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INTERSECTION THEORY ON TORIC VARIETIES

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The operational Chow cohomology classes of a complete toric variety are identified with certain functions, called Minkowski weights, on the corresponding fan. The natural product of Chow cohomology classes makes the Minkowski weights into a commutative ring; the product is computed by a displacement in the lattice, which corresponds to a deformation in the toric variety. We show that, with rational coefficients, this ring embeds in McMullen's polytope algebra, and that the polytope algebra is the direct limit of these Chow rings, over all compactifications of a given torus. In the nonsingular case, the Minkowski weight corresponding to the Todd class is related to a certain Ehrhart polynomial. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

Any algebraic variety X has Chow "homology" groups $A_*(X) = \oplus_k A_k(X)$ and Chow "cohomology" groups $A^*(X) = \oplus_k A^k(X)$. The latter are the operational groups defined in [1]. These cohomology groups have a natural ring structure, written with a cup product, and $A_*(X)$ is a module over $A^*(X)$, written with a cap product. These satisfy functorial properties similar to homology and cohomology groups in topology, and vector bundles have Chern classes in $A^*(X)$. If X is complete, one has a Kronecker duality homomorphism

$$\mathscr{D}_X : A^k(X) \to Hom(A_k(X), \mathbf{Z})$$

that takes c to the map $a \mapsto \deg(c \cap a)$. For general varieties, even if nonsingular, these groups are very large and hard to compute, and $\mathscr{D}_X$ is far from being an isomorphism.

If X is a toric variety, however, $A_k(X)$ is generated by the orbit closures $V(\sigma)$, as σ varies over the cones of codimension k in the fan Δ in a lattice N corresponding to X. The relations are given by the divisors of torus-invariant rational functions on $V(\tau)$, for τ a cone of codimension $k + 1$. This gives an explicit presentation of the Chow groups (Proposition 2.1). In addition, if X is complete, the Kronecker duality homomorphism $\mathscr{D}_X$ is an isomorphism. This identifies Chow cohomology classes with certain functions on the set of cones in Δ; a function c corresponds to a class in $A^k(X)$ if it satisfies the linear equations in formula (1) below. We call these functions *Minkowski weights*.

The Chow homology of a complete toric variety can have torsion, and some of these groups can vanish (Example 2.3). From the duality isomorphism $\mathscr{D}_X$, we see that the Chow cohomology groups are always torsion free. We include a description of these groups for the toric varieties corresponding to hypersimplices (Proposition 3.6).

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There is a canonical homomorphism from $Pic(X)$ to $A^1(X)$, which is not always an isomorphism, but is an isomorphism when X is a complete toric variety. This is a general result of Brion [2] for spherical varieties, given a simple proof here in terms of Minkowski weights (Corollary 3.4) for the toric case.

The ring structure on $A^*(X)$ makes the Minkowski weights into a commutative, associative ring. We prove the following formula for the product.

THEOREM. *For $c \in A^p(X)$, $\tilde{c} \in A^q(X)$, the product $c \cup \tilde{c}$ in $A^{p+q}(X)$ is given by the Minkowski weight that assigns to a cone γ of codimension $p + q$ the value*

$$(c \cup \tilde{c})(\gamma) = \sum m^{\gamma}_{\sigma,\tau} \cdot c(\sigma) \cdot \tilde{c}(\tau).$$

The sum is over a certain set of cones σ and τ of codimension p and q that contain γ, determined by the choice of a generic vector v in N: σ and τ appear when $\sigma + v$ meets τ. The coefficient $m^{\gamma}_{\sigma,\tau}$ is the index $[N : N_\sigma + N_\tau]$ where $N_\sigma := \mathbf{Z}(N \cap \sigma)$ and $N_\tau := \mathbf{Z}(N \cap \tau)$.

The present paper is a sequel to [3], where the general results stated before the theorem were proved for varieties on which a solvable linear algebraic group acts with a finite number of orbits. In particular, it was shown in [3] that the cup product is given by a formula as in the theorem, where the coefficients $m^{\gamma}_{\sigma,\tau}$ are obtained by expressing the diagonal class of $V(\gamma)$ as a linear combination of the cases of $V(\sigma) \times V(\tau)$. What is new here is a much more explicit combinatorial description of the numbers $m^{\gamma}_{\sigma,\tau}$ for toric varieties. In order to prove the theorem, we carry out an explicit rational deformation, flowing in the direction corresponding to the vector v (Section 4).

The Chow rings $A^*(X)$ can be embedded in the polytope algebra of McMullen [4–6]. This is carried out by relating our Minkowski weights, which depend only on the lattice, to McMullen's notion of weights on a polytope, which depend on metric geometry in Euclidean space. In fact, we show (Section 5) that the polytope algebra in the direct limit of all the Chow rings, taken with rational coefficients, as X varies over all compactifications of a fixed torus. Our formula for multiplying Minkowski weights is shown to be equivalent to a mixed volume decomposition of McMullen (Proposition 5.3).

Any variety has a Todd class in its Chow homology group with rational coefficients. If X is the toric variety of a simplicial fan, this Todd class is Poincaré dual to a class Td_X in the Chow cohomology, and hence to a $\mathbf{Q}$-valued Minkowski weight. If $v_1, \dots, v_d$ are the primitive lattice points along the edges of the fan, and $D_1, \dots, D_d$ are the corresponding divisors, and $D = a_1 D_1 + \cdots + a_d D_d$ is a Cartier divisor whose line bundle is generated by its sections, the degree of $\exp(D) \cdot Td_X$ is the number of lattice points $u \in M$ such that $\langle u, v_i \rangle \geqslant -a_i$ for $1 \leqslant i \leqslant d$. When X is nonsingular, this is a polynomial of degree n in the variables $a_1, \dots, a_d$. This is discussed in Section 6.

2. CHOW GROUPS AND THE CHOW COHOMOLOGY RING

The Chow group $A_k(X)$ of an arbitrary variety X is generated by all k-dimensional closed subvarieties of X, with relations generated by divisors of rational functions on $(k + 1)$-dimensional subvarieties. If X is a toric variety, then there is a finite presentation of $A_k(X)$ in terms of torus-invariant subvarieties and torus-invariant divisors. This is the content of Proposition 2.1 below. Our notations concerning toric varieties are as in [7].

Let $X = X(\Delta)$ be the toric variety corresponding to a fan Δ in a lattice N of dimension n. The torus-invariant closed subvarieties of X are of the form $V(\sigma)$, as σ varies over the cones in Δ, with $\dim(V(\sigma)) = \operatorname{codim}(\sigma) = n - \dim(\sigma)$. Each cone τ determines a sublattice $M(\tau) = \tau^{\perp} \cap M$ of the lattice M dual to N. Each nonzero element $u \in M(\tau)$ determines a rational function x^u on $V(\tau)$. The divisor of this rational function is

$$[\operatorname{div}(x^u)] = \sum_{\sigma} \langle u, n_{\sigma,\tau} \rangle \cdot [V(\sigma)] \tag{1}$$

where the sum is over all cones σ that contain τ with $\dim(\tau) + 1$, and $n_{\sigma,\tau}$ is a lattice point in σ whose image generates the one-dimensional lattice N_σ/N_τ. Here N_σ and N_τ are the sublattices of N generated by $\sigma \cap N$ and $\tau \cap N$, respectively.

Proposition 2.1. (a) *The Chow group $A_k(X)$ of a toric variety $X = X(\Delta)$ is generated by the classes $[V(\sigma)]$ where σ runs over all cones of codimension k in the fan Δ.*

(b) *The group of relations on these generators is generated by all relations* (1) *where τ runs over cones of codimension $k + 1$ in Δ, and u runs over a generating set of $M(\tau)$.*

Proof. Part (a) is the proposition on p. 96 in [7, Section 5.1]. Part (b) follows directly from Theorem 1 in [3]. □

If X and Y are arbitrary varieties, then there is a *Künneth map* $A_k(X) \otimes A_l(Y) \to A_{k+l}(X \times Y)$, which is defined by sending $[V] \otimes [W]$ to the class of the product $[V \times W]$. In the special case where X and Y are toric, this map is an isomorphism.

Corollary 2.2. *If X and Y are toric varieties, then the Künneth map $A_*(X) \otimes A_*(Y) \to A_*(X \times Y)$ is an isomorphism.*

Proof. This is a special case of Theorem 2 in [3]. In our toric situation Corollary 2.2 may also be derived directly from the presentation in Proposition 2.1, using the fact that $V(\sigma \times \tau) = V(\sigma) \times V(\tau)$. □

If X is a nonsingular toric variety, then the Chow groups $A_k(X)$ are free abelian, and they are never trivial for complete X (see [7, Section 5.2]). For general toric varieties these groups may have torsion, and they may be trivial even if X is complete.

Example 2.3. Let Δ be the fan over the faces of the cube with vertices at $(\pm 1, \pm 1, \pm 1)$ in $\mathbf{Z}^3$. Let N be the lattice spanned by these vertices, that is, $N = \{(x, y, z) \in \mathbf{Z}^3 : x \equiv y \equiv z \pmod 2\}$. For any positive integer k, let Δ_k be the complete nonprojective fan of the same combinatorial type obtained from Δ by replacing the generator $(1, 1, 1)$ by $(1, 1, 2k + 1)$. An explicit computation using Proposition 2.1 shows that $A_1(X(\Delta_k)) = \mathbf{Z}/k\mathbf{Z}$. In particular, we have

$$A_1(X(\Delta)) = \mathbf{Z}, \qquad A_1(X(\Delta_1)) = 0 \quad \text{and} \quad A_1(X(\Delta_2)) = \mathbf{Z}/2\mathbf{Z}.$$

For some examples of Proposition 2.1 when $k = n - 1$ see [7, Section 3.4]. If Δ is complete, then Proposition 2.1 implies that $A_0(X(\Delta)) = \mathbf{Z}$. Then isomorphism from $A_0(X(\Delta))$ to $\mathbf{Z}$ is the degree map (to be considered in (2) below). It satisfies $\deg([V(\sigma)]) = 1$ for all n-dimensional cones $\sigma \in \Delta$. If Δ is not complete, then Proposition 2.1 implies that $A_0(X(\Delta)) = 0$.

When X is a nonsingular variety, there is a natural ring structure on the Chow homology $A_*(X) = \bigoplus_k A_k(X)$. For a general singular variety X, there is no natural ring structure on the Chow homology, just as there is no natural ring structure on the singular homology $H_*(X, \mathbf{Z})$. Instead, there are *operational Chow cohomology groups* $A^k(X)$. These groups, which were introduced by the first author and MacPherson in [1], fit together to form a ring suitable for doing intersection theory on X. An element of $A^k(X)$ is a compatible collection of homomorphisms of Chow groups from $A_i(Y)$ to $A_{i-k}(Y)$ for all varieties Y mapping to X and all integers $i \geqslant k$. (See [8] or [1] for precise definitions and details.) Composition of these homomorphisms defines the multiplication $A^k(X) \otimes A^l(X) \to A^{k+l}(X)$ which turns the Chow cohomology $A^*(X) = \bigoplus_k A^k(X)$ into a graded commutative $\mathbf{Z}$-algebra.

For any complete variety X there is a degree homomorphism $\deg: A_0(X) \to \mathbf{Z}$. An element of the Chow cohomology group $A^k(X)$ gives a homomorphism of Chow groups from $A_k(X)$ to $A_0(X)$, and, by composition with "deg", it gives a homomorphism from $A_k(X)$ to $\mathbf{Z}$. The resulting homomorphism of abelian groups is denoted

$$\mathscr{D}_X : A^k(X) \to Hom(A_k(X), \mathbf{Z}). \tag{2}$$

For a general complete variety X the map $\mathscr{D}_X$ can have a large kernel, even if X is nonsingular. On the other hand, it was shown in [3, Theorem 3] that $\mathscr{D}_X$ is an isomorphism if X is a complete scheme on which a connected solvable group acts with finitely many orbits. This class of schemes includes toric varieties and their closed torus-invariant subschemes, whence we get the following result.

PROPOSITION 2.4. *If X is a complete toric variety, then the map $\mathscr{D}_X$ is an isomorphism. More generally, if Y is closed and torus-invariant in X, then $\mathscr{D}_Y$ is an isomorphism.*

Totaro [9] has shown that, for a complex toric variety X, the canonical map from $A_k(X)$ to Borel–Moore homology $H_{2k}^{BM}(X)$ maps $A_k(X)_{\mathbf{Q}}$ isomorphically onto the smallest weight space $W_{2k}H_{2k}^{BM}(X)$. He proves this in [9] for a class of varieties containing all spherical varieties. In the toric case, Totaro gives the following simple description for these weight spaces. For any positive integer m, let $t_m : X \to X$ be the morphism determined by multiplication by m on the underlying lattice N. Then $W_{2k}H_{2k}^{BM}(X)$ is the subspace of $H_{2k}^{BM}(X)$ of classes α such that $(t_m)_*(\alpha) = m^k\,\alpha$ for all m. For X complete, this identifies $A^*(X)_{\mathbf{Q}}$ with the corresponding weight space of $H^*(X; \mathbf{Q})$, which refines the result of Batyrev [10].

3. MINKOWSKI WEIGHTS

Let Δ be a complete fan in a lattice N, corresponding to a complete toric variety $X = X(\Delta)$. Let $\Delta^{(k)}$ denote the subset of all cones of codimension k. An integer-valued function on $\Delta^{(k)}$ is called a *weight* of codimension k on Δ. We say that c is a *Minkowski weight* if it satisfies the relation

$$\sum_{\sigma \in \Delta^{(k)} : \sigma \supset \tau} \langle u, n_{\sigma,\tau} \rangle \cdot c(\sigma) = 0 \tag{3}$$

for every cone τ in $\Delta^{(k+1)}$ and every element u of the lattice $M(\tau)$. Equivalently, c is a Minkowski weight if $\sum_\sigma c(\sigma) n_{\sigma,\tau}$ lies in N_τ for all $\tau \in \Delta^{(k+1)}$. As in (1) the lattice point $n_{\sigma,\tau}$ is any representative in σ for the generator of the one-dimensional lattice N_σ/N_τ. We have the following combinatorial description of the Chow cohomology groups.

THEOREM 3.1. *The Chow cohomology group $A^k(X)$ of a complete toric variety $X = X(\Delta)$ is canonically isomorphic to the group of Minkowski weights of codimension k on Δ.*

Proof. This follows immediately from combining Propositions 2.1 and 2.4. □

Remark 3.2. Theorem 3.1 holds for every closed and torus-invariant subscheme Y of a complete toric variety X as well. The Chow cohomology of such a Y is isomorphic to the group of Minkowski weights that is supported on the cones $\sigma \in \Delta$ with $V(\sigma) \subset Y$.

We shall rewrite the defining relation for Minkowski weights of codimension 1 on a complete fan Δ. Given cones $\rho \supset \sigma$ of codimension 0 and 1 in Δ, let $m_{\rho,\sigma}$ denote the unique generator of the one-dimensional lattice $M(\sigma)$ which is nonnegative on ρ. If τ is a codimension 2 cone in Δ, then the star of τ in Δ is given by a sequence of cones

$$\sigma_1 \subset \rho_1 \supset \sigma_2 \subset \rho_2 \supset \cdots \supset \sigma_t \subset \rho_t \supset \sigma_{t+1} = \sigma_1 \tag{4}$$

with $\operatorname{codim}(\sigma_i) = 1$ and $\operatorname{codim}(\rho_i) = 0$.

LEMMA 3.3. *A weight c on the codimension 1 cones in a complete fan Δ is a Minkowski weight if and only if $\sum_{i=1}^{t} c(\sigma_i) \cdot m_{\rho_i,\sigma_i} = 0$ for all cones τ of codimension 2.*

Proof. We identify both $M(\tau)$ and its dual lattice N/N_τ with $\mathbf{Z}^2$. The primitive lattice vectors m_{ρ_i,σ_i} and $n_{\rho_i,\tau}$ are orthogonal to each other. There exists a consistent orientation around the star of τ, such that if m_{ρ_i,σ_i} has coordinates (x_i, y_i), then $n_{\sigma_i,\tau} = 0$ in N/N_τ. This is equivalent to the condition $\sum_{i=1}^{t} c(\sigma_i) \cdot m_{\rho_i,\sigma_i} = 0$ in $M(\tau)$. □

For any variety X there is a canonical homomorphism from $Pic(X)$ to $A^1(X)$. It takes a line bundle L to the operator $\alpha \mapsto c_1(f^*L) \cap \alpha$ for $f: Y \to X$, $\alpha \in A_*(Y)$. This can fail to be an isomorphism [8, Example 17.4.9], but this does not happen for toric varieties.

COROLLARY 3.4 (Brion [2]). *If X is a complete toric variety, then the canonical map $Pic(X) \to A^1(X)$ is an isomorphism.*

This result was proved for projective spherical varieties in [2], and more recently Brion extended it to arbitrary complete spherical varieties. What follows is a simple proof for the toric case.

Proof. By Proposition 2.4 we may compose with the isomorphism $\mathscr{D}_X$ and show that the resulting map $Pic(X) \to A^1(X) \simeq Hom(A_1(X), \mathbf{Z})$ is an isomorphism. This map takes a line bundle L to the linear functional which assigns to a curve C on X the degree of L on C. We know [7, Section 3.4] that the group $Pic(X)$ of isomorphism classes of line bundles equals the group of T-Cartier divisors modulo principal Cartier divisors. To specify an element L in $Pic(X)$, we must define elements $u(\rho)$ in M for every n-dimensional cone ρ in the fan of X, such that whenever two cones ρ and ρ' intersect in a codimension 1 cone σ, then $u(\rho) - u(\rho')$ lies in $M(\sigma)$. In this case there exists a unique integer $c(\sigma)$ such that

$$u(\rho) - u(\rho') = c(\sigma) \cdot m_{\rho,\sigma} = -c(\sigma) \cdot m_{\rho',\sigma}. \tag{5}$$

The integer $c(\sigma)$ is the degree of L on the invariant curve $C = V(\sigma)$.

We claim that the assignment $u \mapsto c$ represents the map $Pic(X) \to A^1(X)$. Indeed, this assignment defines a homomorphism of abelian groups from the T-Cartier divisors to the wrights of codimension 1 on the fan of X. The kernel of the map $u \mapsto c$ equals the group M of principal Cartier divisors, and Lemma 3.3 verifies that its image lies in the subgroup of Minkowski weights.

To complete the proof of Corollary 3.4, it suffices to construct the inverse map $A^1(X) \to Pic(X)$, $c \mapsto u$. Given any Minkowski weight c of codimension 1, we set $u(\rho_0) := 0$ for one fixed n-dimensional cone ρ_0. Then we define $u(\rho)$ for all other cones σ using the relation (5) along any path of adjacent n-dimensional cones in the fan of X. Since any two such paths with the same endpoints differ by a sum of cycles like (4), Lemma 3.3 guarantees that this definition is independent of the chosen path. $\square$

Example 3.5 (*Minkowski weights on the hypersimplex*). An important family of singular toric varieties arises from the action of the torus $T = (\mathbf{C}^*)^n$ on the Grassmannian $Gr_k(\mathbf{C}^n)$. We let $X_{k,n}$ denote the closure of the T-orbit of a generic point in $Gr_k(\mathbf{C}^n)$. This is an $(n-1)$-dimensional projective toric variety, which is singular for $2 \leqslant k \leqslant n-2$. In what follows we describe its Chow cohomology groups.

The polytope associated with the Plücker embedding of $X_{k,n}$ is the *hypersimplex*

$$\Delta(k, n) = \operatorname{conv}\{e_{i_1} + e_{i_2} + \cdots + e_{i_k} : 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\}.$$

The complete fan of the toric variety $X_{k,n}$ is the normal fan of $\Delta(k, n)$, considered with respect to the lattice $N = \mathbf{Z}^n/\mathbf{Z}(1, 1, \ldots, 1)$. The hypersimplex $\Delta(k, n)$ is an important polytope which appears in many different contexts; for instance, see [11, Section 2.1].

We identify the cones of dimension d in this fan with the faces of codimension d of $\Delta(k, n)$. Here are a few basic facts about the hypersimplex:

- The hypersimplices $\Delta(1, n)$ and $\Delta(n-1, n)$ are regular $(n-1)$-simplices.
- The polytopes (k, n) and $\Delta(n-k, n)$ are isomorphic. Both have $\binom{n}{k} = \binom{n}{n-k}$ vertices.
- For $2 \leqslant k \leqslant n-2$, $\Delta(k, n)$ is an $(n-1)$-dimensional polytope with $2n$ facets. Its facet normals are the directed unit vectors $\pm e_i$ considered modulo $\operatorname{span}(e_1 + \cdots + e_n)$.

The positive-dimensional faces of $\Delta(k, n)$ are labeled by pairs (I, J) where $I, J \subset [n] := \{1, 2, \ldots, n\}$, $I \cap J = \emptyset$, $|I| < k$, and $|J| < n-k$. The face with label (I, J) equals

$$\mathscr{F}_{I,J} = \operatorname{conv}\left\{\left(\sum_{i \in I} e_i\right) + e_{v_1} + \cdots + e_{v_{k-|I|}} : \{v_1, \ldots, v_{k-|I|}\} \in \binom{[n] - (I \cup J)}{k - |I|}\right\}.$$

Thus, the face $\mathscr{F}_{I,J}$ is affinely isomorphic to the hypersimplex $\Delta(k - |I|, n - |I| - |J|)$. In particular, $\operatorname{codim}(\mathscr{F}_{I,J}) = |I| + |J|$, and hence the number of faces of codimension d of $\Delta(k, n)$ equals

$$f(d, k, n) = \sum_{i = \max(0, k+d+1-n)}^{\min(k-1, d)} \binom{n}{i, d-i, n-d} \quad \text{for } 0 < d < n-1. \tag{6}$$

Proposition 3.6. *The Chow cohomology group $A^{n-d-1}(X_{k,n})$ is isomorphic to the space of integer-valued functions c on the faces of codimension d of $\Delta(k, n)$, which satisfy the following relations for all codimension $d-1$ faces $\mathscr{F}_{I,J}$ and all $r, s \in [n] - (I \cup J)$:*

(a) $c(\mathscr{F}_{I \cup \{r\}, J}) + c(\mathscr{F}_{I, J \cup \{s\}}) = c(\mathscr{F}_{I \cup \{s\}, J}) + c(\mathscr{F}_{I, J \cup \{r\}})$, *if* $|I| < k-1$ *and* $|J| < n-k-1$;

(b) $c(\mathscr{F}_{I, J \cup \{s\}}) = c(\mathscr{F}_{I, J \cup \{r\}})$, *if* $|I| = k-1$ *and* $|J| < n-k-1$;

(c) $c(\mathscr{F}_{I \cup \{r\}, J}) = c(\mathscr{F}_{I \cup \{s\}, J})$, *if* $|I| = k-1$ *and* $|J| = n-k-1$.

The proof of this proposition is straightforward using the facet normals $\pm e_i \in N$ of $\Delta(k, n)$ and the fact that all faces of a hypersimplex are again hypersimplices.

We do not know whether there exists a nice general formula for the *Chow Betti numbers* $\beta_{r,k,n} = \operatorname{rank}(A^r(X_{k,n})) = \operatorname{rank}(A_r(X_{k,n}))$. However, it is easy to see that $\beta_{r,1,n} = \beta_{r,n-1,n} = 1$ (because it is a simplex), $\beta_{0,k,n} = \beta_{1,k,n} = 1$ (because all 2-faces are triangles), and $\beta_{n-2,k,n} = n+1$ for $2 \leqslant k \leqslant n-2$. Using direct computation we also determined these numbers

- for $X_{2,4}$: the Chow Betti numbers are $(1, 1, 5, 1)$,
- for $X_{2,5}$: $(1, 1, 6, 6, 1)$,
- for $X_{2,6}$: $(1, 1, 7, 7, 7, 1)$,
- for $X_{3,6}$: $(1, 1, 7, 22, 7, 1)$.

Generalizing the example of the hypersimplex, it would be interesting to study the Chow cohomology of the orbit closures of the torus action on a general flag variety G/P (see e.g. [12]). Do the Chow Betti numbers of these toric varieties have combinatorial or representation-theoretic significance? □

Returning to the case of general toric varieties, we shall briefly discuss the contravariant behavior of the Chow cohomology in terms of Minkowski weights. Let $\psi: N' \to N$ be a homomorphism of lattices, Δ a complete fan in N, and Δ' a complete fan in N', such that each cone τ' in Δ' is mapped under ψ onto a subset of some cone τ in Δ. These data define an equivariant morphism of complete toric varieties $f: X(\Delta') \to X(\Delta)$, and conversely every such morphism arises in this way. On the level of Chow cohomology there is an induced ring homomorphism f^* from $A^*(X(\Delta))$ to $A^*(X(\Delta'))$ [8, p. 324].

If c is a Minkowski weight on Δ, then f^*c is a Minkowski weight on Δ', and the problem is to express f^*c in terms of c. In what follows we consider the special case where $\psi \otimes \mathbf{Q}$ is surjective, or, equivalently, where f is dominant. The general case, which is more difficult, will be addressed in the next section.

PROPOSITION 3.7. *Let $f: X(\Delta') \to X(\Delta)$ be a dominant morphism of complete toric varieties as above, and let $c \in A^k(X(\Delta))$ be a Minkowski weight of codimension k on Δ. Let $\tau' \in \Delta'$ with* $\operatorname{codim}(\tau') = k$, *and let τ be the smallest cone of Δ that contains $\psi(\tau')$. Then*

$$(f^*c)(\tau') = \begin{cases} c(\tau)\cdot[N:\ \psi(N') + N_\tau] & \text{if } \operatorname{codim}(\tau) = k \\ 0 & \text{if } \operatorname{codim}(\tau) < k. \end{cases}$$

Proof. The projection formula for Chow cohomology [8, p. 325] states that

$$f_*(f^*c \cap [V(\tau')]) = c \cap f_*([V(\tau')]) \quad \text{in } A_0(X(\Delta)). \tag{7}$$

By [8, Theorem I.1.4] and [7, p. 56], we have

$$f_*([V[\tau']]) = \begin{cases} [R(V(\tau')):\ R(V(\tau))]\cdot[V(\tau)] & \text{if } \operatorname{codim}(\tau) = k \\ 0 & \text{if } \operatorname{codim}(\tau) < k. \end{cases} \tag{8}$$

The degree of the field extension equals

$$[R(V(\tau')) : R(V(\tau))] = [N/N_\tau : \psi_*(N'/N_{\tau'})] = [N : \psi(N') + N_\tau] \tag{9}$$

where ψ_* is the induced homomorphism from $N'/N_{\tau'}$ to N/N_τ. We substitute (9) into (8) and then into the right-hand side of (7). The assertion now follows by applying the degree homomorphism to both sides of (7). □

4. MULTIPLICATION OF CHOW COHOMOLOGY CLASSES

In this section we describe the ring structure of the Chow cohomology $A^*(X)$ of a complete toric variety $X = X(\Delta)$ in terms of Minkowski weights on its fan Δ. We first recall the relevant results from [3]. Let γ be a cone in $\Delta^{(k)}$ and $V(\gamma)$ the corresponding k-dimensional invariant subvariety of X. We consider the diagonal embedding $\delta: X \to X \times X$ and its restriction to $V(\gamma)$. Using the isomorphism in Corollary 2.2, we can write the diagonal in $V(\gamma) \times V(\gamma)$ as a $\mathbf{Z}$-linear combination

$$[\delta(V(\gamma))] = \sum_{\sigma,\tau} m^{\gamma}_{\sigma,\tau} \cdot [V_\sigma \times V_\tau] \quad \text{in } A_k(V(\gamma) \times V(\gamma)) \tag{10}$$

where the sum is over all pairs σ, $\tau \in \Delta$ such that $\gamma \subset \sigma$, $\gamma \subset \tau$ and $\operatorname{codim}(\sigma) + \operatorname{codim}(\tau) = \operatorname{codim}(\gamma) = \mu$. The coefficients $m^{\gamma}_{\sigma,\tau}$ are generally not unique. It was demonstrated in [3] that knowledge of such coefficients characterizes both the action of the Chow cohomology on the Chow groups—the *cap product*—and the multiplication within the Chow cohomology—the *cup product*. In what follows, we identify elements of $A^*(X)$ with Minkowski weights on Δ, and we set $m^{\gamma}_{\sigma,\tau} = 0$ if $\gamma \not\subset \sigma$ or $\lambda \not\subset \tau$.

PROPOSITION 4.1 [3, Theorem 4]). (a) *If $c \in A^p(X)$ and $\gamma \in \Delta^{(k)}$, then the cap product $c \cap [V(\gamma)]$ in $A_{k-p}(X)$ equals*

$$c \cap [V(\gamma)] = \sum_{(\sigma \cdot \tau) \in \Delta^{(p)} \times \Delta^{(k-p)}} m^{\gamma}_{\sigma,\tau} \cdot c(\sigma) \cdot [V(\tau)].$$

(b) *If $c \in A^p(X)$ and $\tilde{c} \in A^q(X)$, then their cup product $c \cup \tilde{c}$ in $A^{p+q}(X)$ is the Minkowski weight given by the formula*

$$(c \cap \tilde{c})(\gamma) = \sum_{(\sigma \cdot \tau) \in \Delta^{(p)} \times \Delta^{(k-p)}} m^{\gamma}_{\sigma,\tau} \cdot c(\sigma) \cdot \tilde{c}(\tau).$$

Note that if $p = k$ in part (a) then the sum on the right-hand side reduces to one term. Taking degrees on both sides we then get $\deg(c \cap [V(\gamma)]) = c(\gamma)$.

Proposition 4.1 shows that the intersection theory on a toric variety $X = X(\Delta)$ is completely determined by a collection of integers $m^{\gamma}_{\sigma,\tau}$ satisfying formula (10). Our objective is thus reduced to the problem of computing $m^{\gamma}_{\sigma,\tau}$ for all ρ, τ, γ as above. This problem is solved by the following theorem, which we call the "fan displacement rule".

THEOREM 4.2. *If γ is any cone in Δ and v a generic element in the lattice N, then*

$$[\delta(V(\gamma))] = \sum_{\sigma,\tau} m^{\gamma}_{\sigma,\tau} \cdot [V(\sigma) \times V(\tau)]$$

where $m^{\gamma}_{\sigma,\tau} = [N : N_\sigma + N_\tau]$, and the sum is over all pairs (σ, τ) of cones in Δ such that σ meets $\tau + v$, $\operatorname{codim}(\sigma) + \operatorname{codim}(\tau) = \operatorname{codim}(\gamma)$, and $\sigma, \tau \supset \gamma$.

Here "generic" means outside a finite union of proper linear subspaces in $N_\gamma \otimes \mathbf{R}$, to be specified further below. Note that the theorem stated in the introduction is implied by Proposition 4.1 and Theorem 4.2.

Example 4.3. We illustrate the fan displacement rule for computing the cup product of Chow cohomology classes in a simple example. Let X be the Hirzebruch surface $\mathbf{F}_m$ with fan

Δ in $N = \mathbf{Z}^2$ having generators $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, m)$, $v_4 = (0, -1)$, where m is a nonnegative integer. Let c and $\tilde{c}$ be elements of $A^1(X)$, that is, Minkowski weights on the rays of Δ. The condition (3) around the zero-dimensional cone $\tau = \{0\}$ states that

$$c(v_1) = c(v_3), \quad c(v_2) + m \cdot c(v_3) = c(v_4), \quad \tilde{c}(v_1) = \tilde{c}(v_3) \text{ and } \tilde{c}(v_2) + m \cdot \tilde{c}(v_3) = \tilde{c}(v_4).$$

We compute the cup product $c \cup \tilde{c} \in A^2(X) = \mathbf{Z}^1$, using Theorem 4.2 and Proposition 4.1(b). It suffices to determine the number $(c \cup \tilde{c})(\{0\})$. We list six different choices of a generic displacement vector $v \in N$, giving six different but equivalent formulas:

(a) $v = (1, 1)$ gives the formula $(c \cup \tilde{c})(\{0\}) = c(v_1) \cdot \tilde{c}(v_4) + c(v_2) \cdot \tilde{c}(v_3)$.
(b) $v = (-1, 1 + m)$ gives $(c \cup \tilde{c})(\{0\}) = c(v_2) \cdot \tilde{c}(v_1) + c(v_3) \cdot \tilde{c}(v_4)$.
(c) $v = (-1 - m, +1)$ gives $(c \cup \tilde{c})(\{0\}) = c(v_2) \cdot \tilde{c}(v_1) + c(v_3) \cdot \tilde{c}(v_2) + m \cdot c(v_3) \cdot \tilde{c}(v_1)$.
(d) $v = (-1, -1)$ gives $(c \cup \tilde{c})(\{0\}) = c(v_3) \cdot \tilde{c}(v_2) + c(v_4) \cdot \tilde{c}(v_1)$.
(e) $v = (1, -1 - m)$ gives $(c \cup \tilde{c})(\{0\}) = c(v_4) \cdot \tilde{c}(v_3) + c(v_1) \cdot \tilde{c}(v_2)$.
(f) $v = (1 + m, -1)$ gives $(c \cup \tilde{c})(\{0\}) = c(v_1) \cdot \tilde{c}(v_2) + c(v_2) \cdot \tilde{c}(v_3) + m \cdot c(v_1) \cdot \tilde{c}(v_3)$.

In order to prove Theorem 4.2, we need a general lemma for computing certain toric deformations on a toric variety. Consider any toric variety $X = X(\Delta)$ of dimension n, with Δ a fan in $N \simeq \mathbf{Z}^n$ and $T = T_N$ the dense torus in X. Let L be a saturated d-dimensional sublattice of N. It corresponds to a subtorus T_L of T. Let Y be the closure of T_L in X. In what follows, we determine the class $[Y]$ in the Chow homology group $A_d(X)$. Our description will depend on the choice of an element v in N. For any $v \in N$ let

$$\Delta(v) := \{\sigma \in \Delta : L_{\mathbf{R}} + v \text{ meets } \sigma \text{ in exactly one point}\}. \tag{11}$$

Note that $\dim(\sigma) \leq n - d$ for all $\sigma \in \Delta(v)$. We say that v is *generic* if $\dim(\sigma) = n - d$ for all $\sigma \in \Delta(v)$; in this case the unique intersection point of $L_{\mathbf{R}} + v$ and σ lies in the relative interior of σ. Note that the set of generic lattice points in N is Zariski dense in $N_{\mathbf{R}}$.

LEMMA 4.4. *If v is a generic lattice point in N, then*

$$[Y] = \sum_{\sigma \in \Delta(v)} m_\sigma \cdot [V(\sigma)] \quad \textit{in } A_d(X), \quad \textit{where } m_\sigma = [N : L + N_\sigma].$$

Proof of Lemma 4.4. We can assume that $L' = L + \mathbf{Z}v = L \oplus \mathbf{Z}v$ is a saturated sublattice of N. Indeed, since $\Delta(v) = \Delta(v + l)$ and $\Delta(r \cdot v) = \Delta(v)$ for all $l \in L$ and $r \in \mathbf{N}$, we can replace any given v by an element of N which is primitive modulo L.

We first consider the case $d = n - 1$. By the discussion in the previous paragraph, we may assume that $N = L \oplus \mathbf{Z}v$. Let M be the lattice dual to N, and let u be an element of M that is perpendicular to L and satisfies $\langle u, v \rangle = -1$. Consider the rational function $f = x^u - 1$ on X. To prove Lemma 4.3 for $d = n - 1$, it suffices to show the following identity of Weil divisors on X:

$$[\mathrm{div}(f)] = [Y] - \sum_{\sigma \in \Delta(v)} m_\sigma \cdot [V(\sigma)]. \tag{12}$$

Here $m_\sigma = [N : L + N_\sigma] = \min\{m \in \mathbf{N} : \exists l \in L \text{ with } mv + l \in \sigma\}$. The two sides of (12) clearly have the same restriction $[T_L]$ to the torus T. Hence, it suffices to show that, for each ray σ of Δ, the irreducible T-divisor $[V(\sigma)]$ appears with the same coefficient on both sides of (12). Note that $\mathrm{ord}_{V(\sigma)}(f)$, the coefficient of $[V(\sigma)]$ in $[\mathrm{div}(f)]$, is a nonpositive integer since f has no zeros outside of T.

If σ lies in $\Delta(v)$, then the unique point in $(L_{\mathbf{R}} + v)\cap\sigma$ has the form $v + (1/m_\sigma)l$ for some $l\in L$. Then $m_\sigma v + l\in N$ is the primitive lattice point which generates σ, and

$$\mathrm{ord}_{V(\sigma)}(f) = \mathrm{ord}_{V(\sigma)}(x^u) = \langle u, m_\sigma v + l\rangle = -m_\sigma$$

which is the required equation in this case. If σ does not lie in $\Delta(v)$, then the generator of the ray σ has the form $l - pv$ for some $l\in L$ and some nonnegative integer p. If $p > 0$ then $\mathrm{ord}_{V(\sigma)}(x^u) = p > 0$, so $\mathrm{ord}_{V(\sigma)}(f) = 0$, as required; if $p = 0$ then x^u restricts to a nonconstant function on $V(\sigma)$, so again $\mathrm{ord}_{V(\sigma)}(f) = 0$. This completes the proof of Lemma 4.4 in the special case $d = n - 1$.

For the general case we consider a saturated sublattice $L' = L + \mathbf{Z}v = L \oplus \mathbf{Z}v$ of N. Let $\Delta_{L'}$ denote the fan in L' whose cones are the intersections $\sigma\cap L'_{\mathbf{R}}$ for $\sigma\in\Delta$. This gives a toric variety $X' = X(\Delta_{L'})$ and a proper map $X' \to X$. Let Y' be the closure of the torus T_L in X'. By the special case proved above, we have an identity of divisor classes

$$[Y'] = \sum_{\sigma'} m_{\sigma'}[V(\sigma')] \quad \text{in } A_d(X') \tag{13}$$

where σ' runs over all rays of $\Delta_{L'}$ that have the form $\sigma\cap L'_{\mathbf{R}}$ for some $\sigma\in\Delta(v)$. The coefficients in (13) are

$$m_{\sigma'} = [L' : L + N_{\sigma'}] = [L' + N_\sigma : L + N_\sigma].$$

To see that these two indices are equal, apply the Second Isomorphism Theorem to the quotient groups and use the fact that $N_{\sigma'} = N_\sigma\cap L'$.

Next we push (13) forward from X' to X. The torus closure Y' maps birationally onto the torus closure Y, so $[Y']$ maps to $[Y]$. Each $V(\sigma')$ maps to $V(\sigma)$, but the degree of this mapping need not be one. To compute its degree, we consider the restriction of this mapping to the open torus $O_{\sigma'}$ in $V(\sigma')$, which maps onto the torus O_σ in $V(\sigma)$. The surjection of tori $O_{\sigma'} \to O_\sigma$ is determined by the inclusion of lattices

$$L'/N_{\sigma'} = L'/(L'\cap N_\sigma) = (L' + N_\sigma)/N_\sigma \hookrightarrow N/N_\sigma.$$

The degree of this map and hence of the map $V(\sigma') \to V(\sigma)$ equals $[N : L' + N_\sigma]$.

We conclude that the push-forward of (13) into $A_d(x)$ gives $[Y] = \sum_{\sigma\in\Delta(v)} m_\sigma[V(\sigma)]$, with coefficients

$$m_\sigma = [N : L' + N_\sigma]\cdot m_{\sigma'} = [N : L' + N_\sigma]\cdot[L' + N_\sigma : L + N_\sigma] = [N : L + N_\sigma].$$

This completes the proof of Lemma 4.4. □

We remark that, in the case when X is projective, Lemma 4.4 can also be derived from Corollary 2.2 in [13]. The proof given there uses the method of Chow forms. If X is projective and P the corresponding polytope then $\Delta(v)$ is interpreted as the *tight coherent face bundle* on P defined by L and v.

We are now prepared to prove our fan displacement rule.

Proof of Theorem 3.2. We apply Lemma 4.4 to the diagonal embedding $X\hookrightarrow X\times X$. This corresponds to the diagonal inclusion of lattices $\delta : N\hookrightarrow N\times N$. Using a generic vector (v, w) in $N\times N$ gives the same formula in Lemma 4.4 as using the vector $(v - w, 0)$. Therefore, it suffices to translate by vectors $(v, 0)$ for $v\in N$. Points of intersection of the cone $\sigma\times\tau$ with $\delta(N_{\mathbf{R}}) + (v, 0)$ correspond to points of intersection of σ with $\tau + v$. For a generic element

$v \in N$ there is at most one such intersection point whenever $\dim(\tau) + \dim(\sigma) = n$. Lemma 4.4 implies that

$$\delta_*([X]) = \sum m_{\sigma,\tau} \cdot [V(\sigma) \times V(\tau)] \tag{14}$$

the sum over the pairs (σ, τ) such that σ meets $\tau + v$ in one point, with coefficient

$$m_{\sigma,\tau} = [N \times N : \delta(N) + (N_\sigma \times N_\tau)] = [N : N_\sigma + N_\tau]. \tag{15}$$

The last equality is seen by mapping $N \times N$ onto N via $(a, b) \mapsto a - b$, which has kernel $\delta(N)$ and maps $N_\sigma \times N_\tau$ onto $N_\sigma + N_\tau$. This proves Theorem 4.2 for the case $\gamma = \{0\}$.

In the general case we apply the formulas (14) and (15) to the toric subvariety $V(\gamma)$ of X. This means we work in the quotient lattice N/N_γ. The multiplicity (15) remains the same:

$$m^\gamma_{\sigma,\tau} = [N/N_\gamma : (N_\sigma + N_\tau)/N_\gamma] = [N : N_\sigma + N_\tau].$$

This completes the proof of Theorem 4.2. □

The Chow cohomology ring $A^*(X)$ operates on the Chow groups $A_*(X')$ for every morphism $f: X' \to X$. We shall describe this operation for a toric morphism f. Let $\psi : N' \to N$ be a homomorphism of lattices, Δ a complete fan in N, and Δ' a complete fan in N', such that each cone σ' in Δ' is mapped under ψ onto a subset of some cone in Δ. If σ is the smallest cone of Δ containing $\psi(\sigma')$, then the corresponding toric morphism $f: X' = X(\Delta') \to X = X(\Delta)$ satisfies

$$f(V(\sigma')) \subseteq V(\sigma). \tag{16}$$

Given an element $c \in A^k(X)$, i.e. a Minkowski weight of codimension k on Δ, we wish to describe the homomorphisms

$$A_p(X') \to A_{p-k}(X'), \quad z \mapsto f^*c \cap z. \tag{17}$$

To this end we prove the following direct generalization of Theorem 4.2. Let $\delta_f : X' \to X' \times X$ denote the graph of f.

THEOREM 4.5. *If γ' is any cone in Δ' and v a generic element of N, then*

$$[\delta_f(V(\gamma'))] = \sum_{\sigma',\tau} m^{\gamma'}_{\sigma',\tau} [V(\sigma')] \otimes [V(\tau)] \quad \textit{in } A_*(X' \times X) = A_*(X') \otimes A_*(X)$$

where the sum is over all pairs of cones $\sigma' \in \Delta'$ and $\tau \in \Delta$ such that $\gamma' \subset \sigma'$, $\psi(\gamma') \subset \tau$, and $\operatorname{codim}(\sigma') + \operatorname{codim}(\tau) = \operatorname{codim}(\gamma')$, *and*

$$m^{\gamma'}_{\sigma',\tau} = \begin{cases} [N : \psi(N_{\sigma'}) + N_\tau] & \textit{if } \psi(\sigma') \textit{ meets } \tau + v \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. It suffices to prove this formula for the special case $\gamma' = \{0\}$. From this the general case is derived by replacing X' by $V(\gamma')$ and X by $V(\gamma)$, where γ is the smallest cone of Δ containing $\psi(\gamma')$. This is possible by eq. (16).

We apply Lemma 4.4 to the fan $\Delta' \times \Delta$ and the sublattice $L = \operatorname{graph}(\psi)$ of $N' \times N$, with displacement vector $(0, -v) \in N' \times N$. Clearly, the graph of f is the closure in $X' \times X$ of the subtorus associated with L. The translated lattice $L_{\mathbf{R}} - (0, v)$ meets a cone $\sigma' \times \tau$ in $(N' \times N)_{\mathbf{R}}$ in exactly one point if and only if $\psi(\sigma')$ meets $\tau + v$ in $N_{\mathbf{R}}$ in exactly one point. Since v is generic, this condition implies that ψ is injective on σ' and $\dim(\psi(\sigma') +$

$\dim(\tau) = \dim(\sigma') + \dim(\tau) = n$. Finally, the lattice index appearing in Lemma 4.4 simplifies to $[N:\psi(N_{\sigma'}) + N_\tau]$, by an argument analogous to the derivation of (15). □

COROLLARY 4.6. *Let $c \in A^k(X)$. The homomorphism* (17) *is given by the formula*

$$f^*c \cap [V(\gamma')] = \sum_{\sigma',\tau} m_{\sigma',\tau}^{\gamma'} \cdot c(\tau) \cdot [V(\sigma')]$$

for any cone γ' of codimension p in Δ'. Here the sum is over all pairs σ', τ as in Theorem 4.5, *subject to the additional condition that* $\operatorname{codim}(\tau) = k$ *and* $\operatorname{codim}(\sigma') = p - k$.

Proof. This follows immediately from Theorem 4.5 and Corollary 1 in [3]. □

By considering the special case $p = k$ and applying the degree homomorphism on both sides, we obtain a general formula for pullbacks of Minkowski weights.

COROLLARY 4.7. *If $f: X' \to X$ is a morphism of complete toric varieties, then the homomorphism $f^*: A^k(X) \to A^k(X')$ is given (in terms of Minkowski weights) by the formula*

$$(f^*c)(\gamma') = \sum_{\sigma',\tau} m_{\sigma',\tau}^{\gamma'} \cdot c(\tau).$$

The sum is as in Corollary 4.6 *subject to* $\operatorname{codim}(\gamma') = \operatorname{codim}(\tau) = k$, *and* $\operatorname{codim}(\sigma') = 0$.

In the special case where $\psi \otimes \mathbf{Q}$ is surjective (so that f is dominant) we recover Proposition 3.7 from Corollary 4.7. In this case σ' and τ are uniquely determined by γ' and v, and the sum collapses to one term.

5. RELATIONS TO THE POLYTOPE ALGEBRA

In this section we study the rational Chow cohomology $A^*(X)_{\mathbf{Q}}$ of a projective toric variety X. Our objective is to relate $A^*(X)_{\mathbf{Q}}$ to the polytope algebra of McMullen [4–6]. Our main result in this section (Theorem 5.2) expresses the polytope algebra as the direct limit of the Chow cohomology rings of all compactifications of a given torus, thus providing a cohomological interpretation of McMullen's theory.

Structures similar to the polytope algebra have been introduced also by Khovanskii-Pukhlikov [14] and Morelli [15]. Here we restrict ourselves to McMullen's theory, with two minor modifications: we work over $\mathbf{Q}$ instead of $\mathbf{R}$, and we replace $\Pi_0 \simeq \mathbf{Z}$ by $\mathbf{Q}$.

We first review the definitions. The *polytope algebra* Π is a $\mathbf{Q}$-algebra, with a generator $[P]$ for every polytope P in $\mathbf{Q}^n$, and $[\emptyset] = 0$. The generators satisfy the relations

(V) $[P \cup Q] + [P \cap Q] = [P] + [Q]$, whenever $P \cup Q$ is a polytope; and

(T) $[P + t] = [P]$, for all translations $t \in \mathbf{Q}^n$.

The multiplication in Π is given by

(M) $[P] \cdot [Q] := [P + Q]$, where $P + Q = \{p + q : P, q \in Q\}$ is the *Minkowski sum*.

The multiplicative unit is the class of a point: $1 = [\{0\}]$. A basic relation in Π states that $([P] - 1)^{n+1} = 0$. This implies that the *logarithm* of a polytope P is well-defined:

$$\log([P]) = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} ([P] - 1)^r. \tag{18}$$

It is shown in [4] that Π is a graded $\mathbf{Q}$-algebra, $\Pi = \bigoplus_{k=0}^{n} \Pi_k$. The kth graded component Π_k is the $\mathbf{Q}$-vector space spanned by all elements of the form $(\log([P]))^k$, where P runs over all polytopes in $\mathbf{Q}^n$.

We now fix a polytope $P \subset \mathbf{Q}^n$, and we define $\Pi(P)$ to be subalgebra of Π generated by all classes $[Q]$, where Q is a *Minkowski summand* of P (i.e. $P = \lambda Q + R$, for some positive rational λ and some polytope R). Let Δ denote the normal fan of P, and let $X = X(\Delta)$ be the corresponding projective toric variety. Here and throughout this section we identify the lattice N with the standard lattice $\mathbf{Z}^n$ inside $\mathbf{Q}^n$. Note that the algebra $\Pi(P)$ depends only on the fan Δ, and hence it is an invariant of the toric variety X.

Every ample line bundle D on X gives rise to a lattice polytope P_D with normal fan Δ. More generally, if $\mathcal{O}(D)$ is generated by its sections, then we get a polytope P_D whose normal fan is equal to or refined by Δ. The latter condition is equivalent to $[P_D] \in \Pi(P)$. We have the following identification of the polytope subalgebra $\Pi(P)$ with a subalgebra of the rational Chow cohomology of X. The exponential of a divisor class D on X is defined by the familiar formula: $\exp(D) = \sum_{r=0}^{\dim(X)} D^r/r!$.

THEOREM 5.1. *There exists a monomorphism of graded* $\mathbf{Q}$*-algebras* $\theta : \Pi(P) \hookrightarrow A^*(X)_{\mathbf{Q}}$ *such that* $\theta([P_D]) = \exp(D)$ *for every ample divisor* D *on* X. *The image of* θ *equals the subalgebra of* $A^*(X)_{\mathbf{Q}}$ *generated by the Picard group* $Pic(X) \otimes \mathbf{Q} = A^1(X)_{\mathbf{Q}}$.

If P is a simple polytope, or equivalently, if Δ is a simplicial fan, or equivalently, if X is a V-manifold, then the Picard group generates $A^*(X)_{\mathbf{Q}}$ as an algebra. Theorem 5.1 implies that in this special case the map θ is an isomorphism. This isomorphism was already established by McMullen in [5, Theorem 14.1].

Proof. In the proof of Theorem 5.1 we shall extend scalars and work over the field of real numbers $\mathbf{R}$. The reason is that some elementary analytic geometry arguments require irrational constants. These irrationals drop out in the final monomorphism θ (see also the discussion in [5, Section 15]). We fix the standard inner product and Euclidean metric on $\mathbf{R}^n$ and the induced inner product and metric on each affine subspace of $\mathbf{R}^n$.

We recall the definition of weights given in [5, Section 5]. Denote by $\mathcal{F}_k(P)$ the set of k-dimensional faces of the polytope P. A *k-weight* on P is mapping $\omega : \mathcal{F}_k(P) \to \mathbf{R}$ which satisfies the *Minkowski relations*:

$$\sum_{F \subset G} \omega(F) \cdot v_{F,G} = 0 \quad \text{for all } G \in \mathcal{F}_{k+1}(P) \tag{19}$$

where the sum is over all k-faces F of G, and $v_{F,G}$ denotes the unit outer normal vector (parallel to the affine span of G) to G at its facet F. The real vector space of all k-weights on P is denoted by $\Omega_k(P)$.

Every polytope Q with $[Q] \in \Pi(P)$ defines a k-weight ω as follows. We set $\omega(F) := \mathrm{vol}_k(F')$, where F' is the face of Q corresponding to the face F of P, and $\mathrm{vol}_k(\cdot)$ denotes the standard k-dimensional volume form on the affine span of F'. Note that $\omega(F')$ may be zero if the normal fan of Q is strictly coarser than Δ. We write $\omega = \mathrm{vol}_k(Q) \in \Omega_k(P)$, and we call ω the kth *volume weight* of Q. The fact that ω is indeed a weight (i.e. that it satisfies (19)) is the content of Minkowski's classical theorem.

It is shown in [5, Theorem 5.1] that the resulting map

$$\phi : \Pi(P) = \bigoplus_{k=0}^{n} \Pi_k(P) \to \Omega(P) := \bigoplus_{k=0}^{n} \Omega_k(P), \quad [Q] \mapsto \oplus_k \mathrm{vol}_k(Q) \tag{20}$$

is a monomorphism of graded vector spaces. The degree k component of $[Q] = \exp(\log[Q])$ equals $[Q]_k = (1/k!)\,(\log[Q])^k$. Therefore, $\phi(\log([Q])^k) = k! \cdot \mathrm{vol}_k(Q)$.

We next show that $\Omega_k(P)$ is canonically isomorphic to $A^k(X)_{\mathbf{R}}$, the space of real-valued Minkowski weights on the codimension k cones in Δ. Let $\tilde{M} = M \otimes \mathbf{R}$, $\tilde{N} = N \otimes \mathbf{R}$, and identify both spaces with $\mathbf{R}^n$. Suppose $F \in \mathscr{F}_k(P)$ and let $\sigma \in \Delta^{(k)}$ be the normal cone to P at F. We identify $\tilde{M}(\sigma) = M(\sigma) \otimes \mathbf{R}$ with the (vector space parallel to the) affine span of F. Let $\{m_1, \dots, m_k\}$ be an orthonormal basis for $\tilde{M}(\sigma)$. The orthogonal projection onto $\tilde{M}(\sigma)$ defines an isomorphism of vector spaces

$$\mathrm{proj}_\sigma : \tilde{N}/\tilde{N}_\sigma \to \tilde{M}(\sigma)$$

$$v \mapsto \sum_{i=1}^{k} \langle v, m_i \rangle \cdot m_i. \tag{21}$$

Let Vol_σ denote the volume form on $\tilde{M}(\sigma)$ which is normalized with respect to the lattice $M(\sigma)$, i.e. $\mathrm{Vol}_\sigma(T) = 1$ for every primitive lattice simplex T in $M(\sigma)$. We define a real constant v_σ using the ratio between the normalized volume and the standard k-volume

$$\mathrm{Vol}_\sigma(\cdot) = v_\sigma \cdot k! \cdot \mathrm{vol}_k(\cdot). \tag{22}$$

Note that v_σ is typically irrational and that $0 < v_\sigma \leqslant 1$.

We claim that the map

$$\psi_k : A^k(X)_{\mathbf{R}} \to \Omega_k(P) \quad \text{given by } \psi_k(c)(\sigma) := c(\sigma)/v_\sigma \tag{23}$$

is well-defined and is a vector space isomorphism. To see this, suppose that $\tau \in \Delta^{(k+1)}$ is contained in σ. The generator $n_{\sigma,\tau}$ of N_σ modulo N_τ satisfies

$$\| \mathrm{proj}_\tau(n_{\sigma,\tau}) \| = v_\tau / v_\sigma. \tag{24}$$

Let $c \in A^k(X)$ be any Minkowski weight. Then the relation (3) translates into

$$\sum_{\sigma \supset \tau} \psi_k(c)(\sigma) \cdot \frac{\mathrm{proj}_\tau(n_{\sigma,\tau})}{\| \mathrm{proj}_\tau(n_{\sigma,\tau}) \|} = \frac{1}{v_\tau} \cdot \sum_{\sigma \supset \tau} c(\sigma) \cdot \mathrm{proj}_\tau(n_{\sigma,\tau}) = 0 \quad \text{for all } \tau \in \Delta^{(k+1)}.$$

This shows that $\psi_k(c)$ satisfies (19), with $G \supset F$ the faces of P polar to $\tau \supset \sigma$. Therefore, $\psi_k(c)$ lies in $\Omega_k(P)$. This argument also shows that ψ_k is an isomorphism.

If we combine the maps ψ_k for all k, then we get a graded vector space isomorphism $\psi : A^*(X)_{\mathbf{R}} \simeq \Omega(P)$. Let $\phi : \Pi(P) \hookrightarrow \Omega(P)$ be as in (20). The composition $\theta := \psi^{-1} \circ \phi$ is a monomorphism of graded vector spaces from $\Pi(P)$ into $A^*(X)_{\mathbf{R}}$.

Let D be any ample divisor on X, and P_D the corresponding lattice polytope in $\tilde{M}$. The element D^k of $A^k(X)$ is represented by the Minkowski weight $\sigma \mapsto \mathrm{Vol}_\sigma(F)$, where F is the k-face of P_D polar to $\sigma \in \Delta^{(k)}$. This follows from the corollary on p. 112 of [7]. From (22) and (23) we derive

$$\psi(D^k) = k! \cdot \mathrm{vol}_k(P_D) = \phi(\log[P_D])^k. \tag{25}$$

This implies that $\theta(\log([P_D])^k) = D^k$, and therefore $\theta([P_D]) = \exp(D)$ in $A^*(X)_{\mathbf{R}}$.

The Picard group $A^1(X)_{\mathbf{R}}$ is spanned by the ample divisors on X. This proves that the degree 1 component of the map θ is surjective, and hence it is a vector space isomorphism:

$$\theta_1 : \Pi_1(P) \simeq A^1(X)_{\mathbf{R}}, \qquad \log([P_D]) \mapsto D. \tag{26}$$

The algebra $\Pi(P)$ is spanned as a vector space by the classes $[P_D]$, where D runs over all ample divisors on X. Our monomorphism θ is multiplicative on these generators:

$$\theta([P_D] \cdot [P_{D'}]) = \theta([P_D + P_{D'}]) = \theta([P_{D+D'}]) = \exp(D + D') = \exp(D) \cdot \exp(D').$$

In view of (26), this completes the proof of Theorem 5.1. $\square$

Let $\mathscr{Y}$ denote the family of all n-dimensional complete toric varieties, that is, all compactifications of an n-dimensional algebraic torus T_N. The corresponding family of Chow cohomology rings $\{A^*(X)\}_{X\in\mathscr{Y}}$ is a directed system: for every equivariant morphism $f: X_1 \to X_2$ of n-dimensional complete toric varieties there is an inclusion of rings $f^*: A^*(X_2) \hookrightarrow A^*(X_1)$; the fact that this is an inclusion follows from the fact that the Chow cohomology groups are torsion free. We can thus form the direct limit $\varinjlim A^*(X) := \varinjlim \{A^*(X)\}_{X\in\mathscr{Y}}$. This limit of Chow cohomology rings can be viewed as a universal ring for intersection theory on all compactifications of a fixed torus. The following theorem shows that this direct limit is an integral version of the polytope algebra.

THEOREM 5.2. *The algebra* $\varinjlim A^*(X)_{\mathbf{Q}}$ *is isomorphic to the polytope algebra* Π.

Proof. Consider a morphism of n-dimensional complete toric varieties $f: X_1 \to X_2$. Let P_i be the polytope associated with X_i, and let $\theta^{(i)}: \Pi(P_i) \hookrightarrow A^*(X_i)_{\mathbf{Q}}$ be the inclusion of Theorem 5.1. Since P_2 is a Minkowski summand of P_1, we have a natural inclusion $i: \Pi(P_2) \hookrightarrow \Pi(P_1)$. We claim that $f^* \circ \theta^{(2)} = \theta^{(1)} \circ i$. To see this, let D be a divisor on X_2 which is generated by its sections, and let P_D be the corresponding lattice polytope. This defines a divisor f^*D on X_1 having the same polytope $P_{f^*D} = P_D$, and therefore $i([P_D]) = [P_D] = [P_{f^*D}]$ in $\Pi(P_1)$. In $A^*(X_1)_{\mathbf{Q}}$ we get the described relation

$$\theta^{(1)}(i([P_D])) = \exp(f^*D) = f^*(\exp(D)) = f^*(\theta^{(2)}([P_D]))$$

Here f^* commutes with "exp" because it is a ring homomorphism.

The polytope algebra Π clearly equals the direct limit of the finitely generated algebras $\Pi(P)$, with respect to the inclusions $\Pi(P_2) \hookrightarrow \Pi(P_1)$ whenever P_2 is a Minkowski summand of P_1. The discussion in the previous paragraph shows that there is a monomorphism θ from the polytope algebra Π into $\varinjlim A^*(X)_{\mathbf{Q}}$. It is compatible with all monomorphisms $\theta^{(i)}$ as in Theorem 5.1. To see that θ is surjective, it suffices to consider equivariant resolution of singularities: for every X_2 there exists a morphism $f: X_1 \to X_2$ such that X_1 is smooth and hence $\theta^{(1)}$ is an isomorphism. This proves that θ is an isomorphism. $\square$

In Section 4 we saw that the cup product in $A^*(X)$ can be computed by a simple rule involving a generic displacement of the fan Δ. We shall now show that in the projective case our rule is equivalent to the following "mixed volume computation", which was introduced by McMullen in [5, Section 5, p. 426]. Let F and G be polytopes in $\mathbf{R}^n$ such that $\dim(F + G) = \dim(F) + \dim(G)$. Then there exists a unique real number $\alpha_{F,G}$, which depends only on (the "angle" between) the affine subspaces spanned by F and G, such that

$$\operatorname{vol}(F + G) = \alpha_{F,G} \cdot \operatorname{vol}(F) \cdot \operatorname{vol}(G) \tag{27}$$

with respect to the standard volume forms in the respective affine subspaces. If P is an n-dimensional polytope in $M_{\mathbf{R}} = \mathbf{R}^n$, then each sufficiently generic linear functional v on M defines a *mixed decomposition* Δ_v of the Minkowski sum $2P = P + P$. We recall the definition of Δ_v; for further details see e.g. [16]. Consider the polytope $\widehat{2P} = \{(p + p', v(p)) \in \mathbf{R}^{n+1}: p, p' \in P\}$. A face of $\widehat{2P}$ is a *lower face* if it has an outer normal vector with negative last coordinate. Each lower face projects onto a subpolytope of $2P$ of the form $F + G$, where F, G are faces of P, and we have $\dim(F + G) = \dim(F) + \dim(G)$ by the genericity of v. The set of all such polytopes $F + G$ is a polyhedral decomposition of $2P$, which we denote by Δ_v and call the *mixed decomposition* of $P + P$ defined by v.

PROPOSITION 5.3 (McMullen [6]). *Let x_1, x_2 be elements in the polytope algebra $\Pi(P)$, given by their weights $\omega_i = \phi(x_i) \in \Omega(P)$. Then the weight of their product $\omega = \phi(x_1 x_2)$*

satisfies

$$\omega(H) = \sum \alpha_{F,G} \cdot \omega_1(F) \cdot \omega_2(G) \quad \textit{for all } H \in \mathscr{F}_{p+q}(P) \tag{28}$$

where the sum is over all faces F, G of H with $\dim(F) + \dim(G) = \dim(H)$ *and* $F + G \in \Delta_v$.

The fact that the formula (28) defines a weight, i.e. that ω satisfies (19) whenever ω_1 and ω_2 do, is a nontrivial statement about polytopes. This was proved by McMullen in [6]. It can also be derived from Theorem 5.1 and our results in Section 4, as follows.

Proof. Let $c \in A^p(X)_{\mathbf{R}}$, $\tilde{c} \in A^q(X)_{\mathbf{R}}$ and $c \cup \tilde{c} \in A^{p+q}(X)_{\mathbf{R}}$ their cup product, be represented by Minkowski weights on Δ. Our fan displacement rule in Theorem 4.2 states

$$(c \cup \tilde{c})(\gamma) = \sum m^{\gamma}_{\sigma,\tau} \cdot c(\sigma) \cdot \tilde{c}(\tau) \quad \text{for all } \gamma \in \Delta^{(p+q)} \tag{29}$$

where $m^{\gamma}_{\sigma,\tau} = [N : N_\sigma + N_\tau]$ and the sum is over all pairs of cones $\sigma \in \Delta^{(p)}$ and $\tau \in \Delta^{(q)}$ such that $\gamma \subset \sigma \cap \tau$ and $(\sigma + v) \cap \tau \neq \emptyset$.

Let F, G, H be the faces of P corresponding to σ, τ, γ. Obviously, the condition $F, G \subset H$ is equivalent to $\gamma \subset \sigma \cap \tau$. The "angle of linear subspaces" defined above satisfies

$$\alpha_{F,G} = m^{\gamma}_{\sigma,\tau} \cdot \frac{v_\sigma \cdot v_\tau}{v_\gamma}. \tag{30}$$

We set $c = \psi(\omega_1)$, $\tilde{c} = \psi(\omega_2)$, where ψ is the isomorphism in (23). Using this isomorphism and (30), we see that (29) is equivalent to (28). The proof of Proposition 5.3 is now completed by the following lemma.

LEMMA 5.4. *Let F, G be faces of P with normal cones σ, τ, and let $v \in N$ be generic. Then $(\sigma + v) \cap \tau \neq \emptyset$ if and only if $F + G$ is a face of the mixed decomposition Δ_v.*

Proof of Lemma 5.4. We have $F + G \in \Delta_v$ if and only if $\widehat{F+G} = \{(p+q, v(p)) \in \mathbf{R}^{n+1} : p \in F, q \in G\}$ is a lower face of $\widehat{2P}$, if and only if there exists a linear functional $l \in N_{\mathbf{R}} = (\mathbf{R}^n)^*$ such that $((l, 1) \in \mathbf{R}^{n+1})^*$ attains its minimum over $\widehat{2P}$ at $\widehat{F+P}$. This holds if and only if l attains its minimum over P at G, and $l + v$ attains its minimum over P at F, or, equivalently, $l \in \sigma$ and $l + v \in \tau$. □

6. THE TODD WEIGHT OF A SMOOTH TORIC VARIETY

The purpose of this section is to illustrate the role of Minkowski weights in the context of a prominent application of toric varieties, namely, counting lattice points (see [7, Section 5.3] and the references given there).

Let X be a nonsingular projective toric variety of dimension n. In this situation the maps $A_*(X) \to H_*(X)$ and $A^*(X) \to H^*(X)$ are both isomorphisms, if we take the ground field to be $\mathbf{C}$. Therefore, the isomorphism in Proposition 2.4 reduces to the Kronecker isomorphism $H^*(X) \simeq Hom(H_*(X), \mathbf{Z})$ from topology. We recall from [8] that any variety X has a Todd homology class td_X in $A_*(X)_{\mathbf{Q}}$. If X is nonsingular, then $td_X = Td_X \cap [X]$, where Td_X is the Todd cohomology class in $A^*(X)_{\mathbf{Q}}$.

Let Δ be the unimodular fan of X in $N \simeq \mathbf{Z}^n$. By Theorem 3.1 the Todd cohomology class Td_X is presented by a Minkowski weight Td_Δ on Δ. We call Td_Δ the *Todd weight*. It is our objective to express Td_Δ in terms of a certain multivariate Ehrhart polynomial Φ.

Let $v_1, \ldots, v_d$ denote the primitive lattice points in N along the rays of Δ. Each v_i corresponds to a divisor D_i on X. Let $\mathscr{K}(\Delta)$ be the family of all lattice polytopes P such that Δ equals or refines the normal fan of P. The polytopes in $\mathscr{K}(\Delta)$ are of the form

$$P = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{x} \cdot v_1 \geqslant -a_1, \ldots, \mathbf{x} \cdot v_d \geqslant -a_d\} \tag{31}$$

where $(a_1, \ldots, a_d)$ runs over all lattice points in a certain closed convex cone in $\mathbf{R}^d$. They are in bijection with the divisors $D = a_1 D_1 + \cdots a_d D_d$ on X whose line bundles are generated by their sections. Let $\Phi = \Phi(a_1, \ldots, a_d)$ denote the number of lattice points in the polytope P in (31). We are interested in this number as a function of the parameters $a_1, \ldots, a_d$.

PROPOSITION 6.1. (a) *The function Φ is polynomial of degree n in the variables $a_1, \ldots, a_d$.*
(b) *If a monomial $a_1^{e_1} a_2^{e_2} \cdots a_d^{e_d}$ appears in the expansion of Φ, then the generators v_j that occur with positive exponent $e_j > 0$ span a cone in Δ.*

Proof. The rational Chow ring of X equals $A^*(X)_{\mathbf{Q}} = \mathbf{Q}[x_1, \ldots, x_d]/I$, where I is the ideal generated by all linear relations $\sum_{i=1}^d \langle m, v_i \rangle \cdot x_i$ where $m \in M$, and all square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$ such that $v_{i_1}, \ldots, v_{i_r}$ do not span a cone in Δ. Let Td_X denote the Todd cohomology class of X. By the Hirzebruch–Riemann–Roch theorem, we have

$$\Phi = \int \exp(D) \cdot Td_X = \sum_{i=0}^{n} \frac{1}{i!} \int D^i \cdot Td_X. \tag{32}$$

This formula implies part (a) of Proposition 6.1 because $\int D^i \cdot Td_X$ is a homogeneous polynomial of degree i in $a_1, \ldots, a_d$. To prove part (b) we consider the expansion

$$D^i = (a_1 x_1 + \cdots + a_d x_d)^i = \sum_{v_1, \ldots, v_i} a_{v_1} \cdots a_{v_i} x_{v_1} \cdots x_{v_i}.$$

After reducing modulo the square-free monomial relations in I, the sum on the right-hand side contains only monomials $a_{v_1} \cdots a_{v_i}$ which are supported on Δ. The Todd class Td_X is a polynomial in $x_1, \ldots, x_d$ with rational coefficients. Therefore, the expression (32) is a $\mathbf{Q}$-linear combination of monomials $a_{v_1} \cdots a_{v_i}$ which are supported on Δ. □

We remark that Proposition 6.1 can also be derived from the results in [14]. For a generalization of the polynomial Φ to arbitrary complete schemes see [8, Example 18.3.6]. In what follows, we identify $A^*(X)_{\mathbf{Q}}$ with the ring of $\mathbf{Q}$-valued Minkowski weights on Δ, and also with the quotient of $\mathbf{Q}[x_1, \ldots, x_d]$ modulo I. The following easy lemma explicates the isomorphism between the two graded algebras.

LEMMA 6.2. *Let p be a homogeneous element of degree i in $\mathbf{Q}[x_1, \ldots, x_d]$ representing a class in $A^i(X)_{\mathbf{Q}}$. Then the corresponding Minkowski weight equals*

$$c: \Delta^{(i)} \to \mathbf{Q}, \qquad \sigma \mapsto \int p(x_1, \ldots, x_d) \cdot \prod_{v_j \in \sigma} x_j. \tag{33}$$

We are now prepared to relate the polynomial Φ to the Todd weight Td_Δ.

THEOREM 6.3. *If σ is a cone of codimension i in the fan Δ, then $Td_\Delta(\sigma)$ equals the coefficient of the square-free monomial $\prod_{v_j \in \sigma} a_j$ in the polynomial $\Phi(a_1, \ldots, a_d)$.*

Proof. Let Td_X^i denote the degree i component of the Todd class, represented by a polynomial of degree i in $x_1, \ldots, x_d$. The coefficient of interest can be computed by applying the differential operator $\prod_{v_j \in \sigma} \partial / \partial a_j$ to the degree $n - i$ component of the polynomial $\Phi(a_1, \ldots, a_d)$. By (32), the degree $n - i$ component equals

$\int (1/(n-i)!)(a_1x_1 + \cdots + a_dx_d)^{n-i} Td_X^i$. By differentiating under the integral sign, we get precisely the expression in (33) for $p = Td_X^i$. This proves the claim. □

COROLLARY 6.4. *The lattice point enumerator $\Phi(a_1, \ldots, a_d)$ is uniquely determined by the coefficients of its square-free terms.*

Proof. This follows from Theorem 6.3 and the formula (32) because the Todd class Td_X is uniquely determined by the values of the Todd weight Td_Δ. □

Consider the polytope P as in (31) and the corresponding divisor D. The Minkowski weight corresponding to $\exp(D)$ is denoted $[P]$ and call the *volume weight*. This notation is consistent with McMullen's polytope algebra. By our results in the previous section, the volume weight satisfies $[P](\sigma) = \mathrm{Vol}_\sigma(P^\sigma)$ for all $\sigma \in \Delta$. Thus, the formula (32) can be rewritten as $\Phi = \int [P] \cdot Td_\Delta$. This means that the polynomial $\Phi(a_1, \ldots, a_d)$ can be computed by the "fan displacement rule" presented in Section 4, namely, by multiplying the Todd weight Td_Δ with the volume weight $[P]$. We illustrate this in a small example.

Example 6.5 ($n = 2$, $d = 4$, *Hirzebruch surfaces*). Consider the two-dimensional fan Δ with four generators $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, m)$ for a nonnegative integer m, and $v_4 = (0, -1)$. Then P is the (possibly degenerate) quadrangle defined by the inequalities $x \geqslant -a_1, a_4 \geqslant y \geqslant -a_2$, and $-x + my \geqslant -a_3$. To ensure that P has the correct normal fan (equivalently, $P \in \mathscr{K}(\Delta)$), we need to assume $a_1 + a_3 \geqslant ma_2$ and $a_2 + a_4 \geqslant 0$.

We wish to count the number $\Phi(a_1, a_2, a_3, a_4)$ of lattice points in P. The volume weight $[P]$ equals

$$\{0\} \mapsto a_1a_2 + a_2a_3 + a_3a_4 + a_4a_1 - \frac{m}{2}a_2^2 + \frac{m}{2}a_4^2,$$

$$\{v_1\} \mapsto a_2 + a_4,\ \{v_2\} \mapsto a_1 + a_3 - ma_2,\ \{v_3\} \mapsto a_2 + a_4,\ \{v_4\} \mapsto a_1 + a_3 + ma_4,$$

$$\{v_1, v_2\} \mapsto 1,\ \{v_2, v_3\} \mapsto 1,\ \{v_3, v_4\} \mapsto 1,\ \{v_1, v_4\} \mapsto 1.$$

The Todd weight Td_Δ equals

$$\{0\} \mapsto 1,\ \{v_1\} \mapsto 1,\ \{v_2\} \mapsto 1 - m/2,\ \{v_3\} \mapsto 1,\ \{v_4\} \mapsto 1 + m/2$$

$$\{v_1, v_2\} \mapsto 1,\ \{v_2, v_3\} \mapsto 1,\ \{v_3, v_4\} \mapsto 1,\ \{v_1, v_4\} \mapsto 1.$$

Applying the "fan displacement rule" to these two weights on Δ, we get the formula

$$\Phi = a_1a_2 + a_2a_3 + a_3a_4 + a_1a_4 - \frac{m}{2}a_2^2 + \frac{m}{2}a_4^2 + a_1 + \left(1 - \frac{m}{2}\right)a_2 + a_3 + \left(1 + \frac{m}{2}\right)a_4 + 1.$$

In closing we comment on the general case when X is singular. The preceding discussion extends to the case where Δ is an arbitrary complete simplicial fan. One still has a Todd cohomology class Td_X in $A^*(X)_\mathbb{Q}$, and the degree of $\exp(D) \cdot Td_X$ counts the number of lattice points in the polytope P, but only when $D = \sum a_iD_i$ is a *Cartier divisor*. For singular varieties, including this simplicial case, the lattice point enumerator Φ is generally not a polynomial but only a quasi-polynomial. If the fan is not simplicial, moreover, there may be no Todd weight at all, as in the following example.

Example 6.6. A three-dimensional complete projective toric variety $X = X(\Delta)$ whose Todd homology class td_X is not in $A^*(X)_\mathbb{Q} \cap [X]$. Thus, there is no Todd weight on Δ.

The Todd homology class of a toric variety satisfies $td_{n-1}(X) = \frac{1}{2}\sum_{i=1}^d [D_i]$, where the D_i are the divisors corresponding to the rays of the fan; see [7, Section 5.3]. Thus, it suffices to find a projective toric variety X such that $\sum_i [D_i]$ is not in the image of $A^1(X)_{\mathbf{Q}} = Pic(X)_{\mathbf{Q}}$ under taking the cap product with the fundamental class $[X]$. This is equivalent to saying that $\sum_i [D_i]$ is not a $\mathbf{Q}$-Cartier divisor, or that there is no piecewise linear function ψ on N whose values on all the primitive generators v_i of the rays of the fan are equal.

Let P be the pyramid in $M = \mathbf{Z}^3$ with apex $(0, 0, 1)$ and a random quadrilateral base, say, with vertices $(2, 1, -1)$, $(1, -1, -1)$, $(-3, -2, -1)$, and $(-1, 1, -1)$. Let Δ be the normal fan of P and $\sigma \in \Delta$ the normal cone at the apex. The primitive generators of σ are $(-4, 2, -3)$, $(0, -2, -1)$, $(6, -4, -5)$, and $(-2, 8, -5)$ in $N = \mathbf{Z}^3$. There is no nonzero element u in M with equal values on these four vectors, since

$$\det \begin{pmatrix} -4 & 2 & -3 & 1 \\ 0 & -2 & -1 & 1 \\ 6 & -4 & -5 & 1 \\ -2 & 8 & -5 & 1 \end{pmatrix} = 176 \neq 0.$$

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