Q1

a) We have seen that the maximal ideals in $\mathbb{C}[x]$ corrispond to points in $\mathbb{C}$.

That is, $\operatorname{maxspec}(\mathbb{C}[x]) = \{(x-a) \mid a \in \mathbb{C}\}$

For $\operatorname{maxspec}(\mathbb{C}[x, 1/x])$, we have seen that

$$\mathbb{C}[x, \tfrac{1}{x}] = \frac{\mathbb{C}[x,y]}{(xy-1)} = \mathbb{C}[V] \text{ where } V = \mathbb{V}(xy-1).$$

(Example 4.16)

Importantly then, points in $V$ corrispond to maximal ideals of $\mathbb{C}[V]$ which are themselves points of $\operatorname{maxspec}(\mathbb{C}[x, 1/x])$.

Note $V = \{(a, 1/a) \in \mathbb{A}^2 \mid a \neq 0\}$, thus the maximal ideals,

$$\mathfrak{m}_{(a, 1/a)} = (x-a, y - 1/a).$$

For completeness, we remark that the ideal $(x-a, y-1/a)$ is principle since, $xy=1$ implies

$$-xa(y - \tfrac{1}{a}) = -xya + x = x - a.$$

Therefor,

$$\operatorname{maxspec}(\mathbb{C}[x, 1/x]) = \{(x-a) \mid a \in \mathbb{C} \setminus \{0\}\}.$$

b) We then have for $\varphi : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$ via $a \mapsto \frac{1}{a}$.

i) $\varphi^*(\frac{1}{x})(a) = \frac{1}{x}(\varphi(a)) = 1/x(\frac{1}{a}) = a$. So $\varphi^*(f) = 1$

ii) $\varphi^*\left(2x^2 + \frac{2x^3+4x}{x^5}\right)(a) = 2\varphi(a)^2 + \frac{2\varphi(a)^3 + 4\varphi(a)}{\varphi(a)^5}$

$$= \frac{2}{a^2} + \frac{2/a^3 + 4/a}{1/a^5}$$

$$= \frac{2}{a^2} + 2a^2 + 4a^4.$$

So, $\varphi^*\left(2x^2 + \frac{2x^3+4x}{x^5}\right) = \frac{2}{y^2} + 2y^2 + 4y^4$.

iii) $\varphi^*(2-x)(a) = 2 - \varphi(a) = 2 - \frac{1}{a} = \frac{2a-1}{a}$

so $\varphi^*(2-x) = \frac{2y-1}{y}$.

## Q2

a) Let $p: \mathbb{A}^3 \to \mathbb{A}^2$ via $(x,y,u) \mapsto (x,u)$.

Note then, $p(x,y,u) = (p_1(x,y,u), p_2(x,y,u))$

$$= (x + 0y + 0u, \ 0x + 0y + u)$$

is a polynomial map so $p$ is a morphism.

Then, define $\bar{p}: \mathbb{A}^2 \to V$ via $(x,u) \mapsto (x, xu, u)$ again this is clearly a polynomial map and so a morphism

For an isomorphism we then need

$$p|_V \circ \bar{p} = Id_{\mathbb{A}^2} \quad \& \quad \bar{p} \circ p|_V = Id_V$$

$$p|_V \circ \bar{p}(x,u) = p|_V(x, xu, u) = (x,u).$$

and $\bar{p} \circ p|_V (x, xu, u) = \bar{p}(x,u) = (x, xu, u).$

so we have an isomorphism

b) Note that the map $q: V \to \mathbb{A}^2$ s.t. $(x,y,u) \mapsto (x,y)$ is not bijective. Consider that $(0,0,u) \mapsto (0,0)$ for all $u \in \mathbb{C}$.

Importantly, if such an inverse existed then since $u = \frac{y}{x}$, the point $(0,0)$ woud have no defined image under an inverse map.

c) Note the set of regular functions $\mathcal{O}_V(D(u))$ are given by the set of polynomial so that $u$ does not vanish. As well as the quotients of polynomial's of the form

$$\frac{f(x,y,u)}{g(x,y,u)} \quad \text{such that } g(x,y,u) \neq 0.$$

I.e.

$$\mathcal{O}_V(D(u)) = \left\{ \frac{f(x,y,u)}{g(x,y,u)} \ ; \ f,g \in \mathbb{C}[x,y,u], \ g(x,y,0) \neq 0 \right\}$$

Q3

Suppose that $\bar{V} = \mathbb{V}(I) \cup \mathbb{V}(J)$ for $\mathbb{V}(I) \neq \bar{V} \neq \mathbb{V}(J)$

That is, suppose $\bar{V}$ is reducible.

Note $\bar{V}$ certainly contains $V$, so

$$V \subseteq \mathbb{V}(I) \cup \mathbb{V}(J).$$

If then $V \subseteq \mathbb{V}(I)$ we are done since $\mathbb{V}(I)$ is certainly closed and so contains the closure of $V$.

Otherwise if $V \subseteq \mathbb{V}(I) \cup \mathbb{V}(J)$ but $V \nsubseteq \mathbb{V}(I)$ or $V \nsubseteq \mathbb{V}(J)$ then, consider that

$$\mathbb{V}(I) \cap V \subseteq V \text{ and}$$

$$\mathbb{V}(J) \cap V \subseteq V.$$

So,

$$V = (\mathbb{V}(I) \cap V) \cup (\mathbb{V}(J) \cap V)$$

That is, $V$ is the union of two varieties contridicting irriducibility of $V$.

Q4

$$\mathbb{V}(y - \sin(x)) = \{(x, \sin x) \mid x \in \mathbb{C}\}$$

We note that in $\mathbb{P}^2$ the variety is then

That is,

$$\mathbb{V}(y - \sin(x)) = \{[x ; \sin(x)] \mid x \in \mathbb{C}^*\}.$$

$$\mathbb{V}(y - \sin(x)) \subseteq U_x \text{ and in fact.}$$

$$\mathbb{V}(y - \sin(x)) = \{[1 ; \tfrac{\sin(x)}{x}] \mid x \neq 0\}.$$

But then, the point $[1 ; \frac{\sin(x)}{x}]$ we may place any point say $x = \pi$ in to see that

$$[1 ; \tfrac{\sin(\pi)}{\pi}] = [1 ; 0] \text{ \& } x = \tfrac{\pi}{2} \text{ to see}$$

$$[1 ; \tfrac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}}] = [1 ; \tfrac{2}{\pi}].$$

The closure of $V$ is then

$$\{[1 ; 0]\} \cup \{[(2k+1)\pi ; 2] \mid k \in \mathbb{Z}\}$$

In particular,

$$\mathbb{V}(y - \sin(x)) = \mathbb{V}(x - 1) \cup \mathbb{V}(x - (2k+1)\pi, y - 2)$$

Q5

$V_t = \mathbb{V}(xy^2 - t)$

Note then,

$V_0 = \mathbb{V}(xy^2 - 0) = \mathbb{V}(xy^2)$

note $\mathbb{R}$ is an integral domain so the Variety is precicely given by the $x$ & $y$ axis.

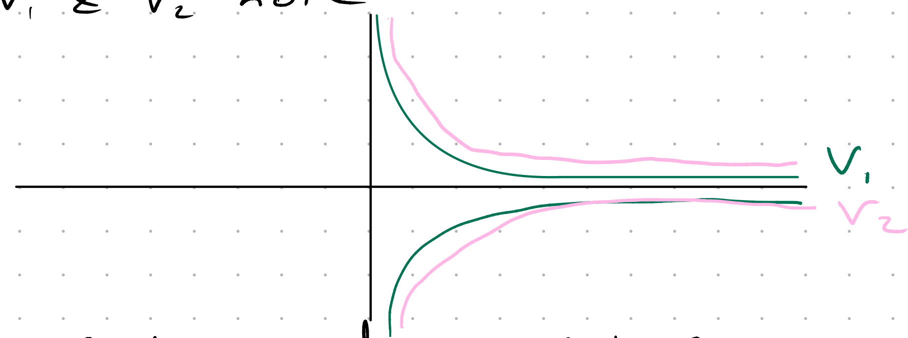
Note that $\nabla(xy^2) = (y^2, 2xy)$.

Then at $(0,0) \in V_0$ we have $\nabla(xy^2)(0,0) = (0,0)$

In particular, $\ker(0,0) = \{0\} \times \mathbb{A}^1$ which has dimention 2.

However, $V_0$ has dimention 1 since it is the ided of one polynomial.

For $V_1$ & $V_2$ note




Note here the grad on $V_1$ has,

$$\nabla(xy^2 - 1)\left(\tfrac{1}{b^2}, b\right) = \left(b^2, \tfrac{2}{b}\right).$$

Then for the kernel notice that,

$$\left(b^2, \tfrac{2}{b}\right)^* \begin{pmatrix} x \\ y \end{pmatrix} = 0 = b^2 x + \frac{2y}{b} \Rightarrow y = \frac{-b^3 x}{2}$$

I.e has dimention 1.

Compairing this to $V_1$ we have that $\mathbb{V}(xy^2 - 1)$ is of dimention 1 since it is the variety of a single polynomial.

For $V_2$ note $\nabla(xy^2 - 2)\Big|_{\left(\frac{2}{b^2}, b\right)} = \left(b^2, \frac{4}{b}\right)$

Evaluating $\ker\left(\left(b^2, \frac{4}{b}\right)^*\right)$ we get

$$\begin{pmatrix} b^2 \\ 4/b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = xb^2 + \frac{4y}{b} = 0$$

$$\Rightarrow x = \frac{-4y}{b^3} \quad \text{i.e. 1 dimensional,}$$

So, $V_2$ is smooth.

Note then both $V_1$ & $V_2$ are smooth and so irriducible. But $V_0 = \mathbb{V}(xy^2) = \mathbb{V}(x) \cup \mathbb{V}(y)$