

# Algebraic Geometry

## Coursework 1

- Available from 12:00 PM on February 4th to 12:00 PM on February 11th, 2025.
- Please submit your work in PDF format on Blackboard.
- If you need clarification or have any questions or concerns, feel free to email me or stop by on Wednesday during the office hour.
- You may discuss only Q5 with each other, but not the rest of the questions. In all cases, please write your solutions in your own words.

Q1. Let  $A \subseteq \mathbb{A}^n$  be a subset.

- (a) **(5 marks)** What is the definition of the closure of  $A$  in  $\mathbb{A}^n$ ?
- (b) **(5 marks)** Prove that  $\mathbb{V}(\mathbb{I}(A))$  equals the Zariski closure of  $A$  in  $\mathbb{A}^n$ .
- (c) **(5 marks)** Give an example of a subset in  $B \subseteq \mathbb{C}$  whose closure in the Zariski topology does not coincide with its closure in the Euclidean topology.

Q2. (a) **(5 marks)** What is the definition of a compact subset of a topological space?

- (b) **(10 marks)** Prove that  $\mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}^2$  is compact in the Zariski topology but not in the Euclidean topology.

Q3. (a) **(5 marks)** Find a curve  $W \subseteq \mathbb{A}^2$  and a morphism  $\varphi : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ , such that  $W$  is irreducible but  $\varphi^{-1}(W)$  is not.

- (b) **(5 marks)** Let  $Y$  be a topological space and consider  $X \subseteq Y$  with the subspace topology. Prove that if  $X$  is irreducible then so is its closure.
- (c) **(5 marks)** Prove that isomorphisms preserve irreducibility and dimension of closed affine algebraic varieties.
- (d) **(10 marks)** Find the irreducible components of  $\mathbb{V}(zx - y, y^2 - x^2(x + 1)) \subseteq \mathbb{A}^3$ . You need to justify why each component is irreducible.

Q4. (a) **(10 marks)** Let  $V \subseteq \mathbb{A}^n$  be a Zariski-closed subset and  $a \in \mathbb{A}^n \setminus V$  be a point. Find a polynomial  $f \in \mathbb{C}[x_1, \dots, x_n]$  such that

$$f \in \mathbb{I}(V), \quad f(a) = 1.$$

(b) **(15 marks)** Let  $I, (g) \subseteq \mathbb{C}[x_1, \dots, x_n]$  be two ideals. Assume that  $\mathbb{V}(g) \supseteq \mathbb{V}(I)$ .

- (i) Prove that if  $I = (f_1, \dots, f_k)$ , then

$$(f_1, \dots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}]. \quad (1)$$

- (ii) By only using Equation (1) and not the nullstellensatz, prove that there exists a positive integer  $m$  such that  $g^m \in I$ .

Q5. Prove at least one implication from each of the following equivalences.

- (a) **(10 marks)** Show that the pullback  $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$  is injective if and only if  $\varphi$  is *dominant*. Recall that a map,  $\varphi$ , is called dominant if its image,  $\varphi(V)$ , is dense in  $W$ .
- (b) **(10 marks)** Prove that the pullback  $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$  is surjective if and only if  $\varphi$  defines an isomorphism between  $V$  and some algebraic subvariety of  $W$ .