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Algebraic Geometry
Coursework 1

1. Let $A \subset \mathbb{A}^n$ be a subset.

   (a) The closure of A in $\mathbb{A}^n$ is the intersection of all closed sets in $\mathbb{A}^n$ containing A.

   (b) The closure of A in $\mathbb{A}^n$ is equal to $\mathbb{V}(\mathbb{I}(A))$.

   Proof. We know that $\mathbb{I}(A)$ consists of all complex polynomials that vanish on A. For any $f \in \mathbb{I}(A)$, note that $\mathbb{V}(f)$ is a closed affine algebraic variety containing A. Thus,

$$A \subseteq \mathbb{V}(\mathbb{I}(A)) = \mathbb{V}\left(\bigcup_{f \in \mathbb{I}(A)} f\right) = \bigcap_{f \in \mathbb{I}(A)} \mathbb{V}(f),$$

   and so $\mathbb{V}(\mathbb{I}(A))$ is the intersection of all closed varieties, and thus closed sets in $\mathbb{A}^n$, containing A i.e. precisely the closure of A. □

   (c) Let $B = \{\frac{1}{n}\}_{n \in \mathbb{N}} \cup \{0\} \subseteq \mathbb{C}$. This is closed in the Euclidean topology, but not in the Zariski topology, and thus its Zariski closure is not the same as its Euclidean closure.

   Proof. Noting that the closed sets in the Zariski topology on $\mathbb{C}$ are precisely the closed affine algebraic varieties, it suffices to show that there exists no polynomial $f \in \mathbb{C}[x]$ such that $f(B) = 0$. However, this follows from the fundamental theorem of algebra: any polynomial in $\mathbb{C}[x]$ has a finite number of roots, but B is not a finite set. Thus, B is not closed in the Zariski topology on $\mathbb{C}$, but is in the Euclidean topology (as shown in any basic analysis class), so their closures in the two topologies are different. □

2. (a) A compact subset of a topological space is a subset such that all open coverings of that subset have a finite subcover.

   (b) The subset $\mathbb{V}(x^2 - y^3) \subset \mathbb{C}^2$ is compact in the Zariski topology, but not in the Euclidean topology.

   Proof. Noting that $V = \mathbb{V}(x^2 - y^3)$ is a closed affine algebraic variety, we know it is closed in th Euclidean topology by Ex. 1.5 in the notes. By Heine-Borel, it suffices to show that V is not bounded and thus not compact in the Euclidean topology. However, this is clear, as the function $f(x) = x^{2/3}$ is entire, and $V = \{(x, f(x)) \mid x \in \mathbb{C}\}$. Thus, V is unbounded, and not compact in the Euclidean topology.

Let $\mathcal{O}_i$ be an open covering of V in the Zariski topology. For each open $\mathcal{O}_i$, let $\{f_i\}_{j(i)}$ be the polynomials in $\mathbb{C}[x,y]$ such that $\mathbb{V}\left(\{f_i\}_{j(i)}\right) = \mathcal{O}_i^c$. Now, note that

$$\begin{aligned} V &\subseteq \bigcup_i \mathcal{O}_i \\ &= \left(\bigcap_i \mathcal{O}_i^c\right)^c \\ &= \left(\bigcap_i \mathbb{V}\left(\{f_i\}_{j(i)}\right)\right)^c \\ &= \mathbb{V}\left(\bigcup_i \{f_i\}_{j(i)}\right)^c. \end{aligned}$$

However, each $\mathbb{V}(\{f_i\}_j)$ is given by a finite set of polynomials, and thus $\bigcup_i \{f_i\}_j$ is finite. Thus, there exists some finite subset containing $f(x,y) = x^2 - y^3$, and this finite subset gives a finite subcover of V. Thus, V is compact in the Zariski topology. □

3. (a) Let $W = \mathbb{V}(x+y)$ and $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ be given by $\varphi(x,y) = x - y^2$. Then W is irreducible but $\varphi^{-1}(W)$ is not.

Proof. First note that W is isomorphic to $\mathbb{A}^1$ and is thus irreducible as it is not finite. Now, note that,

$$\begin{aligned} \varphi^{-1}(W) &= \{(x,y) \in \mathbb{A}^2 \mid \varphi(x,y) \in W\} \\ &= \{(x,y) \in \mathbb{A}^2 \mid x - y^2 \in W\} \\ &= \{(-y,y) \in \mathbb{A}^2 \mid y - y^2 = 0\} \\ &= \{(0,0), (-1,1)\}. \end{aligned}$$

This is clearly reducible as the points $(0,0)$ and $(-1,1)$ are closed proper subsets of $\varphi^{-1}(W)$. □

(b) Let $X \subset Y$ be equipped with the subspace topology of the space Y. If X is irreducible, then so is its closure.

Proof. Assume, striving for contradiction, that while X is irreducible, its closure $\overline{X}$ is not, and thus $\overline{X} = U \cup V$ for two closed proper subsets $U, V \subsetneq \overline{X}$. Now, if $X \subset U$, then $\overline{X} \subseteq U$ and therefore $\overline{X} = U$, a contradiction. Thus, $X \not\subseteq U$, and similarly so for V. However, $X \subset \overline{X} = U \cup V$, while not being properly contained in either U or V, and thus is not irreducible. Having resulted in our contradiction, the claim holds. □

(c) Isomorphisms between closed affine algebraic varieties preserve irreducibility and dimension.

Proof. Let X, Y be closed affine algebraic varieties. Let $\varphi : X \to Y$ be an isomorphism with inverse $\psi : Y \to X$.

Let X be irreducible, and, striving for contradiction, that Y is not. Thus, there exists proper closed subsets $U, V \subset Y$ such that $Y = U \cup V$. Now, we consider the sets

$\varphi^{-1}(U) = \psi(U)$ and $\varphi^{-1}(V) = \psi(V)$. As φ is continuous, we know that these two sets are closed in X, and further, as φ is a bijection, we know they are proper subsets of X. Finally, note

$$\begin{aligned} X &= \psi(Y) \\ &= \psi(U \cup V) \\ &= \psi(U) \cup \psi(V). \end{aligned}$$

Thus, X can be decomposed into two proper closed subsets such that their union is X i.e. it is not irreducible, and our desired contradiction arises.

Now, let X be irreducible with $\dim X = n$. As shown, $Y = \varphi(X)$ is irreducible. Let

$$X = X_n \supsetneq X_{n-1} \supsetneq \cdots \supsetneq X_0 = \{a\}$$

for some point a, with irreducible X_i. Note now that each $\varphi(X_i)$ is irreducible, and induce a chain of subvarieties of Y, with exactly n subvarieties. Thus, $\dim Y = n$, and so isomorphisms preserve dimension. □

(d) The irreducible components of $V = \mathbb{V}(zx - y, y^2 - x^2(x+1))$ are $\mathbb{V}(x, y)$, $\mathbb{V}(z^2 - x + 1)$ and $\mathbb{V}(y^2 - x^2(x+1))$.

Proof. Letting $f(x,y,z) = zx - y$ and $g(x,y,z) = y^2 - x^2(x+1)$, we see that $f(x,y,z) = 0$ if $x = 0, y = 0$ or $y = zx$. For the first case, we note that $g(0,0,z) = 0$ anyway. We can then see that in the second case,

$$\begin{aligned} & g(x, zx, z) = 0 \\ \implies & (zx)^2 - x^2(x+1) = 0 \\ \implies & x^2(z^2 - x + 1) = 0, \end{aligned}$$

and either $x = 0, y = 0$, or $z^2 = x - 1$.

Now, observe that

$$\begin{aligned} & g(x, y, z) = 0 \\ \implies & y^2 - x^2(x+1) = 0 \\ \implies & y^2 - x^3 - x^2 = 0, \end{aligned}$$

which is an irreducible affine curve in $\mathbb{A}^2$. Thus,

$$V = \mathbb{V}(x,y) \cup \mathbb{V}(z^2 - x + 1) \cup \mathbb{V}(y^2 - x^2(x+1)).$$

We now further verify that the first two of these components are also irreducible.

First, with $V_1 = \mathbb{V}(x,y) = \{(0,0,z) \mid z \in \mathbb{C}\}$, we clearly see this is irreducible, as we cannot split the z-axis into closed proper subsets. As for the second, note that the polynomial is irreducible, and thus $\mathbb{I}(\mathbb{V}(z^2 - x + 1))$ is prime. Thus, the second, and all, components as given as irreducible. □

4. (a) Let $V \subset \mathbb{A}^n$ be a Zariski-closed subset and $a \in \mathbb{A}^n \setminus V$ a point. Let $f \in \mathbb{C}[x_1, ..., x_n]$ be a polynomal such that.

Proof. $\mathbb{I}(V) = \{f \in \mathbb{C}[x_1, ..., x_n] \mid f(x) = 0, \forall x \in V\}$ $f(a) = 1$ □

(b) Let $I, (g) \subset \mathbb{C}[x_1, ..., x_n]$ be ideals. Assume that $\mathbb{V}(g) \supset \mathbb{V}(I)$.

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5. (a) If the pullback $\varphi* : \mathbb{C}[W] \to \mathbb{C}[V]$ is injective, the morphism φ dominant i.e. the image $\varphi(V)$ is dense in W.

Proof. □

(b) If the map φ is an isomorphism between V and some algebraic subvariety $U \subset W$, the pullback $\varphi* : \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective.

Proof. □