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Coursework 1

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Q1. Let $A \subseteq \mathbb{A}^n$ be a subset.

- (a) (5 marks) What is the definition of the closure of A in \mathbb{A}^n ?
 (b) (5 marks) Prove that $V(\mathbb{I}(A))$ equals the Zariski closure of A in \mathbb{A}^n .
 (c) (5 marks) Give an example of a subset in $B \subseteq \mathbb{C}$ whose closure in the Zariski topology does not coincide with its closure in the Euclidean topology.

(a) For a topological space X , the closure, \bar{Y} , of $Y \subseteq X$, is the smallest closed set in X such that $Y \subseteq \bar{Y}$, i.e. $\bar{Y} = \bigcap_{B \text{ closed}} B$ s.t. $Y \subseteq B$.

In particular, for \mathbb{A}^n , the closure of $A \subseteq \mathbb{A}^n$ is defined to be

$$\bar{A} = \bigcap_{\mathbb{I}(f)} X_0$$

s.t. each X_0 is closed & $A \subseteq X_0$ for all f .



(b) By definition, we have that,

$$\mathbb{I}(A) = \{f \in \mathbb{F}[x_1, \dots, x_n] : f(a) = 0 \forall a \in A\}.$$

Hence,

$$V(\mathbb{I}(A)) = V(\{f_j \in \mathbb{F}[x_1, \dots, x_n] : f_j(a) = 0 \forall a \in A\}_{j \in J})$$

$$= \bigcap_{j \in J} (V(f_j) \cup \{x_0, \dots, x_n\} : f_j(a) = 0 \forall a \in A)$$

On the other hand, $\bar{A} = \bigcap_{\mathbb{I}(f)} X_0$ s.t. X_0 are closed & $A \subseteq X_0 \forall f \in \mathbb{I}(A)$. Thus since $A \subseteq X_0 \forall f$, we have that

$$\bigcap_{\mathbb{I}(f)} X_0 = \bigcap_{\mathbb{I}(f)} (V(f) \cup \{x_0, \dots, x_n\} : f(a) = 0 \forall a \in A)$$

$$= V(\{f_j \in \mathbb{F}[x_1, \dots, x_n] : f_j(a) = 0 \forall a \in A\})$$

$$A = V(\bar{y}) \Rightarrow A = \bar{A}$$

(c) Let $\bar{A} = \{(z_0) : z_0 \in \mathbb{C}\} \subseteq \mathbb{A}^2$. Then \bar{A} in \mathbb{A}^2 can be described by the polynomial $y=0$, i.e. $\bar{A} = V(y)$. However, $\bar{A} = V(y) = \{(z_0) : z_0 \in \mathbb{C}\} \cup \{(z_0, y) : z_0, y \in \mathbb{C}\} = \mathbb{A}^2$. Thus $\bar{A} = \mathbb{A}^2$.

On the other hand, for \bar{A} in \mathbb{C}^2 under the Euclidean topology, we have that $\bar{A} = A$ since A is closed under the Euclidean topology.

A is closed since any open neighbourhood of some point $(z_0, 0) \in A$ is wholly contained in A .

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Q2. (a) (5 marks) What is the definition of a compact subset of a topological space?
 (b) (10 marks) Prove that $V(x^2 - y^2) \subseteq \mathbb{C}^2$ is compact in the Zariski topology but not in the Euclidean topology.

(a) In a topological space, X , a subset $U \subseteq X$ is compact if for all open coverings, there is a finite subcovering.



Whereas a collection of open sets, $\{O_j\}_{j \in J} \subseteq X$ each O_j is open, is an open covering of $\bigcup_j O_j = X$. For $J \subseteq I$, $\{O_j\}_{j \in J}$ is a subcovering of it is still an open covering.

(b) Suppose $\{O_i\}_{i \in I}$ is an open covering for $V = V(x^2 - y^2) \subseteq \mathbb{C}^2$. Then, let

$$C = \{U : U = \bigcup_{i \in I} O_i\}$$

i.e. C is a set of open sets that can be found in our open cover for V . If $V \notin C$, then there exists an infinite ascending chain of open sets namely $U_1 \subseteq U_2 \subseteq \dots$ for $U_i \in C$. However, this implies that $V \supseteq U_2 \supseteq \dots$ is an infinite descending chain

of closed sets & then by Hilbert's correspondence this descending chain corresponds to $\mathbb{I}(V) \subseteq \mathbb{I}(V) \subseteq \dots$ being an infinite ascending chain, but this is a contradiction since polynomial rings are Noetherian by Hilbert's Basis Theorem.

Therefore, every ascending chain of open sets stabilises & so $V \in C$, & in particular there exists a finite subcovering of V given by $\{U_i\}_{i \in I}$. Thus V is compact in the Zariski topology.

On the other hand, $\{O_i\}_{i \in I}$ is an open covering for V in the Euclidean topology, then as before, let

$$C = \{U : U = \bigcup_{i \in I} O_i\}$$

Suppose that $V \in C$, then $\exists U \in C$ s.t. $V \subseteq U$. However, V is unbounded since $V = V(x^2 - y^2) = \{(z, y) : z^2 = y^2\}$ & so for any $z \in \mathbb{C}$ yet s.t. $(z, y) \in V$, in particular we can choose infinitely many $(z, y) \in V$ & so there does not exist an open ball which wholly contains V .



Thus, \exists infinitely many $(z, y) \in V$ s.t. $(z, y) \notin C$, & so $V \notin C$, i.e. V is not compact in the Euclidean topology.

Why is this sufficient?

Q3. (a) (5 marks) Find a curve $W \subseteq \mathbb{A}^2$ and a morphism $\varphi : \mathbb{A}^2 \rightarrow \mathbb{A}^2$, such that W is irreducible but $\varphi^{-1}(W)$ is not. $W = V(y-x^2)$ also works!

- (b) (5 marks) Let Y be a topological space and consider $X \subseteq Y$ with the subspace topology. Prove that if X is irreducible then so is its closure.

- (c) (5 marks) Prove that isomorphisms preserve irreducibility and dimension of closed affine algebraic varieties.

- (d) (10 marks) Find the irreducible components of $V(zx - y, y^2 - x^2(x+1)) \subseteq \mathbb{A}^3$. You need to justify why each component is irreducible.

(e) Let $W = V(x-y)$ & let $\varphi : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ be given by $(x, y) \mapsto (x^2, y^2)$. Then W is clearly irreducible but $\varphi^{-1}(W)$ is not, since φ is not injective, & thus,

$$\varphi^{-1}(V(x-y)) = V(x^2 - y^2) = V(xy) \cup V(x-y).$$

Therefore, $\varphi^{-1}(W)$ is irreducible.

(f) Let Y be a topological space & $X \subseteq Y$ be endowed with the subspace topology. Let X be irreducible. Assume for contradiction that \bar{X} is reducible, i.e. $\bar{X} = X_1 \cup X_2$ for $X_1, X_2 \subseteq Y$ closed sets. However, we have that

$$X = X \cap \bar{X}$$

$$= X \cap (X_1 \cup X_2)$$

$$= (X \cap X_1) \cup (X \cap X_2)$$

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Q1 V is irreducible. Assume for contradiction that V is reducible, i.e. $V = X_1 \cup X_2$ for $X_1, X_2 \subseteq A^n$ closed sets. However we have that

$$\begin{aligned} V &= X_1 \bar{X}_2 \\ &= X_1 \cap (X_1 \cup X_2) \\ &= (X_1 \cap X_1) \cup (X_1 \cap X_2) \end{aligned}$$

which contradicts V being irreducible unless $X_1 \cap X_2 = \emptyset$ & in this case $\bar{X} = X$ which is another contradiction. Therefore, if V is irreducible then $\bar{V} \neq V$. ✓

what if $V \subseteq \psi^{-1}(W)$?

- 3/5 (c) Let $\psi: A^n \rightarrow A^m$ be an isomorphism & suppose that $V \subseteq A^n$ is an irreducible closed affine algebraic variety. Suppose for contradiction that $\psi(V)$ is reducible i.e. $\psi(V) = W_1 \cup W_2$ for W_1, W_2 closed proper subsets of A^m . But since ψ is an isomorphism, we have that $\psi^{-1}(W_1 \cup W_2) = \psi^{-1}(W_1) \cup \psi^{-1}(W_2) \Rightarrow V = \psi^{-1}(W_1) \cup \psi^{-1}(W_2)$ for $\psi^{-1}(W_1), \psi^{-1}(W_2)$ closed proper subsets of A^n , which is a contradiction. Thus, $\psi(V)$ is irreducible, & hence isomorphisms preserve irreducibility.

Let $V \subseteq A^n$ be a c.a.v.s.t. $\dim(V) = n$, i.e. $V \supseteq V_{n-2} \supseteq \dots \supseteq V_0$. Then, since ψ (as described above) is an isomorphism, $\psi(W)$ is a c.a.v.s.t. for any c.a.v.s.t. $W \subseteq A^m$. So, if $V \subseteq V_j$ then $\psi(V) \subseteq \psi(V_j)$, since if $v \in V$, then $\psi(v) \in \psi(V) \subseteq \psi(\psi^{-1}(\psi(V)))$. Therefore, isomorphisms preserve closure & dimension, e.g. if $V \supseteq V_{n-2} \supseteq \dots \supseteq V_0$, then $\psi(V) \supseteq \psi(V_n) \supseteq \dots \supseteq \psi(V_0)$, & in particular if $\dim(V) = n$, then $\dim(\psi(V)) = n$.

- (d) Let $V = V(zx-y, y^2-x^2(x+1))$, then $y = zx$ for all pts V . So we have that

$$\begin{aligned} V(zx-y, y^2-x^2(x+1)) &= V(zx-y, (zx)^2-x^2(x+1)) \\ &= V(zx-y, x^2(z^2-(x+1))) \\ &= V(zx-y, z^2) \cup V(zx-y, z^2-(x+1)) \end{aligned}$$

Thus, the irreducible components of V are $V_1 = V(zx-y, z^2)$ & $V_2 = V(zx-y, z^2-(x+1))$.

To see that these components are irreducible, we construct the following isomorphisms:

$$\varphi_1: A^1 \rightarrow V_1, \quad \varphi_2: A^1 \rightarrow V_2$$

$$s \mapsto (0, s) \quad t \mapsto (t, \sqrt{t+1}, \sqrt{t+1}).$$

Since, by (c), isomorphisms preserve irreducibility & A^1 is irreducible, it follows that V_1 & V_2 are irreducible.

Note that φ_1 & φ_2 are indeed isomorphisms since there exists morphism

$$\psi_1: V_1 \rightarrow A^1, \quad \psi_2: V_2 \longrightarrow A^1$$

$$(0, s) \mapsto s \quad (t, \sqrt{t+1}, \sqrt{t+1}) \mapsto t$$

s.t. $\psi_1^{-1} \circ \varphi_1 = \varphi_1 \circ \psi_1^{-1}$ & $\psi_2^{-1} \circ \varphi_2 = \varphi_2 \circ \psi_2^{-1}$ are both identity maps on V .

- 2/25 Q4. (a) (10 marks) Let $V \subseteq A^n$ be a Zariski-closed subset and $a \in A^n \setminus V$ be a point. Find a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ such that

$$f \in I(V), \quad f(a) = 1.$$

- (b) (15 marks) Let $I, (g) \subseteq \mathbb{C}[x_1, \dots, x_n]$ be two ideals. Assume that $V(g) \supseteq V(I)$.

- (i) Prove that if $I = (f_1, \dots, f_k)$, then

$$(f_1, \dots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}]. \quad (1)$$

- (ii) By only using Equation (1) and not the nullstellensatz, prove that there exists a positive integer m such that $g^m \in I$.

- (c) To find $J \in I(V)$ s.t. $J(a) = 1$, we want $J(a) = 0 \vee V \subseteq V$, so

$$\text{we can take } J(a) = \frac{(x-v_1)(x-v_2) \cdots (x-v_n)}{(x-a)(x-a)} = \prod_{i \neq a} \frac{(x-v_i)}{(x-a)}, \text{ for } x = (x_1, x_2, \dots, x_n).$$

This holds since $a \notin V$ & so $a - v_i \neq 0$ for any $v_i \in V$, & clearly

$$J(a) = \prod_{i \neq a} \frac{(x-v_i)}{(x-a)} = 1.$$

- (d) Let $I, (g) \subseteq \mathbb{C}[x_1, \dots, x_n]$ be two ideals such that $V(I) \subseteq V(g)$. Suppose $I = (f_1, \dots, f_k)$.

- (e) Let $J = (f_1, \dots, f_k, x_{n+1}g - 1)$. Then clearly $J \subseteq \mathbb{C}[x_1, \dots, x_{n+1}]$.

To show that $\mathbb{C}[x_1, \dots, x_n] \subseteq J$, we want to write any $q \in \mathbb{C}[x_1, \dots, x_n]$ in terms of the generators of J , that is for $p_i \in \mathbb{C}[x_1, \dots, x_{n+1}]$

$$q = p_1 f_1 + \dots + p_k f_k + p_{n+1} (x_{n+1}g - 1).$$

Since $V(I) \subseteq V(g)$, we have that f_1, \dots, f_k & g all vanish at the same time. Therefore, in the case they all vanish, we have $q = p_{n+1}$, & thus $q \in J$.

Since any polynomial that vanishes on $V(I)$ can be written as a linear combination of f_1, \dots, f_k, g , we can always find a polynomial that makes $p_1 f_1 + \dots + p_k f_k$ vanish,

& when all f_i vanish so does g . Therefore, we can always find $q = p_{n+1} \in J$, i.e.

any polynomial $q \in \mathbb{C}[x_1, \dots, x_n]$ is contained in J , & thus $\mathbb{C}[x_1, \dots, x_{n+1}] \subseteq J$ & so $(f_1, \dots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}]$.

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- (e) Since $(f_1, \dots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}]$

$$= (1)$$

we can write

$$p_1 f_1 + \dots + p_k f_k + p_{n+1} (x_{n+1}g - 1) = 1$$

for any $p_i \in \mathbb{C}[x_1, \dots, x_{n+1}]$. If we take $p_{n+1} = -1$, then we get

$$p_1 f_1 + \dots + p_k f_k - x_{n+1}g + 1 = 1$$

$$\Rightarrow p_1 f_1 + \dots + p_k f_k = x_{n+1}g.$$

Then, g can be written as a linear combination of f_i , & hence $g \in I$.

Attempt both directions for (a) & the 3rd for (b). ✓

Q5. Prove at least one implication from each of the following equivalences.

- (a) (10 marks) Show that the pullback $\varphi^*: \mathbb{C}[W] \rightarrow \mathbb{C}[V]$ is injective if and only if φ is dominant. Recall that a map, φ , is called dominant if its image, $\varphi(V)$, is dense in W .

- (b) (10 marks) Prove that the pullback $\varphi^*: \mathbb{C}[W] \rightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W .

Suppose that $V \subseteq A^n, W \subseteq A^m$ are two c.a.v.s, & let

$\varphi: V \rightarrow W$ be a morphism.

- (a) First, we note that if $\varphi(V)$ is dense in W , then it has nonempty intersection with all open sets of W .

Suppose that $\varphi(V) \subseteq W$ is dense, then for any open set $U \subseteq W$,

$\varphi(V) \cap U \neq \emptyset$. In particular the open set $U = (x: f(x)) + \mathbb{C}^k$ where

$f \in \mathbb{C}[W]$, has nonempty intersection with $\varphi(V)$. Therefore $x \in \varphi(V) \cap U$,

we have that $(\varphi^*(f))(x) \neq 0$, & thus $\varphi^*(f) \neq 0$. But this in turn

implies that for $f^* = 0$, since f was arbitrary taken from $\mathbb{C}[W]$, &

hence φ^* is injective.

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Suppose that $\varphi(V) \leq W$ is not dense. Then there exists some open subset U such that $\varphi(V) \cap U = \emptyset$. In particular we have that $\varphi(V) \subseteq U^c$. Since $U^c \subseteq W$, it follows that there exists some nonzero $y \in C(W)$ such that U^c is contained in some set of the form $V(y)$. Therefore, we must have that $\varphi^*(y) \geq 0$, & so φ^* must not be negative.

Thus we have proved the contrapositive & so we have that φ^* is negative then $\varphi(V)$ is dense in W .

b) Suppose that $\varphi: V \rightarrow X$ is an epimorphism where $X \leq W$ is a subobject of W . Then there exists an epicoreciprocal map $\psi: X \rightarrow W$ & we have the map $\varphi \circ \psi: V \rightarrow X \rightarrow W$. The composition of maps has pullback $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*: C(W) \rightarrow C(V) \rightarrow C(X)$, & since φ is an epimorphism, φ^* is an epimorphism. Thus $\psi^* \circ \varphi^*: C(W) \rightarrow C(X)$ is surjective, then the pullback $\varphi^* \circ \psi^*$ is surjective. But φ^* is clearly surjective since φ is epicoreciprocal & $X \leq W$. Therefore, the pullback $\varphi^* \circ \psi^*: C(W) \rightarrow C(V)$ is surjective.