Algebraic Geometry: Assessed Homework 1

Matthew Byrne

February 2025

Question 1. Let $A \subseteq \mathbb{A}^n$.

(a) What is the definition of the closure of A in $\mathbb{A}^n$?

(b) Prove that $\mathbb{V}(\mathbb{I}(A)) = \overline{A}$ in $\mathbb{A}^n$.

(c) Give an example of a subset $B \subseteq \mathbb{A}^1$ whose closure in $\mathbb{A}^1$ does not coincide with its closure as a subset of $\mathbb{C}$.

Solution. (a) The closure of A in $\mathbb{A}^n$ is the intersection of all of the closed subsets of $\mathbb{A}^n$ containing A, which is the smallest closed subset containing A.

(b) We know both that $\mathbb{I}(A)$ is the set of all functions that vanish on A, and that the hypersurfaces $\{\mathbb{V}(g) \mid g \in \mathbb{C}[X_1, \dots, X_n]\}$ form a basis for the closed sets of the Zariski topology. Then

$$\overline{A} = \bigcap_{\substack{V \supseteq A \\ \text{closed}}} V = \bigcap_{\substack{f \in \mathbb{C}[\underline{X}] \\ f(A)=\{0\}}} \mathbb{V}(f) = \bigcap_{f \in \mathbb{I}(A)} \mathbb{V}(f) = \mathbb{V}\left(\bigcup_{f \in \mathbb{I}(A)} \{f\} \right) = \mathbb{V}(\mathbb{I}(A)).$$

(c) Consider the Euclidean open unit disc $B = \{z \in \mathbb{C} \mid z\overline{z} < 1\}$. Then the closure of B in the Euclidean topology is the closed unit disc $\widetilde{B} = \{z \in \mathbb{C} \mid z\overline{z} \leq 1\}$. Now consider the closure of B in the Zariski topology. We know this to be $\mathbb{V}(\mathbb{I}(B))$, and so suppose that $f \in \mathbb{I}(B)$. Then f is zero on all of the open unit disc B, hence f has infinitely many roots. Since every non-constant polynomial $f \in \mathbb{C}[X_1, \dots, X_n]$ has at most $\deg(f) < \infty$ many roots, we know that f must be constant. That is, $f = 0$. Hence $\mathbb{I}(B) = \{0\}$, so $\overline{B} = \mathbb{V}(\mathbb{I}(B)) = \mathbb{V}(0) = \mathbb{A}^1$. □

Question 2.

(a) What is the definition of a compact subset of a topological space?

(b) Prove that $\mathbb{V}(X^2 - Y^3) \subseteq \mathbb{A}^2$ is compact, but it is not compact as a subset of $\mathbb{C}^2$.

Solution. (a) For X a topological space, a subset $K \subseteq X$ is compact if every open cover $\{U_\alpha\}_{\alpha \in I}$ of K admits a finite subcover $\{U_{\alpha_n} \mid 1 \leq n \leq k\}$. Equivalently, we can phrase this in terms of closed sets by taking complements of everything. That is, a subset $K \subseteq X$ is compact if every collection $\{C_\alpha\}_{\alpha \in I}$ such that $\bigcap_\alpha C_\alpha \subseteq X \setminus K$ admits a finite subset $\{C_{\alpha_n} \mid 1 \leq n \leq k\}$ such that $\bigcap_{n=1}^k C_{\alpha_n} \subseteq X \setminus K$.

(b) Consider $V = \mathbb{V}(X^2 - Y^3) \subseteq \mathbb{A}^2$, and suppose that $\{C_\alpha\}_{\alpha \in I}$ is a collection of closed sets so that $\bigcap_\alpha C_\alpha \subseteq \mathbb{A}^2 \setminus V$, which without loss of generality we can take to be a collection of hypersurfaces. Let $C_\alpha = \mathbb{V}(f_\alpha)$. Then

$$\bigcap_{\alpha \in I} C_\alpha = \bigcap_{\alpha \in I} \mathbb{V}(f_\alpha) = \mathbb{V}(\{f_\alpha\}_{\alpha \in I}).$$

By the ascending chain property of Noetherian rings (in particular $\mathbb{C}[X, Y]$ is Noetherian by the Hilbert Basis Theorem), there is a finite subset $\{f_m \mid 1 \leq m \leq k\} \subseteq \{f_\alpha\}_{\alpha \in I}$ so that $(\{f_\alpha\}_{\alpha \in I}) = (f_1, \dots, f_k)$, and thus

$$\bigcap_{m=1}^{k} \mathbb{V}(f_m) = \mathbb{V}(\{f_m \mid 1 \leq m \leq k\}) = \mathbb{V}(\{f_\alpha\}_{\alpha \in I}) = \bigcap_{\alpha \in I} C_\alpha \subseteq \mathbb{A}^2 \setminus V,$$

and so V is compact in the Zariski topology. Consider now V as a subset of $\mathbb{C}^2$, then in particular V is not bounded, since for every $\varepsilon > 1$ the point $(\varepsilon^3, \varepsilon^2) \in V$ and has norm $\sqrt{\varepsilon^6 + \varepsilon^4} > \sqrt{2\varepsilon^4} = \varepsilon^2\sqrt{2} > \varepsilon$. Since every compact subset of a metric space is bounded, V cannot be a compact subset of $\mathbb{C}^2$. □

Question 3.

(a) Find a curve $W \subseteq \mathbb{A}^n$ and a morphism $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ such that W is irreducible but $\varphi^{-1}(W)$ is not.

(b) Let Y be a topological space and consider $X \subseteq Y$ a subspace. Prove that if X is irreducible then so too is its closure.

(c) Prove that isomorphisms of varieties preserve irreducibility and dimension.

(d) Find the irreducible components of $V = \mathbb{V}(XZ - Y, Y^2 - X^2(X+1)) \subseteq \mathbb{A}^3$. You need to justify why each component is irreducible.

Solution. (a) Consider the curve $W = \mathbb{V}(X - Y) \subseteq \mathbb{A}^2$ and the morphism $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$; $(x, y) \mapsto (x^2, y^2)$. Then $\mathbb{C}[W] = \mathbb{C}[X, Y]/(X - Y) \cong \mathbb{C}[X]$ is an integral domain, so $(X - Y) = \mathbb{I}(W)$ is prime, and so W is irreducible. But note that $\varphi^{-1}(W) = \mathbb{V}(X + Y) \cup \mathbb{V}(X - Y)$, so $\varphi^{-1}(W)$ is reducible.

(b) We aim to prove the contrapositive. Suppose that $\overline{X}$ is reducible in Y. Then by definition there exists closed subsets $C_1, C_2 \subsetneq Y$ so that $\overline{X} = C_1 \cup C_2$. But then

$$X = \overline{X} \cap X = (C_1 \cup C_2) \cap X = (C_1 \cap X) \cup (C_2 \cap X),$$

by de Morgan's laws. Then by definition of the subspace topology on X, $C_i \cap X$ is closed in X for each $i = 1, 2$, and so X is reducible. Taking the contrapositive, if X is irreducible, then we must have that $\overline{X}$ is irreducible.

(c) We note first that an isomorphism $\varphi : V \xrightarrow{\sim} W$ induces an isomorphism $\varphi^* : \mathbb{C}[W] \xrightarrow{\sim} \mathbb{C}[V]$ of co-ordinate rings, and similarly any isomorphism $\Psi : \mathbb{C}[W] \xrightarrow{\sim} \mathbb{C}[V]$ admits an isomorphism $\psi : V \xrightarrow{\sim} W$ so that $\psi^* = \Psi$. Using this, we aim to translate both irreducibility and dimension into the language of co-ordinate rings.

We know that V is irreducible if and only if $\mathbb{I}(V)$ is prime, and this happens if and only if $\mathbb{C}[V] := \mathbb{C}[X_1, \dots, X_n]/\mathbb{I}(V)$ is an integral domain. Since isomorphisms of rings preserve zero divisors, we thus have that if $V \simeq W$ then $\mathbb{C}[V] \cong \mathbb{C}[W]$ and so

$$V \text{ irred.} \iff \mathbb{C}[V] \text{ an ID} \iff \mathbb{C}[W] \text{ an ID} \iff W \text{ irred.}$$

Similarly, since irreducible varieties correspond to prime ideals of $\mathbb{C}[X_1, \dots, X_n]$ and $\mathbb{I}(-)$ reverses inclusions, the correspondence theorem yields that the dimension $\dim(V)$ of V is exactly the Krull dimension $\dim(\mathbb{C}[V])$ of the co-ordinate ring $\mathbb{C}[V]$. So since isomorphisms of rings preserve Krull dimension, we have that if $V \simeq W$ then $\mathbb{C}[V] \cong \mathbb{C}[W]$ and so

$$\dim(V) = \dim(\mathbb{C}[V]) = \dim(\mathbb{C}[W]) = \dim(W).$$

(d) We note that for $(x, y, z) \in V$, $y = xz$ and so

$$y^2 - x^2(x+1) = 0 \iff (xz)^2 - x^2(x+1) = 0 \iff x^2(x - z^2 + 1) = 0.$$

Thus we have that

$$\begin{aligned} V = \mathbb{V}(XZ - Y, Y^2 - X^2(X+1)) &= \mathbb{V}(XZ - Y, X^2(X - Z^2 + 1)) \\ &= \mathbb{V}(XZ - Y) \cap \big(\mathbb{V}(X) \cup \mathbb{V}(X - Z^2 + 1)\big) \\ &= \mathbb{V}(X, XZ - Y) \cup \mathbb{V}(X - Z^2 + 1, XZ - Y) \\ &= \mathbb{V}(X, Y) \cup \mathbb{V}(X - Z^2 + 1, XZ - Y) \end{aligned}$$

We note that $\mathbb{C}[X, Y, Z]/(X, Y) \cong \mathbb{C}[Z]$ is an integral domain, so (X, Y) is prime, so $\mathbb{V}(X, Y)$ is irreducible. Similarly, consider the map

$$\begin{aligned} \mathbb{C}[X, Y, Z] &\xrightarrow{\alpha} \mathbb{C}[T] \\ X &\longmapsto T^2 - 1 \\ Y &\longmapsto T^3 - T \\ Z &\longmapsto T \end{aligned}$$

which is clearly surjective (change of variables $T \mapsto Z$ gives a section for α), and has kernel

$$\ker(\alpha) = (XZ - Y, X - Z^2 + 1)$$

by construction. So since $\mathbb{C}[X, Y, Z]/(XZ - Y, X - Z^2 + 1) \cong \mathbb{C}[T]$ is an integral domain, $(XZ - Y, X - Z^2 + 1)$ is prime, and so $\mathbb{V}(XZ - Y, X - Z^2 + 1)$ is irreducible. Hence V broken down into irreducibles as:

$$V = \mathbb{V}(X, Y) \cup \mathbb{V}(X - Z^2 + 1, XZ - Y). \qquad \square$$

Question 4.

(a) Let $V \subseteq \mathbb{A}^n$ be a Zariski closed subset and $a \in \mathbb{A}^n \setminus V$. Find a polynomial $f \in \mathbb{C}[X_1, \ldots, X_n]$ so that $f \in \mathbb{I}(V)$ and $f(a) = 1$.

(b) Let $I, (g) \subseteq \mathbb{C}[X_1, \ldots, X_n]$ be two ideals. Assume that $\mathbb{V}(g) \supseteq \mathbb{V}(I)$.

(i) Prove that if $I = (f_1, \ldots, f_k)$ then

$$(f_1, \ldots, f_k, X_{n+1}g - 1) = \mathbb{C}[X_1, \ldots, X_{n+1}].$$

(ii) By using only the preceding equation and not the Nullstellensatz, prove that there exists a positive integer m such that $g^m \in I$.

Solution. (a) We have shown in class that each such V is the intersection of finitely many hypersurfaces, i.e. there exists $g_1, \ldots, g_m \in \mathbb{C}[X_1, \ldots, X_n]$ so that $V = \mathbb{V}(g_1, \ldots, g_m)$. Since $a \notin V$, we thus have that $a \notin \mathbb{V}(g_1, \ldots, g_m)$ and so there is some $1 \leq j \leq m$ such that $g_j(a) \neq 0$. Let $\lambda = g_j(a) \in \mathbb{C}^\times$. Then

$$g_j \in (g_1, \ldots, g_m) = \mathbb{I}(V) \implies f := \lambda^{-1} g_j \in \mathbb{I}(V),$$

and we have that $f(a) = \lambda^{-1} g_j(a) = \lambda^{-1}\lambda = 1$.

(b) (i) Since $\mathbb{V}(g) \supseteq \mathbb{V}(I)$ and $I = (f_1, \ldots, f_k)$, we have that in $\mathbb{A}^n$

$$\mathbb{V}(g) \supseteq \mathbb{V}(f_1, \ldots, f_k) = \mathbb{V}(f_1) \cap \cdots \cap \mathbb{V}(f_k).$$

We now consider these as varieties in $\mathbb{A}^{n+1}$, for which the same inclusion applies. That is, if $a \in \mathbb{A}^{n+1}$ is such that $f_j(a) = 0$ for all $1 \leq j \leq k$, then $g(a) = 0$. Consider then $h(a)$ for such a and for $h = X_{n+1}g - 1$. Writing $a = (a_1, \ldots, a_{n+1})$, we have that

$$h(a) = a_{n+1}g(a) - 1 = -1 \neq 0,$$

and so if $f_j(a) = 0$ for all j, then $h(a) \neq 0$. So $\mathbb{V}(h) \cap \mathbb{V}(f_1, \cdots, f_k) = \varnothing$. So when $J = (f_1, \ldots, f_k, h) = I + (h)$, we thus have

$$\mathbb{V}(J) = \mathbb{V}\big((f_1, \ldots, f_k) + (h)\big) = \mathbb{V}(f_1, \ldots, f_k) + \mathbb{V}(h) = \varnothing.$$

Thus, by the Nullstellensatz

$$\sqrt{J} = \mathbb{I}(\mathbb{V}(J)) = \mathbb{I}(\varnothing) = \mathbb{C}[X_1, \ldots, X_{n+1}].$$

We know thus have that $1 \in \sqrt{J}$, and hence $1 \in J$. So for $R = \mathbb{C}[X_1, \ldots, X_{n+1}]$, we have that $R \subseteq RJ \subseteq J$, and hence $J = R$. That is,

$$(f_1, \ldots, f_k, X_{n+1}g - 1) = J = R = \mathbb{C}[X_1, \ldots, X_{n+1}],$$

as required.

(ii) Since $(f_1, \dots, f_k, X_{n+1}g - 1) = \mathbb{C}[X_1, \dots, X_{n+1}]$, it must contain 1, and hence there must be $r_j, s \in \mathbb{C}[X_1, \dots, X_{n+1}]$ so that

$$1 = r_1 f_1 + \cdots + r_k f_k + s(X_{n+1}g - 1),$$

and thus this equality holds for any $x = (x_1, \dots, x_{n+1}) \in \mathbb{A}^{n+1}$. In particular, consider all points of the form $a = (x_1, \dots, x_n, 1/g(x_1, \dots, x_n))$, and write $\overline{a} = (x_1, \dots, x_n) \in \mathbb{A}^n$ which can be thought of as an arbitrary point of $\mathbb{A}^n$. Evaluation at a thus gives

$$1 = \sum_{j=1}^{k} r_j(x_1, \dots, x_n) f_j(x_1, \dots, x_n) = \frac{\sum_j \widetilde{r_j}(x_1, \dots, x_n) f_j(x_1, \dots, x_n)}{g(x_1, \dots, x_n)^m},$$

for some polynomials $\widetilde{r_j} \in \mathbb{C}[X_1, \dots, X_n]$ and m the maximal degree of X_{n+1} in each of the $r_j f_j$. Thus for each $(x_1, \dots, x_n) \in \mathbb{A}^n$ we have that

$$1 = \frac{\sum_j \widetilde{r_j}(x_1, \dots, x_n) f_j(x_1, \dots, x_n)}{g(x_1, \dots, x_n)^m} \implies g(x_1, \dots, x_n)^m = \sum_{j=1}^{k} \widetilde{r_j}(x_1, \dots, x_n) f_j(x_1, \dots, x_n),$$

for all $(x_1, \dots, x_n) \in \mathbb{A}^n$. Thus we have an equality as polynomials $g^m = \sum_j \widetilde{r_j} f_j \in (f_1, \dots, f_k) =: I$, as required. □

Question 5. Prove at least one implication from each of the following equivalences.

(a) Show that the pullback $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is injective if and only if φ is dominant, that is, if and only if $\varphi(V)$ is dense in W.

(b) Prove that the pullback $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety $U \subseteq W$.

Solution. Let $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$, and $\varphi : V \to W$ a morphism, and write $R = \mathbb{C}[X_1, \dots, X_m]$ and $S = \mathbb{C}[X_1, \dots, X_n]$.

(a) By elementary ring theory, $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is injective if and only if $\ker(\varphi^*)$ is trivial. By definition of quotient rings, this $\ker(\varphi^*)$ happens exactly when the induced map $\psi : R \to \mathbb{C}[V]$ given by $f \mapsto f \circ \varphi$ has kernel $\mathbb{I}(W)$. But $\ker(\psi)$ is exactly the the inverse image of $\mathbb{I}(V)$ under the map of polynomial rings $\eta : R \to S$; $f \mapsto f \circ \varphi$. We thus have that φ^* is injective if and only if $\eta^{-1}(\mathbb{I}(V)) = \mathbb{I}(W)$. We note that

$$\eta^{-1}(\mathbb{I}(V)) = \{f \in R \mid f \circ \varphi(v) = 0, \forall v \in V\} = \{f \in R \mid f(w) = 0, \forall w \in \varphi(V)\} = \mathbb{I}(\varphi(V)),$$

and so it follows that φ^* is injective if and only if $\mathbb{I}(\varphi(V)) = \mathbb{I}(W)$. Applying the $\mathbb{V}(-)$ operator, $\mathbb{I}(\varphi(V)) = \mathbb{I}(W)$ if and only if $\mathbb{V}(\mathbb{I}(\varphi(V))) = W$. Finally, by a previous question $\mathbb{V}(\mathbb{I}(\varphi(V))) = \overline{\varphi(V)}$ is the closure in the Zariski topology, and so putting together our chain of equivalences gives that

$$\varphi^* \text{ inj.} \iff \ker(\varphi^*) = \{0\} \iff \mathbb{I}(\varphi(V)) = \mathbb{I}(W) \iff \overline{\varphi(V)} = W,$$

but the statement that the closure of $\varphi(V)$ is W is exactly the statement that $\varphi(V)$ is dense in W. So φ^* is injective if and only if φ is dominant.

(b) By the homomorphism theorem, we have that

$$\frac{\mathbb{C}[W]}{\ker(\varphi^*)} \cong \operatorname{im}(\varphi^*)$$

We now appeal to the correspondence theorem, which tells us that the preimage of $\ker(\varphi^*)$ under the quotient map $R \to \mathbb{C}[W]$ is an ideal $J \trianglelefteq R$ containing $\mathbb{I}(W)$, and that

$$\frac{R}{J} \cong \frac{\mathbb{C}[W]}{\ker(\varphi^*)} \cong \operatorname{im}(\varphi^*).$$

Since $\mathbb{I}(W) \subseteq J$, we have that $W = \mathbb{V}(\mathbb{I}(W)) \supseteq \mathbb{V}(J) =: U$. Since J is the preimage of $\mathbb{I}(V)$ under the map

$$\begin{aligned} R &\longrightarrow S \\ f &\longmapsto f \circ \varphi \end{aligned}$$

we have in particular that J is radical, hence $\mathbb{I}(U) = \mathbb{I}(\mathbb{V}(J)) = \sqrt{J} = J$ by the Nullstellensatz, hence we have that $\mathbb{C}[U] = R/J$. Therefore, if φ^* is surjective, we have isomorphisms

$$\mathbb{C}[U] = \frac{R}{J} \cong \frac{\mathbb{C}[W]}{\ker(\varphi^*)} \cong \operatorname{im}(\varphi^*) = \mathbb{C}[V],$$

and hence we obtain an isomorphism $V \simeq U$ of varieties. $\square$