CONTINUITY OF THE SUPERPOTENTIALS AND SLICES OF TROPICAL CURRENTS

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Abstract.

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1. Introduction

Moving lemma, fan displacement etc.

Let X be a complex manifold of dimension n, and p, q non-negative integers with $n = p + q$. We denote by $\mathcal{C}^q(X) = \mathcal{C}_p(X)$ the cone of positive closed bidegree (q, q), or bidimension $(p, p)-$ currents on X. We also consider $\mathcal{D}^q(X) = \mathcal{D}_p(X)$, the $\mathbb{R}$-vector space spanned by $\mathcal{C}^q(X)$. It is well-known that the intersection of two positive closed currents is not always defined. The main intial progress were due to the works of Federer [Fed69] and Bedford and Taylor in [BT82]. Federer define the a generic slicing theory of currents, that is for a dominant holomorphic map $f : X \longrightarrow Y$, and a positive closed currents $\mathcal{T} \in \mathcal{C}_p(X)$, or more generally, a *flat current*, a *slice*

$$\mathcal{T} \wedge [f^{-1}(y)]$$

is well-defined for a generic $y \in Y$. Bedford and Taylor suggested that $\mathcal{S} = dd^c u$ is a bidegree $(1, 1)$-current, then

$$\mathcal{S} \wedge \mathcal{T} := dd^c(u\mathcal{T}),$$

can be defined when

- The potential, u, is bounded
- u is unbounded but its unbounded locus has a small intersection with $\mathrm{supp}(\mathcal{T})$.

may be Demailly and Fornaess-Sibony (Oka inequality)

For instance, when $\mathcal{S} = dd^c \log|f|, \mathcal{T}$ are two integration currents, such that their supports intersect in the expected dimension, then

$$\mathcal{S} \wedge \mathcal{T} = \sum_1 c_i [C_i],$$

and

where each C_i is a component of the intersection, and c_i is the corresponding vanishing number. This intersection coincides with the slicing of integration currents.

Demailly in [Dem92] asked the question of generalising the intersection theory to the case where $\mathcal{T}$ is of a higher bidegree. In several works, Dinh and Sibony introduced *superpotential theory* and *density* of currents to answer this question. In this article we adopt the approach of ~~Dinh and Sibony~~ for our intersection theory. See also the works of Anderson, Eriksson, Kalm, Wulcan and Yger [AESK+21], and [ASKW22] a non-proper intersection theory. the second author

In [DS09] completely discussed the situation where X is a homogeneous space, and in [DS10] investigated the intersection theory for currents with continuous superpotentials, which is a generalisation of the case of bounded potentials in bidegree $(1,1)$. Once the intersections are defined one can ask the following *continuity problem*:

Problem 1.1. Let $\mathcal{T}_k$ be a sequence of positive closed currents on X converging to $\mathcal{T}$. Let $\mathcal{S}$ be also a positive close current on X. Find sufficient conditions such that

$$\lim_{k\to\infty}(\mathcal{S}\wedge\mathcal{T}_k) = \mathcal{S}\wedge(\lim_{k\to\infty}\mathcal{T}_k).$$

Roughly speaking, we say that $\mathcal{S}$ is a current on a compact Kähler manifold with a *continuous superpotential*, when for a current $\mathcal{T}$, the wedge product

$$\mathcal{S}\wedge\mathcal{T} := \lim_{n\to\infty}(\mathcal{S}\wedge\mathcal{T}_n),$$

is independent of the choice of smooth approximation $\mathcal{T}_n \longrightarrow \mathcal{T}$. Consequently, by the
[...] regularisation theorem of ~~Dinh and Sibony~~ for any bidegree we can partially answer Problem 1.1.

Proposition 1.2. Let X be a compact Kähler manifold, $\mathcal{T}_k \longrightarrow \mathcal{T}$ be a convergent sequence in $\mathcal{D}^p(X)$. If a current $\mathcal{S}$ has a continuous superpotential, then

$$\mathcal{S}\wedge\mathcal{T}_n \longrightarrow \mathcal{S}\wedge\mathcal{T}.$$

of bounded mass
Proof. The main result ~~of Dinh and Sibony's result~~ in [DS04] implies any current $\mathcal{D}^q(X)$ can be weakly approximated by a difference of smooth closed positive of bidegree (p,p)-forms. The result then follows from the definition of continuity of super-potentials. □

mention SP-convergence Problem 1.1 becomes more difficult when one considers continuity for slices, and the current $\mathcal{S}$ is an integration current. Borrowing ideas in tropical geometry, we discuss this problem for the very specific case where $\lim_{k\to\infty}\mathcal{T}_k$ is a *complex tropical current* [Bab14, BH17]. (Complex) tropical currents are closed currents on complex tori $(\mathbb{C}^*)^n$ or on a toric variety associated to a *tropical cycle*. Recall that a tropical cycle is a weighted polyhedral complex satisfying the *balancing condition* (see Definition 3.1). For a tropical cycle $\mathcal{C} \subseteq \mathbb{R}^n$, of dimension p, the associated tropical current $\mathcal{T}_{\mathcal{C}} \in \mathcal{D}_p((\mathbb{C}^*)^n)$, is a closed current with support $\text{Log}^{-1}(\mathcal{C})$, where

$$\text{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \quad (z_1,\dots,z_n) \longmapsto (-\log|z_1|,\dots,-\log|z_n|).$$

The tropical current $\mathcal{T}_{\mathcal{C}}$ can be naturally presented as a locally fibration of $\text{Log}^{-1}(\mathcal{C})$, and we say $\mathcal{C}$ is compatible with the fan Σ, if the fibres of $\overline{\mathcal{T}}_{\mathcal{C}}$ intersect the toric invariant divisors of the toric variety X_Σ transversely. Here $\overline{\mathcal{T}}_{\mathcal{C}}$ denotes the extension by zero of $\mathcal{T}_{\mathcal{C}}$ to the toric variety X_Σ.

let be

Theorem 1.3. Let X_Σ be a smooth projective toric variety, and $\mathcal{C}$ is a tropical cycle compatible with Σ. Then $\overline{\mathcal{T}}_\mathcal{C}$ has a continuous super-potential.

The preceding theorem allows for defining the intersection product of a tropical current with any current on a compatible toric variety, we can then restrict the intersection product to the complex torus $T_N \subseteq X_\Sigma$ and using the $T_N \simeq (\mathbb{C}^*)^n$ define the intersection product of two tropical currents in $(\mathbb{C}^*)^n$. On the tropical geometry side, there exists a *stable intersection theory* of tropical cycles. The word stable here precisely corresponds to the continuity of the definition with respect to generic translations of tropical cycles. With the stable intersection and natural addition of tropical cycles, we have the ring of tropical cycles.

Theorem 1.4. The assignment $\mathcal{C} \longmapsto \mathcal{T}_\mathcal{C}$ induces a $\mathbb{Z}$-algebra homomorphism between

(a) The $\mathbb{Z}$-algebra of tropical cycles in $\mathbb{R}^n$ with the natural addition (Definition ??) and stable intersection (Definition 3.3) as the multiplication.

(b) The $\mathbb{Z}$-algebra of tropical currents on $(\mathbb{C}^*)^n$ with the usual addition of currents and the wedge product of currents.

We also address Problem 1.1 in a very particular case of slicing of currents converging to a tropical current. The theorem is inspired by works in [BJS⁺07], [OP13] and [Jon16]. Add more theorems.

Theorem 1.5. Let $D, W \subseteq (\mathbb{C}^*)^n$ be an algebraic subtorus, and an algebraic subvariety respectively. Assume that $\operatorname{Log}(D)$ intersects $\operatorname{trop}(W)$ properly. Then,

$$\lim_{m\to\infty} \Big(\frac{1}{m^{n-p}}\Phi_m^*[W] \wedge [D]\Big) = \Big(\lim_{m\to\infty} \frac{1}{m^{n-p}}\Phi_m^*[W]\Big) \wedge [D],$$

where $\Phi_m : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$ is the m-th power map $(z_1, \ldots, z_n) \longmapsto (z_1^m, \ldots, z_n^m)$. $\mapsto$

The proof relies on a theorem of Berteloot and Dinh [BD20] that the limit of slices satisfies a certain continuity harmonic functions, and we can use Fourier analysis to prove the theorems about tropical currents. the second author

2. Tools from Superpotential Theory

Let (X, ω) be a compact Kähler manifold of dimension n. Assume that $\mathcal{S}$ is either a positive or a negative current of bidegree (q,q) on X. The quantity

$$|\langle \mathcal{S}, \omega^{n-q} \rangle|$$

is referred to as the *total mass* of $\mathcal{S}$. For $0 \leq r \leq n$, we consider the de Rham cohomology groups $H^r(X,\mathbb{C}) = H^r(X,\mathbb{R}) \otimes_\mathbb{R} \mathbb{C}$ with coefficients in $\mathbb{C}$. Recall that Hodge theory provides the following decomposition of the de Rham cohomology group into Dolbeault cohomology groups: Hodge ?

$$H^r(X,\mathbb{C}) \simeq \bigoplus_{p+q=r} H^{p,q}(X,\mathbb{C}).$$

We denote by $\mathcal{C}^q(X)$ the cone of positive closed bidegree (q,q)-currents or bidimension $(n-q, n-q)$ in X. We denote by $\mathcal{D}^q(X) = \mathcal{D}_{n-q}(X)$ the $\mathbb{R}$-vector space spanned by $\mathcal{C}^q(X)$, which is the space of closed real currents of bidegree (q,q). Every current $\mathcal{T} \in \mathcal{D}^q(X)$ has a cohomology class: Dolbeault is for $\bar\partial$

$$\{\mathcal{T}\} \in H^{q,q}(X,\mathbb{R}) = H^{q,q}(X,\mathbb{C}) \cap H^{2q}(X,\mathbb{R}).$$

We define $\mathcal{D}^{q,0}(X) = \mathcal{D}^0_{n-q}(X)$ to be the subspace of $\mathcal{D}^q(X)$, consisting of currents with vanishing cohomology. The $*$-topology on $\mathcal{D}^q(X)$ is defined by the norm:

$$\|\mathcal{S}\|_* := \min(\|\mathcal{S}^+\| + \|\mathcal{S}^-\|),$$

where the minimum is taken over positive currents $\mathcal{S}^+$ and $\mathcal{S}^-$ in $\mathcal{C}^q(X)$ that satisfy $\mathcal{S} = \mathcal{S}^+ - \mathcal{S}^-$. We say that $\mathcal{S}_n$ converges to $\mathcal{S}$ in $\mathcal{D}^q(X)$ if $\mathcal{S}_n$ converges weakly to $\mathcal{S}$ and moreover, $\|\mathcal{S}_n\|$ is bounded by a constant independent of n.

* Let $h := \dim H^{q,q}(X,\mathbb{R})$, and fix a set of smooth forms $\alpha = (\alpha_1, \dots, \alpha_{\beta_h})$ such that their cohomology classes $\{\alpha\} = (\{\alpha_1\}, \dots, \{\alpha_h\})$ form a basis for $H^{q,q}(X,\mathbb{R})$. By Poincaré duality, there exists a set of smooth forms $\alpha^\vee = (\alpha_1^\vee, \dots, \alpha_h^\vee)$ such that their cohomology classes $\{\alpha^\vee\}$ form the dual basis of $\{\alpha\}$, with respect to the cup-product. By adding $U_\mathcal{S}$ to a suitable combination of $\alpha_i^\vee$, we can assume that $\langle U_\mathcal{S}, \alpha_i\rangle = 0$, for all $i = 1, \dots, h$. In this case, we say that $U_\mathcal{S}$ is *α-normalised.*

Definition 2.1. Let $\mathcal{T} \in \mathcal{D}^q(X)$ and $\mathcal{S}$ be a smooth form in $\mathcal{D}^{n-q+1,0}(X)$.

(i) The *α-normalised super-potential* $\mathcal{U}_\mathcal{T}$ of $\mathcal{T}$ is given by the function

$$\mathcal{U}_\mathcal{T} : \{\mathcal{S} \in \mathcal{D}^{n-q+1,0}(X) : \text{smooth}\} \longrightarrow \mathbb{R}$$
$$\mathcal{S} \longmapsto \langle \mathcal{T}, U_\mathcal{S}\rangle,$$

where $U_\mathcal{S}$ is the α-normalised potential of $\mathcal{S}$.

(ii) We say $\mathcal{T}$ has a *continuous super-potential,* if $\mathcal{U}_\mathcal{T}$ can be extended to a function on $\mathcal{D}^{n-q+1,0}$ which is continuous with respect to the $*$-topology.

(X) In general, consider $\mathcal{T} \in \mathcal{D}^q(X)$ and $\mathcal{T} \in \mathcal{D}^r(X)$. Assume that $q + r \leq n$ and $\mathcal{T}$ has a continuous super-potential. Let $\mathcal{U}_\mathcal{T}$ be the α-normalised super-potential of $\mathcal{T}$. Let $\beta \in \mathrm{Span}_\mathbb{R}\{\alpha\}$ such that $\{\beta\} = \{\mathcal{T}\}$. We define

$$\langle \mathcal{T} \wedge \mathcal{S}, \varphi\rangle := \mathcal{U}_\mathcal{T}(\mathcal{S} \wedge dd^c\varphi) + \langle \beta \wedge \mathcal{S}, \varphi\rangle. \tag{1}$$

← φ smooth form?

Now assume that if $f : X \longrightarrow Y$, is a biholomorphism between smooth compact Kähler manifolds, then we have

$$f_*\mathcal{U}_{\mathcal{R}_1} = \mathcal{U}_{f_*\mathcal{R}_1}, \quad f^*\mathcal{U}_{\mathcal{R}_2} = \mathcal{U}_{f^*\mathcal{R}_2},$$

for $\mathcal{R}_1 \in \mathcal{D}^q(X)$ and $\mathcal{R}_2 \in \mathcal{D}^q(Y)$.

There is a problem of α-normalization. If $R_1 \in \mathcal{D}^{q,0}$, $R_2 \in \mathcal{D}^{q,0}$ then $\mathcal{U}_{R_1}, \mathcal{U}_{R_2}$ are independent of the choice of α. So the identities are OK.

Definition 2.2. Let $(\mathcal{T}_n)$ be a sequence of currents in $\mathcal{D}^q(X)$ weakly converging to $\mathcal{T}$. Let $\mathcal{U}_\mathcal{T}$ and $\mathcal{U}_{\mathcal{T}_n}$ be their α-normalised super-potentials. If $\mathcal{U}_{\mathcal{T}_n}$ converges to $\mathcal{U}_\mathcal{T}$ uniformly on any $*$-bounded sets of smooth form in $\mathcal{D}^{n-q+1,0}(X)$, then the convergence is called *SP-uniform.*

It is shown in [DS10, Proposition 3.2.8] that any current with continuous super-potentials can be SP-uniformly approximated by smooth forms. Moreover, currents with continuous super-potentials have other nice properties:

Theorem 2.3 ([DNV18, Theorem 1.1]). Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two positive currents in $\mathcal{D}_q(X)$, such that $\mathcal{T} \leq \mathcal{T}'$, *i.e.*, $\mathcal{T}' - \mathcal{T}$ is a positive current. Then, if $\mathcal{T}'$ has a continuous super-potential, then so does $\mathcal{T}$.

Theorem 2.4. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two positive closed currents, and $\mathcal{T}_1$ has a continuous superpotentials, then $\mathcal{T}_1 \wedge \mathcal{T}_2$ is well-defined. Moreover, if $\mathcal{T}_2$ has also a continuous superpotential, then

(a) [DS10, Proposition 3.3.3] $\mathcal{T}_1 \wedge \mathcal{T}_2$ has a continuous superpotential;
(b) [DS10, Proposition 3.3.3] This wedge product is continuous with respect to the SP-uniform convergence.
(c) [DS09, Theorem 4.2.4] $\mathrm{supp}(\mathcal{T}_1 \wedge \mathcal{T}_2) \subseteq \mathrm{supp}(\mathcal{T}_1) \cap \mathrm{supp}(\mathcal{T}_2)$.

Theorem 2.5 ([DS10, Proposition 3.3.4]). Assume that $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are closed positive currents, and $\mathcal{T}_1$ and $\mathcal{T}_2$ have continuous superpotentials. Then,

$$\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_2 \wedge \mathcal{T}_1 \quad \text{and} \quad (\mathcal{T}_1 \wedge \mathcal{T}_2) \wedge \mathcal{T}_3 = \mathcal{T}_1 \wedge (\mathcal{T}_2 \wedge \mathcal{T}_3).$$

Proposition 2.6. Let X be a compact Kähler manifold, $S_n \longrightarrow S$ be a convergent sequence in $\mathcal{D}^q(X)$. If a current $\mathcal{T}$ has a continuous superpotential, then

$$\mathcal{T} \wedge \mathcal{S}_n \longrightarrow \mathcal{T} \wedge \mathcal{S}.$$

Proof. The main result of Dinh and Sibony in [DS04] implies any current $\mathcal{T} \in \mathcal{D}^p(X)$ can be weakly approximated by a difference of smooth closed positive of bidegree (p,p)-forms. The result then follows from the definition of continuity of super-potentials. □

Lemma 2.7. Let $\mathcal{T}, \mathcal{T}'$ be positive closed currents such that $\mathcal{T}_{|_U} = \mathcal{T}'_{|_U}$ in an open subset $U \subseteq X$, and both $\mathcal{T}$ and $\mathcal{T}'$ have continuous super-potentials. Then, for any $\mathcal{S} \in \mathcal{D}^r(X)$,

$$(\mathcal{T} \wedge \mathcal{S})_{|_U} = (\mathcal{T}' \wedge \mathcal{S})_{|_U}.$$

Proof. In [DS10], for any current $\mathcal{S}$ with continuous super-potential, a family $\{\mathcal{T}_\theta\}_{\theta \in \mathbb{C}^*}$ is constructed that $\mathcal{T}_\theta$ converges SP-uniformly to $\mathcal{S}$ as $|\theta| \to 0$. Therefore, by the hypothesis of the lemma, we can construct families of smooth forms $\mathcal{T}_n, \mathcal{T}'_n$ converging *SP*-uniformly to $\mathcal{T}, \mathcal{T}'$ respectively. Moreover, \rm

$$\mathcal{T}_{n|_{U_\epsilon}} = \mathcal{T}'_{n|_{U_\epsilon}},$$

U_ε slightly smaller than U? where U_ϵ is an ϵ-neighbourhood of U. Now, for a $(n-q-r, n-q-r)$ smooth form φ with compact support on U

$$(\mathcal{T}_n \wedge \mathcal{S}) \wedge \varphi = (\mathcal{T}'_n \wedge \mathcal{S}) \wedge \varphi,$$

together with Theorem 2.4(b), implies the assertion.

□

We also have a very useful local version of Theorem 2.3.

Lemma 2.8. If $\mathcal{T}$ is a positive closed current on a compact Kähler manifold X, which is locally bounded by a product of positive closed bidegree $(1,1)$-currents of continuous potentials, resp. Hölder continuous potentials, then $\mathcal{T}$ has continuous superpotentials, resp. respectively Hölder continuous potentials in X.

Proof. Fix a point a in X. In an open neighbourhood of a that we identify with the ball $B(0,2)$ in $\mathbb{C}^n$, we have

$$\mathcal{T} \leq dd^c u_1 \wedge \cdots \wedge dd^c u_p$$

with u_i continuous or Hölder continuous. Without loss of generality, we can assume that these functions are strictly negative. On $B(0,1)$, define

$$u'_i := \max(u_i, A \log \|z\|)$$

with A sufficiently large so that $u_i' = u_i$ on $B(0, 1/2)$. Observe that $u_i' = A \log \|z\|$ near $\partial B(0,1)$. Hence we can extend it to a function which is smooth in a neighbourhood of $X \setminus B(0,1)$. Thus, this function is quasi-plurisubharmonic. We have

$$\mathcal{T} \leq (B\omega + dd^c u_1') \wedge \cdots \wedge (B\omega + dd^c u_p')$$

in a neighbourhood W_a of a if B is large enough.

Since we can cover X using a finite number of open sets W_{a_k}, we can add up all obtained quasi-plurisubharmonic functions together and obtain a quasi-plurisubharmonic function u. It is clear that

$$\mathcal{T} \leq (C\omega + u)^p$$

if C is large enough. The function u is continuous or Hölder continuous, and we deduce by Theorem 2.3.

□

Corollary 2.9. Let $q : \widehat{X} \longrightarrow X$, be the blowing up of the compact Kähler manifold X. along a submanifold ? Assume that $\mathcal{T} \in \mathcal{D}_p(\widehat{X})$ is such that the support of $\mathcal{T}$ does not intersect the exceptional divisors of $\widehat{X}$. If the current ~~of~~ $\mathcal{T}$ has a continuous superpotential then $q_* \mathcal{T}$ has the same property.

? *Proof.* This is an easy corollary of previous lemma, since q is biholomorphic near $\mathrm{supp}(\mathcal{T})$. □

Theorem 2.10. For two complex manifolds X and Y, consider two convergent sequences of currents $\mathcal{T}_n \longrightarrow \mathcal{T}$ in $\mathcal{D}^q(X)$ and $\mathcal{S}_n \longrightarrow \mathcal{S}$ in $\mathcal{D}^r(Y)$. We have that

$$\mathcal{T}_n \otimes \mathcal{S}_n \longrightarrow \mathcal{T} \otimes \mathcal{S},$$

weakly in $\mathcal{D}^{q+r}(X \times Y)$.

Sketch of the proof. Let us denote by (x, y) the coordinates on $X \times Y$. Using local coordinates and a partition of unity and Weierstrass theorem we can approximate any smooth forms on $X \times Y$ with forms with polynomial coefficients in (x, y). The approximation is in C^∞. As a result, the convergence, we only need test forms with monomial coefficients. Thus, the variables x, y are separated and the convergence of the tensor products becomes the convergence of each factor.

□

2.1. **Semi-continuity of slices.** Let $f : X \longrightarrow Y$ be a dominant holomorphic map between complex manifolds, not necessarily compact, of dimension n and m respectively. Let $\mathcal{T}$ be a positive closed current on X of bi-dimension (p, p) with $p \geq m$. Then a slice

$$\mathcal{T}_y = \langle \mathcal{T} | f | y \rangle$$

obtained by restricting $\mathcal{T}$ to $f^{-1}(y)$ exists for almost every $y \in Y$; see [Dem, Page 171]. This is a positive closed current of bi-dimension $(p-m, p-m)$ on X supported by $f^{-1}(y)$. If Ω is a smooth form of maximal bi-degree on Y and α a smooth $(q-m, q-m)$-form with compact support in X, then we have

$$\langle \mathcal{T}, \alpha \wedge f^*(\Omega) \rangle = \int_{y \in Y} \langle \mathcal{T}_y, \alpha \rangle \Omega(y).$$

In general, if $\mathcal{T}$ and $\mathcal{T}'$ are such that $\mathcal{T}_y = \mathcal{T}'_y$ for almost every y, we do not necessarily have $\mathcal{T} = \mathcal{T}'$. However, the following is true: Let $f_1, \ldots, f_k$ be dominant holomorphic maps from X to $Y_1, \ldots, Y_k$. Consider the vector space spanned by all the differential forms of type $\alpha \wedge f_i^*(\Omega_i)$ for some α as above and for some smooth form Ω_i on Y_i of maximal degree. Assume this space is equal to space of all (q,q)-forms of compact support in X. Then if $\langle \mathcal{T}|f_i|y_i\rangle = \langle \mathcal{T}'|f_i|y_i\rangle$ for every i and almost every $y_i \in Y_i$, we have $\mathcal{T} = \mathcal{T}'$. The proof is a consequence of the above discussion.

Let $U \subseteq \mathbb{C}^m$ and $V \subseteq \mathbb{C}^n$ be two bounded open sets. Assume that $\pi_1 : U \times V \longrightarrow U$ and $\pi_2 : U \times V \longrightarrow V$ are the canonical projections. Consider two closed positive currents $\mathcal{T}$ and $\mathcal{S}$ on $U \times V$ of bi-dimension (m,m) and (n,n) respectively. We say that $\mathcal{T}$ horizontal-like if $\pi_2(\mathrm{supp}(\mathcal{T}))$ is relatively compact in V. Similarly, if $\pi_1(\mathrm{supp}(\mathcal{S}))$ is relatively compact in U, $\mathcal{S}$ is called vertical-like.

Theorem 2.11 ([BD20, Lemma 3.7]). Let $(\mathcal{T}_n) \longrightarrow \mathcal{T}$ be a convergent sequence of horizontal-like positive closed currents to a horizontal-like current $\mathcal{T}$ in $U \times V$. Let $a \in U$ and assume that the sequence of measures $(\langle \mathcal{T}_n, \pi_1, a\rangle)_n$ is also convergent. Then,

$$\langle \lim_{n\to\infty} \mathcal{T}_n|\pi_1|a\rangle(\phi) \leq \langle \mathcal{T}|\pi_1|a\rangle(\phi)$$

for every plurisubharmonic function ϕ on $\mathbb{C}^n$.

There is an simple version of the above theorem for supports which will be useful later.

Lemma 2.12. Assume that $\mathcal{T}_i$'s, $\mathcal{S}$ and $\mathcal{T}$ are all closed positive currents, and $\mathcal{T}_i \wedge \mathcal{S}$ and $\mathcal{T} \wedge \mathcal{S}$ are well-defined. If we have the following weak convergence, together with the convergence of supports in the Hausdorff metric, that is,

$$\mathcal{T}_i \longrightarrow \mathcal{T}, \quad \mathrm{supp}(\mathcal{T}_i) \longrightarrow \mathrm{supp}(\mathcal{T}).$$

Then, $\mathrm{supp}(\lim(\mathcal{T}_i \wedge \mathcal{S})) \subseteq \mathrm{supp}(\mathcal{T}) \cap \mathrm{supp}(\mathcal{S})$.

Proof. For a point x outside the support of $\mathcal{T}$, There is a sufficiently small radius ϵ, such that for a sufficiently large i, $\mathcal{T}_i$ vanishes on the ball $B_\epsilon(x)$ centred at x. It follows that any limit of $\mathcal{T}_i \wedge \mathcal{S}$ vanishes on $B_\epsilon(x)$. So its support does not contain a. Moreover, its support does not contain any point outside $\mathrm{supp}(S)$.

$\mathcal{S}$

□

3. Tropical Varieties, Tori, Tropical Currents

In this section, we recall the definition of tropical cycles and note that with the natural addition of tropical cycles and their *stable intersection*, the tropical cycles form a ring.

3.1. Tropical varieties. A linear subspace $H \subseteq \mathbb{R}^n$ is said to be *rational* if there exists a subset of $\mathbb{Z}^n$ that spans H. A *rational polyhedron* is the intersection of finitely many rational half-spaces defined by

$$\{x \in \mathbb{R}^n : \langle m, x\rangle \geq c, \text{ for some } m \in \mathbb{Z}^n,\ c \in \mathbb{R}\}.$$

A *rational polyhedral complex* is a polyhedral complex consisting solely of rational polyhedra. The polyhedra in a polyhedral complex are also referred to as *cells*. A *fan* is a polyhedral complex whose cells are all cones. If every cone in a fan Σ is contained