2051901

<u>coursework 1</u>

(1) First we notice that:

$\mathbb{V}((f)+(g)) = \mathbb{V}((f)) \cap \mathbb{V}((g))$, $\mathbb{V}((f)\cap(g)) = \mathbb{V}((f)) \cup \mathbb{V}((g))$ and $\mathbb{V}(fg) = \mathbb{V}(f) \cup \mathbb{V}(g)$.

From a remark in the notes, we also note that $\mathbb{V}(\{f_i\}) = \mathbb{V}((\{f_i\}))$.

Therefore, we deduce:

$\mathbb{V}(f+g) \subseteq \mathbb{V}(f) \cap \mathbb{V}(g) = \mathbb{V}((f)+(g)) \subseteq \mathbb{V}(fg) = \mathbb{V}((f)\cap(g))$.

(2) (a) $\bar{A}$ of $A \subset \mathbb{A}^n$ is the smallest closed set in $\mathbb{A}^n$ such that $A \subseteq \bar{A}$. We have that $\bar{A} = \mathbb{V}(\{f \in \mathbb{C}[x_1,\dots,x_n] \mid f(A)=0\})$.

(b) We know that $A \subseteq \bar{A}$. We also have $\mathbb{I}(A) \supseteq \mathbb{I}(\bar{A})$ and $\mathbb{V}(\mathbb{I}(A)) \subseteq \mathbb{V}(\mathbb{I}(\bar{A}))$ as these are order-reversing. $\bar{A}$ is closed by definition, so $\bar{A}$ is a c.a.a.v. This means we have $\mathbb{V}(\mathbb{I}(\bar{A})) = \bar{A}$, so we have $\mathbb{V}(\mathbb{I}(A)) \subseteq \bar{A}$. If we have $a \in A$, then $f(a)=0$ $\forall f \in \mathbb{I}(A) \Rightarrow a \in \mathbb{V}(\mathbb{I}(A))$. $\therefore$ $A \subseteq \mathbb{V}(\mathbb{I}(A))$. We know this is a closed set, and $\bar{A}$ is the smallest such containing A, so $\bar{A} \subseteq \mathbb{V}(\mathbb{I}(A))$. $\therefore$ $\bar{A} = \mathbb{V}(\mathbb{I}(A))$.

(c) Take $B = \{x \in \mathbb{A}^1 \mid x = n \in \mathbb{N}\}$ and $C = \{x \in \mathbb{A}^1 \mid x = z \in \mathbb{Z}\}$. Since B and C are both infinite, we see $\mathbb{V}(\mathbb{I}(B)) = \mathbb{V}(\mathbb{I}(C)) = \mathbb{C}$, and also that $B \subsetneq C$, as required.

d) Consider $W = \mathbb{V}(x^2-y)$ and the morphism $(x,y) \mapsto (x, xy)$. This satisfies $\varphi^{-1}(W)$ being reducible.

(3) (a) A subset S of a topological space X is compact iff given $\{A_i\}_{i\in I}$ a collection of open sets of X such that $S \subseteq \bigcup_{i\in I} A_i$, there exists $J \subseteq I$ such that $S \subseteq \bigcup_{i\in J} A_i$ with J finite. [every open cover has a finite subcover].

(b) It is clear to see that $\mathbb{V}(x^2-y)$ is closed in the Zariski topology, as it is a c.a.a.v.
Additionally, $\mathbb{V}(x^2-y) = \{(x,y) \in \mathbb{A}^2 \mid x^2-y=0\}$ which is clearly a bounded set also, with Zariski.
We could also see from "An Invitation to Algebraic Geometry" that a Zariski closed set in $\mathbb{A}^n$ is compact in the Zariski topology, and we would be done, noting this is because $\mathbb{C}[x_1,\dots,x_n]$ is Noetherian.
However, in the Euclidean topology, the bounded condition fails.

(4) (a) An extension $\bar{K}$ of K is the algebraic closure of K if $K \subset \bar{K}$ is algebraic, and $\bar{K}$ is algebraically closed. Additionally, an algebraic closure of a field is a field that contains all the roots of all the polynomials over that field.

(b) Nullstellensatz $\Rightarrow$ $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$.

$\mathbb{I}(\mathbb{V}(I)) \neq \mathbb{I}(\emptyset) = \{ f \in K[x_0, \ldots, x_n] \mid f(p) = 0 \ \forall p \in \emptyset \} \subseteq \bar{K}^n$.

so $I \neq (1)$, as this is the whole thing, $K[x_1, \ldots, x_n]$.

(5) (a) "$\Rightarrow$" injective $\Rightarrow$ dominant:

Suppose $\varphi^*: \mathbb{C}[W] \to \mathbb{C}[V]$ is injective. Then if we have $p \in \mathbb{C}[W]$ (non-zero), we see that $\varphi^*(p) \neq 0$. This tells us that $\exists v \in V$ such that $(\varphi^*(p))(v) \neq 0$. This implies that $\varphi(V)$ is not contained in any $\mathbb{V}(p)$ for $p \neq 0$. As every non-trivial open subset of W contains the open set where p doesn't vanish (for $p \neq 0$), this tells us that $\varphi(V)$ intersects every non-empty open subset of W, and $\therefore$ dense.

"$\Leftarrow$" dominant $\Rightarrow$ injective:

the open set where g doesn't vanish

Suppose $\varphi(V) \subset W$ is dense. For any non-zero $g \in \mathbb{C}[W]$, consider $\varphi(V) \cap D(g)$. This is non-empty, due to the density. So, take $y \in \varphi(V) \cap D(g)$. Then $(\varphi^*(g))(y) \neq 0$, so $\varphi^*(g) \neq 0$.

$\therefore$ we deduce $\ker \varphi^* = 0$, and φ^* is injective.

(b) "$\Leftarrow$" φ defines an isomorphism $\Rightarrow$ φ surjective.

Suppose that $\varphi: V \to W$ is an isomorphism onto a subvariety X of W. Then we have:

$V \xrightarrow{\sim} X \hookrightarrow W$. Looking at the pullbacks, we get $\mathbb{C}[V] \leftarrow \mathbb{C}[X] \leftarrow \mathbb{C}[W]$.

The map $\mathbb{C}[X] \to \mathbb{C}[V]$ is an isomorphism since $V \to X$ is an isomorphism.

Since we chose X to be a subvariety of W, the map $\mathbb{C}[W] \to \mathbb{C}[X]$ is surjective.

So, we see that $\varphi^*: \mathbb{C}[W] \to \mathbb{C}[V]$ is a composition of surjective maps, so itself is surjective.