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Algebraic Geometry CW1.

1. a) The closure of A in $\mathbb{A}^n$ is the smallest Zariski-closed subset of $\mathbb{A}^n$ containing A. A Zariski-closed subset of $\mathbb{A}^n$ is the common zero locus of a collection of polynomials in $\mathbb{C}^n$, so the closure of A is the smallest common zero locus of a collection of polynomials in $\mathbb{C}^n$ containing A.

b) Prove $\bar{A} = \mathbb{V}(\mathbb{I}(A))$

First observe that by definition $\mathbb{I}(A)$ consists of all polynomials that vanish on A, so by definition of $\mathbb{V}(\mathbb{I}(A)) = \{x \in \mathbb{A}^n \mid f(x) = 0 \ \forall f \in \mathbb{I}(A)\}$. we have that $A \subseteq \mathbb{V}(\mathbb{I}(A))$. $\mathbb{V}(\mathbb{I}(A))$ is clearly closed in the Zariski topology. ~~So we need only show that~~

This means that $A \subseteq \bar{A} \subseteq \mathbb{V}(\mathbb{I}(A))$ by the minimality of the closure.

Now suppose B is any ~~other~~ closed set such that $A \subseteq B$, so every polynomial that vanishes on B must vanish on A. ie $\mathbb{I}(A) \supseteq J$. for some ideal J where $B = \mathbb{V}(J)$.

So $\mathbb{V}(\mathbb{I}(A)) \subseteq \mathbb{V}(J) = B$. so $\mathbb{V}(\mathbb{I}(A))$ is minimal, ie $\mathbb{V}(\mathbb{I}(A)) = \bar{A}$.

c). Let $B = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

The Euclidean closure $\bar{B} = B \cup \{0\}$ as $(\frac{1}{n})_{n \in \mathbb{N}}$ has limit point 0.

The Zariski closure is all of $\mathbb{C}$ as $\mathbb{I}(B)$ must vanish on all of B. In one dimension this can only happen if a polynomial is either 0 everywhere or of infinite degree, but the degree must be finite, so $\bar{B}$ is generated by the 0 polynomial. The zero locus of the zero polynomial is all of $\mathbb{C}$ so $\bar{B} = \mathbb{C}$.

2. a) Let $(X, \mathcal{T})$ be a topological space. A set $Y \subset X$ is compact if every open cover of Y has a finite subcover. That is, if $\{U_\alpha\}_{\alpha \in A}$ is an open cover, there is a $\{U_{\alpha_n}\} \subset \{U_\alpha\}_{\alpha \in A}$ which covers Y.

b). It holds in general for any closed affine algebraic variety V, that V is compact in the Zariski topology. We prove this below.

Suppose V is covered by $\{U_\alpha\}_{\alpha \in A}$. Each U_α is open so $U_\alpha = V \cap N(f_\alpha)$ where $N(f_\alpha) = \{p \in X \mid f_\alpha(p) \neq 0\}$. for some polynomial f_α

As $\{U_\alpha\}_{\alpha\in A}$ is a cover for X,

$$\cancel{V} = \bigcup_{\alpha\in A} (\cancel{V} \cap N(f_\alpha)).$$

So $\forall p \in X$, $\exists$ some α s.t $p \in N(f_\alpha)$, ie. $f_\alpha(p) \neq 0$.

~~Now consider the ideal $I = (f_\alpha \mid \alpha\in A) \subseteq \mathbb{C}[x_1, \dots x_n]$.~~

Since $\mathbb{C}[x_1, \dots x_n]$ is ~~finitely generated~~ Noetherian by Hilbert's Basis theorem, every ideal is finitely generated.

So $\{f_\alpha\}_{\alpha\in A}$ has a finite subset $\{f_1 \dots f_m\}$ s.t $V(f_\alpha \mid \alpha \in A) = V(f_1 \dots f_m)$.

so $X = (\cancel{V} \cap N(f_1)) \cup \dots \cup (\cancel{V} \cap N(f_m)) =: U_1 \cup \dots \cup U_m$.

ie $\{U_1 \dots U_m\}$ is a finite subcover for V.

So $\mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}^2$ is compact in the Zariski topology.

It isn't compact in the Euclidean topology as it isn't even bounded. (since in Euclidean spaces compact $\Rightarrow$ bounded).

3. a) Consider the curve $W = y^2 - x$. and the morphism $\varphi(x, y) = (x^2, y)$.

Then $\varphi^{-1}(W) = y^2 - x^2 = (y-x)(y+x)$ so $\varphi^{-1}(W)$ is reducible.

But W itself is irreducible as it has no linear factors. φ is a morphism as it is a polynomial, thus the required condition is satisfied.

b). $X \subseteq Y$ is reducible if $X = (G_1 \cap X) \cup (G_2 \cap X)$ where G_1, G_2 are closed subsets of Y and neither G_1 nor G_2 contains X.

We are to prove X is irreducible $\Rightarrow \overline{X}$ is irreducible, by using the contrapositive.
So assume $\overline{X} = (G_1 \cap \overline{X}) \cup (G_2 \cap \overline{X})$. and neither G_1 nor G_2 contains X.

Then $X \subseteq \overline{X}$ ie $X = X \cap ((G_1 \cap \overline{X}) \cup (G_2 \cap \overline{X})) = (G_1 \cap X) \cup (G_2 \cap X)$ where neither G_1 nor G_2 contains $\overline{X}$.

We need that neither G_1 nor G_2 contains $\overline{X} \Rightarrow$ neither contains X. We again take the contrapositive: One contains $X \Rightarrow$ one contains $\overline{X}$.

Suppose WLOG that $X \subseteq G_1$. Then $\overline{X} \subset G_1$ as G_1 is closed and any limit point of X must therefore be a limit point of G_1 (minimality of closure).

So we have that $X = (G_1 \cap X) \cup (G_2 \cap X)$ where $G_1, G_2 \subseteq Y$ are closed and neither G_1 nor G_2 contains X, i.e. X is reducible.

c). Suppose V, W are closed affine algebraic varieties, and $\varphi : V \to W$ an isomorphism.

1. Suppose V is irreducible, and $\varphi(V)$ is reducible for the sake of contradiction

ie $\varphi(V) = A \cup B$ where A, B are proper closed subsets of $\varphi(V)$. disjoint.

Then $\varphi^{-1}(A), \varphi^{-1}(B) \subset V$ and are closed by continuity of φ. In fact, $V = \varphi^{-1}(A) \cup \varphi^{-1}(B)$.

Since V is irreducible, either $\varphi^{-1}(A)$ or $\varphi^{-1}(B)$ must be all of V, assume WLOG $\varphi^{-1}(A) = V$, then $\varphi(V) \subset A$, contradicting that $B \neq \emptyset$ if A, B disjoint. So $\varphi(V)$ must be irreducible. $\therefore$ W is irreducible as $\varphi(V) = W$ due to surjectivity of isomorphism.

2. By definition, $\dim(V) = \max\{d \in \mathbb{N} \mid V = V_d \supset V_{d-1} \supset \dots \supset V_0,\ V_i \subseteq V$ irred subvarieties$\}$.

So suppose $\dim(V) = d$, ie. $V = V_d \supset V_{d-1} \dots \supset V_0$.

Then $\varphi(V) = \varphi(V_d) \supseteq \varphi(V_{d-1}) \supseteq \dots \supseteq \varphi(V_0)$. trivially, and the equalities are eliminated since φ, being an isomorphism, is injective.

So $\varphi(V) = \varphi(V_d) \supset \varphi(V_{d-1}) \supset \dots \supset \varphi(V_0)$, that is, $\dim(\varphi(V)) = \dim(V) = d$.

But as φ is an isomorphism onto W, $\varphi(V) = W$ so $\dim(W) = \dim(V)$.

d). Find the irreducible components of $\mathbb{V}(zx - y,\ y^2 - x^2(x+1)) \subseteq \mathbb{A}^3$.

we let both equal zero to obtain $zx = y$ and $y^2 = x^2(x+1)$.

So $z^2x^2 = y^2 = x^2(x+1)$. so either $x = 0$ or $z^2 = x+1$.

and when $x = 0$ $y = 0$

when $z^2 = x+1$, $y = z^3 - z$ so the components are

$V_1 = \mathbb{V}(x, y)$ and $V_2 = \mathbb{V}(x+1-z^2,\ y - zx)$.

V_1 is irreducible since it (x,y) is a prime ideal. and ~~$\mathbb{C}$~~
$\mathbb{C}[x,y,z]/(x,y) \cong \mathbb{C}[z]$.
By part c) irreducibility is preserved by isomorphism so V_1 is irreducible.

$V_2 = \mathbb{V}(z^2-x-1, y-zx)$, so it's coordinate ring is

$$R = \frac{\mathbb{C}[x,y,z]}{(z^2-x-1, y-zx)} \cong \frac{\mathbb{C}[x,z]}{(z^2-x-1)} \quad \text{as } y = zx.$$

z^2-x-1 is irreducible over $\mathbb{C}[x,z]$ ~~so~~ as it has no linear factors, so V_2 is an irreducible variety.

4. a). Take a polynomial $g \in \mathbb{C}[x_1, \dots x_n]$ that vanishes on V, ie $\forall x \in V$, $g(x) = 0$, ~~and such that $g(a)$~~ $g(a) \neq 0$ since $a \in \mathbb{A}^n \setminus V$.

Then define $f(x) = \dfrac{g(x)}{g(a)}$. Then when $g(x) = 0$, ie when $x \in V$,

$f(x) = \dfrac{0}{g(a)} = 0$. so $f \in \mathbb{I}(V)$.

But $f(a) = \dfrac{g(a)}{g(a)} = 1$. ~~the~~

b). i). Let $I = (f_1 \dots f_k)$, $(g) \subseteq \mathbb{C}[x_1 \dots x_n]$ be ideals such that $\mathbb{V}(g) \supseteq \mathbb{V}(I)$.

We need to prove that $(f_1 \dots f_k, x_{n+1} g - 1) = \mathbb{C}[x_1, \dots x_{n+1}]$.
Call $(f_1, \dots, f_k, x_{n+1} g - 1)$ J.

The condition $\mathbb{V}(g) \supseteq \mathbb{V}(I)$ means every common zero of the polynomials in I is also a zero of g. So if $f_i(a_1, \dots a_n) = 0$, $\forall i$, we have $g(a_1, \dots a_n) = 0$.

~~To show the ideal $J = \mathbb{C}[x_1, \dots, x_{n+1}]$ we need only show that $1 \in J$, so suppose for the sake of contradiction that $1 \notin J$.~~

Suppose for the sake of contradiction that $J \neq \mathbb{C}[x_1, \dots x_{n+1}]$. Then J is contained in a maximal ideal m. Maximal ideals ~~corres~~ correspond to points in $\mathbb{C}^{n+1}$, meaning there exists a point $(a_1, \dots a_{n+1})$ such that (1) $\forall i$, $f_i(a_1, \dots a_n) = 0$, ~~but~~ and (2) $x_{n+1} g(a_1, \dots, a_n) - 1 = 0$.

But (1) implies $(a_1, \dots a_n) \in \mathbb{V}(I)$, which as explained above implies $g(a_1, \dots a_n) = 0$.
But this means $x_{n+1} \cdot 0 - 1 = 0$, an obvious impossibility and hence a contradiction, so $J = \mathbb{C}[x_1, \dots, x_{n+1}]$.

ii). Trivially $g^m \in J$ as $J = \mathbb{C}[x_1, \dots x_{n+1}]$.

So we can write $g^m = \sum_{i=1}^{k} h_i f_i + x_{n+1} h_{k+1}$.

for polynomials $h_1 \dots h_{k+1} \in \mathbb{C}[x_1, \dots x_{n+1}]$.

But since $(g) \subseteq \mathbb{C}[x_1, \dots x_n]$, g cannot depend on x_{n+1} and so neither can g^m, so $g^m = \sum_{i=1}^{k} h_i f_i$, ie. $g^m \in I$.

5. a) $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is injective $\iff$ φ is dominant (ie $\varphi(V)$ dense in W).

ie. every point of W is arbitrarily close to a point of $\varphi(V)$.

($\Leftarrow$): Suppose $\varphi(V) \subset W$ is dense. For any $g \in \mathbb{C}[W]$ nonzero, consider the set $\varphi(V) \cap N(g)$ where as in 2b, $N(g)$ is the open set where g doesn't vanish. Since $\varphi(V)$ is dense this must be nonempty.

Now choose $y \in \varphi(V) \cap N(g)$. Then $(\varphi^*(g))(y) \neq 0$ so $\varphi^*(g) \neq 0$. This means $\ker \varphi^* = \{0\}$, which is equivalent to φ^* being injective.

($\Rightarrow$): We use the contrapositive. Suppose $\varphi(V) \subset W$ is not dense. Then there exists an open subset U such that $\varphi(V) \cap U = \emptyset$. which implies $\varphi(V) \subseteq W \setminus U$, as U is open, $W \setminus U$ is a proper closed subset of W, so is a variety contained in a maximal ideal, ie $V(p)$ for some $p \in \mathbb{C}[W]$ nonzero nonunit. Then $F^*(p) = 0$ for some $p \neq 0$ so $\ker \varphi \neq \{0\}$. ie φ is not injective.

b). $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective $\iff$ φ defines an isomorphism $V \to X$ a subvariety of W.

($\Leftarrow$) Suppose $\varphi : V \to W$ is an isomorphism onto $X \subseteq W$. Ie. there are maps $\phi : V \to X$ isomorphism and $\psi : X \to W$ the inclusion map, $\varphi = \psi \circ \phi$. Taking pullbacks gives $\varphi^* = (\psi \circ \phi)^* = \phi^* \circ \psi^* : \mathbb{C}[W] \to \mathbb{C}[X] \to \mathbb{C}[V]$. ϕ^* is an isomorphism as ϕ is, and since X is a subvariety of W, ψ^* is surjective as any polynomial $f : X \to \mathbb{A}^1$ can be extended (using the same polynomial) to $W \to \mathbb{A}^1$. $\therefore \varphi^*$ is a composition of surjections so it is itself a surjection.

($\Rightarrow$) Suppose $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective. Consider $\varphi(V) \subseteq W$.

The Zariski closure of $\varphi(V)$, $\overline{\varphi(V)}$ is by definition a subvariety of W.

So we have maps $V \to \overline{\varphi(V)} \xrightarrow{\text{inclusion}} W$

~~W~~ and the corresponding pullbacks are $\mathbb{C}[W] \to \mathbb{C}[\overline{\varphi(V)}] \to \mathbb{C}[V]$.

The map $V \to \overline{\varphi(V)}$ is dominant so $\mathbb{C}[\overline{\varphi(V)}] \to \mathbb{C}[V]$ is injective by part a).

Since by assumption $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective, the extension to
$\mathbb{C}[\overline{\varphi(V)}] \to \mathbb{C}[V]$ is also surjective.

$\therefore$ the map $\mathbb{C}[\overline{\varphi(V)}] \to \mathbb{C}[V]$ is both injective and surjective so is an isomorphism.

It follows that $\varphi : V \to \overline{\varphi(V)}$ is an isomorphism, satisfying the claims since $\overline{\varphi(V)}$ is a subvariety of W.