

Assessed 2

23 April 2024 16:06

Q1. (a) (10 marks) Find all the elements of $\text{maxSpec}(\mathbb{C}[x])$ and $\text{maxSpec}(\mathbb{C}[x, 1/x])$, respectively.

(b) (10 marks) Consider the isomorphism $\varphi : \mathbb{A}^1 \setminus \{0\} \rightarrow \mathbb{A}^1 \setminus \{0\}$, $a \mapsto b = 1/a$, and the pullback map on the coordinate rings $\varphi^* : \mathbb{C}[x, 1/x] \rightarrow \mathbb{C}[y, 1/y]$. Compute $\varphi^*(1/x)$, $\varphi^*(2x^2 + \frac{2x^3+4x}{x^5})$, $\varphi^*(2 - x)$.

$$\text{a)} \quad \text{maxSpec}(\mathbb{C}[x]) = \{m \subseteq \mathbb{C}[x] \mid m \text{ a maximal ideal}\}$$

$$(\mathbb{C} \text{ a field} \Rightarrow \mathbb{C}[x] \text{ a P.I. D}) = \{(\mathfrak{f}) \subseteq \mathbb{C}[x] \mid (\mathfrak{f}) \text{ maximal}\}$$

$$(\mathbb{C}[x] \text{ an integral domain}) = \{(\mathfrak{f}) \subseteq \mathbb{C}[x] \mid \mathfrak{f} \text{ irreducible}\}$$

$$(\mathbb{C} \text{ algebraically closed}) = \{(\mathfrak{f}) \subseteq \mathbb{C}[x] \mid \deg \mathfrak{f} = 1\}$$

$$= \{(x-\alpha) \subseteq \mathbb{C}[x] \mid \alpha \in \mathbb{C}\}.$$

Consider the \mathbb{C} -algebra homomorphism:

$$\Psi : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, 1/x]; x \mapsto x, y \mapsto 1/x.$$

$$\Rightarrow \Psi(xy) = x \cdot \frac{1}{x} = 1, \Rightarrow \Psi(xy-1) = 0.$$

$\Rightarrow \ker \Psi = (xy-1)$, so by Homomorphism theorem:

$$\text{im } \Psi = \mathbb{C}[x, 1/x] \cong \frac{\mathbb{C}[x, y]}{\ker \Psi} = \frac{\mathbb{C}[x, y]}{(xy-1)} =: \mathbb{C}[V].$$

Since $\mathbb{I}(V) = (xy-1)$, and $\mathbb{V}(\mathbb{I}(V)) = V$, we determine that

$$V = \mathbb{V}((xy-1)) = \mathbb{V}(xy-1).$$

$$\alpha \cdot \beta - 1 = 0 \Rightarrow \alpha \beta = 1 \Rightarrow \beta = \alpha^{-1} \Rightarrow \alpha, \beta \in \mathbb{C}^*.$$

$\text{So } V = \mathbb{V}(xy-1) = \{(\alpha, \alpha^{-1}) \mid \alpha \in \mathbb{C}^*\}$

So maximal ideals of $\mathbb{C}[V]$ are of the form $(x-\alpha, y-\alpha^{-1})$ for $\alpha \in \mathbb{C}^*$ (since we consider $\mathbb{C}[x, y]|_V$).

$$xy-1=0 \Rightarrow xy=1, \text{ so } (x-\alpha, y-\alpha^{-1})$$

$$y(x-\alpha) = 1 - \alpha y \Rightarrow -\alpha^{-1}y(x-\alpha) = \alpha^{-1}(xy-1) = y - \alpha^{-1}.$$

$$\Rightarrow y - \alpha^{-1} \in (x-\alpha) \Rightarrow (x-\alpha) = (x-\alpha, y-\alpha^{-1}).$$

$$\text{Thus, } \text{maxSpec}(\mathbb{C}[V]) = \text{maxSpec}(\mathbb{C}[x, y]_{(xy-1)}) = \{(x-\alpha) \mid \alpha \in \mathbb{C}^*\}.$$

Thus, $\text{maxSpec}(\mathbb{C}[V]) = \text{maxSpec}\left(\frac{\mathbb{C}[x,y]}{(x^2-y)}\right) = \{(x-\alpha) \mid \alpha \in \mathbb{C}^*\}$,

So $\text{maxSpec}(\mathbb{C}[x, \frac{1}{x}]) = \{(x-\alpha) \mid \alpha \in \mathbb{C}^*\}$, by the isomorphism $\varphi|_V$. ✓

b) Let $\alpha \in \mathbb{A}^1 \setminus \{0\}$.

$$\varphi^*\left(\frac{1}{x}\right)(\alpha) = \frac{1}{x} \circ \varphi(\alpha) = \frac{1}{x}(\frac{1}{\alpha}) = \frac{1}{1/\alpha} = \alpha.$$

$$\text{So } \varphi^*\left(\frac{1}{x}\right) = \text{id}_{\mathbb{A}^1 \setminus \{0\}} \Rightarrow \varphi^*\left(\frac{1}{x}\right) = y \in \mathbb{C}[y, \frac{1}{y}]$$

Given that $\varphi^*\left(\frac{1}{x}\right) = y$, and that φ^* is a \mathbb{C} -algebra homomorphism:

$$\varphi^*\left(\frac{1}{x}\right) = y \Rightarrow \varphi^*(x) = \varphi^*\left(\left(\frac{1}{x}\right)^{-1}\right) = \varphi^*\left(\frac{1}{x}\right)^{-1} = y^{-1} = \frac{1}{y}.$$

Thus, additionally given that $\varphi^*(x) = \frac{1}{y}$:

$$\begin{aligned} \varphi^*\left(2x^2 + \frac{2x^3+4x}{x^5}\right) &= \varphi^*\left(2x^2 + \frac{2x^3}{x^5} + \frac{4x}{x^5}\right) = \varphi^*\left(2x^2 + 2 \cdot \frac{1}{x^3} + 4 \cdot \frac{1}{x^4}\right) \\ &= 2\varphi^*(x)^2 + 2\varphi^*\left(\frac{1}{x}\right)^2 + 4\varphi^*\left(\frac{1}{x}\right)^4 \\ &= 2\left(\frac{1}{y}\right)^2 + 2y^2 + 4y^4 = 4y^4 + 2y^2 + 2y^{-2}, \end{aligned}$$

and $\varphi^*(2-x) = \varphi^*(2) - \varphi^*(x) = 2 - \frac{1}{y}$. ✓ □

Q2. Consider the affine algebraic hypersurface $V := \mathbb{V}(y - ux) \subseteq \mathbb{A}^3$.

- (a) (10 marks) Prove that the projection $\mathbb{A}^3 \rightarrow \mathbb{A}^2$, $(x, y, u) \mapsto (x, u)$ restricts to an isomorphism between V and \mathbb{A}^2 .
- (b) (10 marks) Prove that the projection $\mathbb{A}^3 \rightarrow \mathbb{A}^2$, $(x, y, u) \mapsto (x, y)$ does not restrict to isomorphism between V and \mathbb{A}^2 .
- (c) (10 marks) Find $\mathcal{O}_V(D(u))$.

a) Let $\Phi: \mathbb{A}^3 \rightarrow \mathbb{A}^2$; $(x, y, u) \mapsto (x, u)$, and $\Phi|_V = \varphi: V \rightarrow \mathbb{A}^2$.

Since Φ is a polynomial mapping, φ is a morphism of algebraic varieties.

Define $\Psi: \mathbb{A}^2 \rightarrow \mathbb{A}^3$; $(\alpha, \beta) \mapsto (x, y, u) := (\alpha, \alpha\beta, \beta)$

$$\begin{aligned} (x, y, u) \in V &\Rightarrow y - ux = 0 \Rightarrow y = ux \Rightarrow (x, y, u) = (x, ux, u) \\ &\Rightarrow \Psi(x, u) = (x, ux, u). \\ &\Rightarrow V \subseteq \text{im } \Psi \end{aligned}$$

$$\begin{aligned} \text{For } (x, y, u) = (\alpha, \alpha\beta, \beta) = \Psi(\alpha, \beta), \quad y - ux = 0 = \alpha\beta - \beta\alpha = 0, \\ &\Rightarrow \Psi(\alpha, \beta) \in V \\ &\Rightarrow \text{im } \Psi \subseteq V \Rightarrow \text{im } \Psi = V. \end{aligned}$$

$$\begin{aligned} &\Rightarrow \Psi(\alpha, \beta) \in V \\ &\Rightarrow \text{Im } \Psi \subseteq V \Rightarrow \text{Im } \Psi = V. \end{aligned}$$

$$\text{So } \Psi: A^2 \rightarrow V.$$

$$\forall (\alpha, \beta) \in A^2, \quad \Psi \circ \Psi(\alpha, \beta) = \Psi(\alpha, \alpha \beta, \beta) = (\alpha, \beta), \text{ so } \Psi \circ \Psi = \text{id}_{A^2}.$$

$$\forall (x, y, u) \in V, \quad y - ux = 0 \Rightarrow y = xu$$

$$\Rightarrow \Psi \circ \Psi(x, y, u) = \Psi(x, u) = (x, xu, u) = (x, y, u), \text{ so } \Psi \circ \Psi = \text{id}_V.$$

So by definition, $\Psi = \Phi|_V$ is an isomorphism between V & A^2 ✓

b) $V = \{(x, y, u) \in A^3 \mid y = ux\}$. Define $\Psi: V \rightarrow A^2; (x, y, u) \mapsto (x, y)$.
 $= (x, ux)$

$(x, y, u) = (0, 0, \alpha)$ satisfies $y = ux \quad \forall \alpha \in A^1$, as $0 = \alpha \cdot 0$.

Therefore $\Psi(0, 0, \alpha) = (0, 0) \quad \forall \alpha \in C$ (Specifically, it is defined).

If for $\alpha \in C$, $\Theta: A^2 \rightarrow V$ s.t. $\Theta \circ \Psi(0, 0, \alpha) = \Theta(0, 0) = (0, 0, \alpha)$,
then $\forall \beta \neq \alpha, \quad \Theta \circ \Psi(0, 0, \beta) = \Theta(0, 0) = (0, 0, \alpha) \neq (0, 0, \beta)$.

Thus, $\exists \Theta: A^2 \rightarrow V$ s.t. $\Theta \circ \Psi = \text{id}_V$, So the restriction of the
projection $A^3 \rightarrow A^2; (x, y, u) \mapsto (x, y)$ to V , Ψ , cannot be an isomorphism. ✓

c) We have that $D(u) = A^3 \setminus V(u)$

$$V(u) = \{(\alpha, 0, 0) \mid \alpha \in C\} \cup \{(0, 0, \beta) \mid \beta \in C\}.$$

$$\text{So } D(u) = \{(x, y, u) \in (A^3 \setminus \{0\})^3 \mid y = ux\}.$$

Q3. (20 marks) Prove that if V is an irreducible affine variety, then so is its projective closure \bar{V} .

$V \subseteq A^n$ irreducible $\Leftrightarrow I(V)$ prime $\Rightarrow I(V)$ radical

$I := (f_1, \dots, f_k)$, for some f_i 's. (Hilbert's Basis theorem)

So $\forall a, b \in C[x_1, \dots, x_n], \quad ab \in I \Rightarrow a \in I \text{ or } b \in I$.

$$\text{We have } \tilde{I} = (\{f \in C[x_0, x_1, \dots, x_n] \mid f \in I\})$$

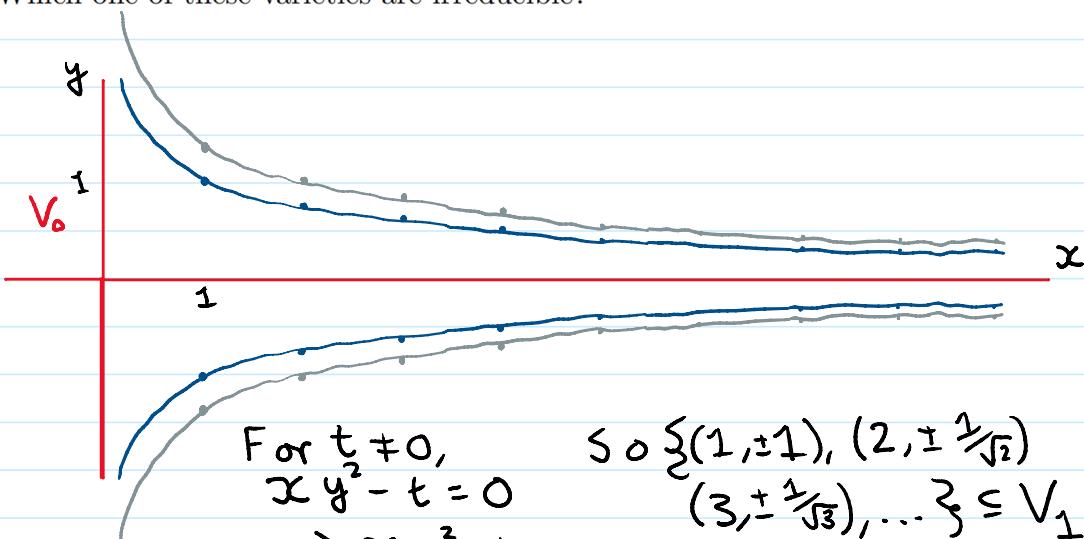
By Ex 3.21, \tilde{I} is radical

Consider $g, h \in \mathbb{C}[x_0, x_1, \dots, x_n]$, such that $gh \in \tilde{I}$.
 $\exists \{f_i\}_{i=1}^k$ such that $gh = \prod_{i=1}^k f_i$.

Q5. (20 marks) Consider the family of algebraic varieties, with parameter $t \in \mathbb{C}$, given by

$$V_t := \mathbb{V}(xy^2 - t) \subseteq \mathbb{A}^2.$$

Sketch the variety of V_0, V_1 , and V_2 in \mathbb{R}^2 . Which one of these varieties are smooth? Which one of these varieties are irreducible?



$$\begin{aligned} \text{For } t \neq 0, \quad & \text{so } \{(1, \pm 1), (2, \pm \frac{2}{\sqrt{3}}), \\ & (3, \pm \frac{3}{\sqrt{3}}), \dots\} \subseteq V_1 \\ & \Rightarrow xy^2 = t \\ & \Rightarrow y^2 = \frac{t}{x} \\ & \Rightarrow y = \pm \sqrt{\frac{t}{x}} \quad \{(1, \pm \sqrt{2}), (2, \pm \frac{\sqrt{2}}{\sqrt{2}}), \\ & (3, \pm \frac{\sqrt{2}}{\sqrt{3}}), \dots\} \subseteq V_2 \end{aligned}$$

$$\begin{aligned} \text{For } t=0, \quad & xy^2 - 0 = 0 \Rightarrow xy^2 = 0 \Rightarrow x=0 \text{ or } y^2=0 \\ & \Rightarrow x=0 \text{ or } y=0. \end{aligned}$$

$$\text{For } f_t = xy^2 - t, \nabla f_t = (y^2, 2xy).$$

$$\begin{aligned} \text{For } t \neq 0 \quad (a, b) \in V_t & \Rightarrow ab^2 - t = 0 \Rightarrow ab^2 = t \neq 0 \\ & \Rightarrow a, b^2 \neq 0 \Rightarrow a, b \neq 0 \\ & \Rightarrow (b^2, 2ab) \neq (0, 0). \end{aligned}$$

So 1 linearly independent vector.

$$\begin{aligned} \text{So } \nabla f_t(a, b) : \mathbb{A}^2 & \rightarrow \mathbb{A}^2; (V_1) \mapsto b^2 V_1 + 2ab V_2 \\ \text{is of rank 1, so} \end{aligned}$$

is of rank 1, so

$$\text{rank} + \text{nullity} = \dim A^2$$
$$1 + \underline{1} = 2$$

Thus $\forall (a,b) \in V_t$, for $t \neq 0$, $\dim \ker \nabla f_t(a,b) = 1$

So $\forall x \in V_t$, $t \neq 0$, V_t is smooth of dimension 1 at x .

Thus, by definition, V_1, V_2 are smooth of dimension 1.

For $t=0$, $(0,0), (0,1) \in V_0$, so:

$$\nabla f_t(0,0) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = (0^2, 2 \cdot 0 \cdot 0) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = (0 \ 0) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

So $\text{rank } \nabla f_t(0,0) = 0$
 $\Rightarrow \dim \ker \nabla f_t(0,0) = 2$

$$\nabla f_t(0,1) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = (1^2, 2 \cdot 0 \cdot 1) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = (1 \ 0) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

So $\text{rank } \nabla f_t(0,1) = 1$
 $\Rightarrow \dim \ker \nabla f_t(0,1) = 1$.

Thus, V_0 is not smooth, as it is at different points in V_0 , it is smooth of differing dimensions.

Since $\deg f_t = 3$, f_t irreducible $\Rightarrow f_t$ has a root

Our graph shows that f_1, f_2 have no roots, so.

f_1, f_2 irreducible $\Rightarrow f_1, f_2$ prime

$\Rightarrow (f_1), (f_2)$ prime

$\Rightarrow (f_1), (f_2)$ radical

$\Rightarrow \mathbb{I}(W(f_1)) = (f_1)$, $\mathbb{I}(W(f_2)) = (f_2)$ prime

$\Leftrightarrow V_1 = W(f_1)$, $V_2 = W(f_2)$ irreducible.