Algorithms:
Greedy Method

Minimum Spanning Tree

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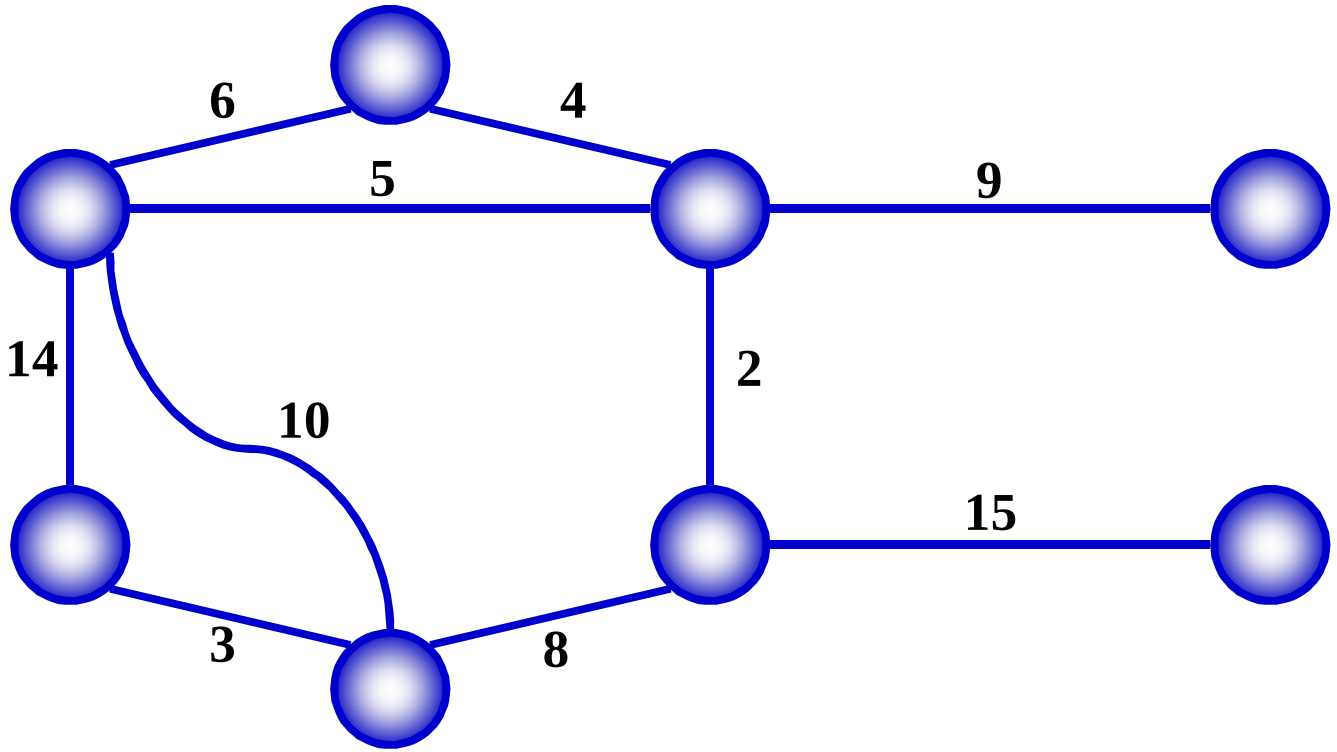
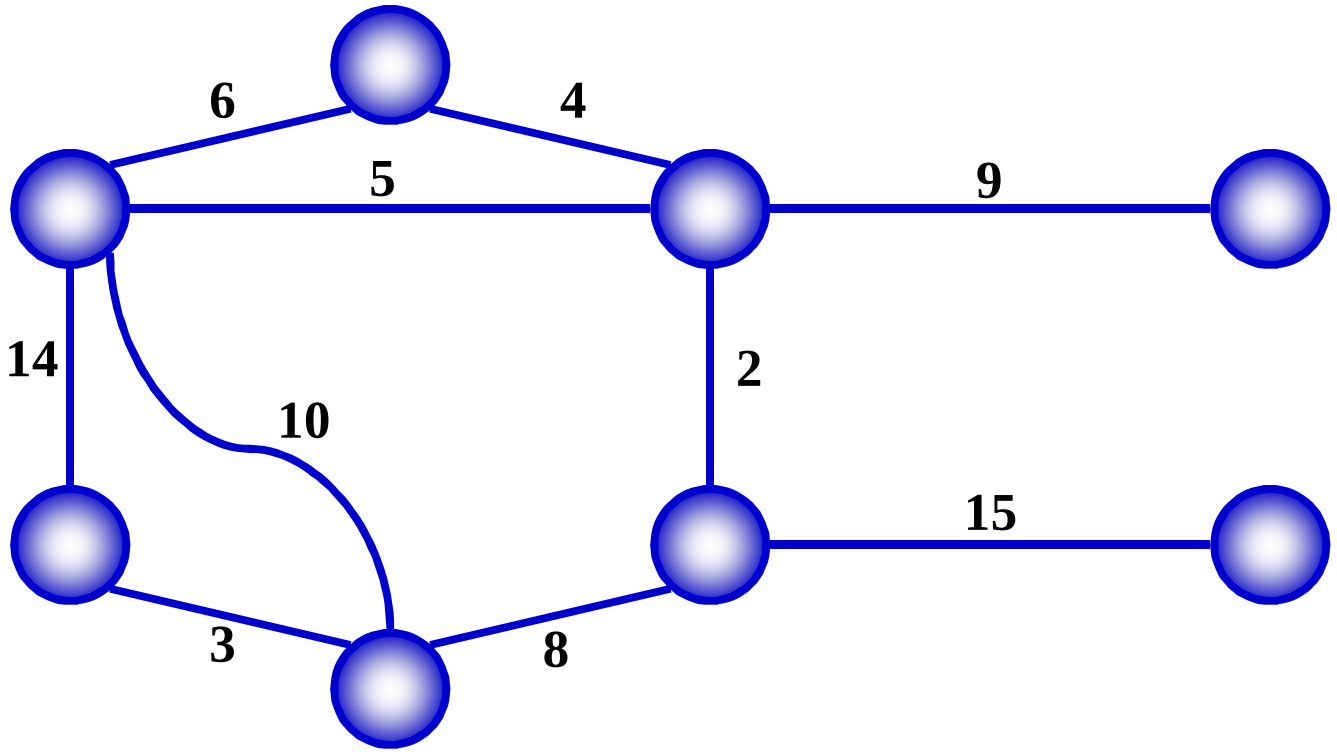
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph:



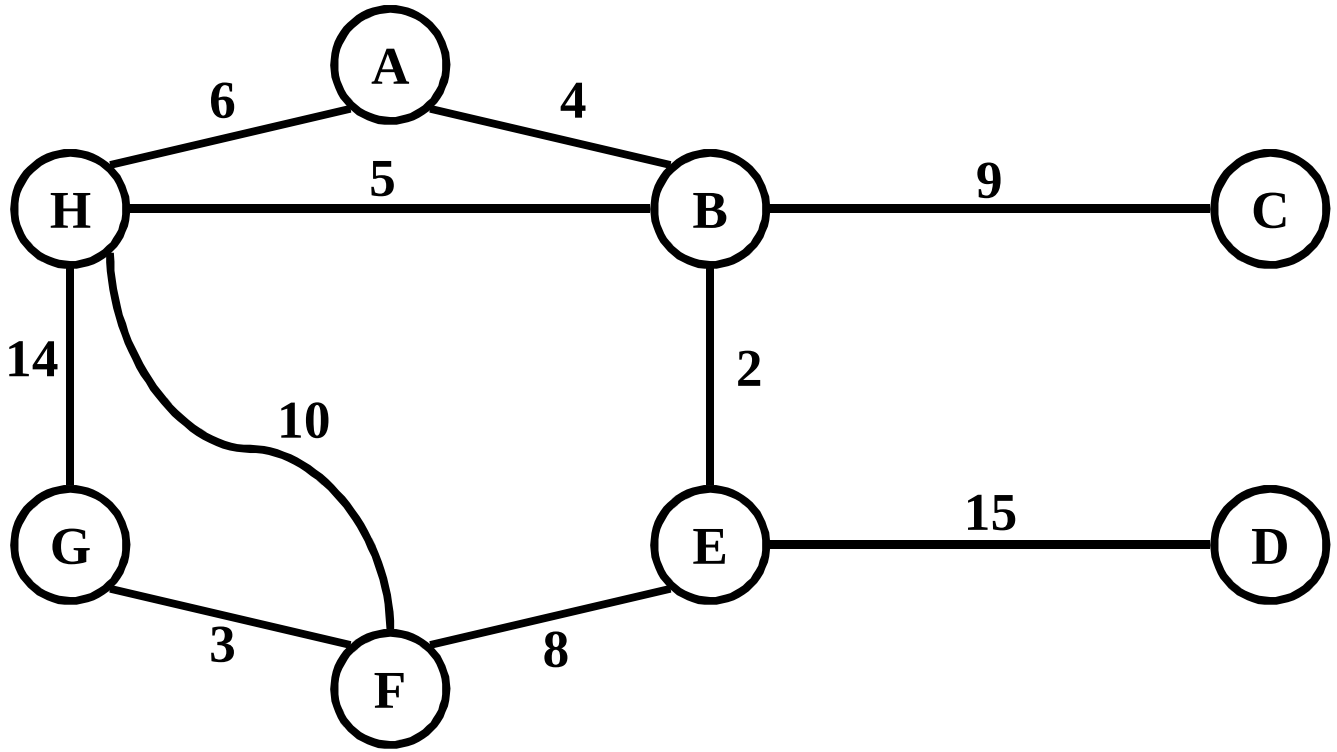
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph, find a *spanning tree* using edges that minimize the total weight



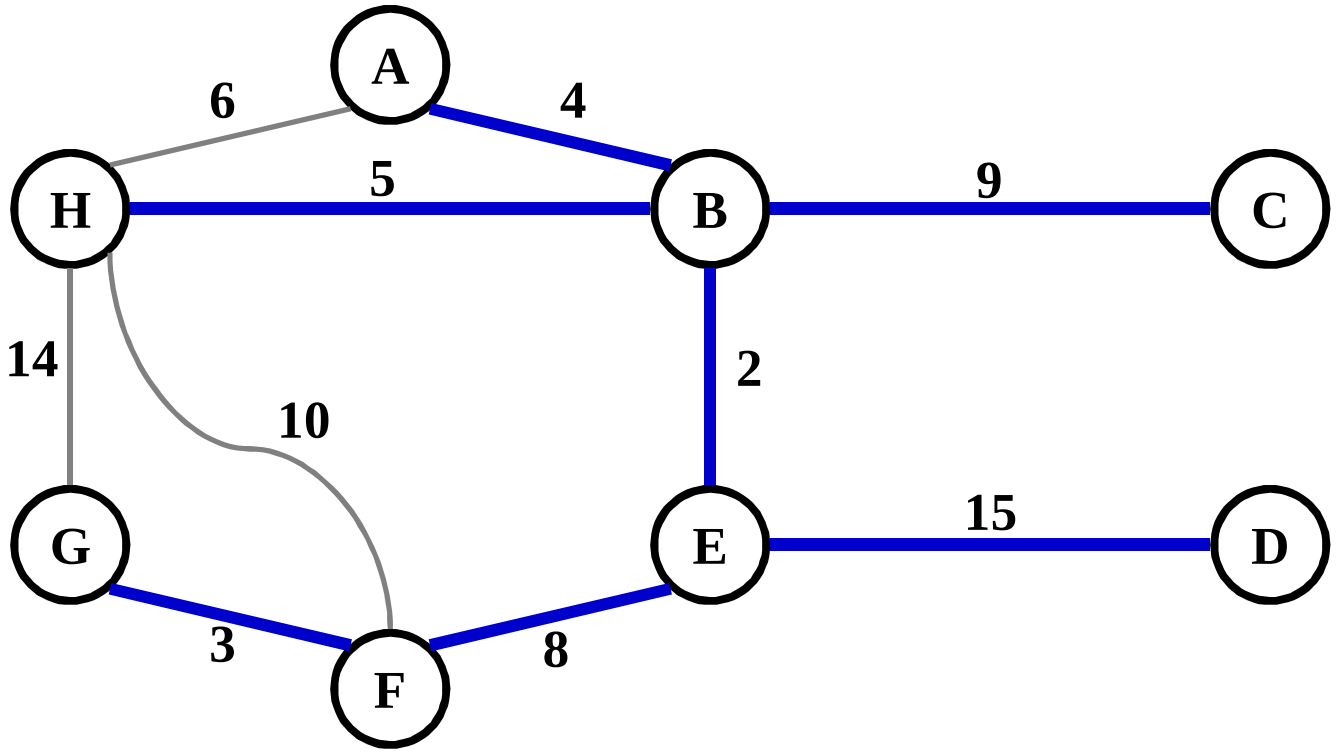
Minimum Spanning Tree

- *Which edges form the minimum spanning tree (MST) of the graph as shown below?*



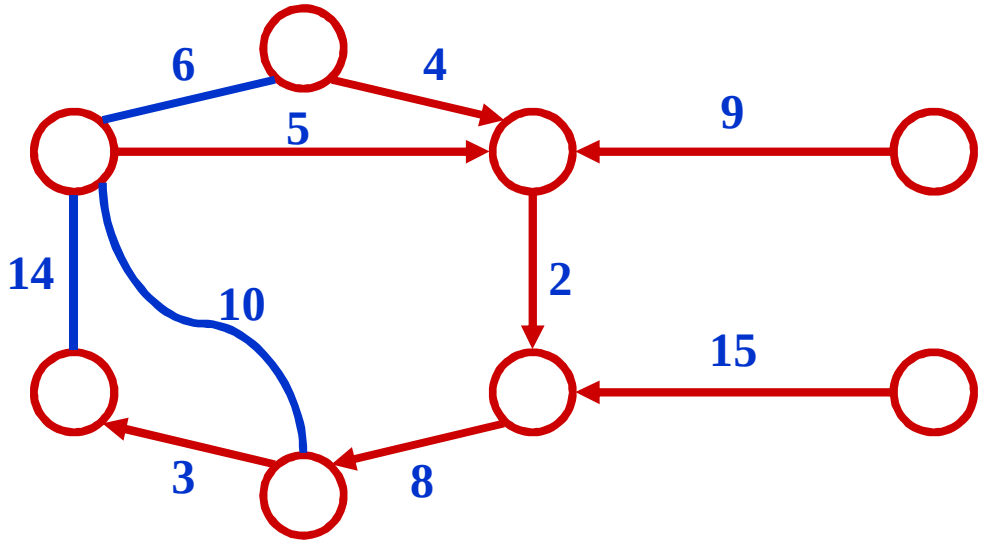
Minimum Spanning Tree

- Answer:



Minimum Spanning Tree

- MSTs satisfy the *optimal substructure* property: an optimal minimum spanning tree is composed of optimal minimum spanning subtrees
  - Let T be an MST of G with an edge (u, v) in the middle
  - Removing (u, v) partitions T into two trees T_1 and T_2
  - Claim: T_1 is an MST of $G_1 = (V_1,E_1)$, and T_2 is an MST of $G_2 = (V_2,E_2)$
    (*Do V_1 and V_2 share vertices? Why?*)
  - Proof: $w(T) = w(u,v) + w(T_1) + w(T_2)$
    (There can't be a better tree than T_1 or T_2. Then T would be suboptimal)



Prim's Algorithm

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MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
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    p[r] = NULL;
    while (Q not empty)
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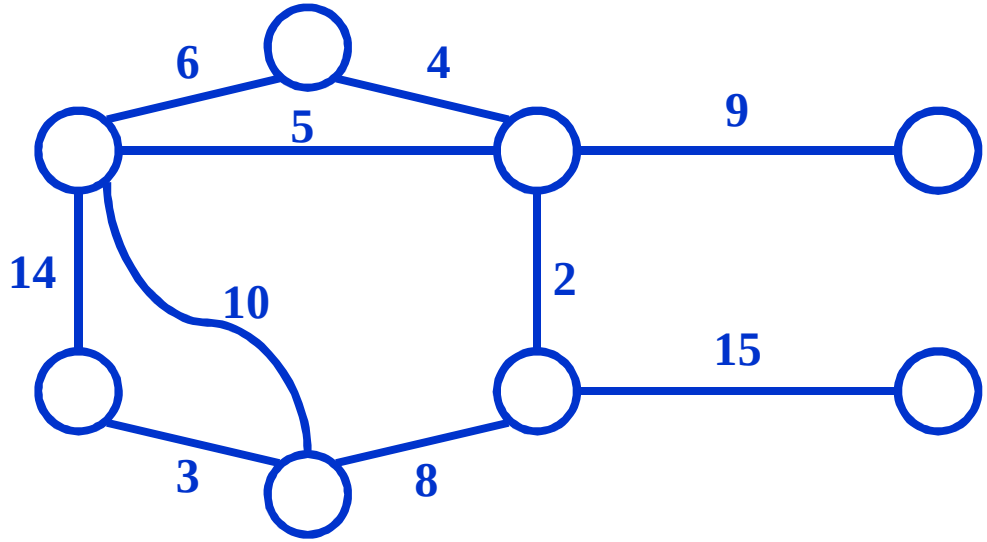

Run on example graph

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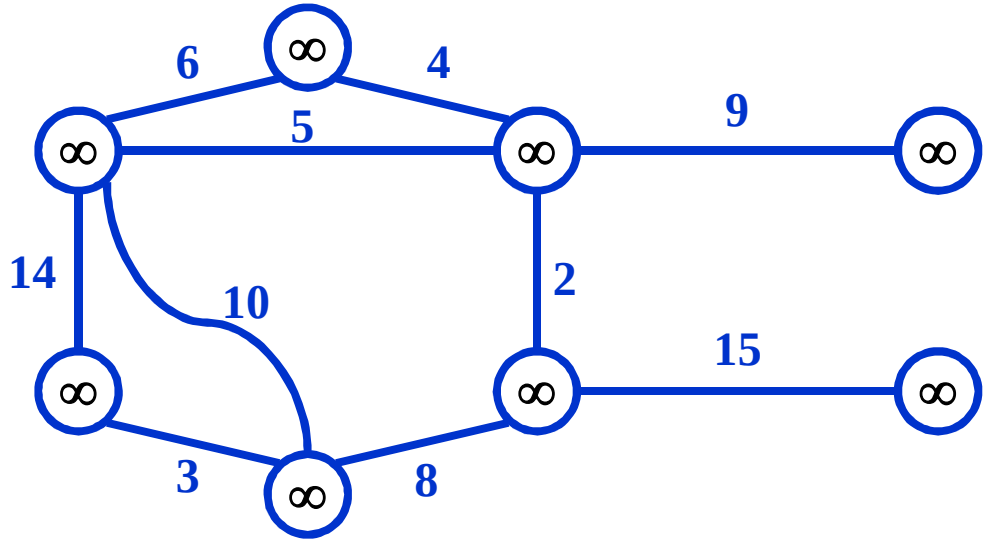

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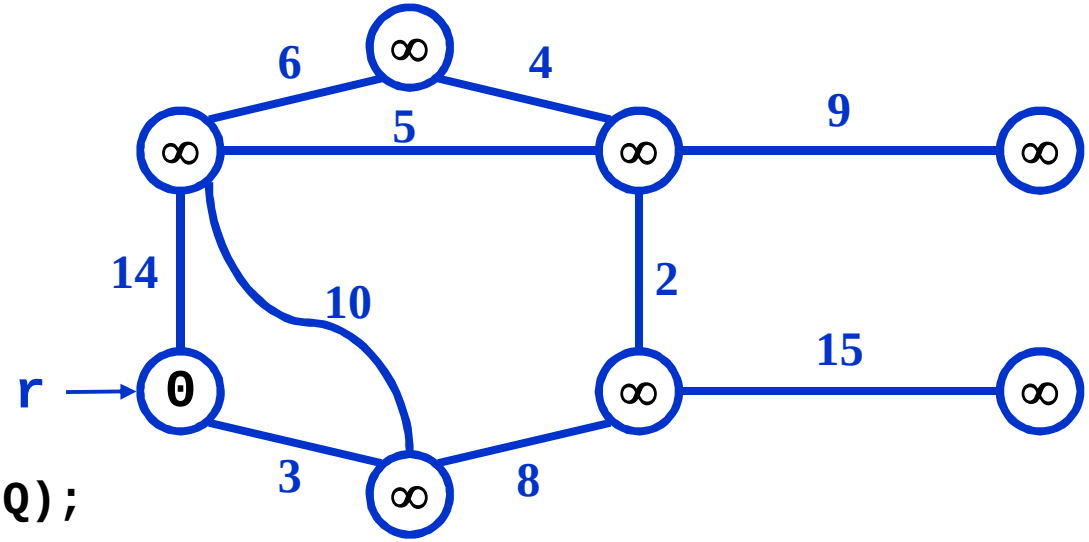

Pick a start vertex r

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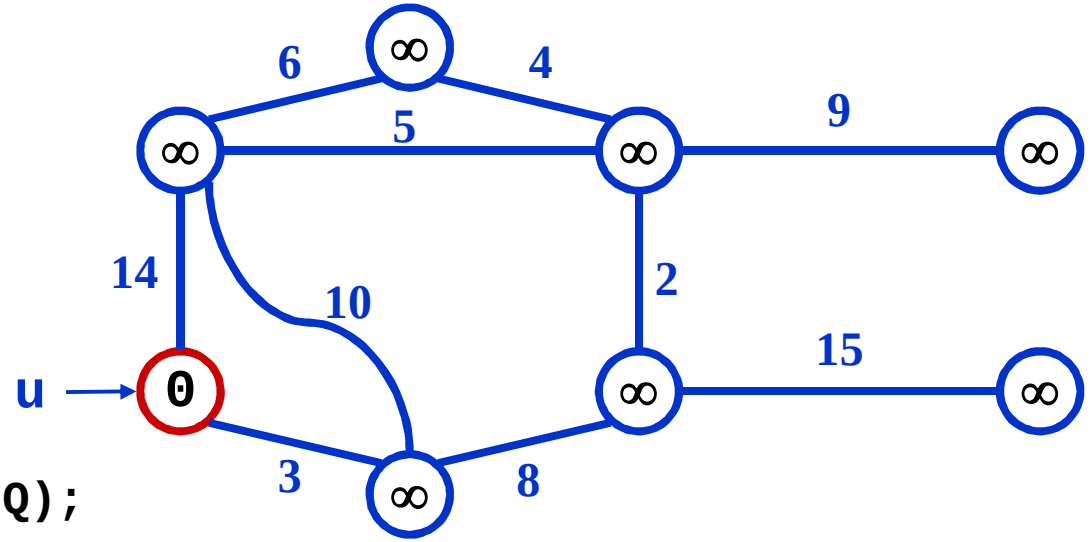

Red vertices have been removed from Q

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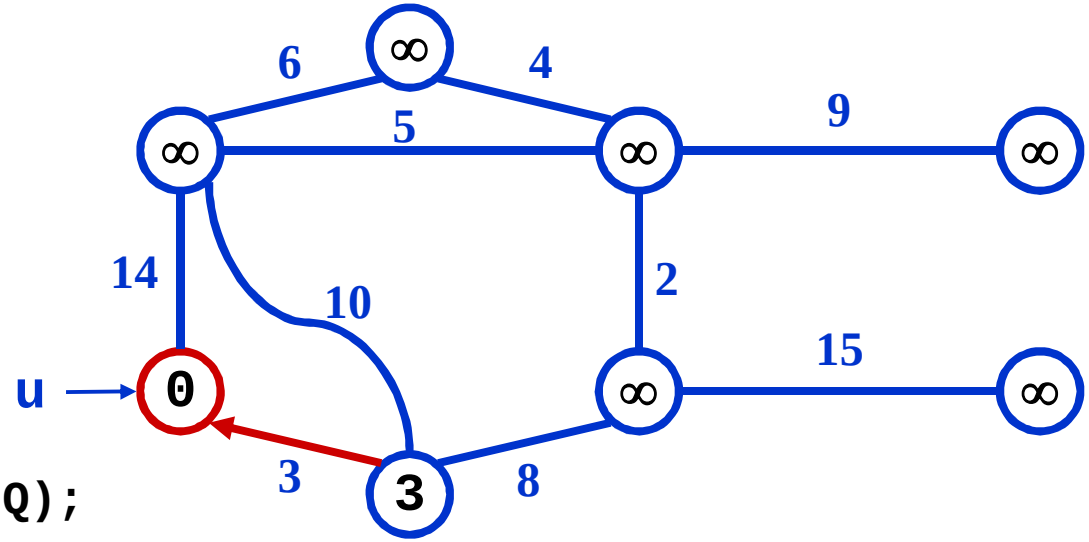

Red arrows indicate parent pointers

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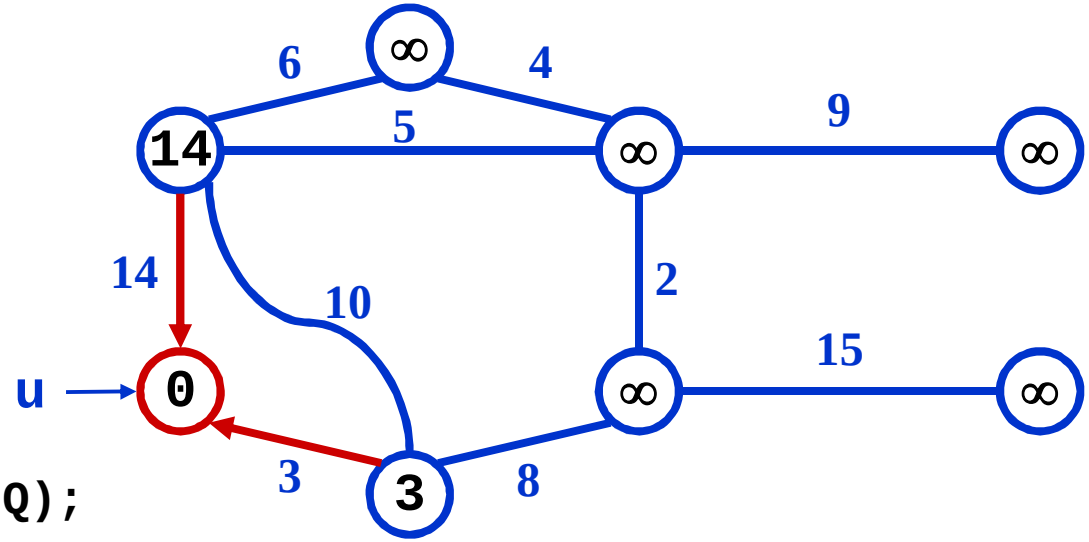

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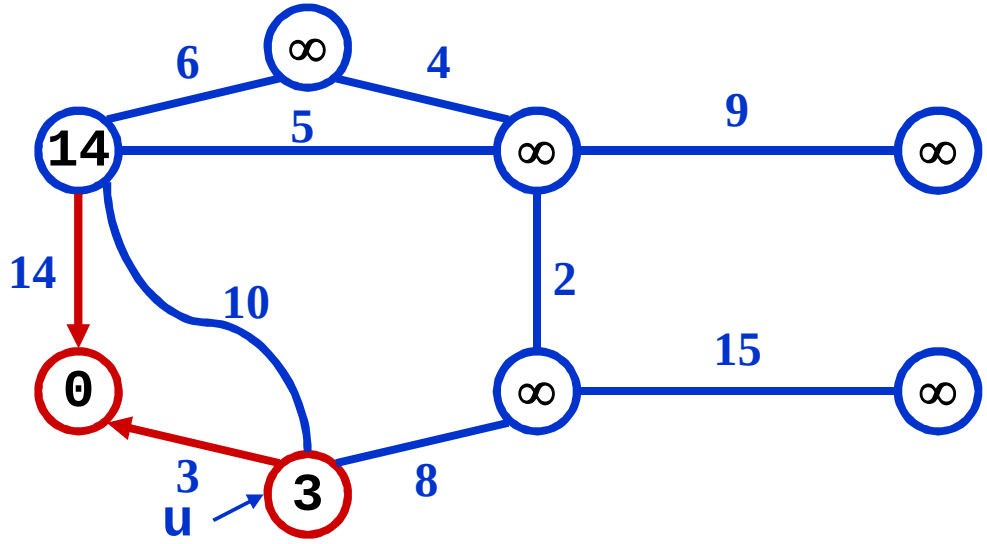

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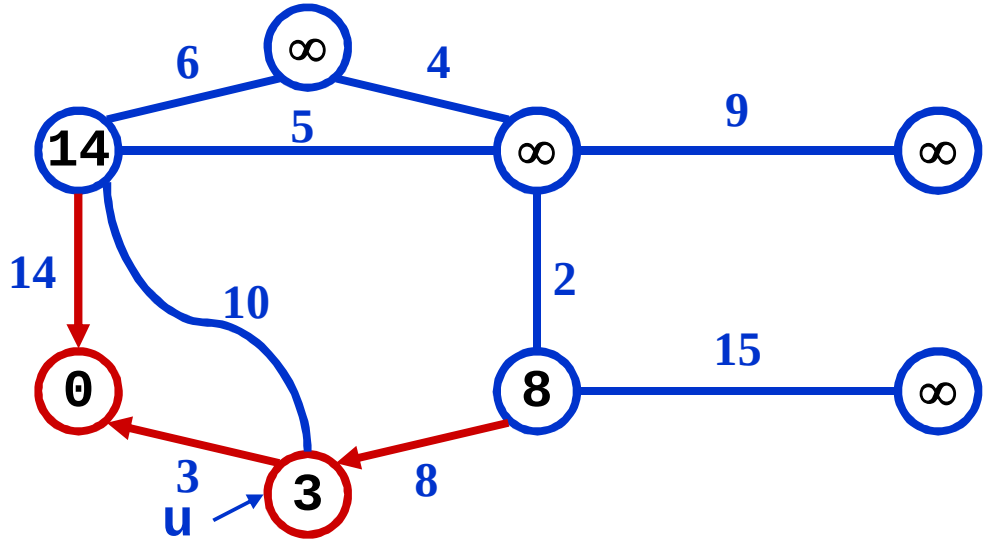

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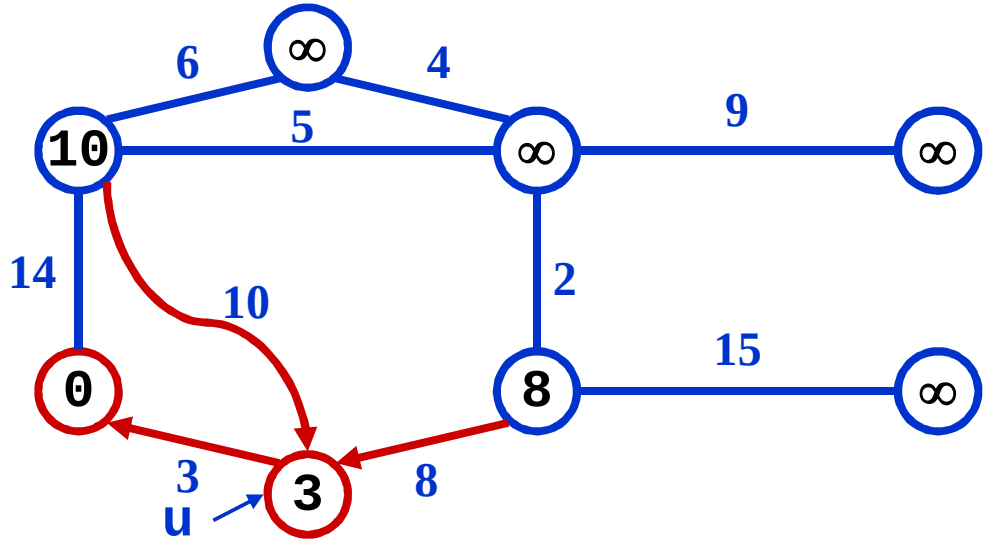

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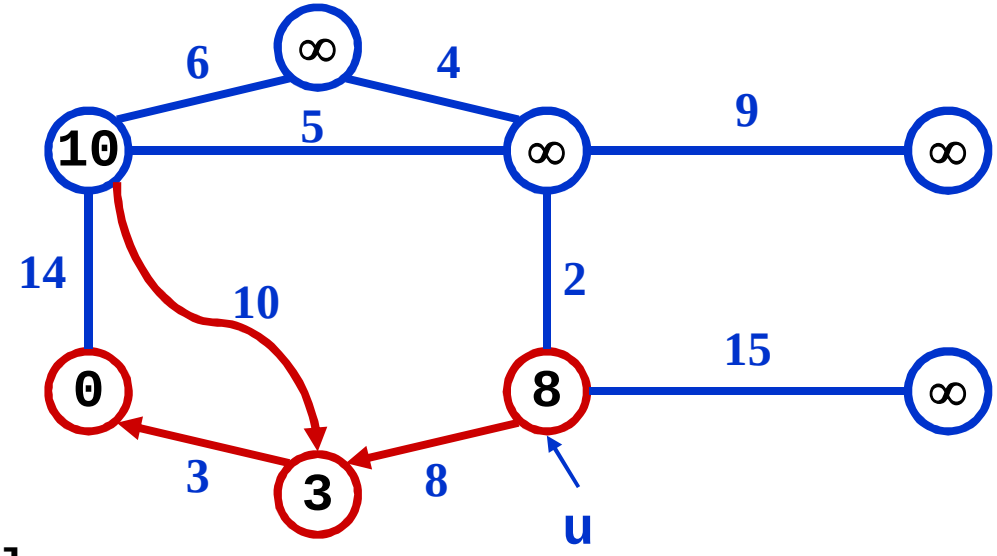

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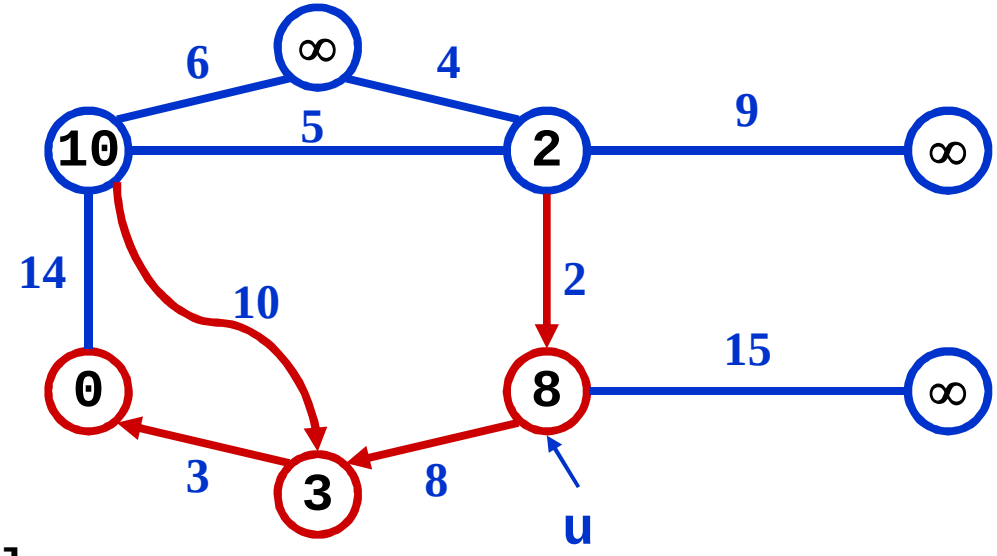

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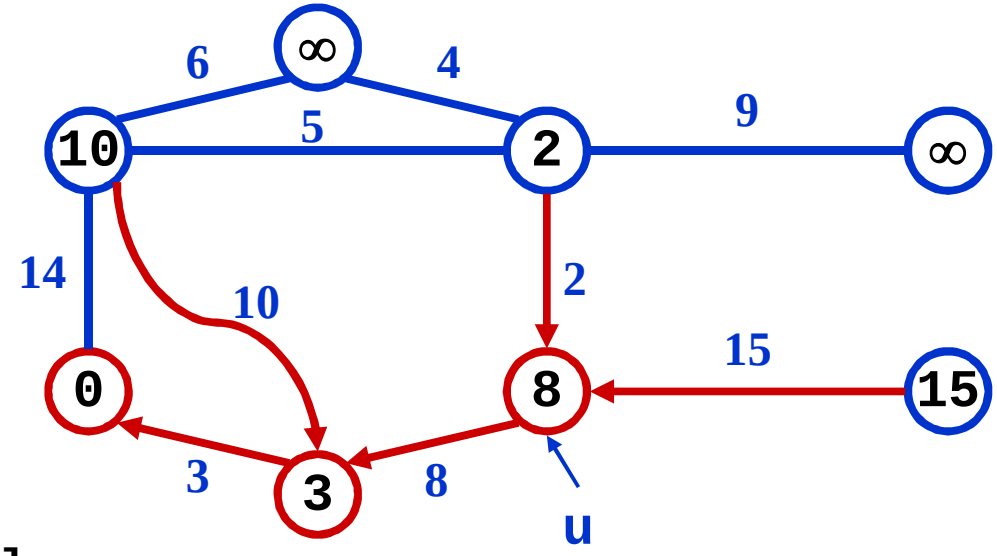

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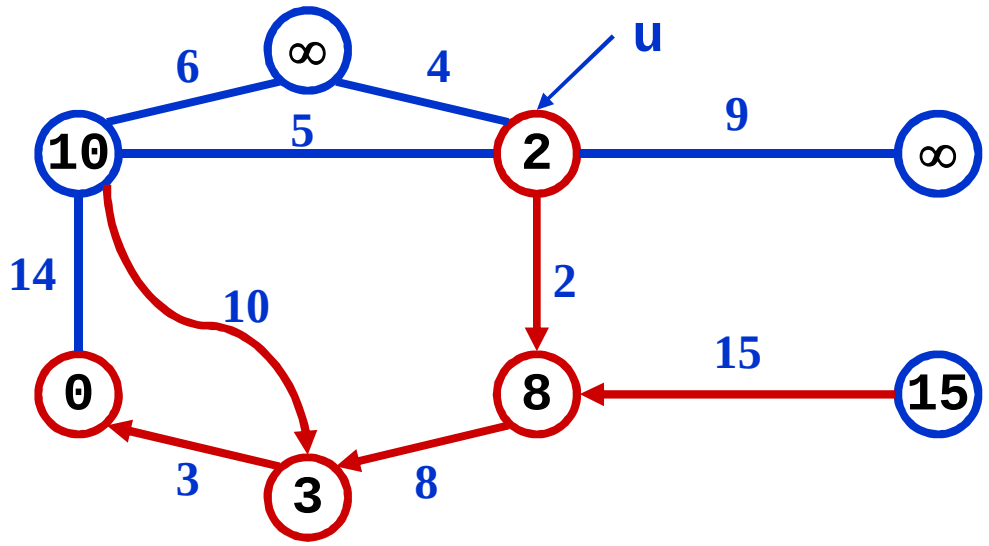

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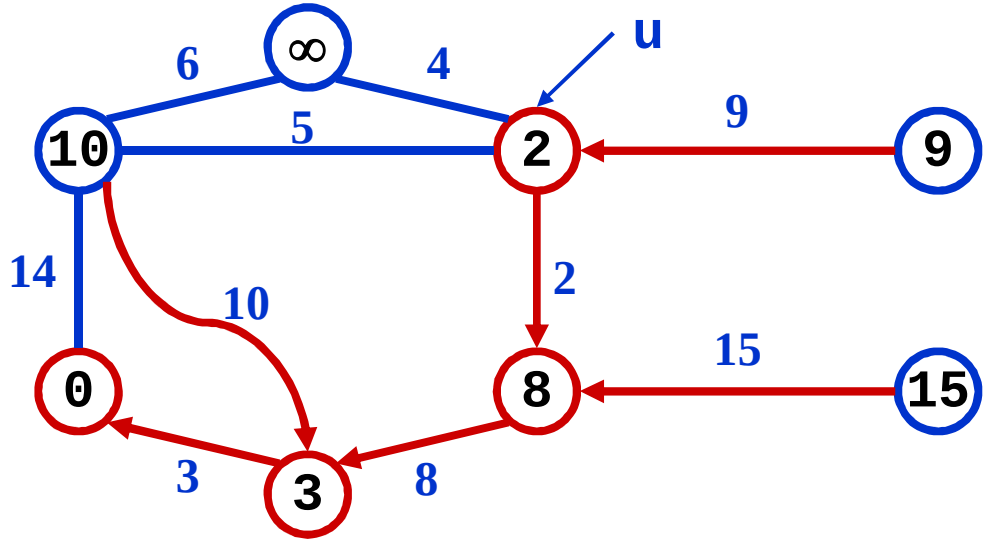

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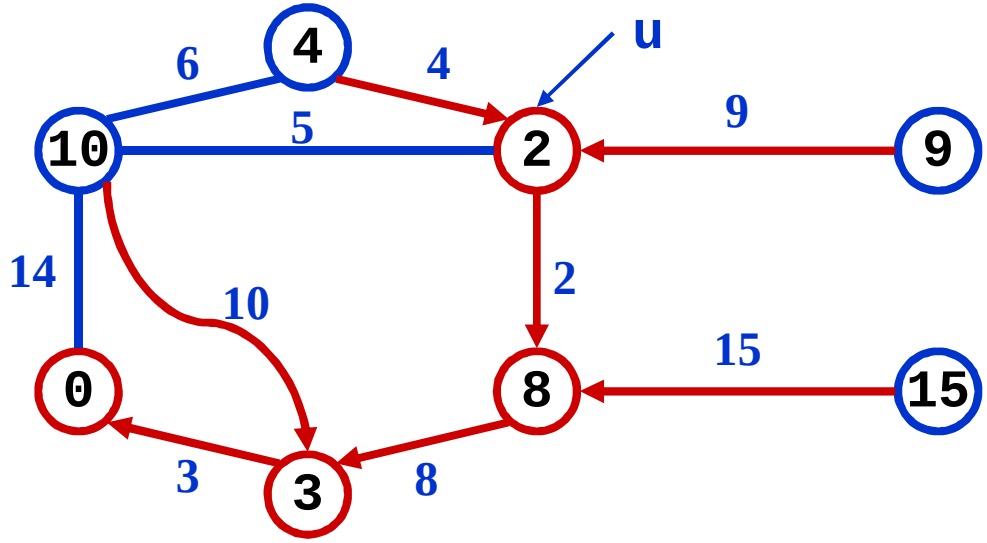

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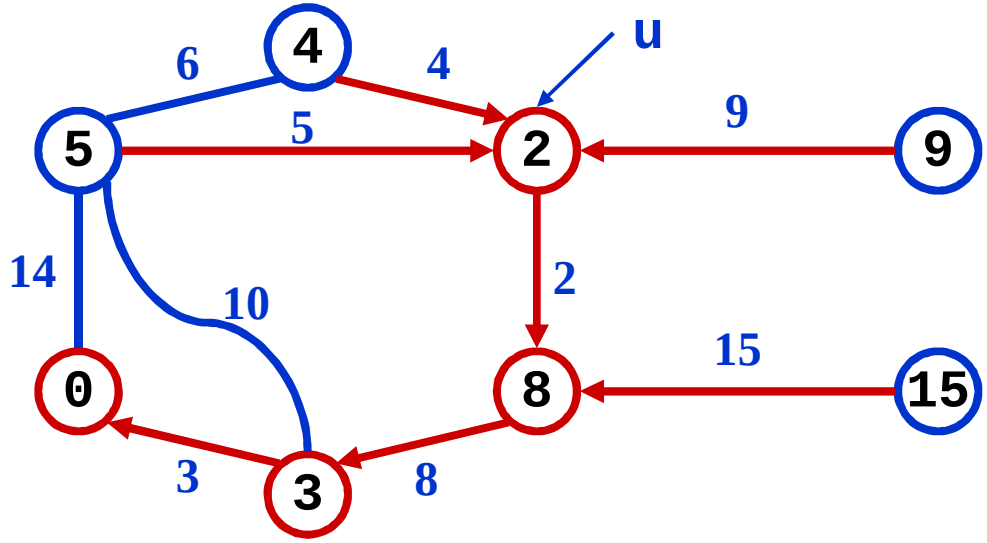

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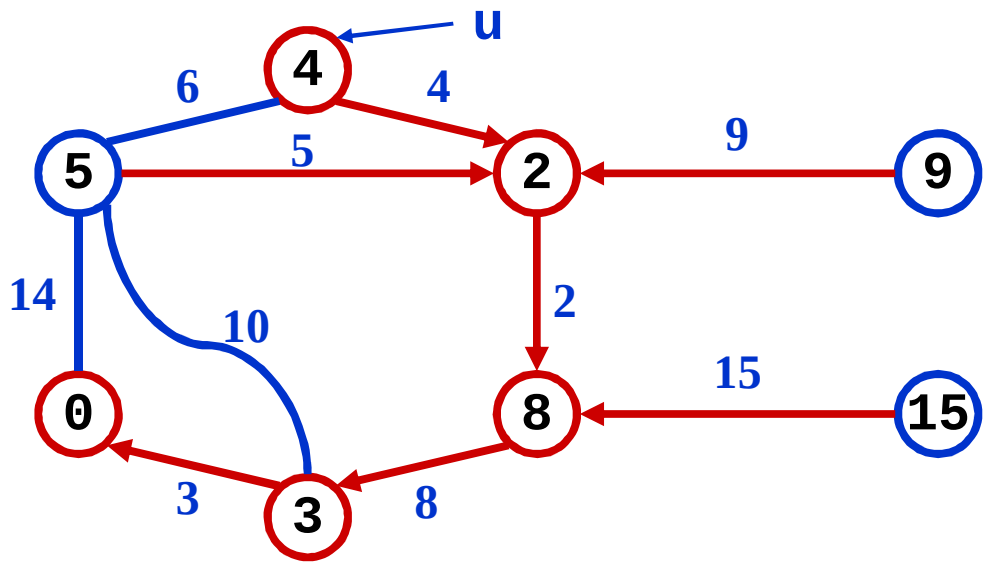

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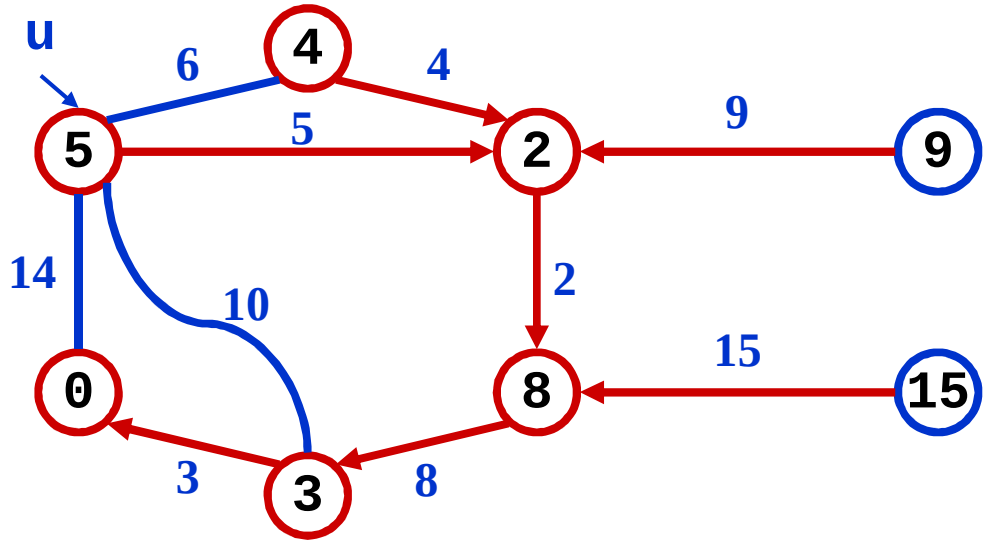

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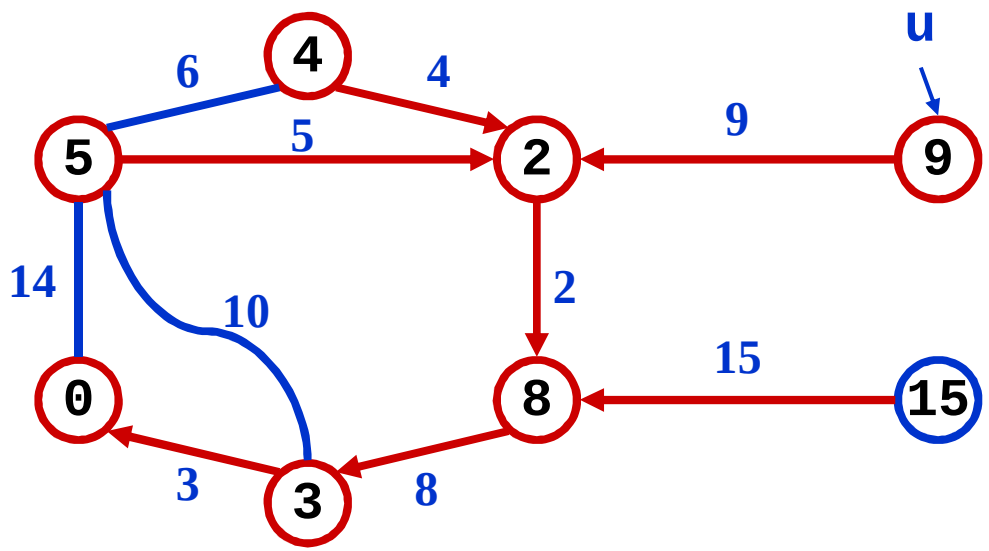

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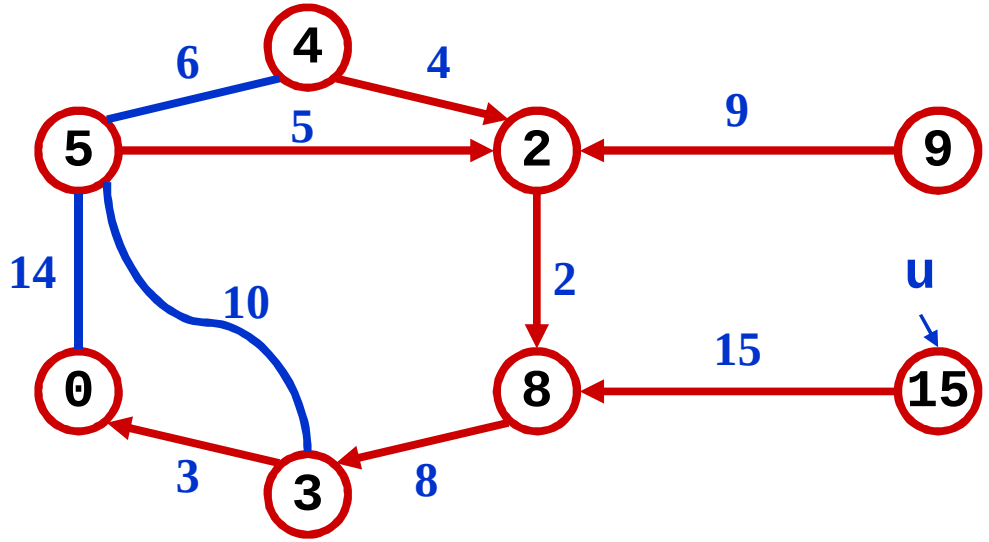

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What is the hidden cost in this code?

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```

How often is ExtractMin() called?

How often is DecreaseKey() called?

Review: Prim's Algorithm

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What will be the running time?

A: Depends on queue

binary heap: O(E lg V)

Fibonacci heap: O(V lg V + E)