

UNIVERSITY OF BRISTOL

School of Mathematics

**Algebraic Geometry**

MATHM0036

(Paper code MATHM0036)

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April/May 2025 2 hour(s) 30 minutes

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The exam contains FOUR questions  
All Four answers will be used for assessment.

Calculators of an approved type (permissible for A-Level examinations) are permitted.

**Candidates may bring ONE hand-written sheet of A4 notes, written double-sided into the examination. Candidates must insert this sheet into their answer booklet(s) for collection at the end of the examination.**

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

*Do not turn over until instructed.*

Q1. (a) (**5 marks**) Show that any polynomial  $f \in \mathbb{C}[x, y, z]$  can be expressed as

$$f = r_1(x^2 - y) + r_2(x^3 - z) + g,$$

for  $r_1, r_2 \in \mathbb{C}[x, y, z]$  and  $g \in \mathbb{C}[x]$ .

(b) (**5 marks**) Define the *twisted cubic*  $V = \mathbb{V}(x^2 - z, x^3 - y)$ , and consider the parametrisation:

$$\begin{aligned} \varphi : \mathbb{A}^1 &\rightarrow \mathbb{A}^3, \\ t &\mapsto (t, t^2, t^3). \end{aligned}$$

Prove that the pullback map

$$\varphi^* : \mathbb{C}[x, y, z] \rightarrow \mathbb{C}[t]$$

induces an isomorphism of  $\mathbb{C}$ -algebras  $\mathbb{C}[V] \simeq \mathbb{C}[t]$ .

(c) (**5 marks**) Explain why the result from part (b) implies that  $V$  is irreducible.

(d) (**5 marks**) We know that the closure of  $V$  in  $\mathbb{P}^3$ , is given by  $\overline{V} = \Phi(\mathbb{P}^1)$  where

$$\begin{aligned} \Phi : \mathbb{P}^1 &\rightarrow \mathbb{P}^3 \\ [t : s] &\mapsto [s^3 : ts^2 : t^2s : t^3]. \end{aligned}$$

Prove that  $\overline{V} = \mathbb{V}(xz - y^2, yw - z^2, xw - yz) \subseteq \mathbb{P}^3$ .

(e) (**5 marks**) Explain why the irreducibility of  $V$  implies that  $\overline{V}$  is also irreducible.

Q2. (a) (**15 marks**) Recall the following definition:

Let  $X, Y$  be two algebraic varieties (*i.e.*, affine, quasi-affine, projective or quasi-projective). A morphism  $\varphi : X \rightarrow Y$ , is a map such that

- $\varphi$  is continuous;
- For any for every open set  $V \subseteq Y$ , and for every regular function  $f \in \mathcal{O}_Y(V)$ ,  $\varphi^*(f) = f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$ .

Prove the following theorem:

Let  $X$  be an algebraic variety,  $Y \subseteq \mathbb{A}^n$  a closed affine algebraic variety, and  $\varphi : X \rightarrow Y$  a map of sets. Then,  $\varphi = (\varphi_1, \dots, \varphi_n)$  is a morphism, if and only if, for all  $i$ , coordinate function  $\varphi_i \in \mathcal{O}_X(X)$ .

(b) (**10 marks**) Let  $V \subseteq \mathbb{A}^n$  and  $W \subseteq \mathbb{A}^m$  be two closed affine algebraic varieties and

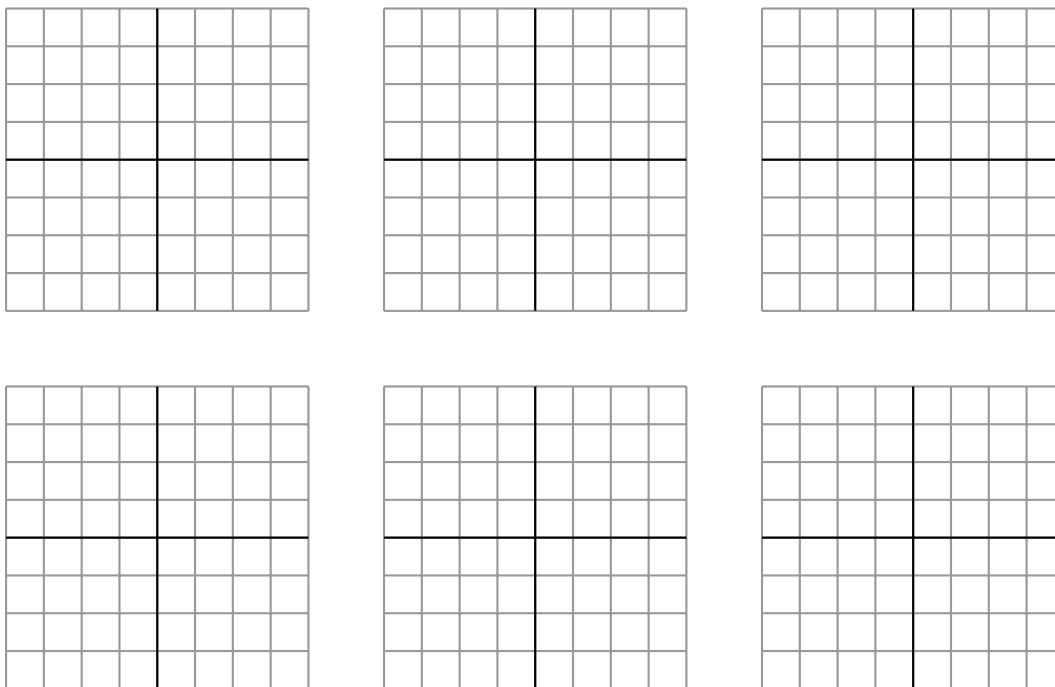
$$\varphi : V \rightarrow W$$

a morphism. Prove that the pullback  $\varphi^* : \mathbb{C}[W] \rightarrow \mathbb{C}[V]$  is surjective if and only if  $\varphi$  defines an isomorphism between  $V$  and some algebraic subvariety of  $W$ .

Q3. Let  $V = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2$ .

- (a) **(5 marks)** Sketch  $V \cap \mathbb{R}^2$  in  $\mathbb{R}^2$ .
- (b) **(5 marks)** Find all the singular point of  $V$ .
- (c) **(10 marks)** Identify the irreducible components of  $\mathbb{V}(y^2 - x^3, xz - y) \subseteq \mathbb{A}^3$ .
- (d) **(5 marks)** Show that  $\mathbb{V}(xz - y) \subseteq \mathbb{A}^3$  is isomorphic to  $\mathbb{A}^2$ .

Q4. Consider the cone  $\sigma = \text{cone}(\{e_1, e_1 + 3e_2\}) \subseteq \mathbb{R}^2$ .



- (a) **(5 marks)** Explain why the affine toric variety  $X_\sigma$  is not smooth. Subdivide  $\sigma$  into a union of smooth two-dimensional cones.
- (b) **(10 marks)** Select two of the two-dimensional cones from your subdivision and denote them by  $\sigma_1$  and  $\sigma_2$ . Let  $\tau = \sigma_1 \cap \sigma_2$ . Describe the the toric varieties  $X_{\sigma_1}$ ,  $X_{\sigma_2}$ ,  $X_\tau$  and their coordinate rings.
- (c) **(2 marks)** Justify why we have the inclusions

$$\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau], \quad \mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau].$$

- (d) **(8 marks)** Explain why  $X_{\sigma_1}$  and  $X_{\sigma_2}$  contain  $X_\tau$  as an open set and describe the glueing of  $X_{\sigma_1}$  and  $X_{\sigma_2}$  along  $X_\tau$ .