Q1. Let $A \subseteq \mathbb{A}^n$ be a subset.

(a) (**5 marks**) What is the definition of the closure of A in $\mathbb{A}^n$?

(b) (**5 marks**) Prove that $\mathbb{V}(\mathbb{I}(A))$ equals the Zariski closure of A in $\mathbb{A}^n$.

(c) (**5 marks**) Give an example of a subset in $B \subseteq \mathbb{C}$ whose closure in the Zariski topology does not coincide with its closure in the Euclidean topology.

Q1 a) the closure of a subset A in $\mathbb{A}^n$ is the intersection of all closed sets containing A

$\bar{A} = \bigcap \{ C \subseteq \mathbb{A}^n \mid A \subseteq C,\ C$ is closed in the Zariski topology $\}$

1b) $\mathbb{V}(\mathbb{I}(A)) = \bar{A}$ proof:

$\mathbb{I}(A) = \{ f \in \mathbb{C}[x_1, \dots, x_n] : f(x) = 0$ for all $x \in A \}$

- If $x \in A$ then $f(x) = 0$ for all $f \in \mathbb{I}(A)$

$\Rightarrow x \in \mathbb{V}(\mathbb{I}(A))$

$\Rightarrow$ if $x \in A$ then $x \in \mathbb{V}(\mathbb{I}(A))$

$\Rightarrow A \subseteq \mathbb{V}(\mathbb{I}(A))$ then because $\mathbb{V}(\mathbb{I}(A))$ is a c.a.a.v it is closed and because the closure of A is the smallest closed set containing A then

$\Rightarrow \bar{A} \subseteq \mathbb{V}(\mathbb{I}(A))$

- $\bar{A}$, the closure of A is closed in the Zariski topology so $\bar{A} = \mathbb{V}(I)$ where I is an ideal of $\mathbb{C}[x_1, \dots, x_n]$

$\Rightarrow A \subseteq \mathbb{V}(I)$ because $A \subseteq \bar{A}$

then $\mathbb{I}(A) \supseteq \mathbb{I}(\mathbb{V}(I))$ because Hilbert's correspondence is inclusion reversing

$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$ (from Nullstellensatz) and $I \subseteq \sqrt{I}$ so $I \subseteq \mathbb{I}(\mathbb{V}(I))$

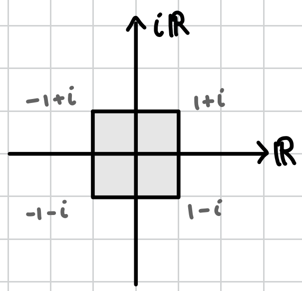
$\Rightarrow I \subseteq \mathbb{I}(\mathbb{V}(I)) \subseteq \mathbb{I}(A)$

$\Rightarrow \mathbb{V}(I) \supseteq \mathbb{V}(\mathbb{I}(A))$

$\Rightarrow \bar{A} \supseteq \mathbb{V}(\mathbb{I}(A))$

so then $\bar{A} \subseteq \mathbb{V}(\mathbb{I}(A)) \subseteq \bar{A} \Rightarrow \mathbb{V}(\mathbb{I}(A)) = \bar{A}$

1c) $B \subseteq \mathbb{C}$, $B = \{ a + bi : -1 \leq a \leq 1, -1 \leq b \leq 1 \}$



the closure of B in the Euclidean topology is B (because B is closed), $\bar{B} = B$

the closure of B in the Zariski topology is $\mathbb{C} = \mathbb{A}^1$, $\bar{B} = \mathbb{C} = \mathbb{A}^1$

because B has infinitely many elements so is not closed in the Zariski topology on $\mathbb{C}$, $\bar{B} \neq B$

and because any infinite set $V \subsetneq \mathbb{A}^1$ is not closed, the smallest closed set containing B is $\mathbb{A}^1$ so $\bar{B} = \mathbb{A}^1$

$\Rightarrow \bar{B}$ in the Zariski topology does not coincide with $\bar{B}$ in the Euclidean topology

Q2. (a) (**5 marks**) What is the definition of a compact subset of a topological space?

(b) (**10 marks**) Prove that $\mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}^2$ is compact in the Zariski topology but not in the Euclidean topology.

Q2a) a subset Y of a topological space X is compact if every open cover of Y has a finite subcover.

if $\{\mathcal{U}_\alpha\}$ is an open cover of Y then Y can be covered by only finitely many of these $\mathcal{U}_\alpha$. There exists $\mathcal{U}_{\alpha_1}, \mathcal{U}_{\alpha_2}, \dots, \mathcal{U}_{\alpha_n} \in \{\mathcal{U}_\alpha\}$ such that $Y \subseteq \bigcup_{i=1}^{n} \mathcal{U}_{\alpha_i}$

Q2b) $V = \mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}^2$ is a closed affine algebraic variety

- V is compact in the Zariski topology:

let $\{\mathcal{U}_\alpha\}$ be an open cover of V, $\mathcal{U}_\alpha$ open subsets of V

if $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \dots$ is a sequence of open subsets of V then

$\mathcal{U}_1^c \supseteq \mathcal{U}_2^c \supseteq \dots$ is a sequence of closed subsets of V, so $\mathcal{U}_i^c$ are c.a.a.Vs

$\Rightarrow \mathbb{I}(\mathcal{U}_1^c) \subseteq \mathbb{I}(\mathcal{U}_2^c) \subseteq \dots$ (Hilbert's correspondence)

So then there exists an integer r such that $\mathbb{I}(\mathcal{U}_r^c) = \mathbb{I}(\mathcal{U}_{r+1}^c) = \dots$ because $\mathbb{C}^2$ is Noetherian

$\Rightarrow$ ascending chains of open subsets of V are finite

if there doesn't exist a finite subcollection of $\{\mathcal{U}_\alpha\}$ such that V is contained in their union then there must be an infinite chain of open subsets of V

But there cannot be an infinite chain because all chains of open subsets of V are finite

$\Rightarrow$ V is contained in the union of a finite subcollection of $\{\mathcal{U}_\alpha\}$

$\Rightarrow$ every open cover of V has a finite subcover

$\Rightarrow$ V is compact in the Zariski topology

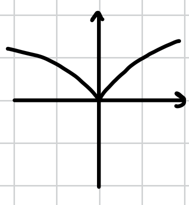
- V is not compact in the Euclidean topology:

$\mathbb{C}^n$ is homeomorphic to $\mathbb{R}^{2n}$ so the Heine-Borel Theorem holds for $\mathbb{C}^n$ with the Euclidean topology

$\Rightarrow$ V is compact if and only if it is closed and bounded

$V = \mathbb{V}(x^2 - y^3)$ is not bounded using the Euclidean metric because it is all the points of $\mathbb{C}^2$ such that $x^2 = y^3$ which is not bounded so for any distance M from the origin there will be a point $z \in \mathbb{V}(x^2 - y^3)$ such that $d(0,z) \geq M$

V is not bounded $\Rightarrow$ V is not compact in the Euclidean topology



Q3. (a) (**5 marks**) Find a curve $W \subseteq \mathbb{A}^2$ and a morphism $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$, such that W is irreducible but $\varphi^{-1}(W)$ is not.

(b) (**5 marks**) Let Y be a topological space and consider $X \subseteq Y$ with the subspace topology. Prove that if X is irreducible then so is its closure.

3a) $W \subseteq \mathbb{A}^2$ such that W irreducible ($\mathbb{I}(W)$ is prime)
and $\varphi^{-1}(W)$ is not irreducible ($\mathbb{I}(\varphi^{-1}(W))$ is not prime)

$\varphi^{-1}(W) = \{(a,b) \in \mathbb{A}^2 : \varphi(a,b) \in W\}$

$J = \mathbb{I}(\varphi^{-1}(W)) = \mathbb{I}(\{(a,b) \in \mathbb{A}^2 : \varphi(a,b) \in W\})$

such that for $(a,b),(c,d) \in \mathbb{A}^2$, $\exists\ (ac, bd) \in J \Rightarrow (a,b) \notin J$ and $(c,d) \notin J$

morphism: $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$
$(a,b) \longmapsto (a^2, b^2)$

curve: $W = \mathbb{V}(x - y^2)$
$\mathbb{I}(W) = \mathbb{I}(\mathbb{V}(x-y^2))$ is prime
$\Rightarrow W$ is irreducible

3b) Y a topological space and $X \subseteq Y$ with subspace topology

X is irreducible $\Rightarrow$ if $X = X_1 \cup X_2 \Rightarrow X \subseteq X_1$ or $X \subseteq X_2$
such that X_1 and X_2 are closed in X

closure of X

Suppose $\overline{X}$ is not irreducible then $\overline{X} = X_1 \cup X_2$ and $\overline{X} \nsubseteq X_1$ or $\overline{X} \nsubseteq X_2$
$X_1, X_2 \subseteq \overline{X}$ closed in subspace topology

if $X \subseteq X_1$ then $\overline{X} \subseteq \overline{X_1} = X_1$ (because X_1 closed) $\Rightarrow \overline{X} \subseteq X_1$ which contradicts $\overline{X} \nsubseteq X_1$
so then $X \nsubseteq X_1$
if $X \subseteq X_2$ then $\overline{X} \subseteq \overline{X_2} = X_2$ (because X_2 closed) $\Rightarrow \overline{X} \subseteq X_2$ which contradicts $\overline{X} \nsubseteq X_2$
so then $X \nsubseteq X_2$

$X \subseteq \overline{X} \Rightarrow X \subseteq \overline{X} = X_1 \cup X_2$ so then $X \nsubseteq X_1, X \nsubseteq X_2$ and $X \subseteq X_1 \cup X_2 \Rightarrow X$ is not irreducible

$\Rightarrow \overline{X}$ is irreducible

so if X is irreducible then so is its closure

(c) (**5 marks**) Prove that isomorphisms preserve irreducibility and dimension of closed affine algebraic varieties.

(d) (**10 marks**) Find the irreducible components of $\mathbb{V}(zx-y, y^2-x^2(x+1)) \subseteq \mathbb{A}^3$. You need to justify why each component is irreducible.

3c) Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be closed affini algebraic varieties and $\varphi : V \to W$ an isomorphism
then there exists a morphism of algebraic varieties $\Psi : W \to V$ such that $\Psi \circ \varphi = id_V$ and $\varphi \circ \Psi = id_W$

- isomorphisms preserve irreducibility:

if V is irreducible then if $V = V_1 \cup V_2$ for V_1, V_2 closed subsets of V then $V = V_1$ or $V = V_2$
$\Rightarrow$ V irreducible, if $V^c = (V_1 \cup V_2)^c = V_1^c \cap V_2^c$ for V_1, V_2 closed subsets of V then $V^c = V_1^c$ or $V^c = V_2^c$

morphisms are continuous $\Rightarrow$ the open sets V^c, V_1^c, V_2^c are mapped to open sets of W by φ
$\Rightarrow \varphi(V^c) = W^c$, $\varphi(V_1^c \cap V_2^c) = \varphi(V_1^c) \cap \varphi(V_2^c) = W_1^c \cap W_2^c$ for W a c.a.a.v in $\mathbb{A}^m$ and W_1, W_2 closed subsets of W
$\Rightarrow W^c = W_1^c \cap W_2^c \Rightarrow W = W_1 \cup W_2$ and if $V^c = V_1^c$ then $\varphi(V^c) = \varphi(V_1^c)$ and if $V^c = V_2^c$ then $\varphi(V^c) = \varphi(V_2^c)$
$\Rightarrow$ if V irreducible then $W = W_1 \cup W_2$ for W_1, W_2 closed subsets of W and $W = W_1$ or $W = W_2$
so W is irreducible so $\varphi(V)$ is irreducible
because φ is an isomorphism it has an inverse so that $\Psi(W)$ also preserves the irreducibility of W in the same way as φ
$\Rightarrow$ isomorphisms preserve irreducibility

- isomorphisms preserve dimension:

dimension: If V is irreducible then $\dim(V)$ is the largest integer d such that there is a chain $V = V_d \supsetneq V_{d-1} \supsetneq \dots \supsetneq V_0 = \{pt\}$
for V irreducible then its irreducibility is preserved by isomorphisms so the dimension can be defined for $\varphi(V)$ for φ an isomorphism
for a chain of closed sets $V = V_d \supsetneq V_{d-1} \supsetneq \dots \supsetneq V_0$ there is a chain of open sets $V^c = V_d^c \subsetneq V_{d-1}^c \subsetneq \dots \subsetneq V_0^c$ and each V_i^c is mapped to one open set so there are not any open sets added to the chain: $\varphi(V^c) = \varphi(V_d^c) \subsetneq \varphi(V_{d-1}^c) \subsetneq \dots \subsetneq \varphi(V_0^c)$
because open sets are mapped to open sets this is a chain of irreducible closed sets which is the complements of each $\varphi(V_i^c)$ so because there are the same number of irreducible $\varphi(V_i^c)$s as V_is, $\dim(V) = \dim(\varphi(V)) = d$.
the inverse Ψ also preserves dimension in the same way
$\Rightarrow$ isomorphisms preserve dimension

3d) $\mathbb{V}(zx-y, y^2-x^2(x+1)) \subseteq \mathbb{A}^3$

$$\left.\begin{array}{l} zx-y=0 \\ y^2-x^2(x+1)=0 \end{array}\right\} \Rightarrow y = zx \Rightarrow (zx)^2 - x^2(x+1) = 0 \Rightarrow (zx)^2 = x^2(x+1) = 0$$

$$\mathbb{V}(x^2(x+1)) = \mathbb{V}(x) \cup \mathbb{V}(x) \cup \mathbb{V}(x+1)$$

irreducible components: $\mathbb{V}(x)$ and $\mathbb{V}(x+1)$

$\mathbb{V}(x)$ and $\mathbb{V}(x+1)$ are both irreducible because they are both isomorphic to $\mathbb{A}^1$

$\mathbb{V}(x)$: $\varphi : \mathbb{A}^3 \to \mathbb{A}^1$, $(x,y,z) \mapsto x$ $\quad \Psi : \mathbb{A}^1 \to \mathbb{A}^3$, $x \mapsto (x,0,0)$ $\quad \Psi(\varphi(x)) = x$

$\mathbb{V}(x+1)$: $\varphi : \mathbb{A}^3 \to \mathbb{A}^1$, $(x,y,z) \mapsto x-1$ $\quad \Psi : \mathbb{A}^1 \to \mathbb{A}^3$, $x \mapsto (x+1,0,0)$ $\quad \Psi(\varphi(x)) = x$

Q4. (a) (**10 marks**) Let $V \subseteq \mathbb{A}^n$ be a Zariski-closed subset and $a \in \mathbb{A}^n \setminus V$ be a point. Find a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that

$$f \in \mathbb{I}(V), \quad f(a) = 1.$$

$V \neq V \cap \{a\}$

$\Rightarrow V^c \neq V^c \cup \{a\}^c$

4a) $\mathbb{I}(V) = \{ f \in \mathbb{C}[x_1, \ldots, x_n] : f(z) = 0$ for all $z \in V \}$

$V = \mathbb{V}(\{g_i\}_{i \in I})$, $\quad a \in V^c = \mathbb{A}^n \setminus V$

$f \in \mathbb{I}(V) = \mathbb{I}(\mathbb{V}(\{g_i\}_{i\in I})) = \sqrt{(\{g_i\}_{i \in I})} = \{ h \in \mathbb{C}[x_1, \ldots, x_n] : h^n \in (\{g_i\}_{i\in I})$ for some $n > 0 \}$

$\Rightarrow f^n \in (\{g_i\}_{i \in I})$ for some $n > 0$

if $\mathbb{I}(V)$ not radical then there exists a non-zero $f \in \mathbb{I}(V)$ which can be so that $f(a) = 1$ as $a \notin V$ and f can be non-zero outside of V

$V \neq V \cap \{a\}$

$\mathbb{I}(V) \neq \mathbb{I}(V \cap \{a\})$

$\mathbb{I}(V) \neq \mathbb{I}(V) \cap \mathbb{I}(\{a\})$

(b) (**15 marks**) Let $I, (g) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be two ideals. Assume that $\mathbb{V}(g) \supseteq \mathbb{V}(I)$.

(i) Prove that if $I = (f_1, \ldots, f_k)$, then

$$(f_1, \ldots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \ldots, x_{n+1}]. \qquad (1)$$

(ii) By only using Equation (1) and not the nullstellensatz, prove that there exists a positive integer m such that $g^m \in I$.

4bi) $I = (f_1, \ldots, f_k) = \{ h_1 f_1 + \ldots + h_k f_k : \text{for positive integer } k, h_i \in \mathbb{C}[x_1, \ldots, x_n] \}$

$$\mathbb{V}(g) \supseteq \mathbb{V}(I) \Rightarrow \mathbb{I}(\mathbb{V}(g)) \subseteq \mathbb{I}(\mathbb{V}(I))$$
$$\Rightarrow \sqrt{(g)} \subseteq \sqrt{I}$$
$$\Rightarrow (g) \subseteq I$$
$$\Rightarrow g = h_1 f_1 + \ldots + h_k f_k \in I$$

$J = (f_1, \ldots, f_k, x_{n+1} g - 1)$

I is not an ideal of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ because multiplying a polynomial in I by the polynomial $x_{n+1} \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ gives a polynomial not in I

adding $x_{n+1}g - 1$ to the generating set to get J makes J an ideal of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ because g is a linear combination of f_i with coefficients $h_i \in \mathbb{C}[x_1, \ldots, x_n]$

plan of proof that didn't work:
- I is not an ideal of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ because it doesn't have the multiplication property of ideals
- J is an ideal because adding $x_{n+1}g - 1$ to the generators means there is now the multiplication property of ideals
- J contains a unit because of $x_{n+1}g - 1$ So $J = \mathbb{C}[x_1, \ldots, x_{n+1}]$

4bii) Equation 1 : $(f_1, \ldots, f_k, x_{n+1} g - 1) = \mathbb{C}[x_1, \ldots, x_{n+1}]$

$$x_{n+1}g - 1 = x_{n+1}(h_1 f_1 + \ldots + h_k f_k) - 1 = 0$$

$$\Rightarrow \quad 1 = x_{n+1}(h_1 f_1 + \ldots + h_k f_k)$$

$\Rightarrow$ x_{n+1} is the multiplicative inverse of g, $g^{-1} = x_{n+1} \notin I$ and $x_{n+1}^{-1} = g$

$$1^m = x_{n+1}^m (h_1 f_1 + \ldots + h_k f_k)^m$$

$$\Rightarrow \quad (x_{n+1}^{-1})^m = (h_1 f_1 + \ldots + h_k f_k)^m$$

$I, (g) \subseteq \mathbb{C}[x_1, \ldots, x_n] \quad x_{n+1}^m \notin I, x_{n+1}^m \notin (g)$

$$g = (h_1 f_1 + \ldots + h_k f_k) \Rightarrow g \in I$$

$$x_{n+1}g - 1 = 0 \quad \Rightarrow x_{n+1}g = 1 \quad \Rightarrow (x_{n+1}g)(x_{n+1}g) = (x_{n+1}g)^2 = 1^2 = 1$$

Q5. Prove at least one implication from each of the following equivalences.

(a) (**10 marks**) Show that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is injective if and only if φ is *dominant*. Recall that a map, φ, is called dominant if its image, $\varphi(V)$, is dense in W.

Q5 a) pullback $\varphi^*: \mathbb{C}[W] \rightarrow \mathbb{C}[V]$ is injective $\Rightarrow$ φ is dominant

<u>proof</u>: Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$

and let $\varphi: V \rightarrow W$ induce the $\mathbb{C}$-algebra homomorphism, the pullback $\varphi^*: \mathbb{C}[W] \rightarrow \mathbb{C}[V]$
$g \mapsto g \circ \varphi$

$\varphi^*: \mathbb{C}[W] \rightarrow \mathbb{C}[V]$ injective $\Rightarrow$ $Ker(\varphi^*) = \{0\}$

$\varphi(V)$ is dense in W if for all $x \in W$ and every neighbourhood $\mathcal{U} \subseteq W$ of x then $\mathcal{U} \cap \varphi(V) \neq \emptyset$

Suppose $\varphi(V)$ is not dense in W then there exists a $x \in \mathcal{U}$ such that $\mathcal{U} \cap \varphi(V) = \emptyset$

$\mathcal{U}$ open $\Rightarrow$ $\mathcal{U}^c$ is a c.a.a.v so then $\mathcal{U}^c = \mathbb{V}(\{f_i\})$

$\Rightarrow$ for all $z \in \mathcal{U}^c$, there exists an $f \in \{f_i\}$ such that $f(z) = 0$

$\Rightarrow$ $\varphi^*(f) = f \circ \varphi = 0$

theres at least one $f_i \in \mathbb{V}(\{f_i\})$ not equal to the zero polynomial because if all of them were $f_i = 0$ then $\mathcal{U}^c = \mathbb{V}(\{f_i\}) = \mathbb{V}(0) = \mathbb{A}^m$ but $\mathcal{U}^c \neq \mathbb{A}^m$ because $(\mathcal{U}^c)^c = \mathcal{U} \neq \emptyset$ because $x \in \mathcal{U}$

$\Rightarrow$ there exists an $f_i \neq 0$ such that $\varphi^*(f_i) = f_i \circ \varphi = 0$

$\Rightarrow$ $f_i \in Ker(\varphi^*)$ which contradicts that $Ker(\varphi^*) = \{0\}$

$\Rightarrow$ it must then be that $\varphi(V)$ is dense in W
$\Rightarrow$ φ is dominant

(b) (**10 marks**) Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W.

5b) pullback $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective $\Rightarrow$ φ defines an isomorphism between V and some algebraic subvariety of W

proof: Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$

$\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ surjective $\Rightarrow$ for all $f \in \mathbb{C}[V]$ there exists some $g \in \mathbb{C}[W]$ such that $\varphi^*(g) = g \circ \varphi = f$

$\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$

$g \mapsto g \circ \varphi$

let φ which induces the pullback φ^* be the morphism $\varphi : V \to \varphi(V)$

φ is an isomorphism between V and $\varphi(V)$ if there exists a morphism $\psi : \varphi(V) \to V$ such that $\psi \circ \varphi = id_V$ and $\varphi \circ \psi = id_{\varphi(V)}$

$\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is a surjective ring homomorphism so by the fundamental isomorphism theorem,

$$\mathbb{C}[W] / Ker(\varphi^*) \cong \mathbb{C}[V] \qquad \left(Ker(\varphi^*) = \{ f \in \mathbb{C}[W] : \varphi^*(f) = 0 \} \right)$$

$$\mathbb{C}[W] / Ker(\varphi^*) = \{ g + Ker(\varphi^*) : g \in \mathbb{C}[W] \} = \{ g + f : f \in ker(\varphi^*) \}$$

for $x \in \varphi(V)$ all the fs are zero so the set is $\{$all polynomials in $\mathbb{C}[x_1, \dots, x_m]$ restricted to $\varphi(V)\}$

$\varphi(V) \subseteq W \subseteq \mathbb{A}^m$ then $\mathbb{C}[\varphi(V)] = \mathbb{C}[x_1, \dots, x_m] / \mathbb{I}(\varphi(V))$ $= \{$all polynomials in $\mathbb{C}[x_1, \dots, x_m]$ restricted to $\varphi(V)\}$

$$\mathbb{I}(\varphi(V)) = \{ f \in \mathbb{C}[x_1, \dots, x_m] : f(x) = 0 \text{ for all } x \in \varphi(V) \}$$

$$\Rightarrow \mathbb{C}[\varphi(V)] = \frac{\mathbb{C}[x_1, \dots, x_m]}{\mathbb{I}(\varphi(V))} = \frac{\mathbb{C}[W]}{Ker(\varphi^*)} \cong \mathbb{C}[V] \quad \Rightarrow \mathbb{C}[\varphi(V)] \cong \mathbb{C}[V]$$

Exercise 2.40: $V \cong V'$ if and only if $\mathbb{C}[V] \cong \mathbb{C}[V']$

$\mathbb{C}[V] \cong \mathbb{C}[\varphi(V)]$ $\Rightarrow$ there's an isomorphism between V and $\varphi(V)$ by Exercise 2.40

$\varphi(V)$ is an algebraic subvariety of W because φ is an isomorphism

$\Rightarrow$ φ defines an isomorphism between V and some algebraic subvariety of W, $\varphi(V)$