Applications of tropical geometry and rigidity theory in algebraic statistics

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Our proposal is to assemble a team of leading experts to develop ambitious new techniques in algebraic statistics by combining methods from both rigidity theory and tropical geometry. Of particular interest to us are *graphical models*, in which the conditional dependencies of a joint distribution of multiple random variables is in some way encoded as a graph. Some commonly used graphical models include directed and undirected Gaussian graphical models, the more general Markov random fields, and Bayesian networks. Graphical models have a wide variety of applications, especially within the area of machine learning, as they allow for abstraction and versitility in probability models.

At the foundations of all of data science are problems of high complexity where it is expensive to sample. This motivates minimising the number of samples required to almost surely guarantee the existence of the maximum likelihood estimator. In the central case of Gaussian graphical models, recent advances in rigidity theory – the study of how and when bar-and-joint frameworks can be deformed – have obtained upper[1] and lower[2] bounds for the notoriously computationally complex *maximum likelihood threshold* of Gaussian graphical models; i.e., the minimum number of samples required to almost surely guarantee the existence of a maximum likelihood estimator. Tropical geometry – the piece-wise linear, combinatorial shadow of algebraic geometry – has also been utilised in recent years to determine the maximum likelihood of closed subvarieties of algebraic tori[3] (algebraic constructions closely linked to scattering equations found in physics) and to study conditional independence in max-linear Bayesian networks[4].

In recent years tropical geometry and rigidity theory have become intertwined, with many concepts in one topic being applicable to the other. We plan to draw on this synthesis to develop new techniques that tackle the following fundamental problems in graphical models and algebraic statistics:

- Construct improved bounds for the maximum likelihood threshold that are computationally cost-effective.
- Determine the exact maximum likelihood thresholds for specific families of graphical models.
- Utilise both tropical geometry and rigidity theory to determine other structural properties of graphical models.

The group indicated below will meet at University of Bristol in early Summer 2023. Selected days will have introductory talks on a variety of related topics. The group will be split each day into two smaller subgroups for focused collaboration on one of two distinct-but-interlinked problems. These daily problems will be chosen from a list of several problems that will be decided by the group in advance of the first day. Each evening the two groups will give presentations of their findings to the other group. There will also be opportunities for contribution and involvement from members of the local department. Specifically, each introductory talk will be open to the local mathematical community, and we expect there to be additional collaboration between attendees and department members (for example, Farhad Babaee).

Proposed Participants

Alongside the two authors of the proposal – Sean Dewar and James Maxwell – we have confirmation of attendance from the following participants: Mathias Drton (TU Munich), Fatemeh Mohammadi (KU Leuven), Anthea Monod (Imperial), Anthony Nixon (Lancaster), Louis Theran (St Andrews) and Josephine Yu (Georiga Tech). The proposed team consists of renowned experts in their fields (Drton, Monod, Yu) and highly active researchers who made key recent contributions on the topic (Mohammadi, Nixon, Theran), as well as 2 postdocs (Dewar, Maxwell). Of the participants, 5 are UK based, 2 are EU based and 1 is non-EU based. We are committed to EDI (Equality, Diversity & Inclusion) principals, and so have striven for a balance of ethnicity and gender minorities across the attendees.

Proposed Budget

We expect the non-local participants to stay for 1 - 2 weeks each. We are requesting £7750, comprising of the following:

- Accommodation - 45 nights at £100 per night - £4500.
- UK travel - 3 return trains - £450 (average £150 per person).
- EU travel - 2 return flights - £800 (average £400 per person).
- Non-EU travel - 1 return flight (Atlanta, USA to UK) - £1500.
- Provision of refreshments and a group dinner - £500.

[1] C. Uhler, *Geometry of maximum likelihood estimation in Gaussian graphical models.* Ann. Statist. 40(1), 2012, 238–261.
[2] D.I. Berstein, S. Dewar, S.J. Gortler, A. Nixon, M. Sitharam, L. Theran, *Maximum likelihood thresholds via graph rigidity.* arXiv:2108.02185, 2021.
[3] D. Agostini, T. Brysiewicz, C. Fevola, L. Kühne, B. Sturmfels, S. Telen, *Likelihood Degenerations.* To appear in Advances in Mathematics, 2023
[4] C. Améndola, C. Klüppleberg, S. Lauritzen, N.M Tran, *Conditional independence in max-linear Bayesian networks.* Ann. Appl. Probab., 2022.