Algebraic Geometry, Assessed Coursework 2

- Available from 12:00 PM on March 11 to 12:00 PM on March 20
- Please submit your work in PDF format directly on Blackboard

Q1. (a) (**15 marks**) Find all the elements of $\mathrm{maxSpec}(\mathbb{C}[x])$, $\mathrm{maxSpec}(\mathbb{C}[x, 1/x])$, and $\mathrm{maxSpec}(\mathbb{C}[x, 1/x, y])$ explicitly.

(b) (**5 marks**) Consider the isomorphism $\varphi : \mathbb{A}^1 \backslash \{0\} \longrightarrow \mathbb{A}^1 \backslash \{0\}$, $a \longmapsto b = 1/a$, and the pullback map on the coordinate rings $\varphi^* : \mathbb{C}[x, 1/x] \longmapsto \mathbb{C}[y, 1/y]$. Compute $\varphi^*(1/x)$, $\varphi^*(2x^2 + \frac{2x^3+4x}{x^5})$, $\varphi^*(2 - x)$.

Q2. (**20 marks**) Consider the affine algebraic hypersurface $V := \mathbb{V}(y - ux) \subseteq \mathbb{A}^3$.

(a) Prove that the projection $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$, $(x, y, u) \longmapsto (x, u)$ restricts to an isomorphism from V to $\mathbb{A}^2$.

(b) Prove that the projection $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$, $(x, y, u) \longmapsto (x, y)$ does not restrict to isomorphism from V to $\mathbb{A}^2$.

Q3. (**25 marks**)

(a) Prove that if $g \in \mathbb{C}[x, y]$ then the projective closure of its variety $\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g}) \subseteq \mathbb{P}^2$ where $\tilde{g} \in \mathbb{C}[x, y, z]$ is the homogenisation of g.

(b) Consider the polynomials $f_1(x, y) = x + y + 1$, $f_2(x, y) = x^2 + 6y^2 + 1$, $f_3(x, y) = x^2 + 3y + 1$, $f_4(x, y) = x^3 + 3xy^2 + 4$. Determine whether or not each of the projective closures includes the points

(i) $[1 : 0 : 0]$;

(ii) $[0 : 1 : 0]$;

(iii) $[0 : 0 : 1]$.

(c) Can you find a general necessary and sufficient condition on $g \in \mathbb{C}[x, y]$ such that its homogenisation $\tilde{g} \in \mathbb{C}[x, y, z]$ does not pass through any of the three points in item (b)?

Q4. (**15 marks**)

(a) Prove that $\mathbb{P}^n$ is compact with respect to the quotient Euclidean topology from $\mathbb{A}^{n+1} \setminus \{0\}$.

(b) What is the projective Zariski-closure of the $\mathbb{V}(y - \sin(x))$ in $\mathbb{P}^2$? How do you compare this to the Chow's Lemma? **Hint.** In Example 3.44 we have seen that this curve is not algebraic.

Q5. (**20 marks**)

(a) The variety of a polynomial of the form $ax + by + cz \in \mathbb{C}[x, y, z]$ for $a, b, c \in \mathbb{C}$ is called a *line* in $\mathbb{P}^2$. Prove that any two distinct lines in $\mathbb{P}^2$ intersect exactly at one point.

(b) Assume that $C_1, C_2 \subseteq \mathbb{A}^2$ are two closed affine algebraic curves.

(i) Prove that we have the inclusion $\overline{C_1 \cap C_2} \subseteq \overline{C}_1 \cap \overline{C}_2$ of projective closures.

(ii) Find two curves such that the above inclusion is strict.

Q6. (**Bonus 10 marks**)

(a) Let Y be a closed affine algebraic variety and $O \subseteq Y$ an open subset. Prove that $\mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra.

(b) A *sheaf* $\mathcal{F}$ of rings associated to a topological space X consists of the following data:

(i) To each open set $U \subseteq X$, it associates a ring $\mathcal{F}(U)$.

(ii) To each inclusion of open sets $U \hookrightarrow V$, there exists a map $\text{res}_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ called the restriction map from $\mathcal{F}(V)$ to $\mathcal{F}(U)$. These maps satisfy the property that $\text{res}_{U,U} = \text{id}_{\mathcal{F}(U)}$ and $\text{res}_{V,U} \circ \text{res}_{W,V} = \text{res}_{W,U}$, where $U \subseteq V \subseteq W$ are open sets.

These data satisfy the following properties:

(iii) Suppose that $f_i \in \mathcal{F}(U_i)$ are a collection of sections that agree on overlaps (formally, $\text{res}_{U_i,U_i \cap U_j} f_i = \text{res}_{U_j,U_i \cap U_j} f_j$ whenever the intersection exists). Then they lift to a section $f \in \mathcal{F}(U)$ which has the property that $\text{res}_{U,U_i} f = f_i$ for all $i \in I$.

(iv) Suppose that $f, f' \in \mathcal{F}(U)$ and that $\text{res}_{U,U_i} f = \text{res}_{U,U_i} f'$ for all $i \in I$. Then $f = f'$.

Let X be an irreducible quasi-projective variety.

(i) Assume that U and V are open subsets of X with $U \subseteq V$. Briefly explain why $f \in \mathcal{O}_X(V)$ implies that $f_{|U} \in \mathcal{O}_X(U)$.

(ii) Briefly explain why the collection of sets of functions $\mathcal{O}_X(U)$, where U ranges over all open subsets of X, forms a sheaf on X.