UNIVERSITY OF BRISTOL

School of Mathematics

# ALGEBRAIC GEOMETRY

MATHM0036

(Paper code MATH–M0036)

May/June 2023   2 hours 30 minutes

**This paper contains one sections.**
**Each section should be answered in a separate booklet.**

The exam contains FOUR questions
All Four answers will be used for assessment.

Calculators of an approved type (permissible for A-Level examinations) are permitted.

**Candidates may bring four sheets of A4 notes written double-sided into the examination.**

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

*Do not turn over until instructed.*

Q3. (a) (**10 marks**) Consider the family of algebraic varieties, with parameter $t \in \mathbb{C}$, given by

$$V_t := \mathbb{V}(xy - t) \subseteq \mathbb{A}^2.$$

Sketch the variety of $V_0, V_1$, and $V_2$ in $\mathbb{R}^2$. Determine whether or not these varieties are smooth. Briefly justify your answers.

(b) (**15 marks**) Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be two affine algebraic varieties, and

$$\varphi : V \longrightarrow W$$

a morphism. Show that the pullback

$$\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$$

is injective if and only if $\varphi$ is *dominant.* Recall that a map $\varphi$ is called dominant if its image, $\varphi(V)$, is dense in $W$.

**Solution.**

(a) Let $f_t = xy - t$. $\nabla f_0 = \nabla f_1 = \nabla f_2 = (y, x)$. Note that the kernel of $\nabla f_i$ is always one dimensional except at $(0,0)$. However, $(0,0)$ is in $V_0$ but not in $V_1$ nor $V_2$. Therefore, $V_0$ is not smooth, but $V_1$ and $V_2$ are.

(b) " $\Longleftarrow$ " Let $f \in \mathbb{C}[W]$. If $\varphi^*(f) = 0$, and $\varphi$ is dominant, then $f \circ \varphi(x) = 0$, for all $x \in V$. Since $\varphi(V)$ is dense in $W$, and $f$ is continuous, $f = 0$ on all $W$, and $f \in \mathbb{I}(W)$.

" $\Longrightarrow$ " Assume that $\varphi$ is not dominant. Then $\overline{\varphi(V)} \subsetneq W$ and by Nullstellensatz $\mathbb{I}(\overline{\varphi(V)}) \supsetneq \mathbb{I}(W)$. Choose $f \in \mathbb{I}(\overline{\varphi(V)}) \setminus \mathbb{I}(W)$. Then, $\varphi^*(f) = 0$, but $f \notin \mathbb{I}(W)$.

*Continued...*

Q2. (**20 marks**) (**Standard - Unseen**) Consider the product varieties $X = \mathbb{P}^1 \times \mathbb{P}^1$, $Y = \mathbb{V}(xw - yz) \subseteq \mathbb{P}^3$ and the *Segre* morphism

$$\begin{aligned} \varphi : X &\longrightarrow Y \\ ([s:t] \times [u:v]) &\longmapsto [su : sv : tu : tv]. \end{aligned}$$

(a) (**10 marks**) Find the inverse map $\psi : Y \to X$.

(b) (**10 marks**) Prove that $\psi$ is a morphism.

**Solution.**

(a) For any variable $f$ let $U_f$ denote a chart with $f \neq 0$. On the chart $U_t \times U_v$, we have the restrictions

$$\begin{aligned} \varphi : U_t \times U_v &\longrightarrow U_w \subseteq Y \\ ([s:1] \times [u:1]) &\longmapsto [su : v : u : 1]. \end{aligned}$$

$$\begin{aligned} \varphi : U_t \times U_u &\longrightarrow U_z \subseteq Y \\ ([s:1] \times [1:v]) &\longmapsto [s : sv : 1 : v]. \end{aligned}$$

Therefore, we define,

$$\begin{aligned} \varphi_w : U_w &\longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \\ [\frac{x}{w} : \frac{y}{w} : \frac{z}{w} : 1] &\longmapsto ([\frac{y}{w} : 1], [\frac{z}{w} : 1]). \end{aligned}$$

$$\begin{aligned} \varphi_z : U_z &\longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \\ [\frac{x}{z} : \frac{y}{z} : 1 : \frac{w}{z}] &\longmapsto ([\frac{x}{z} : 1], [1 : \frac{w}{z}]). \end{aligned}$$

Similarly on other charts. Note that on $U_w \cap U_z \cap \mathbb{V}(xw - yz)$ the values of the above functions coincide, and therefore we can define a map $\psi$, given as above in charts $U_x, \ldots, U_w$.

(b) We have to check that (1) $\psi$ is continuous, which is clear since it is locally a polynomial map, (2) for $f \in \mathcal{O}_X(U)$, $\psi^*(f) \in \mathcal{O}_Y(\varphi^{-1}(U))$. However, this is clear, since by a theorem in the notes we have to only check that the coordinate functions of $\psi$.

Q1. (**20 marks**) (**Standard**) A *sheaf* $\mathcal{F}$ of rings associated to a topological space $X$ consists of the following data:

(i) To each open set $U \subseteq X$, it associates a ring $\mathcal{F}(U)$.

(ii) To each inclusion of open sets $U \hookrightarrow V$, there exists a map $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ called the restriction map from $\mathcal{F}(V)$ to $\mathcal{F}(U)$. These maps satisfy the property that $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ and $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$, where $U \subseteq V \subseteq W$ are open sets.

These data satisfy the following properties:

(iii) Suppose that $f_i \in \mathcal{F}(U_i)$ are a collection of sections which agree on overlaps (formally, $\mathrm{res}_{U_i,U_i\cap U_j} f_i = \mathrm{res}_{U_j,U_i\cap U_j} f_j$ whenever the intersection exists). Then they lift to a section $f \in \mathcal{F}(U)$ which has the property that $\mathrm{res}_{U,U_i} f = f_i$ for all $i \in I$.

(iv) Suppose that $f, f' \in \mathcal{F}(U)$ and that $\mathrm{res}_{U,U_i} f = \mathrm{res}_{U,U_i} f'$ for all $i \in I$. Then $f = f'$.

Let $X$ be an irreducible quasi-projective variety.

(a) (**10 marks**) Prove that any regular function $f \in \mathcal{O}_X(X)$ is continuous.

(b) (**10 marks**) Show that the set of regular functions $\{\mathcal{O}_X(U)\}$, for $U \subseteq X$, forms a sheaf on $X$

**Solution.**

(a) Let $a \in \mathbb{A}^1$, and $f \in \mathcal{O}_X(X)$. It is sufficient to show that $f^{-1}(a) \subseteq X$ is closed. We can cover $X = \cup U_i$ by a union of open affine sets, such that on each $U_i$, $f = \frac{k_i}{h_i}$. Note that $U_i \cap f^{-1}(a) = \{x : f^{-1}(x) = \frac{k_i(x)}{h_i(x)} = a\} = U_i \cap \{x : ah_i(x) - k_i(x) = 0\} = U_i \cap \mathbb{V}(ah_i - g_i)$. The function $ah_i - g_i$ is homogenous, therefore it defines a closed subvariety of $X$, and we have proved that $f^{-1}(a)$ is closed in each $U_i$, and therefore it is closed in $X$.

(b) Properties (i), (ii) are clear. For (iii), we just define $f|_{U_i} = f_i$ and since $f_i$'s agree on the intersection, $f$ is well-defined. (iv) is also clear.

Q4. (**30 marks**)(**Standard - Workbook**) Let $\Sigma$ be the fan consisting of

- $\sigma_1$ cone spanned by $\{(1,0),(0,1)\}$;
- $\sigma_2$ cone spanned by $\{(1,0),(1,-1)\}$;
- $\tau$ cone spanned by $\{(1,0)\}$.

(a) (**6 marks**) Determine whether or not the toric variety $X_\Sigma$ has the following properties. Justify your answer.

(i) Smooth;

(ii) Complete.

(b) (**9 marks**) Describe the coordinate rings of $X_{\sigma_1}$, $X_{\sigma_2}$, and $X_\tau$.

(c) (**15 marks**) Explain

(i) $\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau]$;

(ii) $\mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau]$;

(iii) The gluing map of $X_{\sigma_1}$ and $X_{\sigma_2}$ along $X_\tau$.

**Solution.**

(a) (i) Yes, since the $\sigma_1 \cap \mathbb{Z}^2$ and $\sigma_2 \cap \mathbb{Z}^2$ both span $\mathbb{Z}^2$.

(ii) No, since $|\Sigma| \subsetneq \mathbb{R}^2$.

(b) We have $\sigma_1^\vee = \text{cone}(\{(1,0),(0,1)\})$, $\sigma_2^\vee = \text{cone}(\{(1,1),(0,-1)\})$ $\tau^\vee = \text{cone}(\{(1,0),(0,1),(0,-1)\})$. Therefore $\mathbb{C}[X_{\sigma_1}] = \mathbb{C}[x,y]$, $\mathbb{C}[X_{\sigma_2}] = \mathbb{C}[xy,y^{-1}]$, $\mathbb{C}[X_\tau] = \mathbb{C}[x,y,y^{-1}] = \mathbb{C}[xy,y,y^{-1}]$.

(c) Therefore, the equalities $\mathbb{C}[X_{\sigma_1}]_y = \mathbb{C}[X_\tau] = \mathbb{C}[X_{\sigma_2}]_{y^{-1}}$. These equalities give rise to the inclusions $X_\tau \subseteq X_{\sigma_1}$ and $X_\tau \subseteq X_{\sigma_2}$. We also have the isomorphisms of $\mathbb{C}$-algebras

$$\begin{aligned} \Phi : \mathbb{C}[X_{\sigma_1}] \supseteq \mathbb{C}[X_\tau] &\longrightarrow \mathbb{C}[X_\tau] \subseteq \mathbb{C}[X_{\sigma_2}] \\ x &\longmapsto xy \\ y &\longmapsto y^{-1}. \end{aligned}$$

The map $\Phi$ provides the information for gluing the coordinate rings, as well as the corresponding varieties $X_\tau \subseteq X_{\sigma_1}$ and $X_\tau \subseteq X_{\sigma_2}$.

*End of examination.*