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Dynamical Tropicalisation

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Abstract
We analyse the dynamics of the pullback of the map $z \longmapsto z^m$ on the complex tori and toric varieties. We will observe that tropical objects naturally appear in the limit, and review several theorems in tropical geometry.

Keywords Equidistribution · Tropicalisation · Tropical currents

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1 Introduction

In this article, we employ several key ideas from tropical geometry to analyse the dynamics of the pullback of the map

$$\begin{aligned} \Phi_m : (\mathbb{C}^*)^n &\longrightarrow (\mathbb{C}^*)^n \\ (z_1, \dots, z_n) &\longmapsto (z_1^m, \dots, z_n^m), \end{aligned}$$

as the positive integer $m \to \infty$. This analysis, in turn, lets us review certain aspects of tropical geometry from a dynamical standpoint. To start with, let us examine the case $n = 1$. Observe that for *any* $z \in \mathbb{C}^*$, the family of sets $\left\{\Phi_m^{-1}(z)\right\}$ *converges* towards a uniform distribution of points on the unit circle S^1 as $m \to \infty$. Formally, this observation can be formulated as the following *equidistribution theorem*: for any $z \in \mathbb{C}^*$,

$$\frac{1}{m}\Phi_m^*(\delta_z) := \frac{1}{m}\sum_{\Phi_m(a)=z} \delta_a \longrightarrow \mu(S^1), \quad \text{as } m \to \infty,$$

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where δ_z is the Dirac measure at z, $\mu(S^1)$ is the Haar measure on S^1, and the limit is in the weak sense of measures. Brolin's remarkable theorem [12] is a generalisation of this observation: for *any* monic algebraic map of $f : \mathbb{C} \longrightarrow \mathbb{C}$ of degree $d \geq 2$, there exists a (harmonic) measure μ_f, such that for any *generic* point $z \in \mathbb{C}$,

$$\frac{1}{d^k}(f^k)^*(\delta_z) \longrightarrow \mu_f, \quad \text{as } k \to \infty,$$

where $f^k := f \circ \cdots \circ f$ is the k-fold composition. This result was extended to $\mathbb{P}^1$ for polynomials in [51] and for rational maps in [38]. Here the genericity of a point in $\mathbb{C}$ and $\mathbb{P}^1$ means that z is outside an invariant set of cardinality less than or equal to 1 and 2, respectively.

In complex dynamics, one naturally seeks the generalisation of Brolin's theorem in higher dimensions and codimensions, and the question can be suitably formulated in the language of *currents*, where one explores weak limits of *pullback* of currents after a correct normalisation. Informally, currents on a complex smooth manifold X are continuous functionals acting on the space of smooth forms with compact support and of the appropriate (bi-)degree. For instance, an algebraic subvariety $Z \subseteq X$ of dimension p defines an *integration current* $[Z]$, which is of bidimension (p, p) and acts on the smooth forms with compact support of bidegree (p, p) by integration:

$$\langle [Z], \varphi \rangle := \int_{Z_{\text{reg}}} \varphi.$$

Generally, given a smooth algebraic variety X, $f : X \longrightarrow X$ a holomorphic endomorphism of algebraic degree d, and $Z \subseteq X$ an algebraic subvariety of dimension p, one is interested in analysing (the existence of) the weak limit of

$$\frac{1}{d^{(n-p)k}}(f^k)^*[Z] := \frac{1}{d^{(n-p)k}}[(f^k)^{-1}(Z)],$$

as $k \to \infty$. We recall basic definitions in the theory of currents in Sect. 2.

On the tropical geometry side, algebraic subvarieties of the torus are degenerated to obtain their *tropicalisation*. For instance, a tropicalisation can be obtained by the *logarithm map*

$$\text{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \quad (z_1, \ldots, z_n) \longmapsto (-\log|z_1|, \ldots, -\log|z_n|).$$

Given an algebraic subvariety $Z \subseteq (\mathbb{C}^*)^n$, the set $\text{Log}(Z)$ is called the *amoeba of* Z. By Bergman's theorem [9] there exists a close subset of $\mathbb{R}^n$ such that

$$\frac{1}{\log|t|}\text{Log}(Z) \longrightarrow \mathcal{C}, \quad \text{as } |t| \to \infty,$$

where the limit is in the Hausdorff metric in the compact sets of $\mathbb{R}^n$, and t is a complex parameter. It is further shown in tropical geometry that $\mathcal{C}$ can be equipped with a structure of a *tropical cycle*; see Definition 3.1 and Sect. 5. With such an induced structure, we set $\mathcal{C}$ to be the tropicalisation of Z, denoted by $\operatorname{trop}(Z)$.

Note that for any point $z \in (\mathbb{C}^*)^n$,

$$\frac{1}{\log|t|}\operatorname{Log}(\{z\}) \longrightarrow \{0\}, \quad \text{as} |t| \to \infty,$$

which corresponds to the above-mentioned equidistribution theorem. More generally, our first theorem shows how naturally tropical objects emerge as geometric objects in holomorphic dynamics.

Theorem A *Let $Z \subseteq (\mathbb{C}^*)^n$ be an irreducible subvariety of dimension p, then*

$$\frac{1}{m^{n-p}}\Phi_m^*[Z] \longrightarrow \mathscr{T}_{\operatorname{trop}(Z)}, \quad \text{as } m \to \infty,$$

where $\mathscr{T}_{\operatorname{trop}(Z)}$ is the complex tropical current *associated to* $\operatorname{trop}(Z)$.

A complex tropical current $\mathscr{T}_{\mathcal{C}}$ is a closed current of bidimension (p, p), associated to the tropical cycle $\mathcal{C} \subseteq \mathbb{R}^n$ of pure dimension p, and with support $\operatorname{Log}^{-1}(|\mathcal{C}|)$. Here $|\mathcal{C}|$ denotes the support of $\mathcal{C}$ which is obtained by forgetting its polyhedral structure. Complex tropical currents were introduced in [5], and we recall their definition in Sect. 4.

Noting that the pullback can be extended to more general currents, we introduce the following definition.

Definition 1.1 Let $\mathscr{T}$ be a closed positive current of bidimension (p, p) on $(\mathbb{C}^*)^n$. We define the *dynamical tropicalisation* of $\mathscr{T}$ as

$$\lim_{m\to\infty} \frac{1}{m^{n-p}}\Phi_m^*(\mathscr{T}),$$

when the limit exists.

Theorem A, therefore, states that the dynamical tropicalisation of an integration current along a subvariety of the torus yields the tropical current associated to the tropicalisation of that subvariety. Even though the preceding theorem is stated only in $(\mathbb{C}^*)^n$, we require a passage to *toric varieties* to provide a proof. Recall that a toric variety is an irreducible variety X such that

(a) $(\mathbb{C}^*)^n$ is a Zariski open subset of X, and
(b) the action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on X.

The action of $(\mathbb{C}^*)^n$ on X partitions X into *orbits*, and we will observe in Sect. 5 that the continuous extension of the endomorphism Φ_m from $(\mathbb{C}^*)^n$ to X gives rise to an equidistribution theorem of points within each orbit; see Proposition 5.1. The above-mentioned tropicalisation corresponds to the tropicalisation with respect to the *trivial valuation*. In essence, this tropicalisation captures the (exponential) directions where a subvariety $Z \subseteq (\mathbb{C}^*)^n$ approaches infinity, and there always exists a toric variety X that can contain those directions at infinity so that

(a) the closure of Z in X is compact, and
(b) the the boundary divisors $X \setminus (\mathbb{C}^*)^n$ intersect the closure of Z in X properly.

This closure is called the *tropical compactification* of Z in X; see Definition 5.6. In Subection 7.1, we extend Theorem A to toric varieties:

Theorem B *Let $Z \subseteq (\mathbb{C}^*)^n$ be a an irreducible subvariety of dimension p, and $\bar{Z}$ be the tropical compactification of Z in the smooth projective toric variety X. Then,*

$$\frac{1}{m^{n-p}} \Phi_m^*[\bar{Z}] \longrightarrow \overline{\mathcal{T}}_{\mathrm{trop}(Z)}, \quad as\ .m \to \infty,$$

where $\Phi_m : X \longrightarrow X$ is the continuous extension of $\Phi_m : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$, and $\overline{\mathcal{T}}_{\mathrm{trop}(Z)}$ is the extension by zero of $\mathcal{T}_{\mathrm{trop}(Z)}$ to X.

Theorem B provides a vivid picture of dynamical tropicalisation which is illustrated in Fig.1. In Sect. 7.2, we observe that dynamical tropicalisation of Poincaré–Lelong equation, see Theorem 2.3, yields the following dynamical version of Kapranov's theorem, which informally states that the tropicalisations of an algebraic hypersurface $V(f)$ coincides with the tropical variety of the tropicalisation of the polynomial f:

Theorem C *Let $V \subseteq (\mathbb{C}^*)^n$, be an irreducible algebraic hypersurface, given as the variety of the polynomial $f(z) = \sum_\alpha c_\alpha z^\alpha \in \mathbb{C}[z]$. We have that*

$$m^{-1} \Phi_m^*[V(f)] \longrightarrow \mathcal{T}_{V_{\mathrm{trop}}(\mathfrak{q})},$$

where $\mathfrak{q} = \max_\alpha\{\langle -\alpha, \cdot \rangle\} : \mathbb{R}^n \longrightarrow \mathbb{R}$, *is the* tropicalisation of f *and* $V_{\mathrm{trop}}(\mathfrak{q})$ *is the* tropical variety associated *to* $\mathfrak{q}$*; see Definition 3.2.*

In the beginning of the introduction, we mentioned the Brolin's theorem and some generalisations. Let us now recall a equidistribution Conjecture of Dinh and Sibony, as well as some of the known special cases. This will allow for viewing Theorems A and B within the larger context of complex dynamics. For an overview of the current trends in Complex Dynamics in Higher Codimensions see Dinh's ICM 2018 survey [23], and Dujardin's ICM 2022 survey [31] on Geometric Methods in Holomorphic Dynamics.

Let $\mathcal{H}_d(\mathbb{P}^n)$ denote the set of holomorphic endomorphisms of degree d on $\mathbb{P}^n$, and assume that $d \geq 2$. The *Green current* of $f \in \mathcal{H}_d(\mathbb{P}^n)$, is defined by

$$\mathcal{T}_f := \lim_{k\to\infty} \frac{1}{d^k}(f^k)^*(\omega),$$

where ω is the Fubini–Study form cohomologous to a hyperplane in $\mathbb{P}^n$.

Conjecture D (Dinh–Sibony [23, 27]) For any $f \in \mathcal{H}_d(\mathbb{P}^n)$, any integer p with $1 \leq p \leq n-1$, and *generic* subvariety $Z \subseteq \mathbb{P}^n$ of dimension p, we have the weak convergence of positive closed currents

$$\frac{1}{\deg Z} \frac{1}{d^{(n-p)k}} (f^k)^*[Z] \longrightarrow \mathcal{T}_f^{n-p}, \quad \text{as } k \to \infty,$$

where $\mathcal{T}_f^{n-p} = \mathcal{T}_f \wedge \cdots \wedge \mathcal{T}_f$, is the $(n-p)$-fold *wedge product* of the Green current of f, and in particular, the limit only depends on f.

Here are the special known cases of the above conjecture.

(a) The case $p = 0$ is shown by Fornæss and Sibony [36], Briend and Duval [11], Dinh and Sibony [24].
(b) The case $p = n - 1$ by Fornæss and Sibony for generic maps [35], and by Favre and Jonsson for any map in dimension 2 in [34].
(c) When $p = n - 1$ was proved by Dinh and Sibony in [25].
(d) In any dimension for generic maps $f \in \mathcal{H}_d(\mathbb{P}^n)$ by Dinh and Sibony in [26].

The above conjecture also predicts exponential rate of convergence which is studied in several works, see for instance [26, 48]. There are also non-archimedean versions of the Brolin's theorem and we refer the reader to works Rivera-Letelier [66], Baker and Rumley [7], and to Johnsson's comprehensive survey of results and references [49].

We remark further that

(a) The map Φ_d is not a generic map in $\mathcal{H}_d(\mathbb{P}^n)$ in the sense of Item (d) above. Furthermore, comparing our result to the above conjecture implies that generic subvarieties of the projective space, up to the degree, have the same tropicalisation. This is not a contradiction, in fact, it is the main result of [67] for which we provide two other explanations in Sect. 7.3 and will observe that the *Julia sets* of Φ_m relate to the support of *Bergman fan of uniform matroids.*
(b) The map Φ_m is indeed versatile. This map was already employed in [65] to find the *logarithmic indicators* of *plurisubharmonic functions* with logarithmic growth. Moreover, Fujino in [39] used Φ_m to prove certain vanishing theorems on toric varieties;
(c) In Sect. 7.2, we see that Φ_m^*, after a normalisation, applied to Lelong–Poincaré Eq. 2.3, yields a version of Kapranov's theorem, where an analogue of Maslov dequantisation naturally appears; see Sect. 3. Finally, it is alluded in the proof of Theorem 6.1 that Φ_m^*, after a normalisation preserves the Chow cohomology classes. We will gather further cohomological implications of the latter fact in a subsequent article [4].

In Sects. 2, 3, 4 and 5 of this manuscript, we provide the background required in proving our main theorems. The entirety of Sect. 6 is devoted to the proof of Theorem A. In Sect. 7 we prove Theorems B and C and discuss the implication of the Dinh–Sibony Conjecture in the case of the map Φ_m.

2 Preliminaries of the Theory of Currents

The content of this section is extracted from the Demailly's book [20], which has always been generously publicly available.

Let X be a complex manifold of dimension n. For a non-negative integer k, we consider the space of smooth complex differential forms of degree k with compact support, denoted by $\mathcal{D}^k(X)$, endowed with the inductive limit topology. The *topological dual* to $\mathcal{D}^k(X)$, constitutes the space of currents of dimension k,

$$\mathcal{D}'_k(X) := \left(\mathcal{D}^k(X)\right)',$$

that is, the space of all continuous linear functionals on $\mathcal{D}^k(X)$. Hence, a current $\mathcal{T} \in \mathcal{D}'_k(X)$ acts on any form $\varphi \in \mathcal{D}^k(X)$ and yields $\langle \mathcal{T}, \varphi\rangle \in \mathbb{C}$. We note that when $k = 0$, the space of k-currents is just the space of distributions as in the theory of partial differential equations. The *support* of $\mathcal{T}$, denoted by $\mathrm{supp}(\mathcal{T})$, is the smallest closed subset $S \subseteq X$, such that $\mathcal{T}$ vanishes on its complement. A k-dimensional current $\mathcal{T}$ is *the weak limit* of a sequence of k-dimensional currents $\mathcal{T}_i$ if

$$\lim_{i\to\infty} \langle \mathcal{T}_i, \varphi\rangle = \langle \mathcal{T}, \varphi\rangle, \quad \text{for all } \varphi \in \mathcal{D}^k(X).$$

We say that $\mathcal{T}$ is a *cluster value* of a sequence $\mathcal{T}_i$ if $\mathcal{T}$ is the weak limit of a subsequence of $\mathcal{T}_i$.

Remark 2.1 Note that if $\mathcal{T}$ is the weak limit of the sequence $\mathcal{T}_i$, and the sets $\mathrm{supp}(\mathcal{T}_i)$ converge in the Hausdorff metric to the set S, then $\mathrm{supp}(\mathcal{T}) \subseteq S$. To see this, assume that $z \notin S$, then for any large $i \gg 0$, $\mathcal{T}_i$'s vanish in a small neighbourhood of z, and therefore $z \notin \mathrm{supp}(\mathcal{T})$. As an example for the strict inclusion $\mathrm{supp}(\mathcal{T}) \subsetneq S$, for any $z \in X$, consider the weak limit $\mathcal{T}_i := i^{-1}\delta_z \longrightarrow 0$. We have $S = \mathrm{supp}(\mathcal{T}_i) = \{z\}$, and $\mathrm{supp}(\mathcal{T}) = \varnothing$.

The exterior derivative of a k-dimensional current $\mathcal{T}$ is the $(k-1)$-dimensional current $d\mathcal{T}$ defined by

$$\langle d\mathcal{T}, \varphi\rangle = (-1)^{k+1}\langle \mathcal{T}, d\varphi\rangle, \quad \varphi \in \mathcal{D}^{k-1}(X).$$

The current $\mathcal{T}$ is called *closed* if $d\mathcal{T} = 0$. The duality of currents and forms with compact support induces the following decompositions for the bidegree and bidimension

$$\mathcal{D}^k(X) = \bigoplus_{p+q=k} \mathcal{D}^{p,q}(X), \quad \mathcal{D}'_k(X) = \bigoplus_{p+q=k} \mathcal{D}'_{p,q}(X).$$

The space of smooth differential forms of bidegree (p, p) contains the cone of *positive differential forms*. By definition, a smooth differential (p, p)-form φ is *positive* if

$$\varphi(x)\restriction_S \text{ is a nonnegative volume form for all complex } p-\text{planes } S \subseteq T_x X \text{and } x \in X.$$

A current $\mathcal{T}$ of bidimension (p, p) is *positive* if

$$\langle \mathcal{T}, \varphi\rangle \geq 0 \;\text{ for every positive differential } (p, p) - \text{form } \varphi \text{ on } X.$$

An important class of positive currents on X is obtained by integrating along the complex analytic subsets of X, giving rise to *integration currents.* More precisely, if

Z is a p-dimensional complex analytic subset of X, then by Lelong's theorem [20, Theorem III.2.7] the integration current along Z is the well-defined (p, p)-dimensional current given by

$$\big\langle [Z], \varphi \big\rangle = \int_{Z_{\mathrm{reg}}} \varphi, \quad \varphi \in \mathcal{D}^{p,p}(X).$$

Remark 2.2 For simplicity, we only mention the notion of positive currents. To review *weakly positive, positive* and *strongly positive* forms and currents see [20].

Let us assume further that X is a Kähler manifold, with the Kähler form ω. One can check that all positive currents have signed measure coefficients. Accordingly, for a positive (p, p)-dimensional current $\mathcal{T}$ the *trace measure* of $\mathcal{T}$ is defined as $\mathcal{T} \wedge \omega^p$. Note that the trace measure of any positive current is a positive measure. The local mass of $\mathcal{T}$ on a Borel set $K \subseteq X$, with respect to ω, is given by

$$\|\mathcal{T}\|_K = \int_X \mathbb{1}_K \mathcal{T} \wedge \omega^p.$$

The differentials $\partial, \bar{\partial}$ and the de Rham's exterior derivative are defined on currents by duality and we also have $d = \partial + \bar{\partial}$, and set $d^c = \frac{\partial - \bar{\partial}}{2i\pi}$, to obtain $dd^c = \frac{\partial\bar{\partial}}{i\pi}$. An important family of $(n-1, n-1)$-dimensional, or $(1, 1)$-bidegree, positive currents are currents of the form $dd^c\psi$ where ψ is a *plurisubharmonic* function. Recall that a function $\psi : \Omega \to [-\infty, \infty)$, for $\Omega \subseteq \mathbb{C}^n$ an open set, is called plurisubharmonic if

(a) ψ is not identically $-\infty$ in any component of Ω;
(b) ψ is upper semicontinuous;
(c) for every complex line $L \subseteq \mathbb{C}^n$, $\psi_{\restriction \Omega \cap L}$ is subharmonic on $\Omega \cap L$.

It is well-known that when $u : \mathbb{R}^p \to \mathbb{R}$ is convex and increasing in each variable and $\psi_1, \ldots, \psi_p$ are plurisubharmonic, then

$$u(\psi_1, \ldots, \psi_p)$$

is also a plurisubharmonic function. Moreover, for any holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$, $\log|f|$ is plurisubharmonic, and one has the following equality of currents, for which we provide a tropical version in Sect. 7.2.1.

Theorem 2.3 (Lelong–Poincaré Equation) *Let f be a non-zero meromorphic function on X, and let $\sum m_j Z_j$ be the divisor of f. The function* $\log|f|$ *is locally integrable on X, and*

$$dd^c \log|f| = \sum m_j [Z_j].$$

Given an analytic subset $E \subseteq X$, and a positive closed (p, p)-current $\mathcal{T}$ in $\mathcal{D}'_{p,p}(X \setminus E)$ it is important to know when $\mathcal{T}$ can be extended by zero to a closed positive current

$\overline{\mathcal{T}} \in \mathcal{D}'_{p,p}(X)$. The theorem of El Mir–Skoda asserts that this extension is possible if $\mathcal{T}$ has a finite mass in a neighbourhood of every point of E; see [20, III.2.3]. The *pushforward* and *pullback* of a current are defined as a dual operation to pullback and *pushforward* of differential forms, respectively. The pushforward of a form is defined by integrating along fibers: consider a submersion $f : X \longrightarrow X'$ between the complex manifolds X and X' with respective complex dimensions m and n. Let φ be a differential form of degree k on X, with L^1_{loc} coefficients such that the restriction $f\restriction_{\text{supp}(\varphi)}$ is proper. Then,

$$f_*(\varphi) := \int_{z\in f^{-1}(y)} \varphi(z) \in \mathcal{D}^{k-2(m-n)}(X').$$

In this situation, for $\mathcal{T} \in \mathcal{D}'_{k-2(n-m)}(X)$ one defines $f^*\mathcal{T} \in \mathcal{D}'_k(X)$, given by

$$\langle f^*(\mathcal{T}), \varphi\rangle := \langle \mathcal{T}, f_*(\varphi)\rangle.$$

Note that for an analytic cycle Z, we have $f^*[Z] = [f^{-1}(Z)]$. The set of seminorms

$$|\cdot|_\varphi : \mathcal{D}'_{p,q}(X) \longrightarrow \mathbb{R}, \quad \mathcal{T} \longmapsto |\langle \mathcal{T}, \varphi\rangle|,$$

for all $\varphi \in \mathcal{D}^{p,q}(X)$, defines a topology on $\mathcal{D}'_{p,q}(X)$. With respect to this *weak topology*, the operations

$$\mathcal{T} \longmapsto d\,\mathcal{T}, \quad \mathcal{T} \longmapsto d^c\,\mathcal{T}, \quad \mathcal{T} \longmapsto f_*\mathcal{T}, \quad \mathcal{T} \longmapsto f^*\mathcal{T},$$

are *weakly continuous*. See [20, Sect. I.2.C.3]. We finally recall that the wedge product or intersection of two positive closed currents is not always admissible, as we cannot always multiply measures. The leading theories are due to Bedford–Taylor [8] and Demailly [22] in codimension one, which we will use in Sect. 5. In higher dimensions, the theory has been developed in [2, 26, 28, 29],

3 Tropical Algebra and Tropical Cycles

Tropical geometry can be viewed as a geometry over *max-plus* or *min-plus* algebra. In this article, we choose $(\mathbb{T}, \otimes, \oplus)$, where $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$, and $\otimes$ and $\oplus$ are the usual sum and maximum, respectively. These operations can be obtained from the following multiplication and addition through the *Maslov dequantisation*: for $x, y \in \mathbb{R}$, let $x \otimes_h y := x + y$ and $x \oplus_h y := h \ln(\exp(x/h) + \exp(y/h))$, as $h \to 0$. See [55] for a survey of the topic touching on questions in thermodynamics, classical and quantum physics, and probability. See also [60, 61] and surveys [47, 72] on how Maslov dequantisation and degeneration of amoebas have played a fundamental role in application of tropical geometry to enumerative problems in algebraic geometry. In comparison to Maslov dequantisation, for $z, w \in \mathbb{C}^*$, we can consider $\log|zw| =$

$\log|z| + \log|w|$, and apply $\frac{1}{m}\Phi_m^*$ to $\log|z+w|$ to obtain:

$$\frac{1}{m}\Phi_m^*\big(\log|z+w|\big) = \frac{1}{m}\log|z^m + w^m| \xrightarrow[m\to\infty]{} \max\log\{|z|, |w|\},$$

in the sense of *distributions* or bidegree (0, 0) currents. We use this observation to derive a dynamical version of Kapranov's theorem in Sect. 7.2. In these notes we follow [58] as our main reference of tropical geometry, and refer the reader to [40, 56, 57] for the recent scheme-theoretic development of tropical algebraic geometry.

3.1 Tropical Cycles

Recall that a linear subspace $H \subseteq \mathbb{R}^n$ is called *rational* if it is spanned by a subset of $\mathbb{Z}^n$. A *rational polyhedron* in $\mathbb{R}^n$ is an intersection of finitely many rational half-spaces which are defined by

$$\{x \in \mathbb{R}^n : \langle m, x\rangle \geq c, \text{ for some } m \in \mathbb{Z}^n,\ c \in \mathbb{R}\}.$$

A *rational polyhedral complex* is a complex with only rational polyhedra. The polyhedra in a polyhedral complex are also called *cells.* A *fan* is a polyhedral complex whose cells are all cones. If any cone of a fan Σ also belongs to another fan Σ', then Σ is a *subfan* of Σ'. The *dimension* of a polyhedron is the dimension of the affine subspace of minimal dimension containing it. All the fans and polyhedral complexes considered in this article are rational. For a given polyhedron σ let $\mathrm{aff}(\sigma)$ be the affine span of σ, and H_σ be the translation of $\mathrm{aff}(\sigma)$ to the origin. Assume that τ is a face of codimension one for the p-dimensional polyhedron σ, and $u_{\sigma/\tau}$ is the unique outward generator of the one dimensional lattice $(\mathbb{Z}^n \cap H_\sigma)/(\mathbb{Z}^n \cap H_\tau)$.

Definition 3.1 (Balancing Condition and Tropical Cycles) Let $\mathcal{C}$ be a rational polyhedral complex of pure dimension p. Assume that all the cells of $\mathcal{C}$ of dimension p are weighted with positive integers. We say that $\mathcal{C}$ satisfies the *balancing condition* at τ if

$$\sum_{\sigma\supset\tau} w_\sigma\ u_{\sigma/\tau} = 0, \quad \text{in } \mathbb{Z}^n/(\mathbb{Z}^n \cap H_\tau),$$

where the sum is over all p-dimensional cells σ in $\mathcal{C}$ containing τ as a codimension 1 face, and w_σ is weight of σ. A weighted complex is *balanced* if it satisfies the balancing condition at each of its codimension one cells. A *tropical cycle* or synonymously a *tropical variety* in $\mathbb{R}^n$ is a balanced weighted complex with finitely many cells.

For a tropical variety $\mathcal{C}$ we denote its support by $|\mathcal{C}|$, which is the underlying set of the $\mathcal{C}$ as a polyhedral complex. In the tropical algebra, *tropical polynomials* are obtained by addition and multiplication of the variables and constants in the tropical semiring and we can associate to each tropical polynomial a tropical hypersurface:

Definition 3.2 For a tropical polynomial

$$\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R},$$

$$x \longmapsto \max\{\langle x, \alpha\rangle + c_\alpha\},$$

where $\alpha \in A \subseteq \mathbb{Z}^n_{\geq 0}$, $|A| < \infty$, and $c_\alpha \in \mathbb{R}$, the associated tropical variety to $\mathfrak{q}$, denoted by $V_{\text{trop}}(\mathfrak{q})$, is defined by:

(a) as a set

$$|V_{\text{trop}}(\mathfrak{q})| = \{x \in \mathbb{R}^n : \mathfrak{q}(x) \text{ is not differentiable at } x\};$$

(b) the weights on an $(n-1)$-dimensional cell σ of $|V_{\text{trop}}(\mathfrak{q})|$ is given by the lattice length of $\alpha_1 - \alpha_2$, where $|\sigma| = \{x \in \mathbb{R}^n : \langle \alpha_1, x\rangle + c_{\alpha_1} = \langle \alpha_2, x\rangle + c_{\alpha_2} = \mathfrak{q}(x)\}$.

Given a tropical polynomial $\mathfrak{q}$, one can verify that $V_{\text{trop}}(\mathfrak{q})$ is indeed balanced, [58, Proposition 3.3.2].

4 Complex Tropical Currents

Introduction of different notions of *tropical currents* as bridges between theory of currents and tropical geometry, to the author's understanding, is indebted to [20, 63, 64]. In [63], Passare and Rullgård considered $(S^1)^n$-invariant plurisubharmonic functions of the form $\text{Log}_+^* f = f \circ \text{Log}_+$ for a convex continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, and

$$\text{Log}_+ : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \quad (z_1, \dots, z_n) \longmapsto (\log|z_1|, \dots, \log|z_n|).$$

By Rashkovskii's formula [65], one has

$$\int_{\text{Log}_+^{-1}(V)} (dd^c f \circ \text{Log}_+)^n = n!\, \text{MA}(f)(V),$$

where $\text{MA}(f)$ is the *real Monge-Ampère measure* of f and $V \subseteq \mathbb{R}^n$ is a Borel set. Recall that when f is smooth, we have

$$\text{MA}(f)(V) = \det(\text{Hess}(f))\mu,$$

where μ is the Lebesgue measure in $\mathbb{R}^n$ and the definition can be extended to the non-smooth case by continuity. Lagerberg in [53], introduced the theory of *superforms* and *supercurrents* where he could also decompose the real Monge–Ampère measures into bidegree $(1, 1)$ components, $dd^\sharp f$, to obtain

$$\int_{\text{Log}_+^{-1}(V)} (dd^c f \circ \text{Log}_+)^n = \int_{V \times \mathbb{R}^n} (dd^\sharp f)^n.$$

Moreover, Lagerberg introduced *tropical supercurrents* in codimension one, which were generalised by Gubler in [44] to higher codimensions. Chambert-Loir and Ducros in [17] followed by Gubler and Künnemann in [45] extended supercurrents to a fully-fledged theory on Berkovich spaces, and used them in application to Arakelov theory. Further, in [14], and in [59] intersection theory and Monge–Ampère operators were investigated; see also [15, 18, 30] for several far-reaching extensions of Rashkovskii's formula. In [13], it was proved that the push-forward of the logarithm map can be defined in order to find a correspondence between a cone of certain $(S^1)^n$-invariant closed positive currents on a complex toric variety and closed positive supercurrents on the tropicalisation of that toric variety. Moreover, one can lift, by the pull-back of the logarithm map, the tropical currents in [44] to *complex tropical currents*.

Complex tropical currents were introduced in [5] with a geometric representation which made it convenient for generalizing Demailly's important example of an *extremal* current that is not an integration current along any analytic set. Recall that this current is given by $dd^c \max\{x_0, x_1, x_2\} \circ \mathrm{Log}_+ \in \mathcal{D}'_{2,2}(\mathbb{P}^2)$, and equals the (complex) tropical current associated to the tropical line in $\mathbb{R}^2$; see [21]. The extremality results in [5] were subsequently improved and extended to toric varieties by Huh and the author in [6], and tropical currents were used to find a non-trivial example of a positive closed current on a smooth projective toric variety which refutes a strong version of the Hodge conjecture for positive currents. Thereafter, Adiprasito and the author in [1] proposed a family of tropical currents which are counter-example to the aforementioned conjecture in any dimension and codimension greater than one. Moreover, tropical currents were also used in application to higher convexity problems and Nisse–Sottile conjecture [62], as well as finding a family of peculiar currents which cannot be regularised to obtain mollified currents with smooth boundaries; see [1, Theorem D].

In all the above-mentioned works though, the notion of tropicalisation of integration currents along algebraic varieties was absent, which is the topic of this article.

4.1 Tropical Currents

In this subsection, we recall the definition of tropical currents, and their basic properties. We refer the reader to [5, 6] for more details. Let N be a finitely generated free abelian group, we define

$$\begin{aligned} T_N &:= \text{the complex algebraic torus } \mathbb{C}^* \otimes_{\mathbb{Z}} N, \\ S_N &:= \text{the compact real torus } S^1 \otimes_{\mathbb{Z}} N, \\ N_{\mathbb{R}} &:= \text{the real vector space } \mathbb{R} \otimes_{\mathbb{Z}} N. \end{aligned}$$

Let $\mathbb{C}^*$ be the group of nonzero complex numbers. As before, the logarithm map is the homomorphism

$$\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \qquad (z_1, \dots, z_n) \longmapsto (-\log|z_1|, \dots, -\log|z_n|),$$

and the *argument map* is

$$\mathrm{Arg} : (\mathbb{C}^*)^n \longrightarrow (S^1)^n, \qquad (z_1, \dots, z_n) \longmapsto (z_1/|z_1|, \dots, z_n/|z_n|).$$

For a rational linear subspace $H \subseteq \mathbb{R}^n$ we have the following exact sequences:

$$0 \longrightarrow H \cap \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n(H) \longrightarrow 0,$$

where $\mathbb{Z}^n(H) := \mathbb{Z}^n/(H \cap \mathbb{Z}^n)$. Moreover,

$$0 \longrightarrow S_{H\cap\mathbb{Z}^n} \longrightarrow (S^1)^n = S^1 \otimes_{\mathbb{Z}} \mathbb{Z}^n \longrightarrow S_{\mathbb{Z}^n(H)} \longrightarrow 0.$$

Define

$$\pi_H : \operatorname{Log}^{-1}(H) \xrightarrow{\operatorname{Arg}} (S^1)^n \longrightarrow S_{\mathbb{Z}^n(H)} .$$

One has

$$\ker(\pi_H) = T_{H\cap\mathbb{Z}^n} \subseteq (\mathbb{C}^*)^n.$$

As a result, when H is of dimension p, the set $\operatorname{Log}^{-1}(H)$ is naturally foliated by copies of $T_{H\cap\mathbb{Z}^n} \simeq (\mathbb{C}^*)^p$.

Definition 4.1 Let H be a rational subspace of dimension p, and μ be the Haar measure of mass 1 on $S_{\mathbb{Z}^n(H)}$. We define a (p, p)-dimensional closed current $\mathcal{T}_H$ on $(\mathbb{C}^*)^n$ by

$$\mathcal{T}_H := \int_{x\in S_{\mathbb{Z}^n(H)}} \left[\pi_H^{-1}(x)\right] \mathrm{d}\mu(x).$$

Example 4.2 It is instructive to explicitly understand the equations of the fibers of $\mathcal{T}_H$, where $H \subseteq \mathbb{R}^n$, is a rational hyperplane given by $H = \{x \in \mathbb{R}^n : \langle \beta, x\rangle = 0\}$ for some $\beta \in \mathbb{Z}^n$. The support of $\mathcal{T}_H$ is given by $\operatorname{Log}^{-1}(H)$. Let $\beta = w_H\alpha$, where $\alpha \in \mathbb{Z}^n$ is a *primitive* vector, *i.e.*, its components have greatest common divisor equal to 1 and $w_H \in \mathbb{Z}_{>0}$. In this case, each fiber $\pi_H^{-1}(x)$ is given by the $(S^1)^n$-translations of the toric set

$$\pi_H^{-1}(1) = \{z \in (\mathbb{C}^*)^n : z^{-\alpha} - 1 = 0\} = \{z \in (\mathbb{C}^*)^n : z^{\alpha_+} - z^{\alpha_-} = 0\},$$

where $\alpha = \alpha^+ - \alpha^-$, and $\alpha^\pm \in \mathbb{Z}^n_{\geq 0}$.

We also note that the tangent space of the fiber $\pi_H^{-1}(1)$ at $w = (1, \ldots, 1) \in (\mathbb{C}^*)^n$ is given by

$$T_w\pi_H^{-1}(1) = \ker \nabla(z^{-\alpha} - 1)(w) = H \otimes_{\mathbb{Z}} \mathbb{C}.$$

It follows that if $H, H' \subseteq \mathbb{R}^n$ are two hyperplanes intersecting transversely in $\mathbb{R}^n$, then all the fibers of $\mathcal{T}_H$ and $\mathcal{T}_{H'}$ intersect transversely in $(\mathbb{C}^*)^n$.

When A is an affine subspace of $\mathbb{R}^n$ parallel to the linear subspace $H = A - a$ for $a \in A$, we define $\mathcal{T}_A$ by translation of $\mathcal{T}_H$. Namely, we define the submersion π_A as the composition

$$\pi_A : \ \mathrm{Log}^{-1}(A) \xrightarrow{e^a} \mathrm{Log}^{-1}(H) \xrightarrow{\pi_H} S_{\mathbb{Z}^n(H)}.$$

For σ a p-dimensional (rational) polyhedron in $\mathbb{R}^n$, we denote

$$\begin{aligned} \mathrm{aff}(\sigma) &:= \text{the affine span of } \sigma, \\ \sigma^\circ &:= \text{the interior of } \sigma \text{ in } \mathrm{aff}(\sigma), \\ H_\sigma &:= \text{the linear subspace parallel to } \mathrm{aff}(\sigma), \\ N(\sigma) &:= \mathbb{Z}^n/(H_\sigma \cap \mathbb{Z}^n), \end{aligned}$$

and for homogeneity, $N(H) := \mathbb{Z}^n(H) = \mathbb{Z}^n/(H \cap \mathbb{Z}^n)$.

Definition 4.3 Let $\mathcal{C}$, be a weighted polyhedral complex of dimension p. The tropical current $\mathcal{T}_{\mathcal{C}}$ associated to $\mathcal{C}$ is given by

$$\mathcal{T}_{\mathcal{C}} = \sum_\sigma w_\sigma \ \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \mathcal{T}_{\mathrm{aff}(\sigma)},$$

where the sum runs over all p-dimensional cells σ of $\mathcal{C}$.

Theorem 4.4 ([5]) *A weighted complex $\mathcal{C}$ is balanced, if and only if, $\mathcal{T}_{\mathcal{C}}$ is closed.*

The reminder of this section is devoted to proving the statements that will be useful in later sections. The idea of the following technical lemma is extensively used in the proof of Theorem 4.4 and extremality results in [5, 6]. For a measure η on the k-dimensional compact torus $(S^1)^k$ let us denote by $\widehat{\eta}(\ell_1, \dots, \ell_k)$ its $(\ell_1, \dots, \ell_k)$-th Fourier measure coefficients.

Proposition 4.5 *For a p-dimensional affine plane A parallel to a rational linear space, assume that the current $\mathcal{T}_A(\eta)$ is given by*

$$\mathcal{T}_A(\eta) = \int_{x \in S_{N(H)}} \left[\pi_A^{-1}(x)\right] \mathrm{d}\eta(x),$$

for a positive measure η. Then, all the Fourier measure coefficients $\widehat{\eta}(\nu)$, for $\nu \in \mathbb{Z}^n$, can be determined by the action of the current $\mathcal{T}_A(\eta)$ on the (p, p)-differential forms of type

$$\omega = \exp(-i\langle \nu, \theta\rangle)\rho(r)\mathrm{d}\theta_I \wedge \mathrm{d}r_I,$$

where $I \subseteq [n]$, with $|I| = p$, $\theta = (\theta_1, \dots, \theta_n)$ and $r = (r_1, \dots, r_n)$ are polar coordinates, and $\rho : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support.

Proof (Sketch of Proof.) Without loss of generality, we can assume that A is a linear subspace of $\mathbb{R}^n$. Let us choose $\{w_1, w_2, \dots, w_p\}$ a $\mathbb{Z}-$basis for $\mathbb{Z}^n \cap A$ and extend it to a $\mathbb{Z}$-basis of of $\mathbb{Z}^n$, $B := \{w_1, \dots, w_p, u_1, \dots, u_{n-p}\}$. One can see that, compare to [5, Eq. (3.3.20)], with the right choice of ρ,

$$\langle \mathcal{T}_A(\eta), \omega \rangle = \delta_{\{\nu \in A^\perp\}}\, \widehat{\eta}(\langle u_1, \nu\rangle, \dots, \langle u_{n-p}, \nu\rangle)\, \det{}_I(w_1, \dots, w_p),$$

where $\det_I$ is the minor of rows corresponding to $I \subseteq [n]$, and

$$\delta_{\{\nu \in A^\perp\}} = \begin{cases} 1 & \text{if } \nu \in A^\perp; \\ 0 & \text{otherwise,} \end{cases}$$

where $A^\perp$ is the orthogonal complement of A in $\mathbb{R}^n$ with respect to usual dot product. As B is a $\mathbb{Z}$-basis, the tuple $(\langle u_1, \nu\rangle, \dots, \langle u_{n-p}, \nu\rangle)$ can assume any value in $\mathbb{Z}^{n-p}$ and therefore η can be fully understood. $\square$

We call a current on $(\mathbb{C}^*)^n$, $(S^1)^n-$invariant if it is invariant under the induced action of multiplication by any $x \in (S^1)^n$. The following proposition is comparable to [13, Example 7.1.7]. See also [18, Theorem 3.6] for an interesting related result in codimension one.

Proposition 4.6 *Assume that $\mathcal{T} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$ is a closed positive $(S^1)^n$-invariant current whose support is given by $\mathrm{Log}^{-1}(|\mathcal{C}|)$, for a polyhedral complex $\mathcal{C} \subseteq \mathbb{R}^n$ of pure dimension p. Then $\mathcal{T}$ is a tropical current.*

Proof By Demailly's second theorem of support, [20, III.2.13], any positive current with support $\mathrm{Log}^{-1}(|\mathcal{C}|)$ can be presented as

$$\mathcal{T} = \sum_{\sigma \in \mathcal{C}} \int_{x \in S_{N(H)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \left[\pi^{-1}_{\mathrm{aff}(\sigma)}(x)\right] d\eta_\sigma(x),$$

for a unique complex measure η_σ for each σ. We need to see that the measures η_σ are indeed Haar measures, but this is simple, since by definition of the fibers π_A each fiber of π_A is a translation of the kernel by the action of $(S^1)^n$, and if $\mathcal{T}$ is invariant under $(S^1)^n$ then each η_σ has to be invariant under the translation in the quotient $S(N(H)) = S_{\mathbb{Z}^n/(H \cap \mathbb{Z}^n)}$. $\square$

Before ending this subsection let us recall the notion of refinement.

Definition 4.7 A p-dimensional weighted complex $\mathcal{C}'$ is called a refinement of $\mathcal{C}$, if $|\mathcal{C}| = |\mathcal{C}'|$, and each cell $\sigma' \in \mathcal{C}'$ of dimension p (and non-zero weight) is contained in some p-dimensional cell $\sigma \in \mathcal{C}$ with

$$w_{\sigma'}(\mathcal{C}') = w_\sigma(\mathcal{C}).$$

It is easy to check that if $\mathcal{C}$ and $\mathcal{C}'$ have a common refinement then their tropical currents $\mathcal{T}_{\mathcal{C}}$ and $\mathcal{T}_{\mathcal{C}'}$ coincide; see [6, Sect. 2.6].

5 The Toric Setting

5.1 The Multiplication Map on Toric Varieties

In this subsection, we intend to analyse the dynamics of Φ_m on the points of toric varieties. The reader may consult [19] for a thorough and the notes [10] for a quick introduction to the theory of toric varieties. As before, let N be a free abelian group of rank n, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$,

$$N_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} N, \quad T_N = \mathbb{C}^* \otimes_{\mathbb{Z}} N, \quad S_N = S^1 \otimes_{\mathbb{Z}} N.$$

For a rational cone σ, we let N_σ be the sublattice of N spanned by the points in $\sigma \cap N$, and $N(\sigma) = N/N_\sigma$. Moreover,

$$\begin{aligned}
M_{\mathbb{R}} &:= \mathbb{R} \otimes_{\mathbb{Z}} M,\\
\sigma^{\vee} &:= \{u \in M_{\mathbb{R}} : \langle u, v\rangle \geq 0, \text{ for all } v \in \sigma\},\\
\sigma^{\perp} &:= \{u \in M_{\mathbb{R}} : \langle u, v\rangle = 0, \text{ for all } v \in \sigma\}.
\end{aligned}$$

Let us fix a rational fan $\Sigma \subseteq N_{\mathbb{R}}$, and recall that Σ defines a toric variety X_Σ which is obtained by gluing the affine varieties $U_\sigma := \mathrm{Spec}\, \mathbb{C}[\sigma^{\vee} \cap M]$ according to the fan structure in Σ. There is a *Cone-Orbit Correspondence* between the cones in Σ and orbits of the continuous action of T_N on X_Σ. Namely, for any integer $0 \leq p \leq n$,

$$\begin{aligned}
\{p - \text{dimensional cones } \sigma \text{ in } \Sigma\} &\longleftrightarrow \{(n-p) - \text{ dimensional } T_N - \text{orbits in } X_\Sigma\}\\
\sigma &\longleftrightarrow \mathcal{O}(\sigma) \simeq T_{N(\sigma)}.
\end{aligned}$$

Therefore, the T_N-orbits give rise to the partition

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{O}(\sigma).$$

Each point of any affine piece U_σ corresponds to a semigroup homomorphism

$$\sigma^{\vee} \cap M \longrightarrow \mathbb{C},$$

where $\mathbb{C}$ is considered as a semigroup under multiplication. The *distinguished point* $z_\sigma \in \mathcal{O}(\sigma) \subseteq U_\sigma$, is then defined to be the unique point which corresponds to the semigroup homomorphism

$$u \in \sigma^{\vee} \cap M \longmapsto \begin{cases} 1 & \text{if } u \in \sigma^{\perp},\\ 0 & \text{if } u \notin \sigma^{\perp}. \end{cases}$$

The distinguished point of the open torus $(\mathbb{C}^*)^n$, for instance, is just $(1, \ldots, 1)$. We also have a Lie group isomorphism $T_{N(\sigma)} \longrightarrow \mathcal{O}(\sigma)$, that can be understood by

$t \longmapsto t \cdot z_\sigma$, and we can devise it to define the *distinguished compact torus* of $\mathcal{O}(\sigma)$, given by

$$S(\sigma) := S_{N(\sigma)} \cdot z_\sigma \,.$$

Now, for any positive integer m, we consider the multiplication map

$$\phi_m : N \longrightarrow N, \quad v \longmapsto mv,$$

which induces the *toric morphism* (see [19, Definition 1.3.13]),

$$\phi_m \otimes_{\mathbb{Z}} 1 : T_N \longrightarrow T_N, \quad t \longmapsto t^m.$$

To justify the latter, note that in terms of semigroup homomorphisms the induced map $\phi_m \otimes_{\mathbb{Z}} 1$, is indeed

$$\gamma \longmapsto \gamma \circ \phi_m = \gamma^m,$$

for any semigroup homomorphism $\gamma : M \longrightarrow \mathbb{C}$. As a result, when $T_N = (\mathbb{C}^*)^n$, we obtain our familiar group endomorphism

$$\Phi_m = \phi_m \otimes_{\mathbb{Z}} 1 : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n, \quad (z_1, \ldots, z_n) \longmapsto (z_1^m, \ldots, z_n^m).$$

We can also extend our multiplication map to $(\phi_m)_{\mathbb{R}} : N_{\mathbb{R}} \longrightarrow N_{\mathbb{R}}$, and note that it maps any cone $\sigma \in \Sigma$ to itself, and as a result it induces a toric endomorphism, which we also denote by $\Phi_m : X_\Sigma \longrightarrow X_\Sigma$; see [19, Theorem 3.3.4]. Since σ°, the relative interior of σ, is invariant under $(\phi_m)_{\mathbb{R}}$, every toric orbit $\mathcal{O}(\sigma)$ also remains invariant under Φ_m. In addition, $(\phi_m)_{\mathbb{R}}^{-1}(\sigma^\circ) = \sigma^\circ$ implies that $\Phi_m^{-1}(\mathcal{O}_\sigma) = \mathcal{O}_\sigma$. Let us understand $\Phi_m\restriction_{\mathcal{O}(\sigma)}$ explicitly by observing that

$$\begin{aligned} \Phi_m\restriction_{\mathcal{O}(\sigma)} : \mathcal{O}(\sigma) &\longrightarrow \mathcal{O}(\sigma), \\ t \cdot z_\sigma &\longmapsto t^m \cdot z_\sigma, \quad t \in T_{N(\sigma)}. \end{aligned}$$

Note that $T_N \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$, and Φ_m is the continuous extension of $t \in T_N \longmapsto t^m \in T_N$, to $X_\Sigma \longrightarrow X_\Sigma$. Moreover, the isomorphism $\mathcal{O}(\sigma) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*)$, implies that $\mathcal{O}(\sigma)$ as a subset of $U_\sigma = \mathrm{Spec}\ \mathbb{C}[\sigma^\vee \cap M]$, is given by the set of semigroup homomorphisms satisfying

$$\begin{aligned} \gamma : \sigma^\vee \cap M &\longrightarrow \mathbb{C}, \\ u &\longmapsto \begin{cases} \gamma(u) \in \mathbb{C}^* & \text{if } u \in \sigma^\perp, \\ 0 & \text{if } u \notin \sigma^\perp. \end{cases} \end{aligned}$$

As a result, the map $\Phi_m : T_N \longrightarrow T_N$, $\gamma \longmapsto \gamma^m$, extends continuously on $\mathcal{O}(\sigma) \longrightarrow \mathcal{O}(\sigma)$, by $\gamma \longmapsto \gamma^m$. Finally, we may write $\gamma = t \cdot z_\sigma$, for some

$t \in T_{N(\sigma)}$, to obtain

$$\gamma = t \cdot z_\sigma \longmapsto \gamma^m = (t \cdot z_\sigma)^m = t^m \cdot z_\sigma .$$

Now we have gathered enough tools to show that we have a separate equidistribution theorem within each toric orbit:

Proposition 5.1 *Let $\Phi_m : X_\Sigma \longrightarrow X_\Sigma$ be the toric endomorphism induced by the multiplication map. For any $z \in \mathcal{O}(\sigma)$, we have the weak convergence*

$$m^{\dim(\sigma)-n} \Phi_m^*(\delta_z) \longrightarrow \mu(S(\sigma)), \quad \text{as } m \to \infty,$$

where $\mu(S(\sigma))$ is the normalised Haar measure on the distinguished compact torus $S(\sigma) = S_{N(\sigma)} \cdot z_\sigma \subseteq \mathcal{O}(\sigma)$.

Proof We have observed that

$$\begin{aligned} \Phi_m \restriction_{\mathcal{O}(\sigma)} : \mathcal{O}(\sigma) &\longrightarrow \mathcal{O}(\sigma), \\ t \cdot z_\sigma &\longmapsto t^m \cdot z_\sigma, \quad t \in T_{N(\sigma)}. \end{aligned}$$

Let

$$\begin{aligned} \xi : T_N &\longrightarrow \mathcal{O}(\sigma), \quad t \longmapsto t \cdot z_\sigma, \\ \varphi_m : T_{N(\sigma)} &\longrightarrow T_{N(\sigma)}, \quad t \longmapsto t^m. \end{aligned}$$

The fact that the pullbacks ξ^* and $(\xi^{-1})^*$ are continuous with respect to the weak topology of currents, see Sect. 2, implies that to prove the proposition it suffices to observe the weak convergence for the conjugate map instead:

$$m^{\dim(\sigma)-n} \varphi_m^*(\delta_{z'}) \longrightarrow \mu(S_{N(\sigma)}),$$

for any $z' \in T_{N(\sigma)}$, with $z' = \xi^{-1}(z)$. Let $q = \dim(\mathcal{O}(\sigma)) = n - \dim(\sigma)$. Note that the isomorphism $N(\sigma) \simeq \mathbb{Z}^q$, induces an isomorphisms of Lie groups $S_{N(\sigma)} \simeq (S^1)^q$, and a biholomorphism of complex manifolds

$$N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^* \xrightarrow{\sim} \mathbb{Z}^q \otimes_{\mathbb{Z}} \mathbb{C}^* = (\mathbb{C}^*)^q .$$

Consequently, by weak continuity of pullback morphisms, to prove the asserted convergence, it suffices to show the analogous statement on $(\mathbb{C}^*)^q$. On $(\mathbb{C}^*)^q$, however, the induced map, which we also denote by Φ_m, is given by

$$(\mathbb{C}^*)^q \longrightarrow (\mathbb{C}^*)^q, \quad (z_1, \ldots, z_q) \longmapsto (z_1^m, \ldots, z_q^m).$$

In this case, for any compactly supported smooth function $f : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$, we have

$$m^{-q}\langle \Phi_m^*(z), f\rangle \longrightarrow \int_{(S^1)^q} f \,\mathrm{d}\mu,$$

as the left hand side is simply tending to the Riemann sum for the integral on the right hand side, since μ is the Haar measure on $(S^1)^q$.

□

Let us end this subsection with the following useful lemma, which can be also proved directly for any toric variety without the smoothness assumption.

Lemma 5.2 *Let $Z \subseteq (\mathbb{C}^*)^n$ be a subvariety, and $\bar{Z}$ be the closure of Z in a smooth toric variety X_Σ. Then*

$$\Phi_m^{-1}(\bar{Z}) = \overline{\Phi_m^{-1}(Z)}\,.$$

Proof When X_Σ is smooth, according to [32, Exercise 18.17], the finiteness property of Φ_m implies that Φ_m is flat. The assertion then follows from [42, Théorème 2.3.10]. □

5.2 Tropical Compactifications

Let X be a smooth, projective toric variety and fix a torus equivariant projective embedding

$$\phi : X \longrightarrow \mathbb{P}^N.$$

Assume that ω_0 is the smooth positive $(1, 1)$-form on X, corresponding to ϕ. We have that by Wirtinger's theorem [41, Page 31], the closure of any closed algebraic variety $Z \subset (\mathbb{C}^*)^n$ in X has a finite mass with respect to ω_0. We can therefore employ the El Mir–Skoda theorem, [20, Sect. III.2.A], and extend $[Z]$ by zero to the closed positive current $\overline{[Z]}$ on X. Similarly, any tropical current $\mathcal{T}_{\mathcal{C}}$ can be extended by zero to any smooth projective toric variety, since any fiber $\pi^{-1}_{\mathrm{aff}(\sigma)}(x)$, for $\sigma \in \mathcal{C}$, has a bounded normalised mass; see [6, Proposition 4.4] for more details where we also treat the tropical currents with non-positive weights. If, moreover, we demand that the support of $\overline{[Z]}$ or the fibers of $\overline{\mathcal{T}}_{\mathcal{C}}$ intersect the torus-invariant divisors of X_Σ properly we need a certain compatibility with Σ, which we present in Theorems 5.5 and 5.7 below, after recalling some important theorems from toric geometry. Recall that a p-dimensional cone $\sigma = \mathrm{cone}(\rho_1, \ldots, \rho_p) \subseteq \mathbb{R}^n \simeq N_{\mathbb{R}}$ is called *unimodular*, if $\{\rho_1, \ldots, \rho_p\}$ can be completed to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Theorem 5.3 *Let $\Sigma \subseteq \mathbb{R}^n$, be a fan. Then,*

- *(a) X_Σ is smooth, if and only if, Σ is* unimodular, *i.e., all cones in Σ are unimodular.*
- *(b) X_Σ is* complete, *if and only if, Σ is* complete, *i.e., $|\Sigma| = \mathbb{R}^n$.*
- *(c) X_Σ is projective, if and only if, Σ is* projective, *i.e., Σ is* dual *to a polytope.*

Recall that, since our base field is the set of complex numbers, a variety is *complete*, if and only if, it is compact in the classical topology. Moreover, we do not require our varieties to be projective in this article, and have added Theorems 5.3.c and 5.4.c for completeness; see [19, Page 67] for the definition of a dual polytope. Further,

Theorem 5.4 *(a) The Toric Resolution of Singularities: any fan Σ can be refined to obtain a unimodular fan; [19, Theorem 11.1.9].*

(b) For any fan Σ there exists complete fan Σ' containing Σ as a subfan; [33, Theorem III.2.8].

(c) The Toric Chow Lemma: every complete fan has a refinement that is projective; [19, Theorem 6.1.18].

Since our main objects in this article are differential forms and currents, we always need our ambient toric varieties to be smooth or equivalently their fan to be unimodular. When X_Σ is smooth, $X_\Sigma \setminus (\mathbb{C}^*)^n$ is a simple normal crossing divisor, and the orbit closures $D_\sigma := \overline{\mathcal{O}(\sigma)}$, for $\sigma \in \Sigma$, are intersections of its irreducible components. As we mentioned in the introduction that $\mathrm{trop}(Z)$ as a set is given by the Hausdorff limit

$$\mathrm{trop}(Z) := \lim_{t\to\infty} \frac{1}{\log|t|}\mathrm{Log}(Z).$$

Further, we know from the Structure Theorem in tropical geometry, [58, Theorem 3.3.5], that $\mathrm{trop}(Z)$ can be regarded as the support of a rational polyhedral complex. In the following paragraphs we recall how to endow a tropical structure on the set $\mathrm{trop}(Z)$, using the following theorem due to Tevelev and Sturmfels. Tevelev's theorem below provides a condition for the closure of Z to be complete in a given toric variety, even when the toric variety is not complete. We also note that one significance of [71] is to consider the case where the compactification is *flat*, see [58, Sect. 6.4] for a definition, which we do not require in this article.

Theorem 5.5 *Let $Z \subseteq (\mathbb{C}^*)^n$ be an irreducible algebraic subvariety of dimension p, and $\Sigma \subseteq N_\mathbb{R}$, a unimodular (rational) fan.*

(a) The closure $\bar{Z}$ of Z in X_Σ is complete, if and only if, $\mathrm{trop}(Z) \subseteq |\Sigma|$; *see [71].*

(b) We have $|\Sigma| = \mathrm{trop}(Z)$, if and only if, for every $\sigma \in \Sigma$ the intersection $\mathcal{O}_\sigma \cap \bar{Z}$ is non-empty and of pure dimension $p - \dim(\sigma)$; see [70].

For a subvariety $Z \subseteq (\mathbb{C}^*)^n$, we can always find a unimodular fan Σ such that $\mathrm{trop}(Z) = |\Sigma|$, this is a consequence of the toric resolution of singularities, Thereom 5.4.a. Suppose now that Σ' is a fan that contains Σ as a subfan. Then, the closure of Z in $X_\Sigma = \bigcup_{\sigma\in\Sigma} \mathcal{O}(\sigma)$ can be identified with the closure of Z in $X_{\Sigma'} = \bigcup_{\sigma\in\Sigma'} \mathcal{O}(\sigma)$. Applying the preceding theorem we obtain that for a p-dimensional cone $\sigma \in \Sigma'$, the intersection $\mathcal{O}(\sigma) \cap \bar{Z}$ is non-empty, if and only if, $\sigma \in \Sigma$, and in this case, the intersection is of dimension $p - \dim(\sigma)$; see also [58, Sect. 6.4] and [43]. Therefore, thanks to [20, Corollary 4.10], for any positive integer $k \leq p$, if $\tau = \mathrm{cone}(\rho_1, \ldots, \rho_k) \in \Sigma$, where $\rho_i \in \Sigma$ are rays, the wedge product

$$[D_\tau] \wedge [\bar{Z}] = [D_{\rho_1}] \wedge \cdots \wedge [D_{\rho_k}] \wedge [\bar{Z}]$$

is admissible, and yields a $(p-k, p-k)$-closed positive current which can be considered in both X_Σ and $X_{\Sigma'}$. When $\dim(\sigma) = p$, the intersection is 0-dimensional, and we set

$$w_\sigma := \int_{X_\Sigma} [D_\sigma] \wedge [\bar{Z}].$$

It is observed in tropical geometry that $\mathrm{trop}(Z)$ with the induced fan structure from Σ, and the induced weights w_σ is a balanced fan; see [58, Theorem 6.7.7]. We also review this induced balancing condition as a consequence of our main convergence theorem by noting that the weak limit of a sequence of closed currents is indeed closed.

Definition 5.6 Suppose $Z \subseteq (\mathbb{C}^*)^n$ is an irreducible subvariety, and Σ is a fan with $|\Sigma| = \mathrm{trop}(Z)$. The closure $\bar{Z}$ of Z in the toric variety X_Σ, or equivalently in any $X_{\Sigma'}$, such that Σ' contains Σ as a subfan, is called a *tropical compactification* of Z.

Let us also investigate the situation for the tropical currents. Recall from Theorem 4.4 when a polyhedral complex $\mathcal{C}$ satisfies the balancing condition, then $\mathcal{T}_\mathcal{C}$ is closed.

Theorem 5.7 *Let $\mathcal{C}, \Sigma \subseteq \mathbb{R}^n$ be two fans, and assume that $\mathcal{C}$ is a p-dimensional tropical variety.*

(a) Let $\tau \in \mathcal{C}$, $\sigma \in \Sigma$, and $\overline{\pi^{-1}_{\mathit{aff}(\tau)}(x)}$ be the closure in U_σ. Then, the intersection

$$D_\sigma \cap \overline{\pi^{-1}_{\mathit{aff}(\tau)}(x)},$$

is non-empty, if and only if, τ contains σ as a face and in this case the intersection is transverse; [6, Lemma 4.10.1].

(b) If $\sigma \in \mathcal{C} \cap \Sigma$ is cone of dimension p with weight w_σ in $\mathcal{C}$, then

$$w_\sigma = \int_{X_\Sigma} [D_\sigma] \wedge \overline{\mathcal{T}}_\mathcal{C}.$$

where $\overline{\mathcal{T}}_\mathcal{C}$ is the extension by zero of $\mathcal{T}_\mathcal{C}$ in X_Σ; [6, Theorem 4.7].

The preceding theorem asserts that the wedge product $[\overline{\pi^{-1}_{\mathrm{aff}(\sigma)}(x)}] \wedge D_\sigma$, for any $x \in S_{N(\sigma)}$ is a well-defined $(0,0)$-dimensional current. Transversality implies that the intersection multiplicity is one, and Part (b) implies that averaging with respect to the Haar measure and multiplying with w_σ will indeed yield the total mass w_σ.

Remark 5.8 Let $Z \subseteq (\mathbb{C}^*)^n$ be an irreducible subvariety and consider Σ a unimodular fan with $\mathrm{trop}(Z) = |\Sigma|$. Since, for any positive integer m

$$\mathrm{Log}(\Phi_m^{-1}(Z)) = \frac{1}{m}\mathrm{Log}(Z),$$

the sets $\mathrm{trop}(\Phi_m^{-1}(Z))$ coincide. As a result, by Theorem 5.5 all $\overline{\Phi_m^{-1}(Z)}$'s are simultaneously complete in X_Σ, and intersect the toric invariant divisors of X_Σ properly.

In consequence, the currents $\left[\,\overline{\Phi_m^{-1}(Z)}\,\right]$ carry no mass on $X_\Sigma \setminus (\mathbb{C}^*)^n$. And,

$$\overline{\left[\Phi_m^{-1}(Z)\right]} = \left[\,\overline{\Phi_m^{-1}(Z)}\,\right], \quad \text{for } m \in \mathbb{Z}_{\geq 0},$$

where $\Phi_0^{-1}(Z) := Z$. Finally, in view of Lemma 5.2, we conveniently obtain

$$\overline{\left[\Phi_m^{-1}(Z)\right]} = \left[\,\overline{\Phi_m^{-1}(Z)}\,\right] = [\Phi_m^{-1}(\bar{Z})].$$

5.3 Extended Tropicalisation

For any rational fan $\Sigma \subseteq N_{\mathbb{R}}$, the logarithm map $\text{Log} : T_N \longrightarrow N_{\mathbb{R}}$, can be continuously extended to

$$\text{Log} : X_\Sigma \longrightarrow N_\Sigma,$$

where N_Σ denotes the quotient $X_\Sigma / S_{N(\{0\})}$; see [50, Sect. 3]. The *tropical toric variety* N_Σ can be regarded as the tropicalisation of toric variety X_Σ and it was studied in [3, 52, 64] and [58, Section 6.2]. The set N_Σ has a natural topology and it is straightforward to see that for a polyhedral complex $\mathcal{C} \subseteq N_{\mathbb{R}} \subseteq N_\Sigma$,

$$\text{Log}^{-1}(\overline{\mathcal{C}}) = \overline{\text{Log}^{-1}(\mathcal{C})}.$$

By [50, Theorem A′] for an algebraic subvariety $Y \subseteq X_\Sigma$, the logarithmic tropicalisation also extends to X_Σ :

$$\frac{1}{\log|t|}\text{Log}(Y) \longrightarrow \text{trop}(Y), \quad \text{as} |t| \to \infty,$$

where $\text{trop}(Y) \subseteq N_\Sigma$ is the tropicalisation of Y in N_Σ. Now assume that $Z \subseteq T_N$ is an algebraic subvariety, and $\bar{Z}$ is the closure of Z in X_Σ, Theorem 6.2.18 in [58] implies that $\text{trop}(\bar{Z})$ coincides with the closure of $\text{trop}(Z) \subseteq N_{\mathbb{R}}$ in N_Σ. In consequence,

$$\text{Log}(\text{supp}(\Phi_m^*[\bar{Z}])) = \frac{1}{m}\text{Log}(\bar{Z}) \longrightarrow \text{trop}(\bar{Z}) = \overline{\text{trop}(Z)}, \tag{I}$$

where the convergence is in the Hausdorff metric of compact sets of N_Σ.

6 A Proof for Theorem A

We are now ready to prove the main theorem of this article. The reader may find viewing the figure in Sect. 7.1 instructive while following the steps of the proof.

Theorem 6.1 *Let $Z \subseteq (\mathbb{C}^*)^n$ be an algebraic variety of dimension p, then we have the weak convergence*

$$m^{p-n}\Phi_m^*[Z] \longrightarrow \mathcal{T}_{\mathrm{trop}(Z)},$$

in $\mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$. Moreover, with the induced weights from any unimodular tropical compactification of Z, trop(Z) *is balanced.*

Proof We proceed in the following steps.

Step 1. The extention of the map Φ_m to an endomorphism of $\mathbb{P}^n$ has degree m, with respect to $O(1)$. It is well-known that m^{p-n} is the correct normalisation factor for Φ_m^* to preserve the normalised total mass of the closure of $[Z]$ in $\mathbb{P}^n$. Therefore, all the elements of the sequence $\{m^{p-n}\Phi_m^*[\bar{Z}]\}$ have an equal normalised total mass, and as a result of the Banach–Alaoglu theorem, the sequence is locally compact with respect to the weak topology of currents. In other words, it has a convergence subsequence in $\mathbb{P}^n$, and by restriction on $(\mathbb{C}^*)^n$.

Step 2. In Sect. 6.1 below, we prove that the support of any cluster value $\mathcal{S}$ of the sequence is given by $\mathrm{Log}^{-1}(\mathrm{trop}(Z))$. Let Σ be a unimodular fan with $|\Sigma| = \mathrm{trop}(Z)$. Since $\mathcal{S}$ is a positive closed current of bidimension (p, p), by Demailly's second theorem of support, [20, III.2.13], it can be represented as

$$\mathcal{S} = \sum_{\sigma\in\Sigma}\int_{x\in S_{N(\sigma)}}\left[\mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\pi^{-1}_{\mathrm{aff}(\sigma)}(x)\right]\mathrm{d}\eta_\sigma(x),$$

for some positive measures $\mathrm{d}\eta_\sigma$.

Step 3. In Sect. 6.2 below, we show that for any cluster value $\mathcal{S}$ obtained in the above steps, the corresponding measures $\mathrm{d}\eta_\sigma$ must be Haar measures. This, together with Proposition 4.6, implies that any cluster value $\mathcal{S}$ is indeed a tropical current with support $\mathrm{Log}^{-1}(\mathrm{trop}(Z))$.

Step 4. Let $\mathcal{Z}_{j_i}$ be a subsequence of $m^{p-n}\Phi_m^*[Z]$ weakly convergent to the tropical current $\mathcal{S}$, we show that in the toric variety X_Σ, we have $\overline{\mathcal{Z}}_{j_i} \longrightarrow \overline{\mathcal{S}}$. As in Step 2, the support of $\mathcal{S}$ is given by $\mathrm{Log}^{-1}(\mathrm{trop}(Z))$. Therefore

$$\mathrm{supp}(\overline{\mathcal{S}}) \supseteq \overline{\mathrm{Log}^{-1}(\mathrm{trop}(Z))} = \mathrm{Log}^{-1}(\overline{\mathrm{trop}(Z)}) = \mathrm{Log}^{-1}(\mathrm{trop}(\bar{Z})),$$

where in the first equality we have used a simple property of $\mathrm{Log} : X_\Sigma \longrightarrow N_\Sigma$, and the second equality is provided by [58, Theorem 6.2.18]. Assume now that $\widetilde{\mathcal{S}}$ is a cluster value of $\overline{\mathcal{Z}}_{j_i}$. By Eq. (I) and Remark 2.1, $\mathrm{Log}(\mathrm{supp}(\widetilde{\mathcal{S}})) \subseteq \mathrm{trop}(\bar{Z})$, therefore

$$\mathrm{supp}(\widetilde{\mathcal{S}}) \subseteq \mathrm{Log}^{-1}(\mathrm{trop}(\bar{Z})) \subseteq \mathrm{supp}(\overline{\mathcal{S}}).$$

Since $\overline{\mathcal{S}}\restriction_{(\mathbb{C}^*)^n} = \widetilde{\mathcal{S}}\restriction_{(\mathbb{C}^*)^n}$, the support of the positive closed (p, p)-dimensional current $\widetilde{\mathcal{S}} - \overline{\mathcal{S}}$ is included in $\bigcup_i D_i \cap \mathrm{supp}(\overline{\mathcal{S}})$, where D_i's are the torus invariant divisors of X_Σ. In view of Theorem 5.7, $\bigcup_i D_i \cap \mathrm{supp}(\overline{\mathcal{S}})$ has a Cauchy–Riemann dimension less than p, and by Demailly's first theorem of support, [20, Theorem III.2.10], we have $\widetilde{\mathcal{S}} = \overline{\mathcal{S}}$.

Step 5. Assume that $\mathcal{Z}_{\ell_i}$ and $\mathcal{Z}_{k_j}$ are two subsequences of $m^{p-n}\Phi_m^*[Z]$ weakly convergent to the tropical currents $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. We intend to show that $\mathcal{S}_1 = \mathcal{S}_2$. Applying Theorem 5.4.b, and the toric resolution of singularities Theorem 5.4.a, we can find a fan Σ' containing Σ as a subfan such that $X_{\Sigma'}$ is a smooth projective toric variety. In view of Theorem 5.7.b, it only remains to show that for any p-dimensional cone $\sigma \in \Sigma'$,

$$w_\sigma(\overline{\mathcal{S}}_1) := \int_{X_{\Sigma'}} [D_\sigma] \wedge \overline{\mathcal{S}}_1 \quad = \quad w_\sigma(\overline{\mathcal{S}}_2) := \int_{X_{\Sigma'}} [D_\sigma] \wedge \overline{\mathcal{S}}_2,$$

where the transversality of intersections are guaranteed by Step 2 and Theorem 5.7. For $\sigma \in \Sigma'$, let

$$w_\sigma(\bar{Z}) = \int_{X_{\Sigma'}} [D_\sigma] \wedge [\bar{Z}].$$

We show that $w_\sigma(\bar{Z}) = w_\sigma(\overline{\mathcal{S}}_1) = w_\sigma(\overline{\mathcal{S}}_2)$, for any $\sigma \in \Sigma'$. By Step 4, in $X_{\Sigma'}$

$$\overline{\mathcal{Z}}_{\ell_i} \longrightarrow \overline{\mathcal{S}}_1, \quad \overline{\mathcal{Z}}_{k_j} \longrightarrow \overline{\mathcal{S}}_2.$$

By Remark 5.8,

$$\overline{\mathcal{Z}}_{\ell_i} = {\ell_i}^{n-p}\Phi_{\ell_i}^*[\bar{Z}], \quad \overline{\mathcal{Z}}_{k_j} = {k_j}^{n-p}\Phi_{k_j}^*[\bar{Z}].$$

Since $\dim(\sigma) = p$, we have $\dim\big(\mathcal{O}(\sigma)\big) = n - p$. As a result, the restriction $m^{p-n}\Phi_m^*\restriction_{\mathcal{O}(\sigma)}$ preserves the mass of zero dimensional integration currents in $\mathcal{O}(\sigma)$. Hence, for any positive integer m,

$$w_\sigma(\bar{Z}) = \int_{X_{\Sigma'}} m^{p-n}\Phi_m^*\restriction_{\mathcal{O}(\sigma)}([D_\sigma] \wedge [\bar{Z}]) = \int_{X_{\Sigma'}} [D_\sigma] \wedge \big(m^{p-n}\Phi_m^*[\bar{Z}]\big),$$

where we have used the invariance of D_σ under Φ_m^{-1} in the latter equality. Now we can use [6, Proposition 4.12], which is heavily based on the Demailly's regularisation theorem, to reduce the calculations of the above masses to intersection of cohomology classes. Namely, whenever $\mathcal{T}$ is a closed positive current such that $[D_\sigma] \wedge \mathcal{T}$ is admissible,

$$\int_{X_{\Sigma'}} [D_\sigma] \wedge \mathcal{T} = \big\{[D_\sigma] \wedge \mathcal{T}\big\} = \{\omega_1\} \wedge \cdots \wedge \{\omega_p\} \wedge \{\mathcal{T}\}.$$

In the above formula,

$\{\,\}$ denotes the Dolbeault cohomology class in $X_{\Sigma'}$,

$D_\sigma = D_1 \cap \cdots \cap D_p$, is the intersection of toric invariant divisors, and

$\omega_1, \ldots, \omega_p$ are the first Chern forms corresponding to $D_1, \ldots, D_p$, respectively.

We note that we have chosen $X_{\Sigma'}$ to be a smooth compact complex manifold in order to employ [6, Proposition 4.12]. Now, using the weak continuity of the cohomology class assignment, and the above formulas,

$$w_\sigma(\bar{Z}) = \{[D_\sigma] \wedge \overline{\mathcal{Z}}_{\ell_i}\} \longrightarrow \{[D_\sigma] \wedge \overline{\mathcal{S}}_1\} = w_\sigma(\overline{\mathcal{S}}_1),$$
$$w_\sigma(\bar{Z}) = \{[D_\sigma] \wedge \overline{\mathcal{Z}}_{k_j}\} \longrightarrow \{[D_\sigma] \wedge \overline{\mathcal{S}}_2\} = w_\sigma(\overline{\mathcal{S}}_2).$$

This concludes the proof of the main convergence theorem.

Step 6. Let us now observe that with the induced fan structure from Σ or Σ', and the induced weights, $\mathrm{trop}(Z)$ is balanced. The weak limit of a sequence of closed currents is a closed current, and Theorem 4.4 asserts that a tropical current associated to any weighted polyhedral complex is closed if and only if, the underlying weighted polyhedral complex is balanced. □

6.1 Convergence of the Supports

In this subsection we show that for any p-dimensional subvariety $Z \subseteq (\mathbb{C}^*)^n$, the support of any cluster value of the sequence $m^{p-n}\Phi_m^*[Z]$ is given by $\mathrm{Log}^{-1}(\mathrm{trop}(Z))$. The inclusion in $\mathrm{Log}^{-1}(\mathrm{trop}(Z))$ is easy but for the converse we need certain volume estimates. Notation-wise, it is slightly lighter to write the proofs of the following statements for the whole sequence rather than a subsequence, acknowledging that the proofs are identical.

Proposition 6.2 *Assume that $\mathcal{S}$ is the weak limit of the sequence $\mathcal{Z}_m := m^{p-n}\Phi_m^*[Z]$. Then,*

$$\mathrm{supp}(\mathcal{S}) = \mathrm{Log}^{-1}(\mathrm{trop}(Z)).$$

We give a proof of the preceding proposition after the proof of Lemma 6.5.

Corollary 6.3 *Assume that $\mathcal{S}$ is the weak limit of the sequence $\mathcal{Z}_m = m^{p-n}\Phi_m^*[Z]$, then*

$$\mathrm{supp}(\mathcal{Z}_m) \longrightarrow \mathrm{supp}(\mathcal{T}_{\mathrm{trop}(Z)}),$$

in the Hausdorff metric of compact sets of $(\mathbb{C}^*)^n$.

Proof We note that by Bergman's theorem and the definition of Φ_m

$$\mathrm{supp}(\mathcal{Z}_m) \longrightarrow \mathrm{Log}^{-1}(\mathrm{trop}(Z)),$$

in the Hausdorff metric and by Proposition 6.2, $\mathrm{Log}^{-1}(\mathrm{trop}(Z)) = \mathrm{supp}(\mathcal{T}_{\mathrm{trop}(Z)})$. □

Lemma 6.4 *For any positive integer m sufficiently large, small $\epsilon > 0$, and $b_m \in \Phi_m^{-1}(Z)$ we have*

$$\int_{\Phi_m^{-1}\left(\Phi_m(B_\epsilon(b_m))\right)} m^{p-n}\Phi_m^*[Z] \wedge \beta^p \geq C\epsilon^{2p},$$

where $\beta = dd^c\|z\|^2$, and C is a positive constant and independent of m.

Proof Let $b_m = (b_{m,1}, \ldots, b_{m,n})$, $a_m = \Phi_m(b_m) = (a_{m,1}, \ldots, a_{m,n}) \in Z$, and note that $|b_{m,j}| = |a_{m,j}|^{1/m}$. For each $j = 1, \ldots, n$, we set $\mathbf{r}_j := (|b_{m,j}| - \frac{\epsilon}{\sqrt{2}})$ and $\mathbf{R}_j := (|b_{m,j}| + \frac{\epsilon}{\sqrt{2}})$. For positive real numbers $\mathbf{r} < \mathbf{R}$, we will denote the annulus

$$C_{\mathbf{r},\mathbf{R}} := \{z \in \mathbb{C} \mid \mathbf{r} < |z| < \mathbf{R}\}.$$

We also set

$$C^*_{\mathbf{r},\mathbf{R}} := C_{\mathbf{r},\mathbf{R}} \setminus \mathbb{R}_+$$

which is a contractible open subset of $C_{\mathbf{r},\mathbf{R}}$. When m is large, $\Phi_m(B_\epsilon(b_m))$ contains the multi-annulus given by

$$B_m := \prod_{j=1}^{n} C_{\mathbf{r}_j^m, \mathbf{R}_j^m}.$$

Let $B^*_m = \prod_{j=1}^n C^*_{\mathbf{r}_j^m, \mathbf{R}_j^m}$. Since Φ_m is a covering and B^*_m is contractible, we can partition $\Phi_m^{-1}\big(B^*_m\big)$ into a disjoint union of open sets $K_{m,1}, \ldots, K_{m,m^n}$ such that for each $i \in [m^n] := \{1, \ldots, m^n\}$,

$$\Phi_m\restriction_{K_{m,i}} : K_{m,i} \longrightarrow B^*_m,$$

is a homeomorphism. Each $K_{m,i}$ can be parametrised in polar coordinates with $(0, 2\pi/m)^n \times \prod_{j\in[n]}(\mathbf{r}_j, \mathbf{R}_j)$. For a fixed m, and any $i \in [m^n]$,

$$\int_{K_{m,i}} \Phi^*_m[Z] \wedge \beta^p = \int_{\Phi_m(K_{m,i})} [Z] \wedge (\Phi_m)_*\beta^p = \int_{B^*_m} [Z] \wedge (\Phi_m^{-1})^*(\beta^p).$$

We set $Z_i = \Phi_m^{-1}(Z) \cap K_{m,i}$. Since by Lelong's theorem [20, Theorem III.2.7], the singular set of Z does not charge any mass, we assume that Z is smooth and consider the parametrisation

$$\boldsymbol{\tau} : (S^1)^p \times U \longrightarrow \Phi_m^{-1}(Z) \cap K_{m,1},$$

for $U \subseteq (\mathbb{R}^+)^p$. Let $\{1, \zeta, ..., \zeta^{m-1}\}$ be all different m-th roots of unity, and for $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n \cap [0, m-1]^n$, define $\lambda^\ell(\zeta) := (\zeta^{\ell_1}, \ldots, \zeta^{\ell_n})$. The component-wise multiplication $\boldsymbol{\tau} \cdot \lambda^\ell(\zeta)$ for different ℓ parametrises all the manifolds $\Phi_m^{-1}(Z) \cap K_{m,i}$ for $i = 1, \ldots, m^n$. Therefore, the equality

$$\big(\boldsymbol{\tau} \cdot \lambda^\ell(\zeta)\big)^* dd^c\|z\|^2 = \boldsymbol{\tau}^* dd^c\|z\|^2,$$

implies that all Z_i's have the same mass. We now claim that for each i

$$[Z_i] \wedge \beta^p \geq C_i \big(\frac{\epsilon^2}{m}\big)^p. \tag{II}$$

for a constant C_i. As we noted before, by Bergman's theorem and the definition of Φ_m we have $\operatorname{supp}(\mathcal{Z}_m) \longrightarrow \operatorname{Log}^{-1}(\operatorname{trop}(Z))$. As a result, by shrinking ϵ, if necessary, there exists $\sigma \in \operatorname{trop}(Z)$, and a fiber $\pi_\sigma^{-1}(x)$ for some $x \in (S^1)^{n-p}$, such that at a point $w \in \pi_\sigma^{-1}(x)$, the projection $\operatorname{proj} : Z_i \longrightarrow T_w \pi_\sigma^{-1}(x)$ is surjective. It is crucial to note that, because of the Hausdorff convergence, for larger m's we do not need to shrink ϵ any further for such surjectivity to hold. Using this projection, we can find coordinates $w = (w', w'')$ and polydiscs $w' \in \Delta' \subseteq \mathbb{C}^p$, $w'' \in \Delta'' \subseteq \mathbb{C}^{n-p}$ of radii $r' = \frac{\epsilon}{\sqrt{2}}$ and r'' with $r' \leq Cr''$ for $C > 0$ large, such that the projection $\pi : Z_i \cap \big(\Delta' \times \Delta''\big) \longrightarrow \Delta'$ is also surjective. Therefore,

$$\int_{Z_i} (dd^c \|w\|^2)^p \geq \int_{Z_i} \pi^*(dd^c \|w'\|^2)^p \geq \int_{\Delta'} (dd^c \|w'\|^2)^p.$$

Since $K_{m,i} \supseteq Z_i$ can be parametrised by $(0, 2\pi/m)^n \times \prod_{j\in[n]} (\mathbf{r}_j, \mathbf{R}_j)$, we also can parametrise Δ', with $(0, 2\pi/m)^p \times (0, \frac{\epsilon}{\sqrt{2}})^p$, such that the Jacobian of the parametrisation does not depend on m. We have

$$\int_{\Delta'} (dd^c \|w'\|^2)^p = \int_{(0,2\pi/m)^p \times (0, \frac{\epsilon}{\sqrt{2}})^p} r_{[p]} dr_{[p]} \mathrm{d}\theta_{[p]} = \big(\frac{\pi\epsilon^2}{2m}\big)^p.$$

We obtain Eq. (II), by taking into account that the change of coordinates $z \mapsto w$, contributes to the calculation of mass by multiplying a constant. Finally, to obtain the main assertion, we note that there are m^n components of $K_{m,i}$, all with the same mass, and the normalising factor is m^{p-n}.

□

Lemma 6.5 *For any positive integer m sufficiently large, small $\epsilon > 0$, and $b_m \in \Phi_m^{-1}(Z)$ we have*

$$\int_{B_\epsilon(b_m)} m^{p-n} \Phi_m^*[Z] \wedge \beta^p \geq C'_{b_m} \epsilon^{2p+n},$$

where $\beta = dd^c \|z\|^2$, and C' is a positive constant depending on b_m.

Proof Following the proof and the notation of Lemma 6.4, we find a lower bound for the mass of $\mathcal{Z}_m = m^{p-n}\Phi_m^*[Z]$ in the ball $B_\epsilon(b_m)$. Let $b_m = (b_{m,1}, \ldots, b_{m,n})$. By previous lemma, the mass of $\mathcal{Z}_m$ in the multi-annulus $A = \prod_{j=1}^n C_{\mathbf{r}_j,\mathbf{R}_j}$, is greater than $C\epsilon^{2p}$. We can assume that $|b_{m,j}| \gg \epsilon$, for all $j = 1, \ldots, n$. Note that for each j, the annulus in the j-th coordinate of the multi-annulus has an angle varying between 0 and 2π, which gives rise to the outer circumference of length $2\pi(|b_j| + \frac{\epsilon}{\sqrt{2}})$. Therefore, in each coordinate, the angle corresponding to 2ϵ portion of the outer circumference can be approximated by

$$\theta_j = \frac{2\pi\epsilon}{2(|b_{m,j}| + \frac{\epsilon}{\sqrt{2}})\pi} \geq \frac{\epsilon}{2|b_{m,j}|}.$$

We intend to estimate the portion of mass of $\mathcal{Z}_m$ around b_m sliced within the angles $\theta_1, \ldots, \theta_n$. Since the mass of $\mathcal{Z}_m$ in A equals $C\epsilon^{2p}$, and components of $\operatorname{supp}(\mathcal{Z}_m) \cap A$ are divided symmetrically in A, we obtain the following lower bound

$$C\epsilon^{2p} \times \prod_{j=1}^{n} \frac{\theta_j}{2\pi} = C\epsilon^{2p} \prod_{j=1}^{n} \Big(\frac{\epsilon}{4\pi|b_{m,j}|}\Big)^n = \frac{C}{(4\pi)^n|b_{m,1}\ldots b_{m,n}|}\epsilon^{2p+n} := C_{b_m}\epsilon^{2p+n}.$$

□

Remark 6.6 As the above lemma suggests, the *Lelong number* of any tropical current at any point of its support is, in fact, zero. See [20, Sect. III.5] for a definition.

Proof of Proposition 6.2 We first show the inclusion $\operatorname{supp}(\mathcal{S}) \subseteq \operatorname{Log}^{-1}\big(\operatorname{trop}(Z)\big)$. We note that if for any sufficiently large integer m, z has a neighbourhood that does not intersect $\operatorname{supp}(\mathcal{Z}_m)$, then $z \notin \operatorname{supp}(\mathcal{S})$. However, by Bergman's theorem

$$\operatorname{Log}\big(\operatorname{supp}(\Phi_m^*[Z])\big) = \frac{1}{m}\operatorname{Log}\big(\operatorname{supp}([Z])\big) \longrightarrow \operatorname{trop}(Z),$$

in the Hausdorff metric. Therefore, any point outside $\operatorname{Log}^{-1}\big(\operatorname{trop}(Z)\big)$ is outside of $\operatorname{supp}(\mathcal{Z}_m)$ for any sufficiently large m. This implies that $\operatorname{supp}(\mathcal{S}) \subseteq \operatorname{Log}^{-1}\big(\operatorname{trop}(Z)\big)$. To prove the converse inclusion $\operatorname{Log}^{-1}(\operatorname{trop}(Z)) \subseteq \operatorname{supp}(\mathcal{S})$, consider $b \in \operatorname{Log}^{-1}\big(\operatorname{trop}(Z)\big)$. We show that for any $\epsilon > 0$, there is a test form $\varphi \in \mathcal{D}^{p,p}((\mathbb{C}^*)^n)$, such that its (compact) support contains the open ball of radius 2ϵ centered at b, $B_{2\epsilon}(b)$, and

$$\langle \mathcal{S}, \varphi \rangle > 0.$$

Let $\beta = dd^c\|z\|^2 = dd^c[|z_1|^2 + \cdots + |z_n|^2]$. For any $\epsilon > 0$, and sufficiently large m, there exists a point

$$b_m \in \Phi_m^{-1}(Z) \subseteq \operatorname{Log}^{-1}\big(\operatorname{Log}(\Phi_m^{-1}(Z))\big),$$

such that $\operatorname{dist}\big(b_m, b\big) < \epsilon$. This implies the inclusion $B_\epsilon(b_m) \subseteq B_{2\epsilon}(b)$. By Lemma 6.5, we have that for any sufficiently large m, there exists a constant $C_{b_m} := \frac{C}{(4\pi)^n|b_{m,1}\ldots b_{m,n}|} > 0$ such that

$$\int_{B_{2\epsilon}(b)} \mathcal{Z}_m \wedge \beta^p > \int_{B_\epsilon(b_m)} \mathcal{Z}_m \wedge \beta^p > C_{b_m}\epsilon^{2p+n}.$$

As $b_m \to b$, the sequence $\{C_{b_m}\} \cup \{C_b\} \subseteq \mathbb{R}$ is compact and has a minimum value, say C', and we have

$$\int_{B_{2\epsilon}(b)} \mathcal{Z}_m \wedge \beta^p \geq C' \epsilon^{2p+n},$$

for all m. Now define the test-form $\varphi := \chi \beta^p$, where χ is a non-negative smooth function with compact support, equal to 1 in $B_{2\epsilon}(b)$ and vanishing outside $B_{3\epsilon}(b)$. As a result, the convergence of the complex sequence $\langle \mathcal{Z}_m, \varphi \rangle \longrightarrow \langle \mathcal{S}, \varphi \rangle$, together with the above inequality implies that

$$\langle \mathcal{S}, \varphi \rangle \geq C' \epsilon^{2p+n}.$$

$\square$

6.2 Any Cluster Value is a Tropical Current

In Lemma 6.2, we used the volume form $(dd^c \|z\|^2)^p$ that adds the mass in different components $K_{m,1}, \ldots, K_{m,m^n}$. In contrast, in the following lemma we use the "Fourier differential forms" which are sensitive to the change of phases in polar coordinates, and we show that Fourier measure coefficients finally vanish at every non-zero degree.

Lemma 6.7 *Let $\mathcal{S}$ be the weak limit of sequence $\mathcal{Z}_m := m^{p-n} \Phi_m^*[Z]$, then $\mathcal{S}$ is a tropical current.*

Proof By Proposition 6.2, the support of $\mathcal{S}$ has the form $\mathrm{Log}^{-1}(\mathrm{trop}(Z))$. Let us set $\mathcal{C} := \mathrm{trop}(Z)$. By Demailly's second theorem of support, [20, III.2.13], there are measures $\mathrm{d}\eta_\sigma$ such that

$$\mathcal{S} = \sum_{\sigma \in \mathcal{C}} \int_{x \in S_{N(\sigma)}} \left[\mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \pi^{-1}_{\mathrm{aff}(\sigma)}(x) \right] \mathrm{d}\eta_\sigma(x),$$

where By definition of tropical currents, we need to prove that all the non-zero degree Fourier coefficients of η_σ for all $\sigma \in \Sigma$ vanish, and therefore are Haar measures. To see this we use Proposition 4.5. We prove this by showing that when m is sufficiently large,

$$\langle \Phi_m[Z], \omega_I \rangle = 0,$$

for any fixed (p, p)-differential form ω given in polar coordinates by

$$\omega_I = \exp(-i \langle \nu, \theta \rangle) \rho(r) \mathrm{d}\theta_I \wedge \mathrm{d}r_I,$$

where $\nu \in \mathbb{Z}^n \setminus \{0\}$, $I \subseteq [n]$, with $|I| = p$, $\theta = (\theta_1, \ldots, \theta_n)$ and $r = (r_1, \ldots, r_n)$ are polar coordinates, and $\rho : \mathbb{R}^n \to \mathbb{R}$ is a smooth and appropriately chosen, and ω has the support given by $\Phi_m^{-1}(B_\epsilon(a_m))$ for $a_m \in Z$. For convenience, we change the coordinates to have $\omega_I = x^\nu \rho(r)\,\mathrm{d}r_I \wedge \mathrm{d}x_I$, with $(x, r) \in (S^1)^n \times (\mathbb{R}^+)^n \simeq (\mathbb{C}^*)^n$. Let us recycle the notation in the proof of Lemma 6.4: $b_m = (b_{m,1}, \dots, b_{m,n})$, $a_m = \Phi_m(b_m) = (a_{m,1}, \dots, a_{m,n}) \in Z$, and we partition $\Phi_m^{-1}(B_m^*)$ into a disjoint union of open sets $K_{m,1}, \dots, K_{m,m^n}$. Since by Lelong's theorem [20, Theorem III.2.7], the singular set of Z does not charge any mass, we assume that Z is smooth and consider the parametrisation

$$\boldsymbol{\tau} : (S^1)^p \times U \longrightarrow \Phi_m^{-1}(Z) \cap K_{m,1},$$

for $U \subseteq (\mathbb{R}^+)^p$. We let $\{1, \zeta, ..., \zeta^{m-1}\}$ be all different m-th roots of unity, and for $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{Z}^n \cap [0, m-1]^n$, define $\lambda^\ell(\zeta) := (\zeta^{\ell_1}, \dots, \zeta^{\ell_n})$. The component-wise multiplication $\boldsymbol{\tau} \cdot \lambda^\ell(\zeta)$ for different ℓ parametrises all the manifolds $\Phi_m^{-1}(Z) \cap K_{m,i}$ for $i = 1, \dots, m^n$. We write

$$\langle \Phi_m^*[Z], \omega_I \rangle = \sum_{i=1}^{m^n} \int_{\Phi_m^{-1}(Z) \cap K_{m,i}} \omega_I = \sum_{\ell \in \mathbb{Z}^n \cap [0,m-1]^n} \int_{(S^1)^p \times U} \big(\boldsymbol{\tau} \cdot \lambda^\ell(\zeta)\big)^* \omega_I.$$

For each $\ell = (\ell_1, \dots, \ell_n)$,

$$\begin{aligned} \big(\boldsymbol{\tau} \cdot \lambda^\ell(\zeta)\big)^*(x_j) &= x_j \circ [\boldsymbol{\tau} \cdot \lambda^\ell(\zeta)] = \zeta^{\ell_j}(\boldsymbol{\tau}^* x_j), \\ \big(\boldsymbol{\tau} \cdot \lambda^\ell(\zeta)\big)^*(dx_j) &= dx_j \circ [\boldsymbol{\tau} \cdot \lambda^\ell(\zeta)] = \zeta^{\ell_j}(\boldsymbol{\tau}^* dx_j). \end{aligned}$$

As a consequence, the term x^ν in ω_I discharges $\zeta^{\langle \ell, \nu \rangle}$, and dx_I produces $\zeta^{\langle \ell, \mathbb{1}_I \rangle}$, where $\mathbb{1}_I$ is the characteristic vector of $I \subseteq [n]$. We conclude

$$\langle \Phi_m^*[Z]\,,\, \omega_I \rangle = \sum_{\ell \in \mathbb{Z}^n \cap [0,m-1]^n} \zeta^{\langle \ell\,,\, \nu + \mathbb{1}_I \rangle} \int_{K_{m,1}} \Phi_m^*[Z] \wedge \omega_I.$$

The final sum, however, vanishes for a fixed $\nu = (\nu_1, \dots, \nu_n)$ and large m, since for $m \not| \nu_j$,

$$\sum_{i=0}^{m-1} (\zeta^i)^{\nu_j} = \frac{(\zeta^m)^{\nu_j} - 1}{\zeta^{\nu_j} - 1} = 0.$$

□

7 Applications

7.1 From Tori to Toric Varieties

Let us prove Theorem B of the introduction.

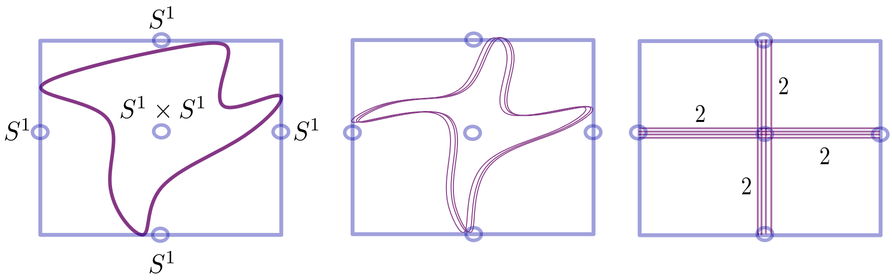

Fig. 1 Dynamical tropicalisation in $\mathbb{P}^1 \times \mathbb{P}^1$. See the discussion following the proof of Theorem 7.1

Theorem 7.1 *Let $Z \subseteq (\mathbb{C}^*)^n$ be an irreducible subvariety of dimension p, and $\bar{Z}$ a tropical compactification of Z in the smooth projective toric variety X_Σ. Then,*

$$\frac{1}{m^{n-p}} \Phi_m^*[\bar{Z}] \longrightarrow \overline{\mathcal{T}}_{\mathrm{trop}(Z)}, \quad \text{as } m \to \infty,$$

where $\Phi_m : X_\Sigma \longrightarrow X_\Sigma$ is the extension of $\Phi_m : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$, and $\overline{\mathcal{T}}_{\mathrm{trop}(Z)}$ is the extension by zero of $\mathcal{T}_{\mathrm{trop}(Z)}$ to X_Σ.

Proof Theorem A asserts that

$$\frac{1}{m^{n-p}} \left(\Phi_m \restriction_{(\mathbb{C}^*)^n}\right)^*[Z] \longrightarrow \mathcal{T}_{\mathrm{trop}(Z)}, \quad \text{as } m \to \infty.$$

As in Step 4 of the proof of Theorem 6.1,

$$\frac{1}{m^{n-p}} \overline{\left(\Phi_m \restriction_{(\mathbb{C}^*)^n}\right)^*[Z]} \longrightarrow \overline{\mathcal{T}}_{\mathrm{trop}(Z)}, \quad \text{as} m \to \infty.$$

Effectively, we only need to observe the equality

$$\overline{\left(\Phi_m \restriction_{(\mathbb{C}^*)^n}\right)^*[Z]} = \Phi_m^*[\bar{Z}],$$

which is addressed in Remark 5.8. □

Let us now elaborate on Fig. 1. The toric variety associated to the square is indeed $\mathbb{P}^1 \times \mathbb{P}^1$. The interior of the square corresponds to the two dimensional orbit $\mathcal{O}(\{0\})$, and the sides of the square to one dimensional orbit closures. On the left, the tropical compactification of a curve $C \subseteq (\mathbb{C}^*)^2$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is considered. The closure $\overline{C}$ intersects the one dimensional toric orbits properly and with multiplicity two at all the intersection points. All the toric orbits remain invariant under Φ_m^{-1}, and under this application the points in each toric orbit move towards the compact torus above the corresponding distinguished points. By applying Φ_m^{-1} to the curve we obtain a local covering. Multiplying the integration current $\frac{1}{\deg(C)}[\Phi_m^{-1}(\overline{C})]$ by $1/m$ normalises the total mass, which is depicted by the thickness of the curves. The limit becomes the associated tropical current whose fibers are transverse to the toric orbits, where the weights account for the intersection multiplicity numbers. Accordingly, the intersection points of multiplicity two converge to the *transverse* (non-normalised) Haar measures on the distinguished circles with mass two.

7.2 Dynamical Kapranov Theorem

Kapranov's theorem [58, Theorem 3.1.3] is a foundational result in tropical geometry for which we provide a dynamical version in the *trivial valuation* case. Let $A \subseteq \mathbb{Z}^n_{\geq 0}$ be a finite set, and $f(z) = \sum_{\alpha \in A} c_\alpha z^\alpha \in \mathbb{C}[z]$ be a complex polynomial. We set the *tropicalisation* of f to be

$$\mathrm{trop}(f) := \max_{\alpha \in A}\{\langle -\alpha, \ \cdot \ \rangle\} : \mathbb{R}^n \longrightarrow \mathbb{R}.$$

We have

$$\frac{1}{m}\Phi_m^*\big(\log|f|\big) = \frac{1}{m}\big(\log\big|\sum_{\alpha \in A} c_\alpha z^{m\,\alpha}\big|\big).$$

Similar to the observation in the beginning of Sect. 3 we obtain

$$\frac{1}{m}\Phi_m^*\big(\log|f|\big) \longrightarrow \mathrm{trop}(f) \circ \mathrm{Log}, \tag{III}$$

in $L^1_{\mathrm{loc}}(\mathbb{C}^n)$. See also [65, Theorem 3.4]. We note that the negative coefficient of α in $\mathrm{trop}(f)$ is to compensate for using $\mathrm{Log} = -\log \otimes 1$ in the preceding formula. The (dynamical) Kapranov's theorem asserts that the tropicalisation of the algebraic hypersurface $V(f)$ coincides with the (tropical current of the) tropical variety of tropicalisation of f. Here is Theorem C of the introduction.

Theorem 7.2 *[Dynamical Kapranov Theorem] Assume that* $[V(f)] = dd^c \log|f|$, *is an integration current in* $\mathcal{D}'_{n-1,n-1}((\mathbb{C}^*)^n)$. *Let* $\mathfrak{q} := \mathrm{trop}(f)$, *defined as above. We have that,*

$$m^{-1}\Phi_m^*[V(f)] \longrightarrow \mathcal{T}_{V_{\mathrm{trop}}(\mathfrak{q})}.$$

Proof We have $\Phi_m^*[V(f)] = \Phi_m^*(dd^c \log|f|) = dd^c(\Phi_m^* \log|f_t|)$. Dividing by m and taking the limit yields

$$m^{-1}\Phi_m^*[V(f)] = dd^c\big[m^{-1}\Phi_m^* \log|f|\big] \longrightarrow dd^c[\mathfrak{q} \circ \mathrm{Log}] = \mathcal{T}_{V_{\mathrm{trop}}(\mathfrak{q})}.$$

For the above convergence, we have used Eq. (III) and the continuity differentials with respect to the weak limit, and for the latter equality we have used the tropical Lelong–Poincaré equation; see Proposition 7.4. □

Remark 7.3 (a) We have observed that the tropicalisation of the Lelong–Poincaré equation leads to a version of Kapranov's theorem. It is therefore tempting to tropicalise the King's formula [20, III.8.18], and wish to obtain a version of the Fundamental Theorem of Tropical Geometry, see [58, Sect. 3.2]. However, Monge–Ampère operators are not in general weakly continuous, see [16, 37, 54], and naively applying the dynamical tropicalisation to the King's formula does not provide useful information.

(b) Let $W \subseteq \mathbb{C}^* \times (\mathbb{C}^*)^n$ be an irreducible subvariety of dimension $p+1$ such that the projection of W onto the first factor is *flat* and surjective. We may therefore regard W as a one-parameter family of algebraic varieties W_t, where each W_t is the p-dimensional fiber above $t \in \mathbb{C}^*$. In tropical geometry, it is shown that the limit in the Hausdorff sense of

$$\frac{1}{\log|t|}\mathrm{Log}(W_t), \quad \text{as } t \to \infty,$$

is a rational polyhedral complex which is balanced. See [63, 68] for the codimension one case, and [50] in any codimensions and the background on the evolution of proofs. The correct analogue of this statement in our setting seems to be proving that the sequence

$$\frac{1}{m^{n-p+1}}\Phi_m^*\big[(e^m, W_{e^m})\big], \quad \text{as} m \to \infty,$$

converges to a tropical current. See also [5, Theorem 5.2.7] for a related result.

7.2.1 Tropical Lelong–Poincaré Equation

Chambert-Loir and Ducros in [17] proved the generalised Lelong–Poincaré equation for Lagerberg currents. We have the following theorem which also appeared in [5].

Proposition 7.4 *For any tropical polynomial* $\mathfrak{q} : \mathbb{R}^n \to \mathbb{R}$*,*

$$dd^c[\mathfrak{q} \circ \mathrm{Log}] = \mathcal{T}_{V_{\mathrm{trop}}(\mathfrak{q})},$$

in $\mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$.

Proof On the one hand, $\mathfrak{q}$ is a piece-wise linear function, and it is easy to see that the support of $dd^c[\mathfrak{q} \circ \mathrm{Log}]$ coincides with the set $\mathrm{Log}^{-1}(|V_{\mathrm{trop}}(\mathfrak{q})|)$, on the other hand, $dd^c[\mathfrak{q} \circ \mathrm{Log}]$ is $(S^1)^n$ invariant. As a consequence of Proposition 4.6, $dd^c[\mathfrak{q} \circ \mathrm{Log}]$ is a tropical current. By Demailly's first theorem of support, [20, Theorem III.2.10], we only need to verify the equality of these $(n-1, n-1)$-dimensional currents on

$$\bigcup_{\sigma,\ \dim(\sigma)=n-1} \mathrm{Log}^{-1}(\sigma^\circ).$$

Therefore, it only remains to see that the weights induced by $dd^c[\mathfrak{q} \circ \mathrm{Log}]$ on any $(n-1)$-dimensional cone in $V_{\mathrm{trop}}(\mathfrak{q})$ coincides with the weights given in Definition 3.2.(b). We can assume, without loss of generality, that σ contains the origin. To prove the equality of the currents, it is sufficient to prove that for any ball B such that $B \cap \mathrm{Log}^{-1}(\sigma^\circ) \neq \varnothing$, $B \cap \partial \mathrm{Log}^{-1}(\sigma^\circ) = \varnothing$, and $B \cap \mathrm{Log}^{-1}\big(|V_{\mathrm{trop}}(\mathfrak{q})| - \sigma\big) = \varnothing$,

$$dd^c[\mathfrak{q} \circ \mathrm{Log}]\restriction_B = \mathcal{T}_{V_{\mathrm{trop}}(\mathfrak{q})}\restriction_B.$$

Assume that $\mathrm{aff}(\sigma) = \{x \in \mathbb{R}^n : \langle \beta_1, x\rangle = \langle \beta_2, x\rangle = \mathfrak{q}(x)\}$, for some $\beta_1, \beta_2 \in \mathbb{Z}^n$. We have that

$$\begin{aligned} dd^c[\mathfrak{q} \circ \mathrm{Log}]\restriction_B &= dd^c \max\{\langle \beta_1, \mathrm{Log}(z)\rangle, \langle \beta_2, \mathrm{Log}(z)\rangle\}\restriction_B, \\ \mathcal{T}_{V_{\mathrm{trop}}(\mathfrak{q})}\restriction_B &= w_\sigma \mathcal{T}_{\mathrm{aff}(\sigma)}\restriction_B . \end{aligned}$$

Therefore, it suffices to prove that

$$w_\sigma \mathcal{T}_{\mathrm{aff}(\sigma)} = dd^c \max\{\langle \beta_1, \mathrm{Log}(z)\rangle, \langle \beta_2, \mathrm{Log}(z)\rangle\}.$$

Recall by Definition 3.2, w_σ is defined to be the lattice length of $\beta_1 - \beta_2$. Let us write

$$\beta_1 - \beta_2 = w_\sigma \alpha = w_\sigma(\alpha_+ - \alpha_-),$$

where α is a primitive vector and $\alpha_\pm \in \mathbb{Z}^n_{\geq 0}$. We have

$$dd^c \max\{\langle \beta_1, \ . \ \rangle, \langle \beta_2, \ . \ \rangle\} \circ \mathrm{Log}(z) = w_\sigma dd^c \max\{\langle \alpha_+, \ . \ \rangle, \langle \alpha_-, \ . \ \rangle\} \circ \mathrm{Log}(z).$$

Finally, to see that the masses of $\mathcal{T}_{\mathrm{aff}(\sigma)}$ and $dd^c \max\{\langle \alpha_+, \ . \ \rangle, \langle \alpha_-, \ . \ \rangle\} \circ \mathrm{Log}(z)$ coincide, we note that $\mathcal{T}_{\mathrm{aff}(\sigma)}$ is an average with respect to the Haar measure of the fibers, and it is enough to show that the mass of each fiber coincides with the mass of $dd^c \max\{\langle \alpha_+, \ . \ \rangle, \langle \alpha_-, \ . \ \rangle\} \circ \mathrm{Log}(z)$. By Example 4.2, for each $x \in (S^1)^n$, the integration current on each fiber is given by

$$\big[\pi^{-1}_{\mathrm{aff}(\sigma)}(\bar{x})\big] = dd^c \log|(x \cdot z)^{\alpha_+} - (x \cdot z)^{\alpha_-}|,$$

where $\bar{x} = x + S_{(H \cap \mathbb{Z}^n)}$ is the element in the quotient $S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}$. Applying $\frac{1}{m}\Phi_m^*$ on both sides of this equation preserves the mass in any compactification of $(\mathbb{C}^*)^n$, but by Eq. (III) the left hand side becomes $dd^c \max\{\langle \alpha_+, \ . \ \rangle, \langle \alpha_-, \ . \ \rangle\} \circ \mathrm{Log}(z)$. □

7.3 The Genericity Condition in $\mathbb{P}^n$

Comparing our convergence results to the conjecture of Dinh and Sibony, we need to verify that there all the *generic* algebraic subvarieties $Z \subseteq \mathbb{P}^n$ of a given degree have the same tropicalisation. This implication, in fact, is the main result of main theorem of Römer and Schmitz in [67]. Roberto Gualdi has proposed a nice and intuitive explanation of this implication in codimension one: if we consider a homogeneous polynomial of degree d in $\mathbb{P}^n$, then all the coefficients are generically non-zero, which in turn, imposes the Newton polytope, hence the tropicalisation (with respect to trivial valuation). The genericity condition in dynamical systems usually amounts to avoiding certain invariant sets of low dimension. In Sect. 5, we observed that all the toric orbits are invariant under Φ_m and Φ_m^{-1}, and the relation between the toric orbits and tropicalisation is already stated in the following proposition by Sturmfels and Tevelev.

Proposition 7.5 ([70, Proposition 3.9]) *Let Σ be a complete (rational) fan in $N_{\mathbb{R}}$ and Z be a p-dimensional subvariety of $(\mathbb{C}^*)^n$. Assume that the closure $\bar{Z} \subseteq X_\Sigma$, does not intersect any of the toric orbits of X_Σ of codimension greater than p. Then,* $\operatorname{trop}(Z)$ *equals the union of all p-dimensional cones $\sigma \in \Sigma$ such that $\mathcal{O}_\sigma$ intersects $\bar{Z}$.*

To deduce that the genericity condition for an algebraic subvariety of $\mathbb{P}^n$ determines its tropicalisation, we will observe that the union of p-dimensional cones in $\mathbb{P}^n$ is the support of a *strongly extremal* fan. Recall that

Definition 7.6 (a) A tropical variety $\mathcal{C}$ is called *strongly extremal* if the support of $\mathcal{C}$ is can be uniquely weighted up to a multiple to become balanced. In other words, $\mathcal{C}$ generates an extremal ray in the cone of positively weighted tropical varieties.

(b) A current $\mathcal{T}$ in the cone of positive closed currents of bidimension (p, p) on X is called *extremal*, if any decomposition $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ in this cone implies that $\mathcal{T}_1$ and $\mathcal{T}_2$ are positive multiples of $\mathcal{T}$. In other words, $\mathcal{T}$ generates an extremal ray in the cone of positive closed currents of bidimension (p, p).

Lemma 7.7 *Let $\mathcal{C}, \tilde{\mathcal{C}}$ be two tropical varieties of same dimension such that $\mathcal{C}$ is strongly extremal, and $|\tilde{\mathcal{C}}| \subseteq |\mathcal{C}|$. Then $|\tilde{\mathcal{C}}| = |\mathcal{C}|$.*

Proof There is a sufficiently large integer k, such that the tropical variety $k\,\mathcal{C} - \tilde{\mathcal{C}}$ is a (positively weighted) tropical variety with support $|\mathcal{C}|$. If $|\tilde{\mathcal{C}}| \subsetneq |\mathcal{C}|$, the weights on $k\,\mathcal{C} - \tilde{\mathcal{C}}$ cannot be a multiple of the weights on $\mathcal{C}$.

□

Essentially the same proof as above gives the following.

Lemma 7.8 *Assume that $\mathcal{C}, \tilde{\mathcal{C}}$ are two tropical varieties such that $\mathcal{T}_{\mathcal{C}}$ is extremal among the tropical currents, and $|\tilde{\mathcal{C}}| \subseteq |\mathcal{C}|$. Then $|\mathcal{C}| = |\tilde{\mathcal{C}}|$.*

For a given polyhedral complex Σ, we denote by $\Sigma(p)$ the union of all p-dimensional cells in Σ. We give two different proofs of the following:

Theorem 7.9 ([67]) *Let Σ be the fan of $\mathbb{P}^n$, and Z be any p-dimensional subvariety of $(\mathbb{C}^*)^n$ of degree d, and generic in the sense that $\bar{Z} \subseteq \mathbb{P}^n$ does not intersect any of toric orbits of $X_\Sigma = \mathbb{P}^n$ with codimension greater than p. Then,* $\operatorname{trop}(Z)$ *is independent of the choice of Z.*

Proof In view of Proposition 7.5 we have the inclusion $|\operatorname{trop}(Z)| \subseteq \bigcup_{\sigma \in \Sigma(p)} \sigma$. By Lemma 7.7, to derive the converse inclusion it is sufficient to observe that the set $\bigcup_{\sigma \in \Sigma(p)} \sigma$ is the support of a strongly extremal tropical variety. Let us justify the extremality in two ways by devising several strong theorems. See also [58, Exercise 6.8.11].

(a) $\bigcup_{\sigma\in\Sigma(p)}\sigma$ is the support of the *Bergman fan of the uniform matroid* $U_{p+1,n+1}$; see [58, Example 4.2.13]. Therefore it is extremal by [46, Theorem 38] and the assertion follows from Lemma 7.7.

(b) Let $H\subseteq(\mathbb{C}^*)^n$ be a generic p-dimensional plane. Note that this genericity condition for H is equivalent to saying that all the maximal minors of the matrix of coefficients for any $n-p$ linear equations defining H are non-zero. By [58, Example 4.2.13] for such a generic p-dimensional plane H, the set $\text{trop}(H)$ is given by support of the Bergman fan of the uniform matroid: $\bigcup_{\sigma\in\Sigma(p)}\sigma$. To see that $\text{trop}(H)$ is extremal, we observe instead that the current $\mathcal{T}_{\text{trop}(H)}$ is extremal among all Φ_m-invariant currents. By Theorem A and [69, Proposition 6.1], for a generic H

$$\mathcal{T}_{\text{trop}(H)}=\lim_{m\to\infty}m^{p-n}(\Phi_m^*[H])=\lim_{m\to\infty}m^{p-n}(\Phi_m^*\omega^{n-p}),$$

where ω is the normalised Fubini–Study form on $\mathbb{P}^n$. Finally, the bidimension (p,p) *Green current* of Φ_m, given by $\lim_{m\to\infty}m^{p-n}\Phi_m^*(\omega^{n-p})$, is extremal among Φ_m-invariant currents, [26, Theorem 1.0.1].

As a result, all generic closed p-dimensional algebraic varieties of $\mathbb{P}^n$, up to the degree, have the same tropicalisation and,

□

The support of the Green current of Φ_m of bidimension (p,p) is called the *Julia set of order* p of Φ_m, denoted by J_p. Therefore, $\text{Log}(J_p)$ coincides with the support of $U_{p+1,n+1}$, *i.e.*, the Bergman fan of uniform matroid of rank $p+1$ on $n+1$ points.

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