As in the exercise, we consider the conjugate gradient method for $\mathbf{A}\mathbf{y} = \mathbf{r}_0$, with $\mathbf{r}_0 = \mathbf{b} - \mathbf{A}\mathbf{x}_0$. Starting with

$$\mathbf{y}_0 = \mathbf{0}, \qquad \mathbf{s}_0 = \mathbf{r}_0 - \mathbf{A}\mathbf{y}_0 = \mathbf{r}_0, \qquad \mathbf{q}_0 = \mathbf{s}_0 = \mathbf{r}_0,$$

one computes, for any $k \geq 0$,

$$\gamma_k := \frac{\mathbf{s}_k^{\mathrm{T}}\mathbf{s}_k}{\mathbf{q}_k^{\mathrm{T}}\mathbf{A}\mathbf{q}_k}, \qquad \mathbf{y}_{k+1} = \mathbf{y}_k + \gamma_k\mathbf{q}_k, \qquad \mathbf{s}_{k+1} = \mathbf{s}_k - \gamma_k\mathbf{A}\mathbf{q}_k,$$

$$\delta_k := \frac{\mathbf{s}_{k+1}^{\mathrm{T}}\mathbf{s}_{k+1}}{\mathbf{s}_k^{\mathrm{T}}\mathbf{s}_k}, \qquad \mathbf{q}_{k+1} = \mathbf{s}_{k+1} + \delta_k\mathbf{q}_k.$$

How are the iterates $\mathbf{y}_k$ and $\mathbf{x}_k$ related? As remarked above, $\mathbf{s}_0 = \mathbf{r}_0$ and $\mathbf{q}_0 = \mathbf{r}_0 = \mathbf{p}_0$. Suppose $\mathbf{s}_k = \mathbf{r}_k$ and $\mathbf{q}_k = \mathbf{p}_k$ for some $k \geq 0$. Then

$$\mathbf{s}_{k+1} = \mathbf{s}_k - \gamma_k\mathbf{A}\mathbf{q}_k = \mathbf{r}_k - \frac{\mathbf{r}_k^{\mathrm{T}}\mathbf{r}_k}{\mathbf{p}_k^{\mathrm{T}}\mathbf{A}\mathbf{p}_k}\mathbf{A}\mathbf{p}_k = \mathbf{r}_k - \alpha_k\mathbf{A}\mathbf{p}_k = \mathbf{r}_{k+1},$$

$$\mathbf{q}_{k+1} = \mathbf{s}_{k+1} + \delta_k\mathbf{q}_k = \mathbf{r}_{k+1} + \frac{\mathbf{r}_{k+1}^{\mathrm{T}}\mathbf{r}_{k+1}}{\mathbf{r}_k^{\mathrm{T}}\mathbf{r}_k}\mathbf{p}_k = \mathbf{p}_{k+1}.$$

It follows by induction that $\mathbf{s}_k = \mathbf{r}_k$ and $\mathbf{q}_k = \mathbf{p}_k$ for all $k \geq 0$. In addition,

$$\mathbf{y}_{k+1} - \mathbf{y}_k = \gamma_k\mathbf{q}_k = \frac{\mathbf{r}_k^{\mathrm{T}}\mathbf{r}_k}{\mathbf{p}_k^{\mathrm{T}}\mathbf{A}\mathbf{p}_k}\mathbf{p}_k = \mathbf{x}_{k+1} - \mathbf{x}_k, \qquad \text{for any } k \geq 0,$$

so that $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_0$.

The problem is

$$\text{minimize} \quad \frac{1}{N}\sum_{i=1}^{N}(d_i - d(x_i, y_i))^2$$

with variable $P \in \mathbf{S}_+^n$. This problem can be rewritten as

$$\text{minimize} \quad \frac{1}{N}\sum_{i=1}^{N}(d_i^2 - 2d_i d(x_i, y_i) + d(x_i, y_i)^2),$$

with variable P (which enters through $d(x_i, y_i)$). The objective is convex because each term of the objective can be written as (ignoring the $1/N$ factor)

$$d_i^2 - 2d_i\left((x_i - y_i)^T P(x_i - y_i)\right)^{1/2} + (x_i - y_i)^T P(x_i - y_i),$$

which is convex in P. To see this, note that the first term is constant and the third term is linear in P. The middle term is convex because it is the negation of the composition of a concave function (square root) with a linear function of P.

7.3 *Probit model.* Suppose $y \in \{0, 1\}$ is random variable given by

$$y = \begin{cases} 1 & a^T u + b + v \leq 0 \\ 0 & a^T u + b + v > 0, \end{cases}$$

where the vector $u \in \mathbf{R}^n$ is a vector of explanatory variables (as in the logistic model described on page 354), and v is a zero mean unit variance Gaussian variable.
Formulate the ML estimation problem of estimating a and b, given data consisting of pairs (u_i, y_i), $i = 1, \ldots, N$, as a convex optimization problem.
Solution. We have

$$\textbf{prob}(y = 1) = Q(a^T u + b), \qquad \textbf{prob}(y = 0) = 1 - Q(a^T u + b) = P(-a^T u - b)$$

where

$$Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{t^2/2}\, dt.$$

The log-likelihood function is

$$l(a, b) = \sum_{y_i = 1} \log Q(a^T u_i + b) + \sum_{y_i = 0} \log Q(-a^T u_i - b),$$

which is a concave function of a and b.

الف)

$$\min \sum_{i=1}^{m} \phi(r_i) \qquad \mathcal{L}(x,r,\nu) = \sum_{i=1}^{m} \phi(r_i) + \nu^T(Ax-b-r)$$
$$r = Ax-b$$

$$g(\nu) = \min_{x,r} \left(\mathcal{L}(x,r,\nu)\right) = \min_r \Big(\underbrace{\min_x \Big(\sum_{i=1}^{m} \phi(r_i) - \nu^T b - \nu^T r + A^T\nu x\Big)}_{g_1(r,\nu)}\Big)$$

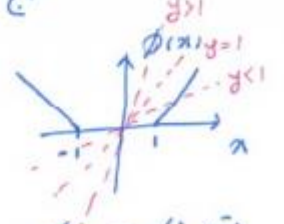
$$= \min_r \left(g_1(r,\nu) = \begin{cases} \sum_{i=1}^{m} \phi(r_i) - \nu^T r - \nu^T b & A^T\nu = 0 \\ -\infty & A^T\nu \neq 0 \end{cases} \right)$$

$$\xrightarrow{A^T\nu = 0} g(\nu) = \min \left(\sum_{i=1}^{m} \phi(r_i) - \sum_{i=1}^{m} \nu_i r_i - b^T\nu \right) = \min \sum_{i=1}^{m} \phi(r_i) - \nu_i r_i - b^T\nu$$

$$= \sum_{i=1}^{m} - \underbrace{\sup(\nu_i r_i - \phi(r_i))}_{\text{تعریف تابع conj}} - b^T\nu = \sum_{i=1}^{m} -\phi^*(\nu_i) - b^T\nu$$

$$\text{dual problem:} \quad \begin{aligned} \max \quad & \sum -\phi^*(\nu_i) - b^T\nu \quad (1) \\ \text{s.t} \quad & A^T\nu = 0 \end{aligned}$$

ب) صرفاً کافی است تابع conj را برای $\phi(x)$ داده شده بیابیم سپس ϕ^* را در (1) جایگذاری می‌کنیم:



$$\phi(x) = \begin{cases} 0 & |x| \le 1 \\ |x|-1 & |x| > 1 \end{cases} \qquad \phi^*(y) = \sup(yx - \phi(x))$$

با توجه به شکل مشخص است که در $\phi^*(y)$ که $\max$ فاصله خط yx با $\phi(x)$ مثبت برای $0 \le y \le 1$ ⇐ برابر y است و برای $y>1$ برابر ∞ می‌شود. مشابهاً این تعبیر برای y های منفی هم تکرار می‌شود لذا داریم

$$\phi^*(y) = \begin{cases} |y| & |y| \le 1 \\ \infty & |y| > 1 \end{cases}$$

لذا داریم

$$\begin{aligned} \max \quad & \sum |\nu_i| - b^T\nu \\ \text{s.t} \quad & A^T\nu = 0, \ |\nu_i| \le 1 \end{aligned} \equiv \begin{aligned} \max \quad & \|\nu\|_1 - b^T\nu \\ \text{s.t} \quad & A^T\nu = 0, \ \|\nu\|_\infty \le 1 \end{aligned}$$

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$$\mathcal{L}(x, y, \nu) = 0.5\|y-y_0\|_2^2 + \sum_{i=1}^{m}\|x_i\|_2 + \sum_{i=1}^{m}\nu_i^T(x_i - C_i y - d_i)$$

$$g(\nu) = \min_y \Big(\underbrace{\min_x \sum_{i=1}^{m}\|x_i\|_2 + \nu_i^T x_i}_{\sum_{i=1}^{m} -\sup(-\|x_i\|_2 - \nu_i^T x_i)} + 0.5\|y-y_0\|_2^2 + \sum_{i=1}^{m} -\nu_i^T C_i y + \sum_{i=1}^{m} -\nu_i^T d_i \Big)$$

$$= 0 \quad \|\nu_i\|_2 \leq 1 \quad i = 1, \dots, m$$

$$= -\infty \quad \text{otherwise}$$

$$g(\nu) = \min_y \Big(0.5\|y-y_0\|_2^2 + \sum_{i=1}^{m} -\nu_i^T C_i y + \sum_{i=1}^{m} -\nu_i^T d_i \Big)$$

$i = 1, \dots, m \quad \|\nu_i\|_2 \leq 1$

$$\nabla_y = 0 \qquad (y - y_0) - \sum C_i^T \nu_i = 0 \implies y = y_0 + \sum C_i^T \nu_i$$

$$g(\nu) = \begin{cases} 0.5\|\sum C_i^T \nu_i\|_2^2 & \\ -\infty & \text{otherwise} \end{cases}$$

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$$\min f(x) \quad \text{s.t.} \quad Ax=b \implies L(x,\nu) = f(x) + \nu^T(Ax-b)$$

$$\nabla L(x,\nu) = \nabla f(x) + A^T\nu \quad (1)$$

$$\arg\min_x f(x) + a\|Ax-b\|_2^2 = x^* \implies \nabla f(x^*) + 2aA^T(Ax^*-b) = 0 \quad (2)$$

به ازای (1), (2): x^* ، $\nu^* = 2a(Ax^*-b)$ ، $L(x,\nu)$ را min کننده است ν^* ، feasible است

$$g(\nu^*) = f(x^*) + \nu^{*T}(Ax^*-b) = f(x^*) + 2a\|Ax^*-b\|_2^2$$

(a) We can rewrite

$$R^{\mathrm{wc}} = \sup_{A\in\mathcal{A}} \sup_{\|b\|_2\le 1} \|A(p(A)b) - b\|_2,$$

and recognize the inner supremum as the definition of the spectral norm of $Ap(A) - I$. If A is symmetric, then $Ap(A) - I$ is also symmetric, and its spectral norm is the largest absolute value of its eigenvalues.

Let QDQ^T be an eigenvalue decomposition of A, with Q orthogonal and D diagonal. Then

$$\begin{aligned} Ap(A) - I &= c_0A + c_1A^2 + \cdots + c_kA^{k+1} - I \\ &= c_0QDQ^T + c_1QD^2Q^T + \cdots + c_kQD^{k+1}Q^T - I \\ &= Q(c_0D + c_1D^2 + \cdots + c_kD^{k+1} - I)Q^T \\ &= Q(Dp(D) - I)Q^T, \end{aligned}$$

which shows that $\lambda p(\lambda) - 1 \in \sigma(Ap(A) - I)$ if $\lambda \in \sigma(A)$.

We can then rewrite

$$\begin{aligned} R^{\mathrm{wc}} &= \sup_{A\in\mathcal{A}} \|Ap(A) - I\|_2 \\ &= \sup_{A\in\mathcal{A}} \sup_{\lambda\in\sigma(A)} |\lambda p(\lambda) - 1| \\ &= \sup_{\lambda\in\Omega} |\lambda p(\lambda) - 1| \\ &= \sup_{\lambda\in\Omega} |c_0\lambda + c_1\lambda^2 + \cdots + c_k\lambda^{k+1} - 1|, \end{aligned}$$

and note that R^{wc} is a convex function of c since, for any λ, $c_0\lambda + c_1\lambda^2 + \cdots + c_k\lambda^{k+1} - 1$ is an affine function of c. We can write the optimization problem as

$$\text{minimize} \quad \sup_{\lambda\in\Omega} |c_0\lambda + c_1\lambda^2 + \cdots + c_k\lambda^{k+1} - 1|.$$

This problem can be converted exactly into an SDP (since the supremum of a polynomial has an LMI representation), but for any practical purpose simple sampling of Ω is fine.