# Linear Algebra: Sheet 6

Present all your answers in complete sentences. There is also a Numbas quiz.

## Hand-in question

Submit your solution on Blackboard by **1pm on Wednesday (Week 8)** for feedback from your tutor.

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$, $x \mapsto Ax$ where
$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \in M_2(\mathbb{R}).$$
   a) Compute the characteristic polynomial of $A$ and so find the eigenvalues of $A$.
   b) For each eigenvalue $\lambda$ of $A$, find the set $E(\lambda)$.
   c) In this case we can diagonalise $A$. To do so, we use $C := (v_1 \ v_2)$ where $\{v_1, v_2\}$ is a basis of eigenvectors.
      (i) Construct the matrix $C$ and directly compute $C^{-1}AC$.
      (ii) Now replace $C$ by $(v_2 \ v_1)$. Directly compute $C^{-1}AC$ in this case.
      (iii) Consider why we get the answers found for (i) and (ii). Write a short explanation of this.[1]

## Additional questions

Try these questions and look at the solutions for feedback. They might also be discussed in your tutorial.

2. Which of the following functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ define an $\mathbb{R}$-linear map? Justify your answer in each case. For those that are linear, find their matrix form.
   a) $f(x, y) = (-y, -x)$ for all $x, y \in \mathbb{R}$.
   b) $f(x, y) = (x + y, x \times y)$ for all $x, y \in \mathbb{R}$.
   c) $f(x, y) = (0, 0)$ for all $x, y \in \mathbb{R}$.
   d) $f(x, y) = (0, 0)$ if $(x, y) \in \mathbb{Q}^2$ and $f(x, y) = (x, y)$ otherwise.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (-x, -y)$ for all $x, y \in \mathbb{R}$.
   a) Write $f$ in standard basis form.
   b) Write $f$ in matrix form.
   c) Using the matrix form, check that $f$ is indeed an $\mathbb{R}$-linear map.

4. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with $g(e_1) = e_2$ and $g(e_2) = -e_1$.
   a) Is $g$ uniquely determined? Justify your answer.
   b) Write $g$ in algebraic form.
   c) Write $g$ in matrix form.
   d) What transformation of $\mathbb{R}^2$ does $g$ represent?
   e) Using the standard basis form, together with linearity, find $g^{-1}$.
   f) Confirm your answer by finding $g^{-1}$ by:
      (i) first finding the inverse of the $2 \times 2$ matrix from (c); and then
      (ii) writing this linear map in standard basis form.

5. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map given by $h(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
   a) Is $h$ uniquely determined? Justify your answer.
   b) Write $h$ in standard basis form.
   c) Write $h$ in algebraic form.
   d) What transformation of $\mathbb{R}^2$ does $h$ represent?
   e) By using the matrix forms of the maps, find $h \circ g$ and $g \circ h$ in algebraic form.

[1] The explanation of diagonalisation in the lecture notes may help here.

6. Let $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates anti-clockwise by $\theta$ radians[2].
   a) Find each of the following, in whichever order you prefer:
      (i) The algebraic form of $f_\theta$.
      (ii) The matrix form of $f_\theta$.
      (iii) The standard basis form of $f_\theta$.
   b) Note, given $\theta, \phi \in \mathbb{R}$, that $f_\theta \circ f_\phi = f_\phi \circ f_\theta = f_{\theta+\phi}$.
      (i) Using (a)(ii), write out the matrix form of $f_{\theta+\phi}$.
      (ii) Using the matrix forms of $f_\theta$ and $f_\phi$, find the matrix form of $f_\theta \circ f_\phi$.
      (iii) Use (i) and (ii) to find identities relating to $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$.
   c) Observe that $f_\theta^{-1} = f_{-\theta}$. By comparing the matrix form of $f_{-\theta}$ and the inverse of the matrix form of $f_\theta$, deduce one property of cos and one of sin.

7. Let $m, n \in \mathbb{N}$ and suppose that $f : \mathbb{Q}^n \to \mathbb{Q}^m$ satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}^n$. Prove that $f$ is a $\mathbb{Q}$-linear map.

8. Let $A \in M_n(\mathbb{C})$ have an eigenvalue $\lambda \in \mathbb{C}$. Note: for (a) and (c), recalling properties of the determinant is helpful.
   a) Show that $\lambda$ is an eigenvalue of $A^t$.
   b) Consider the matrix $A$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ sending $e_1$ to $e_1$ and $e_2$ to $e_1 + e_2$. Are the eigenvectors of $A$ and $A^t$ the same?
   c) Is $-\lambda$ is an eigenvalue of $-A$? (Consider the cases that $n$ is even and odd.)
   d) Using that $(A^t)^t = A$ and $-(-A) = A$, comment on the set of eigenvalues of $A^t$ and $-A$ compared to the set of eigenvalues for $A$.
   e) Must the eigenvectors of $A$ and $-A$ be the same?
   f) Imagine that $A$ is antisymmetric (i.e., $A^t = -A$). What can we say about the set of eigenvalues of $A$?
   g) If $A \in M_n(\mathbb{C})$ is antisymmetric and $n$ odd, show that $\det(A) = 0$. Explain why $A$ therefore has zero as an eigenvalue.

9. Let $A \in M_n(\mathbb{C})$ have the property that the sum of elements in every row equals the same number $r$, i.e., $\sum_j a_{ij} = r$ for each $i \in \{1, \ldots, n\}$.
   a) Show that $r$ is an eigenvalue of $A$ and find a corresponding eigenvector[3].
   b) If $B \in M_n(\mathbb{C})$ instead has the property that the sum of elements in every column equals the same number $r$, must $r$ be an eigenvalue of $B$?

10. Consider the matrix
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$
   a) Compute the characteristic polynomial of $A$ and the eigenvalues of $A$.
   b) Compute a set of eigenvectors of $A$.
   c) Is $A$ diagonalisable? If so, diagonalise $A$.

11. Consider the matrix
$$B = \begin{pmatrix} 1 & -1 & 5 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}.$$
   a) Compute the characteristic polynomial of $B$ and the eigenvalues of $B$.
   b) Compute a set of eigenvectors of $B$.
   c) Is $B$ diagonalisable? If so, diagonalise $B$.

[2] It is helpful to know that the point given by rotating $e_1$ by $\theta$ radians is $(\cos\theta, \sin\theta)$.
[3] **Hint:** we want to consider a special vector to use as the eigenvector here, that somehow uses the property of $A$ that we are given.