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by Arron Willis

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An Introduction to Large Cardinals and the HOD Dichotomy

Arron Willis
Supervisor: Philip Welch

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Abstract

We introduce the concept of a large cardinal property. constructing and ascending the hierarchy of large cardinal axioms under consistency strength, with a focus on the use of elementary embeddings. We work our way all the way up the traditional hierarchy until Kunen's inconsistency is met, which we prove. We also introduce rank-into-rank hypotheses, which lie just below the realm of proven inconsistency, but have themselves yet to be rendered inconsistent.

We place these ideas in context by working towards an understanding of the HOD conjecture and various results surrounding it.

We then discuss some large cardinal concepts on the brink of consistency, such as the choiceless Reinhardt and Berkeley cardinals and their variants, which we place in a hierarchy of their own. We also discuss exacting cardinals, a new large cardinal concept due to Bagaria et al., which can provide a refutation of the HOD conjecture while still being compatible with choice.

Acknowledgement of Sources

For all ideas taken from other sources (books, articles, internet), the source of the ideas is mentioned in the main text and fully referenced at the end of the report.

All material which is quoted essentially word-for-word from other sources is given in quotation marks and referenced.

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Signed ______________________________

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0 Preliminaries

In this text, we assume some familiarity with the basic concepts of set theory and first-order logic. Here we will provide a brief review of some key concepts for easy reference.

0.1 Some Set Theory

We assume the reader has already taken a course in set theory, so we leave out the basics, but here we will provide a very brief review of some important concepts that we will need later.

The Axioms

Recall the axioms of ZFC[†]. In Chapter 4, we will work in ZF, that is, without the axiom of choice, owing to the inconsistency of the axiom of choice with the strongest of large cardinal properties.

(i) (Axiom of Extensionality) $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \leftrightarrow x = y)$

(ii) (Axiom of Pair Set) For all sets x and y, $\{x, y\} \in V$

(iii) (Axiom of Union) For any set x, $\bigcup x \in V$

(iv) (Foundation Scheme) For any term a: $a \neq \emptyset \rightarrow \exists x (x \in a \wedge x \cap a = \emptyset)$

(v) (Separation Scheme) For every term a: $x \cap a \in V$

(vi) (Replacement Scheme) For every term f: $\text{Fun}(f) \rightarrow f''x \in V$

(vii) (Axiom of Infinity) $\exists x (\emptyset \in x \wedge \forall (y \in x \rightarrow y \cup \{y\} \in x))$

(viii) (Axiom of Power Set) $\mathcal{P}(x) \in V$

(ix) (Axiom of Choice) $\text{Fun}(f) \wedge \text{dom}(f) \in V \wedge \emptyset \notin \text{ran}(f) \rightarrow \prod f \neq \emptyset$

Recall that, among several other equivalent statements, the axiom of choice is equivalent to the well-ordering theorem - the statement that every set can be well-ordered. Another equivalent is the trichotomy theorem, the statement that the cardinalities of sets are always comparable, i.e. for all sets A and B, either $|A| < |B|$, $|A| = |B|$, or $|A| > |B|$. Note that in the absence of the axiom of choice this trichotomy still holds for well-orderable sets, but not in general.

[†] Authors tend to vary slightly but this list is fairly standard. The two redundant axioms (pair set and separation) have been retained for ease of use. Some authors include an empty set axiom, but this is unnecessary since by separation we can show the empty set exists given any other set exists. There must exist some set since by the laws of first-order logic something must exist, which in the language of set theory must be a set.

Cardinal Arithmetic

Recall that a limit cardinal is one that cannot be reached by repeated application of the successor operation. Contrast this to the following, stronger notion:

Definition 0.1 (Strong limit cardinal). *A cardinal κ is a strong limit cardinal if for all $\lambda < \kappa$, $2^\lambda < \kappa$.*

Recall also the concept of cofinality[1] (p. 31).

Definition 0.2. *(i) Let $\alpha, \beta > 0$ be limit ordinals. A β-sequence $\langle \alpha_\xi \mid \xi < \beta \rangle$ is cofinal in α if $lim_{\xi \to \beta} \alpha_\xi = \alpha$,*

(ii) If α is an infinite limit ordinal, then the cofinality of α, $cf(\alpha)$ is the least such limit ordinal β,

(iii) A cardinal κ is called regular if $cf(\kappa) = \kappa$, and is called singular otherwise.

Filters and Ideals

See Jech[1](pp. 73-86) for more detail on this particular subject.

Definition 0.3 (Filter). *A filter F on a nonempty set S is a collection of subsets of S such that:*

(i) $S \in F$ and $\emptyset \notin F$,

(ii) If $X \in F$ and $Y \in F$, then $X \cap Y \in F$,

(iii) If $X, Y \subset S$, $X \in F$, and $X \subset Y$, then $Y \in F$.

Definition 0.4 (Ideal). *An ideal I on a nonempty set S is a collection of subsets of S such that:*

(i) $\emptyset \in I$ and $S \notin I$,

(ii) If $X \in I$ and $Y \in I$, then $X \cup Y \in I$,

(iii) If $X, Y \subset S$, $X \in I$, and $Y \subset X$, then $Y \in I$.

We can think of a filter as characterising the large sets, and an ideal as characterising the small sets. We say that filters and ideals are dual to each other for the following reason: If F is a filter on S, then the set $I = \{S \setminus X \mid X \in F\}$ is an ideal on S, and similarly, if I is an ideal on S, then the set $F = \{S \setminus X \mid X \in I\}$ is a filter on S.

Here are listed some additional properties that a filter may admit. We will discover their uses later.

Definition 0.5 (Filter Properties). *Let F be a filter on an infinite cardinal κ. Then, the following are common properties.*

(i) (Non-principality) $\forall \xi < \kappa, \xi \notin F$,

(ii) *(Uniformity)* $X \in F \to |X| = \kappa$,

(iii) *(κ-completeness)* $\forall \xi < \kappa \forall \langle X_\zeta \mid \zeta < \kappa \rangle (\forall \zeta (X_\zeta \in F) \to \bigcap_{\zeta < \xi} X_\zeta \in F)$, *i.e. F is closed under the intersection of less than κ sets.*

(iv) *(Normality)* $\forall \langle X_\zeta \mid \zeta < \kappa \rangle (\forall \zeta (X_\zeta \in F) \to \triangle_{\zeta < \kappa} X_\zeta \in F)$, *i.e. F is closed under diagonal intersections.*

NB: A filter that is ω_1-complete will usually be called σ-complete, or countably complete.

Definition 0.6 (Ultrafilter)**.** *A filter U on a set S is an ultrafilter if for every $X \subset S$, either $X \in U$ or $S \setminus X \in U$*

Again there is a dual notion to this: An ideal is prime if it satisfies the same condition.

Stationary Sets

Recall the following definitions:

Definition 0.7 (Limit point)**.** *Let X be a set of ordinals and α be a limit ordinal. Then α is a limit point of X if $sup(X \cap \alpha) = \alpha$.*

Definition 0.8 (Club set)**.** *Let κ be a regular uncountable cardinal. A set $C \subset \kappa$ is a closed unbounded subset (abbreviated club subset) if it is unbounded in κ and it contains all its limit points less than κ.*

Closure as defined in the above definition is equivalent to the condition that every increasing sequence in C of length less than κ attains its limit within C.

Definition 0.9 (Club Filter)**.** *Let κ be an uncountable regular cardinal. Then the club filter F on κ is defined by the following:*

$$X \in F \leftrightarrow \exists C \subseteq \kappa (\text{"}C \text{ is a club"} \wedge C \subseteq X)$$

Definition 0.10 (Stationary set)**.** *A set $S \subset \kappa$ is stationary if $S \cap C \neq \emptyset$ for all club subsets of $C \subset \kappa$.*

A simple lemma is the following:

Lemma 0.11. *Let C and D be club sets. Then $C \cap D$ is also a club set.*

Proof. That $C \cap D$ is closed is immediate. To show $C \cap D$ is unbounded, observe that by unboundedness of C and D, for any $\alpha < \kappa$, there exists an $\alpha_1 \in C$ and $\alpha_2 \in D$ such that $\alpha < \alpha_1 < \alpha_2$. This process can be repeated to construct an increasing sequence $\alpha < \alpha_1 < \alpha_2 < \alpha_3 ...$ with alternating terms in C and D. Then the limit of this sequence β satisfies $\beta < \kappa$, $\beta \in C$ and $\beta \in D$, so $C \cap D$ is a club set. ☐

This lemma can be generalised to the intersection of fewer than κ club sets. See Jech[1] (p. 92) for the proof. It follows from this result that the club filter on κ is κ-complete.

The following is an important result that we will use in several results.

Theorem 0.12 (Solovay Splitting Theorem). *Let κ be a regular uncountable cardinal. Then every stationary subset of κ can be partitioned into κ stationary sets.*

The proof is not difficult but relies on a few other results so we omit it here. For more on stationary sets, including the aforementioned proof, see Jech[1] (pp. 91-103). The following is also a standard result, which we will invoke later.

Proposition 0.13. *Suppose $\eta < \lambda$ is regular, $S \subseteq \{\xi < \lambda \mid cf(\lambda) = \eta\}$ is stationary in λ, and suppose C is η-closed unbounded in λ. Then $S \cap C \neq \emptyset$.*

0.2 Model Theory and Consistency

Here we give a brief review of some important definitions and results from model theory we will need in later chapters. These should be found in any course on mathematical logic or basic model theory. See, for example, Chang and Keisler for a more complete treatment of the subject[2].

Definition 0.14 (Elementary Submodel). *$\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$, denoted by $\mathfrak{A} \prec \mathfrak{B}$, if $\mathfrak{A} \subset \mathfrak{B}$ and for all formulas $\phi(x_1, ..., x_n)$ of $\mathcal{L}$ and n-tuples $a_1, ..., a_n \in A$, we have:*

$$\mathfrak{A} \models \phi(a_1, ..., a_n) \text{ if and only if } \mathfrak{B} \models \phi(a_1, ..., a_n)$$

Definition 0.15 (Elementary Embedding). *A mapping $j : A \to B$ is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$, written as $j : \mathfrak{A} \prec \mathfrak{B}$ if for all formulas $\phi(x_1, ..., x_n)$ of $\mathcal{L}$ and n-tuples $a_1, ..., a_n \in A$, we have:*

$$\mathfrak{A} \models \phi(a_1, ..., a_n) \text{ if and only if } \mathfrak{B} \models \phi(j(a_1), ..., j(a_n))$$

One can think of elementary embeddings as truth-preserving maps. It is easy to see that an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$ is just an isomorphism of $\mathfrak{A}$ onto an elementary submodel of $\mathfrak{B}$. As we shall explore in Chapter 3, elementary embeddings with domain V, the universe of sets, are important in the study of large cardinals. Note that a common theme in proofs in this text involves justifying a claim "by elementarity" which simply refers to the use of equivalence in the above definition.

Lévy Hierarchy

Recall the Lévy Hierarchy of first-order formulas in the language of set theory:

Definition 0.16. *(i) A formula is $\Delta_0 = \Sigma_0 = \Pi_0$ if it has no unbounded quantifiers.*

(ii) *A formula* φ *is* Σ_{n+1} *if* $\varphi = \exists x\psi$ *where* ψ *is* Π_n

(iii) *A formula* φ *is* Π_{n+1} *if* $\varphi = \exists x\psi$ *where* ψ *is* Σ_n

Note that sometimes statements may be classified like this but with a top index that refers to the order of the formula. E.g. a Π_1^0 formula is equivalent to a first-order formula as defined above. A second-order formula would look like Σ_1^1. See [3] (p. 152) for more detail on this and in particular the (second-order) analytical hierarchy.

The Ultraproduct Construction

The ultraproduct is a construction used to build models with many diverse applications across mathematics. For example, the ultraproduct is used to construct the hyperreals, an extension of the real numbers which includes certain classes of infinite and infinitesimal numbers, and has uses in analysis[4]. It has many uses in model theory - for example, it is used in an elegant proof of the compactness theorem for first-order logic. For our purposes, this will be a useful tool for working with models and elementary embeddings.

Definition 0.17 (Ultraproduct)**.** *[5] Let* U *be an ultrafilter over a non-empty set* I*, and for each* $i \in I$*,* A_i *is a non-empty set. Let* $C = \prod_{i\in I} A_i$ *be the cartesian product of these sets. Then* C *is the set of all functions with domain* I *such that for all* $i \in I, f(i) \in A_i$*. So for two functions* $f, g \in C$*, we define* f *and* g *as* U*-equivalent,* $f =_U g$*, if* $\{i \in I \mid f(i) = g(i)\} \in U$*. Note that* $=_U$ *is an equivalence relation over* C*.*

Finally, we define the U*-equivalence class of* f *as* $f_U = \{g \in C \mid f =_U g\}$ *and use this to define the ultraproduct* $\prod_U A_i$ *as the set of* U*-equivalence classes:*

$$\prod_U A_i = \{f_U \mid f \in \prod_{i\in I} A_i\}$$

Additionally, we can define the ultrapower $\prod_U A$ in the natural way - as the ultraproduct $\prod_U A_i$ where each A_i is equal.

The ultraproduct finds its use in model theory when we take an ultraproduct of $\mathcal{L}$-structures:

Definition 0.18. *Let* U *be an ultrafilter over a nonempty set* I *and let* $\mathfrak{A}_i$ *be* $\mathcal{L}$*-structures with universal sets* A_i *respectively. Then the ultraproduct* $\prod_U \mathfrak{A}_i$ *is the unique* $\mathcal{L}$*-structure* $\mathfrak{B}$ *such that:*

(i) *The universe of* $\mathfrak{B}$ *is* $B = \prod_U A_i$

(ii) *For every atomic formula* $\varphi(x_1, ..., x_k)$ *with at most one symbol from* $\mathcal{L}$*, and for each* $f_1, ..., f_k \in \prod_{i\in I} A_i\}$*,*

$$\mathfrak{B} \models \varphi(f_{1U}, ..., f_{kU}) \text{ iff } \{i \mid \mathfrak{A} \models \varphi(f_1(i), ..., f_k(i))\} \in U$$

The following theorem, also known as the fundamental theorem of ultraproducts, is the reason for the utility of the ultraproduct construction.

Theorem 0.19 (Łoś's Theorem). *Let U be an ultrafilter over a non-empty set I, and let $\mathfrak{A}_i$ be an $\mathcal{L}$-structure for all $i \in I$. Then for each formula $\varphi(x_1, ..., x_k)$ of $\mathcal{L}$ and each for each $f_1, ..., f_k \in \prod_{i \in I} A_i\}$,*

$$\prod_U \mathfrak{A}_i \models \varphi(f_{1U}, ..., f_{kU}) \text{ iff } \{i \mid \mathfrak{A}_i \models \varphi(f_1(i), ..., (f_k(i))\} \in U$$

The proof of Łoś's Theorem follows by induction on the complexity of terms and formulae and can be found in full in [1] (pp. 159-161). See also [2](pp. 164-176) for a more gentle introduction to ultraproducts and Łoś's Theorem. Our reason for using ultrapowers is the following:

Corollary 0.20. *An ultrapower of a model $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{A}$.*

Consistency

Recall that a theory T is consistent, $Con(T)$, if it does not lead to a logical contradiction, that is, if both φ and $\neg\varphi$ are contained in the set of consequences of T. The following notions are of great import:

Definition 0.21 (Relative Consistency). *Let S and T be formal theories. Then T is consistent relative to S if it holds that T is consistent under the assumption of the consistency of S, that is $Con(S) \to Con(T)$.*

Definition 0.22 (Equiconsistency). *Two theories T and S are called equiconsistent if the consistency of one implies the other, that is, if $Con(T) \leftrightarrow Con(S)$.*

An important example of this is the equiconsistency of ZF and ZFC, which Gödel proved in his 1940[8] paper using his inner model L, which we will outline in the next section. We do not know whether or not ZF or ZFC are consistent (though we would like to believe that they are), but we know that they are equiconsistent.

In many cases such as this, we are unable to assert that a theory is consistent, but we can find and prove results relating to their relative consistency. The following notion follows naturally from these discussions:

Definition 0.23 (Consistency Strength). *We say that a theory T has greater consistency strength than a theory S if it is known that S is consistent relative to T, but the converse is not known.*

0.3 Inner Models of Set Theory

Recall that a class X is transitive, Trans(X) if $\forall z \in X(z \subseteq X)$, i.e. every element is a subset. The transitive closure is the class term given by $TC(x) = x \cup \bigcup\{TC(y) \mid y \in x\}$[6](p. 10).

Definition 0.24 (Inner Model). *Let T be a theory (such as ZF or ZFC). A model M is an inner model of T if it is transitive, contains the ordinals, and T holds in M.*

The Constructible Hierarchy

A first example of an inner model is Gödel's constructible hierarchy, L, which is in fact the smallest inner model, as we shall soon show. First, recall the iterative construction of V:

(i) $V_0 = \emptyset$,

(ii) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,

(iii) $\mathrm{Lim}(\lambda) \to V_\lambda = \bigcup_{\alpha<\lambda} V_\lambda$,

(iv) $V = \bigcup_{\alpha\in\mathrm{On}} V_\alpha$.

We construct L in a similar way, except in place of the regular power set, we use the definable power set. Recall that a set y is definable over a model $\mathfrak{A}$ if there exist a formula φ and constants $a_1, ..., a_n \in A$ such that $y = \{x \in A \mid \mathfrak{A} \models \varphi[x, a_1, ...a_n]\}$.

Definition 0.25 (Definable Power Set)**.** *The definable power set, Def(x), is given by the following:*

$$Def(x) = \{y \in x \mid y \textit{ is definable over } (x, \in)\}.$$

This can be properly formalised, but it is much more complicated, and for our needs the above will suffice. We use this to define the constructible hierarchy as follows:

Definition 0.26. *(i)* $L_0 = \emptyset$,

(ii) $L_{\alpha+1} = Def(L_\alpha)$,

(iii) $Lim(\lambda) \to L_\lambda = \bigcup_{\alpha<\lambda} L_\lambda$,

(iv) $L = \bigcup_{\alpha\in On} L_\alpha$.

It is a fairly routine but somewhat time-consuming exercise to show that L is indeed an inner model of ZF, so we omit the proof for the sake of space.

The axiom of choice also holds in L, in a very strong sense - it is provable! That is, we don't need to assert it as an additional axiom. This is because we can construct a *global* well-order on the whole of L, and so any set we can form using elements from L can inherit part of this global well-order (see e.g. [7] pp. 71-76). It was by using this construction that Gödel demonstrated in [8] that the axiom of choice is consistent with ZF - there is a model L of ZF that exhibits AC.

Note also that the generalised continuum hypothesis, GCH, holds in L (recall that the GCH asserts that $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$ for all ordinals α). This was also originally proved by Gödel in [8]; for a modern exposition of this and the proofs involved, see [1] chapter 13.

We call the statement $V = L$ the axiom of constructibility. For various reasons, most set theorists today do not accept it. In the words of Shelah[9]:

> "It is useful, it decides many problems, it also is a natural statement. Just one problem: why the hell should it be true?".

For much further discussion on this topic see for example the work of Maddy (discussion of $V = L$ can be found in [10] and [11]).

It can be shown that L is in fact the smallest inner model of ZF, this follows as a simple corollary of the following proposition. For further details, we again refer to the exposition in Jech[1] chapter 13.

Proposition 0.27. *If $N \subseteq M$ is an inner model, then $L^N = L^M$. In particular, if we set $M = V$ then $L^N = L$ so L is absolute between inner models[12] (p. 33).*

Proof. The statement $x = L_\alpha$ can be expressed as a Σ^0_1 sentence and so is upwards absolute. By definition $L = \bigcup_{\alpha \in \mathrm{On}} L_\alpha$, so as M and N agree on L_α for all α, they must agree on the whole of L, that is $L^N = L^M$. □

We often like to picture the set-theoretic universe as having an upside-down cone shape (see figure 1 below). The idea of the cumulative hierarchy is that we begin with the empty set V_0 and gradually build more layers, called "ranks" V_α on top that become wider and wider as we ascend (as each time we take the power set or the union, the new rank is much larger than the previous). We think of V as being a very wide cone as it by definition contains every set. Since L, which is built up in a similar way, is the smallest inner model, we can think of L as being a narrow cone. How narrow it is in comparison to V is an obvious question to ask, but not one to which we have a satisfactory answer. Of course, $V = L$ asserts that L fully fills V, but it is hard to justify, and so if this is not the case, how close is L to V? We discuss this in chapter 1.

Relative Constructibility

Somewhere in between the two above pictures is the following inner model:

Definition 0.28 ($L(A)$-Hierarchy)**.** *[1] (p. 193)*

(i) $L_0(A) = A \cup \{A\}$,

(ii) $L_{\alpha+1}(A) = \mathit{Def}(\langle L_\alpha(A, \in \rangle)$,

(iii) $\mathit{Lim}(\lambda) \to L_\lambda(A) = \bigcup_{\alpha<\lambda} L_\lambda(A)$,

(iv) $L(A) = \bigcup_{\alpha \in On} L_\alpha(A)$.

Gödel Operations [1](p. 178) The replacement scheme asserts that given any term a and any set x, there exists a set $x \cap a$. So there is a set $y = \{u \in x \mid \varphi(u)\}$ for a given formula φ. If this formula is Δ_0, this set y can be constructed using finitely many operations. These are called Gödel operations.

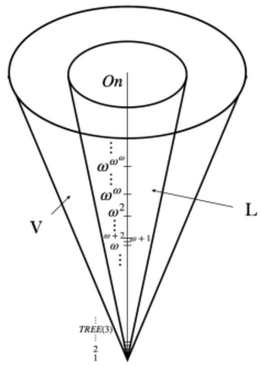

Figure 1: V and L

Definition 0.29 (Gödel Operations)**.**

$$
\begin{aligned}
&G_1(X,Y) = \{X,Y\},\\
&G_2(X,Y) = X \times Y,\\
&G_3(X,Y) = \{(u,v) \mid u \in X \wedge v \in Y \wedge u \in v\},\\
&G_4(X,Y) = X - Y,\\
&G_5(X,Y) = X \cap Y,\\
&G_6(X) = \bigcup X,\\
&G_7(X) = dom(X),\\
&G_8(X) = \{(u,v) \mid (v,u) \in X\},\\
&G_9(X) = \{(u,v,w) \mid (u,w,v) \in X\},\\
&G_{10}(X) = \{(u,v,w) \mid (v,w,u) \in X\}.
\end{aligned}
$$

Definition 0.30. *A transitive class M is almost universal if for all subsets $X \subset M$, there is some $Y \in M$ such that $X \subset Y$.*

The following theorem will prove useful in categorising inner models.

Theorem 0.31. *A transitive class M is an inner model of ZF iff it is closed under Gödel operations, and is almost universal.*

A full proof can be found in [13](pp. 35-38).

Another useful result using the Gödel operations is the following. Recall that def(M) is the set of subsets of M definable over $\langle M, \in \rangle$.

Lemma 0.32. *For every transitive set M, $def(M) = cl(M \cup \{M\}) \cap \mathcal{P}(M)$ where $cl(M)$ is the closure of M under the Gödel operations.*

Finally recall the Mostowski collapse: Recall that a relation R on a class P is well-founded if every nonempty set $x \subseteq P$ has an R-minimal element, and the extension of R, $ext_R = \{z \in P \mid zRx\}$ is a set for all $x \in P$. A well-founded relation is extensional if $ext_E(x) \neq ext_E(y)$ for all elements $x \neq y$ of P. We say that a class P is extensional if $\in$ is an extensional relation on P, i.e. $\forall x, y \in P$ distinct, $x \cap P \neq y \cap P$. We then have the following:

Theorem 0.33 (Mostowski Collapse)**.** *If R is a well-founded extensional relation on a class P, then there exists a unique transitive class M and a unique isomorphism $\pi : \langle P, R\rangle \to \langle M, \in\rangle$.*

See e.g. Jech [1] p.69 for a proof.

1 Introduction

To paint a picture of the current dilemma faced in inner model theory, we will compare two important inner models. The first is L, which we defined in the preliminaries. The second is HOD, the namesake of this text, which we outline below.

Definition 1.1. *[1](p. 194) A set is ordinal-definable if there is a formula φ such that $X = \{u : \varphi(u, \alpha_1, ..., \alpha_n)\}$. Equivalently, we can define the class OD of ordinal-definable sets as*

$$OD = \bigcup_{\alpha \in On} cl\{V_\beta \mid \beta < \alpha\}$$

where $cl(X)$ denotes the closure under the Gödel operations.

Proposition 1.2. *The class OD admits a definable well-ordering.*

Proof Sketch. [1](p. 194) By the standard result providing a well-order for L, we can obtain a well-ordering of $cl\{M\}$ given a well-ordering of M. For every ordinal α, the set $\{V_\beta \mid \beta < \alpha\}$ is well-ordered, and so therefore so is $cl\{V_\beta \mid \beta < \alpha\}$. It therefore follows that we can construct a well-ordering on OD. □

Definition 1.3 (HOD)**.** *HOD denotes the class of hereditarily ordinal-definable sets. I.e. $HOD = \{x \mid TC(\{x\}) \subset OD\}$, or equivalently $z \in HOD \leftrightarrow z \in OD \wedge TC(z) \subseteq OD$.*

By the previous proposition, we can also well-order HOD, by simply restricting the well-order from OD to elements of HOD. The following result is straightforward but important.

Theorem 1.4. *[1](p. 195) The class HOD is an inner model of ZFC.*

Proof. HOD is clearly transitive by its definition. By Theorem 0.31, to show HOD is an inner model of ZF, we need only show that it is closed under Gödel operations and is almost universal. The proof of the former is tedious but straightforward. For example, for $G_1(X, Y) = \{X, Y\}$, let $X, Y \in$ HOD, that is $X, Y \in$ OD and $TC(X), TC(Y) \subseteq$ OD. Since OD was defined as a Gödel closure, immediately $\{X, Y\} \in$ OD. Now consider

$$TC(\{X, Y\}) = \{X, Y\} \cup \bigcup \{TC(z) \mid z \in \{X, Y\}\} = \{X, Y\} \cup TC(X) \cup TC(Y).$$

All of these are subsets of OD so $\{X, Y\} \subset$ HOD, i.e. HOD is closed under G_1. We omit the remaining nine Gödel operations for the sake of brevity.

It remains to show that HOD is almost universal. We claim that it suffices to show that for all ordinals α, $V_\alpha \cap \text{HOD} \in \text{HOD}$. Trivially $V_\alpha \cup \text{HOD} \subseteq$ HOD, so we need only show $V_\alpha \cup$ HOD is ordinal-definable. This is true since $V_\alpha \cup$ HOD is defined by the formula below.

$$u \in V_\alpha \wedge (\forall z \in TC(\{u\}))\exists\beta(z \in cl\{V_\gamma \mid \gamma < \beta\}).$$

So by Theorem 0.31, HOD is an inner model of ZF.

Finally, to show that HOD satisfies the axiom of choice, and therefore that it is an inner model of ZFC, observe that the global well-order defined for OD can be restricted to HOD, and so choice must hold in HOD. □

Proof of Claim. The claim in the above theorem is that $V_\alpha \cap \mathrm{HOD} \in \mathrm{HOD}\ \forall\alpha$ implies that $\forall X \subset \mathrm{HOD}, \exists Y \in \mathrm{HOD}(X \subset Y)$. So assume that $V_\alpha \cap \mathrm{HOD} \in \mathrm{HOD}\ \forall\alpha$. Then since for all sets $X \subset \mathrm{HOD}$, $X \subset V_\alpha$ for some α, and trivially $X = X \cap \mathrm{HOD}$, we have that

$$X \cap \mathrm{HOD} = X \subset V_\alpha \cap \mathrm{HOD} \in \mathrm{HOD},$$

thus satisfying the claim. □

We know that L is, in a sense, the minimal inner model by proposition 0.27, so we can think of the L as the narrowest inner model. Contrast this to HOD, which can be thought of as the widest. L admits only "small" large cardinals (we will see what this means in chapter 2), whereas all traditional large cardinal notions can hold in HOD. For those acquainted with forcing (which in this text will be avoided except for the occasional mention), another striking comparison is that even simple sets such as $0^\#$ (which we discuss in section 3.6), may not be set-generic over L, but every set is set-generic over HOD.

The precise definitions of these large cardinal properties, and of $0^\#$ are not relevant at this introductory stage but will be introduced over the next two chapters. For now, the important thing to know is that the existence of $0^\#$ is unprovable in ZFC, but its existence/nonexistence induces a polarising dichotomy as a corollary to the following theorem.

Theorem 1.5 (Jensen's Covering Theorem)**.** *[1](p. 329) If $0^\#$ doesn't exist, then for every uncountable set of ordinals X, there exists a constructible set $Y \supset X$ such that $|X| = |Y|$.*

The following corollary is the important consequence of this for our purposes.

Corollary 1.6 (L Dichotomy Theorem)**.** *[14]*

(i) If $0^\#$ does not exist, then L is correct about singular cardinals and correctly computes their successors. That is, if κ is singular, (κ is singular)^L and $(\kappa^+)^L = \kappa^+$.

(ii) If $0^\#$ does exist, then every uncountable cardinal is inaccessible in L.

These are two radically different possibilities. In the first, L is, in a sense, close to V in that it correctly identifies much of the structure of V. In the second, L is very far from V, as it perceives that ω_1 is an inaccessible cardinal (see definition 2.3), which is clearly not true in V.

Later, Woodin proved there is a similar dichotomy for HOD.

Theorem 1.7 (HOD Dichotomy Theorem)**.** *[14] Suppose that κ is an extendible cardinal. Then exactly one of the following holds:*

(i) For every singular cardinal $\gamma > \kappa$, γ is singular in HOD and $(\gamma^+)^{HOD} = \gamma^+$.

(ii) Every regular cardinal $\gamma \geq \kappa$ is a measurable cardinal in HOD.

Again, this presents two radically different alternatives: in the first of which HOD is close to V, as it correctly identifies the structure of singular cardinals in V, but in the second, HOD is far from V. It is important to note this second dichotomy is not tied to the existence of a set like $0^{\#}$. If $0^{\#}$ exists, we know which side of the L dichotomy we are on, but no such set is known that does this for the HOD dichotomy. Certainly, no traditional large cardinal axiom asserts enough to force us to the far side of the problem since all traditional large cardinals can hold within HOD, that is, are compatible with $V =$ HOD, and so can't show the difference between the two. The possibility of a higher analogue of $0^{\#}$ is still extant, and perhaps such a set will be found that makes the HOD dichotomy look more like the L dichotomy.

There are two programs in set-theoretic research which correspond to the two sides of the HOD dichotomy. For the close side, the program is that of inner model theory. Recent work by Woodin has been on establishing an ultimate inner model "Ultimate-L". In his paper "In Search of Ultimate-L" [15], he wrote:

> "If there is an ultimate version of L which is compatible with all large cardinals and which must exist in a version that is very close to V, then perhaps there is some axiom that V is an ultimate version of L that is arguably true."

There is a natural conjecture concerning the existence of Ultimate-L. If this conjecture turns out to be true, then the first side must hold - the HOD dichotomy is no longer a dichotomy, and HOD is close to V. We discuss this somewhat further in chapter 6.

The other program representing the other side of the conjecture is that of large cardinals beyond choice, which we will heavily discuss in chapters 4 to 6.

2 A Hierarchy of Large Cardinals

2.1 Motivations

Views vary as to the epistemological status of large cardinal axioms, and whilst this discussion is not our primary purpose here, it is worth looking briefly at this before we embark on our journey up the large cardinal hierarchy. Whether we regard our axioms as self-evident or as nothing but definitions outlining a chosen subject matter can be debated, but in general the theory ZFC is accepted by mathematicians as the standard theory, and so we retain this as our starting point.

However, many questions we may have, even those as fundamental and simple to state as the continuum hypothesis[†], are formally unsolvable in ZFC. We can, however, use additional axioms to derive results concerning these sorts of questions, but the difficulty comes in selecting the appropriate axioms. Large cardinal axioms are a certain class of these.

There is no formal definition of a large cardinal property, but informally, we can think of one as an assertion which says that a cardinal with a certain property exists, and where this existence is not provable under our standard theory (be it ZFC or ZF). Obviously then, even though by our usual standards $\aleph_0$ for example is very large (being of course infinite), it is far too small to be a large cardinal in the set-theoretic sense. Since these properties are not provable in ZFC, we must assert them as additional axioms in order to use them. If we suppose stronger and stronger axioms, we will be able to use them to prove stronger results, so the stronger the axioms we add, the more powerful they will be as tools to us.

The problem we face here is in justifying the addition of these stronger axioms. The axioms of ZFC are largely simple and uncontroversial, for example, it doesn't seem like a big jump to assert that if you take two sets and put them together you can make another set that is their pair set. In contrast to this, the large cardinal assertions which we will give shortly are not usually particularly intuitive. They can, however, be partially justified on the basis that the strength of a large cardinal needed to resolve a question about the universe reflects the degree to which they are solvable, and to not look into large cardinals on the basis of them being unintuitive would prevent us from shedding this type of light on the questions we have. To quote Woodin[17]:

> "The development of Set Theory, after Cohen, has led to the realization that formally unsolvable problems have degrees of unsolvability which can be calibrated by large cardinal axioms."

This perspective is a fairly neutral one: whether or not the underlying assertions are in a sense "true", large cardinals can be used to make proving results easier, and the stronger the large cardinal axiom we need to assert to prove something,

[†]The independence of CH from ZFC was famously shown by Cohen in 1963[16], proving that ZFC+$\neg$CH is consistent, complementing Gödel's proof of the consistency of ZFC+CH in 1940[8].

the further away we are from proving it in ZFC alone. Under this perspective, one could reject any sense of reality of large cardinals, but still find use in their study.

Even if these axioms are not intuitive, and cannot be justified on intrinsic grounds, perhaps their strength alone is sufficiently motivating. As Gödel put it[18]:

> "There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory."

But why are large cardinals specifically of interest, over other potential types of strong axiom? In some sense, they follow from the ideas used in constructing set theory in a more coherent way than other proposed axioms. Ontological maximalism is rather compelling, in that if a set, particularly a large cardinal in this case, can exist, then it should. Why should sets large enough to satisfy a large cardinal property be excluded? Is that not against the very idea of the transfinite that Cantor had in mind?

Cantor proposed the transfinite to extend upwards indefinitely, and that the universe of set theory itself should be so large that no arbitrary property should delimit it. Cantor thought of the ordinals as being "absolutely infinite", larger than any conceivable set, as exemplified by the reflection principle. The reflection principle states that for any property of V, there is a corresponding rank of the universe V_α which satisfies that property. This idea reformulates Cantor's original belief above as it effectively states that the universe can't be uniquely distinguished from any of its layers by any structural property - any property of V we could conceive of is "reflected" by some rank. Weaker versions of the principle are actually theorems of ZFC, the well-known Montague-Lévy reflection principle should spring to mind, but more powerful versions of this idea have been used to motivate the existence of large cardinals. For further discussion of reflection principles and possible justifications see for example Koellner[20].

Whilst arguing against $V = L$ in his 1964 paper[19], Gödel called the axiom a "minimum property" and said of it:

> "Note that only a maximum property would seem to harmonize with the concept of set..."

Perhaps this idea applies equally to large cardinals, being a form of "maximum property". We shouldn't reject large cardinals outright because to do so would, like asserting $V = L$, be asserting a minimum property. For much further discussion of these debates, see Feferman[21] and Maddy[10][11][22][23].

Of course, not having a good reason to reject something a priori doesn't make the statement "true" in a sense which we perhaps would like them to be. There is debate surrounding the extent to which we can say that these axioms are "true". An interesting distinction is between the universe view and the multiverse view of set theory. The universe view is the view that there is one single background concept of set, and in which every statement about the universe of set theory has a definite truth value. Adherents of this view are looking to devices such as large cardinal axioms in the hope that they can be used to shed some light on absolute statements about the universe. The decades of struggle against independence are seen as due to the weakness of our theories - that the theories we have come up with thus far have not been strong enough to yield definitive answers to questions such as that of CH, not that such questions are fundamentally unanswerable.

The multiverse perspective holds the idea that there are simply several possible set-theoretic universes, with correspondingly distinctive concepts of set, under which the truth values of certain statements like CH differ. For a more full look into this subject, see Hamkins[24].

In the next few chapters we will build up a hierarchy of these large cardinal axioms. It is important to note that they will be ordered by the consistency strength of the theories including said axioms, not by the size of the cardinals supposed by them. There is often a correlation, but the relationship is not as simple as one might want.

2.2 "Small" Large Cardinals

Definition 2.1 (Weakly inaccessible cardinal)**.** *[25](p. 16) An uncountable cardinal κ is weakly inaccessible if it is a regular limit cardinal.*

The following proposition gives an idea of just how absurdly large even the smallest of our large cardinals are.

Proposition 2.2. *Let α be weakly inaccessible. Then $\alpha = \aleph_\alpha$.*

Proof. Suppose α is weakly inacessible. Then α is a limit cardinal and therefore $\alpha = \aleph_\lambda$ for some limit ordinal λ. Since $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$ and by the regularity of α, it follows that $\alpha = \lambda$ and so $\alpha = \aleph_\alpha$. □

Definition 2.3 ((Strongly) inaccessible cardinal)**.** *An uncountable cardinal κ is (strongly) inaccessible if it is a regular strong limit cardinal.*

Here, we enclose the word "strongly" in parentheses since later on, we will usually refer to strongly inaccessible cardinals as simply inaccessible cardinals, as is the modern convention. The idea is that a cardinal is inaccessible if it can't be "accessed" from below by the usual operations. Had we not supposed κ to be uncountable in our definitions, then ω would be the first inaccessible cardinal, so intuitively an inaccessible mirrors some of the properties ω has in relation to the finite cardinals. For example, much as no finite sequence of integers can approach ω, if κ is inaccessible, no sequence shorter than κ of elements less than κ can attain κ as a limit.

A potential choice of axiom asserting the existence of inaccessibles is the following, due to Tarski [3](p. 68).

Definition 2.4 (Axiom of Inaccessible Cardinals). *Let α and β be ordinals.*

$$AxInacc = \forall\alpha\exists\beta(\text{"}\beta \text{ is inaccessible"} \wedge \alpha < \beta).$$

To see in a sense how natural inaccessible cardinals are despite their overwhelming vastness compared to everything else we have seen so far, consider the following process (which is discussed in [26], section 3). Let Z_0 be the theory comprising the axioms of extensionality, union, foundation, separation and choice. Let Z_1 be the same but with pair set and power set added, and let ZC be the Z_1 with the axiom of infinity added (so all that is missing from ZFC is the replacement scheme).

Theorem 2.5. *[3](p. 107)*

(i) For every ordinal α, V_α is a model of Z_0.

(ii) For limit ordinals λ, V_λ is a model of Z_1.

(iii) If $\alpha > \omega$, the axiom of infinity holds in V_α; in particular $V_{\omega+\omega}$ is a model of ZC.

Proof. (i) First observe that $\mathrm{Trans}(V_\alpha)$ for all α, and extensionality and foundation are Δ_0-expressible. Since Δ_0 formulae are absolute between transitive structures (a standard result), these hold in V_α by downwards absoluteness from V.

With the exception of choice, the remaining axioms depend on the legitimacy of certain terms in V_α. The axiom of union holds in V_α since $\mathrm{rank}(\bigcup\alpha) \leq \mathrm{rank}(\alpha)$ implies that $x \in V_\alpha \to \bigcup\alpha \in V_\alpha$. For the axiom of separation, by the definability of satisfaction, $b = \{x \in a \mid V_\alpha \models \varphi(x)\}$ is a set and so, since $b \subset a$, if $a \in V_\alpha$, then $b \in V_\alpha$.

For the axiom of choice, observe that though we cannot define our choice function, if a is a disjointed set and b is a choice set for a, we can assume $b \subset \bigcup a$. The statement that b is a choice set for a can be written using a Δ_0 formula

$$\forall y \in a(\exists z(z \in y) \to \exists z \in y(z \in b \wedge \forall w \in y(w \in b \to w = z)))$$

and so the property is downwards absolute from V to V_α. Therefore, part (i) holds.

(ii) Given the results above we only show the additional two axioms. We first consider the pair set $\{a, b\}$. Suppose $a, b \in V_\alpha$ for some ordinal α. Then $\mathrm{rank}(\{a, b\}) = \max(\mathrm{rank}(a), \mathrm{rank}(b)) + 1 = \alpha + 1$ and so the pair set $\{a, b\} \notin V_\alpha$ in general, but is in $V_{\alpha+1}$, so the axiom only holds if α is a limit ordinal. The power set axiom has a Π_1 definition, and so is downwards absolute from V, so if $y = \mathcal{P}(x)$ holds in V, it must hold in all V_α. However, since $\mathrm{rank}(\mathcal{P}(x)) = \mathrm{rank}(x) + 1$, we obtain similarly that such a y can only exist in V_α for all x if $\mathrm{Lim}(\alpha)$.

(iii) is trivial as the set $\omega \in V_{\omega+1} \subseteq V_\alpha$ for all $\alpha > \omega$. □

The following theorem continues the pattern described above.

Theorem 2.6. *[3](p. 109) If κ is strongly inaccessible, then $V_\kappa \models ZFC$.*

Proof. As a consequence of the previous theorem, we need only show that the replacement scheme holds in V_κ since by definition, an inaccessible cardinal is a limit ordinal greater than ω. Briefly, this holds since as κ is regular, if $A \in V_\kappa$ and $F : A \to V_\kappa$ is definable over V_κ then $F''A$ has bounded rank below κ, therefore $F''A \in V_\kappa$. □

In fact, the above theorem has a much more general formulation, which we will make use of in chapter 4. Here ZFC_2 refers to second-order ZFC, which we will discuss further at a later stage.

Theorem 2.7 (Zermelo, Shepherdson)**.** *κ is inaccessible iff $V_\kappa \models ZFC_2$.*

See Kanamori[25], p. 19 for the proof.

The following is an important corollary of the first-order version of this theorem. As mentioned in the previous section, this is somewhat of a defining characteristic of a large cardinal.

Corollary 2.8. *Unless ZFC is inconsistent, we cannot prove the existence of a strongly inaccessible cardinal within ZFC.*

Proof. This follows from Gödel's second incompleteness theorem. We have that

$$\mathrm{ZFC} \vdash \exists\kappa(\text{"}\kappa \text{ is inaccessible"}) \to \exists M(M \models ZFC).$$

But of course, since ZFC cannot prove its own consistency, the conclusion must hold. □

Continuing in the way given by theorems 2.5 and 2.6, we can formulate axioms that assert the existence of larger and larger levels of the universe. Cutting off the universe at these levels gives us models which satisfy the axioms needed to assert the existence of the smaller levels.

Another way of looking at this is through the lens of a reflection principle. To borrow an argument from Reinhardt[27], Cantor's absolute, Ω, is inacessible and greater than any ordinal β, and so for any β there must be an inaccessible cardinal $\kappa > \beta$ as this cardinal reflects the above property of Ω. It follows that there are Ω-many inaccessibles below Ω, and so again by the reflection principle there should be an inaccessible cardinal κ with κ inaccessibles below it.

Inaccessible cardinals were first formulated by Hausdorff in his 1908 paper. It was only a few years after this that Mahlo, inspired by Hausdorff's work, extended the concept of an inaccessible cardinal by constructing the following hierarchy. Again see [25](p. 16) for more historical details.

Definition 2.9. *(i) κ is 0-weakly inaccessible if κ is regular.*

(ii) κ is $(\alpha+1)$-inaccessible if κ is a regular limit of α-weakly inaccessible cardinals.

(iii) *For limit ordinals δ, κ is δ-weakly inaccessible if κ is α-weakly inaccessible for every $\alpha < \delta$.*

This process can also be extended. In proposition 2.2, we characterised weakly inaccessible cardinals as fixed points of the aleph function. We can find higher large cardinals by considering higher fixed points.

Definition 2.10. *Let θ_κ be the κ-th inaccessible cardinal. A cardinal κ is hyper-inaccessible if $\theta_\kappa = \kappa$.*

In a similar vein to the theorems above, we obtain the following proposition:

Proposition 2.11. *Suppose κ is the first hyper-inaccessible. Then*

$$V_\kappa \models ZFC + AxInacc + \text{"there are no hyper-inaccessible cardinals."}$$

Proof. We already have by theorem 2.6 that $V_\kappa \models ZFC$. By definition of a hyper-inaccessible cardinal, AxInacc holds in V_κ. Finally, since V_κ accurately recognises that a cardinal $\lambda < \kappa$ is inaccessible, it holds that V_κ doesn't recognise a hyper-inaccessible cardinal, since the least of which in V_κ would have to be κ itself, an obvious impossibility. □

We can keep going like this by defining hyper-hyper-inaccessible cardinals.

Definition 2.12. *Let θ^2_κ be the κ-th hyper-inaccessible cardinal. A cardinal κ is hyper-hyper-inaccessible if $\theta^2_\kappa = \kappa$.*

Proposition 2.13. *Suppose κ is the first hyper-hyper-inaccessible. Then*

$$V_\kappa \models ZFC + \forall\alpha\exists\beta > \alpha(\text{"}\beta \text{ is hyper-inaccessible"}) +$$

$$\text{"there are no hyper-hyper-inaccessible cardinals"}$$

The proof follows completely analogously to the previous one concerning hyper-inaccessibles.

The above proposition generalises in the obvious way. We can continue like this forever, but to see interesting results we will have to do something qualitatively different. This led Mahlo to make a different approach.

Mahlo Cardinals

The new approach he took was to use stationary sets.

Definition 2.14. *For $\kappa > \omega$, κ is weakly Mahlo if $\{\rho < \kappa \mid \rho \text{ is regular}\}$ is stationary in κ.*

Proposition 2.15 (Mahlo 1911)**.** *[25](p. 17) If κ is weakly Mahlo, then κ is κ-weakly inaccessible.*

Proof. Set $R = \{\xi < \kappa \mid \xi \text{ is regular}\}$. Now define the club sets in $C_\alpha \subsetneq \kappa$ recursively by the following process: Set $C_0 = \kappa$, and $C_{\alpha+1}$ as the set of limit points of $C_\alpha \cap R$ other than κ. As κ is weakly Mahlo, each $C_\alpha \cap R$ is stationary and so $C_{\alpha+1}$ is club. For the limit stage, let $C_\lambda = \bigcap_{\alpha<\lambda} C_\alpha$. By induction, one can easily show that for $\alpha < \kappa$, $C_\alpha \cap R$ is made of the α-weakly inaccessible cardinals below κ. $\square$

In the same way as before, Mahlo then iterated this process.

Definition 2.16. *(i) κ is 0-weakly Mahlo if κ is regular,*

(ii) κ is $(\alpha+1)$-weakly Mahlo if $\{\gamma < \kappa \mid \gamma$ is α-weakly Mahlo$\}$ is stationary in κ,

(iii) For limit ordinals δ, κ is δ-weakly Mahlo if κ is α-weakly Mahlo for every $\alpha < \delta$.

Definition 2.17. *κ is (strongly) Mahlo if $\{\rho < \kappa \mid \rho$ is inaccessible$\}$ is stationary in κ.*

Again, strongly is placed in parentheses as from here onwards we will refer to strong Mahlo cardinals simply as Mahlo cardinals, as is the modern convention. One can also analogously define α-Mahlo cardinals, however, this is of little interest. To go qualitatively further, we will need yet another approach.

2.3 Measurable Cardinals

As suggested by the name, measurable cardinals arose from the study of measurability. Measurable cardinals turn out to be one of the most important types of large cardinal, and we shall see much more of them throughout this text.

Measure theory began with Lebesgue's 1902 thesis, where he proposed the following question[25](p .22).

Definition 2.18 (The Measure Problem)**.** *Is there a function μ that maps every bounded set of reals X to a non-negative real number $\mu(X)$ such that the following hold?*

(i) μ is not identically 0.

(ii) μ is translation-invariant, that is, if r is a real number such that $Y = \{x + r \mid x \in X\}$, then $\mu(X) = \mu(Y)$.

(iii) μ is countably additive, that is, if $\{X_n \mid n \in \omega\}$ is pairwise disjoint with bounded union, then $\mu(\bigcup_n X_n) = \sum_n \mu(X_n)$.

The now ubiquitous Lebesgue measure was Lebesgue's proposal as a solution to this problem. However in 1905, Giuseppe Vitali constructed a non-Lebesgue measurable set of reals, see [28] for the details. His proof assumed the existence of a well-ordering of the reals so his result depends on the axiom of choice. In 1970, Solovay used forcing to construct a model of ZF +

"There exists an inaccessible cardinal" in which all sets of reals are Lebesgue measurable[29].

Banach suggested that the measure problem be generalised by replacing translation-invariance with the idea that singletons should have measure zero. In removing translation, the geometry of $\mathbb{R}$ was effectively removed, allowing the problem to be applied much more generally - to an arbitrary set.

Definition 2.19 (The Measure Problem (Generalised)). *Is there a nonempty set S and a function $\mu : \mathcal{P}(S) \to [0,1]$ such that the following hold?*

(i) $\mu(S) = 1$,

(ii) $\mu(\{x\}) = 0$ *for all* $x \in S$,

(iii) μ *is countably additive, that is, if* $\{X_n \mid n \in \omega\} \subseteq \mathcal{P}(S)$ *is pairwise disjoint, then* $\mu(\bigcup_n X_n) = \sum_n \mu(X_n)$.

Note that the first condition here has been changed too, since if μ isn't identically zero, it must be the case that $\mu(S) > 0$, and so for the sake of simplicity, we can normalise to the case $\mu(S) = 1$. Such a function is a measure on $\mathcal{P}(S)$, but usually we simply refer to it as a measure on S.

A class of measures that will be important to us is the following.

Definition 2.20. *[1](p. 126) A measure μ on S is two-valued if either $\mu(X) = 0$ or $\mu(X) = 1$ for all $X \subset S$.*

The following simple proposition connects the concepts of filters and measures.

Proposition 2.21. *(i) If μ is a two-valued measure on S, then $U = \{X \subset S \mid \mu(X) = 1\}$ is a σ-complete ultrafilter on S.*

(ii) If U is a σ-complete ultrafilter on S, the following is a two-valued measure on S:

$$\mu(X) = \begin{cases} 1 & \text{if } X \in U, \\ 0 & \text{if } X \notin U. \end{cases}$$

Finally, we can now use this to define some large cardinal properties. We first need the following definition.

Definition 2.22. *[30](p. 150) Let κ, λ be uncountable cardinals, and μ a measure on κ. Then μ is called λ-additive if $\mu(\bigcup S) = 0$ for every family S of subsets of κ such that $|S| < \lambda$ and $\mu(X) = 0$ for all $X \in S$.*

Definition 2.23 (Real-valued measurable cardinal). *An uncountable cardinal κ is real-valued measurable if there exists a κ-additive measure μ on κ.*

If such a measure is two-valued, then by the above proposition, we get the following definition.

Definition 2.24 (Measurable cardinal). *An uncountable cardinal κ is measurable if there exists a κ-complete non-principal ultrafilter U on κ.*

Before proving some results, we first use the following minor proposition.

Proposition 2.25. *If U is a κ-complete ultrafilter on κ satisfying $\bigcap U = \emptyset$, then U is a uniform ultrafilter on κ, that is $|A| = \kappa$ for all $A \in U$.*

Lemma 2.26. *[30](pp. 150-151) If κ is real-valued measurable, it is regular.*

Proof. For a contradiction suppose κ is not regular, i.e. it is singular, and let $\langle \xi_\alpha \mid \alpha < \mathrm{cf}(\kappa)\rangle$ be an increasing sequence of ordinals cofinal in κ. Suppose that μ is a κ-additive probability measure on κ such that $\mu(\{x\}) = 0$ for all singletons $\{x\}$. By the above proposition, $\mu(\xi_\alpha) = 0$ for all $\alpha < \mathrm{cf}(\kappa)$. It then follows that $\mu(\kappa) = \mu(\bigcup\{\xi_\alpha \mid \alpha < \mathrm{cf}(\kappa)\}) = 0$, but of course this is a contradiction to μ being a probability measure on κ. □

Theorem 2.27. *[30](p. 151) Let κ be a measurable cardinal. Then κ is strongly inaccessible.*

Proof. Suppose κ is a measurable cardinal. Every measurable cardinal is trivially real-valued measurable, and so by the previous lemma, κ is regular. It therefore suffices to prove only that κ is a strong limit cardinal, i.e. that $2^\alpha < \kappa$ for all $\alpha < \kappa$.

We prove this by contradiction: assume that κ is measurable, and $\alpha < \kappa$ but $2^\alpha \geq \kappa$. Suppose $f : \kappa \to {}^\alpha 2$ is injective, and let U be a κ-complete nonprincipal ultrafilter on κ. Define the following for each $\beta < \alpha$ and $i \in 2$:

$$X^i_\beta = \{\xi \in \kappa \mid f(\xi)(\alpha) = i\}$$

Then for all κ, $X^0_\beta \cap X^1_\beta = \emptyset$ and $X^0_\beta \cup X^1_\beta = \kappa$, so exactly one of these sets is in U. We can now define another function $g : \alpha \to 2$ by letting $g(\beta)$ be such that $X^{g(\beta)}_\beta \in U$. As U is κ-complete, it follows that $X = \bigcap_{\beta<\alpha} X^{g(\beta)}_\beta \in U$. However, it follows from our construction of X and f that we have that $X = f^{-1}\{g\}$, and as f is injective this means X must be a singleton. This is a contradiction to the assumption that U is nonprincipal. □

2.4 Partition Cardinals

The cardinals discussed in this section are not of much relevance to our topic here, but we include them in this hierarchy of large cardinals for the sake of completeness. Because of this, we will remain brief, and omit proofs. For a much more comprehensive introduction to partition cardinals, see chapters 7-8 of Drake[3] or chapter 9 of Jech[1].

The following definitions are required:

Definition 2.28. *Let X be a set and n be a natural number, then*

$$[X]^n = \{Y \subset X : |Y| = n\},$$

i.e. $[X]^n$ is the set of subsets of X of cardinality n.

Definition 2.29. *Let $\{A_i \mid i \in I\}$ be a partition of $[X]^n$. A set $H \subset X$ is homogeneous for the partition if $[H]^n$ is included in some X_i, i.e. if the n-element subsets of H all lie in the same part of the partition.*

Recall the pigeonhole principle: If sufficiently many objects are distributed over some number of classes, at least one of these classes must have several objects. Ramsey's theorem, discovered by Frank Ramsey in 1930 and now well-known, is the following statement, which generalises the above principle.

Theorem 2.30 (Ramsey's Theorem)**.** *Let $n, k \in \mathbb{N}$. Every partition $\{A_1 ... A_k\}$ of $[\omega]^n$ has an infinite homogeneous set. So equivalently if $F : [\omega]^n \to \{1, ..., k\}$ is a partition, there is an infinite set $H \subset \omega$ such that F is constant on $[H]^n$.*

(See Jech[1] pp. 107-109 for a proof.)

This theorem can be generalised much further into the higher infinite, using the following terse notation, due to Erdös and Rado[31].

Definition 2.31 (Arrow Notation)**.** *Let $n \in \mathbb{N}$, m be any cardinal, and κ and λ be infinite cardinals. We write*

$$\kappa \to (\lambda)^n_m$$

to denote the following statement: Every partition of $[\kappa]^n$ into m pieces has a homogeneous set of size λ, i.e. every $F : [\kappa]^n \to m$ is constant on $[H]^n$ for some $H \subset \kappa$ with $|H| = \lambda$.

Under this formalism, Ramsey's theorem looks like this:

Theorem 2.32 (Ramsey's Theorem)**.** *For all $n, k \in \omega$, $\aleph_0 \to (\aleph_0)^n_k$.*

We now briefly give a menagerie of partition results.

Lemma 2.33. *[1](p. 110) For all cardinals κ, $2^\kappa \not\to (\omega)^2_\kappa$.*

Lemma 2.34. *[1](p. 110) For all cardinals κ, $2^\kappa \not\to (\kappa^+)^2_2$.*

Recall the beth numbers:

Definition 2.35 (Beth Numbers)**.** *[3](p. 207) Let κ be a cardinal.*

(i) $\beth_0(\kappa) = \kappa$,

(ii) $\beth_{\alpha+1}, (\kappa) = 2^{\beth_\alpha(\kappa)}$

(iii) For limit ordinals, λ, $\beth_\lambda(\kappa) = \bigcup_{\alpha<\lambda} \beth_\alpha(\kappa)$.

This is the more general form of the usual beth numbers, in which $\beth_0 = \aleph_0$, following the same process as above. An important partition statement is the following:

Theorem 2.36 (Erdös-Rado)**.** *[3](p. 207) If κ is an infinite cardinal and $n < \omega$, then $(\beth_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa$.*

We can now use partition properties to define some large cardinal properties.

Definition 2.37 (Weakly Compact Cardinal). *[1](p. 113) A cardinal κ is weakly compact if it is uncountable and satisfies $\kappa \to (\kappa)^2_2$.*

These are called weakly compact cardinals because they satisfy a compactness theorem for a certain class of languages, we omit the details here but refer the reader to the [1](pp. 292-294). The following lemma will form part of the structure of our hierarchy.

Lemma 2.38. *If κ is weakly compact, it is inaccessible.*

Proof. [1](p. 113) Suppose κ is a weakly compact cardinal. Recall that to be inaccessible, κ must be regular and a strong limit.

By assumption, we can assume κ is the disjoint union of some sets A_γ for $\gamma < \lambda$, and such that for all γ, $|A_\gamma| < \kappa$. Define a partition $F : [\kappa]^2 \to 2$ by $F(\{\alpha, \beta\}) = 0$ iff α and β are in the same A_γ. Then if H is homogeneous for the partition F, it must be that either $H \subset A_\gamma$ for some $\gamma < \lambda$, or that for all $\gamma < \lambda$, $|H \cap A_\gamma| \leq 1$. In either of these cases, it must be that $\kappa \not\to (\kappa)^2_2$.

It follows from lemma 2.34 that κ is a strong limit cardinal. Consider the following: If $\kappa \leq 2^\lambda$ for some $\lambda < \kappa$, then it follows from lemma 2.34 that $\kappa \not\to (\lambda^+)^2_2$ and so therefore $\kappa \not\to (\kappa)^2_2$. $\square$

We also have the following result, establishing that weakly compact cardinals are between measurable and inaccessble cardinals in our hierarchy. This result supersedes theorem 2.27.

Theorem 2.39. *If κ is measurable, then it is weakly compact.*

Proof. Assume κ is measurable and let $c : [\kappa] \to 2$ be a colouring (a colouring $f_P : [X]^\alpha \to I$ is the unique function f such that $f^{-1}\{i\} = P_i$ for all $i \in I$ for some partition $P = \{P_i \mid i \in I\}$). We need to show that there is an $X \in [\kappa]^\kappa$ such that X is homogeneous for c. Suppose U is a κ-complete nonprincipal ultrafilter on κ. By proposition 2.25, each set in U has cardinality κ. Thus for all $\alpha \in \kappa$, the set of ordinals $\alpha < \beta < \kappa$ is in U. Define the following:

$$X^i_\alpha = \{\beta \mid \alpha < \beta < \kappa \wedge c(\{\alpha, \beta\}) = i\}.$$

Then similar to in the proof of theorem 2.27, exactly one of the (disjoint) X^0_α, X^1_α is in U. This lets us induce a colouring $d : \kappa \to 2$ such that $d(\alpha)$ satisfies $X^{d(\alpha)}_\alpha \in U$. We define the following sequence: $\langle \alpha(\xi) \mid \xi < \kappa \rangle$ by $\alpha(0) = 0$ and $\alpha(\eta) = \min(\bigcap_{\xi<\eta} X^{d(\alpha(\xi))}_{\alpha(\eta)} \setminus (\alpha(\xi) + 1))$ for $\eta > 0$. At each stage of this construction, we take an intersection of fewer than κ elements of U, and so the resulting sequence is well-defined by κ-completeness. The sequence is trivially also strictly increasing and comprised of ordinals less than κ. Note also that if $\xi < \eta$, then $\alpha(\xi) \in X^{d(\alpha(\xi))}_{\alpha(\xi)}$.

It follows from the above considerations that $c(\{\alpha(\xi), \alpha(\eta)\}) = d(\alpha(\xi))$ for all $\xi < \eta < \kappa$. It then follows from the pigeonhole principle that there exists a $Y \in [\kappa]^\kappa$ and $i \in \{0,1\}$ such that $d(\alpha(\xi)) = i$ for all $\xi \in Y$. We can now conclude that $Y = \{\alpha(\xi) \mid \xi \in X\}$ is homogeneous for c.

□

Before moving away from the topic of partition cardinals, let's consider a generalisation of the Erdös and Rado's arrow notation. Before doing so we define the following:

Definition 2.40. *Let X be a set. Then we define $[X]^{<\omega} = \bigcup_{n=0}^{\infty}[X]^n$ where $[X]^n$ is as defined in definition 2.28. More generally we can define this for any infinite cardinal κ: $[X]^{<\kappa} = \{Y \subset X : |Y| < \kappa\}$*

Many authors write $\mathcal{P}_\kappa(X)$ for $[X]^{<\kappa}$, and since this is the standard when discussing supercompact cardinals, we will use this notation outside of the discussion of partition properties.

Definition 2.41. *Let κ be an infinite cardinal, m a cardinal such that $1 < m < \kappa$, and α an infinite limit ordinal such that $\alpha < \kappa$. Then we write:*

$$\kappa \to (\alpha)^{<\omega}_m$$

to denote the following statement: For every partition F of the set $[\kappa]^{<\omega}$ into m pieces, there is a $H \subset \kappa$ of order-type α such that for each $n \in \omega$, F is constant on $[H]^n$.

We can use this to define a large cardinal notion.

Definition 2.42 (Erdös Cardinal)**.** *[25](p. 80) For ordinals $\alpha \geq \omega$, the Erdös cardinal, denoted $\kappa(\alpha)$, is the least λ such that $\lambda \to (\alpha)^{<\omega}_2$.*

In common with some of the previous "small" large cardinals, the following large cardinal arises as a fixed point to the sequence of Erdös cardinals.

Definition 2.43 (Ramsey Cardinal)**.** *[25](p. 81) κ is a Ramsey cardinal if $\kappa \to (\kappa)^{<\omega}_2$.*

It is trivial to see that Ramsey cardinals are weakly compact. By an argument due to Erdös and Hajnal (see [25] pp. 83-84), measurable cardinals are Ramsey, and therefore, we have established that Ramsey cardinals lie between weakly compact and measurable cardinals in the hierarchy.

Another large cardinal we will make use of later on is that of a Jónsson cardinal, which has both combinatorial and model-theoretic characterisations. The original definition was as follows:

Definition 2.44. *A cardinal κ is Jónsson iff every algebra of cardinality κ has a proper subalgebra of the same cardinality.*

This can be formulated as a partition property in-keeping with the notation of this section, or as a model-theoretic assertion which we will expand on in Chapter 5.

Proposition 2.45. *[25](p. 93) The following are equivalent:*

(i) κ is a Jónsson cardinal.

(ii) $\kappa \to [\kappa]^{<\omega}_{\kappa}$.

(iii) Any structure for a countable first-order language with a domain of cardinality κ has a proper elementary substructure with a domain of the same cardinality.

We omit the proof. We also omit the proof of the following result which can also be found in [25](p. 93). Note that by (ii), we obtain that if κ is a Ramsey cardinal, then κ is a Jónsson cardinal.

Proposition 2.46 (Keisler-Rowbottom 1965)**.** *If there is a Jónsson cardinal, then $V \neq L$.*

2.5 Supercompact Cardinals

We defined weakly compact cardinals in the previous section. Here, we give two strengthenings of this. These are signficant strengthenings, which give us our first examples of cardinals higher than measurability in consistency strength. The first of these is that of a strongly compact cardinal. There are several equivalent definitions, but we give the following since it is reminiscent of a measurable cardinal.

Definition 2.47 (Strongly compact cardinal)**.** *Let κ be an uncountable regular cardinal. Then κ is strongly compact if for any given set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.*

It is clear that every strongly compact cardinal κ must be measurable, since it is trivial to verify that every ultrafilter that extends the filter $F = \{X : |\kappa \setminus X| < \kappa\}$ is non-principal.

An even stronger version of this is the notion of a supercompact cardinal. We first introduce some preliminary definitions.

Definition 2.48 (Fine Measure)**.** *Let F be the filter on $\mathcal{P}_\kappa(A)$ generated by the sets $\{Q \in \mathcal{P}_\kappa(A) \mid P \subset Q\}$. This filter is κ-complete, and if κ is strongly compact then by definition F can be extended to a κ-complete ultrafilter U. We call such a U extending F a fine measure.*

Definition 2.49 (Normal fine measure)**.** *A fine measure U on $\mathcal{P}_\kappa(A)$ is normal if whenever $f : \mathcal{P}_\kappa(A) \to A$ is such that $f(P) \in P$ for all $P \in X \in U$, then f is constant on a set in U, or equivalently if it is closed under diagonal intersections $\triangle_{a\in A} X_a = \{x \in \mathcal{P}_\kappa(A) \mid x \in \bigcap_{a\in x} X_a\}$.*

We can use this to formulate the following.

Definition 2.50 (Supercompact cardinal)**.** *Let κ be an uncountable cardinal. Then κ is supercompact if for every set A such that $|A| \geq \kappa$, there is a normal fine measure on $\mathcal{P}_\kappa(A)$.*

For now, we don't have much to say about either strongly compact or supercompact cardinals, but we will return to them in the next chapter once we have developed the technique of elementary embeddings, which will allow different characterisations of large cardinals, and unlock many more results.

$$
\begin{tikzcd}
\vdots \arrow[d, "\text{to strongly compact, supercompact etc.}"] & \\
\text{measurable} \arrow[d] & \\
\text{Ramsey} \arrow[d] & \\
\text{Jónsson} \arrow[d] & {} \\
\text{weakly compact} \arrow[d] & \text{increasing consistency strength} \arrow[u] \\
\text{Mahlo} \arrow[d] & {} \arrow[u] \\
\text{strongly inaccessible} \arrow[d] & \\
\text{weakly inaccessible} &
\end{tikzcd}
$$

Figure 2: The large cardinal hierarchy we have built up so far. Here the downward arrows represent the implication "every A is a B".
All hierarchies and commutative diagrams have been made with quiver[32].

3 Embedding Properties

In this section, we introduce elementary embeddings, utilising them to greatly extend the hierarchy we began to build in the previous chapter, and allowing us to derive some key results in the theory of large cardinals.

3.1 Elementary Embeddings and Ultrapowers

We will mostly work with elementary embeddings j from the universe V into some inner model M, that is $j : V \prec M$. As is standard throughout the literature, we always make the implicit assumption that j is not the identity, as this trivial elementary embedding would be very uninteresting.

As j is not the identity, the following proposition holds:

Proposition 3.1. *[33]*

(i) For all ordinals α, $j(\alpha) \geq \alpha$,

(ii) There exists some ordinal β such that $j(\beta) \neq \beta$, i.e. j moves some ordinal.

Proof. (i) follows by transfinite induction: 0 must be sent to at least 0 trivially. Suppose that $j(\beta) \geq \beta$. Then $j(\beta^+) = j(\beta)^+$ by elementarity. So $j(\beta^+) = j(\beta)^+ > j(\beta) \geq \beta$, so $j(\beta^+) \geq \beta^+$. For the limit case, let $\mathrm{Lim}(\lambda)$, and for all $\beta < \lambda$, $j(\beta) \geq \beta$. Then for all β, $j(\lambda) > j(\beta)$ by elementarity, so $j(\lambda) > j(\beta) \geq \beta$ for all β, therefore $j(\lambda) \geq \lambda$.

(ii) Let x be a set of least rank $\delta = rank(x)$ such that $j(x) \neq x$. Such a set exists by assumption of the nontriviality of j (and well-ordering of the ordinals). Assume for the sake of contradiction that $j(\delta) = \delta$, then let $y \in j(x)$. Then we have that $rank(y) < rank((j(x)) = j(rank(x)) = \delta$, where once again the first equality follows from elementarity. By the minimality of $rank(x)$, it follows that $j(y) = y$ and by elementarity it follows that $j(y) \in j(x) \leftrightarrow y \in x$. However, by the axiom of extensionality, this simply means that $j(x) = x$, a contradiction. □

This motivates the following definition:

Definition 3.2 (Critical Point)**.** *An ordinal δ is called the critical point of j, $crit(j)$, if it is the least ordinal moved by j.*

A related proposition is the following:

Lemma 3.3. *[1](p. 289) Let $j : V \prec M$ be an elementary embedding, and let $\kappa = crit(j)$. For all club subsets of κ, $\kappa \in j(C)$.*

Proof. For all $\alpha < \kappa$, $j(\alpha) = \alpha$ by definition of a critical point, so it must be that $j(C) \cap \kappa = C$. This is of course unbounded in κ. Closure of $j(C)$ follows by elementarity from the fact that C is closed, and so it must be that $\kappa \in j(C)$. □

We will later make use of the following, related definition:

Definition 3.4 (Critical Sequence). *[34] Let j be an elementary embedding with $\kappa = crit(j)$. Now let $\kappa_0 = \kappa$ and $\kappa_{n+1} = j(\kappa_n)$. We call the sequence $\langle \kappa_n \mid n \in \omega \rangle$ the critical sequence of j.*

For convenience, some authors use the following notation.

Definition 3.5. *Let M be a transitive set. Let $\mathscr{E}(M)$ denote the set of all nontrivial elementary embeddings $j : M \to M$.*

Ultrapowers of the Universe

The ultraproduct construction (see definition 0.17 for the general statement) can be generalised to construct ultrapowers of the entire universe V.

Definition 3.6 (Ultrapower of V). *[1](p. 285) Let U be an ultrafilter on a set S and consider the class of all functions with domain S. Define the following:*

$$f =^* g \text{ iff } \{x \in S \mid f(x) = g(x)\} \in U,$$

$$f \in^* g \text{ iff } \{x \in S \mid f(x) \in g(x)\} \in U.$$

So $[f] = \{g \mid f =^* g \wedge \forall h(h =^* f \to rank(g) \leq rank(h))\}$ is the equivalence class of f in $=^*$, and we write $[f] \in^* [g]$ when $f \in^* g$.

Let $Ult_U(V)$ be the class comprised of elements $[f]$ for all functions f on S. We use the following notation: $Ult = (Ult_U(V), \in^*)$.

Importantly, Łoś's Theorem still applies in this context:

Theorem 3.7. *Let $\varphi(x_1, ..., x_n)$ be a formula in the language of set theory. Then*

$$Ult \models \varphi([f_1], ..., [f_n] \text{ iff } \{x \in S \mid \varphi(f_1(x), ..., f_n(x)\}).$$

Lemma 3.8. *[1](p. 286) Let U be a σ-complete ultrafilter. Then $(Ult, \in^*)$ is a well-founded model.*

Proof. To show well-foundedness, we show there is no infinite descending $\in^*$-sequence in Ult. Suppose that $\langle f_n \mid n \in \omega \rangle$ is such a sequence. Then for every $n \in \omega$, the set $X_n = \{x \in S \mid f_{n+1}(x) \in f_n(x)\}$ is in the ultrafilter. By σ-completeness, $X = \bigcap_{n=0}^{\infty} X_n$ is also a member of U. But this means that $f_0(x) \ni f_1(x) \ni ...$ is an infinite descending $\ni$-sequence, which cannot exist, so the contradiction proves the claim. □

The pivotal moment in the development of large cardinals was when Scott proved the forward direction of the following theorem. The reverse direction is due to Keisler.

Theorem 3.9 (Scott, Keisler). *[25] (pp. 49-50) A cardinal κ is measurable if and only if there exists an inner model M and an elementary embedding $j : V \prec M$ such that $crit(j) = \kappa$.*

Proof (Forward direction). Let κ be a measurable cardinal, U a κ-complete ultrafilter and let the corresponding embedding be $j : V \prec M \cong \mathrm{Ult}_U(V)$. We seek to prove that $\mathrm{crit}(j) = \kappa$. First, we show that for all $\alpha < \kappa$, $j(\alpha) = \alpha$. By proposition 3.1, $j(\alpha) \geq \alpha$, so we need only show that $j(\alpha) \not> \alpha$.

Let $\alpha < \kappa$ be the least ordinal such that $j(\alpha) > \alpha$. Then if $[f] = \alpha$, $\{\xi < \kappa \mid f(\xi) < \alpha\} \in U$, and so by κ-completeness of U, there must be a $\beta < \alpha$ such that $\{\xi < \kappa \mid f(\xi) = \beta\} \in U$. So this means that $[f] = j(\beta) = \beta = \alpha$ with the final equality being an obvious contradiction. Therefore it must be that $j(\alpha) = \alpha$.

Now, again by κ-completeness of U, it contains no bounded subsets of κ by a standard result: If B is bounded in κ, then $\kappa \setminus \{\beta\} \in U$ for all $\beta \in B$, so

$$\kappa \setminus B = \bigcap_{\beta \in B} (\kappa \setminus \{\beta\}) \in U$$

and therefore $B \notin U$.

Now let $\mathrm{id} : \kappa \to \kappa$ be the identity map on κ, using the above statement, we have that for any α, $\{\xi < \kappa \mid \alpha < \xi < \kappa\} \in U$ implies that $\alpha = j(\alpha) < [id] < j(\kappa)$. So we have that $\kappa \leq [id] < j(\kappa)$. Therefore κ is the first ordinal moved by j, i.e. $\kappa = \mathrm{crit}(j)$. $\square$

Proof (Reverse direction). Let $j : V \prec M$ for some inner model M. We need to show that $\kappa = \mathrm{crit}(j)$ is a measurable cardinal.

By proposition 3.1, an elementary embedding must move some ordinal so such a critical point must exist. Trivially such a κ must be greater than ω since every ordinal up to ω is preserved by elementarity (since 0, $n+1$ and ω are absolute). Now we define U by the following:

$$X \in U \leftrightarrow X \subseteq \kappa \wedge \kappa \in j(X).$$

To show that κ is measurable, we need to show that U as defined above is a κ-complete non-principal ultrafilter. We first verify U is a filter: note that since $\kappa < j(\kappa)$, $\kappa \in j(\kappa)$ by definition, and so $\kappa \in U$. We also have that $\emptyset \notin U$ since $j(\emptyset) = \emptyset$ by elementarity, so clearly $\kappa \notin j(\emptyset)$. Clearly, by elementarity, $j(X \cap Y) = j(X) \cap j(Y)$ and $X \subset Y \to j(X) \subset j(Y)$, so U satisfies the definition of a filter. To show that U is an ultrafilter, we similarly see that $j(\kappa \setminus X) = j(\kappa) \setminus j(X)$.

For all $\alpha < \kappa$, we have $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}$, where the first equality is by elementarity and the second is because $\alpha < \kappa$. Therefore, $\kappa \notin j(\{\alpha\})$, so $\{\alpha\} \notin U$. This means U is non-principal.

Finally, we show that U is κ-complete. Let $\gamma < \kappa$ and let $S = \langle X_\alpha \mid \alpha < \gamma\rangle$ be a sequence of subsets of κ with the property that $\kappa \in j(X_\alpha)$ for all $\alpha < \gamma$. Since $j(\alpha) = \alpha$ for all $\alpha < \gamma$ and $j(\gamma) = \gamma$, and since the $j(\alpha)$-th term of $j(S)$ must be $j(X_\alpha)$, it must be that $j(S) = \langle j(X_\alpha \mid \alpha > \gamma\rangle$. It follows that $X = \bigcap_{\alpha<\gamma} X_\alpha \to j(X) = \bigcap_{\alpha<\gamma} j(X_\alpha)$, so $\kappa \in j(X)$ and so $X \in U$. Therefore U satisfies all the conditions for κ to be measurable. $\square$

The following corollary is also of great importance, marking out the boundary between "small" large cardinals that can consistently exist in L, and the larger large cardinals which are too strong to coexist in L.

Corollary 3.10 (Scott). *If there is a measurable cardinal, then $V \neq L$.*

Proof. Suppose that κ is the least measurable cardinal, and assume for the sake of contradiction that $V = L$. Let $j : V \prec M \cong \mathrm{Ult}_U(V)$. Then $M \subseteq V = L$ so $M = L$ by the minimality of L (see proposition 0.27). By elementarity of j, this means that ("$j(\kappa)$ is the least measurable cardinal")^M, i.e. M accurately states that κ is measurable. However by the above theorem κ is the critical point of j, so $j(\kappa) > \kappa$. □

We also have the following result.

Lemma 3.11. *[1](pp. 26-27) Let $j : V \prec M$ be a nontrivial elementary embedding, $crit(j) = \kappa$, and let U be the ultrafilter defined in the proof of the theorem above:*

$$X \in U \leftrightarrow X \subseteq \kappa \wedge \kappa \in j(X).$$

Define $j_U : V \prec Ult$ as the canonical embedding of V into $\mathrm{Ult}_U(V)$. Then there is an elementary embedding $k : Ult \prec M$ satisfying $k(j_U(a)) = j(a)$ for all a. That is, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "j_U"] \arrow[dr, "j"'] & Ult \arrow[d, "k"] \\ & M \end{tikzcd}$$

Proof. Define the following for all $[f] \in \mathrm{Ult}$:

$$k([f]) = j(f)(\kappa).$$

We will use this notation quite a bit later, so it is important to be clear about what it means. f is a function in V, and $j(f)$ is a function in M by elementarity, so $j(f)(\kappa)$ is the function $j(f)$ applied to κ, so though it looks strange at first, there is nothing untoward happening here.

Returning to the proof itself. We first observe that the choice of f representing $[f]$ doesn't matter. Let $f =_U g$, then the set $X = \{\alpha \mid f(\alpha) = g(\alpha)\} \in U$ and so $\kappa \in j(X) = \{\alpha < j(\kappa) \mid j(f)(\alpha) = j(g)(\alpha)\}$, i.e. $j(f)(\kappa) = j(g)(\kappa)$.

We next show that k, as defined above, is an elementary embedding. Suppose $\varphi(x)$ is a formula such that $\mathrm{Ult} \models \varphi([f])$, then the set $Y = \{\alpha \mid \varphi(f(\alpha))\} \in U$ and so

$$\kappa \in j(Y) = \{\alpha < j(\kappa) \mid M \models \varphi(j(f)(\alpha))\}$$

. Therefore as $j(f)(\kappa) = k([f])$, it follows that $M \models \varphi(k([f]))$, i.e. k is elementary.

It remains only to prove that $k(j_U(a)) = j(a)$. Since $j_U(a) = [c_a]$, where c_a is the constant function on κ giving value a, it follows that $k(j_U(a)) = j(c_a)(\kappa)$. But by elementarity, $j(c_a)$ is also the constant function on κ giving value a, and so $k(j_U(a)) = j(a)$. □

We can also investigate elementary embeddings of other transitive models.

Definition 3.12. *[1](p. 323) Suppose M is a transitive model of ZFC, and κ is a cardinal in M. An M-ultrafilter, U, on κ satisfies:*

(i) $\kappa \in U$ and $\emptyset \notin U$,

(ii) If $X \in U$ and $Y \in U$, then $X \cap Y \in U$,

(iii) If $X \in U$ and $X \subset Y \in M$, then $Y \in U$,

(iv) For all sets $X \subset \kappa$ satisfying $X \in M$, either $X \in U$ or $\kappa \setminus X \in U$.

We can define κ-completeness, nonprincipality and normality in the expected way. We also have the following result.

Lemma 3.13. *Let $j : M \prec N$ be an elementary embedding with critical point κ. Then κ is a regular uncountable cardinal in M, and $U = \{X \in P^M(\kappa) \mid \kappa \in j(X)\}$ is a nonprincipal normal κ-complete M-ultrafilter on κ.*

Scott's theorem linked the largely combinatorial constructions of large cardinal definitions given in Chapter 2 to the more global picture given by an elementary embedding. This discovery led to a period of rapid development of stronger and stronger large cardinal hypotheses using elementary embeddings with increasingly tight constraints. In the next section we will explore these. Before this, we will revisit supercompactness using the embedding tools we now have at our disposal.

Recall that supercompact cardinals were originally characterised in terms of normal measures. The following gives a reformulation in terms of elementary embeddings.

Lemma 3.14. *[1](p. 375) Let $\lambda \geq \kappa$. There exists a normal measure on $P_\kappa(A)$ iff there exists an elementary embedding $j : V \prec M$ such that*

(i) $j(\gamma) = \gamma$ for all $\gamma < \kappa$,

(ii) $j(\kappa) > \lambda$,

(iii) $M^\lambda \subset M$, i.e. all sequences $\langle \beta_\alpha \mid \beta < \lambda \rangle$ of elements of M are members of M.

This equivalence yields the following:

Definition 3.15. *A cardinal is λ-supercompact if it satisfies the above three conditions. A cardinal is supercompact if it is λ-supercompact for all λ.*

Before proving the above equivalence, we give the following. Recall the diagonal function: $d(x) = x$.

Lemma 3.16. *[1](p. 374) Suppose U is a normal measure on $\mathcal{P}_\kappa(\lambda)$. Then $[d] = \{j(\gamma) \mid \gamma < \lambda\} = j''\lambda$, and so we have that for all subsets $X \subset \mathcal{P}_\kappa(\lambda)$,*

$$X \in U \text{ iff } j''\lambda \in j(X).$$

Proof. If $\gamma < \lambda$, then for almost all x, $\gamma \in x$ and therefore $j(\gamma) \in [d]$. If $[f] \in [d]$, then $f(x) \in x$ for almost all x, and therefore by normality there is a $\gamma < \lambda$ satisfying $[f] = j(\gamma)$. □

Proof of Lemma 3.14. We begin with the forward direction. It follows from the previous lemma that

$$[f] = [g] \text{ iff } j(f)(j''\lambda) = j(g)(j''\lambda),$$

and

$$[f] \in [g] \text{ iff } j(f)(j''\lambda) \in j(g)(j''\lambda),$$

hence $[f] = j(f)(j''\lambda)$ for all functions f on $\mathcal{P}_\kappa(\lambda)$. Note that the order-type of $j''\lambda$ is λ and so therefore λ is represented in the ultrapower by the function $x \to \mathrm{ot}(x)$. Since $\mathrm{ot}(x) < \kappa$ we must have that $j(\kappa) > \lambda$, i.e (ii) must hold. Furthermore, by the κ-completeness of U, we have that $j(\gamma) = \gamma$ for all $\gamma < \kappa$, i.e. (i) holds.

To prove (iii), it suffices to show that if $\langle a_\alpha \mid \alpha < \lambda\rangle$ is defined such that $a_\alpha \in M$ for all $\alpha < \lambda$, then $\langle a_\alpha \mid \alpha < \lambda\rangle \in M$. Define functions f_α for $\alpha < \lambda$ representing the elements of M, so $[f_\alpha] \in M$. Now define another function f on $\mathcal{P}_\kappa(\lambda)$ such that $f(x) = \{f_\alpha(x) \mid \alpha \in x\}$. The following claim yields (iii), thus giving the forward result.

Claim . $[f] = \{a_\alpha \mid \alpha < \lambda\}$.

Proof of Claim. Firstly, if $\alpha < \lambda$, then for almost all x, $\alpha \in x$ and so $[f_\alpha] \in [f]$. Next observe that if $[g] \in [f]$, then for almost all x, $g(x) = f_\alpha(x)$ for some $\alpha \in x$. It follows from normality that there must be some $\gamma < \lambda$ such that for almost all x, $g(x) = f_\gamma(x)$ and so $[g] = a_\gamma$, giving us the desired result. □

The reverse direction is somewhat simpler. Suppose $j : V \prec M$ is an elementary embedding satisfying $(i) - (iii)$. By (iii), we have that $\{j(\gamma) \mid \gamma < \lambda\} \in M$ and so the following gives us an ultrafilter on $\mathcal{P}_\kappa(\lambda)$:

$$X \in U \text{ iff } j''\lambda \in j(X)$$

One can easily prove that this is a κ-complete ultrafilter. Furthermore, as for all $\alpha \in \lambda$, $\{x \mid \alpha \in x\} \in U$, U is a fine measure.

Now suppose that for almost all x, $f(x) \in x$, then $j(f)(j''\lambda) \in j''\lambda$. This means that $j(f)(j''\lambda) = j(\gamma)$ for some $\gamma < \lambda$ and so therefore for almost all x, we have $f(x) = \gamma$ and therefore U is normal. This completes the proof. □

The above characterisation is the standard way of defining supercompact cardinals using elementary embeddings, but there are alternatives. The following is due to Magidor, and is more explicitly related to the stronger large cardinals of the next section, particularly extendible cardinals.

Theorem 3.17 (Magidor)**.** *([14], or see [35] for Magidor's original work) A cardinal δ is supercompact iff for all $\kappa > \delta$ and $a \in V_\kappa$, there exist $\bar{\delta} < \bar{\kappa} < \delta$, $\bar{a} \in V_{\bar{\kappa}}$ and an elementary embedding $j : V_{\bar{\kappa}+1} \prec V_{\kappa+1}$ such that crit$(j) = \bar{\delta}$, $j(\bar{\delta}) = \delta$ and $j(\bar{a}) = a$.*

Proof. Suppose $\gamma = |V_{\kappa+1}|$ and $j : V \prec M$ is an elementary embedding witnessing the fact that δ is γ-supercompact. It follows that $j \upharpoonright V_{\kappa+1} \in M$, and that j witnesses the statement that "There exist $\bar{\delta} < \bar{\kappa} < j(\delta)$, $\bar{a} \in V_{\bar{\kappa}}$ and an elementary embedding $i : V_{\bar{\kappa}+1} \prec V_{j(\kappa)+1}$ such that crit$(i) = \bar{\delta}$, $i(\bar{\delta}) = j(\delta)$ and $i(\bar{a}) = j(a)$." By the elementarity of j, this yields Magidor's formulation.

Now for the converse suppose $\gamma > \delta$ is as given above. If one applies the above using $\kappa = \gamma + \omega$ to yield $\bar{\kappa}, \bar{\delta}$ and j as specified. Now take $\bar{\gamma}$ such that $j(\bar{\gamma}) = \gamma$. We have that $j[\bar{\gamma}] \in V_{\kappa+1}$ and therefore it induces a normal fine ultrafilter $\bar{U}$ on $\mathcal{P}_{\bar{\delta}}(\bar{\gamma})$. Clearly, $\bar{U} \in V_{\bar{\gamma}+\omega}$, and so if we define $U = j(\bar{U})$, elementarity of j ensures that in $V_{\kappa+1}$, U is a normal fine ultrafilter on $\mathcal{P}_\delta(\gamma)$. Due to the choice of κ, $V_{\kappa+1}$ is correct about this and so δ is γ-supercompact. □

3.2 Beyond Supercompactness

In the previous chapter, we ascended the large cardinal hierarchy up to the level of supercompact cardinals. Using elementary embeddings, we can now develop large cardinal properties that are even stronger than the notion of a supercompact cardinal, by imposing stronger and stronger constraints on the model. This was largely realised in the work of Solovay and Reinhardt. Here, we pick up where we left off.

Definition 3.18 (α-extendible cardinal)**.** *[25](p. 311) A cardinal κ is α-extendible if there is a β and an elementary embedding $j : V_{\kappa+\alpha} \prec V_\beta$ such that crit$(j) = \kappa$ and $\alpha < j(\kappa)$.*

Following the now very familiar pattern we obtain the following definition from the above:

Definition 3.19 (Extendible cardinal)**.** *[1](p. 379) A cardinal κ is extendible if for every $\alpha > \kappa$, there exists an ordinal β and an elementary embedding $j : V_\alpha \prec V_\beta$.*

This is clearly equivalent to the following definition, which we will use later.

Definition 3.20. *A cardinal κ is extendible if for all α, there exists a β and an elementary embedding*

$$j : V_{\kappa+\alpha} \prec V_{j(\kappa)+\beta}$$

such that crit$(j) = \kappa$ and $\alpha < j(\kappa)$.

Theorem 3.21. *[1](pp. 379-380) Let κ be an extendible cardinal.*

(i) κ is supercompact.

(ii) There is a normal measure D on κ such that $\{\alpha < \kappa \mid \alpha$ is supercompact$\} \in D$.

Proof. (i): Suppose that κ is extendible. It suffices to show that κ is λ-supercompact for all regular $\lambda < \alpha$ where $\alpha > \kappa$ is a limit cardinal such that if $V_\alpha \models \kappa$ is λ-supercompact for all λ, then κ is supercompact. By the reflection principle, such an α does exist, therefore, we have only to prove that κ is λ-supercompact for all regular $\lambda < \alpha$.

Suppose $j : V_\alpha \prec V_\beta$ witnesses the extendibility of κ. Consider its critical sequence $\langle \kappa_n \mid n \in \omega \rangle$. By a known result it holds that either there is an n such that $\kappa_n < \alpha < j(\kappa_n)$, or $\alpha = \lim_{n\to\infty}\kappa_n$. It therefore suffices to prove by induction on n that κ is λ-supercompact for each regular $\lambda < \kappa_n$.

For the base case, observe that clearly κ is λ-supercompact for all $\lambda < \kappa_1$. For the inductive step, assume that κ is λ-supercompact for all $\lambda < \kappa_n$. Now apply j, giving $V_\beta \models j(\kappa)$ is λ-supercompact for all regular $\lambda < \kappa_{n+1}$. Therefore $V_\beta \models \kappa$ is γ-supercompact for all $\gamma < j(\kappa)$ The conclusion follows from the following claim:

Claim . *Suppose $\lambda \geq \kappa$ is regular and κ is λ-supercompact. Then for $\alpha < \kappa$, if α is γ-supercompact for all $\gamma < \kappa$, then α is λ-supercompact.*

Proof of Claim. Let U be a normal measure on $\mathcal{P}_\kappa(\lambda)$, and let $j_U : V \prec \mathrm{Ult}_U$. As $j(\alpha) = \alpha$, it must be that $\mathrm{Ult} \models \alpha$ is γ-supercompact for all $\gamma < j(\kappa)$, and $\mathrm{Ult} \models \alpha$ is λ-supercompact. So there is a normal measure D such that $\mathrm{Ult} \models D$ is a normal measure on $\mathcal{P}_\alpha(\lambda)$. As $|\mathcal{P}_\alpha(\lambda)| = \lambda$ and $\mathrm{Ult}^\lambda \subset \mathrm{Ult}$, every subset of $\mathcal{P}_\alpha(\lambda)$ is in Ult. Therefore we can conclude that D is a normal measure on $\mathcal{P}_\alpha(\lambda)$. □

For (ii), take α to be a limit ordinal such that $\alpha > \kappa$ and let $j : V_\alpha \prec V_\beta$ be such that $\mathrm{crit}(j) = \kappa$. Define $D = \{X \subset \kappa \mid \kappa \in j(X)\}$. By the previous part, κ is supercompact, and therefore $V_\beta \models \kappa$ is γ-supercompact for all $\gamma < j(\kappa)$. It follows that

$$\{\alpha < \kappa \mid \alpha \text{ is } \gamma\text{-supercompact for all } \gamma < \kappa\} \in D.$$

By the claim, we can conclude that every $\alpha \in A$ is supercompact, yielding (ii) □

The following is an axiom schema.

Definition 3.22 (Vopěnka's Principle (VP))**.** *Let C be a proper class of models of the same language. Then there exist two members $\mathfrak{A}$ and $\mathfrak{B}$ of C such that there is an elementary embedding $j : \mathfrak{A} \prec \mathfrak{B}$.*

Lemma 3.23. *[1](p. 380) Assume VP. Then there exists an extendible cardinal.*

Proof. Let A be the class of limit ordinals α such that the following hold:

(i) $\mathrm{cf}(\alpha) = \omega$.

(ii) For every $\kappa < \alpha$, if $V_\alpha \models$ "κ is extendible", then κ is extendible. For $\kappa < \gamma < \alpha$, if there is an elementary embedding $j : V_\gamma \prec V_\delta$ with $\mathrm{crit}(j) = \kappa$, then $V_\alpha \models$ there is an elementary embedding.

By the reflection principle, such a class A must be a proper class.

Now let C be comprised of the models $\langle V_{\alpha+1}, \in \rangle$ for all $\alpha \in A$. By VP, there are some $\alpha, \beta \in A$ and an elementary embedding $j : V_{\alpha+1} \prec V_{\beta+1}$. From this, we have that $j(\alpha) = \beta$, and so j moves some ordinal. However $\alpha \neq \mathrm{crit}(j)$ as the critical point must be measurable, and $\mathrm{cf}(\alpha) = \omega$. So let $\kappa = \mathrm{crit}(j)$. It follows that for all $\gamma < \alpha$, $j \restriction V_\gamma$ reflects to a witness to extendibility, and so therefore we can conclude from the construction of A that κ is extendible. $\square$

Definition 3.24 (Huge cardinal)**.** *A cardinal κ is huge if there exists an elementary embedding $j : V \prec M$ with crit$(j) = \kappa$ such that $M^{j(\kappa)} \subset M$.*

The result below provides some sort of continuation to the program we started with inaccessible cardinals in the previous chapter.

Lemma 3.25. *Let κ be huge. Then $(V_\kappa, \in)$ is a model of VP.*

Proof. It suffices to show that if C is a set of models, and $\mathrm{rank}(C) = \kappa$, then there exist $\mathfrak{A}, \mathfrak{B} \in C$ and an elementary embedding $h : \mathfrak{A} \prec \mathfrak{B}$.

So suppose that $j : V \prec M$ is an elementary embedding with $\mathrm{crit}(j) = \kappa$ and $M^{j(\kappa)} \subset M$, i.e. suppose j witnesses the hugeness of κ. Because $\mathrm{rank}(C) = \kappa$, there must be an $\mathfrak{A}_0 \in j(C)$ satisfying $\mathfrak{A}_0 \notin C$, i.e. $j(\mathfrak{A}) \neq \mathfrak{A}_0$.

Now let $i_0 = j \restriction \mathfrak{A}_0$. It trivially follows that $i_0 : \mathfrak{A}_0 \prec j(\mathfrak{A}_0)$, and as $|\mathfrak{A}_0| < j(\kappa)$, it follows that $i_0 \in M$. Therefore, the following holds:

$$M \models \text{"there exists an } \mathfrak{A} \in j(C), \mathfrak{A} \neq j(\mathfrak{A}_0), \text{ and there}$$

$$\text{is an elementary embedding i: } \mathfrak{A} \prec j(\mathfrak{A}_o)\text{."}$$

Therefore, there exists some $\mathfrak{A} \in C$ such that $\mathfrak{A} \neq \mathfrak{A}_0$, and there exists an elementary embedding $i : \mathfrak{A} \prec \mathfrak{A}_0$. So therefore

$$M \models \text{"}i \text{ is an elementary embedding } \mathfrak{A} \prec \mathfrak{A}_0\text{"} .$$

Since $\mathrm{rank}(\mathfrak{A}) < \kappa$, it must be that $\mathfrak{A} = j(\mathfrak{A})$, and hence $\mathfrak{A} \in j(C)$, so therefore finally we have

$$M \models \text{"there exist distinct } \mathfrak{A}, \mathfrak{B} \in j(C) \text{ and an embedding } h : \mathfrak{A} \prec \mathfrak{B}\text{"}$$

If follows that there do in fact exist distinct $\mathfrak{A}, \mathfrak{B} \in C$, and an elementary embedding $h : \mathfrak{A} \prec \mathfrak{B}$, yielding the result. $\square$

One may also see huge cardinals defined hierarchically, as by now we have done many times.

Definition 3.26. *[25](p. 331) Suppose $n \in \omega$.*

(i) κ is n-huge if there exists a $j : V \prec M$ with crit(j) $= \kappa$ and $\lambda = j^n(\kappa)$ and M is closed under all its sequences of length λ

(ii) κ is huge if it is 1-huge.

(iii) κ is almost huge if there is a $j : V \prec M$ with crit(j) $= \kappa$ and is closed under all its sequences of length less than λ.

There are many more large cardinal concepts and many more variants of large cardinals, such as more variants of huge cardinals which we haven't mentioned in this text. A useful online resource compiling these is Cantor's Attic[36], a community project started by Joel David Hamkins and Victoria Gitman.

Definition 3.27 (λ-strong cardinal)**.** *A cardinal κ is λ-strong for some $\lambda \geq \kappa$ if there exists an elementary embedding $j : V \prec M$ with crit(j) $= \kappa$ such that $j(\kappa) > \lambda$ and $V_\lambda \subset M$.*

The following is equivalent to being λ-strong for all $\lambda \geq \kappa$

Definition 3.28 (Strong cardinal)**.** *A cardinal κ is a strong cardinal if for every set x, there is an elementary embedding $j : V \prec M$ with crit(j) $= \kappa$ such that $x \in M$.*

One can easily verify that if $\alpha \leq \beta$ and κ is β-strong, then κ is also α-strong. One can also easily show that κ is measurable iff κ is 0-strong iff κ is 1-strong.

We further have the following:

Definition 3.29. *A cardinal κ is superstrong if there exists an elementary embedding $j : V \prec M$ with crit(j) $= \kappa$ and $V_{j(\kappa)} \subseteq M$.*

These hypotheses are actually weaker analogues of some that we have already mentioned. If κ is supercompact, then κ is stong, and if κ is huge, then κ is superstrong.

Finally, the following is a somewhat more complex hypothesis that plays a significant role in set theory, having connections to descriptive set theory and determinacy. We will not touch on these areas in this text, but we will briefly return to Woodin cardinals in chapter 6.

Definition 3.30 (Woodin Cardinal)**.** *[25](p. 360) A cardinal κ is a Woodin cardinal if for any function $f \in {}^\kappa\kappa$, there is an $\alpha < \kappa$ such that $f''(\alpha) \subseteq \alpha$ and an elementary embedding $j : V \prec M$ with crit(j) $= \alpha$ and $V_{j(f)(\alpha)} \subseteq M$.*

Finally, putting an end to the program searching for stronger and stronger large cardinal hypotheses, Reinhardt proposed the ultimate elementary-embedding-type large cardinal hypothesis:

Definition 3.31 (Reinhardt Cardinal)**.** *A cardinal κ is Reinhardt if there exists a non-trivial elementary embedding $j : V \prec V$ with critical point κ.*

Is such an outrageous hypothesis even consistent with our axioms? Kunen quickly gave us an answer in 1971[37]. We give an account of this in the following section.

3.3 The Kunen Inconsistency

Kunen's result is given below. It says that under ZFC a nontrivial elementary embedding from V to V, and therefore a Reinhardt cardinal, cannot exist, thus putting an end to the pattern of stronger and stronger large cardinal hypotheses of the previous section. However, all known proofs depend on choice, and it is still unknown whether or not Reinhardt cardinals are consistent with ZF in the absence of choice. Given the speed with which Kunen was able to demonstrate their inconsistency with ZFC, and the fact it has now been over 50 years, if it is the case that Reinhardt cardinals are inconsistent with ZF, it certainly isn't obvious. We will explore Reinhardt cardinals without choice in detail in the next chapter.

Theorem 3.32 (Kunen Inconsistency)**.** *Suppose that $j : V \prec M$. Then $M \neq V$.*

An important thing to notice here is that the above theorem is not formalisable in a first-order theory like ZFC. Explicitly, one is saying that $\neg\exists j(j : V \prec V)$, however, we can't quantify over j. many of our large cardinals defined in terms of embeddings are characterised in this way, but have equivalent first-order formulations. In the case of a Reinhardt cardinal, there cannot be a first-order formulation of the definition: Suppose there is, then if κ is the least Reinhardt cardinal, so is $j(\kappa)$ by elementarity, but this is an immediate contradiction since $j(\kappa) > \kappa$ by the definition of a Reinhardt cardinal and proposition 3.1. So it follows that there is no first-order concept that fully captures the idea of a Reinhardt cardinal.

Originally Kunen formulated his result in Morse-Kelley (MK) set theory (see [37]), but the slightly weaker von Neumann-Bernays-Gödel (NBG) set theory suffices. Both of these theories are second order. We will not go into too much detail in defining these theories but it is important to know how they relate to our usual first-order theories ZF and ZFC. The power gained by working in NBG or MK over ZF/ZFC is that these theories can speak about proper classes, something excluded by ZF and its first-order relatives. This allows us to work with results such as Kunen's inconsistency. When we need this, we will always use NBG in this text, for the following reason: NBG is a conservative extension of ZF.

Definition 3.33. *[38] (p. 66) A supertheory S of a theory T is a conservative extension if every theorem of T is a theorem of S and every theorem of S in the language of T is already a theorem of T.*

What this amounts to in our present context is that everything NBG says about sets, ZFC already says about sets (the same holds with the choice version of NBG and ZFC), so by extending to NBG we don't gain anything as regards sets, however NBG can deal with assertions surrounding proper classes such as Kunen's inconsistency, which we would not be able to fully formulate in ZFC.

Morse-Kelley set theory, however, is stronger than both ZFC and NBG: unlike NBG it can prove more things about sets themselves. In fact, MK is strong enough to prove Con(ZFC).

It is possible to formulate Kunen's result as a schema within a first-order theory: we can assert that no first-order formula defines a nontrivial embedding $j : V \prec V$. However, this only eliminates definable embeddings (see [39] for further discussion of this).

There are several proofs of Kunen's result given in the literature; here we state the one due to Woodin [25](p. 320), which is perhaps the easiest to follow.

Proof of Kunen's Inconsistency. Let $\kappa = crit(j)$, and $\lambda = sup(\{j^n(\kappa) \mid n \in \omega\})$, the supremum of its critical sequence. By the Solovay splitting theorem (theorem 0.12), there exists an $S : \kappa \to \mathcal{P}(\lambda^+)$ such that $ran(S)$ is a partition of $\{\xi < \lambda^+ \mid cf(\xi) = \omega\}$ into sets stationary in λ^+.

By the definition above, we have that $j(\lambda) = j(sup(\{j^n(\kappa) \mid n \in \omega\}) = sup(\{j^{n+1}(\kappa) \mid n \in \omega\}) = \lambda$. Since $j(\lambda) = \lambda$ and by the elementarity of j, $j(\lambda^+) = \lambda^{+M}$.

Hence, by elementarity, $j(S) : j(\kappa) \to \mathcal{P}(\lambda^+)$. We have that $S(\kappa)$ is stationary by Solovay's result above, so $j(S)(\kappa)$ is stationary in λ^{+M}, i.e.

$$(j(S)(\kappa) \subseteq \{\xi < \lambda^+ \mid cf(\xi) = \omega\} \text{ is stationary in } \lambda^+)^M.$$

So since as just shown, $\lambda^+ = \lambda^{+M}$, this is equivalent to:

$$(j(S)(\kappa) \subseteq \{\xi < \lambda^+ \mid cf(\xi) = \omega\} \text{ is stationary}^M \text{ in } \lambda^+.$$

Now assume for the sake of contradiction that $V = M$. Trivially, this gives that the above statement says that $j(S)(\kappa)$ really is stationary in λ^+. Because of this, $j(S)(\kappa)$ must have a stationary intersection with at least one of the stationary sets $S(\alpha)$ in the partition, so $j(S)(\kappa) \cap S(\alpha_0)$ is stationary in λ^+ for some $\alpha_0 < \kappa$.

The following set is easily shown to be ω-closed unbounded in λ^+.

$$C = \{\xi < \lambda^+ \mid j(\xi) = \xi \wedge \mathrm{cf}(\xi) = \omega\}.$$

By a standard result (proposition 0.13), we have that there is a $\xi_0 \in j(S)(\kappa) \cap S(\alpha_0) \cap C$. This gives that $\xi_0 = j(\xi_0) \in j(S(\alpha_0)) = j(S)(\alpha_0)$ by elementarity. But this gives that $\xi_0 \in j(S)(\kappa) \cap j(S)(\alpha_0)$, which by elementarity entails a contradiction since the range of $j(S)$ must be comprised of pairwise disjoint sets. □

The following corollary is a useful refinement of Kunen's inconsistency, and the first of the two refinements below is the form in which we shall most often see Kunen's inconsistency from now onwards.

Corollary 3.34. *[25](p. 322)*

(i) *For any ordinal α, there is no $j : V_{\alpha+2} \prec V_{\alpha+2}$.*

(ii) *Suppose $j : V \prec M$ is a nontrivial elementary embedding such that α is the least ordinal above $crit(j)$ such that $j(\alpha) = \alpha$, then $j''\alpha \notin M$*

Note that the consequence of the first result above is that the only nontrivial elementary embeddings $j : V_\alpha \prec V_\alpha$ are those of the form $j : V_\lambda \prec V_\lambda$, and $j : V_{\lambda+1} \prec V_{\lambda+1}$ where λ is a limit ordinal, since for any other ordinal α, $V_\alpha = V_{\beta+2}$ for some ordinal β.

(i). Kunen's original proof of this is potentially easier to follow, (see [25] pp. 319-320). However, since we used Woodin's proof of the main result , we give the version that follows from Woodin's proof.

The idea here is that everything we used to derive the contradiction can be faithfully coded into $V_{\lambda+2}$ so the same contradiction can be reached using just these codes.

Subsets of λ^+ can be coded as elements of $V_{\lambda+2}$ using well-orders and a pairing function.

For a well-ordering $R \subset \lambda \times \lambda$, let its order type be $\alpha_R < \lambda^+$ and $e_R : \alpha_R \to \lambda$ be the associated order isomorphism (recall that the order type of an ordinal is the unique ordinal to which it is order isomorphic). Define also the pairing function, $p : (\lambda \times \lambda) \times \lambda \to \lambda$, the bijection that allows ordered pairs in λ to be encoded in λ.

A subset $X \subset \lambda^+$ can therefore be encoded in $V_{\lambda+2}$ using the following representation:

$$\{p''(R \times e''_R(X \cap \alpha_R)) \mid R \text{ well-orders } \lambda\} \in V_{\lambda+2}.$$

This is preserved under elementary embeddings which is the criterion we need for this to be useful to us, as the proof relies on this.

We can use the above notion to encode into $V_{\lambda+2}$ both S and C as defined in the proof of Kunen's theorem above, allowing us to reach the same contradiction using only sets within $V_{\lambda+2}$. For (ii), see [25](p. 322) □

Many attempts have been made at finding stronger inconsistencies, refuting large cardinals at lower levels than that of a Reinhardt cardinal (in ZFC), but notably none have been found. It is possible that deeper inconsistency proofs are yet to be discovered, but it is usually the case that when something is inconsistent, it isn't too difficult to show. Kunen's result is relatively simple, and yet it remains the best we have.

The higher in consistency strength a notion is, the more powerful it is as a proving tool, and so it should be easier to prove inconsistent with ZFC. So the best course of action should surely be to look just underneath the notion of a Reinhardt cardinal and see if we can pull the inconsistency result down that little bit. Of course, the first problem one encounters is how exactly to approach the strength of a Reinhardt cardinal without actually reaching it. We will explore this first program in the next section.

Another path towards an inconsistency proof is to push even stronger, beyond the level of a Reinhardt cardinal. Kunen's result depends on choice, and it still remains open whether the existence of a Reinhardt cardinal can be refuted without the axiom of choice. Because increasing the consistency strength of the cardinals we are studying should, in theory, make it easier to derive an inconsistency, pushing further upwards we should reach a point where we can prove a consistent limit like Kunen's without choice, for otherwise, it would seem that such outrageously strong notions are consistent. We introduce the choiceless hierarchy in chapter 4, to show the progress made in this second program.

Generalisations There are several generalisations of Kunen's inconsistency, some of which involve HOD. These results are due to various authors, and have been chronicled in [39] by Hamkins, Kirmayer and Perlmutter.

Theorem 3.35 (Woodin). *There is no nontrivial elementary embedding $j : V \prec HOD$.*

Proof Sketch. See [39] (pp. 16-17) for a complete proof. This result largely follows Woodin's proof of the classic Kunen inconsistency that we gave above. It is notable that in this modification, AC need not be assumed as HOD satisfies AC. By a result from [39], the embedding has some critical point κ, which is measurable in V. We can construct the critical sequence in the usual way, and as in the original, we have that $j(\lambda) = \lambda$ and $j(\lambda^+) = \lambda^+$.

As before, by the Solovay splitting theorem 0.12, we can partition $\{\xi < \lambda^+ \mid \mathrm{cf}(\xi) = \omega\}$ into λ^+-many disjoint stationary sets $\vec{S} = \langle S_\alpha \mid \alpha < \lambda^+\rangle$. Let $\vec{T} = j(\vec{S}) = \langle T_\alpha \mid \alpha < \lambda^+\rangle$. It is proved in [39] that for $\xi < \lambda^+$, $\xi \in \mathrm{ran}(j)$ iff T_ξ is stationary in V. This claim gives that $j''\lambda^+$ is definable from $\vec{T} \in \mathrm{HOD}$, and so therefore we also have that $j''\lambda^+ \in \mathrm{HOD}$. From this, we can also define $j \restriction \lambda^+$ and therefore C, as defined in the original proof, is also in HOD. One can then continue in a similar way to this, deriving a contradiction much like in the original proof. □

Theorem 3.36. *Suppose $j : V \prec M$ is an elementary embedding into an inner model M, then $V = HOD(M)$. I.e. every object in V is definable using a parameter from M.*

This result can be viewed as a corollary to the previous theorem, or vice versa, so we omit the proof (refer to [39], pp. 17-18). From this we obtain the following corollary:

Corollary 3.37. *Suppose $j : V \prec M$ is an elementary embedding and M is a transitive class such that $M \subseteq HOD$. Then $V = HOD$.*

3.4 Flying Close to the Sun

We have established that we cannot go as far as to suppose a Reinhardt cardinal exists in ZFC, but exactly how far can we go? How close to inconsistency can we go without hitting Kunen's wall?

Definition 3.38. *The following are called the rank-into-rank axioms[25] (p. 325) (see also [40]):*

(I3) *For some* λ, *there is a nontrivial elementary embedding* $j : V_\lambda \prec V_\lambda$.

(I2) *For some* λ, *there is a nontrivial elementary embedding* $j : V \prec M$ *such that* $V_\lambda \subseteq M$ *where* $crit(j) < \lambda$ *and* $j(\lambda) = \lambda$.

(I1) *For some* λ, *there is a nontrivial elementary embedding* $j : V_{\lambda+1} \prec V_{\lambda+1}$.

Observe the apparent ambiguity in these definitions. In I3, it is not immediately obvious what values λ can take. However, λ must be at least the supremum of the critical sequence, as the range of j is a subset of V_λ, but by the corollary to the Kunen inconsistency, λ must be at most the supremum of the critical sequence $+1$. Generally, we will assume that λ is the supremum of the critical sequence.

These hypotheses have been ordered in terms of increasing consistency strength, as will be shown in the following propositions. The rank-into-rank axioms are strictly stronger in consistency strength than all of the previous large cardinal definitions given up to this point, with the obvious exception of the proven-to-be-inconsistent Reinhardt cardinals. These strong assertions are not known to be inconsistent, though it is very possible that they are. Perhaps a stronger proof of inconsistency than Kunen's will be found, making these assertions only possible in a choiceless universe, as was done to Reinhardt cardinals. At least for now, no such proof has been found and so these statements may still be asserted as very strong hypotheses.

I3 embeddings are closely related to the following large cardinal concept:

Definition 3.39 (ω-huge cardinal)**.** *A cardinal* κ *is* ω*-huge if there exists a* $\lambda > \kappa$ *and a nontrivial elementary embedding* $j : V_\lambda \prec V_\lambda$ *satisfying* $\kappa = crit(j)$ *and* λ *is the limit of the critical sequence of* j*, i.e.*

$$\lambda = \kappa_\omega(j) = sup_{n\in\omega}\kappa_n$$

where $\kappa_0 = \kappa$ *and* $\kappa_{n+1} = j(\kappa_n)$ *for all* $n \in \omega$.

The following is the canonical way of extending an I3 embedding:

Definition 3.40. *[40] Let* λ *be a limit ordinal, and* $f : V_\lambda \to V_\lambda$ *some function. Define* $f^+ : V_{\lambda+1} \to V_{\lambda+1}$ *by the following* $\forall A \subset V_\lambda$*:*

$$f^+(A) = \bigcup\{f(A \cap V_\alpha) \mid \alpha < \lambda\}.$$

When j is an I3 embedding, it is not necessarily true that j^+ is an elementary embedding. However, it is true that every I1 embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ is equal to $(j \restriction V_\lambda)^+$[40]. The following proposition is simple:

Proposition 3.41. *Suppose* $j : V_\lambda \prec V_\lambda$ *is an I3 embedding. Then* $j^+ : V_{\lambda+1} \to V_{\lambda+1}$ *is a* Σ_0*-elementary embedding.*

Proof. It follows from the definition that the truth of all formulae when restricted to V_λ are preserved. For all subsets of $V_{\lambda+1}$, their membership relation is determined by elements of V_λ and so Σ_0 statements, which use only bounded quantifiers, must be preserved, i.e. $j^+ : V_{\lambda+1} \prec_{\Sigma_0} V_{\lambda+1}$. □

Proposition 3.42. *I2 implies I3.*

Proof. Suppose j satisfies I2. Then $j : V \prec M$ can be restricted to $j \upharpoonright V_\lambda$, but by elementarity $j(V_\lambda) = V_\lambda$ so $j \upharpoonright V_\lambda : V_\lambda \prec V_\lambda$ satisfying I3. ☐

Proposition 3.43. *I1 implies I2.*

Proof. [25](p. 327) Suppose j satisfies I1. Let $\bar{j} = j \upharpoonright V_\lambda$. Then $\bar{j} : V_\lambda \prec V_\lambda$, and for any $R \subseteq V_\lambda$, $\bar{j}^+(R) = j(R)$. If R is a well-founded relation in V, then it is in the sense of $V_{\lambda+1}$, as all functions $f : \omega \to V_\lambda$ are contained in $V_{\lambda+1}$, so since R is wellfounded, by elementarity so is $j(R)$. See [25](pp. 326-327) to see why this is sufficient. ☐

Can we push any further than I1? We established in the previous section that as a corollary to the Kunen inconsistency, there is no elementary embedding from $V_{\lambda+2}$ to itself, so we cannot extend the definition of I1 in the obvious way. We can, however, edge closer by expanding $V_{\lambda+1}$ to $L(V_{\lambda+1})$, as was first proposed by Woodin in 1984. He went on to show that the existence of such an embedding implies $\mathrm{AD}^{L(\mathbb{R})}$ (the statement that the axiom of determinacy holds in $L(\mathbb{R})$, the smallest model that contains all the reals).

Definition 3.44. *(I0) For some λ, there is a nontrivial elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\lambda > crit(j)$.*

Proposition 3.45. *I0 implies I1.*

Proof. Suppose j satisfies I0. Then the map $j \upharpoonright V_{\lambda+1}$ is an elementary embedding from $V_{\lambda+1}$ into some $M \subseteq L(V_{\lambda+1})$, but this M must be $V_{\lambda+1}$ by elementarity. So $j \upharpoonright V_{\lambda+1} : V_{\lambda+1} \prec V_{\lambda+1}$ satisfying I1. ☐

Additionally, since $\lambda > \mathrm{crit}(j)$, $\mathrm{crit}(j) \in V_{\lambda+1}$ and so $\mathrm{crit}(j) = \mathrm{crit}(j \upharpoonright V_{\lambda+1})$ and so $\lambda > \mathrm{crit}(j \upharpoonright V_{\lambda+1})$.

It is not known whether or not these axioms are consistent with ZFC, but as with Reinhardt cardinals without choice, they have been around for decades now, and an inconsistency hasn't yet been found. Of course, this doesn't mean that they are consistent, or even that set theorists necessarily think they are. In fact, the "I" in I0-I3 stands for "Inconsistency", so at least at first, the prevailing view was one of scepticism, but yet they still stand.

Another interesting aspect of rank-into-rank embeddings is the surprising algebraic properties which arise from the study of iterated embeddings. This investigation was initiated by Laver, see [41] for an accessible introduction. The shift in perspective that led to Laver's work was to move away from investigating large cardinals as critical points of embeddings to the embeddings themselves being an object of interest. Let $\mathscr{E}_\lambda = \{j \mid j : V_\lambda \prec V_\lambda\}$. Two operations of interest are composition: if $j, k \in \mathscr{E}_\lambda$, then $j \circ k \in \mathscr{E}_\lambda$, and application: if $j, k \in \mathscr{E}_\lambda$, then $j \cdot k = \bigcup_{\alpha<\lambda} j(k \cap V_\alpha) \in \mathscr{E}_\lambda$ with $\mathrm{crit}(j \cdot k) = j(\mathrm{crit}(k))$. For more on the algebraic properties arising from these operations. see Laver [42], or Dehornoy [43].

Is I0 as close as we can get to Kunen's wall of fire? Is it possible to construct something still stronger than I0 without falling into inconsistency? There certainly isn't much more room above I0. Again, we cannot extend its definition in the obvious way as it follows from Kunen's inconsistency that an embedding $j : L(V_{\lambda+2}) \prec L(V_{\lambda+2})$ doesn't exist since if it did, so would $j \upharpoonright V_{\lambda+2} : V_{\lambda+2} \prec V_{\lambda+2}$, but we know by Kunen's result that this cannot be the case. If we can't extend $L(V_{\lambda+1})$ all the way to $L(V_{\lambda+2})$ by adding in all the subsets of $V_{\lambda+1}$, is there some set of subsets that we can add without triggering Kunen's result? We call such a set an Icarus set.

Definition 3.46 (Icarus set)**.** *[40] (p. 61) A set $X \subseteq V_{\lambda+1}$ is an Icarus set if there is a nontrivial elementary embedding $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with crit$(j) < \lambda$.*

For example, if we assume I0, then trivially $V_{\lambda+1}$ is Icarus, but a well-order of $V_{\lambda+1}$ is not Icarus.

3.5 The Hierarchy of Large Cardinals

See figure 3 for a visualisation of the hierarchy we have built up over this chapter.

One striking observation that can be made about the hierarchy of large cardinal principles is that it is surprisingly linear. This is a well-known phenomenon that has frequently been discussed and is considered somewhat of a mystery of the foundations of mathematics: our mathematical theories seem to be linearly ordered by consistency strength.

It doesn't always seem particularly intuitive that given two of our large cardinal hypotheses , generally one will prove the consistency of the other, especially when such properties can be defined in different ways (there are obvious exceptions to this where such an implication is clear). For example, Ramsey cardinals arose from infinite combinatorics, measurable cardinals from measure theory, and compact cardinals from a compactness theorem for infinitary languages. With these concepts arising from such contrasting motivations, why should they relate to each other in such a straightforward way?

There are exceptions to this supposed linearity, but many set theorists consider such exceptions to be unnatural and claim that the "natural" large cardinal assertions are in fact linearly ordered. See Hamkins[47] for further discussion of nonlinearities in the large cardinal hierarchy, including a discussion of what exactly set theorists mean when referring to examples here as being "natural" or not.

As we observed in the beginning of the previous chapter, it is also interesting that various other extensions of ZFC that have been proposed have turned out to be equivalent to a large cardinal axiom. This observation prompted John Steel in [48] to formulate the following (somewhat vague) conjecture: First let $S \leq_{\mathrm{Con}} T$ abbreviate $\mathrm{ZFC} \vdash \mathrm{Con}(T) \rightarrow \mathrm{Con}(S)$, and define $\geq_{\mathrm{Con}}$ and $\equiv_{\mathrm{Con}}$ in the obvious way.

Conjecture 3.47. *Suppose T is a natural extension of ZFC, then there is an extension H axiomatised by a large cardinal hypothesis such that $T \equiv_{Con} H$. Additionally $\leq_{Con}$ is a pre-well-order*[†] *of the natural extensions of ZFC.*

$$\begin{tikzcd}
\text{Icarus} \arrow[d] & \\
\text{I0} \arrow[d] & \uparrow \\
\text{I1} \arrow[d] & \text{approaching inconsistency} \\
\text{I2} \arrow[d] & \uparrow \\
\text{I3} \arrow[d] & \\
n\text{-huge} \arrow[d] & \uparrow \\
\text{huge} \arrow[d] & \text{increasing consistency strength} \\
\text{extendible} \arrow[d] & \uparrow \\
\text{supercompact} \arrow[d] & \\
\text{strongly compact} \arrow[d] & \\
\text{strong} \arrow[d] & \\
\text{measurable} \arrow[d, "\text{to "small" large cardinals}"] & \\
\vdots &
\end{tikzcd}$$

Figure 3: A continuation of our large cardinal hierarchy, as expanded in this chapter.

[†] A relation that induces a well-ordering on its equivalence classes.

3.6 Elementary Embeddings of L and $0^{\#}$

In chapter 1, we introduced an important dichotomy for L, and compared it to the HOD Dichotomy, which we shall cover in the next section. Here, we will work on building a more proper understanding of the L Dichotomy. To do this properly, we need to understand what exactly $0^{\#}$ is, which we glossed over in the first chapter. We could spend a fair amount of time on this, but we keep it relatively brief as our focus is on HOD, not L. We first require the following definition:

Definition 3.48 (Indiscernibles). *Suppose $\mathfrak{A}$ is a structure and X is a linearly ordered subset of A, then $\langle X, < \rangle$ is a set of indiscernibles for $\mathfrak{A}$ if for all n, any two n-element subsets of X satisfy all of the same formulae in $\mathfrak{A}$. More specifically, suppose $x_1 < ... < x_{n-1}$ and $y_1 < ... < y_{n-1}$ are any two n-element subsets of X and ϕ is any formula in the language of $\mathfrak{A}$ with at most $v_0, ..., v_{n-1}$ free, then*

$$\mathfrak{A} \models \phi(x_0, ...x_n) \text{ iff } \mathfrak{A} \models \phi(y_0, ...y_n).$$

Informally, $0^{\#}$ is the set of true formulae about indiscernibles in L. In the following definition, we encode this as a subset of the natural numbers using Gödel numbering (see e.g [49]).

Definition 3.49. *[3](p. 257) If there exists a κ such that $\kappa \rightarrow (\omega_1)^{<\omega}$, then we define $0^{\#} = \{\ulcorner \phi \urcorner \mid \langle L, E(L) \rangle \models \phi[f]\}$ where $f(n) = \omega_{n+1}$ for $n < \omega$ (here $E(x) = \{\langle y, z \rangle \mid y, z \in x \wedge y \in z\}$).*

The reason why we suppose that $\kappa \rightarrow (\omega_1)^{<\omega}$ in order to define $0^{\#}$ is because it is under this supposition that the existence of such a set has important consequences. These are given in the following theorem:

Theorem 3.50. *(see [3] pp. 253-254 for a proof) Suppose there is a cardinal κ such that $\kappa \rightarrow (\omega_1)^{<\omega}$. Then, there is a proper class X of ordinals such that the following hold:*

(i) X is closed under the order topology, and for all uncountable cardinals λ, $X_\lambda = X \cap \lambda$ has cardinality λ.

(ii) Suppose $\alpha, \beta \in X$ and $\alpha < \beta$, then $L_\alpha \prec L_\beta$ and $\langle X \cap \alpha, < \rangle$ is a set of indiscernibles for L_α.

(iii) Every set in L is definable in L from parameters in X.

(iv) If $\alpha \in X$, then $L_\alpha \prec L$ and $\langle X, < \rangle$ is a class of indiscernibles for L.

(v) The relation $L \models \phi(x_0, ..., x_{n-1})$ for $x_0, ..., n_{n-1} \in L$ can be expressed by a single formula of ZF with parameters $\ulcorner \phi \urcorner$ and $\langle x_0, ..., x_{n-1} \rangle$.

(vi) Every set in L definable without parameters is countable.

Most importantly, we have the following result:

Theorem 3.51 (Kunen). *$0^\#$ exists iff there is a nontrivial elementary embedding $j : L \prec L$.*

See Jech[1] (pp. 323-327), for the full proof. By the above theorem, we have that if $0^\#$ exists, $V \neq L$. This is because if they were equal then there would be an elementary embedding $j : V \prec V$, contradicting Kunen's theorem. The question naturally arises from this, if $0^\#$ exists, we know $V \neq L$, but how far apart are V and L?

We omit the proofs of the following, but it was from investigation into this that the L dichotomy was reached.

Corollary 1.6. *(i) If $0^\#$ does not exist, then L is correct about singular cardinals and correctly computes their successors. I.e. if κ is singular, $(\kappa$ is singular$)^L$ and $(\kappa^+)^L = \kappa^+$.*

(ii) If $0^\#$ does exist, then every uncountable cardinal is inaccessible in L.

We can also generalise $0^\#$ to define sharps of any set, where $0^\#$ is the sharp of $0 = \emptyset$.

Definition 3.52. *[3](p. 257) If there exists a κ such that $\kappa \to (\omega_1)^{<\omega}$, then we define $x^\# = \{\ulcorner\phi\urcorner \mid \langle L(x), E(L(x)), x\rangle \models \phi[f]\}$ where $f(n) = \omega_{n+1}$ for $n < \omega$.*

In analogy to Kunen's theorem above, we have the following:

Theorem 3.53. *$x^\#$ exists iff there is a nontrivial elementary embedding $j : L(x) \prec L(x)$.*

We next bring our investigation back to the HOD dichotomy.

3.7 The HOD Dichotomy

We are now in a position to properly cover the HOD dichotomy, which we introduced in Chapter 1. In this section we will cover some results originating in Woodin's almost 600-page paper "Suitable Extender Models" [44][45], which covers a very wide range of content relating to inner model theory. Much of this is not particularly relevant to our purpose here seeing as our focus is on building up to results concerning HOD. For a more accessible account of this aspect of Woodin's work, see [14], which we are highly indebted to for this section.

First, recall the HOD dichotomy from chapter 1:

Theorem 1.7. *Suppose that κ is an extendible cardinal. Then exactly one of the following holds:*

(i) For every singular cardinal $\gamma > \kappa$, γ is singular in HOD and $(\gamma^+)^{HOD} = \gamma^+$.

(ii) Every regular cardinal $\gamma \geq \kappa$ is a measurable cardinal in HOD.

Before proceeding, we require the following definition:

Definition 3.54. *Let λ be an uncountable regular cardinal. We say λ is ω-strongly measurable in HOD iff there is a $\kappa < \lambda$ such that $(2^\kappa)^{HOD} < \lambda$ and there is no partition $\langle S_\alpha \mid \alpha < \kappa \rangle$ of $cf(\omega) \cap \lambda$ into stationary sets such that $\langle S_\alpha \mid \alpha < \kappa \rangle \in HOD$.*

Lemma 3.55. *If λ is ω-strongly measurable in HOD, then*

$$HOD \models \text{"}\lambda \text{ is a measurable cardinal."}$$

Proof. Assume the following claim: There exists a stationary set $S \subseteq \mathrm{cf}(\omega) \cap \lambda$ such that $S \in \mathrm{HOD}$ and there doesn't exist a partition of S into two stationary sets that belong to HOD.

Now let F be the club filter restricted to S, i.e.

$$F = \{X \subseteq S \mid \exists C \text{ club} \subseteq \lambda(C \cap S \subseteq X)\}.$$

Consider $G = F \cap \mathrm{HOD}$. G is of course trivially in HOD, and so since the club filter is λ-complete, we have that

$$\mathrm{HOD} \models \text{"}G \text{ is a } \lambda\text{-complete filter on } S\text{"}.$$

However, by the above claim, it follows that

$$\mathrm{HOD} \models \text{"}G \text{ is an ultrafilter on } S\text{"}.$$

Since G is also non-principal, the lemma follows by definition of a measurable cardinal. □

Proof of Claim. We prove this claim by contradiction. Let $\kappa < \lambda$ be a cardinal such that $(2^\kappa)^{\mathrm{HOD}} < \lambda$ and such that there doesn't exist a partition $\langle S_\alpha \mid \alpha < \kappa \rangle$ of $\mathrm{cf}(\omega) \cap \lambda$ into stationary sets such that $\langle S_\alpha \mid \alpha < \kappa \rangle \in \mathrm{HOD}$. Using this we can define a subtree T of ${}^{\leq\kappa}2$ with height $\kappa + 1$ and a sequence $\langle S_r \mid r \in T \rangle$ that belongs to HOD such that the following properties hold:

(i) $S_{\langle\rangle} = \mathrm{cf}(\omega) \cap \lambda$.

(ii) For all $r \in T$, 1. S_r is stationary , 2. $r^\frown\langle 0 \rangle$ and $r^\frown\langle 1 \rangle$ belong to T. 3. S_r is the disjoint union of $S_{r^\frown\langle 0 \rangle}$ and $S_{r^\frown\langle 1 \rangle}$, 4. if $\mathrm{dom}(r)$ is a limit ordinal, then $S_r = \bigcap\{S_{r\restriction\alpha} \mid \alpha \in \mathrm{dom}(r)\}$.

(iii) If $\beta \leq \kappa$ is a limit ordinal and $r \in {}^{\beta}2 \setminus T$, if $r \restriction \alpha \in T$ for all $\alpha < \beta$, then $\bigcap_{\alpha<\beta} S_{r\restriction\alpha}$ is nonstationary.

Firstly, observe that even though it might not have the same meaning in HOD, $\mathrm{cf}(\omega) \cap \lambda \in \mathrm{HOD}$. In a similar fashion, $\{S \subseteq \lambda \mid S \in \mathrm{HOD} \text{ and } S \text{ is stationary}\} \in \mathrm{HOD}$, although it may be that some of the sets stationary in HOD aren't truly stationary.

By the supposition that the claim fails, there is a partition of S into two sets stationary in V. One can choose these sets in a uniform way using a well-ordering of the set of stationary subsets $S \subseteq \lambda$ satisfying $S \in \mathrm{HOD}$. So we have constructed the successor stages of our subtree. Next suppose β is a limit ordinal, and that $\langle S_r \mid r \in T \cap {}^{<\beta}2\rangle$, then by (iii) above, we have that except for a nonstationary set, $\mathrm{cf}(\omega) \cap \lambda$ is equal to

$$\bigcup\{\bigcap\{S_{r\restriction\alpha} \mid \alpha < \beta\} \mid r \text{ is a } \beta\text{-branch of } T \cap {}^{<\beta}2 \text{ and } r \in \text{HOD}\}.$$

As the club filter on λ is λ-complete, and as $|{}^{\beta}2|^{\text{HOD}} < \lambda$, there must be some r where the intersection in the set above is stationary. In this case, define $r \in T \cap {}^{\beta}2$ and $S_r = \bigcap\{S_{r\restriction\alpha} \mid \alpha < \beta\}$. This gives us the limit case, and so we have constructed the tree.

Now take any $r \in T$ such that $\mathrm{dom}(r) = \kappa$. Then

$$S_r = \bigsqcup\{S_{r\restriction\alpha+1} \setminus S_{r\restriction\alpha} \mid \alpha < \kappa\}.$$

This contradicts our initial choice of κ, thus the claim must hold. □

The following conjecture says that HOD is not "far" from V, i.e. if δ is an extendible cardinal, then the conjecture is equivalent to the failure of the second statement in the HOD dichotomy.

Conjecture 3.56 (HOD Hypothesis)**.** *There is a proper class of regular cardinals that are not ω-strongly measurable in HOD.*

Note that if the first statement of the HOD dichotomy holds, then then the HOD hypothesis holds, since $\{\gamma^+ \mid \gamma \in \mathrm{On} \wedge \text{"}\gamma \text{ is a singular cardinal"}\}$ is a proper class of regular cardinals that are not ω-strongly measurable in HOD.

There are three main variations on the HOD conjecture - the conjecture that the above hypothesis holds. Recall the idea that stronger large cardinals correspond to having more proving power, for example in our discussion of stronger inconsistencies than Kunen's. The usual HOD conjecture holds under the suppositions given in the HOD dichotomy (here $\exists\kappa(\text{ext.})$ is shorthand for there is an extendible cardinal).

Conjecture 3.57 (HOD Conjecture)**.** *ZFC* $+ \exists\kappa(\mathit{ext.}) \vdash$ *HOD hyp.*

An obvious strengthening is the following:

Conjecture 3.58 (Strong HOD Conjecture)**.** *ZFC* $\vdash$ *HOD hyp.*

And the following is a weakening. We will discuss these conjectures further in chapter 6.

Conjecture 3.59 (Weak HOD Conjecture)**.**

$$\mathit{ZFC} + \exists\kappa\ (\mathit{ext.})\ \text{with a huge}\ \lambda > \kappa\ \vdash\ \mathit{HOD\ hyp.}$$

In inner model theory, an extender is "any object that captures the essence of a given large cardinal property" [14]. Traditionally this refers to a system of ultrafilters, but this is not always the case. We are interested in particular in the following:

Definition 3.60 (Weak Extender Models)**.** *[14] Let N be a transitive class model of ZFC. N is called a weak extender model for δ supercompact iff for every $\gamma > \delta$ there exists a normal fine measure U on $\mathcal{P}_\delta(A)$ such that*

(i) Concentration: $N \cap \mathcal{P}_\delta(\gamma) \in U$ and

(ii) Amenability: $U \cap N \in N$.

The following definition was introduced by Hamkins[46]:

Definition 3.61. *Let δ be a regular uncountable cardinal and N a transitive class model of ZFC. Then*

(i) We say N has the δ-covering property if for every $\sigma \subseteq N$ such that $|\sigma| < \delta$, there exists a $\tau \in N$ such that $|\tau| < \delta$ and $\sigma \subseteq \tau$.

(ii) We say N has the δ-approximation property if for every cardinal κ such that $cf(\kappa) \geq \delta$ and every $\subseteq$-increasing sequence of sets $\langle \tau_\alpha \mid \alpha < \kappa \rangle$ from N, we have $\bigcup \tau_\alpha \in N$.

The proof of the following lemma uses the theorem below, due to Solovay, which is proved in [14] (pp. 12-13).

Theorem 3.62 (Solovay)**.** *Let δ be a supercompact cardinal and let $\gamma > \delta$ be regular. Then there exists a set $X \subseteq \mathcal{P}_\delta(\gamma)$ such that the sup function is injective on X and every γ-supercompactness measure contains X.*

Such a set X satisfying the above may be referred to as a Solovay set.

Lemma 3.63. *[14] If N is a weak extender model for δ supercompact, then it has the δ-covering property.*

Proof. It suffices to prove the δ-covering property for sets of ordinals. Suppose we have some $\tau \subseteq \gamma$ satisfying $|\tau| < \gamma$. Let U be a γ-supercompactness measure (i.e. a normal fine δ-complete ultrafilter on $\mathcal{P}_\delta(\gamma)$) with the properties $N \cap \mathcal{P}_\delta(\gamma) \in U$ and $U \cap N \in N$ as in the definition above. As U is fine, we have that for all $\alpha < \gamma$, $\{\sigma \in \mathcal{P}_\delta(\gamma) \mid \alpha \in \sigma\} \in U$. Therefore, since U is δ-complete, and since $|\tau| < \kappa$, it follows that $\{\sigma \in \mathcal{P}_\delta(\gamma) \mid \tau \subseteq \sigma\} \in U$. Finally, as $N \cap \mathcal{P}_\delta(\gamma) \in U$, there must be a $\sigma \in N \cap \mathcal{P}_\delta(\gamma)$ and $\sigma \subseteq \tau$, satisfying the definition. □

Lemma 3.64. *[14] If N is a weak extender model for δ supercompact and $\gamma > \delta$ such that $N \models$ "γ is a regular cardinal", then $|\gamma| = cf(\gamma)$.*

Proof. Trivially, we have that $\mathrm{cf}(\gamma) \leq |\gamma|$. To show equality it therefore suffices to prove that $\mathrm{cf}(\gamma) \geq |\gamma|$.

By the previous lemma, since N is a weak extender model for δ supercompact, it satisfies the δ-covering property. Therefore, since γ is regular in N. we have that $\mathrm{cf}(\gamma) \geq \delta$.

Now suppose U is a γ-supercompactness measure satisfying concentration and amenability on N. Since N believes that γ is regular, we can apply theorem

3.62 within N, yielding a Solovay set $X \in N$, i.e. the sup function is injective on X and $X \in U$.

Let $D \subseteq \gamma$ be a club set of order type $\mathrm{cf}(\gamma)$, and define $A = \{\sigma \in \mathcal{P}_\delta(\gamma) \mid \sup(\sigma) \in D\}$. One can easily show that $A \in U$ by constructing the ultrapower induced by U. Now define $\beta = \sup(j[\gamma])$. Firstly, observe that $j(D)$ is a club set in $j(\gamma)$. As D is unbounded in γ, it must be that $j[\gamma] \cap j(D)$ is unbounded in β. Therefore as $j(D)$ is closed we have that $\beta \in j(D)$, and so $\{\sigma \in X \mid \sup(\sigma) \in D\} \in U$.

As U is fine, we have that $\gamma = \bigcup\{\sigma \in X \mid \sup(\sigma) \in D\}$. As the sup function is injective on X, we obtain that $|\gamma| \leq \delta|D|$. However, we know that the order type of D is $\mathrm{cf}(\gamma)$, and so $|\gamma| \leq \delta|\mathrm{cf}(\gamma)|$, but as stated near the beginning of the proof, $\delta \leq \mathrm{cf}(\gamma)$, and we finally obtain $|\gamma| \leq \mathrm{cf}(\gamma)$, yielding the desired result. □

From this, we can easily obtain the following corollary:

Corollary 3.65. *Suppose N is a weak extender model for δ supercompact and $\gamma > \delta$ is a singular cardinal. Then*

(i) $N \models$ *"γ is a singular cardinal" and*

(ii) $\gamma^+ = (\gamma^+)^N$.

We can now give a result characterising the HOD hypothesis, giving two different ways in which HOD can be said to be close to V.

Theorem 3.66. *Suppose δ is an extendible cardinal. The following are equivalent:*

(i) The HOD hypothesis (3.56).

(ii) HOD is a weak extender model for δ supercompact.

(iii) Every singular cardinal $\gamma > \delta$ is singular in HOD and $\gamma = (\gamma^+)^{HOD}$.

Proof. We already demonstrated that $(iii) \to (i)$ after introducing the HOD hypothesis: if (iii) holds, then then the HOD hypothesis holds, since $\{\gamma^+ \mid \gamma \in \mathrm{On} \wedge$ "γ is a singular cardinal"$\}$ is a proper class of regular cardinals that are not ω-strongly measurable in HOD. $(ii) \to (iii)$ is just a restatement of the above corollary, so the only thing that remains to be proved is that $(i) \to (ii)$, i.e. that supposing the HOD hypothesis, HOD is a weak extender model for δ supercompact.

With this in mind, suppose $\zeta > \delta$. We need to show that there is a ζ-supercompactness measure U such that $U \cap \mathrm{HOD} \in \mathrm{HOD}$ and $\mathcal{P}_\delta(\gamma) \cap \mathrm{HOD} \in U$. Towards this end, let $\gamma > 2^\zeta$ be such that $|V_\gamma|^{\mathrm{HOD}} = \gamma$ and set a regular cardinal λ such that $\lambda > 2^\gamma$ and λ is not ω-strongly measurable in HOD. Furthermore, select a $\eta > \lambda$ such that the formula defining HOD is absolute for V_η, and therefore $\mathrm{HOD}^{V_\eta} = \mathrm{HOD} \cap V_\eta$. As by the supposition of the theorem, δ is extendible, we have an elementary embedding $j : V_{\eta+1} \prec V_{j(\eta)+1}$. We make the following claim.

Claim . $j[\gamma] \in HOD^{V_{j(\eta)}}$.

Proof of Claim. Since λ is not ω-strongly measurable in HOD, and since in V and therefore also in HOD, $2^\gamma < \lambda$, there is by definition a partition $\langle S_\alpha \mid \alpha < \gamma\rangle$ of $\mathrm{cf}(\omega) \cap \lambda$ into stationary sets such that $\langle S_\alpha \mid \alpha < \gamma\rangle \in \mathrm{HOD}$, and therefore also $\langle S_\alpha \mid \alpha < \gamma\rangle \in \mathrm{HOD}^{V_\eta}$.

By the elementarity of j, we have that

$$\langle S^*_\alpha \mid \alpha < j(\lambda)\rangle = j(\langle S_\alpha \mid \alpha < \gamma\rangle) \in \mathrm{HOD}^{V_{j(\eta)}}.$$

If we now let $\beta = \sup(j[\lambda])$ and $\bar{\beta} = \sup(j[\gamma])$, then by a result due to Solovay given in [14] (pp. 10-11), it follows that

$$j[\gamma] = \{\alpha < \bar{\beta} \mid S^*_\alpha \cap \beta \text{ is stationary in } \beta\}.$$

By this it follows that $j[\gamma]$ is ordinal-definable in $V_{j(\eta)}$, and therefore since $V_{j(\eta)}$ is correct about stationarity in β, we have that $j[\gamma] \in \mathrm{HOD}^{V_{j(\eta)}}$, as is $j[\zeta]$. □

Trivially, it follows from the claim that $j[\gamma] \in \mathrm{HOD}$ as $\mathrm{HOD}^{V_{j(\eta)}} \subset \mathrm{HOD}$. Since $|V_\gamma|^{\mathrm{HOD}} = \gamma$, we can define a bijection $e \in \mathrm{HOD}$ such that $e : \gamma \to V_\gamma^{\mathrm{HOD}}$. It follows that $j(e)[j[\gamma]] = j[V_\gamma \cap \mathrm{HOD}]$ and therefore $j[V_\gamma \cap \mathrm{HOD}] \in \mathrm{HOD}$. Finally, we have that $j \restriction (V_\gamma \cap \mathrm{HOD}) \in \mathrm{HOD}$ as $j \restriction (V_\gamma \cap \mathrm{HOD})$ is the inverse of the Mostowski collapse.

We can now use j to derive an ultrafilter on $\mathcal{P}_\delta(\zeta)$, i.e. define for $A \subseteq \mathcal{P}_\delta(\zeta)$, $A \in U$ if $j[\zeta] \in j(A)$. We can now derive the desired properties:

(i) Since $j[\zeta] \in \mathrm{HOD}^{V_{j(\eta)}} = j(\mathrm{HOD} \cap V_\eta)$, we have that $\mathrm{HOD} \cap \mathcal{P}_\delta(\zeta) \in U$, i.e. U concentrates on HOD.

(ii) Since $j \restriction (V_\gamma \cap \mathrm{HOD}) \in \mathrm{HOD}$ and $\gamma > 2^\zeta$, we have that $U \cap \mathrm{HOD} \in \mathrm{HOD}$, i.e. U is amenable to HOD.

Therefore we have shown that HOD is a weak extender model for δ supercompact. □

Finally, we obtain our now familiar HOD dichotomy theorem as a corollary of the above theorem.

Theorem 1.7. *Suppose that κ is an extendible cardinal. Then exactly one of the following holds:*

(i) For every singular cardinal $\gamma > \kappa$, γ is singular in HOD and $(\gamma^+)^{HOD} = \gamma^+$.

(ii) Every regular cardinal $\gamma \geq \kappa$ is a measurable cardinal in HOD.

Consequences

What results could we obtain if the HOD hypothesis were proved from ZFC? The following theorem gives us an idea.

Theorem 3.67. *(ZF) Assume that ZFC proves the HOD hypothesis, and that δ is an extendible cardinal. Then for every $\lambda > \delta$, there is no nontrivial elementary embedding $j : V_{\lambda+2} \prec V_{\lambda+2}$.*

This follows as a corollary of a stronger result (theorem 228 in [44]). This theorem states that if ZFC proves the HOD hypothesis, we almost have Kunen's inconsistency without the need for the axiom of choice. If such a result could be obtained, it would destroy all the work we will cover in the next two chapters, where we will explore the "far" side of the dichotomy. But will such a feat be possible?

4 Reinhardt and Berkeley Cardinals

4.1 Choiceless Large Cardinals

In this chapter, we will introduce some large cardinal properties which are so strong that they are inconsistent with the axiom of choice (therefore, from now on, we work in ZF, or choiceless NBG when dealing with proper classes). We will build up a hierarchy of these large cardinals just as we did in chapters 2 and 3. However, before we begin, we need to reformulate some of the traditional large cardinal definitions we gave in those chapters to make them compatible with a choiceless environment.

There is one caveat to be aware of here: Several of these definitions that are equivalent when allowing choice may not be equivalent without choice. Here we follow the formulations given by Bagaria, Koellner and Woodin[50].

We begin, as before, at the bottom of our hierarchy with the notion of an inaccessible cardinal. In Chapter 2, we defined a cardinal κ as strongly inaccessible if for all $\alpha < \kappa$, $|V_\alpha| < \kappa$. But without the axiom of choice, we cannot guarantee that a well-order exists on V_α, so it is possible that $|V_\alpha|$ and κ are incomparable. However, the following definition is equivalent to the above under ZFC, and doesn't depend on the well-ordering theorem.

Definition 4.1 (Strongly inaccessible cardinal). *A cardinal κ is strongly inaccessible if for all $\alpha < \kappa$ there does not exist a function $f : V_\alpha \to \kappa$ with range unbounded in κ.*

In Chapter 2 we showed that if κ is strongly inaccessible then $V_\kappa \models$ ZFC (Theorem 2.6). In this choiceless setting, it still holds that if κ is inaccessible, then $\langle V_\kappa, V_{\kappa+1}\rangle \models ZF_2$ where ZF_2 is "the second-order version of ZF (with second-order Separation, Collection and Replacement)" according to [50], i.e. where $\langle V_\kappa, V_{\kappa+1}\rangle$ is a KMCC model (see [51]).

We also need to reformulate supercompact and extendible cardinals in the choiceless settings. Before this, we require the following definition:

Definition 4.2. *Let $V_\alpha \prec_{\Sigma_1} V$. We define $V_\alpha \prec_{\Sigma_1^*} V$ if for all $\beta < \alpha$, for all $a \in V_\alpha$, and for all Σ_0-formulae $\varphi(x,y)$,*

$$\text{There exists } b \in V \text{ such that } {}^{V_\alpha}b \subseteq b \text{ and } V \models \varphi[a,b]$$

implies that

$$\text{There exists } b \in V_\alpha \text{ such that } {}^{V_\alpha}b \subseteq b \text{ and } V \models \varphi[a,b].$$

Note that under AC, the definitions of $V_\alpha \prec_{\Sigma_1} V$ and $V_\alpha \prec_{\Sigma_1^*} V$ coincide.

The following definition is Magidor's formulation of a supercompact cardinal (recall theorem 3.17), in the absence of choice we call this weakly supercompact.

Definition 4.3 (Weakly supercompact cardinal). *A cardinal κ is weakly supercompact if for all $\gamma > \kappa$ and for all $a \in V_\alpha$, there exists a $\bar{\gamma} < \kappa$ and $\bar{a} \in V_{\bar{\gamma}}$, and an elementary embedding $j : V_{\bar{\gamma}+1} \to V_{\gamma+1}$ such that $crit(j) = \bar{\kappa}$ such that $\bar{\kappa} < \kappa$, $j(\bar{\kappa}) = \kappa$ and $j(\bar{a}) = a$.*

We define supercompact cardinals in the choiceless setting using the following slight strengthening. Note that these two definitions coincide under choice.

Definition 4.4 (Supercompact cardinal)**.** *A cardinal κ is weakly supercompact if for all $\gamma > \kappa$ such that $V_\gamma \prec_{\Sigma_1^*} V$ and for all $a \in V_\alpha$, there exists a $\bar{\gamma} < \kappa$ and $\bar{a} \in V_{\bar{\gamma}}$, and an elementary embedding $j : V_{\bar{\gamma}+1} \prec V_{\gamma+1}$ such that crit$(j) = \bar{\kappa}$ such that $\bar{\kappa} < \kappa$, $j(\bar{\kappa}) = \kappa$, $j(\bar{a}) = a$ and $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$.*

We can do a similar thing to extendible cardinals, where the definition given in definition 3.20 becomes the following in the absence of choice.

Definition 4.5 (Weakly extendible cardinal)**.** *A cardinal is weakly extendible if for all ordinals α there exists an ordinal β and an elementary embedding*

$$j : V_{\kappa+\alpha} \prec V_{j(\kappa)+\beta}$$

such that crit$(j) = \kappa$ and $\alpha < j(\kappa)$.

Once again, under choice this is equivalent to the following:

Definition 4.6 (Extendible cardinal)**.** *Let A be the class of γ such that $V_\gamma \prec_{\Sigma_1^*} V$. Then we call κ extendible if it is A-extendible, i.e. if for all α there exists a β and an elementary embedding*

$$j : V_{\kappa+\alpha} \prec V_{j(\kappa)+\beta}$$

such that crit$(j) = \kappa$, $\alpha < j(\kappa)$ and for all $\beta < \alpha$, $j(A \cap V_{\kappa+\beta}) = A \cap V_{j(\kappa)+j(\beta)}$.

Consistency and Reflection

As we ascended the traditional large cardinal hierarchy, we connected the various large cardinal notions we defined by means of relative consistency, but it is possible to gain stronger results than this. In this chapter, we will build up the choiceless hierarchy not just in terms of relative consistency but also in terms of the following reflection properties.

Here are three types of reflection which we will use in this chapter.

Definition 4.7. *Suppose ϕ_1 and ϕ_2 are large cardinal principles.*

(i) We say ϕ_1 reflects ϕ_2 if for all κ such that $\phi_1(\kappa)$ holds, there exists a $\bar{\kappa} < \kappa$ such that $\phi_2(\bar{\kappa})$.

(ii) We say ϕ_1 rank-reflects ϕ_2 if for all κ such that $\phi_1(\kappa)$ holds, there exists $\bar{\kappa} < \gamma \leq \kappa$ such that $\langle V_\gamma, V_{\gamma+1}\rangle \models ZF_2 + \phi_2(\bar{\kappa})$.

(iii) We say ϕ_1 strongly rank-reflects ϕ_2 if for all κ such that $\phi_1(\kappa)$ holds, there exists $\bar{\kappa} < \gamma < \kappa$ such that $\langle V_\gamma, V_{\gamma+1}\rangle \models ZF_2 + \phi_2(\bar{\kappa})$.

Note that the last two differ only in that the second inequality becomes strict for strong rank-reflection.

4.2 Reinhardt Cardinals

Now that we abandon choice, the Kunen inconsistency no longer applies. It is not known at present if Reinhardt cardinals are inconsistent in the absence of choice. We now build a hierarchy of strong choiceless cardinals: the Reinhardt hierarchy, building up from where, under AC, we were forced to stop. Recall the definition of a Reinhardt cardinal:

Definition 3.31. *A cardinal k is Reinhardt if there exists a non-trivial elementary embedding $j : V \prec V$ with critical point κ.*

This concept can be strengthened in the following way, in a way that is reminiscent of the definition of strong cardinals.

Definition 4.8 (Super Reinhardt cardinal)**.** *[50] A cardinal κ is super Reinhardt if for all ordinals λ there exists a non-trivial elementary embedding $j : V \prec V$ with $crit(j) = \kappa$ and $j(\kappa) > \lambda$.*

The following is an even stronger version.

Definition 4.9 (A-super Reinhardt cardinal)**.** *Let A be a proper class. A cardinal κ is A-super Reinhardt if for all ordinals λ there exists a nontrivial elementary embedding $j : V \prec V$ such that $crit(j) = \kappa$, $j(\kappa) > \lambda$, and $j(A) = \bigcup_{\alpha \in On} j(A \cap V_\alpha) = A$.*

Definition 4.10 (Totally Reinhardt cardinal)**.** *A cardinal κ is totally Reinhardt if for every $A \in V_{\kappa+1}$, the following holds:*

$$\langle V_\kappa, V_{\kappa+1} \rangle \models ZF_2 + \text{"There is an A-super Reinhardt cardinal."}$$

It follows straight from the definitions that totally Reinhardt cardinals rank-reflect super Reinhardt cardinals. The next result completes our mini-hierarchy.

Theorem 4.11. *If κ is a super Reinhardt cardinal, then there exists a $\gamma < \kappa$ such that*

$$\langle V_\kappa, V_{\kappa+1} \rangle \models ZF_2 + \text{"There is a Reinhardt cardinal."}$$

Proof. Suppose $j : V \prec V$ is a nontrivial elementary embedding with $\mathrm{crit}(j) = \kappa$, as in definition 4.9. As we did in the proof of Kunen's inconsistency, let $\langle \kappa_n \mid n \in \omega \rangle$ be the critical sequence of j, i.e. $\kappa_0 = \kappa$ and $\kappa_{n+1} = j(\kappa_n)$. Let $\lambda = \sup(\langle \kappa_n \mid n \in \omega \rangle)$, and observe that $j(\lambda) = \lambda$.

As κ is super Reinhardt, there also exists an elementary embedding $i : V \prec V$ with $\mathrm{crit}(i) = \kappa$ and $i(\kappa) > \lambda$. Since κ is a limit of inaccessibles, so is $i(\kappa)$. Now let γ_0 be the least inaccessible above λ. Since γ_0 is definable from λ and since $j(\lambda) = \lambda$, it follows that $j(\gamma_0) = \gamma_0$. It follows that

$$\langle V_{\gamma_0}, V_{\gamma_0+1} \rangle \models ZF_2 + \text{"}\kappa \text{ is a Reinhardt cardinal."}$$

by 2.7, and witnessed by $j \restriction V_{\gamma_0}$. Finally, we achieve the desired claim by applying i^{-1}, since $\gamma_0 < i(\kappa)$. ∎

4.3 Berkeley Cardinals

We now explore even further heights of the choiceless hierarchy.

Definition 4.12 (Proto-Berkeley cardinal). *δ is a proto-Berkeley cardinal if for all transitive sets M such that $\delta \in M$, there exists $j : M \prec M$ (i.e. $j \in \mathscr{E}(M)$), with crit$(j) < \delta$.*

The following proposition is immediate.

Proposition 4.13. *(stated as remark in [52])*

(i) If δ is proto-Berkeley, then for any $\lambda > \delta$ there exists a nontrivial embedding $j : V_\lambda \prec V_\lambda$ with crit$(j) < \delta$ (cf. I3 from definition 3.38).

(ii) If δ_0 is the least proto-Berkeley cardinal then every ordinal greater than δ_0 is also a Berkeley cardinal.

The following lemma gives a reformulation of the notion of a Berkeley cardinal. The lemma effectively states that if δ is proto-Berkeley, there is an embedding with critical point below δ fixing any given set.

Lemma 4.14. [50] *The following are equivalent:*

(i) δ is a proto-Berkeley cardinal.

(ii) For all sets a, and for all transitive sets M such that $a, \delta \in \mathscr{E}(M)$ such that $j(a) = a$ and crit$(j) < \delta$.

Proof. $(ii) \to (i)$ follows immediately if we take $a = \emptyset$. $(i) \to (ii)$: Suppose a is a set and M is a transitive set satisfying $a, \delta \in M$. We require the following claim:

Claim . *For any set a, there exists a transitive set M such that $a \in M$, and a is definable without parameters in M.*

Proof of Claim. Trivially, there exists some λ such that $\mathrm{Lim}(\lambda)$ and $a \in V_\lambda$, and so we define

$$M = V_\lambda \cup \{\{\langle a, x\rangle \mid x \in V_\lambda\}\}.$$

One can easily establish that Trans(M): If $y \in M$ then either $y \in V_\lambda$ or $y = \{\langle a, x\rangle \mid x \in V_\lambda\}$. If $y \in V_\lambda$ then of course $y \subset V_\lambda$. In the case $y = \{\langle a, x\rangle \mid x \in V_\lambda\}$, every element of y is an element of $V_\lambda \subseteq M$ and so $y \subset M$ Therefore, Trans(M) holds.

We can define a in M without parameters as it is the element of all of the pairs in the set of highest rank. □

By the claim, we can let N be a transitive set such that a and M are definable in N. As δ is proto-Berkeley, by definition, there is an elementary embedding $i : N \prec N$ such that crit$(i) < \delta$. Since a and M are definable in N, $i(a) = a$ and $i(M) = M$. Therefore $i \restriction M \in \mathscr{E}(M)$ and so i satisfies (ii). □

Theorem 4.15. *Let δ_0 be the least proto-Berkeley cardinal. Then for all transitive sets M such that $\delta_0 \in M$, and for all $\eta < \delta_0$, there exists $j \in \mathscr{E}(M)$ such that*

$$\eta < crit(j) < \delta_0.$$

Proof. We prove this by contradiction. So assume that there is a transitive set M such that $\delta_0 \in M$ and there exist ordinals $\eta < \delta_0$ such that there doesn't exist a $j \in \mathscr{E}(M)$ satisfying $\eta < \text{crit}(j) < \delta_0$S.

Now let η_0 be the least ordinal such that there exists a transitive set M such that $\delta_0 \in M$ and there doesn't exist a $j \in \mathscr{E}(M)$ with $\eta_0 < \text{crit}(j) < \delta_0$. We make the following claim:

Claim . *η_0 is a proto-Berkeley cardinal.*

Proof of Claim. Suppose M is a transitive set with $\eta \in M$ and let N be such that $\langle M_0, M, \eta_0 \rangle$ is definable in N. As $\delta_0 \in N$, there must exist some elementary embedding $i \in \mathscr{E}(N)$ with $\text{crit}(i) < \delta_0$. As M_0, M and η_0 are all definable in N, they must all be fixed by i. It follows that $i \restriction M_0 \in \mathscr{E}(M_0)$, and $\text{crit}(i \restriction M_0) \leq \eta_0$ by the definition of η_0. But as i fixes η_0, we must have that $\text{crit}(i \restriction M_0) < \eta_0$. The same is true of M, and so we also have $\text{crit}(i \restriction M) < \eta_0$. However M was any transitive set such that $\eta_0 \in M$, satisfying the definition of a proto-Berkeley cardinal. □

By the claim, we have that η_0 is proto-Berkeley, but this implies that $\eta_0 = \delta_0$, contradicting the assumption that $\eta_0 < \text{crit}(j) < \delta_0$. □

The following is a generalisation of proto-Berkeley cardinals.

Definition 4.16 (α-proto-Berkeley cardinal)**.** *Let α and δ be ordinals. Then δ is an α-proto-Berkeley cardinal if for all transitive sets M with $\delta \in M$, there exists a $j \in \mathscr{E}(M)$ with $\alpha < crit(j) < \delta$.*

Theorem 4.17. *Let δ_0 be the least α-proto-Berkeley cardinal. Then for all transitive sets M such that $\delta_0 \in M$, and for all $\eta < \delta_0$, there exists $j \in \mathscr{E}(M)$ such that*

$$\eta < crit(j) < \delta_0.$$

That is to say, theorem 4.15 generalises to α-proto-Berkeley cardinals. The proof is almost identical to that of 4.15, so we omit it.

We established earlier that every ordinal after the least proto-Berkeley cardinal is itself a proto-Berkeley cardinal. This motivates us to question what distinguishes δ_0 from every ordinal above it. The key feature is is that for δ_0, the critical points of the associated embeddings are cofinal in δ_0. This is a property not shared by the majority of proto-Berkeley cardinals, in fact, the next ordinal with this property after δ_0 is the least δ_0-proto-Berkeley cardinal, and the pattern continues. This leads us to the definition of a Berkeley cardinal.

Definition 4.18 (Berkeley cardinal)**.** *A cardinal δ is a Berkeley cardinal if for every transitive set M with $\delta \in M$, and every ordinal $\eta < \delta$, there exists a $j \in \mathscr{E}(M)$ with $\eta < crit(j) < \delta$.*

Proposition 4.19. *[53] If δ is the least proto-Berkeley cardinal, then it is also a Berkeley cardinal.*

Proof. Suppose this is false for the sake of contradiction. Then there must be some $\alpha < \delta$ such that for some transitive M containing δ, every elementary embedding $j \in \mathscr{E}(M)$ has $\mathrm{crit}(j) \leq \alpha$.

Let N be some transitive set containing α, and M' be some transitive set in which N, M, α are all definable, and $j : M' \prec M'$ witnesses the fact that δ is proto-Berkeley. Since j restricts to M, it must satisfy $\mathrm{crit}(j) \leq \alpha$. However, j must fix α, and so $\mathrm{crit}(j) < \alpha$ and therefore $j \restriction N$ witnesses the fact that α is proto-Berkeley, contradicting the minimality of δ. I.e. we have shown that if δ is the minimal proto-Berkeley cardinal but is not Berkeley, it is not minimal, an absurdity. Therefore, it must be a Berkeley Cardinal. $\square$

This proposition also generalises to higher levels: For all $\alpha \in \mathrm{On}$, the least α-proto-Berkeley cardinal is a Berkeley cardinal.

It is now natural to ask how Berkeley cardinals interact with the large cardinal hierarchies we have already discussed. The following starts us along that path.

Theorem 4.20. *[50] Let δ_0 be the least Berkeley cardinal. Then there are no extendible cardinals $\kappa \leq \delta_0$, i.e. Berkeley cardinals do not reflect extendible cardinals.*

Proof. For the sake of contradiction assume that there is an extendible $\kappa < \delta_0$. By a standard result that still applies without choice, we have that $V_\kappa \prec_{\Sigma_3} V$. One can easily verify that the sentence saying that δ_0 is a Berkeley cardinal is Π_2, and since by assumption V satisfies the statement that there exists some Berkeley cardinal, this statement is Σ_3. It follows by the Σ_3-elementarity result above that V_κ agrees that there is a Berkeley cardinal. However, since V_κ is correct in identification of Berkeley cardinals, there is a Berkeley cardinal below κ, which contradicts the minimality of δ_0. $\square$

It follows from this result that Berkeley cardinals also don't reflect super Reinhardt cardinals. However, as we shall now establish, they strongly rank-reflect Reinhardt cardinals, in conjunction with many of the traditional large cardinal concepts from the earlier chapters.

First we require the following lemma:

Lemma 4.21. *Suppose δ_0 is the least Berkeley cardinal. Then for a tail of limit ordinals λ, if an elementary embedding $j \in \mathscr{E}(V_\lambda)$ satisfies $\mathrm{crit}(j) < \delta_0$, then*

(i) $j(\delta_0) = \delta_0$, and

(ii) $\{\alpha < \delta_0 \mid j(\alpha) = \alpha\}$ is cofinal in δ_0.

Proof. (i) : We aim to show that for a tail of limit ordinals λ, V_λ can recognise that δ_0 is the least Berkeley cardinal.

Recall that the least Berkeley cardinal is also the least proto-Berkeley cardinal. For all ordinals $\delta < \delta_0$, which are obviously not proto-Berkeley, there must exist an M_δ such that $\delta \in M_\delta$ and there is no $j \in \mathscr{E}(M_\delta)$ with $\mathrm{crit}(j) < \delta$. Now let β_δ be the minimum ordinal such that V_{β_δ} contains M_δ. I.e. the minimum rank at which a counterexample occurs. Let λ be a limit ordinal greater than both δ_0 and β_δ for every $\delta < \delta_0$.

Then as λ is a limit ordinal greater than δ_0, V_λ (correctly) believes that δ_0 is a Berkeley cardinal. Further, V_λ also (correctly) believes that any $\delta < \delta_0$ is not proto-Berkeley as it, by definition, contains all the counterexamples M_δ. It therefore follows that V_λ correctly recognises that δ_0 is the least proto-Berkeley and therefore least Berkeley cardinal.

If such a λ as above is in the tail described, then for any $j \in \mathscr{E}(V_\lambda)$ with $\mathrm{crit}(j) < \delta_0$, it follows that $j(\delta_0) = \delta_0$, as δ_0 can be defined as the least Berkeley cardinal within V_λ.

(ii) : Suppose λ is contained in the above tail, and suppose that $j \in \mathscr{E}(V_\lambda)$ satisfies $\mathrm{crit}(j) < \delta_0$. Assume that (ii) fails for the sake of contradiction.

Let $\xi_0 = \sup\{\alpha < \delta_0 \mid j(\alpha) = \alpha\}$, and for all $i \in \omega$, let $\xi_{i+1} = j(\xi_i)$. It therefore follows that $\delta_0 = \sup\{\xi_i \mid i \in \omega\}$.

Now let $M_0 \in V_\lambda$ be a witness to the fact that ξ_0 is not proto-Berkeley cardinal, i.e. let M_0 be transitive such that $\xi_0 \in M_0$ and there does not exist a $j \in \mathscr{E}(M_0)$ with $\mathrm{crit}(j) < \xi_0$. For all $i \in \omega$, let $M_{i+1} = j(M_i)$. It follows by elementarity that M_{i+1} also witnesses that ξ_{i+1} is not proto-Berkeley.

So for a tail of limit ordinals λ, it follows that $\langle M_i \mid i \in \omega \rangle \in V_\lambda$. We have by lemma 4.14 that there is some λ for which there exists a $j' \in \mathscr{E}(M_0)$ such that

$$j'(\langle M_i \mid i \in \omega \rangle) = \langle M_i \mid i \in \omega \rangle$$

and $\mathrm{crit}(j) < \delta_0$. If we choose an i such that $\mathrm{crit}(j') < \xi_i$, we obtain that $j'(M_i) = M_i$ and therefore $j' \restriction M_i \in \mathscr{E}(M_i)$ and $\mathrm{crit}(j' \restriction M_i) < \xi_i$. This contradicts the fact that M_i witnesses that ξ_i is not proto-Berkeley, yielding the result. $\square$

We can use this same proof to prove the result for the nth Berkeley cardinal, or in fact more generally the α-th proto-Berkeley cardinal for any α definable in V_λ.

This lemma allows us to prove the following theorem, establishing the rank-reflection between Reinhardt cardinals combined with a traditional large cardinal. We follow the example set in [50] by choosing an ω-huge cardinal, but this theorem generalises to many other of the traditional large cardinal axioms. See, for example, Cutolo[52] p. 14 for a version of this result with an extendible cardinal replacing the ω-huge cardinal below.

Theorem 4.22. *[50] Suppose that δ_0 is the least Berkeley cardinal. Then there exists a $\gamma < \delta_0$ such that*

$$\langle V_\gamma, V_{\gamma+1} \rangle \models ZF_2 + \text{"there exists a Reinhardt cardinal witnessed by } j \text{ and an } \omega\text{-huge cardinal above } \kappa_\omega(j)\text{"}.$$

Proof. Firstly, it follows from the previous lemma that that given a tail of limit ordinals β, if $j \in \mathscr{E}(V_\beta)$ has $\text{crit}(j) < \delta_0$, then $j(\delta_0) = \delta_0$ and the subset $\{\alpha < \delta_0 \mid j(\alpha) = \alpha\}$ is cofinal in δ_0.

Let β be fixed in this tail. Since δ_0 is a Berkeley cardinal, there is a $j \in \mathscr{E}(V_\beta)$ such that $\text{crit}(j) < \delta_0$. As usual, let $\kappa = \text{crit}(j)$, and let $\lambda = \kappa_\omega(j)$. If we restrict j to some appropriate V_γ for $\gamma < \delta_0$, we will have an embedding witnessing that $\langle V_\gamma, V_{\gamma+1}\rangle$ believes κ is Reinhardt. We need to show that such a situation as this exists in which there is an ω-huge cardinal above λ.

So since δ_0 is a Berkeley cardinal, we have embeddings $j'' \in \mathscr{E}(V_\beta)$ with $\lambda < \text{crit}(j'') < \delta_0$. As β was selected from the tail of limit ordinals above, we have that for any j''. $\lambda'' = \kappa_\omega(j'') < \delta_0$. So there exist several ω-huge cardinals between λ and δ_0. By setting $k = j'' \restriction V_{\lambda''}$, we have shown that in V_β, there is a λ'' such that there is a $k \in \mathscr{E}(V_{\lambda''})$ satisfying $\lambda < \text{crit}(k) < \delta_0$ and $\kappa_\omega(k) = \lambda'' < \delta_0$.

Now let λ' be the least such λ'' defined above, and let $j' \in \mathscr{E}(V_\lambda)$ be one of the associated embeddings k. Finally let γ be the least inaccessible above λ'. We have that λ' and γ are definable in V_β with parameters λ and δ_0. So since $j(\lambda) = \lambda$ and $j(\delta_0) = \delta_0$, it follows that $j(\lambda') = \lambda'$ and $j(\gamma) = \gamma$. This gives that $j \restriction V_\gamma \in \mathscr{E}(V_\gamma)$, and $j' \in \mathscr{E}(V_{\lambda'})$ witness the statement in the theorem:

$$\langle V_\gamma, V_{\gamma+1}\rangle \models \text{ZF}_2 + \text{"there exists a Reinhardt cardinal witnessed by } j \restriction V_\gamma \\ \text{and an } \omega\text{-huge cardinal above } \kappa_\omega(j \restriction V_\gamma)\text{"}.$$

$\square$

We now give some generalisations of Berkeley cardinals.

Definition 4.23. *A cardinal δ is club Berkeley if δ is regular and for all club sets $C \subseteq \delta$ and for all transitive sets M with $\delta \in M$ there exists $j \in \mathscr{E}(M)$ with $crit(j) \in C$.*

Theorem 4.24. *[50] If δ is a club Berkeley cardinal, then it is a totally Reinhardt cardinal.*

Proof. Suppose δ is a club Reinhardt cardinal, and let $A \subseteq V_\delta$ be fixed. To prove the theorem we need to show that

$$\langle V_\kappa, V_{\kappa+1}\rangle \models ZF_2 + \text{"There is an A-super Reinhardt cardinal."}$$

By the claim in lemma 4.14, there are transitive sets M such that $V_{\delta+1} \in M$ and A is definable in M. We make the following claim:

Claim . *Let M be a transitive set such that $V_{\delta+1} \in M$ and A is definable in M. Then there exists a $\kappa < \delta$ such that for all $\alpha < \delta$, there exists a $j \in \mathscr{E}(M)$ such that the following hold:*

(i) $crit(j) = \kappa$,

(ii) $j(\kappa) > \alpha$,

(iii) $j(A) = A$.

To complete the proof, take M to be as in the claim. Then the result follows as witnessed by κ and the embeddings $j \upharpoonright V_\delta$ from the claim. $\square$

Proof of Claim. For a contradiction, suppose the claim is false, i.e. that M is transitive and $V_{\delta+1} \in M$, A is definable in M, and for all $\kappa < \delta$ there is an $\alpha < \delta$ such that there is no $j \in \mathscr{E}(M)$ satisfying $(i) - (iii)$ above.

Now define α_κ as the least α satisfying the above. Define the following:

$$C = \{\gamma < \delta \mid \forall \kappa < \gamma(\alpha_\kappa < \gamma)\}.$$

This set is called the set of "no crossover points" in [50]. SInce δ is regular, C is a club set in δ. Further, since δ is a club Berkeley cardinal, there is a $j \in \mathscr{E}(M)$ satisfying the following (this follows for similar reasons to the claim in lemma 4.14):

(i) $\mathrm{crit}(j) \in C$,

(ii) $j(C) = C$,

(iii) $j(A) = A$.

It follows that $\kappa = \mathrm{crit}(j) \in C$ and hence $j(\kappa) \in j(C) = C$. But since additionally $\kappa < j(\kappa)$, by the definition of C it follows that $\alpha_\kappa < j(\kappa)$, but this contradicts the definition of α_κ and therefore the claim must hold. $\square$

Definition 4.25. *A cardinal δ is a limit club Berkeley cardinal if it is a club Berkeley cardinal and a limit of Berkeley cardinals.*

Theorem 4.26. *[50] Suppose δ is a limit club Berkeley cardinal. Then*

$$\langle V_\delta, V_{\delta+1}\rangle \models ZF_2 + \text{"there exists a Berkeley cardinal that is super Reinhardt."}$$

The proof of this theorem (which can be found in [50], p. 296) is very similar to that of theorem 4.24, so we omit it. Considering these sorts of results as a whole, we arrive at a hierarchy much like what we constructed in chapters 2 and 3.

$$
\begin{tikzcd}
\text{limit club Berkeley} \arrow[d, "\subseteq"] \arrow[ddr, bend left, "SRR{,}R"] & \\
\text{club Berkeley} \arrow[d, "\subseteq"] & \\
\text{totally Reinhardt} \arrow[d, "RR"] & \text{Berkeley} \arrow[ddl, bend left, "SRR"] \\
\text{super Reinhardt} \arrow[d, "SRR"] & \\
\text{Reinhardt} &
\end{tikzcd}
$$

Figure 4: The choiceless hierarchy as given so far.
Here $\subseteq$ denotes the usual "every A is a B" relation used to build the previous hierarchies, R, RR and SRR denote reflection, rank-reflection and strict rank-reflection as defined in 4.7.

4.4 More Choiceless Cardinals

Here we outline some other flavours of choiceless large cardinals that are less mainstream than the original Reinhardt and Berkeley cardinals. For the sake of brevity, we will not go into detail.

The following is a weakening of Reinhardt cardinals introduced by Farmer Schlutzenberg in [54].

Definition 4.27 (Rank Berkeley cardinal)**.** *(see [55]) A cardinal λ is rank Berkeley if for all $\alpha < \lambda \leq \beta$, there is an elementary embedding $j \in \mathscr{E}(V_\beta)$ with $\alpha < crit(j) < \lambda$.*

The distinguishing advantage of working with rank-Berkeley cardinals over Reinhardt cardinals is that they are first-order definable.

The following concept was first introduced in a MathOverflow thread by Noah Schweber in 2012[56].

Definition 4.28 (Uniformly supercompact cardinal)**.** *[57] A cardinal κ is uniformly supercompact if it is the critical point of an elementary embedding $j : V \prec M$ such that for all ordinals α, $M^\alpha \subseteq M$.*

He posed two questions, which remain open.

Open Question 4.29. *(i) Must a uniformly supercompact cardinal be a Reinhardt cardinal?*

(ii) What is the consistency strength of the assertion that there is a uniformly supercompact cardinal?

In [57], Goldberg introduced two further variants of Reinhardt cardinals prompted by Schweber's questions.

Definition 4.30 (Weakly Reinhardt cardinal)**.** *A cardinal κ is weakly Reinhardt if it is the critical point of an elementary embedding $j : V \prec M$ such that for all ordinals α, $j \restriction \mathcal{P}(A) \in M$.*

Definition 4.31 (Ultrafilter Reinhardt cardinal)**.** *Let $\beta(X)$ denote the set of ultrafilters on X. A cardinal κ is weakly Reinhardt if it is the critical point of an elementary embedding $j : V \prec M$ such that for all ordinals α, $M^\alpha \subseteq M$ and $\beta(\alpha) \subseteq M$.*

In [57], Goldberg showed the following (we omit the proofs).

Proposition 4.32. *If κ is ultrafilter Reinhardt, it is weakly Reinhardt.*

Theorem 4.33. *If there is a proper class of weakly Reinhardt cardinals, then for all sufficiently large cardinals ν, the inner model N_ν (defined in [57]) contains a proper class of Reinhardt cardinals.*

The following question remains open.

Open Question 4.34. *Are Reinhardt cardinals and weakly Reinhardt cardinals equiconsistent?*

5 Exacting Cardinals

In this chapter, we cover recent work by Juan P. Aguilera, Joan Bagaria and Philipp Lücke, in particular their recent paper introducing exacting cardinals [58].

5.1 $C^{(n)}$-Cardinals

Throughout the current chapter we will require the following notion:

Definition 5.1 ($C^{(n)}$-cardinals)**.** *Let $n \in \mathbb{N}$. Let $C^{(n)}$ denote the club proper class of ordinals α that are "Σ_n-correct" in V, that is, $V_\alpha \prec_{\Sigma_n} V$.*

The following few facts are trivial but useful to be aware of:

(i) An ordinal α is Σ_n-correct iff it is Π_n-correct.

(ii) If α is Σ_n-correct and φ is a Σ_{n+1} sentence with parameters in V_α that also holds in V_α, then φ holds in V.

(iii) Conversely, if ψ is a Π_{n+1} sentence with parameters in V_α and holding in V, then it holds in V_α.

(iv) The class $C^{(0)}$ is the class On of all ordinals.

(v) The class $C^{(1)}$ is the class of ordinals α such that $V_\alpha = H_\alpha$ (recall $H_\kappa = \{x : |TC(x)| < \kappa\}$).

There has been significant investigation into $C^{(n)}$ cardinals, and we only briefly touch upon the subject here as a preliminary to the main focus of this chapter. For a more complete introduction to $C^{(n)}$ cardinals, see Bagaria[59].

Suppose one is investigating elementary embeddings $j : V \prec M$ where M is some transitive model. One is particularly interested in finding a j such that $V_{j(\kappa)}$ reflects some property of V. More generally, we would like $j(\kappa)$ to belong to some definable club class of ordinals. The club sets $C^{(n)}$ form a basis for these classes, and so the question we need to ask is the following: given an $n \in \omega$, when can we have $j(\kappa) \in C^{(n)}$?

This motivates the following:

Definition 5.2. [59] *A cardinal κ is $C^{(n)}$-measurable if there is an elementary embedding $j : V \prec M$ where M is some transitive class, such that $crit(j) = \kappa$ and $j(\kappa) \in C^{(n)}$.*

We can link this back to the traditional notion of measurability.

Proposition 5.3. *If κ is a measurable cardinal, then κ is $C^{(n)}$-measurable for all $n \in \omega$.*

One can similarly define $C^{(n)}$-versions of many large cardinals. Once again see Bagaria[59] for a thorough look into this.

5.2 Exacting Cardinals

The following definitions involving n-exactness were introduced in Bagaria and Lücke's paper "Huge Reflection"[60]. The formulation we use follows [58].

Definition 5.4 (n-exact embedding)**.** *Let λ be a limit cardinal and n a natural number. Let η be a cardinal such that $\lambda < \eta \in C^{(n)}$, and let X be an elementary submodel of V_η with $V_\lambda \cup \{\lambda\} \subseteq X$. Given another cardinal ζ such that $\lambda < \zeta \in C^{(n+1)}$, we say an elementary embedding $j : X \prec V_\zeta$ is an n-exact embedding at λ if '$j(\lambda) = \lambda$ and $j \restriction \lambda \neq id_\lambda$.*

Notice that X in the above cannot be transitive, as this would lead to a violation of Kunen's inconsistency because $V_{\lambda+2} \in X$.

Definition 5.5 (n-exact cardinal)**.** *Let λ be a limit cardinal and n a natural number.*

(i) Let $\vec{\lambda} = \langle \lambda_i \mid i < \omega \rangle$ be a strictly increasing sequence of cardinals with supremum λ. A cardinal $\kappa < \lambda_0$ is n-exact for $\vec{\lambda}$ if for every set $A \in V_{\lambda+1}$, there exists an n-exact embedding $j : X \prec V_\zeta$ at λ with $A \in ran(j)$, $j(\kappa) = \lambda_0$ and $j(\kappa_i) = \lambda_{i+1}$ for all $i < \omega$.

(ii) If, in addition to the above, $j(crit(j)) = \kappa$, we call κ parametrically n-exact for $\vec{\lambda}$.

Proposition 5.6. *[58] Suppose $j : X \prec V_\zeta$ is a 1-exact elementary embedding at a cardinal λ, then $\lambda^+ \not\subseteq X$, and $[\lambda]^\omega \not\subseteq X$.*

Proof. For the sake of contradiction, assume that $\lambda^+ \subseteq X$. As $\zeta \in C^{(2)}$, it must be that $\lambda^+ < \zeta$ and λ^+ is definable in V_ζ from the parameter λ. This gives that $\lambda^+ \in \mathrm{ran}(j)$ and because of this we can use the correctness of X to conclude that $\lambda^+ \in X$ and $j(\lambda^+) = \lambda^+$. By the same reasoning, one can show that $H_{\lambda^{++}} \in X$. It is a known result that the existence of a partition of $S^{\lambda^+}_\omega = \{\gamma < \lambda^+ \mid \mathrm{cf}(\gamma) = \omega\}$ into $\mathrm{crit}(j)$-many sets can be stated within $H_{\lambda^{++}}$ using only λ as a parameter. Therefore, because of this, there is a partition $\vec{S} = \langle S_\alpha \mid \alpha < \mathrm{crit}(j) \rangle \in X$. By elementarity we have that $j(\vec{S}) = \langle T_\beta \mid \beta < j(\mathrm{crit}(j)) \rangle$ partitions $S^{\lambda^+}_\omega$ into $j(\mathrm{crit}(j))$-many stationary sets. Define another set $C = \{\gamma \in S^{\lambda^+}_\omega \mid j(\gamma) = \gamma\}$.

Claim . *C is an ω-closed unbounded subset of λ^+.*

Proof of Claim. Suppose $\gamma < \lambda^+$ and let $\langle \gamma_n \mid n \in \omega \rangle$ be the sequence defined by the following rule: $\gamma_0 = \gamma$, $\gamma_{n+1} = j(\gamma_n + 1)$ for all $n \in \omega$. Now define $\gamma_\omega = \sup_{n \in \omega} \gamma_n \in S^{\lambda^+}_\omega \subseteq X$. By the correctness properties of X, we therefore have that X contains a strictly increasing sequence $\langle \beta_m \mid m \in \omega \rangle$ that is cofinal in γ_ω. It trivially holds that $j(\beta_m) < \gamma_\omega$ for all $m \in \omega$. By elementarity we have that $\langle j(\beta_m) \mid m \in \omega \rangle$ is cofinal in $j(\gamma_\omega)$. Hence, $\gamma_\omega \in C$. Now choose a limit point $\gamma \in S^{\lambda^+}_\omega$ of C. Let $m \in \omega$ be given, then there exists a $\bar{\gamma} \in C$ with $\beta_m < \bar{\gamma} < \gamma$, implying that $j(\beta_m) < \bar{\gamma} < \gamma$. Finally, as $\langle j(\beta_m) \mid m \in \omega \rangle$ is cofinal in $j(\gamma)$, we obtain that $\gamma \in C$, which yields the claim. □

By the above claim, there exists a $\gamma \in C \cap T_{\mathrm{crit}(j)}$. Therefore there exists an $\alpha < \mathrm{crit}(j)$ such that $\gamma \in S_\alpha$ and $\gamma = j(\gamma) \in T_\alpha \cap T_{\mathrm{crit}(j)} = \emptyset$. This is clearly a contradiction. If one now assumes that $[\lambda]^\omega \subseteq X$, we can follow a similar argument to the above, and obtain a surjection $[\lambda]^\omega \to \lambda^+$. This implies that $\lambda^+ \subseteq X$, which leads to the contradiction we just demonstrated, therefore, it must be that $[\lambda]^\omega \nsubseteq X$. $\square$

The following lemma establishes an important fact which allows us to define exacting cardinals.

Lemma 5.7. *[58] Let λ be a limit cardinal, x any set, and n a natural number. The following are equivalent:*

(i) There is an n-exact embedding $j : X \prec V_\zeta$ at λ with $x \in X$ and $j(x) = x$.

(ii) For every $\zeta > \lambda$ such that $x \in V_\zeta$, there is an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda, x\} \subseteq X$, and an elementary embedding $j : X \prec V_\zeta$ such that $j(\lambda) = \lambda$, $j(x) = x$ and $j \restriction \lambda \neq id_\lambda$.

(iii) For every $\zeta > \lambda$ such that $x \in V_\zeta$, and for every $\alpha < \lambda$, there is an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda, x\} \subseteq X$, and an elementary embedding $j : X \prec V_\zeta$ such that $j(\lambda) = \lambda$, $j(x) = x$, $j \restriction \lambda \neq id_\lambda$ and $j \restriction \alpha = id_\alpha$.

Proof. (iii) $\to$ (ii) is trivial since (iii) is just the same as (ii) but with the extra requirement that $j \restriction \alpha = id_\alpha$.

(ii) $\to$ (i) is also simple: Choose a $\zeta \in C^{(n+1)}$ which satisfies $x \in V_\zeta$. The embedding assumed to exist in (ii) satisfies $j(\lambda) = \lambda$ and $j \restriction \lambda \neq id_\lambda$ and so is an n-exact embedding.

(i) $\to$ (iii) is more involved. We prove this by contradiction, i.e. assume (i) and not (iii). Let $\xi > \lambda$ be the minimal such ζ satisfying $x \in V_\zeta$ where there is an $\alpha < \lambda$ such that for all elementary submodels X of V_ζ with $V_\lambda \cup \{\lambda, x\} \subseteq X$ there doesn't exist an elementary embedding $j : X \prec V_\zeta$ with $j(\lambda) = \lambda$, $j(x) = x$, $j \restriction \lambda \neq id_\lambda$ and $j \restriction \alpha = id_\alpha$.

It follows from the above condition that the set $\{\xi\}$ is definable by a Σ_2 formula using x and λ as parameters. Likewise let β be the minimal choice of α in the above. Then the set $\{\beta\}$ is also definable by a Σ_2 formula using x and λ as parameters.

Since we have (i), we can find a cardinal $\lambda < \eta \in C^{(n)}$, an elementary submodel X of V_η such that $V_\lambda \cup \{\lambda, x\} \subseteq X$, another cardinal $\lambda < \zeta \in C^{(n+1)}$ and an elementary embedding $j : X \prec V_\zeta$ with $j(\lambda) = \lambda$, $j(x) = x$ and $j \restriction \lambda \neq id_\lambda$.

Since $n + 1 \geq 2$ for all n, V_ζ is at least Σ_2-correct, and so because of the Σ_2-definability of $\{\xi\}$ and $\{\beta\}$, we have that both $\{\xi\}$ and $\{\beta\}$ are definable in V_ζ. It also follows that ξ and β must therefore be smaller than ζ.

Observe also that again by the Σ_2-correctness of X, we have that since $j(\lambda) = \lambda$ and $j(x) = x$, it must be the case that both ξ and β are elements of X such that $j(\xi) = \xi$, and $j(\beta) = \beta$.

We now construct an embedding to derive a contradiction from Kunen's inconsistency. Set $Y = X \cap V_\xi$, and let $i = j \upharpoonright Y$. By elementarity, we have that $V_\xi \in X$ and $j(V_\xi) = V_\xi$. and this means that Y is therefore an elementary submodel of V_ξ satisfying $V_\lambda \cup \{\lambda, x\} \subseteq Y$. It is simple to verify that $i : Y \to V_\xi$ is an elementary embedding satisfying $i(\lambda) = \lambda$, $i(x) = x$ and $i \upharpoonright \lambda \neq id_\lambda$. We assumed that no such embedding satisfies these conditions and $i \upharpoonright \beta = id_\beta$, so it follows that $i \upharpoonright \beta \neq id_\beta$.

This allows us to conclude that $\beta > \omega$ and $i(V_{\beta+2}) = V_{\beta+2}$, however this gives us that $i \upharpoonright V_{\beta+2} : V_{\beta+2} \to V_{\beta+2}$ is a nontrivial elementary embedding, which is clearly a contradiction by Kunen's result. □

We now define the main concept of this chapter:

Definition 5.8. *[58] A cardinal λ is exacting if for every $\zeta > \lambda$ and for every $\alpha < \lambda$, there exists an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j : X \prec V_\zeta$ with $j(\lambda) = \lambda$, $j \upharpoonright \lambda = id_\lambda$ and $j \upharpoonright \alpha = id_\alpha$.*

If we apply the above lemma to the case $n = 1$, $x = \emptyset$, we derive the following corollary:

Corollary 5.9. *A cardinal λ is exacting iff there is a 1-exact embedding at λ.*

Recall proposition 2.45, where we gave three equivalent ways of characterising a Jónsson cardinal. Here, we show that an exacting cardinal is in a sense a strengthening of a Jónsson cardinal.

Proposition 5.10. *Suppose λ is an exacting cardinal and let $\mathcal{C}$ be a class of structures in a countable first-order language that is definable by a formula with parameters in $V_\lambda \cup \{\lambda\}$. If B is a structure in $\mathcal{C}$ with cardinality λ, then there exist structures A and C of cardinality λ in $\mathcal{C}$ such that A is isomorphic to a proper elementary substructure of B and likewise B is isomorphic to a proper elementary substructure of C.*

Proof. Let $n > 0$, $\phi(v_0, v_1, v_2)$ be a Σ_n-formula and $z \in V_\lambda$ such that $\mathcal{C} = \{A \mid \phi(A, \lambda, z)\}$. We can choose a $\lambda < \zeta \in C^{(n+3)}$ and by using lemma 5.7, we can obtain an elementary submodel X of V_ζ such that $V_\lambda \cup \{\lambda\} \subseteq X$, and an elementary embedding $j : X \prec V_\zeta$ such that $j(\lambda) = \lambda, j(z) = z$ and $j \upharpoonright \lambda \neq id_\lambda$.

Now assume for the sake of contradiction that $\mathcal{C}$ contains a structure with cardinality λ that doesn't have a proper elementary substructure with equal cardinality isomorphic to a structure in $\mathcal{C}$. Since X is Σ_3-correct, we have that X contains a structure B with this property. Further, as $\lambda \subseteq X$, we have that $\mathrm{dom}(B) \subseteq X$ and this means that j induces an elementary embedding $B \prec j(B)$. Suppose $b \in X$ is a binection $b : \lambda \to \mathrm{dom}(B)$. Then given some $\alpha < \lambda$, we have that

$$j(b(\alpha)) = j(b)(j(\alpha)) \in j(b)[j[\lambda]],$$

and since $j[\lambda] \subset \lambda$, we have that the aforementioned elementary embedding induced by j is not surjective, as the image of B under j is a proper elementary submodel of $j(B)$ of cardinality λ isomorphic to $B \in \mathcal{C}$, but this contradicts the properties of B.

Again assume for the sake of contradiction that $\mathcal{C}$ contains a structure of cardinality λ not isomorphic to a proper elementary substructure of a structure of cardinality λ in $\mathcal{C}$. Just as above,we can find a structure $B \in X$ with this property, but then $j(B) \in \mathcal{C}$ is also a structure in C and j induces an elementary embedding $B \prec j(B)$ that also is not surjective. This contradicts the properties of B. $\square$

Returning briefly to our longstanding theme of studying HOD, we have the following generalisation of some now-very-familiar concepts.

Definition 5.11. *Suppose Γ is a class. Then the class OD_Γ is the class of all sets definable from parameters in $\Gamma \cup On$, and similarly HOD_Γ is the class of all sets x such that $tc(\{x\}) \subset OD_\Gamma$.*

Theorem 5.12. *If λ is an exacting cardinal, then λ is regular in HOD_{V_λ}. It therefore follows that $HOD_{V_\lambda} \neq V$.*

Proof. We prove this by contradiction. Assume that λ is singular in HOD_{V_λ}. Then there is an $x \in V_\lambda$ where λ is singular in $\mathrm{HOD}_{\{x\}}$. Choose $\lambda < \zeta \in C^{(3)}$ and an $\alpha < \lambda$ such that $x \in V_\alpha$ and $\mathrm{cf}(\lambda)^{\mathrm{HOD}_{\{x\}}} < \alpha$. Because λ is exacting, there is an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j : X \prec V_\zeta$ such that $j(\lambda) = \lambda$ and $j \restriction \alpha = id_\alpha$.

Define a function c as the least cofinal function

$$c : \mathrm{cf}(\lambda)^{\mathrm{HOD}_{\{x\}}} \to \lambda$$

in $\mathrm{HOD}_{\{x\}}$, with respect to the canonical well-ordering for HOD. Note that this well-ordering is Σ_2 definable. It therefore follows that since $\zeta \in C^{(3)}$, c retains the above property in V_ζ. Therefore $c \in X$, and is defined by the same property with the same parameters.

As j fixes λ, $\mathrm{cf}(\lambda)^{\mathrm{HOD}_{\{x\}}}$ and x, it follows that j also fixes c. Furthermore, since $j \restriction \alpha = id_\alpha$, we obtain that $j(c(\xi)) = c(\xi)$ for all $\xi < \mathrm{cf}(\lambda)^{\mathrm{HOD}_{\{x\}}}$.

We derive a contradiction via Kunen's inconsistency, as since $V_\lambda \subseteq X$, we have that λ is the supremum of the critical sequence, and only boundedly many ordinals are not moved by j. But as the range of c is cofinal in λ with $j \restriction \mathrm{ran}(c) = id_{\mathrm{ran}(c)}$, leading to a contradiction. $\square$

This theorem leads to a failure of the axiom of choice. The existence of a well-ordering of V definable using parameters in V_λ can be shown to be equivalent to the statement $V = \mathrm{HOD}_{V_\lambda}$. Therefore this form of the axiom of choice fails under the assumption of an exacting cardinal. However, as shown in [58], the existence of an exacting cardinal is in fact consistent with the assumption $V = \mathrm{HOD}_{V_{\lambda+1}}$.

The following corollary is derived from the above theorem. The proof uses some forcing arguments so we omit it, but it can be found in [58].

Corollary 5.13. *Suppose the theory ZFC+ "there exists an I0 embedding" is consistent, then this theory doesn't prove the existence of an exacting cardinal.*

5.3 Ultraexacting Cardinals

We can generalise exactness in the following way:

Definition 5.14 (n-ultraexact embedding)**.** *[58] An n-ultraexact embedding is an n-exact embedding $j : X \prec V_\zeta$ at λ such that $j \restriction V_\lambda \in X$.*

Definition 5.15 (n-ultraexact cardinal)**.** *[58]*

(i) Suppose $\vec{\lambda} = \langle \lambda_i, i \in \omega \rangle$ is a strictly increasing sequence of cardinals with supremum λ. Then a cardinal $\kappa < \lambda_0$ is n-ultraexact for $\vec{\lambda}$ if for all sets $A \in V_{\lambda+1}$, there exists an n-ultraexact embedding $j : X \prec V_\zeta$ at λ with $A \in ran(j)$, $j(\kappa) = \lambda_0$ and $j(\lambda_i) = \lambda_{i+1}$ for all $i \in \omega$.

(ii) If, in addition to the above, $j(crit(j)) = \kappa$, we call κ parametrically n-ultraexact for $\vec{\lambda}$.

The following lemma is a direct analogue of lemma 5.7. We omit the proof as it is only a slight modification of that of the aforementioned lemma.

Lemma 5.16. *[58] Let λ be a limit cardinal, x any set, and n a natural number. The following are equivalent:*

(i) There is an n-ultraexact embedding $j : X \prec V_\zeta$ at λ with $x \in X$ and $j(x) = x$.

(ii) For every $\zeta > \lambda$ such that $x \in V_\zeta$, there is an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda, x\} \subseteq X$, and an elementary embedding $j : X \prec V_\zeta$ such that $j(\lambda) = \lambda$, $j(x) = x$ and $j \restriction \lambda \neq id_\lambda$, $j \restriction V_\lambda \in X$.

(iii) For every $\zeta > \lambda$ such that $x \in V_\zeta$, and for every $\alpha < \lambda$, there is an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda, x\} \subseteq X$, and an elementary embedding $j : X \prec V_\zeta$ such that $j(\lambda) = \lambda$, $j(x) = x$, $j \restriction \lambda = id_\lambda$, $j \restriction \alpha = id_\alpha$ and $j \restriction V_\lambda \in X$.

We can now define ultraexacting cardinals, in a way very similar to how we defined exacting cardinals in definition 5.8.

Definition 5.17. *[58] A cardinal λ is ultraexacting if for every $\zeta > \lambda$ and for every $\alpha < \lambda$, there exists an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j : X \prec V_\zeta$ with $j(\lambda) = \lambda$, $j \restriction \lambda = id_\lambda$, $j \restriction \alpha = id_\alpha$.*

Similarly we obtain a corollary to the above lemma by setting $n = 1$ and $x = \emptyset$.

Corollary 5.18. *A cardinal κ is ultraexacting iff there is a 1-ultraexact embedding at λ.*

Recall definition 3.40. The following lemma shows that ultraexact embeddings induce I1 embeddings.

Lemma 5.19. *Suppose $j : X \to V_\zeta$ be a 1-ultraexact embedding λ, then the map*

$$(j \restriction V_\lambda)^+ : V_{\lambda+1} \to V_{\lambda+1}$$

is an I1 embedding in X satisfying

$$j \restriction (V_{\lambda+1} \cap X) = (j \restriction V_\lambda)^+ \restriction (V_{\lambda+1} \cap X)$$

Proof. Let $i = (j \restriction V_\lambda)^+$. It follows directly from our assumptions about X and j that $V_{\lambda+1}, i \in X$ and $j(V_{\lambda+1}) = V_{\lambda+1}$. Additionally, since X contains an ω-sequence cofinal in λ, we have by elementarity that $i \restriction (V_{\lambda+1} \cap X) = j \restriction (V_{\lambda+1} \cap X)$.

If we now suppose that $a_0, ..., a_{n-1} \in V_{\lambda+1} \cap X$ and that $\phi(v_0, .., v_{n-1})$ is a formula in the language of set theory such that in X, $\phi(a_0, ..., a_{n-1})$ holds in $V_{\lambda+1}$. By the elementarity of j and the earlier part of the proof, we have that $\phi(i(a_0), ..., i(a_{n-1}))$ holds in $V_{\lambda+1}$. Since each $i(a_k) \in X$ and $V_{\lambda+1} \in X$, we can conclude that $\phi(i(a_0), ..., i(a_{n-1}))$ holds in $V_{\lambda+1}$. □

Later we will make use of the lemma below to extend ultraexact embeddings.

Lemma 5.20. *Suppose $j : X \to V_\zeta$ is a 1-ultraexact embedding at λ, and suppose γ is an ordinal in X such that there is a surjection $s : V_{\lambda+1} \to \gamma$ in X with $j(s) \in X$. Then there exists a unique function $j \in X$ such that $j_\gamma : \gamma \to j(\gamma)$ satisfying*

$$j \restriction (X \cap \gamma) = j_\gamma \restriction (X \cap \gamma).$$

Proof. Suppose $x, y \in V_{\lambda+1} \cap X$ and $s(x) = s(y)$. We can apply the previous lemma to obtain

$$j(s)((j \restriction V_\lambda)^+(x)) = j(s)(j(x)) = j(s(y)) = j(s)(j(y)) = j(s)((j \restriction V_\lambda)^+(y)).$$

So it holds that there is a well-defined function $j_\gamma : \gamma \to j(\gamma)$ satisfying $j_\gamma(s(x)) = j(s)((j \restriction V_\lambda)^+(x))$ for all $x \in V_{\lambda+1}$. By the correctness of X, j_γ has all these same properties in V.

Suppose $\beta \in X \cap \gamma$. Then there is an $x \in V_{\lambda+1} \cap X$ such that $s(x)$ and again by the previous lemma we have that

$$j_\gamma(\beta) = j(s)((j \restriction V_\lambda)^+(x)) = j(s)(j(x)) = j(s(x)) = j(\beta).$$

Thus, we have established existence, we must now prove uniqueness. So let $k : \gamma \to j(\lambda)$ be a function in X such that $k \restriction (X \cap \gamma) = j \restriction (X \cap \gamma)$. It then follows that $k(\beta) = j(\beta) = j_\gamma(\beta)$ for all $\beta \in (X \cap \gamma)$. It then follows by elementarity that $k = j_\gamma$, thus establishing uniqueness.

□

We will make frequent use of the following definition:

Definition 5.21. *The ordinal Θ_X is the supremum of all ordinals α such that there exists a surjection $f : X \to \alpha$.*

This is a useful tool when relativised to inner models of ZF containing said set X. For our purposes, we will be interested in those of the form $\Theta_{V_{\lambda+1}}^{L(V_{\lambda+1},E)}$ for some set E.In practice, since we will only be interested in those with $X = V_{\lambda+1}$, we will omit the bottom index. We will later make use of the following lemmata:

Lemma 5.22. *Suppose $j : X \to V_\zeta$ is a 1-ultraexact embedding at λ and $E \in V_{\lambda+2} \cap X$ satisfies $j(E) = E$. Then $\Theta^{L(V_{\lambda+1},E)} \in X$ and $j(\Theta^{L(V_{\lambda+1},E)}) = \Theta^{L(V_{\lambda+1},E)}$.*

Proof. Observe that $\Theta^{L(V_{\lambda+1},E)}$ has a Σ_2 definition with parameters E and λ. As $\zeta \in C^{(2)}$, it must therefore be that $\Theta^{L(V_{\lambda+1},E)} < \zeta$ and can be defined in V_ζ using the same Σ_2-formula. Since $j(\lambda) = \lambda$ ajd $j(E) = E$, we have by elementarity of j that $\Theta^{L(V_{\lambda+1},E)} \in X$ and $j(\Theta^{L(V_{\lambda+1},E)}) = \Theta^{L(V_{\lambda+1},E)}$, since Σ_2-formulae are upwards absolute from X to V. □

Lemma 5.23. *Suppose $j : X \to V_\zeta$ is a 1-ultraexact embedding at λ and $E \in V_{\lambda+2} \cap X$ satisfies $j(E) = E$ and $\gamma \in X \cap \Theta^{L(V_{\lambda+1},E)}$ with $j(\gamma) \in X$. Then there is a surjection $s : V_{\lambda+1} \to \gamma$ in X such that $j(s) \in X$.*

Proof. We define $S : \mathrm{On} \times V_{\lambda+1} \to L(V_{\lambda+1}, E)$ to be the canonical class surjection onto $L(V_{\lambda+1}, E)$: From the iterative construction of $L(V_{\lambda+1})$ we can construct a canonical surjection onto $L(V_{\lambda+1})$, but since there is a bijection from $L(V_{\lambda+1})$ to $L(V_{\lambda+1}, E)$, we can obtain S as above. By the correctness of X, and since $\gamma \in X \cap \Theta^{L(V_{\lambda+1},E)}$, there is an $x \in V_{\lambda+1} \cap X$ such that there exists an ordinal β where $S(\beta, x)$ is a surjection from $V_{\lambda+1}$ onto γ. We define the required surjection s as $S(\alpha, x)$ where α is the minimal ordinal such that $S(\alpha, x)$ is a surjection.

Such an α is Σ_1-definable using parameters $E, V_{\lambda+1}, x$ and γ. Therefore by the correctness of X, we have that $\alpha \in X$ and, therefore $s \in X$ too. Furthermore, since $j(E) = E$, we have that $j(\alpha)$ is the minimal ordinal β such that $S(\beta, j(x))$ is a surjection from $V_{\lambda+1}$ to $j(\gamma)$, and so we similarly obtain that $j(\alpha)$ is Σ_1-definable with parameters $E, V_{\lambda+1}, j(x)$ and $j(\gamma)$. By lemma 5.19, we now have that $j(x) = (j \restriction V_\lambda)^+(x) \in X$, and since $j(\gamma) \in X$, $j(\alpha) \in X$ by the correctness of X. Therefore we can conclude that $j(s) = S(j(\alpha), j(x))$. □

Recall the definition of an ω-strongly measurable cardinal (definition 3.54).

Theorem 5.24. *[58](p. 16) (the reader acquainted with forcing should cf. [44] p. 298) If $j : Y \to V_\xi$ is a 2-ultraexact embedding at λ and $E \in V_{\lambda+2} \cap Y$ is such that $j(E) = E$, then every regular cardinal in the interval $(\lambda, \Theta^{L(V_{\lambda+1},E)})$ is ω-strongly measurable in HOD.*

Proof. By a result in [58] (lemma 3.12, which itself follows from the above lemma (5.23)), for a regular cardinal $\kappa < \lambda < \Theta^{L(V_{\lambda+1},E)}$, we can find a surjection $s : V_{\lambda+1} \to \kappa$ and a 2-ultraexact embedding $j : X \to V_\zeta$ such that $E, s, \kappa \in X, j(E) = E, j(\kappa) = \kappa$ and $j(s) \in X$. Let $\delta = \mathrm{crit}(j) < \lambda$. Then $(2^\delta)^{\mathrm{HOD}} < \lambda < \kappa$.

For the sake of contradiction, assume that HOD contains a partition of $S^\kappa_\omega = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \omega\}$ into δ-many stationary subsets. Let $\vec{S} = \langle S_\alpha \mid \alpha < \delta \rangle$

be the least partition with this property (with respect to the canonical well-ordering of HOD).

As both HOD and its wellordering are Σ_2-definable, we have that $\vec{S}$ is definable from parameters κ and δ. By the correctness of X it follows that $\vec{S} \in X$. Now, by elementarity, and since $\zeta \in C^{(3)}$, we have that $j(\vec{S})$ is the least partition of S^κ_ω into $j(\delta)$-many stationary sets (again in the canonical well-ordering of HOD). Which similarly gives us that $j(\vec{S})$ is Σ_2-definable from parameters κ and $j(\delta)$. As $j(\delta) \in X$, we have that $j(\vec{S}) = \langle T_\beta \mid \beta < j(\delta) \rangle \in X$ too.

Now define the following set, where j_κ is defined as in lemma 5.20:

$$C_{j,\kappa} = \{\beta < \kappa \mid j_\kappa(\beta) = \beta\}.$$

By lemma 3.9 of [58], this set is an ω-closed unbounded subset of κ. Now, by elementarity, we have that $T_\delta \subseteq S^\kappa_\omega$ is stationary, and hence it follows that $C \cap T_\delta \neq \emptyset$, and that C and T_δ belong to X implies that $\gamma = \min(C \cap T_\delta) \in S^\kappa_\omega \cap X$. It follows that there is an $\alpha < \delta$ such that $\gamma \in S_\alpha$, and since $\gamma \in C$, it follows from lemma 5.20 that

$$\gamma = j_\kappa(\gamma) = j(\gamma) = j(S_\alpha) \cap T_\delta = T_\alpha \cap T_\delta = \emptyset$$

which cannot hold by elementarity, thus proving the result by contradiction. □

We obtain the following as a particular case of this theorem:

Corollary 5.25. *If λ is an ultraexacting cardinal, then λ^+ is ω-strongly measurable in HOD.*

5.4 Icarus Sets and Ultraexacting Cardinals

Recall the definition of an Icarus set from definition 3.46. The following is a strengthening of this notion.

Definition 5.26. *Let λ be a limit ordinal. Then a set $E \subseteq V_{\lambda+1}$ is a strong Icarus set if there exists a nontrivial elementary embedding $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $crit(j) < \lambda$ and $j(E) = E$.*

The existence of such an embedding has a first-order definition using E as a parameter (as given in [45] p. 123).

Our aim in this section is to prove the following theorem:

Theorem 5.27. *Let $j : X \to V_\xi$ be a 2-ultraexact embedding at a cardinal λ, and let $E \in V_{\lambda+2} \cap X$ satisfy $j(E) = E$. If $(V_{\lambda+2}, E)^\#$ exists, then there is an elementary embedding i: $L(V_{\lambda+1}, E) \to L(V_{\lambda+1}, E)$ such that $i \restriction V_\lambda = j \restriction V_\lambda$ and $i(E) = E$.*

This theorem effectively states that if there is an ultraexact embedding at λ, and certain sharps exist, then many subsets of $V_{\lambda+1}$ turn out to be strong Icarus sets.

To prove this theorem we will require the following lemmata:

Lemma 5.28. *[58](p. 18) Let $j : X \to V_\xi$ be a 2-ultraexact embedding at a cardinal λ, and let $E \in V_{\lambda+2} \cap X$ satisfy $j(E) = E$. Let $\Theta = \Theta^{L(V_{\lambda+1},E)}_{V_{\lambda+1}}$. If X contains a surjection $f : V_{\lambda+1} \to \Theta$ with $j(f) \in X$ then there is a unique map $j_E : L_\Theta(V_{\lambda+1}, E) \to L_\Theta(V_{\lambda+1}, E)$ in X satisfying the following:*

$$j \restriction (L_\Theta(V_{\lambda+1}, E) \cap X) = j_E \restriction (L_\Theta(V_{\lambda+1}, E) \cap X)$$

Proof. By lemma 5.22, we have already that $\Theta \in X$ and $j(\Theta) = \Theta$, which gives that $L_\Theta(V_{\lambda+1}, E) \in X$ and $j(L_\Theta(V_{\lambda+1}, E)) = L_\Theta(V_{\lambda+1}, E)$. By the construction in the proof of lemma 5.23, we can define a surjection $S : \Theta \times V_{\lambda+1} \to L_\Theta(V_{\lambda+1}, E)$ that is Σ_1-definable over $L_\Theta(V_{\lambda+1}, E)$ with parameters E and $V_{\lambda+1}$. By the correctness of X, we therefore have that $S \in X, j(S) \in X$ and S retains the same properties in X.

Now let $\alpha, \beta \in X \cap \Theta$, and let $x, y \in V_{\lambda+1} \cap X$ such that $S(\alpha, x) = S(\beta, y)$. Then by lemmata 5.19 and 5.20, we have that

$$\begin{aligned} S(j_\Theta(\alpha), (j \restriction V_\lambda)^+(x)) &= j(S)(j(\alpha), j(x)) = j(S(\alpha, x)) = j(S(\beta, y)) \\ &= j(S)(j(\beta), j(y)) = S(j_\Theta(\beta), (j \restriction V_\lambda)^+(y)). \end{aligned}$$

Therefore, in X there is a function $j_E : L_\Theta(V_{\lambda+1}, E) \to L_\Theta(V_{\lambda+1}, E)$ such that

$$j_E(S(\alpha, x)) = S(j_\Theta(\alpha), (j \restriction V_\lambda)^+(x))$$

for all $\alpha < \Theta$ and $x \in V_{\lambda+1}$. By the correctness of X, this function j_E has the same property in V.

Now take some $z \in L_\Theta(V_{\lambda+1}, E) \cap X$. As the map S is by definition a surjection $\Theta \times V_{\lambda+1} \to L_\Theta(V_{\lambda+1}, E)$, there exist an $\alpha \in X \cap \Theta$ and $x \in V_{\lambda+1} \cap X$ such that $S(\alpha, x) = z$. Therefore

$$j_E(z) = S(j_\Theta(\alpha), (j \restriction V_\lambda)^+(x)) = j(S)(j(\alpha), j(x)) = j(S(\alpha, x)) = j(z).$$

If we now assume that there is a function $k : L_\Theta(V_{\lambda+1}, E) \to L_(V_{\lambda+1}, E)$ in X where $k \neq j_E$ and

$$k \restriction (L_\Theta(V_{\lambda+1}, E) \cap X) = j \restriction (L_\Theta(V_{\lambda+1}, E) \cap X),$$

then since $k, j_E \in X$, we can obtain a $z \in L_\Theta(V_{\lambda+1}, E) \cap X$ where $k(z) \neq j_E(z)$. This is a contradiction since by assumption, $k(z) = j(z) = j_E(z)$, thus completing our proof. □

Lemma 5.29. *[58](p. 18) Let $j : X \to V_\xi$ be a 2-ultraexact embedding at a cardinal λ, and let $E \in V_{\lambda+2} \cap X$ satisfy $j(E) = E$. Let $\Theta = \Theta^{L(V_{\lambda+1},E)}_{V_{\lambda+1}}$. If X contains a surjection $f : V_{\lambda+1} \to \Theta$ with $j(f) \in X$ then there is an elementary embedding $i : L(V_{\lambda+1}, E) \prec L(V_{\lambda+1}, E)$.*

Proof. We prove this by constructing an ultrapower. This is quite lengthy, but it uses many of the tools we have worked on acquiring throughout this text, so it is potentially quite instructive. For convenience, we abbreviate $N = L(V_{\lambda+1}, E)$ and $D = V^N_{\lambda+2}$.

Because $D \in L_\Theta(V_{\lambda+1}, E) = dom(j_E)$, we define the following:

$$U = \{A \in D \mid j \upharpoonright V_\lambda \in j_E(A)\}.$$

Then $U \in X$ because $D, j \upharpoonright V_\lambda, j_E \in X$. One can easily verify that U is an N-ultrafilter on $V_{\lambda+1}$ in both V and X.

We also can prove that the ultrafilter is non-principal (in X and therefore in V): $\mathrm{crit}(j) \notin \mathrm{ran}(j)$ as the range of j must skip $\mathrm{crit}(j)$, including all ordinals before, and sending $\mathrm{crit}(j)$ to a different ordinal by definition. This implies that $j \upharpoonright V_\lambda \notin \mathrm{ran}(j)$, and therefore we have that $j \upharpoonright V_\lambda \notin j_E(\{a\}) = j(\{a\}) = \{j(a)\}$ for all $a \in V_{\lambda+1} \cap X$, i.e. U is nonprincipal.

Now we can define the ultrapower $\mathrm{Ult}_U(N)$ and the embedding $i : \langle N, \in\rangle \to \langle \mathrm{Ult}_N(U), \in_U\rangle$ induced by the ultrapower. We now prove some properties of this ultrapower and show the induced embedding is elementary.

Claim 1. *Let $f, g \in N \cap X$ be functions $V_{\lambda+1} \to N$. Then the following hold:*

(i) $[f]_U = [g]_U$ *iff* $j(f)(j \upharpoonright V_\lambda) = j(g)(j \upharpoonright V_\lambda)$,

(ii) $[f]_U \in [g]_U$ *iff* $j(f)(j \upharpoonright V_\lambda) \in j(g)(j \upharpoonright V_\lambda)$.

Proof of Claim 1. (i) Let $A = \{x \in V_{\lambda+1} \mid f(x) = g(x)\} \in D \cap X$. Then we have the following series of equivalences:

$$[f]_U = [g]_U \leftrightarrow A \in U \leftrightarrow j \upharpoonright V_\lambda \in j_E(A) = j(A) \leftrightarrow j(f)(j \upharpoonright V_\lambda) = j(g)(j \upharpoonright V_\lambda).$$

The equivalences are all trivial applications of the definitions, and the equality in the middle, $j_E(A) = j(A)$, holds due to lemma 5.28. (ii) is similar. We define $B = \{x \in V_{\lambda+1} \mid f(x) \in g(x)\} \in D \cap X$, and similarly we obtain

$$[f]_U \in [g]_U \leftrightarrow B \in U \leftrightarrow j \upharpoonright V_\lambda \in j_E(B) = j(B) \leftrightarrow j(f)(j \upharpoonright V_\lambda) \in j(g)(j \upharpoonright V_\lambda)$$

giving us the desired result. □

Claim 2. *The ultrapower $\langle Ult_U(N), \in_U\rangle$ is well-founded.*

Proof of Claim 2. We prove this by contradiction, so assume that $\mathrm{Ult}_U(N)$ is ill-founded. This means that there must be a sequence $\langle f_n : V_{\lambda+1} \to N \mid n \in \omega\rangle \subset N$ such that $[f_{n+1}]_U \in [f_n]_U$ for all $n \in \omega$. Such a sequence exists in X due to its correctness properties (recall the definition of an ultraexact embedding). By claim 1(ii), we have that

$$j(f_{n+1})(j \upharpoonright V_\lambda) \in j(f_n)(j \upharpoonright V_\lambda)$$

for all values $n \in \omega$, which is an obvious contradiction as there cannot be any infinite descending $\in$-chains. □

Claim 3. *(Cf. claim 1) Let $f, g \in N \cap X$ be functions $V_{\lambda+1} \to N$. Then the following hold:*

(i) $[f]_U = [g]_U$ *iff* $j_E(f)(j \upharpoonright V_\lambda) = j_E(g)(j \upharpoonright V_\lambda)$,

(ii) $[f]_U \in [g]_U$ *iff* $j_E(f)(j \upharpoonright V_\lambda) \in j_E(g)(j \upharpoonright V_\lambda)$.

Proof of Claim 3. Observe that $f, g \in L_\Theta(V_{\lambda+1}, E)$, by lemma 5.28, in X this reduces to the result in Claim 1. This is carried upwards to V by the correctness of X. □

Now for some further notation. Given some $y \in V_{\lambda+1}$, denote by A_y the set of all I3 embeddings $k : V_\lambda \to V_\lambda$ satisfying $k^+(k) = j \upharpoonright V_\lambda$ and $y \in \operatorname{ran}(k^+)$. So $A_y \in D$ for all $y \in V_{\lambda+1}$ and if additionally $y \in X$, we have $A_y \in X$. This brings us to our next claim.

Claim 4. *For all* $y \in V_{\lambda+1}$, $A_y \in U$.

Proof of Claim 4. Fix $y \in V_{\lambda+1} \cap X$. By lemma 5.28, we have that $j_E(A_y) = j(A_y)$ is the set of I3 embeddings k such that $k^+(k) = j(j \upharpoonright V_\lambda)$ and $j(y) \in \operatorname{ran}(k^+)$. By lemma 5.19, we have that

$$(j \upharpoonright V_\lambda)^+(j \upharpoonright V_\lambda) = j(j \upharpoonright V_\lambda) \text{ and } (j \upharpoonright V_\lambda)^+(y) = j(y).$$

It follows that $j \upharpoonright V_\lambda \in j(A_y)$ and so by definition, $A_y \in U$. As usual, by this point this means the claim holds in X and by correctness it therefore also holds in V. □

We now define $f_y : V_{\lambda+1} \to V_{\lambda+1}$ for a given $y \in V_{\lambda+1}$ by

$$f_y(k) = \begin{cases} x & \text{if } k \in A_y \text{ and } k^+(x) = y, \\ \emptyset & \text{otherwise.} \end{cases}$$

Trivially, it follows that $f_y \in L_\Theta(V_{\lambda+1}, E)$ and if $y \in V_{\lambda+1} \cap X$ then $f_y \in X$.

Claim 5. $y \in V_{\lambda+1} \to j_E(f_y)(j \upharpoonright V_\lambda) = y$

Proof of Claim 5. This is very similar to that of claim 4. We again fix $y \in V_{\lambda+1} \cap X$, and use lemma 5.19 and lemma 5.28 to yield $j \upharpoonright V_\lambda \in j_E(A_y)$ and $(j \upharpoonright V_\lambda)^+(y) = j(y)$. Because $j_E(f_y)(j \upharpoonright V_\lambda = j(f_y)(j \upharpoonright V_\lambda))$, we have that $j(f_y)(j \upharpoonright V_\lambda) = y$ and so the claim holds in X and therefore also in V. □

We can prove Łoś's Theorem for $\mathrm{Ult}_U(N)$. We first require the following:

Claim 6. *Let* $h : V_{\lambda+1} \to N$ *be a function in* N *such that* $\{x \in V_{\lambda+1} \mid h(x) \neq \emptyset\} \in U$. *Then there exists a function* $f \in N$ *such that* $f : V_{\lambda+1} \to N$ *satisfying* $[f]_U \in_U [h]_U$.

Proof of Claim 6. We first establish that the claim holds with the modification that we require only $[f_x]_U \in_U [g]_U$. Assume that $g : V_{\lambda+1} \to N$ is a function in $N \cap X$ such that the set $A = \{x \in V_{\lambda+1} \mid \emptyset \notin g(x) \in V_{\lambda+2}\} \in U$. So $A \in N \cap X$ and because of this we have that $j \upharpoonright V_\lambda \in j(A)$ and therefore

$$\emptyset \neq j(g)(j \upharpoonright V_\lambda) = j_E(g)(j \upharpoonright V_\lambda) \in V_{\lambda+2} \cap X$$

by lemma 5.28 as usual. So we can find an $x \in X \cap j(g)(j \upharpoonright V_\lambda) \in V_{\lambda+1}$ such that $x = j_E(f_x)(j \upharpoonright V_\lambda)$ by claim 5. Due to lemma 5.28 again it follows that

$$j_E(f_x)(j \upharpoonright V_\lambda) = x \in j(g)(j \upharpoonright V_\lambda) = j_E(g)(j \upharpoonright V_\lambda)$$

and thus by claim 3, $[f_x]_U \in_U [g]_U$.

Now assume that h is as assumed in the claim. From the iterative construction of $L(V_{\lambda+1})$, we can construct a class surjection $S : \mathrm{On} \times V_{\lambda+1} \to N$. Since $L(V_{\lambda+1})$ is bijective to $L(V_{\lambda+1}, E)$, and so there is a class surjection $S : \mathrm{On} \times V_{\lambda+1} \to N$ definable over N by a Σ_1-formula with parameters $V_{\lambda+1}$ and E.

We define $g : V_{\lambda+1} \to D$ by $g(y) = \{x \in V_{\lambda+1} \mid \exists \xi \in \mathrm{On}(S(\xi, x) \in h(y))\}$. It follows that $g \in N \cap X$, and because $j(h)(j \upharpoonright V_\lambda) \neq \emptyset$, it follows that $j(g)(j \upharpoonright V_\lambda) \neq \emptyset$ and so $\{x \in V_{\lambda+1} \mid g(x) \neq \emptyset\} \in U$. By the previous part, it therefore follows that there is an $x \in V_{\lambda+1} \cap X$ such that $[f_x]_U \in_U [g]_U$.

We now define two further functions $o : V_{\lambda+1} \to \mathrm{On}$ and $f : V_{\lambda+1} \to N$ by

$$o(y) = \begin{cases} \min\{\xi \in \mathrm{On} \mid S(\xi, f_x(y)) \in h(y)\} & \text{if } f_x(y) \in g(y), \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f(y) = S(o(y), f_x(y)).$$

It follows that $f \in N \cap X$ and it follows from the fact that $[f_x]_U \in_U [g]_U$ that there is a $\xi \in \mathrm{On}$ such that $S(\xi, j(f_x)(j \upharpoonright V_\lambda) \in j(h)(j \upharpoonright V_\lambda)$. So it follows that

$$j(f)(j \upharpoonright V_\lambda) = S(j(o)(j \upharpoonright V_\lambda), j(f_x)(j \upharpoonright V_\lambda)) \in j(h)(j \upharpoonright V_\lambda)$$

and so by claim 1, we conclude that $[f]_U \in_U [g]_U$. □

By the previous claim, we can now prove Łoś's theorem for $\mathrm{Ult}_U(N)$:

Claim 7. *For every formula $\varphi(v_0, ..., v_{n-1})$ in the language of set theory and for all functions $f_0, ..., f_{n-1} : V_{\lambda+1} \to N$ in N, the following are equivalent:*

(i) $\langle Ult_U(N), \in_U\rangle \models \varphi([f_0]_U, ..., [f_{n-1}]_U)$,

(ii) $\{y \in V_{\lambda+1} \mid \langle N, \in\rangle \models \varphi(f_0(y), ... f_{n-1}(y))\} \in U$.

Proof of Claim 7. Recall that the standard proof of Łoś's theorem proceeds by induction on formulae: We prove the result for atomic formulae, then prove the induction step for $\neg$, $\wedge$ and $\exists$. These steps are mostly trivial, with the exception of the (ii)-(i) in the inductive step for $\exists$. We prove this here but omit the remaining steps.

First let $\varphi(v_0, ..., v_n)$ be a formula in the language of set theory and let $f_0, ..., f_{n-1} : V_{\lambda+1} \to N$ be functions in N such that

$$A = \{y \in V_{\lambda+1} \mid \langle N, \in\rangle \models \exists z \varphi(f_0(y), ..., f_{n-1}(y), z)\} \in U$$

We now define the functions $o : V_{\lambda+1} \to \mathrm{On}$ and $h : V_{\lambda+1} \to N$ as follows:

$$o(y) = \begin{cases} \min\{\xi \in \mathrm{On} \mid \langle N, \in\rangle \models \exists z \in V_\xi \varphi(f_0(y), ..., f_{n-1}(y), z)\} & \text{if } y \in A, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$h(y) = \{z \in V_{o(y)} \mid \langle N, \in\rangle \models \varphi(f_0(y), ..., f_{n-1}(y), z)\}.$$

So h is an element on N with $h(y) \neq \emptyset$ for all $y \in A$. It follows by claim 6, there is a function $f : V_{\lambda+1} \to N$ in N such that $[f]_U \in_U [h]_U$. Assume for the sake of a contradiction that (i) doesn't hold. The induction hypothesis gives us that the set

$$B = \{y \in A \mid f(y) \in h(y) \text{ such that } \langle N, \in\rangle \models \neg\varphi(f_0(y), ..., f_{n-1}(y), f(y))\}$$

is a nonempty element of U. This yields a contradiction by the definition of h. $\square$

We can conclude from Łoś's theorem that $i : \langle N, \in\rangle \to \langle \mathrm{Ult}_U(N), \in_U\rangle$ is an elementary embedding.

The following claim establishes some normality properties for U.

Claim 8. *Let $y \in V_{\lambda+1}$. Then the following hold*

(i) If $x \in y$, then $[f_x]_U = [f_y]_U$.

(ii) If $f : V_{\lambda+1} \to N$ is an element of N such that $[f]_U \in_U [f_y]_U$, then there is an $x \in y$ with $[f]_U = [f_x]_U$.

Proof of Claim 8. (i) Let $x \in y$. By claim 5 and the transitivity of $V_{\lambda+1}$, $j_E(f_x)(j \restriction V_\lambda) \in j_E(f_y)(j \restriction V_\lambda)$, and so by claim 3, $[f_x]_U \in_U [f_y]_U$.

(ii) Let $y \in V_{\lambda+1} \cap X$ and let $f : V_{\lambda+1} \to N$ be a function in $N \cap X$ with $[f]_U \in_U [f_y]_U$. Let $x = j(f)(j \restriction V_\lambda) = j_E(f)(j \restriction V_\lambda) \in X$. So by the same claims as in (i), it follows that $x \in j_E(f_y)(j \restriction V_\lambda) = y$ and $j_E(f_x)(j \restriction V_\lambda) = x = j_E(f)(j \restriction V_\lambda)$, and so $[f_x]_U = [f]_U$, so (ii) holds in X and therefore, by correctness, also in V. $\square$

It follows from this that $\langle \mathrm{Ult}_U(N), \in_U\rangle$ is well-founded and extensional. This allows us to take the Mostowski collapse: $\pi : \langle \mathrm{Ult}_U(N), \in_U\rangle \to \langle M, \in\rangle$. By the previous two claims, for all $y \in V_{\lambda+1}$, $\pi([f_y]_U) = y$ holds and so therefore we have that $V_{\lambda+1} \subseteq M$. We therefore have that $(\pi \circ i) : L(V_{\lambda+1}, E) \to M$ is an elementary embedding with $V_{\lambda+1} \subseteq M$. To complete our proof, we need only show that $M = L(V_{\lambda+1}, E)$, and that $(\pi \circ i) \restriction L_\Theta(V_{\lambda+1}, E) = j_E \restriction L_\Theta(V_{\lambda+1}, E)$.

Claim 9. $(\pi \circ i) \restriction V_{\lambda+1} = j_E \restriction V_{\lambda+1}$.

Proof of Claim 9. We define the following for all $y \in V_{\lambda+1} \cap X$.

$$B_y = \{x \in V_{\lambda+1} \mid f_{j_E(y)}(x) = y\}.$$

Observe that $B_y \in D \cap X$, and

$$j_E(B_y) = j(B_y) = \{x \in V_{\lambda+1} \mid j(f_{j_E(y)}(x) = j(y)\}.$$

and by lemma 5.28, we have

$$j(f_{j_E(y)}(j \restriction V_\lambda)) = j_E(f_{j_E(y)}(j \restriction V_\lambda)) = j_E(y) = j(y).$$

so therefore $B_y \in U$ in X. By the correctness properties of X it follows that $B_y \in U$ for all $y \in V_{\lambda+1}$. This gives that $i(y) = [f_{j_E(y)}]_U$ and so by the aforementioned point following from the previous two claims we have that

$$(\pi \circ i)(y) = \pi([f_{j_E(y)}]_U) = j_E(y)$$

holds for all $y \in V_{\lambda+1}$ and so the claim holds. □

This claim gives that $(\pi \circ i)(\lambda) = j_E(\lambda) = j(\lambda) = \lambda$ and therefore by elementarity it follows that $(\pi \circ i)(V_{\lambda+1}) = V_{\lambda+1}$.

Claim 10. $(\pi \circ i) \restriction D = j_E \restriction D$.

Proof of Claim 10. Let $A \in D \cap X$ be fixed and denote by c_A the constant function on $V_{\lambda+1}$ with value A. We have that $c_A \in L_\Theta(V_{\lambda+1}, E) \cap X$ and

$$j_E(c_A)(j \restriction V_\lambda) = j(c_A)(j \restriction V_\lambda) = j(A) = j_E(A)$$

by the usual results. By claim 5, it follows that

$$y \in j_E(A) \leftrightarrow j_E(f_y)(j \restriction V_\lambda) \in j_E(c_A)(j \restriction V_\lambda) \leftrightarrow j(f_y)(j \restriction V_\lambda) \in j(c_A)(j \restriction V_\lambda).$$

If we additionally suppose that $y \in X$, then by claim 3, this is equivalent to

$$[f_y]_U \in_U [c_A]_U \leftrightarrow \pi([f_y]_U) \in (\pi \circ i)(A) \leftrightarrow y \in (\pi \circ i)(A)$$

and so we conclude that $j_E(A) = (\pi \circ i)(A)$ holds in X, and therefore by correctness in V. So in particular, $(\pi \circ i)(E) = E$, and by elementarity $M = N = L(V_{\lambda+1}, E)$. □

Claim 11. $(\pi \circ i) \restriction \Theta = j_E \restriction \Theta$.

Proof of Claim 11. Suppose $A \in D \cap X$ codes a pre-well-ordering of $V_{\lambda+1}$ with order type γ. Then 5.28 gives that $j_E(A)$ codes a pre-well-ordering of $V_{\lambda+1}$ with order type $j_E(\gamma)$ By the correctness of X, it follows that this holds for $A \in D$ instead of $A \in D \cap X$, and therefore by elementarity of $(\pi \circ i)$ and the previous claim, the conclusion holds. □

Claim 12. $(\pi \circ i) \restriction L_\Theta(V_{\lambda+1}, E) = j_E \restriction L_\Theta(V_{\lambda+1}, E)$.

Proof of Claim 12. Define $S : \Theta \times V_{\lambda+1} \to L_\Theta(V_{\lambda+1}, E)$ be the canonical surjection that is Σ_1-definable over $L_\Theta(V_{\lambda+1}, E)$ using E and $V_{\lambda+1}$ as parameters. By 5.28 and by the correctness of X, it follows that for all $\alpha < \Theta$ and for all $x \in V_{\lambda+1}$we have $(\pi \circ i)(S(\alpha, x)) = S(j_E(\alpha), j_E(x))$. By elementarity $(\pi \circ i)(S(\alpha, x)) = S((\pi \circ i)(\alpha), (\pi \circ i)(x))$ for all $\alpha < \Theta$ and $x \in V_{\lambda+1}$, and so the claim follows using the above claims. $\square$

So to conclude, we now have that $(\pi \circ i) : L(V_{\lambda+1}, E) \prec L(V_{\lambda+1}, E)$ is an elementary embedding extending j_E.

$$\begin{tikzcd}
L_\Theta(V_{\lambda+1}, E) \arrow[rr, "j_E"] \arrow[d, "id"'] & & L_\Theta(V_{\lambda+1}, E) \arrow[d, "id"] \\
L(V_{\lambda+1}, E) \arrow[r, "i"'] & \mathrm{Ult}_U(L(V_{\lambda+1}, E)) \arrow[r, "\pi"'] & L(V_{\lambda+1}, E)
\end{tikzcd}$$

$\square$

Proof of Theorem 5.27. We first set $F = (V_{\lambda+1}, E)^\# \in V_{\lambda+2}$. By the definability of sharps and the correctness of X, we have $F \in X$ and $j(F) = F$. We require one final claim:

Claim . $\Theta^{L(V_{\lambda+1},E)} < \Theta^{L(V_{\lambda+1},F)}$

Proof of Claim. First note that $L(V_{\lambda+1}, E) \subseteq L(V_{\lambda+1}.F)$ and this gives that $\Theta^{L(V_{\lambda+1},E)} \leq \Theta^{L(V_{\lambda+1},F)}$. $\Theta^{L(V_{\lambda+1},F)}$ is regular (by an extension of an argument due to Solovay, see [25], pp. 398-399) in $L(V_{\lambda+1}, F)$ and $L(V_{\lambda+1}, E)^\#$ exists and so therefore in $L(V_{\lambda+1}, F)$ we can define an elementary embedding $i : L(V_{\lambda+1}, E) \prec L(V_{\lambda+1}, E)$ satisfying $\mathrm{crit}(i) < \lambda + 2$ and $i(\mathrm{crit}(i)) = \Theta^{L(V_{\lambda+1},F)}$. By elementarity this means that $i(\Theta^{L(V_{\lambda+1},E)}) = \Theta^{L(V_{\lambda+1},E)}$, and therefore $\Theta^{L(V_{\lambda+1},E)} \neq \Theta^{L(V_{\lambda+1},F)}$. $\square$

Finally to complete the proof of the theorem, observe that by lemma 5.22, we have that $\Theta^{L(V_{\lambda+1},E)} \in X$ and $j(\Theta^{L(V_{\lambda+1},E)}) = \Theta^{L(V_{\lambda+1},E)}$By the above claim, lemma 5.23 provides a surjection $s : V_{\lambda+1} \to \Theta^{L(V_{\lambda+1},E)}$ in X with $j(s) \in X$. Therefore by lemma 5.29, there is an elementary $L(V_{\lambda+1}, E) \prec L(V_{\lambda+1}, E)$ that extends $j \restriction V_\lambda$ and fixes E. $\square$

We have thus shown that ultraexact embeddings combined with the existence of sharps yields many strong Icarus sets. For more on this area, see [58] (pp. 23-27).

Another striking result from [58] is the following, for which we omit the proof.

Theorem 5.30. *Let λ be an ultraexacting cardinal and suppose that $V_{\lambda+1}^\#$ exists. Then λ is a limit of cardinals $\bar\lambda$ where there is an IO embedding $j : L(V_{\bar\lambda+1}) \prec L(V_{\bar\lambda+1})$. So in particular, V_λ is a set-model of ZFC satisfying "there is a proper class of I0 embeddings."*

This challenges what may be called the "linear-incremental" view of large cardinals that has long been held, in which adding stronger and stronger large cardinals incrementally increases strength. The presence of an ultraexact cardinal seems to distort the properties of the traditional large cardinals, potentially throwing our sense of order into chaos.

5.5 Exacting Cardinals and Structural Reflection

Here we show that the existence of ultraexact cardinals correspond with the truth of very strong structural reflection principles. This deep connection between structural reflection principles and large cardinals has been explored extensively in the work of Bagaria. We give a brief account of structural reflection principles here, but see [61] for a more in-depth introduction.

As touched upon in chapter 2, the best reflection principle we have as a theorem of ZFC is the Montague-Lévy Reflection Theorem. Stronger principles were explored in the early 1960s by Lévy (see e.g. [62][63]), which he showed to be equivalent to the existence of what we called the "small" large cardinals in chapter 2 - inaccessibles, Mahlo cardinals, hyper-Mahlo cardinals etc. The problem was that continuing up the hierarchy with new reflection properties proved to be difficult.

Structural reflection is somewhat different to the classical reflection of Lévy et al. These new reflection principles explored by Bagaria et al. were motivated by the following quote of Gödel (from Wang[64] p. 280):

> "The universe of sets cannot be uniquely characterized (i.e., distinguished from all its initial segments) by any internal structural property of the membership relation in it which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number."

The suggestion implicit in this quote is not that the transcendence of V should be interpreted via some Montague-Lévy-type reflection principle, but that it should be seen via some reflection of some of structural properties of the membership relation. That is, instead of reflecting the *theory* of V, we reflect the *structural content* of V.

The general informal principle of structural reflection as given by Bagaria is the following:

Definition 5.31 (Structural Reflection Principle)**.** *[61] For every definable class $\mathcal{C}$ (in the first-order language of set theory, possibly with parameters) of relational structures of the same type there exists an ordinal α that reflects $\mathcal{C}$, i.e., for every A in $\mathcal{C}$ there exists B in $\mathcal{C} \cap V_\alpha$ and an elementary embedding from B into A.*

It is emphasised here that what is reflected is not the formula $\varphi(x)$, but the class of structures defined by $\varphi(x)$. Of course this principle, like the classical reflection principles of Lévy et al., is not necessarily what Gödel had in mind, but the advantage of this type of reflection is that it allows us to formulate reflection principles corresponding to much larger cardinals. For the details of this program, extending upwards from the level of supercompact cardinals, see [61]. For a further exploration of structural reflection, including at lower levels in the large cardinal hierarchy, see [65].

We now return to our original topic of ultraexacting cardinals. As previously mentioned there is a correspondence between ultraexact cardinals and very strong structural reflection principles, the first of which is the following:

Definition 5.32 (Exact Structural Reflection). *[60] Let $\mathcal{L}$ be a first-order language containing unary predicate symbols $\vec{P} = \langle \dot{P}_m \mid m \in \omega \rangle$.*

(i) Let $\vec{\lambda} = \langle \lambda_m \mid m \in \omega \rangle$ be a sequence of cardinals with supremum λ. An $\mathcal{L}$-structure A has type $\vec{\lambda}$ with respect to $\vec{P}$ if the universe of A has rank λ and $\mathrm{rank}(\dot{P}_m^A) = \lambda_m$ for all $m \in \omega$.

(ii) Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures and let $\vec{\lambda} = \langle \lambda_m \mid m \in \omega \rangle$ be a strictly increasing sequence of cardinals. Then let $ESR_{\mathcal{C}}(\vec{\lambda})$ denote the statement that for all structures B in $\mathcal{C}$ of type $\langle \lambda_{m+1} \mid m \in \omega \rangle$, there exists an elementary embedding of a structure A in $\mathcal{C}$ into B where A is of type $\langle \lambda_m + \mid m \in \omega \rangle$.

(iii) Suppose Γ is a definability class (i.e one of the Σ_n or Π_n [60], and P a class. Then we let $\Gamma(P) - ESR(\vec{\lambda})$ denote the statement that $ESR_{\mathcal{C}}(\vec{\lambda})$ holds for every class $\mathcal{C}$ of structures of the same type that is Γ-definable with parameters in P.

The following theorem shows the connection between exactness and exact structural reflection. See section 9 of [60] for more in depth results in this area.

Theorem 5.33. *[58] Suppose $n > 0$ is a natural number and $\vec{\lambda} = \langle \lambda_m \mid m \in \omega \rangle$ a strictly increasing sequence of cardinals with supremum λ.*

(i) The cardinal λ_0 is n-exact for $\langle \lambda_{m+1} \mid m \in \omega \rangle$ iff $\Sigma_{n+1}(\{\lambda\}) - ESR(\vec{\lambda})$ holds.

(ii) If λ_0 is parametrically n-exact for $\langle \lambda_{m+1} \mid m < \eta \rangle$, then $\Sigma_{n+1}(V_{\lambda_0} \cup \{\lambda\}) - ESR(\vec{\lambda})$ holds.

We can strengthen this to produce a stronger structural reflection principle corresponding to ultraexact cardinals. We first require the following definition:

Definition 5.34 (Square root). *Let λ be a limit ordinal and suppose $f : V_\lambda \to V_\lambda$ is a function. Then a square root of f is a function $r : V_\lambda \to V_\lambda$ such that $r^+(r) = f$.*

These functions find some use in the study of rank-into-rank embeddings (see e.g. section 2 of [42]).

Definition 5.35 (Square Root Exact Structural Reflection). *[58]*

(i) *Let $\mathcal{L}$ be a first-order language containing unary predicate symbols $\vec{P} = \langle \dot{P}_m \mid m \in \omega\rangle$. Suppose $\mathcal{C}$ is a class of $\mathcal{L}$-structures and $\vec{\lambda} = \langle \lambda_m \mid m \in \omega\rangle$ is a strictly increasing sequence of cardinals with supremum λ. We denote by $\sqrt{ESR_{\mathcal{C}}}(\vec{\lambda})$ the statement that there is a function $f : V_\lambda \to V_\lambda$ with the property that for every structure B in $\mathcal{C}$ of type $\langle \lambda_{m+1} \mid m \in \omega\rangle$, there exists structure A in $\mathcal{C}$ of type $\langle \lambda_m \mid m \in \omega\rangle$ and a square root r of f such that $r \restriction A : A \prec B$.*

(ii) *Suppose Γ is a definability class and P is a class. Then $\Gamma(P) - \sqrt{ESR}(\vec{\lambda})$ denotes the statement that $\sqrt{ESR_{\mathcal{C}}}(\vec{\lambda})$ holds for every class $\mathcal{C}$ of structures of the same type that is Γ-definable with parameters in P.*

The following gives an ultraexact analogue of theorem 5.32:

Lemma 5.36. *Suppose $n > 0$ is a natural number and $\vec{\lambda} = \langle \lambda_m \mid m \in \omega\rangle$ a strictly increasing sequence of cardinals with supremum λ.*

(i) *Suppose λ_0 is n-ultraexact for $\langle \lambda_{m+1} \mid m \in \omega\rangle$, then $\Sigma_{n+1}(\{\lambda\}) - \sqrt{ESR}(\vec{\lambda})$ holds.*

(ii) *Suppose λ_0 is parametrically n-ultraexact for $\langle \lambda_{m+1} \mid m \in \omega\rangle$, then $\Sigma_{n+1}(V_{\lambda_0} \cup \{\lambda\}) - \sqrt{ESR}(\vec{\lambda})$ holds.*

The following lemma provides a converse to (i) of the above lemma. We first require a preliminary definition:

Definition 5.37. *Let $\mathcal{U}_n$ denote the class of $\mathcal{L}$-structures A for which there exists a strictly increasing sequence $\vec{\lambda} = \langle \lambda_m \mid m \in \omega\rangle$ of cardinals with supremum λ such that the following hold:*

(i) *The reduct of A to the language of set theory is $\langle V_\lambda, \in\rangle$.*

(ii) *There is a $\lambda < \zeta \in C^{(n)}$, an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda, \dot{f}^A\} \subseteq X$ and a bijection $\tau : X \to V_\lambda$ with $\tau(\lambda) = \langle 0, 0\rangle$, $\tau(x) = \langle 1, x\rangle$ for all $x \in V_\lambda$, and*

$$x \in y \leftrightarrow \tau(x) \dot{E}^A \tau(y)$$

for all $x, y \in X$.

Lemma 5.38. *Suppose $\vec{\lambda} = \langle \lambda_m \mid m \in \omega\rangle$ is a strictly increasing sequence of cardinals such that $\sqrt{ESR_{\mathcal{U}_n}}(\vec{\lambda})$ holds. Then λ_0 is n-ultraexact for $\langle \lambda_{m+1} \mid m \in \omega\rangle$.*

We omit the proof, which is in [58] (pp. 29-30).

An interesting aspect of ultraexacting cardinals and structural reflection is the many connections to the choiceless large cardinals we discussed in the previous chapter. Here we give a sample of these results, without proof. All of these, along with the proofs and further detail, can be found in [58], section 5.

The following shows that Reinhardt cardinals are parametrically ultraexact for all $n \in \omega$.

Theorem 5.39. *Suppose $j : V \prec V$ is a nontrivial elementary embedding with critical sequence $\langle \lambda_m \mid m \in \omega \rangle$. Then λ_0 is parametrically n-ultraexact for $\langle \lambda_{m+1} \mid m \in \omega \rangle$, for all $n > 0$.*

As a corollary, we obtain the following.

Corollary 5.40. *Suppose $j : V \prec V$ is a nontrivial elementary embedding with critical sequence $\langle \lambda_m \mid m \in \omega \rangle$ and $\lambda = \lambda_\omega$. Then $\sqrt{ESR_{\mathcal{C}}})(\vec{\lambda})$ holds for all classes $\mathcal{C}$ of $\mathcal{L}$-structures definable with parameters in $V_{\lambda_0} \cup \{\lambda\}$.*

We also have a stronger, analogous result for super Reinhardt cardinals.

Theorem 5.41. *Suppose κ is a super Reinhardt cardinal. Then there exists a proper class of sequences $\vec{\lambda} = \langle \lambda_m \mid m \in \omega \rangle$ where $\sqrt{ESR_{\mathcal{C}}})(\vec{\lambda})$ holds for all classes $\mathcal{C}$ of $\mathcal{L}$-structures definable with parameters in V_κ.*

Finally, we give the following theorem, linking Berkeley cardinals to ultraexact cardinals.

Theorem 5.42. *Let $n \in \omega$, and suppose δ is a Berkeley cardinal. Then unboundedly many cardinals below δ are parametrically n-ultraexact for some sequence of cardinals below δ.*

6 Conclusions

6.1 The HOD Conjecture

In chapter 3, we introduced three different variations on the HOD conjecture. The weakest (and therefore most plausible), was the following:

Conjecture 3.59 (Weak HOD Conjecture)**.**

$$\textit{ZFC} + \exists\kappa \text{ (ext.) with a huge } \lambda > \kappa \vdash \text{ HOD hyp.}$$

What makes this version so compelling is that it is formalisable as a Σ^0_1 formula. This means that it must be decidable - if it is independent, it is false. The significance of this is appreciated if it is contrasted to the background of set-theoretic research over the past several decades - where many investigations have resulted in little more than independence results. For example, the case of CH, one of the most fundamental questions a set theorist may ask, is possibly the most salient: It is independent of even the strongest large cardinal hypotheses. We have previously discussed increasingly strong large cardinals as categorising increasingly powerful proving ability, but it is a striking result that for CH, we haven't yet found a level at which CH is decided (still, CH is decided by hypotheses of other flavours such as $V = L$, PFA or $\diamond$).

Returning to the weak HOD conjecture, as it is decidable this means that if the necessary large cardinals exist, the close side of the HOD dichotomy must hold, i.e HOD must be close to V.

Here is another angle from which to look at large cardinals and how close models are to V: Suppose we construct a model for some large cardinal property. Eventually, we isolate a much stronger large cardinal property, in a sense reflecting properties of the universe that don't hold in our first model. But this means that there is some way in which our first model is far from V. However, as we progress up the hierarchy, something unexpected happens:

Corollary 3.65. *Suppose N is a weak extender model for δ supercompact and $\gamma > \delta$ is a singular cardinal. Then*

(i) $N \models$ *"γ is a singular cardinal" and*

(ii) $\gamma^+ = (\gamma^+)^N$.

What this effectively says is that a weak extender model N is already close to V, whether or not further large cardinals are supposed to exist in V. This is closely related to the so-called "universality" of weak extender models.

Theorem 6.1 (Universality)**.** *[50](p. 306) Suppose that N is a weak extender model of the supercompactness of κ, and suppose that $\alpha > \kappa$ and we have the following elementary embedding*

$$j : N \cap V_{\alpha+1} \prec N \cap V_{j(\alpha)+1}$$

where $crit(j) \geq \kappa$. Then $j \in N$.

This result has been used to formulate theorems such as the following:

Theorem 6.2. *[50] If N is a weak extender model of the supercompactness of κ and $\gamma > \kappa$ is supercompact, then N is a weak extender model of the supercompactness of γ.*

Theorem 6.3. *[50] If N is a weak extender model of the supercompactness of κ and $\gamma > \kappa$ is extendible, then γ is extendible in N.*

Results like this have been formulated for cardinals all the way up the traditional large cardinal hierarchy. Weak extender models for compactness are so close to V, that they inherit all traditional large cardinals from V, many of which are much stronger than supercompactness. The struggle for inner model theorists is finding such a model, but the progress made here is in establishing that there is no need to go further.

Finally, we give the following wider selection of equivalences related to the HOD dichotomy.

Theorem 6.4. *[50](p. 307) Suppose there is an extendible cardinal κ. Then the following are equivalent:*

(i) The HOD hypothesis holds.

(ii) There is a regular cardinal $\gamma \geq \kappa$ which is not ω-strongly measurable in HOD.

(iii) No regular cardinal $\gamma \geq \kappa$ is ω-strongly measurable in HOD.

(iv) There is a cardinal $\gamma \geq \kappa$ such that $(\gamma^+)^{HOD} = \gamma^+$.

(v) HOD is a weak extender model of the supercompactness of κ.

(vi) There is a weak extender model N for the supercompactness of κ such that $N \subseteq HOD$.

(vii) There is a weak extender model N for the supercompactness of κ such that $N \models V = HOD$

6.2 The Ultimate-L Conjecture

It is natural to expect that if sufficient large cardinals exist, there is a weak extender model for the supercompactness of κ, and it is also natural to expect this model to satisfy $V = \text{HOD}$. What we lack, according to [50] is a "specific target" when it comes to isolating a particular inner model.

As has been discussed, the inner model program began with Gödel's L. Kunen later elevated this by constructing inner models for measurability and weak compactness. This was later extended to the level of strong cardinals by Mitchell and Steel, and Woodin cardinals by Neeman[67] (see [68] for a history of inner models, or [69] for an earlier history).

As our models M become more complex, the associated axiom $V = M$ become more and more difficult to express. However, notably $V = L$ can be characterised using the following lemma, which means that effectively we could have stated the axiom $V = L$ without even constructing L itself. This idea is motivating in our approach to higher models, since actually constructing them will be very complicated, but perhaps there will be an equivalent statement to the assertion that such a model exists which, like $V = L$, can be stated without an explicit construction of M. Recall that ZFC− is the theory of ZFC without the power set axiom (see [70] for an exploration of this).

Definition 6.5. *For all ordinals α, let*

$$N_\alpha = \bigcap\{M \mid Trans(M) \wedge M \models ZFC- \wedge On^M = \alpha\}.$$

Lemma 6.6. *[50](p. 308) The following are equivalent:*

(i) $V = L$.

(ii) For each Σ_2-sentence φ, if φ holds in V, then there exists a countable ordinal α such that $N_\alpha \models \varphi$.

Similarly we can construct an axiom V = Ultimate-L without explicitly constructing the inner model = Ultimate-L. For much more on Ultimate-L, including the motivations leading to the following, see Woodin's "In Search of Ultimate-L[15]. Firstly, recall the following:

Definition 6.7 (Universally Baire set)**.** *[1](p. 623) A set $A \subset \mathbb{R}$ is called universally Baire if for any compact Hausdorff space X and every continuous function $f : X \to \mathbb{R}$ the set $f^{-1}(A)$ has the Baire property in X.*

Definition 6.8. *[50] V = Ultimate-L is the conjunction of the following:*

(i) There is a proper class of Woodin cardinals.

(ii) For each Σ_2-sentence φ, if φ holds in V, then there exists a universally Baire set $A \subseteq \mathbb{R}$ such that $HOD^{L(A,\mathbb{R})} \models \varphi$.

One of the consequences of the existence of a proper class of Woodin cardinals is projective determinacy. Let $A \subseteq {}^\omega\omega$. One can associate with A a two-player game of infinite length. Players I and II take turns choosing natural numbers, eventually generating a set $x \in {}^\omega\omega$. Player I wins if $x \in A$, Player II wins otherwise. Such a game, and therefore the subset A, is said to be determined if either of the players has a winning strategy.

Definition 6.9 (Projective Determinacy)**.** *Every projective subset of ${}^\omega\omega$ is determined.*

Here are some further consequences:

Theorem 6.10. *[50](p. 309) Assume V = Ultimate-L. Then the following hold:*

(i) CH holds.

(ii) The Ω conjecture holds (see [71].

(iii) $V = HOD$

(iv) V is the minimum universe of the Generic-Multiverse (see [72]).

In a similar vein to what we did for the HOD conjecture, we have the following weakening:

Conjecture 6.11 (Weak Ultimate-L conjecture)**.**

$$ZFC\ + \exists\kappa\ (ext.)\ \text{with a huge}\ \lambda > \kappa\ \vdash \exists N (\text{a weak extender model of the supercompactness of}\ \kappa)\ \text{such that}\ N \models V = \text{Ultimate-}L.$$

Much like the weak HOD conjecture, this is a Σ^0_1 statement and so can't be undecidable. It is immediate from the definitions and equivalences given above that the weak Ultimate-L conjecture implies the weak HOD conjecture. This implication aligns the two conjectures, and paints a broader picture of what the close side of the HOD dichotomy could look like. This is one of the "two futures" given in Large Cardinals Beyond Choice[50].

The other future concerns the choiceless hierarchy we built up in chapter 4. If such axioms are consistent, then the weak Ultimate-L conjecture must fail, and our best hope of pushing us into the close side of the dichotomy is lost. Furthermore, if Berkeley cardinals are consistent, then it may be possible to construct new large cardinal axioms that push us into the far side of the HOD dichotomy, providing us with some form of higher anologue of $0^{\#}$.

To summarise, the HOD dichotomy gives us two radically different potential futures. The first, where HOD is close to V, represents order. It has several nice properties like CH, and cannot be refuted by a traditional large cardinal. It would result in us having an "L-like paradise"[50], within which deep analysis would be possible. It could help make a case for the axiom $V =$ Ultimate-L, which would be more justifiable than the traditional $V = L$. Contrast this to the second future, in which not only would Ultimate-L fail, but inner model theory itself may be seen to have failed. If one cannot prove there is a weak extender model for supercompactness yielding $V = \text{HOD}$, then what would remain to be achieved with inner models? With the consistency of the choiceless hierarchy established, many new questions would be opened, most of which we don't yet seem to have the tools to answer, or worse still perhaps they will be found to be completely undecidable.

We must make clear at this stage, that this second future doesn't necessarily follow from the consistency of choiceless cardinals, but their consistency does open up a potential path to establishing this. It has been a few decades now, and despite the audacious strength of the cardinals at or above a Reinhardt cardinal, no further consistency proof has been proved. Either a proof much deeper than Kunen's is possible, and we have yet to find it, or perhaps the way forward is via consistency. Perhaps this pushes us into the chaotic far side of the HOD dichotomy.

6.3 Exacting Cardinals and the HOD Conjecture

In this section, we use the concepts covered in chapter 5 to discuss a potential failure of the HOD Conjecture.

Definition 6.12. *A cardinal κ is $C^{(n)}$-Reinhardt if there is a nontrivial elementary embedding $j : V \prec V$ with critical point $\kappa \in C^{(n)}$.*

In the final section of [58], the following result is proved. We will not cover the proof here because it involves forcing.

Theorem 6.13. *If there is a $C^{(3)}$-Reinhardt cardinal and a supercompact cardinal greater than the supremum of the critical sequence, then there exists a set-sized model of ZFC + "there is an ultraexacting cardinal that is a limit of extendible cardinals".*

The following conclusions about extendible and ultraexact cardinals are drawn at the end of [58] (see for proof).

Theorem 6.14. *Let T be the theory ZFC + "there exists an exacting cardinal above an extendible cardinal". Then*

(i) $T \vdash$ *"there is a huge cardinal above an extendible cardinal and all sufficiently large regular cardinals are ω-strongly measurable in HOD."*

(ii) PA + con(T) disproves the weak HOD conjecture and therefore also the Ultimate-L conjecture.

(iii) Choiceless NBG + "there is a $C^{(3)}$-Reinhardt cardinal and a supercompact cardinal greater than the supremum of the critical sequence" proves con(T).

Where does all of this leave the HOD conjecture? Both sides of the dichotomy remain possible, and we could still find ourselves in either of the two divergent futures. At this point, what exacting cardinals add to the mix is some further evidence for the "far", chaotic side. This is after a period in which "Many people thought, until now, that the HOD Conjecture was probably true" (Bagaria†). Establishing a reasonable justification for the consistency of exacting cardinals will prove difficult, if not impossible, and so these results do not close the dichotomy. They do, however, reveal some complexities hidden below the surface that were hitherto unknown. To close, we quote Aguilera:

> "Maybe the structure of infinity is more intricate than we thought, and this warrants deeper and more careful exploration."
>
> "Maybe this is just the beginning."

Clearly then, there remains much more to be discovered.

†Both Aguilera and Bagaria were interviewed for [73], a rather oversimplified popular science article on the contents of [58].

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