UNIVERSITY OF BRISTOL

School of Mathematics

ALGEBRAIC GEOMETRY

MATHM0036

(Paper code MATHM0036R (Summer Resit)

Reassessment 2025   2 hour 30 minutes

This paper contains four questions

All answers will be used for assessment.

In multi-part questions, even if you are unable to complete proofs of some parts, you may use the results of those parts in other parts.

Any calculator may be used.

Candidates may bring **ONE hand-written** sheet of A4 notes written double-sided into the examination. Candidates must insert this sheet into their answer booklet(s) for collection at the end of the examination.

Do not turn over until instructed.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM.

Q1. (a) (**15 marks**) Show that $G := \mathrm{GL}_n(\mathbb{C})$, the set of invertible $n \times n$ matrices with entries in $\mathbb{C}$ is isomorphic to an affine algebraic variety.

(b) (**10 marks**) Find $\mathcal{O}_G(G)$.

Q2. Consider the *Veronese* map

$$\begin{aligned} \varphi : \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ [s:t] &\longmapsto [s^3 : s^2t : st^2 : t^3] \end{aligned}$$

(a) (**15 marks**) Prove that φ is a morphism. (Hint. Describe the map φ in some affine charts.)

(b) (**10 marks**) Find the homogeneous ideal $\mathbb{I}\big(\varphi(\mathbb{P}^1)\big)$.

Q3. (a) (**10 marks**) Consider the family of algebraic varieties, with parameter $t \in \mathbb{C}$, given by

$$V_t := \mathbb{V}(xy - t) \subseteq \mathbb{A}^2.$$

Sketch the variety of V_0, V_1, and V_2 in $\mathbb{R}^2$. For each of these varieties verify smoothness.

(b) (**15 marks**) Prove that the locus of singular points of a quasi-projective *hypersurface* V forms proper closed subset of V. Recall that a variety is called a hypersurface if it can be given with only one equation.

Continued...

Q4. Let Σ be the fan consisting of

- σ_1 cone spanned by $\{(-1,-1),(0,1)\}$;
- σ_2 cone spanned by $\{(0,1),(1,0)\}$;
- τ cone spanned by $\{(0,1)\}$.

(a) (**6 marks**) Determine whether or not the toric variety X_Σ has the following properties. Briefly justify your answer.

(i) smooth;

(ii) complete.

(b) (**9 marks**) Describe the coordinate rings of X_{σ_1}, X_{σ_2}, and X_τ.

(c) (i) (**5 marks**) Explain why we have the inclusions $\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau]$, $\mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau]$;

(ii) (**5 marks**) Describe the gluing of X_{σ_1} and X_{σ_2} along X_τ.

End of examination.