IDEAS ON LOG-CONCAVE SEQUENCES.

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Eric Katz and June Huh in [HK12] prove that the coefficients of Characteristic polynomial of a matroid is log-concave in the following way:

- Since M is realisable, there is a subvariety Y of $(\mathbb{C}^*)^n$ whose tropicalisation is the Bergman fan B_M (See [AK06]).
- In a compatible compactification $(\mathbb{C}^*)^n \subseteq X_\Sigma$, the coefficients of the characteristic polynomial of this matroid are given by

$$\mu^k = \operatorname{trop}(Y) \smile \alpha^k \smile \beta^{p-k}$$

  for some α and β are NEF in cohomology of X_Σ. Here $\operatorname{trop}(Y)$ can be understood as the cohomology class of $\overline{V}$ in X_Σ. (See [Kat09])
- Deduce by Khovanskii–Tessier inequality that

$$(\mu^i)^2 \geq \mu^{i-1}\mu^{i+1}.$$

Now consider the tropical current associated the Bergman fan of $\mathcal{T}_{B_M}$. Note that if there are irreducible analytic currents $\lambda_i[W_i]$ such that $\lambda_i[W_i] \longmapsto \mathcal{T}_{B_M}$ we can deduce the above log-concavity for every matroid. Therefore, June Huh and the author have propose the following question:

Question. Is the tropical current associated with every matroid approximable by irreducible analytic currents?

- A *positive* answer to the above statement would have far-reaching consequences in matroid theory such as re-proving the results of [AHK18] as well as results on the Lorentzian polynomials. Moreover, it brings the results about log-concavity of dynamical degrees into the same context.
- A *negative* answer to the above statement is also very interesting. If there is a matroid whose tropical current is not approximable by irreducible integration current, we find more counterexamples on Demailly's generalised Hodge Conjecture for positive currents, however, unlike [BH17] and [AB19], this counterexample would not have the similar cohomological obstruction by Hodge-Riemann relations, and the obstruction needs to be analytic.

Strategy. In [BHM$^+$22], Braden, Huh and Matherne prove that the deletion operation on matroids behaves like *semi-small* maps in algebraic geometry in the sense of [dCM02], and preserves the Hodge–Riemann relations. Based on this observation, the authors prove the results of [AHK18], by a sequence of deletion operations, to transform the matroid into a Boolean matroid where the Hodge–Riemann relations hold. The inverse operation to a deletion operation is given by *tropical modification*, see [Sha13] for details. It appears that tropical modifications can be defined on tropical currents in a consistent way, to obtain the following:

$$\mathcal{T}_{\widetilde{\mathcal{C}}} = (A - B) \wedge \pi^*(\mathcal{T}_{\mathcal{C}}),$$

where A and B are two tropical currents associated to tropical hypersurfaces, and $\pi : (\mathbb{C}^*)^{n+1} \longrightarrow (\mathbb{C}^*)^n$ is the projection onto the first n-coordinates. We can give a positive answer to the above question, if A and B can be chosen in such a way that $\mathcal{T}_{\mathcal{C}}$ can be approximated by analytic currents $\lambda_n[W_n]$ such that $W_n \cap \operatorname{supp}(B) = \varnothing$.

To Karim. Let δ be the deletion operation from a matroid. It is shown in [Sha13] There exists a rational tropical function $f =$ "g/h" $= g - h$, such that δ^* can be understood as the modification along the divisor along f. Then, in the above $A = \mathcal{T}_{(V_{\text{trop}}(\text{"}zh-g\text{"}))}$ and $B = \mathcal{T}_{V_{\text{trop}}}(h)$.

Question 0.1. Let $V_1 = (V_{\text{trop}}(\text{"}zh - g\text{"}))$ and $V_2 = \mathcal{T}_{V_{\text{trop}}}(h)$. We know that the stable intersection $\pi^{-1}(C) \wedge (V_1 - V_2)$ is an effective tropical cycle which is our matroid. Can we choose V_1 and V_2 such that there exists $b \in \mathbb{R}^n$ and

$$\big(\pi^{-1}(C) + \epsilon v\big) \wedge (V_1 - V_2) \quad \text{is effective for all } \epsilon > 0?$$

Let X_Σ be a smooth projective toric variety. Assume that $\lambda_i[W_i] \in \mathcal{D}'_{(p,p)}(X_\Sigma)$ is a sequence integration currents along algebraic varieties converging to the tropical current $\mathcal{T}_{\mathcal{C}}$. Assume that $\mathcal{T}_{\mathcal{C}'}$ is a tropical current associated with a tropical hypersurface $\mathcal{C}'$ that is *not necessarily positively weighted*. If Question 0.1 can be answered positively then $\mathcal{T}_{\mathcal{C}'} \wedge e^{\epsilon b}\lambda_n[W_n]$ is a positive current for each $n > N$. Then, then $\mathcal{T}_{\mathcal{C}.\mathcal{C}'} = \mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}$ is a weak limit of positive integration currents.

Proof.

(a) Since $\mathcal{T}_{\mathcal{C}'}$ is a tropical current, it has a continuous superpotential and we have the convergence of the following well-defined currents:

$$\mathcal{T}_{\mathcal{C}'} \wedge \lambda_n[W_n] \longrightarrow \mathcal{T}_{\mathcal{C}'} \wedge \mathcal{T}_{\mathcal{C}}.$$

(b) For $n > N$, we can view the current $\mathcal{S}_n := \mathcal{T}_{\mathcal{C}'} \wedge \lambda_n[W_n]$ as a positive closed current in the smooth projective manifold W_n. Clearly, $\mathcal{S}_n$ has a Hodge class in W_n, and therefore by Demailly's Hodge Conjecture statement in codimension 1, each $\mathcal{S}_n$ is a weak limit of positive analytic currents $\{\mu_{n_i}[V_{n_i}]\}_i$ in W_n.

(c) By a diagonal argument, we find a sequence $\mu_{n_j}[V_{n_j}] \longrightarrow \mathcal{T}_{\mathcal{C}'} \wedge \mathcal{T}_{\mathcal{C}}$ as currents in X_Σ.

$\square$

References

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