Value Distribution for Sequences of Rational Mappings and Complex Dynamics

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ABSTRACT. We study pre-images under the iterates P^k of a rational (not necessarily holomorphic) mapping P of $\mathbb{P}^n$. We show, assuming a condition on the topological degree λ of P, that there is a probability measure μ on $\mathbb{P}^n$ and a pluripolar set $\mathcal{E} \subset \mathbb{P}^n$ such that $\lambda^{-k} P^{k*}\nu \to \mu$ for all probability measures ν on $\mathbb{P}^n \setminus \mathcal{E}$. We also obtain results on the asymptotic equidistribution of the pre-images of linear subspaces for sequences of rational mappings between projective spaces.

1. INTRODUCTION

In this paper, we investigate the asymptotic equidistribution of pre-images under the iterates of a rational mapping of $\mathbb{P}^n$ and also under an arbitrary sequence of rational mappings with rapidly increasing degrees. The study of value distribution for sequences of mappings may be regarded as an analogue of Nevanlinna theory; instead of studying the asymptotic behavior of the area of a pre-image of an analytic set in a ball when its radius tends to infinity, one investigates the asymptotics of the pre-images of an analytic set under a sequence of mappings. The prototype for our investigation is the following theorem on the convergence of the iterated pull-backs of a measure by a rational function, proved independently by Lyubich [Ly] and by Freire-Lopes-Mañé [FLM] in 1983 (see also [ES, HP]):

Let $R(z)$ be a rational function of degree $\lambda \geq 2$, let R^k denote its k-th iterate, and let $\mathcal{E}$ be the set of points of $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ that have only finitely many iterated pre-image points. Then there is a probability measure μ on $\mathbb{P}^1$, such that for all probability measures ν on $\mathbb{P}^1 \setminus \mathcal{E}$,

$$\frac{1}{\lambda^k} R^{k*}\nu \to \mu \, .$$

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This result was proved by Brolin [Br] in 1965 for iterates of polynomials. The limit measure μ is the unique probability measure on $\mathbb{P}^1 \setminus \mathcal{E}$ that is invariant in the sense that $R^*\mu = \lambda\mu$. (The support of μ is the Julia set of R.) Recall that $\mathcal{E}$ contains at most 2 points; in fact, $\mathcal{E}$ is empty unless R is conjugate to a polynomial or to the map $z \mapsto z^{-\lambda}$. In particular, if we let ν be the delta measure at any point of $\mathbb{P}^1 \setminus \mathcal{E}$, we obtain the asymptotic equidistribution of the pre-images of points (with at most 2 exceptional values).

Several recent papers on holomorphic dynamics in several complex variables (see, for example, [BS1, BS2, BS3, BLS1, BLS2, FS1]) have dealt with various extensions of the Lyubich-Freire-Lopes-Mañé Theorem to polynomial biholomorphisms of $\mathbb{C}^2$. For holomorphic mappings P of complex projective n-space $\mathbb{P}^n$ of degree $\delta \geq 2$, it was shown by Hubbard and Papadopol [HP] and by Fornaess and Sibony [FS2, FS3, FS4] that $\delta^{-k}\log|P^k|$ (where P^k is the k-th iterate of P) converges uniformly to a Green function G on $\mathbb{C}^{n+1}$; see also Ueda [Ue]. Since $dd^c \log|P^k|^2 = P^{k*}\omega$ where ω is the Kähler form of the Fubini-Study metric on $\mathbb{P}^n$ (normalized so that $\int_{\mathbb{P}^n}\omega^n = 1$), it follows by Bedford-Taylor [BT] that

$$\frac{1}{\delta^{nk}}P^{k*}\omega^n \to (2dd^cG)^n = \mu\ ,$$

where μ is a probability measure on $\mathbb{P}^n$ which is invariant in the sense that $\delta^{-n}P^*\mu = \mu$. It then easily follows from the equidistribution theorem in [RS] (see below) that there is a pluripolar set $\mathcal{E} \subset \mathbb{P}^n$ such that for all probability measures ν on $\mathbb{P}^n \setminus \mathcal{E}$,

$$\frac{1}{\delta^{nk}}P^{k*}\nu \to \mu\ .$$

Fornaess and Sibony [FS4] also showed that for certain meromorphic maps $P : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of degree δ, the sequence $\{\delta^{-k}P^{k*}\omega\}$ converges to an invariant positive $(1,1)$-current.

However, a different approach is needed to obtain convergence of iterated pull-back measures for rational, non-holomorphic maps. Let P be a rational map of $\mathbb{P}^n$ given by polynomials of degree δ, and let λ denote its topological degree. One easily sees that the total mass of the measure $P^{k*}\omega^n$ equals λ^k. If P is not holomorphic, then $\lambda < \delta^n$ (see Lemma 4.4) so that $\delta^{-nk}P^{k*}\omega^n \to 0$, and it follows that if the Green function G exists, then $(dd^cG)^n$ vanishes on any domain where the iterates P^k are holomorphic (see Examples 2 and 4 in Section 2). Thus the methods of [HP, FS4, RS] do not yield a useful limit measure in the meromorphic case. Our main result (Theorem 1.1) gives the existence of a limit measure μ and a pluripolar set $\mathcal{E}$ such that $\lambda^{-k}P^{k*}\nu \to \mu$ for all probability measures ν on $\mathbb{P}^n \setminus \mathcal{E}$, provided that $\lambda > \delta^{n-1}$. A new ingredient in our proof of Theorem 1.1 is a second derivative bound (Lemma 5.1) on a "proximity function" given as a fiber integral of a pull-back of the Levine form. In fact, Theorem 1.1 has a slightly weaker hypothesis on λ expressed in terms of an "intermediate dynamic degree" of P, which we now describe.

Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map. For $1 \leq \ell \leq \min(n,m)$, we let $\delta_\ell(P)$ denote the degree of $P^{-1}(W)$ for a generic projective $(m-\ell)$-plane $W \subset \mathbb{P}^m$. (For generic W, the algebraic variety $P^{-1}(W)$ has pure codimension ℓ and its degree is the number of points in its intersection with a generic ℓ-plane in $\mathbb{P}^n$.) We also set $\delta_0(P) = 1$. One easily sees that $\delta_1(P)$ is the degree of the polynomials in a representation of P (using homogeneous coordinates). We now suppose that $n = m$; then $\delta_n(P)$ is the topological degree of P; i.e., the number of pre-images of a generic point. If P is holomorphic (i.e., regular), then $\delta_\ell(P^k) = \delta_1(P)^{\ell k}$ for $0 \leq \ell \leq n,\ k \geq 1$. In Section 4, we give an analytic description of the "intermediate degrees" $\delta_\ell(P)$ and we show that $\delta_\ell(P^{j+k}) \leq \delta_\ell(P^j)\delta_\ell(P^k)$ and $\delta_{k+\ell}(P) \leq \delta_k(P)\delta_\ell(P)$. Thus we can define the "intermediate dynamic degrees"

$$\lambda_\ell(P) = \lim_{k\to\infty} \delta_\ell(P^k)^{1/k} = \inf_{k\geq 1} \delta_\ell(P^k)^{1/k} \leq \delta_\ell(P) \leq \delta_1(P)^\ell \, .$$

Intermediate degrees were also considered by Friedland [Fr] in the context of their relationship with topological entropy.

The following is our main result on limit measures and equidistribution for the iterates of a rational mapping of projective space:

***Theorem* 1.1.** *Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map, let $\lambda = \delta_n(P)$ denote the topological degree of P, and let P^k denote the k-th iterate of P. If $\lambda > \lambda_{n-1}(P)$, then there exist a probability measure μ on $\mathbb{P}^n$ and a pluripolar subset $\mathcal{E} \subset \mathbb{P}^n$ such that, for all probability measures ν on $\mathbb{P}^n$ with $\nu(\mathcal{E}) = 0$,*

$$\frac{1}{\lambda^k} P^{k*}\nu \to \mu$$

weakly (*in the space of measures*) *as $k \to \infty$.*

A subset S of $\mathbb{P}^n$ is said to be pluripolar if it it is contained in the $-\infty$ locus of a quasi-plurisubharmonic function on $\mathbb{P}^n$ (or equivalently, if the cone over S is contained in the $-\infty$ locus of a plurisubharmonic function on $\mathbb{C}^{n+1}$.) The pull-back measures $P^{k*}\nu$ of Theorem 1.1 are defined as follows. For a rational map $Q : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, we write

$$E_Q = \{W \in \mathbb{P}^n : \dim Q^{-1}(W) > 0\} \, ,$$

where $Q^{-1}(W)$ denotes the set of $Z \in \mathbb{P}^n$ such that (Z, W) is in the graph of Q. If ν is a finite measure on $\mathbb{P}^n$ with $\nu(E_Q) = 0$, there is a unique finite measure $Q^*\nu$ on $\mathbb{P}^n$ given by

$$(Q^*\nu, \varphi) = \int_{\mathbb{P}^n \setminus E_Q} Q_*\varphi d\nu \, , \quad \varphi \in \mathcal{C}(\mathbb{P}^n) \, .$$

The function $Q_*\varphi \in \mathcal{C}(\mathbb{P}^n \setminus E_Q)$ is given by $Q_*\varphi(W) = \sum_j \varphi(Z_j)$, where $Z_1, \ldots, Z_{\delta_n(Q)}$ are the points of $Q^{-1}(W)$ repeated according to their multiplicities. (Alternately, $Q_*\varphi$ can be given by first lifting φ to the graph of Q and then pushing it forward as a current.) The exceptional set $\mathcal{E}$ in Theorem 1.1 should be taken to include the algebraic subvarieties E_{P^k} (which have codimension at least 2) so that the pull-back measures $P^{k*}\nu$ are well-defined.

Theorem 1.1 gives an extension of the Lyubich-Freire-Lopes-Mañé Theorem to n variables with a pluripolar exceptional set $\mathcal{E}$ replacing the exceptional set of at most two points in the one-variable case. In general, one cannot expect $\mathcal{E}$ to be finite when $n \geq 2$, since for the meromorphic case, $\mathcal{E}$ must contain all the algebraic subvarieties E_{P^k}, $k \geq 1$. Simple examples of holomorphic maps give $\mathcal{E}$ as a finite union of hyperplanes and it is an open problem whether $\mathcal{E}$ can always be chosen to be algebraic if P is holomorphic. (See Example 1 and Question 1 in Section 2.)

By letting ν be the delta measure δ_W of a point W, we obtain the asymptotic equidistribution of the pre-images of points,

$$\frac{1}{\lambda^k} P^{k*}\delta_W \to \mu\ , \quad W \in \mathbb{P}^n \setminus \mathcal{E}\ , \tag{1}$$

where $P, \mu, \mathcal{E}$ are as in Theorem 1.1. If f is an $\mathcal{L}^1$ function on $\mathbb{P}^n$, we similarly obtain

$$\frac{1}{\lambda^k} P^{k*}(f\omega^n) \to c_f\mu \tag{2}$$

where $c_f = \int f\omega^n$. In particular,

$$\frac{1}{\lambda^k} P^{k*}\omega^n \to \mu\ . \tag{3}$$

Write $\mu_k = \lambda^{-k}P^{k*}\omega^n$; then $P^{k*}\mu_k = \lambda\mu_{k+1}$. If $\mu(E_P) = 0$, we can then let $k \to \infty$ to conclude that μ is invariant in the sense that $P^*\mu = \lambda\mu$. If $\mu(\mathcal{E}) = 0$, we can further conclude that μ is the unique invariant measure that does not charge $\mathcal{E}$. It is an open problem whether the measure μ in Theorem 1.1 can charge the set E_P.

The *Fatou set* Ω_P of P can be defined to be the maximal open set in $\mathbb{P}^n$ on which $\{P_k\}$ is a normal family of holomorphic maps. One easily sees that a subsequence of $\{P^{k*}\omega^n\}$ converges in Ω_P to a smooth $2n$-form, and it then follows from (3) that μ vanishes on Ω_P. (For $n \geq 2$, μ is usually supported on a proper subset of $\mathbb{P}^n \setminus \Omega_P$; see Section 2.) Ueda [Ue] showed that $\operatorname{Supp} dd^cG = \mathbb{P}^n \setminus \Omega_P$ for P holomorphic; however, equality does not always hold for meromorphic P (see Example 2 in Section 2).

Our proof of Theorem 1.1 is based on a study of equidistribution for sequences of rational mappings of projective spaces. This approach was introduced in 1992 by Sodin [So], who considered an arbitrary sequence $\{R_k\}$ of rational functions (on $\mathbb{P}^1$) with rapidly increasing degrees and proved that the pre-images under $\{R_k\}$ of points w outside an exceptional set $\mathcal{E}$ are equidistributed in a certain sense. Sodin's result was generalized in [RS] to sequences of polynomial mappings $\mathbb{C}^n \to \mathbb{P}^m$. The equidistribution phenomenon in [RS] can be described as follows:

Suppose that $\{P_k\}$ is a sequence of polynomial mappings $\mathbb{C}^n \to \mathbb{P}^m$ such that $\sum 1/(\delta_1(P_k)) < \infty$. Then:

(i) *the pre-images of all but an exceptional pluripolar set of complex hyperplanes in $\mathbb{P}^m$ are equidistributed with the pull-backs $P_k^*\omega$;*

(ii) *if all P_k are non-degenerate, then the pre-images of all but a pluripolar set of points in $\mathbb{P}^m$ are equidistributed with $P_k^*\omega^m$ $(m \le n)$.*

In this paper we also obtain more general equidistribution results for pre-images of linear subspaces of any dimension. We let $\mathbb{G}(\ell, m)$ denote the Grassmannian of projective-linear subspaces of codimension ℓ in $\mathbb{P}^m$. Note that $\mathbb{G}(m,m) = \mathbb{P}^m$. We shall prove the following theorem on the equidistribution of pre-images under an arbitrary sequence of rational maps of projective spaces:

Theorem 1.2. *Let $\{P_k\}$ be a sequence of rational mappings from $\mathbb{P}^n$ to $\mathbb{P}^m$. Let $1 \le \ell \le \min(n,m)$, and let $\{a_k\}$ be a sequence of positive numbers such that*

$$\sum_{k=1}^{\infty} \frac{\delta_{\ell-1}(P_k)}{a_k} < +\infty \,.$$

Then there exists a pluripolar subset $\mathcal{E}$ of $\mathbb{G}(\ell, m)$ such that

$$\frac{1}{a_k}\left(P_k^*[W] - P_k^*\omega^\ell\right) \to 0$$

(weakly in $\mathcal{D}'^{\ell,\ell}(\mathbb{P}^n)$) as $k \to \infty$, for all $W \in \mathbb{G}(\ell,m) \setminus \mathcal{E}$.

The pull-back $P_k^*\omega^\ell$ is smooth off the indeterminacy locus of P_k and has locally integrable coefficients. For $W \in \mathbb{G}(\ell,m)$, $[W]$ denotes the current of integration over W, which is a closed positive (ℓ,ℓ)-current on $\mathbb{P}^m$. (If $\ell = m$, $[W] = \delta_W$.) For generic W, $P_k^*[W]$ is given by integration over $P_k^{-1}(W)$. Precise definitions of the pull-back currents $P_k^*[W], P_k^*\omega^\ell \in \mathcal{D}'^{\ell,\ell}(\mathbb{P}^n)$ are given in Section 3. (The exceptional set $\mathcal{E}$ includes those W for which $P_k^*[W]$ is not defined.)

The proof of Theorem 1.2 uses the Levine current Λ_W^ℓ, which is a positive current that satisfies the equation $dd^c\Lambda_W^\ell = \omega^\ell - [W]$ ([Le, GK, Sh]). We consider the "proximity function"

$$m_P^\ell(W) = (P^*\Lambda_W^\ell, \omega^{n-\ell+1})$$

and use the "First Main Theorem"

$$|(P^*[W] - P^*\omega^\ell, \varphi)| = |(P^*\Lambda_W^\ell, dd^c\varphi)| \le c_\varphi m_P^\ell(W)$$

to define the exceptional set $\mathcal{E}$ in terms of the "proximity sequence" $\{m_{P_k}^\ell\}$ (see Section 6), as in [RS]. In order to prove that $\mathcal{E}$ is pluripolar, we apply a new technique based on an approach of Skoda [Sk1, Sk2] to obtain the estimate $dd^c m_P^\ell \le \ell\delta_{\ell-1}(P)\omega$ on $\mathbb{G}(\ell,m)$ (Lemma 5.1). This new explicit bound on the quasi-plurisubharmonicity of the function $-m_P^\ell$ is a crucial ingredient of our proof.

A consequence of the case $\ell = m$ of Theorem 1.2 is the following result on the equidistribution of the pull-backs of measures by sequences of mappings:

***Corollary* 1.3.** *Let $\{P_k\}$ be a sequence of rational mappings from $\mathbb{P}^n$ to $\mathbb{P}^m$, where $n \geq m$. If*

$$\sum_{k=1}^{\infty} \frac{\delta_{m-1}(P_k)}{\delta_m(P_k)} < +\infty \,,$$

then there exists a pluripolar subset $\mathcal{E}$ of $\mathbb{P}^m$ such that for all probability measures ν on $\mathbb{P}^m$ with $\nu(\mathcal{E}) = 0$,

$$\frac{1}{\delta_m(P_k)} \left(P_k^*\nu - P_k^*\omega^m\right) \to 0$$

as $k \to \infty$,

The proof of Corollary 1.3 is given in Section 7.

Remark. Convergence in Corollary 1.3 can be taken to be weak convergence in $\mathcal{D}'^{m,m}(\mathbb{P}^n)$, or since the positive currents $\delta_m(P_k)^{-1}P_k^*\nu$ have mass 1 (see Section 6), we conclude more generally that

$$\frac{1}{\delta_m(P_k)} \left(P_k^*\nu - P_k^*\omega^m, \varphi\right) \to 0$$

for all $(n-m, n-m)$-forms φ on $\mathbb{P}^n$ with continuous coefficients. In the case $m = n$, the currents $\delta_m(P_k)^{-1}P_k^*\nu$ are probability measures and we have weak convergence in the space of measures. However, in Theorem 1.2 (and in Corollary 1.4 below) we do not always have convergence in the space of measures; see the remark at the end of Section 7

We use Theorem 1.2 with $\ell = m$ in the proof of Theorem 1.1. For general ℓ, Theorem 1.2 yields the following result on the equidistribution of the iterated pre-images of linear subspaces:

***Corollary* 1.4.** *Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational mapping, and let P^k denote the k-th iterate of P. Let $1 \leq \ell \leq n$. If $a > \lambda_{\ell-1}(P)$, then*

$$\frac{1}{a^k}\left(P^{k*}[W] - P^{k*}\omega^\ell\right) \to 0$$

(weakly in $\mathcal{D}'^{\ell,\ell}(\mathbb{P}^n)$) for all $W \in \mathbb{G}(\ell, m)$ outside a pluripolar set.

This corollary follows immediately from Theorem 1.2 by noting that for $\varepsilon > 0$,

$$\delta_{\ell-1}(P^k) \leq [\lambda_{\ell-1}(P) + \varepsilon]^k \quad \text{for} \quad k \gg 0 \,. \tag{4}$$

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2. Examples and open questions

We begin by describing several sequences of iterated mappings of the complex projective plane $\mathbb{P}^2$ in order to illustrate our results. Besides equidistribution, we are also going to look at invariance properties of the limit currents and measures. The dynamics of a rational map in more than one variable is usually quite difficult to analyze, so we consider very simple examples of extensions to $\mathbb{P}^2$ of proper polynomial mappings of $\mathbb{C}^2$. We also give a preliminary study of the properties of a particular map arising from Newton's method (Example 4).

In this section, we identify $(z, w) \in \mathbb{C}^2$ with $(1, z, w) \in \mathbb{P}^2$ so that $\mathbb{P}^2 = \mathbb{C}^2 \cup H_\infty$, where H_∞ is the hyperplane (or line) at infinity, $\{(0, z, w)\}$.

Example 1. (This example was discussed in [RS].) We first restrict our attention to $\mathbb{C}^2$ and consider the iterates of the polynomial map $P(z, w) = (z^\delta, w^\delta) : \mathbb{C}^2 \to \mathbb{C}^2$, where δ is a positive integer. It is easy to see that the uniform convergence $(1/\delta^k)\log(1 + |P^k(z, w)|^2)^{1/2} \to G(z, w)$ takes place, where

$$G(z, w) = \sup(\log^+ |z|, \log^+ |w|)$$

is the plurisubharmonic Green function of the unit bidisk with logarithmic growth at infinity. It follows that

$$2dd^c G = \lim_{k\to\infty} \frac{1}{\delta^k} P^{k*}\omega$$

(where $d^c = (4\pi\sqrt{-1})^{-1}(\partial - \overline{\partial})$). The current $T = 2dd^cG$, which is positive and has unit mass, is concentrated on the boundary of the unit bidisk and on the set $\{|z| = |w| > 1\}$. This current is invariant under the mapping P in the sense that

$$P^*T = \delta T \; .$$

(The pull-backs of closed positive $(1, 1)$-currents by holomorphic and meromorphic maps are defined in Section 3.) According to the results of [RS], this current is the limit of pre-images of all nonexceptional hyperplanes (lines).

Every hyperplane of the form $z = c$ or $w = c$ has pre-images of similar form. These pre-images tend towards the cylinder $|z| = 1$ or $|w| = 1$, respectively. Thus all such hyperplanes are exceptional. A third family of exceptional hyperplanes consists of those passing through the origin. Pre-images of these hyperplanes tend towards the "cone" $\{|z| = |w|\}$. We let $\mathbb{P}^{2*} \approx \mathbb{P}^2$ denote the parameter space of hyperplanes in $\mathbb{P}^2$; the point $(\zeta_0, \zeta_1, \zeta_2) \in \mathbb{P}^{2*}$ represents the hyperplane $\{(z, w) \in \mathbb{C}^2 : \zeta_0 + \zeta_1 z + \zeta_2 w = 0\}$. (The point $(1, 0, 0)$ represents the hyperplane at infinity, H_∞.) Thus we see that the set of exceptional hyperplanes consists of the three pencils in $\mathbb{P}^{2*}$: $\{\zeta_0 = 0\}$, $\{\zeta_1 = 0\}$, $\{\zeta_2 = 0\}$. We note that, besides T, there are at least five more (linearly independent) invariant closed $(1,1)$-currents (on $\mathbb{C}^2$), $T_1 = dd^c \log^+ |z|^2$, $T_2 = dd^c \log^+ |w|^2$, $T_3 = dd^c \log \max(|z|^2, |w|^2)$, $T_4 = [H_1] = dd^c \log |z|^2$, $T_5 = [H_2] = dd^c \log |w|^2$ (where $H_1 = \{z = 0\}, H_2 = \{w = 0\}$) with the same property

$$P^* T_i = \delta T_i.$$

In fact, they are limits of pre-images of the corresponding exceptional hyperplanes. (These currents are invariant on $\mathbb{P}^2$ as well as on $\mathbb{C}^2$, and we have the additional invariant current $T_6 = [H_\infty]$ on $\mathbb{P}^2$.)

Now consider pre-images of points. The measure $\mu = T^2 = (2dd^c G)^2$, being the limit of $(1/\delta^{2k}) P^{k*} \omega^2$, is concentrated on the distinguished boundary of the unit bidisk; hence pre-images of most points must tend to the torus by Theorem 2 of [RS]. However, all points of the form $(0, c)$ or $(c, 0)$ have pre-images of the form $(0, c')$ (respectively $(c', 0)$) and are definitely exceptional (as well as the points of H_∞). So in this case the exceptional set is the same union of 3 hyperplanes in $\mathbb{P}^2$.

The measure μ possesses invariance properties, $P^* \mu = \delta^2 \mu$. Note that we also have $\mu = T_1 \wedge T_2$.

The map P extends to a rational mapping $Q(t, z, w) = (t^\delta, z^\delta, w^\delta) : \mathbb{P}^2 \to \mathbb{P}^2$, which we call the *projectivization* of P. For this example, Q is holomorphic. The situation then becomes symmetric with respect to all variables, and because of the absence of the indeterminacy set, no difficulty occurs. We consider the "projectivizations" of the currents defined on $\mathbb{C}^2$: instead of $dd^c \log^+ |z|^2 = dd^c \log(|z|^2 \vee 1)$, one has $dd^c \log(|z|^2 \vee |t|^2)$, where $\vee$ stands for maximum, and so on. The projectivizations of the currents $T, T_1, \ldots, T_5$ (as well as $[H_\infty]$), remain invariant on $\mathbb{P}^2$. There are 3 super-attracting fixed points $(1,0,0), (0,1,0), (0,0,1)$ (and no other attracting periodic points), and their respective basins $\Omega_1, \Omega_2, \Omega_3$ comprise the entire Fatou set Ω_Q. Note that $\operatorname{Supp} \mu = \bigcap_i \partial \Omega_i$ (the distinguished boundary of the bidisk), while $\operatorname{Supp} T = \bigcup_i \partial \Omega_i = \mathbb{P}^2 \setminus \Omega_Q$.

Example 2. The situation changes if we consider the iterates of the mapping $P(z, w) = (z^{d_1}, w^{d_2}) : \mathbb{C}^2 \to \mathbb{C}^2$ with $d_1 \neq d_2$. For simplicity, let $P = (z^2, w^3)$. It is easy to see that in this case $(1/3^k) \log(1 + |P^k(z, w)|^2)^{1/2} \to G(z, w)$, where now $G(z, w) = \log^+ |w|$. It follows as before that

$$2dd^c G = \lim_{k \to \infty} \frac{1}{3^k} P^{k*} \omega \; ;$$

however, this time the current $T = 2dd^cG$ is concentrated on the cylinder $\{|w| = 1\}$. It is invariant for the mapping P:

$$P^*T = 3T .$$

Applying the results of [RS], we see that this current is the limit of the pre-images of all nonexceptional hyperplanes.

The set of exceptional hyperplanes in $\mathbb{C}^2$ consists of the two pencils in $\mathbb{P}^{2*}$: $\{\zeta_0 = 0\}$, $\{\zeta_2 = 0\}$. Besides T, there are at least three more invariant currents: $T_1 = dd^c \log^+ |z|^2$, $T_2 = dd^c \log |z|^2$, $T_3 = dd^c \log |w|^2$. However this time

$$P^*T_1 = 2T_1 .$$

(Note that for any smooth form S on $\mathbb{C}^2$, we have $P_*P^*S = 6S$; by smoothing and taking limits, one sees that this identity is also valid if S is a positive $(1,1)$-current on $\mathbb{C}^2$. Hence $P_*T_1 = 3T_1,\ P_*T = 2T$.) The currents T_i are the limits of pre-images of the corresponding families of exceptional hyperplanes. The coefficient 2 for T_1 makes the situation somewhat different as we shall see.

Now consider pre-images of points. The results of [RS] and [FS4] do not provide useful information since now $T^2 = (2dd^cG)^2 \equiv 0$ on $\mathbb{C}^2$. According to Theorem 1.1, this is not a surprise, since $\delta_2(P) = 6$, not 9, and we have a limit

$$\mu = \lim_{k\to\infty} \frac{1}{6^k} P^{k*}\omega^2,$$

which is the same measure (concentrated on the distinguished boundary of the unit bidisk) as in Example 1. So pre-images of most points must concentrate there. Again, the exceptional set for points is the same union of 3 hyperplanes in $\mathbb{P}^2$. The measure μ possesses the invariance property $P^*\mu = 6\mu$. Note that we also have $\mu = T \wedge T_1$.

Consider the projectivization of this mapping: $Q(t,z,w) = (t^3, tz^2, w^3)$: $\mathbb{P}^2 \to \mathbb{P}^2$ and the sequence Q^k of its iterations. Note that although P is holomorphic, Q is only meromorphic and has one indeterminacy point $(0,1,0)$ which is the reason for all "anomalies." Since H_∞ is contracted by Q to the fixed point $(0,0,1)$, the map Q is "generic" in the sense of Fornaess and Sibony [FS3, FS4]. Also, the graph of the mapping Q is singular, and thus it is necessary to resolve the singularities of the graph in order to define all our currents correctly (see Section 3). Note that we have $\delta_1(Q^k) = 3^k$, $\delta_2(Q^k) = 6^k < \delta_1(Q^k)^2$. In accordance with the results of [FS3, FS4], there is an invariant $(1,1)$-current $T = dd^c \log(|w|^2 \vee |t|^2)$ which is the projectivization of the above mentioned current on $\mathbb{C}^2$. The projectivization of the other invariant current, T_1, is no longer invariant, since now $Q^*T_1 = 2T_1 + [H_\infty]$.

There are only 2 super-attracting fixed points $(1,0,0), (0,0,1)$. As noted in [FS4, Example 2.1], the Fatou set Ω_Q consists of the basins of attraction

$$\Omega_1 = \{|z| < |t|, |w| < |t|\} \text{ (the unit bidisk) }, \quad \Omega_2 = \{|t| < |w|\}$$

of these points together with a third component $\Omega' = \{|w| < |t| < |z|\}$ attracted by the indeterminacy point $(0,1,0)$, and thus

$$\operatorname{Supp} T = \partial\Omega_2 \subsetneq \partial\Omega_1 \cup \partial\Omega_2 = \mathbb{P}^2 \setminus \Omega_Q .$$

The measure μ can be written homogeneously on $\mathbb{P}^2$ as $\mu = [dd^c \log(|z|^2 \vee |w|^2 \vee |t|^2)]^2$. We have $\mu = T \wedge T_1$ as before, and μ is invariant on $\mathbb{P}^2$ ($Q^*\mu = 6\mu$) although T_1 is not invariant. We now compute the product T^2 on $\mathbb{P}^2$. First we note that outside the indeterminacy point $(0,1,0)$, $T^2 = \lim 9^{-k} P^{k*}\omega^2 = 0$. The Green function (on $\mathbb{C}^3$) of Q is given by $G = \log(|w| \vee |t|)$; hence, using the coordinates $w' = w/z$, $t' = t/z$ in $\mathbb{P}^2 \setminus \{z = 0\} \approx \mathbb{C}^2$, we see that $T^2 = [2dd^c \log(|w'| \vee |t'|)]^2 = [(0,1,0)]$. (Since G is locally bounded off the line $\{(0,z,0)\} \subset \mathbb{C}^3$, the product T^2 is well-defined according to [De, Cor. 2.11] or [FS5, Cor. 3.6].)

Remark. The current $T = 2dd^cG$ in Example 2 has mass 1, i.e., $\|T\|(\mathbb{P}^2) = (T,\omega) = 1$ (see Section 6). It follows that $\|T^2\|(\mathbb{P}^2) = 1$, which gives an alternate proof that $T^2 = [(0,1,0)]$. In fact, if $T \in \mathcal{D}'^{1,1}(\mathbb{P}^n)$, $U \in \mathcal{D}'^{q,q}(\mathbb{P}^n)$ $(1 \le q < n)$ are closed positive currents of mass 1 such that the product $T \wedge U$ is well-defined as in [De] or [FS5], then $T \wedge U$ has mass 1 by [FS5, Th. 4.4]. A simple proof of this "general Bezout theorem" is as follows: Since $(T, \omega^{n-1}) = 1$, we can write $T = dd^cu + \omega$. The (quasi-plurisubharmonic) function u can be realized as the pointwise limit of smooth functions $u_k \ge u$ on $\mathbb{P}^n$ such that $dd^cu_k \ge -\omega$. Let $T_k = dd^cu_k + \omega$. Then on any affine open set, $T_k = dd^c(u_k + h)$, where $dd^ch = \omega$. By [De] or [FS5], $T_k \wedge U \to T \wedge U$. Since $(T_k \wedge U, \omega^{n-q-1}) = (U, T_k \wedge \omega^{n-q-1}) = (U, \omega^{n-q}) = 1$, we conclude that

$$\|T \wedge U\|(\mathbb{P}^n) = (T \wedge U, \omega^{n-q-1}) = \lim(T_k \wedge U, \omega^{n-q-1}) = 1 .$$

Example 3. We now consider the sequence of iterations of the map $P(z,w) = (w^3, z^2)$. This example is not generic in the sense of Fornaess and Sibony [FS3, FS4], since H_∞ is contracted by (the projectivization of) P to the indeterminacy point $(0,1,0)$, so we cannot conclude from [FS3, FS4] that the sequence of currents $(1/\delta_1(P^k))P^{k*}\omega$ converges. Instead we have two subsequences

$$\{P^{2k} = (z^{6^k}, w^{6^k})\}\ , \quad \{P^{2k+1} = (w^{3\cdot 6^k}, z^{2\cdot 6^k})\} .$$

The first subsequence is the same as in Example 1 with $\delta = 6$, and the corresponding current subsequence thus converges to

$$T_1 = dd^c \log^+(|z|^2 \vee |w|^2).$$

The second subsequence of currents converges to

$$T_2 = dd^c \log^+(|w|^2 \vee |z|^{4/3}) .$$

These currents are responsible for the distribution of pre-images of hyperplanes. As for the invariance properties, neither T_1 nor T_2 are invariant since $P^*T_1 = 3T_2$, $P^*T_2 = 2T_1$. However, the current $S = dd^c \log(|z|^2 \vee |w|^{\sqrt{6}})$ is invariant on $\mathbb{C}^2$ with $P^*S = \sqrt{6}S$, but S does not extend to an invariant current on $\mathbb{P}^2$. (The current $[H_\infty]$ seems to be the only invariant closed positive $(1,1)$-current on $\mathbb{P}^2$.) Note that $\delta_1(P) = 3$ but $\lambda_1(P) = \sqrt{6} < \delta_1(P)$, which is another way to conclude that P is not Fornaess-Sibony generic (see Section 4).

For the codimension two case, we have $\delta_2(P^k) = \delta_2(P)^k = 6^k$. So there is a limit measure which is the same as in the two previous examples and is invariant.

Example 4. One of the early motivations for the study of iterations came from Newton's method, which provides an interesting example of a rational mapping of $\mathbb{P}^2$, as follows. If one applies Newton's method to find the zeros of the function $F : \mathbb{C}^2 \to \mathbb{C}^2$ given by $F(z,w) = (z - w^2, w - z^2)$, one then must iterate the map

$$(z,w) \mapsto \left(\frac{w(w+2z^2)}{4zw-1}, \frac{z(z+2w^2)}{4zw-1} \right) ,$$

which extends to the rational mapping of $\mathbb{P}^2$,

$$Q(t,z,w) = \big(t(4zw - t^2), w(tw + 2z^2), z(tz + 2w^2)\big) \ .$$

The mapping Q has 4 super-attracting fixed points (which are the zeros of F) and 5 indeterminacy points: $I_Q = I' \cup I_\infty$, where $I' = \{(-2, \alpha, \alpha^2) : \alpha^3 = 1\}$ and $I_\infty = \{(0,1,0), (0,0,1)\}$. (The restriction of Q to $H_\infty \setminus I_\infty$ is the identity map, and thus the points of $H_\infty \setminus I_\infty$ are non-attracting fixed points of Q.) It is interesting to note that Q does not contract any curves in $\mathbb{P}^2$. Precisely,

$$\begin{aligned} Q^{-1}(1,a,b) &= \big\{(1, a \pm \sqrt{a^2 - b}, b \pm \sqrt{b^2 - a})\big\} \ , \\ Q^{-1}(0,a,b) &= \big\{(0,a,b), (2ab, b^2, a^2), (0,1,0), (0,0,1)\big\} \ . \end{aligned} \tag{5}$$

Since no curves are contracted, $\lambda_1(Q) = \delta_1(Q) = 3$. We see from (5) that $\delta_2(P) = 4$. (Note that $\delta_1(P)^2 - \delta_2(P)$ equals the number of indeterminacy points; see the comment following Lemma 4.4.)

Let Ω_j, $j = 1, \dots, 4$, denote the respective basins of attraction of the 4 super-attracting fixed points. By (5), each fixed point is completely invariant, so the Ω_j are connected. Since the Ω_j are components of the Fatou set Ω_Q, μ does not charge the Ω_j. (It is an open question whether Ω_Q contains any other components.) It then follows from Theorem 1.1 applied to the delta measures of (non-exceptional) points of Ω_j that

$$\operatorname{Supp} \mu \subset \bigcap_{j=1}^{4} \partial \Omega_j \ .$$

In particular the Ω_j have common boundary points.

Since $E_Q = \emptyset$, μ is invariant: $Q^*\mu = 4\mu$. The 4 super-attracting fixed points, as well as the 2 infinite indeterminacy points, are exceptional. In fact all the points of H_∞ are apparently exceptional. To see this, consider a point $W = (0, a, b)$, $ab \neq 0$. We iterate (5) to conclude that

$$\frac{1}{4^k} Q^{k*} \delta_W(I_\infty) = \frac{2(4^k - 1)}{3 \cdot 4^k} \to \frac{2}{3}\,.$$

On the other hand, from computer experiments using Maple, it appears that μ does not charge I_∞ (although it looks like $I_\infty \subset \operatorname{Supp}\mu$), as we expect (see Question 2 below), and thus W is exceptional. We have not found any other exceptional points. In particular, if we assume that the finite pre-image $(2ab, b^2, a^2)$ of W is nonexceptional, we can conclude from (5) that

$$\frac{1}{4^k} Q^{k*} \delta_W \to \frac{1}{3}[I_\infty] + \frac{1}{3}\mu\,.$$

Let us assume that the points of I' are nonexceptional, which appears to be the case from computer experiments. It then follows that the backwards orbit of each point of I' is an infinite set with closure containing the support of μ. We can write $I_{Q^k} = I_\infty \cup I'_k$ where $I'_k = I' \cup Q^{-1}(I') \cup \cdots \cup Q^{-(k-1)}(I')$, and it follows that the indeterminacy sets of the iterates of Q accumulate along $\operatorname{Supp}\mu$.

Let $G_k = 3^{-k} dd^c \log|Q^k|$. According to [FS3, FS4], $\{G_k\}$ converges to a Green function G, and $\{Q^{k*}\omega\}$ converges to an invariant (1,1)-current $T = 2dd^cG$. One can show that

$$(2dd^cG_k)^2 = \frac{1}{9^k}\left(Q^{k*}\omega^2 + [I'_k] + n_k[I_\infty]\right)\,, \tag{6}$$

where $n_k = \frac{1}{2}(9^k + 1 - 2\cdot 4^k)$. The discrete part of the measure in (6) is obtained by noting that the points of I'_k are simple indeterminacy points (see the comment following Lemma 4.4); the value of n_k is determined by the requirement that $(2dd^cG_k)^2$ have mass 1. (See the remark following Example 2.) Letting $k \to \infty$ in (6), we conclude that $(2dd^cG_k)^2 \to \frac{1}{2}[I_\infty]$.

We state here some open problems:

Question 1. In the Lyubich-Freire-Lopes-Mañé Theorem, the exceptional set contains at most two points; in Examples 1–3, the exceptional sets are unions of at most 3 hyperplanes in $\mathbb{P}^2$. *Can we further describe the exceptional set in Theorem 1.1? For a holomorphic map, is the exceptional set algebraic?*

Question 2. As mentioned in Section 1, if $\mu(E_P) = 0$, then $P^*\mu$ would be well defined and we would have $P^*\mu = \lambda\mu$. *Does $\mu(E_P) = 0$ for all rational maps satisfying the hypotheses of Theorem 1.1? Indeed, can μ charge an algebraic subvariety of $\mathbb{P}^n$?*

Question 3. If $P : \mathbb{P}^n \to \mathbb{P}^n$ is holomorphic, then $\{\delta_\ell(P)^{-k}P^{k*}\omega^\ell\}$ converges to $(2dd^cG)^\ell$, where G is the Green function. Fornaess and Sibony [FS4] showed that if P is a meromorphic map of $\mathbb{P}^2$ with $\delta_1(P) = \lambda_1(P)$, then $\{\delta_1(P)^{-k}P^{k*}\omega\}$ converges to a positive $(1,1)$-current of mass 1 (provided that $G \not\equiv -\infty$). *Can one interpolate between this result of Fornaess-Sibony* [FS4] *and Theorem 1.1, as in the holomorphic case? In particular, does* $\{\delta_\ell(P)^{-k}P^{k*}\omega^\ell\}$ *converge if* $\delta_\ell(P) = \lambda_\ell(P) > \lambda_{\ell-1}(P)$, *for* $1 < \ell < n$ *?*

Question 4. Theorem 1.2 easily generalizes to the case where $[W]$ is replaced with $(1/d)[A]$, where A is a (nonexceptional) complete intersection of degree d; the conclusion also holds with W replaced with a product of closed positive smooth (1,1)-forms of mass 1. *Does Theorem 1.2* (*or Corollary 1.4*) *hold for* $[W]$ *replaced with a "general" closed positive* (ℓ,ℓ)*-current of mass 1?* (For the case of $P : \mathbb{P}^2 \to \mathbb{P}^2$ holomorphic and $\ell = 1$, Fornaess and Sibony [FS4, Th. 4.16] gave a sufficient condition on P for Corollary 1.4 to hold for *all* closed positive (1,1)-currents of mass 1.)

3. Notation and terminology

We let $\mathcal{E}^{p,q}(X)$, $\mathcal{D}^{p,q}(X)$, $\mathcal{D}'^{p,q}(X)$ denote the spaces of (complex-valued) $\mathcal{C}^\infty$ forms, compactly supported $\mathcal{C}^\infty$ forms, and currents, respectively, of bidegree (p,q) on a complex manifold X and we use the standard differentials $d = \partial + \overline{\partial}$, $d^c = (4\pi\sqrt{-1})^{-1}(\partial - \overline{\partial})$. Points in complex projective n-space $\mathbb{P}^n$ are identified with their representations $z = (z_0, z_1, \dots, z_n)$ in homogeneous coordinates. We shall regard the Grassmannian $\mathbb{G}(\ell, m)$ of projective linear subspaces of codimension ℓ in $\mathbb{P}^m$ as a subvariety of $\mathbb{P}(\bigwedge^{m+1-\ell}\mathbb{C}^{m+1})$.

If $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is a non-constant meromorphic map, it is a well-known consequence of Chow's theorem that P must be rational, i.e., P can be written in the form $P = (P_0, \dots, P_n)$ where $P_j \in \mathbb{C}[z_0, \dots, z_n]$ and $\deg P_0 = \cdots = \deg P_m$. We can assume that the P_j have no common factors; we then say that $\deg P = \deg P_0$. (This notion of degree should not be confused with the *topological degree* of an equidimensional rational map, which is the number of points in the preimage of a generic point in the range.) We let $I_P \subset \mathbb{P}^n$ denote the indeterminacy locus of P (the points where P is not holomorphic); I_P is an algebraic subvariety of codimension ≥ 2.

Suppose $f : Y \to X$ is a holomorphic mapping of complex manifolds. If α is a current on X, $f^*\alpha$ is not always defined. However, we shall define $f^*\alpha$ in two special cases: First, we suppose $\alpha = u\gamma$ where $\gamma \in \mathcal{E}^{p,q}(X)$ is a smooth form and u is the difference of plurisubharmonic functions. Assume further that $f(Y)$ is not contained in the $\pm\infty$ locus of u. Then $u \circ f$ is the difference of plurisubharmonic functions on Y and hence is in $\mathcal{L}^1_{\mathrm{loc}}(Y)$. We define $f^*\alpha = (u \circ f)f^*\gamma$, which is clearly independent of the representation $\alpha = u\gamma$. The second case we consider is that of a current of the form $[D] \wedge \gamma$, where $[D]$ is the current of integration over a divisor D on X and γ is a smooth form as before. We assume also that $f(Y) \not\subset \operatorname{Supp} D$ so that f^*D is a divisor on Y. We then define $f^*([D] \wedge \gamma) = [f^*D] \wedge f^*\gamma$. These two definitions are consistent in the following way. Suppose $\alpha = \log|g|^2 \cdot \gamma$ where g is a meromorphic function on X such that neither the zeroes nor the poles of g contain $f(Y)$ and γ is a closed (p,q)-form on X. Then $dd^c\alpha = [D] \wedge \gamma$ where $D = \operatorname{Div}(g)$. Hence

$$f^*dd^c\alpha = [f^*D] \wedge f^*\gamma = dd^c(\log|g \circ f|^2 \cdot f^*\gamma) = dd^c f^*\alpha \,. \tag{7}$$

If ψ is a current on Y and $f|_{\operatorname{Supp}\psi}$ is proper, one defines the push-forward current $f_*\psi$ by $(f_*\psi, \varphi) = (\psi, f^*\varphi)$; if ψ is of order 0, then so is $f_*\psi$. Furthermore, if ψ has $\mathcal{L}^1_{\mathrm{loc}}$ coefficients, then so does $f_*\psi$, provided that f is surjective. (The latter statement follows from the fact that the coefficients of ψ are in $\mathcal{L}^1_{\mathrm{loc}}$ if and only if ψ is of order 0 and the total variation measure $\|\psi\|$ is absolutely continuous with respect to volume measure. Note that if ψ is a smooth form, then the coefficients of $f_*\psi$ are generally not smooth and may even be unbounded.)

Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map. For a smooth (p,q)-form $\eta \in \mathcal{E}^{p,q}(\mathbb{P}^m)$ we define the pull-back current $P^*\eta \in \mathcal{D}'^{p,q}(\mathbb{P}^m)$ as follows: We let $G_P \subset \mathbb{P}^n \times \mathbb{P}^m$ denote the graph of P (which is an irreducible algebraic subvariety of $\mathbb{P}^n \times \mathbb{P}^m$) and we consider a desingularization $\widetilde{G} \xrightarrow{\rho} G_P$. We have the commutative diagram:

$$\begin{tikzcd}
 & \widetilde{G} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{P}^n \arrow[rr, dashed, "P"] & & \mathbb{P}^m
\end{tikzcd} \tag{8}$$

We then define

$$P^*\eta = \pi_{1*}\pi_2^*\eta \,.$$

The pull-back $P^*\eta$ has coefficients in $\mathcal{L}^1_{\mathrm{loc}}$, since $\pi_2^*\eta$ is smooth. Furthermore $P^*\eta|_{\mathbb{P}^n \setminus I_P}$ is the usual (smooth) pull-back $(P|_{\mathbb{P}^n \setminus I_P})^*\eta$, and thus $P^*\eta$ does not depend on the choice of desingularization of G_P.

We consider the current of integration $[W] \in \mathcal{D}'^{\ell,\ell}(\mathbb{P}^m)$ and define the pull-back $\pi_2^*[W]$ to be the current of integration over the algebraic $(n-\ell)$-cycle π_2^*W on $\widetilde{G}$ (using the diagram (8)), whenever $\dim \pi_2^{-1}(W) = n - \ell$. If we represent W as the intersection of hyperplanes $H_1, \dots, H_\ell$ in $\mathbb{P}^m$, then π_2^*W is the intersection of divisors $\pi_2^*H_1 \cap \cdots \cap \pi_2^*H_\ell$. For a definition of this intersection, which is a formal sum of the irreducible components of $\pi_2^{-1}(W)$ with positive integer coefficients, see [Ha, Appendix A] or Definitions 2.3 and 2.4.2 (or Example 7.1.10) in [Fu]. This pull-back, or intersection, can also be defined analytically as follows. Let g_j be a local defining function for $\pi_2^*H_j$, for $1 \le j \le \ell$, and write $g = (g_1, \dots, g_\ell)$.

Then by Griffiths and King [GK, 1.10] (see also [Sh, I.12, Th. 3]), we have the local formula

$$\pi_2^*[W] = dd^c \left(\log|g|^2 (dd^c \log|g|^2)^{\ell-1}\right) .$$

(Alternately, $\pi_2^*[W] = dd^c \log|g_1|^2 \wedge \ldots \wedge dd^c \log|g_\ell|^2$, where the existence of this product of currents is guaranteed by Demailly [De].)

We now state the Poincaré-Lelong formula for linear subspaces of $\mathbb{P}^m$ and describe its pull-backs by a rational map $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$. Let $W \in \mathbb{G}(\ell, m)$ be an $(m-\ell)$-plane in $\mathbb{P}^m$. For each $W \in \mathbb{G}(\ell, m)$, we consider the *Levine current*

$$\Lambda_W^\ell = \log \frac{|\zeta|^2 |W|^2}{|\zeta \wedge W|^2} \sum_{j=0}^{\ell-1} (d_\zeta d_\zeta^c \log|\zeta \wedge W|^2)^j \wedge \omega_\zeta^{\ell-1-j} \in \mathcal{D}'^{\ell-1,\ell-1}(\mathbb{P}_\zeta^m) ,$$

which (by definition) has locally integrable coefficients. We have the *Poincaré-Lelong formula* for W [Le] (see also [GK, 1.15] or [Sh, II.6, pp. 68-69]),

$$dd^c \Lambda_W^\ell = \omega^\ell - [W] . \tag{9}$$

Now let $W = H_1 \cap \cdots \cap H_\ell$ such that $\dim \pi_2^{-1}(W) = n - \ell$, where we use the notation of (8). In order to "pull back" the identity (9) by the meromorphic map P, we first apply the generalized Poincaré-Lelong formula ([GK, 1.15] or [Sh, II.6, pp. 68-69]) to the divisors $\pi_2^* H_1, \ldots, \pi_2^* H_\ell$ of the lifted hyperplane-section bundle $\pi_2^* \mathcal{O}_{\mathbb{P}^m}(1)$ with Chern form $\pi_2^* \omega$ to obtain

$$dd^c \pi_2^* \Lambda_W^\ell = \pi_2^* \omega^\ell - \pi_2^*[W] , \tag{10}$$

where $\pi_2^* \Lambda_W^\ell \in \mathcal{D}'^{\ell-1,\ell-1}(\widetilde{G})$ is given by

(11) $(\pi_2^* \Lambda_W^\ell)(\widetilde{z})$

$$\stackrel{\text{def}}{=} \log \frac{|\pi_2(\widetilde{z})|^2 \; |W|^2}{|\pi_2(\widetilde{z}) \wedge W|^2} \sum_{j=0}^{\ell-1} (dd^c \log|\pi_2(\widetilde{z}) \wedge W|^2)^j \wedge \pi_2^* \omega^{\ell-1-j} \in \mathcal{D}'^{\ell-1,\ell-1}(\widetilde{G}).$$

In particular, $\pi_2^* \Lambda_W^\ell$ has $\mathcal{L}^1_{\text{loc}}$ coefficients and is smooth on $\widetilde{G} \setminus \pi_2^{-1}(W)$. We define the currents on $\mathbb{P}^n$

$$\begin{aligned} P^* \Lambda_W^\ell &= \pi_{1*} \pi_2^* \Lambda_W^\ell , \\ P^*[W] &= \pi_{1*} \pi_2^*[W] . \end{aligned} \tag{12}$$

The current $P^* \Lambda_W^\ell$, being the push-forward by π_1 of a current with $\mathcal{L}^1_{\text{loc}}$ coefficients, also has coefficients in $\mathcal{L}^1_{\text{loc}}$. By applying π_{1*} to (10), we then obtain the Poincaré-Lelong formula for $P^*[W]$,

$$dd^c P^* \Lambda_W^\ell = P^* \omega^\ell - P^*[W]. \tag{13}$$

Note that for generic W, $\pi_2^*[W]$ has multiplicity identically 1 and contains no components inside the exceptional locus of π_1, and thus $P^*[W]$ is the current of integration over the closure of $(P|_{\mathbb{P}^n \setminus I_P})^{-1}(W)$.

4. The intermediate degrees of a rational map

In this section, we give some properties of the intermediate degrees $\delta_\ell(P)$ of a rational map P, which we also describe analytically and topologically. We use the following consequence of Bertini's theorem:

Lemma 4.1. *Let Y be a projective algebraic manifold and let $f : Y \to \mathbb{P}^m$ be a nonconstant holomorphic map. Then for a generic hyperplane $H \subset \mathbb{P}^m$, the divisor f^*H is smooth and has multiplicity 1.*

Proof. Apply Bertini's Theorem (see, for example, [GH, p. 137]) to the complete linear system of f^*H. □

We now denote the normalized Fubini-Study Kähler form on $\mathbb{P}^m$ by ω_m. For a subvariety $V \subset \mathbb{P}^m$, we write $P^{-1}(V) = \pi_1(\pi_2^{-1}(V))$ (using the notation in (8)). We let $\#(S)$ denote the cardinality of a set S. We begin with a formula for the integral of certain singular forms on $\mathbb{P}^n$:

Lemma 4.2. *Let $P_j : \mathbb{P}^n \dashrightarrow \mathbb{P}^{m_j}$, $1 \le j \le n$, be rational maps, and let $I = I_{P_1} \cup \cdots \cup I_{P_n}$.*

(i) *For generic hyperplanes $H_1 \subset \mathbb{P}^{m_1}, \ldots , H_n \subset \mathbb{P}^{m_n}$,*

$$\int_{\mathbb{P}^n \setminus I} P_1^*\omega_{m_1} \wedge \cdots \wedge P_n^*\omega_{m_n} = \#\Big(\bigcap_{j=1}^{n} P_j^{-1}(H_j) \setminus I \Big) ,$$

(ii)

$$\int_{\mathbb{P}^n \setminus I} P_1^*\omega_{m_1} \wedge \cdots \wedge P_n^*\omega_{m_n} \le \prod_{j=1}^{n} \deg P_j .$$

To verify Lemma 4.2, we first give a topological description of the integral. Let $X = \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n}$ and write $\mathcal{P} = (P_1, \ldots , P_n) : \mathbb{P}^n \dashrightarrow X$; then $I = I_{\mathcal{P}}$. Consider the commutative diagram

$$\begin{tikzcd} & \widetilde{G} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\mathcal{P}"] & & X \end{tikzcd}$$

where $\widetilde{G}$ is a desingularization of the graph of $\mathcal{P}$ and π_1, π_2 are the projections. Let $p_j : X \to \mathbb{P}^{m_j}$ denote the projection to the j-th factor and let $\widetilde{P}_j = p_j \circ \pi_2 : \widetilde{G} \to \mathbb{P}^{m_j}$, for $1 \le j \le n$. Let t_m denote the positive generator of $H^2(\mathbb{P}^m, \mathbb{Z})$, and write

$$t_{m_1} \times \cdots \times t_{m_n} = p_1^* t_{m_1} \bullet \cdots \bullet p_n^* t_{m_n} \in H^{2n}(X, \mathbb{Z}) ,$$

where $\bullet$ denotes the cup product in the cohomology ring.

Lemma 4.3. *Using the notation of Lemma 4.2,*

$$\int_{\mathbb{P}^n\setminus I} P_1^*\omega_{m_1}\wedge\cdots\wedge P_n^*\omega_{m_n} = \left(\pi_2^*(t_{m_1}\times\cdots\times t_{m_n}),\widetilde{G}\right)$$
$$= (\widetilde{P}_1^* t_{m_1}\bullet\cdots\bullet\widetilde{P}_n^* t_{m_n},\widetilde{G})\in\mathbb{Z}\,.$$

Proof. Since the de Rham class of the Kähler form ω_m on $\mathbb{P}^m$ equals t_m, we have by Section 3,

$$\begin{aligned}\int_{\mathbb{P}^n\setminus I} P_1^*\omega_{m_1}\wedge\cdots\wedge P_n^*\omega_{m_n} &= \int_{\mathbb{P}^n\setminus I_{\mathcal{P}}}\mathcal{P}^*(\omega_{m_1}\times\cdots\times\omega_{m_n})\\ &= (\pi_{1*}\pi_2^*(\omega_{m_1}\times\cdots\times\omega_{m_n}),1)\\ &= (\pi_2^*(\omega_{m_1}\times\cdots\times\omega_{m_n}),1)\\ &= \left(\pi_2^*(t_{m_1}\times\cdots\times t_{m_n}),\widetilde{G}\right)\ .\end{aligned}$$

The second equality follows from our definitions. □

Proof of Lemma 4.2. Let $E\subset\widetilde{G}$ be the exceptional locus of π_1. By Lemma 4.1 applied to $\widetilde{P}_1:\widetilde{G}\to\mathbb{P}^{m_1}$, there is a hyperplane $H_1\subset\mathbb{P}^{m_1}$ such that the divisor $\pi_2^*(H_1\times\mathbb{P}^{m_2}\times\cdots\times\mathbb{P}^{m_n})$ is a smooth hypersurface $Y_1\subset\widetilde{G}$ of multiplicity 1 with $\dim Y_1\cap E<n-1$. By Lemma 4.1, we can inductively find hyperplanes $H_2\subset\mathbb{P}^{m_2},\dots,H_n\subset\mathbb{P}^{m_n}$ such that, writing

$$Y_j=\pi_2^{-1}(H_1\times\cdots\times H_j\times\mathbb{P}^{m_{j+1}}\times\cdots\times\mathbb{P}^{m_n})\ ,$$

Y_j is a smooth submanifold of $\widetilde{G}$ of dimension $n-j$, $\dim Y_j\cap E<n-j$, and the divisor $(\widetilde{P}_j|_{Y_{j-1}})^*H_j$ (on Y_{j-1}) has multiplicity 1, or equivalently, $\widetilde{P}_j^{-1}(H_j)$ intersects Y_{j-1} transversely. In particular, $\dim Y_n=0$ and $Y_n\cap E=\emptyset$. Write $H_j'=p_j^{-1}(H_j)$, for $1\le j\le n$, so that

$$Y_j=\pi_2^{-1}(H_1'\cap\cdots\cap H_j')=\pi_2^{-1}(H_j')\cap Y_{j-1}\ .$$

A codimension j submanifold S of a complex manifold Y determines the current of integration $[S]\in\mathcal{D}'^{j,j}(Y)$ given by $([S],\varphi)=\int_S\varphi$ for a test form φ. We also let $[S]$ denote the de Rham class in $H^{2j}(Y,\mathbb{R})$ containing the current $[S]$. (If Y is compact of dimension n, then the cohomology class $[S]$ is the Poincaré dual of the $(2n-2j)$-cycle S.) If two submanifolds S_1, S_2 intersect transversely, then $[S_1\cap S_2]=[S_1]\bullet[S_2]$ in the cohomology ring of Y. In particular, $[H_j]=t_j\in H^2(\mathbb{P}^{m_j},\mathbb{Z})\subset H^2(\mathbb{P}^{m_j},\mathbb{R})$ and

$$[H_1\times\cdots\times H_n]=[H_1'\cap\cdots\cap H_n']=t_{m_1}\times\cdots\times t_{m_n}\ .$$

Furthermore, our construction of the H_j,Y_j implies that

$$[Y_j]=[Y_{j-1}]\bullet\pi_2^*[H_j']=\pi_2^*[H_1']\bullet\cdots\bullet\pi_2^*[H_j']=\pi_2^*[H_1'\cap\cdots\cap H_j']\ .$$

Therefore,

$$\pi_2^*(t_{m_1} \times \cdots \times t_{m_n}) = \pi_2^*[H_1 \times \cdots \times H_n] = [Y_n] \tag{14}$$

(where the points of Y_n have multiplicity 1).

Let $A_j : \mathbb{C}^{m_j+1} \to \mathbb{C}$ be a linear map defining the hyperplane H_j, and consider the polynomial

$$Q_j = A_j(P_{j0}, \ldots, P_{jm_j}) \in \mathbb{C}[z_0, \ldots, z_n] ,$$

where $P_j = (P_{j0}, \ldots, P_{jm_j})$, for $1 \le j \le n$. We then have

$$\{z \in \mathbb{P}^n : Q_j(z) = 0\} = \pi_1\left(\pi_2^{-1}(H_j')\right) \supset I_{P_j} . \tag{15}$$

Since $Y_j \subset \widetilde{G} \setminus E$ and π_1 maps $\widetilde{G} \setminus E$ bijectively to $\mathbb{P}^n \setminus I_{\mathcal{P}}$, we have

$$\begin{aligned}\pi_1(Y_n) &= \pi_1\left(\pi_2^{-1}(H_1' \cap \cdots \cap H_n')\right) \setminus I_{\mathcal{P}} \\ &= \bigcap_{j=1}^{n} \pi_1\left(\pi_2^{-1}(H_j')\right) \setminus I_{\mathcal{P}} = \bigcap_{j=1}^{n} P_j^{-1}(H_j) \setminus I_{\mathcal{P}} \\ &= \{z \in \mathbb{P}^n \setminus I_{\mathcal{P}} : Q_1(z) = \cdots = Q_n(z) = 0\} .\end{aligned}$$

Thus by (14),

$$\left(\pi_2^*(t_{m_1} \times \cdots \times t_{m_n}), \widetilde{G}\right) = \left([Y_n], \widetilde{G}\right) = \#(Y_n) .$$

By Bézout's Theorem,

$$\#(Y_n) = \#(\pi_1(Y_n)) = \#\Big(\bigcap_{j=1}^{n} P_j^{-1}(H_j) \setminus I_{\mathcal{P}}\Big) \le \prod_{j=1}^{n} \deg Q_j = \prod_{j=1}^{n} \deg P_j .$$

The conclusion follows from Lemma 4.3. □

Definition. Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map. We define the *intermediate degrees* $\delta_\ell(P)$ of P by the formula

$$\delta_\ell(P) = \int_{\mathbb{P}^n \setminus I_P} P^* \omega_m^\ell \wedge \omega_n^{n-\ell}$$

for $0 \le \ell \le \min(n, m)$.

We shall show in Lemma 4.4 below that the intermediate degrees are also given by the geometric definition in the introduction; in particular, if $m = n$, then $\delta_n(P)$ is the *topological degree* of P, which is defined as the cardinality of $P^{-1}(x)$, for a generic point $x \in \mathbb{P}^n$. Note that $\delta_0(P) = 1$. Clearly, $\delta_\ell(P) > 0$ if and only if rank $P \ge \ell$. It is easy to verify that $\delta_1(P) = \deg P$, and if P is holomorphic (this can happen only if $m \ge \operatorname{rank} P = n$), then $\delta_\ell(P) = (\deg P)^\ell$, for $\ell \le n$.

It follows from Lemma 4.2 applied to the maps $P_1 = \cdots = P_\ell = P$, $P_{\ell+1} = \cdots = P_n = \mathrm{Id}_{\mathbb{P}^n}$ that in general,

$$\delta_\ell(P) \le (\deg P)^\ell , \quad \text{for } 1 \le \ell \le \min(n,m) . \tag{16}$$

We shall give a more general inequality in Lemma 4.7 below.

***Lemma* 4.4.** *Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map, and let $1 \le \ell \le \min(n,m)$. Then*

$$\delta_\ell(P) = \deg P^{-1}(W) \le (\deg P)^\ell$$

for generic $W \in \mathbb{G}(\ell, m)$, with equality if and only if $\operatorname{codim} I_P > \ell$. *In particular, $\delta_n(P) \le (\deg P)^n$, with equality if and only if P is holomorphic.*

Proof. We shall apply Lemma 4.2 with $P_1 = \cdots = P_\ell = P$, $P_{\ell+1} = \cdots = P_n = \mathrm{Id}_{\mathbb{P}^n}$. By part (ii) of the lemma, $\delta_\ell(P) \le (\deg P)^\ell$. For generic $W = H_1 \cap \cdots \cap H_\ell$, $\pi_2^{-1}(W)$ is of codimension ℓ and has no components contained in the exceptional locus of π_1, and thus $P^{-1}(W)$ has pure dimension $n - \ell$ and $\dim P^{-1}(W) \cap I_P < n - \ell$. Hence for generic hyperplanes $H_1, \ldots, H_n$, we have

$$\bigcap_{j=1}^{n} P_j^{-1}(H_j) \setminus I_P = P^{-1}(W) \cap H_{\ell+1} \cap \cdots \cap H_n ,$$

where $W = H_1 \cap \cdots \cap H_\ell$, and thus by part (i),

$$\delta_\ell(P) = \# \left(P^{-1}(W) \cap H_{\ell+1} \cap \cdots \cap H_n \right) = \deg P^{-1}(W) .$$

Furthermore, using the notation in the proof of Lemma 4.2,

$$\{ z \in \mathbb{P}^n : Q_1(z) = \cdots = Q_\ell(z) = 0 \} = P^{-1}(W) \cup I_P .$$

Since $\deg Q_1 = \cdots = \deg Q_\ell = \deg P$, it follows from Bézout's theorem (see for example [Fu, Example 8.4.6]) that $\delta_\ell(P) < (\deg P)^\ell$ if $\dim I_P \ge n - \ell$. If $\dim I_P < n - \ell$, then $P^{-1}(W) \supset I_P$ (by dimension considerations) and Bézout's theorem gives equality. $\square$

In fact, if $n = 2$ in Lemma 4.4, then $\delta_2(P) = (\deg P)^2 - q$, where q is the number of points of I_P counting multiplicity. This is illustrated by the following example.

Example. Let $P : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the "quadratic transform" given by

$$P(z_0, z_1, z_2) = (z_1 z_2, z_0 z_2, z_0 z_1) = \left(\frac{1}{z_0}, \frac{1}{z_1}, \frac{1}{z_2} \right) .$$

Then $\delta_1(P) = 2,\ \delta_2(P) = 1$. Note that in this example I_P consists of the three points $(1,0,0), (0,1,0), (0,0,1)$.

Lemma 4.5. *Suppose that $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is a rational map and $L : \mathbb{C}^{m+1} \to \mathbb{C}^{M+1}$ is a linear map such that* $\operatorname{Image} P \not\subset \mathbb{P}(L^{-1}(0))$. *Let $P_L = \widehat{L} \circ P : \mathbb{P}^n \dashrightarrow \mathbb{P}^M$, where $\widehat{L} : \mathbb{P}^m \dashrightarrow \mathbb{P}^M$ is the map induced from L. Then*

$$\int_{\mathbb{P}^n \setminus I} (P_L^* \omega)^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell} \leq \delta_\ell(P)$$

for $1 \leq k \leq \ell \leq n$, where $I = I_P \cup I_{P_L}$.

Proof. We can assume without loss of generality that $M = m$. We first consider the case where L is nonsingular and thus $\widehat{L}$ is biholomorphic. Therefore

$$\int_{\mathbb{P}^n \setminus I} (P_L^* \omega)^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell} = \int_{\mathbb{P}^n \setminus I} (P^* \omega')^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell}$$

where $\omega' = \widehat{L}^* \omega$. Since ω' and ω are in the same de Rham class, it follows from Section 3 (or by the proof of Lemma 4.3) that

$$\int_{\mathbb{P}^n \setminus I} (P^* \omega')^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell} = \int_{\mathbb{P}^n \setminus I} (P^*\omega)^{\ell} \wedge \omega^{n-\ell} = \delta_\ell(P) \ .$$

We now suppose that L is singular. Choose a sequence $\{L_\nu\}$ of nonsingular linear operators on $\mathbb{C}^{m+1}$ such that $L_\nu \to L$. We can write

$$(P_{L_\nu}^* \omega)^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell} = f_\nu \omega^n \ , \quad (P_L^* \omega)^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell} = f \omega^n$$

where f_ν, f are non-negative $\mathcal{C}^\infty$ functions on $\mathbb{P}^n \setminus I$. Then $f_\nu \to f$ pointwise on $\mathbb{P}^n \setminus I$, and hence by Fatou's Lemma,

$$\begin{aligned} \int_{\mathbb{P}^n \setminus I} (P_L^* \omega)^k \wedge (P^*\omega)^{\ell-k} \wedge \omega^{n-\ell} &= \int_{\mathbb{P}^n \setminus I} f\omega^n \leq \liminf_{\nu \to \infty} \int_{\mathbb{P}^n \setminus I} f_\nu \omega^n \\ &= \delta_\ell(P) \ . \end{aligned}$$

□

Lemma 4.6. *Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, $Q : \mathbb{P}^m \dashrightarrow \mathbb{P}^r$ be rational maps. Then*

$$\delta_\ell(Q \circ P) \leq \delta_\ell(P)\delta_\ell(Q) \ .$$

Proof. Let $\eta = Q^*\omega^\ell \in \mathcal{D}'^{\ell,\ell}(\mathbb{P}^n)$. We smooth η by an approximate identity $\{\psi_\varepsilon\}_{\varepsilon>0}$ with respect to a Haar measure h on $GL(n+1, \mathbb{C})$ to obtain

$$\eta_\varepsilon \stackrel{\text{def}}{=} \int_{GL(n+1,\mathbb{C})} (g^*\eta)\psi_\varepsilon(g) dh(g) \in \mathcal{E}^{\ell,\ell}(\mathbb{P}^n) \ .$$

Then $\eta_\varepsilon \to \eta$ pointwise as $\varepsilon \to 0$, $\eta_\varepsilon \geq 0$, and we have the identity in de Rham cohomology,

$$[\eta_\varepsilon] = [\eta] = \delta_\ell(Q)[\omega^\ell] \in H^{2\ell}(\mathbb{P}^n, \mathbb{Z}) \ .$$

Using the commutative diagram (8), we then have

$$\int_{\mathbb{P}^n} P^*\eta_\varepsilon \wedge \omega^{n-\ell} = \int_{\widetilde{G}} \pi_2^*\eta_\varepsilon \wedge \pi_1^*\omega^{n-\ell}$$
$$= \int_{\widetilde{G}} \delta_\ell(Q)\pi_2^*\omega^\ell \wedge \pi_1^*\omega^{n-\ell} = \delta_\ell(Q)\delta_\ell(P) \, .$$

Therefore, by Fatou's lemma,

$$\delta_\ell(Q\circ P) = \int_{\mathbb{P}^n\setminus I_{Q\circ P}} P^*\eta \wedge \omega^{n-\ell}$$
$$\leq \liminf_{\varepsilon\to 0} \int_{\mathbb{P}^n} P^*\eta_\varepsilon \wedge \omega^{n-\ell} = \delta_\ell(Q)\delta_\ell(P) \, . \qquad \square$$

In particular, if $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, we have $\delta_\ell(P^{j+k}) \leq \delta_\ell(P^j)\delta_\ell(P^k)$. Hence the sequence $c_k = \log\delta_\ell(P^k)$, $k = 1, 2, \ldots$, is subadditive and thus $c_k/k \to \inf_k c_k/k$ (see, for example, [PS, I.98]). This enables us to introduce the following definition.

Definition. Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a meromorphic map, and let P^k denote the k-th iterate of P. We define the *intermediate dynamic degrees* $\lambda_\ell(P)$ of P by

$$\lambda_\ell(P) = \lim_{k\to\infty} \delta_\ell(P^k)^{1/k} = \inf_{k\geq 1} \delta_\ell(P^k)^{1/k} \, ,$$

for $1 \leq \ell \leq n$.

The intermediate dynamic degrees were considered by Friedland [Fr]; $\lambda_1(P)$ is called the "dynamic degree" in [Di]. We note that $\lambda_\ell(P) \leq \delta_\ell(P)$, and $\lambda_n(P) \leq \delta_n(P)$. If P is holomorphic, then $\lambda_\ell(P) = \delta_\ell(P)$. The map $P : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is "generic" in the sense of Fornaess and Sibony [FS3, FS4] if and only if $\lambda_1(P) = \delta_1(P)$ [FS4, Di].

Lemma 4.7. *Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map. Then*

$$\delta_{k+\ell}(P) \leq \delta_k(P)\delta_\ell(P) \, .$$

Proof. Let $\eta = P^*\omega^\ell \in \mathcal{D}'^{\ell,\ell}(\mathbb{P}^m)$ and consider the smooth forms η_ε as in the above proof. As before,

$$[\eta_\varepsilon] = \delta_\ell(P)[\omega^\ell] \in H^{2\ell}(\mathbb{P}^n, \mathbb{Z}) \, ,$$

and

$$\int_{\mathbb{P}^n} P^*\omega^k \wedge \eta_\varepsilon \wedge \omega^{n-k-\ell} = \int_{\widetilde{G}} \pi_2^*\omega^k \wedge \pi_1^*(\eta_\varepsilon \wedge \omega^{n-k-\ell})$$
$$= \delta_\ell(P) \int_{\widetilde{G}} \pi_2^*\omega^k \wedge \pi_1^*\omega^{n-k} = \delta_\ell(P)\delta_k(P) \, .$$

The conclusion follows as above by letting $\varepsilon \to 0$ and applying Fatou's Lemma. □

Note that it follows immediately from Lemma 4.7 that

$$\lambda_{k+\ell}(P) \leq \lambda_k(P)\lambda_\ell(P) .$$

5. The proximity function

Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map, and let $1 \leq \ell \leq \min(n,m)$. Recall that if $\operatorname{codim} \pi_2^{-1}(W) = \ell$, where $\pi_2 : \widetilde{G} \to \mathbb{P}^m$ is given by the commutative diagram (8), then equations (11), (12) define a current $P^*\Lambda_W^\ell \in \mathcal{D}'^{\ell-1,\ell-1}(\mathbb{P}^n)$ with locally integrable coefficients. Hence we can define the *proximity function* $m_P^\ell : \mathbb{G}(\ell,m) \to [0,+\infty]$ by

$$m_P^\ell(W) = \begin{cases} (P^*\Lambda_W^\ell, \omega^{n-\ell+1}) = \displaystyle\int_{\mathbb{P}^n \setminus [I_P \cup P^{-1}(W)]} P^*\Lambda_W^\ell \wedge \omega^{n-\ell+1} & \\ & \text{if } \operatorname{codim} \pi_2^{-1}(W) = \ell\,, \\ +\infty & \text{if } \operatorname{codim} \pi_2^{-1}(W) < \ell \end{cases}$$

for $W \in \mathbb{G}(\ell,m)$. We give $\mathbb{G}(\ell,m)$ the Kähler metric ω induced from the natural embedding $\mathbb{G}(\ell,m) \subset \mathbb{P}(\bigwedge^{m+1-\ell}\mathbb{C}^{m+1})$.

The following key estimate is used in our proof of Theorem 1.2.

***Lemma* 5.1.** *If $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is a rational map, then $m_P^\ell \in \mathcal{L}^1(\mathbb{G}(\ell,m))$ and*

$$dd^c m_P^\ell \leq \ell\delta_{\ell-1}(P)\omega ,$$

for $1 \leq \ell \leq \min(n,m)$.

Before proving Lemma 5.1, we give a brief outline of our approach, which was inspired by Skoda [Sk1, Sk2]. We consider the (n,n)-current on $\mathbb{P}_z^n \times \mathbb{G}(\ell,m)_W$,

$$\Omega = \log \frac{|P(z)|^2|W|^2}{|P(z)\wedge W|^2}\omega_z^{n-\ell+1} \wedge \sum_{j=0}^{\ell-1}(dd^c \log|P(z)\wedge W|^2)^j \wedge (dd^c \log|P(z)|^2)^{\ell-1-j} . \tag{17}$$

We shall show that Ω has locally integrable coefficients and that

$$m_P^\ell = \pi_*\Omega , \tag{18}$$

where $\pi : \mathbb{P}^n \times \mathbb{G}(\ell,m) \to \mathbb{G}(\ell,m)$ is the projection. We then differentiate (17) and obtain the inequality

$$\begin{aligned} dd^c m_P^\ell &= \pi_* dd^c\Omega \\ &\leq \pi_*\Big[\omega_W \wedge \omega_z^{n-\ell+1} \wedge \textstyle\sum_0^{\ell-1}(dd^c \log|P(z)\wedge W|^2)^j \\ &\qquad \wedge (dd^c \log|P(z)|^2)^{\ell-1-j}\Big] . \end{aligned}$$

Finally, we apply Lemma 4.5 to obtain the desired bound on $dd^c m_P^\ell$.

In order to avoid various difficulties with multiplying and pulling back currents, we shall define Ω as the pull-back (by a rational map) of a current $\widehat{\Omega}$ on a larger product space X. The details are as follows.

Proof of Lemma 5.1. We assume first that $\ell \geq 2$. (The estimate of Lemma 5.1 is straightforward for the hyperplane case $\ell = 1$; we give the argument for this case at the end of this proof. See also [RS] for a complete treatment of pre-images of hyperplanes.) Write $E = \bigwedge^{m+1-\ell} \mathbb{C}^{m+1}$, $\widetilde{E} = \bigwedge^{m+2-\ell} \mathbb{C}^{m+1}$ and let $\lambda_0 : E \to \mathbb{C}$, $\lambda_1 : \widetilde{E} \to \mathbb{C}$ be linear functions of unit norm. For $\zeta \in \mathbb{C}^{m+1}$, $W \in E$, we define the *augmented exterior product*

$$\zeta \widetilde{\wedge} W = (\zeta_0 W_0, \zeta \wedge W) \in \mathbb{C} \oplus \widetilde{E}$$

where $W_0 = \lambda_0(W)$. By making a linear change of coordinates in $\mathbb{P}^m$ we can assume without loss of generality that $\operatorname{Image} P \not\subset \{\zeta \in \mathbb{P}^m : \zeta_0 = 0\}$; we also assume that λ_1 is chosen so that $\lambda_1(P(z) \wedge W) \not\equiv 0$.

We write

$$X = \mathbb{P}^n_z \times \mathbb{G}(\ell, m)_W \times \mathbb{P}^m_\zeta \times \mathbb{P}(\widetilde{E})_\theta \times \mathbb{P}(\mathbb{C} \oplus \widetilde{E})_\eta \,.$$

(The subscripts z, W, ζ, θ, η serve to identify the variables used in this discussion.) Let $Q : \mathbb{P}^n \times \mathbb{G}(\ell, m) \dashrightarrow X$ be the meromorphic (rational) map given by

$$Q(z, W) = (z, W, P(z), P(z) \wedge W, P(z) \widetilde{\wedge} W) \,. \tag{19}$$

Let $I_Q \subset \mathbb{P}^n \times \mathbb{G}(\ell, m)$ denote the indeterminacy locus of Q; I_Q is an algebraic subvariety of codimension ≥ 2. Write $U = \mathbb{P}^n \times \mathbb{G}(\ell, m) \setminus I_Q$, and let $Q_0 = Q|_U : U \to X$.

We further write $\theta_1 = \lambda_1(\theta)$ for $\theta \in \widetilde{E}$. If $\eta = (c, \theta) \in \mathbb{C} \oplus \widetilde{E}$, we write $\eta_0 = c$, $\eta_1 = \theta_1 = \lambda_1(\theta)$. We consider the current

$$\widehat{\Omega} = \log \frac{|\zeta|^2 \, |W|^2 \, |\theta_1|^2 \, |\eta_0|^2}{|\zeta_0|^2 \, |W_0|^2 \, |\theta|^2 \, |\eta_1|^2} \omega_z^{n-\ell+1} \wedge \sum_{j=0}^{\ell-1} \omega_\theta^j \wedge \omega_\zeta^{\ell-1-j} \in \mathcal{D}'^{n,n}(X) \,.$$

Since

$$\frac{|\zeta| \, |W| \, |\theta_1| \, |\eta_0|}{|\zeta_0| \, |W_0| \, |\theta| \, |\eta_1|} \circ Q = \frac{|P(z)| \, |W|}{|P(z) \wedge W|} \,,$$

it follows that $Q_0^* \widehat{\Omega}$ is the smooth form on U given by the right side of equation (17).

We now show that $Q_0^* \widehat{\Omega}$ extends to a current Ω with locally integrable coefficients on all of $\mathbb{P}^n \times \mathbb{G}(\ell, m)$. Let

$$Y \xrightarrow{\rho} \operatorname{Image} Q \subset X$$

be a desingularization of the image of Q. (Note that Image Q is an algebraic subvariety of X and can be identified with the graph of Q.) Consider the projection $\pi' : X \to \mathbb{P}^n \times \mathbb{G}(\ell, m)$ and write $\rho_1 = \pi' \circ \rho$, $\rho_2 = \pi \circ \rho_1$, so that we have the commutative diagram:

$$\begin{tikzcd}
\mathbb{P}^n \times \mathbb{G}(\ell,m) \arrow[d, dashed, "Q"'] & Y \arrow[l, "\rho_1"'] \arrow[dl, "\rho"'] \arrow[d, "\rho_1"] \arrow[dr, "\rho_2"] & \\
X \arrow[r, "\pi'"] & \mathbb{P}^n \times \mathbb{G}(\ell,m) \arrow[r, "\pi"] & X
\end{tikzcd}$$

We then define $\Omega = \rho_{1*}\rho^*\widehat{\Omega}$, where $\rho^*\widehat{\Omega}$ is given as in Section 3. Since ρ_1 maps $\rho_1^{-1}(U)$ biholomorphically onto U, we have $Q_0 = \rho \circ \rho_1^{-1}|_U$, and therefore

$$\Omega|_U = (\rho_1^{-1}|_U)^*\rho^*\widehat{\Omega} = Q_0^*\widehat{\Omega} \, .$$

Since Ω has coefficients in $\mathcal{L}^1_{\text{loc}}$, it is the extension to $\mathbb{P}^n \times \mathbb{G}(\ell, m)$ of $Q_0^*\widehat{\Omega}$ with zero mass on I_Q.

We now verify equation (18):

***Lemma* 5.2.** $m_P^\ell = \rho_{2*}\rho^*\widehat{\Omega} = \pi_*\Omega \in \mathcal{L}^1(\mathbb{G}(\ell, m))$.

Proof. By the definition of Ω we have

$$\rho_{2*}\rho^*\widehat{\Omega} = \pi_*\rho_{1*}\rho^*\widehat{\Omega} = \pi_*\Omega \, . \tag{20}$$

Since Ω has coefficients in $\mathcal{L}^1_{\text{loc}}$, $\pi_*\Omega \in \mathcal{L}^1(\mathbb{G}(\ell, m))$ and

$$(\pi_*\Omega)(W) = \int_{\mathbb{P}^n_z \times \{W\} \setminus I_Q} \Omega < +\infty \, , \quad \text{for a.a. } \; W \in \mathbb{G}(\ell, m) \, . \tag{21}$$

Therefore

$$m_P^\ell(W) = \int_{\mathbb{P}^n \setminus [I_P \cup P^{-1}(W)]} P^*\Lambda_W^\ell \wedge \omega^{n-\ell+1} = \int_{\mathbb{P}^n_z \times \{W\} \setminus I_Q} \Omega \tag{22}$$

whenever $\operatorname{codim} \pi_2^{-1}(W) = \ell$ and thus for a.a. $W \in \mathbb{G}(\ell, m)$. The desired identity follows from (20), (21), and (22). □

We are now ready to compute $dd^c m_P^\ell$. We let H_ζ^0, H_W^0, H_θ^1, H_η^0, H_η^1 denote the hyperplanes in X given by the divisors of $\zeta_0, W_0, \theta_1, \eta_0, \eta_1$ respectively, and we let

$$D = H_\zeta^0 + H_W^0 - H_\theta^1 + H_\eta^1 - H_\eta^0 = \operatorname{Div}\left(\frac{\zeta_0 W_0 \eta_1}{\theta_1 \eta_0}\right) \, .$$

We have

$$dd^c\widehat{\Omega} = (\omega_\zeta + \omega_W - \omega_\theta - D) \wedge \omega_z^{n-\ell+1} \wedge \sum_{j=0}^{\ell-1} \omega_\theta^j \wedge \omega_\zeta^{\ell-1-j} \tag{23}$$

$$= \left[\omega_\zeta^\ell - \omega_\theta^\ell + (\omega_W - D) \wedge \sum_{j=0}^{\ell-1} \omega_\theta^j \wedge \omega_\zeta^{\ell-1-j}\right] \wedge \omega_z^{n-\ell+1} .$$

Lemma 5.3. *The divisor $\rho^* D$ is effective (i.e., is locally the divisor of a holomorphic function).*

Proof. (Our choices of ζ_0 and θ_1 guarantee that $\rho(Y) \not\subset \operatorname{Supp} D$ so that $\rho^* D$ is defined.) Let $y_0 \in Y$ be arbitrary, and let

$$X' = \{(z, W, \zeta, \theta, \eta) \in X : z_i \neq 0, \lambda(W) \neq 0, \zeta_j \neq 0, \lambda'(\theta) \neq 0\}$$

where $0 \leq i \leq n$, $0 \leq j \leq m$ and $\lambda : E \to \mathbb{C}, \lambda' : \widetilde{E} \to \mathbb{C}$ are linear functions chosen so that $\rho(y_0) \in X'$. Let

$$g = \frac{\widetilde{\zeta}_0 \widetilde{W}_0}{\widetilde{\theta}_1} \frac{\eta_1}{\eta_0} \in \operatorname{Mer}(X')$$

where $\widetilde{\zeta}_0 = \zeta_0/\zeta_j$, $\widetilde{W}_0 = W_0/\lambda(W) = (\lambda_0/\lambda)(W)$, $\widetilde{\theta}_1 = \theta_1/\lambda'(\theta) = (\lambda_1/\lambda')(\theta)$. Then $\operatorname{Div}(g) = D|_{X'}$. We must show that $g \circ \rho$ is holomorphic at y_0. Now

$$g \circ Q = \frac{P_0(z)}{P_j(z)} \frac{W_0}{\lambda(W)} \frac{\lambda'(P(z) \wedge W)}{\lambda_1(P(z) \wedge W)} \frac{\lambda_1(P(z) \wedge W)}{P_0(z) W_0}$$

$$= \frac{\lambda'(P(z) \wedge W)}{P_j(z)\lambda(W)} = \frac{\lambda'(P(\widetilde{z}) \wedge \widetilde{W})}{P_j(\widetilde{z})}$$

where $\widetilde{z} = z_i^{-1} z \in \mathbb{C}^{n+1}$, $\widetilde{W} = \lambda(W)^{-1} W \in E$. Write, for $y \in Y$,

$$\rho_1(y) = (z(y), W(y)) ,$$

$\widetilde{z}(y) = z_i^{-1}(y) z(y), \widetilde{W}(y) = \lambda(W(y))^{-1} W(y)$. Since ρ is holomorphic and $\rho(y_0) \notin \operatorname{Div}(\zeta_j)$, there is a neighborhood Y_0 of y_0 so that $P_\mu(\widetilde{z}(y)) = \varphi(y) f_\mu(y)$ for $y \in Y_0$ and $0 \leq \mu \leq m$, where $\varphi, f_0, \ldots, f_m \in \mathcal{O}(Y_0), \varphi \not\equiv 0$, and $f_j(y_0) \neq 0$. Thus

$$g \circ \rho = g \circ Q \circ \rho_1 = \frac{\lambda'(\varphi F \wedge \widetilde{W})}{\varphi f_j} = \frac{\lambda'(F \wedge \widetilde{W})}{f_j}$$

on Y_0, where $F = (f_0, \ldots f_m)$. Since $f_j(y_0) \neq 0$, $g \circ \rho$ is holomorphic at y_0. $\square$

We now complete the proof of Lemma 5.1: By Lemma 5.3 and (23), we have

$$\rho^* dd^c \widehat{\Omega} \leq \rho^* \left[\left(\omega_\zeta^\ell + \omega_W \wedge \sum_{j=0}^{\ell-1} \omega_\theta^j \wedge \omega_\zeta^{\ell-1-j}\right) \wedge \omega_z^{n-\ell+1}\right] .$$

By Section 3, $\rho^* dd^c\widehat{\Omega} = dd^c\rho^*\widehat{\Omega}$. Hence by Lemma 5.2,

$$dd^c m_P^\ell = dd^c(\rho_{2*}\rho^*\widehat{\Omega}) = \rho_{2*}\rho^* dd^c\widehat{\Omega}$$

$$\leq \rho_{2*}\rho^*\left[(\omega_\zeta^\ell + \omega_W \wedge \sum_{j=0}^{\ell-1}\omega_\theta^j \wedge \omega_\zeta^{\ell-1-j}) \wedge \omega_z^{n-\ell+1}\right] .$$

For any smooth form η on X, we define as before $Q^*\eta = \rho_{1*}\rho^*\eta$, which is the unique extension with $\mathcal{L}^1_{\text{loc}}$ coefficients of the form $Q_0^*\eta$, and we have $\rho_{2*}\rho^*\eta = \pi_* Q^*\eta$. Since

$$Q_0^*(\omega_\zeta^\ell \wedge \omega_z^{n-\ell+1}) = (dd^c \log|P(z)|^2)^\ell \wedge \omega_z^{n-\ell+1} = 0 ,$$

we obtain

$$dd^c m_P^\ell \leq \sum_{j=0}^{\ell-1} \pi_* Q^*\left(\omega_W \wedge \omega_\theta^j \wedge \omega_\zeta^{\ell-1-j} \wedge \omega_z^{n-\ell+1}\right)$$

$$= \sum_{j=0}^{\ell-1}\left(\int_{\mathbb{P}_z^n\times\{W\}\setminus I_Q} (dd^c \log|P(z)\wedge W|^2)^j \wedge (P^*\omega_\zeta)^{\ell-1-j} \wedge \omega_z^{n-\ell+1}\right)\omega_W .$$

By Lemma 4.5 with $L : \mathbb{C}^{m+1} \to \widetilde{E}$ given by $L(\zeta) = \zeta \wedge W$,

$$\int_{\mathbb{P}_z^n\times\{W\}\setminus I_Q} (dd^c \log|P(z)\wedge W|^2)^j \wedge (P^*\omega_\zeta)^{\ell-1-j} \wedge \omega_z^{n-\ell+1} \leq \delta_{\ell-1}(P) ,$$

and the desired bound on $dd^c m_P^\ell$ follows.

We now modify (and simplify) the above argument for the case $\ell = 1$: Identify

$$G(1, m) = \mathbb{P}(\textstyle\bigwedge^m \mathbb{C}^{m+1}) = \mathbb{P}(\mathbb{C}^{m+1*}) = \mathbb{P}^{m*} .$$

Let $X = \mathbb{P}_z^n \times \mathbb{P}_W^{m*} \times \mathbb{P}_\zeta^m$ and consider the current

$$\widehat{\Omega} = \log\frac{|\zeta|^2|W|^2}{|(W,\zeta)|^2}\omega_z^n \in \mathcal{D}'^{n,n}(X) .$$

Then

$$dd^c\widehat{\Omega} = (\omega_\zeta + \omega_W - D) \wedge \omega_z^n ,$$

where $D = \operatorname{Div}(W,\zeta)$. Then $m_P^1 = \pi_* Q^*\widehat{\Omega}$, where

$$Q = (z, W, P(z)) : \mathbb{P}_z^n \times \mathbb{P}_W^{m*} \to X .$$

We conclude as before that $dd^c m_P^1 = \pi_* Q^* dd^c\widehat{\Omega} \leq \omega_W$. □

Remark. We could use equalities (11), (12) to define $m_P^\ell(W)$ for all $W \in \mathbb{P}(\bigwedge^{m+1-\ell}\mathbb{C}^{m+1})$; then Lemma 5.1 remains valid on $\mathbb{P}(\bigwedge^{m+1-\ell}\mathbb{C}^{m+1})$.

6. Description of the exceptional set

In this section we prove Theorem 1.2, giving a description of the exceptional set $\mathcal{E}$ in terms of the proximity function as follows. For a rational map $P : \mathbb{P}^n \to \mathbb{P}^m$, we write

$$B_P^\ell = \left\{ W \in \mathbb{G}(\ell, m) : \operatorname{codim} \pi_2^{-1}(W) < \ell \right\} , \tag{24}$$

where π_2 is given by (8). The set B_P^ℓ is a proper algebraic subvariety (possibly empty) of $\mathbb{G}(\ell, m)$. Recall that $P^*\Lambda_\ell^W$ and $P^*[W]$ are defined by (12) and the Poincaré-Lelong formula (13) holds whenever $W \in \mathbb{G}(\ell, m) \setminus B_P^\ell$. By the definition of the proximity function at the beginning of Section 5, $m_P^\ell(W) = +\infty$ if and only if $W \in B_P^\ell$. Now let $\mathcal{P} = \{P_k\}$ be a sequence of rational mappings from $\mathbb{P}^n$ to $\mathbb{P}^m$ as in Theorem 1.2 and fix a sequence $\mathcal{A} = \{a_k\}$ of positive numbers. We define the *exceptional set*

$$\mathcal{E}_\ell(\mathcal{P}, \mathcal{A}) = \left\{ W \in \mathbb{G}(\ell, m) : \limsup_{k \to \infty} \frac{m_{P_k}^\ell(W)}{a_k} > 0 \right\} . \tag{25}$$

(In particular, $\mathcal{E}_\ell(\mathcal{P}, \mathcal{A})$ contains the set $\bigcap_k \bigcup_{j=k}^\infty B_{P_j}^\ell$.)

The following two propositions yield Theorem 1.2 with $\mathcal{E} = \mathcal{E}_\ell(\mathcal{P}, \mathcal{A})$.

***Proposition* 6.1.** *Let $\mathcal{P} = \{P_k\}$ be a sequence of rational mappings from $\mathbb{P}^n$ to $\mathbb{P}^m$ and let $\mathcal{A} = \{a_k\}$ be a sequence of positive numbers. Let $1 \le \ell \le \min(n, m)$. Then for all $W \in \mathbb{G}(\ell, m) \setminus \mathcal{E}_\ell(\mathcal{P}, \mathcal{A})$,*

$$\frac{1}{a_k} \left(P_k^*[W] - P_k^* \omega^\ell \right) \to 0$$

as $k \to \infty$.

***Proposition* 6.2.** *Let $\mathcal{P} = \{P_k\}$, $\mathcal{A} = \{a_k\}$ be as in Proposition 6.1. If*

$$\sum_{k=1}^\infty \frac{\delta_{\ell-1}(P_k)}{a_k} < +\infty ,$$

then $\mathcal{E}_\ell(\mathcal{P}, \mathcal{A})$ is pluripolar in $\mathbb{G}(\ell, m)$.

Before proving Propositions 6.1 and 6.2, we note a corollary to Theorem 1.2 on the equidistribution of pre-images for subsequences of a given sequence of rational mappings. This corollary uses the following terminology: For a current $T \in \mathcal{D}'^{p,p}(\mathbb{P}^n)$ of order 0, we let $\|T\|$ denote the total variation measure of T, which is the regular measure on $\mathbb{P}^n$ given by

$$\|T\|(U) = \sup \left\{ |(T, \varphi)| : \varphi \in \mathcal{D}^{n-p, n-p}(U), \|\varphi\| \le 1 \right\}$$

for U open in $\mathbb{P}^n$. Here, $\|\varphi\|$ denotes the comass norm of a compactly supported form φ (see Federer [Fe, 1.8.1, 4.1.7]). The quantity $\|T\|(\mathbb{P}^n)$ is called the mass of T. If T is positive, then it follows from Wirtinger's inequality (see for example, [Fe, 1.8.2]) that $\|T\|(\mathbb{P}^n) = (T, \omega^{n-p})$. In particular, $\|\omega^p\|(\mathbb{P}^n) = 1$.

Corollary 6.3. *Let $\mathcal{P} = \{P_k\}$ be as in Theorem 1.2, let $1 \le \ell \le \min(n, m)$, and suppose that*

$$\frac{\delta_{\ell-1}(P_k)}{\delta_\ell(P_k)} \to 0 \, .$$

Let $\mathcal{M} \subset \mathcal{D}'^{\ell,\ell}(\mathbb{P}^n)$ be the set of weak cluster points of $\{(1/\delta_\ell(P_k))P^{k*}\omega^\ell\}$. Then $\mathcal{M} \neq \emptyset$, and for every current $\eta \in \mathcal{M}$,

(i) *η is a positive current*
(ii) *η has mass 1*
(iii) *there is a subsequence $\{P'_k\}$ of $\{P_k\}$ and a pluripolar set $\mathcal{E}$ such that*

$$\frac{1}{\delta_\ell(P'_k)} P'^*_k[W] \to \eta$$

for all $W \in \mathbb{G}(\ell, m) \setminus \mathcal{E}$.

Proof (*assuming Theorem 1.2*). Since

$$\left\| \frac{1}{\delta_\ell(P_k)} P_k^* \omega^\ell \right\| (\mathbb{P}^n) = \frac{1}{\delta_\ell(P_k)} \int P_k^* \omega^\ell \wedge \omega^{n-\ell} = 1,$$

it follows that $\mathcal{M} \neq \emptyset$. Let

$$\eta = \lim_{k\to\infty} \frac{1}{\delta_\ell(P'_k)} P'^*_k \omega^\ell \in \mathcal{M}$$

for some subsequence $\{P'_k\}$ of $\mathcal{P}$. Then (i) is obvious, and (ii) follows from

$$\|\eta\|(\mathbb{P}^n) = (\eta, \omega^{\ell-1}) = \lim_{k\to\infty} \frac{1}{\delta_\ell(P'_k)} (P'^*_k \omega^\ell, \omega^{n-\ell}) = 1 \, .$$

Finally, choose a subsequence $\{P''_k\}$ of $\{P'_k\}$ such that

$$\sum \frac{\delta_{\ell-1}(P''_k)}{\delta_\ell(P''_k)} < +\infty,$$

and apply Theorem 1.2 with $a_k = \delta_\ell(P''_k)$ to obtain (iii). □

Proof of Proposition 6.1. Let $\varphi \in \mathcal{D}^{n-\ell,n-\ell}(\mathbb{P}^n)$ be an arbitrary real form and choose a constant c_φ such that $-c_\varphi \omega^{n-\ell+1} \le dd^c\varphi \le c_\varphi \omega^{n-\ell+1}$. Then for all $W \in \mathbb{G}(\ell, m) \setminus B^\ell_{P_k}$, we have by the Poincaré-Lelong formula (13),

$$|(P_k^*[W] - P_k^*\omega^\ell, \varphi)| = |(P_k^* \Lambda^\ell_W, dd^c\varphi)| \le c_\varphi m^\ell_{P_k}(W) \, . \tag{26}$$

The conclusion follows from the definition of $\mathcal{E}_\ell(\mathcal{P}, \mathcal{A})$. □

Proof of Proposition 6.2. Let $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map. By (22) we can write

$$P^* \Lambda_W^\ell \wedge \omega_z^{n-\ell+1} = f(z, W)\omega_z^n ,$$

where

(i) $0 \le f(z, W) < +\infty$,
(ii) $f \in \mathcal{L}^1(\mathbb{P}^n \times \mathbb{G}(\ell, m))$,
(iii) f is continuous on $\mathbb{P}^n \times \mathbb{G}(\ell, m) \setminus I_Q$.

Here Q is the map given by equation (19) in Section 5. We set $f(z, W) = 0$ for $(z, W) \in I_Q$ so that f is lower semi-continuous on $\mathbb{P}^n \times \mathbb{G}(\ell, m)$, and we write

$$\widehat{m}_P^\ell(W) = \int_{\mathbb{P}_z^n} f(z, W)\omega_z^n ,$$

for $W \in \mathbb{G}(\ell, m)$. Since f is lower semi-continuous, it follows immediately from Fatou's Lemma that

$$\widehat{m}_P^\ell(W_0) \le \liminf_{W \to W_0} \widehat{m}_P^\ell(W)$$

for $W_0 \in \mathbb{G}(\ell, m)$, and thus $\widehat{m}_P^\ell$ is lower semi-continuous on $\mathbb{G}(\ell, m)$. By the definition of the proximity function at the beginning of Section 5,

$$m_P^\ell(W) = \widehat{m}_P^\ell(W) , \quad \text{for} \quad W \in \mathbb{G}(\ell, m) \setminus B_P^\ell .$$

A function u on a complex manifold Y (with values in $\mathbb{R} \cup -\{\infty\}$) is said to be *quasi-plurisubharmonic* if u is locally equal to the sum of a $\mathcal{C}^\infty$ function and a plurisubharmonic function; an element $\widetilde{u} \in \mathcal{L}^1_{\text{loc}}$ (has a representative that) is quasi-plurisubharmonic if and only if $dd^c\widetilde{u}$ is bounded below by a continuous real $(1,1)$-form. By Lemma 5.1, $-m_P^\ell$ is equal almost everywhere to a (pointwise-defined) quasi-plurisubharmonic function $-\widetilde{m}_P^\ell$ on $\mathbb{G}(\ell, m)$. Since $-\widehat{m}_P^\ell$ is upper semi-continuous and equal almost everywhere to $-\widetilde{m}_P^\ell$, it follows that $-\widehat{m}_P^\ell \ge -\widetilde{m}_P^\ell$. (If an upper semi-continuous function g is equal almost everywhere to a quasi-plurisubharmonic function $\widetilde{g}$, then it follows immediately by reducing to the plurisubharmonic case that $g \ge \widetilde{g}$.)

We now let $\mathcal{P} = \{P_k\}$, $\mathcal{A} = \{a_k\}$ be as in Proposition 6.2 and suppose that $\sum(\delta_{\ell-1}(P_k))/a_k < +\infty$. Let

$$\mathcal{E}'_\ell(\mathcal{P}, \mathcal{A}) = \left\{ W \in \mathbb{G}(\ell, m) : \sum_{k \to \infty} \frac{m_{P_k}^\ell(W)}{a_k} = +\infty \right\} \supset \mathcal{E}_\ell(\mathcal{P}, \mathcal{A}) .$$

We shall show that $\mathcal{E}'_\ell(\mathcal{P}, \mathcal{A})$ is pluripolar; it suffices to show that it is pluripolar in an arbitrary affine open set

$$G' \stackrel{\text{def}}{=} \{W \in \mathbb{G}(\ell, m) : \lambda(W) \ne 0\}$$

given by a linear function $\lambda : \bigwedge^{m+1-\ell} \mathbb{C}^{m+1} \to \mathbb{C}$.

We write

$$m_k = \widetilde{m}_{P_k}^\ell \ge \widehat{m}_{P_k}^\ell$$

and we consider the functions $u_k : G' \to \mathbb{R} \cup \{-\infty\}$ given by

$$u_k = v - \frac{m_k}{\ell \delta_{\ell-1}(P_k)}, \tag{27}$$

where

$$v(W) = \log \frac{|W|^2}{|\lambda(W)|^2}. \tag{28}$$

We assume that $\|\lambda\| = 1$, so $v(W) \geq 0$. By Lemma 5.1, u_k is plurisubharmonic on G'. Next, construct the series

$$u = \sum_{k=1}^{\infty} \frac{\delta_{\ell-1}(P_k)}{a_k} u_k = Sv - \frac{1}{\ell} \sum_{k=1}^{\infty} \frac{m_k}{a_k}, \tag{29}$$

where

$$S = \sum_{k=1}^{\infty} \frac{\delta_{\ell-1}(P_k)}{a_k}.$$

Since

$$m_k(W) \geq m^{\ell}_{P_k}(W) \quad \text{for } W \in \mathbb{G}(\ell, m) \setminus B^{\ell}_{P_k}, \tag{30}$$

it follows from (29) that

$$\mathcal{E}'_{\ell}(\mathcal{P}, \mathcal{A}) \cap G' \subset \{W \in G' : u(W) = -\infty\} \cup \Big(\bigcup_k B^{\ell}_{P_k}\Big).$$

Since the sets $B^{\ell}_{P_k}$ are pluripolar it suffices to show that u is plurisubharmonic. We can represent u as a limit of the sequence

$$\tau_k = Sv - \frac{1}{\ell} \sum_{j=1}^{k} \frac{m_j}{a_j}.$$

Since $m_k \geq 0$, $\{\tau_k\}$ is a decreasing sequence of plurisubharmonic functions on G', so the limit u is either plurisubharmonic or identically $-\infty$. To see that the latter case is impossible, we shall average $u(W)$ over all W with respect to Haar probability measure σ on $\mathbb{G}(\ell, m)$. It is well known (e.g., see [Sh, Ch. 2, § 4, Th. 7]) that

$$\int_{\mathbb{G}(\ell,m)} \Lambda^{\ell}_W(z)\, d\sigma(W) = c_{\ell,m} \omega^{\ell-1}(z) \tag{31}$$

for some constant $c_{\ell,m}$. Therefore, by Fubini's theorem and the pull-back of (31) by P_k,

$$\int_{\mathbb{G}(\ell,m)} m^\ell_{P_k}(W)\, d\sigma(W) = \int_{\mathbb{G}(\ell,m)} \left(\int_{\mathbb{P}^n} P_k^* \Lambda^\ell_W \wedge \omega^{n-\ell+1} \right) d\sigma(W)$$
$$= c_{\ell,m} \int_{\mathbb{P}^n} P_k^* \omega^{\ell-1} \wedge \omega^{n-\ell+1} = c_{\ell,m}\delta_{\ell-1}(P_k)\ .$$

Hence

$$\int u(W)\, d\sigma(W) \geq -\frac{1}{\ell} \sum_{k=1}^{\infty} \frac{1}{a_k} \int m_k(W)\, d\sigma(W)$$
$$= -\frac{1}{\ell} \sum_{k=1}^{\infty} \frac{1}{a_k} \int m^\ell_{P_k}(W)\, d\sigma(W)$$
$$= -\frac{1}{\ell} \sum_{k=1}^{\infty} \frac{c_{\ell,m}\delta_{\ell-1}(P_k)}{a_k} = -\frac{c_{\ell,m}}{\ell} S > -\infty\ .$$

Thus u is plurisubharmonic. □

7. Limit measures for iterates of rational maps

In this section we prove Theorem 1.1 and Corollary 1.3. We also show that the iterated pull-backs of certain measures converge in the distribution sense at an exponential rate (Proposition 7.1). We continue to use the notation from Section 6.

Proof of Corollary 1.3. Let $\{P_k\}$ be given as in the corollary, and let $\mathcal{E}$ be the exceptional pluripolar set of Theorem 1.2 with $\ell = m$ and $a_k = \delta_m(P_k)$. We can assume that $\mathcal{E} \supset \bigcup_k B^m_{P_k}$, where $B^m_{P_k}$ is given by (24). Let ν be a probability measure with $\nu(\mathcal{E}) = 0$. We pull back the tautological identity

$$\nu = \int_{\mathbb{P}^n} [W] d\nu(W)$$

to obtain

$$P_k^* \nu = \int_{\mathbb{P}^n \setminus B^m_{P_k}} P_k^*[W] d\nu(W)\ .$$

Let $\varphi \in \mathcal{D}^{n-m,n-m}(\mathbb{P}^n)$ be arbitrary. We then have

$$\frac{1}{\delta_m(P_k)} \left(P_k^*\nu - P_k^*\omega^m, \varphi\right) = \int_{\mathbb{P}^n \setminus \mathcal{E}} f_k d\nu\ , \tag{32}$$

where

$$f_k(W) = \frac{1}{\delta_m(P_k)} (P_k^*[W] - P_k^*\omega^m, \varphi)$$

for $W \in \mathbb{P}^m \setminus \mathcal{E}$. By (13), for $W \in \mathbb{P}^m \setminus B^m_{P_k}$ we have

$$(P_k^*[W], \omega^{n-m}) = (P_k^*\omega^m, \omega^{n-m}) = \delta_m(P_k) ,$$

and thus

$$|f_k(W)| \le \frac{1}{\delta_m(P_k)} (|(P_k^*[W], \varphi)| + |(P_k^*\omega^m, \varphi)|) \le 2 \sup \|\varphi\| . \tag{33}$$

By Theorem 1.2, $f_k \to 0$ pointwise, and thus by (33) we can let $k \to \infty$ in (32) to conclude that

$$\frac{1}{\delta_m(P_k)} (P_k^*\nu - P_k^*\omega^m, \varphi) \to 0 . \qquad \square$$

Proof of Theorem 1.1. Assume that $P : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, $\lambda = \delta_n(P)$ are as in Theorem 1.1. Since $\delta_{n-1}(P^k) \le [\lambda_{n-1}(P) + o(1)]^k$ and $\delta_n(P^k) = \lambda^k$, by Corollary 1.3 (and the remark following the corollary) it suffices to show that the sequence of measures $\{\lambda^{-k} P^{k*}\omega^n\}$ converges weakly to a measure μ. Since the measures $\lambda^{-k} P^{k*}\omega^n$ are probability measures, it suffices to show that the sequence converges weakly in $\mathcal{D}'^{n,n}(\mathbb{P}^n)$.

Let

$$h = \frac{1}{\lambda} P^*\omega^n \in \mathcal{D}'^{n,n}(\mathbb{P}^n).$$

By the definition of the topological degree $\delta_n(P)$, $\int_{\mathbb{P}^n} h = 1$. We claim that if f is a quasi-plurisubharmonic function on $\mathbb{P}^n$, then

$$\int_{\mathbb{P}^n} fh > -\infty . \tag{34}$$

(Note that $h \ge 0$; since f is bounded above, the claim is equivalent to saying that fh is $\mathcal{L}^1$.) To verify the claim, we again consider the commutative diagram (8). We then have

$$\int fh = \frac{1}{\lambda} \int_{\widetilde{G}} (f \circ \pi_1) \pi_2^* \omega^n > -\infty$$

since $f \circ \pi_1$ is quasi-plurisubharmonic on $\widetilde{G}$ and hence is in $\mathcal{L}^1$, verifying (34).

Choose $a > 0$ such that $\lambda_{n-1}(P) < a < \lambda$. By (4), the sum

$$S \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{\delta_{n-1}(P^k)}{a^k} < +\infty.$$

Applying Proposition 6.2 with $\ell = n$, $\mathcal{P} = \{P^k\}$, and $\mathcal{A} = \{a^k\}$, we conclude that

$$\frac{m^n_{P^k}(W)}{a^k} \to 0 \tag{35}$$

for points W of $\mathbb{P}^n$ outside a pluripolar set. Since pluripolar sets have Lebesgue measure zero, (35) is valid for a.a. $W \in \mathbb{P}^n$. Write

$$f = \sum_{k=1}^{\infty} \frac{m^n_{P^k}}{a^k} \,.$$

By the proof of Proposition 6.2, f is equal almost everywhere to the function

$$\widetilde{f} \stackrel{\text{def}}{=} nSv - nu \,,$$

where v, u are given by (28),(29); the function u is plurisubharmonic and v is $\mathcal{C}^\infty$ on an (arbitrarily chosen) affine open set in $\mathbb{G}(\ell, m)$. Hence $-\widetilde{f}$ is quasi-plurisubharmonic on $\mathbb{G}(\ell, m)$. Therefore, by (34),

$$\int_{\mathbb{P}^n} fh = \int_{\mathbb{P}^n} \widetilde{f}h < +\infty \tag{36}$$

Let $\varphi \in \mathcal{D}^{0,0}(\mathbb{P}^n)$ be arbitrary. Since $\int_{\mathbb{P}^n} h = 1$,

$$\begin{aligned} \frac{1}{a^k}(P^{k*}\omega^n - P^{k*}h, \varphi) &= \frac{1}{a^k}\int (P^{k*}\omega^n - P^{k*}[W], \varphi)h(W) \\ &= \int \frac{1}{a^k}(P^{k*}\Lambda^n_W, dd^c\varphi)h(W) \,, \end{aligned} \tag{37}$$

where the second equality follows from the Poincaré-Lelong formula (13). By (26),

$$\left| \frac{1}{a^k}(P^{k*}\Lambda^n_W, dd^c\varphi) \right| \le \frac{c_\varphi m^n_{P^k}(W)}{a^k} \le c_\varphi f(W) \,, \tag{38}$$

and hence by (35),

$$\frac{1}{a^k}(P^{k*}\Lambda^n_W, dd^c\varphi) \to 0 \quad \text{for } h\text{-a.a. } W \,. \tag{39}$$

By (36), (38), and (39), we can let $k \to +\infty$ in (37) and apply Lebesgue's dominated convergence theorem to conclude that

$$\frac{1}{a^k}(P^{k*}\omega^n - P^{k*}h, \varphi) \to 0 \tag{40}$$

as $k \to +\infty$.

We note that

$$\frac{1}{\lambda^{k+1}}(P^{k+1})^*\omega^n = \frac{1}{\lambda^k}P^{k*}h,$$

as an identity of currents with $\mathcal{L}^1_{\text{loc}}$ coefficients. Hence by (40),

$$\begin{aligned} \left| \left(\frac{1}{\lambda^{k+1}}(P^{k+1})^*\omega^n - \frac{1}{\lambda^k}P^{k*}\omega^n, \varphi \right) \right| &= \frac{1}{\lambda^k} \left| \left(P^{k*}h - P^{k*}\omega^n, \varphi \right) \right| \\ &\le o\left((a/\lambda)^k \right) \,. \end{aligned} \tag{41}$$

Therefore the sequence

$$\left\{\left(\frac{1}{\lambda^k}P^{k*}\omega^n,\varphi\right)\right\}$$

is Cauchy. Since φ is arbitrary, it follows that the sequence $\{(1/\lambda^k)P^{k*}\omega^n\}$ converges weakly in $\mathcal{D}'^{n,n}(\mathbb{P}^n)$. □

It follows from the above proof that for each smooth test function φ on $\mathbb{P}^n$, the sequence $\{(\lambda^{-k}P^{k*}\delta_W-\mu,\varphi)\}$ converges to 0 at an exponential rate. Precisely, we have the following result:

***Proposition* 7.1.** *Let $P,\lambda>\lambda_{n-1}(P),\mu$ be as in Theorem 1.1. Then there is a pluripolar set $\mathcal{E}\subset\mathbb{P}^n$ such that for all $\mathcal{C}^2$ functions φ on $\mathbb{P}^n$:*

(i) *if $W\in\mathbb{P}^n\setminus\mathcal{E}$, then*

$$\limsup_{k\to\infty}\left|(\lambda^{-k}P^{k*}\delta_W-\mu,\varphi)\right|^{1/k}\le\frac{\lambda_{n-1}(P)}{\lambda}\ ;$$

(ii) *if $g\in\mathcal{L}^\infty(\mathbb{P}^n)$, then*

$$\limsup_{k\to\infty}\left|(\lambda^{-k}P^{k*}(g\omega^n)-c_g\mu,\varphi)\right|^{1/k}\le\frac{\lambda_{n-1}(P)}{\lambda}$$

where $c_g=\int g\omega^n$.

Proof. By Theorem 1.2, we can find a pluripolar set $\mathcal{E}$ such that (35) holds for all $a>\lambda_{n-1}(P)$ and for all $W\in\mathbb{P}^n\setminus\mathcal{E}$. Let $a>\lambda_{n-1}(P)$ be arbitrary. We conclude from (41) that

$$\left|\left(\frac{1}{\lambda^k}P^{k*}\omega^n-\mu,\varphi\right)\right|\le o\left((a/\lambda)^k\right)\ . \tag{42}$$

Part (i) of the proposition follows immediately from Corollary 1.4 and (42). To prove part (ii), we can assume without loss of generality that $g\ge0$ and $c_g=1$. By repeating the part of the proof of Theorem 1.1 from equation (36) to equation (40) with h replaced by $g\omega^n$, we conclude that

$$\frac{1}{a^k}\left(P^{k*}\omega^n-P^{k*}(g\omega^n),\varphi\right)\to0\ , \tag{43}$$

and (ii) then follows from (42) and (43). □

Remark. Proposition 7.1 does not hold for continuous test functions φ. For example, let $P:\mathbb{C}\to\mathbb{C}$ be the map $P(z)=z^\lambda$, $\lambda\ge2$. Let φ be the continuous function on $\mathbb{C}\cup\{\infty\}$ given by $\varphi(z)=\psi(|z|)$ where

$$\psi(r)=\left(\left|\ \log|\log r|\ \right|+1\right)^{-1}\ .$$

Then μ is Haar measure on the circle and $(\mu,\varphi)=0$. Let $w\in\mathbb{C}\setminus\{0\}$ with $|w|\neq 1$, and write $\mu_w^k=\lambda^{-k}P^{k*}\delta_w$. We have

$$(\mu_w^k-\mu,\varphi)=(\mu_w^k,\varphi)=\psi\left(|w|^{1/\lambda^k}\right)=\frac{1}{k\log\lambda+c}$$

for $k\gg 0$, and thus (i) does not hold for this $\mathcal{C}^0$ function φ. This example also shows that convergence in the space of measures does not always hold in Corollary 1.4 if $a<\lambda_\ell(P)$ or in Theorem 1.2.

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