Algebraic Geometry

Farhad Babaee

University of Bristol

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Plan for today

- Finish the proof about homogenisation
- Some important theorems in Algebraic Geometry
- Regular functions

Theorem

Let $V \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ *be a closed affine algebraic variety, and* $I := \mathbb{I}(V) \subseteq \mathbb{C}[x_1, \ldots, x_n]$. *Define the homogenised ideal*

$$\tilde{I} = \{\tilde{f} \in \mathbb{C}[x_0, \ldots, x_n] : f \in I\}.$$

Then,

$$\overline{V} = \mathbb{V}(\tilde{I}) \subseteq \mathbb{P}^n.$$

Proof.

- If $f \in \mathbb{I}(V)$ then $\overline{V} \subseteq \mathbb{V}(\tilde{f})$.

- If $G \in \tilde{I}$, then $g := G(1, x_1, \ldots, x_n) \in I$ (Why?).
  Do we have $\tilde{g} = G$?

□

Example

The twisted cubic is given by $C = \mathbb{V}(y - x^2, z - xy)$. $C \subseteq \mathbb{A}^3$ can be paramertrised by $\mathbb{A}^1 \ni t \longrightarrow (t, t^2, t^3) \in \mathbb{A}^3$. Homogenisation of the generators of this ideal are $wy - x^2$ and $wz - xy$.

- Check that $\mathbb{V}(wy - x^2) \cap \mathbb{V}(wz - xy) \supseteq \{[x : y : z : w] \in \mathbb{P}^3 : w = x = 0\}$.

  This shows that

Morphisms of Projective Varieties

Definition

Let $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^n$ be projective algebraic varieties. We say that the map $\varphi : V \longrightarrow W$ is a *morphism of projective varieties* if for each $p \in V$, there exist

(a) an open subset $U \subseteq V$ with $p \in U$;
(b) homogeneous polynomials $\varphi_0, \ldots, \varphi_m : U \longrightarrow W$ of the same degree,

such that $\varphi_{|_U} = [\varphi_0 : \cdots : \varphi_m]$.

- (Exercise 3.28) Prove that $\mathbb{V}(y) \subseteq \mathbb{A}^2$ and $\mathbb{V}(y - x^3) \subseteq \mathbb{A}^2$ are isomorphic, but their projective closures are not.

Why do we care about Projective Varieties?

Theorem (Chow Lemma)

Assume that $X \subseteq \mathbb{P}^n$ is an analytic subvariety of $\mathbb{P}^n$, that is, X is locally given by an analytic equation. Then $X \subseteq \mathbb{P}^n$ is algebraic.

Why do we care about Projective Varieties?

Theorem (Chow Lemma)

Assume that $X \subseteq \mathbb{P}^n$ *is an analytic subvariety of* $\mathbb{P}^n$, *that is,* X *is locally given by an analytic equation. Then* $X \subseteq \mathbb{P}^n$ *is algebraic.*

Theorem (Bézout Theorem)

Let $f_1, f_2 \in \mathbb{C}[x_0, x_1, x_2]$ *two homogeneous polynomials of degree* d_1 *and* d_2, *respectively. Let* $Z_1 = \mathbb{V}(f_1) \subseteq \mathbb{P}^2$ *and* $Z_2 = \mathbb{V}(f_2) \subseteq \mathbb{P}^2$, *be the projective curves associated to* f_1 *and* f_2. *Then, the number of intersection points of* Z_1 *and* Z_2 *counted with multiplicity is given by* $d_1 d_2$.

Definition

(a) The dimension of an irreducible projective variety is the dimension of any affine open subsets.

(b) The *degree* of an irreducible projective variety $Y \subseteq \mathbb{P}^n$ is the number of intersection points (counted with multiplicity) of V with any linear subvariety $L \subseteq \mathbb{P}^n$ such that $\dim(L) + \dim(Y) = n$.

Quasi-Affine and quasi-projective varieties

Definition

(a) Any open subset of an affine algebraic variety is called a *quasi-affine variety*.

(b) Any open subset of a projective variety is called a *quasi-projective variety*.

A Basis for Zariski Topology of Affine Varieties

Recall that a basis for a topology is a collection $\mathcal{B}$ of open subsets of a topological space X such that every open set U in X can be written as a union of elements from $\mathcal{B}$. Note that for any polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, the set

$$D(f) := \mathbb{A}^n \setminus \mathbb{V}(f),$$

is an open subset in $\mathbb{A}^n$.

Claim 1. The collection of open sets $D(f)$ for $f \in \mathbb{C}[x_1, \ldots, x_n]$ forms a basis for Zariski on $\mathbb{A}^n$.

Claim 2. If $V \subseteq \mathbb{A}^n$, is a c.a.a.v. then the open sets of the form $D(g) = V \setminus \mathbb{V}(g)$, where $g \in \mathbb{C}[V]$ form a basis for the Zariski topology on V.

Proof of claim 2

Proof.

- Any open set is of the form $V \setminus \mathbb{V}(J)$ for some $J \subseteq \mathbb{C}[V]$.
- We can find $g_1, \ldots, g_\ell \in \mathbb{C}[V]$ such that $J = (g_1, \ldots, g_\ell)$.

Observe. $\big(V \setminus \mathbb{V}(g_1)\big) \cup \big(V \setminus \mathbb{V}(g_2)\big) = \ldots\ldots$

- Use induction.

Regular functions

Definition

Let $V \subseteq \mathbb{A}^n$, a (closed) affine algebraic variety, and $U \subseteq V$ open. A function $f : U \longrightarrow \mathbb{C}$, is called *regular at a point* $p \in V$, if there is an open neighbourhood $U' \subseteq U$, and polynomials $g, h \in \mathbb{C}[x_1, \ldots, x_n]$, such that $h(p) \neq 0$, for any $p \in U'$, and $f_{|_{U'}}(p) = \frac{g(p)}{h(p)}$. We say that f is *regular* on U if it is regular at every point of U. The set of regular functions on $U \subseteq V$ is denoted by $\mathcal{O}_V(U)$.

Examples of regular functions

(a) The function

$$f_1 : \mathbb{A}^1 \setminus \{0,1\} \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{(z-2)(z-3)}{(z-1)}$$

is a in $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1 \setminus \{0,1\})$.

(b) Let $f_2 : \mathbb{A}^1 \longrightarrow \mathbb{C}$,

$$f_2(z) = \begin{cases} \frac{(z-1)(z-3)}{(z-1)} & z \in \mathbb{A}^1 \setminus \{1\} \\ \frac{(z-2)(z-3)}{(z-2)} & z \in \mathbb{A}^1 \setminus \{2\} \end{cases} .$$

Then $f_2 \in$ ...... We can see that the values of $f_2(z)$ coincides with .... $\in \mathbb{C}[\mathbb{A}^1]$.

(c) Let $g = xy - 1 \in \mathbb{C}[x,y]$. Give two examples of a regular function on $\mathcal{O}_{\mathbb{V}(g)}(\mathbb{V})(g)$ and a non-example.

Lemma

A regular function $f \in \mathcal{O}_V(U)$ is continuous when $\mathbb{C}$ is identified with $\mathbb{A}^1$.

Proof.

- It suffices to show that $f^{-1}(a)$ is closed, for $a \in \mathbb{A}^1$, because

- For every point $p \in U$ there exists U_p such that .....

- A set V is closed, if and only if, $V \cap U_i$ is closed in U_i, where $\bigcup U_i$ is an open cover for V.

$\square$

Two remarks

Example

The twisted cubic is given by $C = \mathbb{V}(y - x^2, z - xy)$. $C \subseteq \mathbb{A}^3$ can be paramertrised by $\mathbb{A}^1 \ni t \longrightarrow (t, t^2, t^3) \in \mathbb{A}^3$. Homogenisation of the generators of this ideal are $wy - x^2$ and $wz - xy$.

- Check that $\mathbb{V}(wy - x^2) \cap \mathbb{V}(wz - xy) = \mathbb{V}(xz - y^2) \cap \mathbb{V}(z(yw - z^2) - w(xw - yz)) \cup \{[x : y : z : w] \in \mathbb{P}^3 : w = x = 0\}$. This shows that $\mathbb{V}(wy - x^2) \cap \mathbb{V}(wz - xy) \neq \overline{C}$.

Remark

Homogenisation of an ideal is the ideal generated by the homogenisation of its elements.

Regular functions on a closed affine algebraic variety

Theorem

Let V be an irreducible Zariski closed subset of $\mathbb{A}^n$. Then

$$\mathcal{O}_V(V) = \mathbb{C}[V].$$

Proof.

- $\mathcal{O}_V(V) \supseteq \mathbb{C}[V]$.

- $\mathcal{O}_V(V) \subseteq \mathbb{C}[V]$.
  - Let $g \in \mathcal{O}_V(V)$. By definition every point $p \in V$ has a neighbourhood U_p such that on $g_{|U_p} = \frac{h}{k}$ where $h, k \in \mathbb{C}[V]$ and k does not vanish on U_p.
  - By making U_p possibly smaller, we can assume that U_p is of the form $D(f)$.
  - We can do this for every $p \in V$, and cover it with open sets, but V is compact with respect to the Zariski topology. We deduce that ....

- On these finitely many open sets, we can write f as.....

- The $\bigcap \mathbb{V}(k_i) = \varnothing$, therefore by Nullstellensatz....

- On $D(f_i) \cap D(f_j)$, we have $g = \ldots\ldots\ldots\ldots\ldots$, therefore on .... on entire V.

- On $D(f_1)$, we have $g = g \cdot 1$.

- If g, G are two regular functions and $g = G$ in $D(f_1)$, then...

Definition

Let $Y \subseteq \mathbb{P}^n$, a projective algebraic variety, and $U \subseteq Y$ open. A function $f : U \longrightarrow \mathbb{C}$, is called *regular at a point* $p \in Y$, if there is an open neighbourhood $U' \subseteq U$, and homogeneous polynomials $g, h \in \mathbb{C}[x_1, \ldots, x_n]$, of the same degree, such that $h(p) \neq 0$, for any $p \in U'$, and $f_{|_{U'}}(p) = \frac{g(p)}{h(p)}$. We say that f is *regular* on U if it is regular at every point of U. The set of regular functions on $U \subseteq Y$ is denoted by $\mathcal{O}_Y(U)$.

Definition

Let X, Y be two algebraic varieties (*i.e.*, affine, quasi-affine, projective or quasi-projective). A morphism $\varphi : X \longrightarrow Y$, a map such that

(a) φ is continuous;

(b) For any for every open set $U \subseteq Y$, and for every regular function $f \in \mathcal{O}_Y(U)$, $\varphi^*(f) = f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(U))$.

Theorem

Let X be an algebraic variety, $Y \subseteq \mathbb{A}^n$ a closed affine algebraic variety, and $\varphi : X \longrightarrow Y$ a map of sets. Then, $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a morphism, if and only if, for all i, $\varphi_i \in \mathcal{O}_X(X)$.

Question

How do you compare this with the isormorphisms between closed affine algebraic varieties?

Global regular functions on projective varieties

Theorem

Let Y be an irreducible Zariski closed subset of $\mathbb{P}^n$. Then

$$\mathcal{O}_Y(Y) = \mathbb{C}.$$

Example

Let $V = \mathbb{V}(xy - 1) \subseteq \mathbb{A}^2$, and $D(x) = \mathbb{A}^1 \setminus \{0\}$. By definition the map

$$\begin{aligned} \psi : V &\longrightarrow D(x) \\ (x, y) &\longmapsto x, \end{aligned}$$

- ψ is an isomorphism.
- $\mathcal{O}_V(D(x)) = ...$, since

- Any open subset $D(f) \subseteq \mathbb{A}^n$ is isomorphic to a closed subset of $\mathbb{A}^{n+1}$.

- Any open subset $D(f) \subseteq V = \mathbb{V}(g_1, \ldots, g_\ell)$ is isomorphic to .... $\subseteq \mathbb{A}^{n+1}$.

Obtaining $\mathbb{P}^1$ with gluing

We can construct $\mathbb{P}^1$ by gluing two copies of $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\}$, by the map $x \longmapsto x^{-1}$. We have,

- $\xi_0 : U_0 \longrightarrow X_0 := \xi_0(U_0)$, $\xi_1 : U_1 \longrightarrow X_1 := \xi_1(U_1)$, are isomorphism. (why?)
- $X_{01} := \xi_0(U_0 \cap U_1) \subseteq X_0$.
- $X_{10} := \xi_1(U_1 \cap U_0) \subseteq X_1$.
- $g_{01} := \xi_1 \circ \xi_0^{-1} : X_{01} \longrightarrow X_{10}, \quad x \longmapsto y = x^{-1}$.

Note that all these sets are open subsets of $\mathbb{P}^1$ and isomorphic to closed affine algebraic varieties. We have

- $\mathbb{C}[X_0] = \mathcal{O}_{X_0}(X_0) = \mathbb{C}[x]$,
- $\mathbb{C}[X_1] =$ ....
- $\mathbb{C}[X_{01}] = \mathcal{O}_{X_0}(X_{01}) = \frac{\mathbb{C}[x,x']}{(xx'-1)} \simeq$ .... $\supseteq \mathbb{C}[x]$.
- $\mathbb{C}[X_{10}] = \mathcal{O}_{X_1}(X_{10}) =$ .... $\simeq \mathbb{C}[y, y^{-1}] \supseteq \mathbb{C}[y]$.

We have now the isomorphism of $\mathbb{C}$-algebras induced by φ :

$$\begin{aligned} g_{01}^{*} : \mathbb{C}[X_{10}] &\longrightarrow \mathbb{C}[X_{01}] \\ f &\longmapsto f \circ g_{01} = f(y^{-1}) \\ y &\longmapsto x = y^{-1}. \end{aligned}$$

Therefore, we can also think of $\mathbb{P}^1$ as $X_0 \simeq \mathbb{A}^1$ and $X_1 \simeq \mathbb{A}^1$, where X_{01} and X_{10} are glued by the isomorphism g_{01}.

Let $[x_0 : x_1 : x_2]$ denote the homogeneous coordinates of the space $\mathbb{P}^2$. It is covered by three coordinate charts:

- U_0 corresponding to $x_0 \neq 0$, with affine coordinates $(\frac{x_1}{x_0}, \frac{x_2}{x_0}) = (a_1, a_2)$.
- U_1 corresponding to $x_1 \neq 0$, with affine coordinates $(\frac{x_0}{x_1}, \frac{x_2}{x_1}) = (a_1^{-1}, \ldots\ldots\ldots)$.
- U_2 corresponding to $x_2 \neq 0$, with affine coordinates $(\frac{x_0}{x_2}, \frac{x_1}{x_2}) = (\ldots\ldots\ldots, \ldots\ldots\ldots)$.

As before, let $X_i = \xi_i(U_i)$, and $X_{ij} = \xi_i(U_i \cap U_j)$. We have

- $\mathbb{C}[X_0] = \mathcal{O}_{X_0}(X_0) = \mathbb{C}[a_1, a_2]$,
- $\mathbb{C}[X_{01}] = \mathcal{O}_{X_0}(X_{01}) = \mathbb{C}[\ldots\ldots, \ldots\ldots, \ldots\ldots\ldots\ldots]$.

and Since on X_1, $a_1 \neq 0$, we can write

$$\mathbb{C}[X_1] = \mathcal{O}_{X_1}(X_1) = \mathbb{C}[a_1^{-1}, a_1^{-1}a_2].$$

As a result,

$$\mathbb{C}[X_{10}] = \mathcal{O}_{X_{10}}(X_{10}) = \mathbb{C}[\ldots\ldots\ldots, a_1^{-1}, a_1^{-1}a_2].$$

The isomorphism from $X_{01} \longrightarrow X_{10}$ by

$$(a_1, a_2) \longmapsto [1 : a_1 : a_2] \longmapsto (1/a_1, a_2/a_1),$$

provides the information for gluing of $X_{01} \simeq \mathbb{C}^* \times \mathbb{C}$ and $X_{10} \simeq X_{01} \simeq \mathbb{C}^* \times \mathbb{C}$ and their corresponding coordinate rings. We can similarly understand the isomorphisms between other charts.

Tangent spaces

Definition (Tangent space)

If $V = \mathbb{V}(I) = \mathbb{V}(f_1, \ldots, f_k) \subseteq \mathbb{A}^n$. For $a \in V$, we define the tangent space of V at a, denoted by T_aV, as

$$\begin{aligned}
T_aV &= \left\{ v \in \mathbb{A}^n : \forall i,\ \frac{\partial f_i}{\partial v}(a) = \left(\frac{d}{d\lambda} f_i(a+\lambda v)\right)_{|_{\lambda=0}} = 0 \right\} \\
&= \{ v \in \mathbb{A}^n : \forall f \in I, \lambda \longmapsto f(a+\lambda v) \text{ has order } \geq 2 \} \\
&= \left\{ v \in \mathbb{A}^n : \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1}(a) & \cdots & \frac{\partial f_k}{\partial x_n}(a) \end{pmatrix} v = (0, \ldots, 0) \in \mathbb{A}^k \right\} \\
&= \left\{ v \in \mathbb{A}^n : \begin{pmatrix} \nabla f_1(a) \\ \vdots \\ \nabla f_k(a) \end{pmatrix} v = (0, \ldots, 0) \in \mathbb{A}^k \right\}.
\end{aligned}$$

Example

- $V = \mathbb{V}(x^2 + y^2 - z^3) \subseteq \mathbb{A}^3$.
- $C = \mathbb{V}(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$.

Smoothness

Definition

Assume that $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, is a radical ideal. Choose the generators $I = (f_1, \ldots, f_k)$. Then the closed affine algebraic subvariety $V = \mathbb{V}(I) \subseteq \mathbb{A}^n$ is *smooth* of dimension d at $x \in V$, if

$$\dim(T_x V) = d.$$

Definition (Smooth Variety)

Let X be a variety (affine, quasi-affine, projective, quasi-projective). Then X is said to be *smooth of dimension d*, if for all $a \in X$, ... $X \supseteq U \ni a$, which is isomorphic to a smooth ....

Smooth + Connected $\implies$ Irreducible

We state the following without proof:

Theorem (Smoothness implies Irreducibilty)

If X is a connected smooth variety of dimension d, then X is irreducible and $\dim(X) = d$ *(as a topological space).*

Intrinsic definition

Let $A := \mathbb{C}[x_1, \ldots, x_n]$, $I = \mathbb{I}(V) \subseteq A$, a radical ideal, and for a point $a = (a_1, \ldots, a_n) \in V \subseteq \mathbb{A}^n$, denote by $\mathfrak{m}_a \subseteq A$, the ideal corresponding to $a \in \mathbb{A}^n$, and $\bar{\mathfrak{m}} := \bar{\mathfrak{m}}_a \subseteq \mathbb{C}[V] = \frac{A}{\mathbb{I}(V)} \simeq \mathcal{O}_V(V)$, the ideal corresponding to $a \in V$. For any vector space V, and subspace $W \subseteq V$, denote its dual subspace by $W^\vee$. We will show that

- $\mathfrak{m}_a/\mathfrak{m}_a^2$ as a $\mathbb{C}$-vector space, can be identified with the dual of $(T_a\mathbb{A}^n)$.
- $\bar{\mathfrak{m}}_a/\bar{\mathfrak{m}}_a^2$ as a $\mathbb{C}$-vector space can be identified with the dual of (T_aV).

Question

What is the (linear) dual of a vector space V?

Frame Title

Lemma

$\langle \cdot, \cdot \rangle$ is bilinear and induces a perfect pairing

$$\langle \cdot, \cdot \rangle : \mathfrak{m}_a/\mathfrak{m}_a^2 \times T_a\mathbb{A}^n \longrightarrow \mathbb{C},$$

of $\mathbb{C}$-vector spaces, i.e., each side can be identified with the dual of the other side.

Proof. By a translation, we can assume that $a = 0$.

(a) Note that $T_a\mathbb{A}^n = \mathbb{A}^n$ is as a $\mathbb{C}$-vector space.

(b) The dual $T_a(\mathbb{A}^n)^\vee$ is, by definition, the set of linear functions $f : T_a(\mathbb{A}^n) \longrightarrow \mathbb{C}$.

(c) $\langle \cdot, \cdot \rangle$ defines a linear map $\Psi : \mathfrak{m}_a \longrightarrow T_a(\mathbb{A}^n)^\vee$, given by

$$f \longrightarrow \langle f, \cdot \rangle.$$

Frame Title

$$\begin{aligned}\Psi(f) = \langle f, \cdot \rangle : T_a\mathbb{A}^n &\longrightarrow \mathbb{C}\\ v &\longmapsto \langle f, v \rangle = \frac{\partial f}{\partial v}(a).\end{aligned}$$

is linear.

(d) We have $\ker(\Psi) = \mathfrak{m}_a^2$. Since, we can write any polynomial, as its Taylor expansion at $a = 0$:
$f = f(a) + \sum b_i x_i +$ (higher degree terms). If $f \in \mathfrak{m}_a$
$f(a) =$ ....., and $\nabla f(a) =$ (...............), and $\mathfrak{m}_a^2$ ....

(e) The induced map $\overline{\Psi} : \mathfrak{m}_a/\mathfrak{m}_a^2 \longrightarrow T_a\mathbb{A}^n$ is injective. As a $\mathbb{C}$-vector space, $\mathfrak{m}_a/{\mathfrak{m}_a}^2$ is n-dimensional and is spanned by $\{x_1, \ldots, x_n\}$ hence

Blowing up

Goal: Start from a variety V which is possibly singular. Apply a procedure to obtain another variety which is isomorphic to V on an open dense subset but less singular.

Definition

A morphism $\pi : X \longrightarrow V$, of varieties is called a *birational morphism* if there are open dense subsets $A \subseteq X$ and $B \subseteq V$, such that $\pi_{|_A} : A \longrightarrow B$ is an isomorphism of algebraic varieties.

Example

- $y^2 - x^2(x-1) = 0$. Take every point $(x, y) \in \mathbb{A}^2$ to $(x, y, \frac{y}{x})$.
- What happens to $\mathbb{A}^2$ after applying this map?
- What is the variety $\mathbb{V}(y = ux) \subseteq \mathbb{A}^3$?

Examples continued

- $\mathbb{V}(y^2 - x^2(x-1))$ in $\mathbb{A}^3$?
- (Total transform) What is the intersection $\mathbb{V}(y^2 - x^2(x-1)) \cap \mathbb{V}(y = ux) \subseteq \mathbb{A}^3$?
- (Blow up) What is the intersection $\mathbb{V}(y^2 - x^2(x-1)) \cap \mathbb{V}(y = ux) \subseteq \mathbb{A}^3$ after we remove the u axis?