Referee Report

Extremal Decompositions of Tropical Varieties and Relations to Rigidity Theory

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1. Summary of Results

The manuscript submitted to *Journal of Symbolic Computation* has three main results concerning extremal tropical varieties. When a tropical variety C cannot be decomposed into tropical varieties $C = C_1 \cup C_2$ with compatible balancing weights, it is called an extremal tropical variety. Theorem 1.1 characterizes extremal tropical hypersurfaces. The graph G dual to the complement of such a tropical hypersurface gives a framework where each edge corresponding to a pair of adjacent regions is perpendicular to the maximal face of the tropical hypersurface separating these regions. It is shown that a tropical hypersurface is extremal if and only if the reciprocal diagram (G, p) is direction rigid: i.e. any other framework (G, q) where each edge of (G, q) is parallel to the corresponding edge in (G, p) is just a scaled and translated copy of (G, p). Corollary 1.2 is the special case of plane tropical curves. In this case, such a curve is extremal if and only if the reciprocal diagram (G, p) is infinitesimally rigid. Theorem 1.3 settles the general case where a tropical variety is extremal if and only if a rigidity matrix associated to this tropical variety has the correct rank. Finally, all balanced weightings of a tropical variety form a strongly convex polyhedral cone and the rays of this cone give rise to a decomposition of the tropical variety into extremal tropical varieties. This gives an efficient method for such a decomposition. There are other noteworthy results as well. For instance, Corollary 4.2 proves that an extremal tropical plane curve must contain a trivalent vertex, and if it contains exactly one such vertex all the other vertices have degree 4.

I am fond of the connections made between tropical geometry and rigidity theory. But, to be honest, I am not very convinced that decomposing a tropical variety into extremal tropical varieties is such an important task. Of course, computing the decomposition of an algebraic variety into irreducible components is an important question. However, typically, we consider tropical variety that come from irreducible algebraic varieties. So I am not so sure how useful the decomposition in the tropical side is so useful.

2. Recommendation

Because of my reservations above, I would recommend a weak accept with some major revisions. Briefly: I don't think the motivations listed in the Introduction make a strong case, and probably should be omitted. Section 2 should be kept to a minimum. For instance, do we need everything in the Example 2.2 ? Can Section 2.2 be shortened? Is everything introduced used later on (such as infinitesimally flexible) ?

3. Further Comments

Here I have pointed a few other things such as typos etc.

• A bunch of terms have British spellings (such as realisation versus realization). I do not know what the policy of the Journal is in this regard.

• (page 2, line 46): are maximal faces always polytopes? I think there are maximal faces that are unbounded (i.e polyhedra).

• (Theorem 1.3, line 48): $\tilde{E}$ is not defined yet.

• (page 4, line 53): to collections OF polynomials.

• (page 4, line 57): polytopes versus polyhedra ?

• (page 5, line 4): maybe pure should be defined as just a complex where every maximal face has the same dimension.

• (page 5, Example 2.2): If this example is going to stay, include the equation that gives rise to Figure 1b.

• (page 6, line 21): tropical HYPERSURFACE.

• The paragraph after Figure 2 is really really dense. Is there anyway the exposition can be lightened ?

• (Top of page 8): I am confused with the wording. I thought in line 6 "not" should be deleted, and in line 7, it should be "does NOT satisfy" and instead of "does fail to hold" it should be "holds". Am I off ?

• (page 10, line 54): We refer interested readers.

• (page 11, line 26): "orangethe" ?

• (page 13, line 51): do you mean a "primitive vector" ?

• (page 14, line 38): I think it should be $\mathrm{Conv}(\mathcal{L}\mathrm{supp}(f))$.