HW1

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A few important facts: ⓪ (I sometime supress the bracket in $\mathbb{V}, \mathbb{I}$).

① For any $S \subseteq \mathbb{C}[x_1, x_2]$: $\mathbb{V}(S) = \mathbb{V}(\langle S \rangle)$ ($\langle S \rangle$ is the ideal generated by S)

② If $I \subseteq J$ are ideals of $k[x_1, \dots x_n]$, then $\mathbb{V}(J) \subseteq \mathbb{V}(I)$

1. $\mathbb{V}(\langle f \rangle \cap \langle g \rangle) \overset{Ex.2.11}{=} \mathbb{V}(\langle f \rangle) \cup \mathbb{V}(\langle g \rangle) \overset{①}{=} \mathbb{V}(f) \cup \mathbb{V}(g)$

($\vec{x}$ satisfies $f(\vec{x})=0$ or $g(\vec{x})=0$ iff $\vec{x}$ satisfies $fg(\vec{x})=0$) $= \mathbb{V}(fg)$

$$\mathbb{V}(f) \cap \mathbb{V}(g) \overset{①}{=} \mathbb{V}(\langle f \rangle) \cap \mathbb{V}(\langle g \rangle) \overset{Ex.2.11}{=} \mathbb{V}(\langle f + g \rangle)$$

Also clearly $\mathbb{V}(f) \cap \mathbb{V}(g) \subseteq \mathbb{V}(f) \cup \mathbb{V}(g)$

And clearly $\mathbb{V}(f) \cap \mathbb{V}(g) \subseteq \mathbb{V}(f+g)$

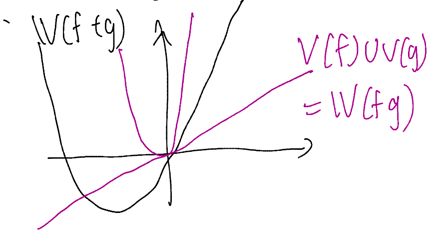
since $f(\vec{x}) = 0$ and $g(\vec{x}) = 0$ implies $f+g\,(\vec{x}) = 0$.

In general $\mathbb{V}(f+g)$ and $\mathbb{V}(f) \cup \mathbb{V}(g)$ is incomparable.

Example: $f := y - x$ $\quad g : y - x^2$.

$f + g = 2y - x - x^2$ so $\mathbb{V}(f+g)$ is the parabola $y = \frac{x(x-1)}{2}$

$\mathbb{V}(f) \cup \mathbb{V}(g)$ is the union of the diagonal and a parabola. $[(y-x)(y-x^2) = 0]$



(For example $(4, 16) \in \mathbb{V}(g)$ but not in $\mathbb{V}(f+g)$.
$(2, 1) \in \mathbb{V}(f+g)$ but not in $\mathbb{V}(f) \cup \mathbb{V}(g)$)

So the required comparison is $\mathbb{V}(\langle f \rangle + \langle g \rangle) = \mathbb{V}(f) \cap \mathbb{V}(g)$
$\subseteq \mathbb{V}(\langle f \rangle \cap \langle g \rangle) = \mathbb{V}(fg)$

and $\mathbb{V}(f) \cap \mathbb{V}(g) \subseteq \mathbb{V}(f+g)$.

2 a) The <u>closure</u> of A is the smallest closed set that contains A.

b) $\mathbb{V}(\mathbb{I}(A))$ is closed in $\mathbb{A}^n$ because it is a closed affine algebraic variety (It is $\mathbb{V}(?)$ of something, so it is by definition)

It also contains A. Reason: let $\vec{a} \in A$, then fix any $f \in \mathbb{I}(A)$, by definition, $f(\vec{a}) = 0$, so $\vec{a} \in \mathbb{V}(\mathbb{I}(A))$ since $\vec{a}$ is in the zero set of all $f \in \mathbb{I}(A)$.

So it remains to show that $\mathbb{V}(\mathbb{I}(A))$ is the smallest such closed set.

Suppose there is closed set $\underline{X}$ such that $A \subseteq \underline{X}$, we want to show $\mathbb{V}(\mathbb{I}(A)) \subseteq \underline{X}$.

Since X is closed write $X = \mathbb{V}(B)$ for some radical ideal B.
Then $A \subseteq X \Rightarrow \mathbb{I}(X) \subseteq \mathbb{I}(A)$
$(\mathbb{I}(X) = \mathbb{I}(\mathbb{V}(B)) = \sqrt{B} = B) \Rightarrow B \subseteq \mathbb{I}(A)$
$\overset{(2)}{\Rightarrow} \mathbb{V}(\mathbb{I}(A)) \subseteq \mathbb{V}(B) = X$. $\divideontimes$

c) $B = \mathbb{Z}$, $C = \mathbb{Q} \subseteq \mathbb{A}^1$ Note that
$\mathbb{I}(B) = \{f \mid f(n) = 0 \text{ all } n \in \mathbb{Z}\}$
$= \{0\}$ since non constant polynomial have only finitely many roots.
Similarly $\mathbb{I}(C) = \{0\}$
$\therefore \mathbb{V}(\mathbb{I}(B)) = \mathbb{V}(\mathbb{I}(C)) = \mathbb{V}(\{0\}) = \mathbb{A}^1$

d) $\varphi: \mathbb{A}^2 \longrightarrow \mathbb{A}^2$, $(x, y) \longmapsto (x^2, y)$. $W \subseteq \mathbb{A}^2$ be $\{(4,0)\} = \mathbb{V}(x-4, y)$
Clearly W is irreducible.

$\varphi^{-1}(W) = \{(x,y) \in \mathbb{A}^2 \mid (x^2, y) = (4,0)\}$
$= \{(2,0), (-2,0)\}$
$= \{(2,0)\} \cup \{(-2,0)\} = \mathbb{V}(x-2, y) \cup \mathbb{V}(x+2, y)$
So $\varphi^{-1}(W)$ is not irreducible

3 a) A set C is compact if for any open cover $\{U_i\}_{i \in I}$ of K, there is a finite subcover $\{U_{i_j}\}_{j=1}^{n}$ of K. (where $i_j \in I$ for all j)

b) i) We assume an open cover $\{U_i\}$ of $\mathbb{V}(x^2-y)$ (with index set I)
$\bigcup_{i \in I} U_i \supseteq \mathbb{V}(x^2-y) \Leftrightarrow \left(\bigcap_{i \in I} U_i^c\right)^c \supseteq \mathbb{V}(x^2-y)$
$\Leftrightarrow \mathbb{V}(x^2-y) \cap \left(\bigcap_{i \in I} U_i^c\right) = \emptyset$
Now U_i^c are closed (by definition) hence it can be written as $\mathbb{V}(I_i)$ for some $I_i \subseteq k[x_1, \dots x_n]$. So
$\bigcup_{i \in} U_i \supseteq \mathbb{V}(x^2-y) \Leftrightarrow \mathbb{V}(x^2-y) \cap \left(\bigcap_{i \in I} \mathbb{V}(I_i)\right) = \emptyset$
$\Leftrightarrow \mathbb{V}(x^2-y) \cap \mathbb{V}\left(\sum_{i \in I} I_i\right) = \emptyset$

Claim: There exists $n \in \mathbb{N}$ and subsequence $\{i_1, \dots i_n\} \subseteq I$ such that:
$$\sum_{i \in I} I_i = \sum_{j=1}^{n} I_{i_j}$$
Reason: Assume not, note that it is always the case that finite sums of $\{I_i\}_{i \in I}$ contains in $\sum_{i \in I} I_i$. So assume that all subsequence $\{\alpha_1 \dots \alpha_M\} \subseteq I$.

$\{I_i\}_{i\in I}$ contains in $\sum_{i\in I} I_i$. So assume that all subsequence $\{\alpha_1 \dots \alpha_M\} \subseteq I$, we have $\sum_{k=1}^{M} I_{\alpha_k} \subsetneq \sum_{i\in I} I_i$. But then, one can build a ascending chain $(0) \subsetneq I_0 \subsetneq I_0 + I_1 \subsetneq I_0 + I_1 + I_2 \dots \subsetneq J$, which contradicts $\mathbb{C}[x_1, \dots x_n]$ being noetherian.

Hence $\bigcup_{i\in} \mathcal{U}_i \supseteq \mathbb{V}(x^2-y) \iff \mathbb{V}(x^2-y) \cap \mathbb{V}(\sum_{j=1}^{n} I_{i_j}) = \emptyset$

By working backwards above: $\iff \bigcup_{j=1}^{n} \mathcal{U}_{i_j} \supseteq \mathbb{V}(x^2-y)$

Since $\mathbb{V}(I_{i_j}) = \mathcal{U}_{i_j}$ by definition. Hence we found an finite subcover.

ii) It is not compact in the Euclidean Topology (which is the metric/product topology $\mathbb{C} \times \mathbb{C}$). The reason is as follows: Consider the cover $\mathcal{U}_n := B_n(0) \times B_n(0)$, where $B_n(0)$ is the set $\{z \in \mathbb{C} \mid |z| = n\}$. This is the open ball, hence open in the Euclidean topology (for $\mathbb{C}$, hence) $\mathbb{C}^2$. Clearly $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n = \mathbb{C}^2$ hence it is a open cover for $\mathbb{V}(y-x^2)$ as well. But there is no finite subcover for $\mathbb{V}(y-x^2)$. Since every finite subcover $\mathcal{U}_{j_1} \cup \mathcal{U}_{j_2} \cdots \cup \mathcal{U}_{j_m}$ is a bounded set (in the metric topolosy sense) as elements have a maximum modulus of j_m, but $\mathbb{V}(y-x^2)$ itself is unbounded (as $(r, r^2) \in \mathbb{V}(y-x^2)$ for all $r \in \mathbb{R}$, say, and the modulus can be arbitrarily large.)

4 a) A field k has algebraic closure $\bar{k}$ when all the polynomials $f \in k[x]$ admits a factorization $f(x) = \lambda(x-\alpha_1)(x-\alpha_2) \cdots (x-\alpha_n) \in \bar{k}[x]$. (where $\alpha_1, \alpha_2 \dots \alpha_n \in \bar{k}$).

b) ($:\Rightarrow$) Suppose $I \neq (1)$ (i.e. $I \subsetneq k[x_1, x_2 \dots x_n]$) Then I is a proper ideal, so there is a maximal ideal m such that $I \subseteq m$, hence $\mathbb{V}(m) \subseteq \mathbb{V}(I)$, but we know

- m has the form $\langle x_1 - a_1, x_2 - a_2, \dots x_n - a_n \rangle \subseteq \bar{k}[x_1, x_2 \dots x_n]$

So $\mathbb{V}(m) = \{(a_1, a_2, \dots a_n)\}$ a one point set in $\bar{k}^n$

So $\mathbb{V}(I)$ is non-empty because it contains $\mathbb{V}(m)$.

($\Leftarrow:$) We proof the contrapositive since it is easy. Suppose $I = (1) = k[x_1, x_2 \dots x_n]$. Then $\mathbb{V}(I) = \emptyset$ since there is no points in $\bar{k}^n$ such that it vanishes on the constant polynomial.

5 We first setup some notation: let $V \subseteq \mathbb{A}^n$ and $\mathbb{C}[V] = \mathbb{C}[x_1, \dots x_n] / \mathbb{I}(V)$

let $W \subseteq \mathbb{A}^m$ and $\mathbb{C}[W] = \mathbb{C}[y_1, \dots y_m] / \mathbb{I}(W)$

$\varphi: V \to W$ takes form $\vec{v} \mapsto (\varphi^1(\vec{v}), \dots, \varphi^m(\vec{v}))$, $\varphi^i \in \mathbb{C}[x_1, \dots x_n]$.

a) We show the forward direction. Assume φ^* is injective so $\ker \varphi^* = \{0\} \subseteq \mathbb{C}[W]$. But note $0 \in \mathbb{C}[W]$ is the equivalence class consist of exactly the polynomials of $\mathbb{I}(W)$. In other words:

$$\varphi^*(f) = 0 \iff f \in \mathbb{I}(W) \quad , \text{ i.e.}$$

$$f \circ \varphi = 0 \iff f \in \mathbb{I}(W) \quad , \text{ i.e.}$$

$$f(\varphi(\vec{x})) = 0 \text{ for all } \vec{x} \in V \iff f \in \mathbb{I}(W). \text{ i.e. } \mathbb{I}(\varphi(V)) = \mathbb{I}(W)$$

Hence we have $\mathbb{V}\mathbb{I}(\varphi(V)) = \mathbb{V}\mathbb{I}(W) = W$ since W is an algebraic variety. To see φ is dominanant, recall Q2 we have $\mathbb{V}\mathbb{I}(\varphi(V))$ is the closure of $\varphi(V)$ under the Zariski Topology, so the above equation literally says the closure of $\varphi(V)$ is W, i.e. $\varphi(V)$ is dense in W as required.

b) We do the backwards direction, Suppose φ is an isomophism from V to a subvariety $W_0 \subseteq W$, then there exists morphism $\psi : W_0 \to V$: takes form

$$\psi(\vec{a}) \mapsto (\psi_1(\vec{a}), \psi_2(\vec{a}) \dots \psi_n(\vec{a})) \text{ for all } \vec{a} \in W_0, \text{ where}$$

$\psi_1, \psi_2 \dots \psi_n$ are polynomials in $\mathbb{C}[y_1, \dots y_m]$.

Now we proceed to show φ^* is surjective. Given any $g \in \mathbb{C}[V]$, viewed as a polynomial function: $V \to k$, we define a polynomial $f \in \mathbb{C}[W]$, viewed as a polynomial function from W to k as follows:

$$f(\vec{w}) := g(\psi_1(\vec{w}), \psi_2(\vec{w}), \dots \psi_n(\vec{w})) = g(\psi(\vec{w}))$$

Note that $(\psi_1(\vec{w}), \dots \psi_n(\vec{w}))$ is really a vector in V (by the definition of ψ) so the map f is well-defined. Also $\varphi^*(f) = f \circ \varphi$ so

$$\varphi^*(f)(\vec{v}) = f(\varphi(\vec{v})) = g(\psi(\varphi(\vec{v}))) = g(\vec{v})$$

since $\psi \circ \varphi = id$ as φ is an isomorphism, hence $\varphi^*(f) = g$, hence φ^* is surjective as required.