APM Report 2024

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We are interested in properties of subgroups of surface groups, specifically subgroups isomorphic to a free group on k generators.

To study these subgroups we use 'critical' maps of finite graphs to our surface. We use curve counting methods to determine the asymptotic behaviour of the number of these critical graph maps of length less than L as L goes to infinity. For each map there is a induced homomorphism from the fundamental group of the graph to the surface group and the image of this homomorphism gives us a subgroup associated to the map. Therefore we can use our results from counting critical maps to give us a notion of growth of the number of subgroups.

In this report I will begin by giving a quick summary of the history of work on curve counting problems and then I will give a brief overview of the main results in the rest of the report. In Section 2 I go into more detail about counting critical maps from graphs into surfaces and then in Section 3 I show how you can move these idea into the setting of subgroups of surface groups. I finish by briefly describing what we are doing to take these results further and generalise them to other settings.

Notation

Throughout this report, unless stated otherwise, we will let Σ be a closed, connected, orientable hyperbolic surface with genus g. There exists a discrete subgroup $\Gamma \subset PSL_2(\mathbb{R})$ which acts properly on the hyperbolic plane $\mathbb{H}^2$ such that $\Sigma = \mathbb{H}^2/\Gamma$.

A geodesic γ is assumed to be closed primitive and unoriented. For any closed curve α its isotopy class $[\alpha]$ contains a unique geodesic representative therefore we may slightly abuse notation by interchangeably letting γ denote a geodesic or the isotopy class of which it is a representative.

1 Introduction

1.1 Background

Geodesic counting problems on hyperbolic surfaces have been studied for decades and are of interest as a intersection of the areas of hyperbolic geometry, dynamics and low-dimensional topology.

Work by Huber [2], Margulis [3] and others in the 1960's gave us one of the first major results in the field, the geodesic prime number theorem, which states that on a hyperbolic surface Σ the number of closed unoriented geodesics of length less than L grows asymptotically exponentially with respect to L. Namely:

$$|\{\gamma \text{ unoriented closed geodesic on } \Sigma \text{ with } \ell_\Sigma(\gamma) \leq L\}| \sim \frac{e^L}{2L}$$

as $L \to \infty$, where $\sim$ indicates that ratio between the two sides tends to 1. Note that for this result asymptotic growth does not depend on the geometry of the surface and is independent of the genus.

Much work has been done since to explore counting results for different classes of geodesics, notably Mirzakhani in her thesis [4] investigated the number of simple closed geodesics on a surface and showed that the asymptotic growth in this setting is polynomial.

Mirzakhani later generalised her result to count all closed curves of a given type [5]. We say two geodesics are of the same type if they belong to the same mapping class group orbit. If we let γ_0 be a closed geodesic on Σ then we have:

$$|\{\gamma \text{ of type } \gamma_0 \text{ with } \ell_\Sigma(\gamma) \leq L\}| \sim c_{\gamma_0} \cdot n_\Sigma \cdot L^{6g-6+2r}$$

as $L \to \infty$, where c_{γ_0} and n_Σ are constants depending the type γ_0 and the geometry of Σ respectively.

1.2 Summary of results

We are interested in studying subgroups of the fundamental group of hyperbolic surface. We can use curve counting techniques to develop statistics about classes of these subgroups.

Fix some $k \geq 2$ and consider the set $G_\Sigma^k = \{\Delta \subset \pi_1(\Sigma) : \Delta \cong F_k\}$ of subgroups of the surface group which are isomorphic to F_k, the free group on k generators. We cannot say much about this set, however if we define some length function then we can ask about growth of the number of subgroups with length less than or equal to L as L grows.

To define our length function let X be a finite, connected, trivalent graph with $\pi_1(X) \cong F_k$. For a continuous map $\phi : X \to \Sigma$ we have a length $\ell_\Sigma(\phi(X))$ given by the hyperbolic metric on the surface. We call a pair (X, ϕ) a k-realisation. For each homotopy class of these maps there exists a map with minimal length which we call a critical realisation. We say a subgroup $\Delta \subset \pi_1(\Sigma)$ has length equal to the minimum length of all the critical realisations with induced image equal to Δ.

Denote by $R_\Sigma^k(L)$ the set of critical k-realisation of length $\leq L$, this set is finite for any $L > 0$ and we know the asymptotic behaviour of the number of critical k-realisations is given by

$$\left|R_\Sigma^k(L)\right| \sim \left(\frac{2}{3}\right)^{3-3k} \cdot \frac{\mathrm{vol}(T^1\Sigma)^{1-k}}{(3k-4)!} \cdot L^{3k-4} \cdot e^L$$

as $L \to \infty$.

Each map $\phi : X \to \Sigma$ induces a homomorphism $\phi_* : \pi_1(X) \to \pi_1(\Sigma)$. We can therefore define a length function $\ell_k : G^k_\Sigma \to \mathbb{R}_{\geq 0}$ as follows

$$\ell_k(\Delta) = \min(\ell(X, \phi) : (X, \phi) \text{ critical k-realisation}, \phi_*(\pi_1(X)) = \Delta)$$

for any $\Delta \in G^k_\Sigma$. This is well-defined because the number of critical realisations with length less than or equal to L is finite.

Counting critical k-realisations gives us a method to count group homomorphisms from the free group F_k to our surface group. Then the image of these homomorphisms are the subgroups that we are interested in.

A difficultly comes from the fact that many homomorphisms may be associated with the same subgroup, however we are able to show that two k-realisations give us the same subgroup only if they differ by a graph homeomorphism. Therefore the ratio of the number of realisations to subgroups with length less than or equal to L is finite to one.

Our aim is to use this to give a precise result of the asymptotic growth of subgroups and understand more clearly the space in which these groups live. It should be possible to adapt this work to generalise the results for surfaces with boundary and punctures. It would also be interesting to use these methods to investigate broader classes of subgroups.

2 Counting Critical Realisations

Many of the ideas in this section were inspired by methods of counting maps of graphs into surfaces introduced by my supervisors Viveka Erlansson and Juan Souto in [1] and results about minimal length trivalent graphs in hyperbolic space by White [6].

Throughout this section, we understand a graph to be a connected, 1-dimensional CW-complex with finitely many cells and so we allow our graphs to contain loops and multiple edges between vertices. Unless stated otherwise we will also assume graphs are trivalent.

Remark. *Note that for any $k \geq 2$, the number of trivalent graphs with fundamental group of rank k is finite. Each graph of this type satisfies $V = 2k - 2$ and $E = 3k - 3$ where V is the number of vertices of the graph and E is the number of edges.*

Definition 2.1. *Let $k \geq 2$. A **k-realisation** is a graph X with rank $\pi_1(X) = k$ along with a (continuous) map $\phi : X \to \Sigma$. A k-realisation is **regular** if the restriction of the map to every edge of the graph is non-constant.*

The set of all k-realisations $\mathcal{R}^k_\Sigma$ can be written as the disjoint union

$$\mathcal{R}^k_\Sigma = \bigsqcup_{X \in \mathbf{X}_k} \{\phi : X \to \Sigma\}$$

where $\mathbf{X}_k$ is the finite set of trivalent graphs X with rank $\pi_1(X) = k$. For a graph $X \in \mathbf{X}_k$ the set of k-realisations $\mathcal{R}_X = \{\phi : X \to \Sigma\}$ can be endowed with the compact-open topology and therefore we inherit a topology on $\mathcal{R}^k_\Sigma$. We note that the connected components of $\mathcal{R}_X$ correspond to the homotopy classes of maps X to Σ.

For a k-realisation (X, ϕ) the image of each edge of X is an arc which has a length given by the hyperbolic metric on the surface. Therefore the space of k-realisations $\mathcal{R}^k_\Sigma$ has a natural length function $\ell : \mathcal{R}^k_\Sigma \to \mathbb{R}_{\geq 0}$ given by

$$\ell(X, \phi) := \sum_{e \in \mathbf{edge}(X)} \ell_\Sigma(\phi(e))$$

for any $(X, \phi) \in \mathcal{R}^k_\Sigma$ and where $\mathbf{edge}(X)$ denotes the set of the edges of the graph X.

Lemma 2.2. *Each connected component of $\mathcal{R}^k_\Sigma$ has a realisation (X, ϕ) of minimal length. Under this realisation, the edges of X map to geodesic arcs in Σ.*

Realisations of minimal length have a nice symmetric property which we later use in order to count the number of minimal length realisations. We first define some properties of realisations.

Definition 2.3. *A k-realisation is ℓ-**long** if the image of every edge in the graph has length greater than ℓ.*

Definition 2.4. *A regular k-realisation is **critical** if for each trivalent vertex the three geodesic segments corresponding to the incident edges meet at angles $\frac{2\pi}{3}$.*

Lemma 2.5. *A regular k-realisation (X, ϕ) is of minimal length in its homotopy class if and only if it is a critical realisation. Moreover this realisation is not homotopic to any other critical k-realisation therefore the minimal length k-realisation in each homotopy class is unique.*

We can pick a cover $\rho : \mathbb{H}^2 \to \Sigma$ of our hyperbolic surface. For a sufficiently large $\ell > 0$ a lift of any ℓ-long critical realisation to the hyperbolic plane $\mathbb{H}^2$ is a quasi-convex tree. We can use this fact along with lattice counting results in order to count critical realisations and we get the following result.

Theorem 2.6. *Let $R^k_\Sigma(L)$ be the set of critical k-realisations with length less than or equal to L. Then we have*

$$\left|R^k_\Sigma(L)\right| \sim \left(\frac{2}{3}\right)^{3-3k} \cdot \frac{\mathrm{vol}(T^1\Sigma)^{1-k}}{(3k-4)!} \cdot L^{3k-4} \cdot e^L$$

as $L \to \infty$.

3 Realisations to Groups

In our working here we must be a bit more careful about basepoints. We fix a point p_0 on the surface Σ to be the basepoint. For our graphs we now add a single leaf and pick the valence 1 vertex x_0 as our basepoint. Homotopy of maps is now considered up to fixing the basepoints. All the working in the previous section can be shown to work in this new setting with only a few adaptions and a slight change to the constants for growth in Theorem 2.6.

In the previous section we looked at maps from a graph X to a surface Σ and the asymptotic behaviour of the number of length minimising maps. We are now going to think about subgroups of $\pi_1(\Sigma, p_0)$, the fundamental group of the surface.

A realisation $\phi : (X, x_0) \to (\Sigma, p_0)$ induces a homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(\Sigma, p_0)$. Homotopic realisations induce the same homomorphism and therefore we can consider homotopy classes rather than individual maps.

We also know that every homomorphism $\pi_1(X, x_0) \to \pi_1(\Sigma, p_0)$ is induced by a map $(X, x_0) \to (\Sigma, p_0)$ unique up to homotopy fixing x_0.

We therefore have a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{homotopy classes of realisations} \\ (X, x_0) \to (\Sigma, p_0) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{homomorphisms} \\ \pi_1(X, x_0) \to \pi_1(\Sigma, p_0) \end{array} \right\}$$

between realisations and homomorphisms. As mentioned in the previous section each homotopy class contains a unique critical realisation and therefore considering a critical realisation to be the representative for its homotopy class we get

$$\left\{ \begin{array}{c} \text{critical realisations} \\ \phi : (X, x_0) \to (\Sigma, p_0) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{homomorphisms} \\ \phi_* : \pi_1(X, x_0) \to \pi_1(\Sigma, p_0) \end{array} \right\}$$

where we are slightly abusing notation to let ϕ_* be the homomorphism induced by $[\phi]$. From now on we will always assume a k-realisation (X, ϕ) is critical unless stated otherwise.

The aim is to now show that using induced homomorphisms we can give a consistent definition of length for subgroups of $\pi_1(\Sigma, p_0)$.

For $k \geq 2$ let $G_\Sigma^k = \{\Delta \subset \pi_1(\Sigma, p_0) : \Delta \cong F_k\}$ be the subgroups of the surface group isomorphic to the free group on k generators. We say $\Delta \in G_\Sigma^k$ has length

$$\ell_k(\Delta) = \min(\ell(X, \phi) : (X, \phi) \text{ critical k-realisation, } \phi_*(\pi_1(X, x_0)) = \Delta)$$

this is well defined as for any L there are a finite number of critical k-realisations with length $\leq L$. Let $G_\Sigma^k(L)$ be the set of subgroups with length $\leq L$.

For a critical k-realisation (X, ϕ) denote by Δ_ϕ the subgroup of $\pi_1(\Sigma, p_0)$ which is the image of $\pi_1(X, x_0)$ in ϕ_*. It is clear by definition that $\ell_k(\Delta_\phi) \leq \ell(X, \phi)$. We have a map

$$\begin{aligned} \Pi_\Sigma^k : R_\Sigma^k &\to G_\Sigma^k \\ (X, \phi) &\mapsto \Delta_\phi \end{aligned}$$

where R^k_Σ is again the set of critical k-realisations. For any $\Delta \in G^k_\Sigma$ we can pick k generators and associate these with loops in the surface, therefore there must exist some k-realisation with image Δ and so the map Π^k_Σ is surjective. We want to use this relationship to count the number of subgroups of length less than or equal to L as $L \to \infty$.

Proposition 3.1. *There exists a constant $\ell_0 > 0$ such that given an ℓ_0-long critical k-realisation $\phi : (X, x_0) \to (\Sigma, p_0)$ and another (not necessarily critical) k-realisation $\psi : (Y, y_0) \to (\Sigma, p_0)$ with $\ell(Y, \psi) \leq \ell(X, \phi)$ and $\Delta_\phi = \Delta_\psi$, then there exists a homeomorphism $F : (Y, y_0) \to (X, x_0)$ such that $\phi \circ F$ is homotopic to ψ.*

Corollary 3.2. *There exists a constant $\ell_0 > 0$ such that for any ℓ_0-long critical k-realisation $(X, \phi) \in R^k_\Sigma$ we have*

$$|\{(Y, \psi) \in R^k_\Sigma : \Delta_\psi = \Delta_\phi\}| \leq |Homeo(X)/\sim| < \infty$$

where two homeomorphisms of a graph X are equivalent if they are homotopic.

We can show that for any $\ell > 0$, almost all critical realisations are ℓ-long. Therefore we can use the above Corollary along with Theorem 2.6 to bound the asymptotic growth of the number of subgroups.

Theorem 3.3. *There exist constants $C_1, C_2 > 0$ such that*

$$C_1 \cdot L^{3k-4} \cdot e^L \preceq |\{\Delta \in G^k_\Sigma : \ell_k(\Delta) \leq L\}| \preceq C_2 \cdot L^{3k-4} \cdot e^L$$

as $L \to \infty$.

Further work

Some aspects of this work that I am interested in studying further involve adapting these result to work when we let our surface Σ have boundary or punctures.

I am also interested in looking at ways to understand broader classes of subgroups using the above methods and understand what else we can think about using these counting methods such as looking at properties of covers of surfaces associated to our subgroups.

References

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