Bézout's Theorem in Tropical Algebraic Geometry

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BÉZOUT'S THEOREM IN TROPICAL ALGEBRAIC GEOMETRY

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Abstract. In this thesis, a version of Bézout's theorem in tropical geometry is studied. The classical Bézout's theorem counts the number of intersections for two curves defined on an algebraically closed field, for which we motivate and define relevant concepts such as intersection numbers. We introduce the basics of tropical geometry, including the arithmetic on the tropical semiring and tropical analogues of polynomials, roots and curves. The algebraic and geometric properties of tropical curves are studied and concluded in Kapranov's theorem and the structure theorem. We construct an analogue of intersection numbers in tropical geometry. A version of tropical Bézout's theorem for one kind of tropical curves is presented and proved, and a generalization on all tropical curves in general position is discussed.

Contents

1. Introduction

1.1. **Classical Bézout's Theorem.** Given two polynomials, it is a natural question to ask what their zeros are and how many common zeros they have. On one hand, we can approach the problem algebraically by equating the polynomials to zero and solving equations. On the other hand, there is a geometric perspective given by the intersection of the curves represented by the polynomials. For example, consider the polynomials $x^2 - y$ and y over the real numbers, it is easy to sketch their zeros and see that the curves intersect at only one point $(0, 0)$ (Figure 1). Another example is the intersections of the curves $x(x+1)(x-1)-y=0$ and $y=0$. They intersect at three points $(-1,0),(0,0)$ and $(1,0)$ as shown in Figure 2. The common zeros of polynomials are in one-to-one correspondence with the intersections of curves represented by these polynomials.

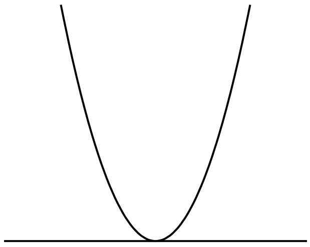

Figure 1. Intersection of the curves x^2-y and y.

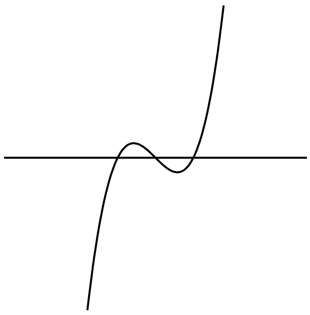

Figure 2. Intersections of the curves $x(x+1)(x-1)-y$ and y.

With regard to the total number of intersections, the Bézout's theorem gives an accurate count of common zeros of two polynomials under desirable conditions (Section 2.4).

Theorem 1.1 (Bézout)**.** *Let C and D be curves represented by polynomials $f(x,y)$ and $g(x,y)$ of degree c and d over an algebraic closed field F. The curves C and D have exactly $c\cdot d$ intersections if*

(1) $f(x,y)$ and $g(x,y)$ do not have common components in their factorizations,
(2) the intersections are considered in the projective plane $\mathbb{FP}^2$, and
(3) the intersections are counted with properly defined multiplicities.

The first condition is the most intuitive one, since curves given by polynomials with common components such as x and $x(y-x)$ overlap at a continuous segment of the curve, resulting in infinitely many intersections.

The second condition deals with intersections at infinity. For example, the curves $x=0$ and $x-1=0$ do not intersect in affine spaces such as $\mathbb{R}^2$ but an intersection exists at $x=\infty$.

The third condition can be motivated both geometrically and algebraically. Consider the example in Figure 1, while the two polynomials have degrees 2 and 1, the curves intersect at only one point $(0,0)$, which seems to contradict the total number of intersections asserted by Bézout's theorem. However, if we move the curve $y=0$ upward, we can see the intersection at $(0,0)$ splits into two points of intersection (Figure 3). Thus, the intersection at $(0,0)$ is considered to have multiplicity 2. Alternatively from the algebraic perspective, equate x^2-y and y to zero and we end up solving $x^2=0$, which has a double root $x=0$.

Indeed, one can argue that if the curve $y=0$ is moved downward, there will be no intersections at all. However, note that Figure 1 is drawn in $\mathbb{R}^2$, so the intersections exist and satisfy the Bézout's theorem in its algebraic closure $\mathbb{C}^2$. This is why the field F is required to be algebraically closed.

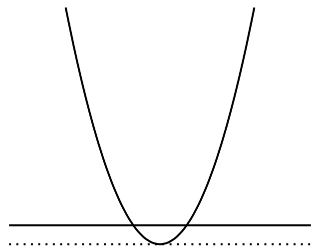

FIGURE 3. Intersection of the curves $x^2 - y$ and y with perturbation.

1.2. **Tropical Geometry.** In this thesis, we will present and prove a version of the Bézout's theorem in tropical geometry. Tropical geometry is the algebraic geometry over the *tropical semiring*

$$\mathbb{R}_{\text{trop}} = (\mathbb{R} \cup \{+\infty\}, \oplus, \odot) = (\mathbb{R} \cup \{+\infty\}, \min, +),$$

in which addition is defined as the min function and multiplication is the regular addition (Section 3.1). Results from tropical geometry are connected to the classical algebraic geometry, polyhedral geometry, toric varieties and mirror symmetry.

Just as in classical algebraic geometry, we want to study "zeros" of polynomials over $\mathbb{R}_{\text{trop}}$. Consider the *tropical polynomial* of degree 2

$$p_1(x) = 3x^2 \oplus 0x \oplus 5 = \min(2x + 3, x + 0, 5). \tag{1}$$

Here, $3x^2$ should be considered tropically, which translates to $2x + 3$ in the regular arithmetic. The idea is similar for $0x$. (Note that $0x$ is the same as x in tropical arithmetic.) We compute that $p_1(x)$ is a piecewise function with the following values:

$$\begin{cases} 2x+3 & x \leq -3 \\ x & -3 \leq x \leq 5 \\ 5 & 5 \leq x \end{cases}$$

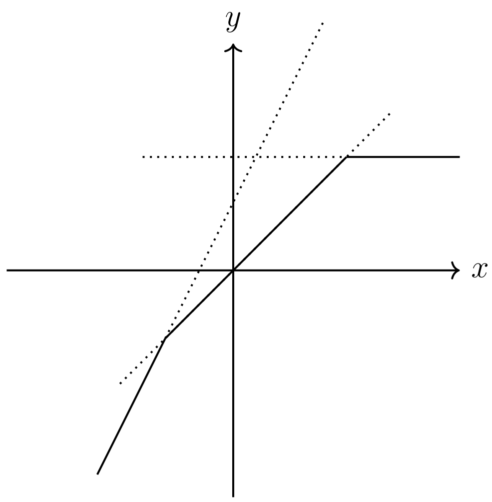

FIGURE 4. The graph of $p_1(x) = 3x^2 \oplus x \oplus 5$.

Geometrically, we observe that the graph of $p_1(x)$ is continuous, piecewise linear and concave as shown in Figure 4, because 1) each summand of $p_1(x)$ is continuous and linear, 2) min preserves continuity and linearity, and 3) min guarantees concavity. These properties hold for tropical polynomials in general.

Naturally, we are interested in the nondifferentiable part of a tropical polynomial $f(x_1, \ldots, x_n)$. We call $(a_1, \ldots, a_n)$ a *root* of $f(x_1, \ldots, x_n)$ if $f(x_1, \ldots, x_n)$ is nondifferentiable at $(a_1, \ldots, a_n)$. For example, the roots of $p_1(x)$ are $x = -3, 5$.

With properly defined multiplicity, there is an analogue of the fundamental theorem of algebra in tropical geometry (Section 3.2). That is, for a tropical polynomial in one variable, the total number of its roots counted with multiplicity is equal to its degree.

One may wonder why roots in tropical geometry are defined differently from their counterparts in classical algebraic geometry, in the sense that we do not simply define the roots as solutions to the equation $f(x_1, \ldots, x_n) = \infty$. Actually, the definitions of tropical roots and classical roots are from the same source, and they are also connected by Kapranov's theorem (Section 3.3.2).

As we move to multivariable tropical polynomials, drawing graphs become difficult. Consider a simple two-variable tropical polynomial

$$p_3(x, y) = 0x \oplus 0y \oplus 0 = \min(x, y, 0)$$

with values:

$$\begin{cases} x & x \leq 0, x \leq y \\ y & y \leq 0, y \leq x \\ 0 & 0 \leq x, 0 \leq y \end{cases}$$

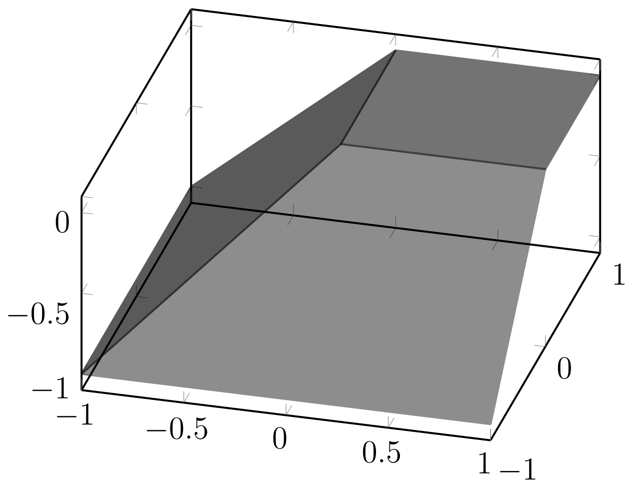

FIGURE 5. The graph of $p_3(x, y) = \min(x, y, 0)$.

Its graph is realized in $\mathbb{R}^3_{\text{trop}}$ as shown in Figure 5. We can see that $p_2(x, y)$ is continuous, piecewise linear and concave just as in the single variable case. As the number of variables and summands of a polynomial increase, graphs will be difficult to draw. Nevertheless, it is easier to draw the roots of a two-variable polynomial on the $\mathbb{R}^2_{\text{trop}}$ plane. That is, the set of (x, y) where $p_3(x, y)$ is nondifferentiable, or equivalently the projection of edges of the graph onto $z = 0$ plane. For example, the zeros of $p_3(x, y)$ are a union of rays starting from $(0, 0)$ and pointing along the x-axis, y-axis and $x - y = 0$ respectively, as shown in Figure 6.

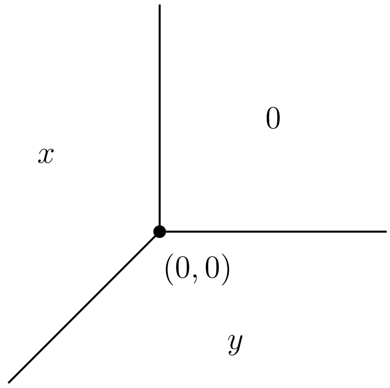

FIGURE 6. Roots of $p_3(x,y) = \min(x,y,0)$.

Definition 1.2. A *tropical plane curve* is the set of all roots of some two-variable tropical polynomial $f(x,y)$.

Tropical plane curves are the main objects studied in this thesis. We will explore both of its algebraic and geometric properties, and furthermore, how these properties make tropical plane curves connected to other subfields of geometry.

On the algebraic side, we will discuss the interaction between classical and tropical algebraic geometry. Classical algebraic geometry usually studies the zeros of polynomials on an addition-multiplication field. This problem will be simplified in tropical geometry, as tropical polynomials are piecewise linear and concave. Thus, our motivation is to study the zeros of a regular polynomial using the tropical world.

Given a field K, a valuation on K is a map val : $K \to \mathbb{R}_{\text{trop}}$ that behaves like a field homomorphism. Examples of valuations, such as the p-adic valuation and the lowest exponent valuation defined on Puiseux series are discussed in Section 3.3.1.

Via a valuation, our first approach is to map polynomials over familiar fields K to tropical polynomials and study their roots, i.e., tropical curves given by these tropical polynomials. Another approach is consider the zeros of a polynomial over K and map the zeros to the tropical world. It turns out that both approach results in the same tropical plane curve as shown in Figure 7. This result is concluded in Kapranov's theorem.

$$\begin{tikzcd}
\text{polynomials over } \mathbb{C} \arrow[r, "V"] \arrow[d, "\text{val}"'] & \text{plane curves} \subset K^2 \arrow[d, "\text{val}"] \\
\text{tropical polynomials} \arrow[r, "V_{\text{trop}}"] & \text{tropical curves} \subset \mathbb{R}^2_{\text{trop}}
\end{tikzcd}$$

FIGURE 7. Two approaches to study zeros of polynomials in $\mathbb{V}$ in the tropical world.

On the geometric side, we wish to connect tropical geometry to polyhedral geometry. Given a tropical polynomial $f(x,y)$, we will present an algorithm that uses the coefficients and exponents of $f(x,y)$ to obtain a polygon subdivision that is the dual graph to the tropical curve up to rescaling. As a dual graph to a polygon subdivision, the tropical curve naturally carries weights on all edges and is balanced at each vertex. We conclude that all tropical curves are balanced rational graphs. In fact, the converse is also true, which gives raise to an equivalent definition of tropical curves. That is, tropical curves are exactly balanced rational graphs and can be determined completely geometrically [3, 5].

1.3. **Tropical Bézout's Theorem.**

The discussion on tropical curves is the basis for Bézout's theorem in tropical geometry. To formulate a tropical version of Bézout's theorem, we should construct analogues of degree of a curve and intersection multiplicity in tropical geometry.

A tropical curve is said to have *degree n* if its defining polynomial $p(x, y)$ includes x^n, y^n, a constant term and no terms with degree higher than n. The *intersection multiplicity* of tropical curves C and D at a point P is defined as

$$I_{\text{trop}}(C \cap D, P) = |w_C w_D (x_C y_D - x_C y_D)|,$$

where w_C, w_D are weights of edges at the intersection and $(x_C, y_C), (x_D, y_D)$ are the primitive integral vectors along the edges.

We provide a version of the statement of the tropical Bézout's theorem as follows.

Theorem 1.3 (Tropical Bézout's theorem)**.** *Let C and D be tropical curves of degree c and d respectively. Suppose C and D intersect at finitely many points. The total number of intersection of C and D is $c \cdot d$ points counting multiplicity.*

We first observe that the theorem is true for curves in a special position, and we will prove the theorem by moving the curves to the special position without changing the total intersection multiplicity. A generalization to tropical curves with no assigned degrees will be discussed as well.

1.4. **Acknowledgement.** First of all, I would like to appreciate my advisor, Dr. Prakash Belkale, for the opportunity to work on this thesis and all his help and support throughout my project. This work would not have been possible without his patient guidance. I would also like to thank the other members of my committee, Dr. Justin Sawon and Dr. Jiuzu Hong, for their insightful questions and suggestions. I would like to thank Dr. Hans Christianson for all his advice on completing this thesis. Lastly, I would like to especially thank Baqiao Liu for all the days (and nights) we spent together working on our theses.

2. Classical Bézout's Theorem

Recall in Section 1.1, Bézout's theorem concerns the total number of intersections of two plane curves. To make sense of the result, this section gives necessary definitions such as plane curves, local rings and intersection numbers. Relevant results such as the Nullstellensatz will be introduced as well.

We consider a field k. For convenience, assume k is algebraically closed, though most of the terms below can be defined without this assumption.

2.1. **The Affine Case.**

2.1.1. *Affine varieties and local rings.* Define the *affine n-space* over k to be the set of n-tuples of elements of k denoted by $\mathbb{A}^n(k)$. We may abbreviate the notation as $\mathbb{A}^n$ if k is clear in the context.

A subset of $\mathbb{A}^n$ is an *algebraic set* if it is the set of all common zeros of some polynomials. We say an algebraic set is *irreducible* if it cannot be written as the union of two strictly smaller algebraic sets. An irreducible algebraic set is called a *variety.*

The map I takes a subset $V \subset \mathbb{A}^n$ and maps it to the ideal of all polynomials in $k[x_1, \ldots, x_n]$ vanishing on V. If V is a nonempty variety, I(V) is a prime ideal.

Let $V \subset \mathbb{A}^n$ be a nonempty variety, so $I(V)$ is a prime ideal and the quotient $\Gamma(V) = k[x_1, \ldots, x_n]/I(V)$ is an integral domain. We call $\Gamma(V)$ the *coordinate ring* of V. Other than as the equivalence classes of polynomials, $\Gamma(V)$ can also be understood as a subset of functions defined on V.

Let $\mathcal{F}(V,k)$ be the set of all function from V to k. A function $f \in \mathcal{F}(V,k)$ is called a *polynomial function* if it coincides with $F \in k[x_1,\dots,x_n]$ on V. The polynomial functions on V form a ring and this ring is exactly $\Gamma(V)$ because the polynomial functions f,g are equal if and only if their corresponding polynomials F,G are equal on V, i.e., $F-G=0$ and $F-G \in I(V)$.

Since $\Gamma(V)$ is an integral domain, we may form its field of fractions $k(V)$, which is called the *field of rational functions* on V. We call $f \in k(V)$ a *rational function*. f is *defined* at P if f can be written as g/h and $h(P) \neq 0$.

Thus, we may consider the subring of rational functions on V that are defined at P. Denote this subring by $\mathcal{O}_P(V)$ and call it the *local ring* of V at P. It is easy to see that $k \subset \Gamma(V) \subset \mathcal{O}_P(V) \subset k(V)$.

2.1.2. *Affine change of coordinates.* Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be varieties and a mapping $\phi : V \to W$ is called a *polynomial map* if there are polynomials $T_1,\dots,T_m \in k[x_1,\dots,x_n]$ such that $\phi(a_1,\dots,a_n) = (T_1(a_1,\dots,a_n),\dots,T_m(a_1,\dots,a_n))$ for all $(a_1,\dots,a_n) \in V$. Let $F \in k[x_1,\dots,x_m]$, denote $F^T = F(T_1,\dots,T_m)$.

An *affine change of coordinates* on $\mathbb{A}^n$ is a polynomial map

$$T = (T_1,\dots,T_n) : \mathbb{A}^n \to \mathbb{A}^n$$

such that each T_i is a polynomial of degree 1 and T is one-to-one and onto. T is an automorphism of $\mathbb{A}^n$.

2.1.3. *Affine plane curves.* Let an affine plane curve to be an equivalence class of non-constant polynomials in $k[x,y]$ under the equivalence relation $F \sim G$ if $F = \lambda G$ for some nonzero $\lambda \in k$. We observe that polynomials in the same equivalence class has the same set of zeros, the same degree and irreducible factors up to a scalar multiple, so we sometimes abuse the notation and represent a plane curve by a defining polynomial of this equivalence class, say, the plane curve $y^3 - x^2$, $y = x^2$, etc.

The *degree* of a curve is the degree of its defining polynomial. Write the curve $F = \Pi_i F_i^{e_i}$ where F_i are its distinct irreducible factors and we say F_i are the *components* of F and e_i is the *multiplicity* of F_i. For curves F and G, we say they *intersect properly* at P if they have no common components passing through P.

For a curve F passing $(0,0)$, we can express it as the sum $F_m + F_{m+1} + \cdots + F_n$ where F_i is a homogeneous polynomial of degree i. Take $P = (0,0)$ and define the *multiplicity* of F to be the lowest nonzero degree m in the sum and denote it as $m_P(F)$. Since k is algebraically closed, we can write $F_m = \Pi_i L_i^{r_i}$ where $\deg(L_i) = 1$. We call L_i a *tangent line* to F at P. We note that if $F = \Pi_i G_i^{e_i}$ is the factorization of F into irreducible components, then $m_P(F) = \Sigma_i e_i m_P(G_i)$, since the lowest degree term of F is the product of the lowest degree terms of its factors.

Naturally, we want to extend these notions to any point P other than $(0,0)$. This can be done simply by taking an affine change of coordinates T that maps $(0,0)$ to P and define $m_P(F)$ to be $m_{(0,0)}(F^T)$. Similarly, we can express F^T as a sum of homogeneous polynomials so the multiplicity of F and the tangent lines to F at P are both well defined.

Theorem 2.1 (Affine Nullstellensatz)**.** *Let I be a ideal in $k[x_1,\dots,x_n]$. If $V(I) \neq \emptyset$, then $I(V(I)) = \operatorname{Rad} I$.*

2.2. **The Projective Case.** Now we want to define analogues of the notions above in the projective space and we will focus on the difference between the projective and affine cases, as most of the notions still apply.

2.2.1. *The projective space.* Define the *projective n-space* over k to be $\mathbb{P}^n(k) = \mathbb{A}^{n+1}(k) - \{0\}/\sim$ where $P \sim Q$ if $P = \lambda Q$ for $\lambda \neq 0$. Similarly, we abbreviate the notation as $\mathbb{P}^n$ when k is clear. We denote a point $P \in \mathbb{P}^n$ as $[x_1 : \cdots : x_{n+1}]$, i.e. the equivalence class of $0 \neq (x_1 : \cdots : x_{n+1}) \in \mathbb{A}^{n+1}$.

Fix i and let

$$U_i = \{[x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n \mid x_i \neq 0\}.$$

We can see that U_i is isomorphic to $\mathbb{A}^n$ under the map $\phi_i : \mathbb{A}^n \to U_i$ by $\phi_i(a_1, \ldots, a_n) = [a_1 : \cdots : a_{i-1} : 1 : a_i : \cdots : a_n]$.

Conventionally, we take $i = n+1$. Consider the complement of U_{n+1} in $\mathbb{P}^n$ and denote it as

$$H_\infty = \{[x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n \mid x_{n+1} = 0\}.$$

We call H_∞ the *hyperplane at infinity.* H_∞ is isomorphic to $\mathbb{P}^{n-1}$ by the map sending $P = [x_1 : \cdots : x_{n+1}] \in H_\infty$ to $[x_1 : \cdots : x_n] \in \mathbb{P}^{n-1}$. Such construction indicates that $\mathbb{P}^n$ is exactly a copy of $\mathbb{A}^n$ and $\mathbb{P}^{n-1}$.

2.2.2. *Projective varieties and local rings.* Now we extend the notion of varieties to the projective space. Since there are could be more than one (affine) point in the equivalence class $P \in \mathbb{P}^n$, we say P is a *zero* of $F \in k[x_1, \ldots, x_{n+1}]$ if every point in P is a zero of F in the affine sense. Specifically, we note that if F is homogeneous, then if any affine point $Q \in P$ is a zero of F, the projective point P is a zero of F. In the other way, if we write $F = F_m + \cdots + F_n$ as a sum of homogeneous polynomials, P is a zero of F if and only if P is a zero of F_i for each i.

With the definition of zero, we can then define the ideal of a set in $\mathbb{P}^n$ in $k[x_1, \ldots, x_{n+1}]$ and the projective algebraic set corresponding an ideal. An ideal I is called *homogeneous* if for every $F = \Sigma_{i=0}^m F_i \in I$ as a sum of homogeneous polynomials, we also have $F_i \in I$. In fact, the ideal of any set in $\mathbb{P}^n$ is homogeneous. Furthermore, we have the notion of *projective varieties*, i.e., irreducible algebraic sets, using the definition in the affine case. There is a one-to-one correspondence between prime ideals in $k[x_1, \ldots, x_{n+1}]$ and projective varieties. Not surprisingly, the Nullstellensatz holds for the projective case, too.

Theorem 2.2 (Projective Nullstellensatz)**.** *Let I be a homogeneous ideal in $k[x_1, \ldots, x_{n+1}]$. If $V(I) \neq \emptyset$, then $I(V(I)) = \operatorname{Rad} I$.*

The projective local ring requires some work. To form the field of fractions of the ring $k[x_1, \ldots, x_{n+1}]/I$, we can see that not all elements in the quotient ring are well-defined functions in the projective space. For example, the values of $\frac{x-y}{x^2-y^2}$ at $(1,0)$ and $(2,0)$ do not agree, even though $(1,0) \sim (2,0)$. However, if the denominator and numerator are homogeneous polynomials of the same degree, they will have consistent value at any $P \in \mathbb{P}^2$. The specific construction is as follows.

Let V be a nonempty projective variety in $\mathbb{P}^n$ and let $\Gamma_h(V) = k[x_1, \ldots, x_{n+1}]/I(V)$. The quotient ring is a domain since $I(V)$ is prime. We call $\Gamma_h(V)$ the *homogeneous coordinate ring* of V.

Let $k_h(V)$ be the field of fractions of $\Gamma_h(V)$ and call it the *homogeneous function field* of V. As stated before, not every element in $\Gamma_h(V)$ is well defined on $\mathbb{P}^n$, which motivates us to define the *function field* of V, written $k(V) = \{z \in k_h(V) \mid z = f/g, f, g \in \Gamma_h(v), \deg(f) = \deg(g) \text{for some} f, g\}$. Again, elements of $k(V)$ are called *rational functions* on V. For $P \in V$, $z \in k(V)$, z is *defined* at P if $z = f/g$ and $g(P) \neq 0$ and the *local ring* of V at P is the subring of elements of $k(V)$ defined at P, denoted as $\mathcal{O}_P(V)$.

2.2.3. *Projective change of coordinates.* Let $T = (T_1, \dots, T_{n+1}) : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ be an affine change of coordinates. Then T takes lines through the origin to lines through the origin. In other words, T takes $P \in \mathbb{P}^n$ to some $Q \in \mathbb{P}^n$ and induces a map $T_p : \mathbb{P}^n \to \mathbb{P}^n$, called a *projective change of coordinates.* Similarly, T_p is an automorphism of $\mathbb{P}^n$.

2.2.4. *Homogenization and dehomogenization.* We now introduce some notations that allow us to transit between homogeneous and nonhomogeneous polynomials. In the proof of Bézout's theorem, the technique of (de)homogenization is especially useful because we want to convert the problem setting in the projective space to one in the affine space, for which we have nice conclusions such as Equation 2.

Let $F \in k[x_1, \dots, x_{n+1}]$ be a homogeneous polynomial and define

$$F_* = F(x_1, \dots, x_n, 1) \in k[x_1, \dots, x_n].$$

F_* dehomogenizes F. Let $f \in k[x_1, \dots, x_n]$ and $\deg(f) = d$. Write $f = f_0 + f_1 + \cdots + f_d$ as a sum of homogeneous polynomials. Define

$$f^* = x_{n+1}^d f(x_1/x_{n+1}, \dots, x_n/x_{n+1}).$$

f^* homogenizes f and has degree d.

We have the following important properties of F_* and f^*:

Proposition 2.3. $(FG)_* = F_*G_*$ *and* $(fg)^* = f^*g^*$.

Proposition 2.4. *If* $F \neq 0$ *and* r *is the highest power of* x_{n+1} *that divides* F, *then* $x_{n+1}^r(F_*)^* = F$ *and* $(f^*)_* = f$.

2.3. **Intersection Numbers.** We will first define the intersection number for affine curves and give an analogue for projective curves. Let $F, G \in k[x, y]$ be two curves and let $P \in \mathbb{A}^2$. Before giving the definition, we will discuss some desired properties of intersection numbers. Actually, these properties guarantee the uniqueness of the definition.

Denote the intersection number F and G at P as $I(P, F \cap G)$. The geometric motivation requires the following properties of $I(P, F \cap G)$:

(1) $I(P, F \cap G) = \infty \iff F$ and G do not intersect properly at P.
(2) $I(P, F \cap G) \in \mathbb{N} \iff F$ and G intersect properly at P.
(3) $I(P, F \cap G) = 0 \iff F$ and G do not intersect at P.

Remark 2.5. Note that if F and G intersect improperly, their common component will give infinite number of solutions to the equation $F = G$.

We also want $I(P, F \cap G)$ to be invariant under certain operations:

(4) $I(P, F \cap G) = I(P, G \cap F)$.
(5) $I(P, F \cap G) = I(Q, F^T \cap G^T)$ for an affine change of coordinates T on $\mathbb{A}^2$ which sends Q to P.
(6) $I(P, F \cap G) = I(P, F \cap (G + AF))$ for any $A \in k[x, y]$.
(7) $I(P, F \cap G) = \Sigma_{i,j} r_i s_j I(P, F_i \cap G_j)$ given $F = \Pi_i F_i^{r_i}$ and $G = \Pi_j G_j^{s_j}$ as irreducible factorizations.

Remark 2.6. Property 5 states that $I(P, F \cap G)$ is invariant under an affine change of coordinates, which means it is enough to consider the intersection number at $P = (0, 0)$ of the curves under a suitable choice of T.

Property 6 indicates that $I(P, F \cap G)$ is invariant under a change of the generators of the ideal (F, G). Indeed, (F, G) is part of the definition of $I(P, F \cap G)$ later in the section.

Property 7 suggests a way to calculate $I(P, F \cap G)$ by decomposing F and G, which will be heavily used in Example 2.8.

Finally, if F and G do not have common tangent lines, we expect $I(P, F \cap G)$ to be exactly $m_P(F)m_P(G)$. This is because at $P = (0,0)$ locally, the higher-degreed terms vanish faster than the terms of degree $m_{(0,0)}(F)$ and $m_{(0,0)}(G)$, so the intersection of F and G at $(0,0)$ can be approximated by the intersection of their lowest degree terms. For points other than $(0,0)$, we can make use of Property 5 and shift that point to $(0,0)$.

(8) $I(P, F \cap G) \geq m_P(F)m_P(G)$. Specifically, $I(P, F \cap G) = m_P(F)m_P(G) \iff F$ and G have no common tangent lines at P.

Theorem 2.7. *$I(P, F \cap G)$ is uniquely defined.*

Proof. The proof uses induction on the value of intersection number. Property 3 guarantees that when the intersection number is sufficiently small (in fact, equals to 0), it is unique. Then, repeatedly use Property 7 so that the intersection number can be written as the sum of smaller intersection numbers, which completes the proof [2, p. 37]. □

We give an example on the computation of intersection numbers.

Example 2.8. Let $F = (x^2 + y^2)^2 + 3x^2y - y^3$ and $G = (x^2 + y^2)^3 - 4x^2y^2$, we want to compute $I(P, F \cap G)$ for $P = (0,0)$ using the properties above.

First, we observe that the lowest degree terms of F and G share the common factor y, so we cannot calculate $I(P, F \cap G)$ using Property 8 directly.

$$
\begin{aligned}
&\text{(Property 6)} & I(P, F \cap G) &= I(P, F \cap (G - (x^2 + y^2)F)) \\
& & &= I(P, F \cap yG') \\
&\text{(Property 7)} & &= I(P, F \cap y) + I(P, F \cap G') \\
&\text{(Property 6)} & &= I(P, F \cap y) + I(P, F \cap (G' + 3F)) \\
& & &= I(P, F \cap y) + I(P, F \cap yG'') \\
&\text{(*, Property 7)} & &= 2I(P, F \cap y) + I(P, F \cap G'') \\
&\text{(Property 8)} & &= 2 \cdot 4 + 2 \cdot 3 \\
& & &= 14,
\end{aligned}
$$

where $G' = (x^2 + y^2)(y^2 - 3x^2) - 4x^2y$ and $G'' = (5x^2 - 3y^2 + 4y^3 + 4x^2y)$. At step (*), we check that the lowest degree terms of F and G'' indeed do not share common factors.

We now give the definition of intersection numbers for curves F and G intersecting at P.

Definition 2.9. In the affine plane, $I(P, F \cap G) = \dim(\mathcal{O}_P(\mathbb{A}^2)/(F, G))$.

This definition indeed satisfies all the desired Properties 1-8 of intersection number [2, p. 38].

There is another important property of intersection numbers induced by a result from commutative algebra. Let I be an ideal in $k[x_1, \ldots, x_n]$ and $V(I) = \{P_1, \ldots, P_N\}$ be finite. There is a natural isomorphism between $k[x_1, \ldots, x_n]/I$ and $\Pi_i \dim_k(\mathcal{O}_{P_i}/I\mathcal{O}_{P_i})$ which results in the following property of intersection numbers.

Corollary 2.10. *If F and G have no common components, then*

$$\Sigma_P I(P, F \cap G) = \dim_k(k[x, y]/(F, G)). \tag{2}$$

Now we consider the projective case. Properties 1-8 still apply in the projective apply, except that Property 5 should concern a projective change of coordinate and Property 6 requires A to be a homogeneous polynomial rather than an arbitrary element in $k[x, y, z]$. Still, Properties 1-8 guarantee the uniqueness of $I(P, F \cap G)$ and we may define it in the similar manner.

Definition 2.11. In the projective plane, $I(P, F \cap G) = \dim\left(\mathcal{O}_P(\mathbb{P}^2)/(F_*, G_*)\right)$.

2.4. **Bézout's Theorem.** Not only that we can calculate the intersection number easily by its properties, the sum of intersection numbers over the projective plane is also fixed given by Bézout's Theorem.

Theorem 2.12 (Bézout's Theorem). *Let F and G be projective plane curves of degree m and n respectively. Assume F and G have no common component. Then*

$$\Sigma_{P\in\mathbb{P}^2} I(P, F \cap G) = mn.$$

Proof. (sketch of proof) First, if F and G intersect at some points in H_∞, we perform a projective change of coordinates and move all the intersections to the affine space sitting inside projective space, i.e., we may consider $I(P, F_* \cap G_*)$ instead of $I(P, F \cap G)$.

Since F and G have no common components, we only need to show

$$\dim(k[x, y]/(F_*, G_*)) = mn \tag{3}$$

by Corollary 2.10. The proof uses $k[x, y, z]/(F, G)$ as an intermediate step and aims to show $\dim k[x, y]/(F_*, G_*) = \dim \Big(k[x, y, z]/(F, G)\Big)_d$ and $\dim \Big(k[x, y, z]/(F, G)\Big)_d = mn$, where $\Big(k[x, y, z]/(F, G)\Big)_d$ denotes the residues of homogeneous polynomials of degree d in $k[x, y, z]$.

Consider the sequence

$$\begin{aligned} 0 \to k[x, y, z]_{d-m-n} &\xrightarrow{\ \psi\ } k[x, y, z]_{d-m} \oplus k[x, y, z]_{d-n} \\ &\xrightarrow{\ \varphi\ } k[x, y, z]_d \xrightarrow{\ \pi\ } \Big(k[x, y, z]/(F, G)\Big)_d \longrightarrow 0. \end{aligned}$$

where $\psi(C) = (GC, -FC)$, $\varphi(A, B) = AF + BG$ is and π is the natural map. We can prove it is exact, which deduces the dimension of $\Big(k[x, y, z]/(F, G)\Big)_d$.

Then the remaining work is to choose a basis of $\Big(k[x, y, z]/(F, G)\Big)_d$ and induce a basis of the same size for $k[x, y]/(F_*, G_*)$ [2, p. 57]. $\square$

3. Tropical Geometry

3.1. **Tropical Semiring.** In tropical geometry, the basic object to study is the *tropical semiring*

$$\mathbb{R}_{\text{trop}} = (\mathbb{R} \cup \{+\infty\}, \oplus, \odot) = (\mathbb{R} \cup \{+\infty\}, \min, +),$$

where $a \oplus b = \min\{a, b\}$ and $a \odot b = a + b$ for $a, b \in \mathbb{R}_{\text{trop}}$. That is, the *tropical sum* of a and b is the minimum of the two numbers and the *tropical product* of a and b is their sum. For example in tropical arithmetic, $3 \oplus 5 = 3$ and $3 \odot 5 = 8$. The tropical multiplication takes precedence when both operations occur in an expression. Just like regular multiplication, we abbreviate the product of n a's, i.e., $a \odot a \odot \cdots \odot a$, as a^n, and $a \odot b$ as ab. When $n = 0$, $a^n = 0$ is the identity of tropical multiplication and when $n < 0$, $a^n = -a^{-n}$. In this thesis, notations such as a^n and ab should be understood regarding the context.

We proceed to check the semiring properties of $\mathbb{R}_{\text{trop}}$. Both tropical addition and multiplication are commutative and associative. The identity element for tropical addition is $+\infty$ since $\min\{a, +\infty\} = a$ for $a \in \mathbb{R}_{\text{trop}}$, and the identity element for tropical multiplication is 0 as in the usual addition. The distributive law also holds, because

$$a \odot (b \oplus c) = a + \min\{b, c\} = \min\{a + b, a + c\} = (a \odot b) \oplus (a \odot c)$$

for $a, b, c \in \mathbb{R}_{\text{trop}}$. For example, $3 \odot (2 \oplus 4) = (3 \odot 2) \oplus (3 \odot 4) = 5$. Finally, multiplication by the tropical additive identity annihilates the semiring since $\infty \odot a = \infty$ for $a \in \mathbb{R}_{\text{trop}}$. Indeed, $\mathbb{R}_{\text{trop}}$ is a commutative semiring.

Moreover, we notice that every element in $\mathbb{R}_{\text{trop}}$ except for $+\infty$ does not have a tropical additive inverse, because the only solution to $a \oplus b = +\infty$ in $\mathbb{R}_{\text{trop}}$ is $a, b = +\infty$. Thus, $\mathbb{R}_{\text{trop}}$ fails to be a ring.

Remark 3.1. $\mathbb{R}_{\text{trop}}$ can also be defined as $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, where $a \oplus b = \max\{a, b\}$ and $\odot$ is the regular addition. In this case, the tropical additive identity becomes $-\infty$. Nevertheless, the new semiring is isomorphic to the one described above by the map $x \mapsto -x$.

3.2. **Tropical Polynomials.** Upon defining the tropical semiring, we now present analogues of mathematical objects studied in the regular addition-multiplication ring, such as polynomials and their zeros.

A *tropical monomial* is a map $p : \mathbb{R}^n_{\text{trop}} \to \mathbb{R}_{\text{trop}}$ of the form

$$m(x_1, \ldots, x_n) = c x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n},$$

where $c \in \mathbb{R}_{\text{trop}}$ and $k_1, \ldots, k_n \in \mathbb{N}$. Using regular arithmetic, m is a linear combination of $x_1, \ldots, x_n$ with natural number coefficients plus a real number. A *tropical polynomial* is the tropical sum of finitely many tropical monomials, which is a map $p : \mathbb{R}^n_{\text{trop}} \to \mathbb{R}_{\text{trop}}$ of the form

$$p(x_1, \ldots, x_n) = \bigoplus_{i=1}^{l} c_i x_1^{k_{i,1}} x_2^{k_{i,2}} \ldots x_n^{k_{i,n}} \tag{4}$$

where $c_1, \ldots, c_l \in \mathbb{R}_{\text{trop}} - \{\inf\}$ and $k_{1,1}, \ldots, k_{l,n} \in \mathbb{N}$. We say a tropical polynomial is of *degree* k for $k = \max_i\{k_{i,1} + \cdots + k_{i,n}\}$.

Example 3.2. We present graphs of the following tropical polynomials. The dotted lines represent the graph of each summand of the polynomial and the line segments represent the graph of the polynomial.

$$p_1(x) = 3x^2 \oplus 0x \oplus 5 = \min\{2x + 3, x, 5\} \tag{5}$$

$$p_2(x) = 3x^2 \oplus 5x \oplus 5 = \min\{2x + 3, x + 5, 5\} \tag{6}$$

For p_1, we break down its value as a piecewise function.

$$\begin{aligned} p_1(x) &= 2x + 3, & x &\le -3 \\ p_1(x) &= x, & -3 \le x &\le 5 \\ p_1(x) &= 5, & 5 &\le x \end{aligned}$$

We observe at the inflection point $x = -3$, $3x^2 = x = -3$, so $p_1(x)$ is continuous at $x = -3$.

When x is large enough, the first summands increase boundlessly so the constant summands will take the minimum. On the contrary, when x is small enough, $3x^2$ decreases faster than x and will eventually take the minimum.

When x is neither too large nor too small (in this case $-3 \le x \le 5$), x takes the minimum in the interval where $x \le 3x^2$ and $x \le 5$. It is possible that this interval is empty, for example, $4x$ is never the minimum of the three summands of $p_2(x)$ because $4x \le 3x^2$ and $4x \le 5$ cannot be satisfied together. Thus, $p_2(x)$ has only one inflection point.

We also notice that it is not possible that two summands may take the minimum alternatively. This is because each summand is a line, and two distinct lines cannot have more than one intersection point.

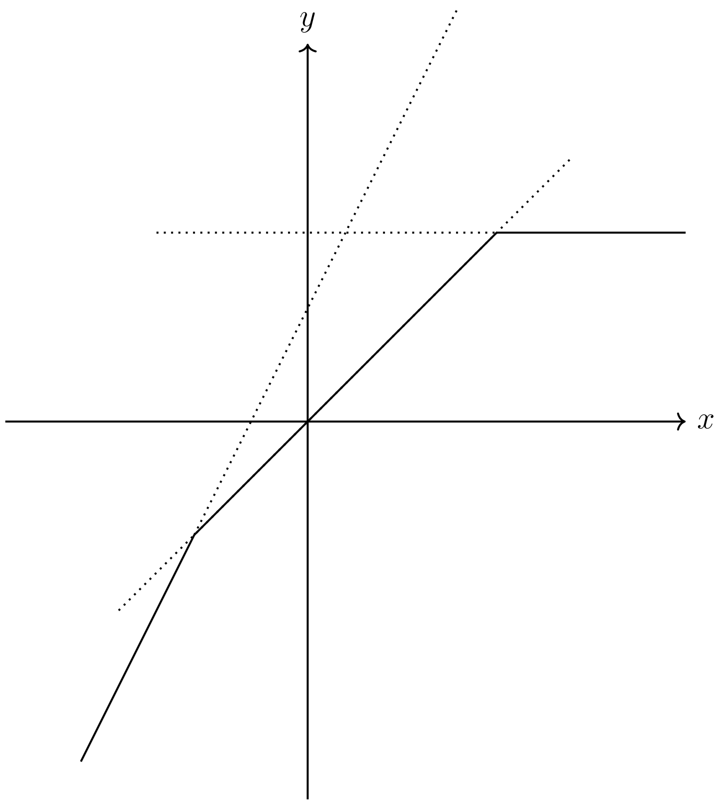

Figure 8. $p_1(x) = \min\{2x + 3, x, 5\}$

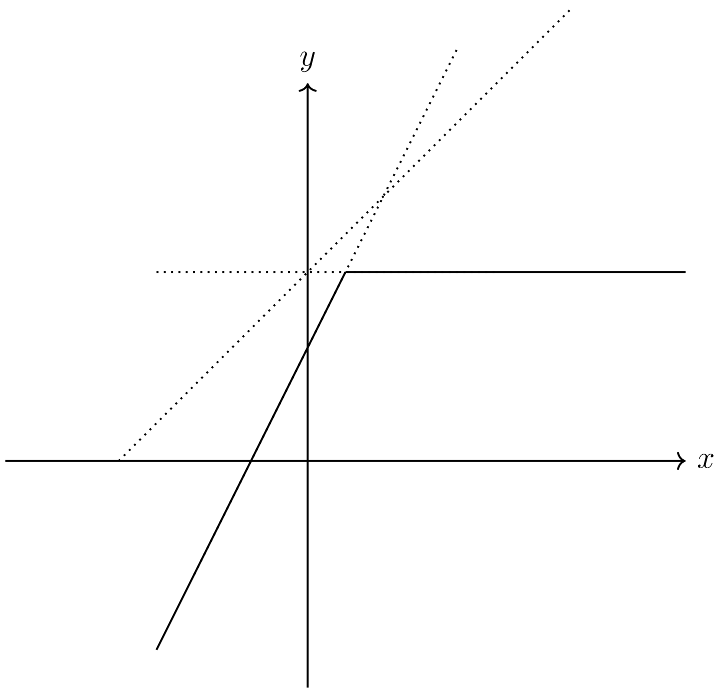

Figure 9. $p_2(x) = \{2x + 3, x + 5, 5\}$

The example shows that for a single-variable tropical polynomial, as x increases, the summands take the minimum in an order that the exponent of x increases. Not all summand may be the minimum.

Summing up the observations illustrated in the examples above, we now conclude the properties of a tropical polynomial. All summands of a tropical polynomial are linear (therefore continuous) and concave, and taking the minimum preserves linearity piecewise and concavity, which results in the following properties of tropical polynomials:

(1) continuity;
(2) piecewise linearity;
(3) concavity.

Naturally, we are interested in the nondifferentiable parts of the graph of a tropical polynomial. In fact, this gives rise to the roots of a tropical polynomial.

Definition 3.3. For a tropical polynomial $f(x_1, \ldots, x_n)$, we say $(a_1, \ldots, a_n)$ is a root of $f(x_1, \ldots, x_n)$ if $f(x_1, \ldots, x_n)$ is nondifferentiable at $(a_1, \ldots, a_n)$.

Equivalently, the roots of $p(x_1, \ldots, x_n)$ are the values at which minimum is attained at least twice. We denote the set of all roots of $p(x_1, \ldots, x_n)$ as $V(p)$ and call it the *tropical hypersurface* defined by $p(x_1, \ldots, x_n)$.

Example 3.4. We give the tropical hypersurface defined by the following polynomial.

$$p_3(x, y) = \min\{2x + 1, x + y, 2y + 1, x, y, 1\} \tag{7}$$

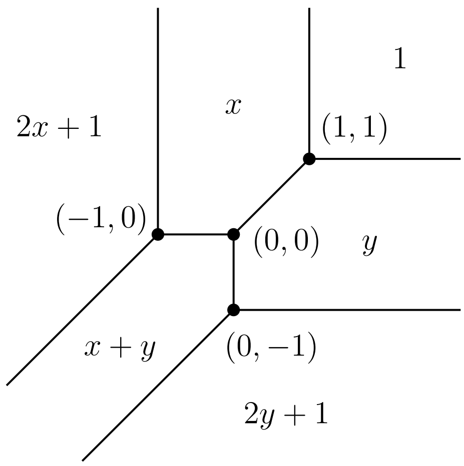

FIGURE 10. Roots of $p_3(x, y) = \min\{2x + 1, x + y, 2y + 1, x, y, 1\}$.

The definition of roots in tropical geometry does not seem intuitive at the first glance. Naively, the analogous zeros would be

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n_{\text{trop}} : p(x_1, \ldots, x_n) = +\infty\}.$$

However, this will make the zero of any nontrivial (i.e., not constantly equal to $+\infty$) tropical polynomial $(x_1, \ldots, x_n) = (+\infty, \ldots, +\infty)$ which does not give us useful information on the polynomials itself.

Instead, we can think of the zeros of a polynomial as the factorization of the polynomial. Recall that for $f(x) \in \mathbb{C}[x]$, x_0 is a root of $f(x)$ with multiplicity k if $f(x)$ can be factored as $(x - x_0)^k g(x)$ such that $g(x_0) \neq 0$. Translating this statement into the tropical world, we define that x_0 is a *root* (or *zero*) of the tropical polynomial $p(x)$ if we can write $p(x) = (x \oplus x_0)^k q(x)$ where $k \in \mathbb{N}$ and $q(x)$ is another tropical polynomial. We say x_0 is a root of *multiplicity* k for the largest possible k.

For example, $p_2(x) = \min\{2x+3, x+5, 5\} = 2 \cdot \min\{x, 1\}$ so 1 is a root of multiplicity 2. For $p_1 = \min\{2x + 3, x, 5\} = \min\{x, -3\} + \min\{x, 5\} + 3$, it has two roots $x = -3, 5$

of multiplicity 1 respectively. From the examples, it is not hard to see that the degree of a single-variable tropical polynomial equals the sum of all roots counting multiplicity, which motivates us to develop an analogue of the fundamental theorem of algebra for tropical polynomials.

Theorem 3.5. *Every nontrivial, single-variable, degree n tropical polynomial has exactly n roots counted with multiplicity.*

3.3. **Connecting the Two Worlds.** Having established analogous concepts such as polynomials and hypersurfaces in the tropical world, we wonder how they interact with their counterparts in algebraic geometry we used to work in. Specifically, let K be a field and we will study the zeros of polynomials in $K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ in the tropical world. We will realize it via a valuation.

3.3.1. *Valuations.* Let K be a field. A *valuation* on K is a map $\mathrm{val} : K \to \mathbb{R}_{\mathrm{trop}}$ satisfying the following conditions:

(1) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$,
(2) $\mathrm{val}(a + b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ for $a, b \in K^\times$, i.e., nonzero elements of K,
(3) $\mathrm{val}(a) = +\infty$ if and only if $a = 0$.

Example 3.6. We give the following valuations.

(1) The *p-adic valuation.* The p-adic valuation is a map from $\mathbb{Q}$ to $\mathbb{R}_{\mathrm{trop}}$. Any element in $\mathbb{Q}$ can be written as a fraction with integral numerator and denominator, and for any prime p and an integer a, we can write $a = p^k b$ where p does not divide the integer b. For a rational number $q = p^k b / p^l c$, we define $\mathrm{val}(q) = b - c$. For example, the 2-adic valuation $\mathrm{val}(8) = 3$ and $\mathrm{val}(3/4) = -2$.
(2) Let K be the field of Puiseux series with complex coefficients. We denote $\mathbb{C}\{\{t\}\} = \Cup_{n \geq 1} \mathbb{C}((t^{1/n}))$ where $\mathbb{C}((t^{1/n}))$ is the field of Laurent series in the variable $t^{1/n}$. This field is particularly useful in algebraic geometry as it is algebraically closed. We define a valuation $\mathrm{val} : \mathbb{C}\{\{t\}\} \to \mathbb{R}_{\mathrm{trop}}$ by taking the lowest exponent of t appeared in the series expansion of $f(t) \in \mathbb{C}\{\{t\}\}$. For example, $\mathrm{val}(3t^2 + 2t) = 1$ and $\mathrm{val}(t^2 + 5t^{-3} + 1) = -3$.

Valuations provide a way to convert numbers from K to $\mathbb{R}_{\mathrm{trop}}$ in a way that somewhat preserves its algebraic structure. The definition itself is very close to a homomorphism, and actually, supported by the following lemma, a valuation acts exactly like a homomorphism in most cases.

Lemma 3.7. *If* $\mathrm{val}(a) \neq \mathrm{val}(b)$, *then* $\mathrm{val}(a + b) = \min\{\mathrm{val}(a), \mathrm{val}(b)\}$.

3.3.2. *Kapranov's Theorem.* With a fixed valuation $\mathrm{val} : K \to \mathbb{R}_{\mathrm{trop}}$, it is easy to realize both paths in Figure 7. Given $f(x_1, \ldots, x_n) \in K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, we can compute the regular hypersurface $V(f) \in K^n$ and map points in $V(f)$ by the valuation map coordinate-wise, which results in the *tropicalization of the hypersurface* defined by f. We denote it by

$$\mathrm{trop}(V(f)) = \{(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) \in \mathbb{R}^n_{\mathrm{trop}} : (x_1, \ldots, x_n) \in V(f)\}.$$

Alternatively, we may "tropicalize" a polynomial in the following way. For $f(x_1, \ldots, x_n) = \Sigma_{i=1}^{l} c_i x_1^{k_{i,1}} x_2^{k_{i,2}} \ldots x_n^{k_{i,n}} \in K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, its *tropicalization* is defined as

$$\mathrm{trop}(f(x_1, \ldots, x_n)) = \bigoplus_{i=1}^{l} \mathrm{val}(c_i) x_1^{k_{i,1}} x_2^{k_{i,2}} \ldots x_n^{k_{i,n}}.$$

That is, when applied to polynomials, trop is a function from $K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ to $\mathbb{R}_{\text{trop}}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ that maps K-coefficients to their valuations and the binary operations $(+, \cdot)$ in K to their tropical counterparts $(\oplus, \odot)$.

Now we want to show that the two tropicalizations described above are equivalent. The result is first proved by Kapranov.

Theorem 3.8 (Kapranov)**.** *For a Laurent polynomial $f(x_1, \ldots, x_n) \in K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, the following sets are equal:*

(1) $\overline{\text{trop}(V(f))}$, the closure of the tropicalization of $V(f)$ in $\mathbb{R}^n_{\text{trop}}$;
(2) $V(\text{trop}(f))$, the tropical hypersurface defined by the tropicalization of f;

Proof. By an application of Lemma 3.7, it is easy to see that set (2) contains (1). To prove the converse, we introduce the initial form of a polynomial and use the conclusion that every zero of an initial form lifts to a zero of the given polynomial [4, p. 92]. □

3.4. **Tropical Curves and Regular Subdivision.** Now we focus on studying *tropical curves*, i.e., tropical hypersurfaces of polynomials of two variables. Recall that for a two-variable tropical polynomial $p(x, y)$, its associated tropical curve is

$$\{(x, y) \in \mathbb{R}^2_{\text{trop}} : p(x, y) \text{ is nonlinear}\} = \{(x, y) \in \mathbb{R}^2_{\text{trop}} : p(x, y) \text{ attains minimum twice}\}$$

Let $G = (V, E)$ be a $\mathbb{R}^2$ graph with edges $e_1, \ldots, e_n$. For our purpose, the edges of a $\mathbb{R}^2$ graph can either be line segments, rays or lines. Define the *faces* of the graph G be the connected components of $\mathbb{R}^2 - E$. Just like edges, faces of a graph can be unbounded. Denote the faces of G as $f_1, \ldots, f_m$. Its *dual graph* G' consists of vertices $v'_1, \ldots, v'_m$ and edges $e'_1, \ldots, e'_n$ such that each e'_i is perpendicular to e_i and if e_i has incident faces f_j and f_k in G, $e'_i = (v'_j, v'_k)$ in G'. An example is presented in Figure 11.

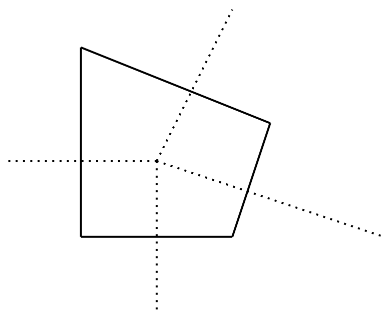

Figure 11. A graph and its (dotted) dual graph.

We have seen in Figure 5 that the graph of a multivariable tropical polynomial is difficult to visualize but comparatively, the tropical curve is rather easy to draw. We now introduce an algorithm that aids to draw the tropical curve. Let $p(x, y) = \bigoplus_{i=1}^{l} c_i x^{d_i} y^{e_i}$ be a tropical polynomial.

(1) Form the convex hull of the set $\{(x, y) \in \mathbb{Z}^2 : x = d_i, y = e_i \text{ for some } i, c_i \neq -\infty\}$. We call this convex hull the *Newton polytope* of $p(x, y)$ and denote it as $\text{Newt}(p)$.
(2) Let $P = \{(x, y, z) \in \mathbb{R}^3 : x = d_i, y = e_i, z = c_i \text{ for some } i\}$ and form its convex hull $\text{conv}(P)$.
(3) We define the *lower faces* of $\text{conv}(P)$ as the faces of $\text{conv}(P)$ looked from $(0, 0, -\infty)$. Project the lower faces of $\text{conv}(P)$ onto the xy-plane. The projection is called the *regular subdivision* of $\text{Newt}(p)$ induced by $p(x, y)$.

(4) Form the dual graph of the regular subdivision of Newt(p). The result is exactly the tropical polynomial up to scaling and after reflection over the line $y = -x$ [4, p. 94].

Example 3.9. We will use $p_3(x, y) = \min\{2x + 1, x + y, 2y + 1, x, y, 1\}$ again as an example of taking the regular triangulation.

(1) Consider the convex hull of the set $\{(2, 0), (1, 1), (0, 2), (1, 0), (0, 1), (0, 0)\}$. That is, Newt$(p_3) = \{(2, 0), (0, 2), (0, 0)\}$ as shown in the undotted part of Figure 12.
(2) Adding the coefficient of p_3, we have
$$P = \{(2, 0, 1), (1, 1, 0), (0, 2, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$
Its convex hull is exactly itself.
(3) Project the lower faces of conv(P), we have the regular subdivision of Newt(p_3) in Figure 12.
(4) Form the dual graph of the regular subdivision. As shown in Figure 13, this is exactly the tropical curve in Figure 10 up to scaling and after reflection over the line $y = -x$.

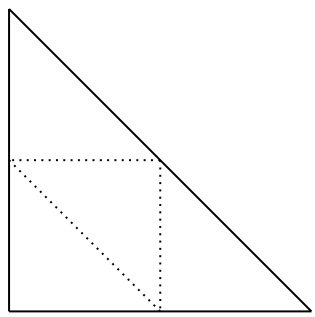

FIGURE 12. A graph and its (dotted) dual graph.

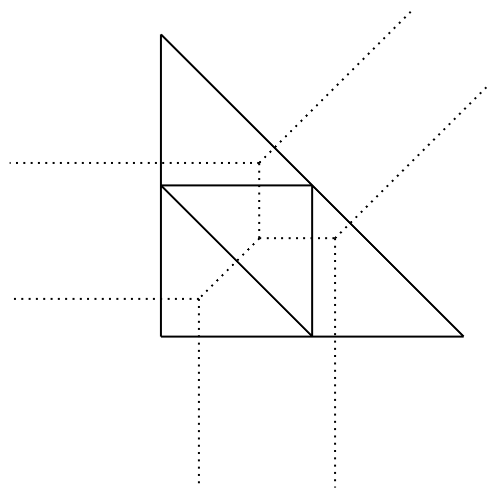

FIGURE 13. Recovering the tropical curve defined by p_3.

The triangulation naturally gives rise to the weights of edges in a tropical curve. Consider a vertex P and its incident edges or rays $e_1, \ldots, e_m$. Let P' be the polygon dual to P and $e'_1, \ldots, e'_m$ be the edges dual to $e_1, \ldots, e_m$ in the regular triagulation of Newt(p) induced by $p(x, y)$. Since P' is a polygon, its edges $e'_1, \ldots, e'_m$ sum up to zero as vectors. Because $e_1, \ldots, e_m$ are perpendicular to $e'_1, \ldots, e'_m$ respectively, we can assign weights to edges or rays in the tropical curve so that the edges can be balanced. Specifically, let $f_1, \dots, f_m$ be the primitive integral vectors along the directions of $e_1, \dots, e_m$ and $w_1, \dots, w_m$ be the length of $e'_1, \dots, e'_m$, where the *length* of an edge is defined as the number of lattice points the edge passes minus 1. The duality guarantees that at each vertex, the tropical curve is balanced by

$$\Sigma_{i=1}^{m} w_i f_i = 0. \tag{8}$$

Example 3.10. Continue the example in Figure 11, we label the weights of edges in the dual graph as shown in Figure 14. The balance is reached by

$$4 \cdot (0, -1) + 1 \cdot (3, -1) + 1 \cdot (2, 5) + 5 \cdot (-1, 0) = 0.$$

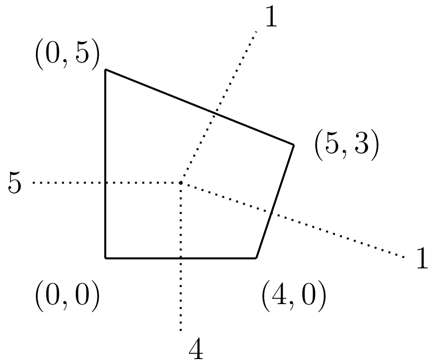

FIGURE 14. Weights of a tropical curve.

Actually, not only that all tropical curves are balanced, the balancing condition also describes a tropical curve. Define a *rational graph* as a subset G of $\mathbb{R}^2_{\text{trop}}$ consisting of finitely many edges and rays, whose vertices are in $\mathbb{Q}^2$ and slopes of their underlying lines are rational. Positive integral weights can be assigned to edges and rays, and the graph is *balanced* if at each vertex, the Equation 8 holds.

Theorem 3.11. *Any balanced rational graph is a tropical curve.*

Proof. The proof uses a theorem which states that any balanced rational graph is the projection of the lower edges of a convex polytope [1]. A sketch of proof can be found in [5]. □

4. Tropical Bézout's Theorem

Recall that the classical Bézout's theorem applies to curves of certain degrees and states that total intersection multiplicity is the product of the degrees of the curves. To formulate Bézout's theorem in tropical geometry, we need to define the degree of tropical curves and an analogue of intersection multiplicity.

4.1. Degrees and Intersection Multiplicity.

We say a tropical curve C has *degree* c if its Newton polytope consists of exactly $(0, 0), (0, c)$ and $(c, 0)$. That is, the defining polynomial of the curve has maximal degree c and has nontrivial coefficients for x^c, y^c and the constant term. If any of the points is missing from the Newton polytope, the degree of the tropical curve is then undefined. Let C, D be tropical curves intersecting at P. Assume that there is exactly one segment of C and D passing P and let the corresponding weights and primitive integral vector of the segments be $w_C, w_D, u_C = (x_C, y_C)$ and $u_D = (x_D, y_D)$. We define the intersection multiplicity of C and D at P as

$$I_{\text{trop}}(C \cap D, P) = |w_C w_D (x_C y_D - x_C y_D)|.$$

We say the curves *intersect transversally* if any vertex of C or D is not in the intersection. Otherwise, we say the curves *intersect non-transversally*.

Example 4.1. We give some examples of the concepts above.

- The tropical curve defined by $p(x, y) = \min\{2x, 2y, 4\}$ has degree 2 and the curve defined by $p(x, y) = \min\{2x, y, 4\}$ has no degree assigned.
- Figure 15 and Figure 16 show different types of intersections.

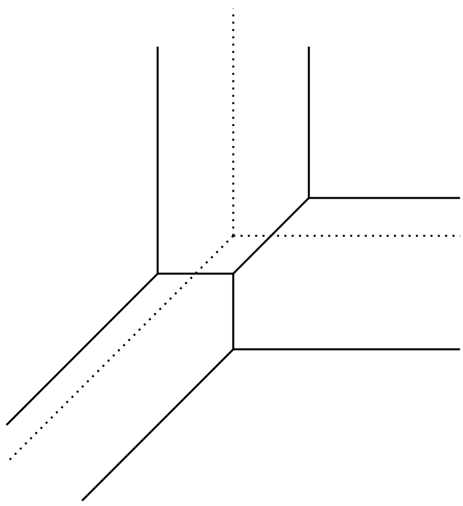

FIGURE 15. A trasversal intersection.

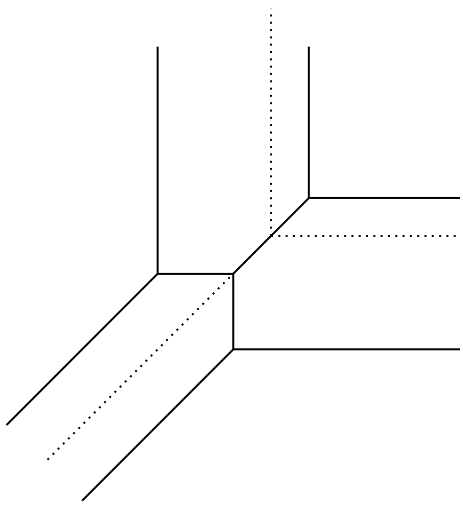

FIGURE 16. A non-trasversal intersection.

4.2. **Proof of Tropical Bézout's Theorem.** Now we state and prove the tropical Bézout's theorem.

Theorem 4.2. *Let C and D be two tropical curves of degrees c and d that intersect in finitely many points. Then for all $P \in C \cap D$,*

$$\sum I_{\text{trop}}(C \cap D, P) = c \cdot d.$$

Proof. The proof has the following structure. We will first show a special case of the theorem as follows.

Claim 1 When C and D intersect only in rays in $x-$ or $y-$direction, they have $c \cdot d$ total intersections counting multiplicity as shown in Figure 17.

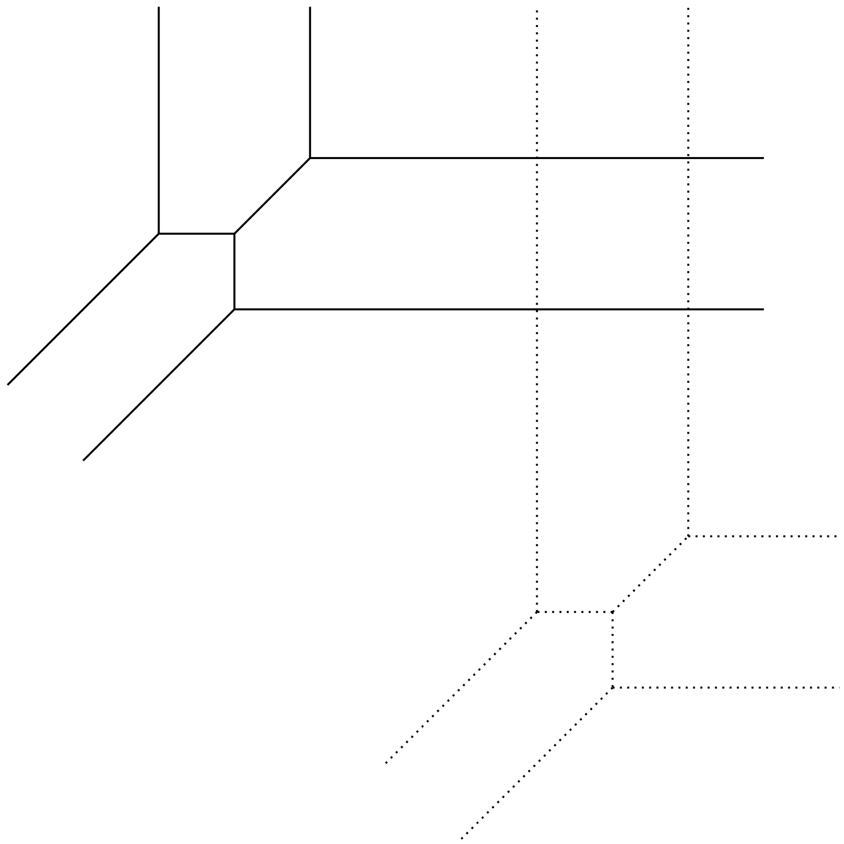

FIGURE 17. C and D intersect in rays in $x-$ or $y-$direction.

Then we proceed to show that for C and D in general position, there exists a way to move the curves so that they end up intersecting in rays in $x-$ or $y-$direction only, and during this process, the total number of intersections does not change.

Without loss of generality, we will move curve C and denote the curve at time t as C_t. C will be moved by translations only (i.e., no rotations, deformations), which means tracking the movement of a fixed point on C is equivalent as tracking the movement of C. During the process, C may intersect D transversally or nontransversally, we will deal with the situations separately with the following claims.

Claim 2 C_t intersects D nontransversally for a finite number of t. Denote the family of such C_t as $\{C_{t_1}, \dots, C_{t_k}\}$. Any vertex of C_{t_i} does not intersect any vertex of D.

Claim 3 The total number of intersections do not change during the interval between t_i and t_{i+1}. That is, $\sum I_{\text{trop}}(C_t \cap D, P)$ stays the same for $t_i < t < t_{i+1}$.

Claim 4 The total number of intersections do not change in an infinitesimal time before and after t_i. That is, $\sum I_{\text{trop}}(C_{t_i - \Delta t} \cap D, P) = \sum I_{\text{trop}}(C_{t_i + \Delta t} \cap D, P)$.

Claim 1: The rays along the x-axis consist of (x, y) where the corresponding polynomial achieves minimum twice at summands without y terms. Thus the rays are always parallel. We notice that as $y \to \infty$, the tropical polynomial is equivalent to a single variable tropical polynomial consisting only terms consisting x, as any nontrival term consisiting y is ∞ and cannot be the minimum. In the discussion of the fundamental theorem of algebra in tropical geometry, we notice that the weight of such rays is equal to the difference between the degrees of x in the minimum summands achieved on the rays. Thus by a computation, the total number of intersections is exactly $c \cdot d$. We also notice that if the Newton polytope of the tropical curve C does not contain, say, $(c, 0)$, there will be fewer total number of intersections due to the lack of higher-ordered terms. This is why degrees are only defined for tropical curves whose Newton polytopes contain exactly $(0,0), (0,c)$ and $(c, 0)$.

Claim 2: To prove this claim, a basic question to ask is when C_t and D intersect nontransversally. As stated before, the position of C_t can be determined by tracking a fixed vertex Q on C. For example, pick $Q = (0,0)$ on the curve given by $p_2(x,y)$, if Q is moved to $(2,1)$, then we know that the curve is moved upward by 2 units and right by 1 unit. Thus, searching for the positions of C_t intersecting nontransversally with D is the same as searching for the coordinates Q after movement that makes the intersection nontransversal.

As shown in Figure 18, the dotted curve C with the fixed point Q cannot be moved to the circled points to prevent a vertex-vertex intersection.

During the movement, we want to forbid any vertex-vertex intersection of C_t and D, and have only finitely many edge-vertex intersections. That is, we may collect the set $S_{\text{v-v}}$ ($S_{\text{e-v}}$) that contains coordinates of Q resulting in vertex-vertex (edge-vertex) intersections of C_t and D, and show that there exists a path of Q avoiding $S_{\text{v-v}}$ and containing finitely many elements of $S_{\text{e-v}}$.

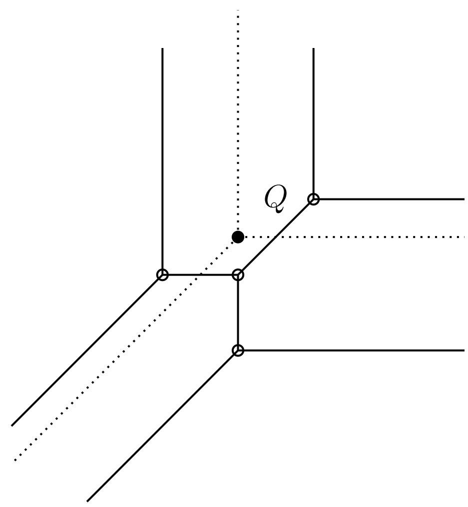

FIGURE 18. The point Q cannot be at the circled points.

First, we consider $S_{\text{v-v}}$. There are only finitely ways for Q to intersect a vertex of D because a tropical curve has finitely many vertex. This is true for other vertices of C as well, and the forbidden coordinates for other vertices of C can be equivalently expressed as elements of $S_{\text{v-v}}$. For example, let $Q = (0,0)$ and $R = (1,1)$ be another vertex in C, and $S = (2,1)$ be a vertex in D. We do not want to move Q or R to $(2,1)$, which is the same as not moving Q to $(2,1)$ or $(1,0)$, since the position of R can be determined by the position of Q. Thus, each vertex of C forbids finitely many positions in the movement of Q, which means $S_{\text{v-v}}$ is finite and is therefore avoidable in the translation of C.

Second, $S_{\text{e-v}}$ is the union of finitely many copies of D located at different positions. For the vertex Q in C, if it is moved to coordinates of points on the curve D, the intersection of the corresponding C_t and D will be nontransversal. Therefore, all points on D are elements of $S_{\text{e-v}}$. For another vertex R in C, the coordinates of R that makes the intersection nontransversal can be reflected by coordinates of Q as well. Therefore,

$S_{\text{e-v}}$ contains points from finitely many copies of D, so that the path of Q can intersect $S_{\text{e-v}}$ at finitely many positions.

Claim 3: As C_t moves transversally, the weights and primitive integral vectors at each intersection do not change, so are the intersection multiplicities. Additionally, during the period of continuous transversal intersections, the number of points of intersections will not change, so the total number of intersection with multiplicity does not change.

Claim 4: Without loss of generality, suppose a vertex P of the curve C intersects a non-vertex segment E of D at t. Let w be the weighted integral vector of segment E and L be the line in the direction of E. Let $u_1, \dots, u_m$ be the weighted integral vectors on one side of L and let $v_1, \dots, v_n$ be those on the other side. As shown in Figure 19, the dashed line segments are the positions of the segment E shortly before or after t.

The equilibrium condition of curve C at vertex D tells us that

$$\Sigma_i u_i + \Sigma_j v_j = 0.$$

Since u_i and v_j are on different sides of L, the determinant of the matrices $[u_i w]$ and $[v_i w]$ always have different signs. Suppose at time $t - \Delta t$ (shortly before the intersection of P and E), E intersects with incident edges of P with weighted integral vectors $u_1, \dots, u_m$. Then at time $t + \Delta t$, E intersects with incident edges of P with weighted integral vectors $v_1, \dots, v_n$. The equilibrium condition guarantees that the intersection multiplicity at P does not change shortly before or after t.

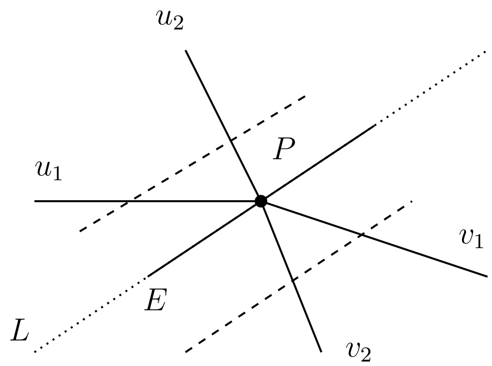

FIGURE 19. An edge-vertex intersection at P.

Thus, the proof is complete.

□

4.3. **One Generalization.** The Bézout's theorem we present only applies to tropical curves whose Newton polytopes contain only $(0,0), (0,c)$ and $(c,0)$. It turns out that such restriction guarantees that all intersections happen in the tropical affine plane $\mathbb{TA}^2 = \{(x,y) : x, y \in K\}$ over some field K. There is a generalization of the theorem for all tropical curves defined in the tropical projective plane. We present the statement of the theorem as follows, and detailed construction and proof can be found in [6].

Theorem 4.3 (Complete Tropical Bézout's)**.** *Let C and D be two tropical projective plane curves of degree c and d respectively. Then C and D stably intersect in $c \cdot d$ points, counting multiplicity.*

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