Assessed Homework 1 (Neat)

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Q1. Let $A \subseteq \mathbb{A}^n$ be a subset.

(a) (**5 marks**) What is the definition of the closure of A in $\mathbb{A}^n$?

(b) (**5 marks**) Prove that $\mathbb{V}(\mathbb{I}(A))$ equals the Zariski closure of A in $\mathbb{A}^n$.

(c) (**5 marks**) Give an example of a subset in $B \subseteq \mathbb{C}$ whose closure in the Zariski topology does not coincide with its closure in the Euclidean topology.

a) The closure of A in $\mathbb{A}^n$ is the set $\bar{A} = A \cup \{\text{limit points of } A\}$ where the limit points of A are the points $a \in \mathbb{A}^n$ such that if $\forall U \subset A$ open with $a \in U$, we have $(U \setminus \{a\}) \cap A \neq \emptyset$

b) We first show that $\bar{A} \subseteq \mathbb{V}(\mathbb{I}(A))$

Take $a \in \bar{A}$ then either $a \in A$ or a is a limit point of A.

Clearly if $a \in A$ then $a \in \mathbb{V}(\mathbb{I}(A))$

if a is a limit point of A, then since polynomials are continuous, polynomials that vanish on A will also vanish on limit points of A. To see this consider a sequence of points that converges to a, say $(a_n)_{n\in\mathbb{N}} \to a$, and a polynomial that vanishes on A, say f. Then $f(a_n) = 0$ $\forall n \in \mathbb{N}$, and so continuity of f implies that $f(a_n) \to f(a) \Rightarrow f(a) = 0$.

For the inverse inclusion ($\mathbb{V}(\mathbb{I}(A)) \subseteq \bar{A}$), Note that $A \subseteq \bar{A}$ so $\mathbb{V}(\mathbb{I}(A)) \subseteq \mathbb{V}(\mathbb{I}(\bar{A})) = \bar{A}$

c) Consider the set $B = \{\frac{1}{n}, n \in \mathbb{N}\}$. In the Euclidean Topology, the only limit point of B is 0 so $\bar{B} = B \cup \{0\}$.

However in the Zariski Topology, we have that $\mathbb{I}(B) = \{0\}$. This is because B has infinitely many distinct points, and a non-zero polynomial cannot have infinitely many distinct roots since it would need an infinite degree, which would not be a polynomial. Then $\mathbb{V}(\mathbb{I}(B)) = \mathbb{V}(0) = \mathbb{C}$.

So by part (b), $\bar{B} = \mathbb{C}$ in the Zariski Topology.

$\mathbb{C} \neq B \cup \{0\}$ so we have our desired example.

Q2. (a) (**5 marks**) What is the definition of a compact subset of a topological space?

(b) (**10 marks**) Prove that $\mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}^2$ is compact in the Zariski topology but not in the Euclidean topology.

a) For $K \subset X$. K is compact if every open cover of K has a finite subcover. So if $\{U_\alpha\}$ is an open cover of K, then only finitely many of the $\{U_\alpha\}$ are needed to cover K: $\exists\, U_{\alpha_1}, \dots, U_{\alpha_n} \in \{U_\alpha\}$ such that $K = \bigcup_{i=1}^{n} U_{\alpha_i}$

b) In the Euclidean Topology, a subset is bounded if it is closed and bounded. $\mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}^2$ is the set

c) In the Euclidean Topology, a subset is bounded if it is closed and bounded. $\mathbb{V}(x^2-y^3) \subseteq \mathbb{C}^2$ is the set of all points such that $x^2-y^3=0$ or $x^2=y^3$ or $x=y^{3/2}$ which has arbitrarily large solutions and so is not bounded and thus not compact.

In the Zariski Topology:

Take an open cover of $V=\mathbb{V}(x^2-y^3)$, say $\{U_\alpha\}_{\alpha\in A}$ and note that each U_α is open and so is the complement of some closed affine algebraic variety, say $U_\alpha = V_\alpha^c$.

Whilst A may be an uncountable index set, we can pick an index and call it 1, then 2, and so on.

This allows us to consider U_1 and it's complement V_1.

So consider $V\cap V_1$. If $V\cap V_1=\emptyset$ then U_1 covers V and we are done. If not consider $(V\cap V_1)\cap V_2$. If $(V\cap V_1)\cap V_2=\emptyset$ then U_1, U_2 cover V and we are done.

If not then continue this process with V_3, and so on. Note that we have the chain of inclusions

$$\mathbb{I}(V_1) \subseteq \mathbb{I}(V_1\cap V_2) \subseteq \dots \subseteq \mathbb{I}(V_1\cap V_2\cap\dots\cap V_m) \subseteq \dots$$

Since $\mathbb{C}^2$ is Noetherian, this chain must stabilise at some point, say at $\mathbb{I}(V_1\cap\dots\cap V_n)$. If we now consider some other $U_\alpha \neq U_1,\dots,U_n$ then if V is not compact, we must have that

$$V_\alpha \cap (V\cap V_1\cap\dots\cap V_n) = V\cap V_1\cap\dots\cap V_n.$$

Since the RHS is $\neq\emptyset$ this contradicts the fact that $\{U_\alpha\}$ is an open cover of V, since all further intersections will be $\neq\emptyset$. We must therefore have that V is compact since there must exist some V_α such that

$$V_\alpha \cap (V\cap V_1\cap\dots\cap V_n)=\emptyset \text{ and so}$$

$\{U_1,\dots,U_n,U_\alpha\}$ covers V

Q3. (a) (**5 marks**) Find a curve $W \subseteq \mathbb{A}^2$ and a morphism $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$, such that W is irreducible but $\varphi^{-1}(W)$ is not.

(b) (**5 marks**) Let Y be a topological space and consider $X \subseteq Y$ with the subspace topology. Prove that if X is irreducible then so is its closure.

(c) (**5 marks**) Prove that isomorphisms preserve irreducibility and dimension of closed affine algebraic varieties.

(d) (**10 marks**) Find the irreducible components of $\mathbb{V}(zx-y, y^2-x^2(x+1)) \subseteq \mathbb{A}^3$. You need to justify why each component is irreducible.

a) Let W be defined by $y=x$ which is irreducible and consider $\varphi(x,y)=(x,y^2)$ which is a morphism

then $\varphi^{-1}(W)=\{(x,y)\in\mathbb{A}^2 : \varphi(x,y)\in W\}$

so we need (x,y) such that $\varphi(x,y)=(x,x)$

then $\varphi^{-1}(W) = \{(x,y) \in \mathbb{A}^2 : \varphi(x,y) \in W\}$

so we need (x,y) such that $\varphi(x,y) = (x,x)$

$\varphi(x,y) = (x, y^2)$ so $y^2 = x \Rightarrow y = \pm\sqrt{x}$

so $\varphi^{-1}(W) = \{(x,y) \in \mathbb{A}^2 : y = \sqrt{x}\} \cup \{(x,y) \in \mathbb{A}^2 : y = -\sqrt{x}\}$

has 2 components and is thus reducible

b) $X \subseteq Y$ is irreducible if when $X = X_1 \cup X_2$ for X_1, X_2 closed, we have that $X \subseteq X_1$ or $X \subseteq X_2$

So take some $X \subseteq Y$ that is irreducible and X_1, X_2 closed, such that $X = X_1 \cup X_2$. By irreducibility $X \subseteq X_1$ or $X \subseteq X_2$.

Now consider $\bar{X}$, we have that $\bar{X} = \overline{X_1 \cup X_2} = \bar{X}_1 \cup \bar{X}_2 = X_1 \cup X_2$ since X_1, X_2 are closed, where the second equality holds since a limit point of $X_1 \cup X_2$ is clearly also a limit point of X_1 and/or X_2 and vice versa. Finally, since $X \subseteq X_1$ or $X \subseteq X_2$ and $\bar{X} \subseteq \bar{X}_1 = X_1$ or $\bar{X} \subseteq \bar{X}_2 = X_2$.

Thus if X can't be reduced by X_1, X_2, neither can $\bar{X}$.

On the other hand, consider V_1, V_2 closed such that $\bar{X} = V_1 \cup V_2$. Then $X \subseteq V_1 \cup V_2$, so $\exists W_1 \subseteq V_1, W_2 \subseteq V_2$ such that $X = W_1 \cup W_2$. Then $X \subseteq W_1$ or $X \subseteq W_2$ by irreducibility. so $X \subseteq V_1$ or $X \subseteq V_2$ and thus X cannot be reduced by V_1, V_2 and therefore neither can $\bar{X}$ by above.

Thus $\bar{X}$ is also irreducible

c) Consider V, W closed affine algebraic varieties, and φ an isomorphism $\varphi: V \to W$

Suppose V is irreducible. Then when $V = V_1 \cup V_2$, we have either $V \subseteq V_1$ or $V \subseteq V_2$. Now consider $\varphi(V) = \varphi(V_1 \cup V_2)$

We claim that $\varphi(V_1 \cup V_2) = \varphi(V_1) \cup \varphi(V_2)$.

To see this consider $x \in \varphi(V_1 \cup V_2)$, by surjectivity of φ $\exists y \in V_1 \cup V_2$ such that $\varphi(y) = x$, and since $y \in V_1 \cup V_2$, $y \in V_1$ and/or, $y \in V_2$ so $\varphi(y) \in V_1$ and/or $\varphi(y) \in V_2$, and thus $\varphi(V_1 \cup V_2) \subseteq \varphi(V_1) \cup \varphi(V_2)$

For the other direction, take $x \in \varphi(V_1) \cup \varphi(V_2)$ then we have that $x \in \varphi(V_1)$ and/or $\varphi(V_2)$ so by surjectivity of φ, $\exists y$ s.t. $\varphi(y) = x$, so $\varphi(y) \in \varphi(V_1)$ and/or $\varphi(y) \in \varphi(V_2)$ so $y \in V_1 \cup V_2$ and thus $\varphi(y) \in \varphi(V_1 \cup V_2)$. Thus $\varphi(V_1 \cup V_2) = \varphi(V_1) \cup \varphi(V_2)$.

Now since $V \subseteq V_1$ or $V \subseteq V_2$. We must have that $\varphi(V) \subseteq \varphi(V_1)$ or $\varphi(V) \subseteq \varphi(V_2)$ since φ is bijective.

Thus whenever $\varphi(V) = \varphi(V_1) \cup \varphi(V_2)$, we have $\varphi(V) \subseteq \varphi(V_1)$ or $\varphi(V) \subseteq \varphi(V_2)$ and thus $\varphi(V)$ is irreducible.

Now suppose that V has dimension d, that is d is the largest integer such that

$$V = V_d \supsetneq V_{d-1} \supsetneq \dots \supsetneq V_0 = \{p\}$$

We show that isomorphisms preserve strict subsets and thus the chain

$$\varphi(V) = \varphi(V_d) \supsetneq \varphi(V_{d-1}) \supsetneq \dots \supsetneq \varphi(V_0)$$

holds and so $\dim(\varphi(V)) = d$.

So consider $A \subsetneq B$, clearly if $a \in A$, then $\varphi(a) \in \varphi(A)$ and $a \in B \Rightarrow \varphi(a) \in \varphi(B)$ so $\varphi(A) \subseteq \varphi(B)$

Then if $b \in B \setminus A$ (which exists since $B \neq A$)

clearly $\varphi(b) \in \varphi(B \setminus A) \subseteq \varphi(B)$

then since φ is bijective, if $\varphi(b) \in \varphi(A)$, then $b \in A$ contradiction. Thus $\varphi(B) \neq \varphi(A)$ and so $\varphi(A) \subsetneq \varphi(B)$ and we are done.

d) The points of $\mathbb{V}(zx-y, y^2-x^2(x+1))$ are given by the solutions to the simultaneous equations

$$zx - y = 0 \qquad ①$$

$$y^2 - x^2(x+1) = 0 \qquad ②.$$

Rearranging ① gives $y = zx$, then substituting this into ② gives

$$z^2x^2 - x^2(x+1) = x^2(z^2 - (x+1)) = 0$$

$$\Rightarrow \quad x^2 = 0 \quad \text{or} \quad z^2 - (x+1) = 0$$

$$\text{so} \quad x = 0 \quad \text{or} \quad z^2 = x+1$$

In the first case $(x=0)$, from ① we have also that $y=0$ and z is a free variable, so the solutions are given by the points $(0,0,z) \in \mathbb{A}^3$

This can be represented by the morphism

$$\varphi : \mathbb{A}^1 \to \mathbb{V}(zx-y, y^2-x^2(x+1))$$
$$t \mapsto (0,0,t)$$

which is an isomorphism with inverse $\varphi^{-1} : \mathbb{V}(zx-y, y^2-x^2(x+1)) \to \mathbb{A}^1$
$$(0,0,z) \mapsto z$$

In the second case $(z^2 = x+1)$ we have that $x = z^2-1$ so from ①
$y = zx = z^3 - z$. So the solutions are given by the points

$$(z^2-1, z^3-z, z) \in \mathbb{A}^3$$

Similarly to above, this can be represented by the isomorphism

$$\psi : \mathbb{A}^1 \to \mathbb{V}(zx-y, y^2-x^2(x+1))$$
$$t \mapsto (t^2-1, t^3-t, t)$$

with inverse

$$\psi^{-1} : \mathbb{V}(zx-y, y^2-x^2(x+1))$$

with inverse

$$\varphi^{-1}: \mathbb{V}(zx-y, y^2-x^2(x+1))$$
$$(z^2-1, z^3-z, z) \mapsto z$$

Therefore we have two components of $\mathbb{V}(zx-y, y^2-x^2(x+1))$. To show that they are irreducible, it suffices to show that $\mathbb{A}^1$ is irreducible by part (c) since the isomorphism would preserve the irreducibility. Since $\mathbb{A}^1$ is the entire space, it is clearly irreducible, as if not: $\mathbb{A}^1 = V_1 \cup V_2$ for V_1, V_2 closed then V_1, V_2 must be finite, so $V_1 \cup V_2$ is finite, but $\mathbb{A}^1$ is infinite. Thus $\mathbb{A}^1$ must be irreducible.

Finally this tells us that

$$\varphi(A) = \{(0,0,t) \in \mathbb{A}^3 : t \in \mathbb{A}\}$$

and $$\varphi(A) = \{(t^2-1, t^3-t, t) \in \mathbb{A}^3 : t \in \mathbb{A}\}$$

are irreducible, and thus irreducible components of $\mathbb{V}(zx-y, y^2-x^2(x+1))$

Q4. (a) (**10 marks**) Let $V \subseteq \mathbb{A}^n$ be a Zariski-closed subset and $a \in \mathbb{A}^n \setminus V$ be a point. Find a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that

$$f \in \mathbb{I}(V), \quad f(a) = 1.$$

(b) (**15 marks**) Let $I, (g) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be two ideals. Assume that $\mathbb{V}(g) \supseteq \mathbb{V}(I)$.

(i) Prove that if $I = (f_1, \ldots, f_k)$, then

$$(f_1, \ldots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \ldots, x_{n+1}]. \tag{1}$$

(ii) By only using Equation (1) and not the nullstellensatz, prove that there exists a positive integer m such that $g^m \in I$.

a) Since V is a closed affine algebraic variety in $\mathbb{A}^n$ which is Noetherian, it must be finitely generated, say by $\{f_i\}$. Then take any of these f_i and consider

$$g(x) = \frac{f_i(x)}{f_i(a)}$$

For $v \in V$, $g(v) = \dfrac{f_i(v)}{f_i(a)} = \dfrac{0}{f_i(a)} = 0$

and for $a \in \mathbb{A}^n \setminus V$, $g(a) = \dfrac{f_i(a)}{f_i(a)} = 1$

b) i) Assume for the sake of contradiction that

$$J = (f_1, \ldots, f_k, x_{n+1}g - 1) \neq \mathbb{C}[x_1, \ldots, x_{n+1}]$$

Then there exists some maximal ideal $m \subseteq \mathbb{C}[x_1, \ldots, x_{n+1}]$ such that $J \subseteq m$. Then since a maximal ideal can be represented by a point, say $\alpha \in \mathbb{A}^{n+1}$, such that $m = (x_1 - a_1, \ldots, x_{n+1} - a_{n+1})$ $\alpha = (a_1, \ldots, a_{n+1})$. [illegible]

=> a point, say $\alpha \in \mathbb{A}^{n+1}$, such that $m = (x_1 - a_1, \dots, x_{n+1} - a_{n+1})$
$\alpha = (a_1, \dots, a_{n+1})$, all polynomials in m must vanish on α (by definition of an ideal). Since $J \subseteq m$ we thus have that $f_i(\alpha) = 0$ $\forall i, 1 \le i \le k$, and $x_{n+1} g(\alpha) - 1 = 0$.

But then since $\mathbb{V}(g) \supseteq \mathbb{V}(I)$ if f_i vanish on α, so must g, so $x_{n+1} g(\alpha) - 1 = x_{n+1} \cdot 0 - 1 = -1 \neq 0$

Thus we have a contradiction and so

$$J = \mathbb{C}[x_1, \dots, x_{n+1}] \text{ as required.}$$

ii) From part (i) we know that

$$(f_1, \dots, f_k, x_{n+1} g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}]$$

So we can write

$$1 = h_1 f_1 + \dots + h_k f_k + h(x_{n+1} g - 1) \qquad (*)$$

for some $h_i, h \in \mathbb{C}[x_1, \dots, x_{n+1}]$

This holds for all x_{n+1}, so consider in particular $x_{n+1} = 1$.

Call $j_i = h_i(x_1, \dots, x_n, 1)$, $j = h(x_1, \dots, x_n, 1)$

Clearly $j_i, j \in \mathbb{C}[x_1, \dots, x_n]$ and $(*)$ becomes

$$1 = j_1 f_1 + \dots + j_k f_k + j(g - 1),$$

an equation for 1 in $\mathbb{C}[x_1, \dots, x_n]$

Thus $(f_1, \dots, f_k, g-1) = \mathbb{C}[x_1, \dots, x_n]$

$\Rightarrow$ $(f_1, \dots, f_k, g) = \mathbb{C}[x_1, \dots, x_n]$

So $\exists\, p_i, p \in \mathbb{C}[x_1, \dots, x_n]$ such that

$$p_1 f_1 + \dots + p_k f_k + pg = 1$$

multiply both sides by g^m

$$g^m p_1 f_1 + \dots + g^m p_k f_k + p g^{m+1} = g^m$$

$$\Rightarrow \quad p g^{m+1} = g^m (1 - p_1 f_1 - \dots - p_k f_k)$$

$$g^m (pg - 1 + p_1 f_1 + \dots + p_k f_k) = 0$$

$$g^m (pg - 1) \equiv 0 \mod I \qquad I = (f_1, \dots, f_k)$$

$\Rightarrow$ $g^m \equiv 0 \mod I$ in this case we are done since $g^m \equiv 0 \mod I \Rightarrow g^m \in I$

or $pg - 1 \equiv 0 \mod I$

if $pg - 1 \equiv 0 \mod I$

then $pg \equiv 1 \mod I$

$\Rightarrow$ g is a unit mod I

$\Rightarrow$ g generates $\mathbb{C}[x_1, \dots x_n] / I$ ran out of time...

Q5. Prove at least one implication from each of the following equivalences.

(a) (**10 marks**) Show that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is injective if and only if φ is *dominant*. Recall that a map, φ, is called dominant if its image, $\varphi(V)$, is dense in W.

(b) (**10 marks**) Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W.

a) Suppose $\varphi(V)$ is dense in W. Take a nonzero $g \in \mathbb{C}[W]$ and then consider $\varphi(V) \cap \mathbb{V}(g)^c$, where $\mathbb{V}(g)^c$ is the open set where g doesn't vanish. Since $\varphi(V)$ is dense $\varphi(V) \cap \mathbb{V}(g)^c \neq \emptyset$, so take some $y \in \varphi(V) \cap \mathbb{V}(g)^c$. Then $(\varphi^*(g))(y) \neq 0$, so $\varphi^*(g)$ cannot be zero. Thus $\ker \varphi^* = 0$ and thus φ^* is injective

b) Suppose that $\varphi^*: \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective. Then consider $\varphi(V) \subseteq W$, and let $\overline{\varphi(V)}$ be its Zariski closure so that $\overline{\varphi(V)}$ is an algebraic subvariety of W. By definition of $\overline{\varphi(V)}$, the map $\varphi: V \to \overline{\varphi(V)}$ is dominant (since the image of V, $\varphi(V)$, is dense in $\overline{\varphi(V)}$ because the closure of the image is the whole set), and so the map $\varphi^*: \mathbb{C}[\overline{\varphi(V)}] \to \mathbb{C}[V]$ is injective by part (a). Then by assumption, we have that $\mathbb{C}[W] \to \mathbb{C}[V]$ is surjective and thus so is $\mathbb{C}[\overline{\varphi(V)}] \to \mathbb{C}[V]$. Thus $\mathbb{C}[\overline{\varphi(V)}] \to \mathbb{C}[V]$ is an isomorphism and hence so is $V \to \overline{\varphi(V)}$ as required.