5.1 EIGEN VALUES AND EIGEN VECTORS

Although a transformation $\mathbf{x} \mapsto A\mathbf{x}$ may move vectors in a variety of directions, it often happens that there are special vectors on which the action of A is quite simple.

EXAMPLE 1 Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

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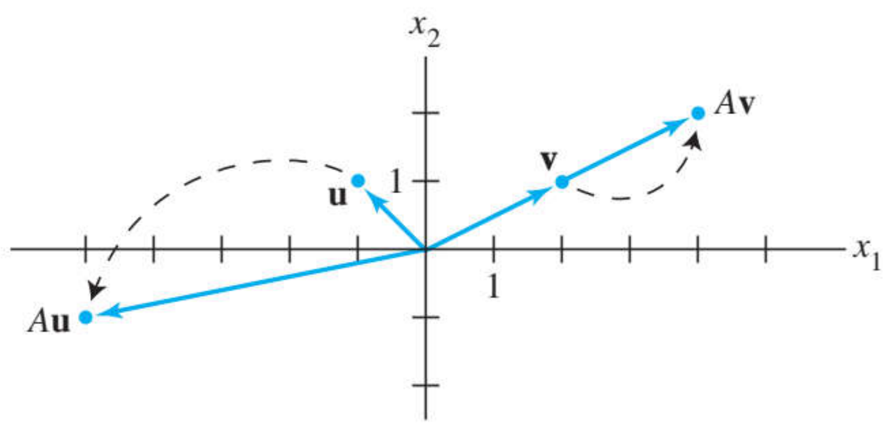

FIGURE 1 Effects of multiplication by A.

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

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It is easy to determine if a given vector is an eigenvector of a matrix. It is also easy to decide if a specified scalar is an eigenvalue.

EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

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$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

EXAMPLE 3 Show that 7 is an eigenvalue of matrix A in Example 2, and find the corresponding eigenvectors.

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$$A\mathbf{x} = 7\mathbf{x} \qquad A\mathbf{x} - 7\mathbf{x} = \mathbf{0} \qquad (A - 7I)\mathbf{x} = \mathbf{0}$$

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

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$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

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has a nontrivial solution. The set of *all* solutions of (3) is just the null space of the matrix $A - \lambda I$. So this set is a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ. The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

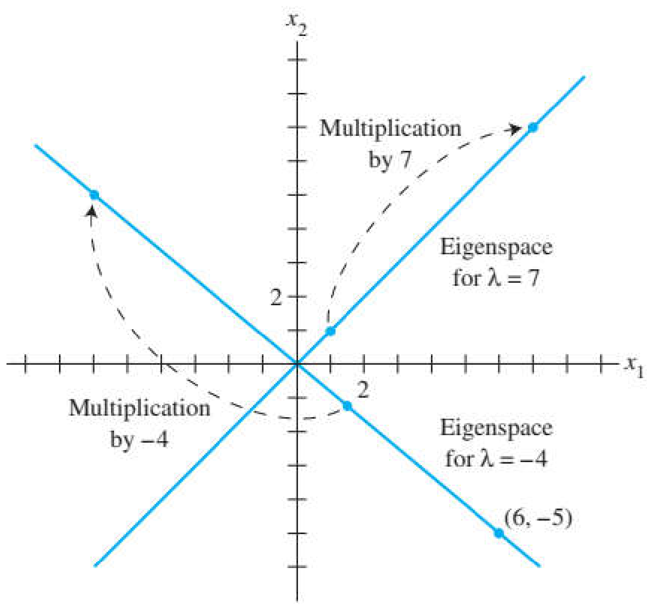

FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

EXAMPLE 4 Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

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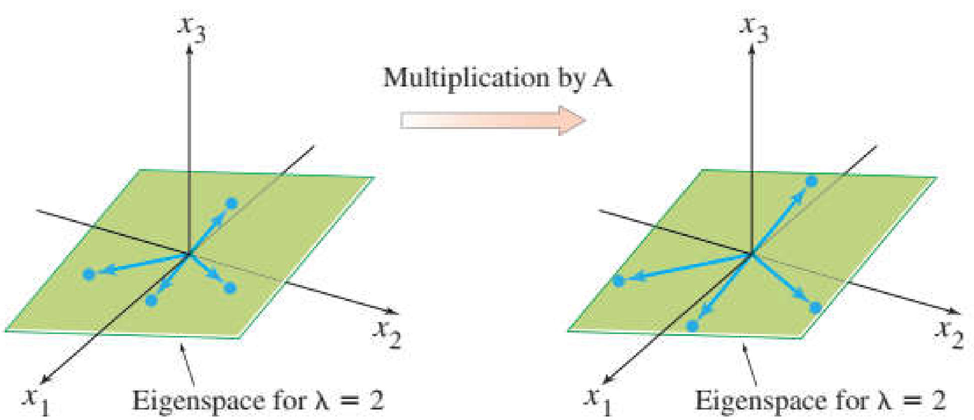
$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$



THEOREM 1 The eigenvalues of a triangular matrix are the entries on its main diagonal.

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$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

Lower triangular??

EXAMPLE 5 Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$. The eigenvalues of A are $3, 0$, and 2. The eigenvalues of B are 4 and 1. ■

What does it mean for a matrix A to have an eigenvalue of 0, such as in Example 5?

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What does it mean for a matrix A to have an eigenvalue of 0, such as in Example 5?

$$A\mathbf{x} = 0\mathbf{x}$$

Thus 0 *is an eigenvalue of* A *if and only if* A *is not invertible*.

THEOREM 2 If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

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Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent.

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{v}_{p+1}$$

$$c_1A\mathbf{v}_1 + \cdots + c_pA\mathbf{v}_p = A\mathbf{v}_{p+1}$$

$$c_1\lambda_1\mathbf{v}_1 + \cdots + c_p\lambda_p\mathbf{v}_p = \lambda_{p+1}\mathbf{v}_{p+1}$$

$$c_1(\lambda_1 - \lambda_{p+1})\mathbf{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\mathbf{v}_p = \mathbf{0}$$

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad (k = 0, 1, 2, \ldots)$$

If A is an $n \times n$ matrix, then (8) is a *recursive* description of a sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$.

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If A is an $n \times n$ matrix, then (8) is a *recursive* description of a sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$.

The simplest way to build a solution of (8) is to take an eigenvector $\mathbf{x}_0$ and its corresponding eigenvalue λ and let

$$\mathbf{x}_k = \lambda^k \mathbf{x}_0 \quad (k = 1, 2, \dots) \tag{9}$$

This sequence is a solution because

$$A\mathbf{x}_k = A(\lambda^k \mathbf{x}_0) = \lambda^k (A\mathbf{x}_0) = \lambda^k (\lambda \mathbf{x}_0) = \lambda^{k+1} \mathbf{x}_0 = \mathbf{x}_{k+1}$$

2. If $\mathbf{x}$ is an eigenvector of A corresponding to λ, what is $A^3\mathbf{x}$?

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$$A\mathbf{x} = \lambda\mathbf{x}$$

$$A^2\mathbf{x} = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$$

$$A^3\mathbf{x} = A(A^2\mathbf{x}) = A(\lambda^2\mathbf{x}) = \lambda^2 A\mathbf{x} = \lambda^3\mathbf{x}$$

$$A^k\mathbf{x} = \lambda^k\mathbf{x}$$

If A is an $n \times n$ matrix and λ is an eigenvalue of A, show that 2λ is an eigenvalue of $2A$.

5.2 CHARACTERISTIC EQUATION

EXAMPLE 1 Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

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$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2 - \lambda & 3 \\ 3 & -6 - \lambda \end{bmatrix}$$

finding all λ such that the matrix $A - \lambda I$ is *not* invertible

determinant is zero

$$\begin{aligned} \det(A - \lambda I) &= (2 - \lambda)(-6 - \lambda) - (3)(3) \\ &= -12 + 6\lambda - 2\lambda + \lambda^2 - 9 \\ &= \lambda^2 + 4\lambda - 21 \\ &= (\lambda - 3)(\lambda + 7) \end{aligned}$$

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

EXAMPLE 3 Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Characteristic polynomial of A.

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$
$$= (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

eigenvalue 5 is said to have *multiplicity* 2

EXAMPLE 4 The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues and their multiplicities.

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$$\lambda^6 - 4\lambda^5 - 12\lambda^4 = \lambda^4(\lambda^2 - 4\lambda - 12) = \lambda^4(\lambda - 6)(\lambda + 2)$$

Because the characteristic equation for an n* n matrix involves an nth-degree polynomial, the equation has exactly n roots, counting multiplicities, provided complex roots are allowed.

Similar matrices

If A and B are n*n matrices, then A is similar to B if there is an invertible matrix P such that

$$P^{-1}AP = B, \text{ or, equivalently, } A = PBP^{-1}$$

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B is also similar to A, and we say simply that A and B **are similar**

Changing A into $P^{-1}AP$ is called a **similarity transformation**

Similar matrices

THEOREM 4

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

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warning

PROOF If $B = P^{-1}AP$, then

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$$

Using the multiplicative property (b) in Theorem 3, we compute

$$\begin{aligned} \det(B - \lambda I) &= \det[P^{-1}(A - \lambda I)P] \\ &= \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P) \end{aligned}$$

EXAMPLE 5 Let $A = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$. Analyze the long-term behavior of the dynamical system defined by $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 0, 1, 2, \ldots)$, with $\mathbf{x}_0 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$.

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$$0 = \det \begin{bmatrix} .95 - \lambda & .03 \\ .05 & .97 - \lambda \end{bmatrix} = (.95 - \lambda)(.97 - \lambda) - (.03)(.05)$$
$$= \lambda^2 - 1.92\lambda + .92$$

$$\lambda = 1 \text{ and } \lambda = .92$$

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$$\begin{aligned} 0 = \det \begin{bmatrix} .95 - \lambda & .03 \\ .05 & .97 - \lambda \end{bmatrix} &= (.95 - \lambda)(.97 - \lambda) - (.03)(.05) \\ &= \lambda^2 - 1.92\lambda + .92 \end{aligned}$$

$$\lambda = 1 \text{ and } \lambda = .92 \qquad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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$$\begin{aligned}\mathbf{x}_1 = A\mathbf{x}_0 &= c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 \\ &= c_1\mathbf{v}_1 + c_2(.92)\mathbf{v}_2\end{aligned}$$

$$\begin{aligned}\mathbf{x}_2 = A\mathbf{x}_1 &= c_1 A\mathbf{v}_1 + c_2(.92)A\mathbf{v}_2 \\ &= c_1\mathbf{v}_1 + c_2(.92)^2\mathbf{v}_2\end{aligned}$$

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$$\mathbf{x}_k = c_1\mathbf{v}_1 + c_2(.92)^k\mathbf{v}_2 \quad (k = 0, 1, 2, \ldots)$$

$$\mathbf{x}_k = .125\begin{bmatrix} 3 \\ 5 \end{bmatrix} + .225(.92)^k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{x}_k \text{ tends to } \begin{bmatrix} .375 \\ .625 \end{bmatrix} = .125\mathbf{v}_1$$

Show that if $A = QR$ with Q invertible, then A is similar to $A_1 = RQ$.

DIAGONALIZATION

$$A = PDP^{-1}$$

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$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \quad \text{for } k \geq 1$$

EXAMPLE 2 Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

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$$A^2 = (PDP^{-1})(PDP^{-1}) = PD\underbrace{(P^{-1}P)}_{I}DP^{-1} = PDDP^{-1}$$

$$= PD^2P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A^3 = (PDP^{-1})A^2 = (PD\underbrace{P^{-1})P}_{I}D^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$$

$$A^k = PD^kP^{-1}$$

A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

THEOREM 5

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

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$$A = PDP^{-1} \qquad AP = PD$$

$$AP = A[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\,] = [\,A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n\,]$$

$$PD = P\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\,\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n\,]$$

EXAMPLE 3 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

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$$\begin{aligned} 0 = \det(A - \lambda I) &= -\lambda^3 - 3\lambda^2 + 4 \\ &= -(\lambda - 1)(\lambda + 2)^2 \end{aligned}$$

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Basis for $\lambda = 1$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Basis for $\lambda = -2$: $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

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$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

EXAMPLE 4 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$$

Basis for $\lambda = 1$: $\quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Basis for $\lambda = -2$: $\quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

EXAMPLE 5 Determine if the following matrix is diagonalizable.

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

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THEOREM 6

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

THEOREM 7

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (*i*) the characteristic polynomial factors completely into linear factors and (*ii*) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

5.4. EIGENVECTORS AND LINEAR TRANSFORMATIONS

Matrix of a Linear Transformation

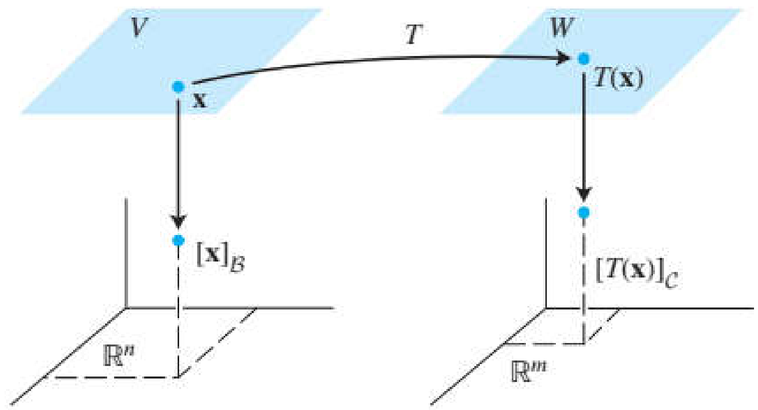

FIGURE 1 A linear transformation from V to W.

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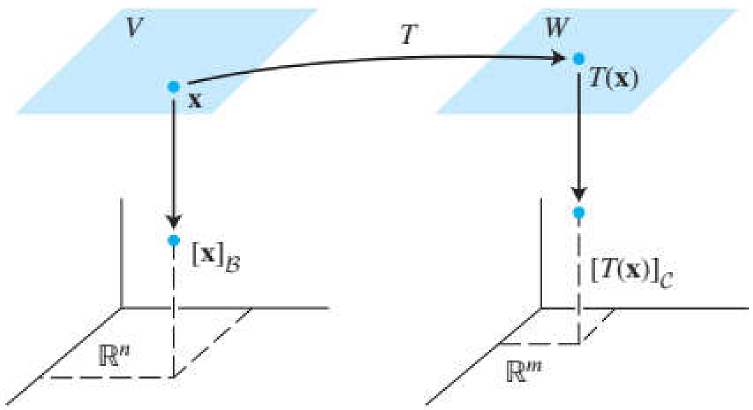

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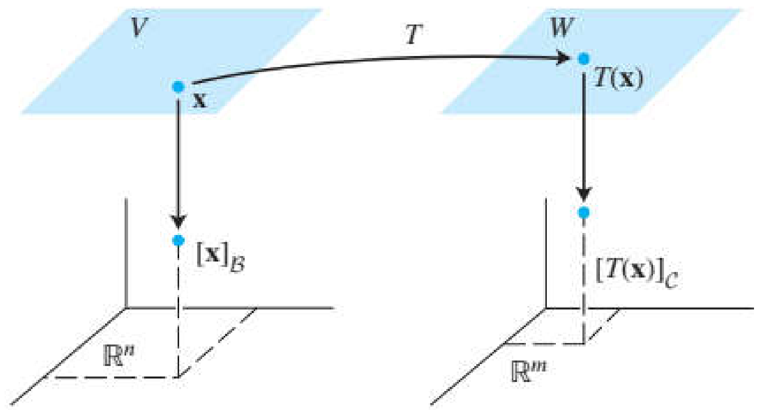
$$\mathbf{x} = r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n$$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

$$T(\mathbf{x}) = T(r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n) = r_1T(\mathbf{b}_1) + \cdots + r_nT(\mathbf{b}_n)$$

Matrix of a Linear Transformation



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$$[\,T(\mathbf{x})\,]_{\mathcal{C}} = r_1[\,T(\mathbf{b}_1)\,]_{\mathcal{C}} + \cdots + r_n[\,T(\mathbf{b}_n)\,]_{\mathcal{C}}$$

$$[\,T(\mathbf{x})\,]_{\mathcal{C}} = M[\,\mathbf{x}\,]_{\mathcal{B}}$$

$$M = \begin{bmatrix} [\,T(\mathbf{b}_1)\,]_{\mathcal{C}} & [\,T(\mathbf{b}_2)\,]_{\mathcal{C}} & \cdots & [\,T(\mathbf{b}_n)\,]_{\mathcal{C}} \end{bmatrix}$$

$$[\,T(\mathbf{x})\,]_{\mathcal{C}} = M[\,\mathbf{x}\,]_{\mathcal{B}}$$

matrix for *T* relative to the bases *B* and *C*

$$\begin{tikzcd}
\mathbf{x} \arrow[r, "T"] \arrow[d] & T(\mathbf{x}) \arrow[d] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "\text{Multiplication}", "\text{by } M"'] & {[T(\mathbf{x})]_{\mathcal{C}}}
\end{tikzcd}$$

EXAMPLE 1 Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for V and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ is a basis for W. Let $T : V \to W$ be a linear transformation with the property that

$$T(\mathbf{b}_1) = 3\mathbf{c}_1 - 2\mathbf{c}_2 + 5\mathbf{c}_3 \quad \text{and} \quad T(\mathbf{b}_2) = 4\mathbf{c}_1 + 7\mathbf{c}_2 - \mathbf{c}_3$$

Find the matrix M for T relative to $\mathcal{B}$ and $\mathcal{C}$.

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Find the matrix M for T relative to $\mathcal{B}$ and $\mathcal{C}$.

$$[\,T(\mathbf{b}_1)\,]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \quad \text{and} \quad [\,T(\mathbf{b}_2)\,]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

$$M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$$

Linear Transformations from V into V

$$
\begin{tikzcd}
\mathbf{x} \arrow[r, "T"] \arrow[d] & T(\mathbf{x}) \arrow[d] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "\text{Multiplication}", "\text{by } {[T]_{\mathcal{B}}}"'] & {[T(\mathbf{x})]_{\mathcal{B}}}
\end{tikzcd}
$$

matrix for T relative to $\mathcal{B}$, or simply the **$\mathcal{B}$-matrix for T**.

$$[\,T\,]_{\mathcal{B}} = \big[\, [\,T(\mathbf{b}_1)\,]_{\mathcal{B}} \quad \cdots \quad [\,T(\mathbf{b}_n)\,]_{\mathcal{B}} \,\big]$$

EXAMPLE 2 The mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$$

a. Find the $\mathcal{B}$-matrix for T, when $\mathcal{B}$ is the basis $\{1, t, t^2\}$.

b. Verify that $[\,T(\mathbf{p})\,]_{\mathcal{B}} = [\,T\,]_{\mathcal{B}}[\,\mathbf{p}\,]_{\mathcal{B}}$ for each $\mathbf{p}$ in $\mathbb{P}_2$.

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b. Verify that $[\,T(\mathbf{p})\,]_{\mathcal{B}} = [\,T\,]_{\mathcal{B}}[\,\mathbf{p}\,]_{\mathcal{B}}$ for each $\mathbf{p}$ in $\mathbb{P}_2$.

$$T(1) = 0$$
$$T(t) = 1$$
$$T(t^2) = 2t$$

$$[\,T(1)\,]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [\,T(t)\,]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\,T(t^2)\,]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$[\,T\,]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

b. For a general $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2$,

$$[\,T(\mathbf{p})\,]_{\mathcal{B}} = [\,a_1 + 2a_2 t\,]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = [\,T\,]_{\mathcal{B}} [\,\mathbf{p}\,]_{\mathcal{B}}$$

$$P[\mathbf{x}]_{\mathcal{B}} = \mathbf{x} \quad \text{and} \quad [\mathbf{x}]_{\mathcal{B}} = P^{-1}\mathbf{x}$$

THEOREM 8

Diagonal Matrix Representation

Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

$$\begin{aligned}
[T]_{\mathcal{B}} &= \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{B}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}} \end{bmatrix} \\
&= \begin{bmatrix} [A\mathbf{b}_1]_{\mathcal{B}} & \cdots & [A\mathbf{b}_n]_{\mathcal{B}} \end{bmatrix} \\
&= [\, P^{-1}A\mathbf{b}_1 \quad \cdots \quad P^{-1}A\mathbf{b}_n \,] \\
&= P^{-1}A[\, \mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n \,] \\
&= P^{-1}AP
\end{aligned}$$

Similarity of Matrix Representations

The proof of Theorem 8 did not use the information that D was diagonal.

$$A = PCP^{-1}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{x} \arrow[r, "\text{Multiplication}", "\text{by } A"'] \arrow[d, "\text{Multiplication}"', "\text{by } P^{-1}"] & A\mathbf{x} \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "\text{Multiplication}", "\text{by } C"'] & {[A\mathbf{x}]_{\mathcal{B}}} \arrow[u, "\text{Multiplication}"', "\text{by } P"]
\end{tikzcd}$$

FIGURE 5 Similarity of two matrix representations: $A = PCP^{-1}$.

Find $T(a_0 + a_1 t + a_2 t^2)$, if T is the linear transformation from $\mathbb{P}_2$ to $\mathbb{P}_2$ whose matrix relative to $\mathcal{B} = \{1, t, t^2\}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

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$$[\,T\,]_{\mathcal{B}} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

Let $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2$ and compute

$$[\,T(\mathbf{p})\,]_{\mathcal{B}} = [\,T\,]_{\mathcal{B}}[\,\mathbf{p}\,]_{\mathcal{B}} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3a_0 + 4a_1 \\ 5a_1 - a_2 \\ a_0 - 2a_1 + 7a_2 \end{bmatrix}$$

So $T(\mathbf{p}) = (3a_0 + 4a_1) + (5a_1 - a_2)t + (a_0 - 2a_1 + 7a_2)t^2$.

5.5. COMPLEX EIGENVALUES

a complex scalar λ satisfies $\det(A - \lambda I) = 0$ if and only if there

is a nonzero vector $\mathbf{x}$ in $\mathbb{C}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$. We call λ a (**complex**) **eigenvalue** and $\mathbf{x}$ a (**complex**) **eigenvector** corresponding to λ.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \lambda^2 + 1 = 0 \qquad \lambda = i \text{ and } \lambda = -i$$

EXAMPLE 2 Let $A = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}$. Find the eigenvalues of A, and find a basis for each eigenspace.

$$0 = \det \begin{bmatrix} .5 - \lambda & -.6 \\ .75 & 1.1 - \lambda \end{bmatrix} = (.5 - \lambda)(1.1 - \lambda) - (-.6)(.75) \qquad \lambda = \tfrac{1}{2}[1.6 \pm \sqrt{(-1.6)^2 - 4}] = .8 \pm .6i$$

$$= \lambda^2 - 1.6\lambda + 1$$

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$$A - (.8 - .6i)I = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix} - \begin{bmatrix} .8 - .6i & 0 \\ 0 & .8 - .6i \end{bmatrix} = \begin{bmatrix} -.3 + .6i & -.6 \\ .75 & .3 + .6i \end{bmatrix}$$

$$\begin{aligned} (-.3 + .6i)x_1 - .6x_2 &= 0 \\ .75x_1 + (.3 + .6i)x_2 &= 0 \end{aligned}$$

$$\begin{aligned} .75x_1 &= (-.3 - .6i)x_2 \\ x_1 &= (-.4 - .8i)x_2 \end{aligned} \qquad \mathbf{v}_1 = \begin{bmatrix} -2 - 4i \\ 5 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -2 + 4i \\ 5 \end{bmatrix}$$

$$\mathbf{x}_1 = A\mathbf{x}_0$$

$$\mathbf{x}_2 = A\mathbf{x}_1$$

$$\mathbf{x}_3 = A\mathbf{x}_2,$$

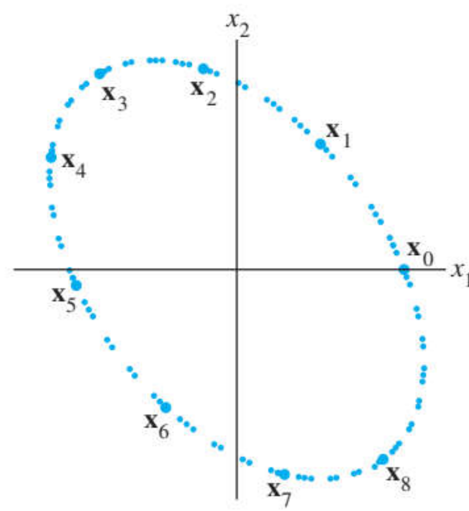

FIGURE 1 Iterates of a point $\mathbf{x}_0$ under the action of a matrix with a complex eigenvalue.

The complex conjugate of a complex vector $\mathbf{x}$ in $\mathbb{C}^n$ is the vector $\overline{\mathbf{x}}$ in $\mathbb{C}^n$ whose entries are the complex conjugates of the entries in $\mathbf{x}$. The **real** and **imaginary parts** of a complex vector $\mathbf{x}$ are the vectors $\operatorname{Re}\mathbf{x}$ and $\operatorname{Im}\mathbf{x}$ in $\mathbb{R}^n$ formed from the real and imaginary parts of the entries of $\mathbf{x}$.

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EXAMPLE 4 If $\mathbf{x} = \begin{bmatrix} 3-i \\ i \\ 2+5i \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + i\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$, then

$$\operatorname{Re}\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \quad \operatorname{Im}\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}, \quad \text{and} \quad \overline{\mathbf{x}} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} - i\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3+i \\ -i \\ 2-5i \end{bmatrix} \quad \blacksquare$$

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$$\overline{r\mathbf{x}} = \overline{r}\,\overline{\mathbf{x}}, \quad \overline{B\mathbf{x}} = \overline{B}\,\overline{\mathbf{x}}, \quad \overline{BC} = \overline{B}\,\overline{C}, \quad \text{and} \quad \overline{rB} = \overline{r}\,\overline{B}$$

Let A be an $n \times n$ matrix whose entries are real. Then $\overline{A\mathbf{x}} = \overline{A}\overline{\mathbf{x}} = A\overline{\mathbf{x}}$. If λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector in $\mathbb{C}^n$, then

$$A\overline{\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$$

Hence $\overline{\lambda}$ is also an eigenvalue of A, with $\overline{\mathbf{x}}$ a corresponding eigenvector.

EXAMPLE 6 If $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a and b are real and not both zero, then the eigenvalues of C are $\lambda = a \pm bi$. (See the Practice Problem at the end of this section.) Also, if $r = |\lambda| = \sqrt{a^2 + b^2}$, then

$$C = r\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

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$$A = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}$$

rotation

EXAMPLE 7 Let $A = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}$, $\lambda = .8 - .6i$, and $\mathbf{v}_1 = \begin{bmatrix} -2 - 4i \\ 5 \end{bmatrix}$, as in Example 2. Also, let P be the 2×2 real matrix

$$P = \begin{bmatrix} \operatorname{Re} \mathbf{v}_1 & \operatorname{Im} \mathbf{v}_1 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix}$$

$$C = P^{-1}AP$$

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$$P = \begin{bmatrix} \operatorname{Re}\mathbf{v}_1 & \operatorname{Im}\mathbf{v}_1 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix}$$

$$C = P^{-1}AP = \frac{1}{20}\begin{bmatrix} 0 & 4 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}\begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}$$

$$A = PCP^{-1} = P\begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}P^{-1}$$

Pure rotation

$$
\begin{tikzcd}
\mathbf{x} \arrow[r, "A"] \arrow[d, "P^{-1}", "\text{Change of variable}"'] & A\mathbf{x} \\
\mathbf{u} \arrow[r, "C", "\text{Rotation}"'] & C\mathbf{u} \arrow[u, "P", "\text{Change of variable}"']
\end{tikzcd}
$$

FIGURE 4 Rotation due to a complex eigenvalue.

$$
\begin{tikzcd}
\mathbf{x} \arrow[r, "A"] \arrow[d, "P^{-1}", "\text{Change of variable}"'] & A\mathbf{x} \\
\mathbf{u} \arrow[r, "C", "\text{Rotation}"'] & C\mathbf{u} \arrow[u, "P", "\text{Change of variable}"']
\end{tikzcd}
$$

FIGURE 4 Rotation due to a complex eigenvalue.

THEOREM 9

Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then

$$A = PCP^{-1}, \quad \text{where} \quad P = [\,\operatorname{Re}\mathbf{v} \quad \operatorname{Im}\mathbf{v}\,] \quad \text{and} \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

5.8. ITERATIVE ESTIMATE FOR EIGENVALUES

Some applications require only a rough approximation to the largest eigenvalue

Power method

The power method applies to an $n \times n$ matrix A with a **strictly dominant eigenvalue** λ_1, which means that λ_1 must be larger in absolute value than all the other eigenvalues. In this case, the power method produces a scalar sequence that approaches λ_1 and a vector sequence that approaches a corresponding eigenvector.

Power method

Assume for simplicity that A is diagonalizable and $\mathbb{R}^n$ has a basis of eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, arranged so their corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ decrease in size, with the strictly dominant eigenvalue first. That is,

$$\underbrace{|\lambda_1| >}_{\text{Strictly larger}} |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \tag{1}$$

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$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \tag{1}$$

Strictly larger

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

$$A^k\mathbf{x} = c_1(\lambda_1)^k\mathbf{v}_1 + c_2(\lambda_2)^k\mathbf{v}_2 + \cdots + c_n(\lambda_n)^k\mathbf{v}_n \quad (k = 1, 2, \dots)$$

Power method

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$$\underbrace{|\lambda_1| > |\lambda_2|}_{\text{Strictly larger}} \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \tag{1}$$

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

$$A^k\mathbf{x} = c_1(\lambda_1)^k\mathbf{v}_1 + c_2(\lambda_2)^k\mathbf{v}_2 + \cdots + c_n(\lambda_n)^k\mathbf{v}_n \quad (k = 1, 2, \ldots)$$

$$\frac{1}{(\lambda_1)^k}A^k\mathbf{x} = c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{v}_n \quad (k = 1, 2, \ldots)$$

$$(\lambda_1)^{-k}A^k\mathbf{x} \to c_1\mathbf{v}_1 \quad \text{as } k \to \infty$$

EXAMPLE 1 Let $A = \begin{bmatrix} 1.8 & .8 \\ .2 & 1.2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} -.5 \\ 1 \end{bmatrix}$. Then A has eigenvalues 2 and 1, and the eigenspace for $\lambda_1 = 2$ is the line through $\mathbf{0}$ and $\mathbf{v}_1$. For $k = 0, \ldots, 8$, compute $A^k\mathbf{x}$ and construct the line through $\mathbf{0}$ and $A^k\mathbf{x}$. What happens as k increases?

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TABLE 1 Iterates of a Vector

k	0	1	2	3	4	5	6	7	8
$A^k\mathbf{x}$	$\begin{bmatrix} -.5 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -.1 \\ 1.1 \end{bmatrix}$	$\begin{bmatrix} .7 \\ 1.3 \end{bmatrix}$	$\begin{bmatrix} 2.3 \\ 1.7 \end{bmatrix}$	$\begin{bmatrix} 5.5 \\ 2.5 \end{bmatrix}$	$\begin{bmatrix} 11.9 \\ 4.1 \end{bmatrix}$	$\begin{bmatrix} 24.7 \\ 7.3 \end{bmatrix}$	$\begin{bmatrix} 50.3 \\ 13.7 \end{bmatrix}$	$\begin{bmatrix} 101.5 \\ 26.5 \end{bmatrix}$

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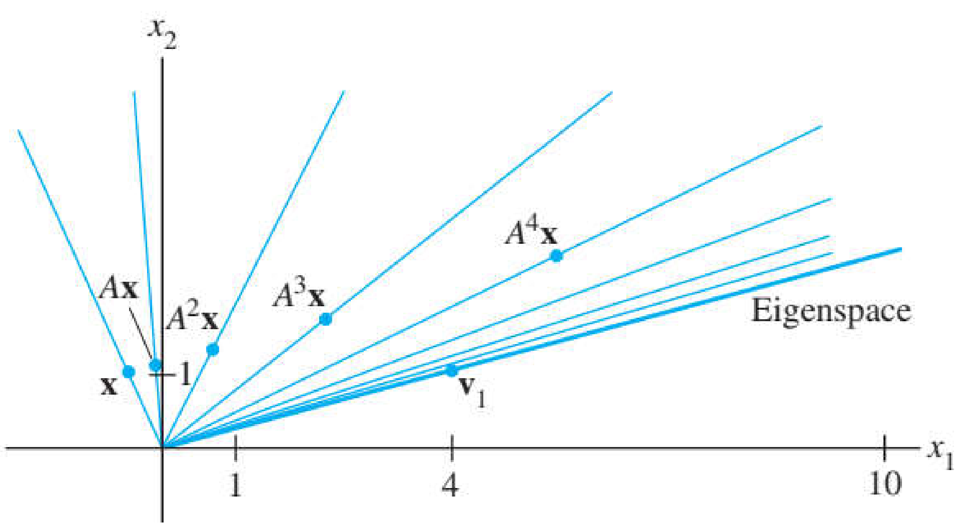

FIGURE 1 Directions determined by $\mathbf{x}$, $A\mathbf{x}$, $A^2\mathbf{x}$, $\ldots$, $A^7\mathbf{x}$.

We cannot scale $A^k\mathbf{x}$ in this way because we do not know λ_1. But we can scale each $A^k\mathbf{x}$ to make its largest entry a 1. It turns out that the resulting sequence $\{\mathbf{x}_k\}$ will converge to a multiple of $\mathbf{v}_1$ whose largest entry is 1.

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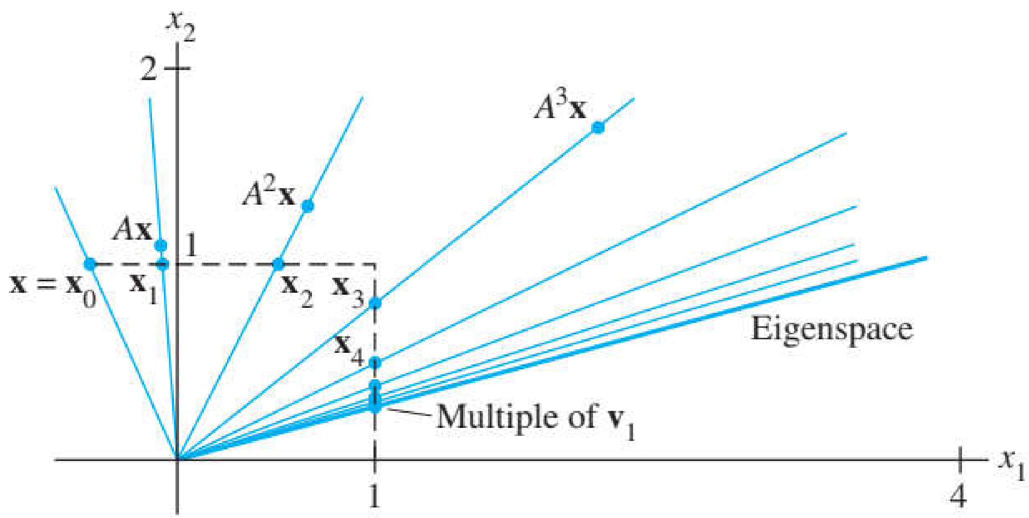

FIGURE 2 Scaled multiples of $\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, \dots, A^7\mathbf{x}$.

$\mathbf{x}_k$ is close to an eigenvector for λ_1, the vector $A\mathbf{x}_k$ is close to $\lambda_1\mathbf{x}_k$, with each entry in $A\mathbf{x}_k$ approximately λ_1 times the corresponding entry in $\mathbf{x}_k$. Because the largest entry in $\mathbf{x}_k$ is 1, the largest entry in $A\mathbf{x}_k$ is close to λ_1.

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THE POWER METHOD FOR ESTIMATING A STRICTLY DOMINANT EIGENVALUE

1. Select an initial vector $\mathbf{x}_0$ whose largest entry is 1.

2. For $k = 0, 1, \ldots,$

a. Compute $A\mathbf{x}_k$.

b. Let μ_k be an entry in $A\mathbf{x}_k$ whose absolute value is as large as possible.

c. Compute $\mathbf{x}_{k+1} = (1/\mu_k)A\mathbf{x}_k$.

3. For almost all choices of $\mathbf{x}_0$, the sequence $\{\mu_k\}$ approaches the dominant eigenvalue, and the sequence $\{\mathbf{x}_k\}$ approaches a corresponding eigenvector.

EXAMPLE 2 Apply the power method to $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ with $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Stop when $k = 5$, and estimate the dominant eigenvalue and a corresponding eigenvector of A.

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TABLE 2 The Power Method for Example 2

k	**0**	**1**	**2**	**3**	**4**	**5**
$\mathbf{x}_k$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .4 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .225 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .2035 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .2005 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .20007 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 8 \\ 1.8 \end{bmatrix}$	$\begin{bmatrix} 7.125 \\ 1.450 \end{bmatrix}$	$\begin{bmatrix} 7.0175 \\ 1.4070 \end{bmatrix}$	$\begin{bmatrix} 7.0025 \\ 1.4010 \end{bmatrix}$	$\begin{bmatrix} 7.00036 \\ 1.40014 \end{bmatrix}$
μ_k	5	8	7.125	7.0175	7.0025	7.00036

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$$A\begin{bmatrix} 1 \\ .2 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ .2 \end{bmatrix} = \begin{bmatrix} 7 \\ 1.4 \end{bmatrix} = 7\begin{bmatrix} 1 \\ .2 \end{bmatrix}$$

Rate of convergence

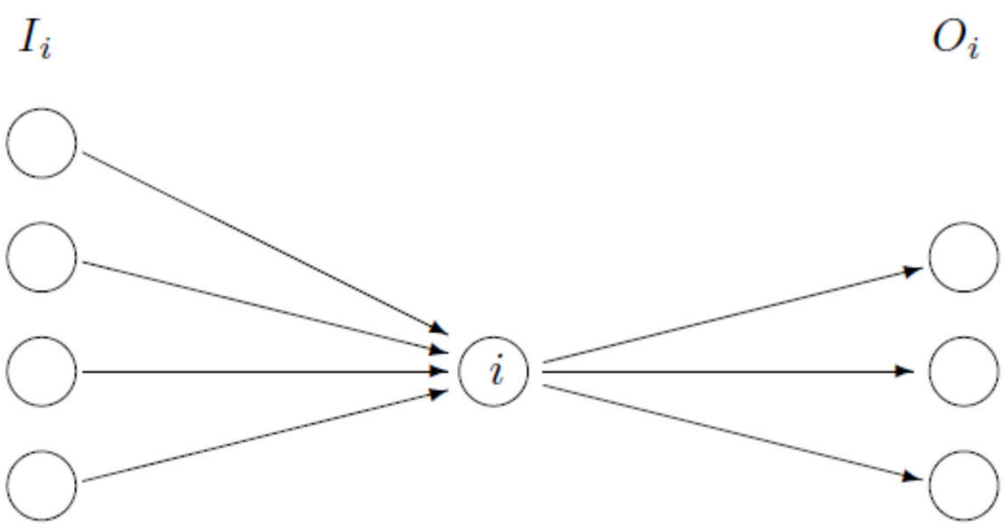
PAGE RANKING FOR A WEB SEARCH ENGINE

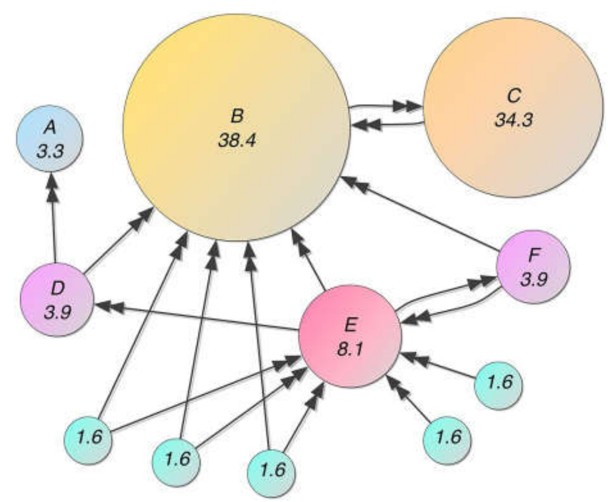
Pagerank

- When a search on the Internet using a search engine, find all the Web pages containing the words of the query.
- massive size of the Web
- some measure to filter out pages less interesting
- Google uses an algorithm for ranking all the Web pages based on link structure of the Web
- Google assigns a high rank to a Web page if it has inlinks from other pages that have a high rank.

Pagerank

- Number of links to and from a page give information about the importance of a page
- Web pages ordered from 1 to *n*, and let *i* be a particular Web page
- O*i* set of pages that *i* is linked to, *outlinks*. number of outlinks denoted *Ni* = |O*i*|.
- set of *inlinks*, denoted *Ii*, are the pages that have an outlink to *i*.

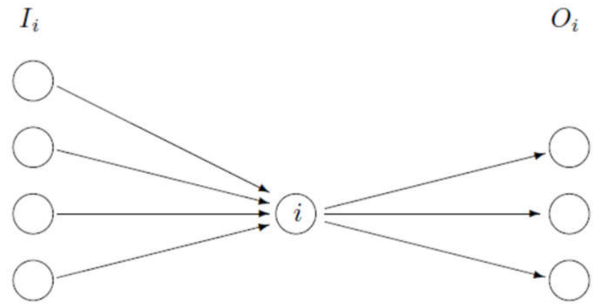




Pagerank

- rank of page *i* is a weighted sum of the ranks of the pages that have outlinks to *i*

$$r_i = \sum_{j \in I_i} \frac{r_j}{N_j}$$



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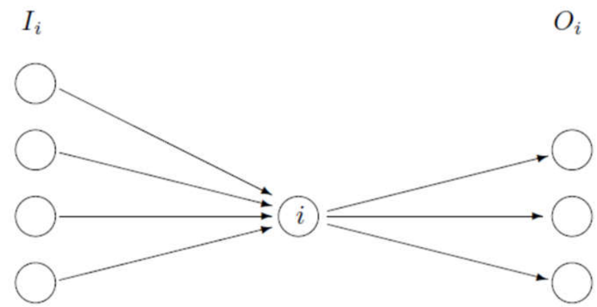

reformulate as an eigenvalue problem for a matrix representing the graph of Internet

$$Q_{ij} = \begin{cases} 1/N_j & \text{if there is a link from } j \text{ to } i, \\ 0 & \text{otherwise.} \end{cases}$$

product of row *I* and the vector *r*

$$\begin{array}{ccc}
 & j & \\
i & \begin{pmatrix} & & * & & \\ & & 0 & & \\ & & \vdots & & \\ 0 & * \;\cdots & * & * & \cdots \\ & & \vdots & & \\ & & 0 & & \\ & & * & & \end{pmatrix} & \leftarrow \text{inlinks} \\
 & \uparrow & \\
 & \text{outlinks} &
\end{array}$$

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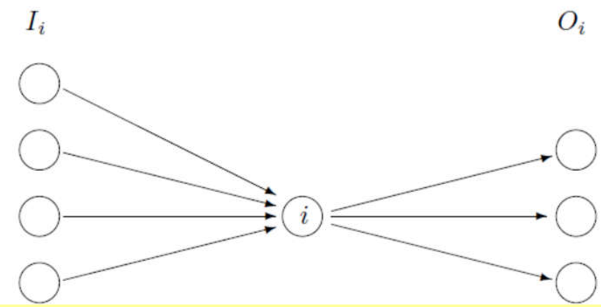

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Pagerank

$$r_i = \sum_{j \in I_i} \frac{r_j}{N_j}$$

$$\lambda r = Qr, \qquad \lambda = 1$$

nonnegative elements, and the elements of each column sum up to 1

column-stochastic matrix

Power Method for Pagerank Computation