DETERMINANTS

INTRODUCTION TO DETERMINANTS

Determinants

2*2 matrix is invertible if and only if its determinant is nonzero.
extend this useful fact to larger matrices
definition for the 3*3 case

$$A = [a_{ij}] \text{ with } a_{11} \neq 0 \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}$$

$$\sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

DETERMINANT

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Determinants

$A = [a_{11}]$—we define $\det A = a_{11}$

$\det A = a_{11}a_{22} - a_{12}a_{21}$

To generalize the definition of the determinant to larger matrices, we'll use 2*2 determinants to rewrite the 3*3 determinant •described above.

$$(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - (a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}) + (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31})$$

$$\Delta = a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

let Aij denote the submatrix formed by deleting the i th row and j th column of A.

Determinants

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix} \quad A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\Delta = a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + a_{13} \cdot \det A_{13}$$

In general, an n*n determinant is defined by determinants of (n-1)*(n-1) submatrices.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

Determinants

EXAMPLE 1 Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\det A = 1\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = \cdots = -2$$

Given $A = [a_{ij}]$, the **(i, j)-cofactor** of A is the number C_{ij} given

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

cofactor expansion across the first row of A

Determinants

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

helpful for computing the determinant of a matrix that contains many zeros

Determinants

EXAMPLE 3 Compute det A, where

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\det A = 3 \cdot \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0 \cdot C_{21} + 0 \cdot C_{31} + 0 \cdot C_{41} + 0 \cdot C_{51}$$

$$\det A = 3 \cdot 2 \cdot \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 3 \cdot 2 \cdot (-2) = -12.$$

If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

Determinants

NUMERICAL NOTE

By today's standards, a 25×25 matrix is small. Yet it would be impossible to calculate a 25×25 determinant by cofactor expansion. In general, a cofactor expansion requires more than $n!$ multiplications, and $25!$ is approximately 1.5×10^{25}.

If a computer performs one trillion multiplications per second, it would have to run for more than 500,000 years to compute a 25×25 determinant by this method. Fortunately, there are faster methods, as we'll soon discover.

example

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

PROPERTIES OF DETERMINANTS

Determinants

THEOREM 3

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

Determinants

EXAMPLE 2 Compute det A, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$

$$\det A = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$= 2 \cdot (1)(3)(-6)(1) = -36$$

Determinants

Suppose a square matrix A has been reduced to an echelon form U

If there are r interchanges, then Theorem shows that

$$\det A = (-1)^r \det U$$

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

THEOREM 4 A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem 4 adds the statement "$\det A \neq 0$" to the Invertible Matrix Theorem

Determinants

det $A = 0$ when the columns of A are linearly dependent

det $A = 0$ when the *rows* of A are linearly dependent.

NUMERICAL NOTES

1. Most computer programs that compute det A for a general matrix A use the method of formula (1) above.
2. It can be shown that evaluation of an $n \times n$ determinant using row operations requires about $2n^3/3$ arithmetic operations. Any modern microcomputer can calculate a 25×25 determinant in a fraction of a second, since only about 10,000 operations are required.

Determinants

EXAMPLE 4 Compute det A, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$..

$$\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

Column Operations

- ✓ operations on the columns of a matrix similar to the row operations
- ✓ column operations have the same effects on determinants as row operations.

THEOREM 5 If A is an $n \times n$ matrix, then $\det A^T = \det A$.

PROOF $n = 1$ true for $k \times k$ $n = k + 1$

cofactor of a_{1j} in A equals the cofactor of a_{j1} in A^T

each statement in Theorem 3 is true when the word *row* is replaced everywhere by *column*.

A row operation on A^T amounts to a column operation on A.

Determinants and Matrix Products

THEOREM 6

Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

EXAMPLE 5 Verify Theorem 6 for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

$$\det AB = 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45$$

$$(\det A)(\det B) = 9 \cdot 5 = 45 = \det AB$$

example

verify that $\det EA = (\det E)(\det A)$, where

E is the elementary matrix shown and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

33. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

34. $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

35. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Proof of theorem 3

If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det EA = (\det E)(\det A)$$

where

$$\det E = \begin{cases} 1 & \text{if } E \text{ is a row replacement} \\ -1 & \text{if } E \text{ is an interchange} \\ r & \text{if } E \text{ is a scale by } r \end{cases}$$

The proof is by induction on the size of A $\quad n = k + 1$

The action of E on A involves either two rows or only one row.

expand det(EA) across an unchanged row by the action of E, say, row i (B=EA)

$$\boxed{\det B_{ij} = \alpha \cdot \det A_{ij} \quad \alpha = 1, -1, \text{or } r}$$

Proof of theorem 3

$$\det EA = a_{i1}(-1)^{i+1} \det B_{i1} + \cdots + a_{in}(-1)^{i+n} \det B_{in}$$

$$= \alpha a_{i1}(-1)^{i+1} \det A_{i1} + \cdots + \alpha a_{in}(-1)^{i+n} \det A_{in}$$

PROOF OF THEOREM 6

A is not invertible, then neither is AB.

$$\det AB = (\det A)(\det B)$$

If A is invertible, then A and the identity matrix I_n are row equivalent

$$A = E_p E_{p-1} \cdots E_1 \cdot I_n = E_p E_{p-1} \cdots E_1$$

$$|AB| = |E_p \cdots E_1 B| = |E_p||E_{p-1} \cdots E_1 B| = \cdots$$

$$= |E_p| \cdots |E_1||B| = \cdots = |E_p \cdots E_1||B|$$
$$= |A||B|$$

CRAMER'S RULE, VOLUME, AND LINEAR TRANSFORMATIONS

Cramer's Rule

For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$.

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \quad \cdots \quad \underset{\substack{\uparrow \\ \text{col } i}}{\mathbf{b}} \quad \cdots \quad \mathbf{a}_n]$$

THEOREM 7

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n \tag{1}$$

Cramer's Rule

PROOF

$$\begin{aligned} A \cdot I_i(\mathbf{x}) &= A\begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} A\mathbf{e}_1 & \cdots & A\mathbf{x} & \cdots & A\mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix} = A_i(\mathbf{b}) \end{aligned}$$

$$(\det A)(\det I_i(\mathbf{x})) = \det A_i(\mathbf{b})$$

$$(\det A) \cdot x_i = \det A_i(\mathbf{b})$$

Example

EXAMPLE 1 Use Cramer's rule to solve the system

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \qquad A_1(\mathbf{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \qquad A_2(\mathbf{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{24 + 16}{2} = 20$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{24 + 30}{2} = 27$$

Example

EXAMPLE 2 Consider the following system in which s is an unspecified parameter. Determine the values of s for which the system has a unique solution, and use Cramer's rule to describe the solution.

$$3sx_1 - 2x_2 = 4$$
$$-6x_1 + sx_2 = 1$$

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}, \quad A_1(\mathbf{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}, \quad A_2(\mathbf{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}$$

Since

$$\det A = 3s^2 - 12 = 3(s+2)(s-2) \qquad s \neq \pm 2$$

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{4s+2}{3(s+2)(s-2)}$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{3s+24}{3(s+2)(s-2)} = \frac{s+8}{(s+2)(s-2)}$$

A Formula for A^{-1}

jth column of A^{-1} is a vector $\mathbf{x}$ that satisfies

$$A\mathbf{x} = \mathbf{e}_j$$

$$\{(i, j)\text{-entry of } A^{-1}\} = \frac{\det A_i(\mathbf{e}_j)}{\det A}$$

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} = \frac{1}{\det A}\operatorname{adj} A$$

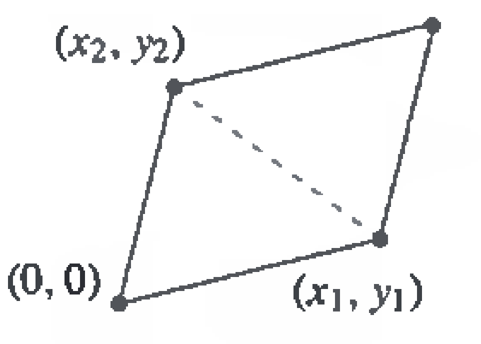
Determinants as Area or Volume

THEOREM 9

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

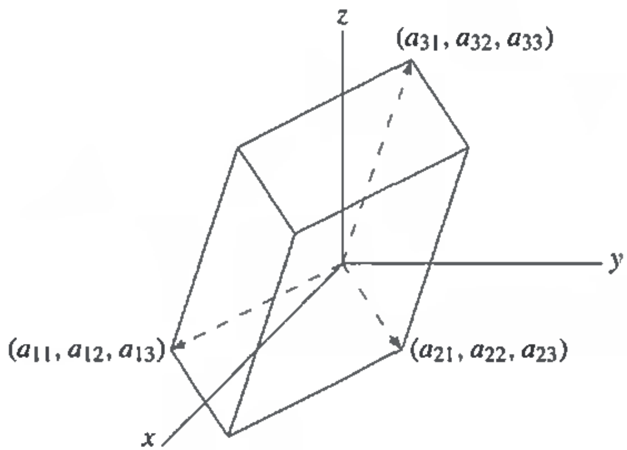
- Area of a parallelogram in $\mathbb{R}^2$ spanned by points $(0,0)$, (a,b), (c,d), $(a+c, b+d)$ is:

$$\left|\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$$



Determinants as Area or Volume

- Volume of a parallelogram in $\mathbb{R}^3$ spanned by points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is $\left| \det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \right|$



Linear Transformation

THEOREM 10

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \tag{5}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\} \tag{6}$$

PROOF $A = [\,\mathbf{a}_1 \quad \mathbf{a}_2\,]$ $\qquad S = \{s_1\mathbf{b}_1 + s_2\mathbf{b}_2 : 0 \le s_1 \le 1,\ 0 \le s_2 \le 1\}$

$$\begin{aligned} T(s_1\mathbf{b}_1 + s_2\mathbf{b}_2) &= s_1T(\mathbf{b}_1) + s_2T(\mathbf{b}_2) \\ &= s_1A\mathbf{b}_1 + s_2A\mathbf{b}_2 \end{aligned}$$

parallelogram determined by the columns of the matrix .
A**b**1 A**b**2

$$\begin{aligned} \{\text{area of } T(S)\} &= |\det AB| = |\det A| \cdot |\det B| \\ &= |\det A| \cdot \{\text{area of } S\} \end{aligned}$$

Linear Transformation

$\mathbf{p} + S$

T transforms $\mathbf{p} + S$ into $T(\mathbf{p}) + T(S)$

The conclusions of Theorem 10 hold whenever S is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.

example

EXAMPLE 5 Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

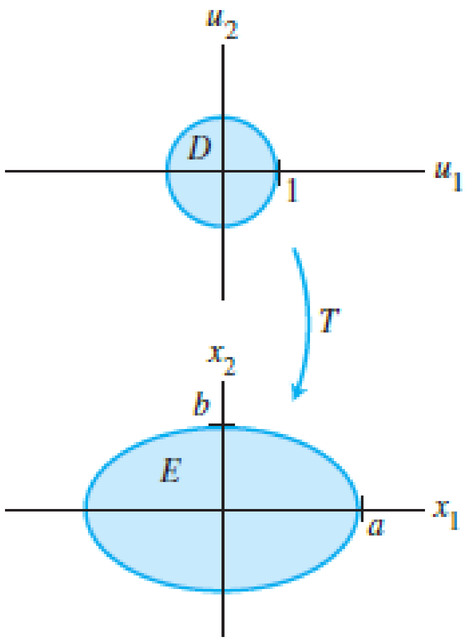


image of the unit disk D

$$\mathbf{x} = A\mathbf{u} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$u_1 = \frac{x_1}{a} \quad \text{and} \quad u_2 = \frac{x_2}{b}$$

$$\begin{aligned} \{\text{area of ellipse}\} &= \{\text{area of } T(D)\} \\ &= |\det A| \cdot \{\text{area of } D\} \\ &= ab \cdot \pi(1)^2 = \pi ab \end{aligned}$$