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ON ARCHIMEDEAN ZETA FUNCTIONS AND NEWTON POLYHEDRA

FUENSANTA AROCA, MIRNA GÓMEZ-MORALES, AND EDWIN LEÓN-CARDENAL

ABSTRACT. Let f be a polynomial function over the complex numbers and let ϕ be a smooth function over $\mathbb{C}$ with compact support. When f is non-degenerate with respect to its Newton polyhedron, we give an explicit list of candidate poles for the complex local zeta function attached to f and ϕ. The provided list is given just in terms of the normal vectors to the supporting hyperplanes of the Newton polyhedron attached to f. More precisely, our list does not contain the candidate poles coming from the additional vectors required in the regular conical subdivision of the first orthant, and necessary in the study of local zeta functions through resolution of singularities.

Our results refine the corresponding results of Varchenko and generalize the results of Denef and Sargos and Kamimoto and co-authors in the real case, to the complex setting.

1. INTRODUCTION

Archimedean local zeta functions were introduced by Gel'fand and Shilov in the 50's in [13]. Take $K = \mathbb{R}$ or $\mathbb{C}$ and take $f(x) = f(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$. Let ϕ be a smooth function with compact support in K^n. The local zeta function attached to (f, ϕ) is the parametric integral

$$Z_\phi(s, f) = \int\limits_{K^n \smallsetminus f^{-1}(0)} \phi(x) \, |f(x)|_K^s \, |dx|,$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, where $|dx|$ is the Haar measure on K^n. For uniformity reasons we will use for $a \in \mathbb{C}$, the convention: $|a|_K = |a|_{\mathbb{C}}^2$, where $|a|_{\mathbb{C}}$ is the standard complex norm.

It is not difficult to show that $Z_\phi(s, f)$ converges on the half plane $\{s \in \mathbb{C} \; ; \; \mathrm{Re}(s) > 0\}$ and defines a holomorphic function there. Furthermore, $Z_\phi(s, f)$ admits a meromorphic continuation to the whole complex plane. This was proved by Bernstein and Gel'fand in [6], then independently by Atiyah in [3], both proofs make use of Hironaka's theorem on resolution of singularities. Later, Bernstein [5] gave a proof by using what is called nowadays $D-$module theory. The main motivation of Gel'fand behind this problem was that the meromorphic continuation of $Z_\phi(s, f)$ implies the existence of fundamental solutions for differential operators with constant coefficients, see e.g. [18, Section 5.5].

Since those days, the theory of local zeta functions has evolved considerably due to multiple connections with many fields of mathematics, such as number theory, representation theory and singularity theory, among others. For instance, it is known that the poles of $Z_\phi(s, f)$ are integer shifts of the roots of the Bernstein–Sato polynomial of f, and therefore, by the classical results of Malgrange, the poles induce eigenvalues of the complex monodromy of f. These relations give some light about the difficult task that represents in general to compute the poles of $Z_\phi(s, f)$.

However, there is a special case in which some of these invariants can be computed effectively: the non-degenerate case.

Varchenko shows in [25] that when f is assumed to be non-degenerate with respect to its Newton polyhedron $NP(f)$, the set of candidate poles of $Z_\phi(s, f)$ can be described explicitly in terms of the normal vectors to the supporting hyperplanes of $NP(f)$. This description is very useful to study, for instance, real oscillatory integrals. Indeed, Varchenko proves that the poles of the local zeta function control the asymptotic behavior of the oscillatory integrals associated to f and ϕ, see [25, §1].

Roughly speaking, Varchenko's idea is to attach a Newton polyhedron $NP(f)$ to the function f and then define a non-degeneracy condition with respect to $NP(f)$. Then one may construct a toric variety associated to the Newton polyhedron, and use the well known toric resolution of singularities to give a list of candidate poles for $Z_\phi(s, f)$. Toric resolution of singularities requires a regular fan subordinated to $NP(f)$ (see Section 2 for the corresponding definitions) and it turns out that the extra rays required to obtain such a regular fan give rise to fake candidate poles, see Remark 2.2. In the case $K = \mathbb{R}$, Denef and Sargos proved in [10] that the poles coming from those extra rays can be discarded, thus reducing the list of candidate poles. In [23] this proof is extended for real zeta functions of analytic mappings. It has been pointed out to us that this work is related to the works of Kamimoto and its co-authors [7, 20] and [19]. In this work, we prove the analogue of the results of Denef and Sargos, and Kamimoto and its co-authors, for the case $K = \mathbb{C}$. This provides a much shorter list of candidate poles, for the complex $Z_\phi(s, f)$, that can be read off directly from the geometry of the Newton polyhedron of f.

We would like to emphasize that our approach to the study of complex zeta functions avoids the use of a toric resolution of singularities, thus the relevant information about the poles of $Z_\phi(s, f)$ relies just in the geometry of the Newton polyhedron of f, more precisely in the geometry of its normal fan (see Remark 2.2). This point of view is on the same line that the work of Gilula in [14], and the work of Kamimoto and its co-authors in [7, 20, 19] for real oscillatory integrals. Perhaps our methods, combined with those of [14] or [7, 20, 19], may give a better estimation and/or asymptotic expansions of complex oscillatory integrals like the ones studied in [26]. Note also that this kind of problems is also an object of study for topological zeta functions and its relatives, see for example [22].

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2. Fans, Monomial transformations and Toric Varieties

In this section, we review some basic results about toric geometry, such as cones, fans and toric varieties. The material presented in this section can be found for instance in [11, 12, 24].

2.1. **Cones and polyhedral fans.** The rational polyhedral cone generated by $u^{(1)}, \ldots, u^{(k)} \in \mathbb{Z}^n$ is the set $\langle u^{(1)}, \ldots, u^{(k)} \rangle = \{t_1 u^{(1)} + \cdots + t_k u^{(k)}; \ t_i \in \mathbb{R}_{\geq 0}, \ i = 1, \ldots, k\} \subset \mathbb{R}^n$. By $\mathcal{L}(\sigma)$ we will denote the minimal linear subspace containing the cone σ. The *dimension* of σ, denoted by $\dim(\sigma)$, is the dimension of $\mathcal{L}(\sigma)$ and the *relative interior* of σ, denoted by $Int_{rel}\sigma$, is the interior of σ as a subset of $\mathcal{L}(\sigma)$.

A rational polyhedral cone is said to be *strongly convex* if it does not contain any non-trivial linear subspace. Note that a cone contained in the first orthant is strongly convex.

A set of generators $\{u^{(1)}, \ldots, u^{(n)}\}$ of a rational cone $\langle u^{(1)}, \ldots, u^{(k)} \rangle$ can be chosen to be primitive, i.e. such that for any i, $\gcd_j(u_j^{(i)}) = 1$. If, furthermore, the set $\{u^{(1)}, \ldots, u^{(k)}\}$ is minimal it would be called the set of *vertices* of the cone $\langle u^{(1)}, \ldots, u^{(k)} \rangle$. For a strongly convex cone, its set of vertices is uniquely determined. We will denote by $\sigma = Cone\,(u^{(1)}, \ldots, u^{(k)}) \subset \mathbb{R}^n$ the rational convex cone with vertices $u^{(1)}, \ldots, u^{(k)} \in \mathbb{Z}^n$; or, for simplicity $\sigma = Cone\,(N)$, where N is the $k \times n$ matrix having the vertices $u^{(1)}, \ldots, u^{(k)}$ of σ as columns. A strongly convex rational polyhedral cone $\sigma = Cone\,(u^{(1)}, \ldots, u^{(k)})$ is said to be *simplicial* if the group $\mathcal{L}(\sigma) \cap \mathbb{Z}^n$ has rank k and, furthermore, *regular* if $\mathcal{L}(\sigma) \cap \mathbb{Z}^n$ is generated by the vertices of σ.

The *dual* $\sigma^\vee$ of a cone σ is the polyhedral cone given by $\sigma^\vee := \{v \in \mathbb{R}^n; v \cdot u \geq 0, \text{ for all } u \in \sigma\}$, where $u \cdot v$ stands for the inner product of the vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. Note that given an n-dimensional rational simplicial cone $\sigma = Cone\,(N)$ there is a matrix $M \in \mathcal{M}(n, \mathbb{Z})$ with columns $v^{(1)}, \ldots, v^{(n)} \in \mathbb{Z}^n$, such that

$$(1) \qquad \sigma^\vee = Cone\,(v^{(1)}, \ldots, v^{(n)}) \text{ and } M\,N^{\mathbf{t}} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} = N^{\mathbf{t}}\,M,$$

with positive integers $\lambda_1, \ldots, \lambda_n$. Here $N^{\mathbf{t}}$ stands for the transpose of the matrix N. Set L_M to be the linear map

$$L_M : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto M\,x.$$

The following properties of L_M are straightforward.

Lemma 2.1. *(i)* $L_M\left((\mathbb{R}_{\geq 0})^n\right) = Cone\,(M)$.

(ii) $L_{N^{\mathbf{t}}}(\sigma^\vee \cap \mathbb{Z}^n) = \{N^{\mathbf{t}} \cdot \beta \; ; \; \beta \in \sigma^\vee \cap \mathbb{Z}^n\} \subset (\mathbb{Z}_{\geq 0})^n$.

(iii) *If* $M, N \in \mathcal{M}(n, \mathbb{Z})$ *satisfy* $N^{\mathbf{t}} M = \Lambda$, *for some diagonal matrix* Λ *with values* $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{>0}$ *along its diagonal, then* $L_{N^{\mathbf{t}}}\,(Cone\,(M)) = (\mathbb{R}_{\geq 0})^n$.

There is a simple test to known when a given $n-$dimensional cone $\sigma = Cone\,(N)$ is regular or not: σ is regular if and only if $N \in GL(n, \mathbb{Z})$ is an unimodular matrix. In this case (1) holds for $M = (N^{-1})^{\mathbf{t}}$ and $\lambda_1 = \cdots = \lambda_n = 1$.

A finite collection $\Sigma = \{\sigma\}_{\sigma \subset \mathbb{R}^n}$ of cones is called a *polyhedral fan* if every face of a cone in Σ is a cone in Σ, and the intersection of any two cones $\sigma, \tau \in \Sigma$ is a face of both σ and τ. The *support* $|\Sigma|$ of a polyhedral fan Σ is the union of its cones. A polyhedral fan is said to be *simplicial* if all of its cones are simplicial and *regular* if all of its cones are regular. In a simplicial polyhedral fan Σ, a cone τ is a face of another cone σ if and only if the set of vertices of τ is a subset of the set of vertices of σ. For a polyhedral fan Σ, its *set of vertices*, denoted $Vert\,(\Sigma)$, is the union of vertices of cones in Σ.

Remark 2.2. *Given a polyhedral fan* Σ, *it is always possible to find a simplicial (respectively, regular) fan* Σ' *refining* Σ, *that is, such that every cone in* Σ *is union of cones in* Σ', *and with the property that every cone in* Σ' *is simplicial (respectively, regular) (see for example [12]). The procedure of finding a simplicial fan refining* Σ *can be done without changing the set of vertices*

in Σ, but obtaining a regular fan Σ' refining Σ would require, in general, the introduction of new vertices.

2.2. **Monomial transformations defined for integer matrices.** Given a non-singular matrix $M \in \mathcal{M}(n, \mathbb{Z})$ with columns $u^{(1)}, u^{(2)}, \ldots, u^{(n)} \in \mathbb{Z}^n$, denote by ψ_M the morphism given by

$$\psi_M : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$$
$$z \longmapsto \zeta,$$

where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $\zeta_k = \prod_{j=1}^n z_j^{u_j^{(k)}}$. This morphism is rational on $\mathbb{C}^n$ and it is bi-rational if $\det M = \pm 1$. When $\det M = \pm d$, the morphism ψ_M is a d-fold covering, see e.g. [24]. Furthermore, for any matrices $M, N \in \mathcal{M}(n, \mathbb{Z})$ we have

$$\psi_M \circ \psi_N = \psi_{NM}; \quad \text{while for } M \in GL(n, \mathbb{Z}) \text{ we have } (\psi_M)^{-1} = \psi_{M^{-1}}.$$

2.3. **The toric variety associated to a simplicial polyhedral fan.** Let us recall the construction of the toric variety associated to a simplicial polyhedral fan Σ of $\mathbb{R}^n$ with support in the first orthant. This process can be found, for example in [11]. Denote by $\mathfrak{M}(\Sigma)$ the set of maximal dimension cones in Σ, i.e. n−dimensional cones. Take a simplicial cone $\sigma = Cone\,(N) \in \mathfrak{M}(\Sigma)$, with $\sigma^\vee = Cone\,(v^{(1)}, \ldots, v^{(n)})$ satisfying (1). Since the first orthant is contained in $\sigma^\vee$, we have that the canonical basis of $\mathbb{R}^n : \{e^{(1)}, \ldots, e^{(n)}\}$, is contained in $\sigma^\vee \cap \mathbb{Z}^n$. Choose $w^{(1)}, \ldots, w^{(\ell)} \in \mathbb{Z}^n$ so that the set $\{v^{(1)}, \ldots, v^{(n)}, e^{(1)}, \ldots, e^{(n)}, w^{(1)}, \ldots, w^{(\ell)}\} \subset \mathbb{Z}^n$ generates the semi-ring $\sigma^\vee \cap \mathbb{Z}^n$. Consider the morphism

$$\varphi_\sigma : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \times (\mathbb{C}^*)^\ell \subset \mathbb{C}^{2n+\ell}$$
$$y \mapsto (y^{v^{(1)}}, \ldots, y^{v^{(n)}}, y^{e^{(1)}}, \ldots, y^{e^{(n)}}, y^{w^{(1)}}, \ldots, y^{w^{(\ell)}}).$$

Set $U_\sigma := \mathrm{Image}(\varphi_\sigma)$. The closure $T_\sigma := \overline{U_\sigma} \subset \mathbb{C}^{2n+\ell}$ is the affine variety associated to the cone σ. Using local coordinates (x, y, z) in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^\ell$, we have:

$$T_\sigma = \left\{(x, y, z) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^\ell; (x, y, z)^{(\alpha, \beta, \gamma)} = (x, y, z)^{(\alpha', \beta', \gamma')}\right\},$$

where $\alpha, \alpha', \beta, \beta' \in (\mathbb{Z}_{\geq 0})^n; \gamma, \gamma' \in (\mathbb{Z}_{\geq 0})^\ell$ and they satisfy

$$\sum_{i=1}^n (\alpha_i - \alpha_i')v^{(i)} + \sum_{j=1}^n (\beta_j - \beta_j')e^{(j)} + \sum_{k=1}^\ell (\gamma_k - \gamma_k')w^{(k)} = 0.$$

The toric variety T_Σ associated to Σ is constructed by glueing the affine toric varieties associated to each of its cones in such a way that $\mathrm{Image}(\varphi_\sigma) = \mathrm{Image}(\varphi_{\sigma'})$ for $\sigma, \sigma' \in \Sigma$. This construction of affine toric varieties is well known, furthermore one may show that T_Σ is a Hausdorff variety.

The following lemma will be used in the proof of our main result. Its proof is a straightforward calculation.

Lemma 2.3. *Let $A^\mathbf{t} \in \mathcal{M}(n, \mathbb{Z})$ be the matrix with columns $u^{(1)}, \ldots, u^{(n)}$, where $u^{(i)} = (u_1^{(i)}, \ldots, u_n^{(i)})^\mathbf{t} \in (\mathbb{Z}_{\geq 0})^n$. Then the Jacobian of the morphism ψ_A at x is given by*

$$\det D(\psi_A)_x = \det A \cdot x_1^{||u^{(1)}||-1} \cdots x_n^{||u^{(n)}||-1},$$

where $||u^{(j)}|| := u_1^{(j)} + \cdots + u_n^{(j)}$.

2.4. **Toric modification.** In this section we will introduce some morphisms defined over T_σ (or more generally over T_Σ) that will be used in the computation of $Z_\phi(s, f)$. For each $\sigma \in \mathfrak{M}(\Sigma)$, set π_σ to be the morphism

$$\pi_\sigma : T_\sigma \subset \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^\ell \longrightarrow \mathbb{C}^n$$

$$(x, y, z) \longmapsto y.$$

It can be verified that the morphisms π_σ are compatible with the glueing, defining a morphism $\pi : T_\Sigma \to \mathbb{C}^n$ such that for each $\sigma \in \mathfrak{M}(\Sigma)$, $\pi|_{T_\sigma} = \pi_\sigma$. Since the support of Σ is the first orthant, the morphism $\pi : T_\Sigma \to \mathbb{C}^n$ is a proper and bi-rational morphism (see for example [8, Thm. 3.4.11]) called the *toric modification* associated to Σ. This morphism π is bi-regular in the complement of the coordinate hyperplanes.
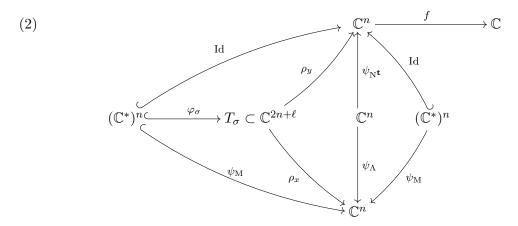
Take $\sigma = Cone(\mathrm{N}) \in \mathfrak{M}(\Sigma)$. Let M be the matrix such that $\sigma^\vee = Cone(\mathrm{M})$ and let Λ be the diagonal matrix $\mathrm{N}^{\mathbf{t}}\, \mathrm{M}$. Let $v^{(1)}, \ldots, v^{(n)}$ be the columns of M, let $w^{(1)}, \ldots, w^{(\ell)} \in \mathbb{Z}^n$ be such that the set

$$\{v^{(1)}, \ldots, v^{(n)}, e^{(1)}, \ldots, e^{(n)}, w^{(1)}, \ldots, w^{(\ell)}\} \subset \mathbb{Z}^n$$

generates the semi-ring $\sigma^\vee \cap \mathbb{Z}^n$, and let W be the $(\ell \times n)-$matrix that has $w^{(1)}, \ldots, w^{(\ell)}$ as columns.

Now, consider the morphism

$$\varphi_\sigma : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^{2n+\ell}$$

$$y \longmapsto (y^{v^{(1)}}, \ldots, y^{v^{(n)}}, y^{e^{(1)}}, \ldots, y^{e^{(n)}}, y^{w^{(1)}}, \ldots, y^{w^{(\ell)}}),$$

and the projections $\rho_x : \mathbb{C}^{2n+\ell} \to \mathbb{C}^n$ and $\rho_y : \mathbb{C}^{2n+\ell} \to \mathbb{C}^n$ defined in local coordinates $(x, y, z) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^\ell$ by $\rho_x(x, y, z) = x \in \mathbb{C}^n$ and $\rho_y(x, y, z) = y \in \mathbb{C}^n$.

The diagram

(2)

commutes. In particular, the restriction $\pi_\sigma := \pi|_{T_\sigma} = \rho_y|_{T_\sigma}$ of the toric modification associated to Σ satisfies the equalities $\rho_x \circ \varphi_\sigma = \psi_{\mathrm{M}}$ and $\pi_\sigma \circ \varphi_\sigma = \mathrm{Id}|_{(\mathbb{C}^*)^n}$.

Definition 2.4. *Take $\sigma = Cone(\mathrm{N}) \in \mathfrak{M}(\Sigma)$. Let M be the matrix such that $\sigma^\vee = Cone(\mathrm{M})$ and let Λ be the diagonal matrix $\mathrm{N}^{\mathbf{t}}\, \mathrm{M}$. Let Φ_σ be given by*

$$\mathbb{C}^n \longrightarrow T_\sigma$$

$$x \longmapsto (\psi_\Lambda(x), \psi_{\mathrm{N}^{\mathbf{t}}}(x), \psi_{\mathrm{N}^{\mathbf{t}}W}(x)).$$

The following diagram illustrates the role of Φ_σ in the precedent construction of morphisms,

(3)

$$
\begin{array}{ccc}
& & \mathbb{C}^n \xrightarrow{\ f\ } \mathbb{C} \\
& \nearrow^{\pi_\sigma} \quad \uparrow^{\psi_{\mathrm{N}\mathbf{t}}} & \\
T_\sigma \xleftarrow{\ \Phi_\sigma\ } \mathbb{C}^n & & \\
& \searrow_{\rho_x} \quad \downarrow^{\psi_\Lambda} & \\
& & \mathbb{C}^n
\end{array}
$$

Note that $\Phi_\sigma|_{(\mathbb{C}^*)^n} = \varphi_\sigma \circ \psi_{\mathrm{N}\mathbf{t}}$ and all the exponents appearing here are positive so we can extend this map to the whole $\mathbb{C}^n$. This shows that Φ_σ is well defined. Also we have that $\Phi_\sigma : (\mathbb{C}^*)^n \longrightarrow T_\sigma^*$ is a finite covering, moreover, it is a local diffeomorphism.

2.5. **Partitions of Unity.** We will use a partial resolution of singularities to decompose the integral $Z_\phi(s, f)$ into monomial integrals. A key ingredient in our proof is the Theorem of existence of partitions of the unity.

Recall that, for any topological space X, given a function $\chi : X \longrightarrow \mathbb{C}$, the *support of* χ is the closed set

$$
Supp(\chi) := \overline{\{x \in X; \chi(x) \neq 0\}}.
$$

Definition 2.5. *Let T_Σ be a toric variety, and consider U a connected open subset of T_Σ. Let $\chi : U \longrightarrow \mathbb{C}$ be a function and take a point x in U. Take $\sigma \in \mathfrak{M}(\Sigma)$ such that $x \in T_\sigma \subset \mathbb{C}^{2n+\ell}$. We will say that χ **is** $\mathcal{C}^\infty$ **at** x if there exists $V \subset \mathbb{C}^{2n+\ell}$ a neighbourhood of x in $\mathbb{C}^{2n+\ell}$, and a $\mathcal{C}^\infty-$function $\Psi : V \longrightarrow \mathbb{C}$ such that $\Psi|_{V \cap U} = \chi|_{V \cap U}$.*

The fact that the definition does not depend on the affine chart T_σ follows from the fact that the glueings defining T_Σ are regular morphisms.

Definition 2.6. *Let X be a topological space and let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets with $X = \bigcup_{\alpha \in \Lambda} U_\alpha$. **A partition of unity subordinated to the cover** $\{U_\alpha\}_{\alpha \in \Lambda}$ is a collection of continuous functions $\xi_\alpha : X \longrightarrow \mathbb{C}$, such that $\sum_{\alpha \in \Lambda} \xi_\alpha(x) = 1$ and the support of each ξ_α is contained in U_α.*

It is well known that partitions of unity subject to a covering exist for the complex n-dimensional space. Moreover, the partitions can be chosen such that the ξ_α's are $\mathcal{C}^\infty$. Then, there exist $\mathcal{C}^\infty-$partitions of unity in the sense of Definition 2.5 for toric varieties.

3. Newton Polyhedra

3.1. **The Newton polyhedron of a function.** In this section we recall the construction and some properties of the Newton polyhedron of a polynomial and the corresponding toric variety associated to its normal fan. Most of the statements in this section may also be found in [2].

In this section we will take $K = \mathbb{R}$ or $\mathbb{C}$. Let f be a K-polynomial function, such that $f(\underline{0}) = 0$. We write

$$(4) \qquad f(x) = \sum_{\mu \in (\mathbb{Z}_{\geq 0})^n} a_\mu x^\mu \quad \text{for} \quad x = (x_1, \ldots, x_n) \in K^n.$$

The *support* or *set of exponents* of f is the set

$$\varepsilon(f) := \{\mu \in (\mathbb{Z}_{\geq 0})^n; a_\mu \neq 0\}.$$

The *restriction* of f to a subset $F \subset (\mathbb{R}_{\geq 0})^n$ is defined as

$$f|_F = \sum_{\mu \in \varepsilon(f) \cap F} a_\mu x^\mu.$$

Example 3.1. *Take $\{e^{(1)}, \ldots, e^{(k)}\}$ from the canonical basis of $\mathbb{R}^n$ for $1 \leq k \leq n$. If $\langle e^{(1)}, \ldots, e^{(k)} \rangle$ denotes the cone generated by these vectors, then*

$$f|_{\langle e^{(1)}, \ldots, e^{(k)} \rangle} = f(x_1, \ldots, x_k, 0, \ldots, 0).$$

Therefore $f(x) = \sum_{\mu \in (\mathbb{Z}_{\geq 0})^n} a_\mu x^\mu$ can be written as

$$f = f|_{\langle e^{(1)}, \ldots, e^{(k)} \rangle} + \sum_{i=k+1}^n x_i \widetilde{f}_i,$$

for some K-polynomial functions $\widetilde{f}_{k+1}, \ldots, \widetilde{f}_n$.

Definition 3.2. *The Newton polyhedron of f is the following convex hull*

$$NP(f) := Conv(\{\mu + (\mathbb{R}_{\geq 0})^n; \mu \in \varepsilon(f)\}) \subset (\mathbb{R}_{\geq 0})^n.$$

Remark 3.3. *The polyhedron $NP(f)$ has only one vertex $\nu \in (\mathbb{Z}_{\geq 0})^n$ if and only if $f(x) = x^\nu h_\nu(x)$ where $h_\nu(x) \in K[[x_1, \ldots, x_n]]$ satisfies $h_\nu(\underline{0}) \neq 0$.*

Let $\mathcal{H}$ be the hyperplane given by $\{x \in \mathbb{R}^n \; ; \; a \cdot x = b\}$. The hyperplane $\mathcal{H}$ determines two closed half-spaces:

$$\mathcal{H}^+ := \{x \in \mathbb{R}^n; a \cdot x \geq b\} \quad \text{and} \quad \mathcal{H}^- := \{x \in \mathbb{R}^n; a \cdot x \leq b\}.$$

We say that $\mathcal{H}$ is *a supporting hyperplane* of $NP(f)$, if $NP(f) \cap \mathcal{H} \neq \emptyset$ and $NP(f) \subset \mathcal{H}^+$ or $NP(f) \subset \mathcal{H}^-$.

A *proper face* of $NP(f)$ is the intersection of the polyhedron with a supporting hyperplane, and the *non-proper* face is the whole $NP(f)$. Faces of dimension $0, 1$, and $n - 1$ are called vertices, edges and facets, respectively.

When F is a face of $NP(f)$, the restriction $f|_F$ is often denoted by f_F and called the *face function*.

Given a supporting hyperplane $\mathcal{H}$ of $NP(f)$ containing a facet, there exists a vector $u \in \mathbb{Z}^n \setminus \{0\}$ which is orthogonal to $\mathcal{H}$ and is directed into the polyhedron. Such a vector is called an *inward normal* to $\mathcal{H}$. When the vector u is chosen to be primitive, it turns out that every facet of $NP(f)$ has a unique primitive inward vector; the set of such vectors is denoted by $\mathcal{I}(NP(f))$.

Now, given $\omega \in (\mathbb{R}_{\geq 0})^n$, the ω-*order* of f is defined as

$$\nu_\omega(f) := min\{\omega \cdot \mu; \mu \in \varepsilon(f)\}.$$

Note that $\mathcal{H}_\omega := \{x \in \mathbb{R}^n \; ; \; \omega \cdot x = \nu_\omega(f)\}$ is a *supporting hyperplane* for $NP(f)$, and the intersection

$$F_\omega := NP(f) \cap \mathcal{H}_\omega$$

is a face of $NP(f)$ called *the first meet locus* of ω or the ω-*face*.

3.2. **Dual fans and fans subordinated to f.** We define an equivalence relation on $(\mathbb{R}_{\geq 0})^n$ by taking

$$\omega \sim \omega' \iff F_\omega = F_{\omega'}.$$

In order to describe the equivalence classes of $\sim$ we define *the cone associated to* a given face F of $NP(f)$, as

$$\sigma_F := \{\omega \in (\mathbb{R}_{\geq 0})^n \ ; \ F = F_\omega\}.$$

Note that $\sigma_{NP(f)} = \{\underline{0}\}$. The other equivalence classes are described in the following Lemma.

Lemma 3.4.   *(i) Let F be a proper face of $NP(f)$, then the topological closure $\overline{\sigma}_F$ of σ_F is a rational polyhedral cone and*

$$\overline{\sigma}_F = \{\omega \in (\mathbb{R}_{\geq 0})^n \ ; \ F_\omega \supset F\}.$$

  (ii) Let $F_1, \ldots, F_k$ be the facets of $NP(f)$ containing F and let $u^{(1)}, \ldots, u^{(k)} \in \mathbb{Z}^n \setminus \{\underline{0}\}$ be the inward normal vectors to $F_1, \ldots, F_k$ respectively. Then $\sigma_F = \{\lambda_1 u^{(1)} + \cdots + \lambda_k u^{(k)} \ ; \ \lambda_i \in \mathbb{R}_{>0}\}$, and $\overline{\sigma}_F = \langle u^{(1)}, \ldots, u^{(k)} \rangle$.
  (iii) $\dim \sigma_F = \dim \overline{\sigma}_F = n - \dim F$.

Definition 3.5. *The collection of cones $\Sigma(f) := \{\overline{\sigma}_F \ ; \ F \text{ is a face of } NP(f)\}$ is called the dual fan of f.*

There is a natural duality between cones in $\Sigma(f)$ and faces of $NP(f)$ given by

$$F \mapsto \overline{\sigma}_F$$

and

$$\sigma \mapsto F_\sigma,$$

where F_σ is the intersection of the facets having as inward vectors the generators of σ.

From the previous construction one has that $\Sigma(f)$ is a polyhedral fan with support $(\mathbb{R}_{\geq 0})^n$. Moreover,

$$Vert\,(\Sigma(f)) = \mathcal{I}(NP(f)).$$

We will say that a fan Σ *is subordinated to f* if it defines a refinement of $\Sigma(f)$. From now on, we will work with simplicial fans subordinated to f.

Proposition 3.6.   *(1) $NP(f) = \bigcap_{\nu \text{ vertex of } NP(f)} \left(\nu + \overline{\sigma}^\vee_{\{\nu\}} \right)$.*
  (2) If Σ is a fan subordinated to f and $\sigma \in \mathfrak{M}(\Sigma)$, the face $\nu := F_\sigma$ of $NP(f)$ is a vertex of $NP(f)$ with $\sigma \subset \overline{\sigma}_\nu$. We have $\varepsilon(f) \subset \nu + \sigma^\vee$, thus

$$f(x) = x^\nu h_\sigma(x),$$

  where $h_\sigma(x) \in K[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]]$ satisfies $\underline{0} \in \varepsilon(h_\sigma) \subset \sigma^\vee \cap \mathbb{Z}^n$.
  (3) Let Σ be a simplicial fan subordinated to f and let $\sigma = Cone(\mathrm{N}) \in \mathfrak{M}(\Sigma)$. If $\nu := F_\sigma$, then

$$L_{\mathrm{N}^t}(\nu) = (\nu_{u^{(1)}}(f), \ldots, \nu_{u^{(n)}}(f))$$

  where $u^{(1)}, \ldots, u^{(n)}$ are the columns of the matrix N.

Proof. All the statements follow easily from the given definitions and Lemma 3.4.   □

3.3. **Newton Polyhedron under monomial modifications.** Let f be a polynomial as in (4). Given $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ and $\mathrm{M} = (u_k^{(j)}) \in \mathcal{M}(n, \mathbb{Z})$, we have $\psi_{\mathrm{M}}(x)^\mu = x^{\mathrm{M} \cdot \mu}$. Hence,

$$(5) \qquad f \circ \psi_{\mathrm{M}}(x) = \sum_{\mu \in \varepsilon(f)} a_\mu x^{L_{\mathrm{M}}(\mu)} \quad \text{and} \quad \varepsilon(f \circ \psi_{\mathrm{M}}) = L_{\mathrm{M}}(\varepsilon(f)).$$

Then, for any subset $F \subset (\mathbb{R}_{\geq 0})^n$,

$$(6) \qquad f|_F \circ \psi_{\mathrm{M}}(x) = (f \circ \psi_{\mathrm{M}}(x))|_{L_{\mathrm{M}}(F)}.$$

Proposition 3.7. *Let Σ be a simplicial fan subordinated to the polynomial f, and let $\sigma = Cone(\mathrm{N}) \in \mathfrak{M}(\Sigma)$, then the following assertions hold.*

(1) *The polyhedron $NP(f \circ \psi_{\mathrm{N}^{\mathfrak{t}}})$ has only one vertex. Furthermore, we have*

$$f \circ \psi_{\mathrm{N}^{\mathfrak{t}}}(x) = x^{L_{\mathrm{N}^{\mathfrak{t}}}(\nu)} h(x),$$

where $h(x) \in K[x_1, \ldots, x_n]$ satisfies $h(\underline{0}) \neq 0$ and $L_{\mathrm{N}^{\mathfrak{t}}}(\nu) \in (\mathbb{Z}_{\geq 0})^n$ with ν vertex of $NP(f)$.

(2) *If τ is the face of σ generated by the ith columns of N, with $i \in J \subset \{1, \ldots, n\}$. Then, we have*

$$(f \circ \psi_{\mathrm{N}^{\mathfrak{t}}})|_{L_{\mathrm{N}^{\mathfrak{t}}}(\nu) + \langle e^{(i)} \ ; \ i \notin J \rangle} = f|_{F_\tau} \circ \psi_{\mathrm{N}^{\mathfrak{t}}},$$

where ν is the vertex of $NP(f)$ such that $\varepsilon(f) \subset \nu + Cone (\mathrm{N})^\vee$.

Proof. (1) It follows from Remark 3.3 and Proposition 3.6.

(2) Let $\mathrm{M} \in \mathcal{M}(n, \mathbb{Z})$ be the matrix with columns $v^{(1)}, \ldots, v^{(n)}$, such that $\sigma^\vee = Cone (\mathrm{M})$ and $\mathrm{N}^{\mathfrak{t}} \mathrm{M} \in \mathcal{M}(n, \mathrm{N})$ is the diagonal matrix with positive values $\lambda_1, \ldots, \lambda_n$ along its diagonal. Then,

$$F_\tau = NP(f) \cap \left(\nu + \left(Cone (\mathrm{N})^\vee \cap \tau^\perp \right) \right)$$
$$= NP(f) \cap \left(\nu + \langle \{v^{(i)}\}_{i \notin J} \rangle \right)$$

and hence,

$$f|_{F_\tau} = f|_{\nu + \langle \{v^{(i)}\}_{i \notin J} \rangle}.$$

By (6),

$$f|_{\nu + \langle \{v^{(i)}\}_{i \notin J} \rangle} \circ \psi_{\mathrm{N}^{\mathfrak{t}}}(x) = (f \circ \psi_{\mathrm{N}^{\mathfrak{t}}}(x))|_{L_{\mathrm{N}^{\mathfrak{t}}} \left(\nu + \langle \{v^{(i)}\}_{i \notin J} \rangle \right)}.$$

The result follows from the fact that

$$L_{\mathrm{N}^{\mathfrak{t}}} \left(\nu + \langle \{v^{(i)}\}_{i \notin J} \rangle \right) = L_{\mathrm{N}^{\mathfrak{t}}}(\nu) + \left\langle L_{\mathrm{N}^{\mathfrak{t}}} \left(v^{(i)} \right) ; i \notin J \right\rangle$$
$$= L_{\mathrm{N}^{\mathfrak{t}}}(\nu) + \left\langle \lambda_i e^{(i)} ; i \notin J \right\rangle = L_{\mathrm{N}^{\mathfrak{t}}}(\nu) + \left\langle e^{(i)} ; i \notin J \right\rangle. \quad \square$$

Under the hypothesis of the second part of Proposition 3.7, the following result holds.

Corollary 3.8. *There exists a polynomial $h_\tau \in K[x_1, \ldots, x_n]$ depending only on the variables x_i for $i \notin J$ with $h_\tau(\underline{0}) \neq 0$ such that*

$$f|_{F_\tau} \circ \psi_{\mathrm{N}^{\mathfrak{t}}}(x) = x^{L_{\mathrm{N}^{\mathfrak{t}}}(\nu)} h_\tau(x).$$

4. Non-degeneracy Condition

Definition 4.1. *Let $f(x)$ be a polynomial, such that $f(\underline{0}) = 0$. We say that f is non–degenerate with respect to a face $F \subseteq NP(f)$ if the system of equations*

$$\{f_F(x) = 0, \nabla f_F(x) = 0\}$$

has no solutions in $(K^*)^n$.

   We say that f is non–degenerate with respect to its Newton polyhedron if it is non–degenerate with respect to any face of $NP(f)$.

   It seems that the non-degeneracy condition was first proposed by Arnol'd in [1], where he uses it to classify critical points of functions. See also [4, 15, 16, 17] and [21].

Proposition 4.2. *Let Σ be a simplicial fan subordinated to the polynomial f and let $\sigma = Cone$ (N) $\in$ $\mathfrak{M}(\Sigma)$. If the function f is non-degenerate with respect to its Newton polyhedron then $f \circ \psi_{\mathrm{N^t}}$ is also non-degenerate with respect to $NP(f \circ \psi_{\mathrm{N^t}})$.*

Proof. By the first part of Proposition 3.7, the polyhedron $NP(f \circ \psi_{\mathrm{N^t}})$ has only one vertex $L_{\mathrm{N^t}}(\nu)$ and its faces are of the form $E_J := L_{\mathrm{N^t}}(\nu) + \langle e^{(i)} \ ; \ i \notin J \rangle$, for some $J \subset \{1, \ldots, n\}$, where ν is the vertex of $NP(f)$ such that $\varepsilon(f) \subset \nu + Cone$ (N)$^\vee$ (that is $\nu = F_\sigma$). Let E_J be a face of $NP(f \circ \psi_{\mathrm{N^t}})$ and assume, without loss of generality, that $J = \{1, 2, \ldots, r\}$. By the second part of Proposition 3.7,

$$(f \circ \psi_{\mathrm{N^t}})\,|_{E_J} = f|_{F_\tau} \circ \psi_{\mathrm{N^t}}(x_1, \ldots, x_n)$$

where τ is the compact face of σ generated by the last $n-r$ columns of N. Since f is non-degenerate, $V(f|_{F_\tau})$, the set of $K-$zeroes of $f|_{F_\tau}$, does not have singularities in the coordinate hyperplanes. The result now follows from the fact that $\psi_{\mathrm{N^t}}$ restricted to $(\mathbb{C}^*)^n$ is a local diffeomorphism from $(\mathbb{C}^*)^n$ to $(\mathbb{C}^*)^n$. $\qquad\square$

Lemma 4.3. *Let $g(x), h(x) \in K[x_1, \ldots, x_n]$ be such that $h(\underline{0}) \neq 0$ and*

$$g(x) = x^\alpha h(x)$$

for some $\alpha \in (\mathbb{Z}_{\geq 0})^n$. If g is non-degenerate with respect to $NP(g)$, then h is non-degenerate with respect to $NP(h)$.

Proof. Note that $F \subseteq NP(g)$ is a face of $NP(g)$ if and only if $F + (-\alpha)$ is a face of $NP(h)$. Suppose that $h|_{F+(-\alpha)}(x) = 0$ for some $x \in V(x_1 \cdots x_n)$. Since $g|_F(x) = x^\alpha h|_{F+(-\alpha)}(x)$, it follows that $g|_F(x) = 0$ and, furthermore, we have

$$\frac{\partial \left(g|_F\right)}{\partial x_i}(x) = x^\alpha \frac{\partial \left(h|_{F+(-\alpha)}\right)}{\partial x_i}(x) + x^{\alpha - e_i} h|_{F+(-\alpha)}(x).$$

Since g is non-degenerate with respect to $NP(g)$, there exists $i \in \{1, \ldots, n\}$ such that

$$0 \neq \frac{\partial \left(g|_F\right)}{\partial x_i}(x) = x^\alpha \frac{\partial \left(h|_{F+(-\alpha)}\right)}{\partial x_i}(x),$$

which finishes the proof. $\qquad\square$

4.1. **Transversality.** Given a point $P \in K^n$ and r polynomials $h_1, \ldots, h_r$, with $r \leq n$ and $h_i(P) = 0$ we say that $V(h_1), \ldots, V(h_r)$ intersect *transversally* at P, when they are all smooth at P and the dimension of the linear subspace generated by $\nabla h_1(P), \ldots, \nabla h_r(P)$ is r. We say that $V(h_1), \ldots, V(h_r)$ intersect *transversally*, when they intersect transversally at every common zero P of $V(h_1), \ldots, V(h_r)$.

Proposition 4.4. *If $V(h_1), \ldots, V(h_r)$ intersect transversally at P, then there exists a neighbourhood U of P, a neighbourhood U' of $\underline{0}$ and a diffeomorphism $\eta : (U', \underline{0}) \longrightarrow (U, P)$ such that $h_i \circ \eta = x_i$ for $i = 1, \ldots, r$.*

And the determinant of the Jacobian of η is a $\mathcal{C}^\infty(K)-$function that is different from zero on U'.

Proof. Let $i_{r+1}, \ldots, i_n \in \{1, \ldots, n\}$ be such that

$$\{\nabla h_1(P), \ldots, \nabla h_r(P), e^{(i_{r+1})}, \ldots, e^{(i_n)}\},$$

is a base of K^n.

Consider the morphism

$$H : \quad (K^n, P) \quad \longrightarrow \quad (K^n, \underline{0})$$
$$x \quad \mapsto \quad \left(h_1(x), \ldots, h_r(x), \left(x_{i_{r+1}} - P_{i_{r+1}}\right), \ldots, (x_{i_n} - P_{i_n})\right).$$

The determinant of the Jacobian of H is different that zero at P then, by the Inverse Function Theorem, there exists a smooth function $H^{-1} : (K^n, \underline{0}) \longrightarrow (K^n, P)$ such that $H \circ H^{-1}$ is the identity. Taking $\eta := H^{-1}$ we have the result. $\qquad\square$

Proposition 4.5. *Let $h \in K[x_1, \ldots, x_n]$ be a non-degenerate polynomial with respect to its Newton polyhedron, with $h(\underline{0}) \neq 0$. The variety $V(h)$ intersects transversally with the coordinate hyperplanes.*

Proof. If $n = 1$, the coordinate hyperplane corresponds to the origin and h does not vanish at 0. For $n \geq 2$, let $z = (z_1, \ldots, z_n) \in h^{-1}(0) \cap V(x_1 \cdots x_n)$. Since $h(\underline{0}) \neq 0$, there exists $i \in \{1, \ldots, n\}$ such that $z_i \neq 0$ and, since $z \in V(x_1 \cdots x_n)$, there exists $j \in \{1, \ldots, n\}$ such that $z_j = 0$. Assume, without loss of generality, that $z_1 \cdots z_r \neq 0$ and $z_k = 0$ for $k \in \{r+1, \ldots, n\}$.

In what follows we will show that $V(x_{r+1}), \ldots, V(x_n)$ and $V(h)$ intersect transversally at z. Since $h(\underline{0}) \neq 0$, the origin is the only vertex of $NP(h)$ and $\langle e^{(1)}, \ldots, e^{(r)} \rangle$ is a face of $NP(h)$. Now, $h|_{\langle e^{(1)}, \ldots, e^{(r)} \rangle}$ is a polynomial in the variables $x_1, \ldots, x_r$, thus

$$\frac{\partial \left(h|_{\langle e^{(1)}, \ldots, e^{(r)} \rangle}\right)}{\partial x_i} = 0 \quad \text{for} \quad i \in \{r+1, \ldots, n\}.$$

Recall from Example 3.1 that $h = h|_{\langle e^{(1)}, \ldots, e^{(r)} \rangle} + \sum_{j=r+1}^n x_j \widetilde{h}_j$, for some polynomials $\widetilde{h}_{r+1}, \ldots, \widetilde{h}_n \in K[x_1, \ldots, x_n]$. Therefore

$$\frac{\partial h}{\partial x_i} = \frac{\partial \left(h|_{\langle e^{(1)}, \ldots, e^{(r)} \rangle}\right)}{\partial x_i} + \sum_{j=r+1}^n x_j \frac{\partial \widetilde{h}_j}{\partial x_i},$$

when $i \in \{1, \ldots, r\}$. Evaluating at $z = (z_1, \ldots, z_r, 0, \ldots, 0)$ we get

$$\frac{\partial h}{\partial x_i}(z) = \frac{\partial \left(h|_{\langle e^{(1)}, \ldots, e^{(r)} \rangle}\right)}{\partial x_i}(z).$$

On the other hand, the non–degeneracy condition over h implies that for any $\varepsilon \in (K^*)^n$ there exists some index $i \in \{1, \ldots, r\}$ such that

$$\frac{\partial \left(h|_{\langle e^{(1)}, \ldots, e^{(r)} \rangle} \right)}{\partial x_i}(\varepsilon) \neq 0.$$

If $(z_{r+1}^*, \ldots, z_n^*) \in (K^*)^{n-r}$, then $(z_1, \ldots, z_r, z_{r+1}^*, \ldots, z_n^*) \in (K^*)^n$ and for $i \in \{1, \ldots, r\}$ we have

$$\frac{\partial h}{\partial x_i}(z) = \frac{\partial h|_{\langle e^{(1)}, \ldots, e^{(r)} \rangle}}{\partial x_i}(z) = \frac{\partial h|_{\langle e^{(1)}, \ldots, e^{(r)} \rangle}}{\partial x_i}(z_1, \ldots, z_r, z_{r+1}^*, \ldots, z_n^*),$$

so there should be some index $i \in \{1, \ldots, r\}$ such that $\frac{\partial h}{\partial x_i}(z) \neq 0$.

Finally, since

$$\{(\nabla x_{r+1})(z), \ldots, (\nabla x_n)(z), \nabla h(z)\} = \{e^{(r+1)}, \ldots, e^{(n)}, (\nabla h)(z)\}$$

and the set on the right hand side generates an $(n - r + 1)-$dimensional space, we obtain the result.   □

Corollary 4.6. *Take $f \in K[x_1, \ldots, x_n]$ and assume that it is non-degenerate with respect to its Newton polyhedron. Let Σ be a simplicial fan subordinated to f and take $\sigma = Cone\,(N) \in \mathfrak{M}(\Sigma)$. Then $f \circ \psi_{N^t} = x^\alpha h$, where $h(\underline{0}) \neq 0$ and $\alpha \in (\mathbb{Z}_{\geq 0})^n$. In addition, the variety defined by h intersects transversally the hyperplane coordinates.*

4.2. **Neighbourhoods and coordinates.**

Proposition 4.7. *Take $h(x) \in K[x_1, \ldots, x_n]$ with $h(\underline{0}) \neq 0$ and assume that h is non-degenerate with respect to its Newton polyhedron. Let $g := x^\alpha h$, for some $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$ and let z be a point in the coordinate hyperplanes. Set $J_z := \{i \; ; \; z_i = 0\}$ and choose $k \notin J_z$. Then there exists a neighbourhood U_z of z, a neighbourhood $\widetilde{U}_z$ of the origin and a diffeomorphism*

$$\eta_z : (\widetilde{U}_z, \underline{0}) \longrightarrow (U_z, z)$$

with $(\eta_z)_j(x) = x_j$ for all $j \in J_z$ and $(\eta_z)_j(x) \neq 0$ for all $j \notin J_z$ for all $x \in \widetilde{U}_z$; and such that one of the following holds

(i) $g \circ \eta_z = \prod_{j \in J_z} x_j^{\alpha_j} \tilde{h}$ *where $\tilde{h}(x) \neq 0$ for all $x \in \widetilde{U}_z$,*

(ii) $g \circ \eta_z = \prod_{j \in J_z} x_j^{\alpha_j} x_k \tilde{h}$ *where $\tilde{h}(x) \neq 0$ for all $x \in \widetilde{U}_z$.*

In addition the determinant of the Jacobian of η_z is a $\mathcal{C}^\infty(K)-$function not vanishing on $\widetilde{U}_z$.

Proof. Let z be a point in the coordinate hyperplanes and set $r := \#J_z$, without loss of generality, suppose that $J_z = \{1, \ldots, r\}$ and $k = r + 1$.

(i) Suppose that $h(z) \neq 0$. Take U_z small enough such that h and $\{x \mapsto x_j \; ; \; j = r + 1, \ldots, n\}$ do not vanish in U_z. Now take: $\eta_z : x \mapsto x + z$, $\widetilde{U}_z := \eta_z^{-1}(U_z)$ and $\tilde{h}(x) := \prod_{j=r+1}^n (x_j + z_j)^{\alpha_j}(h \circ \eta_z)$. Then $\tilde{h}$ does not vanish on $\widetilde{U}_z$,

$$g \circ \eta_z(x) = \prod_{j=1}^r x_j^{\alpha_j} \tilde{h}(x),$$

and the determinant of the Jacobian of η_z is one.

(ii) Suppose that $h(z) = 0$. Then, by Proposition 4.5, the varieties

$$V(x_1), \dots, V(x_r), V(h)$$

intersect transversally. By Proposition 4.4, there exists a neighbourhood U of z, a neighbourhood U' of the origin, and a diffeomorphism

$$\eta = (\eta_1, \dots, \eta_n) : (U', \underline{0}) \longrightarrow (U, z)$$

such that

(7) $$\eta_i(x) = x_i \quad \text{for} \quad i = 1, \dots, r \quad \text{and} \quad h \circ \eta(x) = x_{r+1}.$$

Since $\eta(\underline{0}) = z$, then $\eta_i(\underline{0}) \neq 0$ for $i = r+1, \dots n$, so, we may choose $\widetilde{U}_z \subset U'$ to be a small enough neighbourhood of $\underline{0}$ such that $\eta_i(x) \neq 0$ for all $x \in \widetilde{U}_z$. Set $U_z := \eta(\widetilde{U}_z)$ and $\tilde{h}(x) := \prod_{j=r+1}^n (\eta_j(x))^{\alpha_j}$. Then $\tilde{h}$ does not vanish on $\widetilde{U}_z$ and

$$g \circ \eta(x) = \prod_{j=1}^r (\eta_j(x))^{\alpha_j} h \circ \eta(x) \tilde{h}(x) \overset{(7)}{=} \prod_{j=1}^r x_j^{\alpha_j} x_{r+1} \tilde{h}(x)$$

where $\tilde{h}$ does not vanish on $\widetilde{U}_z$.

$\square$

Corollary 4.8. *Let f be a polynomial and assume that f is non–degenerate with respect to its Newton polyhedron. Let Σ be a simplicial fan subordinated to f and take $\sigma = Cone(\mathrm{N}) \in \mathfrak{M}(\Sigma)$. With the notation of the previous Proposition we have that one of the following holds:*

(1) $f \circ \psi_{\mathrm{N}^t} \circ \eta_z = \prod_{j \in J_z} x_j^{\alpha_j} \tilde{h}$ where $\tilde{h}(x) \neq 0$ for all $x \in \widetilde{U}_z$.

(2) $f \circ \psi_{\mathrm{N}^t} \circ \eta_z = \prod_{j \in J_z} x_j^{\alpha_j} x_k \tilde{h}$ where $\tilde{h}(x) \neq 0$ for all $x \in \widetilde{U}_z$.

In addition, the determinant of the Jacobian of η_z is a $\mathcal{C}^\infty(K)$−function not vanishing on $\widetilde{U}_z$ and $\alpha_i = \nu_{u^{(i)}}(f)$, where $u^{(i)}$ stands for the i-th column of N.

5. Local Zeta Functions

5.1. **Some Integrals.** In this section we present some results about integrals that will be used later on. The first Lemma is about meromorphic continuation of integrals attached to monomials over the complex numbers. These results are easy variations of the ones presented in [13, App. B, Sections 2.2 and 2.9]. The real version of this results are presented in e.g. [10, Lemme 3.1] and [13, Ch. III, Sect. 4.4.]. Next we present a powerful Lemma about not–injective changes of variables in integrals, one may consult [9, Thm. 1.6.24] for a real version. The same proof for the real version can be adapted to the complex setting.

Lemma 5.1. *Let g be a polynomial function over $\mathbb{C}$ and let ϕ be a smooth function with compact support in some neighbourhood of the origin of $\mathbb{C}^n$. Assume that g has no zeroes in the support of ϕ. Define for $\mathrm{Re}(s) > 0$, $m = (m_1, \dots, m_n) \in \mathbb{N}^n$ and $\nu = (\nu_1, \dots, \nu_n) \in (\mathbb{N} \setminus \{0\})^n$, the following integral*

$$I(s) = \int_{\mathbb{C}^n} \phi(x) \, x^{2sm+\nu-1} |g(x)|^{2s} \, dx.$$

Then the following assertions hold:

(1) $I(s)$ is convergent and defines a holomorphic function on

$$\mathrm{Re}(s) > \max\{-1, -\nu_1/2m_1, \dots, -\nu_n/2m_n\};$$

(2) $I(s)$ admits a meromorphic continuation to the whole complex plane, with poles of order at most n. Furthermore, the poles belong to

$$\bigcup_{1 \leq i \leq n} \left(-\frac{\nu_i + \mathbb{N}}{2m_i} \right) \cup \left(-\frac{1 + \mathbb{N}}{2} \right).$$

(3) Let κ be a positive integer and let s_0 be a candidate pole of $I(s)$ with $s_0 \notin -(1 + \mathbb{N})/2$ (resp. $s_0 \in -(1 + \mathbb{N})/2$). A necessary condition for s_0 to be a pole of $I(s)$ of order κ, is that

$$Card \left\{ i \; ; \; s_0 \in -\frac{\nu_i + \mathbb{N}}{2m_i} \right\} \geq \kappa \text{ (resp. } \geq \kappa - 1).$$

Lemma 5.2. *Suppose that U and V are open subsets of $\mathbb{C}^n$ and $\Psi : U \to V$ is a map of class $\mathcal{C}^\infty$. For a point $v \in V$, denote by $\#\Psi^{-1}(v)$ the number of $\Psi-$preimages of v. Then for any $\mathcal{C}^\infty$-function $g : V \to \mathbb{C}$ we have*

$$\int_U g(\Psi(u)) \, |\det \Psi'(u)| \, du = \int_V g(v) \#\Psi^{-1}(v) \, dv,$$

provided that both integrals exist. The integrals may not exist, but if any of them exists, then the other exists as well.

5.2. **Poles of Complex Local Zeta Functions.** Recall that for a given $\omega \in (\mathbb{R}_{\geq 0})^n$, the ω-*order* of $f(x)$ is defined as $\nu_\omega(f) := min\{\omega \cdot \mu \; ; \; \mu \in \varepsilon(f)\}$. Now, for any $u = (u_1, \ldots, u_n) \in \mathbb{N}^n \setminus \{\underline{0}\}$ satisfying $\nu_u(f) \neq 0$, we define the following arithmetic progression

$$\mathcal{P}(u) = \left\{ -\frac{||u|| + k}{2\nu_u(f)} \; ; \; k \in \mathbb{N} \right\}.$$

The remoteness of $NP(f)$ (also called by Varchenko the distance from the origin to $NP(f)$) is defined as

$$\nu_0(f) = \min_{u \in Vert(\Sigma(f))} \left\{ \frac{||u||}{2\nu_u(f)} \right\}.$$

From Varchenko's work, we have that the number $\nu_0(f)$ has a nice geometric interpretation: if $(t_0, \ldots, t_0)$ is the intersection point of the diagonal $\{(t, \ldots, t) \in \mathbb{R}^n \; ; \; t \in \mathbb{R}\}$ with the boundary of $NP(f)$, then $\nu_0(f) = 1/t_0$.

Theorem 5.3. *Let f be a polynomial over the complex numbers, satisfying $f(\underline{0}) = 0$. Let $NP(f)$ be the Newton polyhedron of f and let Σ be a simplicial fan subordinated to f. Assume that f is non-degenerate with respect to $NP(f)$, then there exists a neighborhood Ω of the origin such that, for every smooth function ϕ with compact support contained in Ω, the following assertions hold.*

(1) *The function $Z_\phi(s, f)$ is holomorphic on the complex half-plane $\text{Re}(s) > \max\{-\nu_0(f), -1/2\}$.*
(2) *The poles of $Z_\phi(s, f)$ belong to the set*

$$\bigcup_{u \in Vert(\Sigma)} \mathcal{P}(u) \cup \left(-\frac{1 + \mathbb{N}}{2} \right).$$

(3) *Let κ be an integer satisfying $1 \leq \kappa \leq n$, and let s_0 be a candidate pole of $Z_\phi(s, f)$ with $s_0 \notin -(1 + \mathbb{N})/2$ (respectively $s_0 \in -(1 + \mathbb{N})/2$). A necessary condition for s_0 to be a pole of $Z_\phi(s, f)$ of order κ, is that there exists a face $F \subset NP(f)$ of codimension κ (respectively of codimension $\kappa - 1$) such that $s_0 \in \mathcal{P}(u)$ for any facet F_u containing F.*

Proof. We will use the notation of Section 3. Let $\Sigma(f)$ be the dual fan of f and take Σ a simplicial fan subordinated to f, i.e. such that $Vert(\Sigma) = Vert(\Sigma(f))$, see Remark 2.2. Let T_Σ be the toric variety associated to Σ and let $\pi : T_\Sigma \longrightarrow \mathbb{C}^n$ be the toric modification associated to $\Sigma(f)$. The restriction

$$\pi : T_\Sigma{}^* := T_\Sigma \setminus \pi^{-1}(V(x_1 \cdots x_n)) \longrightarrow (\mathbb{C}^*)^n$$

is a diffeomorphism. By using $\pi : T_\Sigma{}^* \to (\mathbb{C}^*)^n$ as a change of variables in the integral

$$Z_\phi(s,f) = \int\limits_{(\mathbb{C}^*)^n \setminus \{f^{-1}(0)\}} \phi(x) \, |f(x)|^{2s} \, |dx|,$$

we have

$$Z_\phi(s,f) = \int\limits_{T_\Sigma{}^* \setminus \{\pi^{-1}(f^{-1}(0))\}} (\phi \circ \pi)(t) \, |(f \circ \pi)(t)|^{2s} \, |J_\pi(t)| \, |dt|,$$

where $|dt|$ is a volume element in $T_\Sigma{}^*$ and J_π denotes the Jacobian matrix of π.

Since $\pi : T_\Sigma \longrightarrow \mathbb{C}^n$ is proper, the support of $\phi \circ \pi$ as a function over T_Σ is compact. We will take $\{\xi_\sigma\}_{\sigma \in \mathfrak{M}(\Sigma)}$ a $\mathcal{C}^\infty$−partition of the unity of T_Σ subordinated to the covering $\{T_\sigma\}_{\sigma \in \mathfrak{M}(\Sigma)}$ (see subsection 2.5). Then

$$(8) \qquad Z_\phi(s,f) = \sum_{\sigma \in \mathfrak{M}(\Sigma)} \int_{T_\Sigma{}^* \setminus \{\pi^{-1}(f^{-1}(0))\}} (\phi \circ \pi)(t)\xi_\sigma(t) \, |(f \circ \pi)(t)|^{2s} \, |J_\pi(t)| \, |dt|.$$

Note that the function $\vartheta_\sigma := (\phi \circ \pi)\xi_\sigma$, is a $\mathcal{C}^\infty$−function with compact support contained in the chart T_σ. For every $\sigma \in \mathfrak{M}(\Sigma)$, we take

$$Z_\phi(s,f)^\sigma := \int\limits_{T_\sigma{}^* \setminus \{\pi_\sigma^{-1}(f^{-1}(0))\}} \vartheta_\sigma(t) \, |(f \circ \pi_\sigma)(t)|^{2s} \, |J_{\pi_\sigma}(t)| \, |dt|.$$

In view of this notation, (8) becomes:

$$(9) \qquad\qquad Z_\phi(s,f) = \sum_{\sigma \in \mathfrak{M}(\Sigma)} Z_\phi(s,f)^\sigma.$$

Now our task is to compute $Z_\phi(s,f)^\sigma$. To do so, take a cone $\sigma = Cone(\mathrm{N}) \in \mathfrak{M}(\Sigma)$ and set M to be the matrix with $\sigma^\vee = Cone(\mathrm{M})$, and such that $\Lambda := \mathrm{N}^{\mathbf{t}} \mathrm{M}$ is diagonal. Recall from Definition 2.4, that $\Phi_\sigma : \mathbb{C}^n \longrightarrow T_\sigma$ is given by $\Phi_\sigma(x) = (\psi_\Lambda(x), \psi_{\mathrm{N}^{\mathbf{t}}}(x), \psi_{\mathrm{N}^{\mathbf{t}}W}(x))$. The restriction of Φ_σ to $(\mathbb{C}^*)^n$ is a finite (locally diffeomorphism) covering, lets say of degree d. By Lemma 5.2,

$$Z_\phi(s,f)^\sigma =$$

$$\frac{1}{d} \int\limits_{(\mathbb{C}^*)^n \setminus \{(f \circ \pi_\sigma \circ \Phi_\sigma)^{-1}(0)\}} \vartheta_\sigma(\Phi_\sigma(x)) \, |(f \circ \pi_\sigma)(\Phi_\sigma(x))|^{2s} \, |J_{\pi_\sigma}(\Phi_\sigma(x))| \, |J_{\Phi_\sigma}(x)| \, |dx|$$

$$= \frac{1}{d} \int\limits_{(\mathbb{C}^*)^n \setminus \{(f \circ \pi_\sigma \circ \Phi_\sigma)^{-1}(0)\}} (\vartheta_\sigma \circ \Phi_\sigma)(x) \, |(f \circ \pi_\sigma \circ \Phi_\sigma)(x)|^{2s} \, |J_{(\pi_\sigma \circ \Phi_\sigma)}(x)| \, |dx|$$

$$\overset{(3)}{=} \frac{1}{d} \int\limits_{(\mathbb{C}^*)^n \setminus \{(f \circ \psi_{\mathrm{N}^{\mathbf{t}}})^{-1}(0)\}} (\vartheta_\sigma \circ \Phi_\sigma)(x) \, |(f \circ \psi_{\mathrm{N}^{\mathbf{t}}})(x)|^{2s} \, |J_{\psi_{\mathrm{N}^{\mathbf{t}}}}(x)| \, |dx|.$$

By Lemma 2.3,

$$Z_\phi(s,f)^\sigma$$

$$= \frac{1}{d} \int\limits_{(\mathbb{C}^*)^n \smallsetminus \{(f \circ \psi_{\mathrm{N}^{\mathbf{t}}})^{-1}(0)\}} (\vartheta_\sigma \circ \Phi_\sigma)(x) \, |(f \circ \psi_{\mathrm{N}^{\mathbf{t}}})(x)|^{2s} \, |\det \mathrm{N}^{\mathbf{t}}| \prod_{i=1}^n |x_i|^{||u^{(i)}||-1} \, |dx|$$

$$= \frac{1}{d} \int\limits_{(\mathbb{C}^*)^n \smallsetminus \{(f \circ \psi_{\mathrm{N}^{\mathbf{t}}})^{-1}(0)\}} \tilde\vartheta_\sigma(x) \, |(f \circ \psi_{\mathrm{N}^{\mathbf{t}}})(x)|^{2s} \prod_{i=1}^n |x_i|^{||u^{(i)}||-1} \, |dx|,$$

where $\tilde\vartheta_\sigma := |\det \mathrm{N}^{\mathbf{t}}| (\vartheta_\sigma \circ \Phi_\sigma)$. Since ϑ_σ has compact support and Φ_σ is a smooth and finite covering, $\tilde\vartheta_\sigma : \mathbb{C}^n \longrightarrow \mathbb{C}$ is a smooth function with compact support.

In the next step we use the knowledge that we have about the term $f \circ \psi_{\mathrm{N}^{\mathbf{t}}}$. Consider a point $z \in V(x_1 \cdots x_n) \cap Supp(\tilde\vartheta_\sigma)$ and let U_z be a neighbourhood of z as in Corollary 4.8. Note that

$$V(x_1 \cdots x_n) \cap Supp(\tilde\vartheta_\sigma) \subset \bigcup_{z \in V(x_1 \cdots x_n) \cap Supp(\tilde\vartheta_\sigma)} U_z,$$

which implies, since $Supp(\tilde\vartheta_\sigma)$ is compact, that there exists a finite subset $W \subseteq V(x_1 \cdots x_n) \cap Supp(\tilde\vartheta_\sigma)$, with

$$V(x_1 \cdots x_n) \cap Supp(\tilde\vartheta_\sigma) \subset \bigcup_{z \in W} U_z.$$

By shrinking $Supp(\phi)$ if necessary, we may assume that $Supp(\tilde\vartheta_\sigma) \subset \bigcup_{z \in W} U_z$ and then we have to deal with integrals of type $Z_\phi(s,f)^\sigma$ over U_z.

Since U_z has been chosen as in Corollary 4.8, there exist: a set of indices $J_z \subseteq \{1, \ldots, n\}$, a marked index $k \in \{1, \ldots, n\} \setminus J_z$, a neighbourhood $\widetilde{U}_z$ of $\underline{0}$ and a diffeomorphism $\eta_z : (\widetilde{U}_z, \underline{0}) \longrightarrow (U_z, z)$ such that

$$f \circ \psi_{\mathrm{N}^{\mathbf{t}}} \circ \eta_z = \prod_{j \in J_z} x_j{}^{\alpha_j} \tilde h \quad \text{or} \quad f \circ \psi_{\mathrm{N}^{\mathbf{t}}} \circ \eta_z = \prod_{j \in J_z} x_j{}^{\alpha_j} x_k \tilde h,$$

where $\tilde h(x) \neq 0$ for all $x \in \widetilde{U}_z$.

By using η_z as a change of variables in our integral $Z_\phi(s,f)^\sigma$ over U_z, we get that $Z_\phi(s,f)^\sigma$ becomes a finite sum of integrals of types $I_1(s)$ and $I_2(s)$, where

$$I_1(s) =$$

$$\int\limits_{\widetilde{U}_z} \left| \left(\prod_{j \in J_z} x_j{}^{\nu_{u(j)}(f)} \right) \tilde h(x) \right|^{2s} \prod_{j \in J_z} |x_j|^{||u^{(j)}||-1} \prod_{j \notin J_z} |(\eta_z)_j|^{||u^{(j)}||-1} \theta_{\sigma,z}(x) \, |J_{\eta_z}(x)| |dx|,$$

and

$$I_2(s) =$$

$$\int\limits_{\widetilde{U}_z} \left| \left(\prod_{j \in J_z} x_j{}^{\nu_{u(j)}(f)} \right) x_l \tilde h(x) \right|^{2s} \prod_{j \in J_z} |x_j|^{||u^{(j)}||-1} \prod_{j \notin J_z} |(\eta_z)_j|^{||u^{(j)}||-1} \theta_{\sigma,z}(x) \, |J_{\eta_z}(x)| |dx|.$$

As the reader may have guessed, we have set $\theta_{\sigma,z} = \tilde{\vartheta}_\sigma \circ \eta_z$. Finally, note that

$$(10) \qquad I_1(s) = \int_{\widetilde{U}_z} \prod_{j \in J_z} |x_j|^{2s\nu_{u^{(j)}}(f) + ||u^{(j)}|| - 1} |\tilde{h}(x)|^{2s} \tilde{\theta}_{\sigma,z}(x) \ |dx|,$$

and

$$(11) \qquad I_2(s) = \int_{\widetilde{U}_z} \prod_{j \in J_z} |x_j|^{2s\nu_{u^{(j)}}(f) + ||u^{(j)}|| - 1} |x_k \tilde{h}(x)|^{2s} \tilde{\theta}_{\sigma,z}(x) \ |dx|.$$

All the assertions of our theorem follows now from (9) and Lemma 5.1 applied to (10) and (11). $\qquad\square$

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Universidad Nacional Autónoma de México, Unidad Cuernavaca. A.P.273-3 C.P. 62251. Cuernavaca, MOR. México.

E-mail address: `fuen@matcuer.unam.mx`

Universidad Nacional Autónoma de México, Unidad Cuernavaca. A.P.273-3 C.P. 62251. Cuernavaca, MOR. México.

E-mail address: `m.gomez@im.unam.mx`

CONACYT – Centro de Investigación en Matemáticas, Unidad Zacatecas. Quantum Ciudad del Conocimiento. Avenida Lasec, Andador Galileo Galilei, Manzana 3 Lote 7. C.P. 98160. Zacatecas, ZAC. México.

E-mail address: `edwin.leon@cimat.mx`