Probability Theory 1: Notes for Mathematicians and Physicists

BalĂĄzs MĂĄrton[1] and BĂĄlint TĂłth[1]

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[1]Budapest University of Technology and Economics

These notes are for an introductory course on Probability Theory, primarily intended for undergraduate mathematics and physics students. So far, they focus on presenting the main theoretical foundations and a few basic examples, with a limited number of exercises and practice problems. The main objective is to provide practical and useful knowledge and methods, but precision and mathematical rigor are also emphasized, serving as a foundation for understanding more advanced topics in probability theory, stochastic processes, and statistics.

Keywords: Probability Theory, Probability, Conditional Probability, Bayes' Theorem, Independence, Random Variable, Distribution, Distribution Transformation, Poisson Process, Joint Distribution, Expectation, Central Limit Theorem, Law of Large Numbers

Preface

These notes are primarily intended for students of the Probability Theory 1 course in the mathematics and physics BSc programs at the Budapest University of Technology and Economics. They currently cover the main theoretical foundations and a few basic examples, with a limited number of exercises and practice problems. These notes *are in no way a substitute for lectures and practice sessions*, but rather serve as a small supplement to them.

Yo, listen up, let me take you to class, Probability Theory, where the knowledge is vast. For math and physics students in the BSc,

=== Weâre breakinâ down the basics, makin' it easy, you'll see!

We got the foundations, yeah, thatâs where we start, A few examples, hittinâ you hard with the smart. Limited exercises, but donât you fear, We're layinâ the groundwork, the path is clear.

Itâs practical and useful, methods so tight, But precision's the mission, gotta get it right! With rigor in the mix, buildin' blocks of the game, For advanced topicsâyeah, remember the name.

Probability, stochastics, and stats in line, We're droppinâ knowledge thatâll sharpen your mind. From Bayes to Poisson, to joint distributions, Expectations rise, it's the math revolution!

We set the base for the central limit flow, And the law of large numbers, now you know how it goes! Itâs a journey, but we got you on the track, So hit those problems, ain't no turnin' back!

Oh, and one last thing, donât you forget, Big thanks to Farhad, youâre the best yet! === These notes are

largely based on Sheldon Ross's book and BĂĄlint TĂłth's handwritten notes and problem sets (see below). The structure and sequence of the material mostly follow the former. However, in many places, we provided a more detailed explanation of the mathematical background and discussed some theorems or phenomena in greater depth. In these cases, we mainly relied on BĂĄlint TĂłth's notes. For examples and problems, we also drew from both sources, as well as other references.

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References:

- S. Ross: *A First Course in Probability*, Seventh Edition, Pearson and Prentice Hall, 1976-2006
- TĂłth B.: *Probability Theory I*, Problem Sets and Handwritten Notes, Budapest University of Technology and Economics
- G. Grimmett: *One Thousand Exercises in Probability*, Oxford University Press, USA, 2001
- A. M. Iaglom, I. M. Iaglom: *Probleme Neelementare Tratate Elementar*, Second Edition, Editura TehnicĂ, Bucharest
- B. A. Sevastyanov, V. P. Chistyakov, A. M. Zubkov: *Problems in Probability Theory*, Textbook Publishing, Budapest, 1987
- J.-nĂŠ BognĂĄr, J. MogyorĂłdi, A. PrĂŠkopa, A. RĂŠnyi, D. SzĂĄsz: *Collection of Problems in Probability Theory*, Second Edition, Textbook Publishing, Budapest, 1974

For any comments regarding these notes, the authors can be contacted at: `balazs@math.bme.hu`, `balint@math.bme.hu`.

0.1. Counting Principles

In probability theory problems, counting tasks often arise, so we begin with a brief combinatorial summary on the topic. The *basic principle of counting* is the following extremely simple statement:

> If an experiment can end in n different outcomes,
> and independently, another experiment has m possible outcomes,
> then together, there are $n \cdot m$ possible outcomes.

Probability theory deals with random experiments. We model their possible outcomes using a set Ω. The above statement can be reformulated formally as follows. Let Ω_1 be the set of outcomes of the first experiment, and

Ω_2 the set of outcomes of the second experiment, and assume that the number of possible outcomes, $|\Omega_1| = n$, and $|\Omega_2| = m$ is finite. Then the two experiments together are modeled by the set of ordered pairs

$$\Omega := \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) : \omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2\}.$$

The basic principle of counting takes the form $|\Omega| = |\Omega_1| \cdot |\Omega_2|$.

It is important to distinguish between the following types of experiments during counting:

- **Permutation:** ordering the elements of a set H.
  - **Permutation without repetition:** Consider the elements of a finite set $H = \{h_1, h_2, \ldots, h_n\}$ as distinct, and the experiment is the ordering of the elements. That is:

    $$\Omega = \{\text{all possible orderings of the elements of } H\}; \qquad |\Omega| = n!.$$

    Thus, there are n *factorial*, $n! = 1 \cdot 2 \cdots n$ permutations without repetition. This can be seen by choosing the first position from n elements, the second from the remaining $n-1$, and so on, until choosing the last from 2 elements, and the final position from 1. Applying the basic principle of counting, we define $0! := 1$.

    0.1. PĂŠlda. *In a horse race with 7 participants,* $\Omega = \{$*all possible finishing orders*$\}$, $|\Omega| = 7! = 5040$.

  - **Permutation with repetition:** The elements of H are not all distinct. We speak of *multisets*: $H = \{h_1, \ldots, h_1, h_2, \ldots, h_2, \ldots, h_r, \ldots, h_r\}$, where h_1 appears n_1 times, h_2 appears n_2 times, and so on, and h_r appears n_r times in H. Then the size of H is $|H| = n = n_1 + n_2 + \cdots + n_r$. The experiment again involves arranging all the elements, without distinguishing between identical elements h_i for a given i:

    $$\Omega = \{\text{all possible orderings of the elements of } H\}; \qquad |\Omega| = \binom{n}{n_1\, n_2\, \ldots\, n_r} = \frac{n!}{n_1! \cdot n_2! \cdots n_r!}. \quad (1)$$

    To see this, imagine that all n elements of H are distinguishable, leading to $n!$ distinct permutations without repetition. However, for each permutation of Ω, we counted identical elements multiple times: specifically, $n_1!$ for identical elements h_i for $i = 1$, $n_2!$ for $i = 2$, and so on, up to $n_r!$ for $i = r$. By the basic principle of counting, each distinct permutation appears $n_1! \cdot n_2! \cdots n_r!$ times, so we divide the number of permutations by this value. The quotient on the right-hand side is known as the *multinomial coefficient*, which we will discuss later.

    0.2. PĂŠlda. *Using the letters M, I, S, S, I, S, S, I, P, P, I, one can form* $\frac{11!}{1!\cdot 4!\cdot 4!\cdot 2!} = 34650$ *different words if each letter is used exactly once.*

- **Variation:** selecting and ordering a subset of elements from a set H (i.e., selection and ordering of the chosen elements).
  - **Variation without repetition:** The elements of a finite set $H = \{h_1, h_2, \ldots, h_n\}$ are considered distinct, and the experiment involves selecting k elements and arranging them. That is:

    $$\Omega = \{\text{words of length } k \text{ formed from the elements of } H\}; \qquad |\Omega| = \frac{n!}{(n-k)!} = n \cdot (n-1) \cdots (n-k+1). \quad (2)$$

    We can form words of length k by taking the first k elements from the permutations of H. Each word appears as many times as we can permute the remaining $n-k$ elements. Therefore, each variation without repetition was counted $(n-k)!$ times in $n!$. The product form can also be understood combinatorially: we can choose n elements for the first position, $n-1$ for the second, and so on, until choosing $n-k+1$ for the k-th position.