Assessed Coursework 1

- Available from 13:00 on February 20th to 13:00 on February 27th
- Please submit your work in PDF format directly on Blackboard
- This exam counts for %7.5 of your final course mark
- Feel free to discuss Q1 to Q3 with each other, but I expect Q4 and Q5 to be the outcome of your sole effort

Q1. (**20 marks**) For $f, g \in \mathbb{C}[x_1, x_2]$ compare the following closed sets with respect to inclusion. You need to justify your answers.

- $\mathbb{V}(f+g)$,
- $\mathbb{V}((f)+(g))$,
- $\mathbb{V}\big((f)\cap(g)\big)$,
- $\mathbb{V}(f)\cap\mathbb{V}(g)$.
- $\mathbb{V}(fg)$.

Q2. (**20 marks**) Let $A \subseteq \mathbb{A}^n$ be a subset.

(a) What is the definition of the closure of A in $\mathbb{A}^n$?

(b) Prove that $\mathbb{V}(\mathbb{I}(A))$ equals the closure of A in $\mathbb{A}^n$.

(c) Give an example of two subsets $B, C \subseteq \mathbb{A}^1$, such that $B \subsetneq C$, but $\mathbb{V}(\mathbb{I}(B)) = \mathbb{V}(\mathbb{I}(C))$.

(d) Find a curve $W \subseteq \mathbb{A}^2$ and a morphism $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$, such that W is irreducible but $\varphi^{-1}(W)$ is not.

Q3. (**20 marks**)

(a) What is the definition of a compact subset of a topological space?

(b) Prove that $\mathbb{V}(x^2-y) \subseteq \mathbb{A}^2$ is compact in the Zariski topology but not in the Euclidean topology.

Q4. (**20 marks**) Let k be a field, and denote by $\overline{k}$ its algebraic closure.

(a) What is the definition of the algebraic closure of a field?

(b) Assume that $I \subseteq k[x_1, \ldots, x_n]$ is an ideal and recall that Nullstellensatz holds over any algebraically closed field. Prove that $I \neq (1)$ if and only if $\mathbb{V}(I) \neq \varnothing$ as a subset of $\overline{k}^n$.

Q5. (**20 marks**) Prove at least one implication from each of the following equivalences.

(a) Show that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is injective if and only if φ is *dominant.* Recall that a map, φ, is called dominant if its image, $\varphi(V)$, is dense in W.

(b) Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W.