(1) محمد کاظم معارف

if Convex $\cap$ line $\Rightarrow$ Convex

if $S \cap$ line $=$ Convex $\overset{?}{\Rightarrow}$ S Convex

$\forall x_1, x_2 \in S \overset{?}{\Rightarrow} \theta x_1 + (1-\theta) x_2 \in S$

$\theta x_1 + (1-\theta) x_2 \in S \cap$ line.

$\Rightarrow \theta x_1 + (1-\theta) x_2 \in S$ ✓

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(2)

$\{x \mid \|x-a\|_2^2 \leq \|x-b\|_2^2\}$

$\Rightarrow (x-a)^T (x-a) \leq (x-b)^T (x-b)$

$\cancel{x^T x} - x^T a - a^T x + a^T a \leq \cancel{x^T x} - x^T b - b^T x + b^T b$

$\Rightarrow 2(b-a)^T x \leq b^T b - a^T a$

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(3) الف

arbitrary line: $\{x_0 + tv \mid v \in \mathbb{R}^N\} = L \qquad x_0 = 0$

$\Rightarrow L \cap C \Rightarrow C = \{tv \mid \underbrace{(tv)^T A (tv) + b^T (tv) + c}_{f(v)} \leq 0\}$

$\Rightarrow v_1, v_2 \in C \overset{?}{\Rightarrow}$

$v_3 = \theta v_1 + (1-\theta) v_2 \in C$

$\bar{x} \mid \bar{x}_2$

$\nabla^2 f(v) = t^2 A \succeq 0 \Rightarrow f$ is a Convex function.

$\Rightarrow f(v_3) \leq \theta f(v_1) + (1-\theta) f(v_2) \leq 0 \Rightarrow v_3 \in C$

$C \cap \{g^T x + h = 0\} \cap \forall$ Line (Convex)?

$\forall \text{Line} = \{ x_0 + tv \mid v \in R^n \}$

$I = \{ x_0 + tv \mid (x_0+tv)^T A (x_0+tv) + b^T(x_0+tv) + c \leq 0,$

$g^T tv + \underbrace{g^T x_0 + h}_{0} = 0 \}$

if $x_0 \in \{g^T x + h = 0\}$

$\Rightarrow$ — $g^T v \neq 0 \Rightarrow t = 0 \Rightarrow x_0^T A x_0 + b^T x_0 + c \leq 0$

\ $g^T v = 0 \Rightarrow I$ is Part (a)

(a) is Convex if $A \succeq 0$ or $\forall v \;\; v^T A v \geq 0$.

$\Rightarrow \exists \lambda \quad A + \lambda g g^T \succeq 0 \Rightarrow v^T A v = v^T \underbrace{(A + \lambda g g^T)}_{\succeq 0} v \geq 0$

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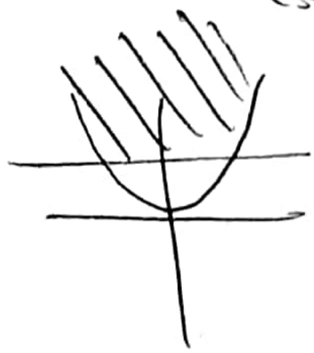
$$C = \{x \mid -x^2 + 1 \le 0\}$$

$$= \{x \mid x^2 \ge 1\}$$

$A = -1$

$b = 0$

$c = +1$



Convex

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$$= \{x \in R^2 \mid x_1 x_2 \ge 1\}$$

$$x \in C,\ y \in C \overset{?}{\Rightarrow} \theta x + (1-\theta) y \in C$$

$$x_1 x_2 \ge 1 \quad y_1 y_2 \ge 1 \overset{?}{\Rightarrow} (\theta x_1 + (1-\theta) y_1)(\theta x_2 + (1-\theta) y_2) \ge 1$$

$$\ge x_1^{\theta}\, y_1^{(1-\theta)} \qquad \ge x_2^{\theta}\, y_2^{(1-\theta)}$$

$$x_1^{\theta}\, x_2^{\theta}\, y_1^{(1-\theta)} \cdot y_2^{(1-\theta)} \ge 1$$

توابع محدب ۱

(1)

الف)

$$f(x)=\frac{x_1}{x_2} \qquad \nabla^2 f=\begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix} \qquad Z=\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$Z^T\nabla^2 f\, Z \overset{?}{\geq} 0 = \frac{-2z_1z_2}{x_2^2}+\frac{2z_2^2x_1}{x_2^3} \Rightarrow$$

نه همواره مثبت

نه همواره منفی

⇐ نه Convex نه Concave.

$$f(x)\leq\alpha \Rightarrow \frac{x_1}{x_2}\leq\alpha \Rightarrow \begin{bmatrix} 1 \\ -\alpha \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 0 \Rightarrow \text{half space}$$

⇒ quasi Convex است.

$$f(x)\geq\alpha \Rightarrow \frac{x_1}{x_2}\geq\alpha \Rightarrow \begin{bmatrix} 1 \\ -\alpha \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0 \Rightarrow$$

quasi Concave است.

ب)

$$f(x)=\frac{1}{x_1x_2} \Rightarrow \nabla^2 f=\begin{bmatrix} \frac{2}{x_1^3x_2} & \frac{1}{x_1^2x_2^2} \\ \frac{1}{x_1^2x_2^2} & \frac{2}{x_1x_2^3} \end{bmatrix}$$

$$x\in R^2_{++}$$

$$|\nabla^2 f|=\frac{3}{x_1^4x_2^4}\geq 0$$

$$\operatorname{trac}(\nabla^2 f)=\frac{2}{x_1x_2}\left(\frac{1}{x_1^2}+\frac{1}{x_2^2}\right)\geq 0$$

$$\Rightarrow \lambda_1,\lambda_2\geq 0$$

$$\nabla^2 f\succeq 0$$

⇐ Convex، quasi Convex، Concave نیست

$\frac{1}{x_1 x_2} \geq \alpha \xrightarrow{R^2_{++}} x_1 x_2 \cancel{\leq} \frac{1}{\alpha}$

; quasi Concave

super level set هاش محدب نیست

= quasi Concave نیست

← تابع مورد نظر

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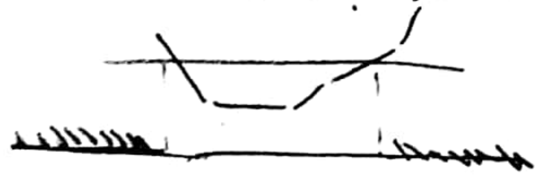
(1) توابع affine ترکیبشان با توابع Convex ، محدب بودن را حفظ می‌کند

← همه توابع Convex هستند. به quasi Convex هستند.

Convex نیست →



مثال نقض ← quasi Concave نیست

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(2)

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ f(x) & f(y) & f(z) \end{vmatrix} \geq 0 \Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-y \\ f(x) & f(y)-f(x) & f(z)-f(y) \end{vmatrix} \geq 0$$

$$\Rightarrow (y-x)(f(z)-f(y)) - (z-y)(f(y)-f(x)) \geq 0$$

$$f(y)\left((z-y)+(y-x)\right) \leq f(z)(y-x) + (z-y)f(x)$$

$$\theta = \frac{y-x}{(z-y)+(y-x)} \Rightarrow y = \theta z + (1-\theta)x$$

$$1-\theta = \frac{z-y}{(z-y)+(y-x)}$$

$f>0$, convex $\xrightarrow{\text{تعریف}}$ $\big(f(\overbrace{\theta x_1+(1-\theta)x_2}^{x_3})\big)^2 \leq \big(\theta f(x_1)+(1-\theta)f(x_2)\big)^2$ ③

$g>0$, Concave. $g(\theta x_1+(1-\theta)x_2) \geq \theta g(x_1)+(1-\theta)g(x_2)$

$$\Rightarrow \frac{f^2(x_3)}{g(x_3)} \leq \underline{\frac{\big(\theta f(x_1)+(1-\theta)f(x_2)\big)^2}{\theta g(x_1)+(1-\theta)g(x_2)}}$$

$$h(y) = \frac{y_1^2}{y_2} \xrightarrow{y_2>0} h(y) \text{ convex}$$

$$\nabla^2 h = \begin{bmatrix} \frac{2}{y_2} & \frac{-2y_1}{y_2^2} \\ \frac{-2x_1}{y_2^2} & \frac{2y_1^2}{y_2^3} \end{bmatrix} \Rightarrow |\nabla^2 h| = 0$$

$$tr(\nabla^2 h) = \frac{2}{y_2}\left(1+\frac{2y_1^2}{y_2^2}\right) > 0$$

$$\underline{\nabla^2 h \succeq 0}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\xrightarrow{h \text{ Convex}} h(\overset{\theta Y_1+(1-\theta)Y_2}{Y_3}) \leq \theta h(Y_1)+(1-\theta)h(Y_2)$$

$$Y_3 = \begin{bmatrix} \theta f(x_1)+(1-\theta)f(x_2) \\ \theta g(x_1)+(1-\theta)g(x_2) \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} f(x_1) \\ g(x_1) \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} f(x_2) \\ g(x_2) \end{bmatrix}$$