Algebraic Geometry, Lecture 1

Farhad Babaee

University of Bristol

Schedule // Assessments // Office HoursSchedule // Assessments // Office Hours item optionsSchedule // Assessments // Office Hours Lectures and Problem Classes

Mondays 15:00 - 17:00 (Fry G.16) Fridays 10:00 - 11:00 (Fry G.16)

Assessment

Problem Classes Presentations (5Final exam (50

Master's students: Final written exam (80

PhD/research students: Oral exam or presentation.

Office Hours (My office Fry 2.13)

Wednesdays 15:30 - 16:30

Admin stuff

- Lectures and Problem Classes
  - Mondays
  - Fridays

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  - Assessed Homework 1, 22.5%, title: Affine Varieties, dates: Feb 04, noon — Feb 11, noon. , Individual upload
  - Assessed Homework 2, 22.5%, title: General varieties, dates: March 04, noon — March 11 noon.

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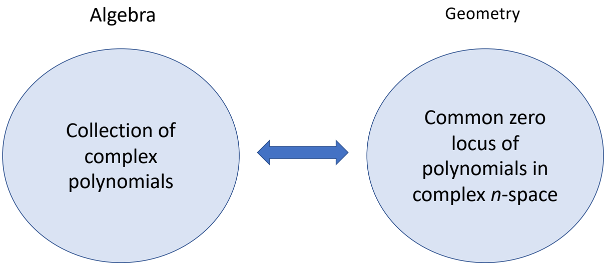
What is the (Complex) Algebraic Geometry?



The goal of our course

- Describe basic objects in algebraic geometry
- Describe dimension, degree, smoothness, etc. in both algebraic and geometric settings
- In toric varieties read off a lot of info from combinatorial data

Some examples

Example

The zero locus of the polynomial $x^2 + y^2 + 1$ is empty in $\mathbb{R}^2$ but non-empty in $\mathbb{C}^2$.

Example (Fermat's last theorem)

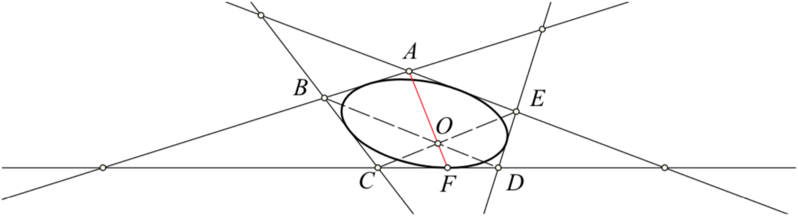
The zero locus of $x^n + y^n + z^n$ is empty in $\mathbb{Q}^3$, for integer $n \geq 3$.

Remark

Algebraic geometry can be done in any field, but for simplicity and intuition, we mostly deal with complex numbers in this course.

Some history

Babylonians (2000-1500BC) seemed to know how to solve $ax^2 + bx = c$. in $\mathbb{R}$. They also knew Pythagoras Theorem!

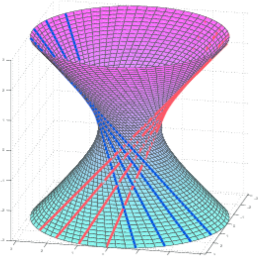


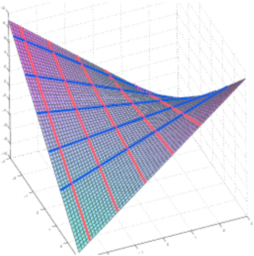
Appolonius (262-190 BC) seemed to know that a non-degenerate plane conic is determined by 5 tangent lines.

Mid-nineteenth century

Bernhard Riemann (1828-1866) showed that compact Riemann surfaces can be described as zero sets of a polynomial function. We later state a generalisation of this statement which called the Chow Theorem.

Mid-nineteenth century





Any quadratic surface (zero set of a degree 2 polynomial in 3 variables) can be covered by lines.

Any cubic surface contains exactly 27 lines!

Turn of the century

- Italian School of Algebraic Geometry asserted many statements, but sometimes they lacked rigor. For example, let us look at the Bézout's Theorem

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Assume that C_1 and C_2 are two curves of degree d_1 and d_2 in the complex projective space $\mathbb{P}^2$. Then, the number of intersection points, counting the multiplicities, is $d_1 d_2$.

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An idea for the proof.

- Proof for $d_1 = 1$, $d_2 = 2$.
- By moving the curves we can assume the intersections take place in $\mathbb{C}^2 \subseteq \mathbb{P}^2$.
- The theorem is true if the defining functions of C_1 and C_2 are of the form $f(x, y) = a_1 y - b_1 x$, and $g(x, y) = (a_1' y - b_1' x)(a_2' x - b_2' y)$.
- Intuitively, the number of intersection points, taking into account the multiplicities, do not change if we perturb the curves.

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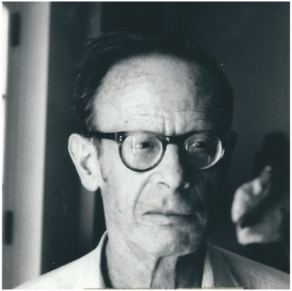
- Not complete!

$\square$

Beginning of 20th century

David Hilbert (1862-1943) and Emmy Noether (1882-1935) set the algebraic foundations for solid algebraic geometry.

20th century

Oscar Zariski 1899-1986, and André Weil 1906-1998 with many others revived the topic and developed it.

Alexander Grothendieck (1928-2014 - Fields Medal 1966) aided by Artin, Mumford (Fields 1974) and many others, introduced Scheme Theory and lifted Algebraic Geometry to a “dizzying heights of abstraction”. This abstraction made algebraic geometry more natural, general, and often simplified.

What do we study in this course?

- Basic foundations and many nice constructions/theorems:
  - Affine, Projective, and Quasi-Projective Algebraic Varieties
  - Degree
  - Gluing
  - Some flavours of Scheme Theory
  - Smoothness and resolution of singularities
  - Some toric geometry

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- What's your name?
- Have you taken any courses on the Geometry of Manifolds? Algebraic Topology? (No problem if you haven't!)