UNIVERSITY OF BRISTOL

School of Mathematics

Algebraic Geometry
MATHM0036
(Paper code MATHM0036R)

August 2024   2 hour(s) 30 minutes

This paper contains two sections: Section A and Section B.
Each section should be answered in a separate booklet.

All FOUR answers will be used for assessment.

Calculators of an approved type (permissible for A-Level examinations) are permitted.

Candidates may bring ONE hand-written sheet of A4 notes, written double sided into the examination. Candidates must insert this sheet into their answer booklet(s) for collection at the end of the examination.

THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Do not turn over until instructed.

Q1. (a) (**15 marks**) Show that $G := \mathrm{GL}_n(\mathbb{C})$, the set of invertible $n \times n$ matrices with entries in $\mathbb{C}$ is isomorphic to an affine algebraic variety.

(b) (**10 marks**) Find $\mathcal{O}_G(G)$.

Q2. Consider the *Veronese* map

$$\begin{aligned} \varphi : \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ [s:t] &\longmapsto [s^3 : s^2 t : s t^2 : t^3] \end{aligned}$$

(a) (**15 marks**) Prove that φ is a morphism. (Hint. Describe the map φ in some affine charts.)

(b) (**10 marks**) Find the homogeneous ideal $\mathbb{I}\big(\varphi(\mathbb{P}^1)\big)$.

Q3. (a) (**10 marks**) Consider the family of algebraic varieties, with parameter $t \in \mathbb{C}$, given by

$$V_t := \mathbb{V}(xy - t) \subseteq \mathbb{A}^2.$$

Sketch the variety of V_0, V_1, and V_2 in $\mathbb{R}^2$. Determine whether or not these varieties are smooth. Briefly justify your answers.

(b) (**15 marks**) Prove that the locus of singular points of a quasi-projective *hypersurface* V forms proper closed subset of V. Recall that a variety is called a hypersurface if it can be given with only one equation.

Continued...

Q4. Let Σ be the fan consisting of

- σ_1 cone spanned by $\{(-1,-1),(0,1)\}$;
- σ_2 cone spanned by $\{(0,1),(1,0)\}$;
- τ cone spanned by $\{(1,1)\}$.

(a) (**6 marks**) Determine whether or not the toric variety X_Σ has the following properties. Briefly justify your answer.

(i) smooth;

(ii) complete.

(b) (**9 marks**) Describe the coordinate rings of X_{σ_1}, X_{σ_2}, and X_τ.

(c) (i) (**5 marks**) Explain why we have the inclusions $\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau]$, $\mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau]$;

(ii) (**5 marks**) Describe the gluing of X_{σ_1} and X_{σ_2} along X_τ.

End of examination.