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ON ZEROS OF PERIODIC ZETA FUNCTIONS

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We consider zeta functions $\zeta(s;\mathfrak{a})$ given by Dirichlet series with multiplicative periodic coefficients and prove that, for some classes of functions F, the functions $F(\zeta(s;\mathfrak{a}))$ have infinitely many zeros in the critical strip. For example, this is true for $\sin(\zeta(s;\mathfrak{a}))$.

1. Introduction

The distribution of zeros of zeta functions is of particular interest in the analytic number theory and, in general, in mathematics. The most important problems are related to the Riemann zeta function $\zeta(s),\ s=\sigma+it$, defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},$$

and analytically continued to the entire complex plane, except a simple pole at the point $s=1$ with residue 1. It is well known that $s=-2m,\ m\in\mathbb{N}$, are so-called trivial zeros of $\zeta(s)$. Moreover, $\zeta(s)\neq 0$, for $\sigma\geq 1$ and for $\sigma\leq 0,\ t\neq 0$. However, the function $\zeta(s)$ has infinitely many complex (nontrivial) zeros in the critical strip $\{s\in\mathbb{C}\colon 0<\sigma<1\}$. The famous Riemann hypothesis (RH) says that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma=\dfrac{1}{2}$. This is equivalent to the nonvanishing of $\zeta(s)$ in the half plane $\left\{s\in\mathbb{C}\colon \sigma>\dfrac{1}{2}\right\}$. The last known result on zero-free regions for $\zeta(s)$ can be formulated as follows: There exists an absolute constant $c>0$ such that $\zeta(s)\neq 0$ in the region

$$\left\{s\in\mathbb{C}\colon \sigma\geq 1-\frac{c}{(\log(|t|+2))^{2/3}(\log\log(|t|+2))^{1/3}}\right\}.$$

G. H. Hardy proved [1] that infinitely many nontrivial zeros lie on the critical line. This result was improved by A. Selberg, N. Levinson, and B. Conrey. The last result in this direction says [2] that at least 41% of all nontrivial zeros of $\zeta(s)$ in a sense of of density are on the critical line. The results of numerical calculations also support the RH: The first 10^{13} nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma=\dfrac{1}{2}$ [3].

A natural generalization of the function $\zeta(s)$ is the periodic zeta function. Let $\mathfrak{a}=\{a_m\colon m\in\mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $k\in\mathbb{N}$. The periodic zeta function $\zeta(s;\mathfrak{a})$ is defined, for $\sigma>1$, by the series

$$\zeta(s;\mathfrak{a})=\sum_{m=1}^{\infty}\frac{a_m}{m^s}.$$

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Moreover, the function $\zeta(s;\mathfrak{a})$ can be analytically continued to the entire complex plane. Indeed, let $\zeta(s,\alpha)$ denote the Hurwitz zeta function with parameter $\alpha,\ 0<\alpha\leq 1,$ given by the series

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s}$$

for $\sigma>1$ and by the analytic continuation everywhere outside this region except a simple pole at $s=1$ with residue 1. Then the periodicity of the sequence $\mathfrak{a}$ implies, for $\sigma>1,$ the equality

$$\zeta(s;\mathfrak{a})=\frac{1}{k^s}\sum_{l=1}^{k}a_l\zeta\left(s,\frac{l}{k}\right).$$

Therefore, by virtue of the remarks made above, the last equality gives the analytic continuation for $\zeta(s;\mathfrak{a})$ to the entire complex plane. If

$$a\overset{\text{df}}{=}\frac{1}{k}\sum_{l=1}^{k}a_l\neq 0,$$

then the function $\zeta(s;\mathfrak{a})$ has a simple pole at $s=1$ with residue a; otherwise, the function $\zeta(s;\mathfrak{a})$ is an entire function.

Obviously, if $a_1=1$ and $k=1,$ then $\zeta(s;\mathfrak{a})=\zeta(s).$

We use the notation

$$a_m^{\pm}=\frac{1}{k}\sum_{l=1}^{k}a_l\exp\left\{\pm 2\pi il\frac{m}{k}\right\}$$

and $\mathfrak{a}^{\pm}=\{a_m^{\pm}\colon m\in\mathbb{N}\}$. Then the sequences of complex numbers $\mathfrak{a}^{\pm}$ are also periodic with period k. In [4], it was proved that the function $\zeta(s;\mathfrak{a})$ satisfies the functional equation

$$\zeta(1-s;\mathfrak{a})=\left(\frac{k}{2\pi}\right)^s\Gamma(s)\left(\exp\left\{\frac{\pi is}{2}\right\}\zeta(s;\mathfrak{a}^-)+\exp\left\{-\frac{\pi is}{2}\right\}\zeta(s;\mathfrak{a}^+)\right),$$

where $\Gamma(s)$, as usual, stands for the Euler gamma function.

In [5], J. Steuding began to study the distribution of zeros for the function $\zeta(s;\mathfrak{a})$. Denote the zeros of the function $\zeta(s;\mathfrak{a})$ by $\rho=\beta+i\gamma$. Moreover, let

$$c_{\mathfrak{a}}=\max(|a_m|\colon 1\leq m\leq k),\qquad m_{\mathfrak{a}}=\min\{1\leq m\leq k\colon a_m\neq 0\},$$

and

$$A(\mathfrak{a})=\frac{m_{\mathfrak{a}}c_{\mathfrak{a}}}{|a_{m_{\mathfrak{a}}}|}.$$

Thus, it was established in [5] that $\zeta(s;\mathfrak{a})\neq 0$ for $\sigma>1+A(\mathfrak{a})$.

Now let

$$\hat{a}_m^{\pm}=\frac{1}{\sqrt{k}}\sum_{l=1}^{k}a_l\exp\left\{\pm 2\pi il\frac{m}{k}\right\},$$

$\hat{\mathfrak{a}}^{\pm} = \{\hat{a}_m^{\pm} \colon m \in \mathbb{N}\}$, and $B(\mathfrak{a}) = \max\{A(\hat{\mathfrak{a}}^{\pm})\}$. Hence, in [5], it was obtained that the function $\zeta(s;\mathfrak{a})$ with $\sigma < -B(\mathfrak{a})$ may have only zeros close to the negative real axis for $m_{\hat{\mathfrak{a}}^+} = m_{\hat{\mathfrak{a}}^-}$ and zeros close to the line

$$\sigma = 1 + \frac{\pi t}{\log \dfrac{m_{\hat{\mathfrak{a}}^-}}{m_{\hat{\mathfrak{a}}^+}}}$$

for $m_{\hat{\mathfrak{a}}^+} \neq m_{\hat{\mathfrak{a}}^-}$. The zeros ρ of $\zeta(s;\mathfrak{a})$ with $\beta < -B(\mathfrak{a})$ are called trivial and the other zeros of $\zeta(s;\mathfrak{a})$ are called nontrivial. Hence, the nontrivial zeros lie in the strip $-B(\mathfrak{a}) \leq \sigma \leq 1 + A(\mathfrak{a})$.

An asymptotic formula for the number of nontrivial zeros ρ of $\zeta(s;\mathfrak{a})$ with $|\gamma| \leq T$ was also obtained in [5]. It was proved that the nontrivial zeros of $\zeta(s;\mathfrak{a})$ are clustered around the critical line.

Suppose that $k > 2$, a_m is not a multiple of the Dirichlet character $\bmod k$ and $a_m = 0$ for $(m,k) > 1$. In this case, it was observed in [6, p. 223] that $\zeta(s;\mathfrak{a})$ has infinitely many zeros in the strip

$$D = \left\{ s \in \mathbb{C} \colon \frac{1}{2} < \sigma < 1 \right\}.$$

Note that, in this case, the sequence $\mathfrak{a}$ is nonmultiplicative (we recall that $\mathfrak{a}$ is multiplicative if $a_1 = 1$ and $a_{mn} = a_m a_n$ for all $m, n \in \mathbb{N}$ $(m,n) = 1$), and the function $\zeta(s;\mathfrak{a})$ does not have the Euler product over the primes.

Our aim is to consider the case of multiplicative sequence $\mathfrak{a}$ and prove that the function $F(\zeta(s;\mathfrak{a}))$ with some F has infinitely many zeros in the strip D. In other words, we construct composite functions for the zeta functions with Euler product such that the RH is not true. This is motivated by the better understanding of the RH problem.

Let G be a region in the complex plane. By $H(G)$ we denote the space of analytic functions on G equipped with the topology of uniform convergence on compact sets. We define some classes of functions $F \colon H(G) \to H(G)$ for certain regions G. Let $V > 0$ be an arbitrary fixed number, let

$$D_V = \left\{ s \in \mathbb{C} \colon \frac{1}{2} < \sigma < 1,\ |t| < V \right\},$$

and let

$$S_V = \{ g \in H(D_V) \colon g(s) \neq 0 \text{ or } g(s) \equiv 0 \}.$$

By U_V we denote the class of continuous functions $F \colon H(D_V) \to H(D_V)$ such that, for any polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S_V$ is nonempty.

It is easy to see that the function

$$F(g) = \sum_{k=1}^{r} c_k g^{(k)}, \qquad g \in H(D_V), \quad c_1, \ldots, c_r \in \mathbb{C} \setminus \{0\},$$

where $g^{(k)}$ is the kth derivative of g, is an element of the class U_V. Indeed, for any polynomial $p(s)$ of degree k, there exists a polynomial $\hat{p}(s)$ of degree $k+1$ such that $\hat{p}(s) \neq 0$ for $s \in D_V$ and $F(\hat{p}) = p$.

Let

$$S = \{ g \in H(D) \colon g(s) \neq 0 \text{ or } g(s) \equiv 0 \}.$$

We now introduce a class of functions F for which the image $F(S)$ is a subset of $H(D)$. For $a_1, \ldots, a_r \in \mathbb{C}$, by $U_{a_1,\ldots,a_r}$ we denote the class of continuous functions $F\colon H(D) \to H(D)$ such that $F(S) \supset H_{a_1,\ldots,a_r;F(0)}(D)$, where

$$H_{a_1,\ldots,a_r;F(0)}(D) = \{g \in H(D)\colon g(s) \neq a_j,\ j = 1, \ldots, r\} \cup \{F(0)\}.$$

Thus, the functions

$$F(g) = \sin g, \quad F(g) = \cos g, \quad F(g) = \sinh g, \quad \text{and} \quad F(g) = \cosh g$$

belong to the class $U_{-1,1}$. To see this, it suffices to solve the equation $F(g) = f$ in $g \in S$. For $F(g) = \cos g$, we get

$$\frac{\mathrm{e}^{ig} + \mathrm{e}^{-ig}}{2} = f.$$

Hence,

$$g_{\pm} = \frac{1}{i} \log\Big(f \pm \sqrt{f^2 - 1}\Big).$$

This means that, for $f \in H_{-1,1;1}(D)$, we can choose, say, the solution g_+, which belongs to S. Therefore, $F \in U_{-1,1}$.

Our last class is very simple. We say that a continuous function $F\colon H(D) \to H(D)$ belongs to the class U, if $s - a \in F(S)$ for all $a \in \left(\frac{1}{2}, 1\right)$.

It is easy to see that the function $F(g) = gg'$, $g \in H(D)$, belongs to the class U. Indeed, as a result of the solution of the equation $gg' = s - a$, we find

$$g = \pm\sqrt{s^2 - 2as + C}$$

with arbitrary constant C. We can choose C such that $s^2 - 2as + C \neq 0$ for $s \in D$. Thus, there exists $g \in S$ satisfying $F(g) = s - a$.

We are now ready to state the theorems on zeros of the function $F(\zeta(s; \mathfrak{a}))$. In the notation used in the Introduction, we suppose that $c_{\mathfrak{a}} < \sqrt{2} - 1$. Note that this inequality implies, for all primes p, the inequality

$$\sum_{\alpha=1}^{\infty} \frac{|a_{p^\alpha}|}{p^{\alpha/2}} \leq c < 1. \tag{1}$$

Theorem 1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative and such that the inequality $c_{\mathfrak{a}} < \sqrt{2} - 1$ is satisfied and that F belongs to at least one of the classes U_V and U with sufficiently large V. Then, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, \mathfrak{a}, F) > 0$ such that, for sufficiently large T (in the case of the class U_V, it is supposed that $T < V$), the function $F(\zeta(s; \mathfrak{a}))$ has more than cT zeros in the rectangle $\{s \in \mathbb{C}\colon \sigma_1 < \sigma < \sigma_2,\ 0 < t < T\}$.*

Theorem 2. *Suppose that $\mathfrak{a}$ is the same sequence as in Theorem 1 and $F \in U_{a_1,\ldots,a_r}$, where $\operatorname{Re} a_j \notin \left(-\frac{1}{2}, \frac{1}{2}\right)$, $j = 1, \ldots, r$. Then the same assertion as in Theorem 1 is true.*

To prove Theorems 1 and 2, we apply the universality property of the function $\zeta(s; \mathfrak{a})$.

2. Universality

The universality property of zeta functions was discovered by S. M. Voronin in 1975. In [8], he proved that the Riemann zeta function $\zeta(s)$ is universal in a sense that its shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions. The last version of the Voronin theorem is contained in the following theorem (see, e.g., [9, p. 225]). Denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$ by $\operatorname{meas}\{A\}$.

Theorem 3. *Let $K \subset D$ be a compact set with connected complement and let $f(s)$ be a continuous nonvanishing function on K analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\operatorname{meas}\left\{\tau \in [0,T]\colon \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

The first result on the universality of the function $\zeta(s;\mathfrak{a})$ was obtained in [10, p. 145]; see also [11]. We state a more general case given in [6, p. 219].

Theorem 4. *Suppose that $k > 2$, a_m is not a multiple of a Dirichlet character $\bmod k$, and $a_m = 0$ for $(m,k) > 1$. Let $K \subset D$ be a compact set with connected complement and let $f(s)$ be a continuous function on K analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\operatorname{meas}\left\{\tau \in [0,T]\colon \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

Note that the sequence $\mathfrak{a}$ in Theorem 4 is not multiplicative. The general case was discussed in [12] and [13]. We now recall the universality theorem for $\zeta(s;\mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$ from [14].

Theorem 5. *Suppose that the sequence $\mathfrak{a}$ is multiplicative and inequality (1) holds. Let K and $f(s)$ be the same as in Theorem 3. Then the same assertion as in Theorem 4 is true.*

Since $f(s)$ is nonvanishing on K, Theorem 5 does not give any information on zeros of the function $\zeta(s;\mathfrak{a})$.

In [7], the first author began to study the universality of $F(\zeta(s;\mathfrak{a}))$ for some classes of functions F. In theorems obtained in the cited paper, the shifts $F(\zeta(s+i\tau;\mathfrak{a}))$ approximate analytic functions not necessarily nonvanishing. Hence, the theorems of this kind give information on the distribution of zeros for the function $F(\zeta(s;\mathfrak{a}))$. To prove Theorems 1 and 2, we use the following universality statements:

Lemma 1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative and inequality (1) holds, $K \subset D$ is a compact set with connected complement, and $f(s)$ is a continuous function on K analytic in the interior of K. Let $V > 0$ be such that $K \subset D_V$ and let $F \in U_V$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\operatorname{meas}\left\{\tau \in [0,T]\colon \sup_{s\in K} |F(\zeta(s+i\tau;\mathfrak{a})) - f(s)| < \varepsilon\right\} > 0.$$

The proof of the lemma can be found in [7].

We now formulate a universality theorem for functions from the class $U_{a_1,\ldots,a_r}$.

Lemma 2. *Suppose that $\mathfrak{a}$ is the same as in Lemma 1 and $F \in U_{a_1,\ldots,a_r}$. For $r = 1$, let $K \subset D$ be a compact set with connected complement and let $f(s)$ be a continuous function on K unequal to a_1 and analytic in the interior of K. For $r \geq 2$, let $K \subset D$ be an arbitrary compact set and let $f \in H_{a_1,\ldots,a_r;F(0)}(D)$. Then the same assertion as in Lemma 1 is true.*

Note that the universality of $F(\zeta(s;\mathfrak{a}))$ with F satisfying a stronger condition $F(S) = H_{a_1,\dots,a_r;F(0)}(D)$ was considered in [7].

Lemma 3. *Suppose that* $\mathfrak{a}$ *is the same as in Lemma 1,* $K \subset D$ *is a compact subset, and* $f \in F(S)$. *Then the same assertion as in Lemma 1 is true.*

Lemmas 2 and 3 are obtained from the limit theorem on weak convergence of probability measures in the space $H(D)$ [14] and from the Mergelyan theorem on the approximation of analytic functions by polynomials [15]; see also [16, p. 436].

3. Remarks to Theorems 1 and 2

Theorems 1 and 2 are, in fact, corollaries of the classical Rouché theorem; see, e.g., [17, p. 246] and Lemmas 1, 3, and 2, respectively.

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