UNIVERSITY OF BRISTOL

School of Mathematics

SOLUTIONS - Algebraic Geometry

MATHM0036

(Paper code MATHMATHM0036

May/June 2024 2 hour(s) 30 minutes

This paper contains two sections: Section A and Section B.
Each section should be answered in a separate booklet.

All FOUR answers will be used for assessment.

Calculators of an approved type (permissible for A-Level examinations) are permitted.

Candidates may bring ONE hand-written sheet of A4 notes, written double sided into the examination. Candidates must insert this sheet into their answer booklet(s) for collection at the end of the examination.

Do not turn over until instructed.

Q1. Assume that V is an affine algebraic variety, and $U, U_1, U_2 \subseteq V$ are two open subsets.

(a) (**15 marks) (Standard - Workbook - practicing definition**) State the definition of the set of regular functions $\mathcal{O}_V(U)$, and prove that $\mathcal{O}_V(U)$ is a $\mathbb{C}$-algebra.

Solution. The easiest way is to prove that $\mathcal{O}_V(U)$ includes $\mathbb{C}$ as a subring. Obviously, $\mathbb{C} \subseteq \mathcal{O}_V(U)$. We show that $\mathcal{O}_V(U)$ has a ring structure. Recall that function $f : U \longrightarrow \mathbb{C}$, is called *regular at a point* $p \in V$, if there is an open neighbourhood $U' \subseteq U$, and polynomials $g, h \in \mathbb{C}[x_1, \dots, x_n]$, such that $h(p) \neq 0$, for any $p \in U'$, and $f_{|_{U'}}(p) = \frac{g(p)}{h(p)}$. We say that f is *regular* on U if it is regular at every point of U. Therefore, it suffices to show that if f, k are regular at $p \in U$ then $f + k$ and fk are also regular at $p \in U$, this is also clear: assume that $U' and V' \subseteq U$ and $f_{|_{U'}}(p) = \frac{g(p)}{h(p)}$ and $k_{|_{V'}}(p) = \frac{g'(p)}{h'(p)}$ then on the open $U' \cap V'$, $f + k = \frac{gh' + hg'}{gg'}$ and $fg = \frac{gg'}{hh'}$.

(b) (**10 marks**) (**Workbook - practicing definition**) Assume further that $f_1 \in \mathcal{O}_V(U_1), f_2 \in \mathcal{O}_V(U_2)$, with $f_{1|_{U_1 \cap U_2}} = f_{2|_{U_1 \cap U_2}}$. Prove that there exists a regular function $f \in \mathcal{O}_V(U_1 \cup U_2)$ such that

$$f_{|_{U_1}} = f_1, \quad f_{|_{U_2}} = f_2.$$

Solution. This question means that the regular functions can be glued. Define the well-defined function f on $U_1 \cup U_2$ as given. It is clear that for $i = 1, 2$, on U_i $f = f_i$ is regular, since for any point $p \in U_1 \cup U_2$ so $f(p) = f_1(p)$ or $f(p) = f_2(p)$ which are regular by assumption.

Q2. (a) (**15 marks**)(**Standard - Unseen**) Let $U = \mathbb{A}^2 \setminus \{0\}$ and $X = \mathbb{A}^2$. Compute $\mathcal{O}_X(U)$, and show that U is not an affine algebraic variety.

Solution. We show that $\mathcal{O}_{A^2}(A^2 \setminus \{0\}) = \mathbb{C}[x_1, x_2]$, implying $\mathcal{O}_X(U) = \mathcal{O}_X(X)$; thus, every regular function on U extends to X. This is, in fact, rephrasing a result in complex analysis: the Removable Singularity Theorem, which ensures every holomorphic function on $\mathbb{C}^2 \setminus 0$ extends holomorphically to $\mathbb{C}^2$. Consider $f \in \mathcal{O}_X(U)$, therefore, f is regular on the two open subsets $D(x_1) = (\mathbb{A}^1 \setminus 0) \times \mathbb{A}^1$ and $D(x_2) = \mathbb{A}^1 \times (\mathbb{A}^1 \setminus 0)$. A discussion in the notes, we know that

$$\mathcal{O}_{\mathbb{A}^2}(D(x_1)) \simeq \mathbb{C}[x_1, x_1^{-1}, x_2], \quad \mathcal{O}_{\mathbb{A}^2}(D(x_2)) \simeq \mathbb{C}[x_1, x_2^{-1}, x_2].$$

If $f_{|D(x_1)} = p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$, is a polynomial, since f is a regular function and therefore continuous, then it has to be equal to $p(x_1, x_2)$ everywhere on $\mathbb{A}^2$ and we are done. Otherwise, by taking common denominators $f_{|D(x_1)} = \frac{g}{x_1^m}$, for $m \in \mathbb{Z}_{\geq 0}$, $g \in \mathbb{C}[x_1, x_2]$ and $x_1 \nmid g$. Similarly, $f_{|D(x_2)} = \frac{h}{x_2^n}$, for $n \in \mathbb{Z}_{\geq 0}$, $h \in \mathbb{C}[x_1, x_2]$ and $x_2 \nmid h$. Therefore on the intersection $D_{x_1} \cap D(x_2)$,

$$\frac{g}{x_1^m} = \frac{h}{x_2^n} \implies x_2^n g = h x_1^n.$$

Note that, $x_2^n g = h x_1^n$ is an equality of polynomials on $D_{x_1} \cap D(x_2)$, and polynomials are continuous therefore the equality also holds for on the closure $\overline{D(x_1) \cap D(x_2)} = \mathbb{A}^2$, but this is a contradiction, since $x_1 \nmid g, x_1 \nmid x_2$. As a result, f has to be a polynomial on $D(x_1)$ or $D(x_2)$ and we are done.

(b) (**10 marks**)(**Standard - Unseen**) Prove that $\mathbb{V}(y) \subseteq \mathbb{A}^2$ and $\mathbb{V}(y-x^2) \subseteq \mathbb{A}^2$ are isomorphic, but their corresponding projective closures in $\mathbb{P}^2$ are not.

Solution. The map $\mathbb{V}(y-x^2) \subseteq \mathbb{A}^2 \longrightarrow \mathbb{V}(y) = \mathbb{A}^1 \times \{0\}, (x,y) \mapsto x$ is an isomorphism with the inverse given by $t \mapsto (t, t^2)$. The projective closure of $\mathbb{V}(y)$, $\overline{V(y)}$ in $\mathbb{P}^2$ is given by the $\{[x:0:z] \in \mathbb{P}^2\}$ while the projective closure of $\mathbb{V}(y-x^2)$ is given by

$$\overline{\mathbb{V}(y-x^2)} = \{[x:y:z] \in \mathbb{P}^2 : yz - x^2\}.$$

On the chart U_x where $x = 1$, $\overline{\mathbb{V}(y-x^2)} \cap U_x$ is given by $yz = 1$. This set, however, is isomorphic to $\mathbb{A}^1 \setminus \{0\} \subseteq \mathbb{A}^1$ and cannot be isomorphic to $\mathbb{A}^1$ itself, since isomorphisms are homemorphisms too.

Q3. (a) (**10 marks**) (**Workbook - Unseen**) Consider the family of algebraic varieties, with parameter $t \in \mathbb{C}$, given by

$$V_t := \mathbb{V}(x^2 + y^2 - t) \subseteq \mathbb{A}^2.$$

Sketch the variety of V_0, V_1, and V_2 in $\mathbb{R}^2$. Determine which one of these three varieties is smooth. Briefly justify your answers.

Solution. Let $f_t = x^2 + y^2 - t$. $\nabla f_0 = \nabla f_1 = \nabla f_2 = (2x, 2y)$. Note that the kernel of ∇f_i is always one dimensional except at $(0,0)$. However, $(0,0)$ is in V_0 but not in V_1 nor V_2. Therefore, V_0 is not smooth, but V_1 and V_2 are.

(b) (**15 marks**) (**Standard - Seen**) Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be two closed affine algebraic varieties, and

$$\varphi : V \longrightarrow W$$

a morphism. Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W.

Solution.

" $\Longrightarrow$ ". We claim that $Z := \mathbb{V}(\ker(\varphi^*))$ is a closed affine algebraic subvariety of W isomorphic to V. Note that $\ker(\varphi^*) = \{g \in \mathbb{C}[W] : g \circ \varphi \in \mathbb{I}(V)\} = \{g \in \mathbb{C}[W] : g \circ \varphi(x) = 0, \text{ for all } x \in V\}$ which includes $\mathbb{I}(W)$. Since φ^* is a homomorphism of $\mathbb{C}$-algebras $\ker(\varphi^*)$ is an ideal, and

$$\mathbb{C}[W]/\ker(\varphi^*) \simeq \mathbb{C}[Z] \simeq \mathbb{C}[V] \implies Z \simeq W.$$

" $\Longleftarrow$ " Assume that φ induces an isomorphism $V \simeq \varphi(V)$. Note that isomorphism are closed maps, so $\varphi(V)$ is a closed affine algebraic variety. Therefore, φ^* is a $\mathbb{C}$-algebra isomorphism between $\mathbb{C}[\varphi(V)] \subseteq \mathbb{C}[W]$ and $\mathbb{C}[V]$.

Q4. Let Σ be the fan consisting of

- σ_1 cone spanned by $\{(-1,-1),(0,1)\}$;
- σ_2 cone spanned by $\{(0,1),(1,0)\}$;
- τ cone spanned by $\{(0,1)\}$.

(a) (**6 marks**)(**Standard - Workbook**) Determine whether or not the toric variety X_Σ has the following properties. Briefly justify your answer.

(i) smooth;
(ii) complete.

Solution.

(i) Yes, since the $\sigma_1 \cap \mathbb{Z}^2$ and $\sigma_2 \cap \mathbb{Z}^2$ both span $\mathbb{Z}^2$.
(ii) No, since $|\Sigma| \subsetneq \mathbb{R}^2$.

(b) (**9 marks**)(**Standard - Workbook**) Describe the coordinate rings of X_{σ_1}, X_{σ_2}, and X_τ.

Solution. We have $\sigma_1^\vee = \text{cone}(\{(-1,1),(-1,0)\})$. $\sigma_2^\vee = \text{cone}(\{(1,0),(0,1)\})$, $\tau^\vee = \text{cone}(\{(0,1),(-1,0),(1,0)\})$. Therefore $\mathbb{C}[X_{\sigma_2}] = \mathbb{C}[x,y]$, $\mathbb{C}[X_{\sigma_1}] = \mathbb{C}[x^{-1}y, x^{-1}]$, $\mathbb{C}[X_\tau] = \mathbb{C}[y,x,x^{-1}] = \mathbb{C}[yx^{-1},x,x^{-1}]$.

(c) (i) (**5 marks**)(**Standard - Workbook**) Explain why we have the inclusions $\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau]$, $\mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau]$;

(ii) (**5 marks**)(**Standard - Workbook**) Describe the gluing of X_{σ_1} and X_{σ_2} along X_τ.

Solution. Therefore, the equalities $\mathbb{C}[X_{\sigma_1}]_x = \mathbb{C}[X_\tau] = \mathbb{C}[X_{\sigma_2}]_{x^{-1}}$. These equalities give rise to the inclusions $X_\tau \subseteq X_{\sigma_1}$ and $X_\tau \subseteq X_{\sigma_2}$. We also have the isomorphisms of $\mathbb{C}$-algebras

$$\begin{aligned} \Phi : \mathbb{C}[X_{\sigma_1}] \supseteq \mathbb{C}[X_\tau] &\longrightarrow \mathbb{C}[X_\tau] \subseteq \mathbb{C}[X_{\sigma_2}] \\ x^{-1} &\longmapsto x \\ x^{-1}y &\longmapsto y. \end{aligned}$$

The map Φ provides the information for gluing the coordinate rings, as well as the corresponding varieties $X_\tau \subseteq X_{\sigma_1}$ and $X_\tau \subseteq X_{\sigma_2}$.

End of examination.