Algebraic Geometry
Coursework 1

- Available from 12:00 PM on February 4th to 12:00 PM on February 11th, 2025.
- Please submit your work in PDF format on Blackboard.
- If you need clarification or have any questions or concerns, feel free to email me or stop by on Wednesday during the office hour.
- You may discuss only Q5 with each other, but not the rest of the questions. In all cases, please write your solutions in your own words.

Q1. Let $A \subseteq \mathbb{A}^n$ be a subset.

(a) (**5 marks**) What is the definition of the closure of A in $\mathbb{A}^n$?

(b) (**5 marks**) Prove that $\mathbb{V}(\mathbb{I}(A))$ equals the Zariski closure of A in $\mathbb{A}^n$.

(c) (**5 marks**) Give an example of a subset in $B \subseteq \mathbb{C}$ whose closure in the Zariski topology does not coincide with its closure in the Euclidean topology.

Q2. (a) (**5 marks**) What is the definition of a compact subset of a topological space?

(b) (**10 marks**) Prove that $\mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}^2$ is compact in the Zariski topology but not in the Euclidean topology.

Q3. (a) (**5 marks**) Find a curve $W \subseteq \mathbb{A}^2$ and a morphism $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$, such that W is irreducible but $\varphi^{-1}(W)$ is not.

(b) (**5 marks**) Let Y be a topological space and consider $X \subseteq Y$ with the subspace topology. Prove that if X is irreducible then so is its closure.

(c) (**5 marks**) Prove that isomorphisms preserve irreducibility and dimension of closed affine algebraic varieties.

(d) (**10 marks**) Find the irreducible components of $\mathbb{V}(zx - y, y^2 - x^2(x+1)) \subseteq \mathbb{A}^3$. You need to justify why each component is irreducible.

Q4. (a) (**10 marks**) Let $V \subseteq \mathbb{A}^n$ be a Zariski-closed subset and $a \in \mathbb{A}^n \setminus V$ be a point. Find a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that

$$f \in \mathbb{I}(V), \quad f(a) = 1.$$

(b) (**15 marks**) Let $I, (g) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be two ideals. Assume that $\mathbb{V}(g) \supseteq \mathbb{V}(I)$.

(i) Prove that if $I = (f_1, \ldots, f_k)$, then

$$(f_1, \ldots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \ldots, x_{n+1}]. \tag{1}$$

(ii) By only using Equation (1) and not the nullstellensatz, prove that there exists a positive integer m such that $g^m \in I$.

Q5. Prove at least one implication from each of the following equivalences.

(a) (**10 marks**) Show that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is injective if and only if φ is *dominant*. Recall that a map, φ, is called dominant if its image, $\varphi(V)$, is dense in W.

(b) (**10 marks**) Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W.