UNIVERSITY OF BRISTOL

School of Mathematics

Algebraic Geometry
MATHM0036
(Paper code MATHM-0036)

Summer 2025 2 hour(s) 30 minutes

The exam contains FOUR questions
All Four answers will be used for assessment.

Any calculator permitted.

Candidates may bring ONE hand-written sheet of A4 notes, written double-sided into the examination. Candidates must insert this sheet into their answer booklet(s) for collection at the end of the examination.

Do not turn over until instructed.

Q1. (a) (**5 marks**) Show that any polynomial $f \in \mathbb{C}[x, y, z]$ can be expressed as

$$f = r_1(x^2 - y) + r_2(x^3 - z) + g,$$

for $r_1, r_2 \in \mathbb{C}[x, y, z]$ and $g \in \mathbb{C}[x]$.

(b) (**5 marks**) Recall that the *twisted cubic* is defined as $V = \mathbb{V}(x^2 - z, x^3 - y)$. Consider the parametrisation:

$$\begin{aligned} \varphi : \mathbb{A}^1 &\to \mathbb{A}^3, \\ t &\mapsto (t, t^2, t^3). \end{aligned}$$

Prove that the pullback map

$$\varphi^* : \mathbb{C}[x, y, z] \to \mathbb{C}[t]$$

induces an isomorphism of $\mathbb{C}$-algebras $\mathbb{C}[V] \simeq \mathbb{C}[t]$.

(c) (**5 marks**) Explain why the result from part (b) implies that V is irreducible.

(d) (**5 marks**) We know that the closure of V in $\mathbb{P}^3$ is given by $\overline{V} = \Phi(\mathbb{P}^1)$ where

$$\begin{aligned} \Phi : \mathbb{P}^1 &\to \mathbb{P}^3 \\ [t : s] &\mapsto [s^3 : ts^2 : t^2 s : t^3]. \end{aligned}$$

Prove that $\overline{V} = \mathbb{V}(xz - y^2, yw - z^2, xw - yz) \subseteq \mathbb{P}^3$.

(e) (**5 marks**) Explain why the irreducibility of V implies that $\overline{V}$ is also irreducible.

Q2. (a) (**15 marks**) Recall the following definition:

Let X, Y be two algebraic varieties (*i.e.,* affine, quasi-affine, projective or quasi-projective). A morphism $\varphi : X \longrightarrow Y$, is a map such that

- φ is continuous;
- For any for every open set $V \subseteq Y$, and for every regular function $f \in \mathcal{O}_Y(V)$, $\varphi^*(f) = f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$.

Prove the following theorem:

Let X be an algebraic variety, $Y \subseteq \mathbb{A}^n$ a closed affine algebraic variety, and $\varphi : X \longrightarrow Y$ a map of sets. Then, $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a morphism, if and only if, for all i, the coordinate function $\varphi_i \in \mathcal{O}_X(X)$.

(b) (**10 marks**) Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be two closed affine algebraic varieties and

$$\varphi : V \longrightarrow W$$

a morphism. Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W.

Q3. (a) (**10 marks**) Let $V = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2$.

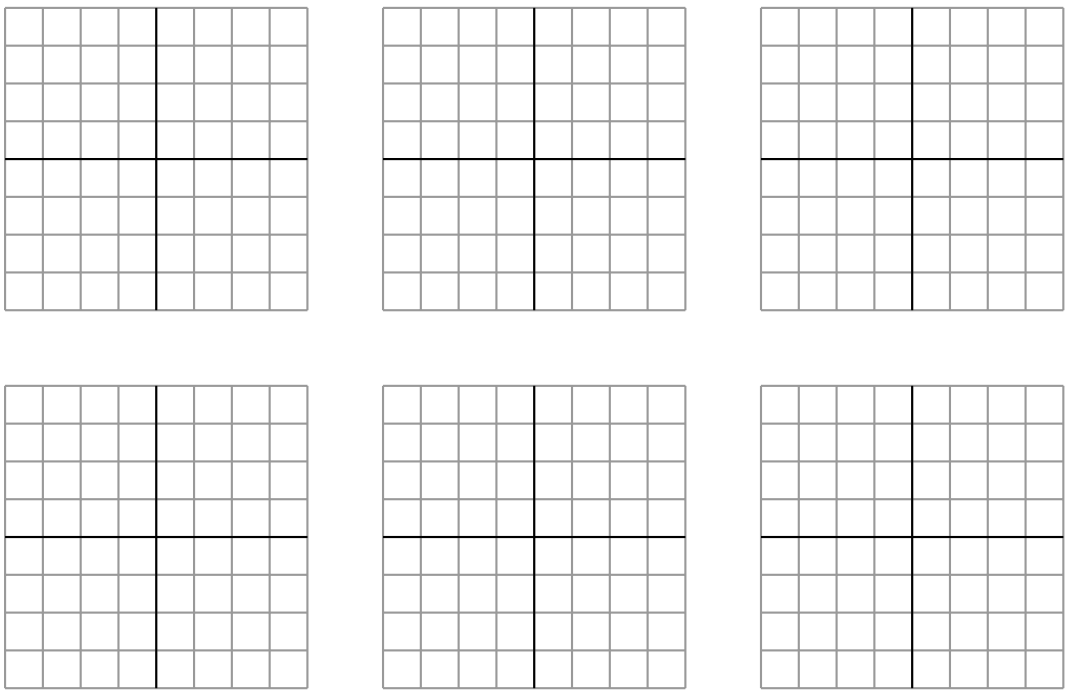
(i) Sketch $V \cap \mathbb{R}^2$ in $\mathbb{R}^2$.

(ii) Find all the singular points of V.

(b) (**10 marks**) Identify the irreducible components of $\mathbb{V}(y^2 - x^3, xz - y) \subseteq \mathbb{A}^3$.

(c) (**5 marks**) Show that $\mathbb{V}(xz - y) \subseteq \mathbb{A}^3$ is isomorphic to $\mathbb{A}^2$.

Q4. Consider the cone $\sigma = \text{cone}(\{e_1, e_1 + 3e_2\}) \subseteq \mathbb{R}^2$. (If you wish, you can use the following grids for calculations.)



(a) (**5 marks**) Explain why the affine toric variety X_σ is not smooth. Subdivide σ into a union of smooth two-dimensional cones.

(b) (**10 marks**) Select two of the two-dimensional cones from your subdivision and denote them by σ_1 and σ_2. Let $\tau = \sigma_1 \cap \sigma_2$. Describe the the toric varieties X_{σ_1}, X_{σ_2}, X_τ and their coordinate rings.

(c) (**2 marks**) Justify why we have the inclusions

$$\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau], \quad \mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau].$$

(d) (**8 marks**) Explain why X_{σ_1} and X_{σ_2} contain X_τ as an open set and describe the glueing of X_{σ_1} and X_{σ_2} along X_τ.

End of examination.