(i) $\mathbb{V}(f) \cap \mathbb{V}(g) \underset{(1)}{=} \mathbb{V}((f)+(g)) \underset{(2)}{\subseteq} \mathbb{V}((f)\cap(g)) \underset{(3)}{=} \mathbb{V}(fg)$

(1):

$$
\begin{aligned}
\mathbb{V}(f) &= \{(x,y) : f(x,y) = 0\} \\
&= \{(x,y) : h(x,y)f(x,y) = 0 \ \forall h \in \mathbb{C}[x_1, x_2]\} \\
&= \{(x,y) : i(x,y) = 0 \ \forall i \in (f)\} \\
&= \mathbb{V}((f))
\end{aligned}
$$

And by Ex 2.11 (b): $\mathbb{V}((f)) \cap \mathbb{V}((g)) = \mathbb{V}((f)+(g))$

(2):

$\mathbb{V}((f)+(g)) = \mathbb{V}((f)) \cap \mathbb{V}((g)) \subseteq \mathbb{V}((f)) \cup \mathbb{V}((g))$

by Ex 2.11 (a): $= \mathbb{V}((f)\cap(g))$

(3):

$$
\begin{aligned}
\mathbb{V}(fg) &= \{(x,y) : f(x,y)g(x,y) = 0\} \\
&= \{(x,y) : f(x,y) \text{ or } g(x,y) \text{ equal } 0\} \\
&= \{(x,y) : f(x,y) = 0\} \cup \{(x,y) : g(x,y) = 0\} \\
&= \mathbb{V}(f) \cup \mathbb{V}(g) \\
&= \mathbb{V}((f)) \cup \mathbb{V}((g))
\end{aligned}
$$

as $\mathbb{C}[x_1, x_2]$ is an integral domain

Lastly, $\mathbb{V}((f)+(g)) \subseteq \mathbb{V}(f+g)$, since $f(x,y)+g(x,y)=0$ (the (x,y) that satisfy) includes the mutual zeroes of f and g, however there may also be a point (x,y) for which $f(x,y) = -g(x,y)$ (e.g: x^2+y and $x-y^2$ @ $(-1,y)$ for $y \in \mathbb{C}$, even though individually $(-1,y)$ will not zero each polynomial, $(-1,y)$ zeros $x^2+y+x-y$).

Considering this point (x,y) for which $f(x,y) = -g(x,y)$, $\mathbb{V}(f+g)$ cannot be contained in $\mathbb{V}((f)\cap(g))$. And, provided (f) and (g) are distinct, $\mathbb{V}(f+g)$ cannot contain $\mathbb{V}((f)\cap(g))$.

(a) The closure of A in $\mathbb{A}^n$ is the intersection of all closed sets (affine algebraic varieties) containing A.

(b) Let $\bar{A}$ be the closure of A, and $\mathbb{I}(A)$ the ideal of $A := \{ f : \forall x \in A \; (f(x) = 0 \}$.

Clearly $A \subseteq \mathbb{V}(\mathbb{I}(A))$, by def^n of $\mathbb{I}(A)$. And as $\mathbb{V}(\mathbb{I}(A))$ is an affine algebraic variety, it is closed and thus must contain $\bar{A}$; $\bar{A} \subseteq \mathbb{V}(\mathbb{I}(A))$.

Next, using the fact $\mathbb{I}$ and $\mathbb{V}$ are order reversing:
$A \subseteq \bar{A} \to \mathbb{I}(A) \supseteq \mathbb{I}(\bar{A}) \to \mathbb{V}(\mathbb{I}(A)) \subseteq \mathbb{V}(\mathbb{I}(\bar{A}))$

And as $\bar{A}$ is a closed set in the Zariski topology, it is already an affine algebraic variety and so $\mathbb{V}(\mathbb{I}(\bar{A})) = \bar{A}$, so in total $\bar{A} \subseteq \mathbb{V}(\mathbb{I}(A))$ and $\mathbb{V}(\mathbb{I}(\bar{A})) \subseteq \bar{A} \to \bar{A} = \mathbb{V}(\mathbb{I}(A))$.

(c) Let $B = \mathbb{A}^1 \setminus \{0\}$ and $C = \mathbb{A}^1$.

$\{0\}$ is a point and thus closed, so $\mathbb{A}^1 \setminus \{0\}$ is open as it is the complement of $\{0\}$.

$\mathbb{V}(\mathbb{I}(C)) = \bar{C} = \bar{\mathbb{A}^1} = \mathbb{A}^1$ from the above q, and $\mathbb{V}(\mathbb{I}(B)) = \bar{B} = \overline{\mathbb{A}^1 \setminus \{0\}} = \mathbb{A}^1$, since the only closed set containing B is $\mathbb{A}^1$ itself, thus $\mathbb{V}(\mathbb{I}(B)) = \mathbb{V}(\mathbb{I}(C))$.

(d) Let $W = \mathbb{V}((x-1)) = \{(x,y) : x-1=0\}$
$= \{(1,y) : y \in \mathbb{C}\}$

And $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ defined by:
$(x,y) \mapsto (x^2, y)$

Then $\varphi^{-1}(W) = \{(1,y) : y \in \mathbb{C}\} \cup \{(-1,y) : y \in \mathbb{C}\}$
$= \mathbb{V}((x-1)) \cup \mathbb{V}((x+1))$

with both ideals $((x-1)$ & $(x+1))$ prime and thus both varieties irreducible.

(7) A compact subset in a topological space is a subset for which every open cover, wrt the subspace topology, has a finite subcover.

(b) Let $X = \mathbb{V}(x^2-y)$; since $\mathbb{C}[x_1, x_2]$ is Noetherian, the ideals of $\mathbb{C}[x_1, x_2]$ satisfies the ascending chain condition: $I_1 \subseteq I_2 \subseteq \dots$ stabilizes for some integer $k \in \mathbb{N}$.

This implies then that $\mathbb{A}^2$ satisfies a descending chain condition for its closed sets. Otherwise, using that $\mathbb{I}$ is order reversing, we would have an ascending chain of (radical) ideals that doesn't stabilize.

Again, this implies every ascending chain of open sets in $\mathbb{A}^2$ stabilises. Otherwise, taking their complements gives a descending chain of closed sets that doesn't stabilise.

So then let ~~&~~ $Y = \{X_\alpha : \alpha \in \mathbb{N}\}$ be any open cover of X, define the following sequence of open sets: ~~X~~ $Y_0 = X_0$, $Y_{i+1} = Y_i \cup X_{i+1}$, we have then that $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots$ and by the above fact, this chain must stabilise, i.e. $\exists k \in \mathbb{N}$ so that $Y_k = \bigcup_{\alpha \in \mathbb{N}} Y_\alpha = \bigcup_{\alpha \in \mathbb{N}} X_\alpha = X$.

In other words every open cover of X has a finite subcover, so X is compact.

In the Euclidean topology, compactness is equivalent to closed and boundedness and $X = \{(x,y) : x^2 = y\}$ is not bounded, eg $\forall M > 0$ $(M, M^2) \in X$ and $|(M, M^2)| = \sqrt{M^2 + M^4} > \sqrt{M^2} = M$. And so not compact in the Euclidean topology.

(a) The algebraic closure of a field k is an algebraic field extension for which is algebraically closed, that is all roots of polynomials over the field extension are in the field extension.

(b) Let J be any ideal of $\bar{k}[x_1, \dots, x_n]$, then, using the Nullstellensatz:

$\bar{\mathbb{V}}(J) = \emptyset = \bar{\mathbb{V}}(1) = \bar{\mathbb{V}}((1))$
$\to \mathbb{I}(\bar{\mathbb{V}}(J)) = \sqrt{J} = (1) = \mathbb{I}(\bar{\mathbb{V}}((1)))$
So $1 \in \sqrt{J} \to \exists n \in \mathbb{N}$ for which $1^n \in J$, and $1^n = 1$ so $1 \in J$. $\to J = \bar{k}[x_1, \dots, x_n] = (1)$.

And $J = (1) \to \bar{\mathbb{V}}(J) = \bar{\mathbb{V}}((1)) = \emptyset$.

So: $\bar{\mathbb{V}}(J) = \emptyset \longleftrightarrow J = (1)$, which is the contrapositive of $J \neq (1) \longleftrightarrow \bar{\mathbb{V}}(J) \neq \emptyset$.

Next, let $I \subseteq k[x_1, \dots, x_n]$ be any ideal, since k is a field, k is Noetherian, and by the HBT $k[x_1, \dots, x_n]$ is Noetherian.

Let $\{f_i\}_{i=1}^{k}$ be the generating set of I and $\bar{I}$ be the ideal generated by $\{f_i\}_{i=1}^{k}$ in $\bar{k}[x_1, \dots, x_n]$.

Clearly $I \neq (1) \longleftrightarrow \bar{I} \neq (1)$ as they share the same generating set, so:

$I \neq (1) \longleftrightarrow \bar{I} \neq (1) \longleftrightarrow \bar{\mathbb{V}}(\bar{I}) \neq \emptyset$
and $\bar{\mathbb{V}}(\bar{I}) = \bar{\mathbb{V}}(\{f_i\}_{i=1}^{k}) = \bar{\mathbb{V}}(I)$ as a subset of $\bar{k}^n$, so:

$I \neq (1) \longleftrightarrow \bar{\mathbb{V}}(I) \neq \emptyset$ as a subset of $\bar{k}^n$.

(5) Suppose φ is not dominant, then there is some open set $X \subseteq W$ for which $\varphi(V) \cap X = \emptyset$, or there is some subvariety $W' \subseteq W$ for which $\varphi(V) \subseteq W'$.

Let $W' = \mathbb{V}(\{f_i\}_{i=1}^k)$, and since $W' \subseteq W$; $\sqrt{(\{f_i\}_{i=1}^k)} \supseteq \mathbb{I}(W)$.

Let $f \in \sqrt{(\{f_i\}_{i=1}^k} \setminus \mathbb{I}(W)$, then:
$\forall v \in V \;\; \varphi^*(f)(v) = f(\varphi(v)) = 0$, since $\varphi(V) \subseteq W'$ and $f \in \mathbb{I}(W')$. Thus there is an element $f \notin \mathbb{I}(W)$ for which $\varphi^*(f)$ is identically zero on V, i.e. $f \in \ker(\varphi^*)$ and the kernel of φ^* is non-trivial i.e. φ^* is not injective.

Next assume φ^* is not injective, that is $\exists\, f \in \ker(\varphi^*) \setminus \mathbb{I}(W)$. Since φ^* is a ~~homomorphism~~ homomorphism, $\ker(\varphi^*)$ is an ideal of $\mathbb{C}[W]$, which then corresponds with some ideal $J \subset \mathbb{C}[x_1, \ldots, x_n]$ properly containing $\mathbb{I}(W)$.

~~Let~~ Let $f \in J$ ($\ker(\varphi^*)$); $0 = \varphi^*(f) = f(\varphi(v))$ for any $v \in V$, i.e. $\varphi(V) \subseteq \mathbb{V}(J) \subset \mathbb{V}(\mathbb{I}(W)) = W$.

Thus there is an open set (the complement of $\mathbb{V}(J)$ in W) X for which we have:

$\varphi(V) \cap X \subseteq \mathbb{V}(J) \cap X = \emptyset$, i.e. $\varphi(V)$ is not dense in W, and so ~~so~~ φ is not dominant.

And the contrapositive of each direction has been proven, so: φ^* is injective iff φ is dominant.

(b) First, assume $\varphi : V \to X \subseteq W$ is an isomorphism, we have:

$$V \xrightarrow[\varphi]{\sim} X \xhookrightarrow[\text{ind.}]{} W$$

and by considering the pullbacks:

$$\begin{tikzcd}
\mathbb{C}[V] & \mathbb{C}[X] \arrow[l, "\tau"'] & \mathbb{C}[W] \arrow[l, "\text{ind.}^*"'] \arrow[ll, bend left, "\varphi^*"]
\end{tikzcd}$$

By considering the coordinate ring of V, X, W as the set of polynomials restricted to V, X, W resp.:

ind.^* is surjective as one can just consider the restricting of the domain W to X, and all such polynomials in $\mathbb{C}[X]$ can be described in this way.

Ex 2.38 (d) implies τ is an isomorphism (since $V \simeq X$) and thus surjective, so:

φ^* being the composition of the two morphisms must also be surjective.

Next, suppose the pullback $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective and let $\varphi(V)$ be the image of φ on V, and $\overline{\varphi(V)}$ its closure (which is a subvariety of W).

Then $\bar{\varphi} : V \to \overline{\varphi(V)}$ is dominant as $\varphi(V)$ is dense in $\overline{\varphi(V)}$ and considering pullbacks & coordinate rings:

$$\begin{tikzcd} \mathbb{C}[V] & \mathbb{C}[\overline{\varphi(V)}] \arrow[l, "\bar{\varphi}^*"'] & \mathbb{C}[W] \arrow[l, "\tau"'] \arrow[ll, bend left, "\varphi^*"] \end{tikzcd}$$

By the previous q, $\bar{\varphi}$ being dominant $\to$ $\bar{\varphi}^*$ is injective, and as $\varphi^* = \bar{\varphi}^* \circ \tau$ is surjective, $\bar{\varphi}^*$ must be too.

So we have found an isomorphism between $\mathbb{C}[V]$ and $\mathbb{C}[\overline{\varphi(V)}]$, which by Ex 2.38 (d) implies $\exists$ an isomorphism $\rho : V \leftrightarrow \overline{\varphi(V)}$
i.e. V is isomorphic to a subvariety of W.

And both sides of the equivalence have been shown true, so $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective iff ~~this~~ V is isomorphic to some subvariety of W.