Personal details

Personal details

First / given name	Fatima
Second given name	
Third given name	
Surname/family name	Ilyas
Date of birth	24 November 1997
Preferred first/given name	Fatima
Previous surname	
Country of birth	Pakistan
Legal nationality	Pakistani
Dual nationality	
Country of residence	Pakistan
Have you previously studied with us at the University of Bristol?	No

Contact details

Home address

Please provide your permanent residential address. If you have another address and would prefer for us to contact you at that address instead you have the opportunity to add a correspondence address in the next section.

Country	Pakistan
Postcode	57000
Address Line 1	Street 1, House 7 Niazi Colony
Address Line 2	Arifwala Road
City	Sahiwal
County	
Telephone	3068163963

If you would like us to send any postal correspondence to an address which is not your home address please enter an alternative address here. If you want us to send correspondence to your home address then please select No.

Do you want to add a correspondence address?	Yes
Country	England
Postcode	W1B 4NF
Address Line 1	211-213 Regent Street
Address Line 2	
City	London
County	
Telephone	

Agent

Agent details

Agency Name	
Email address	

Other information

Additional Documents

Please upload required documents as outlined in your admissions statement

Mode of study

How would like to study this programme?	Full Time

Qualifications

Qualifications

Institution	Qualification	Type	Subject	Actual/predicted	Grade	Start date	End date
Comsat University Islamabad	Master's Degree (PG)	Academic Qualification	Mathematics	Actual	Pass	01/Oct/2020	01/Jul/2022
Bahauddin Zakariya University	First degree BA/BSC etc	Academic Qualification		Actual	Pass	07/May/2016	05/Jun/2018

If these qualifications have altered since your last application please note the changes in the free text box here.

English Language

Is English your first language? No

What is your first language? Urdu

Did you study at school/university where you were taught in English? Yes

For how many years? 10

Have you sat a relevant English language test? Yes

TOEFL (internet-based)

Registration number

Date of TOEFL test

TOEFL reading score

TOEFL listening score

TOEFL speaking score

TOEFL writing score

TOEFL total score

IELTS (International English Language Testing System)

Test report form (TRF) number

UKVI number (if applicable)

Date of IELTS test

IELTS listening score

IELTS reading score

IELTS writing score

IELTS speaking score

IELTS total score

Pearson Test of English

Score report code

Date of Pearson test

Pearson listening score

Pearson reading score

Pearson speaking score

Pearson writing score

Pearson overall score

Other English Language test

Name of course	Higher Secondary
Registration number	3420721314
Date of test	17 September 2016
Listening score	
Writing score	
Reading score	
Total score	79

Experience

Current Employer

Employer name and address	Divisional Public School and College M37J+MFC, DPS Rd, Sahiwal, Sahiwal District, Punjab 57000
Job title and main duties	Senior Teacher Mathematics
Full time/Part time	Full time
Date of Appointment	03 April 2023
End date (if applicable)	

Previous employment 1

Employer name and address	POST GRADUATE COLLEGE SAHIWAL
Job title and main duties	MATHEMATICS LECTURER
Full time/Part time	Full time
Date of Appointment	15 September 2020
End date (if applicable)	31 March 2022

Previous employment 2

Employer name and address	COMSAT UNIVERSITY ISLAMABAD
Job title and main duties	TEACHER ASSISTANTSHIP
Full time/Part time	Full time
Date of Appointment	22 August 2020
End date (if applicable)	14 July 2022

Previous employment 3

Employer name and address	
Job title and main duties	
Full time/Part time	
Date of Appointment	
End date (if applicable)	

Other Experience

Do you have any other relevant work experience to support your application?	
Please provide details	

Personal statement

Personal details

Do you have a personal statement to upload?	Yes
Please type your personal statement in the box	

Research proposal

Research proposal

Proposed supervisor 1	
Proposed supervisor 1	
Proposed project title (max 150 chars)	

Passport and visa

Visa required

Do you require a visa to study in the UK? Yes

Please fill out your passport details below. If you are unable to provide these at the current time you will have another opportunity to upload your passport after you submit the form. If you do not provide us with this information we will be unable to issue you with your confirmation of acceptance number and you will be unable to obtain a visa.

Passport details

Passport number HX1744451

Further details

Have you previously studied in the UK? No

What was the highest level of study in the UK?

Please confirm the total length of your UK study in years

Referees

Referee 1

Do you have a reference to upload? Yes

Type of reference

Referee title

Forename

Surname

Position

Institution/Company

Email address

Country

Referee 2

Do you have a second reference to upload? Yes

Type of reference

Referee title

Forename

Surname

Position

Institution/Company

Email address

Country

Funding

Funding 1

What is your likely source of funding?	Yourself/family
Please give the name of your scholarship or Studentship	
Please specify	
Percentage from this source	100
Is this funding already secured?	Yes

Funding 2

What is your likely source of funding?	
Please give the name of your scholarship or Studentship	
Please specify	
Percentage from this source	
Is this funding already secured?	

Funding 3

What is your likely source of funding?	
Please give the name of your scholarship or Studentship	
Please specify	
Percentage from this source	
Is this funding already secured?	

Other funding

I would like to be considered for other funding opportunities	Yes

Submission

Documents

Document type	File name
Personal statement	Fatima SOP BRISTOL.docx
Research proposal	Fatima Research Proposal.pdf
Degree certificate	Fatima UG Degree.pdf
Transcript	Fatima UG Marksheet.pdf
Admissions documents (Miscellaneous)	SIUK authority form.pdf
Passports and visas	Fatima PPT.pdf
References	Fatima LOR.pdf
References	Fatima LOR 1.pdf
Language qualification	Fatima 12TH.pdf
Curriculum vitae	Fatima CV.pdf
Degree certificate	Fatima MSc Degree.pdf
Transcript	Fatima MSc Transcript.pdf

By ticking the checkbox below and submitting your completed online application form, you acknowledge the University of Bristol will use the information provided from time to time, along with any further information about you the University may hold, for the purposes set out in the University's full Data Protection Statement. Applicants applying to the collaborative programmes of doctoral training should also read the Data Protection Statement for collaborative programmes of doctoral training.

The information that you provided on your application form will be used for the following purposes:

- To enable your application for entry to be considered and allow our Admissions Advisors, where applicable, to assist you through the application process;
- To enable the University to compile statistics, or to assist other organisations to do so. No statistical information will be published that would identify you personally;
- To enable the University to initiate your student record should you be offered a place at the University.

All applicants should note that the University reserves the right to make without notice changes in regulations, courses, fees etc at any time before or after a candidate's admission. Admission to the University is subject to the requirement that the candidate will comply with the University's registration procedure and will duly observe the Charter, Statutes, Ordinances and Regulations from time to time in force.

By ticking the checkbox below and submitting your completed online application form, you are confirming that the information given in this form is true, complete and accurate and that no information requested or other material information has been omitted. You are also confirming that you have read the Data Protection Statement and you confirm the statement below.

I can confirm that the information I have provided is true, complete and accurate. I accept that the information given in my application will be stored and processed by the University of Bristol, in accordance with the *UK General Data Protection Regulation and Data Protection Act 2018*, in order to:

- Consider my application and operate an effective and impartial admissions process;
- Monitor the University’s applicant and student profile;
- Comply with all laws and regulations;
- Ensure the wellbeing and security of all students and staff;
- If my application is successful to form the basis of the statement made within my application.

If the University of Bristol discovers that I have made a false statement or omitted signification information from my application, for example examination results, I understand that it may have to withdraw or amend its offer or terminate my registration, according to circumstances.

europass

FATIMA ILYAS

Date of birth: 24/11/1997 | **Nationality:** Pakistani | **Gender:** Female | **Phone number:** (+92) 3068163963 (Home) | **Email address:** fatimanov1997@outlook.com |

Address: Street # 1, House # 7, Niazi Colony, Arifwala Road., 57000, Sahiwal, Pakistan (Home)

WORK EXPERIENCE

03/04/2023 – CURRENT Sahiwal, Pakistan
SENIOR TEACHER MATHS DIVISIONAL PUBLIC SCHOOL AND COLLEGE

15/09/2020 – 31/03/2022 Sahiwal, Pakistan
MATHEMATICS LECTURER POST GRADUATE COLLEGE SAHIWAL.

22/08/2020 – 14/07/2022 Sahiwal, Pakistan
TEACHER ASSISTANTSHIP COMSAT UNIVERSITY ISLAMABAD, SAHIWAL CAMPUS.

01/04/2022 – 02/2023 Sahiwal, Pakistan
SENIOR TEACHER MATHS ATM'S HIGH SCHOOL.

2019 – 2022 Jeddah, Saudi Arabia
ONLINE FOREIGN COACHING SPECIALST

EDUCATION AND TRAINING

2020 – 2022 Sahiwal, Pakistan
MASTER OF SCIENCE IN MATHEMATICS Comsat University Islamabad, Sahiwal Campus.

2018 – 2020 Sahiwal, Pakistan
MASTER IN MATHEMATICS Comsat University Islamabad, Sahiwal Campus.

2016 – 2018 Multan, Pakistan
BACHELOR OF SCIENCE Bahauddin Zakariya University.

2014 – 2016 Sahiwal
INTERMEDIATE OF SCIENCE Punjab Group of Colleges, Sahiwal Campus.

08/08/2023 – 14/08/2023 Lahore,, Pakistan
EFFECTIVE TEACHING STRATEGIES IN MATHEMATICS Oxford University Press

Website https://oup.com.pk/

01/08/2023 – 06/08/2023 Sahiwal,, Pakistan
ASSESSMENT AND EVALUATION Book Wise

Website bookwisesolutions.com

24/07/2023 – 29/07/2023 Sahiwal,, Pakistan
CLASS ROOM MANAGEMENT Book Wise

Website bookwisesolutions.com

LANGUAGE SKILLS

Mother tongue(s): **PANJABI; PUNJABI** | **URDU**

Other language(s):

	UNDERSTANDING		SPEAKING		WRITING
	Listening	Reading	Spoken production	Spoken interaction	
ENGLISH	B1	B2	A2	A2	B2

Levels: A1 and A2: Basic user; B1 and B2: Independent user; C1 and C2: Proficient user

DIGITAL SKILLS

LaTeX | Zoom, | Microsoft Word | Microsoft Excel | Microsoft PowerPoint

ADDITIONAL INFORMATION

PUBLICATIONS

Ansari, K. J., Asma, Ilyas, F., Shah, K., Khan, A., & Abdeljawad, T. (2023). On new updated concept for delay differential equations with piecewise Caputo fractional-order derivative. Waves in Random and Complex Media, 1-20. (I. F.= 3. 41)

– 2023

HONOURS AND AWARDS

2019
PEEF Scholarship – Punjab Government

2020
Bronze Medalist in MSc(16 years) – Comsat University Islamabad, Sahiwal Campus

2022
Silver Medalist in MS(18 years) – Comsat University Islamabad, Sahiwal Campus

2016
Achieved Star Performance Award in Academics – Punjab Group of Colleges, Sahiwal Campus.

2015
Achieved Certificate and Trophy in Quiz Competition at District Level – Punjab Group of Colleges, Sahiwal Campus.

HOBBIES AND INTERESTS

Content Writing

Badminton

Novel Reading

COMSATS University Islamabad, Sahiwal Campus
(Student Support Center)
COMSATS Road, Off G.T. Road, Sahiwal
Ph: Off: 040-4305005 Web: www.sahiwal.comsats.edu.pk

NO. CUI/SWL/SSC/TA/FA20/13

March 22, 2024

To Whom It May Concern

This is to certify that **Ms. FATIMA ILYAS** D/O **MUHAMMAD ILYAS** Registration No. **CIIT/FA20-RMT-025/SWL** has been a student of MS in Mathematics under department of Mathematics. She has availed TA ship Scholarship for **Fall 2020** to **Spring 2022 semesters.**

She has worked on tasks assigned by her supervisor. She displayed professional traits during her TAship period and managed to complete all assigned tasks well in time. She was hardworking, dedicated and committed.

We wish her a successful future career.

This letter has been issued on her written request.

Safdar Ali
Incharge Student Support Center
COMSATS University Islamabad (CUI),
Sahiwal Campus

COMSATS UNIVERSITY
CUI Sahiwal Campus

Dated: March 28, 2023

TO WHOM IT MAY CONCERN

This is to certify that **Miss Fatima Ilyas** d/o **Muhammad Ilyas** has been working at ATMS as a **Teacher** for the period as follows:

April 2022 TO February 2023.

During this period, she performed her duties efficiently and effectively. She is a hardworking, regular, intelligent and conscientious worker with positive attitude towards life and official duties. She is a good team member who generally meets her deadline related to assigned tasks. She is willing worker and remains cheerful with her subordinates. She bears a good moral character and I wish her thorough success in her future career.

Atiq Akbar Ch.

Director,

ATMS High School

Serial No: 038393

Registration No: 2016-PCS-22

Roll No: 42585

BAHAUDDIN ZAKARIYA UNIVERSITY
MULTAN-PAKISTAN

SESSION 2018

THIS IS TO CERTIFY THAT

FATIMA ILYAS

Daughter of **MUHAMMAD ILYAS**

of PUNJAB COLLEGE SAHIWAL

has passed the *Annual Examination* of the academic session held in April - May , 2018 securing 655 /*800* marks and has been placed in FIRST Division.

Having fulfilled the requirements she has been admitted to the degree of

BACHELOR OF SCIENCE

in this University.

The examination was taken as a whole

Controller of Examinations

Chancellor

Multan,

Serial No: 078060

Registration No: CIIT/FA18-MMT-066/SWL

COMSATS University Islamabad

FATIMA ILYAS d/o MUHAMMAD ILYAS

Of

Sahiwal Campus

has been conferred upon the degree of

Master of Mathematics

Given on this Eighth day of October two thousand and Twenty at Islamabad

Date of Issuance: 8th October 2020

Controller of Examinations

Rector

Registrar

Serial No: 098486

Registration No: CIIT/FA20-RMT-025/SWL

COMSATS University Islamabad

FATIMA ILYAS d/o MUHAMMAD ILYAS

Of

Sahiwal Campus

has been conferred upon the degree of

Master of Science in Mathematics

Given on this Fifteenth day of September two thousand and Twenty Two at Islamabad

Date of Issuance: 15th September 2022

Controller of Examinations

Rector

Registrar

STATEMENT OF PURPOSE

UNIVERSITY OF BRISTOL

MSc BY RESEARCH MATHEMATICS

Introduction:

I am Fatima Ilyas, a dedicated educator with a passion for mathematics and a commitment to fostering a dynamic learning environment. With a Master's degree in Mathematics from Comsat University Islamabad, Sahiwal Campus, and a wealth of experience in teaching at various educational institutions, I bring forth a profound understanding of mathematical concepts coupled with effective pedagogical strategies. My journey in academia has been marked by continuous learning, scholarly achievements, and a relentless pursuit of excellence in education.

Education Background:

My academic journey commenced with a Bachelor of Science degree from Bahauddin Zakariya University in Multan, Pakistan, followed by a Master's in Mathematics from Comsat University Islamabad, Sahiwal Campus. Building upon this foundation, I pursued advanced studies, earning a Master of Science in Mathematics from the same esteemed institution. Alongside my formal education, I have actively participated in professional development programs, including specialized training in effective teaching strategies, assessment, and classroom management offered by reputable institutions such as Oxford University Press and Book Wise Solutions.

Work Experience and Research Experience:

In terms of work experience, I have had the privilege to serve as a Mathematics Lecturer at Post Graduate College Sahiwal, where I imparted knowledge and inspired students towards academic excellence. Subsequently, I assumed the role of a Senior Teacher in Mathematics at Divisional Public School and College, contributing to the intellectual growth of young minds. Additionally, my tenure as a Teacher Assistant at COMSAT University Islamabad, Sahiwal Campus, provided me with invaluable insights into higher education dynamics.

Furthermore, I had the opportunity to extend my expertise beyond borders, working as an Online Foreign Coaching Specialist in Jeddah, Saudi Arabia, nurturing students globally in the realm of mathematics. My dedication and proficiency in teaching were acknowledged as I took on the position of a Senior Mathematics Teacher at ATM's High School in Sahiwal, Pakistan.

My previous research work includes Fractional Calculus which is the branch of calculus that generalizes the derivative of a function to non-integer order. During M.Sc. Mathematics research, I delve into the intricate realm of fractional calculus, a burgeoning field with diverse applications. Under the supervision of Dr. Asma, I have drafted my thesis entitled '' Stability analysis of implicit fractional boundary value problem with anti-periodic integral boundary conditions''. In this work, I investigate problem concerning to nonlinear implicit fractional order differential equations with special type of boundary conditions.

It is essential to note that in a real-world scenario when discrete pulses or sequences are identified, the data collected will determine which piecewise fractional differential operators to use. The innovative piecewise mathematical modeling notion that defines differential operators piecewise was recently suggested by Atangana, an African mathematician. I have examined a class of delay fractional differential equations using the Caputo sense of piecewise derivative in my MS research work. My expertise extends to research, notably culminating in a publication titled "On new updated concept for delay differential equations with piecewise Caputo fractional-order derivative" in the esteemed international journal, 'Waves in Random and Complex Media,' boasting an impressive impact factor of 3.41. My final year project on "Mathematical Analysis of Delay Type Piecewise Fractional Differential Equations" underscores my dedication to advancing the frontiers of mathematical knowledge.

Reasons to study in the United Kingdom:

Studying in the United Kingdom for my research degree offers unparalleled academic excellence, with prestigious universities and research institutions renowned worldwide. Here, I can collaborate with leading experts, access cutting-edge facilities, and engage in innovative research that pushes boundaries.

The UK's multicultural environment fosters diverse perspectives and ideas, enriching my academic experience and broadening my horizons. Networking opportunities abound, connecting me with professionals globally and paving the way for potential collaborations and career prospects.

Generous research funding and support systems in the UK ensure I have the resources to pursue my research ambitions, including scholarships, grants, and fellowships. This support empowers me to focus on my studies and make meaningful contributions to my field.

Additionally, the UK offers a rich cultural experience, blending history, tradition, and modernity. Exploring historic landmarks, immersing myself in the arts, and experiencing vibrant cultural scenes complement my academic journey and enhance my personal growth.

In essence, studying for my research degree in the United Kingdom aligns perfectly with my academic goals, offering academic excellence, global networking, robust support, and a rich cultural experience. It is a decision that promises to shape my future endeavors profoundly.

Reasons to study at University of Bristol:

As I contemplate my future academic journey, pursuing my PhD at the University of Bristol emerges as a compelling choice. The university's prestigious ranking, consistently hailed for academic excellence and research innovation, beckons me to join a community devoted to advancing knowledge and pushing the boundaries of discovery.

Considering the University of Bristol's prime location in the vibrant city of Bristol, I envision immersing myself in a culturally rich and diverse environment. Here, amidst the city's historical significance and thriving social scene, I anticipate a holistic experience that extends beyond academia, nurturing personal growth and enrichment.

Exploring my research interests further, I am drawn to Bristol's extensive research facilities and vibrant academic groups in Algebra, Analysis, and Geometry. From Applied Probability to Bayesian Modelling and Analysis, from Combinatorial Algebraic Geometry to Complexity Science, Bristol offers a fertile ground where I can cultivate my intellectual pursuits with ample support and expertise.

As I look ahead, the prospect of embarking on my PhD journey at the University of Bristol fills me with excitement and anticipation. It promises to be a transformative chapter where I can delve deep into my academic passions, collaborate with esteemed scholars, and contribute meaningfully to the world of research.

Reasons to study MSc by Research Mathematics:

Choosing to pursue an MSc by Research in Mathematics at the University of Bristol School of Mathematics is an exciting prospect, especially for someone like me who has already completed an MS in Mathematics and now seeks to deepen their engagement in research.

At the University of Bristol, the School of Mathematics has undergone significant growth, boasting a substantial faculty of approximately 130 academic staff and a thriving community of 90 postgraduates. This expansion speaks to the institution's commitment to fostering a vibrant research environment and providing ample opportunities for intellectual exploration and collaboration.

Ranked 4th in the UK for Mathematical Sciences in the prestigious Times Higher Education analysis of the Research Excellence Framework (REF) 2021, the School of Mathematics at Bristol stands as a beacon of excellence in the field. The academic staff, recognized as among the world's best in their respective areas, have garnered distinguished prizes, fellowships, and major grants, underscoring the caliber of expertise and mentorship available to students.

As an MSc by Research student, you'll benefit from personalized supervision that aligns with the School's esteemed research merits. The program is designed to cultivate cutting-edge mathematical skills and facilitate engagement in international collaborations, offering you a unique opportunity to contribute meaningfully to the advancement of mathematical knowledge.

Furthermore, the School's partnerships, such as the Heilbronn Institute for Mathematical Research—a collaboration with GCHQ—provide additional avenues for research and networking opportunities, particularly in areas like Discrete Mathematics, number theory, combinatorics, quantum information, and computational statistics.

With collaborative programs extending across various disciplines within the University of Bristol and beyond, including Biological, Computer, Earth, and Medical Sciences, Chemistry, Engineering, and Philosophy, you'll have access to a rich tapestry of interdisciplinary research opportunities, further enriching your academic experience.

In summary, pursuing an MSc by Research in Mathematics at the University of Bristol promises to be a transformative journey, offering unparalleled academic rigor, mentorship, and opportunities for intellectual growth and collaboration.

Motivation and Proposed Research Focus:

I am deeply motivated to pursue research in the realm of differential equations, with a particular emphasis on fractional and piecewise fractional differential equations. My aim is to unravel the intricate dynamics of complex systems, thereby contributing to the development of robust mathematical frameworks with practical applications. By understanding the behaviors encoded within these equations, I seek to deepen our comprehension of fundamental principles while addressing real-world challenges across diverse domains. I am particularly intrigued by piecewise fractional differential equations, which present fascinating challenges due to their discontinuous nature.

Fractional order derivatives and integrals offer a distinct advantage over integer order counterparts by capturing global characteristics rather than just local ones. The widespread adoption of fractional integrals and differential operators across modern technology and engineering underscores their significance in fields such as fluid mechanics, physics, automation, and biotechnology. Moreover, the emerging concept of piecewise fractional differential equations is gaining traction, notably in the study of infection disorders like Covid-19 and CAT-T cells-SARS-2 virus, where they can elucidate abrupt behavioral changes in dynamic processes. Additionally, the study of delay-type problems in fractional calculus is attracting considerable attention from researchers.

I am also eager to engage in interdisciplinary collaborations, leveraging differential equations to tackle pressing challenges in physics, biology, and engineering. Through my research, I aspire to advance knowledge in the fields of differential equations and fractional calculus, fostering transformative solutions to real-world problems and promoting interdisciplinary dialogue and collaboration.

Future Plans:

Embarking on an M.Sc. by research program in the domain of differential equations and fractional calculus marks the beginning of an exciting academic journey for me. During this program, I aim to explore various subfields, deepen my understanding, and refine my analytical skills through hands-on research experiences and collaborative projects with esteemed faculty members.

Upon successful completion of my M.Sc. by research, I intend to pursue a Ph.D. in the same field, ideally in Pakistan. The University of Bristol's esteemed reputation and its rich academic resources make it an ideal environment for furthering my academic and research pursuits. With access to state-of-the-art facilities and a vibrant academic community, I am confident that pursuing a Ph.D. here will provide ample opportunities to engage in cutting-edge research, collaborate with leading scholars, and contribute meaningfully to the advancement of knowledge in the field.

I am wholeheartedly committed to making the most of the opportunities offered by the University of Bristol and am enthusiastic about the prospect of joining the esteemed department. I sincerely appreciate your consideration of my application and eagerly await a favorable response.

Regards,

Fatima Ilyas

On new updated concept for delay differential equations with piecewise Caputo fractional-order derivative

ABSTRACT

It's worth mentioning that within a realistic situation when distinct pulses or sequences are detected, the choice of piecewise fractional differential operators will be guided by the data collected. Atangana, an African mathematician, recently proposed a novel piecewise mathematical modeling idea that defines differential operators piecewise. Short memory fractional order differential equations play significant roles in the description of many real world problems. Therefore, in this work, we have considered a class of delay fractional differential equations under the concept of piecewise derivative in the Caputo sense. We have expanded some results about the uniqueness, existence, and stability analysis for our considered problem. The results were obtained by utilizing the fixed point concept and nonlinear functional analysis tools. Sufficient circumstances have been set up to ensure the existence of at least one solution and its uniqueness to the suggested problem. Nonlinear tools of analysis were also used to determine its stability. We can observe that these derivatives can better describe an abrupt behavioral change in the dynamics of various processes of real problems. As a result, we concluded that this sort of calculus has now attracted researchers more. More research into how to deal with piecewise fractional-order differential boundary value problems will be conducted in the future.

Keywords:- Piecewise fractional differential equation, Piecewise Caputo type derivative; existence result; stability result; nonlinear analysis

Introduction

Mathematicians can create mathematical equations called partial and ordinary differential equations using the theory of the rate of change. In recent decades, certain kinds of differential calculus have received a lot of attention. In order to solve those equations, numerical and analytic methods have been devised. These mathematical equations have proven to be extremely effective in simulating real-world phenomena. However, these mathematical domains of integro differential calculus have consistently failed to recreate the physical phenomena multiple times due to the complexity of several real-world problems. To address these challenges, many classes of fractional derivatives were proposed. Additionally, the concept of piecewise derivatives of fractional orders have recently used in some papers to investigate some dynamical systems. It is also worth mentioning that delay-type problems have gotten a lot of attention in the subject of fractional calculus. Many researchers have introduced short memory fractional derivatives and a short memory fractional modeling procedure to resolve the issue. Hence, inspired from the applicability of piecewise fractional-order problems and delay differential equations, we consider the following class described as:

$$\begin{aligned} {}^{PCF}D_t^{\alpha} y(t) &= f(t, y(t), y(\lambda t), y(\tau - t)), \quad t \in J = [0, \mathrm{T}], \\ y(0) &= \varphi(y), \end{aligned} \tag{1}$$

where $0 < \alpha \leq 1, 0 < \lambda < 1$ and $\tau > 0$ the notion ${}^{PCF}D_t^{\alpha}$ stands for piecewise fractional derivative.Here $f : \mathrm{J} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\varphi \in C(J)$ is continuous functions.

The **piecewise fractional derivative** of order α is defined as,

$${}^{PCF}D_t^{\alpha} f(t) = \begin{cases} f'(t), & \text{if } 0 \leq \mathrm{t} \leq t_1, \\ {}^{C}D_t^{\alpha} f(t), & \mathrm{t}_1 \leq t \leq T, \end{cases}$$

here $f'(t)$ represents classical derivative on $t \in [0, t_1]$ and ${}^{C}D_t^{\alpha}$ represent Caputo fractional derivative on $t \in [t_1, T]$.

In 1967, Michele Caputo proposed the **Caputo fractional derivative** which Is as follows;

$${}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\zeta)}{(t-\zeta)^{\alpha+1-m}} d\zeta, \quad t > 0,$$

and $m = [\alpha] + 1$.

Consider the f to be a continuous function. The piecewise **integral** of f on Is given as;

$$
{}^{PPL}\Im_t^{\alpha} f(t) = \begin{cases} \int_0^{t_1} f(\zeta)d\zeta, & \text{if} \quad 0 \le \mathrm{t} \le t_1, \\ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{T} f(\zeta)(\mathrm{t}-\zeta)^{\alpha-1}d\zeta, & \text{if} \ \mathrm{t}_1 \le t \le T, \end{cases}
$$

Utilizing elementaryfractional calculus results, first, we convert the suggested Problem (1) into its integral foam such as;

$$
y(t) = \begin{cases} \varphi(y) + \int_0^{t_1} f(\zeta, y(\zeta), y(\lambda\zeta), y(\tau-\zeta))d\zeta, & \text{if} \quad 0 \le \mathrm{t} \le t_1, \\ y(t_1) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{T} f(\zeta, y(\zeta), y(\lambda\zeta), y(\tau-\zeta))(\mathrm{t}-\zeta)^{\alpha-1}d\zeta, & \text{if} \ \mathrm{t}_1 \le t \le T, \end{cases}
$$

Further, sufficient and necessary conditions are established to analyze the associated existence theory by using fixed point results and keeping in mind the importance of stability theory in numerical analysis and optimization theory, we establish some adequate results for U-H-type stability of suggested problem(1)(to see results[37]).

Background

Predicting and understanding physical phenomena is one of the most effective tools whichhave been developed in recent decades. Certainly, mathematics has proven its effectiveness in this respect, as seen by its widespread usage in modern scientific domains such as epidemiological, physics, and biology. In the past few decades, researchers have paid a lot of attention to fractional calculus. For instance, author has employed parameter estimation for fractional dynamical models arising in biology, modeling multiple electrochemical processes with fractional differential equations. Keeping in mind the above-mentioned, details, it was proved that fractional-order derivative in comparison with the classical derivative, has an advantage that the initial values take the same form as that for classicalinteger-order differential equations, which is more applicable for mathematical modeling. Many real world problems, exhibit some unpredictability that traditional mathematical models are unable to capture. In recent decades, the idea of stochastic mathematical differential equations has been proposed and used extensively with some significant achievements. But, instead of the following randomness, some problems follow non-locality trends, such as long-range dependence, fractal process, power law processes, and crossover behaviors, which means physical phenomena exhibit many behaviors. To address these challenges, a class of fractional derivatives was proposed, including fractional differential operators of singular type kernels fractional differential operators with singular type kernels, fractal fractional operators , and differential operators with regard to other functions. Caputo and Fabrizio developed an improved class of derivatives known as nonsingular in 2015, which has received a lot of attention from researchers [12]. In 2016, Atangana and his co-authors also generalized the above-mentioned operator by changing the exponential function with a Mittag–Leffler function. But there is one thing to bear in mind is that many real-world phenomena do not have a single behavior and rather exhibit a variety of behaviors. Moreover, employing the Caputo Fabrizio and Atangana Baleanu derivatives, real-world issues exhibiting various processes indicated by the Mittag–Leffler function and exponential function cannot be replicated. However, those operators are still ineffective in describing crossover behavior. For instance, authors [13] have studied some impulsive fractional-order problems and discussed the exact solutions, integral equations, and short-memory cases. On the basis of mentioned concept, authors [14] introduced and applied for the first time short memory fractional-order differential equations was introduced. It is remarkable that variable-order fractional differential equations are the natural extension that has also been studied in Refs [15-16]. By modeling various memory phenomena, usually the memory process is divided in two stages. One is related with a short memory with permanent retention, while the other one is governed by a simple model of fractional derivative (see Ref. [17]). Therefore, to address the problem with short memory, we need to use equations with piecewise fractional-order derivative. Authors [18] have used the concept of piecewise fractional differential equations to investigate some dynamical systems. Many researchers are using this idea to investigate infectious diseases models like the CAT-T cells-SARS-2 disease model investigated by using the said concept in Ref. [19]. A Covid-19 model has been studied using the mentioned concept in Ref. [20]. Authors [21] have considered the third wave of the Covid-19 outbreak in three countries. The writers of several articles will employ the concept of piecewise operators to calculate the numerical results of the food web model also (see Ref. [22]). More recently, authors [23] established some qualitative results devoted to the existence and uniqueness of solutions. In the same line authors [24] studied the existence theory for a problem under the piecewise concept of the derivative using the Caputo-Fabrizio operator. Here we refer to some work for more details on delay-type problems as, (an overview of delay-type differential equations and their applications in science [25], delay-type differential equations theory [26], oscillations and stability in population dynamics delay type differential equations [27], a new approach to fractional delay type differential equations [28].

Statement of Problem

Short memory fractional-order differential equations play significant roles in the description of many real-world problems. Therefore, in this work, we have considered a class of delay fractional differential equations under the concept of piecewise derivative in the Caputo sense. We have expanded some results about the uniqueness, existence, and stability analysis for our considered problem. We can observe that these derivatives can better describe an abrupt behavioral change in the dynamics of various processes of real problems. As a result, we concluded that this sort of calculus has now attracted researchers more. More research into how to deal with piecewise fractional-order differential boundary value problems will be conducted in the future.

Project objectives

The objectives of the research are:

- To introduce a class of delay fractional differential equations under the concept of piecewise derivative in the Caputo sense.
- The crossover behaviors of piecewise fractional derivatives are demonstrated by using examples and used tools are defined in Refs [33–36].
- Under the considered hypothesis and lemmas the existence and stability results for suggested problem have been devised.
- To solve some instances of piecewise delay fractional differential equations under the concept of Caputo derivative.

Justification

Results coming from this method are better than previous ones and can be handled easily.

Research Methodology

- Riemann-Liouville integral
- Caputo derivative
- Piecewise fractional integral and derivative
- Delay type piecewise fractional differential equations
- Existence theory and stability
- Latex
- Boundary value problems

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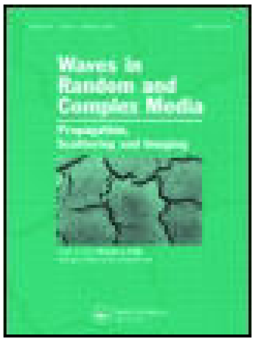

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On new updated concept for delay differential equations with piecewise Caputo fractional-order derivative

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On new updated concept for delay differential equations with piecewise Caputo fractional-order derivative

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ABSTRACT

This study introduces some updated results for piecewise mixed delay equations with Caputo-type fractional-order derivatives. We establish some necessary results for the aforementioned problem devoted to the existence and uniqueness of the solution and different forms of Ulam Hyers (U-H) type stability. To obtain the required results, fixed point theorems described by Krasnoselskii and Banach are used. Furthermore, the results related to U-H stabilities are determined by using the basic concept of nonlinear analysis. To illustrate our results, we provide a relevant test problem.

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1. Introduction

Predicting and understanding physical phenomena is one of the most effective tools which have been developed in recent decades to predict and avoid many of the worst-case scenarios. Certainly, mathematics has proven its effectiveness in this respect, as seen by its widespread usage in modern scientific domains such as epidemiological [1], physics [2], and biology [3]. In the past few decades, researchers have paid a lot of attention to fractional calculus. Infact, the properties of the differentiable functions are used to determine the derivatives of integer orders only in an infinitesimal neighborhood of the considered point. As a result, ordinary differential equations with respect to time cannot describe processes with a dynamic long memory. Due to this reason, to describe dynamic memory, it is possible to use the theory of fractional calculus with derivatives and integrals of fractional orders. Also, the concerned field has a variety of significant applications in many fields. For instance, author [4] has employed parameter estimation for fractional dynamical models arising in biology, modeling multiple electrochemical processes with fractional differential equations [5]. The physical understanding of the fractional derivative, the most famous and known one is the continuous time random walk [6], the reader should see the detailed theory about the area and applications in Ref. [7]. Keeping in mind the above-mentioned

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details, it was proved that fractional-order derivative in comparison with the classical derivative, has an advantage that the initial values take the same form as that for classical integer-order differential equations, which is more applicable for mathematical modeling. Mathematicians can create mathematical equations called partial and ordinary differential equations using the theory of the rate of change. In recent decades, certain kinds of differential calculus have received a lot of attention. In order to solve those equations, numerical and analytic methods have been devised. These mathematical equations have proven to be extremely effective in simulating real-world phenomena. However, these mathematical domains of integro-differential calculus have consistently failed to recreate the physical phenomena multiple times due to the complexity of several real-world problems.

For example, many real-world problems, exhibit some unpredictability that these mathematical models are unable to capture. In recent decades, the idea of stochastic mathematical differential equations has been proposed and used extensively with some significant achievements. But, instead of the following randomness, some problems follow nonlocality trends, such as long-range dependence, fractal process, power law processes, and crossover behaviors, which means physical phenomena exhibit many behaviors. To address these challenges, a class of fractional derivatives was proposed, including fractional differential operators of singular type kernels [8] fractional differential operators with singular type kernels [9], fractal fractional operators [10], and differential operators with regard to other functions [11]. So these mathematical models are given birth to a variety of nonlinear ordinary and partial mathematical differential and integral problems that have been successfully applied to a variety of issues. Despite this, the concern of crossover behaviors has yet to be resolved.

Caputo and Fabrizio developed an improved class of derivatives known as nonsingular in 2015, which has received a lot of attention from researchers [12]. This differential operator involves an exponential type of nonsingular kernel. In 2016, Atangana and his co-authors also generalized the above-mentioned operator by changing the exponential function with a Mittag–Leffler function. Despite of power law kernel, the Mittag–Leffler functions and exponential function exhibit crossover behavior in fractional differential operators. For instance, Using Mittag–Leffler stability theory a simple active control strategy for a new chaotic dynamical system with fractional derivatives has been discussed in Ref. [13], to investigate and solve a model of nonlinear fractional differential equations characterizing e deadly and destructive virus known as coronavirus disease under Mittag–Leffler operators [14] and see an extension of the exponential function to describe crossover nature in Ref. [15]. But there is one thing to bear in mind is that many real-world phenomena do not have a single behavior and rather exhibit a variety of behaviors. For example, economic fluctuations, earthquakes, and comparable molecular dynamics behaviors etc. To achieve better results in the aforementioned procedure, researchers are employing the various operators described earlier. Moreover, employing the Caputo Fabrizio and Atangana–Baleanu derivatives, real-world issues exhibiting various processes indicated by the Mittag–Leffler function and exponential function cannot be replicated. However, those operators are still ineffective in describing crossover behavior. To describe the mentioned behavior, the concept of short memory fractional-order derivative was introduced for the first time. For instance, authors [16] have studied some impulsive fractional-order problems and discussed the exact solutions, integral equations, and short-memory cases. On the basis of mentioned concept, authors [17] introduced and applied for the first time short memory fractional-order differential equations was introduced. It is remarkable that variable-order fractional differential equations are the natural extension that has also been studied in Refs [18,19].

In addition, fractional derivatives include the memory and genetic effects that play a crucial part in investigations of many real-world dynamical problems. Particularly, the use of fractional-order derivatives in mathematical models of the spread of infectious diseases which makes the situation more plausible. The standard fractional calculus is long memory effects and results in difficulties in long-term calculation problems. Furthermore, the power-law long memory is described by using the mathematical tools of classical fractional calculus, which involves the derivatives and integrals of fractional orders. Here, we remark that modeling various memory phenomena, usually the memory process is divided in two stages. One is related with a short memory with permanent retention, while the other one is governed by a simple model of fractional derivative (see Ref. [20]). Therefore, to address the problem with short memory, we need to use equations with piecewise fractional-order derivative. Because short memory can be used to improve performance and efficiency to explain physical phenomena more clearly (we refer [21] for details). Additionally, the concept of piecewise derivatives of fractional orders have recently used in some papers. For instance, authors [22] have used the concept of piecewise fractional differential equations to investigate some dynamical systems. Such operators will indeed be utilized to solve problems that exhibit crossover behavior. The concept is quickly getting popularity among researchers. Many researchers are using this idea to investigate infectious diseases models like the CAT-T cells-SARS-2 disease model investigated by using the said concept in Ref. [23]. A Covid-19 model has been studied using the mentioned concept in Ref. [24]. Authors [25] have considered the third wave of the Covid-19 outbreak in three countries, Turkey, Spain, and Czechia, which has been modeled using piecewise differential and integral operators. The writers of several articles will employ the concept of piecewise operators to calculate the numerical results of the food web model also (see Ref. [26]). More recently, authors [27] established some qualitative results devoted to the existence and uniqueness of solutions to a coupled system of nonlinear Cauchy problems. In the same line authors [28] studied the existence theory for a problem under the piecewise concept of the derivative using the Caputo-Fabrizio operator.

It is also worth mentioning that delay-type problems have gotten a lot of attention in the subject of fractional calculus. It is a significant branch in this field. A number of researchers have been working on delay-type problems at an incredible rate. Here we refer to some work for more details on delay-type problems as, (an overview of delay-type differential equations and their applications in science [29], delay-type differential equations theory [30], oscillations and stability in population dynamics delay type differential equations [31], a new approach to fractional delay type differential equations [32]). Further, delay when inserted in a dynamical problem, for instance, we include a delay to fractional-order epidemic model to take into account the incubation period of the disease.

As we have mentioned that models involve ordinary fractional-order derivative need large storage space and cause poor efficiency. Therefore, researchers have introduced short-memory fractional derivatives and a short memory fractional modeling procedure to resolve the issue. Hence, inspired from the applicability of piecewise fractional-order problems and delay differential equations, we consider the following class described as

$$\begin{cases} {}^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0+}\mathfrak{y}(\mathfrak{s}) = \mathfrak{u}(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \mathfrak{y}(\lambda\mathfrak{s}), \mathfrak{y}(\tau - \mathfrak{s})), \quad \mathfrak{s} \in \mathfrak{J} = [0, \mathcal{T}], \\ \mathfrak{y}(0) = \phi(\mathfrak{y}), \end{cases} \tag{1}$$

where $0 < \mathfrak{p} \leq 1, 0 < \lambda < 1$ and $\tau > 0$ the notion ${}^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0+}$ stands for piecewise fractional derivative. Let $\phi \in \mathcal{C}(\mathfrak{J})$ and $\mathfrak{u} : \mathfrak{J} \times \mathfrak{R} \times \mathfrak{R} \longrightarrow \mathfrak{R}$ is a continuous function. We develop the existence theory and stability results for the considered problem. Utilizing elementary fractional calculus results, first, we convert the suggested problem into an integral form of piecewise. Further, sufficient and necessary conditions are established to analyze the associated existence theory by using fixed point results [32]. Keeping in mind the importance of stability theory in numerical analysis and optimization theory, we establish some adequate results for U-H-type stability. Although many stability notions, such as exponential, Lyapunov, and Mittag -Leffler types have been introduced in the literature. Since U-H type stability has recently received a lot of attention. Therefore, we investigate the said stability results for the considered problem. Author [33] used the Laplace transformation to determine the stability of the linear Caputo- Fabrizio differential equations. Also, some generalized U-H stability results in quaternionic analysis for the initial boundary value problem of fractional differential equations are obtained in Ref. [34], U-H stability for Hilfer–Hadamard fractional differential equations [35] and U-H stability for implicit impulsive problems has been discussed in Ref. [36]. Using nonlinear analysis, we develop certain sufficient requirements for different forms of U-H type stabilities for the presented problem. Further, the crossover behaviors of piecewise fractional derivatives are demonstrated by using examples. For the required results, we use tools defined in Refs [37–40].

Our paper is organized as: Section 1 is devoted to introduction. In Section 2, we recall some preliminaries. In Section 3, existence theory is derived. Section 4 is devoted to stability results. Section 5 is devoted to example. Section 6 is related to a brief conclusion of the paper.

2. Preliminaries

The given results are re-collected.

Definition 2.1 ([17]): Let $\mathfrak{y} \in L^1([0, \mathcal{T}], \mathfrak{R})$ Riemann-Liouville integral of order $\mathfrak{p} > 0$ is defined by

$$\mathfrak{J}^{\mathfrak{p}}_{0+}\mathfrak{y}(\mathfrak{s}) = \frac{1}{\Gamma(\mathfrak{p})} \int_0^t (t - \xi)\, d\xi, \quad s \in [0, \mathcal{T}]$$

provided that integral on right exists.

Definition 2.2 ([17]): Let $\mathfrak{y}$ be differentiable, then Caputo derivative of order $\mathfrak{p} \in (0, 1]$ is defined by

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}^{\mathfrak{p}}_{0+}\mathfrak{y}(\mathfrak{s}) = \dfrac{1}{\Gamma(1 - \mathfrak{p})} \displaystyle\int_0^t (t - \xi)^{-\mathfrak{p}}\mathfrak{y}'(\xi)\, d\xi, & s \in [0, \mathcal{T}],\ 0 < \mathfrak{p} < 1, \\ \dfrac{d\mathfrak{y}(\mathfrak{s})}{d\mathfrak{s}}, & \text{if } \mathfrak{p} = 1. \end{cases}$$

Definition 2.3 ([22]): The piecewise fractional integral of a continuous function $\mathfrak{y}$ with fractional order $\mathfrak{p} \in (0,1]$ is defined as

$$ {}^{\mathcal{PCF}}\mathfrak{J}_{0+}^{\mathfrak{p}}\mathfrak{y}(\mathfrak{s}) = \begin{cases} \displaystyle\int_0^{\mathfrak{s}_1} \mathfrak{y}(\xi)\,\mathrm{d}\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \displaystyle\frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi)(\mathfrak{s}-\xi)^{\mathfrak{p}-1}\,\mathrm{d}\xi, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}, \end{cases} \tag{2} $$

where ${}^{\mathcal{PCF}}\mathfrak{J}_{0+}^{\mathfrak{p}}$ represents classical integral on $0 \le \mathfrak{s} \le \mathfrak{s}_1$ and power law integral on $\mathfrak{s}_1 \le \mathfrak{s} \le \mathcal{T}$.

Definition 2.4 ([22]): Let $\mathfrak{g}$ be a differentiable function, then the piecewise fractional derivative of a function $\mathfrak{u}$ with fractional and classical derivative involving power-law kernel is defined as

$$ {}^{\mathcal{PCF}}\mathfrak{D}_{0+}^{\mathfrak{p}}\mathfrak{y}(\mathfrak{s}) = \begin{cases} \mathfrak{y}(\mathfrak{s}), & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ {}^{\mathcal{C}}\mathfrak{D}_{\mathfrak{s}}^{\mathfrak{p}}\mathfrak{y}(\mathfrak{s}), & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}, \end{cases} \tag{3} $$

where ${}^{\mathcal{C}}\mathfrak{D}_{0+}^{\mathfrak{p}}$ represents integer-order derivative on $0 \le \mathfrak{s} \le \mathfrak{s}_1$ and Caputo derivative on $\mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}$.

Lemma 2.1 ([22,25]): *Keeping the aforementioned definition in mind, the solution of the given piecewise problem with fractional derivative*

$$ \begin{cases} {}^{\mathcal{PCF}}\mathfrak{D}_{0+}^{\beta}\mathfrak{y}(\mathfrak{s}) = \mathfrak{h}(\mathfrak{s}), \quad \beta \in (0,1], \\ \mathfrak{y}(0) = \phi(\mathfrak{y}), \end{cases} $$

is equal to

$$ \mathfrak{y}(\mathfrak{s}) = \begin{cases} \mathfrak{y}(0) + \displaystyle\int_0^{t_1} \mathfrak{h}(\xi)\,\mathrm{d}\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \displaystyle\frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{h}(\xi)(\mathfrak{s}-\xi)^{\mathfrak{p}-1}\,\mathrm{d}\xi, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{4} $$

Here we defining a Banach space $\mathcal{H}$ as discussed in Ref. [37] *contains functionals of the form $\mathfrak{y}(\mathfrak{s})$ from $\mathfrak{J}$ to R equipped with a norm $\|y\|_{\mathcal{H}} = \sup_{\mathfrak{s}\in\mathfrak{J}} |y(\mathfrak{s})|$. Let consider a cone $\mathcal{K}$ contains functionals of the form $\mathfrak{y}(\tau - \mathfrak{s})$ in the space $\mathcal{H}$, where $\mathcal{K} = \{\mathfrak{y} \in \mathcal{H} : \mathfrak{y}(\mathfrak{s}) \ge s^l\|\mathfrak{y}\|, l \in [0,1]\}$. Then, we see that*

$$ s^l\|\mathfrak{y}\| \le \mathfrak{y}(s) \le \|\mathfrak{y}\|. $$

Theorem 2.5 ([37]): *If $\mathcal{K} \subset \mathcal{H}$ is a closed, convex and non-empty subset of $\mathcal{H}$, then $\mathcal{P}, \mathcal{Q}$ are two operators with,*

(1) $\mathcal{P}\mathfrak{x} + \mathcal{Q}\mathfrak{y} \in \mathcal{K}$ *where* $\mathfrak{x}, \mathfrak{y} \in \mathcal{K}$;
(2) $\mathcal{P}$ *is contraction.*
(3) $\mathcal{Q}$ *is completely continuous.*

Then there exist at least one fixed point $z \in \mathcal{K}$ exists such that $z = \mathcal{P}z + \mathcal{Q}z$.

3. Existence theory

Existence theory is an important consequence of mathematical analysis of various real-world problems. It is important that verify the existence of solution to a dynamical problem by using some useful mathematical tools. In this regard, for classical fractional-order problems plenty of research works have been done. Researchers have used the fixed point theory is a powerful tool to investigate the existence theory for the aforementioned area. However, problems involving piecewise integrals and derivatives, the aforesaid area has very rarely studied. Therefore, it is needed to derive some sufficient conditions for the existence of solution to the considered problems by extending the usual classical fixed point theory. The importance of tools of nonlinear analysis can be read in Refs [38,39].

Lemma 3.1: *By using aforementioned Lemma* 2.1, *the solution of piecewise problem*

$$\begin{aligned} &{}^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0^+}\mathfrak{y}(\mathfrak{s}) = \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi)), \quad \mathfrak{s} \in [0, \mathcal{T}], \\ &\mathfrak{y}(0) = \phi(\mathfrak{y}), \end{aligned} \tag{5}$$

is calculated as

$$\mathfrak{y}(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{y}) + \displaystyle\int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))\, \mathrm{d}\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \dfrac{1}{\Gamma(\mathfrak{p})}\displaystyle\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, \mathrm{d}\xi, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{6}$$

Proof: Inview of Lemma 2.1, from (5), we can easily obtain the result as

$$\mathfrak{y}(\mathfrak{s}) = \begin{cases} \mathfrak{y}(0) + \displaystyle\int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))\, \mathrm{d}\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \dfrac{1}{\Gamma(\mathfrak{p})}\displaystyle\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, \mathrm{d}\xi, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{7}$$

Using the result $\mathfrak{y}(0) = \phi(\mathfrak{y})$ in above result, we obtain

$$\mathfrak{y}(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{y}) + \displaystyle\int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))\, \mathrm{d}\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \dfrac{1}{\Gamma(\mathfrak{p})}\displaystyle\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, \mathrm{d}\xi, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{8}$$

Hence the proof is completed. ∎

Corollary 3.1: *Thankful to Lemma* 3.1, *the solution of our suggested problem* (1) *is find as*

$$\mathfrak{y}(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{y}) + \displaystyle\int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))\, \mathrm{d}\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \dfrac{1}{\Gamma(\mathfrak{p})}\displaystyle\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, \mathrm{d}\xi, & \text{if} \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}. \end{cases}$$

To demonstrate the existence of solutions to suggested problem (1), *we have to define operator* $\mathbf{Z} : \mathcal{H} \to \mathcal{H}$ *by*

$$(\mathbf{Z}\mathfrak{y})(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{y}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))\, d\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \dfrac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, d\xi, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{9}$$

$$(\mathbf{Z}\mathfrak{x})(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{x}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))\, d\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{x}(\mathfrak{s}_1) + \dfrac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, d\xi, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{10}$$

We need the given hypothesis to be held:

(F_1) Under the continuity of $\mathfrak{u}$ and $\mathfrak{U}_1, \mathfrak{U}_2, \mathcal{V}_1, \mathcal{V}_2, \mathcal{W}_1, \mathcal{W}_2 \in \mathfrak{R}$ there exist some constants $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ such that,

$$|\mathfrak{g}(\mathfrak{s}, \mathfrak{U}_1, \mathcal{V}_1, \mathcal{W}_1) - \mathfrak{g}(\mathfrak{s}, \mathfrak{U}_2, \mathcal{V}_2, \mathcal{W}_2)| \le \mathcal{M}_1 |\mathfrak{U}_1 - \mathfrak{U}_2| + \mathcal{M}_2 |\mathcal{V}_1 - \mathcal{V}_2| + \mathcal{M}_3 |\mathcal{W}_1 - \mathcal{W}_2|.$$

(F_2) Let $\mathfrak{U}, \mathcal{V} \in \mathfrak{R}$ and $\mathbf{C}_\phi > 0$ be any constant such that

$$|\phi(\mathfrak{U}) - \phi(\mathcal{V})| \le \mathbf{C}_\phi |\mathfrak{U} - \mathcal{V}|.$$

Theorem 3.1: *Under the hypothesis* (F_1), (F_2), *our suggested problem* (1) *has a unique solution if the condition*

$$\mathbf{D} = \max \left\{ \mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}} \mathfrak{s}_1, \mathbf{M}_{\mathfrak{u}} \frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p} + 1)} \right\} < 1$$

holds.

Proof: We have to use the principle of Banach space contraction to show that $\mathbf{Z} : \mathcal{H} \to \mathcal{H}$ defined in Corollary 3.1 has a fixed point [28]. Firstly we have to show that $\mathbf{Z}$ is a contraction mapping. Therefore, we have

$$\|\mathbf{Z}(\mathfrak{y}) - \mathbf{Z}(\mathfrak{x})\| \le \sup_{\mathfrak{s} \in \mathfrak{J}} \begin{cases} |\phi(\mathfrak{y}) - \phi(\mathfrak{x})| + \int_0^{\mathfrak{s}_1} |\mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi)) \\ \quad - \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))|\, d\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ |\mathfrak{y}(\mathfrak{s}_1) - \mathfrak{x}(\mathfrak{s}_1)| + \dfrac{1}{\Gamma(\mathfrak{p})} \\ \quad \Big[\int_{\mathfrak{s}_1}^{\mathfrak{s}} |\mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi)) \\ \quad - \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))|(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, d\xi \Big] \end{cases}$$

which yields

$$\|\mathbf{Z}(\mathfrak{y}) - \mathbf{Z}(\mathfrak{x})\| \leq \sup_{\mathfrak{s}\in\mathfrak{I}} \begin{cases} \mathbf{C}_\phi|\mathfrak{y}-\mathfrak{x}| + (\mathcal{M}_1|\mathfrak{y}(\xi)-\mathfrak{x}(\xi)| \\ \quad +\mathcal{M}_2|\mathfrak{y}(\lambda\xi)-\mathfrak{x}(\lambda\xi)| \\ \quad +\mathcal{M}_3|\mathfrak{y}(\tau-\xi)-\mathfrak{x}(\tau-\xi)|)\mathfrak{s}_1 & \text{if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \dfrac{(\mathcal{T}-\mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}(\mathcal{M}_1|\mathfrak{y}(\xi)-\mathfrak{x}(\xi)| \\ \quad +\mathcal{M}_2|\mathfrak{y}(\lambda\xi)-\mathfrak{x}(\lambda\xi)| \\ \quad +\mathcal{M}_3|\mathfrak{y}(\tau-\xi)-\mathfrak{x}(\tau-\xi)|), & \text{if } \mathfrak{s}_1 < \mathfrak{s} \leq \mathcal{T}. \end{cases}$$

By taking $\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 = \mathbf{M}_{\mathfrak{u}}$, then on further simplification, one has

$$\|\mathbf{Z}(\mathfrak{y}) - \mathbf{Z}(\mathfrak{x})\| \leq \sup_{\mathfrak{s}\in\mathfrak{I}} \begin{cases} (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1)|\mathfrak{y}-\mathfrak{x}|, & \text{if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \mathbf{M}_{\mathfrak{u}}\dfrac{(\mathcal{T}-\mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}|\mathfrak{y}-\mathfrak{x}|, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \leq \mathcal{T}. \end{cases}$$

In short, we can write

$$\|\mathbf{Z}(\mathfrak{y}) - \mathbf{Z}(\mathfrak{x})\| \leq \mathbf{D}\|\mathfrak{y}-\mathfrak{x}\|,$$

where $\mathbf{D} = \max\{\mathbf{C}_\phi + \mathbf{M}_u\mathfrak{s}_1, \mathbf{M}_u\frac{(\mathcal{T}-\mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\}$. Since $\mathbf{D} < 1$, thus from (F_3) the operator $\mathbf{Z} : \mathcal{H} \to \mathcal{H}$ is a contraction mapping. So suggested problem (1) has a unique solution. ■

Let the given hypothesis holds:

(F_3) There exist some constants such that $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 > 0$ and $\mathbf{L}_{\mathfrak{u}} > 0$, then

$$|\mathfrak{u}(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \mathfrak{y}(\lambda\mathfrak{s}), \mathfrak{y}(\tau-\mathfrak{s}))| \leq \mathcal{N}_1(\mathfrak{s})|\mathfrak{y}(\mathfrak{s})| + \mathcal{N}_2(\mathfrak{s})|\mathfrak{y}(\lambda\mathfrak{s})| + \mathcal{N}_3(\mathfrak{s})|\mathfrak{y}(\tau-\mathfrak{s})| + \mathbf{L}_{\mathfrak{u}}(\mathfrak{s}).$$

Also we denote

$$\mathbf{N}^* = \sup_{\mathfrak{s}\in\mathfrak{I}}|\mathbf{N}_{\mathfrak{u}}(\mathfrak{s})|, \quad \mathbf{L}^* = \sup_{\mathfrak{s}\in\mathfrak{I}}|\mathbf{L}_{\mathfrak{u}}(\mathfrak{s})|.$$

Theorem 3.2: *Under the assumptions* (F_1)–(F_3)*, our suggested problem* (1) *has at least one solution if the given condition hold, which is stated*

Proof: Firstly, we define the operator as

$$(\mathbf{A}\mathfrak{y})(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{y}), & \text{if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1), & \text{if } \mathfrak{s}_1 < \mathfrak{s} \leq \mathcal{T}, \end{cases} \tag{11}$$

and

$$(\mathbf{B}\mathfrak{y})(t) = \begin{cases} \displaystyle\int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi))\,d\xi, & \text{if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \dfrac{1}{\Gamma(\mathfrak{p})}\displaystyle\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi))(\mathfrak{s}-\xi)^{\mathfrak{p}-1}\,d\xi, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \leq \mathcal{T}. \end{cases} \tag{12}$$

Step 1 Assume a set $\Delta = \{\mathfrak{y} \in \mathcal{H} : \|\mathfrak{y}\| \le \mathfrak{r}\}$ as ϕ and f both are continuous functions so is **A**. Then we have to show that **A** is contraction mapping. Let $\mathfrak{x}, \mathfrak{y} \in \Delta$, so it is easy to check the inequality

$$\|\mathbf{A}(\mathfrak{y}) - \mathbf{A}(\mathfrak{x})\| \le \sup_{\mathfrak{s}\in\mathfrak{I}} \begin{cases} |\phi(\mathfrak{y}) - \phi(\mathfrak{x})|, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ |\mathfrak{y}(\mathfrak{s}_1) - \mathfrak{x}(\mathfrak{s}_1)|, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}, \end{cases}$$

then

$$\|\mathbf{A}(\mathfrak{y}) - \mathbf{A}(\mathfrak{x})\| \le \sup_{\mathfrak{s}\in\mathfrak{I}} \begin{cases} \mathbf{C}_\phi \|\mathfrak{y} - \mathfrak{x}\|, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ 0, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}. \end{cases}$$

Using the assumption (F_2) and condition $\mathbf{C}_\phi < 1$, we have

$$\|\mathbf{A}(\mathfrak{y}) - \mathbf{A}(\mathfrak{x})\| \le \mathbf{C}_\phi \|\mathfrak{y} - \mathfrak{x}\|.$$

So, it is clear that **A** is contraction mapping.

Step 2 To show that **B** is completely continuous, we proceed as $\mathfrak{y} \in \Delta$ then one has

$$\|\mathbf{B}(y)\| \le \sup_{\mathfrak{s}\in\mathfrak{I}} \begin{cases} \displaystyle\int_0^{\mathfrak{s}_1} |\mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))| \, d\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \displaystyle\frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} |\mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))| (\mathfrak{s} - \xi)^{\mathfrak{p}-1} \, d\xi, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathfrak{T}. \end{cases} \tag{13}$$

By using hypothesis (F_3), we have

$$\|\mathbf{B}(\mathfrak{y})\| \le \sup_{\mathfrak{s}\in\mathfrak{I}} \begin{cases} \displaystyle\int_0^{\mathfrak{s}_1} [(|\mathcal{N}_1(\mathfrak{s})| + \mathcal{N}_2(\mathfrak{s})| + \mathcal{N}_3(\mathfrak{s})|)\mathfrak{r} + |\mathbf{L}_\mathfrak{u}(\mathfrak{s})|], & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \displaystyle\frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} [(|\mathcal{N}_1(\mathfrak{s})| + \mathcal{N}_2(\mathfrak{s})| + \mathcal{N}_3(\mathfrak{s})|)\mathfrak{r} + |\mathbf{L}_\mathfrak{u}(\mathfrak{s})|], & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{14}$$

Also we have

$$\mathbf{N}_1^* = \sup_{\mathfrak{s}\in\mathfrak{I}} |\mathcal{N}_1(\mathfrak{s})|,$$

$$\mathbf{N}_2^* = \sup_{\mathfrak{s}\in\mathfrak{I}} |\mathcal{N}_2(\mathfrak{s})|,$$

$$\mathbf{N}_3^* = \sup_{\mathfrak{s}\in\mathfrak{I}} |\mathcal{N}_3(\mathfrak{s})|.$$

Further, we take $\mathbf{N}_1^* + \mathbf{N}_2^* + \mathbf{N}_3^* = \mathbf{N}^*$, thus (14) yields

$$\|\mathbf{B}(\mathfrak{y})\| \le \sup_{\mathfrak{s}\in\mathfrak{I}} \begin{cases} \mathfrak{s}_1(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*), & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \displaystyle\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*), & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}. \end{cases}$$

Using

$$\mathbf{R}^* = \max\left\{\mathfrak{s}_1(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*), \frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*)}{\Gamma(\mathfrak{p}+1)}\right\}.$$

one has

$$\|\mathbf{B}(\mathfrak{y})\| \le \mathbf{R}^*.$$

Therefore, **B** is uniformly bounded.

Step 3 Now for equi-continuity, assume $\mathfrak{s}_2 < \mathfrak{s}_3 \in \mathfrak{I}$, then

$$\begin{aligned}
|\mathbf{B}\mathfrak{y}(\mathfrak{s}_3) - \mathbf{B}\mathfrak{y}(\mathfrak{s}_2)| &= \Bigg| \frac{1}{\Gamma(\mathfrak{p})} \Bigg[\int_{\mathfrak{s}_1}^{\mathfrak{s}_3} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s}_3 - \xi)^{\mathfrak{p}-1}\, d\xi \\
&\quad - \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s}_2 - \xi)^{\mathfrak{p}-1}\, d\xi \Bigg] \Bigg| \\
&\le \Bigg| \frac{1}{\Gamma(\mathfrak{p})} \Bigg[\int_{\mathfrak{s}_2}^{\mathfrak{s}_3} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s}_3 - \xi)^{\mathfrak{p}-1}\, d\xi \\
&\quad + \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))[(\mathfrak{s}_3 - \xi)^{\mathfrak{p}-1} - (\mathfrak{s}_2 - \xi)^{\mathfrak{p}-1}]\, d\xi \Bigg] \Bigg| \\
&\le \frac{(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*)}{\Gamma(\mathfrak{p})} \\
&\quad \times \Bigg[\int_{\mathfrak{s}_2}^{\mathfrak{s}_3} (\mathfrak{s}_3 - \xi)^{\mathfrak{p}-1}\, d\xi + \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} [(\mathfrak{s}_3 - \xi)^{\mathfrak{p}-1} - (\mathfrak{s}_2 - \xi)^{\mathfrak{p}-1}]\, d\xi \Bigg] \\
&\le \frac{(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*)}{\Gamma(\mathfrak{p}+1)} [(\mathfrak{s}_3 - \mathfrak{s}_1)^{\mathfrak{p}} - (\mathfrak{s}_2 - \mathfrak{s}_1)^{\mathfrak{p}}] \\
&\le \frac{(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*)}{\Gamma(\mathfrak{p}+1)} [(\mathfrak{s}_3 - \mathfrak{s}_1)^{\mathfrak{p}} - (\mathfrak{s}_2 - \mathfrak{s}_1)^{\mathfrak{p}}] \to 0, \quad \text{as } \mathfrak{s}_3 \to \mathfrak{s}_2.
\end{aligned}$$

As **B** is continuous and bounded on $\mathfrak{I}$, so it is uniformly continuous. Thus, we have

$$\|\mathbf{B}\mathfrak{y}(\mathfrak{s}_3) - \mathbf{B}\mathfrak{y}(\mathfrak{s}_2)\| \to 0, \quad \text{as } \mathfrak{s}_3 \to \mathfrak{s}_2.$$

Hence **B** is equi-continuous. Therefore, $\mathbf{B}(\Delta)$ is also compact. Using Arzelá Ascoli theorem, **B** is uniformly continuous operator. The theorem dependent on Krasnoselkii's fixed point result has now achieved its conclusion. So there exist at least one solution for the problem (1) on $\mathfrak{I}$. ■

4. Stability results

Stability is important concept which needed in numerical as well as in optimization theory of differential equations (see Ref. [40]). Here, we use U-H concept to develop some necessary results related to stability theory for the considered problem under the concept of piecewise equations. Our goal is to develop criteria for

- U-H stability
- Generalized U-H stability
- U-H Rassias stability
- Generalized U-H Rassias stability.

Definition 4.1: Our suggested problem (1) is U-H stable, if for any $\epsilon > 0$, the following relation given as

$$|^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0+}\mathfrak{y}(\mathfrak{s}) - \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))| < \epsilon, \quad \text{forall } \mathfrak{s} \in \mathfrak{I} \tag{15}$$

holds. Keep the requirement for a unique solution $\mathfrak{x}$ of problem (1) with $\mathcal{K}_{\mathfrak{u}} > 0$ as

$$\|\mathfrak{y} - \mathfrak{x}\| < \mathcal{K}_{\mathfrak{u}}\epsilon, \quad \text{forall } \mathfrak{s} \in \mathfrak{I}.$$

For a non-decreasing map $\psi_{\mathfrak{u}} : [0, \infty) \to \mathfrak{R}^+$, the following relation

$$\|\mathfrak{y} - \mathfrak{x}\| < \mathcal{K}_{\mathfrak{u}}\varphi(\epsilon), \quad \mathfrak{s} \in \mathfrak{I},$$

holds with $\psi(0) = 0$, then the suggested results is said to be generalized U-H stable.

Definition 4.2: Our suggested problem (1) is U-H Rassias stable equivalent to the function $\varphi \in \mathcal{C}(\mathfrak{I}, \mathfrak{R})$, if for any $\epsilon > 0$, the following relation

$$|^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0+}\mathfrak{y}(\mathfrak{s}) - \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))| < \epsilon\varphi(\mathfrak{s}), \quad \mathfrak{s} \in \mathfrak{I}, \tag{16}$$

holds. Keep the requirement for a unique solution $\mathfrak{x}$ of problem (1) with $\mathcal{K}_{\mathfrak{u}} > 0$ as it is

$$\|\mathfrak{y} - \mathfrak{x}\| < \mathcal{K}_{\mathfrak{u},\varphi}\epsilon\varphi(\mathfrak{s}), \quad \mathfrak{s} \in \mathfrak{I}.$$

Again a function $\varphi : [0, \infty) \to \mathfrak{R}^+$, such that

$$|^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0+}\mathfrak{y}(\mathfrak{s}) - \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))| < \varphi(\mathfrak{s}), \quad \mathfrak{s} \in \mathfrak{I},$$

and there is a unique solution $\mathfrak{x}$ of problem with $\mathcal{K}_{\mathfrak{u},\varphi} > 0$, such that

$$\|\mathfrak{y} - \mathfrak{x}\| < \mathcal{K}_{\mathfrak{u},\varphi}\varphi(\mathfrak{s}), \mathfrak{s} \in \mathfrak{I}.$$

holds, then the suggested problem (1) is said to be generalized U-H Rassias stable.

Prior to deriving the main findings of stability, we make the following observations

Remark 4.1: The inequality (8) has a solution $\mathfrak{y} \in \mathcal{H}(\mathfrak{I}, \mathfrak{R})$, if and only if a function $\phi \in \mathcal{H}(\mathfrak{I}, \mathfrak{R})$ exist such that

(1) $|\phi(\mathfrak{s})| \leq \epsilon, \mathfrak{s} \in \mathfrak{I}$.
(2) $^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0+}\mathfrak{y}(\mathfrak{s}) = \mathfrak{g}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi)) + \phi(\mathfrak{s}), \mathfrak{s} \in \mathfrak{I}$.

Lemma 4.1: *Again consider the solution of our suggested problem* (1)

$$\begin{aligned} &{}^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0^+}\mathfrak{y}(\mathfrak{s}) = \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi)), \quad \mathfrak{s} \in [0, \mathcal{T}], \\ &\mathfrak{y}(0) = \phi(\mathfrak{y}). \end{aligned} \tag{17}$$

is computed as

$$\mathfrak{y}(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{y}) + \displaystyle\int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))\, d\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \dfrac{1}{\Gamma(\mathfrak{p})} \displaystyle\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}}\, d\xi, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{18}$$

The above solution moreover fulfils the following criteria using (8)

$$\|\mathfrak{y} - \mathbf{Z}\mathfrak{x}\| \le \begin{cases} \mathfrak{s}_1\epsilon, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \left[\dfrac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}-1}}{\Gamma(\mathfrak{p}+1)}\right]\epsilon = \Omega\epsilon, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{19}$$

Proof: Thank full to Lemma 3.1 and Remark 4.1, solution of (17) can be generated easily. On further simplification the relation of Equation (15) is easily straight forward. ∎

Theorem 4.3: *Thank full to Lemma* 4.1*, the generated solution of suggested problem* (1) *is U-H stable and on further simplification the problem* (1) *is also generalized U-H stable by using the condition* $(\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{g}}\mathfrak{s}_1) < 1$.

Proof: We can divide our results into two categories. Consider, $\mathfrak{I}_1 = [0, \mathfrak{s}_1]$ and $\mathfrak{I}_2 = [\mathfrak{s}_1, \mathcal{T}]$ and then proceed as,

Case 1 When $\mathfrak{s} \in \mathfrak{I}_1$,then

$$\begin{aligned} \|\mathfrak{y} - \mathfrak{x}\| &= \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - \left(\phi(\mathfrak{x}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))\, d\xi\right) \right| \\ &\le \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - \left(\phi(\mathfrak{y}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi)\, d\xi\right) \right| \\ &\quad + \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \phi(\mathfrak{y}) - \phi(\mathfrak{x}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))\, d\xi \right. \\ &\quad \left. - \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))\, d\xi \right| \\ &\le \sup_{\mathfrak{s}\in\mathfrak{I}} (\mathfrak{s}_1\epsilon + \mathbf{C}_\phi|\mathfrak{y} - \mathfrak{x}| + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1|\mathfrak{y} - \mathfrak{x}|) \\ &\le \mathfrak{s}_1\epsilon + (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{g}}\mathfrak{s}_1)\|\mathfrak{y} - \mathfrak{x}\|. \end{aligned}$$

The further simplification gives

$$\|\mathfrak{y} - \mathfrak{x}\| \le \left(\frac{\mathfrak{s}_1}{1 - (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1)}\right)\epsilon. \tag{20}$$

Case 2 When $\mathfrak{s} \in \mathfrak{I}_2$,then

$$\|\mathfrak{y}-\mathfrak{x}\| = \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - \left(\mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau-\xi))(\mathfrak{s}-\xi)^{\mathfrak{p}-1}\,d\xi\right)\right|$$

$$\leq \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - \left(\mathfrak{y}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi))(\mathfrak{s}-\xi)^{\mathfrak{p}-1}\,d\xi\right)\right|$$

$$+ \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y}(\mathfrak{s}_1) - \mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi))(\mathfrak{s}-\xi)^{\mathfrak{p}-1}\,d\xi \right.$$

$$\left. -\frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau-\xi))(\mathfrak{s}-\xi)^{\mathfrak{p}-1}\,d\xi \right|$$

$$\leq \left[\frac{(\mathcal{T}-\mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\right]\epsilon + \mathbf{C}_\phi\|\mathfrak{y}-\mathfrak{x}\| + \mathbf{M}_{\mathfrak{u}}\frac{(\mathcal{T}-\mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\|\mathfrak{y}-\mathfrak{x}\|$$

$$\leq \Omega\epsilon + (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\Omega)\|\mathfrak{y}-\mathfrak{x}\|,$$

where

$$\left[\frac{(\mathcal{T}-\mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\right] = \Omega$$

Hence we get

$$\|\mathfrak{y}-\mathfrak{x}\| \leq \left(\frac{\Omega}{1-(\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\Omega)}\right)\epsilon. \tag{21}$$

If we combine **Case 1** and **Case 2**, we can at the following conclusion

$$\mathcal{K}_1 = \max\left\{\frac{\mathfrak{s}_1}{1-(\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1)}, \frac{\Omega}{1-(\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\Omega)}\right\}.$$

Then we have

$$\|\mathfrak{y}-\mathfrak{x}\| \leq \mathcal{K}_1\epsilon, \quad \text{forall } \mathfrak{s} \in \mathfrak{I}. \tag{22}$$

Hence, problem (1) is U-H stable. If we take $\psi(\epsilon) = \epsilon$, and $\psi(0) = 0$, then problem (1) is also generalized U-H stable. ∎

Prior to derive the findings about Rassias stability, we make the following observations:

Lemma 4.2: *Again consider the solution of our suggested problem* (1),

$$\begin{aligned} {}^{\mathcal{PCF}}\mathfrak{D}_{0^+}^{\mathfrak{p}}\mathfrak{y}(\mathfrak{s}) &= \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi)), \quad \mathfrak{s} \in [0, \mathcal{T}], \\ \mathfrak{y}(0) &= \phi(\mathfrak{y}). \end{aligned} \tag{23}$$

satisfies the following relation

$$\|\mathfrak{y}-\mathbf{Z}\mathfrak{x}\| \leq \begin{cases} \mathfrak{s}_1\varphi(\mathfrak{s})\epsilon, & \text{if} 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \left[\dfrac{(\mathcal{T}-\mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\right]\epsilon = \Omega\varphi(\mathfrak{s})\epsilon, & \text{if } \mathfrak{s}_1 < \mathfrak{s} \leq \mathcal{T}. \end{cases} \tag{24}$$

Proof: Thank full to Lemma 3.1 and Remark 4.1, solution of (23) can be generated easily. On further simplification the relation of Equation (24) is easily straight forward. ■

Theorem 4.4: *Inview of* (F_1)–(F_3) *Lemma* 4.2, *the generated solution of suggested problem* (1) *is U-H Rassias stable by using the condition* $\Omega\mathbf{M}_{\mathfrak{u}} < 1$.

Proof: We can divide our results into two categories.

Case 1 When $\mathfrak{s} \in \mathfrak{J}_1$,then

$$\begin{aligned}
\|\mathfrak{y} - \mathfrak{x}\| &= \sup_{\mathfrak{s}\in\mathfrak{J}} \left|\mathfrak{y} - \left(\phi(\mathfrak{x}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))\, d\xi\right)\right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{J}} \left|\mathfrak{y} - \left(\phi(\mathfrak{y}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))\, d\xi\right)\right| \\
&\quad + \sup_{\mathfrak{s}\in\mathfrak{J}} \left|\phi(\mathfrak{y}) - \phi(\mathfrak{x}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))\, d\xi \right. \\
&\quad \left. - \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))\, d\xi\right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{J}} (\mathfrak{s}_1\epsilon + \mathbf{C}_\phi|\mathfrak{y} - \mathfrak{x}| + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1|\mathfrak{y} - \mathfrak{x}|) \\
&\leq \mathfrak{s}_1\varphi(\mathfrak{s})\epsilon + (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1)\|\mathfrak{y} - \mathfrak{x}\|.
\end{aligned}$$

The further simplification gives,

$$\|\mathfrak{y} - \mathfrak{x}\| \leq \left(\frac{\mathfrak{s}_1}{1 - (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1)}\right)\varphi(\mathfrak{s})\epsilon. \tag{25}$$

Case 2 When $\mathfrak{s} \in \mathfrak{J}_2$,then

$$\begin{aligned}
\|\mathfrak{y} - \mathfrak{x}\| &= \sup_{\mathfrak{s}\in\mathfrak{J}} \left|\mathfrak{y} - \left(\mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, d\xi\right)\right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{J}} \left|\mathfrak{y} - \left(\mathfrak{y}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, d\xi\right)\right| \\
&\quad + \sup_{\mathfrak{s}\in\mathfrak{J}} \left|\mathfrak{y}(\mathfrak{s}_1) - \mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, d\xi \right. \\
&\quad \left. - \frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, d\xi\right| \\
&\leq \left[\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\right]\epsilon + \mathbf{C}_\phi\|\mathfrak{y} - \mathfrak{x}\| + \mathbf{M}_{\mathfrak{u}}\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\|\mathfrak{y} - \mathfrak{x}\| \\
&\leq \Omega\varphi(\mathfrak{s})\epsilon + (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\Omega)\|\mathfrak{y} - \mathfrak{x}\|,
\end{aligned}$$

where

$$\left[\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\right] = \Omega.$$

Hence we get

$$\|\mathfrak{y}-\mathfrak{x}\| \leq \left(\frac{\Omega}{1-(\mathbf{C}_\phi+\mathbf{M}_\mathfrak{u}\Omega)}\right)\varphi(\mathfrak{s})\epsilon. \tag{26}$$

If we combine **Case 1** and **Case 2**, we can at the following conclusion,

$$\mathcal{K}_2 = \max\left\{\frac{\mathfrak{s}_1}{1-(\mathbf{C}_\phi+\mathbf{M}_\mathfrak{u}\mathfrak{s}_1)}, \frac{\Omega}{1-(\mathbf{C}_\phi+\mathbf{M}_\mathfrak{u}\Omega)}\right\}.$$

then we have

$$\|\mathfrak{y}-\mathfrak{x}\| \leq \mathcal{K}_2\varphi(\mathfrak{s})\epsilon, \quad \text{forall } \mathfrak{s}\in\mathfrak{I}. \tag{27}$$

Hence, problem (1) is U-H Rassias stable. ■

Theorem 4.5: *Inview of* (F_1)–(F_3) *Lemma* 4.2*, the generated solution of suggested problem* (1) *is generalized U-H Rassias stable by using condition* $\Omega\mathbf{M}_\mathfrak{u} < 1$.

Proof: We can divide our results into two categories.

Case 1 When $\mathfrak{s}\in\mathfrak{I}_1$,then

$$\begin{aligned}
\|\mathfrak{y}-\mathfrak{x}\| &= \sup_{\mathfrak{s}\in\mathfrak{I}}\left|\mathfrak{y}-\left(\phi(\mathfrak{x})+\int_0^{\mathfrak{s}_1}\mathfrak{u}(\xi,\mathfrak{x}(\xi),\mathfrak{x}(\lambda\xi),\mathfrak{x}(\tau-\xi))\,\mathrm{d}\xi\right)\right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{I}}\left|\mathfrak{y}-\left(\phi(\mathfrak{y})+\int_0^{\mathfrak{s}_1}\mathfrak{u}(\xi,\mathfrak{y}(\xi),\mathfrak{y}(\lambda\xi),\mathfrak{y}(\tau-\xi))\,\mathrm{d}\xi\right)\right| \\
&\quad+\sup_{\mathfrak{s}\in\mathfrak{I}}\left|\phi(\mathfrak{y})-\phi(\mathfrak{x})+\int_0^{\mathfrak{s}_1}\mathfrak{u}(\xi,\mathfrak{y}(\xi),\mathfrak{y}(\lambda\xi),\mathfrak{y}(\tau-\xi))\,\mathrm{d}\xi\right. \\
&\quad\left.-\int_0^{\mathfrak{s}_1}\mathfrak{u}(\xi,\mathfrak{x}(\xi),\mathfrak{x}(\lambda\xi),\mathfrak{x}(\tau-\xi))\,\mathrm{d}\xi\right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{I}}(\mathfrak{s}_1\varphi(\mathfrak{s})+\mathbf{C}_\phi|\mathfrak{y}-\mathfrak{x}|+\mathbf{M}_\mathfrak{u}\mathfrak{s}_1|\mathfrak{y}-\mathfrak{x}|) \\
&\leq \mathfrak{s}_1\varphi(\mathfrak{s})+(\mathbf{C}_\phi+\mathbf{M}_\mathfrak{u}\mathfrak{s}_1)\|\mathfrak{y}-\mathfrak{x}\|.
\end{aligned}$$

The further simplification gives,

$$\|\mathfrak{y}-\mathfrak{x}\| \leq \left(\frac{\mathfrak{s}_1}{1-(\mathbf{C}_\phi+\mathbf{M}_\mathfrak{u}\mathfrak{s}_1)}\right)\varphi(\mathfrak{s}). \tag{28}$$

Case 2 When $\mathfrak{s} \in \mathfrak{I}_2$,then

$$\begin{aligned}
\|\mathfrak{y} - \mathfrak{x}\| &= \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - \left(\mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, d\xi\right) \right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - \left(\mathfrak{y}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, d\xi\right) \right| \\
&\quad + \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y}(\mathfrak{s}_1) - \mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, d\xi \right. \\
&\quad \left. - \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}\, d\xi \right| \\
&\leq \left[\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\right] \varphi(\mathfrak{s}) + \mathbf{C}_\phi \|\mathfrak{y} - \mathfrak{x}\| + \mathbf{M}_{\mathfrak{u}} \frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)} \|\mathfrak{y} - \mathfrak{x}\| \\
&\leq \Omega\varphi(\mathfrak{s}) + (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\Omega)\|\mathfrak{y} - \mathfrak{x}\|,
\end{aligned}$$

where

$$\left[\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\right] = \Omega.$$

Hence we get

$$\|\mathfrak{y} - \mathfrak{x}\| \leq \left(\frac{\Omega}{1 - (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\Omega)}\right) \varphi(\mathfrak{s}). \tag{29}$$

If we combine **Case 1** and **Case 2**, we look at the following conclusion,

$$\mathcal{K}_{\mathfrak{u},\varphi} = \max\left\{\frac{\mathfrak{s}_1}{1 - (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1)}, \frac{\Omega}{1 - (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\Omega)}\right\}.$$

Then we have

$$\|\mathfrak{y} - \mathfrak{x}\| \leq \mathcal{K}_{\mathfrak{u},\varphi}\varphi(\mathfrak{s}), \quad \text{forall } \mathfrak{s} \in \mathfrak{I}. \tag{30}$$

Hence, problem (1) is generalized U-H Rassias stable. ∎

5. Example

Example 5.1: Consider the following problem

$$\begin{cases}
{}^{\mathcal{PCF}}\mathfrak{D}_{0^+}^{0.8}\mathfrak{y}(\xi) = \dfrac{\exp(-\pi \sin\xi)}{62 + \xi^4} \\
\quad + \dfrac{1}{74}\left(\dfrac{2\pi|\mathfrak{y}(\xi)|}{(1 + |\mathfrak{y}(\xi)|)} + \dfrac{|\mathfrak{y}(\lambda\xi)| \tan^{-1}(\xi)}{\sqrt{\xi^2 + 64}} + \sin^{-1} \ln|\mathfrak{y}(\tau - \xi|\right), \quad \xi \in [0, 1], \\
\mathfrak{y}(0) = \dfrac{\cos|\mathfrak{y}|}{102}.
\end{cases} \tag{31}$$

Taking $\mathfrak{p} = 0.8$

$$g(\xi, \mathfrak{y}) = \frac{\exp(-\pi \sin \xi)}{62 + \xi^4} + \frac{1}{74}\left(\frac{2\pi |\mathfrak{y}(\xi)|}{(1 + |\mathfrak{y}(\xi)|)} + \frac{|\mathfrak{y}(\lambda\xi)| \tan^{-1}(\xi)}{\sqrt{\xi^2 + 64}} + \sin^{-1} \ln |\mathfrak{y}(\tau - \xi|\right).$$

Let $\mathfrak{y}, \mathfrak{z} \in \mathfrak{I}$,

$$\begin{aligned}
&|g(\xi, \mathfrak{y}) - g(\xi, \mathfrak{z})| \\
&= \left|\left[\frac{\exp(-\pi \sin \xi)}{62 + \xi^4} + \frac{1}{74}\left(\frac{2\pi |\mathfrak{y}(\xi)|}{(1 + |\mathfrak{y}(\xi)|)} + \frac{|\mathfrak{y}(\lambda\xi)| \tan^{-1}(\xi)}{\sqrt{\xi^2 + 64}} + \sin^{-1} \ln |\mathfrak{y}(\tau - \xi|\right)\right]\right. \\
&\quad \left. - \left[\frac{\exp(-\pi \sin \xi)}{62 + \xi^4} + \frac{1}{74}\left(\frac{2\pi |\mathfrak{z}(\xi)|}{(1 + |\mathfrak{z}(\xi)|)} + \frac{|\mathfrak{z}(\lambda\xi)| \tan^{-1}(\xi)}{\sqrt{\xi^2 + 64}} + \sin^{-1} \ln |\mathfrak{z}(\tau - \xi|\right)\right]\right| \\
&\leq \frac{81\pi}{2368} \|\mathfrak{y} - \mathfrak{z}\|.
\end{aligned}$$

For assumption (F_4)

$$\begin{aligned}
|g(\xi, \mathfrak{y})| &= \left|\frac{\exp(-\pi \sin \xi)}{62 + \xi^4} + \frac{1}{74}\left(\frac{2\pi |\mathfrak{y}(\xi)|}{(1 + |\mathfrak{y}(\xi)|)} + \frac{|\mathfrak{y}(\lambda\xi)| \tan^{-1}(\xi)}{\sqrt{\xi^2 + 64}} + \sin^{-1} \ln |\mathfrak{y}(\tau - \xi|\right)\right| \\
&\leq \frac{1}{62} + \frac{81\pi}{2368} |\mathfrak{y}(\xi)|.
\end{aligned}$$

We can see that $\mathbf{M}_{\mathfrak{u}} = \frac{81\pi}{2368}, \mathbf{N}_{\mathfrak{u}} = \frac{81\pi}{2368}, \mathbf{C}_{\phi} = \frac{1}{102}$, and $\mathbf{L}_{\mathfrak{u}} = \frac{1}{62}$. Taking $\mathcal{T} = 1$ and $\xi_1 = 0.9$, now applying Theorem 3.1 to get

$$\begin{aligned}
\mathbf{B} &= \max\left\{\mathbf{C}_{\phi} + \mathbf{M}_{\mathfrak{u}}\xi_1, \mathbf{M}_{\mathfrak{u}} \frac{(\mathcal{T} - \xi_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p} + 1)}\right\} \\
&= \max\{0.10646, 0.09646\} = 0.10646 < 1.
\end{aligned}$$

By using Theorem 3.1, the given problem has a unique solution. Now inview of Theorem 3.2, we can see that

$$\mathbf{C}_{\phi} = \frac{1}{102} < 1,$$

hence the condition of Theorem 3.2 is satisfied. So the given problem has at least one solution. Further, the condition for U-H stable and generalized U-H stable is also fulfilled because we can see that $\Omega = 0.1705$, $(\mathbf{C}_{\phi} + \mathbf{M}_{\mathfrak{u}}\xi_1) = 0.10646 < 1$. Moreover, if we take $\varphi(\xi) = \xi$, then the condition for U-H Rassias stable and U-H generalized Rassias stable is also satisfied.

6. Concluding remarks

Short memory fractional-order differential equations play significant roles in the description of many real-world problems. Therefore, in this work, we have considered a class of delay fractional differential equations under the concept of piecewise derivative in the

Caputo sense. We have expanded some results about the uniqueness, existence, and stability analysis for our considered problem. The results were obtained by utilizing the fixed point concept and nonlinear functional analysis tools. Sufficient circumstances have been set up to ensure the existence of at least one solution and its uniqueness to the suggested problem. Nonlinear tools of analysis were also used to determine its stability. We can observe that these derivatives can better describe an abrupt behavioral change in the dynamics of various processes of reals problems. As a result, we concluded that this sort of calculus has now attracted researchers more. More research into how to deal with piecewise fractional-order differential boundary value problems will be conducted in the future.

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Data availability statement

The data used has been included within the paper.

Disclosure statement

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Mathematical Analysis of Piece Wise Fractional Differential Equations

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Declaration

I hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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FATIMA ILYAS

CIIT/FA20-RMT-025/SWL

Certificate

It is certified that FATIMA ILYAS has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Sahiwal Campus and the work fulfills the requirement for award of MS degree.

Date:_____________

Head of Department:	Supervisor:
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DEDICATION TO

My loving parents, teachers and
my family.

ACKNOWLEDGEMENTS

Praise to be ALLAH, the Cherisher and Lord
of the World, Most Gracious and Most Merciful

Person is not perfect in all the contexts of this life. He has a limited mind and minor thinking approaches. It is the guidance from the almighty ALLAH that shows the man light in the darkness and the person finds his way in the light. Without this helping light, person is nothing but a helpless creature. Same is the case with us, as we experience all these phenomena during the completion of this project and have been successful in fulfilling this duty assigned to us only because of the help of ALLAH. The teaching of the Holy Prophet Muhammad (PBUH) were also the continuous source of guidance for us especially. His order of getting knowledge and fulfilling one duty honestly was the key for motivating us. Credit of my humble efforts goes to my respectable teacher Dr.Asma, under whose supervision I was able to complete this task.

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ABSTRACT

In this thesis we intend to work on a particular type of dynamical system including piece wise fractional differential and integral operators. This study introduces a new concept of piece wise fractional derivatives with the intent of modelling real world problems with crossover behaviors. Fractional calculus is always being studied by well known researchers due to its wide applications in many fields. The present research work has considered a class of delay type problems with piece wise fractional derivative. We proceed with certain fundamental results that will be used to create the integral equation in our scenario. To check the uniqueness and existence of our problem's solution, we use well known fixed point theorems as Krasnoselskii's fixed point theorem and Arzela Ascoli fixed point theorem. The principles of fractional differential equations are influenced by stability analysis. We use a few distinct types of Ulam Hyers(U-H) stabilities such as U-H stability, Generalized U-H stability, U-H Rassias stability and Generalized U-H stability for piece wise fractional differential equations. Here we'll also use several illustrations to back up all of our theoretical conclusions.

List of symbols

Symbols	Denoted
$\mathbb{R}$	Set of real numbers
$\mathcal{H}$	Banach space
$\parallel \mathfrak{y} \parallel$	Norm
Γ	Gamma Function
${}^{\mathcal{PCF}}\mathfrak{D}_{0^+}^{\zeta}$	$\mathcal{PC}$ Fractional Derivative
${}^{\mathcal{PCF}}\mathcal{I}_{0^+}^{\zeta}$	$\mathcal{PC}$ Fractional Integral
${}^{\mathcal{RL}}\mathfrak{D}_{0^+}^{\zeta}$	$\mathcal{RL}$ Fractional Derivative
${}^{\mathcal{C}}\mathfrak{D}_{0^+}^{\zeta}$	Caputo Fractional Derivative

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Chapter 1

History and Introduction

1 History and Introduction

Fractional calculus is thought to have been born on September $30, 1695$. This approach is as old as our traditional integer order calculus. Fractional differential and integral equations theory has now become a popular research topic in past few decades. The reason for this growing interest is that, when compared to integer ordered differential equations, it produces more accurate results. Many remarkable mathematicians, such as Liouville, Laplace, Euler, Riemann, Letnikov, and Grunwald, have expressed interest in this area. In the recent decades, piece wise fractional derivative is getting popular to solve many problems that exhibit crossover behavior.

1.1 Historical Survey

Ideas about fractional calculus are as old as ordinary calculus. In the 17^{th} century, Newton and Leibnitz established differential calculus. For each positive integer m, Leibnitz used the notation $\frac{d^m z}{dt^m}$ which is also used for the m^{th} order derivative with respect to t. Now the question arises of how to continue the process for a function $z(t)$ if the order of derivative and integral is not a positive integer. One of the people that asked such a fascinating question was L'Hopital." This appears to be a paradox from which a valuable conclusion can indeed be drawn one day, " Leibnitz replied on September $30, 1695$. Fractional calculus was developed from this concept. In 1819, Lacroix is considered the first one who introduced integer order derivatives. The Gamma function was initially introduced in 1783 by Euler's Gamma. This function is used to solve numerous fractional analysis problems.

Researchers have diligently worked in this topic, producing several definitions and significant conclusions. The fractional derivative and integral of Riemann Liouville and Caputo have been used in a lot of research. Caputo and Fabrizio developed an improved class of derivative known as non singular in 2015, that has received a lot of attention from researchers [1]. This differential operator involving exponential type of non singular kernel. In 2016, Atangana and his co authors also generalized the above mentioned operator by changing exponential function with a Mittag Leffler function. Modeling real life physical phenomena's with crossover behavior has indeed proven to be a difficult task for modelers. The theories of piece wise fractional integral and differential operators were postulated by Atangana and his co-author Seda [2]. These operators will be developed for solving problems that exhibit crossover behavior.

1.2 Applications

Predicting and understanding physical phenomena is one of the most effective tools humans have developed in recent decades to predict and avoid many of the worst case scenarios. Certainly, mathematics has proven its effectiveness in this respect, as seen by its widespread usage in modern scientific domains such as epidemiological [3], physics [4], and biology [5]. In past few decades, researchers have paid a lot of attention to fractional calculus. This is due to the fact that it has a variety of significant applications in many fields including, the parameter that are used for estimation of fractional dynamical models arising from biology [6], modeling multiple electrochemical processes with fractional differential equations [7], several physics related applications found in [8], the reader should see detailed theory about the area and applications in [9].

Financial mathematics, physics, control theory, chemistry, optical and thermoelectric systems, bioscience, signal processing and structure of identification and so on have all used the fractional derivative system as just a mathematical model to characterized the many processes and phenomena. Fractional derivatives can be used to develop population growth models. These fractional derivatives are also used to develop the Newton law of cooling and the blood alcohol models. Many scientists are using the idea of piece wise fractional derivative to explore infectious diseases like CAT-T cells-SARS-2 [10] virus and Covid-19 [11]. Using fractional calculus the third wave of the Covid-19 outbreak in three countries, Turkey, Spain, and Czechia, was modelled using piece wise differential and integral operators [12]. The writers of several articles will employ the concept of piece wise operators to calculate the numerical results of the food web model [13].

1.3 Our Proposed Problem

Mathematicians can create mathematical equations called partial and ordinary differential equations using the theory of rate of change. In recent decades, differential calculus have received a lot of attention. In order to solve those equations, numerical and analytic methods have been devised. These mathematical equations have proven to be extremely effective in simulating real world phenomena. However, these mathematical domains of integro differential calculus, have consistently failed to recreate the physical phenomena multiple times due to the complexity of several real world problems.

For example, many real world problems, exhibit some unpredictability that many mathematical models are unable to capture. In recent decades, the idea of stochastic mathematical differential equations has been proposed and used extensively with some significant achievements. But, instead of following randomness, some problems are following non locality trends, such as long range dependence, fractal process, power law processes and crossover behaviours, which means a physical phenomena exhibits many behaviours. To address these challenges, a class of fractional derivatives were proposed, including fractional differential operators of singular type kernels, [14] fractional differential operators with of singular type kernels, [15], fractal fractional operators [16] and differential operators with regard to other functions [17]. So these mathematical models are given a birth to variety of nonlinear ordinary and partial mathematical differential and integral problems that've been successfully applied to a variety of issues. Despite this, the concern of crossover behaviours have yet to be resolved.

Caputo and Fabrizio developed an improved class of derivative known as non singular in 2015, that has received a lot of attention from researchers. This differential operator involving exponential type of non singular kernel. In 2016, Atangana and his co authors also generalized the above mentioned operator by changing exponential function with a Mittag Leffler function. Despite of power law kernel, the Mittag Leffler functions and exponential function exhibit crossover behaviour in fractional differential operators. For instance, Using Mittag-Leffler stability theory a simple active control strategy for a new chaotic dynamical system with fractional derivatives has been discussed in [18], to investigate and solve a model of nonlinear fractional differential equations characterising the deadly and destructive virus known as corona virus (COVID-19) under Mittag Leffler operators [19] and see an extension of the exponential function in crossover in [20]. But there is one thing to bear in mind is that many real world phenomena do not have a single behaviour and rather than exhibit a variety of behaviours.

Moreover, employing the Caputo Fabrizio and Atangana Baleanu derivatives, real world issues exhibiting various processes indicated by the Mittag Leffler function and exponential function cannot be replicated. However, those operators are still ineffective in describing crossover behaviour. To further understand the aforementioned characteristics, authors recently established the concept of piece wise fractional differential equations and provided detailed characteristics. Such operators will indeed be utilised to solve problems that exhibit crossover behavior. Atangana and his co author Seda proposed the theories of piece wise fractional differential equations.

It's also worth mentioning that delay type problems have got a lot of attention in the subject of fractional calculus. It is a significant branch in this field. Number of researchers have been working on delay type problems at an incredible rate. Here we refer some work in for more details on delay type problems as, (an overview of delay type differential equations and their applications in science [21], delay type differential equations theory [22], oscillations and stability in population dynamics delay type differential equations [23], a new approach to fractional delay type differential equations [24]).

As a conclusion, we will develop existence and stability analysis results for the following delay type problem using the concept of piece wise fractional operators, keeping in mind the aforementioned requirement and importance as,

$$\begin{cases} {}^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0+}\mathfrak{y}(\mathfrak{s}) = \mathfrak{u}(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \mathfrak{y}(\lambda\mathfrak{s}), \mathfrak{y}(\tau - \mathfrak{s})), \quad \mathfrak{s} \in \mathfrak{J} = [0, \mathcal{T}], \\ \mathfrak{y}(0) = \phi(\mathfrak{y}), \end{cases} \tag{1}$$

where $0 < \mathfrak{p} \leq 1$, $0 < \lambda < 1$ and $\tau > 0$ the notion ${}^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0+}$ stands for piece wise fractional derivative. Let $\phi \in \mathcal{C}(\mathfrak{J})$ and $\mathfrak{u} : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Utilizing elementary fractional calculus results, first we convert the suggested problem into integral form of piece wise. Further, sufficient and necessary conditions are established to analyse the associated existence theory by using fixed point results [25]. Using nonlinear analysis, we develop certain sufficient requirements for different forms of U-H type stabilities for the presented problem. Further, the crossover behaviors of piece wise fractional derivatives are demonstrated by using examples.

The chapter 1 is dedicated for the introduction and background material. In chapter 2, we provide some basic notations, elementary results, lemmas etc. Then we discuss the existence and uniqueness of our problem's solution by constructing some reasonable conditions in chapter 3. The conclusion and summary of our thesis are discussed in 4 chapter. The final chapter is devoted to references.

Chapter 2

Fundamental Terminologies and Concepts

2 Fundamental Terminologies

This chapter covers certain elementary results, basic definitions, lemmas, preliminaries being used in this thesis. This section also includes fundamental results and useful definition from functional analysis and some fixed point techniques which is essential to solve the differential equation of an arbitrary orders [26][27]. We use these fundamental results to derive our main conclusions. We start by defining several fundamental functions such as the Gamma function. We also expand on basic definitions of piece wise fractional integral and derivatives.

2.1 Special Functions

Gamma function is one of the most useful and special functions in fractional calculus theory. The gamma function is essential in calculus, complex analysis, differential equations and statistics because it is useful for modelling the situations involving continuous change. In this section we will elaborate the gamma functions briefly.

2.1.1 Gamma Function

In the 18^{th} century, Swiss mathematician Leonhard Euler developed the gamma function, which is a non integer generalization of the factorial function. It shows an important function in a wide range of applied sciences. There are too many methods for obtaining the gamma function. The Euler limit approach is the most practical way to describes the gamma function.

Definition 1. The integral transformation of gamma function is given as,

$$\Gamma(\zeta) = \int_0^\infty \mathfrak{u}^{\zeta-1} e^{-\mathfrak{u}} d\mathfrak{u}, \quad \zeta \in \mathbb{R}^+.$$

Properties of Gamma Function

- $\Gamma(\zeta+1) = \zeta!, \quad \zeta \in \mathbb{N}.$

- $\Gamma(\zeta+1) = \zeta\Gamma(\zeta), \quad \zeta \in \mathbb{R}^+.$

- $\Gamma(\zeta) = \frac{\Gamma(\zeta+1)}{\zeta}, \quad$ for -ve values of ζ.
- $\Gamma(\zeta)\Gamma(1-\zeta) = \frac{\pi}{\sin(\pi\zeta)}$.
- $\Gamma\left(\zeta + \frac{1}{2}\right) = \frac{(2\zeta-1)!}{2^{\zeta}}\sqrt{\pi}$.
- $\Gamma\left(\frac{1}{2}+\zeta\right)\Gamma\left(\frac{1}{2}-\zeta\right) = \pi \sec \pi z$.

By using above relations we get

- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$
- $\Gamma\left(\frac{-3}{2}\right) = \frac{\Gamma\left(\frac{-3}{2}+1\right)}{\frac{-3}{2}} = \frac{4}{3}\sqrt{\pi}$

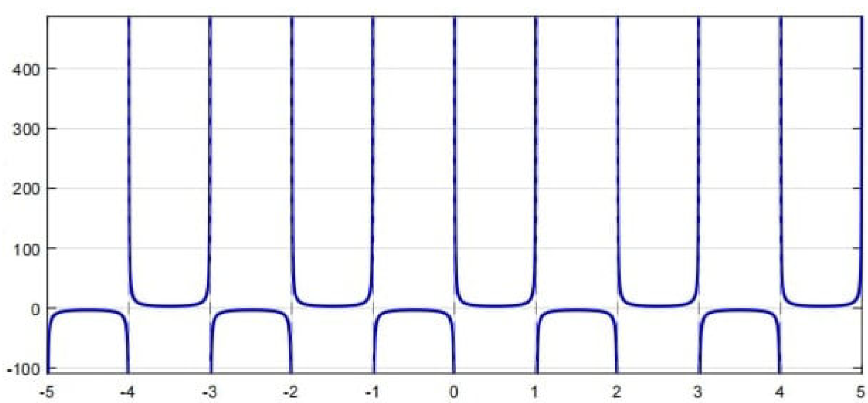

Figure 1: Gamma Function

2.2 Basic Definitions for Fractional Derivatives and Integrals

In this section, we elaborate Riemann Liouville, Caputo and Piece wise fractional derivatives and integrals. We derive our main results by using these results.

Definition 2. [28][29]⟨ *RiemannLiouville Fractional Integral* ⟩

The Riemann Liouville fractional integral of order ζ for function $\mathfrak{u}$ is defined by

$$^{\mathcal{RL}}\mathcal{J}_{0^+}^{\zeta}\mathfrak{u}(\mathfrak{s}) = \frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}} \frac{\mathfrak{u}(t)}{(\mathfrak{s}-t)^{1-\zeta}}dt, \quad m-1 < \zeta \leq m,$$

here $\Gamma(\zeta)$ is a gamma function.

Definition 3. [30][31]$\langle$ *RiemannLiouville Fractional Derivative* $\rangle$.

The Riemann Liouville fractional derivative of order ζ for function $\mathfrak{u}$ is defined by

$$^{\mathcal{RL}}\mathfrak{D}_{0^+}^{\zeta}\mathfrak{u}(\mathfrak{s}) = \frac{1}{\Gamma(m-\zeta)}\left(\frac{d}{d\mathfrak{s}}\right)^m \int_0^{\mathfrak{s}} (\mathfrak{s}-t)^{m-\zeta-1}\mathfrak{u}(t)dt, \quad m-1 < \zeta \le m,$$

where Γ is the Gamma function.

Example: Consider a function $\mathfrak{u}(\mathfrak{s}) = (\mathfrak{s}^2+4)^n \quad n \ge 0$ and $0 < \zeta \le 1$. The $\mathcal{RL}$ fractional derivative of above function is,

$$\mathfrak{D}_{0^+}^{\zeta}(\mathfrak{s}^2+4)^n = \frac{\Gamma(n+1)}{\Gamma(n-\zeta+1)}(\mathfrak{s}^2+4)^{n-\zeta}$$

Solution:

$$\begin{aligned}\mathfrak{D}_{0^+}^{\zeta}(\mathfrak{s}^2+4)^n &= \mathfrak{D}^1\left(\mathfrak{D}^{\zeta-1}(\mathfrak{s}^2+4)^n\right)\\ &= \mathcal{D}^1\left(\frac{\Gamma(n+1)}{\Gamma(n-\zeta+2)}(\mathfrak{s}^2+4)^{n-\zeta+1}\right)\\ &= \frac{\Gamma(n+1)}{\Gamma(n-\zeta+2)}2s(\mathfrak{s}^2+4)^{n-\zeta}.\end{aligned}$$

Put $\zeta = \frac{1}{2}$ and $n = 0, 1, 2$ in above result then we have

$n = 0$

$$\mathfrak{D}_{0^+}^{\frac{1}{2}}(\mathfrak{s}^2+4)^0 = \frac{4s}{\sqrt{\pi(\mathfrak{s}^2+4)}}$$

$n = 1$

$$\mathfrak{D}_{0^+}^{\frac{1}{2}}(\mathfrak{s}^2+4)^1 = \frac{16s}{3}\sqrt{\frac{(\mathfrak{s}^2+4)}{\pi}}$$

$n = 2$

$$\mathfrak{D}_{0^+}^{\frac{1}{2}}(\mathfrak{s}^2+4)^2 = \frac{32s}{5}\sqrt{\frac{(\mathfrak{s}^2+4)^3}{\pi}}$$

Definition 4. [32][33]$\langle$ *Caputo Fractional Derivative* $\rangle$.

Let $\mathfrak{u}$ be any continuous and m differentiable function. The Caputo fractional derivative of order ζ is defined as,

$$^{\mathcal{C}}\mathfrak{D}_{0^+}^{\zeta}\mathfrak{u}(\mathfrak{s}) = \frac{1}{\Gamma(m-\zeta)}\int_0^{\mathfrak{s}} (\mathfrak{s}-t)^{m-\zeta-1}\left(\frac{d}{dt}\right)^m \mathfrak{u}(t)dt, \quad m-1 < \zeta \le m,$$

Example: Consider function $\mathfrak{u}(\mathfrak{s}) = \mathfrak{s}^4$ of order $0 < \zeta \leq 1$ by using above definition of caputo fractional derivative we have

$$^{\mathcal{C}}\mathfrak{D}_{0^+}^{\frac{1}{2}}\mathfrak{s}^4 = \frac{4\mathfrak{s}^3}{\Gamma(\frac{1}{2})}\int_0^{\mathfrak{s}}(\mathfrak{s}-t)^{-\frac{1}{2}}dt$$

Solution: By using properties of Gamma function and rules of integration,

$$\begin{aligned} ^{\mathcal{C}}\mathfrak{D}_{0^+}^{\frac{1}{2}}\mathfrak{s}^4 &= \frac{4\mathfrak{s}^3}{\sqrt{\pi}}\int_0^{\mathfrak{s}}(\mathfrak{s}-t)^{-\frac{1}{2}}dt \\ &= 8\sqrt{\frac{\mathfrak{s}^7}{\pi}} \end{aligned}$$

Definition 5. [2]⟨ *Piece wise ($\mathcal{PC}$) Fractional Derivative* ⟩.

Let $\mathfrak{u}$ be a differentiable function, then the piece wise fractional derivative of function $\mathfrak{u}$ with fractional and classical derivative involving powerlaw kernel is defined as,

$$^{\mathcal{PCF}}\mathfrak{D}_{0^+}^{\zeta}\mathfrak{u}(\mathfrak{s}) = \begin{cases} \mathfrak{u}\,'(\mathfrak{s}), \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ ^{\mathcal{C}}\mathfrak{D}_{\mathfrak{s}}^{\zeta}\mathfrak{u}(\mathfrak{s}), \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}, \end{cases}$$

where $^{\mathcal{PCF}}\mathfrak{D}_{0^+}^{\zeta}$ represents classical derivative on $0 \leq \mathfrak{s} \leq \mathfrak{s}_1$ and Caputo derivative of fractional order on $\mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}$.

Example: Consider a function $\mathfrak{u}(\mathfrak{s}) = \mathfrak{s}^{\vartheta}$,then piece wise fractional order derivative of $\mathfrak{u}(\mathfrak{s})$ by applying the Caputo derivative and using properties of gamma function is given as,

$$^{\mathcal{PCF}}\mathfrak{D}_{0^+}^{\zeta}\mathfrak{u}(\mathfrak{s}) = \begin{cases} \vartheta\mathfrak{s}^{\vartheta-1}, \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \frac{\Gamma(1+\vartheta)}{\Gamma(1+\vartheta-\zeta)}\mathfrak{s}^{\vartheta-\zeta}, \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}, \end{cases}$$

Definition 6. ⟨ *$\mathcal{PC}$ Fractional Integral* ⟩.[2]

The piece wise fractional integral of a continuous $\mathfrak{u}$ with $\mathfrak{p} \in (0,1]$ is defined as,

$$^{\mathcal{PCF}}\mathcal{J}_{0^+}^{\zeta}\mathfrak{u}(\mathfrak{s}) = \begin{cases} \int_0^{\mathfrak{s}_1}\mathfrak{u}(\xi)d\xi, \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{s}_1}^{\mathfrak{s}}\mathfrak{u}(\xi)(\mathfrak{s}-\xi)^{\zeta-1}d\xi, \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}, \end{cases}$$

where $^{\mathcal{PCF}}\mathcal{J}_{0^+}^{\zeta}$ represents classical integral on $0 \leq \mathfrak{s} \leq \mathfrak{s}_1$ and powerlaw integral on $\mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}$.

Example: Consider a function $\mathfrak{u}(\mathfrak{s}) = \mathfrak{s}^{\vartheta}$,then piece wise fractional integral of $\mathfrak{u}(\mathfrak{s})$ is given as

$$
{}^{\mathcal{PCF}}\mathcal{J}^{\zeta}_{0^+}\mathfrak{u}(\mathfrak{s}) = \begin{cases} \int_0^{\mathfrak{s}_1} \xi^{\vartheta} d\xi, \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \frac{1}{\Gamma(\zeta)} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \xi^{\vartheta}(\mathfrak{s}-\xi)^{\zeta-1} d\xi, \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}, \end{cases}
$$

hence we have

$$
{}^{\mathcal{PCF}}\mathcal{J}^{\zeta}_{0^+}\mathfrak{u}(\mathfrak{s}) = \begin{cases} \mathfrak{u}(0) + \frac{\mathfrak{s}_1^{\vartheta+1}}{\vartheta+1}, \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \mathfrak{u}(\mathfrak{s}_1) + \frac{\Gamma(\vartheta+1)}{\Gamma(\zeta+\vartheta+1)} \mathfrak{s}^{\zeta+\vartheta}, \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}, \end{cases}
$$

Lemma 2.1. *For $\mathcal{PC}$ fractional differential equations, the following result holds,*

$$
{}^{\mathcal{PC}}\mathcal{J}^{\zeta}{}_{0^+}{}^{\mathcal{PC}}\mathfrak{D}^{\zeta}_{0^+}\mathfrak{u}(\mathfrak{s}) = \mathfrak{u}(\mathfrak{s}) + \mathfrak{b}_o + \mathfrak{b}_1\mathfrak{s} + \mathfrak{b}_2\mathfrak{s}^2 + ... + \mathfrak{b}_{m-1}\mathfrak{s}^{m-1}.
$$

for arbitrary constants $\mathfrak{b}_m$, $m = 0, 1, 2, ...m-1$, where $m = [\zeta] + 1$ and $[\zeta]$ represent the integer part.

2.3 Basic Terminologies from Nonlinear Analysis

In this section, we will cover some of the most significant observations from nonlinear analysis, as well as the fixed point theory that have been used throughout the dissertation.

Definition 7. Consider a mapping from a set to itself $\mathcal{W} : \mathcal{V} \to \mathcal{V}$, $\mathfrak{u} \in \mathcal{V}$ which is kept fixed by $\mathcal{W}$ is referred to as a fixed point, as seen here.

$$
\mathcal{W}(\mathfrak{u}) = \mathfrak{u}
$$

and $\mathcal{W}(\mathfrak{u})$ corresponds to $\mathfrak{u}$.

Definition 8. Let we have a metric space $\mathcal{V}$. If there exists a positive real number $\zeta < 1$ for all $\mathfrak{u}, \mathrm{v} \in \mathcal{V}$ then the mapping $\mathcal{W} : \mathcal{V} \to \mathcal{V}$ from metric space to metric space is termed as contraction.

$$
\parallel \mathcal{W}(\mathfrak{u}) - \mathcal{W}(\mathrm{v}) \parallel \leq \zeta \parallel \mathfrak{u} - \mathrm{v} \parallel
$$

Definition 9. [34] If every open cover of $\mathcal{B}_1$ has a finite subcover, the subset $\mathcal{B}_1$ of $\mathcal{V}$ is said to be compact.

Definition 10. Let $\mathcal{B}_1 \subset \mathcal{V}$ is relatively compact if any arbitrary subset of a compact set is compact.

Definition 11. [34] Let $\mathcal{V}$,$\mathcal{U}$ be two Banach spaces. A mapping $\mathcal{W} : \mathcal{V} \rightarrow \mathcal{U}$ is continuous if for every $\mathfrak{u} \in \mathcal{V}$ and $\epsilon > 0$ there exist a $\delta = \delta(\epsilon)$ such that,

$$\| \mathrm{v} - \mathfrak{u} \|_{\mathcal{V}} < \delta \Rightarrow \| \mathcal{V}(\mathrm{v}) - \mathcal{V}(\mathfrak{u}) \|_{\mathcal{W}} < \epsilon.$$

Definition 12. Let $\mathcal{L} : \mathcal{V} \rightarrow \mathcal{U}$ is a set of mapping from Banach space to Banach space, then this mapping is equicontinuous on $\mathcal{B}$, where $\mathcal{B} \subset \mathcal{V}$, if for each $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ such that for every pair of element $\mathrm{v}, \mathfrak{u} \in \mathcal{B}$, we have

$$\| \mathrm{v} - \mathfrak{u} \|_{\mathcal{V}} < \delta \Rightarrow \| \mathcal{L}(\mathrm{v}) - \mathcal{L}(\mathfrak{u}) \|_{\mathcal{U}} < \epsilon.$$

Definition 13. A Banach space $\mathcal{H} = \mathcal{C}[\mathfrak{I}, \mathbb{R}]$ is equipped with a norm defined as,

$$\|\mathfrak{y}\|_{\mathcal{H}} = \sup_{\mathfrak{s} \in \mathfrak{I}} |\mathfrak{y}(\mathfrak{s})|, \quad \mathfrak{s} \in \mathfrak{I}$$

.

Theorem 2.2. *[33, 36]⟨ Banach Fixed Point Theorem⟩*
Let us consider a metric space $\mathcal{V}$ where $\mathcal{V}$ is not an empty set. Let $\mathcal{B}$ be any closed and convex subset of $\mathcal{V}$ and $\mathcal{W} : \mathcal{V} \rightarrow \mathcal{V}$ be a contraction on $\mathcal{W}$. Then $\mathcal{W}$ has one fixed point if,

$$\|\mathcal{W}(\mathrm{v}) - \mathcal{W}(\mathfrak{u})\| \leq \zeta \|\mathrm{v} - \mathfrak{u}\|, \quad \forall \quad \mathrm{v}, \mathfrak{u} \in \mathcal{W}, \quad 0 < \zeta < 1$$

holds.

Theorem 2.3. *⟨Arzela-Ascoli Theorem⟩*
Let $\mathcal{V}$ be any subset of $\mathcal{H} = \mathcal{C}[\mathfrak{I}, \mathbb{R}]$ and $\mathcal{V}$ is relatively compact that is $\tilde{\mathcal{V}}$ is compact if:

- *$\mathcal{V}$ is uniformly bounded.*
- *$\mathcal{V}$ is equi-continuous.*

Theorem 2.4. $\langle$ *Krasnoselskii Fixed Point Theorem* $\rangle$

If $\mathcal{K} \subset \mathcal{H}$ is a closed, convex and non empty subset of $\mathcal{H}$, then $\mathcal{P}, \mathcal{Q}$ are two operators with,

(1) $\mathcal{P}\mathfrak{x} + \mathcal{Q}\mathfrak{y} \in \mathcal{K}$ where $\mathfrak{x}, \mathfrak{y} \in \mathcal{K}$;

(2) $\mathcal{P}$ is contraction.

(3) $\mathcal{Q}$ is completely continuous.

Then there exist at least one fixed point $z \in \mathcal{K}$ exists such that $z = \mathcal{P}z + \mathcal{Q}z$.

2.4 Stability Analysis

The results of stability for our suggested problem are covered in this chapter. On both the research and application sides, stability analysis is the most significant factor. The stability results must be explored in order to establish different numerical techniques and procedures. As a result, many stability notions, such as exponential, Laypunov and Mittag Leffler type have been introduced in research. For fractional differential equations, U-H type stabilities have recently received a lot of attention. The author of [35] used the Laplace transformation to determine the stability of the linear Caputo Fabrizio differential equations, some generalized U–H stability conclusions in quaternionic analysis for the initial boundary value problem of fractional differential equations are obtained in [36], U-H stability for Hilfer-Hadamard fractional differential equations [37] and U-H stability for implicit impulsive problems has been discussed in [38]. The U-H stability idea was first proposed by Ulam in 1940, and after formally defined by Hyers in 1941 for functional problem in Banach space. The U-H stability idea has grown in popularity due to the simplicity and wide range of applications. It's a brand new way to solving differential and integral equations. U-H stability, Generalized U-H stability, U-H Rassias stability, and Generalized U-H Rassias stability are all examples of U-H stability. For further explanation about these definitions one can refer to [39], [40], [41]. We define four main types of U-H stability linked to our problem in this section.

- U-H stable
- Generalized U-H stable
- U-H Rassias stable
- Generalized U-H Rassias sable

Definition 14. ⟨ *U-H Stable* ⟩

Our suggested problem (1) is U-H stable, if for any $\epsilon > 0$ the following relation is given by,

$$|{}^{\mathcal{PCF}}\mathfrak{D}_{0+}^{\mathfrak{p}}\mathfrak{y}(\mathfrak{s}) - \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))| < \epsilon, \quad \text{for all } \mathfrak{s} \in \mathfrak{I}, \tag{2}$$

keep the requirement for a unique solution $\mathfrak{x}$ of problem (1) with $\mathcal{K}_{\mathfrak{u}} > 0$ as it is,

$$\|\mathfrak{y} - \mathfrak{x}\| < \mathcal{K}_{\mathfrak{u}}\epsilon, \quad \text{for all } \mathfrak{s} \in \mathfrak{I}$$

Definition 15. ⟨ *Generalized U-H Stable* ⟩ Our suggested problem (1) is generalized U-H stable if there exists a a non decreasing map $\psi_{\mathfrak{u}} : [0, \infty) \to \mathbb{R}^+$ for following given relation,

$$\|\mathfrak{y} - \mathfrak{x}\| < \mathcal{K}_{\mathfrak{u}}\varphi(\epsilon), \ \mathfrak{s} \in \mathfrak{I},$$

and $\psi(0) = 0$, then the suggested results will be generalized U-H stable.

Remark 2.5. *[42]*

The inequality (2) *has a solution* $\mathfrak{y} \in \mathcal{H}(\mathfrak{I}, \mathbb{R})$, *if and only if a function* $\phi \in \mathcal{H}(\mathfrak{I}, \mathbb{R})$ *exist such that,*

1. $|\phi(\mathfrak{s})| \leq \epsilon, \mathfrak{s} \in \mathfrak{I}$.
2. ${}^{\mathcal{PCF}}\mathfrak{D}_{0+}^{\mathfrak{p}}\mathfrak{y}(\mathfrak{s}) = \mathfrak{g}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi)) + \phi(\mathfrak{s}), \mathfrak{s} \in \mathfrak{I}$.

Definition 16. ⟨ *U-H Rassias Stable* ⟩

Our suggested problem (1) is U-H Rassias stable equivalent to the function $\varphi \in \mathcal{C}(\mathfrak{I}, \mathbb{R})$, if for any $\epsilon > 0$ the following relation is given by,

$$|^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0+}\mathfrak{y}(\mathfrak{s}) - \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))| < \epsilon\varphi(\mathfrak{s}),\ \mathfrak{s} \in \mathfrak{I}, \tag{3}$$

keep the requirement for a unique solution $\mathfrak{x}$ of problem (1) with $\mathcal{K}_{\mathfrak{u}} > 0$ as it is,

$$\|\mathfrak{y} - \mathfrak{x}\| < \mathcal{K}_{\mathfrak{u},\varphi}\epsilon\varphi(\mathfrak{s}),\ \mathfrak{s} \in \mathfrak{I}.$$

Definition 17. ⟨ *Generalized U-H Rassias Stable* ⟩

Our suggested problem (1) is generalized U-H Rassias stable if a function $\varphi : [0, \infty) \to \mathbb{R}^+$ exist for the following relation,

$$|^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0+}\mathfrak{y}(\mathfrak{s}) - \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))| < \varphi(\mathfrak{s}),\ \mathfrak{s} \in \mathfrak{I},$$

keep the requirement for a unique solution $\mathfrak{x}$ of problem and $\mathcal{K}_{\mathfrak{u},\varphi} > 0$ such that,

$$\|\mathfrak{y} - \mathfrak{x}\| < \mathcal{K}_{\mathfrak{u},\varphi}\varphi(\mathfrak{s}), \mathfrak{s} \in \mathfrak{I}.$$

then the suggested problem (1) will be generalized U-H Rassias stable.

Chapter 3

Mathematical Analysis of Piece Wise Fractional Differential Equations

3 Mathematical Analysis of Piece Wise Fractional Differential Equations

A class of piece wise fractional differential equations has arisen from the complexity of physical methods in mathematics. This category of issues has gained a lot of attention lately. We examine the piece wise fractional derivative with boundary value problems in our research. We derive the integral lemma for our model problem in section 3.1. Then we check for our problem's existence, uniqueness, and boundedness. Using several U-H stability results, we analyse the stability analysis of our problem. We analyse our topic by using many examples. $\mathcal{H} = \mathcal{C}[\mathfrak{J}, \mathbb{R}]$ be a Banach space with norm $\| \mathfrak{y} \| = \max_{\mathfrak{s}\in\mathfrak{J}} |\mathfrak{y}(\mathfrak{s})|$.

3.1 Mathematical Analysis of Model Problem

Lemma 3.1. *Keeping the definition of piece wise fractional derivative in mind, the answer to the given piece wise fractional differential problem,*

$$\begin{cases} {}^{\mathcal{PCF}}\mathfrak{D}^{\beta}_{0^+}\mathfrak{x}(\mathfrak{s}) = \mathfrak{h}(\mathfrak{s}), \quad \beta \in (0,1], \\ \mathfrak{x}(0) = \phi(\mathfrak{x}), \end{cases}$$

is find as,

$$\mathfrak{x}(\mathfrak{s}) = \begin{cases} \mathfrak{x}(0) + \int_0^{\mathfrak{s}_1} \mathfrak{h}(\xi)d\xi, \ \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{h}(\xi)(\mathfrak{s}-\xi)^{\mathfrak{p}-1}d\xi, \ \text{if } \mathfrak{s}_1 \le \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{4}$$

As we have a Banach space $\mathcal{H} = \mathcal{C}[\mathfrak{J}, \mathbb{R}]$ equipped with a norm $\|\mathfrak{y}\|_{\mathcal{H}} = \sup_{\mathfrak{s}\in\mathfrak{J}} |\mathfrak{y}(\mathfrak{s})|$.

Lemma 3.2. *By using aforementioned Lemma* 3.1*, the solution of piece wise problem*

$$\begin{aligned} &{}^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0^+}\mathfrak{y}(\mathfrak{s}) = \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi)), \quad \mathfrak{s} \in [0, \mathcal{T}], \\ &\mathfrak{y}(0) = \phi(\mathfrak{y}), \end{aligned} \tag{5}$$

is calculated as

$$\mathfrak{y}(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{y}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))d\xi, & \textit{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}d\xi, & \textit{if } \mathfrak{s}_1 \le \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{6}$$

Proof. When piece wise type integration is used on both sides of (1), the result is

$$\mathfrak{y}(\mathfrak{s}) = \begin{cases} \mathfrak{y}(0) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))d\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}d\xi, & \text{if } \mathfrak{s}_1 \le \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{7}$$

Using the result $\mathfrak{y}(0) = \phi(\mathfrak{y})$ in above result, we obtain

$$\mathfrak{y}(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{y}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))d\xi, & \text{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}d\xi, & \text{if } \mathfrak{s}_1 \le \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{8}$$

Hence the proof is completed. □

Corollary 3.3. *Thankful to Lemma* 3.2*, the solution of our suggested problem* (1) *is find as*

$$\mathfrak{y}(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{y}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))d\xi, & \textit{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}d\xi, & \textit{if } \mathfrak{s}_1 \le \mathfrak{s} \le \mathcal{T}. \end{cases}$$

To demonstrate the existence of solutions to suggested problem (1) *, we have to define operator* $\boldsymbol{Z} : \mathcal{H} \to \mathcal{H}$ *by*

$$(\boldsymbol{Z}\mathfrak{y})(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{y}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))d\xi, & \textit{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}d\xi, & \textit{if } \mathfrak{s}_1 \le \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{9}$$

$$(\boldsymbol{Z}\mathfrak{x})(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{x}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))d\xi, & \textit{if } 0 \le \mathfrak{s} \le \mathfrak{s}_1, \\ \mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}d\xi, & \textit{if } \mathfrak{s}_1 \le \mathfrak{s} \le \mathcal{T}. \end{cases} \tag{10}$$

3.2 Existence and Uniqueness of Piece Wise Fractional Differential Equation

In this section we are going to discuss the results of existence and uniqueness of our delay type problem involving piecewise fractional derivative.

Theorem 3.4. *If the following assumptions* (F_1)–(F_3) *assumptions are hold, then prove that our suggested problem* (1) *has unique solution.*

(F_1) *Under the continuity of* $\mathfrak{u}$ *and* $\mathfrak{U}_1, \mathfrak{U}_2, \mathcal{V}_1, \mathcal{V}_2, \mathcal{W}_1, \mathcal{W}_2 \in \mathbb{R}$ *there exist some constants* $\mathcal{M}_1, \mathcal{M}_2$ *and* $\mathcal{M}_3$ *such that,*

$$| \mathfrak{g}(\mathfrak{s}, \mathfrak{U}_1, \mathcal{V}_1, \mathcal{W}_1) - \mathfrak{g}(\mathfrak{s}, \mathfrak{U}_2, \mathcal{V}_2, \mathcal{W}_2)| \leq \mathcal{M}_1|\mathfrak{U}_1 - \mathfrak{U}_2| + \mathcal{M}_2|\mathcal{V}_1 - \mathcal{V}_2| + \mathcal{M}_3|\mathcal{W}_1 - \mathcal{W}_2|.$$

(F_2) *Let* $\mathfrak{U}, \mathcal{V} \in \mathbb{R}$ *and* $\boldsymbol{C}_\phi > 0$ *be any constant such that*

$$|\phi(\mathfrak{U}) - \phi(\mathcal{V})| \leq \boldsymbol{C}_\phi |\mathfrak{U} - \mathcal{V}|.$$

(F_3)

$$\boldsymbol{D} = \max\left\{ \boldsymbol{C}_\phi + \boldsymbol{M}_{\mathfrak{u}}\mathfrak{s}_1, \boldsymbol{M}_{\mathfrak{u}} \frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)} \right\} < 1$$

.

Proof. We have to use the principle of Banach space contraction to show that $\mathbf{Z} : \mathcal{H} \to \mathcal{H}$ defined in Corollary 3.3 has a fixed point [43]. Firstly we have to show that $\mathbf{Z}$ is a contraction mapping. Therefore we have

$$\|\mathbf{Z}(\mathfrak{y}) - \mathbf{Z}(\mathfrak{x})\| \leq \sup_{\mathfrak{s} \in \mathfrak{I}} \begin{cases} |\phi(\mathfrak{y}) - \phi(\mathfrak{x})| + \int_0^{\mathfrak{s}_1} |\mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi)) - \\ \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))| d\xi, \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ |\mathfrak{y}(\mathfrak{s}_1) - \mathfrak{x}(\mathfrak{s}_1)| + \frac{1}{\Gamma(\mathfrak{p})} \Big[\int_{\mathfrak{s}_1}^{\mathfrak{s}} |\mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi)) \\ -\mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))| (\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi \Big], \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}. \end{cases}$$

Which yields

$$\|\mathbf{Z}(\mathfrak{y})-\mathbf{Z}(\mathfrak{x})\| \leq \sup_{\mathfrak{s}\in\mathfrak{I}} \begin{cases} \mathbf{C}_\phi| \mathfrak{y}-\mathfrak{x} | + (\mathcal{M}_1| \mathfrak{y}(\xi)-\mathfrak{x}(\xi) | + \mathcal{M}_2| \mathfrak{y}(\lambda\xi)-\mathfrak{x}(\lambda\xi) | + \\ \mathcal{M}_3| \mathfrak{y}(\tau-\xi)-\mathfrak{x}(\tau-\xi) |)\mathfrak{s}_1 \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \frac{(\mathcal{T}-\mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}(\mathcal{M}_1| \mathfrak{y}(\xi)-\mathfrak{x}(\xi) | + \mathcal{M}_2| \mathfrak{y}(\lambda\xi)-\mathfrak{x}(\lambda\xi) | + \\ \mathcal{M}_3| \mathfrak{y}(\tau-\xi)-\mathfrak{x}(\tau-\xi) |), \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}, \end{cases}$$

by taking $\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3 = \mathbf{M}_{\mathfrak{u}}$,

then on further simplification

$$\|\mathbf{Z}(\mathfrak{y})-\mathbf{Z}(\mathfrak{x})\| \leq \begin{cases} (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1)\| \mathfrak{y}-\mathfrak{x} \|, \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \mathbf{M}_{\mathfrak{u}}\frac{(\mathcal{T}-\mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\| \mathfrak{y}-\mathfrak{x} \|, \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}. \end{cases}$$

In short we can write

$$\|\mathbf{Z}(\mathfrak{y})-\mathbf{Z}(\mathfrak{x})\| \leq \mathbf{D}\| \mathfrak{y}-\mathfrak{x} \|.$$

Where $\mathbf{D} = \max\left\{\mathbf{C}_\phi + \mathbf{M}_u\mathfrak{s}_1, \mathbf{M}_u\frac{(\mathcal{T}-\mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\right\}$. Since $\mathbf{D} < 1$, thus from (F_3) the operator $\mathbf{Z} : \mathcal{H} \to \mathcal{H}$ is a contraction mapping. So suggested problem (1) has unique solution. □

Theorem 3.5. *Consider the assumption (F_1)–(F_3) are hold. Further we assume that following (F_1) holds, then our suggested problem* (1) *has at least one solution.*

(F_4) *There exist some constants such that $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 > 0$ and $\boldsymbol{L}_{\mathfrak{u}} > 0$, then*

$$|\mathfrak{u}(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \mathfrak{y}(\lambda\mathfrak{s}), \mathfrak{y}(\tau-\mathfrak{s}))| \leq \mathcal{N}_1(\mathfrak{s})|\mathfrak{y}(\mathfrak{s})| + \mathcal{N}_2(\mathfrak{s})|\mathfrak{y}(\lambda\mathfrak{s})| + \mathcal{N}_3(\mathfrak{s})|\mathfrak{y}(\tau-\mathfrak{s})| + \boldsymbol{L}_{\mathfrak{u}}(\mathfrak{s}),$$

further we denote

$$\boldsymbol{N}^* = \sup_{\mathfrak{s}\in\mathfrak{I}} |\boldsymbol{N}_{\mathfrak{u}}(\mathfrak{s})|, \boldsymbol{L}^* = \sup_{\mathfrak{s}\in\mathfrak{I}} |\boldsymbol{L}_{\mathfrak{u}}(\mathfrak{s})|.$$

Proof. Firstly we define the operator as,

$$(\mathbf{A}\mathfrak{y})(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{y}), \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1), \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}, \end{cases} \tag{11}$$

and

$$(\mathbf{B}\mathfrak{y})(t) = \begin{cases} \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))d\xi, \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1}d\xi, \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}. \end{cases} \tag{12}$$

Step 1 Assume a set $\Delta = \{\mathfrak{y} \in \mathcal{H} : \|\mathfrak{y}\| \leq \mathfrak{r}\}$ as ϕ and f both are continuous functions so is $\mathbf{A}$. Then we have to show that $\mathbf{A}$ is contraction mapping. Let $\mathfrak{x}, \mathfrak{y} \in \Delta$, so it is easy to check the inequality

$$\|\mathbf{A}(\mathfrak{y}) - \mathbf{A}(\mathfrak{x})\| \leq \sup_{\mathfrak{s} \in \mathfrak{I}} \begin{cases} |\phi(\mathfrak{y}) - \phi(\mathfrak{x})|, \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ |\mathfrak{y}(\mathfrak{s}_1) - \mathfrak{x}(\mathfrak{s}_1)|, \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}, \end{cases}$$

then

$$\|\mathbf{A}(\mathfrak{y}) - \mathbf{A}(\mathfrak{x})\| \leq \begin{cases} \mathbf{C}_\phi \| \mathfrak{y} - \mathfrak{x} \|, \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ 0, \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}. \end{cases}$$

Using the assumption (F_2) and condition $\mathbf{C}_\phi < 1$, we have

$$\|\mathbf{A}(\mathfrak{y}) - \mathbf{A}(\mathfrak{x})\| \leq \mathbf{C}_\phi \| \mathfrak{y} - \mathfrak{x} \|.$$

So, it is clear that $\mathbf{A}$ is contraction mapping.

Step 2 To show that $\mathbf{B}$ is completely continuous, we proceed as $\mathfrak{y} \in \Delta$ then one has,

$$\|\mathbf{B}(y)\| \leq \sup_{\mathfrak{s} \in \mathfrak{I}} \begin{cases} \int_0^{\mathfrak{s}_1} |\mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))|d\xi, \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} |\mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))|(\mathfrak{s} - \xi)^{\mathfrak{p}-1}d\xi, \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}. \end{cases} \tag{13}$$

By using hypothesis (F_4), we have

$$\|\mathbf{B}(\mathfrak{y})\| \leq \sup_{\mathfrak{s} \in \mathfrak{I}} \begin{cases} \int_0^{\mathfrak{s}_1} [(|\mathcal{N}_1(\mathfrak{s})| + \mathcal{N}_2(\mathfrak{s})| + \mathcal{N}_3(\mathfrak{s})|)\mathfrak{r} + |\mathbf{L}_\mathfrak{u}(\mathfrak{s})|], \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} [(|\mathcal{N}_1(\mathfrak{s})| + \mathcal{N}_2(\mathfrak{s})| + \mathcal{N}_3(\mathfrak{s})|)\mathfrak{r} + |\mathbf{L}_\mathfrak{u}(\mathfrak{s})|], \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}, \end{cases}$$

by our assumption

$$\mathbf{N}_1^* = \sup_{\mathfrak{s} \in \mathfrak{I}} |\mathcal{N}_1(\mathfrak{s})|, \mathbf{N}_2^* = \sup_{\mathfrak{s} \in \mathfrak{I}} |\mathcal{N}_2(\mathfrak{s})|, \mathbf{N}_3^* = \sup_{\mathfrak{s} \in \mathfrak{I}} |\mathcal{N}_3(\mathfrak{s})|,$$

further we take

$$\mathbf{N}_1^* + \mathbf{N}_2^* + \mathbf{N}_3^* = \mathbf{N}^*,$$

which yields,

$$\|\mathbf{B}(\mathfrak{y})\| \leq \begin{cases} \mathfrak{s}_1(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*), \text{ if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \frac{(\mathcal{T}-\mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*), \text{ if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}. \end{cases}$$

Putting

$$\mathbf{R}^* = \max\left\{\mathfrak{s}_1(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*), \frac{(\mathcal{T}-\mathfrak{s}_1)^{\mathfrak{p}}(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*)}{\Gamma(\mathfrak{p}+1)}\right\}.$$

So, by take the foam

$$\|\mathbf{B}(\mathfrak{y})\| \leq \mathbf{R}^*,$$

Therefore, $\mathbf{B}$ is uniformly bounded.

Step 3 Now for equi-continuity, assume $\mathfrak{s}_2 < \mathfrak{s}_3 \in \mathfrak{J}$ then,

$$\begin{aligned} |\mathbf{B}\mathfrak{y}(\mathfrak{s}_3) - \mathbf{B}\mathfrak{y}(\mathfrak{s}_2)| = & \Bigg| \frac{1}{\Gamma(\mathfrak{p})}\Bigg[\int_{\mathfrak{s}_1}^{\mathfrak{s}_3} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi))(\mathfrak{s}_3-\xi)^{\mathfrak{p}-1}d\xi \\ & - \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi))(\mathfrak{s}_2-\xi)^{\mathfrak{p}-1}d\xi\Bigg]\Bigg| \\ \leq & \Bigg| \frac{1}{\Gamma(\mathfrak{p})}\Bigg[\int_{\mathfrak{s}_2}^{\mathfrak{s}_3} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi))(\mathfrak{s}_3-\xi)^{\mathfrak{p}-1}d\xi \\ & + \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi))[(\mathfrak{s}_3-\xi)^{\mathfrak{p}-1} - (\mathfrak{s}_2-\xi)^{\mathfrak{p}-1}]d\xi\Bigg]\Bigg| \\ \leq & \frac{(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*)}{\Gamma(\mathfrak{p})}\left[\int_{\mathfrak{s}_2}^{\mathfrak{s}_3}(\mathfrak{s}_3-\xi)^{\mathfrak{p}-1}d\xi + \int_{\mathfrak{s}_1}^{\mathfrak{s}_2}[(\mathfrak{s}_3-\xi)^{\mathfrak{p}-1} - (\mathfrak{s}_2-\xi)^{\mathfrak{p}-1}]d\xi\right] \\ \leq & \frac{(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*)}{\Gamma(\mathfrak{p}+1)}[(\mathfrak{s}_3-\mathfrak{s}_1)^{\mathfrak{p}} - (\mathfrak{s}_2-\mathfrak{s}_1)^{\mathfrak{p}}] \\ \leq & \frac{(\mathbf{N}^*\mathfrak{r} + \mathbf{L}^*)}{\Gamma(\mathfrak{p}+1)}[(\mathfrak{s}_3-\mathfrak{s}_1)^{\mathfrak{p}} - (\mathfrak{s}_2-\mathfrak{s}_1)^{\mathfrak{p}}] \to 0, \text{ as } \mathfrak{s}_3 \to \mathfrak{s}_2. \end{aligned}$$

As $\mathbf{B}$ is continuous and bounded on $\mathfrak{J}$ so it is uniformly continuous. Thus we have

$$\|\mathbf{B}\mathfrak{y}(\mathfrak{s}_3) - \mathbf{B}\mathfrak{y}(\mathfrak{s}_2)\| \to 0, \text{ as } \mathfrak{s}_3 \to \mathfrak{s}_2.$$

Hence **B** is equi-continuous. So, $\mathbf{B}(\Delta)$ is also compact. Using Arzela Ascoli theorem **B** is uniformly continuous operator. The theorem dependent on Krasnoselkii's fixed point has now achieved its conclusion. So there exist at least one fixed point for problem (1) on $\mathfrak{I}$. $\square$

3.3 Stability Results

Stability analysis has an important role in numerical methods and optimization theory. Furthermore, the stability results must be explored in order to established different numerical techniques and procedures. Stabilities of many types are introduced in fractional calculus. In our delay type fractional differential problem we use U-H stability, Generalized U-H stability, U-H Rassias stability and Generalized U-H Rassis stability. Prior to deriving the main findings of stability, we make the following observations:

Remark 3.6. The inequality (2) has a solution $\mathfrak{y} \in \mathcal{H}(\mathfrak{I}, \mathbb{R})$, if and only if a function $\phi \in \mathcal{H}(\mathfrak{I}, \mathbb{R})$ exist such that,

1. $|\phi(\mathfrak{s})| \leq \epsilon, \mathfrak{s} \in \mathfrak{I}$.
2. ${}^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0^+}\mathfrak{y}(\mathfrak{s}) = \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi)) + \phi(\mathfrak{s}), \mathfrak{s} \in \mathfrak{I}$.

Lemma 3.7. *Again consider the integral solution of our suggested problem* (1)*,*

$$\begin{aligned} &{}^{\mathcal{PCF}}\mathfrak{D}^{\mathfrak{p}}_{0^+}\mathfrak{y}(\mathfrak{s}) = \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi)), \quad \mathfrak{s} \in [0, \mathcal{T}], \\ &\mathfrak{y}(0) = \phi(\mathfrak{y}). \end{aligned} \tag{14}$$

is computed as

$$\mathfrak{y}(\mathfrak{s}) = \begin{cases} \phi(\mathfrak{y}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))d\xi, \ \textit{if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \mathfrak{y}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}} d\xi, \ \textit{if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}. \end{cases} \tag{15}$$

The above solution moreover fulfils the following criteria by (3.3)*,*

$$\|\mathfrak{y} - \boldsymbol{Z}\mathfrak{x}\| \leq \begin{cases} \mathfrak{s}_1\epsilon, \ \textit{if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \left[\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}-1}}{\Gamma(\mathfrak{p}+1)}\right]\epsilon = \Omega\epsilon, \ \textit{if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}. \end{cases} \tag{16}$$

Proof. Thank full to Lemma 3.1 and Remark 3.6 , the integral solution of (14) can be generated easily. On further simplification the relation of the above equations is easily straight forward. □

Theorem 3.8. *Thank full to Lemma* 4 *, the generated solution of suggested problem* (1) *is U-H stable and on further simplification the problem* (1) *is also generalized U-H stable by using the condition* $(\boldsymbol{C}_\phi + \boldsymbol{M}_{\mathfrak{u}}\mathfrak{s}_1) < 1$.

Proof. We can divide our results into two categories. Consider, $\mathfrak{I}_1 = [0, \mathfrak{s}_1]$ and $\mathfrak{I}_2 = [\mathfrak{s}_1, \mathcal{T}]$ and then proceed as,

Case 1 When $\mathfrak{s} \in \mathfrak{I}_1$,then

$$\begin{aligned}
\|\mathfrak{y} - \mathfrak{x}\| &= \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - (\phi(\mathfrak{x}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau-\xi))d\xi) \right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - (\phi(\mathfrak{y}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi)d\xi) \right| \\
&+ \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \phi(\mathfrak{y}) - \phi(\mathfrak{x}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi))d\xi - \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau-\xi))d\xi \right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{I}} (\mathfrak{s}_1\epsilon + \mathbf{C}_\phi | \mathfrak{y} - \mathfrak{x} | + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1 | \mathfrak{y} - \mathfrak{x} |) \\
&\leq \mathfrak{s}_1\epsilon + (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{g}}\mathfrak{s}_1) \| \mathfrak{y} - \mathfrak{x} \|.
\end{aligned}$$

The further simplification gives,

$$\| \mathfrak{y} - \mathfrak{x} \| \leq \left(\frac{\mathfrak{s}_1}{1 - (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1)} \right) \epsilon. \tag{17}$$

Case 2 When $\mathfrak{s} \in \mathfrak{J}_2$,then

$$
\begin{aligned}
\| \mathfrak{y} - \mathfrak{x} \| &= \sup_{\mathfrak{s}\in\mathfrak{J}} \left| \mathfrak{y} - (\mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi) \right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{J}} \left| \mathfrak{y} - (\mathfrak{y}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi) \right| \\
&+ \sup_{\mathfrak{s}\in\mathfrak{J}} \left| \mathfrak{y}(\mathfrak{s}_1) - \mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi \right. \\
&\left. - \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi \right| \\
&\leq \left[\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)} \right] \epsilon + \mathbf{C}_\phi \| \mathfrak{y} - \mathfrak{x} \| + \mathbf{M}_{\mathfrak{u}} \frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)} \| \mathfrak{y} - \mathfrak{x} \| \\
&\leq \Omega\epsilon + (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\Omega) \| \mathfrak{y} - \mathfrak{x} \|,
\end{aligned}
$$

where

$$\left[\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)} \right] = \Omega$$

Hence we get

$$\| \mathfrak{y} - \mathfrak{x} \| \leq \left(\frac{\Omega}{1 - (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\Omega)} \right) \epsilon. \tag{18}$$

If we combine **Case 1** and **Case 2**, we can at the following conclusion,

$$\mathcal{K}_1 = \max \left\{ \frac{\mathfrak{s}_1}{1 - (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1)}, \frac{\Omega}{1 - (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\Omega)} \right\}.$$

Then we have

$$\| \mathfrak{y} - \mathfrak{x} \| \leq \mathcal{K}_1\epsilon, \text{ for all } \mathfrak{s} \in \mathfrak{J}. \tag{19}$$

Hence, problem (1) is U-H stable. If we take $\psi(\epsilon) = \epsilon$, and $\psi(0) = 0$, then problem (1) is also generalized U-H stable. □

Prior to deriving the findings about Rassias stability, we make the following observations:

Lemma 3.9. *Again consider the solution of our suggested problem* (1),

$$
\begin{aligned}
&{}^{\mathcal{PCF}}\mathfrak{D}_{0^+}^{\mathfrak{p}} \mathfrak{y}(\mathfrak{s}) = \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi)), \quad \mathfrak{s} \in [0, \mathcal{T}], \\
&\mathfrak{y}(0) = \phi(\mathfrak{y}).
\end{aligned} \tag{20}
$$

The above solution of (1) *moreover fulfils the following criteria by* (3.3),

$$\|\mathfrak{y}-\boldsymbol{Z}\mathfrak{x}\| \leq \begin{cases} \mathfrak{s}_1\varphi(\mathfrak{s})\epsilon, \ \textit{if } 0 \leq \mathfrak{s} \leq \mathfrak{s}_1, \\ \left[\frac{(\mathcal{T}-\mathrm{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\right]\epsilon = \Omega\varphi(\mathfrak{s})\epsilon, \ \textit{if } \mathfrak{s}_1 \leq \mathfrak{s} \leq \mathcal{T}. \end{cases} \tag{21}$$

Proof. Thank full to Lemma 3.1 and Remark 3.6 , the integral solution of (20) can be generated easily. On further simplification the relation of the above equations is easily straight forward. □

Theorem 3.10. *Inview of* (F_1)–(F_3) *Lemma* 3.9, *the generated solution of suggested problem* (1) *is U-H Rassias stable by using the condition* $\Omega\boldsymbol{M}_{\mathfrak{u}} < 1$.

Proof. We can divide our results into two categories.

Case 1 When $\mathfrak{s} \in \mathfrak{I}_1$,then

$$\begin{aligned}
\| \mathfrak{y}-\mathfrak{x} \| &= \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - (\phi(\mathfrak{x}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau-\xi))d\xi) \right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - (\phi(\mathfrak{y}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi))d\xi) \right| \\
&+ \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \phi(\mathfrak{y}) - \phi(\mathfrak{x}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau-\xi))d\xi - \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau-\xi))d\xi \right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{I}} (\mathfrak{s}_1\epsilon + \mathbf{C}_\phi | \ \mathfrak{y}-\mathfrak{x} \ | + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1 | \ \mathfrak{y}-\mathfrak{x} \ |) \\
&\leq \mathfrak{s}_1\varphi(\mathfrak{s})\epsilon + (\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1) \| \ \mathfrak{y}-\mathfrak{x} \ \|.
\end{aligned}$$

The further simplification gives,

$$\| \ \mathfrak{y}-\mathfrak{x} \ \| \leq \left(\frac{\mathfrak{s}_1}{1-(\mathbf{C}_\phi + \mathbf{M}_{\mathfrak{u}}\mathfrak{s}_1)} \right) \varphi(\mathfrak{s})\epsilon. \tag{22}$$

Case 2 When $\mathfrak{s} \in \mathfrak{I}_2$,then

$$\begin{aligned}
\|\mathfrak{y} - \mathfrak{x}\| &= \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - (\mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi) \right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - (\mathfrak{y}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi) \right| \\
&+ \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y}(\mathfrak{s}_1) - \mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi \right. \\
&\left. - \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi \right| \\
&\leq \left[\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)} \right] \epsilon + \mathbf{C}_\phi \| \mathfrak{y} - \mathfrak{x} \| + \mathbf{M}_\mathfrak{u} \frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)} \| \mathfrak{y} - \mathfrak{x} \| \\
&\leq \Omega\varphi(\mathfrak{s})\epsilon + (\mathbf{C}_\phi + \mathbf{M}_\mathfrak{u}\Omega) \| \mathfrak{y} - \mathfrak{x} \|,
\end{aligned}$$

where

$$\left[\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)} \right] = \Omega.$$

Hence we get

$$\| \mathfrak{y} - \mathfrak{x} \| \leq \left(\frac{\Omega}{1 - (\mathbf{C}_\phi + \mathbf{M}_\mathfrak{u}\Omega)} \right) \varphi(\mathfrak{s})\epsilon. \tag{23}$$

If we combine **Case 1** and **Case 2**, we can at the following conclusion,

$$\mathcal{K}_2 = \max \left\{ \frac{\mathfrak{s}_1}{1 - (\mathbf{C}_\phi + \mathbf{M}_\mathfrak{u}\mathfrak{s}_1)}, \frac{\Omega}{1 - (\mathbf{C}_\phi + \mathbf{M}_\mathfrak{u}\Omega)} \right\}.$$

then we have

$$\| \mathfrak{y} - \mathfrak{x} \| \leq \mathcal{K}_2\varphi(\mathfrak{s})\epsilon, \quad \text{for all } \mathfrak{s} \in \mathfrak{I}. \tag{24}$$

Hence, problem (1) is U-H Rassias stable. □

Theorem 3.11. *Inview of* (F_1)–(F_3) *Lemma* 3.9*, the generated solution of suggested problem* (1) *is generalized U-H Rassias stable by using condition* $\Omega \boldsymbol{M}_\mathfrak{u} < 1$.

Proof. We can divide our results into two categories.

Case 1 When $\mathfrak{s} \in \mathfrak{I}_1$,then

$$
\begin{aligned}
\parallel \mathfrak{y} - \mathfrak{x} \parallel &= \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - (\phi(\mathfrak{x}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))d\xi) \right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - (\phi(\mathfrak{y}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))d\xi) \right| \\
&+ \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \phi(\mathfrak{y}) - \phi(\mathfrak{x}) + \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))d\xi - \int_0^{\mathfrak{s}_1} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))d\xi \right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{I}} (\mathfrak{s}_1 \varphi(\mathfrak{s}) + \mathbf{C}_\phi | \mathfrak{y} - \mathfrak{x} | + \mathbf{M}_\mathfrak{u} \mathfrak{s}_1 | \mathfrak{y} - \mathfrak{x} |) \\
&\leq \mathfrak{s}_1 \varphi(\mathfrak{s}) + (\mathbf{C}_\phi + \mathbf{M}_\mathfrak{u} \mathfrak{s}_1) \parallel \mathfrak{y} - \mathfrak{x} \parallel.
\end{aligned}
$$

The further simplification gives,

$$
\parallel \mathfrak{y} - \mathfrak{x} \parallel \leq \left(\frac{\mathfrak{s}_1}{1 - (\mathbf{C}_\phi + \mathbf{M}_\mathfrak{u} \mathfrak{s}_1)} \right) \varphi(\mathfrak{s}). \tag{25}
$$

Case 2 When $\mathfrak{s} \in \mathfrak{I}_2$,then

$$
\begin{aligned}
\parallel \mathfrak{y} - \mathfrak{x} \parallel &= \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - (\mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi) \right| \\
&\leq \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y} - (\mathfrak{y}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi) \right| \\
&+ \sup_{\mathfrak{s}\in\mathfrak{I}} \left| \mathfrak{y}(\mathfrak{s}_1) - \mathfrak{x}(\mathfrak{s}_1) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{y}(\xi), \mathfrak{y}(\lambda\xi), \mathfrak{y}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi \right. \\
&\left. - \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_1}^{\mathfrak{s}} \mathfrak{u}(\xi, \mathfrak{x}(\xi), \mathfrak{x}(\lambda\xi), \mathfrak{x}(\tau - \xi))(\mathfrak{s} - \xi)^{\mathfrak{p}-1} d\xi \right| \\
&\leq \left[\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)} \right] \varphi(\mathfrak{s}) + \mathbf{C}_\phi \parallel \mathfrak{y} - \mathfrak{x} \parallel + \mathbf{M}_\mathfrak{u} \frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)} \parallel \mathfrak{y} - \mathfrak{x} \parallel \\
&\leq \Omega \varphi(\mathfrak{s}) + (\mathbf{C}_\phi + \mathbf{M}_\mathfrak{u} \Omega) \parallel \mathfrak{y} - \mathfrak{x} \parallel,
\end{aligned}
$$

where

$$
\left[\frac{(\mathcal{T} - \mathfrak{s}_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)} \right] = \Omega.
$$

Hence we get

$$
\parallel \mathfrak{y} - \mathfrak{x} \parallel \leq \left(\frac{\Omega}{1 - (\mathbf{C}_\phi + \mathbf{M}_\mathfrak{u} \Omega)} \right) \varphi(\mathfrak{s}). \tag{26}
$$

If we combine **Case 1** and **Case 2**, we can at the following conclusion,

$$\mathcal{K}_{\mathfrak{u},\varphi} = \max\left\{ \frac{\mathfrak{s}_1}{1-(\mathbf{C}_\phi + \mathbf{M}_\mathfrak{u}\mathfrak{s}_1)}, \frac{\Omega}{1-(\mathbf{C}_\phi + \mathbf{M}_\mathfrak{u}\Omega)} \right\}.$$

Then we have

$$\parallel \mathfrak{y} - \mathfrak{x} \parallel \leq \mathcal{K}_{\mathfrak{u},\varphi}\varphi(\mathfrak{s}), \quad \text{for all } \mathfrak{s} \in \mathfrak{I}. \tag{27}$$

Hence, problem (1) is generalized U-H Rassias stable. □

3.4 Examples

In this section, we verify our theoretical findings with different examples.

Example 3.12. *Consider the following problem,*

$$\begin{aligned} {}^{\mathcal{PCF}}\mathfrak{D}^{0.8}_{0^+}\mathfrak{x}(\xi) &= \frac{exp(-\pi\sin\xi)}{62+\xi^4} + \frac{1}{74}\left(\frac{2\pi|\mathfrak{x}(\xi)|}{(1+|\mathfrak{x}(\xi)|)} + \frac{|\mathfrak{x}(\lambda\xi)|\tan^{-1}(\xi)}{\sqrt{\xi^2+64}} + \sin^{-1}\ln|\mathfrak{x}(\tau-\xi| \right), \quad \xi \in [0,1] \\ \mathfrak{x}(0) &= \frac{\cos|\ \mathfrak{x}\ |}{102}. \end{aligned} \tag{28}$$

Taking $\mathfrak{p} = 0.8$

$$g(\xi,\mathfrak{x}) = \frac{exp(-\pi\sin\xi)}{62+\xi^4} + \frac{1}{74}\left(\frac{2\pi|\mathfrak{x}(\xi)|}{(1+|\mathfrak{x}(\xi)|)} + \frac{|\mathfrak{x}(\lambda\xi)|\tan^{-1}(\xi)}{\sqrt{\xi^2+64}} + \sin^{-1}\ln|\mathfrak{x}(\tau-\xi| \right).$$

Let $\mathfrak{x}, \mathfrak{z} \in \mathfrak{I}$,

$$\begin{aligned} &|g(\xi,\mathfrak{x}) - g(\xi,\mathfrak{z})| \\ &= \Bigg| \left[\frac{exp(-\pi\sin\xi)}{62+\xi^4} + \frac{1}{74}\left(\frac{2\pi|\mathfrak{x}(\xi)|}{(1+|\mathfrak{x}(\xi)|)} + \frac{|\mathfrak{x}(\lambda\xi)|\tan^{-1}(\xi)}{\sqrt{\xi^2+64}} + \sin^{-1}\ln|\mathfrak{x}(\tau-\xi| \right) \right] \\ &- \left[\frac{exp(-\pi\sin\xi)}{62+\xi^4} + \frac{1}{74}\left(\frac{2\pi|\mathfrak{z}(\xi)|}{(1+|\mathfrak{z}(\xi)|)} + \frac{|\mathfrak{z}(\lambda\xi)|\tan^{-1}(\xi)}{\sqrt{\xi^2+64}} + \sin^{-1}\ln|\mathfrak{z}(\tau-\xi| \right) \right] \Bigg| \\ &\leq \frac{81\pi}{2368} \parallel \mathfrak{x} - \mathfrak{z} \parallel. \end{aligned}$$

For assumption (F_4)

$$\begin{aligned} |g(\xi,\mathfrak{x})| &= \left| \frac{exp(-\pi\sin\xi)}{62+\xi^4} + \frac{1}{74}\left(\frac{2\pi|\mathfrak{x}(\xi)|}{(1+|\mathfrak{x}(\xi)|)} + \frac{|\mathfrak{x}(\lambda\xi)|\tan^{-1}(\xi)}{\sqrt{\xi^2+64}} + \sin^{-1}\ln|\mathfrak{x}(\tau-\xi| \right) \right| \\ &\leq \frac{1}{62} + \frac{81\pi}{2368}|\ \mathfrak{x}(\xi)\ |. \end{aligned}$$

We can see that $\boldsymbol{M}_{\mathfrak{u}} = \frac{81\pi}{2368}, \boldsymbol{N}_{\mathfrak{u}} = \frac{81\pi}{2368}, \boldsymbol{C}_{\phi} = \frac{1}{102}$, *and* $\boldsymbol{L}_{\mathfrak{u}} = \frac{1}{62}$. *Taking* $\mathcal{T} = 1$ *and* $\xi_1 = 0.9$, *now applying Theorem* **??** *to get,*

$$\begin{aligned}\boldsymbol{D} &= \max\left\{\boldsymbol{C}_{\phi} + \boldsymbol{M}_{\mathfrak{u}}\xi_1, \boldsymbol{M}_{\mathfrak{u}}\frac{(\mathcal{T} - \xi_1)^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\right\} \\ &= \max\{0.10646, 0.09646\} = 0.10646 < 1.\end{aligned}$$

By using Theorem 3.4 *the given problem has unique solution. Now inview of Theorem* 3.5, *we can see that*

$$\boldsymbol{C}_{\phi} = \frac{1}{102} < 1,$$

hence the condition of Theorem 3.5 *is satisfied. So the given problem has at least one solution. Further the condition for U-H stable and generalized U-H stable is also fulfilled because we can see that* $\Omega = 0.1705$, $(\boldsymbol{C}_{\phi} + \boldsymbol{M}_{\mathfrak{u}}\xi_1) = 0.10646 < 1$. *Moreover if we take* $\varphi(\xi) = \xi$, *then the condition for U-H Rassias stable and U-H generalized Rassias stable is also satisfied.*

Chapter 4

Summary and Conclusion

4 Summary and Conclusion

This section of thesis is reserved for summary and conclusion part. Here we give review about thesis and research work we done using different fractional calculus approaches.

4.1 Summary

Our thesis work is organised into five chapters. The first chapter explores the history and applications of fractional calculus in detail. In this section we also discussed our proposed problem. We introduced several special functions utilized in fractional calculus in second chapter. We also cover the definitions of fractional integral and derivative, as well as stability results and some nonlinear dynamic and fixed point theorem results that are important in the study of fractional differential equations. In chapter three, we construct some reasonable assumptions and using fixed point techniques to prove the existence and uniqueness of our problem's solution. In this chapter, we also presented the U-H stability results for our proposed topic. The conclusion and summary of our thesis are discussed in the following chapter. The final chapter is devoted to references.

4.2 Conclusion

Recently, a new idea of piece wise fractional differential equations was introduced. We have expanded some results about uniqueness, existence and stability analysis for our delay type problem, acknowledging the importance of fractional calculus in recent years. The results were obtained by utilizing the fixed point concept and non linear functional analyses concepts. Sufficient circumstances have been set up to ensure the existence of at least one solution and its uniqueness to the suggested problem. Nonlinear analytic tools were also used to determine its stability. We can observe that these derivatives better describe the abrupt behavioral change. As a result, we conclude that this sort of calculus will open up a new area of study in the near future. More research into how to deal with piece wise fractional order differential boundary value problems will be conducted in the future.

Chapter 5

References

5 References

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Serial No: 351008

Registration No: **2016-PCS-22**

Result Declared on: 01-08-2018.

Roll No: **42585**

Bahauddin Zakariya University, Multan (Pakistan)

Controller of Examinations

TRANSCRIPT

PASS

Bachelor of SCIENCE , Annual Examination 2018

Held in April - May, 2018.

Can be Verified at http://www.bzu.edu.pk

Name of the Candidate: **FATIMA ILYAS**

Father's Name: **MUHAMMAD ILYAS**

The candidate mentioned above is hereby informed that she has **PASSED** B.A/B.Sc. Annual Examination, 2018 obtaining 655 /800 marks and has been placed in **FIRST** division.

The marks obtained by her in various subjects are given below:

SUBJECTS	MARKS OBTAINED		
	Marks Detail (A+B+C+D+P)	*In Figures*	*In Words*
1.English Language (Compulsory)	54	54 /100	*Fifty Four*
2.Islamic Studies/Ethics(Compulsory)		41 /60	*Forty One*
3.Pakistan Studies(Compulsory)		28 /40	*Twenty Eight*
4.Physics	35+ 33+38+38+ 31	175 /200	*One Hundred Seventy Five*
5 Mathematics-A Course	86+96	182 /200	*One Hundred Eighty Two*
6 Mathematics-B Course	81+94	175 /200	*One Hundred Seventy Five*
	Total:	655/800	*Six Hundred Fifty Five*

NOTE: This result card is issued as a notice only; errors and omissions are excepted. An entry appearing in it does not in itself confer any right or privilege independently to the grant of a proper Certificate/ Degree which will be issued under the Regulations in due course.

Checked by

Asst. Controller of Examinations
for Controller.

343926

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COMSATS University Islamabad

TRANSCRIPT

Date: **15-Sep-2022**

Certified that **Fatima Ilyas** d/o **Muhammad Ilyas** Registration No. **CIIT/FA20-RMT-025/SWL** Department of **Mathematics** has completed and passed all the requisite courses / examinations for the degree of **Master of Science in Mathematics** , on **15th July 2022** , taught as a **Regular Mode** of study from **Sahiwal Campus.**
Date of Birth : 24-Nov-1997

The details of the courses passed are as follows:

Course Code	Semester / Course Title	Cr.	LG	CP	Course Code	Semester / Course Title	Cr.	LG	CP
	Fall 2020					**Fall 2021**			
MTH612	Numerical Solutions of PDEs I	3	A+	12.00	MTH800	Thesis		IP	
MTH619	Commutative Algebra	3	A	11.70					
MTH658	Direct and Inverse Problems in Wave Propagation	3	A+	12.00		**Spring 2022**			
MTH661	Viscous Fluids I	3	A	11.55	MTH800	Thesis	6^{+}	Approved	
	Spring 2021								
MTH611	Integral Inequalities	3	A	11.10					
MTH618	Advanced Topics in Graph Theory	3	A+	12.00					
MTH651	Symmetry Methods in Differential Equations	3	A	10.65					
MTH662	Viscous Fluids II	3	A	11.25					

Thesis Title: **Mathematical Analysis of Piece Wise Fractional Differential Equations**

Total Credit Hours Registered:	30
*Discounted Credit Hours:	0
Net Credit Hours Passed:	30
Total CP:	92.25
CGPA^{+}:	3.84

$^{+}$Credit Hours allocated to Thesis are not counted for calculation of CGPA

Controller of Examinations

Errors/Omissions Excepted

(Please Turn Over)

COMSATS University Islamabad

TRANSCRIPT

0072176

Date: **08-Oct-2020**

Certified that **FATIMA ILYAS** d/o **MUHAMMAD ILYAS** Registration No. **CIIT/FA18-MMT-066/SWL** Department of **Mathematics** has completed and passed all the requisite courses / examinations for the degree of **Master of Mathematics**, on **4th September 2020**, taught as a **Regular Mode** of study from **Sahiwal Campus.**
Date of Birth : 24-Nov-1997

The details of the courses passed are as follows:

Course Code	Semester / Course Title	Cr.	LG	CP	Course Code	Semester / Course Title	Cr.	LG	CP
	Fall 2018					**Fall 2019**			
MTH105	Multi Variable Calculus	3	B+	9.9	MTH352	Differential Geometry	3	A	12.0
MTH251	Set Topology	3	B+	9.9	MTH374	Optimization	3	A-	11.1
MTH321	Real Analysis I	3	A	12.0	MTH415	Rings and Modules	3	A	12.0
MTH483	Mathematical Methods of Physics	3	A	12.0	MTH471	Integral Equations	3	A	12.0
STA365	Mathematical Statistics	3	A	12.0	MTH486	Fluid Mechanics	3	A	12.0
					MTH499 *	Project Report	3	IP	0.0
	Spring 2019								
MTH322	Real Analysis II	3	A-	11.1		**Spring 2020**			
MTH324	Complex Analysis	3	A	12.0	MTH382	Analytical Dynamics	3	A-	11.1
MTH327	Functional Analysis	3	A	12.0	MTH442	Graph Theory	3	A-	11.1
MTH343	Partial Differential Equations	3	A	12.0	MTH480	Introductory Quantum Mechanics	3	B	9.0
MTH375	Numerical Computations	3	A-	11.1	MTH487	Continuum Mechanics	3	B+	9.9
MTH376	Algebra	3	A-	11.1	MTH499	Project Report	6	A	24.0

Total Credit Hours Registered:	**69**
* Discounted Credit Hours:	**3**
Net Credit Hours Passed:	**66**
Total CP:	**249.3**
CGPA:	**3.78**

Controller of Examinations

Errors/Omissions Excepted

(Please Turn Over)

BOARD OF INTERMEDIATE & SECONDARY EDUCATION, SAHIWAL

Sr.No. 5407738

Roll No. 308032

Result Pass

Registration No. 3420721314

PROVISIONAL RESULT INTIMATION

INTERMEDIATE (PART-I / II) ANNUAL EXAMINATION, 2016

GROUP PRE-ENGINEERING

NAME FATIMA ILYAS

FATHER'S NAME MUHAMMAD ILYAS

INSTITUTION/DISTRICT PUNJAB COLLEGE FOR GIRLSSAHIWAL

He/She has secured the marks as detailed below against each subject.

Sr. No.	Name of Subjects	Maximum Marks	Marks Obtained				Status	
			Theory		Practical	Total	I	II
			Part-I	Part-II				
1	URDU	200	83	75		158	P	P
2	ENGLISH	200	83	76		159	P	P
3	ISLAMIC EDUCATION	50	48			48	P	
4	PAKISTAN STUDIES	50		46		46		P
5	PHYSICS	200	93	97	B	190	P	P
6	CHEMISTRY	200	86	95	A+	181	P	P
7	MATHEMATICS	200	83	92		175	P	P
8								
	TOTAL/OVERALL GRADE	1100				957	Grade	A+

The candidate has passed and obtained marks Nine Hundred Fifty Seven.

NOTE:

(I) This provisional result intimation is issued as notice only. Any entry appearing in it does not itself confer any right or privilege independently for the grant of proper certificate which will be issued later on under the rules.

(II) The star (*) indicates that the candidate has passed the subject/s with concessional marks under Rule of the Board's Calendar. In case he/she is not willing to accept the concessional marks, necessary permission to reappear in the subject/s may be obtained within the schedule for submitting the admission forms of the next examination. The candidate will have to attach the attested copy of revised result intimation with the Admission Form and an affidavit that he/she will not claim for the previous result in case of failure.

(III) If the result intimation is lost, duplicate result card may be obtained by the candidate on payment of prescribed fee.

(Error & Omissions are excepted)

نوٹ: جو امیدواران ری چیکنگ کیلئے درخواست دینا چاہیں وہ رزلٹ کی اشاعت کے پندرہ یوم تک مروجہ فیس جمع کروا کر درخواست دے سکتے ہیں۔

Checked by

Dated 17-SEPTEMBER-2016

CONTROLLER OF EXAMINATIONS

POSTAL ADDRESS: ROLL NO.

6354

CamScanner

BOARD OF INTERMEDIATE & SECONDARY EDUCATION, SAHIWAL

Sr.No. 5130474 | Roll No. 101388

Result Pass | Registration No. 3412233412

PROVISIONAL RESULT INTIMATION

SECONDARY SCHOOL CERTIFICATE PART (I/II) (ANNUAL) EXAMINATION, 2014

Group : SCIENCE

Name : FATIMA ILYAS

Father's Name : MUHAMMAD ILYAS

Date of Birth : 24-11-1997 (TWENTY FOURTH NOVEMBER NINETEEN HUNDRED NINETY SEVEN)

Institution/District : BEACON MODEL GIRLS HIGH SCHOOL,SABIR TOWN SAHIWAL

Has secured the marks as detailed below against each subject.

Sr. No.	Name of Subjects	Maximum Marks	Marks Obtained				Status	
			Theory		Practical	Total	9Th	10Th
			9Th	10Th				
1	URDU	150	61	61		122	P	P
2	ENGLISH	150	71	65		136	P	P
3	ISLAMIYAT(COMPULSORY)	100	48	49		97	P	P
4	PAKISTAN STUDIES	100	44	47		91	P	P
5	MATHEMATICS	150	68	75		143	P	P
6	PHYSICS	150	73	72	A+	145	P	P
7	CHEMISTRY	150	68	68	A+	136	P	P
8	BIOLOGY	150	65	65	A+	130	P	P
	TOTAL/OVERALL GRADE	1100				1000		A+

The candidate has passed and obtained marks Ten Hundred .

NOTE:

(I) This provisional result intimation is issued as notice only. Any entry appearing in it does not itself confer any right or privilege independently for the grant of proper certificate which will be issued later on under the rules.

(II) The star (*) indicates that the candidate has passed the subject/s with concessional marks under Rule of the Board's Calendar. In case he/she is not willing to accept the concessional marks, necessary permission to reappear in the subject/s may be obtained within the schedule for submitting the admission forms of the next examination. The candidate will have to attach the attested copy of revised result intimation with the Admission Form and an affidavit that he/she will not claim for the previous result in case of failure.

(III) If the result intimation is lost, duplicate result card may be obtained by the candidate on payment of prescribed fee.

(Error & Omissions are excepted)

نوٹ: جوامیدواران ری چیکنگ کیلئے درخواست دینا چاہیں وہ رزلٹ کی اشاعت کے پندرہ یوم تک مروجہ فیس جمع کروا کر درخواست دے سکتے ہیں۔

Checked by ______ Dated 25-July-2014

CONTROLLER OF EXAMINATIONS

POSTAL ADDRESS: ROLL NO.

MOVE FORWARD. BE GREAT.

AUTHORITY FORM

I am requesting a change of agent for providing me with counseling/advice regarding my university application.

When completed, this form must be submitted to the college I am seeking enrolment at.

First Name	Fatima
Last Name	Ilyas
Date of Birth:	24/11/1997
Country of Nationality:	Pakistan
Your Current Agent or 'Applied Direct'	-
Address of Your Current Agent:	-
Your Proposed Agent:	SI-UK
Address of Your Proposed Agent:	3rd Floor, 211-213 Regent St, Mayfair, London W1B 4NF
Reason(s) for change of Agent:	Through SI-UK I have been able to obtain expert information on UK universities, courses, degree requirements, pre-sessional, application procedures, funding, visas and accommodation.

Upon receipt of my completed form, I am aware and authorize the university to remove my current agent and change it in the student database to my proposed Agent (SI-UK). Based on my previous Privacy Declaration, (as applicable) all future correspondence will be sent to SI-UK. Any commission payment due in relation to my enrolment will be paid to SI-UK.

Signature: ____________________ Date: 10/05/2024

COMSATS UNIVERSITY ISLAMABAD,

SAHIWAL COMPUS.

Department of Mathematics

(Recommendation Letter)

I am pleased to write this recommendation letter in support of **Ms. Fatima Ilyas** as one of my MS students at the department of Mathematics, Comsat University Islamabad, Sahiwal Campus. She possesses good research qualities and excellent academic record throughout in her career and has performed well during her MS Mathematics.

Ms. Ilyas performance as a student has always been commendable to me. I taught her Set Topology and Introductory Quantum Mechanics in 1^{st} and 4^{th} semester of M.Sc Mathematics, and she got **A** grade in both subjects. She worked as a assistant in MS Mathematics in my under.

She proved to be quite keen and adept in learning and using new abstruse concepts. The strongest point of her personality in her diligence and passion for Mathematics. She actively participated in class discussion by illustrating her views confidently. She is very active minded and energetic participant in co-curricular activities.

I would like to mention her that I place her among top 1% students in Mathematics. I have seen a keen urge in her to understand concepts of Mathematics beyond the theoretical purview reflecting her potential for research. I very strongly recommended her for higher studies and I confident that she will prove a valuable asset to your program.

Being her teacher, I wish him a bright future. I sincerely recommend **Ms. Fatima Ilyas** for the position she is applying for.

Sincerely,
Yours faithfully,

Dr. Muhammad Raza,
Associate Professor,
Incharge Academics,
Department of Mathematics,
COMSATS University Islamabad Sahiwal
Campus, Pakistan.
Contact # +92-3056901272
E-mail: mraza@cuisahiwal.edu.pk

Admin no: 517

COMSATS University Islamabad

Sahiwal Campus

COMSATS Road off G.T. Road, Sahiwal

Ph: 040-4305001-5 **Fax:** 040-4305006 **Web:** https://sahiwal.comsats.edu.pk/

TO WHOM IT MAY CONCERN
RECOMMENDATION LETTER

It gives me great pleasure to recommend **Ms. Fatima Ilyas** for undertaking next degree in your esteemed University this year. I have known her from her first semester of the M.Sc. Mathematics programme at COMSATS University Islamabad Sahiwal Campus (2018). She completed her MSc Mathematics research project under my supervision in the field of Fractional Differential Equations. Miss Fatima is an academically exceptional student. She always seeks out challenging schoolwork and completes it with diligence. She also has an excellent talent for taking care of her responsibilities.

I have been her research supervisor during MS Mathematics also. I found her as a motivated and devoted student having command in his domain of work, that is, fractional differential equations. Her thesis work is on a special type of piecewise fractional differential equations

Miss Fatima has learned to keep her options open for the future while maintaining an organized approach to achievement. During her academic year, I tutored her a subject, and she excelled, earning an A. Her success will undoubtedly make her a highly desirable candidate for the best graduate programme nationally and internationally.

Intellectually, she is one the very best students I have ever worked with. She has the ability to carry out research independently, she is extremely disciplined in her work, her analytical skills are highly developed, and she is always through in what she does. She demonstrated a strong aptitude for learning and applying new abstract topics.

In conclusion, I heartily suggest Miss Fatima for further study at your prestigious institution. Her suggested line of study, in my opinion, is appropriate and builds on her advantages. She is especially well-suited to take advantage of this chance because of her inspiring spirit and methodical learning style.

Dr. Asma,
Assistant Professor,
Department of Mathematics,
COMSATS University Islamabad Sahiwal Campus, Pakistan.
Contact # +92-3317057067
E-mail: asma@cuisahiwal.edu.pk

ISLAMIC REPUBLIC OF
PAKISTAN
PASSPORT

P	PAK	HX1744451
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F	SAHIWAL, PAK	
ILYAS, MUHAMMAD		
21 FEB 2023		PAKISTAN
20 FEB 2028		10701675234 R0790446

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Ministry of Interior,
Government of Pakistan
requires and requests
in the name of
The President
Islamic Republic of Pakistan
all those to whom it may concern
to allow the bearer
to pass freely without let or hindrance
and to afford the bearer such assistance
and protection as may be necessary

Director General
Immigration and Passports.

R0790446