|_2^s) = c_{\mathcal{E},S_{-}}(|\,|_2^{2-s}),$$

and is holomorphic outside its poles at $s=0,2$ in characteristic zero and at $q^s=1,q^2$ in positive characteristic.

The zeta integral $\zeta_{\mathcal{E},S,}(f,|\,|_2^s)$ absolutely converges on $\Re(s)>2$. The same is true for the zeta integral $\zeta(g,|\,|_2^s)$ for every $g\in Q^0_{\mathbb{A}\times\mathbb{A}}$.

Proof: From the definitions of the local measures in 30 and the zeta integrals in 39 we get at nonsingular x of a fibre $\star$

$$\zeta_{x,\star}\big(char_{(t_1^{c_{x,\star}}O_{x,\star},t_1^{c'_{x,\star}}O_{x,\star})},|\,|_2^s\big)=q_{x,\star}^{d_{x,\star}-(c_{x,\star}+c'_{x,\star})s}\left(\frac{1}{1-q_{x,\star}^{-s}}\right)^2$$

and we get

$$\begin{aligned}\zeta_{x,\star}\big(q_x^{-1}char_{(\mathrm{O}_{x,\star},\mathrm{O}^{\perp}_{x,\star})},|\,|_2^s\big)&=\int_{T_{x,z}}q_x^{-1}char_{(O_{x,z},t_{1\,x,z}^{-1}O_{x,z})}|\,|^s\,d\mu_{K^{\times}_{x,z}\times K^{\times}_{x,z}}\\&=q_x^{-1+s}\left(\frac{1}{1-q_x^{-s}}\right)^2\end{aligned}$$

at singular x of $\star$.

The definition of $c_{x,z}$ above implies that $\prod_{x\in\star,\mathrm{ns}}q_{x,z}^{d_{x,z}-(c_{x,z}+c'_{x,z})s}\prod_{x\in\star,\mathrm{s}}q_x^{-1+s}=\mathfrak{c}_\star^{1-s}$.

For a singular fibre $\star$ of $\mathcal{E}\to B$ we get $g_y=0$ for all $y\subset\star$ and so $\mathfrak{c}_\star=\prod_{y\subset\star}q_y^2\prod_{x\in\star,\mathrm{s}}q_x^{-1}$. If $\mathcal{E}\longrightarrow B$ is a minimal proper regular model with normal crossings then it is easy to check directly the equality $\mathfrak{c}_\star=|k(b)|^{f_b+m_b-1}$ using the well known description of fibres of minimal proper regular models with normal crossings, e.g. 10.2 of [37]. Using 9.3.4 of [37] we then get the equality for every proper regular model with normal crossings $\mathcal{E}\longrightarrow B$, since blowing up at a point of such a model which gives another such model results in the same factor to appear on the left and right hand sides of the desired equality.

If y is a horizontal curve in characteristic zero then its zeta integral is the product of the local zeta integrals for nonarchimedean data

$$\zeta(char_{(t_1^{c_{x,y}}O_{x,y},t_1^{c'_{x,y}}O_{x,y})},|\,|_2^s,\mu_{x,y})=q_{x,y}^{d_{x,y}-(c_{x,y}+c'_{x,y})s}\left(\frac{1}{1-q_{x,y}^{-s/2}}\right)^2,$$

and for archimedean data

$$\zeta(f_{\omega,y},|\,|_2^s,\mu_{\omega,y})=\begin{cases}\big(\mathfrak{c}_y^{-s/4n}\pi^{-s/4}\Gamma(s/4)\big)^2 & \text{if } \omega \text{ is real}\\ \big(\mathfrak{c}_y^{-s/2n}(2\pi)^{1-s/2}\Gamma(s/2)\big)^2 & \text{if } \omega \text{ is complex.}\end{cases}$$

Hence the zeta integral $\zeta_y(f,|\,|_2^s)$ is proportional (up to a power of 2π) to the square of the appropriate one-dimensional zeta integral, defined in [25], p.446, at $s/2$ on

$k(y)$. Since $\mathcal{F}(f_y)(0) = f_y(0)$ and $|\nu_y| = 1$, the one-dimensional theory [57] implies that $\zeta_y(f,|\,|_2^s) = \zeta_y(f,|\,|_2^{2-s})$.

Let y be a horizontal curve in characteristic p. We will use the fact that the integral over T_y equals the double integral over M and over $T_{1,y}$, this directly follows from the same property at the residue level since we are in the unramified case, for more general results see 43. Compare the integral

$$\zeta_y(f,|\,|_2^s) = \int_{\mathfrak{T}_y} f(\alpha)|\alpha|^{s/2} d\mu(\alpha)$$

with another integral

$$\zeta_y^a(f,|\,|_2^s) = \int_{T_y} f(\alpha)|\alpha|^{s/2} d\mu(\alpha).$$

The latter is the product of the local integrals $\int_{T_{x,y}} f(\alpha)|\alpha|^{s/2} d\mu_{K_{x,y}^\times \times K_{x,y}^\times}(\alpha)$ to calculate which we can use the formula in the previous paragraph. Hence $\zeta_y^a(f,|\,|_2^s) = \mathfrak{c}_y^{1-s}\zeta_y(s/2)^2$, and it has the meromorphic continuation and the functional equation with respect to $s \to 2-s$ and it is holomorphic outside its poles of multiplicity two at $q^s = 1, q^2$. We can also write $\zeta_y^a(f,|\,|_2^s)$ as a series $\sum_{n\in N_y} c_n n^{-s/2}$ with $c_n = \int_{T_{1,y}} f(\mathfrak{m}_n\gamma)\, d\mu_{(\mathbb{A}_y\times\mathbb{A}_y)^\times}(\gamma)$, where $\mathfrak{m}_n \in \mathfrak{T}_y$, $||\mathfrak{m}_n|| = n$. Now, returning back to the original integral we deduce from the displayed formulas that $\zeta_y(f,|\,|_2^s) = \sum_{n\in N_y} c_{n^2} n^{-s}$. It is easy to show that then the latter satisfies the functional equation with respect to $s \to 2-s$ and is holomorphic outside its poles of multiplicity two at $q_y^s = 1, q_y^2$, as required.

Continuing the proof notice that for $\mathcal{S} = \mathcal{E}$ the factor $\mathfrak{c}_\star$ equals 1 for every nonsingular fibre. The definition of f implies

$$\zeta_{S_\prime}(f,|\,|_2^s) = c_{\mathcal{E},S_\prime}(|\,|_2^s)\zeta_{\mathcal{E}}(s)^2, \qquad c_{\mathcal{E},S_\prime}(|\,|_2^s) = c_{\mathcal{E},S_\shortmid}(|\,|_2^s) c_{\mathcal{E},S_-}(|\,|_2^s),$$

where $c_{\mathcal{E},S_\shortmid}(|\,|_2^s) = \prod_{\star\in S_\shortmid} \mathfrak{c}_\star^{1-s}$. For $y \in S_-$ the y-part of $c_{\mathcal{E},S_-}(|\,|_2^s)$ is $\zeta_y(|\,|_2^s)$ described above and so $c_{\mathcal{E},S_-}(|\,|_2^s) = c_{\mathcal{E},S_-}(|\,|_2^{2-s})$.

The well known absolute convergence of $\zeta_{\mathcal{E}}(\chi)$ for $\Re(s(\chi)) > 2$ implies the assertion about the convergence of the zeta integral.

To deduce the last assertion note that if $g = \otimes g_{x,\star} \in Q^0_{\mathbb{A}\times\mathbb{A}}$, then for almost all $x \in y \in S_\prime$ the x,y-factor of $\zeta(g,|\,|_2^s)$ equals the x,y-factor of $\zeta(f,|\,|_2^s)$. $\square$

Remark 1 The order of the pole of $\zeta(f,|\,|_2^s)$ at $s = 2$ equals $2 + 2|S_-|$.

Remark 2 The definitions can be extended to to the more general case of proper regular models of elliptic curves without the restriction of normal crossings and with the analogous comparison formulas between the fibre zeta integrals and the zeta functions, at least if the wild part of the conductor of the curve is trivial. In such an extension for an arbitrary singular point x of a fibre $\star$ over b one can define an object Q_x, extending the definition in 36, and the integration over $T_{x,\star}$, and then work with the corresponding fibre zeta integral $\zeta_\star$. If the wild part of the conductor at b is trivial, as an extension of the formulas in the previous theorem it is expected that we would get

$$\zeta_\star(f, |\,|_2^s) = \mathfrak{c}_\star^{1-s}\zeta_\star(s)^2, \qquad |k(b)|^{f_b+m_b-1} = \mathfrak{c}_\star,$$

which in particular gives a new formula for the tame part of the conductor. It is very interesting to investigate whether there is an adelic interpretation of the wild part of the conductor.

Remark 3 For regular models of curves of higher genus the factor $\mathfrak{c}_\star \neq 1$ for almost all fibres $\star$ and hence $\prod_{\star\in S_1}\mathfrak{c}_\star$ diverges, so one needs to renormalize the zeta integral, see 57.

3.5. Subgroups of T and associated integrals

In order to perform the second calculation of the zeta integral we use the filtration $T > T_1 > T_0$ where T_0 is the lift of units of the function ring of $\star$. The integration over T_0 will differ from the lift of the counting integration at the residue level: we will rescale the counting discrete measure on the function fields of fibres and then work with the limit of the rescaled measures.

41. In this section we define a local-global subgroup T_0 of T which in certain sense plays the role of the group of global elements in the one-dimensional case.

DEFINITION OF $\mathfrak{x}$, $\mathfrak{z}$, $\mathfrak{u}$, $\mathfrak{v}$, $T_\star^r$, $\mathcal{T}_\star$. Using the local maps of 36, denote by $\mathfrak{x}$ the homomorphism from $\big(\prod_{x\in\star,\mathrm{ns}}\mathcal{O}_{x,\star}\times\mathcal{O}_{x,\star}\times\prod_{x\in\star,\mathrm{s}}Q_x\times Q_x\big)\cap\mathbb{A}_\star\times\mathbb{A}_\star$ to $\mathbb{A}_\star\times\mathbb{A}_\star$, which is the local $\mathfrak{x}$ at every local singular data and is the identity map elsewhere. Similarly define the homomorphism $\mathfrak{z}$. As in 36, we get the isomorphisms $\mathfrak{v}$,$\mathfrak{u}$ between the images of $\mathfrak{x}$ and $\mathfrak{z}$.

Denote

$$\mathcal{T}_\star = \mathfrak{x}\Big(\prod_{x\in\star,\mathrm{ns}} T_{x,\star}\times\prod_{x\in\star,\mathrm{s}} Q_x^\times\times Q_x^\times\Big)\cap\big(\mathbb{A}_\star\times\mathbb{A}_\star\big)^\times,$$

this is a subgroup of $T_\star$.

Put

$$T_\star^r := {\prod_{x\in\star}}' T_{x,z},$$

following the notation in 24.

Then $\mathfrak{u}(\mathcal{T}_\star)$ is a subgroup of $T_\star^r$, and using 36 we obtain $p(\mathfrak{u}(\mathcal{T}_\star)) = p(T_\star^r)$.

Definition of T_0. Denote

$$T_0 := \mathfrak{x}(\mathbb{B}^\times \times \mathbb{B}^\times),$$

this group is isomorphic to $\mathbb{B}^\times \times \mathbb{B}^\times$. For a subset S_o of $S_,$ denote $T_{0,S_o} = T_0 \cap T_{S_o}$.

Definition of T_1, M. Using $|\,|$ defined in 36 put

$$T_1 = \{\alpha \in T : |\alpha| = 1\}.$$

For a subset S_o of $S_,$ introduce

$$T_{1,S_o} = \{\alpha \in T_{S_o} : |\alpha|_{S_o} = 1\}$$

which coincides with the kernel of $|\,|$ on T_{S_o}. In particular, for a fibre or a horizontal curve $\star$ the definition of $T_{1,\star}$ is compatible with the definition given in 36.

Using the definition of $S_,$ in 36 and 40 (S_- contains the zero section y_0) choose a subgroup M of T_{y_0} which is in bijection with N via $|\,|$. This gives the splitting

$$T = M \times T_1.$$

Definition of UT. Denote by UT the intersection of T_1 with the product of the nonarchimedean part of $T \cap \prod T_{1x,z}$ and the archimedean part of T (which corresponds to the data for all (ω, y), where ω runs through the archimedean places of the horizontal curves $y \in S_,$). The group UT is open in T_1.

Lemma (a) *T_0 is a subgroup of T_1.*
(b) *The homomorphism $\mathfrak{t}$ of* 37 *induces homomorphisms $T_1 \longrightarrow J^1$, $T_0 + UT \longrightarrow P + UJ$ and a surjective homomorphism*

$$T_1/(T_0 + UT) \longrightarrow J^1/(P + UJ) \simeq CH^0(\mathcal{E})^0,$$

the latter isomorphism comes from Lemma 35.
(c) *The diagramme of Lemma* 36 *induces a commutative diagramme*

$$\begin{tikzcd}
 & \mathbb{B}^\times \otimes \mathbf{B}_{S_,}^\times/(\mathbf{B}_{S_,}^\times \cap \mathbf{VA}_{S_,}^\times) \arrow[d] \arrow[dr] & \\
T_0 \arrow[r] & \mathbb{B}^\times \times \mathbb{B}^\times/(\mathbb{B}^\times \cap V\mathbb{A}^\times) \arrow[r] & P/(P \cap VJ_S),
\end{tikzcd}$$

where the diagonal map is the symbol map, and the maps are the induced restrictions of the appropriate maps in the diagramme of Lemma 36.

Proof: (a) follows from the definitions. The homomorphism $\mathfrak{t}$ induces the isomorphism $T_1/UT \longrightarrow J^1/UJ$. Since $\mathfrak{t}(\alpha,\beta) \equiv \{\alpha\beta,t_2\} \mod UK_2^t(K_{x,y})$, we have $\mathfrak{t}(T_0 + UT) \subset P + UJ$, and (b), (c) follow from Lemma 36. □

Remark Notice that $\mathfrak{t}^{-1}(P + UJ) \cap T$ is different from $T_0 + UT$ and $\mathfrak{t}^{-1}(P) \cap T$ is different from T_0.

42. DEFINITION OF $\mathfrak{T}_1$, $\mathfrak{T}_0$, $\mathfrak{M}$. Using the twisted module $||\,||$ defined in 36 put

$$\mathfrak{T}_1 = \{\alpha \in \mathfrak{T} : ||\alpha|| = 1\}.$$

Similarly, for a subset S_o of S, define

$$\mathfrak{T}_{1,S_o} = \{\alpha \in \mathfrak{T}_{S_o} : ||\alpha||_{S_o} = 1\}.$$

Then $\mathfrak{T}_{1,S_o} = T_{1,S_o} \cap \mathfrak{T}_{S_o}$ if and only if $S_o \subset S_\mathrm{l}$ or $S_o \subset S_h$.

We have $\mathfrak{T}_{1,\star} = T_{1,\star}$ for every fibre or curve $\star$. Since $T_{0,\star} < T_{1,\star}$, we define $\mathfrak{T}_0 = T_0$.

Denote

$$\mathfrak{M} = M^2$$

The definition of M in the previous section implies that $\mathfrak{M} < \mathfrak{T}_{y_0}$. We get $||\mathfrak{M}|| = |M| = N$. If $k(y_0)$ is in characteristic zero then $\mathfrak{M} = M$.

The choices give the splitting

$$\mathfrak{T} = \mathfrak{M} \times \mathfrak{T}_1.$$

DEFINITION. For a horizontal curve y endow the group $N_y < \mathbb{R}$ defined in 36 with the induced ordering. If N_y is discrete, generated by $q_y > 1$, then we say that a function $f : N_y \longrightarrow \mathbb{R}$ is nonincreasing (resp. nondecreasing) at $n \in N_y$ if $f(q_y^{-1}n) \geq f(n)$ (resp. $f(q_y^{-1}n) \leq f(n)$).

DEFINITION OF $\wr\wr$, $T_{1,1}$. Let $\wr\wr$ be the homomorphism

$$(\mathbb{A} \times \mathbb{A})^\times \longrightarrow N \times N, \quad (\alpha_1,\alpha_2) \mapsto (||\alpha_1||,||\alpha_2||).$$

Denote $T_{1,1} = \ker \wr\,\wr$.

Similarly for a horizontal curve y define $\wr\wr_y : T_y \longrightarrow N_y \times N_y$ and $T_{1,1,y} = \ker \wr\,\wr_y$. Similarly define $\wr\wr_{x,z}$.

DEFINTION (CHOICE OF REPRESENTATIVES $m_n \in M$ OF N). Choose elements $m_n \in T_{y_0}$, such that $M = \{m_n : n \in N\}$ and $|m_n|_{y_0} = n \in N_{y_0} = N$, and each of the two components of $\wr\,\wr_{y_0}(m_n)$ is a nondecreasing function at n and $\wr\,\wr_{y_0}(m_n) = (\sqrt{n},\sqrt{n})$ if n is a square in N. In characteristic zero in addition we choose m_n such that the components of $\wr\,\wr_{x,y_0}(m_n)$ for nonarchimedean points $x \in y_0$ are 1 and the components of $\wr\,\wr_{\omega,y_0}(m_n)$ for all archimedean places ω of $k(y_0)$ are equal to each other positive real numbers.

DEFINITION OF $\mathfrak{m}$, $\mathfrak{m}_n$. For $m \in M$ define $\mathfrak{m} = m^2 \in \mathfrak{M}$. In particular $\mathfrak{m}_n = m_n^2$ and $\mathfrak{M} = \{\mathfrak{m}_n : n \in N\}$.

43. Now we define the integral $\int_{T_0}$. Note that for a nonsingular fibre $\star$ and the function f^{pr} defined in 40 the value of the integral $\int_{(\mathbb{B}_\star\times\mathbb{B}_\star)^\times} f^{pr}\, d\mu_{(\mathbb{B}_\star\times\mathbb{B}_\star)^\times}$ is $\geq (q_\star - 1)^2$, $q_\star$ is defined in 36, so the product of all such fibre integrals diverges. Therefore we have to rescale the measure $\mu_{(\mathbb{B}_\star\times\mathbb{B}_\star)^\times}$ fibrewise in order to have a useful measure on T_0. We will also define the integral $\int_{\mathfrak{T}/T_0}$ which connects the former and $\int_{\mathfrak{T}}$.

DEFINITION OF $\mu_N, \mu_M, \mu_{\mathfrak{M}}$. Let μ_N be the appropriate measure on the group N, as in the one-dimensional theory, corresponding to the counting measure in positive characteristic and corresponding to the induced from $\mathbb{R}^\times, dn/n$ in characteristic zero.

Let the measure μ_M on M be $|\,|^*\mu_N$ where $|\,|\colon M \longrightarrow N$. Let the measure $\mu_{\mathfrak{M}}$ on $\mathfrak{M}$ correspond to μ_N via the restriction of the map $||\,||\colon \mathfrak{M} \longrightarrow N$, i.e. $\mu_{\mathfrak{M}} = ||\,||^*\mu_N$. Raising to the second power gives an isomorphism $\mathfrak{s}\colon M \longrightarrow \mathfrak{M}$. Thus we get $\mu_M = \mathfrak{s}^*\mu_{\mathfrak{M}}$.

DEFINITION OF $\int_{\mathfrak{T}_1} g$. Let g be an element of $R^0_{(\mathbb{A}\times\mathbb{A})^\times}$, the space defined in 37. Let S^o be a finite subset of nonsingular curves in $S_\prime$ such that $y_0 \in S^o$. We get $\mathfrak{T}_{S^o} = \mathfrak{M} \times \mathfrak{T}_{1,S^o}$. Denote $S_o = S_\prime \setminus S^o$. Let $\mu_{\mathfrak{T}_{1,S^o}}$ be the lift of the Haar measure on its residue image $p(\mathfrak{T}_{1,S^o})$, the latter measure is normalized in such a way that the Haar measure on $p(\mathfrak{T}_{S^o})$ is the product of it and the measure $p_*\mu_{\mathfrak{M}}$ on $p(\mathfrak{M})$. For $\alpha \in \mathfrak{T}_{S_o}$ introduce

$$g^o(\alpha) = \int_{\mathfrak{T}_{1,S^o}} g(\alpha^0\gamma)\, d\mu_{\mathfrak{T}_{1,S^o}}(\gamma)$$

where $\alpha^0 \in \mathfrak{T}_{S^o}$ is such that $||\alpha^0||_{S^o} = ||\alpha||_{S_o}^{-1}$. The definitions imply that $g^o \in R^0_{(\mathbb{A}_{S_o}\times\mathbb{A}_{S_o})^\times}$.

Define

$$\int_{\mathfrak{T}_1} g := \int_{\mathfrak{T}_{S_o}} g^o,$$

where we restrict g^o defined in 37 to T_{S_o}. It is easy to check that the integral does not depend on the choice of S^o.

DEFINITION OF d_y, d_{S_o}. Put $d_\star = 1$ for horizontal curves and singular fibres. For a nonsingular fibre $\star$ denote

$$d_\star = (q_\star - 1)^{-2},$$

where $q_\star$ is defined in 36, it is the cardinality of the maximal finite subfield of $k(\star)$.

For a finite subset $S_o \subset S_\prime$ put

$$d_{S_o} = \prod_{\star \in S_o} d_\star.$$

DEFINITION OF $\mu'_{(\mathbb{B}\times\mathbb{B})^\times}$. Recall that the measure $\mu_{(\mathbb{B}\times\mathbb{B})^\times}$, defined in 31, is induced from the measure $\mu_{\mathbb{B}\times\mathbb{B}}$.

For a finite subset $S_o \subset S_\prime$ define

$$\mu'_{(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times} = d_{S_o}\,\mu_{(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times}.$$

In particular, if S_o includes nonsingular fibres only then $\mu'_{(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times}((\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times\cap (O\mathbb{A}_\star \times O\mathbb{A}_\star)^\times) = 1$, with $O\mathbb{A}_\star$ defined in 25.

Now define

$$\mu'_{(\mathbb{B}\times\mathbb{B})^\times} = \lim_{S_o \subset S_\prime} \mu'_{(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times} = \otimes_{\star\in S_\prime}\mu'_{(\mathbb{B}_\star\times\mathbb{B}_\star)^\times}.$$

The ring of measurable subsets in $(\mathbb{B}\times\mathbb{B})^\times$ with respect to $\mu'_{(\mathbb{B}\times\mathbb{B})^\times}$ is generated by $\prod_{\star\in S_\prime} B_\star$, $B_\star$ are measurable subsets of $(\mathbb{B}_\star\times\mathbb{B}_\star)^\times$ almost all of which equal to $(\mathbb{B}_\star\times\mathbb{B}_\star)^\times \cap (O\mathbb{A}_\star \times O\mathbb{A}_\star)^\times$.

Thus, the measure $\mu'_{(\mathbb{B}\times\mathbb{B})^\times}$ is the limit of the rescaled measures and differs significantly from $\mu_{(\mathbb{B}\times\mathbb{B})^\times}$.

DEFINITION OF SPACES $Q^*_{\mathbb{A}\times\mathbb{A}}$, $Q^*_{\mathbb{A}_{S_o}\times\mathbb{A}_{S_o}}$. Let $g = \otimes_\star g_\star \in Q^0_{\mathbb{A}\times\mathbb{A}}$. Consider two restrictions (1) and (2).

Restriction (1) is that there is a finite set of fibres and horizontal curves $S_g \subset S_\prime$ which includes all singular fibres such that the integral $d_y \int_{(\mathbb{B}_y\times\mathbb{B}_y)^\times} g_\star\, d\mu_{(\mathbb{B}_y\times\mathbb{B}_y)^\times} = 1$ for all $y \in S_\prime \setminus S_g$.

Let $r_\star$ be the group of roots of unity of $\mathbb{B}_\star^\times$, it is isomorphic to the group of roots of unity of $k(\star)$. Restriction (2) is that for every vertical $\star$ the action of $T_\star$ on $g_\star$, defined in 37, factorizes through the action of $T_\star/r_\star \times r_\star$, i.e. for every $\alpha \in T_\star$ the value $g_\star\big((\theta_1,\theta_2)\alpha\big)$ is constant when θ_i run through roots of unity in $k(\star)$. Then we have

$$g_\star(\alpha) = d_\star \sum_{(\theta_1,\theta_2)\in r_\star\times r_\star} g_\star\big((\theta_1,\theta_2)\alpha\big).$$

Denote the subspace of $Q^0_{\mathbb{A}\times\mathbb{A}}$ generated by the functions g satisfying (1) and (2) by $Q^*_{\mathbb{A}\times\mathbb{A}}$. Similarly define $Q^*_{\mathbb{A}_{S_o}\times\mathbb{A}_{S_o}}$.

For example, f^{pr}, f defined in 40 belong to the space $Q^*_{\mathbb{A}\times\mathbb{A}}$.

DEFINITION OF $\int_{T_0} g$. Using the isomorphism $\mathfrak{x}$ from $(\mathbb{B}\times\mathbb{B})^\times$ onto T_0 define for $g=\otimes g_\star \in Q^*_{\mathbb{A}\times\mathbb{A}}$

$$\int_{T_{0,S_o}} g := d_{S_o}\int_{(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times} g\circ\mathfrak{x}\, d\mu_{\mathbb{B}\times\mathbb{B}},\quad \int_{T_0} g := \int_{(\mathbb{B}\times\mathbb{B})^\times} g\circ\mathfrak{x}\, d\mu'_{(\mathbb{B}\times\mathbb{B})^\times} = \lim_{S_o\subset S_\prime}\int_{T_{0,S_o}} g.$$

Thus, $\int_{T_0} g = d_{S_g}\int_{(\mathbb{B}_{S_g}\times\mathbb{B}_{S_g})^\times} g\circ\mathfrak{x}\, d\mu_{\mathbb{B}\times\mathbb{B}}$, and the integral does not depend on the choice of S_g. We get $\int_{T_0} g = \prod_{\star\in S_o}\int_{T_{0,\star}} g_\star$.

Extend by linearity to the vector space generated by such functions g.

Keeping in mind the definition of $\mu_{(\mathbb{B}\times\mathbb{B})^\times}$ in 31, for a function $g\in Q^*_{\mathbb{A}\times\mathbb{A}}$ which is the pullback of a Bruhat–Schwartz function at the residue level we obtain

$$\int_{T_{0,\star}} g = d_\star \sum_{p(\beta)\in k(\star)^\times\times k(\star)^\times} g(\mathfrak{x}(\beta))$$

where β runs through a set of representatives of $k(\star)^\times\times k(\star)^\times$ in $(\mathbb{B}_\star\times\mathbb{B}_\star)^\times$.

DEFINITION OF $\mu'_{(\mathbb{A}\times\mathbb{A})^\times/(\mathbb{B}\times\mathbb{B})^\times}$. For the group $(\mathbb{A}_{k(\star)}\times\mathbb{A}_{k(\star)})^\times/(k(\star)\times k(\star))^\times$ choose a translation invariant measure $\mu_{(\mathbb{A}_{k(\star)}\times\mathbb{A}_{k(\star)})^\times/(k(\star)\times k(\star))^\times}$ such that the one-dimensional normalized measure $\mu_{(\mathbb{A}_{k(\star)}\times\mathbb{A}_{k(\star)})^\times}$ equals the product of it and of the discrete counting measure $\mu_{(k(\star)\times k(\star))^\times}$.

Let $\mu_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times/(\mathbb{B}_\star\times\mathbb{B}_\star)^\times}$ be its pullback with respect to $p_\star$, $p_\star$ is defined in 31. So measurable sets belong to the ring of sets generated by $p_\star^{-1}(A_1,A_2)$ where A_i are measurable subsets of $\mathbb{A}^\times_{k(\star)}/k(\star)^\times$, and $\mu_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times/(\mathbb{B}_\star\times\mathbb{B}_\star)^\times}(p_\star^{-1}(A_1,A_2))$ is equal to the product of the measures of A_1 and A_2.

Put

$$\mu'_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times/(\mathbb{B}_\star\times\mathbb{B}_\star)^\times} = d_\star^{-1}\mu_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times/(\mathbb{B}_\star\times\mathbb{B}_\star)^\times}.$$

Then the weak measure $\mu^w_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times}$ defined in Remark 1 of 31 is the tensor product of it and the measure $\mu'_{(\mathbb{B}_\star\times\mathbb{B}_\star)^\times}$.

Define

$$\mu'_{(\mathbb{A}\times\mathbb{A})^\times/(\mathbb{B}\times\mathbb{B})^\times} = \otimes_{\star\in S_\prime}\mu'_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times/(\mathbb{B}_\star\times\mathbb{B}_\star)^\times}.$$

Remark The measure $\mu'_{(\mathbb{B}\times\mathbb{B})^\times}$ is not the lift of the discrete counting measure on $\prod_{\star\in S_\prime}(k(\star)\times k(\star))^\times$, due to the nontrivial factor $d_\star$ for all nonsingular vertical curves. Compare this with the relation between the measures on the idele and adele groups in dimension one. Similarly, the measure $\mu'_{(\mathbb{A}\times\mathbb{A})^\times/(\mathbb{B}\times\mathbb{B})^\times}$ is not the lift of the measure $\otimes_{\star\in S_\prime}\mu_{(\mathbb{A}_{k(\star)}\times\mathbb{A}_{k(\star)})^\times/(k(\star)\times k(\star))^\times}$.

DEFINITION OF $\int_{\mathfrak{T}_\star/T_{0,\star}} g_\star$. Let $g_\star$ be the pullback of an integrable function on $(\mathbb{A}_{k(\star)}\times\mathbb{A}_{k(\star)})^\times$ at the residue level, such that $g_{\star\beta}=g_\star$ for every $\beta\in T_{0,\star}$.

If $\star$ is a nonsingular fibre or a horizontal curve, define

$$\int_{\mathfrak{T}_\star/T_{0,\star}} g_\star := \int g_\star char_{\mathfrak{T}_\star/T_{0,\star}}\, d\mu'_{(\mathbb{A}_\star\times\mathbb{A}_\star)^\times/(\mathbb{B}_\star\times\mathbb{B}_\star)^\times}.$$

Hence $\int_{\mathfrak{T}_\star/T_{0,\star}}\int_{T_{0,\star}}=\int_{\mathfrak{T}_\star}$.

Let $\star$ be a singular fibre. Then $\mathfrak{T}_\star=T_\star$. Denote by $\mu_{T^r_\star}$ the measure on the group $T^r_\star$ which is the tensor product of the local normalized measures on $T_{x,z}$. The subgroup $\mathfrak{u}(\mathcal{T}_\star)$ of $\mu_{T^r_\star}$ gets the induced measure $\mu_{\mathfrak{u}(\mathcal{T}_\star)}$: its measurable subsets are $S'=S\cap\mathfrak{u}(\mathcal{T}_\star)$ where S runs through measurable subsets with respect to$\mu_{T^r_\star}$ and $\mu_{\mathfrak{u}(\mathcal{T}_\star)}(S')=\mu_{T^r_\star}(S)$, $S=p^{-1}(p(S'))$.

We can use the ring isomorphism $\mathfrak{u}$ to transfer the additive and multiplicative measures.

Using $\mu_{\mathfrak{u}(\mathcal{T}_\star)}$ define the translation invariant measure $\mu_{\mathcal{T}_\star}=\mathfrak{u}^*\mu_{\mathfrak{u}(\mathcal{T}_\star)}$ on $\mathcal{T}_\star$. Note that if for two $\mu_{\mathcal{T}_\star}$-measurable sets B_1,B_2 we have $p(B_1)=p(B_2)$, then $p(\mathfrak{u}(B_1))=p(\mathfrak{u}(B_2))$ and their $\mu_{\mathfrak{u}(\mathcal{T}_\star)}$-measures are equal, hence $\mu_{\mathcal{T}_\star}(B_1)=\mu_{\mathcal{T}_\star}(B_2)$. So we get the well defined measure $p_*\mu_{\mathcal{T}_\star}$ on $p(\mathcal{T}_\star)$.

The image $p(\mathcal{T}_\star)$ equals $p(T_\star)$. Denote by $\mathcal{T}_{0,\star}$ the preimage in $\mathcal{T}_\star$ of $p(T_{0,\star})$, so $\mathcal{T}_{0,\star}=T_{0,\star}\cap\mathcal{T}_\star$. Consider the measure $\mu_{\mathcal{T}_{0,\star}}$ which lifts with respect to p the discrete counting measure $\mu_{p(T_{0,\star})}$ on $p(T_{0,\star})$. Choose a splitting of $1\to\mathcal{T}_{0,\star}\to\mathcal{T}_\star\to\mathcal{T}_\star/\mathcal{T}_{0,\star}\to 1$ and denote by $\mathcal{R}_\star$ the image of $\mathcal{T}_\star/\mathcal{T}_{0,\star}$. Using the one-dimensional approximation property we can assume that for all singular $x\in\star$ the $(x,\star)$-components of all elements of $\mathcal{R}_\star$ belong to $O^\times_{x,\star}$.

Let $\mu_{\mathcal{R}_\star}$ be the translation invariant measure on $\mathcal{R}_\star$ which lifts the translation invariant measure μ on $p(\mathcal{R}_\star)$ via the map $p\colon\mathcal{T}_\star\to p(\mathcal{R}_\star)$, for which $\mu_{p(\mathcal{T}_\star)}=\mu\otimes\mu_{p(T_{0,\star})}$. Then we get the formula

$$\int_{\mathcal{T}_\star} g\,d\mu_{\mathcal{T}_\star}=\int_{\mathcal{R}_\star}\int_{\mathcal{T}_{0,\star}} g(\gamma\beta)\,d\mu_{\mathcal{T}_{0,\star}}(\beta)\,d\mu_{\mathcal{R}_\star}(\gamma)$$

for functions $g\in Q^*_{\mathbb{A}_\star\times\mathbb{A}_\star}$.

Now define

$$\int_{\mathfrak{T}_\star/T_{0,\star}} g_\star=\int_{T_\star/T_{0,\star}} g_\star := \int_{\mathcal{R}_\star} g_\star\,d\mu_{\mathcal{R}_\star}.$$

DEFINITION OF $\int_{\mathfrak{T}/T_0} g$, $\int_{\mathfrak{T}_1/T_0} g$. Let $g=\otimes_{\star\in S_\prime} g_\star$, where $g_\star$ is the pullback of an integrable function on $(\mathbb{A}_{k(\star)}\times\mathbb{A}_{k(\star)})^\times$ at the residue level, such that $g_{\star\beta}=g_\star$ for every $\beta\in T_{0,\star}$ and $\star\in S_\prime$. Define

$$\int_{\mathfrak{T}/T_0} g := \prod_{\star\in S_\prime}\int_{\mathfrak{T}_\star/T_{0,\star}} g_\star,$$

where the factor integrals are defined as above. Extend to linear combinations.

Similar to the definition of $\int_{\mathfrak{T}_1} g$ and using g^o defined above, define $\int_{\mathfrak{T}_1/T_0} g := \int_{\mathfrak{T}_{S_o}/T_{0,S_o}} g^o$. This does not depend on the choice of $S^o = S_\prime \setminus S_o$.

Lemma *Let $g \in R^0_{(\mathbb{A}\times\mathbb{A})^\times}$ and suppose that the integral $\int_{\mathfrak{T}} g$ absolutely converges. Then*

$$\int_{\mathfrak{T}} g = \int_{\mathfrak{M}} \int_{\mathfrak{T}_1} g(\mathfrak{m}\alpha)\, d\mu(\alpha)\, d\mu_{\mathfrak{M}}(\mathfrak{m}) = \int_M \int_{\mathfrak{T}_1} g(\mathfrak{m}\alpha)\, d\mu(\alpha)\, d\mu_M(m).$$

Let $g = \otimes g_\star \in Q^*_{\mathbb{A}\times\mathbb{A}}$. such that at every singular $x \in \star$ the $(x,\star)$-component of g is $char_{(\mathfrak{r}(t)^i \mathrm{O}_{x,\star}, \mathfrak{r}(t)^j \mathrm{O}_{x,\star})}$. Let $\pi = ||\,||^s \colon \mathfrak{T} \longrightarrow \mathbb{C}^\times$, $s \in \mathbb{C}$. Suppose that the integral $\int_{\mathfrak{T}} g\pi$ absolutely converges. Then

$$\begin{aligned} \int_{\mathfrak{T}} g\pi &= \int_{\mathfrak{T}/T_0} \int_{T_0} g(\gamma\beta)\, d\mu(\beta)\, \pi(\gamma)\, d\mu(\gamma) \\ &= \int_M \int_{\mathfrak{T}_1/T_0} \int_{T_0} g(\mathfrak{m}\gamma\beta)\, d\mu(\beta)\, \pi(\gamma)\, d\mu(\gamma)\, d\mu_M(m), \\ \int_{\mathfrak{T}_1} g &= \int_{\mathfrak{T}_1/T_0} \int_{T_0} g(\gamma\beta)\, d\mu(\beta)\, d\mu(\gamma). \end{aligned}$$

Proof: The formula $\int_{\mathfrak{T}} = \int_{\mathfrak{T}/T_0} \int_{T_0}$ follows from $\int_{\mathfrak{T}_\star} = \int_{\mathfrak{T}_\star/T_{0,\star}} \int_{T_{0,\star}}$ which for a singular fibre $\star$ follows from

$$\begin{aligned} \int_{\mathfrak{T}_\star} g_\star(\alpha)\, ||\alpha||^s &= \int_{\mathfrak{u}(\mathcal{T}_\star)} g_\star^\diamond(\alpha)\, |\alpha|^s\, d\mu_{\mathfrak{u}(\mathcal{T}_\star)}(\alpha) = \int_{\mathcal{T}_\star} g_\star(\alpha)\, |\alpha|^s\, d\mu_{\mathcal{T}_\star}(\alpha) \\ &= \int_{\mathcal{R}_\star} \left(\int_{\mathcal{T}_{0,\star}} g_\star(\gamma\beta)\, d\mu(\beta) \right) |\gamma|^s\, d\mu_{\mathcal{R}_\star}(\gamma) \\ &= \int_{\mathcal{R}_\star} \left(\int_{T_{0,\star}} g_\star(\gamma\beta)\, d\mu(\beta) \right) |\gamma|^s\, d\mu_{\mathcal{R}_\star}(\gamma), \end{aligned}$$

here we use that $g^\diamond = g \circ \mathfrak{v}$ on $\mathfrak{u}(\mathcal{T}_\star)$. Thus, we get the formula for the singular fibre $\star$. $\square$

3.6. Two-dimensional theta formula

The one-dimensional theta formula relates the integral over $\mathrm{k}^\times$ with the integral over $\partial \mathrm{k}^\times = \mathrm{k} \setminus \mathrm{k}^\times = \{0\}$. In dimension one the set-theoretical difference between the additive and multiplicative structures, the latter is of course important for class field theory, is relatively small, and the classical theta formula is very close to the summation formula. In dimension two the difference between the additive and class field theory structures (i.e. K_2 or $K_1 \times K_1$ in the unramified theory) is much larger. So is the difference between the two-dimensional summation formula in 32, which follows from additive analytic duality, and the two-dimensional theta formula of this section. The latter will be derived for an integral over T_0 of a certain combination of functions, which naturally originate in the study of the zeta integral and is partially motivated by class field theory. The two-dimensional theta formula takes into account some class field theoretical structures and the summation formulas for all infinitely many finite subsets of $S_\prime$. It also glues together differently scaled data on vertical and horizontal curves. The theta formula will play a fundamental role in the study of analytic properties of the zeta integral.

The homomorphism $\mathfrak{r}$ defined in 41 induces $\mathfrak{r}\colon \mathbb{B}\times\mathbb{B} \longrightarrow \mathbb{A}\times\mathbb{A}$. By abuse of notation we do not write $\mathfrak{r}$ in the formulas for $\int_{T_0}$ and $\int_{\partial T_0}$ in the next section.

44. First we define an integral $\int_{\partial T_0}$ over the (weak) boundary ∂T_0 of T_0.

Recall that locally the closure of the set $\mathcal{O}_{x,y}^\times$ in the topology of $K_{x,y}$ is $\mathcal{O}_{x,y}$ (see 15). However, $\mathbb{A}_\star^\times$ is not a dense subset in the topology of $\mathbb{A}_\star$: already in the one-dimensional case the subset of ideles is not dense in adeles. In dimension one the weak topology on adeles is the weakest topology in which every continuous character of adeles is continuous. Ideles are dense in adeles with respect to the weak topology and the closure $\mathrm{Ck}^\times$ of $\mathrm{k}^\times$ with respect to the weak topology on adeles is k, so $\{0\} = \mathrm{k}\setminus\mathrm{k}^\times = \mathrm{Ck}^\times\setminus\mathrm{k}^\times$ can be viewed in this sense as the (weak) boundary, $\partial\mathrm{k}^\times$ of $\mathrm{k}^\times$.

DEFINITION OF THE WEAK TOPOLOGY ON $\mathbb{A}_{S_o}$ FOR $S_o \subset S_\prime$. The weak topology on $\mathbb{A}_{S_o}$ is the weakest topology in which every continuous character of adeles is continuous.

It is easy to see that the set $\mathbb{A}_\star^\times$ is a dense subset in the weak topology of $\mathbb{A}_\star$. Using Lemma 29 we derive that the closure $\mathrm{C}\mathbb{B}_\star^\times$of $\mathbb{B}_\star^\times$ in the weak topology of $\mathbb{A}_\star$ is $\mathbb{B}_\star$.

DEFINITION OF ∂T_0. For a finite subset $S_o \subset S_\prime$ we define

$$\partial T_{0,S_o} := \mathbb{B}_{S_o}\times\mathbb{B}_{S_o}\setminus(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times.$$

Hence $\partial T_{0,S_o} = \mathrm{C}(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times\setminus(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times$ is the boundary of T_{0,S_o} in the weak topology of $\mathbb{A}_{S_o}\times\mathbb{A}_{S_o}$.

Define the 'boundary' of T_0

$$\partial T_0 := \bigcup_{S_o\subset S_\prime} \partial T_{0,S_o}\times T_{0,S_\prime\setminus S_o},$$

where S_o runs through all finite subsets of $S_\prime$.

DEFINITION. For a non empty subset S_o of $S_\prime$ put

$$\Delta_{S_o} = \partial(\mathbb{A}_{S_o} \times \mathbb{A}_{S_o})^{\times} = \mathbb{A}_{S_o} \times \mathbb{A}_{S_o} \setminus (\mathbb{A}_{S_o} \times \mathbb{A}_{S_o})^{\times}.$$

For every $y \in S_1$ we get

$$\partial T_{0,y} = \Delta_{S_y} \cap \mathbb{B}_y \times \mathbb{B}_y = \mathcal{M}_y \times \mathcal{O}_y \cup \mathcal{O}_y \times \mathcal{M}_y,$$

hence

$$\begin{aligned}\partial T_{0,S_o} &= \Delta_{S_o} \cap \mathbb{B} \times \mathbb{B} \\ &= \Big(\bigcup_{y_1 \in S_o, y_1 \in S_1} \prod_{y \neq y_1, y \in S_o, y \in S_1} (\mathcal{O}_y \times \mathcal{O}_y) \times (\mathcal{M}_{y_1} \times \mathcal{O}_{y_1} \cup \mathcal{O}_{y_1} \times \mathcal{M}_{y_1}) \Big) \cap \mathbb{B}_{S_o} \times \mathbb{B}_{S_o}\end{aligned}$$

for every finite S_o.

By 43 for a function $g \in Q^*_{\mathbb{A}\times\mathbb{A}}$ we have

$$\int_{T_0} g = \int_{T_0} g\, \mu'_{(\mathbb{B}\times\mathbb{B})^\times} = \lim_{S_o \subset S_\prime} d_{S_o} \int_{(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times} g\, d\mu_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}}.$$

Then it is natural to introduce

DEFINITION OF $\mu'_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}}$. For a finite subset $S_o \subset S_\prime$ put $\mu'_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}} = d_{S_o}\mu_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}}$.

We can also define $\mu'_{\mathbb{B}\times\mathbb{B}} = \otimes_{\star\in S_\prime}\mu'_{\mathbb{B}_\star\times\mathbb{B}_\star}$, even though unlike $\mu'_{(\mathbb{B}\times\mathbb{B})^\times}$ this object is less useful, since $\mu'_{\mathbb{B}_\star\times\mathbb{B}_\star}(\mathbb{B}_\star\times\mathbb{B}_\star) \neq 1$ for infinitely many $\star$.

DEFINITION OF $\int_{\partial T_0} g$. Let $g \in Q^*_{\mathbb{A}\times\mathbb{A}}$. For a finite subset $S_o \subset S_\prime$ define

$$\int_{\partial T_{0,S_o}} g := \int_{\partial T_{0,S_o}} g\, d\mu'_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}} = d_{S_o} \int_{\partial(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times} g\, d\mu_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}}.$$

Put

$$\int_{\partial T_0} g := \lim_{S_o\subset S_\prime} \int_{\partial T_{0,S_o}} g.$$

The limit is not finite already for the function f defined in 40.

Let f be as fixed in 40. Using the defining property of T_0 in 41 and 43 and the definition of f we obtain that if $\star$ is a nonsingular fibre then

$$\int_{T_{0,\star}} f(\beta)\, d\mu(\beta) = \int_{T_{0,\star}} f(\nu^{-1}\beta)\, d\mu(\beta) = 1,$$

ν is defined in 40.

DEFINITION OF S_α. Recall that all $m \in M$ lie on one horizontal curve y_0, the zero section.

For each $\alpha = (\alpha^{(1)}_{x,z}, \alpha^{(2)}_{x,z}) \in T$ choose a finite set S_α of fibres and horizontal curves in S, such that

(a) S_α includes all horizontal curves in S, and all singular fibres.

(b) $\alpha^{(m)}_{x,z} \in O^\times_{x,z}$ for all $z \in \star(x)$, $x \in \star \notin S_\alpha$. In particular $|\alpha_\star|_\star = 1$ if $\star \notin S_\alpha$.

(c) $S_\alpha = S_{\alpha^{-1}}$.

In particular,

$$\int_{T_{0,y}} f(\alpha\beta)\,d\mu(\beta) = \int_{T_{0,y}} f(\nu^{-1}\alpha^{-1}\beta)\,d\mu(\beta) = \int_{T_{0,y}} |\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta)\,d\mu(\beta) = 1$$

for all $y \notin S_\alpha$. So the integrals $\int_{T_0}$ of f_α and $f_{\nu^{-1}\alpha}$ are defined, and in their calculation it is sufficient to use any finite set S_o which contains S_α.

Theorem (two-dimensional theta formula on T_0) *Let $\mathcal{S} = \mathcal{E}$ as above. Let f be as in* 40.

Then for $\alpha \in \mathfrak{T}$ the integral $\int_{\partial T_0} \big(|\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta) - f(\alpha\beta)\big)\,d\mu(\beta)$ exists and

$$\int_{T_0} \big(f(\alpha\beta) - |\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta)\big)\,d\mu(\beta) = \int_{\partial T_0} \big(|\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta) - f(\alpha\beta)\big)\,d\mu(\beta).$$

Equivalently,

$$\int_{T_0} \big(f_\alpha(\beta) - \mathcal{F}(f_\alpha)(\beta)\big)\,d\mu(\beta) = \int_{\partial T_0} \big(\mathcal{F}(f_\alpha)(\beta) - f_\alpha(\beta)\big)\,d\mu(\beta).$$

Proof: For a fixed $\alpha \in \mathfrak{T}$ each summand in the first line of the displayed formula in the statement of the Theorem is the product of fibre factors almost all of which are 1, and so the proof is reduced to the case of the finite set S_α as above, and then we can use the summation formula of 32. Of course, when α runs through all T, the set of finitely many fibres where those factors are not 1 can be arbitrarily large.

Let $\alpha = (\alpha_\star)$, $\alpha_\star \in \mathfrak{T}_\star$.

If y is a nonsingular fibre or a horizontal curve then $\mathfrak{T}_{0,y} = T_{0,y}$. Using the summation formula in 32 and the formula for the transform of f in 40 we obtain

$$\begin{aligned}\int f(\alpha_y\beta)\,d\mu_{\mathbb{B}_y\times\mathbb{B}_y}(\beta) &= |\alpha_y|^{-1}\int \mathcal{F}(f)(\alpha_y^{-1}\beta)\,d\mu_{\mathbb{B}_y\times\mathbb{B}_y}(\beta)\\ &= |\alpha_y|^{-1}\int f(\nu_y^{-1}\alpha_y^{-1}\beta)\,d\mu_{\mathbb{B}_y\times\mathbb{B}_y}(\beta).\end{aligned}$$

If $\star$ is a singular fibre then apply the formulas in 32 to the function $\beta \mapsto f(\alpha_\star\beta)$ on $\mathbb{B}_\star \times \mathbb{B}_\star$ similarly to the above calculation for a nonsingular fibre. It will contain $|\alpha_\star|^{-1}$ as the factor on the right hand side.

Now for $g(\beta) = f(\alpha\beta)$ and $g(\beta) = |\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta)$ we get

$$\int_{T_0} g + \int_{\partial T_{0,S_o}} g = \int_{T_{0,S_o}} g + \int_{\partial T_{0,S_o}} g = d_{S_o} \int g \, d\mu_{\mathbb{B}_{S_o} \times \mathbb{B}_{S_o}}$$

for every finite S_o which contains S_α. Thus, for every such finite subset S_o of S, we obtain

$$\int_{\partial T_{0,S_o}} \big(|\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta) - f(\alpha\beta)\big) \, d\mu(\beta) = \int_{T_0} \big(f(\alpha\beta) - |\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta)\big) \, d\mu(\beta).$$

□

3.7. Second calculation of the zeta integral

The second calculation of the zeta integral uses the decomposition $\mathfrak{T} = \mathfrak{T}_1 \times \mathfrak{M}$ and the two-dimensional theta formula. The boundary term which we will get at the end of the second calculation has a very different structure in comparison to the structure of the boundary term in dimension one.

45. To slightly simplify the notation it will be convenient to work with the zeta integral

$$\zeta(g,\pi) = \int_{\mathfrak{T}} g\pi,$$

where $g \in Q^0_{\mathbb{A}\times\mathbb{A}}$ and $\pi : \mathfrak{T} \longrightarrow \mathbb{C}^\times$ is a continuous homomorphism such that $\pi(T_0) = 1$ and $\pi \in R^0_{(\mathbb{A}\times\mathbb{A})^\times}$. Denote

$$\hat{\pi} = \pi^{-1} |||\,|^2.$$

The definitions and Remark 41 imply that in the general case of ramified χ the quasi-character $\chi_{\mathfrak{t}}$ does not vanish on T_0, but in the unramified case we get

Lemma *Let g, χ be as in* 39. *Then* $\zeta(g, |\,|_2^s) = \zeta(g, |||\,|^s)$.

Till the end of this section we fix $\pi = |\,|_{2\mathfrak{t}}^s = |||\,|^s$.

Thus, the zeta integral $\zeta(f,\pi)$ exists and absolutely converges for $\Re(s) > 2$, by the preceding Lemma and Theorem 40.

For $m \in M$ denote

$$\zeta_m(f,\pi) := \int_{\mathfrak{T}_1} f(\mathrm{m}\alpha)\pi(\mathrm{m}\alpha) \, d\mu(\alpha).$$

As in the one-dimensional theory, the integral $\zeta_m(f,\pi)$ absolutely converges for any $s = s(\pi)$, since it converges for some s.

The second calculation starts with the following representation of the zeta integral

$$\zeta(f,\pi) = \int_M \zeta_m(f,\pi)\, d\mu_M(m),$$

for $\Re(s(\pi)) > 2$; here we have applied Lemma 43.

Note that the zeta integral was defined as the product of the fibre factors. The second calculation involves integrals over T_1 which are not the product of the fibre factors.

DEFINITION. Introduce

$$M^{\pm} = \{m \in M : \pm(|m|-1) \geq 0\}$$

with the measure induced from M on $M \setminus M \cap T_1$ and half of the measure μ_M on $M \cap T_1$ for each of M^+ and M^-. So the measure space M is the disjoint union of the spaces M^- and M^+ which are mapped to each other by the involution $m \mapsto m^{-1}$.

Similarly define measure spaces $N^{\pm}$, $\mathfrak{M}^{\pm}$. In particular, in characteristic zero N^- is $(0,1]$ with the measure dx/x and in positive characteristic N^- is $\{q^k, k \leq 0\}$, q as in 36, with $\mu_{N^-}(\{q^k\}) = 1$ for $k < 0$ and $\mu_{N^-}(\{1\}) = 1/2$.

Similar to the one-dimensional case, see e.g. [57], we observe that the absolute convergence of $\zeta(f,\pi)$ for $\Re(s(\pi)) > 2$ implies the absolute convergence of the integral $\int_{M^+} \zeta_m(f,\pi)\, d\mu_{M^+}(m)$ in the same area, and therefore this integral and the integral $\int_{M^+} \zeta_m(f,\widehat{\pi})\, d\mu_{M^+}(m)$ absolutely converge for all $s = s(\pi)$.

Define

$$\omega_m(\pi) := \zeta_m(f,\pi) - \zeta_{m^{-1}}(f,\widehat{\pi}).$$

Note that

$$\begin{aligned}\int_{\mathfrak{T}_1} f(\mathrm{m}\alpha)\pi(\mathrm{m}\alpha)\, d\mu(\alpha) &= \int_{\mathfrak{T}_1} f(\mathrm{m}\alpha\nu^{-1})\pi(\mathrm{m}\alpha\nu^{-1})\, d\mu(\alpha)\\ &= \int_{\mathfrak{T}_1} f(\mathrm{m}\alpha^{-1})\pi(\mathrm{m}\alpha^{-1})\, d\mu(\alpha).\end{aligned}$$

In particular, we obtain

$$\begin{aligned}\omega_m(\pi) &= \zeta_m(f,\pi) - \zeta_{m^{-1}}(f,\widehat{\pi})\\ &= \int_{\mathfrak{T}_1} f(\mathrm{m}\alpha)\pi(\mathrm{m}\alpha)\, d\mu(\alpha) - \int_{\mathfrak{T}_1} f(\mathrm{m}^{-1}\alpha^{-1})\widehat{\pi}(\mathrm{m}^{-1}\alpha^{-1})\, d\mu(\alpha)\\ &= \int_{\mathfrak{T}_1} \big(f(\mathrm{m}\alpha) - |m|^{-2} f(\mathrm{m}^{-1}\alpha^{-1})\big)\pi(\mathrm{m}\alpha)\, d\mu(\alpha)\\ &= \int_{\mathfrak{T}_1} \big(|m|^2 f(\mathrm{m}\alpha) - f(\mathrm{m}^{-1}\alpha^{-1})\big)\widehat{\pi}(\mathrm{m}^{-1}\alpha)\, d\mu(\alpha)\end{aligned}$$

We get

$$\int_{M^-} \zeta_m(f,\pi)\, d\mu_{M^-}(m) = \int_{M^-} \big(\zeta_{m^{-1}}(f,\widehat{\pi}) + \omega_m(\pi)\big)\, d\mu_{M^-}(m).$$

Proposition *Let $\pi = |\,|_{2\mathfrak{t}}^s = ||\,||^s$. Then on the right half plane $\Re(s) > 2$*

$$\zeta(f,\pi) = \int_{M^+} \zeta_m(f,\pi)\, d\mu_{M^+}(m) + \int_{M^-} \zeta_m(f,\pi)\, d\mu_{M^-}(m) = \xi(\pi) + \xi(\widehat{\pi}) + \omega(\pi),$$

where

$$\xi(\pi) = \int_{M^+} \zeta_m(f,\pi)\, d\mu_{M^+}(m)$$

absolutely converges for all s and therefore extends to an entire function on the whole complex plane.

The boundary term $\omega(\pi)$ for $\Re(s(\pi)) > 2$ is given by

$$\omega(\pi) = \int_{M^-} \omega_m(\pi)\, d\mu_{M^-}(m),$$

where

$$\begin{aligned}
\omega_m(\pi) &= |m|^s \int_{\mathfrak{T}_1/T_0} \int_{T_0} f(\mathfrak{m}\gamma\beta)\, d\mu(\beta)\, d\mu(\gamma) \\
&\qquad - |m|^{s-2} \int_{\mathfrak{T}_1/T_0} \int_{T_0} f(\mathfrak{m}^{-1}\nu^{-1}\gamma^{-1}\beta)\, d\mu(\beta)\, d\mu(\gamma) \\
&\qquad = |m|^s \int_{\mathfrak{T}_1/T_0} \int_{T_0} \big(f(\mathfrak{m}\gamma\beta) - |m|^{-2} f(\mathfrak{m}^{-1}\nu^{-1}\gamma^{-1}\beta)\big)\, d\mu(\beta)\, d\mu(\gamma),
\end{aligned}$$

which due to the two-dimensional theta formula is equal to

$$\begin{aligned}
\omega_m(\pi) &= |m|^{s-2} \int_{\mathfrak{T}_1} \big(|\alpha|^{-1} - 1\big) f(\mathfrak{m}^{-1}\alpha^{-1})\, d\mu(\alpha) \\
&\qquad + |m|^s \int_{\mathfrak{T}_1/T_0} \int_{\partial T_0} \big(|\mathfrak{m}\gamma|^{-1} f(\mathfrak{m}^{-1}\nu^{-1}\gamma^{-1}\beta) - f(\mathfrak{m}\gamma\beta)\big)\, d\mu(\beta)\, d\mu(\gamma),
\end{aligned}$$

and, alternatively,

$$\begin{aligned}
\omega_m(\pi) &= |m|^s \int_{\mathfrak{T}_1} \big(1 - |\alpha|\big) f(\mathfrak{m}\alpha)\, d\mu(\alpha) \\
&\qquad + |m|^s \int_{\mathfrak{T}_1/T_0} \int_{\partial T_0} \big(|m|^{-2} f(\mathfrak{m}^{-1}\nu^{-1}\gamma^{-1}\beta) - |\gamma| f(\mathfrak{m}\gamma\beta)\big)\, d\mu(\beta)\, d\mu(\gamma).
\end{aligned}$$

Proof: Using Lemma 43 and making change of variables $\alpha \mapsto \alpha\nu$ we get the first portion of equalities for $\omega_m(\pi)$.

Rearranging terms, we obtain

$$\begin{aligned}\omega_m(\pi) &= |m|^{s-2}\int_{\mathfrak{T}_1/T_0}\big(|\gamma|^{-1}-1\big)\int_{T_0} f(\mathfrak{m}^{-1}\nu^{-1}\gamma^{-1}\beta)\,d\mu(\beta)\,d\mu(\gamma)\\ &\quad+|m|^s\int_{\mathfrak{T}_1/T_0}\int_{T_0}\big(f(\mathfrak{m}\gamma\beta)-|\mathfrak{m}\gamma|^{-1}f(\mathfrak{m}^{-1}\nu^{-1}\gamma^{-1}\beta)\big)\,d\mu(\beta)\,d\mu(\gamma)\\ &=|m|^s\int_{\mathfrak{T}_1/T_0}\big(1-|\gamma|\big)\int_{T_0} f(\mathfrak{m}\gamma\beta)\,d\mu(\beta)\,d\mu(\gamma)\\ &\quad+|m|^s\int_{\mathfrak{T}_1/T_0}\int_{T_0}\big(|\gamma|f(\mathfrak{m}\gamma\beta)-|m|^{-2}f(\mathfrak{m}^{-1}\nu^{-1}\gamma^{-1}\beta)\big)\,d\mu(\beta)\,d\mu(\gamma).\end{aligned}$$

By Theorem 44 we get

$$\begin{aligned}&\int_{T_0} f(\mathfrak{m}\gamma\beta)\,d\mu(\beta)-|m|^{-2}|\gamma|^{-1}\int_{T_0} f(\mathfrak{m}^{-1}\gamma^{-1}\nu^{-1}\beta)\,d\mu(\beta)\\ &\quad=\int_{\partial T_0}\big(|m|^{-2}|\gamma|^{-1}f(\mathfrak{m}^{-1}\gamma^{-1}\nu^{-1}\beta)-f(\mathfrak{m}\gamma\beta)\big)\,d\mu(\beta).\end{aligned}$$

Now we deduce the formulas of the second set, and similarly of the third set. □

Remark The advantage of the second set of formulas for ω_m is that the integral $\int_{M^-}|m|^{s-2}\int_{\mathfrak{T}_1} f(\mathfrak{m}^{-1}\alpha^{-1})\big(|\alpha|^{-1}-1\big)\,d\mu(\alpha)\,d\mu_{M^-}(m)$ extends to an entire function on the complex plane. Indeed, this integral equals

$$-\int_{M^+}|m|^{2-s}\int_{\mathfrak{T}_1} f(\mathfrak{m}\alpha)\,d\mu(\alpha)\,d\mu_{M^+}(m)+\int_{M^+}|m|^{2-s}\int_{\mathfrak{T}_1} f(\mathfrak{m}\alpha)|\alpha|\,d\mu(\alpha)\,d\mu_{M^+}(m),$$

the first integral absolutely converges as noted above, and the second integral absolutely converges since $|\alpha| = |\alpha|_-^{1/2}$ for $\alpha \in \mathfrak{T}_1$ and there is $c > 0$ such that $f(m\alpha)|\alpha| \le cf(m\alpha)$ uniformly for all $m \in M^+$, $\alpha \in \mathfrak{T}_1$. Indeed, $e^{-|\alpha|_-^2}|\alpha|_-^{1/2}$ goes to zero very fast when $|\alpha|_-^{1/2} \to \infty$ in characteristic zero and $f(\alpha) = 0$ for all sufficiently large $|\alpha|_-$ in positive characteristic.

Thus, the analytic properties of the zeta integral, including the meromorphic continuation and functional equation, are reduced via the second calculation to the analogous properties of the boundary term $\omega(\pi)$. The structure of the boundary term is more complicated and richer than that in dimension one, see the next part.

From now on we assume that χ is trivial on J^1, so $\chi = |\,|_2^s$. Then

$$\zeta(f,\chi) = \zeta(f,|\,|_{2\mathfrak{t}}^s) = \zeta(f,||\,||^s).$$

Denote

$$\xi(|\,|_2^s) := \xi(|\,|_{2\mathfrak{t}}^s), \quad \omega(|\,|_2^s) := \omega(|\,|_{2\mathfrak{t}}^s), \quad \zeta_m(f,|\,|_2^s) := \zeta_m(f,|\,|_{2\mathfrak{t}}^s), \quad \omega_m(|\,|_2^s) = \omega_m(|\,|_{2\mathfrak{t}}^s).$$

Using the Proposition we obtain

Theorem *Let $\mathcal{S} = \mathcal{E}$ and f be as in* 40. *On the half plane $\Re(s) > 2$*

$$\zeta(f,|\,|_2^s) = \xi(|\,|_2^s) + \xi(|\,|_2^{2-s}) + \omega(|\,|_2^s),$$

where

$$\xi(|\,|_2^s) = \int_{M^+} \zeta_m(f,|\,|_2^s)\, d\mu_{M^+}(m).$$

The integral $\xi(|\,|_2^s)$ absolutely converges for all s and is an entire function on the complex plane. The boundary term $\omega(|\,|_2^s)$ for $\Re(s) > 2$ is given by

$$\omega(|\,|_2^s) = \int_{M^-} \omega_m(|\,|_2^s)\, d\mu_{M^-}(m),$$
$$\omega_m(|\,|_2^s) = |m|^s \int_{\mathfrak{T}_1} \big(f(\mathrm{m}\alpha) - |m|^{-2} f(\mathrm{m}^{-1}\alpha^{-1})\big)\, d\mu(\alpha),$$

with the further formulas for $\omega_m(|\,|_2^s)$ as in the previous Proposition.

These two calculations of the zeta integral form a two-dimensional analogue of the (unramified part of the) theory of Tate and Iwasawa for $\mathcal{E}$. For the case of fibered regular models of hyperbolic curves over global fields see 57. Due to the connections between the zeta integral and zeta function $\zeta_{\mathcal{E}}(s)$ (see 40) and the L-function of E (see the introduction), the study of the appropriate analytic properties of the zeta function and L-function is reduced to their study for the boundary term. In particular, all the information about the meromorphic continuation, functional equation and poles of the zeta integral is contained in the boundary term $\omega(|\,|_2^s)$ whose study is a new aspect of the two-dimensional theory in comparison to the classical theory.

4. The boundary term and first applications

Except 57 we will assume that $\mathcal{S} = \mathcal{E}$ as in 40.

From now on *let $\chi = |\,|_2^s$ be the unramified quasi-character*.

In this part we try to build bridges from the study of the zeta integral and its calculation in part 3 to several important directions of the arithmetic of elliptic curves: the meromorphic continuation and functional equation of the zeta function $\zeta_{\mathcal{E}}(s)$, the location of its poles, and its behaviour at the central point.

Almost all the material of part 4 can be read more or less independently of the previous parts, just using the description of the boundary term in 46. There are already several papers which further develop the material of this part, see [54]–[56].

In 46 we describe an integral representation of the boundary term as $\int_{N^-} h(n) n^{s-2} d\mu_{N^-}(n)$ where $h(n)$ is given by an adelic integral. We easily get a functional equation for $h(n)$. A more explicit description of $h(n)$ will be obtained in Proposition 49, Proposition 50 and their proofs, in section 51 (the fourth derivative of $h(e^{-t})$) and sections 52–54.

In 47–48 we discuss the analytic shape of the function h. It is expected that a function H which is obtained from h via the exponential change of variable is a mean-periodic function in an appropriate functional space. The mean-periodicity would imply that the zeta integral, after change of variable $s \to s+1$, is the sum of a symmetric entire function and the Laplace–Carleman transform of an odd mean-periodic function. The latter transform is a symmetric meromorphic function. All this would imply the meromorphic continuation and functional equation of the zeta integral and hence of the square of the zeta function of E, see 48. The recent work [56] shows that, in turn, the functional equation of the zeta function and the analytic shape of its denominator (i.e. the L-function of E) imply the mean-periodicity of the associated function H. Thus, in the first approximation, the meromorphic continuation and functional equation of the zeta integral is equivalent to the mean-periodicity of H in the space of smooth functions of exponential growth on the real line. The recent work [55], assuming the cuspidality property of the L-function, derives a homogeneous convolution equation for a function related to h and a function originating naturally in the theory of Connes and Soulé in their spectral interpretation of zeros of L-functions, thus exhibiting some duality between the two theories.

In 50–55 we assume without loss of generality that the set $S_\prime$, fixed in 36 and 40, contains only one horizontal curve: the image of the zero section. We describe the behaviour of $h(n)$ and its derivatives near zero. In 51 we introduce hypothesis $(*)$ which says that the fourth derivative of $H(t)$ keeps its sign for all sufficiently large t. We discuss various analytic and computational aspects of this hypothesis in 52–53, relating it to the behaviour of associated Bessel series. In 54 we prove that hypothesis $(*)$ and the real part of the Riemann hypothesis imply the full Riemann hypothesis for the zeta integral. Results in the opposite direction are established in [54]. In 55, interpreting the boundary term as the Laplace–Carleman transform, we derive some implications for the analytic continuation of the zeta integral, zeta function and L-function and their functional equations.

We briefly indicate in 57 how to treat the case of curves of higher genus.

Section 58 contains a sketch of a new method aiming to relate the analytic and arithmetic ranks of elliptic curve over a global field via the boundary term and the previous theory.

4.1. The boundary function h

46. DEFINITION. For every $n \in N$ we use the definition of $\mathfrak{m}_n$ in 42, make in appropriate places change of variable $\gamma \to \gamma^{-1}$ and $\gamma \to \nu\gamma$, and apply Lemma in 43 to define and deduce the following equalities

$$
\begin{aligned}
h(n) &:= \int_{\mathfrak{T}_1} \big(n^2 f(\mathfrak{m}_n\gamma) - f(\mathfrak{m}_n^{-1}\gamma)\big)\, d\mu(\gamma) = \int_{\mathfrak{T}_1} \big(n^2 f(\mathfrak{m}_n\gamma) - f(\mathfrak{m}_n^{-1}\gamma^{-1})\big)\, d\mu(\gamma)\\
&= \int_{\mathfrak{T}_1} \big(n^2 f(\mathfrak{m}_n\gamma) - f(\mathfrak{m}_n^{-1}\nu^{-1}\gamma^{-1})\big)\, d\mu(\gamma)\\
&= \int_{\mathfrak{T}_1/T_0} \left(\int_{T_0} \big(n^2 f(\mathfrak{m}_n\gamma\beta) - f(\mathfrak{m}_n^{-1}\nu^{-1}\gamma^{-1}\beta)\big)\, d\mu(\beta)\right) d\mu(\gamma).
\end{aligned}
$$

The function $h(n)$ does not depend on the choice of m_n corresponding to n.

Using the two-dimensional theta formula of 44, as in 45 we get

$$
\begin{aligned}
h(n) &= h_1(n) + h_2(n),\\
h_1(n) &= \int_{\mathfrak{T}_1} \big(|\alpha|^{-1} - 1\big) f(\mathfrak{m}_n^{-1}\alpha^{-1})\, d\mu(\alpha)\\
h_2(n) &= n^2 \int_{\mathfrak{T}_1/T_0} \int_{\partial T_0} \big(|\mathfrak{m}_n\gamma|^{-1} f(\mathfrak{m}_n^{-1}\nu^{-1}\gamma^{-1}\beta) - f(\mathfrak{m}_n\gamma\beta)\big)\, d\mu(\beta)\, d\mu(\gamma).
\end{aligned}
$$

The integral $\int_{N^-} h_1(n) n^{s-2}\, d\mu_{N^-}(n)$ extends to an entire function on the complex plane, see Remark 45.

Remark The integral over the boundary ∂T_0 of T_0 of the function $|\mathfrak{m}_n\gamma|^{-1} f(\mathfrak{m}_n^{-1}\nu^{-1}\gamma^{-1}\beta) - f(\mathfrak{m}_n\gamma\beta)$ and then the subsequent external integral $\int_{N^-} n^s \int_{\mathfrak{T}_1/T_0}$ contains the full information on the meromorphic continuation, functional equation and location of poles of the zeta integral.

Proposition *The boundary term* $\omega(|\,|_2^s)$ *of Theorem* 45 *has the following integral presentation for* $\Re(s) > 2$

$$
\omega(|\,|_2^s) = \int_{N^-} h(n) n^{s-2}\, d\mu_{N^-}(n).
$$

Hence $\omega(|\,|_2^s)$ *is equal to* (*Laplace or Laplace–Stieltjes transform in variable* t) *for*

$\Re(s) > 2$

$$\int_{N^-} h(n) n^{s-2}\, d\mu_{N^-}(n) = \int_0^\infty e^{-st}\, dj(t)$$
$$= \begin{cases} \displaystyle\int_0^1 n^{-2} h(n)\cdot n^s \frac{dn}{n} = \int_0^\infty e^{2t} h(e^{-t})\cdot e^{-st}\, dt & \textit{in characteristic } 0, \\ \displaystyle h(1)/2 + \sum_{k\geq 1} h(q^{-k}) q^{-k(s-2)} & \textit{in characteristic } p, \end{cases}$$

where q is as in 36, $j(t)$ which depends on h is defined as

$$j(t) = \begin{cases} \displaystyle\int_0^t e^{2u} h(e^{-u})\, du & \textit{in characteristic zero,} \\ \displaystyle\frac{h(1)}{2}\lambda(t) + \sum_{k\geq 1} h(q^{-k}) q^{2k} \lambda(t - k\log q) & \textit{in positive characteristic,} \end{cases}$$

where the classical

$$\lambda(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0. \end{cases}$$

The function $h(n)$ is equal to $l(n)n^2 - l(n^{-1})$ where the function $l(n)$ is the inverse transform of $\zeta(f, |\,|_2^s)$:

$$\int_N l(n) n^s\, d\mu_N(n) = \zeta(f, |\,|_2^s).$$

Proof: Use the formula for $\omega_m(\pi)$ in 45 and the definition of μ_M in 43. The representation for $h(n)$ as the inverse Mellin transform of the zeta integral follows from 45. □

For several more explicit formulas for $h(n)$ see 51–53.

Lemma *We have*

$$h(n^{-1}) = -n^{-2} h(n), \quad n \in N.$$

Proof: Use the definition (the first line) of h. □

Remark Singularities of the zeta integral correspond to singularities of the boundary term and the boundary term is given in its integral representation of adelic type. Notice the very different nature of this description of poles of the zeta function of the two-dimensional $\mathcal{E}$ from the known integral representation formulas in dimension one for poles of the Dedekind zeta function: the classical formula

$$-\frac{\zeta'(s)}{s\zeta(s)} = \int_0^\infty \psi(e^t) e^{-st}\, dt,$$

where $\zeta(s)$ is the Riemann zeta function and $\psi(x) = \sum_{n\leq x} \Lambda(n)$, $\Lambda(p^k) = \log p$ if $k > 0$, p is prime, and equals 0 otherwise, is far away to be of adelic type.

4.2. *Mean-periodicity and meromorphic continuation of the zeta integral*

47. Lemma 46 shows that functions $e^t h(e^{-t})$ in characteristic zero and $q^t h(q^{-t})$ in positive characteristic are odd functions of t. So $\omega(|\,|_2^{1+s})$ is the Laplace–Stieltjes transform of the odd function. It is therefore natural to ask the following question: *for which subclass of the class of odd infinitely differentiable functions* $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ *with finite exponential growth bound* $\inf\{c \in \mathbb{R} : \sup_{t\geq 0}|e^{-ct}g(t)| < \infty\}$ *their Laplace–Stieltjes transform* $G(s)$ *extends to a meromorphic function on the plane satisfying* $G(s) = G(-s)$? *Mean-periodic functions in appropriate spaces* X *of functions on* $\mathbb{R}$ satisfy this property. Note that in general the mean-periodicity is only a sufficient but not a necessary condition for a function to have its Laplace–Stieltjes transform extending to a meromorphic function on the plane. However, assuming the standard hypothetical behaviour of the L-function of E one can show that the functions above are indeed mean-periodic in those spaces, as was shown by Suzuki and Ricotta in [56]. In this section we discuss the concept of mean-periodicity, which is supposed to play a much more prominent role in number theory, approaching the significance of the role played by automorphicity.

Let X be a space of complex valued functions on $\mathbb{R}$ or on $\mathbb{Z}$ endowed with its appropriate topology. Below X will be the inductive or projective limit of Fréchet spaces. Denote by X^* the dual space of X. Classical examples of functional spaces relevant for the zeta functions are:

– the space $\mathcal{C}(\mathbb{R})$ of continuous functions on the real line endowed with the topology of uniform convergence on compact subsets and the corresponding set of seminorms, then X^* is the space of Radon measures with compact support,

– the space $\mathcal{E}(\mathbb{R}) = \mathcal{C}^\infty(\mathbb{R})$ of infinitely differentiable functions, then X^* is the space of distributions with compact support,

– the space $\mathcal{C}_{\exp}(\mathbb{R})$ of continuous functions of (not more than) exponential growth at $\pm\infty$, then X^* is the space of measures with more than exponential decay at $\pm\infty$,

– the space $\mathcal{C}^\infty_{\exp}(\mathbb{R})$ of infinitely differentiable functions of (not more than) exponential growth at $\pm\infty$, then X^* is the space of distributions with more than exponential decay;

– the space $\mathfrak{F}(\mathbb{Z})$ of functions on $\mathbb{Z}$ and the space $\mathfrak{F}_{\exp}(\mathbb{Z})$ of functions of exponential growth.

DEFINITION OF A MEAN-PERIODIC FUNCTION IN X. A complex valued function $g \in X$ is called mean-periodic in X if one of the following two equivalent conditions (in the presence of the Hahn–Banach theorem) is satisfied:

(a) there exists a closed proper translation invariant linear subspace of X which contains g;

(b) g is a solution of a homogeneous convolution equation $g * \tau = 0$ where τ is a nonzero element in the dual space X^*.

Homogeneous convolution equations can be viewed as a natural extension of homogeneous differential equations with constant coefficients; already Euler used exponential polynomials to find solutions of differential equations with constant coefficients.

Harmonic analysis studies analogues of Fourier series for mean-periodic functions. Spectral analysis is said to hold in X if every translation invariant subspace of X contains a finite dimensional translation invariant subspace. Spectral synthesis is said to hold in X if every translation invariant subspace of X is generated by its finite dimensional translation invariant subspaces.

The theory of mean-periodic functions initiated by Delsarte was developed by Schwartz, Kahane and many other mathematicians. For the theory of mean-periodic functions in $\mathcal{C}(\mathbb{R})$ see [26] and a short review in [38], p.169–181; for mean-periodic functions in $\mathcal{E}(\mathbb{R})$ see e.g. Ch. 6 of [2], [3]; for the general case of functional spaces see [42], [43]. It is known that spectral synthesis and spectral analysis hold in all of the listed above spaces X; in particular for $X = \mathcal{C}^\infty_{\exp}(\mathbb{R})$ it follows from results of [17], see also p.202 of [42].

In all these functional spaces on $\mathbb{R}$ finite dimensional translation invariant subspaces are generated by exponential monomials $x^j e^{z_i x}$, $0 \le j \le m_i$, $z_i \in \mathbb{C}$. Spectral synthesis means that every mean-periodic function g can be approximated by an appropriately grouped series of exponential polynomials each of which belongs to the closure of the space generated by translations of g. Such series of exponential polynomials generalize Fourier series. The class of mean-periodic functions in $\mathcal{C}(\mathbb{R})$ is an extension of the class of periodic functions; it is related to but does not contain the class of almost periodic functions.

Define the so called causal function g^+ associated to a function g on the real line:

$$g^+(t) = g(t) \text{ for } t > 0, \quad g^+(0) = g(0)/2, \quad g^+(t) = 0 \text{for } t < 0.$$

If g is mean-periodic in X and if g is of finite exponential growth then $g^+ \circ \tau \in X^*$ and for sufficiently large $\Re(s)$ its Laplace–Stieltjes transform $\int_0^\infty g(t)e^{-st}\,dt$ equals

$$G(s) = \frac{\displaystyle\int_{-\infty}^{\infty} g^+ \circ \tau(t) e^{-st}\,dt}{\displaystyle\int_{-\infty}^{\infty} \tau(t) e^{-st}\,dt}.$$

This does not depend on the choice of $\tau \neq 0$. Both the numerator and denominator extend to entire functions on the plane, and hence $G(s)$ is a meromorphic function on the complex plane. It is called the *Laplace–Stieltjes–Carleman transform* of g. It serves as the meromorphic continuation of the Laplace–Stieltjes transform of the mean-periodic function g.

In the case of an odd mean-periodic function g the function $G(s)$ is a symmetric: $G(-s) = G(s)$, and $G(s)$ coincides with the Laplace–Carleman transform of g, the latter is defined in 55.

The set of poles of $G(s)$ is a subset of the set of zeros of the denominator. The complex numbers z_i above are some of the poles of $G(s)$ and its principal part at z_i is a polynomial of degree m_i.

It is known that $G(s)$ is entire if and only if the function g is zero.

Now we state the following 6 years old

Hypothesis *The zeta integral $\zeta(f, |\,|_2^{1+s})$ is the sum of an entire symmetric function and a symmetric meromorphic function which is the Laplace–Stieltjes–Carleman transform of a mean-periodic function of an appropriate functional space.*

If such a functional space is fixed, then these mean-periodic function and entire function are then uniquely determined.

More precisely, the spaces $X = \mathcal{C}_{\exp}(\mathbb{R})$ and $X = \mathcal{C}^{\infty}_{\exp}(\mathbb{R})$ as well as some other spaces (but not $\mathcal{C}(\mathbb{R})$, $\mathcal{C}^{\infty}(\mathbb{R})$, see Remark 5.12 of [56]) should do the job in characteristic zero and the space $X = \mathfrak{F}_{\exp}(\mathbb{Z})$ does the job in positive characteristic. See also Remarks 2 and 4 in 48.

If the hypothesis holds true, then by the above discussion the zeta integral (and hence the square of the zeta function of $\mathcal{E}$, see 48) extend meromorphically to the complex plane and satisfy the appropriate functional equations.

Keeping in mind Theorem 45, to verify the hypothesis one only needs to show that the boundary term $\omega(|\,|_2^{1+s})$ is the sum of a symmetric entire function and the Laplace–Stieltjes–Carleman transform of an odd mean-periodic function in the appropriate functional space; see also 55.

48. Define $\zeta_E(s) = \prod_{b \in B_0} \zeta_{E_b}(s)$ where $\zeta_{E_b}(s)$ is the zeta function of the minimal Weierstrass model of E at b; if the class number of k is 1 then $\zeta_E(s)$ is the zeta function of the minimal Weierstrass model of E. The zeta function of E determines the L-function via

$$\zeta_E(s) = \frac{\zeta_{\mathbb{P}^1(B)}(s)}{L_E(s)} = \frac{\zeta_B(s)\zeta_B(s-1)}{L_E(s)}.$$

This historically first approach to the zeta and L-function of E does not require to specify explicitly the (mysteriously looking in its traditional definition) bad factors at all.

Using Thms 3.7, 4.35 in Ch. 9 and section 10.2.1 in Ch. 10 of [37], one can easily deduce that the quotient $\zeta_{\mathcal{E}}(s)/\zeta_E(s)$ is the product of finitely many zeta functions of affine lines over finite extensions of the residue field of b, where b runs through those closed points of the base for which the special fibre $\mathcal{E}_b$ is singular. More precisely, denote by m_b the number of components in the reduced part of the geometric fibre $\mathcal{E} \times_B k(b)^{\mathrm{sep}}$ of $\mathcal{E}$ over a closed point b of B; so $m_b = 1$ for almost all b. Then

$$\zeta_{\mathcal{E}}(s) = n_{\mathcal{E}}(s)\zeta_E(s), \quad n_{\mathcal{E}}(s) = \prod_{b \in B_0, 1 \le i \le n_b} (1 - |k(b)|^{n_{i,b}(1-s)})^{-1}.$$

Here if $m_b \neq 1$, i.e. the special fibre $\mathcal{E}_b$ is singular, then $n_{i,b}$ are certain positive integers, $1 \le i \le n_b$, such that $\sum_{1 \le i \le n_b} n_{i,b} = m_b - 1$, the number n_b is the number of components of the special fibre $\mathcal{E}_b$ with the component intersecting the zero section excluded. For other related points of view see e.g. [50], [51], [58], and for a cohomological interpretation see e.g. [4], p.300. In particular, the functions $n_{\mathcal{E}}(s)$ and $n_{\mathcal{E}}(s)^{-1}$ are holomorphic for $\Re(s) > 1$.

Now we obtain

Theorem *Let $\mathcal{S} = \mathcal{E}$ be as in* 40. *Suppose that the function*

$$H(t) = \begin{cases} h(e^{-t}), t \in \mathbb{R}, & \text{in characteristic zero,} \\ h(q^{-t}), t \in \mathbb{Z}, & \text{in positive characteristic,} \end{cases}$$

is a mean-periodic function in an appropriate functional space (e.g. $\mathcal{C}^{\infty}_{\exp}(\mathbb{R})$ and $\mathfrak{F}_{\exp}(\mathbb{Z})$).

Then the zeta integral $\zeta(f, |\,|_2^s)$ of $\mathcal{E}$, the zeta function ζ_E and the L-function of E meromorphically extend to the plane and satisfy the functional equations:

$$\begin{aligned} \zeta_{\mathcal{E}}(f, |\,|_2^s) &= \zeta_{\mathcal{E}}(f, |\,|_2^{2-s}), \\ c_{\mathcal{E}}(s)\zeta_{\mathcal{E}}(s)^2 &= c_{\mathcal{E}}(2-s)\zeta_{\mathcal{E}}(2-s)^2, \\ m_E(s)\zeta_E(s)^2 &= \zeta_E(2-s)^2, \end{aligned}$$

where $c_{\mathcal{E}}(s) = c_{\mathcal{E},S_{\text{I}}}(|\,|_2^s)$ is the vertical part of $c_{\mathcal{E},S}(|\,|_2^s)$ defined in Theorem 40,

$$m_E(s) = \frac{c_{\mathcal{E}}(s)}{c_{\mathcal{E}}(2-s)} \frac{n_{\mathcal{E}}(s)^2}{n_{\mathcal{E}}(2-s)^2} = \mathfrak{c}_E^{2-2s},$$

where $\mathfrak{c}_E$ is the norm of the conductor of E.

Remark 1 The conjectural constants in the conjectural functional equations of $\zeta_E(s)$ and $\zeta_{\mathcal{E}}(s)$ do not depend on the archimedean data and do not involve Γ-functions; the same is true for the zeta functions of abelian varieties over global fields. In characteristic zero the completed L-function $\Lambda_E(s)$ is the product of $L_E(s)$ and a certain factor $\Gamma_E(s)$, see e.g. [51]. However, the ratio $\Gamma_E(s)/\Gamma_{\mathbb{P}^1(B)}(s)$, where $\Gamma_{\mathbb{P}^1(B)}(s) = \Gamma_k(s)\Gamma_k(s-1)$, is a simple rational function; for example equal to $(s-1)/(4\pi)$ if $\mathrm{k} = \mathbb{Q}$. Its square is invariant with respect to $s \to 2-s$. Thus, the factor $\Gamma_E(s)$ in the functional equation of the denominator $L_E(s)$ of the zeta function is essentially due to the factor $\Gamma_{\mathbb{P}^1(B)}(s)$ in the functional equation of the numerator of the zeta function. See also Remark 57 about Γ-factors for zeta functions of arithmetic surfaces. Using the formula for $c_{\mathcal{E}}(s)$ in Theorem 40, we see that the mean-periodicity of $H(t)$ implies the functional equation of ζ_E^2 with its conjectured exponential factor exactly as described in [51], for more details see 5.4 of [56].

Remark 2 In characteristic zero the material of [56] shows that if the L-function of E extends to a holomorphic function of order one and satisfies the functional equation then the function H is mean-periodic in $\mathcal{C}_{\exp}(\mathbb{R})$, $\mathcal{C}^\infty_{\exp}(\mathbb{R})$ and other spaces. The proof uses convexity bounds for Dedekind zeta functions and L-functions of elliptic curves. For modular curves the convolutor for H can be obtained using the spectral interpretation of zeros of $GL(2)$ cuspidal automorphic representations, see [55].

More generally, [56] demonstrates new links between zeta functions which extend meromorphically and satisfy the functional equation and mean-periodic functions. In particular, a rescaled completed zeta function of an arithmetic scheme extends to a meromorphic function of order 1 of the expected analytic shape with the functional equation with respect to $s \to 1-s$ and the sign ϵ if and only if the function $f(e^{-t}) - \epsilon e^t f(e^t)$ is a mean-periodic function in $\mathcal{C}^\infty_{\exp}(\mathbb{R})$, where $f(x)$ is the inverse Mellin transform of the product of an appropriate sufficiently large positive power of the completed Riemann zeta function and the completed and rescaled zeta function.

Of course, when complex multiplication is available and in the case of elliptic curves over $\mathbb{Q}$ the well known theorems imply the holomorphic continuation and the functional equation of L_E and hence they imply the mean-periodicity of H in $\mathcal{C}^\infty_{\exp}(\mathbb{R})$.

Remark 3 If $\mathfrak{f}$ is a modular function of weight two, then the function $\mathfrak{g}(z) = \mathfrak{f}(iz)$ satisfies the two functional equations: $\mathfrak{g}(z^{-1}) = -z^2\mathfrak{g}(z)$ and $\mathfrak{g}(z+i) = \mathfrak{g}(z)$. The analogue of the first equation for the function h is $h(n^{-1}) = -n^{-2}h(n)$ in Lemma 46. The analogue of the second equation is the mean-periodicity of the function $H(t)$. It is expected that to prove the mean-periodicity of $H(t)$ and get the meromorphic continuation and functional equation of the zeta function of $\mathcal{E}$ is easier than to get the full automorphic properties of its factor $L_E(s)$.

The study of relations between mean-periodic and automorphic properties can bring many fruits. One of the main open problems is how far one can go from the mean-periodicity of H to the automorphic properties of L_E.

Remark 4 Let K be of positive characteristic. Using the rationality (as functions of q^{-s}) and the functional equations for the zeta function $\zeta_{\mathcal{E}}(s)$ and zeta functions of global fields of positive characteristic it is straightforward to deduce that the function $H(t)$ in mean-periodic in the spaces $\mathfrak{F}(\mathbb{Z})$ and $\mathfrak{F}_{\exp}(\mathbb{Z})$.

Namely, using the above facts and Theorem 40 we can write the zeta integral as $\mathcal{L}(u)/\mathcal{L}(v)$ where $\mathcal{L}$ is the Laplace–Stielties transform as in 46, and u, v are complex valued functions on $\{q^{-k} : k \in \mathbb{Z}\}$ with finite support and such that $u(n) = n^{-2}u(n^{-1})$, $v(n) = n^{-2}v(n^{-1})$. Then the function H satisfies the homogeneous convolution equation $H * w = 0$ with $w(n) = v(n^{-1})$.

Remark 5 For another approach towards the analytic continuation and functional equation of the boundary term see 55.

4.3. Monotone behaviour of log derivatives of the boundary function and poles of the zeta integral

Choose the set $S_{\prime}$ such that it contains exactly one horizontal curve y_0, the image of the zero section. Then we get the decomposition $\mathfrak{T} = \mathfrak{T}_{y_0} \times \mathfrak{T}_{S_{\mathrm{I}}}$. The zeta integral in this case lifts the square of the zeta function $\zeta_{\mathcal{E}}(s)$ to a much better from analytic point of view object $\sum_{\mathfrak{n}\in\mathfrak{c}_{\mathcal{E}}\mathbb{N}} c(\mathfrak{n}^2) A_{\mathfrak{n}^2}(x)$ in characteristic zero, which we now describe. Recall that $\zeta_{\mathcal{E}}(s)^2 \mathfrak{c}_{\mathcal{E}}^{1-s}$ is the vertical part of the zeta integral by Theorem 40, $\mathfrak{c}_{\mathcal{E}}$ is defined in 40. Write

$$\zeta_{\mathcal{E}}(s)^2 \mathfrak{c}_{\mathcal{E}}^{1-s} = \sum_{\mathfrak{n}\in\mathfrak{c}_{\mathcal{E}}\mathbb{N}} c(\mathfrak{n}^2)\mathfrak{n}^{-s}$$

as a generalized Dirichlet series with coefficients $c(\mathfrak{n}^2)$. Then the zeta integral

$$\zeta_{\mathcal{E},S_{\prime}}(f,|\,|_2^s) = \mathfrak{e} \sum_{\mathfrak{n}\in\mathfrak{c}_{\mathcal{E}}\mathbb{N}} c(\mathfrak{n}^2) \int_0^\infty Y_{\mathfrak{n}^2}(n) n^s \frac{dn}{n}, \text{ where } Y_{\mathfrak{n}}(n) = \int_0^\infty y_{a,\mathfrak{n}a^{-1}}(n) \frac{da}{a},$$

where $\mathfrak{e}$ is the square of the normalized classical measure of the idele class group of k,

$$y_{a,b}(n) = (\Theta(n^2a^2) - 1)(\Theta(n^2b^2) - 1)$$

and Θ is the theta-function of k, see 51. Compare with the classical case $\mathrm{k} = \mathbb{Q}$, where one similarly passes from the zeta function $\sum_{\mathfrak{n}\in\mathbb{N}} \mathfrak{n}^{-s}$ to the integral $\int_0^\infty n^s(\theta(n^2) - 1)\frac{dn}{n}$.

Similarly we deduce

$$
\begin{aligned}
h(n) &= -\mathfrak{e} \sum_{\mathfrak{n} \in \mathfrak{c}_{\mathcal{E}} \mathbb{N}} c(\mathfrak{n}^2) V_{\mathfrak{n}^2}(n), \\
V_{\mathfrak{n}}(n) &= \int_0^\infty w_{a,\mathfrak{n}a^{-1}}(n) \frac{da}{a}, \\
w_{a,b}(n) &= (\Theta(n^{-2}a^2)-1)(\Theta(n^{-2}b^2)-1) - n^2(\Theta(n^2a^2)-1)(\Theta(n^2b^2)-1),
\end{aligned}
$$

see 51.

We look at the behaviour of the first two logarithmic derivatives of h in 49 and 50 and check that it is monotone near 0. In 51 we state hypothesis $(*)$ that the third logarithmic derivative of h is monotone near $0+$. This hypothesis is partially motivated by the fact that the boundary ∂T_0 in dimension two is very large and integrals over it are expected to result in 'nicely behaved' functions, unlike the dimension one case where the boundary $\partial \mathrm{k}^\times$ is so small. In characteristic zero we include more formulas for the fourth logarithmic derivative of h and its approximation in 52. We discuss hypothesis $(*)$ in the case of characteristic zero in 52, and also include there a reference to some computational data, and we discuss the hypothesis in the case case of positive characteristic in 53.

In 54 we show that hypothesis $(*)$ implies the following: if the zeta integral extends meromorphically to the half-plane $\Re(s) > 1$ and if it does not have real poles in the open interval $(1,2)$ then the zeta integral does not have complex poles in the strip $\Re(s) \in (1,2)$. The real part of the generalized Riemann hypothesis (i.e. no real poles in $(1,2)$) is easy to check computationally for a modular elliptic curve, so in this case $(*)$ implies the GRH for the zeta integral. Conversely, [54] shows that the generalized Riemann hypothesis and some technical condition imply hypothesis $(*)$, see Remark 2 in 54. In 55 we show that if $(*)$ holds and the zeta integral extends meromorphically without new real poles to the half-plane $\Re(s) > 1$ and the Carleman spectrum of the Laplace–Carleman transform of a function p related to h is not dense on the real line, then the zeta integral extends meromorphically to the complex plane and satisfies the functional equation.

49. In this section we derive some first information about a function $w_\gamma(n)$. We will use this for the study of the function h which can be written as an integral whose integrand involves the function $w_\gamma(n)$, see the last part of 51.

Recall that in 42 we denote by $y_0 \in S_\prime$ the image of the zero section, and we choose representatives m_n of $n \in N$ in T_{y_0}.

Proposition *For $\gamma \in T$ put*

$$u_\gamma(n) = \int_{T_{0,y_0}} f(\mathfrak{m}_n^{-1}\gamma\beta)\,d\mu(\beta),$$
$$v_\gamma(n) = -n^2 \int_{T_{0,y_0}} f(\mathfrak{m}_n\gamma\beta)\,d\mu(\beta),$$
$$w_\gamma(n) = u_\gamma(n) + v_\gamma(n).$$

In characteristic zero for all n for every $\gamma \in T$ the functions $u_\gamma(n)$, $v_\gamma(n)$ are nondecreasing. The function $w_\gamma(n)$ is nondecreasing nonpositive for $n \le 1$.

In positive characteristic for all sufficiently small n for every $\gamma \in T$ the function $w_\gamma(n)$ is nonpositive.

Proof: Denote

$$\mathfrak{m}_n = (n_1,n_2), \quad \gamma = (\gamma_1,\gamma_2), \quad \wr\wr_{y_0}(\mathfrak{m}_n) = (r_1,r_2), \quad \wr\wr_{y_0}(\gamma) = (c_1,c_2),$$

with $n_i, \gamma_i \in \mathbb{A}_{y_0}^\times$, $\wr\wr$ was defined in 42.

In positive characteristic we get

$$u_\gamma(n) = (|L(n_1^{-1}\gamma_1)|-1)(|L(n_2^{-1}\gamma_2)|-1), \quad v_\gamma(n) = -n^2(|L(n_1\gamma_1)|-1)(|L(n_2\gamma_2)|-1),$$

where $L(\delta)$ is the classical space L for the divisor on y_0 which corresponds to the idele δ. Denote by g the genus of y_0. The one-dimensional Riemann–Roch theorem, which is part of the adelic analysis theory in dimension one, implies

$$|L(\delta)|-1 = \begin{cases} q^{1-g}|\delta|^{-1}-1, & \text{if } |\delta| < q^{2-2g} \\ 0, & \text{if } |\delta| > 1, \end{cases}$$

where $q = q_{y_0}$ is as in 36.

This implies in the case $g=0$ that the functions $u_\gamma(n)$, $v_\gamma(n)$ are nondecreasing. Since $w_\gamma(1) = 0$, the function $w_\gamma(n)$ is nonpositive for $n \le 1$. For the rest of the discussion of positive characteristic in this proof we assume that $g \ge 1$.

Denote $b = \max_{q^{2-2g} \le |\delta| \le 1}(|L(\delta)|-1)$. Impose the following condition on n:

$$n \le \min\big(q^{-2}, q^{-1-g}, q^{-g}(b^2+q^{1-g})^{-1/2}\big).$$

We will show that $w_\gamma(n) \le 0$ for all γ.

Due to the definition of m_n in 42 we get $r_i \le qn$. In particular, $r_i < 1$.

If $r_1^{-1}c_1 > 1$ or $r_2^{-1}c_2 > 1$ then $u_\gamma(n) = 0$ and $w_\gamma(n) \le 0$.

Suppose that $c_1 \le r_1, c_2 \le r_2$, then $c_i r_i \le r_i^2 < q^{2-2g}$ and

$$-v_\gamma(n) = (q^{1-g}c_1^{-1} - r_1)(q^{1-g}c_2^{-1} - r_2).$$

If $r_1^{-1}c_1, r_2^{-1}c_2 < q^{2-2g}$ then

$$u_\gamma(n) = (q^{1-g}c_1^{-1}r_1 - 1)(q^{1-g}c_2^{-1}r_2 - 1).$$

Since $q^{1-g}c_i^{-1} - r_i \geq q^{1-g}c_i^{-1}r_i - 1$, we deduce $-v_\gamma(n) \geq u_\gamma(n)$.

If $1 \geq r_1^{-1}c_1 \geq q^{2-2g}$ then the first factor in $u_\gamma(n)$ is $\leq b$. Since $c_1 \leq r_1$ and $r_1 \leq q^{1-g}(b^2 + q^{1-g})^{-1/2} \leq (-b + (b^2 + 4q^{1-g})^{1/2})/2$, we get $q^{1-g}c_1^{-1} - r_1 \geq q^{1-g}r_1^{-1} - r_1 \geq b$. This and the previous argument imply $-v_\gamma(n) \geq u_\gamma(n)$ for all values of c_2.

Thus, $w_\gamma(n) \leq 0$ for all γ. See also Proposition 50 which gives another proof in positive characteristic.

In characteristic zero the function $u_\gamma(n)$ is obviously nondecreasing.

Consider first the case of $\mathrm{k} = \mathbb{Q}$. To show that $v_\gamma(n)$ is nondecreasing, we look at the function

$$v_1(x) = -x^2\big(\vartheta(x^2) - 1\big)^2, \qquad \text{where } \vartheta(x) = \sum_{k\in\mathbb{Z}} \exp\big(-\pi k^2 x\big).$$

Note that the function $y(x) = x\big(\vartheta(x) - 1\big)^2$ is nonincreasing for $x \geq 1/\pi$: indeed $y(x) = \sum_{km\neq 0} x\exp\big(-\pi(k^2 + m^2)x\big)$, its derivative is $\sum_{km\neq 0}\big(1 - \pi x(k^2 + m^2)\big)\exp\big(-\pi(k^2 + m^2)x\big)$, and $1 - \pi x(k^2 + m^2) \leq 0$ for $\pi x \geq 1$, $km \neq 0$. Hence $v_1(x)$ is nondecreasing for $x \geq 1/\sqrt{\pi}$. To show that

$$z(x) = x\big(\vartheta(x^2) - 1\big)$$

is nonincreasing for $x \leq \sqrt{2\pi/3}$ use the classical $x\vartheta(x^2) = \vartheta(x^{-2})$ for $x > 0$ to deduce $z(x) = \vartheta(x^{-2}) - x$. The derivative of $\vartheta(x^{-2})$ equals $\sum_k 2\pi k^2 x^{-3}\exp\big(-\pi k^2 x^{-2}\big)$ whose derivative equals $\sum_k 2\pi k^2 x^{-3}\big(-3x^{-1} + 2\pi k^2 x^{-3}\big)\exp\big(-\pi k^2 x^{-2}\big)$ and $-3x^{-1} + 2\pi k^2 x^{-3} \geq 0$ for $x \leq \sqrt{2\pi/3}$, $k \neq 0$. Therefore it is sufficient to check that $\sum_k 2\pi k^2 x^{-3}\exp\big(-\pi k^2 x^{-2}\big) \leq 1$ for $x = \sqrt{2\pi/3}$. But we already know that this is so for $x \geq 1/\sqrt{\pi} < \sqrt{2\pi/3}$. Thus, $v_1(x)$ is nondecreasing for all $x > 0$. We have also proved that the function $z(x)$ is nonincreasing positive for all $x > 0$. Then the function $v_\gamma(x) = -c_1^{-1}c_2^{-1}z(xc_1)z(xc_2)$ is nondecreasing for $x > 0$.

In the general case of k, due to the definition of f, we get

$$v_1(x) = -x^2\big(\Theta(x^2) - 1\big)^2$$

and the argument goes exactly as above. Here $\Theta(x^2)$ is the theta function associated to k as in [25], $\Theta = \Theta_{O_\mathrm{k}}$ where for a fractional ideal I of k

$$\Theta_I(x) = \sum_{\alpha\in I} \exp\Big(-\pi\, \mathfrak{c}_\mathrm{k}^{1/n}\, |\alpha|^2 x\Big),$$

where, similar to 40, $\mathfrak{c}_{\mathrm{k}}^{-1}$ is the discriminant of k, n is the degree of k, $|\alpha|^2 = \sum_\omega e_\omega |\omega(\alpha)|^2$ where ω runs through archimedean places of k and we use the notation e_ω of Example 2 of 30. The functional equation is $\Theta_{\mathcal{O}_{\mathrm{k}}}(x^2) = x^{-1}\Theta_{\mathfrak{d}_{\mathrm{k}}^{-1}\mathcal{O}_{\mathrm{k}}}(\mathfrak{c}_k^{-2/n}x^{-2})$, $x > 0$, where $\mathfrak{d}_{\mathrm{k}}$ is the different of k.

Since $w_\gamma(1) = 1$, the nondecreasing property of $w_\gamma(n)$ implies its nonpositivity for $n \le 1$. □

50. *In sections* 50–55 *we assume that* $|S_-| = 1$. In this section we study the asymptotic behaviour of $h(n)$ near 0.

DEFINITION. For $n \in N$ define

$$\mathrm{Log}(n) = \begin{cases} \log n & \text{in characteristic zero,} \\ \log_q(n) & \text{in positive characteristic,} \end{cases}$$

where q is as in 36.

DEFINITION. In positive characteristic define the derivative of a function $g\colon \mathbb{Z} \longrightarrow \mathbb{R}$ as $g'(k) = g(k) - g(k-1)$.

Proposition *Let* $\mathcal{S} = \mathcal{E}$ *be as in* 40 *and let* $|S_-| = 1$ *as in the beginning of this section.*

Then there is a polynomial $\mathfrak{w}$ *of degree 3 with the leading coefficient c such that for the function* $H(t)$ *defined in Theorem* 48 ($t \in \mathbb{Z}$ *in positive characteristic*) *we have*

$$H(t)^{(i)} - \mathfrak{w}(t)^{(i)} \to 0 \quad \textit{when} \quad t \to \infty, \quad \textit{for all} \quad i \ge 0.$$

If we use the classical property of $L_E(s)$ *to extend holomorphically to the half plane* $\Re(s) > 3/2$, *then in characteristic zero the left hand sides in the above formulas are* $o(e^{-t(1/2-\varepsilon)})$ *for any* $\varepsilon > 0$.

In positive characteristic the third derivative $h(n) - 3h(nq) + 3h(nq^2) - h(nq^3)$ *tends to* $6c$ *when* $n \to 0$. *In characteristic zero the third derivative* $h(e^{-t})'''$ *tends to* $6c$ *when* $t \to +\infty$.

In particular, $h(n)$ *and its first two derivatives are monotone functions for all sufficiently small* n.

Proof: The singular behaviour of $\int_{N^-} h(n) n^{s-2} d\mu_{N^-}(n)$ at $s = 2$ (resp. $q^s = q^2$ in positive characteristic) corresponds to the singular behaviour of the zeta integral, and hence by Remark 1 of 40 its principal part at $s = 2$ is $\sum_{1 \le i \le 4} a_i (s-2)^{-i}$ with nonzero a_4. Let $\mathfrak{w}(t) = \sum_{0 \le i \le 3} c_i t^i$ be a polynomial of degree 3 such that $\int_{N^-} \mathfrak{w}(-\mathrm{Log}\, n) n^{s-2} d\mu_{N^-}(n)$ equals this principal part. Put $c = c_3$.

In characteristic zero the analytic functions

$$(s-2)^i \int_{N^-} \big(h(n) - \mathfrak{w}(-\operatorname{Log} n)\big) n^{s-2} \, d\mu_{N^-}(n)$$

are holomorphic in the half-plane $\Re(s) > 2 - \varepsilon$ for some $\varepsilon > 0$. For $i \geq 1$ we can rewrite the integrals as $-\sum_{0 \leq j \leq i-1} (s-2)^{i-1-j} \big(h(e^{-t}) - \mathfrak{w}(t)\big)^{(j)}(0) + \int_0^\infty e^{-t(s-2)} d\big(h(e^{-t}) - \mathfrak{w}(t)\big)^{(i-1)}$. Since they are holomorphic near $s = 2$, the latter integrals converge at $s = 2$, so the standard properties of the Laplace transform imply $h(e^{-t})^{(i)} - \mathfrak{w}(t)^{(i)} \to 0$ when $t \to \infty$, for all i (e.g. [64], Ch.5, Lemma 5.2). If we use the cited above property of the L-function and zeta function of E, see 38, then those integrals are holomorphic for $\Re(s) > 3/2 + \varepsilon$. This together with loc.cit. imply the corresponding statement in the Proposition.

In positive characteristic proceed similarly, multiplying by $(1 - q^{2-s})^i$.

□

51. In this section we get more explicit information about the function $h(n)$ and then state hypothesis $(*)$.

Recall that we assume that the set $S_\prime$ consists of all the vertical curves in $S_\shortmid$ and one horizontal curve y_0, the image of the zero section.

For each $\epsilon \in \mathfrak{T}_{S_\shortmid}$ choose an $\varepsilon \in \mathfrak{T}_{y_0}$ such that the element (ε, ϵ) of $\mathfrak{T}_{S_\prime}$ belongs to $\mathfrak{T}_{1,S_\prime}$, the latter was defined in 42. So $|\epsilon|_{S_\shortmid} = ||\epsilon||_{S_\shortmid} = ||\varepsilon||_{y_0}^{-1}$. Each time when we have (ϵ, ε) in this section, they will satisfy this relation.

We get the following decomposition

$$\mathfrak{T}_{1,S_\prime} = \{(\gamma, 1) : \gamma \in \mathfrak{T}_{1,y_0}\} \times \{(\varepsilon, \epsilon) : \epsilon \in \mathfrak{T}_{S_\shortmid}\}.$$

Denote

$$\mathfrak{e} = \int_{T_{1,1,y_0}/T_{0,y_0}} d\mu = \mu(T_{1,1,y_0}/T_{0,y_0}),$$

$T_{1,1,y}$ was defined in 42. The number $\mathfrak{e}$ is the square of the normalized one-dimensional measure of the idele class group $I_{1,k(y_0)}/k(y_0)^\times$.

Recall that $\mathfrak{T}_{1,y_0} = T_{1,y_0}$. The integral over $T_{1,y_0}/T_{0,y_0}$ of $f(\mathfrak{m}_n \gamma)$ equals $\mathfrak{e}$ times the integral over $T_{1,y_0}/T_{1,1,y_0}$, and $T_{1,y_0}/T_{1,1,y_0} \simeq N$.

Using the decomposition of $\mathfrak{T}_{1,S_\prime}$ above and the definition of the integral over

$\mathfrak{T}_1$ in 43, we derive

$$\int_{\mathfrak{T}_{1,S'}} f(\mathfrak{m}_n\gamma)\,d\mu(\gamma) = \int_{\mathfrak{T}_{S_|}} \left(\int_{T_{1,y_0}} f(\mathfrak{m}_n\gamma\varepsilon)\,d\mu(\gamma)\right) f(\epsilon)\,d\mu(\epsilon)$$
$$= \int_{\mathfrak{T}_{S_|}} \left(\int_{T_{1,y_0}/T_{0,y_0}} u_{\gamma\varepsilon}(n^{-1})\,d\mu(\gamma)\right) f(\epsilon)\,d\mu(\epsilon)$$
$$= \mathfrak{e}\int_{\mathfrak{T}_{S_|}} \left(\int_{T_{1,y_0}/T_{1,1,y_0}} u_{\gamma\varepsilon}(n^{-1})\,d\mu(\gamma)\right) f(\epsilon)\,d\mu(\epsilon).$$

For $\mathfrak{n} \in N_{S_|}$ denote

$$c(\mathfrak{n}^2) = \int_{\mathfrak{T}_{1,S_|}} f(\epsilon\gamma)\,d\mu(\gamma) \ge 0,$$

where $|\epsilon|_{S_|} = \mathfrak{n}^{-1}$, so for the corresponding $\varepsilon \in \mathfrak{T}_{y_0}$ as above we have $|\varepsilon|_{y_0} = \mathfrak{n}^2$.

Using the definitions from 40 and 45 we get

$$\zeta_{\mathcal{E}}(s)^2\mathfrak{c}_{\mathcal{E}}^{1-s} = \zeta_{\mathcal{E},S_|}(f,|\,|_2^s) = \sum_{\mathfrak{n}\in I_{\mathcal{E}}} c(\mathfrak{n}^2)\mathfrak{n}^{-s}$$

for an appropriate index-set $I_{\mathcal{E}} \subset \mathbb{Q}$.

The definition of f in 40 shows that if $f(\epsilon) \ne 0$ for $\epsilon \in T_{S_|}$ then $|\epsilon|_\star \le \mathfrak{c}_\star^{-1}$ for each fibre $\star \in S_|$, $\mathfrak{c}_\star$ was defined in 40, and so $||\epsilon||_{S_|} = |\epsilon|_{S_|} \le \mathfrak{c}_{\mathcal{E}}^{-1}$ and so $||\varepsilon||_{y_0} \ge \mathfrak{c}_{\mathcal{E}}$, $\mathfrak{c}_{\mathcal{E}}$ was defined in 40. If $\mathfrak{n} < \mathfrak{c}_{\mathcal{E}}$ then, as we have seen, $c(\mathfrak{n}^2) = 0$. Put $I_{\mathcal{E}} = \mathfrak{c}_{\mathcal{E}}\mathbb{N} \cap N_{S_|}$, the definitions show that $c(\mathfrak{n}^2) = 0$ if $\mathfrak{n} \notin I_{\mathcal{E}}$.

Recall that $\zeta_{\mathcal{E},S'}(f,|\,|_2^s) = \int_N n^s \int_{\mathfrak{T}_1} f(\mathfrak{m}_n\gamma)\,d\mu(\gamma)\,d\mu_N(n)$. So in characteristic zero, using $u_\gamma(n^{-1}) = y_{a,b}(n) = (\Theta(n^2a^2)-1)(\Theta(n^2b^2)-1)$ from 49, where $\wr\gamma\wr_{y_0} = (a,b)$, we obtain

$$\zeta_{\mathcal{E},S'}(f,|\,|_2^s) = \mathfrak{e}\int_0^\infty n^s\left(\sum_{\mathfrak{n}\in I_{\mathcal{E}}} c(\mathfrak{n}^2)Y_{\mathfrak{n}^2}(n)\right)\frac{dn}{n}, \qquad Y_{\mathfrak{n}}(n) = \int_0^\infty y_{a,\mathfrak{n}a^{-1}}(n)\,\frac{da}{a},$$

$Y_{\mathfrak{n}}(n)$ corresponds to applying $\int_{T_{1,y_0}/T_{1,1,y_0}}$. This is the formula at the beginning of 4.3.

In every characteristic we similarly deduce

$$
\begin{aligned}
h(n) &= \int_{\mathfrak{T}_{1,S'}} \big(n^2 f(\mathfrak{m}_n\gamma) - f(\mathfrak{m}_n^{-1}\gamma)\big)d\mu(\gamma)\\
&= \int_{\mathfrak{T}_{S_1}} \Big(\int_{T_{1,y_0}} \big(n^2 f(\mathfrak{m}_n\gamma\varepsilon) - f(\mathfrak{m}_n^{-1}\gamma\varepsilon)\big)d\mu(\gamma)\Big) f(\epsilon)\,d\mu(\epsilon)\\
&= -\int_{\mathfrak{T}_{S_1}} \Big(\int_{T_{1,y_0}/T_{0,y_0}} w_{\gamma\varepsilon}(n)\,d\mu(\gamma)\Big) f(\epsilon)\,d\mu(\epsilon)\\
&= -\mathfrak{e}\int_{\mathfrak{T}_{S_1}} \int_{T_{1,y_0}/T_{1,1,y_0}} w_{\gamma\varepsilon}(n)\,d\mu(\gamma)\, f(\epsilon)\,d\mu(\epsilon),
\end{aligned}
$$

where $w_\delta(n)$ is defined in 48. Denote

$$
W_\mathfrak{n}(n) = \int_{T_{1,y_0}/T_{0,y_0}} w_{\gamma\varepsilon}(n)\,d\mu(\gamma), \quad B_\mathfrak{n}(n) = \int_{T_{1,y_0}/T_{1,1,y_0}} w_{\gamma\varepsilon}(n)\,d\mu(\gamma), \quad ||\varepsilon||_{y_0} = \mathfrak{n},
$$

then

$$
h(n) = -\int_{N_{S_1}} c(\mathfrak{n}^2)\,W_\mathfrak{n}(n)\,d\mu_{N_{S_1}}(\mathfrak{n}) = -\mathfrak{e}\sum_{\mathfrak{n}\in I_\varepsilon} c(\mathfrak{n}^2)\,B_\mathfrak{n}(n).
$$

In particular, the description of $w_\gamma(n)$ in Proposition 49 implies that the function $h(n)$ is nonnegative for all sufficiently small n.

In the previous section we observed the monotone behaviour of the function $H(t) - \mathfrak{w}(t)$ and its two derivatives when $t \to +\infty$. It is natural to look at the behaviour of the third logarithmic derivative of $h(n)$. As we will see later in 54, its monotone behaviour for all small n is closely related to the Riemann hypothesis for the zeta function $\zeta_\mathcal{E}(s)$. On the other hand, we will see in 52 and 53 that the Riemann hypothesis and some other condition imply the monotone behaviour of the third logarithmic derivative.

Hypothesis $((*) = (*)_\mathcal{E})$ *The fourth derivative of the function $H(t)$ keeps its sign for all sufficiently large t.*

52. Now we discuss hypothesis $(*)$ in characteristic zero. First we define several functions.

DEFINITION OF $w_{a,b}$, $V_\mathfrak{n}$, $Z_\mathfrak{n}$. Let k be a number field. Following 49 and 51, let Θ be the the theta function associated to k. For positive real a,b denote

$$
w_{a,b}(n) = \big(\Theta(n^{-2}a^2)-1\big)\big(\Theta(n^{-2}b^2)-1\big) - n^2\big(\Theta(n^2a^2)-1\big)\big(\Theta(n^2b^2)-1\big).
$$

For $\mathfrak{n} > 0$ denote

$$
V_\mathfrak{n}(x) = \int_0^\infty w_{a,\mathfrak{n}a^{-1}}(x)\frac{da}{a}, \qquad Z_\mathfrak{n}(e^{-t}) = V_\mathfrak{n}(e^{-t})'''',
$$

the derivative is taken with respect to t. So

$$Z_{\mathfrak{n}}(x) = \left(x\frac{d}{dx}\right)^4 V_{\mathfrak{n}}(x).$$

Recall that $c(\mathfrak{n}^2)$, $\mathfrak{n} \in I_{\mathcal{E}}$, are the coefficients of $\zeta_{\mathcal{E},S_{\mathfrak{l}}}(f,|\,|_2^s) = \sum_{\mathfrak{n}\in I_{\mathcal{E}}} c(\mathfrak{n}^2)\mathfrak{n}^{-s}$.

DEFINITION OF $Z(\{c(\mathfrak{n})\})$. For a set of coefficients $c(\mathfrak{n}^2)$, $\mathfrak{n}$ runs through a subset $I = c\mathbb{N}$, define the function

$$Z(\{c(\mathfrak{n})\}) = \sum_{\mathfrak{n}\in I} c(\mathfrak{n}^2)\, Z_{\mathfrak{n}^2}(x).$$

When $c(\mathfrak{n}^2)$ are the coefficients associated to $\mathcal{E}$ as in 51, we use the notation $Z_{\mathcal{E}}(x)$.

Since $V_{\mathfrak{n}^2}(x)$ equals $B_{\mathfrak{n}}(x)$ defined in 51, we obtain

$$h(n) = -\mathfrak{e}\sum_{\mathfrak{n}\in I_{\mathcal{E}}} c(\mathfrak{n}^2)\, V_{\mathfrak{n}^2}(x).$$

and so

$$h(e^{-t})'''' = -\mathfrak{e}\, Z_{\mathcal{E}}(e^{-t})$$

which gives a more explicit description of the fourth logarithmic derivative of h.

Thus, in characteristic zero hypothesis $(*)$ is equivalent to the single sign property of $Z_{\mathcal{E}}(x)$ near 0: *for all sufficiently small positive x the function $Z_{\mathcal{E}}(x)$ keeps its sign*. From Proposition 50 we also know that for every $\varepsilon > 0$ $Z_{\mathcal{E}}(x) = o(x^{1/2-\varepsilon})$ when $x \to 0$.

Discussion 1 Similar to the zeta integral description in 51, the series $Z_{\mathcal{E}}(x)$ involves a modification of the Dirichlet series associated to $\mathcal{E}$ using relatively nicely behaved functions including K_0-, K_1-Bessel functions, coming from a horizontal curve on $\mathcal{E}$. The following description appeared after conversations with Zagier and Suzuki in 2004.

Let, for simplicity, $\mathrm{k} = \mathbb{Q}$.

Let K_0 be the Bessel function

$$K_0(x) = \frac{1}{2}\int_0^\infty e^{-x\left(t+\frac{1}{t}\right)/2}\,\frac{dt}{t}.$$

It is easy to see that

$$\int_0^\infty \big(\vartheta(xa^2)-1\big)\big(\vartheta(x\mathfrak{n}^2a^{-2})-1\big)\frac{da}{a} = 4\sum_{l_1,l_2\geq 1} K_0(2\pi l_1 l_2 \mathfrak{n}x) = 4\sum_{l\geq 1}\sigma_0(l)K_0(2\pi l\mathfrak{n}x),$$

where σ_0 is the number of positive divisors. Using this, we get

$$V_{\mathfrak{n}}(x) = 4\sum_{l\geq 1}\sigma_0(l)\big(K_0(2\pi l\mathfrak{n}x^{-2}) - x^2K_0(2\pi l\mathfrak{n}x^2)\big).$$

Thus we have a more explicit description of the function $h(x) = -\mathfrak{e}\sum_{\mathfrak{n}\in\mathfrak{c}_\mathcal{E}\mathbb{N}\cap N_{S_1}} c(\mathfrak{n}^2)V_{\mathfrak{n}^2}(x)$ for $\mathrm{k} = \mathbb{Q}$.

Since

$$\left(x\frac{d}{dx}\right)^4\big(x^2K_0(ax^2)\big) = a^{-1}\mathcal{K}_1(ax^2), \quad \left(x\frac{d}{dx}\right)^4\big(K_0(ax^{-2})\big) = \mathcal{K}_2(ax^{-2}),$$

where

$$\mathcal{K}_1(x) = \big(16x + 288x^3 + 16x^5\big)K_0(x) - \big(64x^2 + 128x^4\big)K_1(x),$$
$$\mathcal{K}_2(x) = \big(64x^2 + 16x^4\big)K_0(x) - 64x^3K_1(x),$$

with K_1-Bessel function involved, we obtain

$$Z_{\mathfrak{n}}(x) = 4\sum_{l\geq 1}\sigma_0(l)\Big(\mathcal{K}_2(2\pi l\mathfrak{n}x^{-2}) - \frac{1}{2\pi l\mathfrak{n}}\mathcal{K}_1(2\pi l\mathfrak{n}x^2)\Big).$$

When $x \to 0$ we get $Z_{\mathfrak{n}}(x) \to 0$. Denote

$$\widetilde{V}_{\mathfrak{n}}(x) = -4x^2\sum_{l\geq 1}\sigma_0(l)K_0(2\pi l\mathfrak{n}x^2), \quad \widetilde{Z}_{\mathfrak{n}}(x) = \left(x\frac{d}{dx}\right)^4\widetilde{V}_{\mathfrak{n}}(x).$$

Then

$$\widetilde{Z}_{\mathfrak{n}}(x) = -\frac{2}{\pi\mathfrak{n}}\sum_{l\geq 1}\frac{\sigma_0(l)}{l}\mathcal{K}_1(2\pi l\mathfrak{n}x^2), \quad \widetilde{Z}_{\mathfrak{n}}(x) = \frac{1}{\mathfrak{n}}\widetilde{Z}_1\big(x\sqrt{\mathfrak{n}}\big), \quad .$$

Define

$$\widetilde{Z}(x) = \sum_{\mathfrak{n}\in I_\mathcal{E}} c(\mathfrak{n})\widetilde{Z}_{\mathfrak{n}}(x) = \sum_{\mathfrak{n}\in I_\mathcal{E}}\frac{c(\mathfrak{n})}{\mathfrak{n}}\widetilde{Z}_1\big(x\sqrt{\mathfrak{n}}\big).$$

The behaviour of $Z_{\mathfrak{n}}(x)$ and $Z(x)$ when $x \to 0$ is determined by the behaviour of $\widetilde{Z}_{\mathfrak{n}}(x)$, $\widetilde{Z}(x)$.

Discussion 2 From the definitions we get

$$Z_\mathcal{E}(x) = \sum_{n\geq 1}c(\mathfrak{c}_\mathcal{E}^2n^2)Z_{\mathfrak{c}_\mathcal{E}^2n^2}(x), \quad \text{where} \quad \sum_{n\geq 1}\frac{c(\mathfrak{c}_\mathcal{E}^2n^2)}{n^s} = \mathfrak{c}_\mathcal{E}\zeta_E(s)^2\prod_{b\in B_{0,i}}(1-|k(b)|^{n_{i,b}(1-s)})^{-2},$$

see 48 and 51.

Denote

$$z_E(x) = \sum_{n\ge 1} c(n^2) Z_{n^2}(x), \quad \text{where} \quad \sum_{n\ge 1} \frac{c(n^2)}{n^s} = \zeta_E(s)^2$$

$$Z_E(x) = \sum_{n\in \mathfrak{c}_E \mathbb{N}} d(n^2) Z_{n^2}(x), \quad \text{where} \quad \sum_{n\in \mathfrak{c}_E \mathbb{N}} \frac{d(n^2)}{n^s} = \mathfrak{c}_E^{1-s}\zeta_E(s)^2,$$

where $\mathfrak{c}_E$ is the conductor of E (we still assume that $\mathrm{k} = \mathbb{Q}$). Similarly define $\widetilde{z}_E(x)$ and $\widetilde{Z}_E(x)$ using $\widetilde{Z}_{n^2}(x)$. Similarly to the above, the function $Z_E(x)$ tends to $\widetilde{Z}_E(x)$ when $x \to 0$, and hence tends to $\mathfrak{c}_E^{-1}\widetilde{z}_E(\mathfrak{c}_E x)$ when $x \to 0$.

As far as the computational data are concerned, for a table of values of function $z_E(x)$ associated to elliptic curves E over $\mathbb{Q}$ with small conductor the reader can inspect [13] and p.311 of [14]. See also Remark 2 in 54.

53. This section includes some material about hypothesis $(*)$ in positive characteristic, which is not such an interesting case since we already know the Riemann Hypothesis for $\zeta_{\mathcal{E}}(s)$. Let $\mathcal{E}$ be as in 40 and let $|S_-| = 1$ as in the beginning of section 50. Let K be of positive characteristic.

Using the notation of 51, abbreviate $W_k(n) = W_{q^{-2k}}(n)$. Then

$$W_k(n) = \int_{T_{1,1,y_0}/T_{0,y_0}} \sum_{a=q^l, l\in\mathbb{Z}} w_{(a,a^{-1})\gamma\varepsilon}(n)\, d\mu(\gamma),$$

where $|\varepsilon|_{y_0} = q^{-2k}$, w was defined in 48.

Put $c_k = c(q^{-2k}) \ge 0$, the coefficient $c(\mathfrak{n})$ is introduced in 51. Then

$$\sum c_k q^{ks} = \zeta_{\mathcal{E},S_1}(f, |\,|_2^s).$$

We have

$$h(n) = -\sum_k c_k W_k(n) = -\sum_{k\le -\log_q \mathfrak{c}_{\mathcal{E}}} c_k W_k(n).$$

Let $n = q^{-r} < \mathfrak{c}_{\mathcal{E}}$. Let $|\varepsilon|_{y_0} = q^{-2k} \ge \mathfrak{c}_{\mathcal{E}}^2$. Then $|\varepsilon|_{y_0} \ge \mathfrak{c}_{\mathcal{E}} > n$, i.e. $r > k$. This implies, in the notation of the proof of Proposition 49, $u_{(a,a^{-1})\gamma\varepsilon}(n) = 0$. To calculate $W_k(n)$ we will use the formulas in the same proof. Let $\wr\ \wr_{y_0}(\mathfrak{m}_n) = (q^{-r_1}, q^{-r_2})$ and $\wr\ \wr_{y_0}((a,a^{-1})\gamma\varepsilon) = (q^{-k_1}, q^{-k_2})$, so $r_1 + r_2 = 2r$, $k_1 + k_2 = 2k$. Choose x_i such that $|x_i| = q^{-i}$. Introduce nonnegative numbers

$$b_i = \mu(I_{1,k(y_0)}/k(y_0)^\times)^{-1} \int_{I_{1,k(y_0)}/k(y_0)^\times} (|L(x_i\beta)| - 1)\, d\mu(\beta),$$

using the one-dimensional measure on the class idele group of $k(y_0)$ and the notation of the same proof. The latter shows $b_i = 0$ if $i < 0$ and $b_i = q^{1-g+i} - 1$ if $i > 2g-2$, where g is the genus of y_0. We then obtain

$$W_k(n) = -\mathfrak{c} q^{2k} e_{k+r},$$

where

$$e_m = \begin{cases} 0, & \text{if } m < 0, \\ q^{-2m} \sum_{0 \le i \le 2m} b_i b_{2m-i}, & \text{if } 0 \le m \le g-1, \\ q^{-2m} \sum_{2(m-g+1) \le i \le 2(g-1)} b_i b_{2m-i} & \\ +2 \sum_{0 \le i \le 2(m-g)} b_i (q^{1-g-i} - q^{-2m}), & \text{if } g-1 < m \le 2g-2, \\ (2m+3-4g)(q^{2-2g} + q^{-2m}) - 2(q^{3-3g} - q^{-2m+g})/(q-1) & \\ +2 \sum_{i=0}^{2g-2} b_i (q^{1-g-i} - q^{-2m}), & \text{if } 2g-2 < m. \end{cases}$$

This gives some more information on $h(n)$.

Denote

$$Z_k(n) = -\mathfrak{c}^{-1} (W_k(n) - 4W_k(nq^{-1}) + 6W_k(nq^{-2}) - 4W_k(nq^{-3}) + W_k(nq^{-4})).$$

Then

$$q^{-2k} Z_k(n) = e^{(4)}_{k+r} = e_{k+r} - 4e_{k+r+1} + 6e_{k+r+2} - 4e_{k+r+3} + e_{k+r+4}.$$

Using the formulas for $W_k(n)$ we deduce that for $n < \mathfrak{c}_{\mathcal{E}}$

$$Z_k(n) = \begin{cases} 0 & \text{if } r+k < -4, \\ \big(2r+2k+B\big) q^{-2r} (1-q^{-2})^4 & \text{if } r+k > 2g-2, \end{cases}$$

where $B = 3 - 4g + \frac{2q^g}{q-1} - \frac{8}{q^2-1} - 2\sum_{i=0}^{2g-2} b_i$.

For $n < \mathfrak{c}_{\mathcal{E}}$ the formula

$$h(n) - 4h(nq^{-1}) + 6h(nq^{-2}) - 4h(nq^{-3}) + h(nq^{-4}) = -\mathfrak{c} \sum_{-r-4 \le k \le -\log_q \mathfrak{c}_{\mathcal{E}}} c_k Z_k(n)$$

gives another description of the fourth derivative of $H(t)$ in positive characteristic.

If we use the rationality in $t = q^s$ of $\zeta_{\mathcal{E}, S_1}(f, |\,|^{\frac{s}{2}})$ and denote by q^{γ_i} all its poles inside the ball of radius q^2, then it is easy to show that

$$h(q^{-r})^{(4)} = g_0(r) + g(r) q^{-2r} + \sum g_i(r) q^{r(\gamma_i - 2)}$$

with certain polynomials g, g_i. Since the order of the pole of the zeta integral at $s = 2$ is 4, the fourth derivative tends to zero when $r \to +\infty$ and $g_0(r) = 0$. It is easy to obtain

Lemma *Suppose that there are no poles of the zeta function $\zeta_{\mathcal{E}}(s)$ inside the strip $1 < \Re(s) < 2$ and that the order of pole at s with $q^s = q$ is strictly greater than the order of pole at any other s with $q^s \neq q$, $\Re(s) = 1$. Then $h(n) - 4h(nq^{-1}) + 6h(nq^{-2}) - 4h(nq^{-3}) + h(nq^{-4})$ keeps its sign for all sufficiently small n, i.e. hypothesis $(*)$ of* 51 *holds.*

Proof: The degree of g_i equals the order of the pole γ_i of the boundary term. Under these assumptions the prevailing term will be the one associated to $\gamma_i = 1$. □

The order of pole of $\zeta_{\mathcal{E}}(s)$ at $s = 1$ is the sum of two, coming from $\zeta_{\mathbb{P}^1(B)}(s)$ (which do not have poles as s with $\Re(s) = 1$ and $q^s \neq q$), plus the order of zero of $L_E(s)$ at $s = 1$, plus the order of poles of $n_{\mathcal{E}}(s)$ at $s = 1$ (which do not have poles as s with $q^s \neq q$). Hence, for the second condition of Lemma to be satisfied it suffices that the order of zero of $L_E(s)$ at $s = 1$ is not smaller than the order of its other zeros on the line $\Re(s) = 1$.

54. Now we discuss applications of the previous theorems.

Theorem *Let $\mathcal{S} = \mathcal{E}$ be as in* 40 *and let $|S_-| = 1$ as in the beginning of section* 50.

In characteristic zero let $r(t)$ be the second derivative with respect to t of $h(n) + c\,\mathrm{Log}(n)^3$, $n = e^{-t}$.

In positive characteristic let

$$\begin{aligned} r(t) = {} & \frac{\mathfrak{h}(1)}{2}\lambda(t) + \Big(\mathfrak{h}(q^{-1}) - \frac{3\mathfrak{h}(1)}{2}\Big)\lambda(t - \log q) + \Big(\mathfrak{h}(q^{-2}) - 3\mathfrak{h}(q^{-1}) \\ & + \frac{3\mathfrak{h}(1)}{2}\Big)\lambda(t - 2\log q) \\ & + \Big(\mathfrak{h}(q^{-3}) - 3\mathfrak{h}(q^{-2}) + 3\mathfrak{h}(q^{-1}) - \frac{\mathfrak{h}(1)}{2}\Big)\lambda(t - 3\log q) \\ & + \sum_{k \geq 4}\Big(\mathfrak{h}(q^{-k}) - 3\mathfrak{h}(q^{-k+1}) + 3\mathfrak{h}(q^{-k+2}) - \mathfrak{h}(q^{-k+3})\Big)\lambda(t - k\log q), \end{aligned}$$

where $\mathfrak{h}(n) = h(n) + c\,(\mathrm{Log}\,n)^3$, c is defined in Proposition 50 *and the function $\lambda(t)$ is defined in Proposition* 46.

Suppose that hypothesis $(*)$ of 51 *holds. Then in every characteristic $r(t)$ is monotone for all sufficiently large t.*

Denote by x_0 the abscissa of convergence of

$$R(s) = \int_0^\infty e^{-(s-2)t}\, dr(t).$$

Then $x_0 < 2$ and x_0 is a real pole of $R(s)$. The boundary term $\omega(|\,|_2^s)$ and zeta integral $\zeta(f, |\,|_2^s)$ extend meromorphically to the right half plane $\Re(s) > x_0$, have a real pole x_0 and do not have poles inside the strip $\Re(s) \in (x_0, 2)$.

Proof: As in the proof of Proposition 50 the integral

$$\int_{N^-} \big(h(n) + c\,\mathrm{Log}(n)^3\big) n^{s-2}\, d\mu_{N^-}(n)$$

multiplied by $(s-2)^3$ in characteristic zero equals

$$\int_0^\infty e^{-(s-2)t}\, dr(t) - \sum_{0\le j\le 2} (s-2)^{2-j}\big(h(e^{-t}) - ct^3\big)^{(j)}(0).$$

This integral multiplied by $(1 - q^{2-s})^3$ in positive characteristic equals $\int_0^\infty e^{-(s-2)t}\, dr(t)$ where $r(t)$ is as in Theorem 53. In each case $R(s)$ is a holomorphic function near $s = 2$ (resp. all s such that $q^s = q^2$ in positive characteristic). Hence its abscissa of convergence x_0 is smaller than 2.

Since we assume that $(*)$ holds, there is t_0 such that 0 does not separate the values of the fourth derivative of $h(e^{-t})$ with respect to t for $t > t_0$. By Proposition 50 $h(e^{-t})''' - 6c \to 0$ when $t \to \infty$. Therefore we deduce that there is t_0 such that 0 does not separate the values of $h(e^{-t})''' - 6c$ for all $t > t_0$. Hence in characteristic zero $r(t)$ is monotone for $t > t_0$.

In positive characteristic hypothesis $(*)$ tells that $h(n) - 4h(nq^{-1}) + 6h(nq^{-2}) - 4h(nq^{-3}) + h(nq^{-4})$ keeps its sign for all sufficiently small n. By Proposition 50 $h(n) - 3h(nq) + 3h(nq^2) - h(nq^3)$ tends to $6c$ when $n \to 0$, therefore for all sufficiently small n we deduce that $h(n) - 3h(nq) + 3h(nq^2) - h(nq^3) - 6c$ keeps its sign, which implies the monotone property of $r(t)$ for t close to $+\infty$ in positive characteristic.

Using classical properties of the Laplace–Stieltjes transform of monotone functions, see e.g. [63], Th. 5a,5b of Ch.II §5, we deduce that $R(s)$ has a real singular point x_0 on its line of convergence $\Re(s) = x_0$ and is holomorphic in $\Re(s) > x_0$. Thus, x_0 is a real pole of $R(s)$. Using the relation between $R(s)$ and $\omega(|\,|_2^s)$, and the zeta integral, we deduce that all these functions are holomorphic inside the strip $\Re(s) \in (x_0, 2)$. □

Using the relation between the zeta integral and zeta function in Theorem 40, the relation between the zeta function and L-function, and the absence of poles for real $s \in (1,2)$ of the function $c_{\mathcal{E},S,}(|\,|_2^s)$ defined in 40, we obtain

Corollary *Let the assumptions of the Theorem hold. Assume that the zeta function $\zeta_{\mathcal{E}}(s)$ (or equivalently, $L_E(s)$) extends to a meromorphic function on the half-plane $\Re(s) > 1$. Suppose that $\zeta_{\mathcal{E}}(s)$ (resp. $L_E(s)$) has no real poles (resp. real zeros) in $(1,2)$. Then the zeta integral $\zeta(f, |\,|_2^s)$ does not have complex poles with $\Re(s) \in (1,2)$.*

Assume, in addition, that the zeta function $\zeta_{\mathcal{E}}(s)$ extends meromorphically on the plane and satisfies the functional equation, then the poles of $\zeta(f, |\,|_2^s)$ inside the critical strip $\Re(s) \in (0,2)$ lie on the critical line $\Re(s) = 1$.

Remark 1 In dimension one it is elementary to show that the zeta function does not have real zeroes in the critical strip outside the critical line. In higher dimensions the similar property is very far from elementary. In particular, the real zeros part of the Riemann hypothesis for the L-function of elliptic curves over number fields is not known in general. However, from the computational point of view it is not difficult to check the real zeros part of the Riemann hypothesis for a given L-function. For computational results on low lying zeros (including real zeros) of L-functions of elliptic curves E over rationals of conductor < 8000, see [R]. They imply the real part of the Riemann hypothesis for those curves.

Thus, if hypothesis $(*)$ *in* 51 *holds for any of those* E *then the Riemann hypothesis holds for poles of* $\zeta_{\mathrm{k}}(s/2)\zeta_{\mathcal{E}}(s)$ *and* $\zeta_{\mathrm{k}}(s/2)\zeta_{\mathrm{k}}(s)\zeta_{\mathrm{k}}(s-1)/L_E(s)$.

In positive characteristic to show the real zeros part of the Riemann hypothesis seems to be as difficult as to show the full Riemann hypothesis, at least using the known methods. If so, the previous Corollary is not very useful in positive characteristic.

Remark 2 For a converse result to Theorem 54 in positive characteristic see Lemma 53. Suzuki proved the following results in [54] which are converse to Theorem 54 in characteristic zero.

Suppose that L_E extends to an entire function satisfying the functional equation and let the Riemann hypothesis hold for the L-function. If all nonreal zeros of $L_{\mathcal{E}}(s) = L_E(s)n_{\mathcal{E}}(s)^{-1}$ on the critical line are of multiplicity strictly smaller than the multiplicity of its zero as $s = 1$ and if the estimate $\sum_{0<\Im(z)\le x}|L'_{\mathcal{E}}(z)|^{-2} = O(x)$ holds, where z runs through all zeros of $L_{\mathcal{E}}(s)$ on the critical line, then the function $Z_{\mathcal{E}}(x)$ is negative for all sufficiently small positive x. Since the factor $n_{\mathcal{E}}(s)$ is always nontrivial, the first condition in the previous sentence holds if the order of zero of $L_E(s)$ at $s = 1$ is not smaller than the order of its any other zeros on the line $\Re(s) = 1$.

In relation to $Z_E(x)$ of 52, if all nonreal zeros of $L_E(s)$ on the critical line are single, $L_E(1) = 0$ and if the estimate $\sum_{0<\Im(z)\le x}|L'_E(z)|^{-2} = O(x)$ holds, where z runs through all zeros of L_E on the critical line, Suzuki proved that the function $Z_E(x)$ is negative for all sufficiently small positive x.

It might be true that the function $Z(\{c(\mathfrak{n})\})(x)$ defined in 52 keeps its sign for a larger class of coefficients $c(\mathfrak{n})$ than those which come from the zeta function of $\mathcal{E}$ and E, see the last section of [54].

Remark 3 In the study of zeros of one-dimensional zeta functions one derives classical formulas for the logarithmic derivative of the Riemann zeta function like the formula in Remark 46. It is very well known that an application of the Ikehara–Wiener tauberian Theorem implies $\psi(x) \sim x$ and hence the prime number

Theorem, see e.g. [63], §17 Ch.V, and [34], Ch.III. Notice that the behaviour of the one-dimensional function $\psi(x) - x$ is not as good as of the expected behaviour of the fourth derivative of the function $h(n) - c\,(\mathrm{Log}\, n)^3$. The latter nice behaviour is expected to be related to the fact that the boundary of T_0 is very large in dimension two, whereas its analogue in dimension one is very small.

55. In this section we describe another approach to verify the analytic continuation, functional equation and Riemann hypothesis for the zeta integral of a proper regular model of elliptic curve over a number field, without assuming its automorphic property or the mean-periodicity of $h(e^{-t})$, but assuming instead hypothesis $(*)$ of 51.

DEFINITION. By Lemma 46 the function $h(n)n^{-1}$ is odd with respect to the multiplicative variable $n \in N$. Define

$$b(n) = \big(h(n) - \mathfrak{w}\big(-|\mathrm{Log}\, n|\big)\big)n^{-1},$$

where $\mathfrak{w}$ is the polynomial of Proposition 50.

So $b(n)$ is an odd function of its multiplicative argument n: $b(n^{-1}) = -b(n)$.

Put

$$p(t) = \begin{cases} b(e^{-t}) & \text{in characteristic zero,} \\ b(q^{-k}) & \text{for } t = k \text{ in positive characteristic.} \end{cases}$$

Then $p(t)$ is an odd function of the variable t. In the notation of Theorem 54 the abscissa of convergence of the Laplace transform of $p(t)$ is $y_0 = x_0 - 1$.

Recall that the *Laplace–Carleman transform* of $p(t)$ is defined as

$$P(s) = \begin{cases} \displaystyle\int_0^\infty p(t)e^{-st}\, dt & \text{for } \Re(s) > y_0, \\ \displaystyle -\int_0^\infty p(-t)e^{st}\, dt & \text{for } \Re(s) < -y_0, \end{cases}$$

see e.g. [20], Ch.1, 5.8 and 12.1.

Then $P(s) = \int_{N^-} b(n)n^s\, d\mu_{N^-}(n)$ for $\Re(s) > y_0$ and $P(-s) = P(s)$ for $\Re(s) > y_0$.

In positive characteristic we can define similarly the Laplace–Stieltjes–Carleman transform $P(s)$.

Now we describe the method. Assume that hypothesis $(*)$ of 51 holds. Check (using computers, for example) that $x_0 = 1$, i.e., in accordance with Theorem 54, the function $R(s)$ does not have real poles in (1,2). Then 54 implies that the zeta function extends meromorphically to $\Re(s) > 1$, and that the abscissa of convergence of $P(s)$ is $x_0 - 1 = 0$, so $P(s)$ is a holomorphic function on $\Re(s) > 0$ and on $\Re(s) < 0$. Recall that the Carleman spectrum sp_c of the transform $P(s)$ of $p(t)$ is the complement on the real line of the set of those y such that the function $P(s)$ has a holomorphic extension in a neighbourhood of iy. Then

$$\{y \in \mathbb{R} : 1 + iy \text{ is a pole of } \zeta(f,|\,|_2^s)\} = \text{ poles part of } \mathrm{sp}_c(P(s))$$

Now suppose that we can check that the closed set of singularities of the Laplace–Stieltjes–Carleman transform $P(s)$ is not the whole imaginary line, e.g. near 0. Since $p(t)$ is an odd function, we deduce that $P(s) = P(-s)$ at nonsingular points s lying on the imaginary line. The value of $P(s)$ at those points on the imaginary line is the *boundary value* of the Laplace–Carleman transform. Thus, we can analytically extend $P(s)$ to the complex plane, the extended function satisfies the equation $P(s) = P(-s)$. Therefore, the zeta integral can be extended to the plane and satisfies the functional equation $\zeta(f,|\,|_2^s) = \zeta(f,|\,|_2^{2-s})$.

In summary we obtain

Proposition *Let $\mathcal{S} = \mathcal{E}$ be as in* 40 *and let $|S_-| = 1$ as in the beginning of section* 50.

Suppose that hypothesis $(*)$ *of* 51 *holds.*

Suppose that the abscissa x_0 of convergence of $R(s)$ in Theorem 54 *is 1, i.e. the zeta integral does not have real poles inside $\Re(s) \in (1,2)$. Suppose that the Carleman spectrum of the Laplace–Stieltjes–Carleman transform of $p(t)$ is not the whole real line.*

Then $\zeta(f,|\,|_2^s)$, $\zeta_{\mathcal{E}}(s)$ and $L_E(s)$ have meromorphic extension to the complex plane and satisfy the functional equations of Theorem 48.

4.4. Supplementary comments

56. *Remark* 1 One can try to develop a two-dimensional version of Weil's interpretation [59] of parts of the work of Tate and Iwasawa in the language of distributions. Recall that in [59] the one-dimensional zeta function $\zeta(s)$ is represented as

$$\zeta(s) = \frac{\zeta(f,|\,|^s)}{\zeta(W(f),|\,|^s)},$$

where W is the operator introduced by Weil, it is the product of local W_v, and for almost all local data

$$W_v(f_v)(\alpha) = f_v(\alpha) - f_v(\pi_v^{-1}\alpha)$$

with a prime π_v with respect to v. If one uses a function f with the property $f(0) = \widehat{f}(0) = 0$, then both the numerator and denominator are entire functions satisfying the functional equations.

In dimension two one has an analogue of the map W, for the local part see Example 3 in 17. The problem is to find a suitable function f such that the zeta integrals $\zeta(f, |\,|_2^s)$ and $\zeta(W(f), |\,|_2^s)$ either satisfy the functional equation and extend meromorphically to the plane or have boundary terms which extend meromorphically. This would give then the meromorphic continuation and functional equation for the zeta integral.

Remark 2 Better understanding of local zeta integrals for ramified characters (see 21) will lead to the theory of adelic zeta integrals for arbitrary quasi-characters, which would also have applications to higher ramification theory.

Remark 3 The one-dimensional theta formula is directly related to the Riemann–Roch formula. In dimension two the theta formula of 44 is related to the Riemann–Roch theorem for zero cycles. Using the notation of 35 write the zeta function $\zeta_{\mathcal{E}}(s)$ as

$$\zeta_{\mathcal{E}}(s) = \sum_{C \in C_0(\mathcal{E}), C \geq 0} |C|_0^s = \sum_{I \geq 0} a_I |I|_0^s,$$

where I runs through representatives of nonnegative cycles with respect to the equivalence relation given by the degree. The study of the zeta function and a recurrent formula for a_I in the positive characteristic is closely related to the study of

$$\sum_{I \geq 0} b_I |I|_0^s, \quad b_I = \int_{T_0} f(\alpha_I \beta)\, d\mu(\beta),$$

where α_I corresponds to the cycle I via the map i of 35. Then the two-dimensional theta formula of 44 gives a certain recurrent relation for b_I and hence for a_I.

Remark 4 The adelic structures $\mathbb{A}$, $(\mathbb{A} \times \mathbb{A})^\times$ take into account the (local and adelic) integral structures of rank two on the surface, whereas the adelic object $\mathbf{A}$ rather takes into account the integral structures of rank one. Depending on applications, one should use either the adelic object $\mathbf{A}$ (geometry and arithmetic) or $\mathbb{A}$ (analysis and arithmetic), or their mixture as in the commutative diagrammes of 36 and 41.

One can keep wondering about a new refined algebraic geometry on arithmetic surfaces, a geometry in which the structure sheaf would take into account more information at the residue level and thus more information about integral structures of rank two. The Riemann–Roch theorem in such a refined algebraic geometry on arithmetic surfaces would be closely related to properties of the zeta function of the surface.

57. We very briefly sketch how to proceed in the general case of proper regular models of curves over k.

Let $\mathcal{S} \longrightarrow B$, as in 24, be a proper regular model of a smooth projective curve of genus g over k. As discussed in 40, in the calculation of the zeta integrals on every nonsingular fibre $\star$ we get the factor $\mathfrak{c}_\star^{1-s}$, which, if $\mathfrak{c}_\star$ is different from 1, makes the product of the factors over all the fibres divergent. In particular, it diverges for $\mathcal{S}_0 = \mathbb{P}^1(B)$ whose zeta function is very simple: it is just $\zeta_B(s)\zeta_B(s-1)$, the product of the one-dimensional zeta functions. Interestingly, from the point of view of the adelic analysis, the simplest case in dimension two is not the projective line over the base, but an elliptic surface.

For proper regular models of hyperbolic curves of genus $g > 1$, to compensate the nontrivial factor for the zeta integrals on almost all the fibres we will work with a *renormalized* zeta integral using the $(g-1)$st power of the zeta integral of $\mathcal{S}_0$ for the fibrewise renormalization.

DEFINITION. Let $S_\prime$ be a subset of curves on $\mathcal{S}$ which includes all the fibres and finitely many horizontal curves.

Fix $\mathfrak{f} = \otimes char_{(O_{x,y}, O_{x,y})}$, a function on the adelic object associated to $\mathbb{P}^1(B)$. Let g be a finite sum of functions of type $\otimes(f_{x,y}^{(1)}, f_{x,y}^{(2)})$, as in the definition of $R_{\mathbb{A}\times\mathbb{A}}$ in 31, but without the condition $\int f_\star \, d\mu_{\mathbb{A}_\star\times\mathbb{A}_\star} = 1$ for almost all $\star \in S_\prime$. Note that unless $\mathcal{S}$ corresponds to an elliptic curve, g does not belong to the space $Q_{\mathbb{A}_{S_\prime}\times\mathbb{A}_{S_\prime}}$, due to the divergence issues. Let $|\,|_{2,\mathcal{S}}\colon J_\mathcal{S} \longrightarrow \mathbb{R}^\times_{>0}$ be as in 35.

Using the definition of $\zeta_{\mathcal{S},y}$ in 39, introduce a (renormalized) zeta integral (depending on the choice of $S_\prime$)

$$
\begin{aligned}
&\widetilde{\zeta}_{\mathcal{S},S_\prime}(f,|\,|^s_{2,\mathcal{S}}) \\
&= \prod_{b\in B_0} \left(\zeta_{\mathbb{P}^1(B),\mathbb{P}^1(B)_b}\left(\mathfrak{f},|\,|^s_{2,\mathbb{P}^1(B)}\right)^{g-1} \zeta_{\mathcal{S},S_b}\left(f,|\,|^s_{2,\mathcal{S}}\right)\right) \cdot \prod_{y\in S_\prime\setminus S_\shortmid} \zeta_{\mathcal{S},y}\left(f,|\,|^s_{2,\mathcal{S}}\right)
\end{aligned}
$$

If $\star$ is a nonsingular fibre over $b \in B_0$, then the $\star$-factor $\mathfrak{c}_\star^{1-s}$ of $\zeta_{\mathcal{S},\star}(f,|\,|^s_{2,\mathcal{S}})$ (calculate similar to Theorem 40) *is cancelled out* by the b-factor of $\zeta_{\mathbb{P}^1(B)}\left(\mathfrak{f},|\,|^s_2\right)^{g-1}$ which is equal to its inverse.

Let now f be similar to the normalized function f in 40. Similar to the calculation in 40 one deduces that for $\Re(s) > 2$ the zeta integral $\widetilde{\zeta}_{\mathcal{S},S_\prime}\left(f,|\,|^s_{2,\mathcal{S}}\right)$ equals the product of $\zeta_{\mathbb{P}^1(B)}(s)^{2g-2}\zeta_\mathcal{S}(s)^2$ and of $c_{\mathcal{S},S_\prime}(|\,|^s_2)$ which is the product of an exponential factor for singular fibres and of appropriate factors for horizontal curves in $S_\prime$.

The second calculation of $\widetilde{\zeta}_{\mathcal{S},S_\prime}\left(f,|\,|^s_{2,\mathcal{S}}\right)$ goes along the lines of 45 using the formula

$$
\widetilde{\zeta}_{\mathcal{S},S_\prime}\left(f,|\,|^s_{2,\mathcal{S}}\right) = \int_M \widetilde{\zeta}_m(|\,|^s_{2,\mathcal{S}})\, d\mu_M(m),
$$

where

$$\widetilde{\zeta}_m(|\,|^s_{2,\mathcal{S}}) := \int_L \zeta_l\big(\mathfrak{f},|\,|^s_{2,\mathbb{P}^1(B)}\big)^{g-1} \zeta_{ml^{-1}}\big(f,|\,|^s_{2,\mathcal{S}}\big)\, d\mu_L(l).$$

Here L is the module value group associated to $\mathbb{P}^1(B)$, and M is the module value group associated to $\mathcal{S}$.

As an analogue of Theorem 48, the functional equation for the zeta integral and zeta function $\zeta_{\mathcal{S}}(s)$ in this general case would be

$$\widetilde{\zeta}_{\mathcal{S},S,}(f,|\,|^s_2) = \widetilde{\zeta}_{\mathcal{S},S,}(f,|\,|^{2-s}_2),$$
$$c_{\mathcal{S}}(s)\big(\zeta_{\mathcal{S}}(s)\zeta_{\mathbb{P}^1(B)}(s)^{g-1}\big)^2 = c_{\mathcal{S}}(2-s)\big(\zeta_{\mathcal{S}}(2-s)\zeta_{\mathbb{P}^1(B)}(2-s)^{g-1}\big)^2,$$

the factor $c_{\mathcal{S}}(s)$ is the product of an exponential vertical factor and the zeta integrals for horizontal curves.

Remark This functional equation, which is not proved in this text, is of course compatible with the conjectural functional equation for $\zeta_{\mathcal{S}}(s)$. Note that the Γ-factors in the functional equation are $(g-1)$st power of the Γ-factors of the product of the one-dimensional zeta functions $\zeta_B(s)\zeta_B(s-1)$. More generally, it is expected that the Γ-factors in the functional equation of the zeta function of every proper regular over B arithmetic scheme come from the Γ-factors of the one-dimensional zeta functions, i.e. there are no new Γ-factors for the zeta functions in higher dimensions.

4.5. Behaviour at the central point

58. Recall that there are three levels of objects associated to $\mathcal{E}$: the adelic objects $\mathbf{A}$ and $\mathbb{A}$ which are the restricted products of two-dimensional local objects, then the adelic objects $\mathbf{C}$, $\mathbf{B}$, $\mathbb{B}$ which have features of both local and global objects, and finally the discrete object K, see 28. In the previous study of the zeta integral of the surface and in class field theory of its field of rational functions the objects related to the first two levels play a dominant role. Using the objects of the third level as well, one can use apply the theory of this work (the boundary term integral representation and $\int_{\partial T_0}$) to the study of the zeta function at $s=1$.

We assume that the boundary term $\omega(|\,|^s_2) = \int_{N^-} h(n) n^{s-2}\, d\mu_{N^-}(n)$, and hence $\zeta_{\mathcal{E}}(s)$ and $L_E(s)$ extend meromorphically and satisfy the functional equation.

Using the formulas in Proposition 45, we get $\omega(|\,|^s_2) = \int_{M^-} \omega_m(|\,|^s_2)\, d\mu_{M^-}(m)$

where

$$\omega_m(|\,|_2^s) = \omega_m^{(1)}(|\,|_2^s) + \omega_m^{(2)}(|\,|_2^s),$$

$$\omega_m^{(1)}(|\,|_2^s) = |m|^{s-2} \int_{\mathfrak{T}_1} f(\mathfrak{m}^{-1}\alpha^{-1})\big(|\alpha|^{-1} - 1\big)\, d\mu(\alpha),$$

$$\omega_m^{(2)}(|\,|_2^s) = |m|^s \int_{\mathfrak{T}_1/T_0} \int_{\partial T_0} \big(|\mathfrak{m}\gamma|^{-1} f(\mathfrak{m}^{-1}\nu^{-1}\gamma^{-1}\beta) - f(\mathfrak{m}\gamma\beta)\big)\, d\mu(\beta)\, d\mu(\gamma).$$

Recall that the integral $\int_{M^-} \omega_m^{(1)}(|\,|_2^s)\, d\mu_{M^-}(m)$ extends to an entire function on the complex plane, see Remark 1 of 45.

Let r be the arithmetic rank of E, the rank of the free part of $E(\mathrm{k})$. Choose horizontal curves y_i, $i \in I$, $|I| = r+1$, the images of sections of $\mathfrak{p}\colon \mathcal{E} \longrightarrow B$ which include the image of the zero section and the curves on the surface, corresponding to a choice of free generators of the group $E(\mathrm{k})$. Denote $S_- = \{y_i : i \in I\}$ and let $S_\prime$ be the union of it and all the fibres.

For every singular fibre $\mathcal{E}_b$ take all the components of its reduced part except one which intersects the zero section and denote them by y_j, $1 \le j \le n_b$, where n_b is as in 48 and b runs through closed points in the base for which the special fibre $\mathcal{E}_b$ is singular. In addition, choose one nonsingular fibre y_*, and if K is of positive characteristic add it to the above curves. Denote the whole collection of curves in this paragraph by y_j, $j \in J$, $|J| = \sum n_b$.

Immediately from the definitions we see that the Picard group of $\mathcal{E}$ is isomorphic to the cokernel of the map

$$K^\times \longrightarrow \frac{\mathbf{B}_{\mathcal{E}}^\times}{\mathbf{B}_{\mathcal{E}}^\times \cap \mathbf{VA}_{\mathcal{E}}^\times},$$

see 28, 36 for the notation. So $\mathbf{B}_{\mathcal{E}}^\times$ is the product of $\mathbf{B}_{\mathcal{E}}^\times \cap \mathbf{VA}_{\mathcal{E}}^\times$, a discrete group, isomorphic to the quotient of $K^\times$, and the image of $\mathrm{Pic}(\mathcal{E})$. The quotient of $\mathrm{Pic}(\mathcal{E})$ by $\mathfrak{p}^*\mathrm{Pic}(B)$ can be presented in adelic form; it follows from the classical considerations that it is a finitely generated group, which in positive characteristic is the quotient of the Neron-Severi group of $\mathcal{E}$ modulo its subgroup generated by y_*. Its rank is equal to $r + 1 + \sum n_b$ (for the functional field case see e.g. [58], and for more detail in the case of the minimal model see [52]), and free generators are classes of y_i, y_j, $i \in I, j \in J$, without $y_\star$.

Now we briefly sketch a method to establish the relation between the order of the pole of the zeta integral at $s = 1$ and the arithmetic rank. This method is based on the previous material. The multiplicative group of the adeles $\mathbf{A}$ and of the adeles $\mathbb{A}$ are blended in the K_2^t-delic group J in the commutative diagramme of 41. Using the previous paragraph, it is easy to see that the group $\mathbf{B}_{\mathcal{E}}^\times$ is generated by the product of $\mathbf{B}_{\mathcal{E}}^\times \cap \mathbf{VA}_{\mathcal{E}}^\times$, of a compact group (corresponding to $\mathfrak{p}^*\mathrm{Pic}(B)$ in characteristic zero and to $\mathfrak{p}^*\mathrm{Pic}^0(B)$ in positive characteristic), of the image of the discrete group $K^\times$, and of the images of $\mathbf{B}_{y_i}^\times$, $i \in I$, and $\mathbf{B}_{y_j}^\times$, $j \in J$. Using the commutative diagramme in Lemma 41 and the above description of $\mathbf{B}_{\mathcal{E}}^\times$, it is expected that T_0 is the product of a group whose (weak) boundary is a one element set, of the product of T_{0,y_i}, T_{0,y_j} $(i \in I,\ j \in J)$, and of a subgroup of finite measure. This description would then be used in the computation of the integral $\int_{M_-}\int_{\mathfrak{T}_1/T_0}\int_{\partial T_0}|m|^s f(\mathfrak{m}\gamma\beta)d\mu(\beta)d\mu(\gamma)d\mu_{M_-}(m)$ whose leading term at $s = 1$ is expected to correspond to the integral where the internal integral is taken over the product of T_{0,y_i}, T_{0,y_j}, and over the subgroup of finite measure. Using the meromorphic continuation of the zeta integral (hence symmetry for integrals of $|m|^{s-2}|\gamma|\,f(\mathfrak{m}^{-1}\nu^{-1}\gamma\beta)$ and $|m|^s f(\mathfrak{m}\gamma\beta)$ when $s \to 1$) would then give that the order of the pole at $s = 1$ of the zeta integral equals the order of pole at $s = 1$ of the product of the squares of the usual one-dimensional zeta integrals along all y_i, y_j, $i \in I, j \in J$, i.e. $2(|I| + |J|)$. If so, then Theorem 40 which compares the zeta integral $\zeta_{\mathcal{E},S,}(f,|\,|_2^s)$ with the zeta function of $\mathcal{E}$ and the well known relation between the zeta function of $\mathcal{E}$ and L-function of E, see 48, would imply that the analytic rank of E equals r.

Remark Section 10 of [14] introduces an object which generalizes the idele class group to dimension two, not from the more narrow point of view of class field theory (the corresponding object is just $J_{\mathcal{E}}/P_{\mathcal{E}}$) but from the wider point of view of the zeta integral and hence the zeta function of $\mathcal{E}$. In particular the object takes into account some information related to the Picard group of $\mathcal{E}$ which at the adelic level is expressed via the quotient of the multiplicative groups at the local-global and global levels. This object glues in some special way the integral structures of rank one and two on $\mathcal{E}$ and shows up naturally in the study of the zeta integral. Functions (in one or another sense) on it should be closely related to $GL(1)$-automorphic functions on $\mathcal{E}$.

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