Linear Algebra: Sheet 6

Present all your answers in complete sentences. There is also a Numbas quiz.

Hand-in question

Submit your solution on Blackboard by **1pm on Wednesday (Week 8)** for feedback from your tutor.

1. Consider the matrix
$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}.$$

a) Compute the characteristic polynomial of A and so find the eigenvalues of A.

b) For each eigenvalue, find the set of eigenvectors.

c) In this case we can diagonalise A. To do so, we use $C := (v_1\ v_2)$ where $\{v_1, v_2\}$ is a basis of eigenvectors.

(i) Construct the matrix C and directly compute $C^{-1}AC$.

(ii) Now replace C by $(v_2\ v_1)$. Directly compute $C^{-1}AC$ in this case.

(iii) Consider why we get the answers found for (i) and (ii). Write a short explanation of this.[1]

Solution:

a) The characteristic polynomial is

$$\begin{aligned} p_A(x) &= \det \begin{pmatrix} 3-x & 1 \\ 0 & 2-x \end{pmatrix} \\ &= (3-x)(2-x) \\ &= (x-2)(x-3) \end{aligned}$$

which gives us that A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$.

b) We find an eigenvector for each eigenvalue given above.

i) For $\lambda_1 = 2$, we wish to solve $(A - 2I)v = 0$. This involves finding $a, b \in \mathbb{R}$ such that

$$(A - 2I)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus an eigenvector satisfies $a + b = 0$. Choosing a as the free variable we see that $b = -a$. Setting $a := 1$ gives us the eigenvector $v_1 = (1, -1)$.

ii) With $\lambda_2 = 3$, we get

$$(A - 3I)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives us the equations $b = 0$ and $-b = 0$, which both amount to the same thing. Thus a is a free variable and setting $a := 1$ we obtain the eigenvector $v_2 = (1, 0)$.

c) i) We continue with our notation above, setting $C = (v_1\ v_2)$, which is invertible because v_1 and v_2 are linearly independent (observable directly or because they relate to different eigenvalues). Now,

$$C = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \text{ and so } C^{-1}AC = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

ii) In this case we have that $C := (v_2\ v_1)$, which leads us to

$$C = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \text{ and so } C^{-1}AC = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

iii) If we replace v_1 and v_2 with eigenvectors relating to the eigenvalues λ_1 and λ_2 respectively, we will see that $C^{-1}AC = \mathrm{diag}(\lambda_1, \lambda_2)$. This is by construction, since the ith column of $C^{-1}AC$ is given by where it sends e_i. In (ii) we swapped the roles of λ_1 and λ_2, and this gives rise to the different order of the eigenvalues seen in the resulting diagonal matrix.

[1] The explanation of diagonalisation in the lecture notes may help here.

Additional questions

Try these questions and look at the solutions for feedback. They might also be discussed in your tutorial.

2. Which of the following functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ define an $\mathbb{R}$-linear map? Justify your answer in each case. For those that are linear, find their matrix form.

   a) $f(x, y) = (-y, -x)$ for all $x, y \in \mathbb{R}$.

   b) $f(x, y) = (x + y, x \times y)$ for all $x, y \in \mathbb{R}$.

   c) $f(x, y) = (0, 0)$ for all $x, y \in \mathbb{R}$.

   d) $f(x, y) = (0, 0)$ if $(x, y) \in \mathbb{Q}^2$ and $f(x, y) = (x, y)$ otherwise.

   Solution:

   One approach to all of these questions is to construct a matrix that agrees with the function on the standard basis vectors e_1 and e_2, and then check if this agrees on every point in $\mathbb{R}^2$. Instead we will check each condition of linearity.

   a) We have wish to show that $f((x, y) + (x', y')) = f(x, y) + f(x', y')$ and $f(\lambda(x, y)) = \lambda f((x, y))$. First,

   $$\begin{aligned} f((x, y) + (x', y')) &= f((x + x', y + y')) \\ &= (-(y + y'), -(x + x')) \\ &= (-y - y', -x - x') \\ &= (-y, -x) + (-y', -x') \\ &= f(x, y) + f(x', y'). \end{aligned}$$

   Similarly,

   $$\begin{aligned} f(\lambda(x, y)) &= f(\lambda x, \lambda y) \\ &= (-\lambda y, -\lambda x) \\ &= \lambda(-y, -x) \\ &= \lambda f((x, y)). \end{aligned}$$

   b) We note that $f(e_1) = f(e_2) = e_1$, but $f(e_1 + e_2) = (2, 1) \neq f(e_1) + f(e_2)$.

   c) This is linear. The same steps taken above apply.

   d) This is not linear. We could either take two irrational numbers that sum to a rational, e.g. $\sqrt{2}, 1-\sqrt{2}$, or could consider an irrational scalar applied to a point in $\mathbb{Q}^2$, e.g. $f(\sqrt{3}e_1) = \sqrt{3}e_1$ but $\sqrt{3}f(e_1) = (0, 0)$.

   The matrix for (a) is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and for (c) is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (-x, -y)$ for all $x, y \in \mathbb{R}$.
   a) Write f in standard basis form.
   b) Write f in matrix form.
   c) Using the matrix form, check that f is indeed an $\mathbb{R}$-linear map.

Solution:

If unsure, do check the examples in the lecture notes before looking at this solution. Here we work with $f(x, y) = (-x, -y)$.

a) We see that $f(e_1) = f(1, 0) = (-1, 0)$. Similarly, $f(e_2) = f(0, 1) = (0, -1)$. Thus $f : e_1 \mapsto -e_1, e_2 \mapsto -e_2$ is f in standard basis form.

b) From (a) we can read off the columns of the matrix representing f. This gives us $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

c) We wish to check that the matrix and function agree on all inputs. Take $(a, b) \in \mathbb{R}^2$. Then

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$$

which works for arbitrary $a, b \in \mathbb{R}$. Hence the matrix and function agree and the function must be linear.

4. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with $g(e_1) = e_2$ and $g(e_2) = -e_1$.
   a) Is g uniquely determined? Justify your answer.
   b) Write g in algebraic form.
   c) Write g in matrix form.
   d) What transformation of $\mathbb{R}^2$ does g represent?
   e) Using the standard basis form, together with linearity, find g^{-1}.
   f) Confirm your answer by finding g^{-1} by:
      (i) first finding the inverse of the 2×2 matrix from (c); and then
      (ii) writing this linear map in standard basis form.

Solution:

Recall the function is $g(e_1) = e_2$, $g(e_2) = -e_1$ and we are told it is linear.

a) The function is uniquely determined, since for any $a, b \in \mathbb{R}$ we have

$$g(ae_1 + be_2) = g(ae_1) + g(be_2) = ag(e_1) + bg(e_2) = ae_2 + b(-e_1) = (-b, a)$$

which means that each input $(a, b) \in \mathbb{R}^2$ has a specific output.

b) We have already somewhat done this in (a), where we found that $g(a, b) = (-b, a)$. Thus $g(x, y) = (-y, x)$ for every $x, y \in \mathbb{R}$.

c) The matrix that sends e_1 to e_2 and e_2 to $-e_1$ must have first column e_2 and second column $-e_1$. Thus the matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

d) The linear transformation that sends e_1 to e_2 and e_2 to $-e_1$ is a rotation by $\pi/2$ radians.

e) We are asked to use the form $g(e_1) = e_2, g(e_2) = -e_1$. Thus g^{-1} sends e_2 to e_1, which is to say that $g^{-1}(e_2) = e_1$, and g^{-1} sends $-e_1$ to e_2. We can finish writing g^{-1} in standard basis form by noting that it is a linear function. Then $-e_1 = (-1)e_1$ and so $g^{-1}(-e_1) = -g^{-1}(e_1) = e_2$. Thus $g^{-1}(e_1) = -e_2$.

f) i) We find the inverse of the matrix from (c): $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

   ii) For standard basis form we simply read off the columns of our matrix. Thus it sends e_1 to $-e_2$ and e_2 to e_1, as expected.

5. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map given by $h(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   a) Is h uniquely determined? Justify your answer.
   b) Write h in standard basis form.
   c) Write h in algebraic form.
   d) What transformation of $\mathbb{R}^2$ does h represent?
   e) By using the matrix forms of the maps, find $h \circ g$ and $g \circ h$ in algebraic form.

Solution:

In this question we work with the matrix form, where the matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

a) For any given $\mathbf{x} = (x, y) \in \mathbb{R}^2$, the matrix A sends $\mathbf{x}$ to a specific point in $\mathbb{R}^2$. Thus h is uniquely determined.

b) The standard basis form can be read off from the columns of the given matrix A. Thus h sends e_1 to e_2 and e_2 to e_1. This would be another way to approach (a), since we could then use that h is linear (from the FACT in the lecture notes) and we know the image of the basis e_1, e_2 of $\mathbb{R}^2$.

c) To find the algebraic form we could either convert from the standard basis form or note, for any $a, b \in \mathbb{R}$, that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

and since this applies to all points in $\mathbb{R}^2$, we have $h(x, y) = (y, x)$ for all $(x, y) \in \mathbb{R}^2$.

d) This transformation sends e_1 to e_2 and e_2 to e_1. Therefore it is a reflection in the line $y = x$.

e) Composition of maps relates to multiplication of matrices. Thus, for any $a, b \in \mathbb{R}$,

$$h \circ g \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$$

and so in algebraic form we have $h \circ g : (x, y) \mapsto (x, -y)$. Similarly,

$$g \circ h \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ b \end{pmatrix}$$

which gives $g \circ h : (x, y) \mapsto (-x, y)$. Note that we could also have completed the computations above by applying each matrix in turn to the vector (a, b).

6. Let $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates anti-clockwise by θ radians[2].

   a) Find each of the following, in whichever order you prefer:

      (i) The algebraic form of f_θ.

      (ii) The matrix form of f_θ.

      (iii) The standard basis form of f_θ.

   b) Note, given $\theta, \phi \in \mathbb{R}$, that $f_\theta \circ f_\phi = f_\phi \circ f_\theta = f_{\theta+\phi}$.

      (i) Using (a)(ii), write out the matrix form of $f_{\theta+\phi}$.

      (ii) Using the matrix forms of f_θ and f_ϕ, find the matrix form of $f_\theta \circ f_\phi$.

      (iii) Use (i) and (ii) to find identities relating to $\sin(\theta+\phi)$ and $\cos(\theta+\phi)$.

   c) Observe that $f_\theta^{-1} = f_{-\theta}$. By comparing the matrix form of $f_{-\theta}$ and the inverse of the matrix form of f_θ, deduce one property of cos and one of sin.

Solution:

There is a nice way to approach this question, which we might cover in class.

a) iii) This requires some trigonometry. We can show that $f_\theta(e_1) = (\cos\theta, \sin\theta)$ and $f_\theta(e_2) = (-\sin\theta, \cos\theta)$, which gives us the standard form of f_θ.

   ii) Using (iii), the matrix form of f_θ is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

   i) We apply the above matrix to a general point $(x, y) \in \mathbb{R}^2$ to obtain that $f_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

b) i) We have that $f_{\theta+\phi}$ is given by the matrix $\begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$.

   ii) Take the product of the matrix forms of f_θ and f_ϕ, so that $f_{\theta+\phi}$ equals

$$\begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -(\cos\theta\sin\phi + \sin\theta\cos\phi) \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix}.$$

   iii) The matrices found are both the matrix form of $f_{\theta+\phi}$. Since a given linear function is represented by one **unique** matrix, it must be that these two matrices are the same. Hence each entry is the same, and we obtain the trigonometric identities $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ and $\sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi$.

c) From $f_\theta^{-1} = f_{-\theta}$, the algebraic form of f_θ^{-1} is

$$f_{-\theta}(x, y) = (x\cos(-\theta) - y\sin(-\theta), x\sin(-\theta) + y\cos(-\theta))$$

and so the matrix form of f_θ^{-1} can be represented in two ways:

$$\begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \text{ and } \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Comparing the entries, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ for all θ, i.e., cos and sin are even and odd functions respectively.

[2]It is helpful to know that the point given by rotating e_1 by θ radians is $(\cos\theta, \sin\theta)$.

7. Let $m, n \in \mathbb{N}$ and suppose that $f : \mathbb{Q}^n \to \mathbb{Q}^m$ satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}^n$. Prove that f is a $\mathbb{Q}$-linear map.

Solution:

To show that f is linear we can prove that

$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{Q}^n, \text{ and} \tag{1}$$

$$f(\lambda x) = \lambda f(x) \text{ for all } x \in \mathbb{Q}^n \text{ and } \lambda \in \mathbb{Q} \tag{2}$$

of which we know that (1) holds from the question. We consider various cases of (2).

For $\lambda = 0$.
We note that $f(0+0) = f(0) + f(0)$ by (1), and so $0 = f(0) - f(0) = f(0)$.

For $\lambda = -1$.
We note that $f(x + (-x)) = f(x) + f(-x)$ by (1), and so $f(0) = f(x) + f(-x)$.

For $\lambda \in \mathbb{N}$.
By applying (1) $(k-1)$-times, we see that

$$f\left(\sum_{i=1}^{k} x_i\right) = \sum_{i=1}^{k} f(x_i),$$

and so $f(kx) = f(\sum_{i=1}^{k} x) = \sum_{i=1}^{k} f(x)$.

For $\lambda = 1/q$, where $q \in \mathbb{N}$.
This follows by applying (1) $(q-1)$-times we obtain the equation $f(\sum_{i=1}^{q} \frac{1}{q}x) = \sum_{i=1}^{q} f(\frac{1}{q}x)$, which simplifies to $f(x) = qf(\frac{1}{q}x)$.

Putting these different cases together, we see that if $p \in \mathbb{N}$ and $q \in \mathbb{N}$, then

$$f(\frac{p}{q}x) = pf(\frac{1}{q}x) = \frac{p}{q}f(x).$$

Similarly, $f(-\frac{p}{q}x) = -f(\frac{p}{q}x) = -pf(\frac{1}{q}x) = -\frac{p}{q}f(x)$, as required.

8. Let $A \in M_n(\mathbb{C})$ have an eigenvalue $\lambda \in \mathbb{C}$. Note: for (a) and (c), recalling properties of the determinant is helpful.

   a) Show that λ is an eigenvalue of A^t.

   b) Consider the matrix A from $\mathbb{R}^2$ to $\mathbb{R}^2$ sending e_1 to e_1 and e_2 to $e_1 + e_2$. Are the eigenvectors of A and A^t the same?

   c) Is $-\lambda$ is an eigenvalue of $-A$? (Consider the cases that n is even and odd.)

   d) Using that $(A^t)^t = A$ and $-(-A) = A$, comment on the set of eigenvalues of A^t and $-A$ compared to the set of eigenvalues for A.

   e) Must the eigenvectors of A and $-A$ be the same?

   f) Imagine that A is antisymmetric (i.e., $A^t = -A$). What can we say about the set of eigenvalues of A?

   g) If $A \in M_n(\mathbb{C})$ is antisymmetric and n odd, show that $\det(A) = 0$. Explain why A therefore has zero as an eigenvalue.

   Solution:

   a) We have that $\lambda \in \mathbb{C}$ satisfies $\det(A-\lambda I) = 0$. A property of the determinant is that $\det(M) = \det(M^t)$. Thus
   $$\det(A - \lambda) = \det((A - \lambda I)^t) = \det(A^t - \lambda I)$$
   using that $(A - \lambda I)^t = A^t - \lambda I$ because λI is diagonal.

   b) For both A and A^t the only eigenvalue is $\lambda_1 = 1$. The eigenvectors are not the same, however. A solution $v = (a, b)$ to $(A - I)v = 0$ requires $b = 0$, meaning all eigenvectors are in $\text{span}\{e_1\}$, whereas a solution $v = (a, b)$ to $(A^t - I)v = 0$ requires $a = 0$ and so eigenvectors of A^t all lie in $\text{span}\{e_2\}$. Since 0 is not an eigenvector, A and A^t have no eigenvectors in common in our case.

   c) Let $M \in M_n(\mathbb{C})$. We note that to obtain $-M$ from M requires n row operations, each of the form $R_i \mapsto -R_i$ where $i \in \{1, \ldots, n\}$. Each of these introduces a factor of -1 to the determinant, and so if n is odd we have $\det(-M) = -\det(M)$ and if n is even we have $\det(-M) = \det(M)$. As suggested in the question, we first work with $A \in M_n(\mathbb{C})$ where n is even. Then
   $$0 = \det(A - \lambda I) = \det(-(A - \lambda I)) = \det(-A - (-\lambda)I)$$
   meaning that $-\lambda$ is an eigenvalue of $-A$. Similarly, if n is odd,
   $$0 = \det(A - \lambda I) = -\det(-(A - \lambda I)) = -\det(-A - (-\lambda)I)$$
   and again $-\lambda$ is an eigenvalue of $-A$.

   d) We have seen that for each eigenvalue λ of A, we have that λ is an eigenvalue of A^t. But then for each eigenvalue μ of A^t we have that μ is an eigenvalue of $(A^t)^t = A$. Similarly for each eigenvalue μ of $-A$ we have that $-\mu$ is an eigenvalue of $-(-A) = A$.

   e) Let v be an eigenvector of A. Then, $(-A)v = -(Av) = -(\lambda v) = (-\lambda)v$ and so v is an eigenvector of $-A$ corresponding to the eigenvalue $-\lambda$. Running the argument for an eigenvector w of $-A$ gives us the same conclusion, but depends on us knowing that for each eigenvalue μ of $-A$ we have that $-\mu$ is an eigenvalue of A.

   f) Let $\text{spec}(A) = \{\lambda_1, \ldots, \lambda_n\}$ (possibly listed with multiplicity). Then, by our work above, $\text{spec}(A^t) = \{\lambda_1, \ldots, \lambda_n\}$ and $\text{spec}(-A) = \{-\lambda_1, \ldots, -\lambda_n\}$. But by assumption $A^t = -A$, and so $\{\lambda_1, \ldots, \lambda_n\} = \{-\lambda_1, \ldots, -\lambda_n\}$, i.e., for each eigenvalue λ of A there is an eigenvalue $-\lambda$ of A.

   g) With n odd, we note that $\det(A) = \det(A^t) = \det(-A) = -\det(A)$ and so $2\det(A) = 0 \Rightarrow \det(A) = 0$. Thus zero is a solution to $\det(A - xI) = 0$, and so $\lambda = 0$ is an eigenvalue of A.

9. Let $A \in M_n(\mathbb{C})$ have the property that the sum of elements in every row equals the same number r, i.e., $\sum_j a_{ij} = r$ for each $i \in \{1, \dots, n\}$.

   a) Show that r is an eigenvalue of A and find a corresponding eigenvector[3].

   b) If $B \in M_n(\mathbb{C})$ instead has the property that the sum of elements in every column equals the same number r, must r be an eigenvalue of B?

   Solution:

   a) The hint says to consider a specific vector for use as an eigenvector. We note that the sum of each row of A is r. Let $v := (1, \dots, 1) \in \mathbb{R}^n$. Then $Av = rv$, i.e., the vector consisting of all 1's is an eigenvector corresponding to the eigenvalue r.

   b) We note that B is the transpose of a matrix which has the property assumed in part (a). Thus, by the previous question, B has an eigenvalue r.

[3] **Hint:** we want to consider a special vector to use as the eigenvector here, that somehow uses the property of A that we are given.

10. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

a) Compute the characteristic polynomial of A and the eigenvalues of A.

b) Compute a set of eigenvectors of A.

c) Is A diagonalisable? If so, diagonalise A.

Solution:

a) We expand along the first column to find the characteristic polynomial:

$$\begin{aligned} p_A(x) &= \det \begin{pmatrix} 1-x & 1 & 1 \\ 0 & -2-x & 0 \\ 1 & 0 & 1-x \end{pmatrix} \\ &= (1-x) \det \begin{pmatrix} -2-x & 0 \\ 0 & 1-x \end{pmatrix} + 1 \det \begin{pmatrix} 1 & 1 \\ -2-x & 0 \end{pmatrix} \\ &= (1-x)^2(-2-x) - (-2-x) \\ &= (x-1)^2(-x-2) - (-x-2) \\ &= -(x-1)^2(x+2) + (x+2) \\ &= -(x+2)[(x-1)^2 - 1] = -(x+2)[x^2 - 2x] = -x(x+2)(x-2). \end{aligned}$$

Therefore we have three distinct eigenvalues, $\lambda_1 = 2$, $\lambda_2 = 0$ and $\lambda_3 = -2$.

b) We now find an eigenvector for each eigenvalue given above. In contrast to the earlier question, we will solve the matrix equations by first applying elementary row operations, but note we could have also solved the simultaneous equations directly.

i) For $\lambda_1 = 2$, we solve $(A - 2I)v = 0$. Simplifying with elementary row operations, we see that

$$(A - 2I) = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & -1 \end{pmatrix} \equiv \begin{pmatrix} -1 & 1 & 1 \\ 0 & -4 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where the last matrix is obtained by the row op $R_3 \mapsto R_3 + R_1$. To solve for $v = (a, b, c)$, observe that the equations corresponding to the second and third row give $b = 0$ and the first row gives $a = c$. We choose a as a free variable and set $a := 1$ to obtain the eigenvector $v_1 = (1, 0, 1)$.

ii) For $\lambda_2 = 0$, we wish to solve $(A - 0I)v = 0$ and note

$$(A - 0I) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

where we have applied $R_3 \mapsto R_3 - R_1$. With $v = (a, b, c)$, we get $b = 0$ and $a = -c$. Thus with a as a free variable we set $a := 1$ and have an eigenvector $v_2 = (-1, 0, 1)$.

iii) For $\lambda_3 = -2$ we apply the row op $R_1 \mapsto R_1 - 3R_3$ to obtain

$$(A + 2I) = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 & -8 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

This gives $b = 8c$ and $a = -3c$, so choosing c as the free variable and setting $c := 1$ gives the eigenvector $v_3 = (-3, 8, 1)$.

c) Since A has three different eigenvalues, any choices made above for v_1, v_2, v_3 will be linearly independent and hence form a basis. Therefore with

$$C = (v_1, v_2, v_3) = \begin{pmatrix} 1 & -1 & -3 \\ 0 & 0 & 8 \\ 1 & 1 & 1 \end{pmatrix} \text{ we obtain } C^{-1}AC = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

11. Consider the matrix

$$B = \begin{pmatrix} 1 & -1 & 5 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}.$$

a) Compute the characteristic polynomial of B and the eigenvalues of B.

b) Compute a set of eigenvectors of B.

c) Is B diagonalisable? If so, diagonalise B.

Solution:

This is analogous to the previous question.

a) We find $p_B(x) = -x(x-1)(x-5)$ and so have three distinct eigenvalues, $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 5$.

b) Eigenvectors v_i corresponding to λ_i are given by $v_1 = (-3, 2, 1)$, $v_2 = (1, 0, 0)$, and $v_3 = (-11, 4, -8)$.

c) These eigenvectors are linearly independent, and therefore form a basis of $\mathbb{R}^3$. We let $C = \begin{pmatrix} -3 & 1 & -11 \\ 2 & 0 & 4 \\ 1 & 0 & -8 \end{pmatrix}$ and obtain $C^{-1}BC = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.