SOME HINTS FOR THE FOCUSED RESEARCH MEETING IN BRISTOL

1. THE (RECENT) PAST

<u>DEF</u>: X complex manifold of dim n,

$A_c^{p,q}(X)$ (p,q)-diff. forms with compact support on X

$D_{p,q}(X) = D^{n-p,n-q}(X)$ is the topological dual of $A_c^{p,q}(X)$

$\uparrow$

space of currents of bidegree $(n-p, n-q)$

/ bidimension (p,q)

Ex: Z closed (orientable) submanifold of X

$\dim Z = p$

$\rightsquigarrow$ current of integration of bid. (p,p)

$$\delta_Z(\omega) := \int_Z \omega \qquad \forall\ \omega \in A_c^{p,p}(X)$$

$([Z])$

(Demailly not.)

Notice that we have a notion for convergence of currents

say that $(T_\ell)_\ell$ of (p,p)-currents

converges to a limit $T \Leftrightarrow$

$$\lim_\ell \langle T_\ell, \omega \rangle = \langle T, \omega \rangle$$

$\forall\ \omega \in A_c^{p,p}(X)$

"Dynamics of currents"

CONJ (Dinh-Sibony, 2010-2018) Let f be a hol. endomorphism of $\mathbb{P}^n$ of degree $d \geq 2$. Fix $p \in \{0, \dots, n\}$.

Then Z be a generic subvariety of $\mathbb{P}^n$ of dimension p, then the sequence of currents

$$\left(\frac{1}{\deg Z} \cdot \frac{1}{d^{(n-p)\cdot \ell}} (f^{\ell})^{*} \delta_Z \right)_{\ell}$$

converge to $\tau_{f,p}$.

current which only depends on f and p.

THM (Babaee, 2023) Let $\Phi_\ell : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ be the ℓ-power map

$$(z_1, \dots, z_n) \longmapsto (z_1^\ell, \dots, z_n^\ell).$$

Let Z be any subvariety of $(\mathbb{C}^*)^n$ of dimension p.

Then $\left(\frac{1}{\ell^{n-p}} (\Phi_\ell^*) \delta_Z \right)_\ell$ converges to $\mathcal{T}_{trop(Z)}$

"tropical current of $trop(Z)$" [Babaee 2014]

fix d consider $(\Phi_d^k)^* \delta_Z = \Phi_{d^k}$

trop(Z) is always a fan

Rem: can extend the Thm to toric varieties

Rem: for a generic choice of a subv. of $\mathbb{P}^n$ of fixed dim, trop (Z) is always the same

$\boxed{Ex}$: $n=1$, $p=0$, $Z=\{a\}$ in $\mathbb{C}^*$

$$\Phi_\ell^* \delta_Z = \sum_{k=1}^{\ell} \delta_{\left\{\sqrt[\ell]{|a|}\cdot e^{2\pi i\cdot \arg(a)\cdot k/\ell}\right\}}$$

as $\ell \to \infty$

$\Rightarrow$ the sequence is conv. to $\mathrm{Haar}_{S^1} = \tau_{\{0\}}$

2. FUTURE

2.1: Extension to monomial maps

column n

A $n\times n$ matrix with integer coeff. $\qquad A = [A_1 \dots A_n]$

$$\rightsquigarrow \quad \varphi_A : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$$

$$(z_1, \dots, z_n) \longmapsto (\underbrace{z_1^{A_{11}} \dots z_n^{A_{n1}}}, \dots, z_1^{A_{1n}} \cdot \ldots \cdot z_n^{A_{nn}})$$

Remark: $A = d \cdot \mathbb{I}_{n\times n} \rightsquigarrow$ "Good ol' power map"

$\boxed{Q}$:

can we predict the asymptotic of (assume φ_A has degree ≥ 2)

$$\left(\frac{1}{d^{(n-p)\cdot \ell}} (\varphi_A^{\ell})^* \delta_Z \right)_\ell \quad \text{for every fixed}$$

find the right factor

subvariety Z of $(\mathbb{C}^*)^n$ of codim. p ?

Insight 1: assume the sequence of currents has a limit

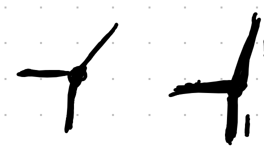
$\Rightarrow$ the trop of their supp has a limit

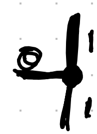
Here the support of $(\varphi_A^{\ell})^* \delta_Z$ has tropicalization

$$(A^{\ell})^{-1} \cdot \mathrm{trop}(Z).$$

$\rightsquigarrow$ $\boxed{\text{understand limits of tropical cycles}}$

⚠ we should also understand weights

$\boxed{\text{EX}}$ $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ apply $(A^{\ell})^{-1}$ to 

$\Rightarrow$ the limit is set-theor.  = 

Insight 2 : $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ act on $(a_1, a_2) \in (\mathbb{C}^*)^2$

the limit of $(\varphi_A^{-\ell})^* \delta_{(a_1, a_2)}$

will be $\mathrm{Haar}_{S^1} \times \{a_2\}$

$\boxed{\text{Idea}}$: we should be careful with eigenvalue 1,

we are going to recover something

tropical only outside the corr. eigenspaces

2.2 COHOMOLOGICAL POV

Obs: $CH^p_{\mathbb{Q}}(\mathbb{P}^n) \simeq \mathbb{Q}$ $\forall p$, the iso is given by the degree map

$\rightsquigarrow$ Dinh-Sibony:

$$\lim_{\ell} \frac{1}{d^{(n-p)\ell}} \cdot (f^{\ell})^* \delta_Z = \deg(Z) \cdot \tau_{f,p}$$

$\forall$ generic Z

CONJ (?) f rational end. of degree ≥ 2 on X of dim n

Fix $p \in \{0, \dots, n\}$, fix $\alpha \in CH^p_{\mathbb{Q}}(X)$.

Then $\forall$ generic Z subv. of X of dim p
and $[Z] = \alpha$ we have

$$\lim_{\ell} \frac{1}{*} (f^{\ell})^* \delta_Z = \tilde{T}_{f,p,\alpha}$$

current of bid (p,p) which only depends on f, p, α

Q: is it checkable on toric varieties?

Insight: cohomology of toric varieties can be seen on the fan.

2.3 NON-ARCHIMEDEAN

largely open in high dimension (equidistr.)

<u>reasons</u>:

① ambient space $\mathbb{P}^{n,an}$ is complicated!

② need a theory of currents on X^{Berk}

this is available!

(pull-back from tropical)

↳ Chambert-Loir – Ducros

↳ Gubler – Künnemann (on analytified)

(+ Burgos Gil, Jell)

Q: can we translate Jarhad's thm to the non-archimedean setting?

Obs: there might be relations between Farhad's currents and BGJK

$\nearrow$

connect complex setting and the non-archimedean case

Obs: hybrid spaces (Jonson-Boucksom): families of varieties over the closed punctured disc "converge" non-archimedeanly to a "tropical degeneration"

$\updownarrow$

skeleton

complex varieties degenerate to non-archimedean