Problem sheet

Q1. Let $A \subseteq \mathbb{A}^n$ be a subset.

(a) What is the definition of the closure of A in $\mathbb{A}^n$?

(b) Prove that $\mathbb{V}(\mathbb{I}(A))$ equals the Zariski closure of A in $\mathbb{A}^n$.

(c) Give an example of a subset in $B \subseteq \mathbb{C}$ whose closure in the Zariski topology does not coincide with its closure in the Euclidean topology.

Q2. (a) What is the definition of a compact subset of a topological space?

(b) Prove that $\mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}^2$ is compact in the Zariski topology but not in the Euclidean topology.

Q3. (a) Find a curve $W \subseteq \mathbb{A}^2$ and a morphism $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$, such that W is irreducible but $\varphi^{-1}(W)$ is not.

(b) Let Y be a topological space and consider $X \subseteq Y$ with the subspace topology. Prove that if X is irreducible then so is its closure.

(c) Prove that isomorphisms preserve irreducibility and dimension of closed affine algebraic varieties.

(d) Find the irreducible components of $\mathbb{V}(zx - y, y^2 - x^2(x+1)) \subseteq \mathbb{A}^3$. You need to justify why each component is irreducible.

Q4. (a) Let $V \subseteq \mathbb{A}^n$ be a Zariski-closed subset and $a \in \mathbb{A}^n \setminus V$ be a point. Find a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that

$$f \in \mathbb{I}(V), \quad f(a) = 1.$$

(b) Let $I, (g) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be two ideals. Assume that $\mathbb{V}(g) \supseteq \mathbb{V}(I)$.

(i) Prove that if $I = (f_1, \ldots, f_k)$, then

$$(f_1, \ldots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \ldots, x_{n+1}]. \tag{1}$$

(ii) By only using Equation (1) and not the nullstellensatz, prove that there exists a positive integer m such that $g^m \in I$.

Remark. Using this exercise, we can prove nullstellensatz from a weaker version. Weak nullstellensatz asserts that if $V \neq \varnothing \iff \mathbb{I}(V) \neq (1)$.

Q5. Prove at least one implication from each of the following equivalences.

(a) Show that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is injective if and only if φ is *dominant*. Recall that a map, φ, is called dominant if its image, $\varphi(V)$, is dense in W.

(b) Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W.

Q6. (a) Find all the elements of $\operatorname{maxSpec}(\mathbb{C}[x])$ and $\operatorname{maxSpec}(\mathbb{C}[x, 1/x])$, respectively.

(b) Consider the isomorphism $\varphi : \mathbb{A}^1 \setminus \{0\} \longrightarrow \mathbb{A}^1 \setminus \{0\}$, $a \longmapsto b = 1/a$, and the pullback map on the coordinate rings $\varphi^* : \mathbb{C}[x, 1/x] \longmapsto \mathbb{C}[y, 1/y]$. Compute $\varphi^*(1/x)$, $\varphi^*(2x^2 + \frac{2x^3+4x}{x^5})$, $\varphi^*(2-x)$.

Q7. Consider the affine algebraic hypersurface $V := \mathbb{V}(y - ux) \subseteq \mathbb{A}^3$.

(a) Prove that the projection $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$, $(x, y, u) \longmapsto (x, u)$ restricts to an isomorphism between V and $\mathbb{A}^2$.

(b) Prove that the projection $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$, $(x, y, u) \longmapsto (x, y)$ does not restrict to isomorphism between V and $\mathbb{A}^2$.

Q8. Show that any radical ideal I in $\mathbb{C}[x_1, \dots, x_n]$ is the intersection of all maximal ideals containing I.

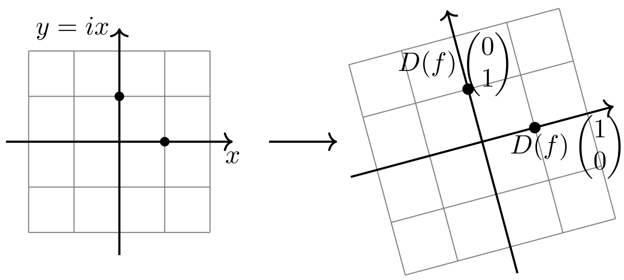
Q9. Let X be a topological space. Prove that $A \subseteq X$ is closed if and only if $A \cap U$ is closed in U (with respect to the subspace topology in U) for every open $U \subseteq X$.

Q10. (a) Derive the Cauchy–Riemann equations from the picture below and conformality.

(b) Write $\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ and $\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$

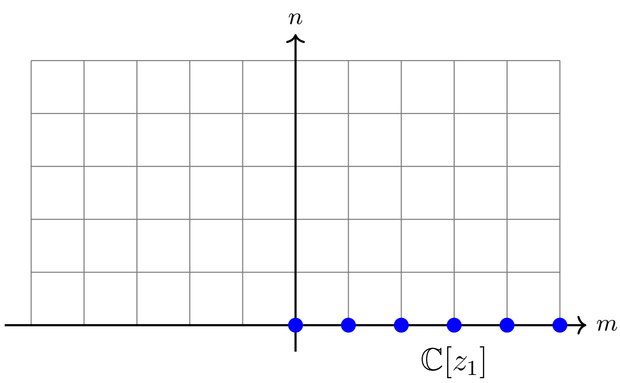
$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$$

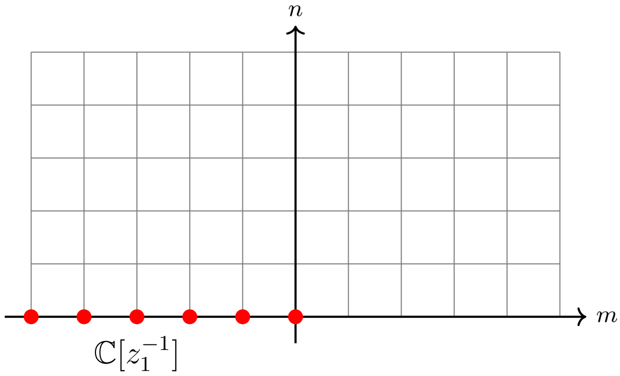
Thus $\frac{\partial f}{\partial \bar{z}}$ measures the extent by which f from being analytic.



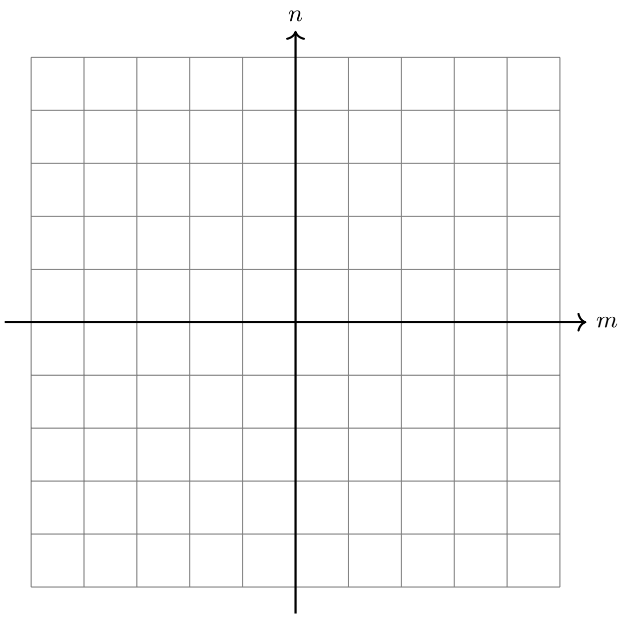
Q11. Rewrite the proof of Theorem 3.6 for yourself when $n = 2$. That is, prove that $\mathbb{P}^2$ is an analytic manifolds. Write down all the charts U_0, U_1, U_2 and all the change of coordinates on the intersections explicitly.

Q12. Look at the *lattice representation* for $\mathbb{C}[z_1]$ **and** $\mathbb{C}[z_1^{-1}]$, below:

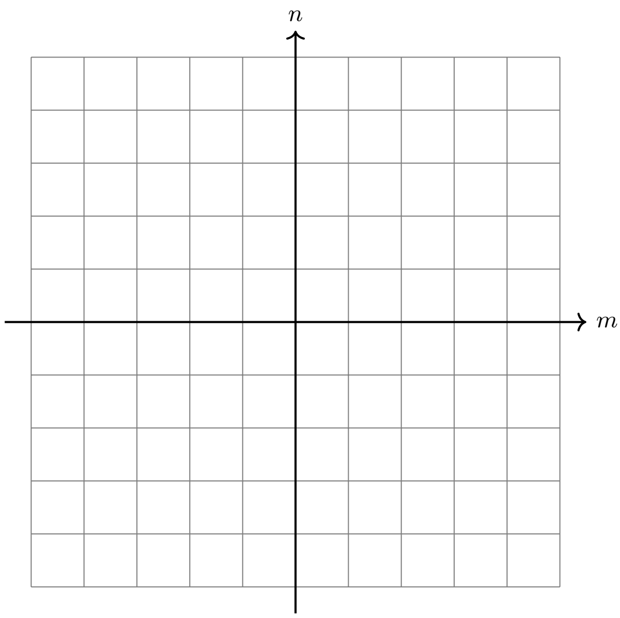




(i) Highlight the lattice points related to $\mathbb{C}[z_1, z_2]$



(ii) Highlight the lattice points associated to $\mathbb{C}[z_1, z_2^2]$



(iii) Highlight the lattice points related to $\mathbb{C}[z_1 z_2^2, z_2^{-1}]$.