Coursework 2

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Q1. (a) (**15 marks**) Find all the elements of maxSpec($\mathbb{C}[x]$), maxSpec($\mathbb{C}[x, 1/x]$), and maxSpec($\mathbb{C}[x, 1/x, y]$) explicitly.

(b) (**5 marks**) Consider the isomorphism $\varphi : \mathbb{A}^1 \backslash \{0\} \longrightarrow \mathbb{A}^1 \backslash \{0\}$, $a \longmapsto b = 1/a$, and the pullback map on the coordinate rings $\varphi^* : \mathbb{C}[x, 1/x] \longmapsto \mathbb{C}[y, 1/y]$. Compute $\varphi^*(1/x)$, $\varphi^*(2x^2 + \frac{2x^3+4x}{x^5})$, $\varphi^*(2-x)$.

(a) By definition, $\text{maxSpec}(\mathbb{C}[x]) = \{\text{maximal ideals in } \mathbb{C}\}$
$= \{(x-a) : a \in \mathbb{C}\}$.

We have that $\text{maxSpec}(\mathbb{C}[V]) = V$ & as we saw in the lectures, $V = \{(x, \frac{1}{x})\} \simeq \mathbb{C}^*$, & thus $\text{maxSpec}(\mathbb{C}[x, \frac{1}{x}]) = \mathbb{C}^*$.

For $\text{maxSpec}(\mathbb{C}[x, \frac{1}{x}, y])$ we again use the fact that $\text{maxSpec}(\mathbb{C}[V]) = V$, & here we claim that $\{(x, \frac{1}{x}, y)\} \simeq \mathbb{C}^* \times \mathbb{C}$.
To see this, let $V = \{(x, \frac{1}{x}, y) : x, y \in \mathbb{C}^*\}$ & $D(x) = \mathbb{A} \backslash \{0\} \times \mathbb{A}$ then we define

$$\varphi : V \longrightarrow D(x)$$
$$(x, y, z) \longmapsto (x, y),$$

which is a morphism since φ is a projection map, hence continuous, & if $U \subseteq D(x)$, & $f \in \mathcal{O}_{D(x)}(U)$ is regular, then $\varphi^*(f) = f \circ \varphi$ is also regular on $\varphi^{-1}(U) = \{(x, y, z) : x \in U, y = \frac{1}{x}, z = y\}$.
Then, the inverse of φ is given by,

$$\psi : D(x) \longrightarrow V$$
$$(x, y) \longmapsto (x, \frac{1}{x}, y),$$

which is also a morphism, since x, $\frac{1}{x}$ & y are indeed regular.
Since we have an isomorphism of varieties, we have that

$$\mathcal{O}_V(V) = \mathbb{C}[V] = \frac{\mathbb{C}[x, y, z]}{\mathbb{I}(V)}.$$

Therefore, $\mathcal{O}_{D(x)}(D(x)) = \varphi^*(\mathcal{O}_V(V)) = \varphi^*\left(\frac{\mathbb{C}[x, y, z]}{\mathbb{I}(V)}\right) = \mathbb{C}[x, \frac{1}{x}, y]$
Since $x, y \in \mathcal{O}_V(V)$ & $\varphi^*(y) = y \circ \varphi = y \circ (x, \frac{1}{x}, y) = \frac{1}{x}$, & $\varphi^*(z) = z \circ \varphi = z \circ (x, \frac{1}{x}, y) = y$.
Finally, since $\mathcal{O}_{D(x)}(D(x)) = \mathbb{A} \backslash \{0\} \times \mathbb{A} \simeq \mathbb{C}^* \times \mathbb{C}$, our claim holds.
Namely, $\text{maxSpec}(\mathbb{C}[x, \frac{1}{x}, y]) = \mathbb{C}^* \times \mathbb{C}$.

(b) Let $\varphi : \mathbb{A}^1 \backslash \{0\} \longrightarrow \mathbb{A}^1 \backslash \{0\}$
$a \longmapsto \frac{1}{a}$,

$\varphi^* : \mathbb{C}[x, \frac{1}{x}] \longrightarrow \mathbb{C}[y, \frac{1}{y}]$
$f \longmapsto f \circ \varphi$.

Then,

$$\varphi^*\left(\frac{1}{x}\right) = \frac{1}{x} \circ \varphi$$
$$= \frac{1}{x} \circ \frac{1}{y}$$
$$= y.$$

Since pullbacks are homomorphisms we have,

$$\varphi^*\left(2x^2 + \frac{2x^3 + 4x}{x^5}\right) = \varphi^*(2x)\left(\varphi^*(x) + \varphi^*\left(\frac{1}{x^2}\right) + \varphi^*\left(\frac{2}{x^4}\right)\right)$$

$$= \varphi^*(2)\varphi^*(x)\left(\varphi^*(x) + \varphi^*\left(\frac{1}{x}\right)^3 + \varphi^*(2)\varphi^*\left(\frac{1}{x}\right)^5\right)$$

$$= \varphi^*(2)\varphi^*(x)\left(\varphi^*(x) + \varphi^*\left(\tfrac{1}{x}\right)^3 + \varphi^*(2)\varphi^*\left(\tfrac{1}{x}\right)^5\right)$$
$$= \left(2\cdot\tfrac{1}{y}\right)\left(\tfrac{1}{y} + y^3 + 2\cdot y^5\right)$$
$$= \frac{2}{y^2} + 2y^2 + 4y^4.$$

Finally $\varphi^*(2-x) = \varphi^*(2) - \varphi^*(x)$
$$= 2 - \frac{1}{y}.$$

Q2. (**20 marks**) Consider the affine algebraic hypersurface $V := \mathbb{V}(y - ux) \subseteq \mathbb{A}^3$.

(a) Prove that the projection $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$, $(x, y, u) \longmapsto (x, u)$ restricts to an isomorphism from V to $\mathbb{A}^2$.

(b) Prove that the projection $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$, $(x, y, u) \longmapsto (x, y)$ does not restrict to isomorphism from V to $\mathbb{A}^2$.

Let $V = \mathbb{V}(y-ux) \subseteq \mathbb{A}^3$.

(a) Let $\varphi : \mathbb{A}^3 \longrightarrow \mathbb{A}^2$
$(x,y,u) \longmapsto (x,u)$.

Then,
$\varphi|_V : V \longrightarrow \mathbb{A}^2$
$(x, ux, u) \longmapsto (x,u)$.

It follows that $\varphi|_V$ is an isomorphism if $\exists$ an inverse morphism $\psi : \mathbb{A}^2 \longrightarrow V$. We claim that $\psi : \mathbb{A}^2 \longrightarrow V$
$(x,u) \longmapsto (x, ux, u)$

suffices.

Clearly $xu \in V$, since $V = \mathbb{V}(y - ux)$, & so one can see that $\psi \circ \varphi|_V : V \longrightarrow V$ & $\varphi|_V \circ \psi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ are the identity map on V & $\mathbb{A}^2$ respectively. Since $\varphi|_V : (x, ux, u) \longmapsto (x,u)$, as the only $y \in V$ is $v = ux$, & then $\psi : (x,u) \longmapsto (x, ux, u)$.

Restructure/reword!

Now, it remains to show that ψ is a morphism, which is the case if x, ux & u are all regular functions by Theorem 4.15. Therefore since x, ux & u are indeed regular, it follows that ψ is a morphism (x & u are clearly regular & ux is a composition of regular functions).

ψ is continuous & for every open set $U \in V$, & every regular function $f \in \mathcal{O}_V(U)$, $\psi^*(f) = f \circ \psi \in \mathcal{O}_{\mathbb{A}^2}(\varphi|_V(U))$.

(b) Let $\varphi : \mathbb{A}^3 \longrightarrow \mathbb{A}^2$
$(x,y,u) \longmapsto (x,y)$.

Then,
$\varphi|_V : V \longrightarrow \mathbb{A}^2$
$(x, ux, u) \longmapsto (x, ux)$.

However, $\varphi|_V$ is not bijective, since $\varphi|_V(0,1,u) = (0,0)$ for any $u \in \mathbb{C}$, & thus $\varphi|_V$ is not injective. Therefore, an inverse mapping of $\varphi|_V$ does not exist, & thus $\varphi|_V$ is not an isomorphism.

Q3. **(25 marks)**

(a) Prove that if $g \in \mathbb{C}[x, y]$ then the projective closure of its variety $\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g}) \subseteq \mathbb{P}^2$ where $\tilde{g} \in \mathbb{C}[x, y, z]$ is the homogenisation of g.

(b) Consider the polynomials $f_1(x, y) = x + y + 1$, $f_2(x, y) = x^2 + 6y^2 + 1$, $f_3(x, y) = x^2 + 3y + 1$, $f_4(x, y) = x^3 + 3xy^2 + 4$. Determine whether or not each of the projective closures includes the points

(i) $[1 : 0 : 0]$;

(ii) $[0 : 1 : 0]$;

(iii) $[0 : 0 : 1]$.

(c) Can you find a general necessary and sufficient condition on $g \in \mathbb{C}[x, y]$ such that its homogenisation $\tilde{g} \in \mathbb{C}[x, y, z]$ does not pass through any of the three points in item (b)?

(a) Let $g \in \mathbb{C}[x,y]$.

• $\overline{\mathbb{V}(g)} \subseteq \mathbb{V}(\tilde{g})$:

If $g \in \mathbb{C}[x,y]$, then $\tilde{g} \in \mathbb{C}[x,y,z]$, & so by setting $z=1$ in $\tilde{g}$, we get g.
Thus, if $(x,y,z) \in \mathbb{V}(\tilde{g})$ it follows that $(x,y,1) \in \mathbb{V}(\tilde{g})$, i.e. $\mathbb{V}(g) \subseteq \mathbb{V}(\tilde{g})$.
Therefore, since $\mathbb{V}(g) \subseteq \mathbb{V}(\tilde{g})$, it follows that $\overline{\mathbb{V}(g)} \subseteq \mathbb{V}(\tilde{g})$.

• $\mathbb{V}(\tilde{g}) \subseteq \overline{\mathbb{V}(g)}$:

To prove this is equivalent to proving $\mathbb{I}(\overline{\mathbb{V}(g)}) \subseteq \mathbb{I}(\mathbb{V}(\tilde{g})) = \sqrt{\tilde{g}} = \tilde{g}$.
By results in the lectures, we have that $\mathbb{I}(\overline{\mathbb{V}(g)})$ is homogeneous & thus has a homogeneous generator, say $G \in \mathbb{I}(\overline{\mathbb{V}(g)})$. But then, G must vanish on $\overline{\mathbb{V}(g)}$, & so any (x,y,z) that makes G vanishes also makes $\tilde{g}$ vanishes, as $\overline{\mathbb{V}(g)} \subseteq \mathbb{V}(\tilde{g})$ as seen above. Therefore, $G \subseteq \tilde{g}$ & since G is a generator for $\mathbb{I}(\overline{\mathbb{V}(g)})$, it follows that $\mathbb{I}(\overline{\mathbb{V}(g)}) \subseteq \tilde{g}$. Thus, $\mathbb{V}(\tilde{g}) \subseteq \overline{\mathbb{V}(g)}$.

Since $\overline{\mathbb{V}(g)} \subseteq \mathbb{V}(\tilde{g})$ & $\mathbb{V}(\tilde{g}) \subseteq \overline{\mathbb{V}(g)}$, it follows that $\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g})$.

(b) The projective closure of each f_i are: $\mathbb{V}(\tilde{f}_i)$ by (a).
For each f_i we have,

$$\tilde{f}_1 = x + y + z,$$
$$\tilde{f}_2 = x^2 + 6y^2 + z^2,$$
$$\tilde{f}_3 = x^2 + 3yz + z^2,$$
$$\tilde{f}_4 = x^3 + 3xy^2 + 4z^3.$$

(i) By substituting $(x,y,z) = (1,0,0)$ into each $\tilde{f}_i$ above, we see that $[1:0:0] \notin \tilde{f}_i$ for any i.

(ii) Likewise, for $[0:1:0]$, we have $[0:1:0] \notin \tilde{f}_1, \tilde{f}_2$, but we do have $[0:1:0] \in \tilde{f}_3, \tilde{f}_4$ since $\tilde{f}_3(0,1,0) = 0 + 3(1)(0) + 0 = 0$ & $\tilde{f}_4 = 0 + 3(0)(1) + 4(0) = 0$.
Thus, $[0:1:0] \in \mathbb{V}(\tilde{f}_3), \mathbb{V}(\tilde{f}_4)$.

(iii) Then, like (i), we have that $[0:0:1] \notin \tilde{f}_i$ for any i.

Therefore only $[0:1:0]$ is in any of the projective closures, namely $\mathbb{V}(\tilde{f}_3)$ & $\mathbb{V}(\tilde{f}_4)$.

(c) For $\tilde{g}$ of $g \in \mathbb{C}[x,y]$ to pass through none of $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, it must be the case that $g = x^d + y^d + a$ where d is any degree of g & $a \in \mathbb{C}$.
This is because, if the monomials x^d & y^d have different powers, say x^{d_1}, y^{d_2}, then $\tilde{g} = x^{d_1} z^{d_2 - d_1} + y^{d_2} + a z^{d_2}$ if $d_2 > d_1$, & in this case $[1:0:0]$ satisfies $\tilde{g}$. On the other hand, if $d_1 > d_2$, then $\tilde{g} = x^{d_1} + y^{d_2} z^{d_1 - d_2} + a z^{d_1}$ & clearly $[0:1:0]$ satisfies $\tilde{g}$ in this case. Finally, $[0:0:1]$ satisfies $\tilde{g}$ if & only if $\tilde{g} = x^{d_1} y^{d_2}$ or $\tilde{g} = x^{d_1} + y^{d_1}$, since in these cases $\tilde{g}$ is invariant of z.

Q4. (**15 marks**)

(a) Prove that $\mathbb{P}^n$ is compact with respect to the quotient Euclidean topology from $\mathbb{A}^{n+1} \setminus \{0\}$.

Makes sense :: $\mathbb{P}^n$ is a quotient space on $x \sim y$ by $x = \lambda y$. Thus it is just the Euclidean topology on non-zero elements of $\mathbb{A}^n$ (quotienting by 0).

(b) What is the projective Zariski-closure of the $\mathbb{V}(y - \sin(x))$ in $\mathbb{P}^2$? How do you compare this to the Chow's Lemma? **Hint.** In Example 3.44 we have seen that this curve is not algebraic.

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(a) To show that $\mathbb{P}^n$ is compact, it suffices to find a continuous surjective map from a compact set to $\mathbb{P}^n$. This is because compactness is preserved under continuous surjective maps:

Let $f: X \to Y$ be a continuous surjective map, & U_i be open sets that form a cover for Y. Then, the sets $f^{-1}(U_i)$ form an open cover of X. If X is compact, then this cover has a finite subcover, say $f^{-1}(U_1), \dots, f^{-1}(U_n)$. Then, since f is surjective, it follows that the corresponding sets $U_1, \dots, U_n$ form a cover for Y. This is because, for each $y \in Y$, $\exists\, x \in X$ with $f(x) = y$, & this x will be in some set $f^{-1}(U_i)$ of the finite cover of X, & so y will be in the corresponding set U_i.

Since the unit sphere in $\mathbb{C}^{n+1}$, S^{n+1}, is closed & bounded w.r.t the Euclidean topology, then it is compact. Then, we can just take the projection map

$$\varphi: S^{n+1} \to \mathbb{P}^n$$
$$(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n).$$

Which is clearly continuous & surjective, since for any $y \in \mathbb{P}^n$, it follows that $y = \frac{y}{\|y\|}$, since we are in the projective space, & hence $\frac{y}{\|y\|} \in S^n \subseteq S^{n+1}$ & thus for any $y \in \mathbb{P}^n$ $\exists\, \varphi^{-1}(y) \in S^{n+1}$, i.e. φ is surjective.

& since φ is a projection map, it is trivially continuous.

Therefore, φ is a continuous surjective map that preserves compactness, & thus $\mathbb{P}^n$ is compact.

(b) The projective closure of $\mathbb{V}(y - \sin x)$ is:

$$\overline{\mathbb{V}(y-\sin x)} = \mathbb{V}(y - \sin x) \sqcup \{[0:1:0]\}.$$

However, this doesn't contradict Chow's Lemma since the point at infinity is not locally analytic, & so $\overline{\mathbb{V}(y - \sin x)} \subseteq \mathbb{P}^2$ is not an analytic subvariety.

Q5. (**20 marks**)

(a) The variety of a polynomial of the form $ax + by + cz \in \mathbb{C}[x, y, z]$ for $a, b, c \in \mathbb{C}$ is called a *line* in $\mathbb{P}^2$. Prove that any two distinct lines in $\mathbb{P}^2$ intersect exactly at one point.

(b) Assume that $C_1, C_2 \subseteq \mathbb{A}^2$ are two closed affine algebraic curves.

(i) Prove that we have the inclusion $\overline{C_1 \cap C_2} \subset \overline{C}_1 \cap \overline{C}_2$ of projective closures.

(a) The variety of a polynomial of the form $ax + by + cz \in \mathbb{C}[x, y, z]$ for $a, b, c \in \mathbb{C}$ is called a *line* in $\mathbb{P}^2$. Prove that any two distinct lines in $\mathbb{P}^2$ intersect exactly at one point.

(b) Assume that $C_1, C_2 \subseteq \mathbb{A}^2$ are two closed affine algebraic curves.

(i) Prove that we have the inclusion $\overline{C_1 \cap C_2} \subseteq \overline{C}_1 \cap \overline{C}_2$ of projective closures.

(ii) Find two curves such that the above inclusion is strict.

(a) Let $f(x,y,z) = a_1x + b_1y + c_1z$, $g(x,y,z) = a_2x + b_2y + c_2z$, for $a_i, b_i, c_i \in \mathbb{C}$, $f(x,y,z) \neq g(x,y,z)$.
Then, it suffices to show that $\mathbb{V}(f,g) = \{(x_0, y_0, z_0)\}$ for some $x_0, y_0, z_0 \in \mathbb{C}$.
We have that $\mathbb{V}(f,g) = \mathbb{V}(f) \cap \mathbb{V}(g)$, &

$$\mathbb{V}(f) = \{(x,y,z) : a_1x + b_1y + c_1z = 0\},\ \mathbb{V}(g) = \{(x,y,z) : a_2x + b_2y + c_2z = 0\}.$$

Therefore,

$$\mathbb{V}(f) \cap \mathbb{V}(g) = \{(x,y,z) : a_1x + b_1y + c_1z = 0,\ a_2x + b_2y + c_2z = 0\}$$

& thus is the set of solutions to the system of equations

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0. \end{cases}$$

Which we can write as,

$$\underbrace{\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

& then the set of solutions are the elements of $\ker A$. However, by the rank-nullity theorem, we have that $\text{rank}(A) + \text{null}(A) = 3$, & since our lines are distinct, our rows are linearly independent, i.e. $\text{rank}(A) = 2$. In particular, $\text{null}(A) = 3 - 2 = 1$, & hence $\dim(\ker(A)) = 1$, i.e. the number of linearly independent vectors in $\ker(A)$ is 1 & since we are working in the projective space, all our vectors are linearly independent, i.e. there is only one solution to this system of linear equations. Therefore, we have that $\mathbb{V}(f,g) = \{(x_0, y_0, z_0)\}$ for some $x_0, y_0, z_0 \in \mathbb{C}$, & there is only one point of intersection for any two distinct lines in $\mathbb{P}^2$.

(b)

(i) Let $C_1, C_2 \subseteq \mathbb{A}^2$ be two closed affine algebraic curves.
Since $C_1 \cap C_2 \subseteq C_1$, we have that $\overline{C_1 \cap C_2} \subseteq \overline{C_1}$, since $\overline{C_1}$ is a closed set that contains $C_1 \cap C_2$. Likewise, $C_1 \cap C_2 \subseteq C_2 \Rightarrow \overline{C_1 \cap C_2} \subseteq \overline{C_2}$, since $\overline{C_2}$ is closed & contains $C_1 \cap C_2$. Therefore, since $\overline{C_1 \cap C_2} \subseteq \overline{C_1}$ & $\overline{C_1 \cap C_2} \subseteq \overline{C_2}$, it follows that $\overline{C_1 \cap C_2} \subseteq \overline{C_1} \cap \overline{C_2}$.

(ii) Let $C_1 = \mathbb{V}(y - x^2)$, $C_2 = \mathbb{V}(y - x^3)$. Then, we have that,
$\overline{C_1} = \mathbb{V}(yz - x^2) = \mathbb{V}(y - x^2) \sqcup \{[0:1:0]\}$ from the lecture notes, &
$\overline{C_2} = \mathbb{V}(yz^2 - x^3) = \mathbb{V}(y - x^3) \sqcup \{[0:1:0]\}$ which one can see easily by following the argument for $\overline{C_1}$.
Therefore, it follows that

$$\begin{aligned} \overline{C_1} \cap \overline{C_2} &= (\mathbb{V}(y - x^2) \cap \mathbb{V}(y - x^3)) \sqcup \{[0:1:0]\} \\ &= \{(0,0), (1,1)\} \sqcup \{[0:1:0]\}. \end{aligned}$$

However, on the other hand,

$$C_1 \cap C_2 = \{(0,0), (1,1)\}$$
$$\Rightarrow \overline{C_1 \cap C_2} = \{(0,0), (1,1)\} \subsetneq \overline{C_1} \cap \overline{C_2}.$$

$C_1 \cap C_2 = \{(0,0),(1,1)\}$

$\Rightarrow \overline{C_1 \cap C_2} = \{(0,0),(1,1)\} \subsetneq \overline{C_1} \cap \overline{C_2}.$

Q6. (**Bonus 10 marks**)

(a) Let Y be a closed affine algebraic variety and $O \subseteq Y$ an open subset. Prove that $\mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra.

(a) To show that $\mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra, it suffices to show that it has a $\mathbb{C}$ vector-space structure, i.e. closed under addition, multiplication & scalar multiplication.
Let $f, g \in \mathcal{O}_Y(O)$, then $f+g$ is defined by pointwise addition & so writing $f = \frac{f_1}{f_2}$, $g = \frac{g_1}{g_2}$, then we have that $f+g = \frac{f_1 g_2 + g_1 f_2}{f_2 g_2} \in \mathcal{O}_Y(O)$.

Likewise, $fg = \frac{f_1 g_1}{f_2 g_2} \in \mathcal{O}_Y(O)$ clearly.

Finally, for $\lambda \in \mathbb{C}$, it follow that $\lambda f = \frac{\lambda f_1}{f_2} \in \mathcal{O}_Y(O)$.

This all holds since $f_1 g_1$, λf_1, $f_1 g_2 + g_1 f_2$ & $f_2 g_2$ are all polynomial functions.
Therefore, $\mathcal{O}_Y(O)$ has a $\mathbb{C}$ vector-space structure & is a ring, & thus $\mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra.

Let X be an irreducible quasi-projective variety.

(i) Assume that U and V are open subsets of X with $U \subseteq V$. Briefly explain why $f \in \mathcal{O}_X(V)$ implies that $f_{|U} \in \mathcal{O}_X(U)$.

(ii) Briefly explain why the collection of sets of functions $\mathcal{O}_X(U)$, where U ranges over all open subsets of X, forms a sheaf on X.

(i) Since $U \subseteq V$, if $f \in \mathcal{O}_X(V)$ then it is regular everywhere in V & hence $f|_U$ is regular everywhere in U, i.e. $f|_U \in \mathcal{O}_X(U)$.

(ii) Since each $\mathcal{O}_X(U)$ is a $\mathbb{C}$-algebra, it is a ring & so $\mathcal{F}(U) = \mathcal{O}_X(U)$.
Then, by (i) if $U \subseteq V$ then there exists a restriction from $\mathcal{O}_X(V)$ to $\mathcal{O}_X(U)$ that satisfy the necessary properties (i.e. self identities & etc.).
Finally, since $\mathcal{O}_X(U)$ is a $\mathbb{C}$-algebra the last two properties hold, i.e. if $f_i \in \mathcal{O}_X(U)$ agree on an intersection, then we can find a regular function that restricts to each f_i.