Assignment 3
Manipulating Images in Frequency Domain

Please remember:

1. What you must hand in includes the assignment report (.pdf), source codes (.m) and output files (.png). Please insert each part in a different folder, and zip them all together into an archive file named according to the following template: HW3_XXXXXXXX.zip
   Where XXXXXXXX must be replaced with your student ID.
2. Your work will be evaluated mostly by the quality of your report. Don't forget to explain what you have done, and provide enough discussions which proves you have realized the subject.
3. 5 points of each homework belongs to compactness, expressiveness and neatness of your codes and report.
4. By default, we assume you implement your codes in MATLAB. If you're using Python, you have to use equivalent functions when it is asked to use specific MATLAB functions.
5. Your codes must be separated for each question, and for each part. For example, you have to create a separate .m file for part b. of question 3. Please name it like p3b.m.
6. "Keywords" will help you find useful information about the problem. They may also include some ideas for solving that problem.
7. Using built-in functions is not allowed, except for simple operations like reading, displaying, converting and saving images, or in cases it is clearly mentioned in "Allowed MATLAB Functions" section of each problem.
8. **Please upload your work in Moodle, before the end of May 9th.**
9. If there is ***any*** question, please don't hesitate to contact me through the following email address: **ali.the.special@gmail.com**
   I'd be glad to help.
10. Unfortunately, it is quite easy to detect copy-pasted or even structurally similar works, no matter being copied from another student or internet sources. Try to send me your own work, without being worried about the grade! ;)

1. Fundamentals of Fourier Transform in 1D Space (20 Pts.)

Keywords: *Fourier Transform, Inverse Fourier Transform, Dirac Delta Function, Duality/Linearity/Time Shift/Convolution Property of Fourier Transform, Nyquist Sampling Rate*

The **Fourier Transform** is arguably one of the greatest insights ever made in the history of science. Theoretically, everything in the world can be represented via a waveform, i.e. a function of time, space or some other variables. The Fourier transform gives us a unique and powerful way of viewing these waveforms, by breaking them into an alternate representation characterized by sine and cosines.

In this problem, you are going to practice your skills and knowledge in Fourier transform and its properties. Please answer the following questions by hand, and include sufficient and clear calculations in your report.

a. Calculate and determine the Fourier transform of the following functions using properties of the Fourier transform and the definitions related to it:

a1. $g(t) = \sin(t-2)$ a2. $g(t) = \dfrac{2t^2+1}{t^2+1}$ a3. $g(t) = \delta(t-1) + \delta(t+2)$

a4. $g(t) = \mathcal{F}^{-1}\left\{\dfrac{1}{f^2+1}\right\}$ a5. $g(t) = \cos t e^{-|t|}$ a6. $g(t) = \cos(4\pi t)\cos(6\pi t)$

b. Specify which – if any – of the real signals shown in figure 1 have Fourier transforms that meet the following conditions:

b1. $\mathfrak{Re}\{X(j\omega)\} = 0$

b2. $\mathfrak{Im}\{X(j\omega)\} = 0$

b3. $\int_{-\infty}^{\infty} X(j\omega) d\omega = 0$

b4. $\int_{-\infty}^{\infty} \omega X(j\omega) d\omega = 0$

b5. $X(j\omega)$ is periodic

b6. There is a real α such that $e^{j\alpha\omega}X(j\omega)$ is real

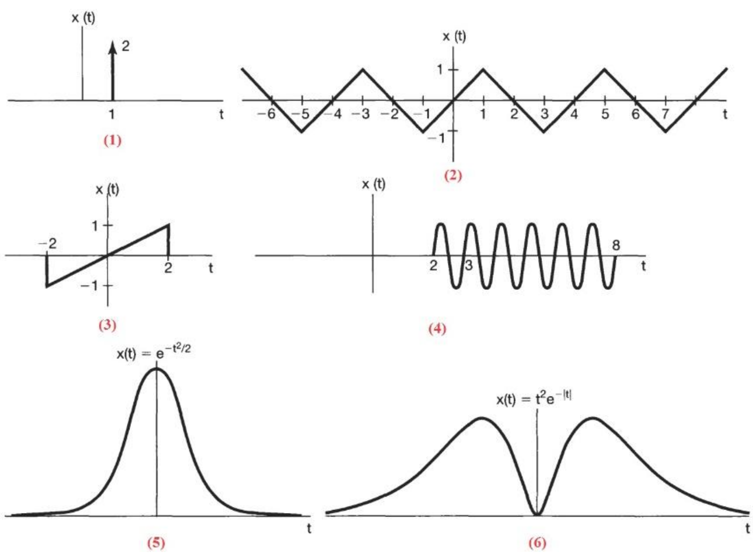

Figure 1 Signals of part b.

c. Find the convolution of each of the following signals $x(t)$ and $h(t)$ by calculating $X(j\omega)$ and $H(j\omega)$. You have to use convolution properties and inverse transforming.

c1. $x(t) = te^{-2t}u(t), \quad h(t) = e^{-4t}u(t)$

c2. $x(t) = te^{-2t}u(t), \quad h(t) = te^{-4t}u(t)$

c3. $x(t) = e^{-t}u(t), \quad h(t) = e^{t}u(-t)$

d. If we want to sample $g(t)$ in a6 and avoid aliasing, what would be the largest sampling step?

2. Some 2D Fourier Transform Calculations (10 Pts.)

Keywords: 2D *Fourier Transform, Convolution Theorem,*

Fourier Transform concept and its properties in 1D space can be generalized to 2D space, where signals are of two dimensions. Before getting involved with their applications in the area of image processing, we are going to practice some basic 2D Fourier transform calculations. Please include clear and significant calculations for each of the following problems.

a. Determine the Fourier transform of the following function: $f(x,y) = \sin 8\pi x + \cos 2\pi y$
b. Perform discrete Fourier transform on the following data: $\{1,2,3,4,5\}$
c. Perform inverse Fourier transform on the result you obtained from part b., and explain your observations.
d. Determine the equivalent filters in the frequency domain for the following spatial filters:

c1: $\begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$ c2: $\begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix}$

e. (Additional point) Mathematically prove that the corresponding filters of c1 and c2 in the frequency domain are low-pass and high-pass, respectively.

3. Understanding Basics of Image Transformation to Frequency Domain (15 Pts.)

Keywords: *Fourier Transform, Frequency Mask*

As can be expected, **Fourier Transform** Could be considered as a useful tool in image processing. It decomposes an image into sine and cosine components, where the output represents the image in the **Fourier** or **Frequency Domain**. In Fourier domain image, each point represents a specific frequency contained in the spatial domain image.
In this problem, you are about to get familiar with basics of image analysis in the frequency domain. You will also perform some manipulations there and see the results back in spatial domain.

a. Based on your knowledge about 2D discrete Fourier transform, guess the properties and the general appearance of the corresponding spectrum of each of the following images. Include sufficient explanations about your answer on your report.
   a1. Horizontal black line in a white background
   a2. Horizontal white line in a black background
   a3. Vertical black line in a white background
   a4. Diagonal black line in a white background
   a5. Multiple horizontal black parallel lines in a white background
   a6. Multiple concentric black circles in a white background
   a7. A grid of black circles in a white background
   a8. A black and white chessboard
   a9. A collection of similar objects (like coffee beans)
   a10. Human fingerprint
   Note: Assume that the above shapes (lines, circles, etc.) are somewhat thick.

b. Read images “sketch.jpg”, “piano_keys.jpg”, “chessboard.jpg” and “trump_and_flag.jpg”. Compute and display the magnitude and phase of the input images, and explain which specific structure in each image yields the results you obtained. Specify the relations between the images in the spatial domain and the frequency domain.

Figure 2 Input images of part b., each with a specific structure

c. Read the image “parallel_man.jpg” and display its corresponding spectrums in frequency domain (you may need to apply histogram equalization), alongside your observations based on the image appearance in spatial domain. Now try to remove the background (vertical lines), by creating an appropriate mask and multiplying it with the Fourier transform of the image. Display the result. Then remove the man in the middle (horizontal lines) in a similar manner.

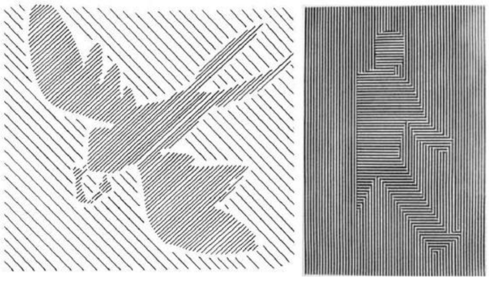

Figure 3 Input images of part c. (right) and part d. (left)

d. (Additional point) Do the same for the image “phoenix.jpg”. Note that the lines forming the image are now diagonal.

Allowed MATLAB functions: `fft2()`, `ifft2()`, `fftshift()`, `abs()`, `angle()`, `log()`, `im2double()`, `imagesec()`, `rgb2gray()`, `histeq()`

4. Using Frequency Filtering for Pattern Matching (10 Pts.)

Keywords: *Pattern Matching, Image Filtering, Mask (Frequency Domain), Image Thresholding*

Pattern Matching (not to be mistaken with “template matching”) in the area of image processing, is the process of checking a given image for the presence of some pre-specified patterns. The theory of Fourier transform offers a powerful tool for doing pattern matching, especially when it comes to binary images, and letters and digits.

In this problem, we want to find all locations in an input image which are occupied by a certain letter. We need an image of that separated letter which can be used as a mask.

Read the image “math.png”. By applying image filtering in frequency domain, you have to find the locations of letter ‘w’ inside the input image. Use “letter_w.png” as a filter mask.

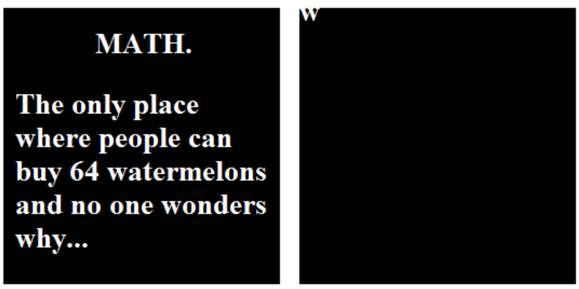

Figure 4 Input image (left) and the mask for letter ‘w’ (right)

a. Transform both the image and the mask into the frequency domain and multiply tem. Then, apply the inverse Fourier transform to the resulting Fourier image and scale the output. Theoretically, the image you obtained so far is identical to the result of convolving the image and the mask in the spatial domain.

Hence, it is expected that the image shows high values at locations where the expected pattern exists. To highlight those locations, image thresholding is probably needed.
Note: It is not necessary to find all the letters. It is also possible to find some incorrect locations.

b. Now let's try a modified method. First, threshold the Fourier image of the mask to specify the most important frequencies which construct the desired letter in the spatial domain. Then, multiply the modified mask with the Fourier image of the text image, and apply inverse Fourier transform to the obtained image. After thresholding the resultant image, compare the performance of this method with the method used in part a.

Note: Display the results you obtain after each step in both parts. Your final result in each part must be an image where the locations of the letter 'w' you found is clearly visible by white dots.

Allowed MATLAB functions: `fft2(), ifft2(), fftshift(), abs(), angle(), log(), im2double(), imagesec(), histeq()`

5. 2D Discrete Fourier Transform and Geometric Transformations (10 Pts.)

Keywords: *Fourier Transform (Properties), Geometric Transformation, Affine Group, Image Reflection, Image Translation, Image Rotation, Image Scaling, Image Shearing*

Image **Geometric Transformation** is the process of applying a function to an image, where the domain and range are sets of points of the image. Geometric transformation is mostly useful in the area of **Image Registration**. When it comes to images, translation, Rotation, Scaling and Shearing are among the most useful geometric transformations.

The goal of this problem is to examine the effects of some geometric transformations on the image Fourier transform result, and the possibility of applying them using the frequency domain.

a. Read the image "café_wall_illusion.png", and display its Fourier spectrum. Apply the following affine transformations on it, and investigate how these transformations affect the image Fourier spectrum.
a1. Reflection
a2. Translation (arbitrary amount)
a3. Rotation (by 45, 90 and 180 degrees)
a4. Scaling (by the factor of 0.5 and 2)
a5. Shearing (arbitrary amount)

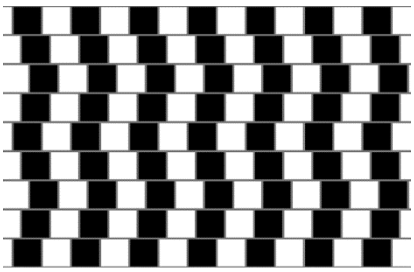

Figure 5 "Café wall illusion". Can you believe that the horizontal lines are parallel?! (link)

b. Now, you have to implement simple geometric transformations using image representation in frequency domain. Please read the image "checker_shadow_illusion.png" and display the spectrum associated with it. Then Implement the following transformations upon it:
b1. Translation (arbitrary amount)
b2. Rotation (by 45, 90 and 180 degrees)
b3. Scaling (by the factor of 0.5 and 2)

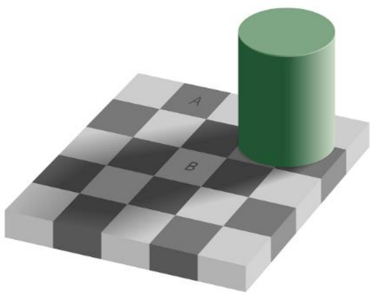

Figure 6 "Checker shadow illusion". The squares A and B are of identical brightness! (link)

Note: You don't have to implement any function.
Allowed MATLAB functions: `imtranslate(), imrotate(), imresize(), maketform(), imtransform(), fft2(), ifft2(), fftshift(), abs(), angle(), log(), im2double(), imagesec(), rgb2gray(), histeq()`

6. A Deeper Look to the Image Filtering in Frequency Domain (10 Pts.)

Keywords: *Convolution Theorem*, Image Filtering, Bandlimiting

Convolution Theorem has provided a powerful tool for **Image Filtering** in frequency domain. According to this theorem, instead of performing a convolution in the spatial domain – which is not computationally efficient - filtering could be easily done in the frequency domain by multiplying two 2D Fourier transforms of the image and the filter.

Figure 7 The input image for low-pass filtering is added with a Gaussian noise

Our goal in this problem is to get familiar with some well-known image filtering techniques in frequency domain. You are going to work with images "farewell.jpg" for low-pass filtering, and "old_friends.jpg" for band-pass filtering.

a. Implement a function to perform ideal low-pass filtering on the input image. It must take the input image and two parameters `n` and `width` as its arguments, and return the frequency domain representation of an ideal low-pass filter of size $n \times n$. For spatial frequencies less than `width`, your filter should be one and zero otherwise. Apply it on "farewell.jpg" with three arbitrary settings, and display the results.

b. Implement a function to perform Gaussian low-pass filtering on the input image. It must take the input image and two parameters `n` and `variance` as its arguments, and return the frequency domain representation of a Gaussian low-pass filter of size $n \times n$. Your filter must be a Gaussian variance of parameter `variance` centred on the zero spatial frequency. Apply it on "farewell.jpg" with three arbitrary settings, and display the results.

c. Do you notice any ringing effect? In which method and for what frequency cut-offs? Explain the reason.

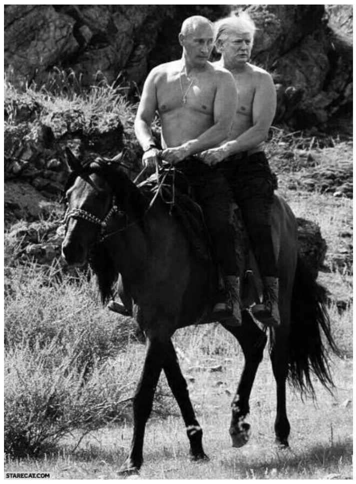

Figure 8 Input image of part d. and part e.

d. Implement a function to perform ideal band-pass filtering on the input image. It must take the input image and three parameters `n`, `centre` and `width` as its arguments, and return the frequency domain representation of an ideal band-pass filter of size $n \times n$. Your filter must be one inside a band of width `width` centred on spatial frequency `centre` and zero otherwise. Apply it on "old_friends.jpg" with three arbitrary settings, and display the results.

e. Implement a function to perform Gaussian band-pass filtering on the input image. It must take the input image and three parameters `n`, `centre` and `variance` as its arguments, and return the frequency domain representation of a Gaussian band-pass filter of size $n \times n$. Your filter must be an annulus with Gaussian cross-section with variance `variance` and mean `centre`. Apply it on "old_friends.jpg" with three arbitrary settings, and display the results.

Allowed MATLAB functions: `fft2(), ifft2(), fftshift(), abs(), angle(), log(), im2double(),imagesec(),histeq()`

7. Smoothing Halftone Image and Removal of Moiré Pattern using FFT (10 Pts.)

Keywords: *Halftone Image, Moiré Pattern, Image Filtering, Notch Filters*

Halftone Technique is the process of applying dots of different size and spacing in the image, so that the image tone looks continuous. As a traditional method of printing books and newspapers, it makes images extracted from old documents look dotted, as can be seen in Figure 9.

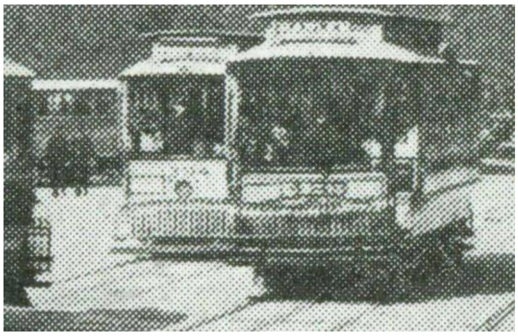

Figure 9 Example of a halftone image, taken from a newspaper. White dots are clearly visible.

Another close, yet not identical, concept in image processing is **Moiré Pattern**, which is a visual perception that happens when viewing a set of lines or dots that is superimposed on another set of lines or dots. These sets differ in relative size, angle or spacing, making strange-looking wavy pattern on the image. Figure 10 demonstrate the effect on some areas of the man's coat. Both effects are usually undesirable, especially when it comes to digital imaging. Your goal in this problem is to deal with these artefacts in the frequency domain, which provides powerful capabilities for handling them.

Figure 10 Moiré pattern can be seen in a large area of the man's coat

a. Read the image "little_donald.png". As you can see, halftone effect is clearly visible, even without zooming in. Transform the image to the frequency domain, and comment on the result. Based on what you have learnt from image filtering in frequency domain, try to implement a method in order to reduce the artefacts in the input image.

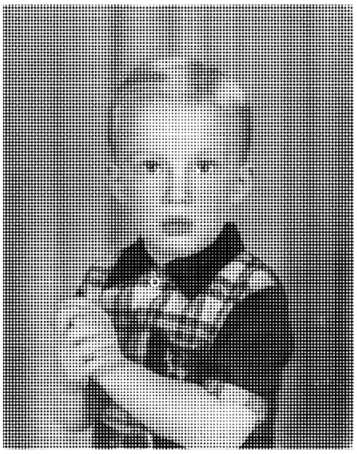

Figure 11 Trump as a child. Halftone effect is artificially added to the image.

b. Moiré pattern is a very common artefact in medical imaging, making **Moiré Pattern Removal** an active research topic in the field of Radiography.
Read the image "trump_x-ray.png", and display the corresponding spectrum in the frequency domain, alongside your observations. Then try to reduce the effect of Moiré Pattern using an appropriate method.

c. (Additional Point) Repeat part b. for the image "finger_x-ray.png". Note that the Moiré pattern in the input image in this case is angled.

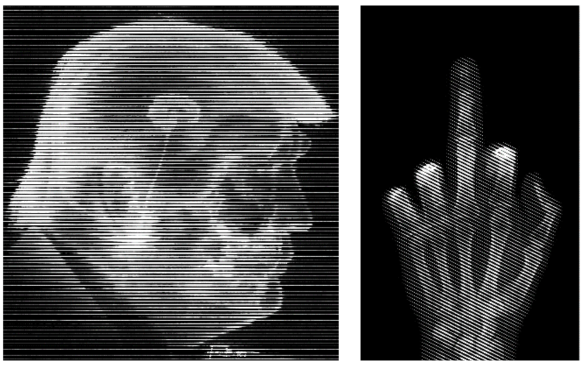

Figure 12 Input radiographic images with Moiré pattern artefacts, corresponding to part b. (left) and part c. (right).

Allowed MATLAB functions: `fft2()`, `ifft2()`, `fftshift()`, `abs()`, `angle()`, `log()`, `im2double()`, `imagesec()`, `histeq()`

8. Some Explanatory Questions (10 Pts.)

Please answer the following questions as clear as possible:

a. What can you infer from the magnitude and phase of the image in frequency domain? Which one is more informative?
b. When it is more preferable to use image filtering in frequency domain rather than the spatial one? Give at least two examples.
c. What is the weak point of low-pass filter when smoothing high frequency noise?
d. Explain how we could calculate the convolution using Fast Fourier Transforms?
e. As you know, **Image Padding** is necessary for image filtering in the frequency domain. Consider two types of padding with equal total numbers of zeros; appending zeros to the ends of rows and columns of the image, and surround the image by a border of zeros. Do you think these two types would make any difference in filtering? Explain.

Good Luck!
Ali Abbasi