Dear Clement,

Thank you for your questions. It's great to see your enthusiasm in revisiting calculus. I'm using ChatGPT with some modifications to respond to your question properly. I double-checked the response and it seems reasonable.

1. Understanding Limits

Your understanding of limits is mostly correct. When we define the gradient (slope) between two points A and B on a curve, we write:

$$m(h) = \frac{f(x+h) - f(x)}{h}$$

You're exactly right that as the distance h between these two points shrinks toward zero (i.e., point B approaches point A), the value $m(h)$ approaches a specific number called the derivative of $f(x)$ at point A. However, the subtlety is in the concept of a limit:

- We never actually set $h = 0$ in the original definition because, as you pointed out, this would make $m(h)$ undefined (division by zero).
- Instead, we observe what happens as h becomes very small—infinitesimally close to zero—and find a unique number that $m(h)$ approaches. This unique number is precisely the derivative at point A.

In short, the derivative is defined by the limiting process, not by directly substituting $h = 0$.

2. Clarifying with an Example: $f(x) = x^2$

Let's carefully examine your example:

Initially, we have:

$$m(h) = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h}$$

At this stage, you're correct; we cannot simply substitute $h = 0$ because we would have division by zero.

However, notice we can factor h out in the numerator:

$$m(h) = \frac{h(2x+h)}{h}$$

Provided $h \neq 0$, we can safely simplify (cancel out h) to get:

$$m(h) = 2x + h$$

Now, we examine the limit as h approaches zero:

$$\lim_{h \to 0} (2x + h) = 2x$$

In this step, we no longer face the division-by-zero issue because we've already simplified the expression correctly. By carefully simplifying first, we transform the problem from one involving division by zero into a simple limit evaluation.

Thus, there's no contradiction:

- Initially substituting $h = 0$ directly is not allowed.
- After simplification (which is valid when $h \neq 0$), we evaluate the limit by examining behavior as h approaches zero, which allows us to directly substitute $h = 0$.

I hope this clarifies the apparent contradiction and helps deepen your understanding of differentiation. Feel free to reach out if you have more questions—it's always a pleasure to help.

Best wishes,
Farhad