(1) max Spec $(\mathbb{C}[x])$ is the set of all maximal ideals of $\mathbb{C}[x]$. By the Nullstellensatz, all the maximal ideals of $\mathbb{C}[x]$ are in (1-1) correspondence with the points of $\mathbb{A}^1$, so:

$$\max \operatorname{Spec}(\mathbb{C}[x]) = \{ m_p \subseteq \mathbb{C}[x] : p \in \mathbb{C} \}$$
$$= \{ (x-p) : p \in \mathbb{C} \}.$$

For $\mathbb{C}[x, \frac{1}{x}]$, it was shown in the lecture notes that $\mathbb{C}[x, \frac{1}{x}]$ can be identified with the coordinate ring of $V = \mathbb{V}(xy-1)$, $\mathbb{C}[V] = \mathbb{C}[x,y]/(xy-1)$, and in particular:
By identifying x with x and y with $1/x$ we have the morphism $\varphi : \mathbb{C}[x,y] \to \mathbb{C}[x, \frac{1}{x}]$; $\varphi(x) = x$, $\varphi(y) = \frac{1}{x}$. We have the polynomial $xy-1$ maps to zero and so $(xy-1)$ is the kernel of φ and $\mathbb{C}[x, \frac{1}{x}] \cong \mathbb{C}[x,y]/(xy-1) = \mathbb{C}[V]$.

It is therefore only necessary to find the ideals of $\mathbb{C}[x,y]$ that are maximal. These are the maximal ideals of $\mathbb{C}[x,y]$ containing $(xy-1)$, which by the Nullstellensatz are in correspondence with the points in V, so:

$$\max \operatorname{Spec}(\mathbb{C}[V]) = \{ m_p \in \mathbb{C}[x,y] : p \in V \}$$
$$= \{ (x-z, y-z^{-1}) : z \in \mathbb{C} - \{0\} \}$$

This is not the reduced form of the ideal, however, as $xy = 1$ on V, and the coordinate ring of V can be understood as the polynomials of $\mathbb{C}[x,y]$ restricted to V, we find that:

$$y - z^{-1} = y - xyz^{-1} = -yz^{-1}(-z+x) = (x-z)\cdot -yz^{-1}$$

So $\max \operatorname{Spec}(\mathbb{C}[V]) = \{ (x-z) : z \in \mathbb{C} - \{0\} \}$.

And so by the isomorphism we have
$\mathrm{maxSpec}(\mathbb{C}[x, \frac{1}{x}]) = \{(x - z) : z \in \mathbb{C} - \{0\}\}$.

(b) Let φ be the isomorphism: $\mathbb{A}^1 - \{0\} \to \mathbb{A}^1 - \{0\}$, $a \mapsto b = 1/a$.

With φ^* the pullback on the ~~coordinate rings~~ algebras we have for $a \in \mathbb{A}^1 - \{0\}$:

$$\varphi^*(1/x)(a) = (1/x)(\varphi(a)) = (1/x)(1/a) = a,$$

in other words $\varphi^*(1/x) = y \in \mathbb{C}[y, 1/y]$.

We then have that since the pullback is a homomorphism of $\mathbb{C}$-algebras:

$$1 = x \cdot 1/x \to \varphi^*(x \cdot \tfrac{1}{x}) = 1 \to \varphi^*(x) \cdot \varphi^*(1/x) = 1$$
$$\to \varphi^*(x) \cdot y = 1 \to \varphi^*(x) = 1/y,$$

and using the fact that φ^* is a homomorphism further:

$$\varphi^*\left(2x^2 + \frac{2x^3 + 4x}{x^5}\right) = 2\varphi^*(x)^2 + \frac{2\varphi^*(x)^3 + 4\varphi^*(x)}{\varphi^*(x)^5}$$
$$= 2y^{-2} + \frac{2y^{-3} + 4y^{-1}}{y^{-5}}$$
$$= 2y^{-2} + (2y^{-3} + 4y^{-1})y^5$$
$$= 2y^{-2} + 2y^2 + 4y^4$$

And, $\varphi^*(2 - x) = 2 - \varphi^*(x) = 2 - y^{-1}$

(2) Let $V := \mathbb{V}(y - ux) \subseteq \mathbb{A}^3$ and φ be the projection of $\mathbb{A}^3$ onto the x-u plane, $(x, y, u) \mapsto (x, u)$.

Then considering the restriction of φ to V;

$$\varphi_{|V}(x, y, u) \underset{\substack{\uparrow \\ y = ux \text{ on } V}}{=} \varphi_{|V}(x, xu, u) = (x, u).$$

This gives the natural inverse map ψ:

$\psi : \mathbb{A}^2 \to \mathbb{A}^3$, $(x, u) \mapsto (x, xu, u)$, and we have:

$$\psi(\varphi_{|V}(x, y, u)) = \psi(\varphi_{|V}(x, xu, u))$$
$$= \psi(x, u) = (x, xu, u) = (x, y, u).$$

$$\varphi_{|V}(\psi(x, u)) = \varphi_{|V}((x, xu, u)) = (x, u).$$

I.e. ψ is indeed the inverse of $\varphi_{|V}$ and so $\varphi_{|V}$ is an isomorphism.

(b) The projection onto the x-y plane, when restricted to V, is not an isomorphism as it is not injective: $\forall u \in \mathbb{C}$ we have $(0, 0, u) \mapsto (0, 0)$.

(3) First, let g and h be two polynomials such that their product is a homogeneous f.

Assume first that either g or h (wlog g in this case) is not homogeneous, that is:

$g = g_{in} + g_{hom}$, g splits into a homogeneous part of degree b, and an inhomogeneous part with at least one monomial of degree $a \neq b$.

Since h is homogeneous, let it be of degree c, and then:
$f = g \cdot h = (g_{in} + g_{hom}) \cdot h = g_{in} h + g_{hom} h$,
with the right term homogeneous of degree $b+c$, and the left term including some monomial of degree $a+c \neq b+c$. I.e. f is not homogeneous, a contradiction.

Similarly for both g and h not homogeneous, we end up with monomials of differing degree and thus implying f is not homogeneous.

So $f = gh$ is homogeneous $\longrightarrow$ $g \wedge h$ are homogeneous.

Next, let V be an irreducible variety, so that $V = \mathbb{V}(I)$ for some prime ideal I.

The projective closure of V, $\bar{V}$ is given by $\mathbb{V}(\tilde{I})$, $\tilde{I}$ being the homogenization of I. Thus if $\tilde{I}$ is prime, $\bar{V}$ is irreducible by the projective Nullstellensatz.

Let $g, h \in \mathbb{C}[x_0, \dots, x_n]$ be such that $g \cdot h = f \in \tilde{I}$. As shown earlier, we have necessarily that g and h are both homogeneous, so we can do the following:

$$g(x_0, x_1, \dots, x_n) \underset{\text{homogenisation}}{\overset{\text{dehomogenisation}}{\rightleftarrows}} g(1, x_1, \dots, x_n).$$

So, dehomogenising the product:

$$g(x_0, x_1, \dots, x_n)\, h(x_0, x_1, \dots, x_n) = f(x_0, x_1, \dots, x_n)$$

$$\downarrow$$

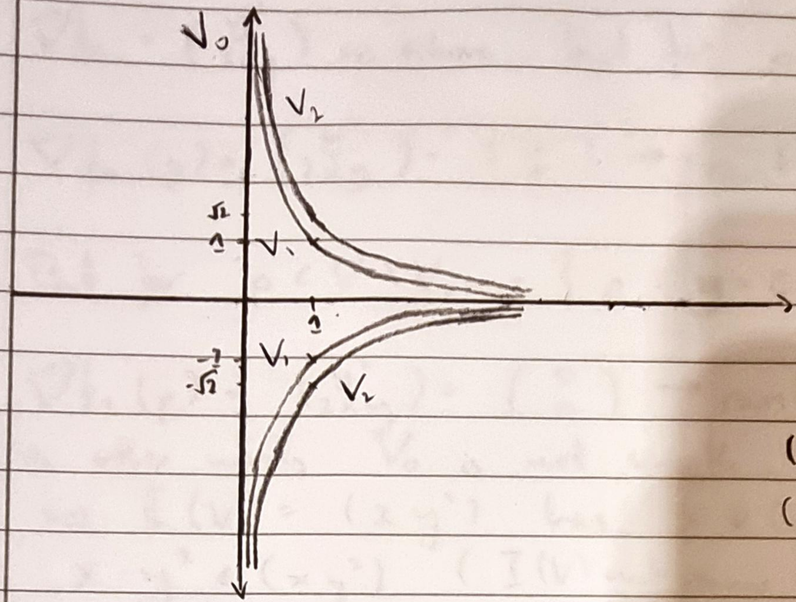
$$g(1, x_1, \dots, x_n)\, h(\cancel{x}1, x_1, \dots, x_n) = \underset{\overset{\uparrow}{I}}{f(1, x_1, \dots, x_n)}$$

We find g or h^* is in I, since I is prime.

$\therefore$ $g(x_0, x_1, \dots, x_n)$ or $h(x_0, x_1, \dots, x_n)$ is the homogenisation of an element of I, so must also be in $\tilde{I}$. $\tilde{I}$ is then prime.

* $g(1, x_1, \dots, x_n)$ or $h(1, x_1, \dots, x_n)$, not the $n+1$ variable polynomials.

(5) $V_t := \mathbb{V}(xy^2 - t)$;



^ (The axes are supposed to be bold to indicate V_0)

(V_1 intersects $(1,1)$ and $(1,-1)$)

(V_2 intersects $(1,\sqrt{2})$ and $(1,-\sqrt{2})$)

We have: $V_0 := \{(x,y) : xy^2 = 0\}$
$= \{(x,y) : x = 0 \vee y = 0\}$

$V_1 := \{(x,y) : xy^2 = 1\}$
$= \{(x,y) : x = \frac{1}{y^2}\}$
$= \{(y^{-2}, y) : y \in \mathbb{C} - \{0\}\}$

$V_2 := \{(x,y) : xy^2 = 2\}$
$= \{(x,y) : x = \frac{2}{y^2}\}$
$= \{(2(y)^{-2}, y) : y \in \mathbb{C} - \{0\}\}$.

For $t \in \{1,2\}$ we have ($f_t := xy^2 - t$):

$\nabla f_t = \begin{pmatrix} \partial f_t/\partial x \\ \partial f_t/\partial y \end{pmatrix} = \begin{pmatrix} y^2 \\ 2xy \end{pmatrix}$. So, for $p \in V_t$, $p = (\frac{t}{y^2}, y)$ and
$\nabla f_t(p) = \begin{pmatrix} y^2 \\ 2\cdot(t/y^2)y \end{pmatrix} = \begin{pmatrix} y^2 \\ 2t/y \end{pmatrix}$ for $y \neq 0$, that is:

rank $Df_t = 1 \rightarrow$ these V_t are smooth of dimension 1 at all points $p \in V_t$, and thus irreducible.

For $t = 0$, however:

$\nabla f_0 = \begin{pmatrix} y^2 \\ 2xy \end{pmatrix}$ as above, but for $p = (x,y) \in V_{0x} := \{p : p \in V \wedge y \neq 0\}$:

$\nabla f_0(p) = \begin{pmatrix} y^2 \\ 2xy \end{pmatrix} = \begin{pmatrix} y^2 \\ 0 \end{pmatrix} \rightarrow \operatorname{rank} \nabla f_0(p)_{V_{0x}} = 1.$

But for $p \in V \setminus V_{0x} = \{p : y = 0\}$:

$\nabla f_0(p) = \begin{pmatrix} y^2 \\ 2xy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \operatorname{rank} \nabla f_0(p)_{V \setminus V_{0x}} = 0,$
in other words V_0 is not smooth, and clearly reducible, as $\mathbb{I}(V) = (xy^2)$ has $x \notin (xy^2)$, $y \notin (xy^2)$, but $x \cdot y^2 \in (xy^2)$ ($\mathbb{I}(V)$ not prime).