MATRIX EQUATION

Matrix Equation

view a linear combination of vectors as the product of a matrix and a vector

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Matrix Equation

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{6}$$

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

example

EXAMPLE 3 Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

Existence of a solution

Theorem: Let A be an $m \times n$ matrix. Then the following four statements are all mathematically equivalent.

1. For each b in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.
2. Each b in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^m$
4. A has a pivot position in every row.

First: 1, 2, 3 are mathematically equivalent

Proof

- So, it suffices to show (for an arbitrary matrix A) that (1) is true iff (4) is true, i.e., that (1) and (4) are either both true or false.
- Given $\boldsymbol{b}$ in $\mathbb{R}^m$, we can row reduce the augmented matrix $[\boldsymbol{A}|\boldsymbol{b}]$ to reduced row echelon form $[\boldsymbol{U}|\boldsymbol{d}]$.
- If statement (4) is true, then each row of $\boldsymbol{U}$ contains a pivot position, and so $\boldsymbol{d}$ cannot be a pivot column.
- So $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution for any $\boldsymbol{b}$, and (1) is true.

Proof

- If (4) is false, then the last row of U is all zeros.
- Let d be any vector with a 1 in its last entry. Then $[U|d]$ represents an inconsistent system.
- The new system $Ax = b$ is also inconsistent, and (1) is false.

example

EXAMPLE 4 Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}$$

Row--Vector Rule for Computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

SOLUTION SETS OF LINEAR SYSTEMS

Homogeneous Linear Systems

Goal: uses vector notation to give explicit and geometric descriptions of such solution sets

Homogeneous Linear Systems

$$A\mathbf{x} = \mathbf{0}$$

$\mathbf{x} = \mathbf{0}$ **trivial solution**

whether there exists a **nontrivial solution**

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Example

Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \qquad - \tfrac{4}{3}x_3 &= 0 \\ x_2 \qquad\quad &= 0 \\ 0 &= 0 \end{aligned}$$

Example

parametric vector form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\mathbf{v}, \quad \text{where } \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

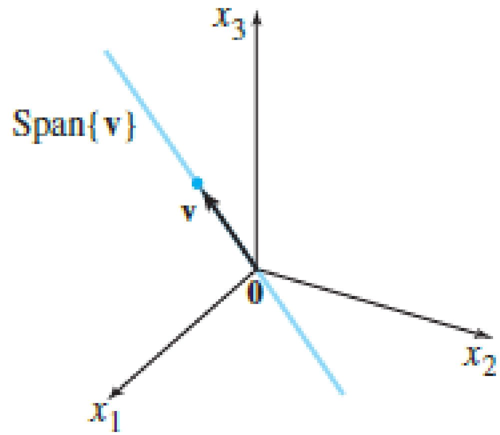


FIGURE 1

Example

Describe all solutions of the homogeneous “system”

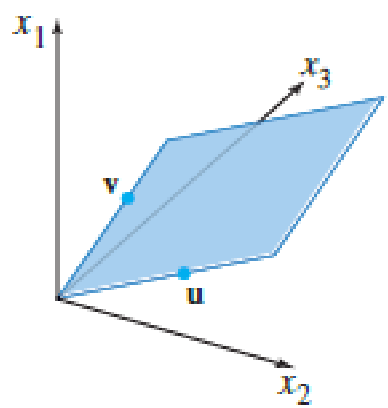
parametric vector form

$$10x_1 - 3x_2 - 2x_3 = 0$$

$x_1 = .3x_2 + .2x_3.$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \underbrace{\begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}} \quad \text{(with } x_2, x_3 \text{ free)}$$

Span $\{\mathbf{u}, \mathbf{v}\}$



Solution set of a homogeneous equation A**x=0** can always be expressed explicitly as Span $\{\mathbf{v}_1; : : : ; \mathbf{v}_p\}$ for suitable vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

Nonhomogeneous Systems

When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system

WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM

1. Row reduce the augmented matrix to reduced echelon form.

2. Express each basic variable in terms of any free variables appearing in an equation.

3. Write a typical solution **x** as a vector whose entries depend on the free variables, if any.

4. Decompose **x** into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Nonhomogeneous Systems

Example: Describe all solutions of A**x** = **b**, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 \quad - \tfrac{4}{3}x_3 &= -1 \\ x_2 \qquad &= 2 \\ 0 &= 0 \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\mathbf{p}} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}}$$

$$\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$$

parametric vector form

parametric
vector form

$$\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$$

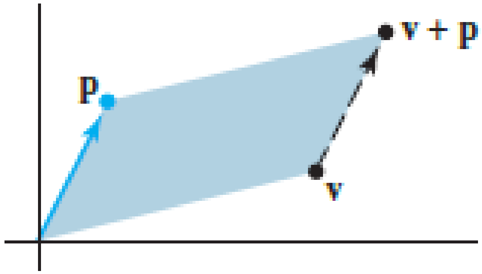
$$\boxed{\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R})}$$

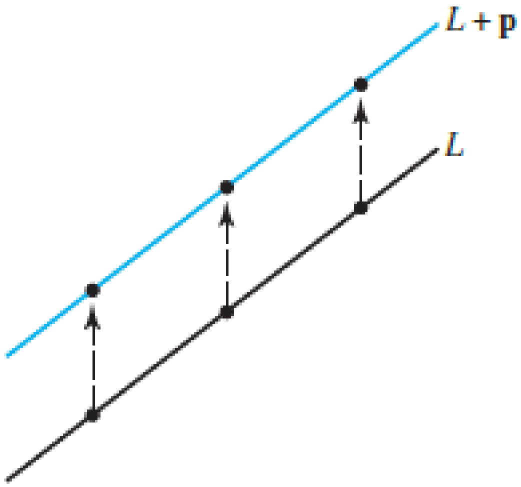
solution set of A**x** = **0** has the parametric vector equation

$$\boxed{\mathbf{x} = t\mathbf{v} \quad (t \text{ in } \mathbb{R})}$$

vector **p** itself is just one particular solution of A**x** = **b**

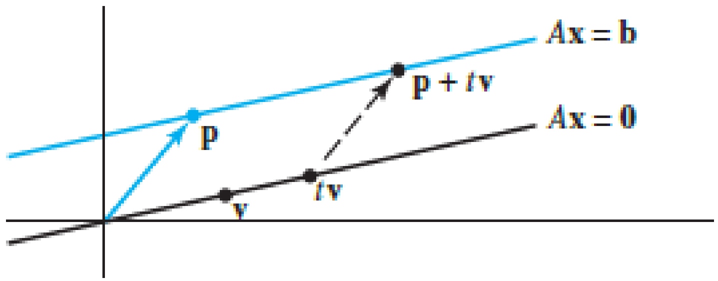
Geometric Descriptions



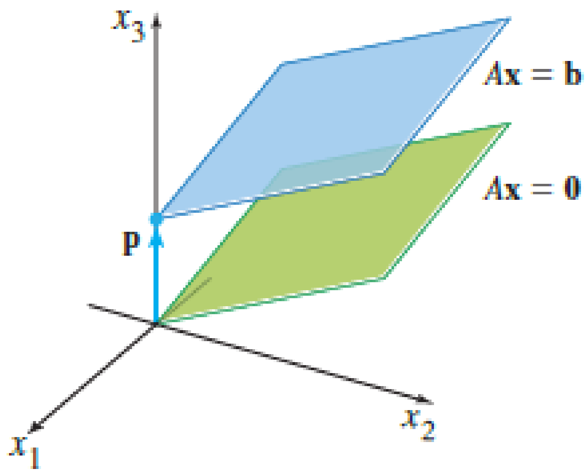


Geometric Descriptions

$$\boxed{\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R})}$$



two free variables



Theorem

THEOREM 6

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

LINEAR INDEPENDENCE & MATRIX EQUATIONS

Linear independency

Studying Linear dependency ⟺ Studying a homogeneous linear system

Linear Independence of Matrix Columns

$$A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n] \qquad x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

$$A\mathbf{x} = \mathbf{0}$$

The columns of a matrix A are linearly independent if and only if the equation A$\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

Example

EXAMPLE 2 Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

Linear independency

An indexed set S of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

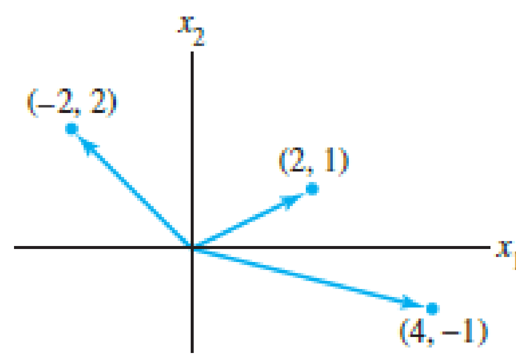
every vector in a linearly dependent set is a linear combination of the preceding vectors.

THEOREM 8 If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Proof : ?

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \longrightarrow \text{Linear dependent} \longleftarrow$$



Linear independency

Theorem 8 says nothing about the case in which the number of vectors in the set does *not* exceed the number of entries in each vector.

THEOREM 9

If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

APPLICATIONS OF LINEAR SYSTEMS: NETWORK FLOW

Network Flow

flow of some quantity through a network:

- pattern of traffic flow in a grid of city streets
- flow through electrical circuits
- …

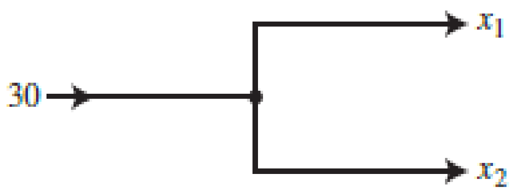
- A network consists of a set of points called junctions, or nodes, with lines or arcs called branches connecting some or all of the junctions.
- The direction of flow in each branch is indicated, and the flow amount (or rate) is either shown or is denoted by a variable.

basic assumption of network flow is that the total flow into the network equals the total flow out of the network

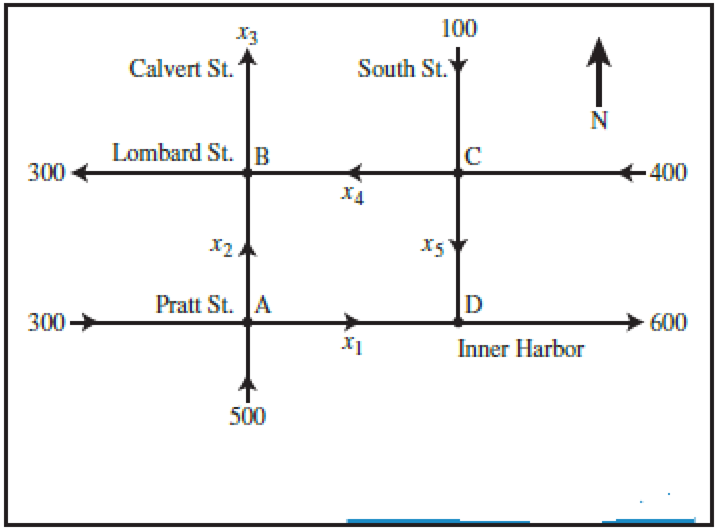
&

total flow into a junction equals the total flow out of the junction

Examples



$$x_1 + x_2 = 30$$



Intersection	Flow in		Flow out
A	$300 + 500$	$=$	$x_1 + x_2$
B	$x_2 + x_4$	$=$	$300 + x_3$
C	$100 + 400$	$=$	$x_4 + x_5$
D	$x_1 + x_5$	$=$	600

$$x_3 = 400$$

Example

$$\begin{aligned}
x_1 + x_2 \quad\quad\quad\quad\quad &= 800 \\
x_2 - x_3 + x_4 \quad\quad &= 300 \\
x_4 + x_5 &= 500 \\
x_1 \quad\quad\quad\quad\quad + x_5 &= 600 \\
x_3 \quad\quad\quad\quad &= 400
\end{aligned}$$

$$\begin{aligned}
x_1 \quad\quad\quad\quad + x_5 &= 600 \\
x_2 \quad\quad\quad - x_5 &= 200 \\
x_3 \quad\quad &= 400 \\
x_4 + x_5 &= 500
\end{aligned}$$

$$\begin{cases}
x_1 = 600 - x_5 \\
x_2 = 200 + x_5 \\
x_3 = 400 \\
x_4 = 500 - x_5 \\
x_5 \text{ is free}
\end{cases}$$

INTRODUCTION TO LINEAR TRANSFORMATIONS

Introduction to linear mappings

- A transformation or function or mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to every x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

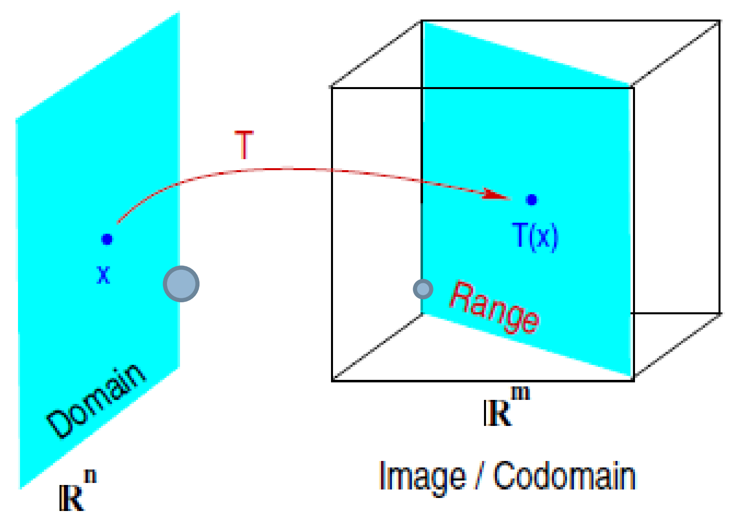
- $\mathbb{R}^n$ is called the domain space of T and $\mathbb{R}^m$ the image space or co-domain of T.

- Notation:

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

- $T(x)$ is the image of x under T

set of all images T(x) is called the **range** of T



example

Example: Take the mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow T(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \\ x_1^2 + x_2^2 \end{pmatrix}$$

difference

Example: Another mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow T(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 + 5x_2 \end{pmatrix}$$

Introduction to linear mappings

Definition A mapping T is linear if:

(i) $T(u+v) = T(u) + T(v)$ for u, v in the domain of T
(ii) $T(\alpha u) = \alpha T(u)$ for all $\alpha \in \mathbb{R}$, all u in the domain of T

➤ If a mapping is linear then $T(0) = 0$. (Why?)

Observation: A mapping is linear if and only if

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

for all scalars α, β and all u, v in the domain of T.

Introduction to linear mappings

- Given an $m \times n$ matrix A, consider the special mapping:

$$\begin{aligned} T: \quad \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ x &\longrightarrow y = Ax \end{aligned}$$

Domain $==$??; Image space $==$?? Linear ?

THEOREM 10

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

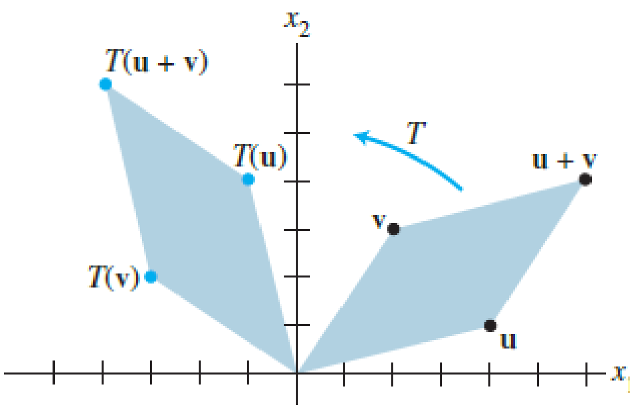
$$A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \tag{3}$$

proof

example

EXAMPLE 3 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle. Find the standard matrix A of this transformation.

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$



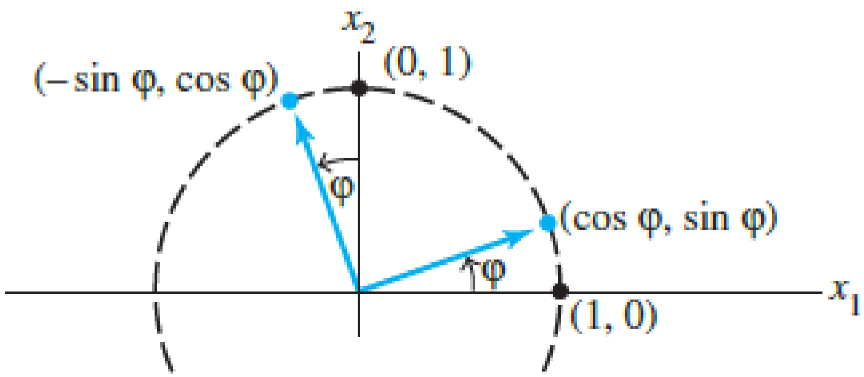
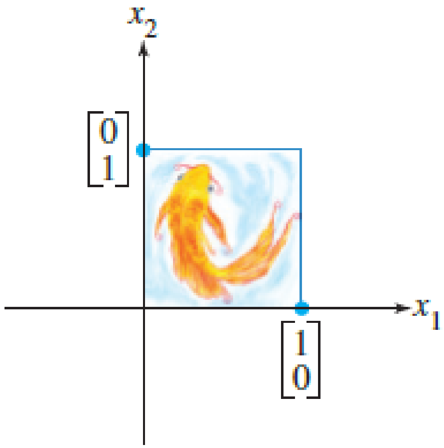
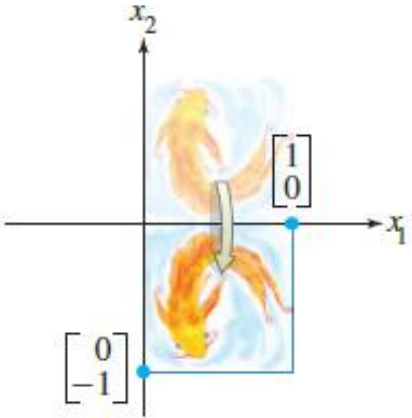


FIGURE 1 A rotation transformation.

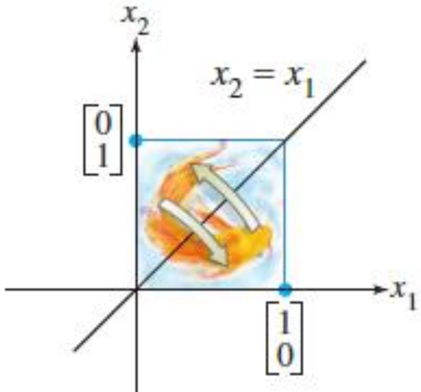
Geometric Linear Transformations of R^2



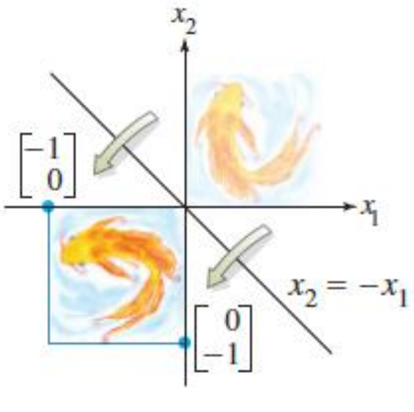
$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$



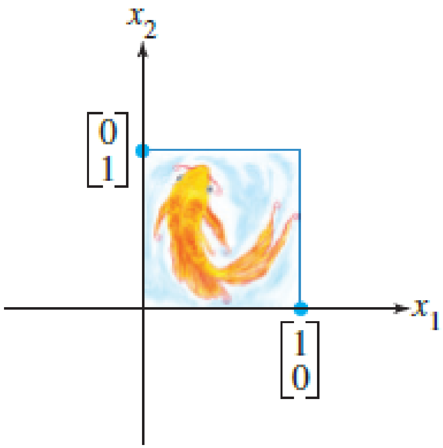
$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$



$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$

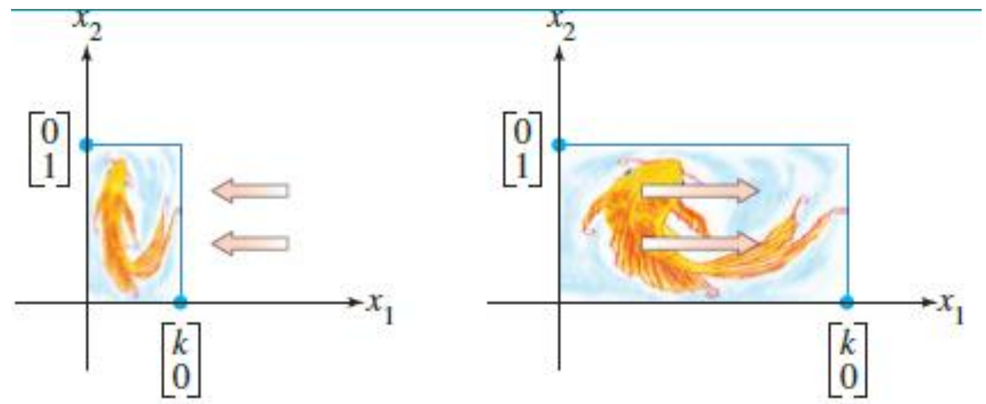


Geometric Linear Transformations of R^2



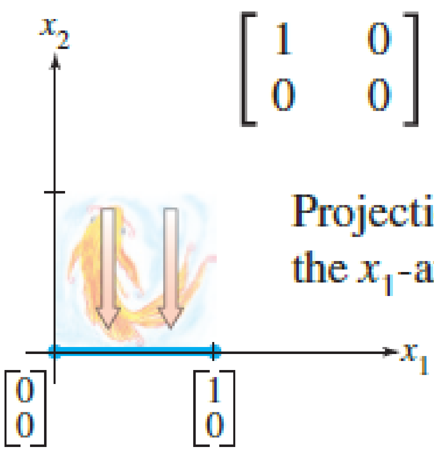
$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Horizontal contraction and expansion



$0 < k < 1$ $\qquad$ $k > 1$

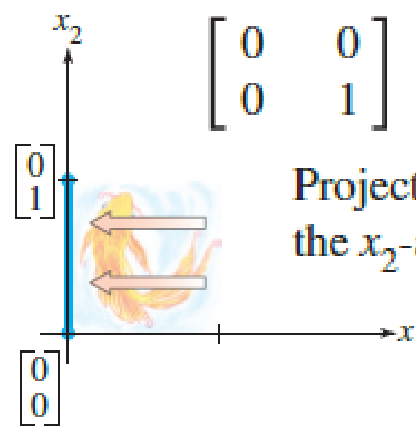
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Projection onto the x_1-axis

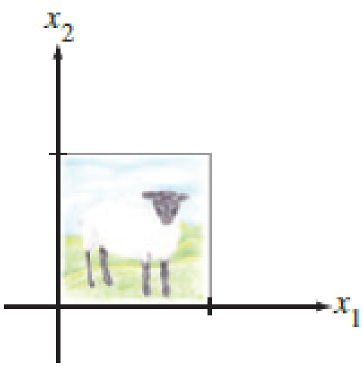
Linear Transform?

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



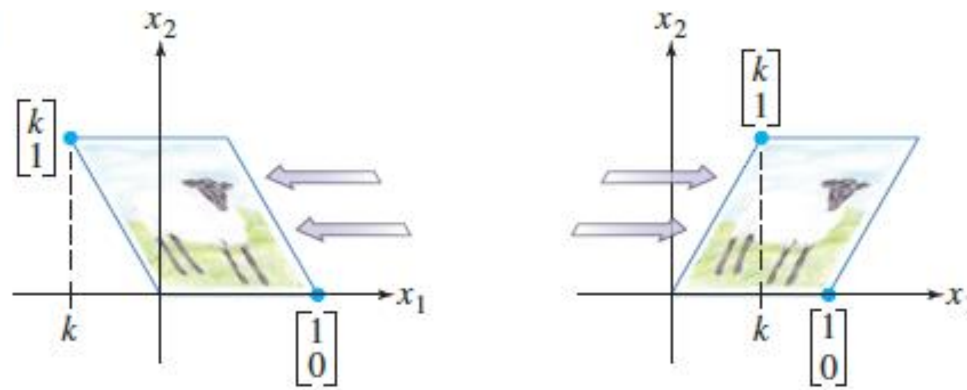
Projection onto the x_2-axis

Geometric Linear Transformations of R^2



Horizontal shear

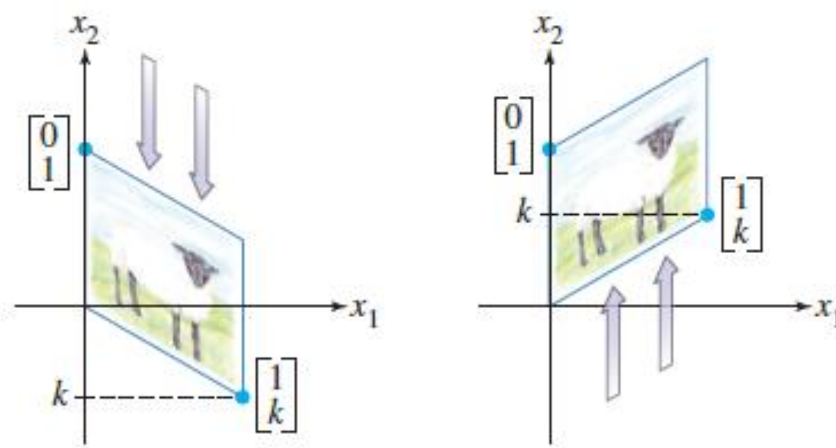
$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$



$k < 0$ $k > 0$

Vertical shear

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

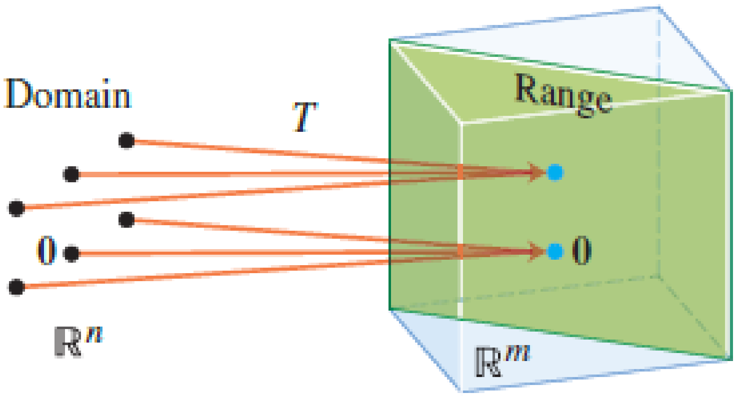


$k < 0$ $k > 0$

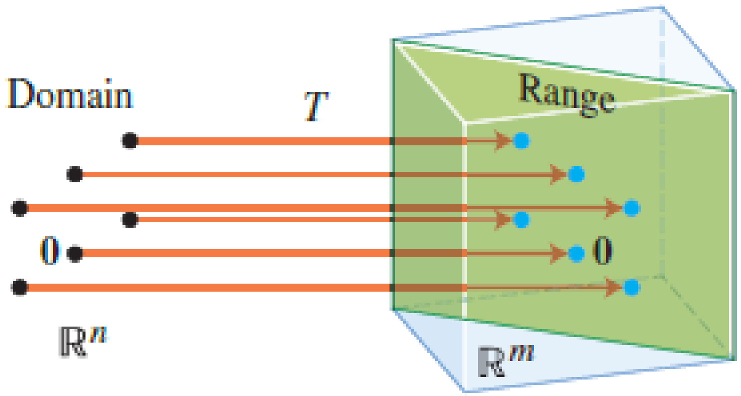
Definitions

A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.



T is *not* one-to-one



T is one-to-one

EXAMPLE 4 Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

THEOREM 11

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

THEOREM 12

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b. T is one-to-one if and only if the columns of A are linearly independent.

EXAMPLE 5 Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

$$T(\mathbf{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$