Algebraic Geometry CW2.

1. a) i) $\text{maxSpec}(\mathbb{C}[x]) = \{\text{maximal ideals of } \mathbb{C}[x]\}$.

The maximal ideals of $\mathbb{C}[x]$ are the ideals of the form $(x-\alpha)$ for $\alpha \in \mathbb{C}$, so $\text{maxSpec}(\mathbb{C}[x]) = \{(x-\alpha) \mid \alpha \in \mathbb{C}\}$.

ii) $\text{maxSpec}(\mathbb{C}[x, 1/x])$

This ring is made of Laurent polynomials in x, where x can take any value other than 0. Just as in $\mathbb{C}[x]$, every maximal ideal consists of an evaluation at some α, except it can't be at $\alpha = 0$, so
$\text{maxSpec}(\mathbb{C}[x, 1/x]) = \{(x-\alpha) \mid \alpha \in \mathbb{C}^*\}$.

iii) $\text{maxSpec}(\mathbb{C}[x, 1/x, y])$

This consists of polynomials in y with coefficients in the Laurent polynomials of 1.
By the first two parts, the maximal ideals are of the form
$(x-\alpha, y-\beta)$ for $(\alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}$.
So $\text{maxSpec}(\mathbb{C}[x, 1/x, y]) = \{(x-\alpha, y-\beta) \mid \alpha \in \mathbb{C}^* \ \beta \in \mathbb{C}\}$

b) The pullback $\varphi^* : g \to g \circ \varphi$.

i) $\varphi^*(1/x) = y$.

ii) $\varphi^*\left(2x^2 + \frac{2x^3+4x}{x^5}\right) = \varphi^*(2x^2 + 2x^{-2} + 4x^{-4})$
$= 2y^{-2} + 2y^2 + 4y^4$.

iii) $\varphi^*(2-x) = 2 - \frac{1}{y}$.

$V = \mathbb{V}(y-ux) \subseteq \mathbb{A}^3$

2. a). Let $\varphi : \mathbb{A}^3 \to \mathbb{A}^2$, $(x,y,u) \mapsto (x,u)$ be the projection.

Then $\varphi \restriction V : (x, ux, u) \to (x,u)$. is a restriction of a polynomial so is a morphism.
$\varphi^{-1} \restriction \varphi(V) : (x,u) \to (x, ux, u)$ exists and is a function
so $\varphi \restriction V$ is bijective so it is an isomorphism

b) Similarly let $\psi : \mathbb{A}^3 \to \mathbb{A}^2$, $(x,y,u) \to (x,y)$.

Then $\psi \restriction V : (x, ux, u) \to (x, ux)$.
~~and similarly $\psi^{-1} \restriction V$:~~
~~But $\psi(0,0,1) = \psi(0,0,2) = (0,0)$ so~~
~~$\psi \restriction V$ isn't~~
$\psi^{-1} \restriction \psi(V) : (u, ux) \to (x, ux, u)$
sends $(0,0) \to (0,0,u)$ $\forall u$ so isn't a function.
$\therefore$ $\psi \restriction V$ cant be a bijection, and therefore isn't an isomorphism.

3. a). Prove that, if $g \in \mathbb{C}[x,y]$, $\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g}) \subseteq \mathbb{P}^2$

Let $g \in \mathbb{C}[x,y]$ be a polynomial. Then
$\mathbb{V}(g) = \{ (x,y) \in \mathbb{C}^2 \mid g(x,y) = 0 \}$.

The homogenisation $\tilde{g}$ of g is $\tilde{g}(x,y,z) = z^d g\left(\frac{x}{z}, \frac{y}{z}\right)$ where d is the degree of g.
The variety $\mathbb{V}(\tilde{g}) = \{ [x:y:z] \in \mathbb{P}^2 \mid \tilde{g}(x,y,z) = 0 \}$.

The projective closure $\overline{V(g)}$ is defined as the Zariski closure of $i(V(g))$ in $\mathbb{P}^2$ where $i : \mathbb{A}^2 \to \mathbb{P}^2$, $(x,y) \mapsto [x:y:1]$, ie the smallest projective variety in $\mathbb{P}^2$ containing all points of $\mathbb{V}(g)$.

So $V(\tilde{g}) \subseteq V(\tilde{g})$. as $g(x,y)=0 \Rightarrow \tilde{g}(x,y,1)=0$.

Suppose there is another projective variety $\mathbb{V}(g) \subseteq X$.
Then the homogeneous polynomial defining X must vanish on $\mathbb{V}(g)$
so is a multiple of $\tilde{g}$, so $\mathbb{V}(\tilde{g})$ is minimal.
$\therefore \overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g})$.

b). $f_1(x,y) = x+y+1$ so $\tilde{f}_1 = x+y+z$.
$f_2(x,y) = x^2+6y^2+1$ so $\tilde{f}_2 = x^2+6y^2+z^2$
$f_3(x,y) = x^2+3y+1$ so $\tilde{f}_3 = x^2+3yz+z^2$.
$f_4(x,y) = x^3+3xy^2+4$ so $\tilde{f}_4 = x^3+3xy^2+4z^3$.

i) $[1:0:0]$ is in none of them. as each $\tilde{f}_n(1,0,0) \neq 0$.
ii) $[0:1:0]$ is in $\tilde{F}_3$ and $\tilde{F}_4$ as $\tilde{f}_1(0,1,0) = \tilde{f}_2(0,1,0) \neq 0$ and
$\tilde{f}_3(0,1,0) = \tilde{f}_4(0,1,0) = 0$.
iii). $[0:0:1]$ is also in none of them as $\tilde{f}_n(0,0,1) \neq 0$.

c). All terms are of maximal degree or degree 0 as this results in no mixed terms.

4. a). Prove $\mathbb{P}^n$ is compact wrt quotient Euclidean topology from $\mathbb{A}^{n+1} \setminus \{0\}$.

Instead of using $\mathbb{A}^{n+1} \setminus \{0\}$, we can use its intersection with the unit sphere. $S^n = \{\underline{x} \in \mathbb{A}^{n+1} \mid \|\underline{x}\| = 1\}$, so instead of $[\underline{x}]$ ~~changing~~ being an entire line, we only consider it's intersection with S^n, which is just two points.
So $\mathbb{P}^n \cong S^n/\sim$ where $\underline{x} \sim -\underline{x}$

The map $\pi: S^n \to \mathbb{P}^n$ is continuous as by definition in the quotient topology induced by π, $U \subseteq \mathbb{P}^n$ is open iff $\pi^{-1}(U) \subseteq S^n$ is open.
π is also a surjection trivially so it follows that $\mathbb{P}^n$ must be compact since the continuous image of a compact subset is compact.

b). Since $y - \sin x$ is not algebraic (as $\sin x$ is transcendental), it's closure in the Zariski topology must be the entire projective plane $\mathbb{P}^2$ since any proper algebraic subvariety would be defined by a polynomial, and no polynomial can globally approximate $y = \sin x$.

Chow's lemma doesn't ~~a~~ apply in this ~~case~~ case as even though $y - \sin x$ is ~~not~~ analytic, it's zero set is not compact in the Euclidean topology.

5. a). Let $a_1 x + b_1 y + c_1 z = 0$ and $a_2 x + b_2 y + c_2 z = 0$ be two lines in $\mathbb{P}^2$.
we need to show that there is an (x_0, y_0, z_0) satisfying the above the above system, and that this is unique.

In matrix form this amounts to showing that $(x_0, y_0, z_0) \in \operatorname{Ker} M$ where $M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$ and $(x_0, y_0, z_0) \neq 0$.

By the rank-nullity theorem, $\operatorname{rank}(M) + \operatorname{nullity}(M) = \dim(\mathbb{P}^2)$, so $\operatorname{nullity}(M) = 3 - \operatorname{rank}(M) \geq 1$ as $\operatorname{rank}(M) \leq 2$.
In fact ~~rank~~ $\operatorname{rank}(M) = 2$ since the lines are distinct, ~~an~~ and therefore $\operatorname{nullity}(M) = 1$.

This means that if two solutions (x_0, y_0, z_0) and (x_1, y_1, z_1) are nonzero, then $(x_0, y_0, z_0) = \lambda(x_1, y_1, z_1)$. It follows that $[x_0, y_0, z_0] = [x_1, y_1, z_1]$, ie the intersection is unique.

b) i). Prove $\overline{C_1 \cap C_2} \subseteq \overline{C_1} \cap \overline{C_2}$

This is a basic topological result, which holds more generally:

$$\overline{\bigcap_{i \in I} C_i} \subseteq \bigcap_{i \in I} \overline{C_i}$$

Proof:

Since $\bigcap_{i\in I} \overline{C_i}$ is an intersection of closed sets it is closed.

Since $C_i \subseteq \overline{C_i} \forall i$, $\bigcap_{i\in I} C_i \subseteq \bigcap_{i\in I} \overline{C_i}$

$\overline{\bigcap_{i\in I} C_i}$ is by definition the smallest closed set X such that $\bigcap_{i\in I} C_i \subseteq X$,

and $\bigcap_{i\in I} \overline{C_i}$ is closed, so it follows that $\overline{\bigcap_{i\in I} C_i} \subseteq \bigcap_{i\in I} \overline{C_i}$

ii) We need an example of two curves where the affine intersection and therefore its closure doesn't capture all projective intersections.
For example consider $y = x$ and $y = 1/x$. These intersect at $(1,1)$ and $(-1,-1)$, but in the projective space they possess an additional intersection at infinity.

To homogenise we get $y = x$ and $yx = z^2$, and so solutions of their intersection are those of form $x^2 = z^2$:
$[1:1:1]$, $[-1:-1:1]$ corresponding to $(1,1)$, and $(-1,-1)$.
The additional intersection at infinity is where $z = 0$, ie at $[1:1:0]$.
Their respective closures are the same so the inclusion is ~~not~~ strict for this example.

6. a). To show that $\mathcal{O}_Y(0)$ is a $\mathbb{C}$-algebra, we show it is a vector space and a ring using the subring test.

The constant function belongs to $\mathcal{O}_Y(0)$, (multiplicative identity) and $\mathcal{O}_Y(0)$ is trivially closed under addition, multiplication and additive inverses since sums and products of rational functions are rational functions $\therefore$ By the subring test $\mathcal{O}_Y(0)$ is a ring (subring of $\mathbb{C}[x_1,\dots x_n]$)

Trivially it is also a ~~vct~~ vector space since if f, g are regular functions, $\lambda f + g$ is too. $\therefore\ \mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra.

b) i). By definition if f is regular on V it is regular on all points in V and therefore trivially also regular on $U \subseteq V$.

ii). By part i) and a) each set of functions $\mathcal{O}_X(U)$ is a $\mathbb{C}$-algebra so is a ring $F(U)$; ~~the second~~ so property (i) holds.
(ii) follows from part i) ~~above~~ above, as do (iii) and (iv), so the collection of such sets forms a Sheaf on X.