Vision and Approach

In my fellowship, I will amplify the theoretical power of tropical geometric methods in mathematics by developing a bold new foundation for linear algebraic structures in tropical geometry, and thereby enable a new vista of applications across pure and applied domains. Blue skies mathematical research today will enable world-changing applications in the next decades.

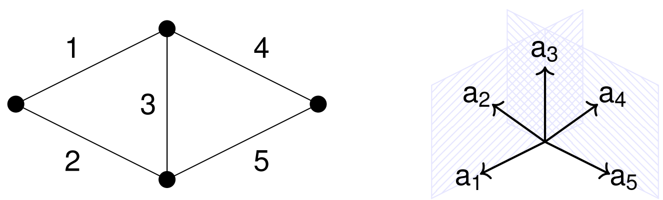
Linear Algebra, whose origins can be traced back to the 179 AD Chinese textbook *"The Nine Chapters on the Mathematical Art"*, is the study vectors, matrices, and systems of linear equations. It is the foundation for all scientific computing and modern engineering, and it underlies much of our technology, e.g., Google's PageRank Algorithm, current AI systems, and quantum computation.

Linear algebra is powerful, but *not* everything in the world is linear. The introduction of **vector bundles** by André Weil allows us to use methods from linear algebra to describe non-linear geometric objects locally. The study of vector bundles is central to many of the breakthroughs of modern geometry in the last 70 years, leading to multiple Fields medals. Starting in the 1980s, the geometry of vector bundles has been enriched by new inputs from theoretical physics, e.g. in the context of gauge theory in quantum field theory. In recent years, profound developments in non-abelian Hodge theory and the geometry of stability conditions have led to another surge of activities in this classical field.

Tropical geometry is a relatively new field of modern mathematics that has seen tremendous growth in the last two decades. Its practitioners number in the hundreds. It has already found numerous applications, for example to phylogenetics in bio-informatics and auction theory in economics. Critical UK financial services are distributed by the Bank of England using tropical methods since the 2007 financial crisis. Recently, tropical geometry has shed new light onto the structure of biochemical reaction networks. It is also becoming increasingly important in the analysis of neural networks in AI and it is used in UK railway scheduling.

The central idea of tropical geometry is to replace the usual addition and multiplication of real numbers with two different operations: the product of two numbers becomes their sum and their sum becomes the minimum. This idea, perhaps seemingly quite outlandish at first, can be traced back to Hungarian-born Brazilian computer scientist Imre Simon. However, it has proved to be extraordinarily fruitful, since it allows us to transform algebraic varieties (geometric objects defined by systems of polynomial equations) into piecewise linear 'origami' objects that break down complicated problems into smaller pieces to be solved on their own and then re-assembled.

Interactions between **linear algebra and tropical geometry** have been profound, as indicated not only by the numerous applications of tropical linear spaces, e.g. in phylogenetics, but also by June Huh's 2022 Fields medal in this field. So far they are limited by a lack of comprehensive foundations. The challenge is the combinatorial complexity of working with **matroids** that generalize collections of vectors, matrices and graphs (see Figure 1). These objects are at same time very accessible in specific situations and provably difficult to understand in full generality.



$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Figure 1: Three different ways of describing the same matroid

The key theoretical barrier is the development of a flexible and comprehensive interface between tropical geometry and linear algebra that incorporates the additional intricacy introduced by the geometry of matroids. A successful implementation of such an interface would not only allow us to explore new applications of tropical methods, e.g. in mathematical physics (see e.g. WP4.4 below), but also help us overcome tropical geometry's current inability to process *vector bundles*.

In my fellowship, I am going to lay the theoretical foundations necessary to overcome this barrier using two new key ingredients: **bimatroids** and **affine buildings**. This will allow me to leverage my unique expertise in combinatorics and arithmetic algebraic geometry in order to develop a **tropical geometry of vector bundles** and, thus, facilitate the next phase in the evolution of tropical geometry. The theoretical aspects of my program will be guided by applications to important open problems in modern geometry, for which vector bundles are a key ingredient. The merging of *a priori* quite different — but equally rich — mathematical theories will lead to several serendipitous developments, including new connections beyond mathematics, e.g. in mathematical physics.

The geometry of **moduli spaces** illuminates properties of mathematical objects that would otherwise be invisible to our mathematical senses. Moduli spaces function like a prism, allowing us the see the full spectrum of light. More precisely, a moduli space is a single geometric meta-object that encodes the behaviour of the full infinite range of individual geometric objects of a certain type. In my project, moduli spaces (in particular, when they are enriched with structure from non-Archimedean and logarithmic geometry) will allow us to perceive tropical geometry as a **piecewise linear limit of classical geometry**. This unique moduli-theoretic perspective distinguishes my project from many previous and contemporary approaches to tropical geometry.

Approach and Work Packages

WP1: Rethinking tropical linear algebra. Tropical linear algebra, in its current state and with a view towards vector bundles, suffers from two essential problems:

1. There is no common language to study tropical vectors, matrices, and linear systems of equations on the one side and the geometry of matroids on the other.
2. Every approach to tropical linear algebra strongly depends on the choice of coordinates.

In order to solve Problem 1, in **WP1.1** I am going to develop the theory of **valuated bimatroids**, which interpolate between tropical matrices and valuated matroids. The central feature of this story will be an algebraic calculus of valuated bimatroids that generalizes the multiplication of matrices in classical linear algebra. A central goal is to give a characterization of affine maps between valuated matroids in terms of valuated bimatroids, expanding [Kun78] beyond the constant coefficients case.

The solution of Problem 2 in **WP1.2** relies on the geometry of **affine buildings**, which are highly symmetric combinatorial and geometric objects that can be used to encode all possible choices of coordinates simultaneously (see Figure 2 for an example). Results from one of my previous articles [BKK+24] show that affine buildings of type A_n may be described as matroids of finite rank on infinite ground sets. This observation will allow me to expand the methods from WP1.1 to this infinite setting in order to find a coordinate-independent way of characterizing linear maps in tropical linear algebra.

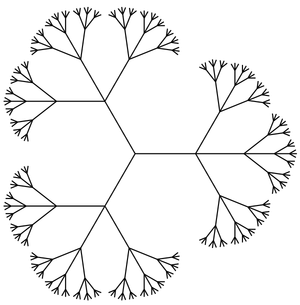


Figure 2: The affine building of $PGL_2(\mathbb{Q}_2)$

A successful completion of these two work packages will, in particular, lead me to introduce and study the tropical geometry of the **classical groups** from a Tannakian point of view. This will constitute a new bridge from tropical geometry to the study of symmetry in classical algebraic geometry.

WP2: Tropical vector bundles. The central objective of this work package is to develop the tropical geometry of vector bundles by combining the matroidal methods from WP1 with techniques from arithmetic algebraic geometry (leveraging my expertise in logarithmic and non-Archimedean geometry). In **WP2.1**, I will lay comprehensive foundations merging several previously unavailable tools:

- my own previous approach to tropical vector bundles via **affine Weyl groups** in [GUZ22] that expands on the very successful geometry of tropical line bundles introduced in [MZ08];
- **non-Archimedean norms** on vector bundles in [CLD12], as championed by Fields medalist Maxim Kontsevich in the context of non-Archimedean stability conditions;
- the **logarithmic Picard group**, brought to perfection by Molcho–Wise in [MW22]; and
- Mumford's **toroidal bordifications** of reductive groups using affine buildings in [KKMSD73] and the study of **toric vector bundles** from the perspective of affine buildings by Kaveh–Manon in [KM22], which already led to another step towards tropical vector bundles in [KM24].

A successful implementation of this objective will enable me to define a flexible notion of tropical vector bundles, whose fibers may be visualized using affine buildings (see Figure 2). Building on this, I will construct **moduli spaces** of vector bundles on degenerate (i.e. broken apart) algebraic varieties in **WP2.2**, using methods from non-Archimedean and logarithmic geometry; the overall goal is a *degeneration formula* for these moduli spaces and, subsequently, their invariants.

WP3: Applications. The goal of WP3 is to connect the theory developed so far to major open problems from algebraic geometry and find applications that transcend tropical geometry.

The central area of applications of tropical geometry, pioneered in the works of Mikhalkin and Nishinou–Siebert, is **enumerate geometry**, i.e. the counting of geometric objects of a certain type. In **WP3.1**, I will revisit vector bundle based enumerative theories, e.g. Donaldson–Thomas theory, from the perspective of tropical geometry. A first goal is the development of a new approach to classical correspondence theorems that likely lead to novel insights beyond the well-known toric cases (e.g. in the case of K3 surfaces, abelian varieties, (log-)Calabi varieties, or Hyperkähler varieties).

Vector bundles and their generalization to coherent sheaves form the basic object of study in the context of **derived categories**. The goal of **WP3.2** is to gain new insights into the derived category of algebraic varieties via degeneration techniques. Promising directions are the study of **Fourier–Mukai duality** (see e.g. my previous article [GKUW24] for first steps) and the open problem of constructing **stability conditions** on higher-dimensional algebraic varieties.

In **WP3.3**, I am going to use the methods developed in WP2 to approach central conjectures in non-abelian Hodge theory, e.g. the **geometric** $P = W$ **conjecture**. This is, in parts, motivated by insights gained in [BKU24], which suggest that the local geometry of this problem on degenerate varieties is solvable. The central difficulty lies in reassembling the local pieces to a global one, a general task in which tropical geometry has excelled in the past.

The aim of **WP3.4** is to generalize the story developed for vector bundles to the case of **principal** G**-bundles**, that is, vector bundles with extra symmetry, employing a Tannakian approach. I expect this to lead to new instances of **mirror symmetry**, a phenomenon of central importance that underlies many of the latest developments in modern geometry and originates from dualities in string theory.

WP4: Serendipities. The highly innovative nature of my program is bound to lead to serendipitous mathematical discoveries along the way. WP4 explores the most promising ones.

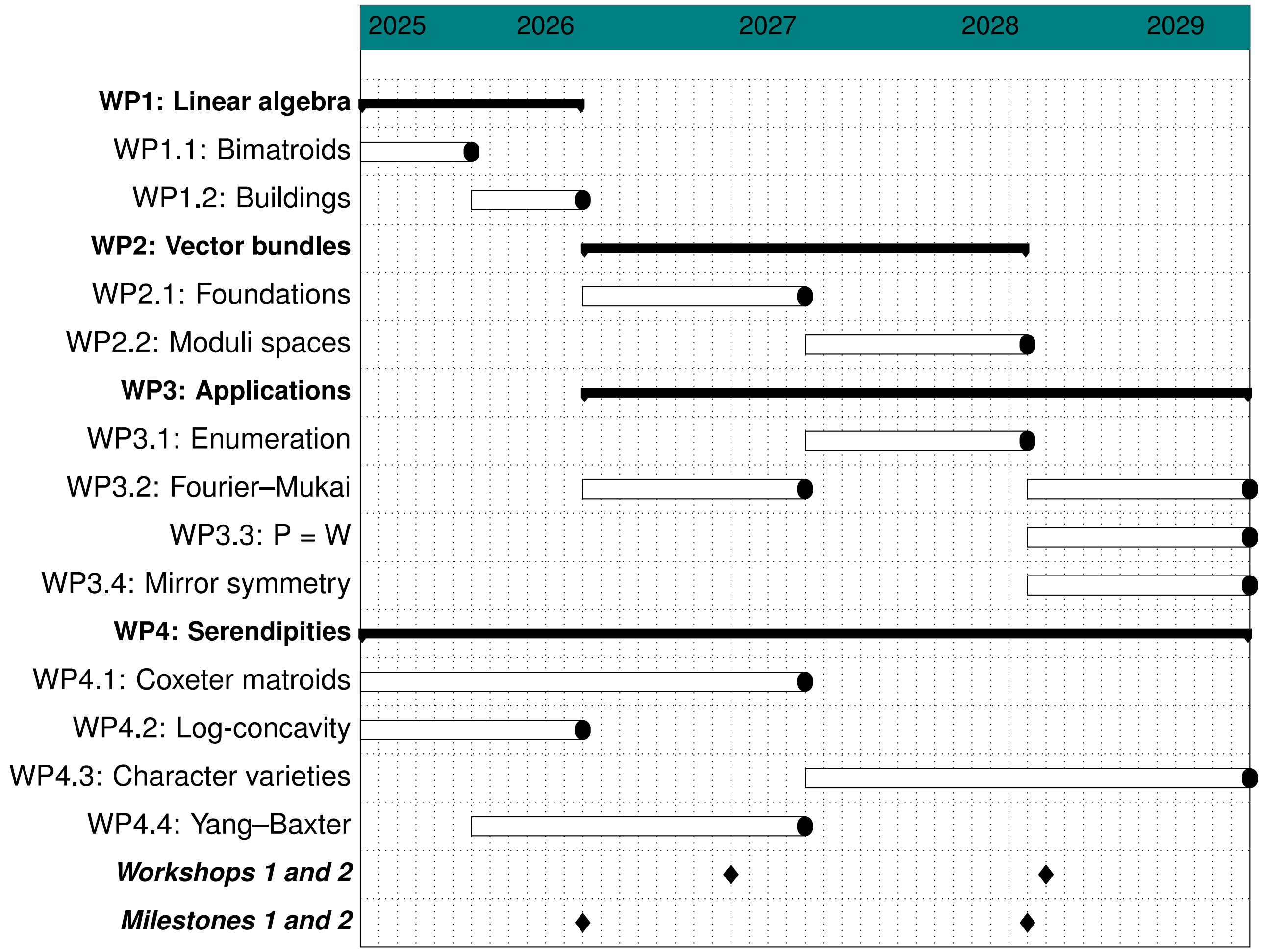
Coxeter matroids are generalizations of matroids that take into account additional symmetries arising from other Dynkin–Lie types. Their generalization to **valuated Coxeter matroids** is an open problem that has seen significant attention in recent years (with many partial solutions). Using insights from the geometry of affine buildings in WP1.2 (where the above-mentioned additional symmetries are well-understood), I will resolve this open problem in **WP4.1** once and for all by providing a Tannakian framework that unifies all the previous approaches.

June Huh's 2022 Fields medal was awarded for the development of techniques (aka *combinatorial Hodge theory*) to prove the **logarithmic concavity** of natural sequences associated to matroids. In **WP4.2**, the developments in WP1.1 would allow me to generalize several of his results to the setting of **valuated matroids** (incorporating ideas from [RU24]). As natural follow-up, I intend to study log-concavity for natural sequences associated to **oriented matroids** and **Coxeter matroids**.

In **WP4.3**, motivated by algebro-geometric phenomena observed in the geometry of parabolic Hitchin fibrations, I will initiate the systematic study of **character varieties** of the fundamental group of an oriented matroid. This is the start of an ambitious long-term program (likely transcending my fellowship), that is best described as the **non-abelian Hodge theory of oriented/phase matroids**. Central insights will emerge from the geometry of parabolic Higgs bundles, expanding on [Sim90].

Tropical geometry may be applied to study the process of dequantization in mathematical physics. In **WP4.4**, tensor products for valuated bimatroids will enable me to propose a tropical analogue of the quantum **Yang–Baxter equation**, a consistency equation in statistical mechanics whose solutions enjoy a particularly rich mathematical structure. On the long run this opens up the path for new interactions of tropical geometry with the rich structure of quantum groups, which, in parts, has been developed in response to the study of quantum Yang–Baxter equations.

Timeline



Work on WP1 will commence right at the beginning of the fellowship and is expected to be completed relatively quickly, culminating in Milestone 1. Once the foundations are laid, I can start working on WP2, which will take up a significant amount of time. Milestone 2 marks its completion. Within WP3, it is reasonable to start working on some aspects of WP3.2 from the beginning, while WP3.1 requires the simultaneous development of WP2.2. The more advanced applications of WP2 in WP3.2, WP3.3, and WP3.4 are scheduled for the last quarter of the fellowship and are meant to be left, in part, open-ended. Within WP4, WP4.1 and WP4.2 can start alongside WP1 and WP4.4, once WP1.1 is completed. The ambitious WP4.3 is scheduled for the second half of the fellowship.

In order to achieve the objectives of this grant, I will hire **two PDRAs**. The first PDRA, who ideally has a background in matroid theory, will be hired for three years starting in Sept. 2025 in order to contribute to the projects in WP4, in particular WP4.1, WP4.3, and WP4.4. The second PDRA should ideally have a background in the geometry of vector bundles that complements my own experience in tropical geometry. PDRA 2 will be hired for three years starting in Sept. 2026 in order to contribute to the projects in WP3, in particular, to WP3.2-3.4.

In order to facilitate progress in this project and to build my reputation as a leading figure in my field, I will organize a workshop titled **"Vector bundles, matroids, and tropical geometry"** in April 2027. In Sept. 2028, I will organize a follow-up to the workshop **"Combinatorial Algebraic Geometry: Highlighting Underrepresented Genders"**, which I am organizing at Goethe University in April 2025. Both PDRAs will be co-organizers of these workshops, which will increase their visibility.

Risk management

While WP1 is relatively low risk (thanks to an extensive literature on both buildings and (bi-)matroids as separate topics), the inclusion of the perspective from logarithmic geometry in WP2 is a high-risk endeavor, since it opens up a completely new area of study. If this does not work out as planned, I will focus more on the perspective coming from non-Archimedean geometry, for which there are already some previous works to rely on, e.g. [CLD12]. This way, I can still approach the applications in WP3, albeit with techniques of a different flavor.

Several of the proposed applications in WP3 concern major open problems of algebraic geometry (e.g. the construction of stability conditions in WP3.2 and the geometric P = W conjecture). Therefore, I expect that even partial results would be received well by the community. While WP4.1 and WP4.2 are relatively low-risk, WP4.3 is a very high-risk/high gain subproject. Due to the open and innovative nature of this project, even partial progress would be quite significant. WP4.4 builds a bridge to the mathematical physics community. Therefore, it is unclear how the results gained in WP4.4 will cross over. I will mitigate this problem by expanding my mathematical network to include researchers from mathematical physics in order to learn and adjust my approach.

While the work of the PDRAs on WP3 and WP4 is an important contribution to this project, possible problems (them leaving for a different position etc.) will not derail the overall success of the project, since I will be responsible for the central foundational parts WP1 and WP2.

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