UNIVERSITY OF BRISTOL

School of Mathematics

Algebraic Geometry
MATHM0036
(Paper code MATHMATHM0036

May/June 2024   2 hour(s) 30 minutes

The exam contains FOUR questions
All Four answers will be used for assessment.

Calculators of an approved type (permissible for A-Level examinations) are permitted.

Candidates may bring ONE hand-written sheet of A4 notes, written double sided into the examination. Candidates must insert this sheet into their answer booklet(s) for collection at the end of the examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

Q1. Assume that V is an affine algebraic variety, and $U, U_1, U_2 \subseteq V$ are open subsets.

(a) (**15 marks**) State the definition of the set of regular functions $\mathcal{O}_V(U)$, and prove that $\mathcal{O}_V(U)$ is a $\mathbb{C}$-algebra.

(b) (**10 marks**) Assume further that $f_1 \in \mathcal{O}_V(U_1), f_2 \in \mathcal{O}_V(U_2)$, with ${f_1}_{|_{U_1 \cap U_2}} = {f_2}_{|_{U_1 \cap U_2}}$. Prove that there exists a regular function $f \in \mathcal{O}_V(U_1 \cup U_2)$ such that

$$f_{|_{U_1}} = f_1, \quad f_{|_{U_2}} = f_2.$$

Q2. (a) (**15 marks**) Let $U = \mathbb{A}^2 \setminus \{0\}$. Compute $\mathcal{O}_{\mathbb{A}^2}(U)$ and show that U is not an affine algebraic variety.

(b) (**10 marks**) Prove that $\mathbb{V}(y) \subseteq \mathbb{A}^2$ and $\mathbb{V}(y - x^2) \subseteq \mathbb{A}^2$ are isomorphic, but their corresponding projective closures in $\mathbb{P}^2$ are not.

Q3. (a) (**10 marks**) Consider the family of algebraic varieties, with parameter $t \in \mathbb{C}$, given by

$$V_t := \mathbb{V}(x^2 + y^2 - t) \subseteq \mathbb{A}^2.$$

Sketch the variety of V_0, V_1, and V_2 in $\mathbb{R}^2$. Determine which one of these three varieties is smooth. Briefly justify your answers.

(b) (**15 marks**) Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be two affine algebraic varieties, and

$$\varphi : V \longrightarrow W$$

a morphism. Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W.

Continued...

Q4. Let Σ be the fan consisting of

- σ_1 cone spanned by $\{(-1,-1),(0,1)\}$;
- σ_2 cone spanned by $\{(0,1),(1,0)\}$;
- τ cone spanned by $\{(1,1)\}$.

(a) (**6 marks**) Determine whether or not the toric variety X_Σ has the following properties. Briefly justify your answer.

(i) smooth;

(ii) complete.

(b) (**9 marks**) Describe the coordinate rings of X_{σ_1}, X_{σ_2}, and X_τ.

(c) (i) (**5 marks**) Explain why we have the inclusions $\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau]$, $\mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau]$;

(ii) (**5 marks**) Describe the gluing of X_{σ_1} and X_{σ_2} along X_τ.

End of examination.