Growth of Vertex-Transitive Graphs and the Dimension of the space of Harmonic Functions

Merlin Haith Rowlatt

September 19, 2024

1 Background and Notation

Often, one can get information from a geometric object by looking at it algebraically or vice versa. A famous example of this is Gromov's theorem [1], which states that a finitely generated group has polynomial growth if and only if it has a nilpotent subgroup of finite index. In Kleiner's proof of Gromov's theorem [4], his approach introduces a link between the geometry of a group and harmonic functions by proving if a group has polynomial growth then the space of harmonic functions with polynomial growth of bounded degree is finite dimensional. Tointon then looked specifically at groups with linear growth, and proved that a finitely generated group having linear growth is equivalent to the space of all harmonic functions on that group being finite dimensional.

Here we look a an analogous problem to Tointon, but expand our approach to vertex-transitive graphs, rather than just group and Cayley graphs.

Note for APM: after proving this, I would like to show that the following conjecture is true if it is true for groups. The conjecture as formulated for groups was originally posed by Meyerovitch and Yadin [5]

Conjecture 1.1. *Let* Γ *be an infinite, connected, locally finite vertex transitive graph. Then the following are equivalent:*

(1) $\dim H^k < \infty$ *for some* $k \in \mathbb{N}$*;*

(2) $\dim H^k < \infty$ *for every* $k \in \mathbb{N}$*;*

(3) Γ *has polynomial growth.*

Throughout, we use $\Gamma = \Gamma(V, E)$ to denote a graph with vertices $V(\Gamma)$ (often we will just write Γ for convenience) and edges $E(\Gamma)$ (strictly speaking each edge is a pair of vertices $\{x, y\}$, but we will also write xy to be the edge between x and y). For $x, y \in \Gamma$, we will write $x \sim y$ if the edge $\{x, y\} \in E(\Gamma)$. We say Γ is *vertex-transitive* if for every $x, y \in \Gamma$, there is some automorphism $g \in \mathrm{Aut}(\Gamma)$ such that $g(x) = y$. Notice that each vertex will have the same number of neighbours in a vertex-transitive graph. We say that Γ is *locally finite* if every vertex has a finite number of neighbours.

Given a graph Γ, we can define a *weight* $\omega : E(\Gamma) \to \mathbb{R}_{>0}$ on the edge set of the graph. We then call the pair (Γ, ω) a *weighted graph.* For a weighted graph (Γ, ω), the *degree* of a vertex x is given by $\deg_\omega(x) = \sum_{x\sim y} \omega(xy)$. Given a function $f : \Gamma \to \mathbb{R}$, We define the *weighted Laplacian* Δ_ω on (Γ, ω) by $\Delta_\omega f(x) = f(x) - \frac{1}{\deg_\omega x} \sum_{y\sim x} \omega(xy)f(y)$. A function f is *harmonic on* (Γ, ω) if $\Delta_\omega f(x) = 0$ for every $x \in \Gamma$. We write $H(\Gamma, \omega)$ for the *space of harmonic functions on* Γ *with respect to* ω. When considering Γ as *unweighted*, we consider these definitions with the weight of every edge having the value 1, and will dispense with the subscript. We write $H(\Gamma)$ for the space of harmonic functions on Γ.

Given a group G, a probability measure μ is a *generating probability measure* if G is generated by the $\operatorname{supp}\mu$, the support of μ. We say that the measure μ is symmetric if $\mu(g) = \mu(g^{-1})$ for every $g \in G$. We define the *Laplacian on* G *with respect to* μ by $\Delta_\mu f(g) = f(g) - \sum_{s\in\operatorname{supp}\mu} \mu(s)f(xs)$. Notice that if μ is symmetric, and Γ is the Cayley graph $\mathrm{Cay}(G, \operatorname{supp}\mu)$ with a weight ω defined by $\omega(\{g, h\}) = \mu(g^{-1}h)$, then we have $\Delta_\mu = \Delta_\omega$.

Remark 1. It is easy to show that $H(\Gamma)$ is indeed a vector space. Notice that sums and scalar multiplication of harmonic functions will still have a Laplacian of value 0.

Given a vertex-transitive graph Γ and a group $G < \mathrm{Aut}(\Gamma)$, we write $G_x \coloneqq \{g \in G : g(x) = x\}$ for the vertex stabiliser of $x \in \Gamma$, and $G_{x\to y} \coloneqq \{g \in G : g(x) = y)\}$. Let $h \in G_{x\to y}$. It is simple to show that $G_{x\to y} = hG_x$.

For any vertex $x \in V(\Gamma)$, $n \in \mathbb{N}$, we write $B_\Gamma(x, n) \coloneqq \{y \in V(\Gamma) : d(x, y) \leq n\}$ and $S_\Gamma(x, n) \coloneqq \{y \in V(\Gamma) : d(x, y) = n\}$ for the *ball of radius*

n, and the *sphere of radius* n respectively. We will write $N(x) = B_\Gamma(x, 1)$, the *neighbourhood* of $x \in \Gamma$, and $N(X) = \bigcup_{x \in X} N(x)$ for the neighbourhood of $X \subset \Gamma$. We will also write $\partial X = N(X) \setminus X$, the *boundary* of $X \subset \Gamma$. Notice that for vertex-transitive graphs, $B_\Gamma(x, n) \simeq B_\Gamma(y, n)$ for every $x, y \in V(\Gamma)$. Because of this symmetry, we can define the *growth function* to be $f_\Gamma(n) \coloneqq |B_\Gamma(x, n)|$, where $x \in \Gamma$ is any starting vertex. We say that a graph Γ has *linear growth* if there is some positive constant c such that $f_\Gamma(n) \le cn$.

Theorem 1.2. *Let* Γ *be a infinite, connected, locally finite vertex-transitive graph. Then* $\dim H(\Gamma) < \infty$ *if and only if* Γ *has linear growth.*

2 Harmonic Functions on Graphs with Linear Growth

In this section we look into the space of harmonic functions on locally finite vertex transitive graphs of linear growth. A primary aim is to prove the easier direction of theorem 1.2.

Proposition 2.1. *Let* Γ *be an infinite, locally finite, connected vertex-transitive graph with linear growth, let* ω *be some weight on the edges of the graph. Then* $\dim H(\Gamma, \omega) < \infty$.

We start the proof by introducing a result of Imrich and Seifter [3] and Halin [2] that will allow us to partition a graph of linear growth into finitely many two-sided infinite paths.

Proposition 2.2. *Let* Γ *be an infinite, connected, locally finite, vertex-transitive graph with linear growth. Then* Γ *is spanned by finitely many disjoint two-ended paths* $P_1, \dots P_l$*, and there is some* $\alpha \in \mathrm{Aut}(\Gamma)$ *of infinite order with no fixed points that leaves the paths invariant.*

Lemma 2.3. *Let* Γ *and* α *be as in proposition 2.2. Then there is some finite subset* T *such that there is a bijection* $f : \Gamma \to T \times \mathbb{Z}$ *such that if* $f(x) = (t, m)$*, we have* $f(\alpha^k(x)) = (t, m + k)$*.*

Proof. We have $\Gamma = \bigsqcup_{i \in [l]} P_i$. Note that each two-ended path P_i is isomorphic to a copy of $\mathbb{Z}$ when viewed as a Cayley graph with the usual generators. For each $i \in [l]$, we can choose a distinguished vertex o_i, and an isomorphism $\zeta : P_i \to \mathbb{Z}$ such that $\zeta(o_i) = 0 < \zeta(\alpha(o_i))$, since α has no fixed points.

Since the paths $P_1, \dots, P_l$ are invariant under α, and α has no fixed points, we have $\alpha|_{P_i}$ is an automophism on P_i with no fixed points. We thus have $\alpha|_{P_i}$ acts on P_i via translation. Notice that for each $i \in [l]$, the set $T_i = \{x \in P_i : \zeta(x) \in \{0, \dots, \zeta(\alpha(o_i)) - 1\}$ is finite, and $P_i = \bigsqcup_{m \in \mathbb{Z}} \alpha^m(T_i)$. Let $T = \bigsqcup_{i \in [l]} T_i$. Then T is finite, and $\Gamma = \bigsqcup_{m \in \mathbb{Z}} \alpha^m(T)$. This means that each $x \in \Gamma$ has a unique $t(x) \in T$, $m(x) \in \mathbb{Z}$ such that $x = \alpha^m(t)$. We can thus define $f : \Gamma \to T \times \mathbb{Z}$ by $f(x) = (t(x), m(x))$. This is clearly a bijection by the fact that the union is disjoint. Now suppose $f(x) = (t, m)$. Then $x = \alpha^m(t)$. We then have $\alpha^k(x) = \alpha^{m+k}(t)$, so $f(\alpha^k(x)) = (t, m + k)$.

□

We will use Lemma 2.3 and the following result to prove Proposition 2.1. The following result is standard.

Lemma 2.4. *Let Γ be a graph, let $A \subset \Gamma$ be a finite subset such that ∂A is nonempty. Let $f_0 : \partial A \to \mathbb{R}$. Then the function $f : N(A) \to \mathbb{R}$ defined by $f(x) = \mathbb{E}_x[f_0(X_{T_{\partial A}})]$ is the unique function that is both harmonic on A, and $f|_{\partial A} = f_0$.*

Proof of Proposition 2.1. Take f, T as in Lemma 2.3. Notice that since T finite, and Γ locally finite, we have $N(T)$ is finite. Then there is some $M \in \mathbb{N}$ such that for every $x \in N(T)$, we have $-M \le m(x) \le M$. Notice also by Lemma 2.3, for every $x' \in N(\alpha^k(T))$ we have that $-M + k \le m(x') \le M + k$.

For each $n \in \mathbb{N}$, let $S_n = \bigsqcup_{-n \le j \le n} \alpha^j(T)$. Then $N(S_n) = \bigcup_{-n \le j \le n} \alpha^j(N(T))$, so for every $x \in N(S_n)$ we have $-n - M \le m(x) \le n + M$. We thus have $N(S_n) \subset S_{n+m}$, thus $\partial S_n \subset S_{n+M} \setminus S_n$, so in particular

$$\partial S_n \subset \left(\bigsqcup_{-n-M \le j \le -n} \alpha^j(T) \right) \cup \left(\bigsqcup_{n \le j \le n+M} \alpha^j(T) \right)$$

so $|\partial S_n| \le 2M|T|$. By Lemma 2.4, we have every harmonic function on $N(S_n)$ is determined by its values on ∂S_n, therefore the dimension of the space of harmonic functions on $N(S_n)$ is at most $2M|T|$. Since $\Gamma = \bigcup_{n=1}^{\infty} S_n$, we thus have the space of harmonic functions on Γ has dimension at most $2M|T|$. □

3 Harmonic Functions on Quotient Graphs

If Γ is a vertex-transitive graph with $K \le \mathrm{Aut}(\Gamma)$ a subgroup, we define $K(v) := \{k(v) : k \in K\}$ and $K(S) := \{K(v) : v \in S\}$. We then define the *quotient graph* Γ/K to be the graph with vertex set $\{K(v) : v \in V(\Gamma)\}$ and edge set $\{\{K(v), K(w)\} : \{v, w\} \in E(\Gamma)\}$.

Lemma 3.1. *Let Γ be a graph, Γ/K a quotient graph. Let $x \in \Gamma$. Then $K(N(x)) = N(K(x))$*

Proof. First, see that $K(N(x)) \subseteq N(K(x))$, since if $\{y, x\} \in E(\Gamma)$, then we have $\{K(y), K(x)\} \in E(\Gamma/K)$.

Now suppose $K(y) \in N(K(x))$. Then there is some $x' \in K(x), y' \in K(y)$ such that $\{y', x'\} \in E(\Gamma)$. Then $x' = k \cdot x$ for some $k \in K$. Since k is an automorphism, we then have $\{k^{-1} \cdot y', x\} \in E(\Gamma)$. This means $k^{-1} \cdot y' \in N(x)$, but then $K(k^{-1}y') \in K(N(x))$. By the same argument, also see that $K(k^{-1}y') = K(y') = K(y)$, so $K(N(x)) = N(K(x))$. □

Lemma 3.2. *Let Γ be a graph, Γ/K a quotient graph. Let $x \in \Gamma$. Then $K(B(x, r)) = B(K(x), r)$*

Proof. First see that for all subsets $X \subset \Gamma$, we have $K(N(X)) = N(K(X))$, since

$$K(N(X)) = K(\bigcup_{x \in X} N(x)) = \bigcup_{x \in X} K(N(x)) = \bigcup_{x \in X} N(K(x)) = N(K(X))$$

Notice for $r = 0$, we have $K(B(x, 0)) = B(K(x), 0) = K(x)$. Now suppose that $K(B(x, r - 1)) = B(K(x), r - 1)$. See that

$$B(x, r) = B(x, r - 1) \cup N(B(x, r - 1))$$

Using the inductive hypothesis, we thus have

$$\begin{aligned} K(B(x, r)) &= K(B(x, r - 1)) \cup K(N(B(x, r - 1))) \\ &= B(K(x), r - 1) \cup N(B(K(x), r - 1)) \\ &= B(K(x), r) \end{aligned}$$

□

Let $G \leq \mathrm{Aut}(\Gamma)$, we then have G acts on $H(\Gamma)$ via $g \cdot f(x) = f(g^{-1}(x))$. Let K be the kernel of this action. Then $K \trianglelefteq G$, and $f(k(x)) = f(x)$ for every $k \in K$. For the remainder of the section, let K be this kernel.

Lemma 3.3. *Let Γ be an infinite, connected, locally finite vertex-transitive weighted graph. Then there is some weight ω on Γ/K such that $H(\Gamma)$ and $H(\Gamma/K, \omega)$ are isomorphic.*

Proof. Let us consider the set $N(x) \cap K(y)$. Since Γ is locally finite, we can denote the set $\{k_1 y, k_2 y, \dots, k_n y\} = N(x) \cap K(y)$ where each $k_i \in K$ and $n = |N(x) \cap K(y)|$. Then for each $k \in K$ we can construct a bijection $k_i w \mapsto k k_i y$ between $N(x) \cap K(y)$ and $N(kx) \cap K(y)$. This is clearly injective. For surjectivity, see that

$$\{kx, y'\} \in N(kx) \cap K(y) \implies \{x, k^{-1} y'\} \in N(x) \cap K(y)$$

Also see that we can construct a bijection between $N(x) \cap K(y)$ and $N(y) \cap K(x)$ via $k_i y \mapsto k_i^{-1} x$. This allows us to define a weight $\omega : E(\Gamma/K) \to \mathbb{R}$ as follows:

$$\omega(\{K(x), K(y)\}) = |N(x) \cap K(y)|$$

Let us define a map $\phi : H(\Gamma) \to H(\Gamma/K, \omega)$ such that

$$\phi(f)(K(v)) = f(v)$$

For every $x, x' \in K(x)$, we have $f(x) = f(x')$, so ϕ is independent of the choice of $x' \in K(x)$.

Choose an arbitrary $x \in \Gamma$. Then by lemma 3.1, we have $N(K(x)) = K(N(x))$. We can thus write $N(K(x)) = \{K(y_1), K(y_2) \dots, K(y_m)\}$ with representatives $y_1, y_2, \dots, y_m \in N(x)$. Since each $K(y_i)$ is distinct, we have $N(x) \cap K(y_1), N(x) \cap K(y_2) \dots, N(x) \cap K(y_m)$ are clearly disjoint, and

$$\bigcup_{i=1}^{m} N(x) \cap K(y_i) = N(x)$$

Given $f \in H(\Gamma)$, we then have

$$\begin{aligned}
\phi(f)(K(x)) &= \frac{1}{|N(x)|} \sum_{y \in N(x)} f(y) \\
&= \frac{1}{|N(x)|} \sum_{i=1}^{m} \sum_{y \in N(x) \cap K(y_i)} f(y_i) \\
&= \frac{1}{|N(x)|} \sum_{i=1}^{m} |N(x) \cap K(y_i)| f(y_i) \\
&= \frac{1}{|N(x)|} \sum_{i=1}^{m} \omega(\{K(x), K(y_i)\}) f(y_i)
\end{aligned}$$

So $\phi(f) \in H(\Gamma/K, \omega)$.

Suppose we have $f, h \in H(\Gamma)$ with $f \neq h$. then there is some $x \in \Gamma$ such that $f(x) \neq h(x)$. We then have $\phi(f)(K(x)) \neq \phi(h)(K(x))$, thus ϕ is injective.

Now consider some $\bar{h} \in H(\Gamma/K, \omega)$. Let us define $h : \Gamma \to \mathbb{R}$ by $h(x) = \bar{h}(K(x))$. Then we have by a similar argument to above,

$$\frac{1}{|N(x)|} \sum_{y \in N(x)} h(x) = \frac{1}{|N(x)|} \sum_{i=1}^{m} \omega(\{K(x), K(y_i)\}) h(y_i) = \bar{h}(K(x)) = h(x)$$

so $h \in H(\Gamma)$, and $\phi(h) = \bar{h}$, thus ϕ is surjective, and thus bijective.

Finally, see that for every $f, g \in H(\Gamma)$, $x \in \mathbb{R}$, we have

$$\phi(f+g)(K(v)) = (f+g)(v) = f(v) + g(v) = \phi(f)(K(v)) + \phi(g)(K(v))$$

and

$$\phi(xf)(K(v)) = xf(v) = x\phi(f)(K(v))$$

so ϕ is a linear map. Thus it is an isomorphism. □

4 Finite Vertex Stabilisers of Quotient Group on Quotient Graph

Here we show that G/K acts discretely on Γ/K.

Lemma 4.1. *If* $\dim H(\Gamma, \omega) < \infty$*, then there is some finite ball* $B \subset \Gamma$ *such that for every* $f \in H(\Gamma, \omega)$*,* $f|_B$ *completely determines* f*.*

Proof. We will prove the contrapositive. Suppose harmonic functions are not uniquely determined by their values on some finite set. See that harmonic functions can take arbitrary values on one vertex.

Now suppose we have points $x_1, x_2, \ldots, x_{n-1} \in \Gamma$ on which harmonic functions can take arbitrary values. Since harmonic functions are not uniquely determined by their values on some finite set, we must have two harmonic functions $f_1, f_2 \in H(\Gamma, \omega)$ with the same value on $x_1, x_2, \ldots, x_{n-1}$, but different values on some $x_n \in \Gamma$. Notice that the function $f_1 - f_2$ is harmonic, has value 0 on x_i where $i \in [n-1]$, but non-zero value on x_n. Then scalar multiples of $f_1 - f_2$ can have arbitrary values on x_n, so we can construct a function with arbitrary values on $x_1, x_2, \ldots, x_n$.

By induction, we can find a function with arbitrary values on $x_1, x_2, \ldots, x_n$ for every $n \in \mathbb{N}$, thus $\dim H(\Gamma, \omega) = \infty$.

This shows that given a finite dimensional space of harmonic functions, there is some finite set of vertices of the graph such that every harmonic function is uniquely determined by its value on that set. Then since Γ is connected, we can find a finite ball B that contains these elements. $\square$

Lemma 4.2. *Let $G < \mathrm{Aut}(\Gamma)$. Then G/K is isomorphic to a subgroup of $\mathrm{Aut}(\Gamma/K)$.*

Proof. Let us define an action $\phi : G \to \mathrm{Sym}(\Gamma/K)$ via $\phi(g)(K(x)) = gK(x)$. See that $K(x) \sim K(y)$ is equivalent to there existing some $x' \in K(x), y' \in K(y)$ such that $x' \sim y'$. Since $g \in G$ is an automorphism, we have $x' \sim y'$ is equivalent to $g(x') \sim g(y')$, but this is equivalent to $gK(x) = gK(x') \sim gK(y') = gK(y)$. This means we have the action is a homomorphism $\phi : G \to \mathrm{Aut}(\Gamma/K)$.

Let us consider $\ker\phi = \{g \in G : \forall x \in \Gamma, gK(x) = K(x)\}$. Take some $g \in \ker\phi$. Then since we have the identity $e \in K$, we must have for each $x \in \Gamma$, there is some $k \in K$ such that $g(x) = k(x)$. But then for every harmonic function $f \in H(\Gamma)$, we have $f(g(x)) = f(k(x)) = f(x)$. We thus have $f(g(x)) = f(x)$ for every $x \in \Gamma$, so $g \in K$. Notice also that for every $k \in K$, we have $kK = K$, so clearly $k \in \ker\phi$, thus $\ker\phi = K$.

Finally, see that $\phi(G) < \mathrm{Aut}(\Gamma/K)$ is isomorphic to $G/\ker\phi = G/K$. $\square$

Given a connected, locally finite, vertex-transitive graph, we can endow $\mathrm{Aut}(\Gamma)$ with the *topology of pointwise convergence.* Given a fixed arbitrary element o, this is the topology induced by the metric d on $\mathrm{Aut}(\Gamma)$ defined by

$$d(g,h) = 2^{-\inf\{r \geq 0 : \exists x \in B(o,r) \text{ such that } g(x) \neq h(x)\}}$$

Given this metric, a subset $A \subset \mathrm{Aut}(\Gamma)$ is *closed* if and only if it contains all the limits of sequences in A, and *discrete* if for every $g \in A$, we have some $\epsilon > 0$ such that $d(g,h) > \epsilon$ for all $h \in A \setminus \{g\}$.

Lemma 4.3. *Let Γ be such that $H(\Gamma) < \infty$. Then the group $G/K < \mathrm{Aut}(\Gamma/K)$ is discrete.*

Proof. By lemma 4.1, we have some finite ball B such that $f|_B$ completely determines f. Then $B = B(o, r)$ for some $o \in \Gamma, r \in \mathbb{N}$. Defining our metric based on the fixed arbitrary element o in the ball, let us take $g, h \in \mathrm{Aut}(\Gamma)$ such that $d(g(o), h(o)) < 2^{-r}$. Then $g(o), h(o) \in B(o, r)$, and $g(x) = h(x)$ for every $x \in B$, so $g^{-1}f(x) = h^{-1}f(x)$ for every $f \in H(\Gamma), x \in B$. But then $g^{-1}f = h^{-1}f$ by lemma 4.1. We thus have $gh^{-1} \in K$, so $gK = hK$.

Now take $g'K, h'K \in G/K$ such that $d(g'K, h'K) < 2^{-r}$. Then $g'K, h'K \in B(K(o), r)$. See that by lemma 3.2, we have $B(K(o), r) = K(B(o, r))$. We thus have $g'K = gK, h'K = hK$ for some $g, h \in B(o, r)$. Then by above, $gK = hK$, so $d(g'K, h'K) = 0$, therefore G/K is discrete. □

Lemma 4.4. *$G/K < Aut(\Gamma/K)$ is closed.*

Proof. Let $(g_n)_{n=1}^{\infty}$ be a sequence in G such that (g_n) converges pointwise to $g \in \mathrm{Aut}(\Gamma)$. Since B is finite, there is some $N \in \mathbb{N}$ such that $g_n(x) = g(x)$ for every $n \geq N$, $x \in B$. For every $f \in H(\Gamma)$ we have that $g_n f(x) = gf(x)$ for every $n \geq N$, $x \in B$. This means that $g_n f = gf$ for every $n \geq N$, thus $g_n^{-1}gf = f$. Notice that $g_n^{-1}g \in K$, so $g_nK = gK$ for every $n \geq N$. This means that $gK \in G/K$, thus $g \in G$.

G is therefore a closed subgroup of $\mathrm{Aut}(\Gamma)$, then G/K is also closed with the quotient topology. Finally notice that this agrees with the subspace topology of G/H when viewed as a subgroup of $\mathrm{Aut}(\Gamma/H)$, therefore $G/K < \mathrm{Aut}(\Gamma/K)$ is closed. □

Lemma 4.5. *[6] Let Γ be a non-empty, connected, locally finite vertex-transitive graph, let $G < \mathrm{Aut}(\Gamma)$ be a closed subgroup. Then the following are equivalent:*

(i) The group G is discrete

(ii) For every $x \in \Gamma$ the stabiliser G_x is finite.

(iii) There exists $x \in \Gamma$ such that G_x is finite.

Proposition 4.6. *The group G/K acts on Γ/K with finite vertex stabilisers.*

Proof. By lemmas 4.3, 4.4, G/K is closed and discrete, thus by lemma 4.5 we have for every $K(x) \in \Gamma/K$, $G_{K(x)}$ is finite. □

5 Finite Vertex Stabilisers Admit an Isomorphism Between Spaces of Harmonic Functions

Lemma 5.1. *Let Γ be a vertex-transitive graph with some vertex labelled o. Suppose that G is a transitive subgroup of* $\mathrm{Aut}(\Gamma)$ *Let $S = \{g \in G : d(g(o), o) \leq 1\}$. Then S is a symmetric generating set for G, containing the identity.*

Definition 5.2. Let (Γ, ω) be a weighted graph, $G < \mathrm{Aut}(G)$. We say that the weight ω is *invariant under* G if for every $g \in G$, we have $\omega(\{x, y\}) = \omega(\{g(x), g(y)\})$

Lemma 5.3. *Let $(\Gamma/K, \omega)$ be the same as defined in lemma 3.3. Then ω is invariant under G/K.*

Proof. First, recall that $G/K < \mathrm{Aut}(\Gamma/K)$ by lemma 4.2. Now see that $\omega(\{K(x), K(y)\}) = |N(x) \cap K(y)|$. Take $gK \in G/K$. We then have $\omega(\{gK(x), gK(y)\}) = |N(g(x)) \cap K(g(y))|$. See that $N(g(x)) \cap K(g(y)) = g(N(x)) \cap gK(y) = g(N(x) \cap K(y))$, and since g is an automorphism of Γ, we have $|N(x) \cap K(y)| = |N(g(x)) \cap K(g(y))|$. Notice that this is true for any representative of gK, so it is well defined. We thus have $\omega(\{K(x), K(y)\} = \omega(\{gK(x), gK(y)\})$. $\square$

Lemma 5.4. *Suppose (Γ, ω) is a vertex-transitive graph with some vertex labelled o. Let $G <$ $\mathrm{Aut}(\Gamma)$ be a transitive subgroup with finite vertex stabilisers such that ω is invariant under G, let $S = \{g \in G : d(g(o), o) = 1\}$, $f \in H(\Gamma, \omega)$. Define $\psi(f) : G \to \mathbb{R}$ by $\psi(f)(g) = f(g(o))$. Then we have $\psi(f) \in H(G, \mu)$ for some probability measure on S. Moreover, the map $\psi : H(\Gamma, \omega) \to H(G, \mu)$ is an isomorphism of vector spaces.*

Proof. $\square$

For each $y \in \Gamma$, let us fix some g_y, such that $g_y(o) = y$. See that $S = \bigcup_{y \in N(o)} g_y G_o$.

Let us define a probability measure μ on S by $\mu(s) = \frac{\omega(\{o, s(o)\}}{|G_o| \deg_\omega o}$.

let $f \in H(G, \mu)$. Define $\psi : \Gamma \to \mathbb{R}$ by $\psi(f)(x) = \sum_{g \in G_{o \to x}} f(g)$. See then that

$$\begin{aligned}
\psi(f)(x) &= \sum_{g\in G_{o\to x}} f(g) \\
&= \sum_{a\in G_o} f(g_x a) \\
&= \sum_{a\in G_o}\sum_{s\in S} \mu(s) f(g_x a s) \\
&= \frac{1}{|G_o| \deg_\omega o} \sum_{a\in G_o}\sum_{s\in S} \omega(\{o, s(o)\}) f(g_x a s) \\
&= \frac{1}{|G_o| \deg_\omega o} \sum_{a\in G_o}\sum_{y\in N(o)}\sum_{b\in G_o} \omega(oy) f(g_x a g_y b) \\
&= \frac{1}{|G_o| \deg_\omega o} \sum_{a\in G_o}\sum_{y\in N(o)}\sum_{g\in G_{o\to g_x a(y)}} \omega(oy) f(g) \\
&= \frac{1}{|G_o| \deg_\omega o} \sum_{a\in G_o}\sum_{y\in N(o)} \omega(oy)\psi(f)(g_x a(y)) \\
&= \frac{1}{\deg_\omega o} \sum_{y\in N(o)} \omega(oy)\psi(f)(g_x(y)) \\
&= \frac{1}{\deg_\omega x} \sum_{y\in N(x)} \omega(xy)\psi(f)(y)
\end{aligned}$$

Thus $\psi(f) \in H(\Gamma, \omega)$.

Suppose we have a $f' \in H(\Gamma, \omega)$. Let us define $f : G \to \mathbb{R}$ by $f(g) = \frac{1}{|G_o|} f'(g(o))$. See then that we have

$$
\begin{aligned}
\sum_{s\in S}\mu(s)f(gs) &= \frac{1}{|G_o|}\sum_{s\in S}\mu(s)f'(gs(o)) \\
&= \frac{1}{|G_o|}\sum_{y\in N(o)}\sum_{a\in G_o}\mu(s)f'(gg_y a(o)) \\
&= \sum_{y\in N(o)}\mu(s)f'(gg_y(o)) \\
&= \frac{1}{|G_o|\deg_\omega o}\sum_{y\in N(o)}\omega(oy)f'(g(y)) \\
&= \frac{1}{|G_o|\deg_\omega o}\sum_{y\in N(o)}\omega(\{g(o),g(y)\})f'(g(y)) \\
&= \frac{1}{|G_o|}f(g(o)) \\
&= f(g)
\end{aligned}
$$

thus ψ is surjective.

See that for every $h,f\in H(G,\mu)$, $\alpha\in\mathbb{R}$, we have

$$
\psi(h+f)(x) = \sum_{g\in G_{o\to x}}(h+f)(g) = \left(\sum_{g\in G_{o\to x}}h(g)\right)+\left(\sum_{g\in G_{o\to x}}f(g)\right)
$$

and

$$
\psi(\alpha h)(x) = \sum_{g\in G_{o\to x}}\alpha h(g) = \alpha\sum_{g\in G_{o\to x}}h(g)
$$

so ψ is a linear map.

We have $\ker\psi = \{f\in H(G,\mu) : \sum_{g\in G_{o\to x}} f(g) = 0 \text{ for every } x\in\Gamma\}$. See that for every $f\in\ker\psi$, $g\in G$, we have

$$
\begin{aligned}
f(g) &= \sum_{s\in S} f(gs) \\
&= \sum_{y\in N(o)}\sum_{a\in G_o} f(gg_y a) \\
&= \sum_{y\in N(o)}\sum_{b\in G_{o\to g(y)}} f(b) \\
&= 0
\end{aligned}
$$

thus $\ker \psi = \{0\}$. We thus have that ψ is an isomorphism of vector spaces.

6 Harmonic Functions and Quotients with Linear Growth

Lemma 6.1. *[7] Let Γ be an infinite, connected, locally finite vertex-transitive graph. There is a finitely generated subgroup $G < \mathrm{Aut}(\Gamma)$ that is transitive.*

The proof of this lemma constructs such a group G as follows: Let $o \in \Gamma$. Since $\mathrm{Aut}(\Gamma)$ is transitive, for each neighbour y of o we have some automorphism $g_y \in \mathrm{Aut}(\Gamma)$ such that $g_y o = y$. Choose one such g_y for each y. Then the group $G = \langle g_y : y \sim o \rangle$ is finitely generated and acts on Γ transitively.

For the rest of this chapter, we will let $G =< S >$ where $S = \{g_y : y \sim o\}$.

Conjecture 6.2. *Let Γ be an infinite, connected, locally finite vertex-transitive graph with a transient random walk. Suppose we have some $K \triangleleft G < \mathrm{Aut}(\Gamma)$ with G as constructed above such that Γ/K has linear growth, but K is not finitely generated. Then Γ admits a positive harmonic function that does not factor through Γ/K.*

We will view the graph Γ as a symmetric directed graph - that is we view each edge as a pair of directed edges, one going in each direction. For a word $\omega = g_1 g_2 \dots g_n$ in S, we will write the directed walk of ω on graph Γ starting at x as a sequence of directed edges of Γ $(e_1, e_2, \dots, e_n)$, where $e_1 = (x, g_1 x)$ and $e_i = (g_1 \dots g_{i-1} x, g_1 \dots g_i x)$ for $i > 1$. For a directed edge $e = (v_1, v_2)$, we will write $e^{-1} = (v_2, v_1)$

We will call such a walk a *word-walk*. If a word-walk starts and ends at the same vertex, we will call it a *closed word-walk*, and we will call the word a *stabilising word*. If a closed word-walk has no repeated edges, so $e_i \neq e_j$ for $i \neq j$, then we call it a *word-circuit* and call the word a *simple stabilising word*.

For a word ω, we will write $c(\omega) = |\{i \in [n] : e_i = e_j, j < i\}|$, the number times any edge is repeated in a word-walk. For ease of notation, for any n-tuple $\nu = (a_1, a_2, \dots, a_n)$, for $i \in [n]$, we will write $\nu(i) = a_i$. We will also writ $\omega(i)$ for the ith letter in a word ω.

Lemma 6.3. *Every reduced stabilising word in G over Γ can be written as a finite product of reduced simple stabilising words.*

Proof. Let ω be a reduced stabilising word of length n. Then it admits a closed word-walk $(e_1, e_2, \dots e_n)$ from some vertex $x \in \Lambda$.

See that if $c(\omega) = 0$, then ω is a reduced simple stabilising word. Now suppose $c(\omega) = c > 0$. Let $a = \min\{i \in [n] : e_i = e_j, j < i\}$. Then the walk $(e_1, \dots e_{a-1})$ has no repeated edges.

We will show that the word ω can be written as a product of two words $\omega_t \omega'$ such that ω_t is a simple stabilising word of length t, and $c(\omega') < c(\omega)$. The result follows by induction on c.

We can construct a new circuit as follows: Let $w_0 = (e_1, \dots e_{a-1})$. For $i \geq 0$, if $w_i(a-1+i)(2) = x$, then we have w_i is a circuit so we are done, otherwise for $i \geq 1$, let $w_i = (w_{i-1}(1), \dots, w_{i-1}(a+i-2), e_{b_i}^{-1})$ where b_i is the largest $j \in [a-1]$ such that both

1. $e_j(1) = w_{i-1}(a+i-2)(2)$, and
2. $e_j^{-1} \neq w_{i-1}(l)$ for every $l \in [a-1+i]$

are satisfied. We know we can choose such a b_i, since $w_{i-1}(a+i-2)$ has a higher in-degree in w_{i-1} than its out-degree, so there must be at least one unused backwards edge leading out of the vertex. Since w has length $a-1$, we must have our process terminates with $w_t(a-1+i)(2) = x$ after at most $t \leq a-1$ steps, giving us a circuit w_t.

We will convert w_t to a simple stabilising word ω_t as follows: The first $a-1$ elements of the word are the elements of ω. for each $i \in [t]$, choose the letter $\omega_t(a-1+i)s \in S$ such that the $s(w_t(a-1+i)(1)) = w_t(a-1+i)(2)$. We can do this since for every element $x \in \Gamma$, we have $N(x) \subset S(x)$. Then the word ω_t is a simple stabilising word.

We then construct the word ω' as follows: for $i \in [t]$ Let $\omega'(i) = \omega_t(a-1+i)^{-1}$. For $i > t$, let $\omega'(i) = \omega(a-1-t+i)$, so the remaining elements of ω' are the elements of ω after the $a-1$th element.

Notice that $\omega_t \omega' = \omega$ as words after reduction, since the end of ω_t and the start of ω' cancel out. Also see that the edge e_a in the word-walk w' of ω' is no longer the first repeated edge of w' since by construction, the edges $e_a \notin \{w'(1), \dots, w'(t)\}$. Also see that $\{w'(1), \dots, w'(t)\} \subset \{e_1, \dots, e_{a-1}\}$, so we must have $c(\omega') < c(\omega)$.

Repeating this process allows us to write ω as a finite product of simple stabilised words. Notice though that the product of the reduction of each of these words is the same as the reduction of the product, so ω is the product of reduced simple stabilising words as required. $\square$

Lemma 6.4. *Let $K < G$. Every element of K can be written as a finite product of reduced simple stabilising words in G over Γ/K.*

Proof. Notice that by the definition of Γ/K, we have every reduced word ω over S with $\omega \in K$ as a group element has the property that ω is a reduced stabilising word over Γ/K, thus can be written as a finite product of reduced simple stabilising words. $\square$

Lemma 6.5. *Let $K \triangleleft G < \mathrm{Aut}(\Gamma)$ be as in the statement of proposition 6.2. Suppose there is finite subgraph of Γ/K for which the connected component of $o \in \Gamma$ contains the whole of Ko. Then Γ admits a positive harmonic function that does not factor through Γ/K*

Proof. Suppose we have some finite subgraph $\Lambda \subset \Gamma/K$ such that the connected component of $o \in \Gamma$ contains Ko. Because Γ/K has linear growth, lemma 2.3 gives us a bijection $f : \Gamma/K :\to T \times \mathbb{Z}$. Since Λ is finite, there must be some $n \in \mathbb{N}$ such that $\Lambda \subset f^{-1}(T \times [-n, n])$. Since the subgraph $f^{-1}(T \times [-n, n]) \subset \Gamma/K$ is finite, it has finitely many circuits. We thus have finitely many simple stabilising words in G over $f^{-1}(T \times [-n, n])$. Let $H < K$ be the subgroup generated by these simple stabilising words. Then H is finitely generated and $Ho = Ko$, so by [7], we have a harmonic function on Γ that is not constant on $Ho = Ko$, thus it does not factor through Γ/K. $\square$

TODO: Construct harmonic function on the graph Γ that doesn't factor through Γ/K. In Matt's paper he uses a function similar to $h_n : f^{-1}(T \times [-n, n]) \to \mathbb{R}$ with

$h_n(x) = \mathbb{P}_x$[the random walk from x reaches $f^{-1}(T \times \{n\})$before$f^{-1}(T \times \{-n\})$ and the random walk doesn't do "much" in $f^{-1}(T \times [-n, -1])$].

7 Vertex-transitive Graphs with Finite Dimensional Spaces of Harmonic Functions

In this section we prove the following:

Theorem 7.1. *Let Γ be an infinite, connected, locally finite vertex-transitive graph with $dim(H(\Gamma)) < \infty$. Then Γ has linear growth.*

Proof of Theorem 7.1. First suppose that the random walk on Γ is recurrent. Then by Varopoulos' theorem, Γ either has linear or quadratic growth. If Γ has linear growth we are done, so suppose Γ has quadratic growth.

TODO: USE TROFIMOV TO PROVE FOR QUADRATIC

Now suppose that the random walk on Γ is transient. By lemma 6.1, there is some subgroup $G < \mathrm{Aut}(\Gamma)$ such that G is finitely generated and transitive. From section 3, we have that if $G < \mathrm{Aut}(\Gamma)$, then G acts on $H(\Gamma)$ via $g \cdot f(x) = f(g^{-1}(x))$. Let K be the kernel of this action. Then lemma 3.3 gives us the existence of some weight ω on Γ/K such that $H(\Gamma)$ is isomorphic to $H(\Gamma/K, \omega)$, and every harmonic function in $H(\Gamma)$ factors through Γ/K. This in particular means that $H(\Gamma/K, \omega)$ is finite-dimensional.

From proposition 4.6, we have that G/K acts on Γ/K with finite vertex stabilisers. By lemma 5.4, we then have some probability measure μ on G/K such that the spaces $H(\Gamma/K, \omega)$ and $H(G/K, \mu)$ are isomorphic, and so $H(G/K, \mu)$ is finite dimensional. The main result of [7] then gives us that G/K has linear growth, and since G/K acts on Γ/K with finite vertex stabilisers, Γ/K has linear growth.

Since every element of $H(\Gamma)$ factors through Γ/K by proposition 3.3 and Γ/K has linear growth, proposition 6.2 gives us that K is not infinitely generated. Since Γ is transient, [7] tells us that if K is infinite and finitely generated, then there is a harmonic function that doesn't factor through Γ/K, which is a contradiction, thus K is finite.

Since K is finite, see that $B(x, n) \subset \bigcup B(Kx, n)$, and $|\bigcup B(Kx, n)| = |K||B(Kx, n)|$. Since Γ/K has linear growth there is some positive constant c such that $|B(Kx, n)| \leq cn$ for every $n \in \mathbb{N}$. We therefore have $|B(x, n)| \leq |K|cn$ for every $n \in \mathbb{N}$, so Γ has linear growth.

□

References

[1] Michael Gromov. Groups of polynomial growth and expanding maps (with an appendix by Jacques Tits). *Publications Mathématiques de*

l'IHÉS, 53:53–78, 1981.

[2] R. Halin. Automorphisms and endomorphisms of infinite locally finite graphs. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 39(1):251–283, September 1973.

[3] W. Imrich and N. Seifter. A note on the growth of transitive graphs. *Discrete Mathematics*, 73(1):111–117, January 1988.

[4] Bruce Kleiner. A new proof of Gromov's theorem on groups of polynomial growth. *Journal of the American Mathematical Society*, 23(3):815–829, December 2009.

[5] Tom Meyerovitch, Idan Perl, Matthew Tointon, and Ariel Yadin. Polynomials and Harmonic Functions on Discrete Groups. *Transactions of the American Mathematical Society*, 369(3):2205–2229, 2017. Publisher: American Mathematical Society.

[6] Romain Tessera and Matthew C. H. Tointon. A Finitary Structure Theorem for Vertex-Transitive Graphs of Polynomial Growth. *Combinatorica*, 41(2):263–298, April 2021.

[7] Matthew C. H. Tointon. Characterisations of algebraic properties of groups in terms of harmonic functions. *Groups, Geometry, and Dynamics*, 10(3):1007–1049, September 2016.