Assignment #2

Bayesian Decision Theory

How TA evaluates your assignments:

Report: half of your score will be graded proportional to the quality of your report. You should provide a distinct section for each problem, include the desired outputs and explain what you've done. Don't forget to discuss your results as well. It is not necessary to accommodate your source codes in your reports unless you want to refer to them. Compactness, expressiveness and neatness are of high importance.

Source Code: create an m-file for any problem and write all your codes there. If a problem consists of several sub-problems, separate them by comments in your code. Finally, name your m-files according to the number of the problems.

As you have to upload you submission electronically, it is of high interest to prepare your reports using Microsoft Office tools or Latex. However, scanned handwritten solutions are also acceptable as long as they are readable, neat and expressive.

What to hand in:

You must submit your report (.pdf) and source codes (m-files) for each assignment. Zip all your files into an archive file and use the following template to name it:

HW2_XXXXX.zip

where XXXXX must be replaced with your student ID. Your file size must not be bigger than 20MB. Send your files to *mohammadhme@gmail.com* with a subject of PR961_HW2_ XXXXX (replace XXXXX with your student number).

The Due Date for This Assignment is: Aban 19th

Problems:

1. Consider the following 2-class classification problem involving a single feature x. Assume equal class priors and 0-1 loss function.

$$p(x \mid w_1) = \begin{Bmatrix} 2x & 0 \le x \le 1 \\ 0 & otherwise \end{Bmatrix} \qquad p(x \mid w_2) = \begin{Bmatrix} 2-2x & 0 \le x \le 1 \\ 0 & otherwise \end{Bmatrix}$$

   a. Sketch the two densities.
   b. State the Bayes decision rule and show the decision boundary.
   c. What is the Bayes classification error?
   d. How will the decision boundary change if the prior for class w_2 is increased to 0.7?

2. For a two-class recognition problem with salmon (ω = 1) and sea bass (ω = 2), suppose we have two features x = (x_1,x_2) and the two class-conditional densities, p(x | ω = 1) and p(x | ω = 2), are 2D Gaussian distributions centered at points (4, 16) and (16, 4) respectively with the same covariance matrix Σ = 4I (with I is the identity matrix). Suppose the priors are P(ω = 1) = 0.6 and P(ω = 2) = 0.4.

   a. Suppose we use a Bayes decision rule, write the two discriminant functions g_1(x) and g_2(x).
   b. Derive the equation for the decision boundary g_1(x) = g_2(x). Draw the boundary on the feature space (the 2D plane).

3. Consider the two-dimensional data points from two classes ω_1 and ω_2 below, and each of them come from a Gaussian distribution $p(x \mid \omega_k) \sim N(\mu_k, \Sigma_k)$.

Table 1: Data points from class ω_1 and ω_2

ω_1	ω_2
$(0,0)$	$(6,9)$
$(0,1)$	$(8,9)$
$(2,2)$	$(9,8)$
$(3,1)$	$(9,9)$
$(3,2)$	$(9,10)$
$(3,3)$	$(8,11)$

   a. What is the prior probability for each class, i.e. $p(\omega_1)$ and $p(\omega_2)$.
   b. Calculate the mean and covariance matrix for each class.
   c. Derive the equation for the decision boundary that separates these two classes, and plot the boundary. (Hint: you may want to use the posterior probability)
   d. Think of the case that the penalties for misclassification are different for the two classes (i.e. not zero-one loss), will it affect the decision boundary, and how?

4. Consider two normal distributions in one dimension: $N(\mu_1, \sigma_1^2)$and $N(\mu_2, \sigma_2^2)$. Imagine that we choose two random samples x_1 and x_2, one from each of the two distributions and calculate their sum $x_3 = x_1 + x_2$. Suppose we do this repeatedly.

   a. Consider the resulting distribution of values of x_3. Show that x_3 possesses the requisite statistical properties and thus its distribution is normal.
   b. What is the mean, μ_3, of the new distribution?
   c. What is the variance, σ_3^2, of the new distribution?

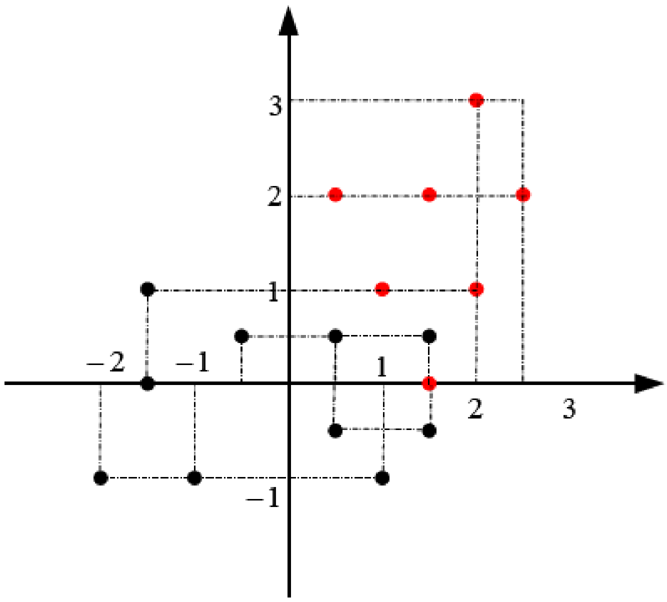
5. Consider a two-category classification problem in two dimensions with

$$p(X \mid \omega_1) \sim N(0, I),\ P(X \mid \omega_2) \sim N\left(\begin{pmatrix}1\\1\end{pmatrix}, I\right) \text{ and } P(\omega_1) = P(\omega_2) = 1/2.$$

   a. Calculate the Bayes decision boundary.
   b. Calculate the Bhattacharyya error bound.
   c. Calculate the Chernoff error bound.
   d. Repeat the above for the same prior probabilities, but

$$p(X \mid \omega_1) \sim N\left(0, \begin{pmatrix}2 & 0.5\\0.5 & 2\end{pmatrix}\right) \text{ and } p(X \mid \omega_2) \sim N\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}5 & 2\\2 & 5\end{pmatrix}\right)$$

6. Find mean and covariance metric for the following **Gaussian data** of two classes. Calculate decision region and plot it.



7. Consider a classification problem with 2 classes and a single real-valued feature vector X. For class 1, $p(x \mid c_1)$ is uniform U(a, b) with a = 2 and b = 4. For class 2, $p(x \mid c_2)$ is exponential with density $\lambda\exp(-\lambda x)$ where $\lambda = 1$. Let $p(c_1) = p(c_2) = 0.5$.

    a) Determine the location of the optimal decision regions.
    b) Draw a sketch of the two class densities multiplied by $P(c_1)$ and $P(c_2)$ respectively, as a function of x, clearly showing the optimal decision boundary (or boundaries).
    c) Compute the Bayes error rate for this problem within 3 decimal places of accuracy.

Computer Projects:

Implement the following projects from the reference book [1]. Make sure that your codes are well-commented. Code-only submissions will gain at most 33/100 pts. So, do not forget to provide a compact, well-documented and informative submission.

1. Implement the 7^{th} computer exercise of chapter 2 from the reference book [1].

Reference:

[1] Duda, Richard O., Peter E. Hart, and David G. Stork. Pattern classification. John Wiley & Sons, 2012.