4. VECTOR SPACE

4.1 VECTOR SPACES AND SUBSAPCES

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

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3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.

   unique
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

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7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

For each $\mathbf{u}$ in V and scalar c,

$$0\mathbf{u} = \mathbf{0} \tag{1}$$

$$c\mathbf{0} = \mathbf{0} \tag{2}$$

$$-\mathbf{u} = (-1)\mathbf{u} \tag{3}$$

Examples

The spaces $\mathbb{R}^n$, where $n \geq 1$

examples

EXAMPLE 4 For $n \geq 0$, the set $\mathbb{P}_n$ of polynomials of degree at most n consists of all polynomials of the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \tag{4}$$

where the coefficients $a_0, \ldots, a_n$ and the variable t are real numbers.

examples

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If $\mathbf{p}$ is given by (4) and if $\mathbf{q}(t) = b_0 + b_1 t + \cdots + b_n t^n$, then the sum $\mathbf{p} + \mathbf{q}$ is defined by

$$\begin{aligned}(\mathbf{p} + \mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t) \\ &= (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n\end{aligned}$$

The scalar multiple $c\mathbf{p}$ is the polynomial defined by

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = ca_0 + (ca_1)t + \cdots + (ca_n)t^n$$

examples

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$\mathbf{f}+\mathbf{g}$ is the function whose value at t in the domain $\mathbb{D}$ is $\mathbf{f}(t)+\mathbf{g}(t)$.

Likewise, for a scalar c and an $\mathbf{f}$ in V, the scalar multiple $c\mathbf{f}$ is the function whose value at t is $c\mathbf{f}(t)$. For instance, if $\mathbb{D}=\mathbb{R}$, $\mathbf{f}(t)=1+\sin 2t$, and $\mathbf{g}(t)=2+.5t$, then

$$(\mathbf{f}+\mathbf{g})(t)=3+\sin 2t+.5t \quad \text{and} \quad (2\mathbf{g})(t)=4+t$$

subspaces

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A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.[2]

b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

examples

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R2 and R3

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R2 and R3

R_2 is not even a subset of R_3

examples

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$$

Subspace of R3

examples

A plane in $\mathbb{R}^3$

a line in $\mathbb{R}^2$

A Subspace Spanned by a Set

EXAMPLE 10 Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V.

A Subspace Spanned by a Set

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The zero vector is in H

take two arbitrary vectors in H, say,

$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) \\ &= (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2 \end{aligned}$$

$$c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1)\mathbf{v}_1 + (cs_2)\mathbf{v}_2$$

A Subspace Spanned by a Set

THEOREM 1 If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

A Subspace Spanned by a Set

THEOREM 1 If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

We call Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Given any subspace H of V, a **spanning** (or **generating**) **set** for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in H such that $H =$ Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

EXAMPLE 11 Let H be the set of all vectors of the form $(a - 3b, b - a, a, b)$, where a and b are arbitrary scalars. That is, let $H = \{(a - 3b, b - a, a, b) : a \text{ and } b \text{ in } \mathbb{R}\}$. Show that H is a subspace of $\mathbb{R}^4$.

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$$\begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} = a \underset{\substack{\uparrow \\ \mathbf{v}_1}}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}} + b \underset{\substack{\uparrow \\ \mathbf{v}_2}}{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$$

An $n \times n$ matrix A is said to be symmetric if $A^T = A$. Let S be the set of all 3×3 symmetric matrices. Show that S is a subspace of $M_{3\times3}$, the vector space of 3×3 matrices.

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a. Observe that the $\mathbf{0}$ in $M_{3\times3}$ is the 3×3 zero matrix and since $\mathbf{0}^t = \mathbf{0}$, the matrix $\mathbf{0}$ is symmetric and hence $\mathbf{0}$ is in S.

b. Let A and B in S. Notice that A and B are 3×3 symmetric matrices so $A^T = A$ and $B^T = B$. By the properties of transposes of matrices, $(A + B)^T = A^T + B^T = A + B$. Thus $A + B$ is symmetric and hence $A + B$ is in S.

c. Let A be in S and let c be a scalar. Since A is symmetric, by the properties of symmetric matrices, $(cA)^T = c(A^T) = cA$. Thus cA is also a symmetric matrix and hence cA is in S.

4.2 NULL SPACE, COLUMN SPACE AND LINEAR TRANSFORMATION

Null space

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

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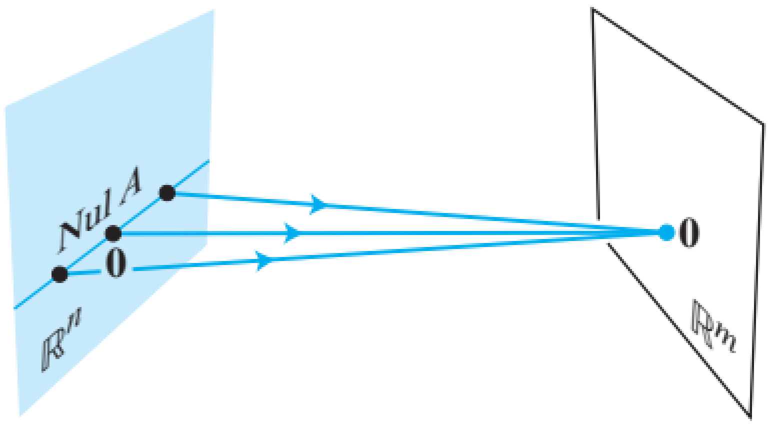

example

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

Determine if **u** belongs to the null space of A

example

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Determine if **u** belongs to the null space of A

$$A\mathbf{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 - 9 + 4 \\ -25 + 27 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

THEOREM 2

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

proof

EXAMPLE 2 Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates a, b, c, d satisfy the equations $a - 2b + 5c = d$ and $c - a = b$. Show that H is a subspace of $\mathbb{R}^4$.

An Explicit Description of Nul A

There is no obvious relation between vectors in Nul A and the entries in A. We say that Nul A is defined *implicitly*, because it is defined by a condition that must be checked.

EXAMPLE 3 Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 - 2x_2 \qquad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

$x_1 = 2x_2 + x_4 - 3x_5,\ x_3 = -2x_4 + 2x_5$, with x_2, x_4, and x_5

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

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$$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$$

$\boxed{\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \text{ is a spanning set for Nul } A.}$

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$$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$$

1. Linearly independent
2. Number of free variables

Column Space of a Matrix

DEFINITION

The **column space** of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\,\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n\,]$, then

$$\text{Col } A = \text{Span}\,\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$$

THEOREM 3

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$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}}$$

range of the linear transformation A**x**.

example

EXAMPLE 4 Find a matrix A such that $W = \text{Col } A$.

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$$

Column space and null space

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

let $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

a. Determine if $\mathbf{u}$ is in Nul A. Could $\mathbf{u}$ be in Col A?

b. Determine if $\mathbf{v}$ is in Col A. Could $\mathbf{v}$ be in Nul A?

Column space and null space

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$$A\mathbf{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Column space and null space

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a. Determine if $\mathbf{u}$ is in Nul A. Could $\mathbf{u}$ be in Col A?

b. Determine if $\mathbf{v}$ is in Col A. Could $\mathbf{v}$ be in Nul A?

$$[\,A \quad \mathbf{v}\,] = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

Column space and null space

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\, A \quad \mathbf{0} \,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Linear transformation

DEFINITION

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

The **kernel** (or **null space**) of such a T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$ (the zero vector in W). The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V

example

EXAMPLE 8 (Calculus required) Let V be the vector space of all real-valued functions f defined on an interval $[a, b]$ with the property that they are differentiable and their derivatives are continuous functions on $[a, b]$. Let W be the vector space $C[a, b]$ of all continuous functions on $[a, b]$, and let $D : V \to W$ be the transformation that changes f in V into its derivative f'. In calculus, two simple differentiation rules are

$$D(f + g) = D(f) + D(g) \quad \text{and} \quad D(cf) = cD(f)$$

That is, D is a linear transformation. It can be shown that the kernel of D is the set of constant functions on $[a, b]$ and the range of D is the set W of all continuous functions on $[a, b]$. ■

4.3. LINEARLY INDEPENDENT SET, BASES

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An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has *only* the trivial solution, $c_1 = 0, \dots, c_p = 0$.[1]

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THEOREM 4

An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

EXAMPLE 1 Let $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, and $\mathbf{p}_3(t) = 4 - t$.

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$\{\sin t \cos t, \sin 2t\}$

EXAMPLE 7 Let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}, \quad \text{and} \quad H = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, and show that Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ = Span $\{\mathbf{v}_1, \mathbf{v}_2\}$.

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Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, and show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Not efficient

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(5\mathbf{v}_1 + 3\mathbf{v}_2) \\ &= (c_1 + 5c_3)\mathbf{v}_1 + (c_2 + 3c_3)\mathbf{v}_2 \end{aligned}$$

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \operatorname{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$$

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EXAMPLE 4 Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the columns of the $n \times n$ identity matrix, I_n. That is,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

EXAMPLE 3 Let A be an invertible $n \times n$ matrix—say, $A = [\, \mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n \,]$.

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the columns of A form a basis for $\mathbb{R}^n$ because they are linearly independent and they span $\mathbb{R}^n$, by the Invertible Matrix Theorem. ■

EXAMPLE 6 Let $S = \{1, t, t^2, \ldots, t^n\}$. Verify that S is a basis for $\mathbb{P}_n$. This basis is called the **standard basis** for $\mathbb{P}_n$.

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SOLUTION Certainly S spans $\mathbb{P}_n$. To show that S is linearly independent, suppose that $c_0, \dots, c_n$ satisfy

$$c_0 \cdot 1 + c_1 t + c_2 t^2 + \cdots + c_n t^n = \mathbf{0}(t) \tag{2}$$

THEOREM 5

The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and let $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

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b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

$$\mathbf{v}_p = a_1\mathbf{v}_1 + \cdots + a_{p-1}\mathbf{v}_{p-1} \tag{3}$$

Given any $\mathbf{x}$ in H, we may write

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_{p-1}\mathbf{v}_{p-1} + c_p\mathbf{v}_p \tag{4}$$

for suitable scalars $c_1, \dots, c_p$. Substituting the expression for $\mathbf{v}_p$ from (3) into (4), it is easy to see that $\mathbf{x}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}$. Thus $\{\mathbf{v}_1, \dots, \mathbf{v}_{p-1}\}$ spans H, because $\mathbf{x}$ was an arbitrary element of H.

b. If the original spanning set S is linearly independent, then it is already a basis for H. Otherwise, one of the vectors in S depends on the others and can be deleted, by part (a). So long as there are two or more vectors in the spanning set, we can repeat this process until the spanning set is linearly independent and hence is a basis for H. If the spanning set is eventually reduced to one vector, that vector will be nonzero (and hence linearly independent) because $H \neq \{\mathbf{0}\}$. ■

Bases for Nul A and Col A

Bases for Nul A?

Bases for Nul A and Col A

Bases for Nul A?

Bases for Col A ?

EXAMPLE 8 Find a basis for Col B, where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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SOLUTION Each nonpivot column of B is a linear combination of the pivot columns. In fact, $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$. By the Spanning Set Theorem, we may discard $\mathbf{b}_2$ and $\mathbf{b}_4$, and $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span Col B. Let

$$S = \{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- Any linear dependence relationship among the columns of A can be expressed in the form Ax =0, where x is a column of weights
- When A is row reduced to a matrix B, the columns of B are often totally different from the columns of A
- However, the equations A**x** **=0** and B**x** =**0** have exactly the same set of solutions.

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However, the equations A**x =0** and B**x** =**0** have exactly the same set of solutions.

$A = [\,\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n\,]$ and $B = [\,\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n\,]$, then the vector equations

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0} \quad \text{and} \quad x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n = \mathbf{0}$$

also have the same set of solutions. That is, the columns of A have *exactly the same linear dependence relationships* as the columns of B.

EXAMPLE 9 It can be shown that the matrix

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix B in Example 8. Find a basis for Col A.

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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SOLUTION In Example 8 we saw that

$$\mathbf{b}_2 = 4\mathbf{b}_1 \quad \text{and} \quad \mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$$

so we can expect that

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$$\boxed{\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}}$$

so we can expect that

$$\mathbf{a}_2 = 4\mathbf{a}_1 \quad \text{and} \quad \mathbf{a}_4 = 2\mathbf{a}_1 - \mathbf{a}_3$$

THEOREM 6

The pivot columns of a matrix A form a basis for Col A.

Two Views of a Basis

a basis is a spanning set that is as small as possible

A basis is also a linearly independent set that is as large as possible.

3. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$. Then every vector in H is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for H?

4. Let V and W be vector spaces, let $T : V \to W$ and $U : V \to W$ be linear transformations, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a basis for V. If $T(\mathbf{v}_j) = U(\mathbf{v}_j)$ for every value of j between 1 and p, show that $T(\mathbf{x}) = U(\mathbf{x})$ for every vector $\mathbf{x}$ in V.

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4. Since $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for V, for any vector $\mathbf{x}$ in V, there exist scalars $c_1, \dots, c_p$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$. Then since T and U are linear transformations

$$\begin{aligned} T(\mathbf{x}) &= T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) \\ &= c_1U(\mathbf{v}_1) + \cdots + c_pU(\mathbf{v}_p) = U(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) \\ &= U(\mathbf{x}) \end{aligned}$$

4.4 COORDINATE SYSTEMS

THEOREM 7

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

proof

DEFINITION

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of x**) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

$$[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of x (relative to $\mathcal{B}$)**, or the **$\mathcal{B}$-coordinate vector of x**. The mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$ is the **coordinate mapping (determined by $\mathcal{B}$)**.[1]

EXAMPLE 1 Consider a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$, where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Suppose an $\mathbf{x}$ in $\mathbb{R}^2$ has the coordinate vector $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\mathbf{x}$.

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SOLUTION The $\mathcal{B}$-coordinates of $\mathbf{x}$ tell how to build $\mathbf{x}$ from the vectors in $\mathcal{B}$. That is,

$$\mathbf{x} = (-2)\mathbf{b}_1 + 3\mathbf{b}_2 = (-2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

■

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■

EXAMPLE 2 The entries in the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the coordinates of $\mathbf{x}$ relative to the *standard basis* $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, since

Coordinates in Rn

EXAMPLE 4 Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\,\mathbf{x}\,]_{\mathcal{B}}$ of $\mathbf{x}$ relative to $\mathcal{B}$.

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SOLUTION The $\mathcal{B}$-coordinates c_1, c_2 of $\mathbf{x}$ satisfy

$$c_1 \underset{\mathbf{b}_1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + c_2 \underset{\mathbf{b}_2}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \underset{\mathbf{x}}{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}$$

or

$$\begin{array}{c} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\ \mathbf{b}_1 \quad \mathbf{b}_2 \end{array} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underset{\mathbf{x}}{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}$$

$$[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

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General case Rn

$$[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Coordinates in Rn

a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$.

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

Coordinates in Rn

a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$.

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

$$P_{\mathcal{B}} = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n\,]$$

$$\boxed{\mathbf{x} = P_{\mathcal{B}}[\,\mathbf{x}\,]_{\mathcal{B}}}$$

change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$

Coordinates in Rn

a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$.

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

$$P_{\mathcal{B}} = [\, \mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n \,]$$

$$\boxed{\mathbf{x} = P_{\mathcal{B}}[\,\mathbf{x}\,]_{\mathcal{B}}}$$

change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$

$$\boxed{P_{\mathcal{B}}^{-1}\mathbf{x} = [\,\mathbf{x}\,]_{\mathcal{B}}}$$

correspondence $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$, produced here by $P_{\mathcal{B}}^{-1}$, is the coordinate mapping

Coordinate Mapping

Choosing a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for a vector space V introduces a coordinate system in V. The coordinate mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$ connects the possibly unfamiliar space V to the familiar space $\mathbb{R}^n$.

THEOREM 8

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

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$$\mathbf{u} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$
$$\mathbf{w} = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n$$

$$\mathbf{u} + \mathbf{w} = (c_1 + d_1)\mathbf{b}_1 + \cdots + (c_n + d_n)\mathbf{b}_n$$

$$[\,\mathbf{u} + \mathbf{w}\,]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\,\mathbf{u}\,]_{\mathcal{B}} + [\,\mathbf{w}\,]_{\mathcal{B}}$$

$$r\mathbf{u} = r(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n) = (rc_1)\mathbf{b}_1 + \cdots + (rc_n)\mathbf{b}_n$$

$$[\,r\mathbf{u}\,]_{\mathcal{B}} = \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix} = r\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = r[\,\mathbf{u}\,]_{\mathcal{B}}$$

If $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are in V and if $c_1, \ldots, c_p$ are scalars, then

$$[\,c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p\,]_{\mathcal{B}} = c_1[\,\mathbf{u}_1\,]_{\mathcal{B}} + \cdots + c_p[\,\mathbf{u}_p\,]_{\mathcal{B}}$$

one-to-one linear transformation from a vector space V onto a vector space W is called an **isomorphism** from V onto W

Every vector space calculation in V *is accurately reproduced in* W, *and vice versa*.

EXAMPLE 5 Let $\mathcal{B}$ be the standard basis of the space $\mathbb{P}_3$ of polynomials; that is, let $\mathcal{B} = \{1, t, t^2, t^3\}$. A typical element $\mathbf{p}$ of $\mathbb{P}_3$ has the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

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$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$[\,\mathbf{p}\,]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

an isomorphism from $\mathbb{P}_3$ onto $\mathbb{R}^4$

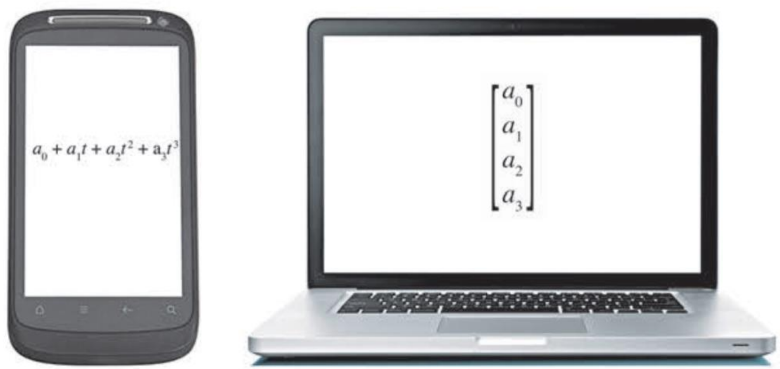

FIGURE 6 The space $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$

EXAMPLE 6 Use coordinate vectors to verify that the polynomials $1 + 2t^2$, $4 + t + 5t^2$, and $3 + 2t$ are linearly dependent in $\mathbb{P}_2$.

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$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a vector space V with a basis *B* containing n vectors is isomorphic to R^n

DIMENSION OF A VECTOR SPACE

a vector space V with a basis *B* containing n vectors is isomorphic to R^n. This section shows that this number n is an intrinsic property (called the dimension) of the space V that does not depend on the particular choice of basis.

THEOREM 9

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

generalizes a well-known result about the vector space R^n

THEOREM 9 If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

PROOF Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a set in V with more than n vectors.

$[\,\mathbf{u}_1\,]_{\mathcal{B}}, \ldots, [\,\mathbf{u}_p\,]_{\mathcal{B}}$ form a linearly dependent set in $\mathbb{R}^n$, because there are more vectors (p) than entries (n) in each vector. So there exist scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1[\,\mathbf{u}_1\,]_{\mathcal{B}} + \cdots + c_p[\,\mathbf{u}_p\,]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{The zero vector in } \mathbb{R}^n$$

$$[\,c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p\,]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

THEOREM 10

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

THEOREM 10

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

PROOF Let $\mathcal{B}_1$ be a basis of n vectors and $\mathcal{B}_2$ be any other basis (of V). Since $\mathcal{B}_1$ is a basis and $\mathcal{B}_2$ is linearly independent, $\mathcal{B}_2$ has no more than n vectors, by Theorem 9. Also, since $\mathcal{B}_2$ is a basis and $\mathcal{B}_1$ is linearly independent, $\mathcal{B}_2$ has at least n vectors. Thus $\mathcal{B}_2$ consists of exactly n vectors. ■

DEFINITION If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

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$$\dim \mathbb{R}^n$$

$$\dim \mathbb{P}_n$$

$$\mathbb{P}$$

Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

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$$\dim H = 2$$

EXAMPLE 4 The subspaces of $\mathbb{R}^3$ can be classified by dimension.

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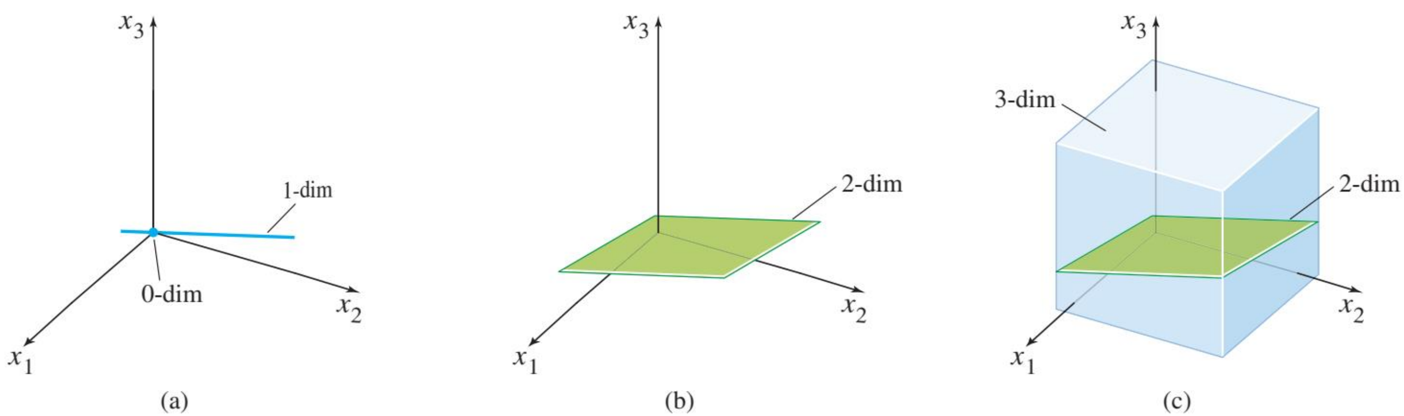

THEOREM 11

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

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$S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be any linearly independent set in H. If S spans H, then S is a basis for H.

$$\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$$

number of vectors in a linearly independent expansion of S can never exceed the dimension of V

THEOREM 12

The Basis Theorem

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

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PROOF By Theorem 11, a linearly independent set S of p elements can be extended to a basis for V. But that basis must contain exactly p elements, since $\dim V = p$. So S must already be a basis for V. Now suppose that S has p elements and spans V. Since V is nonzero, the Spanning Set Theorem implies that a subset S' of S is a basis of V. Since $\dim V = p$, S' must contain p vectors. Hence $S = S'$. ■

The Dimensions of Nul A and Col A

The dimension of Nul A is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col A is the number of pivot columns in A.

EXAMPLE 5 Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three free variables—x_2, x_4, and x_5. Hence the dimension of Nul A is 3. Also, dim Col $A = 2$ because A has two pivot columns. ■

Here V is a nonzero finite-dimensional vector space.

a. If $\dim V = p$ and if S is a linearly dependent subset of V, then S contains more than p vectors.

b. If S spans V and if T is a subset of V that contains more vectors than S, then T is linearly dependent.

RANK

Row space

- ✓ A is an m*n matrix
- ✓ set of all linear combinations of the row vectors is called the **row space** of A and is denoted by Row A.
- ✓ Each row has n entries, so Row A is a subspace of R_n.
- ✓ Since the rows of A are identified with the columns of AT , we could also write Col AT in place of Row A.

Row space

EXAMPLE 1 Let

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad \text{and} \quad \begin{aligned} \mathbf{r}_1 &= (-2, -5, 8, 0, -17) \\ \mathbf{r}_2 &= (1, 3, -5, 1, 5) \\ \mathbf{r}_3 &= (3, 11, -19, 7, 1) \\ \mathbf{r}_4 &= (1, 7, -13, 5, -3) \end{aligned}$$

The row space of A is the subspace of $\mathbb{R}^5$ spanned by $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$. That is, Row A = Span $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$. It is natural to write row vectors horizontally; however, they may also be written as column vectors if that is more convenient. ■

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Bases for row space, independent rows, row reduction??

THEOREM 13

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

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If B is obtained from A by row operations, the rows of B are linear combinations of the rows of A. It follows that any linear combination of the rows of B is automatically a linear combination of the rows of A. Thus the row space of B is contained in the row space of A. Since row operations are reversible, the same argument shows that the row space of A is a subset of the row space of B. So the two row spaces are the same. If B is in echelon form, its nonzero rows are linearly independent because no nonzero row is a linear combination of the nonzero rows below it. the nonzero rows of B form a basis of the (common) row space of B and A.

EXAMPLE 2 Find bases for the row space, the column space, and the null space of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Basis for Col A: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$

Basis for Row A: $\{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)\}$

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_3 - x_5, x_2 = 2x_3 - 3x_5, x_4 = 5x_5$, with x_3 and x_5 free variables.

$$\text{Basis for Nul } A: \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

Warning: Although the first three rows of B in Example 2 are linearly independent, it is wrong to conclude that the first three rows of A are linearly independent. (In fact, the third row of A is 2 times the first row plus 7 times the second row.) Row operations may change the linear dependence relations among the *rows* of a matrix.

DEFINITION

The **rank** of A is the dimension of the column space of A.

THEOREM 14

The Rank Theorem

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n$$

EXAMPLE 3

a. If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A?

b. Could a 6×9 matrix have a two-dimensional null space?

Invertible matrices

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

Invertible matrices

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.
n. $\text{Col } A = \mathbb{R}^n$
o. $\dim \text{Col } A = n$
p. $\text{rank } A = n$
q. $\text{Nul } A = \{\mathbf{0}\}$
r. $\dim \text{Nul } A = 0$

The matrices below are row equivalent.

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find rank A and dim Nul A.

2. Find bases for Col A and Row A.

3. What is the next step to perform to find a basis for Nul A?

4. How many pivot columns are in a row echelon form of A^T?

4.7 CHANGE OF BASES

we study how $[\,\mathbf{x}\,]_{\mathcal{C}}$ and $[\,\mathbf{x}\,]_{\mathcal{B}}$ are related for each $\mathbf{x}$ in V

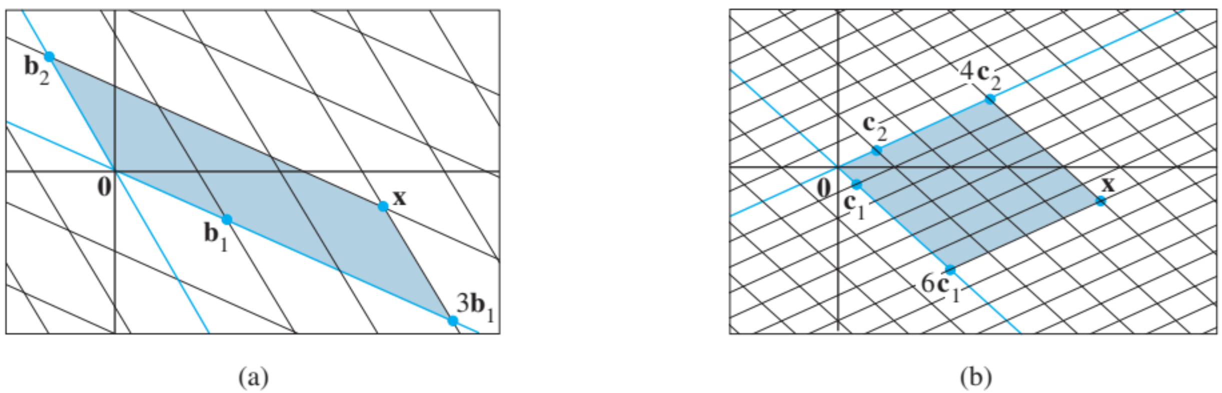

FIGURE 1 Two coordinate systems for the same vector space.

EXAMPLE 1 Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2 \tag{1}$$

Suppose

$$\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2 \tag{2}$$

That is, suppose $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\,\mathbf{x}\,]_{\mathcal{C}}$.

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Suppose

$$\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2 \tag{2}$$

That is, suppose $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\,\mathbf{x}\,]_{\mathcal{C}}$.

$$\begin{aligned} [\,\mathbf{x}\,]_{\mathcal{C}} &= [\,3\mathbf{b}_1 + \mathbf{b}_2\,]_{\mathcal{C}} \\ &= 3[\,\mathbf{b}_1\,]_{\mathcal{C}} + [\,\mathbf{b}_2\,]_{\mathcal{C}} \end{aligned}$$

$$[\,\mathbf{x}\,]_{\mathcal{C}} = \begin{bmatrix} [\,\mathbf{b}_1\,]_{\mathcal{C}} & [\,\mathbf{b}_2\,]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$[\,\mathbf{x}\,]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\,\mathbf{x}\,]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\,\mathbf{x}\,]_{\mathcal{B}} \tag{4}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \tag{5}$$

change-of-coordinates matrix from *B* to *C*

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ such that

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change-of-coordinates matrix from *B* to *C*

$$(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P})^{-1}[\,\mathbf{x}\,]_{\mathcal{C}} = [\,\mathbf{x}\,]_{\mathcal{B}}$$

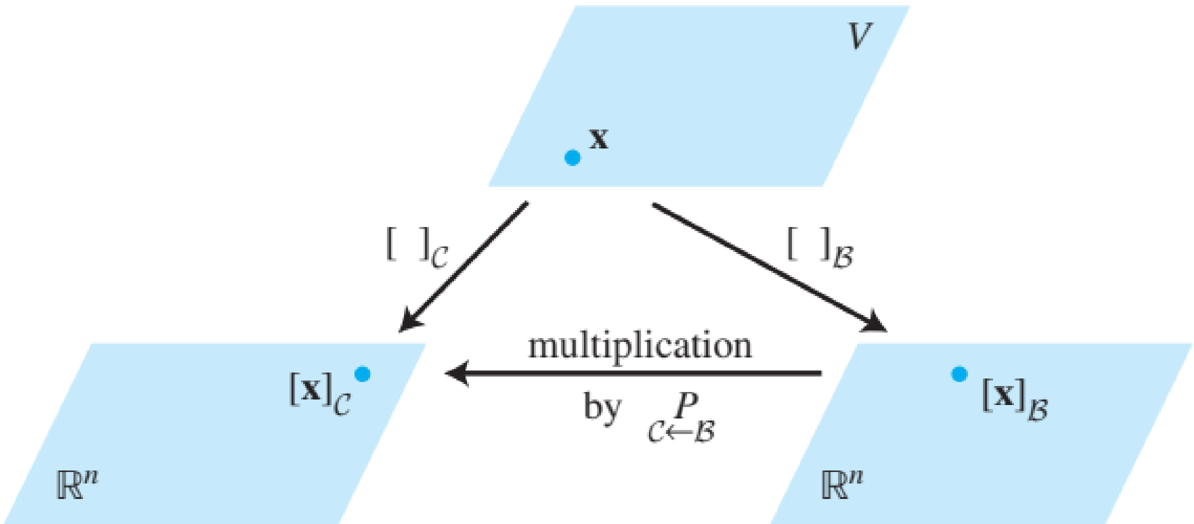

FIGURE 2 Two coordinate systems for V.

Change of Basis in $\mathbb{R}^n$

If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{E}$ is the *standard basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, then $[\mathbf{b}_1]_{\mathcal{E}} = \mathbf{b}_1$, and likewise for the other vectors in $\mathcal{B}$. In this case, $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ is the same as the change-of-coordinates matrix $P_{\mathcal{B}}$ introduced in Section 4.4, namely,

$$P_{\mathcal{B}} = [\, \mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n \,]$$

EXAMPLE 2 Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

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$[\,\mathbf{b}_1\,]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\,\mathbf{b}_2\,]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then, by definition,

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}_1 \quad \text{and} \quad \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{b}_2$$

$$\left[\begin{array}{cc:cc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

Thus

$$[\,\mathbf{b}_1\,]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \quad \text{and} \quad [\,\mathbf{b}_2\,]_{\mathcal{C}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\,\mathbf{b}_1\,]_{\mathcal{C}} & [\,\mathbf{b}_2\,]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

$$\boxed{\left[\ \mathbf{c}_1 \quad \mathbf{c}_2 \ \middle\vdots\ \mathbf{b}_1 \quad \mathbf{b}_2\ \right] \sim \left[\ I \ \middle\vdots\ \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\ \right]}$$

$$P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}, \quad P_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \mathbf{x}, \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x}$$

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$