

95/100

8/10 1.a) $\mathbb{C}[x] = \frac{\mathbb{C}[x]}{(0)} = \frac{\mathbb{C}[x]}{\mathbb{I}(A^2)} = \mathbb{C}[A^2]$, And we know $\text{Max Spec } \mathbb{C}[V]$ has a 1-to-1 correspondence with V for any closed affine algebraic variety V , by

$$\begin{aligned} \text{Max Spec } \mathbb{C}[V] &\longleftrightarrow V \\ m_{\vec{p}} &\longleftrightarrow \vec{p} \in A^n \end{aligned}$$

Where $m_{\vec{p}}$ is the ideal $(x_1 - p_1, x_2 - p_2, \dots, x_n - p_n)$ with

$\mathbb{C}[V] = \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbb{I}(V)}$ and $\vec{p} = (p_1, p_2, \dots, p_n)$, ideal generated by \vec{p} .

So for $V = A^2$, $\text{Max Spec } \mathbb{C}[V] = \text{Max Spec } \mathbb{C}[x] = \{ (x-p) \mid p \in \mathbb{C} \}$.

For $\mathbb{C}[x, \frac{1}{x}] = \mathbb{C}[V(xy=1)]$

So $\text{Max Spec } \mathbb{C}[x, \frac{1}{x}] \longleftrightarrow V(xy=1)$ is 1-to-1 correspondence.

So $\text{Max Spec } \mathbb{C}[x, \frac{1}{x}] = \{ (x-a_0, \frac{1}{x}-a_1) : a_0, a_1 \in \mathbb{C}, a_0 a_1 - 1 = 0 \}$.
Is there a simpler way of expressing this?

b) 10/10 The pull back is sending x to $x \circ \varphi$ and $x \circ \varphi(y) = x \circ \frac{y}{y} = \frac{y}{y}$
(As a function: $A^1 \setminus \{0\} \rightarrow A^1 \setminus \{0\}$)

sending $\frac{1}{x}$ to $\frac{1}{x} \circ \varphi$ and $\frac{1}{x} \circ \varphi(y) = \frac{1}{x} \circ \frac{y}{y} = \frac{1}{y} = y$

So that determines the \mathbb{C} -algebra homomorphism.

For example $\varphi^*(2x^2 + \frac{2x^3+4x}{x^5}) = \varphi^*(2x^2 + \frac{2x^3+4x}{x^5})$

$$\begin{aligned} &= \varphi^*\left(\frac{1}{x}\right) \varphi^*\left(2x^2 + 2x^3 + 4x\right) \\ &= \left(\varphi^*\left(\frac{1}{x}\right)\right)^5 \left[2\varphi^*(x)^2 + 2\varphi^*(x)^3 + 4\varphi^*(x)\right] \\ &= y^5 \left[2\left(\frac{1}{y}\right)^2 + 2\left(\frac{1}{y}\right)^3 + 4\left(\frac{1}{y}\right)\right] \\ &= \frac{2}{y^2} + 2y^2 + 4y^4 \end{aligned}$$

Similarly, $\varphi^*(2-x) = 2\varphi^*(1) - \varphi^*(x) = 2 - \frac{1}{y}$.

(φ^* is a \mathbb{C} -algebra homomorphism, so it always send 1 to 1.)

2. a) Let $\varphi: V \rightarrow A^2$ $(x, y, u) \xrightarrow{\varphi} (x, u)$

10/10 Define $\psi: A^2 \rightarrow V$ $(w, z) \xrightarrow{\psi} (w, wz, z)$

It is clear that ψ is well defined and is a morphism of varieties,
(as $w \cdot z - (wz) = 0$, so $\psi(w, z) \in V$)
(Only polynomials inside the brackets)

And one can check $\psi \circ \varphi = \text{id}_V$ and $\varphi \circ \psi = \text{id}_{A^2}$

$$(x, y, u) \xrightarrow{\varphi} (x, u) \xrightarrow{\psi} (x, xu, u) \quad (xu=y \text{ since } (x, y, u) \in V) \\ = (x, y, u)$$

$$(w, z) \xrightarrow{\psi} (w, wz, z) \xrightarrow{\varphi} (w, z)$$

So φ is an isomorphism as required.

b) Let $\widetilde{\varphi}: V \rightarrow A^2$ $(x, y, u) \mapsto (x, y)$

Suppose by contradiction There is morphism $\chi: A^2 \rightarrow V$ such that

$$(w, z) \mapsto (\gamma_1(w, z), \gamma_2(w, z), \gamma_3(w, z)) , \gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}[t_1, t_2]$$

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Suppose by contradiction. There is morphism $\varphi: A^2 \rightarrow V$ such that $(w, z) \mapsto (\gamma_1(w, z), \gamma_2(w, z), \gamma_3(w, z))$, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}[t_1, t_2]$

such that $\gamma_1 \circ \varphi = \text{id}_V$ and $\gamma_2 \circ \varphi = \text{id}_{A^2}$.

Since $\gamma_1 \circ \varphi = \text{id}_V$ $(\gamma_1(x, y), \gamma_2(x, y), \gamma_3(x, y)) = (x, y, u)$
for all $x, y \in \mathbb{C}$,

This implies $\gamma_1(x, y) = x$, $\gamma_2(x, y) = y$ (as polynomials).

Now since $(x, y, u) \in V$, $x \cdot \gamma_3(x, y) = y$ for all $x, y \in \mathbb{C}$.

But this can't hold for $x=0, y=1$. Contradiction.

So our assumption that there is an isomorphism φ must be wrong.

c) Given $(x, y, u) \in A^3$, note that there is an isomorphism

$V \cong A^2$, so there are isomorphisms

$$V \setminus W(u) \cong A^2 \setminus W(u) \cong W(zu-1) \subseteq A^3.$$

$$(x, y, u) \mapsto (x, u) \mapsto (x, u, zu)$$

$$(x, xu, u) \leftarrow (x, u) \leftarrow (x, u, zu)$$

$$\therefore \mathbb{C}[V \setminus W(u)] \cong \mathbb{C}[W(zu-1)] = \frac{\mathbb{C}[x, u, z]}{(zu-1)} \cong \mathbb{C}[x, u, z].$$

since $\mathcal{O}_V(D(u))$ is the set of regular function on $V \setminus D(u)$

$$\text{so } \mathcal{O}_{V \setminus W(u)}(V \setminus W(u)) = \boxed{\mathbb{C}[V \setminus W(u)]} \cong \mathbb{C}[x, u, z].$$

What is $\mathbb{C}[U]$ when U is not closed
affine?

Q3. Without loss of generality, suppose $V \subseteq A^n$ is identified with $\varphi(V)$

where $\varphi: A^n \rightarrow \mathbb{P}^n$, $\varphi(x_1, \dots, x_n) = [1 : x_1 : x_2 : \dots : x_n]$.

Note that φ is a homeomorphism to its image $\{[x_0 : x_1 : \dots : x_n] \in \mathbb{P}^n \mid x_0 \neq 0\}$.

Suppose the contrary that $\overline{V} = W_1 \cup W_2$, where W_1, W_2 are closed sets in \mathbb{P}^n . Then consider $\varphi^{-1}(W_1)$, since W_1 is closed, φ is continuous,

so $\varphi^{-1}(W_1)$ is closed in A^n . Similarly $\varphi^{-1}(W_2)$ are closed,

but since $\varphi^{-1}(W_1) \cup \varphi^{-1}(W_2) = \varphi^{-1}(W_1 \cup W_2) = \varphi^{-1}(\overline{V}) = V$.

and V is irreducible (in A^n), $\varphi^{-1}(W_1) = V$ or $\varphi^{-1}(W_2) = V$.

Say (without loss of generality) $\varphi^{-1}(W_1) = V$, so $\varphi(V) \subseteq W_1$,

since W_1 is closed by assumption, $W_1 = \overline{V}$, similarly if $\varphi^{-1}(W_2) = V$,
 $W_2 = \overline{V}$, so \overline{V} is irreducible as required.

Q4. We first compute the closure of $W(y - \sin x)$ in A^2 , then compute the projective closure. Claim: Closure of $W(y - \sin x)$ in A^2 is the whole A^2 .

Proof: Suppose $W(y - \sin x) \subseteq W(f)$ for some polynomial $f \in \mathbb{C}[x, y]$.

Now we use the fact that \sin is surjective on \mathbb{C} with period 2π .

So for any $\beta \in \mathbb{C}$ there exist $\alpha \in \mathbb{C}$ such that $\beta = \sin(\alpha + 2\pi n)$ for all $n \in \mathbb{Z}$.

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Now we use the fact that \sin is surjective on \mathbb{C} with period 2π .
 So for any $\beta \in \mathbb{C}$ there exist $\alpha \in \mathbb{C}$ such that $\beta = \sin(\alpha + 2\pi n)$ for all $n \in \mathbb{Z}$.
 So $f(\alpha + 2\pi n, \beta) = 0$ for all $n \in \mathbb{Z}$. Now if I write,

$$f(x, y) = g_n(y)x^n + g_{n-1}(y)x^{n-1} + \dots + g_0(y) \text{ for } g_n, \dots, g_0 \in \mathbb{C}[y].$$

$$\text{Define } F(x) := f(x, \beta) = g_n(\beta)x^n + g_{n-1}(\beta)x^{n-1} + \dots + g_0(x) \in \mathbb{C}[x]$$

$$\text{and } F(\alpha + 2\pi n) = f(\alpha + 2\pi n, \beta) = 0 \text{ for all } n \in \mathbb{Z}, \text{ so } F \equiv 0$$

(since a single-variable non-constant polynomial had only finitely many roots.)

$$\text{So } g_n(\beta), g_{n-1}(\beta) \dots, g_0(\beta) = 0 \text{ for all } \beta \text{ by comparing coefficients.}$$

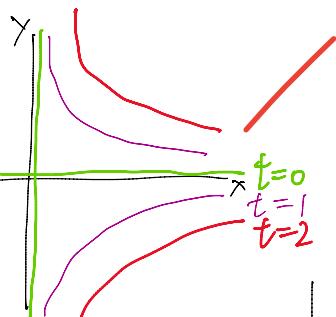
$$\text{Hence } g_n, g_{n-1} \dots, g_0 \equiv 0 \text{ so } f(x, y) \equiv 0. \text{ So } V(f) = V(0) = A^2.$$

So the projective closure is $\overline{A^2} = \mathbb{P}^2$.

This does not contradict the Chow lemma, as $V(y \cdot \sin x)$ is an analytic affine subvariety of A^2 , but not an analytic projective subvariety of \mathbb{P}^2 .

5.

18/20



To compute smoothness we fix $\vec{p} \in V_t$.

$$\begin{aligned} T_{\vec{p}} V_t &:= \left\{ \vec{v} \in A^n \mid \left(\frac{\partial(xy^2-t)}{\partial x}, \frac{\partial(xy^2-t)}{\partial y} \right)_{\vec{p}} \vec{v} = 0 \right\} \\ &:= \left\{ \vec{v} \in A^n \mid (y^2, 2xy)_{\vec{p}} \vec{v} = 0 \right\}. \end{aligned}$$

$$\text{If } \vec{p} = (x_0, y_0) \in V_t,$$

$$:= \left\{ \vec{v} \in A^n \mid (y_0^2, 2y_0) \vec{v} = 0 \right\}.$$

Now, as long as the vector $(y_0^2, 2y_0)$ is non-zero, this is a rank 1 matrix hence $\dim T_{\vec{p}} V_t = 2 - 1 = 1$ in this case.

Else, when $(y_0^2, 2y_0) = 0$, $\dim T_{\vec{p}} V_t = 2$.

Also, since V_t is a hypersurface ~~hyperplane~~ (for all $t \in \mathbb{C}$), it has dimension ~~$\dim(A^2) - 1$~~
 ~~$= 2 - 1 = 1$~~ .

Now notice $(y_0, 2y_0)$ is the zero vector if and only if $y_0 = 0$

and $(x_0, 0) \in V_t$ if and only if $t = 0$.

So indeed V_0 is not smooth (at $\vec{p} = (0, 0)$), while V_1, V_2 are smooth.

V_0 is NOT irreducible since $xy^2 = 0 \iff x=0$ or $y=0$

$$\text{so } V_0 = V(x) \cup V(y) \text{ and } V(x) \subseteq V_0, V(y) \subseteq V_0.$$

We claim that the ideal $(xy^2 - t)$ when $t = 1, 2$ is prime, (then it is radical), so $\mathbb{I}(V_t) = \mathbb{I}(V(xy^2 - t)) = (xy^2 - t)$ (by Nullstellensatz) is prime, hence V_t is irreducible.

To show the claim, since $\mathbb{C}[x, y]$ is a UFD. It suffice to check

that $xy^2 - t$ is irreducible. Suppose $xy^2 - t = F(x,y) G(x,y)$, since x has degree 1, one of the polynomial (say G), must solely depend on y .

$$xy^2 - t = (G_1(y)x + G_2(y)) \cdot G_3(y)$$

$$= G_1(y) G_3(y)x + G_2(y) G_3(y)$$

By comparing coefficient and the fact that $t \neq 0$, both polynomial G_2, G_3 must be non-zero constant functions.

$$\text{so } xy^2 - t = F(x,y) G(x,y) \Rightarrow G \text{ must be constant}$$

i.e. $xy^2 - t$ is irreducible for $t \neq 0$, hence U_1, U_2 are irreducible as required.