CONTINUITY OF THE SUPERPOTENTIALS AND SLICES OF TROPICAL CURRENTS

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Abstract.

Contents

1. Introduction

Moving lemma, fan displacement etc.

Let X be a complex manifold of dimension n, and p, q non-negative integers with $n = p + q$. We denote by $\mathcal{C}^q(X) = \mathcal{C}_p(X)$ the cone of positive closed bidegree (q, q), or bidimension $(p, p)-$ currents on X. We also consider $\mathcal{D}^q(X) = \mathcal{D}_p(X)$, the $\mathbb{R}$-vector space spanned by $\mathcal{C}^q(X)$. It is well-known that the intersection of two positive closed currents is not always defined. The main intial progress were due to the works of Federer [Fed69] and Bedford and Taylor in [BT82]. Federer define the a generic slicing theory of currents, that is for a dominant holomorphic map $f : X \longrightarrow Y$, and a positive closed currents $\mathcal{T} \in \mathcal{C}_p(X)$, or more generally, a *flat current*, a *slice*

$$\mathcal{T} \wedge [f^{-1}(y)]$$

is well-defined for a generic $y \in Y$. Bedford and Taylor suggested that $\mathcal{S} = dd^c u$ is a bidegree $(1, 1)$-current, then

$$\mathcal{S} \wedge \mathcal{T} := dd^c(u\mathcal{T}),$$

can be defined when

- The potential, u, is bounded
- u is unbounded but its unbounded locus has a small intersection with $\mathrm{supp}(\mathcal{T})$.

For instance, when $\mathcal{S} = dd^c \log|f|, \mathcal{T}$ are two integration currents, such that their supports intersect in the expected dimension, then

$$\mathcal{S} \wedge \mathcal{T} = \sum c_i[C_i],$$

where each C_i is a component of the intersection, and c_i is the corresponding vanishing number. This intersection coincides with the slicing of integration currents.

Demailly in [Dem92] asked the question of generalising the intersection theory to the case where $\mathcal{T}$ is of a higher bidegree. In several works, Dinh and Sibony introduced *superpotential theory* and *density* of currents to answer this question. In this article we adopt the approach of Dinh and Sibony for our intersection theory. See also the works of Anderson, Eriksson, Kalm, Wulcan and Yger [AESK+21], and [ASKW22] a non-proper intersection theory.

In [DS09] completely discussed the situation where X is a homogeneous space, and in [DS10] investigated the intersection theory for currents with continuous superpotentials, which is a generalisation of the case of bounded potentials in bidegree $(1, 1)$. Once the intersections are defined one can ask the following *continuity problem*:

Problem 1.1. Let $\mathcal{T}_k$ be a sequence of positive closed currents on X converging to $\mathcal{T}$. Let $\mathcal{S}$ be also a positive close current on X. Find sufficient conditions such that

$$\lim_{k\to\infty}(\mathcal{S}\wedge\mathcal{T}_k) = \mathcal{S}\wedge(\lim_{k\to\infty}\mathcal{T}_k).$$

Roughly speaking, we say that $\mathcal{S}$ is a current on a compact Kähler manifold with a *continuous superpotential*, when for a current $\mathcal{T}$, the wedge product

$$\mathcal{S}\wedge\mathcal{T} := \lim_{n\to\infty}(\mathcal{S}\wedge\mathcal{T}_n),$$

is independent of the choice of smooth approximation $\mathcal{T}_n \longrightarrow \mathcal{T}$. Consequently, by the regularisation theorem of Dinh and Sibony for any bidegree we can partially answer Problem 1.1.

Proposition 1.2. Let X be a compact Kähler manifold, $\mathcal{T}_k \longrightarrow \mathcal{T}$ be a convergent sequence in $\mathcal{D}^p(X)$. If a current $\mathcal{S}$ has a continuous superpotential, then

$$\mathcal{S}\wedge\mathcal{T}_n \longrightarrow \mathcal{S}\wedge\mathcal{T}.$$

Proof. The main result of Dinh and Sibony's result in [DS04] implies any current $\mathcal{D}^q(X)$ can be weakly approximated by a difference of smooth closed positive of bidegree (p, p)-forms. The result then follows from the definition of continuity of super-potentials. □

mention SP-convergence Problem 1.1 becomes more difficult when one considers continuity for slices, and the current $\mathcal{S}$ is an integration current. Borrowing ideas in tropical geometry, we discuss this problem for the very specific case where $\lim_{k\to\infty}\mathcal{T}_k$ is a *complex tropical current* [Bab14, BH17]. (Complex) tropical currents are closed currents on complex tori $(\mathbb{C}^*)^n$ or on a toric variety associated to a *tropical cycle*. Recall that a tropical cycle is a weighted polyhedral complex satisfying the *balancing condition* (see Definition 3.1). For a tropical cycle $\mathcal{C} \subseteq \mathbb{R}^n$, of dimension p, the associated tropical current $\mathcal{T}_{\mathcal{C}} \in \mathcal{D}_p((\mathbb{C}^*)^n)$, is a closed current with support $\mathrm{Log}^{-1}(\mathcal{C})$, where

$$\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \quad (z_1, \ldots, z_n) \longmapsto (-\log|z_1|, \ldots, -\log|z_n|).$$

The tropical current $\mathcal{T}_{\mathcal{C}}$ can be naturally presented as a locally fibration of $\mathrm{Log}^{-1}(\mathcal{C})$, and we say $\mathcal{C}$ is compatible with the fan Σ, if the fibres of $\overline{\mathcal{T}}_{\mathcal{C}}$ intersect the toric invariant divisors of the toric variety X_Σ transversely. Here $\overline{\mathcal{T}}_{\mathcal{C}}$ denotes the extension by zero of $\mathcal{T}_{\mathcal{C}}$ to the toric variety X_Σ.

Theorem 1.3. Let X_Σ be a smooth projective toric variety, and $\mathcal{C}$ is a tropical cycle compatible with Σ. Then $\overline{\mathcal{T}}_{\mathcal{C}}$ has a continuous super-potential.

The preceding theorem allows for defining the intersection product of a tropical current with any current on a compatible toric variety, we can then restrict the intersection product to the complex torus $T_N \subseteq X_\Sigma$ and using the $T_N \simeq (\mathbb{C}^*)^n$ define the intersection product of two tropical currents in $(\mathbb{C}^*)^n$. On the tropical geometry side, there exists a *stable intersection theory* of tropical cycles. The word stable here precisely corresponds to the continuity of the definition with respect to generic translations of tropical cycles. With the stable intersection and natural addition of tropical cycles, we have the ring of tropical cycles.

Theorem 1.4. The assignment $\mathcal{C} \longmapsto \mathcal{T}_{\mathcal{C}}$ induces a $\mathbb{Z}$-algebra homomorphism between

(a) The $\mathbb{Z}$-algebra of tropical cycles in $\mathbb{R}^n$ with the natural addition (Definition **??**) and stable intersection (Definition 3.3) as the multiplication.

(b) The $\mathbb{Z}$-algebra of tropical currents on $(\mathbb{C}^*)^n$ with the usual addition of currents and the wedge product of currents.

We also address Problem 1.1 in a very particular case of slicing of currents converging to a tropical current. The theorem is inspired by works in [BJS⁺07], [OP13] and [Jon16]. Add more theorems.

Theorem 1.5. Let $D, W \subseteq (\mathbb{C}^*)^n$ be an algebraic subtorus, and an algebraic subvariety respectively. Assume that $\mathrm{Log}(D)$ intersects $\mathrm{trop}(W)$ properly. Then,

$$\lim_{m\to\infty}\Big(\frac{1}{m^{n-p}}\Phi_m^*[W]\wedge[D]\Big) = \Big(\lim_{m\to\infty}\frac{1}{m^{n-p}}\Phi_m^*[W]\Big)\wedge[D],$$

where $\Phi_m : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$ is the m-th power map $(z_1,\dots,z_n) \longrightarrow (z_1^m,\dots,z_n^m)$.

The proof relies on a theorem of Berteloot and Dinh [BD20] that the limit of slices satisfies a certain continuity harmonic functions, and we can use Fourier analysis to prove the theorems about tropical currents.

2. Tools from Superpotential Theory

Let (X,ω) be a compact Kähler manifold of dimension n. Assume that $\mathcal{S}$ is either a positive or a negative current of bidegree (q,q) on X. The quantity

$$\langle \mathcal{S}, \omega^{n-q}\rangle$$

is referred to as the *total mass* of $\mathcal{S}$. For $0 \le r \le n$, we consider the de Rham cohomology groups $H^r(X,\mathbb{C}) = H^r(X,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$ with coefficients in $\mathbb{C}$. Recall that Hodge theory provides the following decomposition of the de Rham cohomology group into Dolbeault cohomology groups:

$$H^r(X,\mathbb{C}) \simeq \bigoplus_{p+q=r} H^{p,q}(X,\mathbb{C}).$$

We denote by $\mathcal{C}^q(X)$ the cone of positive closed bidegree (q,q)-currents or bidimension $(n-q,n-q)$ in X. We denote by $\mathcal{D}^q(X) = \mathcal{D}_{n-q}(X)$ the $\mathbb{R}$-vector space spanned by $\mathcal{C}^q(X)$, which is the space of closed real currents of bidegree (q,q). Every current $\mathcal{T} \in \mathcal{D}^q(X)$ has a cohomology class:

$$\{\mathcal{T}\} \in H^{q,q}(X,\mathbb{R}) = H^{q,q}(X,\mathbb{C}) \cap H^{2q}(X,\mathbb{R}).$$

We define $\mathcal{D}^{q,0}(X) = \mathcal{D}^0_{n-q}(X)$ to be the subspace of $\mathcal{D}^q(X)$, consisting of currents with vanishing cohomology. The $*$-topology on $\mathcal{D}^q(X)$ is defined by the norm:

$$\|\mathcal{S}\|_* := \min(\|\mathcal{S}^+\| + \|\mathcal{S}^-\|),$$

where the minimum is taken over positive currents $\mathcal{S}^+$ and $\mathcal{S}^-$ in $\mathcal{C}^q(X)$ that satisfy $\mathcal{S} = \mathcal{S}^+ - \mathcal{S}^-$. We say that $\mathcal{S}_n$ converges to $\mathcal{S}$ in $\mathcal{D}^q(X)$ if $\mathcal{S}_n$ converges weakly to $\mathcal{S}$ and moreover, $\|\mathcal{S}_n\|$ is bounded by a constant independent of n.

Let $h := \dim H^{q,q}(X,\mathbb{R})$, and fix a set of smooth forms $\alpha = (\alpha_1, \dots, \alpha_{\beta_h})$ such that their cohomology classes $\{\alpha\} = (\{\alpha_1\}, \dots, \{\alpha_h\})$ form a basis for $H^{q,q}(X,\mathbb{R})$. By Poincaré duality, there exists a set of smooth forms $\alpha^\vee = (\alpha_1^\vee, \dots, \alpha_h^\vee)$ such that their cohomology classes $\{\alpha^\vee\}$ form the dual basis of $\{\alpha\}$, with respect to the cup-product. By adding $U_\mathcal{S}$ to a suitable combination of $\alpha_i^\vee$, we can assume that $\langle U_\mathcal{S}, \alpha_i\rangle = 0$, for all $i = 1, \dots, h$. In this case, we say that $U_\mathcal{S}$ is *α-normalised*.

Definition 2.1. Let $\mathcal{T} \in \mathcal{D}^q(X)$ and $\mathcal{S}$ be a smooth form in $\mathcal{D}^{n-q+1,0}(X)$.

(i) The *α-normalised super-potential* $\mathcal{U}_\mathcal{T}$ of $\mathcal{T}$ is given by the function

$$\begin{aligned} \mathcal{U}_\mathcal{T} : \{\mathcal{S} \in \mathcal{D}^{n-q+1,0}(X) : \text{smooth}\} &\longrightarrow \mathbb{R} \\ \mathcal{S} &\longmapsto \langle \mathcal{T}, U_\mathcal{S}\rangle, \end{aligned}$$

where $U_\mathcal{S}$ is the α-normalised potential of $\mathcal{S}$.

(ii) We say $\mathcal{T}$ has a *continuous super-potential*, if $\mathcal{U}_\mathcal{T}$ can be extended to a function on $\mathcal{D}^{n-q+1,0}$ which is continuous with respect to the $*$-topology.

In general, consider $\mathcal{T} \in \mathcal{D}^q(X)$ and $\mathcal{T} \in \mathcal{D}^r(X)$. Assume that $q + r \le n$ and $\mathcal{T}$ has a continuous super-potential. Let $\mathcal{U}_\mathcal{T}$ be the α-normalised super-potential of $\mathcal{T}$. Let $\beta \in \mathrm{Span}_\mathbb{R}\{\alpha\}$ such that $\{\beta\} = \{\mathcal{T}\}$. We define

$$\langle \mathcal{T} \wedge \mathcal{S}, \varphi\rangle := \mathcal{U}_\mathcal{T}(\mathcal{S} \wedge dd^c\varphi) + \langle \beta \wedge \mathcal{S}, \varphi\rangle. \tag{1}$$

Now assume that if $f : X \longrightarrow Y$, is a biholomorphism between smooth compact Kähler manifolds, then we have

$$f_*\mathcal{U}_{\mathcal{R}_1} = \mathcal{U}_{f_*\mathcal{R}_1}, \quad f^*\mathcal{U}_{\mathcal{R}_2} = \mathcal{U}_{f^*\mathcal{R}_2},$$

for $\mathcal{R}_1 \in \mathcal{D}^q(X)$ and $\mathcal{R}_2 \in \mathcal{D}^q(Y)$.

Definition 2.2. Let $(\mathcal{T}_n)$ be a sequence of currents in $\mathcal{D}^q(X)$ weakly converging to $\mathcal{T}$. Let $\mathcal{U}_\mathcal{T}$ and $\mathcal{U}_{\mathcal{T}_n}$ be their α-normalised super-potentials. If $\mathcal{U}_{\mathcal{T}_n}$ converges to $\mathcal{U}_\mathcal{T}$ uniformly on any $*$-bounded sets of smooth form in $\mathcal{D}^{n-q+1,0}(X)$, then the convergence is called *SP-uniform*.

It is shown in [DS10, Proposition 3.2.8] that any current with continuous super-potentials can be SP-uniformly approximated by smooth forms. Moreover, currents with continuous super-potentials have other nice properties:

Theorem 2.3 ([DNV18, Theorem 1.1])**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two positive currents in $\mathcal{D}_q(X)$, such that $\mathcal{T} \le \mathcal{T}'$, *i.e.*, $\mathcal{T}' - \mathcal{T}$ is a positive current. Then, if $\mathcal{T}'$ has a continuous super-potential, then so does $\mathcal{T}$.

Theorem 2.4. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two positive closed currents, and $\mathcal{T}_1$ has a continuous superpotentials, then $\mathcal{T}_1 \wedge \mathcal{T}_2$ is well-defined. Moreover, if $\mathcal{T}_2$ has also a continuous superpotential, then

(a) [DS10, Proposition 3.3.3] $\mathcal{T}_1 \wedge \mathcal{T}_2$ has a continuous superpotential;
(b) [DS10, Proposition 3.3.3] This wedge product is continuous with respect to the SP-uniform convergence.
(c) [DS09, Theorem 4.2.4] $\operatorname{supp}(\mathcal{T}_1 \wedge \mathcal{T}_2) \subseteq \operatorname{supp}(\mathcal{T}_1) \cap \operatorname{supp}(\mathcal{T}_2)$.

Theorem 2.5 ([DS10, Proposition 3.3.4])**.** Assume that $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are closed positive currents, and $\mathcal{T}_1$ and $\mathcal{T}_2$ have continuous superpotentials. Then,

$$\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_2 \wedge \mathcal{T}_1 \quad \text{and} \quad (\mathcal{T}_1 \wedge \mathcal{T}_2) \wedge \mathcal{T}_3 = \mathcal{T}_1 \wedge (\mathcal{T}_2 \wedge \mathcal{T}_3).$$

Proposition 2.6. Let X be a compact Kähler manifold, $S_n \longrightarrow S$ be a convergent sequence in $\mathcal{D}^q(X)$. If a current $\mathcal{T}$ has a continuous superpotential, then

$$\mathcal{T} \wedge \mathcal{S}_n \longrightarrow \mathcal{T} \wedge \mathcal{S}.$$

Proof. The main result of Dinh and Sibony in [DS04] implies any current $\mathcal{T} \in \mathcal{D}^p(X)$ can be weakly approximated by a difference of smooth closed positive of bidegree (p,p)-forms. The result then follows from the definition of continuity of super-potentials. $\square$

Lemma 2.7. Let $\mathcal{T}$, $\mathcal{T}'$ be positive closed currents such that $\mathcal{T}_{|_U} = \mathcal{T}'_{|_U}$ in an open subset $U \subseteq X$, and both $\mathcal{T}$ and $\mathcal{T}'$ have continuous super-potentials. Then, for any $\mathcal{S} \in \mathcal{D}^r(X)$,

$$\big(\mathcal{T} \wedge \mathcal{S}\big)_{|_U} = \big(\mathcal{T}' \wedge \mathcal{S}\big)_{|_U}.$$

Proof. In [DS10], for any current $\mathcal{S}$ with continuous super-potential, a family $\{\mathcal{T}_\theta\}_{\theta \in \mathbb{C}^*}$ is constructed that $\mathcal{T}_\theta$ converges SP-uniformly to $\mathcal{S}$ as $|\theta| \to 0$. Therefore, by the hypothesis of the lemma, we can construct families of smooth forms $\mathcal{T}_n, \mathcal{T}'_n$ converging SP-uniformly to $\mathcal{T}, \mathcal{T}'$ respectively. Moreover,

$$\mathcal{T}_{n|_{U_\epsilon}} = \mathcal{T}'_{n|_{U_\epsilon}},$$

where U_ϵ is an ϵ-neighbourhood of U. Now, for a $(n-q-r, n-q-r)$ smooth form φ with compact support on U

$$\big(\mathcal{T}_n \wedge \mathcal{S}\big) \wedge \varphi = \big(\mathcal{T}'_n \wedge \mathcal{S}\big) \wedge \varphi,$$

together with Theorem 2.4(b), implies the assertion.

$\square$

We also have a very useful local version of Theorem 2.3.

Lemma 2.8. If $\mathcal{T}$ is a positive closed current on a compact Kähler manifold X, which is locally bounded by a product of positive closed bidegree $(1,1)$-currents of continuous potentials, resp. Hölder continuous potentials, then $\mathcal{T}$ has continuous superpotentials, resppectively Hölder continuous potentials in X.

Proof. Fix a point a in X. In an open neighbourhood of a that we identify with the ball $B(0,2)$ in $\mathbb{C}^n$, we have

$$\mathcal{T} \leq dd^c u_1 \wedge \cdots \wedge dd^c u_p$$

with u_i continuous or Hölder continuous. Without loss of generality, we can assume that these functions are strictly negative. On $B(0,1)$, define

$$u_i' := \max(u_i, A \log \|z\|)$$

with A sufficiently large so that $u'_i = u_i$ on $B(0,1/2)$. Observe that $u'_i = A\log\|z\|$ near $\partial B(0,1)$. Hence we can extend it to a function which is smooth in a neighbourhood of $X \setminus B(0,1)$. Thus, this function is quasi-plurisubharmonic. We have

$$\mathcal{T} \leq (B\omega + dd^c u'_1) \wedge \cdots \wedge (B\omega + dd^c u'_p)$$

in a neighbourhood W_a of a if B is large enough.

Since we can cover X using a finite number of open sets W_{a_k}, we can add up all obtained quasi-plurisubharmonic functions together and obtain a quasi-plurisubharmonic function u. It is clear that

$$\mathcal{T} \leq (C\omega + u)^p$$

if C is large enough. The function u is continuous or Hölder continuous, and we deduce by Theorem 2.3.

□

Corollary 2.9. Let $q : \widehat{X} \longrightarrow X$, be the blowing up of the compact Kähler manifold X. Assume that $\mathcal{T} \in \mathcal{D}_p(\widehat{X})$ is such that the support of $\mathcal{T}$ does not intersect the exceptional divisors of $\widehat{X}$. If the current on $\mathcal{T}$ has a continuous superpotential then $q_*\mathcal{T}$ has the same property.

Proof. This is an easy corollary of previous lemma, since q is biholomorphic near $\operatorname{supp}(\mathcal{T})$. □

Theorem 2.10. For two complex manifolds X and Y, consider two convergent sequences of currents $\mathcal{T}_n \longrightarrow \mathcal{T}$ in $\mathcal{D}^q(X)$ and $\mathcal{S}_n \longrightarrow \mathcal{S}$ in $\mathcal{D}^r(Y)$. We have that

$$\mathcal{T}_n \otimes \mathcal{S}_n \longrightarrow \mathcal{T} \otimes \mathcal{S},$$

weakly in $\mathcal{D}^{q+r}(X \times Y)$.

Sketch of the proof. Let us denote by (x,y) the coordinates on $X \times Y$. Using local coordinates and a partition of unity and Weierstrass theorem we can approximate any smooth forms on $X \times Y$ with forms with polynomial coefficients in (x,y). The approximation is in C^∞. As a result, the convergence, we only need test forms with monomial coefficients. Thus, the variables x, y are separated and the convergence of the tensor products becomes the convergence of each factor.

□

2.1. **Semi-continuity of slices.** Let $f : X \longrightarrow Y$ be a dominant holomorphic map between complex manifolds, not necessarily compact, of dimension n and m respectively. Let $\mathcal{T}$ be a positive closed current on X of bi-dimension (p,p) with $p \geq m$. Then a slice

$$\mathcal{T}_y = \langle \mathcal{T} | f | y \rangle$$

obtained by restricting $\mathcal{T}$ to $f^{-1}(y)$ exists for almost every $y \in Y$; see [Dem, Page 171]. This is a positive closed current of bi-dimension $(p-m, p-m)$ on X supported by $f^{-1}(y)$. If Ω is a smooth form of maximal bi-degree on Y and α a smooth $(q-m, q-m)$-form with compact support in X, then we have

$$\langle \mathcal{T}, \alpha \wedge f^*(\Omega) \rangle = \int_{y \in Y} \langle \mathcal{T}_y, \alpha \rangle \Omega(y).$$

In general, if $\mathcal{T}$ and $\mathcal{T}'$ are such that $\mathcal{T}_y = \mathcal{T}'_y$ for almost every y, we do not necessarily have $\mathcal{T} = \mathcal{T}'$. However, the following is true: Let $f_1, \dots, f_k$ be dominant holomorphic maps from X to $Y_1, \dots, Y_k$. Consider the vector space spanned by all the differential forms of type $\alpha \wedge f_i^*(\Omega_i)$ for some α as above and for some smooth form Ω_i on Y_i of maximal degree. Assume this space is equal to space of all (q, q)-forms of compact support in X. Then if $\langle \mathcal{T}|f_i|y_i\rangle = \langle \mathcal{T}'|f_i|y_i\rangle$ for every i and almost every $y_i \in Y_i$, we have $\mathcal{T} = \mathcal{T}'$. The proof is a consequence of the above discussion.

Let $U \subseteq \mathbb{C}^m$ and $V \subseteq \mathbb{C}^n$ be two bounded open sets. Assume that $\pi_1 : U \times V \longrightarrow U$ and $\pi_2 : U \times V \longrightarrow V$ are the canonical projections. Consider two closed positive currents $\mathcal{T}$ and $\mathcal{S}$ on $U \times V$ of bi-dimension (m, m) and (n, n) respectively. We say that $\mathcal{T}$ horizontal-like if $\pi_2(\mathrm{supp}(\mathcal{T}))$ is relatively compact in V. Similarly, if $\pi_1(\mathrm{supp}(\mathcal{S}))$ is relatively compact in U, $\mathcal{S}$ is called vertical-like.

Theorem 2.11 ([BD20, Lemma 3.7])**.** Let $(\mathcal{T}_n) \longrightarrow \mathcal{T}$ be a convergent sequence of horizontal-like positive closed currents to a horizontal-like current $\mathcal{T}$ in $U \times V$. Let $a \in U$ and assume that the sequence of measures $(\langle \mathcal{T}_n, \pi_1, a\rangle)_n$ is also convergent. Then,

$$\langle \lim_{n\to\infty} \mathcal{T}_n|\pi_1|a\rangle(\phi) \leq \langle \mathcal{T}|\pi_1|a\rangle(\phi)$$

for every plurisubharmonic function ϕ on $\mathbb{C}^n$.

There is an simple version of the above theorem for supports which will be useful later.

Lemma 2.12. Assume that $\mathcal{T}_i$'s, $\mathcal{S}$ and $\mathcal{T}$ are all closed positive currents, and $\mathcal{T}_i \wedge \mathcal{S}$ and $\mathcal{T} \wedge \mathcal{S}$ are well-defined. If we have the following weak convergence, together with the convergence of supports in the Hausdorff metric, that is,

$$\mathcal{T}_i \longrightarrow \mathcal{T}, \quad \mathrm{supp}(\mathcal{T}_i) \longrightarrow \mathrm{supp}(\mathcal{T}).$$

Then, $\mathrm{supp}(\lim(\mathcal{T}_i \wedge \mathcal{S})) \subseteq \mathrm{supp}(\mathcal{T}) \cap \mathrm{supp}(\mathcal{S})$.

Proof. For a point x outside the support of $\mathcal{T}$, There is a sufficiently small radius ϵ, such that for a sufficiently large i, $\mathcal{T}_i$ vanishes on the ball $B_\epsilon(x)$ centred at x. It follows that any limit of $\mathcal{T}_i \wedge \mathcal{S}$ vanishes on $B_\epsilon(x)$. So its support does not contain a. Moreover, its support does not contain any point outside $\mathrm{supp}(S)$.

□

3. Tropical Varieties, Tori, Tropical Currents

In this section, we recall the definition of tropical cycles and note that with the natural addition of tropical cycles and their *stable intersection*, the tropical cycles form a ring.

3.1. **Tropical varieties.** A linear subspace $H \subseteq \mathbb{R}^n$ is said to be *rational* if there exists a subset of $\mathbb{Z}^n$ that spans H. A *rational polyhedron* is the intersection of finitely many rational half-spaces defined by

$$\{x \in \mathbb{R}^n : \langle m, x\rangle \geq c, \text{ for some } m \in \mathbb{Z}^n, \ c \in \mathbb{R}\}.$$

A *rational polyhedral complex* is a polyhedral complex consisting solely of rational polyhedra. The polyhedra in a polyhedral complex are also referred to as *cells*. A *fan* is a polyhedral complex whose cells are all cones. If every cone in a fan Σ is contained in another fan Σ', then Σ is called a *subfan* of Σ. The one-dimensional cones of a fan are often called *rays*. Throughout this article, all fans and polyhedral complexes are assumed to be *rational*.

For a given polyhedron σ, and a finitely generated abelian group N, we denote by

$$\begin{aligned} \operatorname{aff}(\sigma) &:= \text{affine span of } \sigma, \\ H_\sigma &:= \text{translation of } \operatorname{aff}(\sigma) \text{ to the origin}, \\ N_\sigma &:= N \cap H_\sigma, \\ N(\sigma) &:= N/N_\sigma. \end{aligned}$$

Consider τ, a codimension one face of a p-dimensional polyhedron σ, and let $u_{\sigma/\tau}$ be the unique outward generator of the one-dimensional lattice $(\mathbb{Z}^n \cap H_\sigma)/(\mathbb{Z}^n \cap H_\tau)$.

Definition 3.1 (Balancing Condition and Tropical Cycles)**.** Let $\mathcal{C}$ be a p-dimensional polyhedral complex whose p-dimensional cones are equipped with integer weights. We say that $\mathcal{C}$ satisfies the *balancing condition* at τ if

$$\sum_{\sigma \supset \tau} w(\sigma)\; u_{\sigma/\tau} = 0, \quad \text{in } \mathbb{Z}^n/(\mathbb{Z}^n \cap H_\tau),$$

where the sum is over all p-dimensional cells σ in $\mathcal{C}$ containing τ as a face. A *tropical variety* in $\mathbb{R}^n$ is a weighted complex with finitely many cells that satisfies the balancing condition at every cone of dimension $p-1$.

3.2. **The $\mathbb{Z}$-algebra of Tropical Cycles.** Recall that, generally speaking, the star of a cone in a complex is the extension of the local p-dimensional fan surrounding it. More precisely:

Definition 3.2 (Star of a Cone)**.** Given a polyhedral complex $\mathcal{C} \subseteq \mathbb{R}^n$ and a cell τ within $\mathcal{C}$, define the star of σ in Σ, denoted by $\operatorname{star}_\Sigma(\tau)$, as a fan in $\mathbb{R}^n$. The cones of $\operatorname{star}_\Sigma(\tau)$ are the *extensions* of cones σ that include τ as a face. Here, by extension, we mean

$$\bar{\sigma} = \{\lambda(x-y) : \lambda \geq 0, x \in \sigma, y \in \tau\}.$$

Definition 3.3 (Stable Intersection)**.**

(a) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two tropical cycles intersecting transversely, then the stable intersection of $\mathcal{C}_1 \cdot \mathcal{C}_2$ is the tropical cycles supported on finitely many zero dimensional cells $\mathcal{C}_1 \cap \mathcal{C}_2$. In this case, the weight of a cell $\sigma_1 \cap \sigma_2$, where $\sigma_1 \in \mathcal{C}_1$ and $\sigma_2 \in \mathcal{C}_2$ are top dimensional cells, we define the weights by

$$w_{\mathcal{C}_1\cdot\mathcal{C}_2}(\sigma_1 \cap \sigma_2) = w_{\sigma_1} w_{\sigma_2} [N : N_{\sigma_1} + N_{\sigma_2}].$$

(b) When $\mathcal{C}_1$ and $\mathcal{C}_2$ do not intersect transverse, then $\mathcal{C}_1 \cdot \mathcal{C}_2$ as a set is the Hausdorff limit of

$$\mathcal{C}_1 \cap (\epsilon b + \mathcal{C}_2), \quad \text{as } \epsilon \longrightarrow 0,$$

for a fixed generic $b \in \mathbb{R}^n$, and the weights are the sum of all the tropical multiplicities of the cells in the transversal intersection $\mathcal{C}_1 \cap (\epsilon b + \mathcal{C}_2)$ which converge to the same zero-dimensional cell in the Hausdorff metric. Equivalently, for top dimensional cones $\sigma_1 in \mathcal{C}_1$ and $\sigma_2 \in \mathcal{C}_2$

$$w_{\mathcal{C}_1\cdot\mathcal{C}_2}(\sigma_1 \cap \sigma_2) = \sum_{\tau_1,\tau_2} w_{\tau_1} w_{\tau_2} [N : N_{\tau_1} + N_{\tau_2}],$$

where the sum is taken over all $\tau_1 \in \mathrm{star}_{\mathcal{C}_1}(\sigma_1 \cap \sigma_2), \tau_2 \in \mathrm{star}_{\mathcal{C}_2}(\sigma_1 \cap \sigma_2)$ with $\tau_1 \cap (v + \tau_2) \neq \varnothing$, for some fixed generic vector $b \in \mathbb{R}^n$.

In tropical geometry, the following theorem is shown which we reprove using superpotential theory.

Theorem 3.4 (Stable Intersection Invariance)**.** The stable intersection, as defined above, does not depend on the choice of a generic vector $b \in \mathbb{R}^n$ and the induced weights satisfy the balancing condition on the support.

We also need the following for turning the set of tropical cycles into a $\mathbb{Z}$-algebra.

Definition 3.5 (Addition of Tropical Cycles)**.** For two p-dimensional tropical cycles $\mathcal{C}_1, \mathcal{C}_2$ in $\mathbb{R}^n$, the addition $\mathcal{C}_1 + \mathcal{C}_2$ is the tropical cycle obtained by the common refinement of the support $|\mathcal{C}_1| \cup |\mathcal{C}_2|$ where the weights of a cone σ in the refinement are determined by $w_{\mathcal{C}_1+\mathcal{C}_2}(\sigma) = w_{\mathcal{C}_1}(\sigma) + w_{\mathcal{C}_2}(\sigma)$.

4. Tropical Currents

Let us briefly recall the definition of tropical currents from [Bab14, BH17]. To fix the notation,

$$\begin{aligned} T_N &:= \text{ the complex algebraic torus } \mathbb{C}^* \otimes_{\mathbb{Z}} N, \\ S_N &:= \text{ the compact real torus } S^1 \otimes_{\mathbb{Z}} N, \\ N_{\mathbb{R}} &:= \text{ the real vector space } \mathbb{R} \otimes_{\mathbb{Z}} N. \end{aligned}$$

Let $\mathbb{C}^*$ be the group of nonzero complex numbers. As before, the logarithm map is the homomorphism

$$\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \qquad (z_1, \dots, z_n) \longmapsto (-\log|z_1|, \dots, -\log|z_n|),$$

and the *argument map* is

$$\mathrm{Arg} : (\mathbb{C}^*)^n \longrightarrow (S^1)^n, \qquad (z_1, \dots, z_n) \longmapsto (z_1/|z_1|, \dots, z_n/|z_n|).$$

For a rational linear subspace $H \subseteq \mathbb{R}^n$ we have the following exact sequences:

$$0 \longrightarrow H \cap \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n/(H \cap \mathbb{Z}^n) \longrightarrow 0,$$

Moreover,

$$0 \longrightarrow S_{H\cap\mathbb{Z}^n} \longrightarrow (S^1)^n = S^1 \otimes_{\mathbb{Z}} \mathbb{Z}^n \longrightarrow S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)} \longrightarrow 0.$$

Define

$$\pi_H : \ \mathrm{Log}^{-1}(H) \xrightarrow{\mathrm{Arg}} (S^1)^n \longrightarrow S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)} \ .$$

Similarly,

$$0 \longrightarrow T_{H\cap\mathbb{Z}^n} \longrightarrow (\mathbb{C}^*)^n = \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Z}^n \longrightarrow T_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)} \longrightarrow 0,$$

We define

$$\Pi_H : (\mathbb{C}^*)^n \simeq \mathbb{C}^* \otimes \big((H \cap \mathbb{Z}^n) \oplus \mathbb{Z}^n/(\mathbb{Z}^n \cap H)\big) \longrightarrow T_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)} \ .$$

One has

$$\ker(\Pi_H) = \ker(\pi_H) = T_{H\cap\mathbb{Z}^n} \subseteq (\mathbb{C}^*)^n.$$

As a result, when H is of dimension p, the set $\operatorname{Log}^{-1}(H)$ is naturally foliated by the $\pi_H^{-1}(x) = T_{H\cap\mathbb{Z}^n}\cdot x \simeq (\mathbb{C}^*)^p$ for $x \in S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}$. For a lattice basis $u_1,\dots,u_p$, of $H\cap\mathbb{Z}^n$, the tori $T_{H\cap\mathbb{Z}^n}.x$ can be parametrised by the monomial map

$$(\mathbb{C}^*)^p \longrightarrow (\mathbb{C}^*)^n, \quad z \longmapsto x.z^{[u_1,\dots,u_p]^t}$$

where $U = [u_1,\dots,u_p]$ is the matrix with column vectors $u_1,\dots,u_p$, and z^{U^t} denotes that $z \in (\mathbb{C}^*)^p$ is taken to have the exponents with rows of the matrix U. Accordingly, one can easily check that

$$T_{H\cap\mathbb{Z}^n}\cdot x = \{z \in (\mathbb{C}^*)^n : z^{m_i} = x^{m_i},\, i = 1,\dots,m-p\}.$$

for any choice of a $\mathbb{Z}$-basis $\{m_1,\dots,m_{n-p}\}$ of $\mathbb{Z}^n/(H\cap\mathbb{Z}^n)$.

Definition 4.1. Let H be a rational subspace of dimension p, and μ be the Haar measure of mass 1 on $S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}$. We define a (p,p)-dimensional closed current $\mathcal{T}_H$ on $(\mathbb{C}^*)^n$ by

$$\mathcal{T}_H := \int_{x\in S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}} \left[\pi_H^{-1}(x)\right]\, d\mu(x).$$

When A is an affine subspace of $\mathbb{R}^n$ parallel to the linear subspace $H = A - a$ for $a \in A$, we define $\mathcal{T}_A$ by translation of $\mathcal{T}_H$. Namely, we define the submersion π_A as the composition

$$\pi_A :\ \operatorname{Log}^{-1}(A) \xrightarrow{e^a} \operatorname{Log}^{-1}(H) \xrightarrow{\pi_H} S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}.$$

We will call $T^A := \pi_A^{-1}(1) = \ker \pi_A = e^{-a}T_{H\cap\mathbb{Z}^n}$, the *distinguished fibre* of $\mathcal{T}_A$.

Definition 4.2. Let $\mathcal{C}$, be a weighted polyhedral complex of dimension p. The tropical current $\mathcal{T}_{\mathcal{C}}$ associated to $\mathcal{C}$ is given by

$$\mathcal{T}_{\mathcal{C}} = \sum_{\sigma} w_\sigma\ \mathbb{1}_{\operatorname{Log}^{-1}(\sigma^\circ)}\mathcal{T}_{\operatorname{aff}(\sigma)},$$

where the sum runs over all p-dimensional cells σ of $\mathcal{C}$.

Theorem 4.3 ([Bab14])**.** A weighted complex $\mathcal{C}$ is balanced, if and only if, $\mathcal{T}_{\mathcal{C}}$ is closed.

Theorem 4.4 ([Bab14])**.** Any tropical current $\mathcal{T}_{\mathcal{C}} \in \mathcal{D}'_{n-1,n-1}((\mathbb{C}^*)^n)$ is of the form $dd^c[\mathfrak{q}\circ \operatorname{Log}]$, where $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$, is a tropical Laurent polynomial, that is $\mathfrak{q}(x) = \max_{\alpha\in A}\{c_\alpha + \langle\alpha, x\rangle\}$, for $A \subseteq \mathbb{Z}^n$ a finite subset and $c_\alpha \in \mathbb{R}$.

Remark 4.5. Note that the support of $dd^c[\mathfrak{q}\circ\operatorname{Log}]$, is given by $\operatorname{Log}^{-1}(\mathcal{V}(\mathfrak{q}))$, where $\mathcal{V}(\mathfrak{q})$ is the set of points $x \in \mathbb{R}^n$ where $\mathfrak{q}$ is not smooth at x. This set can be balanced with natural weights which coincides with the weights of the closed current $dd^c[\mathfrak{q}\circ\operatorname{Log}]$ and it is called the tropical variety associated to $\mathfrak{q}$.

Proposition 4.6 ([Bab23, Proposition 4.6])**.** Assume that $\mathcal{T} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$ is a closed positive $(S^1)^n$-invariant current whose support is given by $\operatorname{Log}^{-1}(|\mathcal{C}|)$, for a polyhedral complex $\mathcal{C} \subseteq \mathbb{R}^n$ of pure dimension p. Then $\mathcal{T}$ is a tropical current.

5. Continuity of superpotentials

Let $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$, be a tropical polynomial function, and $\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$, as before. The current $dd^c[\mathfrak{q} \circ \mathrm{Log}] \in \mathcal{D}'_{n-1,n-1}(\mathbb{C}^*)^n$ has a bounded potential, and by Bedford–Taylor theory, for any positive closed current $\mathcal{T} \in \mathcal{D}'_{p,p}(\mathbb{C}^*)^n$, the product

$$dd^c[\mathfrak{q} \circ \mathrm{Log}] \wedge \mathcal{T} = dd^c([\mathfrak{q} \circ \mathrm{Log}]\, \mathcal{T}),$$

is well-defined. See [Dem, Section III.3]. In higher codimensions though, to prove that any two tropical currents have a well-defined wedge product, we utilise Dinh and Sibony's superpotential theory [DS09] on a compact Kähler manifold, and as a result, we extend the tropical currents to smooth compact toric varieties.

5.1. **Tropical Currents on Toric Varieties.** In a toric variety X_Σ, for a cone $\sigma \in \Sigma$, we denote by $\mathcal{O}_\sigma$, the toric orbit associated with σ. We have

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{O}_\sigma.$$

We also set D_σ to be the closure of $\mathcal{O}_\sigma$ in the X_Σ. $\Sigma(p)$ p-dimensional skeleton.

Fibers of tropical currents are algebraic varieties with finite degrees and can be extended by zero to any toric variety, in consequence, any tropical current can be extended by zero to toric varieties. Moreover, with the following compatibility condition, we can ask for the extension of the fibres to intersect the toric invariant divisors transversally.

Definition 5.1. (i) For a polyhedron σ, its *recession cone* is the convex polyhedral cone

$$\mathrm{rec}(\sigma) = \{b \in \mathbb{R}^n : \sigma + b \subseteq \sigma\} \subseteq H_\sigma.$$

(ii) Let $\mathcal{C}$ be a p-dimensional balanced weighted complex in $\mathbb{R}^n$, and Σ a p-dimensional fan. We say that $\mathcal{C}$ is *compatible* with Σ, if $\mathrm{rec}(\sigma) \in \Sigma$ for all $\sigma \in \mathcal{C}$.

(iii) We say the tropical current $\mathcal{T}_\mathcal{C}$ is *compatible* with X_Σ, if all the closures of the fibers $\pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ in X_Σ of $\mathcal{T}_\mathcal{C}$ intersect the torus invariant divisors of X_Σ transversely.

Theorem 5.2. Let $\mathcal{C}$ be a p-dimensional tropical cycle Σ be a fan. Assume that $\sigma \in \mathcal{C}$ is a p-dimensional polyhedron and $\rho \in \Sigma$ is a one-dimensional cone. Then

(a) The intersection $D_\rho \cap \overline{\pi^{-1}_{\mathrm{aff}(\sigma)}(x)}$ is non-empty and transverse, if and only if, $\rho \in \mathrm{rec}(\sigma)$. Here $\overline{\pi^{-1}_{\mathrm{aff}(\sigma)}(x)}$ corresponds the closure of a fiber of $\mathcal{T}_{\mathrm{aff}(\sigma)}$ in the toric variety X_Σ.

(b) In particular, if $\mathcal{C}$ is compatible with Σ, if and only if, $\mathcal{T}_\mathcal{C}$ is compatible with X_Σ.

Proof. See Lemma [BH17, Lemma 4.10]. □

For a tropical current $\mathcal{T}_\mathcal{C} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$, and given a toric variety X_Σ we denote its extension by zero $\overline{\mathcal{T}}_\mathcal{C} \in \mathcal{D}'_{p,p}(X_\Sigma)$.

Proposition 5.3. For every tropical variety $\mathcal{C}$, a smooth projective toric fan Σ compatible with a subdivision of $\mathcal{C}$.

Proof. By [BS11], for $\mathcal{C}$ there is a refinement $\mathcal{C}'$, and a complete fan $\Sigma_1 \subseteq \mathbb{R}^n$ such that $\mathcal{C}'$ is compatible with Σ_1. Applying the toric Chow lemma [CLS11, Theorem 6.1.18] and the toric resolution of singularities [CLS11, Theorem 11.1.9] we can find a fan Σ which is a refinement of Σ_1 that defines a smooth projective variety X_Σ. The tropical variety $\mathcal{C}''$ which is the refinement of $\mathcal{C}'$ induced by Σ, satisfies the statement. □

Remark 5.4. When $\mathcal{C}'$ is a refinement of a tropical variety $\mathcal{C}$, then $\mathcal{C}'$ is a tropical variety with natural induced weights. It is also easy to check that we have the equality of currents $\mathcal{T}_{\mathcal{C}} = \mathcal{T}_{\mathcal{C}'}$ in $(\mathbb{C}^*)^n$; see [BH17, Section 2.6].

Lemma 5.5. Let $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tropical Laurent polynomial and X_Σ be a smooth projective toric variety compatible with a subdivision of $V_{\text{trop}}(\mathfrak{q})$. Let $\rho \in \Sigma(1)$. Assume that $\zeta_0 \in D_\rho \cap \text{supp}(\overline{dd^c[\mathfrak{q} \circ \text{Log}]})$, and Ω is a sufficiently small neighbourhood of ζ_0. Then, $\mathfrak{q} \circ \text{Log} \in \text{PSH}(\Omega \setminus D_\rho) \cap \mathcal{C}^0(\Omega \setminus D_\rho)$ can be extended to a function $u : \Omega \longrightarrow \mathbb{R}$, such that

(a) In Ω, $u = g + \kappa \log |f|$, where g is a continuous function, f is the local equation for D_ρ, and κ is a negative integer.
(b) Restricted to Ω, we have $dd^c u = \overline{\mathcal{T}}_{V_{\text{trop}(\mathfrak{q})}} + c[D_\rho]$.
(c) In Ω, we have $\overline{\mathcal{T}}_{\mathcal{C}} = dd^c g$. In particular, $\overline{\mathcal{T}}_{V_{\text{trop}(\mathfrak{q})}}$ has a continuous superpotential.

Proof. Assume that $\mathfrak{q} = \max_{\alpha \in A}\{c_\alpha + \langle \alpha, x\rangle\}$. Recall that

$$\text{Log} = (-\log|\cdot|, \dots, -\log|\cdot|).$$

We write

$$\mathfrak{q} \circ \text{Log} = \log \max_\alpha \{|e^{c_\alpha} z^{-\alpha}|\}.$$

Assume that near ζ_0, $\mathfrak{q} \circ \text{Log}$ is given by $\max\{|e^{c_\beta} z^{-\beta}, |e^{c_\gamma} z^{-\gamma}|\}$. This implies that in $\text{Log}(\Omega \setminus D_\rho)$, $\mathfrak{q}$ is given by $\max\{c_\beta + \langle \beta, x\rangle, c_\gamma + \langle \gamma, x\rangle\}$. For $\mathfrak{q} = \max_{\alpha \in A}\{c_\alpha + \langle \alpha, x\rangle\}$ we set $\text{rec}(\mathfrak{q}) = \max_{\alpha \in A}\{\langle \alpha, x\rangle\}$. It is not hard to check that

$$\text{rec}\big(V_{\text{trop}}(\mathfrak{q})\big) = V_{\text{trop}}(\text{rec}(\mathfrak{q}));$$

see [MS15, Page 132].

We now show that by extending each $z^{-\alpha}$ as a rational function to X_Σ, the compatibility condition implies that $\mathfrak{q} \circ \text{Log}$ extends to X_Σ. By [CLS11, Proposition 4.1.2] the divisor of the extension of a character z^α in X_Σ is given by

$$\text{Div}(z^\alpha) = \sum_{\rho \in \Sigma(1)} \langle \alpha, n_\rho\rangle D_\rho, \tag{2}$$

where n_ρ the is the minimal generator of ρ. By assumption,

$$D_\rho \cap \text{supp}(\overline{dd^c[\mathfrak{q} \circ \text{Log}]}) \neq \varnothing.$$

Theorem 5.2 implies that

$$n_\rho \in \text{rec}(V_{\text{trop}(\mathfrak{q})}).$$

Moreover, if $\zeta_1 \in D_\rho \cap \text{supp}(\overline{dd^c[\text{rec}(\mathfrak{q}) \circ \text{Log}]})$, then in a small neighbourhood of $\text{Log}(\zeta_1)$, $\text{rec}(\mathfrak{q})(x) = \max\{\langle \beta, x\rangle, \langle \gamma, x\rangle\}$. By definition

$$n_\rho \in \text{rec}(V_{\text{trop}(\mathfrak{q})}) \quad \text{if and only if} \quad \kappa := \langle \beta, n_\rho\rangle = \langle \gamma, n_\rho\rangle.$$

This, together with Equation 2 implies that the extension of $z^{-\beta}$ and $z^{-\gamma}$ as rational functions to X_Σ have the same vanishing order along D_ρ, and we write $z^{-\beta} = f^\kappa \frac{g_1}{h_1}$ and $z^{-\gamma} = f^\kappa \frac{g_2}{h_2}$. Now note that in $\Omega \setminus D_\rho$

$$\mathfrak{q} \circ \mathrm{Log} = \max \log\{|e^{c_\beta} z^{-\beta}|, |e^{c_\gamma} z^{-\gamma}|\} = \kappa \log|f| + \max\{|e^{c_\beta}\frac{g_1}{h_1}|, |e^{c_\gamma}\frac{g_2}{h_2}|\},$$

we must have $\kappa < 0$, otherwise $q \circ \mathrm{Log} = -\infty$ in $\Omega \setminus D_\rho$. Consequently, $\mathfrak{q} \circ \mathrm{Log} : \Omega \setminus D_\rho \longrightarrow \mathbb{R}$, can be extended to

$$u := \kappa \log|f| + \max\{|e^{c_\beta}\frac{g_1}{h_1}|, |e^{c_\gamma}\frac{g_2}{h_2}|\}$$

on Ω. Setting

$$g = \max\{|e^{-c_\beta}\frac{g_1}{h_1}|, |e^{-c_\gamma}\frac{g_2}{h_2}|\},$$

implies (a).

We have

$$dd^c[\mathfrak{q} \circ \mathrm{Log}]_{|\Omega \setminus D_\rho} = \big(dd^c \log|f|^\kappa \; dd^c \log|g|\big)_{|\Omega \setminus D_\rho} = dd^c \log|g|_{|\Omega \setminus D_\rho},$$

since $dd^c \log|f|^\kappa$ is holomorphic in $\Omega \setminus D_\rho$. As a result of compatibility with X_Σ, $\overline{dd^c[\mathfrak{q} \circ \mathrm{Log}]}$ does not charge any mass in D_ρ, and we obtain

$$\overline{dd^c[\mathfrak{q} \circ \mathrm{Log}]} = dd^c \log|g|.$$

This together with Theorem 4.4 implies (c) and (b).

□

Lemma 5.6. Assume that σ is p-dimensional and $\mathrm{aff}(\sigma) = H_1 \cap \cdots \cap H_{n-p}$, is given as the transveral intersection hyperplanes $H_i \subseteq \mathbb{R}^n$. If Σ is a smooth projective fan compatible with $\bigcup_i H_i$, then

$$\overline{\mathcal{T}}_{\mathrm{aff}(\sigma)} \leq \overline{\mathcal{T}_{H_1} \wedge \cdots \wedge \mathcal{T}_{H_{n-p}}} \leq \overline{\mathcal{T}}_{H_1} \wedge \cdots \wedge \overline{\mathcal{T}}_{H_{n-p}}.$$

Proof. By the definition of tropical currents we have the inequality

$$\mathcal{T}_{\mathrm{aff}(\sigma)} \leq \mathcal{T}_{H_1} \wedge \cdots \wedge \mathcal{T}_{H_{n-p}},$$

as currents in $(\mathbb{C}^*)^n$, since the right hand side might have multiplicities but the currents have the same support. Now, the wedge products in X_Σ are well-defined by Lemma 5.5 and Theorem 2.4. As both currents on both sides of the equation coincide on $(\mathbb{C}^*)^n$, the support of the current on the right-hand side contains the closure of the support of $\mathcal{T}_\mathcal{C}$ in X_Σ. □

Should I modify this for non-positive tropical cycles too? Since they can be written as a difference of two positive cycles, this is easy.

Theorem 5.7. Let $\mathcal{C}$ be a positively weighted tropical cycle of dimension p compatible with a smooth, projective fan Σ, then $\overline{\mathcal{T}}_\mathcal{C}$ has a continuous superpotential in X_Σ.

We need the following definition.

Definition 5.8. We define the affine extension p-dimensional a tropical cycle $\mathcal{C}$, by as the addition of tropical cycles

$$\widehat{\mathcal{C}} := \sum_{\sigma \in \mathcal{C}} w_\sigma \mathrm{aff}(\sigma).$$

It is clear that if $\mathcal{C}$ is a positively weighted tropical cycle, then $\mathscr{T}_{\widehat{\mathcal{C}}} - \mathscr{T}_{\mathcal{C}} \geq 0$.

Proof of 5.7 . Let $\widehat{\mathcal{C}}$ be the affine extension of $\mathcal{C}$, and $\widehat{\Sigma}$ be a smooth projective fan which is a refinement of Σ and compatible with $\widehat{\mathcal{C}}$. By the preceding lemma and repeated application of Theorem 2.4 for any $\sigma \in \mathcal{C}$, $\overline{\mathscr{T}}_{\mathrm{aff}(\sigma)}$ has a bounded superpotential, which implies this property for $\overline{\mathscr{T}}_{\widehat{\mathcal{C}}}$. Now, since $\mathscr{T}_{\widehat{\mathcal{C}}} - \mathscr{T}_{\mathcal{C}}$ is a positive closed tropical current in $(\mathbb{C}^*)^n$,

$$\overline{\mathscr{T}_{\widehat{\mathcal{C}}} - \mathscr{T}_{\mathcal{C}}} = \overline{\mathscr{T}}_{\widehat{\mathcal{C}}} - \overline{\mathscr{T}}_{\mathcal{C}} \geq 0$$

in $X_{\widehat{\Sigma}}$. Continuity of the superpotential of $\overline{\mathscr{T}}_{\mathcal{C}}$ in $X_{\widehat{\Sigma}}$ follows from Theorem 2.3.

We now show that $\overline{\mathscr{T}}_{\mathcal{C}}$ has also a continuous super-potential on X_Σ as well. We consider the proper map $f : X_{\widehat{\Sigma}} \longrightarrow X_\Sigma$, which can be understood as a composition of multiple blow-ups along toric points with exceptional divisors D_ρ for any ray $\rho \in \widehat{\Sigma} \setminus \Sigma$. These divisors satisfy $D_\rho \cap \mathrm{supp}(\overline{\mathscr{T}}_{\mathcal{C}}) = \varnothing$. We deduce by Corollary 2.9.

□

Proposition 5.9. In a toric variety X_Σ compatible with the tropical cylce $\mathcal{C}_1 + \mathcal{C}_2$,

$$\overline{\mathscr{T}_{\mathcal{C}_1} \wedge \mathscr{T}_{\mathcal{C}_2}} = \overline{\mathscr{T}}_{\mathcal{C}_1} \wedge \overline{\mathscr{T}}_{\mathcal{C}_2}.$$

Proof. The proof is clear since both $\overline{\mathscr{T}}_{\mathcal{C}_1}$ and $\overline{\mathscr{T}}_{\mathcal{C}_2}$ have continuous superpotentials with no mass on the boundary divisors $X_\Sigma \setminus T_N$. □

Proposition 5.10. For any two tropical currents $\mathcal{C}_1$ and $\mathcal{C}_2$, the intersection product

$$\mathscr{T}_{\mathcal{C}_1} \wedge \mathscr{T}_{\mathcal{C}_2} := \overline{\mathscr{T}}_{\mathcal{C}_1} \wedge \overline{\mathscr{T}}_{\mathcal{C}_2}|_{(\mathbb{C}^*)^n},$$

does not depend on the choice of a smooth projective toric variety of the fan Σ compatible with $\mathcal{C}_1 + \mathcal{C}_2$, where $(\mathcal{C}^*)^n$ is identified with $T_N \subseteq X_\Sigma$. Moreover, this product coincides with the definition of wedge products with bi-degree $(1,1)$ tropical currents in Bedford–Taylor Theory in $(\mathbb{C}^*)^n$.

Proof. This is a consequence of Lemma 2.7, and the fact that intersection product with a bidegree $(1,1)$ current in super-potential theory, in an open set of compact Kähler manifold, coincides with the Bedford–Taylor theory. □

5.2. Proof of $\mathscr{T}_C \wedge \mathscr{T}_{C'} = \mathscr{T}_{\mathcal{C}\cdot\mathcal{C}'}$.

Theorem 5.11. For two tropical varieties $\mathcal{C}$ and $\mathcal{C}'$ with complementary dimensions the notion of stable intersection is well-defined and we have

$$\mathscr{T}_{\mathcal{C}} \wedge \mathscr{T}_{C'} = \mathscr{T}_{\mathcal{C}\cdot\mathcal{C}'}.$$

Proposition 5.12 ([Kat09, Propositions 6.1]). Let $H_1, H_2 \subseteq \mathbb{R}^n$ be two rational planes of dimension p and q with $p + q = n$ that intersect transversely. Then, the complex tori $T_{H_1 \cap \mathbb{Z}^n}$ and $T_{H_2 \cap \mathbb{Z}^n}$ intersect at $[N : N_{H_1} + N_{H_2}]$ distinct points.

Proof of Theorem 5.11. Note that $\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}$ is well-defined by Proposition 5.10. We proceed with the following steps:

(a) $\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'} = \mathcal{T}_{\mathcal{C}\cdot\mathcal{C}'}$ in the transversal case.
(b) $\mathrm{Log}(\mathrm{supp}(\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}))$ is 0-dimensional in the general case.
(c) $\mathrm{supp}(\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}) = \mathrm{Log}^{-1}(\mathcal{C}.\mathcal{C}')$.
(d) Proof of Theorem 3.4.

$\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}$ is invariant under the action of $(S^1)^n$, and therefore it is a tropical current. To see (a), when $\mathcal{C} \cap \mathcal{C}'$ is transverse. Assume that $x \in \mathcal{C} \cap \mathcal{C}'$ and the intersection is transverse. Since $\mathcal{C}$ and $\mathcal{C}'$ are of complementary dimensions, we can choose a small ball $B_\epsilon(x) \in \mathbb{R}^n$ such that x is an isolated point of intersection in B. Now, by Lemma 2.7

$$\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}|_{\mathrm{Log}^{-1}(B)} = w_\sigma w_{\sigma'} \mathbb{1}_{\mathrm{Log}^{-1}(B)} \int_{(x,x')\in(S^1)^n} [\pi_\sigma^{-1}(x)] \wedge [\pi_{\sigma'}^{-1}(x')]\, d\mu_\sigma(x) \otimes d\mu_{\sigma'}(x').$$

Transversality of the fibers implies

$$[\pi_\sigma^{-1}(x)] \wedge [\pi_{\sigma'}^{-1}(x')] = [\pi_\sigma^{-1}(x) \cap \pi_{\sigma'}^{-1}(x')].$$

By Proposition 5.12, we have $\kappa = [N : N_{\mathrm{aff}(\sigma)} + N_{\mathrm{aff}(\sigma')}]$ distinct intersection points covering $\mathrm{Log}^{-1}(x)$. When $(x,x') \in (S^1)^n$ vary with respect to normalised Haar measure, these κ points cover $(S^1)^n$ with speed κ. As a result,

$$\int_{(x,x')\in(S^1)^n} [\pi_\sigma^{-1}(x)] \wedge [\pi_{\sigma'}^{-1}(x')]\, d\mu_\sigma(x) \otimes d\mu_{\sigma'}(x') = \int_{y\in(S^1)^n} \kappa[\pi_{\sigma\cap\sigma'}^{-1}(y)] d\mu_{\sigma\cap\sigma'}(y).$$

This proves (a). To prove (b) note that if $\mathcal{C} + b$ is the translation of the the tropical variety by $b \in \mathbb{R}^n$, then $(e^b)^*\mathcal{T}_{\mathcal{C}} = \mathcal{T}_{\mathcal{C}+b}$. Moreover, we have the SP-convergence of currents with continuous superpotentials.

$$(e^{\varepsilon b})^*\mathcal{T}_{\mathcal{C}} \longrightarrow \mathcal{T}_{\mathcal{C}}\,, \qquad \text{as } \varepsilon \to 0.$$

Therefore, by Theorem 2.4,

$$(e^{\varepsilon b})^*\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'} = \mathcal{T}_{\mathcal{C}+\varepsilon b} \wedge \mathcal{T}_{\mathcal{C}'} \longrightarrow \mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}\,, \qquad \text{as } \varepsilon \to 0. \tag{3}$$

Considering the support, we obtain the Hausdorff limit

$$\lim \mathrm{supp}((e^{\varepsilon b})^*\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}) \supseteq \mathrm{supp}(\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}).$$

We now note that for all ϵ, the number of intersection points in $(\mathcal{C} + \epsilon b) \cap \mathcal{C}'$ is uniformly bounded by the number of p-dimensional cells in $\mathcal{C}$ and q-dimensional cells in $\mathcal{C}$, the Hausdorff limit of $(\mathcal{C} + b\epsilon) \cap \mathcal{C}'$ is also zero dimensional. To prove (c), by definition of $\mathcal{C} \cdot \mathcal{C}'$ it suffices to show that

$$\lim \mathrm{supp}((e^{\varepsilon b})^*\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}) = \mathrm{supp}(\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}),$$

for any fixed generic b. This is also easy. Let $x_\epsilon \in \mathcal{C} + \epsilon b \cap \mathcal{C}'$. Since the translation by ϵb does not change slopes of the cells, as $x_\epsilon \to x$, the multiplicity for all x_ϵ remains constant for $\epsilon > 0$, therefore the mass $\lim (e^{\varepsilon b})^*\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}$ has a non-zero mass at $\mathrm{Log}^{-1}(x)$. Now Part (d) is deduced from Equation (3) since we can choose b generically and Parts (a),(b),(c).

∎

Theorem 5.13. Stable intersection of tropical cycles is associative and commutative, and $\mathcal{C} \mapsto \mathcal{T}_{\mathcal{C}}$ induces an isomorphism of $\mathbb{Z}$-algebras between effective tropical cycles and positive tropical currents on $(\mathbb{C}^*)^n$.

Proof. This is the application of Theorem 5.7 and Theorem 2.5, and Theorem 5.11. □

5.2.1. *Calculating Intersection Multiplicities Using Monge-Ampère Measures.* In this section, we explain how to calculate intersection multiplicities in two different ways. Note that by the equality of the supports in the previous section, we only need to prove the intersection multiplicities in the transversal case locally.

5.2.2. *Real Monge–Ampeȑe Measures.* Let $\Omega \subseteq \mathbb{R}^n$ be an open subset and $u : \Omega \longrightarrow \mathbb{R}$ be a convex (hence continuous) function. The *generalised gradient* of u at $x_0 \in \Omega$ is defined by

$$\nabla u(x_0) = \big\{\xi \in (\mathbb{R}^n)^* : u(x) - u(x_0) \geq \langle \xi, x - x_0\rangle, \text{ for all } x \in \Omega\big\}.$$

In the above, $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{R}^n$, and $(\mathbb{R}^n)^*$ is the dual. The real Monge-Ampére measure associated to a convex polynomial of a Borel set $E \subseteq \Omega$, is given by

$$\mathrm{MA}[u](E) = \mu\Big(\bigcup_{y\in E} \nabla u(y)\Big),$$

where μ is the Lebesgue measure on $\mathbb{R}^n$.

It is interesting that for the tropical polynomials, one can compute the associate real Monge–Ampére measures explicitly. Recall that, for any tropical polynomial, there is a natural subdivision of its Newton polytope which is dual to the tropical variety of it. See Figure for an example and [BS14, MS15] for details.

Lemma 5.14 ([Yge13, Page 59],[BGPS14, Proposition 2.7.4])**.** Let $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tropical polynomial $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$ with associated tropical variety $\mathcal{C} = V_{\text{trop}}(\mathfrak{q})$, one has

$$\mathrm{MA}[\mathfrak{q}] = \sum_{a\in\mathcal{C}(0)} \mathrm{Vol}\big(\{a\}^*\big)\delta_a,$$

where $\mathcal{C}(0)$ is the 0-dimensional skeleton of $\mathcal{C}$, and $\{a\}^*$ is the dual of the vertex $a \in \mathcal{C}(0)$.

A detailed discussion of the preceding theorem can be also found in [Bab14].

5.3. **Polarisation.** For n convex functions $u_1, \dots, u_n : \mathbb{R}^n \longrightarrow \mathbb{R}$, their *mixed Monge-Ampére measure* is defined by

$$\widetilde{\mathrm{MA}}[u_1, \dots, u_n] = \frac{1}{n!}\sum_{k=1}^{n} \sum_{1\leq j_1<\cdots<j_k\leq n} (-1)^{n-k}\,\mathrm{MA}[u_{j_1} + \cdots + u_{j_k}].$$

Recall that this is how the *mixed volume* of n convex bodies can be defined from the n-dimensional volume. Moreover, it is easy to check that for a convex function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathrm{MA}[u] = \widetilde{\mathrm{MA}}[u, \dots, u]$.

The following statements are clear from 5.14 by taking the total mass.

Proposition 5.15. Let $\mathfrak{q}, \mathfrak{q}_1, \dots, \mathfrak{q}_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ be tropical polynomials. We have the following facts:

(a) $\mathrm{MA}[\mathfrak{q}](\mathbb{R}^n) = \mathrm{Vol}_n(\Delta_{\mathfrak{q}})$, where $\Delta_{\mathfrak{q}}$ is the Newton polytope of $\mathfrak{q}$.
(b) (Tropical Bernstein Theorem) $\widetilde{\mathrm{MA}}[\mathfrak{q}_1, \ldots, \mathfrak{q}_n](\mathbb{R}^n) = \widetilde{\mathrm{Vol}}(\Delta_{\mathfrak{q}_1}, \ldots, \Delta_{\mathfrak{q}_n})$, where $\widetilde{\mathrm{Vol}}$ is the mixed volume.

Corollary 5.16. Assume that $\alpha_i, \beta_i \in \mathbb{Z}^n$ for $i = 1, \ldots, n$. Let $\mathfrak{q}_i = \max\{\langle \alpha_i, x\rangle, \langle \beta_i, x\rangle\}$ be n tropical polynomials. Then,

$$n!\, \widetilde{\mathrm{MA}}[\mathfrak{q}_1, \ldots, \mathfrak{q}_n] = \kappa\, \delta_0,$$

where κ is given by the volume *zonotope* of the Minkowski sum of the vectors $\sum_{i=1}^n [\alpha_i - \beta_i]$.

Proof. Note that $\Delta_{\mathfrak{q}_i}$ is the line segment between α_i and β_i. Moreover, in the definition of $\widetilde{\mathrm{MA}}[\mathfrak{q}_1, \ldots, \mathfrak{q}_n]$ only $\mathrm{Vol}(\sum_{i=1}^n [\alpha_i - \beta_i])$ possibly has a non-zero n-dimensional volume. Finally, the origin is the only 0-dimensional cell of the tropical variety of polynomial $\mathfrak{q}_1 + \cdots + \mathfrak{q}_n$, if and only if, $\{\alpha_1 - \beta_1, \cdots, \alpha_n - \beta_n\}$ forms a linearly independent set. Therefore, $n!\, \mathrm{MA}[\mathfrak{q}_1 + \cdots + \mathfrak{q}_n] = \kappa\, \delta_0$. □

6. Slicing tropical currents

Proposition 6.1. Let $\mathcal{C}$ be a p-dimensional tropical cycle in $\mathbb{R}^n$, and $S \subseteq (\mathbb{C}^*)^n$ be an algebraic hypersurface with transversal intersection with $\mathcal{T}_{\mathcal{C}}$. Then, $[S] \wedge \mathcal{T}_{\mathcal{C}}$ is admissible and it is a closed positive current of bidimension $(p-1, p-1)$ given by

$$[S] \wedge \mathcal{T}_{\mathcal{C}} = \sum_{\sigma \in \mathcal{C}} w_\sigma \int_{x \in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[S \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\big]\; d\mu(x).$$

Proof. The idea of the proof is similar to that of [BH17, Proposition 4.11]. Let f be the equation of S in $(\mathbb{C}^*)^n$. Assume that $\mathrm{Log}^{-1}(\sigma^\circ) \cap S \neq \varnothing$, for a p-dimensional cone $\sigma \in \mathcal{C}$, then for each fiber, $\pi_\sigma^{-1}(x)$ the transversality assumption allows for application of the Lelong–Poincaré formula to deduce

$$\begin{aligned} &dd^c\big(\log|f| \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[\pi_\sigma^{-1}(x)\big]\big) \\ &\qquad\qquad\qquad = \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\big[S \cap \pi_\sigma^{-1}(x)\big] + \mathcal{R}_\sigma(x). \end{aligned}$$

where $\mathcal{R}_\sigma(x)$ is a $(p-1, p-1)$-bidimensional current. The support of $\mathcal{R}_\sigma(x)$ lies in the boundary of $\mathrm{Log}^{-1}(\sigma)$, as $\mathcal{R}_\sigma(x)$ is the difference of two currents that coincide in any set of form $\mathrm{Log}^{-1}(B)$, where $B \subseteq \mathbb{R}^n$ is a small ball with

$$B \cap \sigma^\circ \neq \varnothing, \quad B \cap \partial\sigma = \varnothing,$$

and both vanish outside $\mathrm{Log}^{-1}(\sigma)$. Integrating along the fibers, and adding for all p-dimensional cones $\sigma \in \mathcal{C}$, we obtain

$$[S] \wedge \mathcal{T}_{\mathcal{C}} = \sum_{\sigma \in \mathcal{C}} w_\sigma \int_{x \in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[S \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\big]\; d\mu(x) + \mathcal{R}_{\mathcal{C}},$$

where $\mathcal{R}_{\mathcal{C}}$ is $(p-1, p-1)$-dimensional current. We claim that $\mathcal{R}_{\mathcal{C}}$ is *normal*, *i.e.* $\mathcal{R}_{\mathcal{C}}$ and $d\mathcal{R}_{\mathcal{C}}$ have measure coefficients; $\mathcal{R}_{\mathcal{C}}$ is a difference of two normal currents, where the first current $[S] \wedge \mathcal{T}_{\mathcal{C}}$ is a positive closed current, and the second current is an addition of normal pieces. Moreover, the support of $\mathcal{R}_{\mathcal{C}}$ is a subset of S as it is a difference of two currents that both vanish outside S. As a result, the current $\mathcal{R}_{\mathcal{C}}$ is supported on

$S \cap \bigcup_\sigma \partial\mathrm{Log}(\sigma)$. This set is a real manifold of Cauchy–Riemann dimension less than $p-1$, therefore by Demailly's first theorem of support the normal current $\mathcal{R}_\mathcal{C}$ vanishes; see also the discussion following [BH17, Proposition 4.11]. $\square$

Corollary 6.2. Let $H \subseteq \mathbb{R}^n$ be a rational plane of dimension r and $A := a + H$, a translation of H for $a \in \mathbb{R}^n$. Assume also that $\mathcal{C} \subseteq \mathbb{R}^n$ is a tropical variety of dimension p that intersects A transversely. Then

$$[(e^{-a})T_{H\cap\mathbb{Z}^n}] \wedge \mathcal{T}_\mathcal{C}$$

can be viewed as a tropical current of dimension $p-(n-r)$ in the complex subtori $T^A := (e^{-a})\, T_{H\cap Z^n} \subseteq (\mathbb{C}^*)^n$.

Proof. Note that the hypothesis implies that the intersection $T^A \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ is transversal for any $x \in S_{N(\sigma)}$. By translation, it is sufficient to prove the statement for $a = 0$. By preceding theorem,

$$[T^A] \wedge \mathcal{T}_\mathcal{C} = \sum_{\sigma\in\mathcal{C}} w_\sigma \int_{x\in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[T^A \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\big]\ d\mu(x).$$

The sets $T^A \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ can be understood as a translation toric sets in T^A and $d\mu_\sigma(x)$ are Haar measures, which imply the assertion. $\square$

Theorem 6.3. Let $M \subseteq (\mathbb{C}^*)^{n-p}$ and $N \subseteq (\mathbb{C}^*)^p$ be two bounded open subsets such that N contains the real torus $(S^1)^p$. Let $\pi : M \times N \longrightarrow M$ be the canonical projection. Let $\mathcal{T}_n$ be a sequence of positive closed (p,p)-bidimensional currents on $M \times N$ such that $\overline{\mathrm{supp}(\mathcal{T}_n)} \cap (M \times \partial\overline{N}) = \varnothing$. Assume that $\mathcal{T}_n \longrightarrow \mathcal{T}$ and $\mathrm{supp}(\mathcal{T}) \subseteq M \times (S^1)^p$. Then we have the following convergence of slices

$$\langle \mathcal{T}_n|\pi|x\rangle \longrightarrow \langle \mathcal{T}|\pi|x\rangle \quad \text{for every } x \in M.$$

Note that all the above slices are well-defined for all $x \in M$.

Proof. Since all the currents $\mathcal{T}_n$ and $\mathcal{T}$ are horizontal-like, the slices are well-defined, and we prove that the slices have the same cluster value. Let $\mathcal{S}$ be any cluster value of $\langle \mathcal{T}_n|\pi|x\rangle$. Note that such $\mathcal{S}$ always exists by Banach–Alaoglu theorem. As both measures $\mathcal{S}$ and $\langle \mathcal{T}|\pi|x\rangle$, are supported $\{x\}\times(S^1)^p$ to prove their equality, it suffices to prove that they have the same Fourier coefficients. By Theorem 2.11, we have

$$\langle \mathcal{S}, \phi\rangle \le \langle \mathcal{T}|\pi|x\rangle(\phi),$$

for every plurisubharmonic function ϕ on $\mathbb{C}^n$, and the mass of $\mathcal{S}$ coincides with the mass of $\langle \mathcal{T}|\pi|x\rangle$. Now, note that if ϕ is pluriharmonic, then $-\phi$ and ϕ are plurisubharmonic. As a result,

$$\langle \mathcal{S}, \phi\rangle = \langle \mathcal{T}|\pi|x\rangle(\phi),$$

for every pluriharmonic function. Recall that if f is a holomorphic function, then $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are pluriharmonic. We now consider the elements of the Fourier basis $f(\theta) = \exp 2\pi i\langle \nu, \theta\rangle$ for $\nu \in \mathbb{Z}^n$. then we have the equality

$$\langle \mathcal{S}, f\rangle = \langle \mathcal{T}|\pi|x\rangle(f)$$

This implies that the Fourier measure coefficients of both $\mathcal{S}$ and $\langle \mathcal{T}|\pi|x\rangle$ coincide.

$\square$

Lemma 6.4. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a tropical variety of dimension p, and L be a rational $(n-p)$-dimensional plane such that L is transveral to all the affine extensions $\mathrm{aff}(\sigma)$ for $\sigma \in \mathcal{C}$. Assume that $\mathcal{T}$ be a positive closed current of bidimension (p,p) on a smooth projective toric variety (X_Σ compatible with $\mathcal{C} + L$ such that $\mathrm{supp}(\mathcal{T}) \subseteq \mathrm{supp}(\mathcal{T}_{\mathcal{C}})$. Further, for all $a \in \mathbb{R}^n$,

$$\overline{\mathcal{T}}_{L+a} \wedge \mathcal{T} = \overline{\mathcal{T}}_{L+a} \wedge \overline{\mathcal{T}}_{\mathcal{C}}\,,$$

then $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$ in $(\mathbb{C}^*)^n$.

Proof. Let us first remark that $\mathrm{rec}(L+a) = \mathrm{rec}(L)$ for all $a \in \mathbb{R}^n$ and therefore, all $\mathcal{T}_{a+L}$ are compatible with X_Σ and have a continuous super-potenrial in X_Σ and as a result, all the above wedge products are well-defined.

By Demailly's second theorem of support [Dem, III.2.13], there are measures μ_σ such that

$$\mathcal{T} = \sum_\sigma \int_{x \in S(Z^n \cap H_\sigma)} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[\pi_\sigma^{-1}(x)\big]\, d\mu_\sigma^{\mathcal{T}}(x).$$

By repeated application of Proposition 6.1,

$$\mathcal{T}_L \wedge \mathcal{T} = \sum_\sigma \int_{(x,y) \in S(\mathbb{Z}^n \cap H_L) \times S(\mathbb{Z}^n \cap H_\sigma)} \big[\pi_H^{-1}(x) \cap \pi_\sigma^{-1}(y)\big]\, d\mu_L(x) \otimes \mu_\sigma^{\mathcal{T}}(y).$$

Applying both sides of the equality $\mathcal{T}_L \wedge \mathcal{T} = \mathcal{T}_L \wedge \mathcal{T}_{\mathcal{C}}$ on test-functions of the form

$$\omega_\nu = \exp(-i\langle \nu, \theta\rangle)\rho(r)$$

where $\rho : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support and $\theta \in [0, 2\pi)^n$, and $\nu \in \mathbb{Z}^n$, completely determines the Fourier coefficients of $\mu_\sigma^{\mathcal{T}}$ which have to coincide with the normalised Haar measures multiplied by the weight of σ, *i.e.*, $\mu_\sigma^{\mathcal{T}} = w_\sigma \mu_\sigma$. ☐

Note that any subtorus of $(\mathbb{C}^*)^n$, can be understood as a fibre of a tropical current. We have the following slicing theorem.

Theorem 6.5. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a tropical variety and $A \subseteq \mathbb{R}^n$ a rational hyperplane intersecting $\mathcal{C}$ transversely. Let Σ be a fan compatible with $\mathcal{C} + A$. Assume that $\overline{\mathcal{S}}_n$ is a sequence of positive closed currents on X_Σ, and denote by $\mathcal{S}_n$ the restriction to T_N. Further,

- $\overline{\mathcal{S}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}}$;
- $\mathrm{supp}(\overline{\mathcal{S}}_n) \longrightarrow \mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}})$.

We have that

$$\lim_{n\to\infty} \big(\mathcal{S}_n \wedge [T^A]\big) = \mathcal{T}_{\mathcal{C}} \wedge [T^A],$$

as currents on $T_N \subseteq X_\Sigma$.

Proof. Assume that $L \subseteq \mathbb{R}^n$ is an $(n-p-1)$-dimensional affine plane intersecting all $\mathrm{aff}(\sigma)$ for all $\sigma \in \mathcal{C} \cap A$ transversely. Then, on a projective smooth toric variety $X_{\Sigma'}$ compatible with $\mathcal{C} + L + A$ the tropical currents $\overline{\mathcal{T}}_{a+L}$, $a \in \mathbb{R}^n$ have continuous super-potentials. Therefore, by Proposition 2.6, we have

$$\lim_{m\to\infty} \big(\overline{\mathcal{S}}_n \wedge \overline{\mathcal{T}}_{a+L}\big) = \overline{\mathcal{T}}_{\mathcal{C}} \wedge \overline{\mathcal{T}}_{a+L}.$$

Now, for any $x \in \mathcal{C} \cap L \cap A$, let $B \subseteq \mathbb{R}^n$ containing x be a bounded open set containing only x as an isolated point of the intersection. By a translation we can assume that $x = 0$. Let H be the linear space parallel to A, and

$$\xi : (\mathbb{C}^*)^n \xrightarrow{\sim} T_{\mathbb{Z}^n/(Z^n \cap H)} \times T_{\mathbb{Z}^n \cap H}$$

be the isomorphism, and π_1 and π_2 be the respective projections. Note that for $x \in S^1_{\mathbb{Z}^n/(\mathbb{Z}^n \cap H)}$, we have $\pi_1^{-1}(1) = T^A$. We now set

$$\begin{aligned} U &:= \pi_1 \circ \xi\big(\mathrm{Log}^{-1}(U) \cap \mathrm{supp}(\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{a+L})\big) \\ V &:= \pi_2 \circ \xi\big(\mathrm{Log}^{-1}(U) \cap T^A\big), \\ \mathcal{T}_n &:= \xi_*\big(\mathcal{S}_n \wedge \mathcal{T}_{a+L}\big), \text{ in } T_N, \\ \mathcal{T} &:= \xi_*\big(\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{a+L}\big). \end{aligned}$$

Therefore, for large n, $\mathcal{T}_n$ and $\mathcal{T}_{\mathcal{C}}$ are horizontal-like. By Theorem 6.5, we obtain

$$\lim_{n\to\infty} \big(\mathcal{S}_n \wedge [T^A]\big) \wedge \mathcal{T}_{a+L} = \mathcal{T}_{\mathcal{C}} \wedge [T^A] \wedge \mathcal{T}_{a+L},$$

for every a. We now deduce the convergence on $X_{\Sigma'}$ by Lemma 6.4. Finally the convergence on $(\mathbb{C}^*)^n \simeq T_N$ follows from restriction.

□

Theorem 6.6. In the situation of Theorem 6.5,

$$\lim_{n\to\infty} \big(\mathcal{S}_n \wedge [\overline{T}^A]\big) = \overline{\mathcal{T}}_{\mathcal{C}} \wedge [\overline{T}^A],$$

where the extension is considered in a smooth projective toric variety X_Σ compatible with $\mathrm{trop}(W) + A$.

Lemma 6.7. Let $U \subseteq \mathbb{C}^n$ be an open subset and D an analytic subset. Assume that we have the convergence of closed positive currents $\mathcal{V}_n \longrightarrow \mathcal{V}$ in $U \setminus D$, and $\mathcal{V}_n$'s and $\mathcal{V}$ have a finite local mass near D. Further, assume that for any cluster value of the sequence $\{\overline{\mathcal{V}}_n\}_n$, $\mathcal{W}$ we have

(a) $\mathrm{supp}(\mathcal{W}) \subseteq \mathrm{supp}(\overline{\mathcal{V}})$,
(b) $\mathrm{supp}(\overline{\mathcal{V}}) \cap D$ has the expected Cauchy–Riemann dimension,

then

$$\overline{\mathcal{V}}_n \longrightarrow \mathcal{W} = \overline{\mathcal{V}}.$$

Proof. $\overline{\mathcal{V}} - \mathcal{W}$ has the Cauchy–Riemann dimension less than or equal to p, therefore, it must be zero. □

Proof of Theorem 6.6. Applying Theorem 5.2 (or [OP13, Proposition 3.3.2] to each fibre of $\overline{\mathcal{T}}_{\mathcal{C}}$ separately), we obtain $\mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}}) \cap \overline{T}_A \cap [D_\rho]$ has the expected Cauchy–Riemann dimension $p-2$. By Demailly's first theorem of support [Dem, Theorem III.2.10] $\overline{\mathcal{S}}_{\mathcal{C}} \wedge [\overline{T}_A] = \overline{\mathcal{T}_{\mathcal{C}} \wedge [T_A]}$. By assumption $\overline{\mathcal{S}}_n \longrightarrow \overline{\mathcal{T}}_{\mathrm{trop}(W)}$ and $\mathrm{supp}(\mathcal{T}_n) \longrightarrow \mathrm{supp}(\overline{\mathcal{T}}_{\mathrm{trop}(W)})$. The observation in Lemma 2.12,

$$\lim_n \mathrm{supp}(\overline{\mathcal{S}}_n \wedge [\overline{T}^A]) \subseteq \mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}} \wedge [\overline{T}^A]).$$

Therefore, any cluster value of $\overline{\mathcal{S}_n \wedge [T^A]} \subseteq \overline{\mathcal{S}}_n \wedge [\overline{T}^A]$ has a support in $\mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}} \wedge [\overline{T}^A])$. Now by setting

(a) $\mathcal{V}_n := \mathcal{S}_n \wedge [\overline{T}^A]$.
(b) $\mathcal{V} := \mathcal{T}_{\mathcal{C}} \wedge [\overline{T}^A]$,
(c) $\mathcal{W}$ a cluster value of $\overline{\mathcal{T}_n \wedge [T^A]}$.

we are in the situation of Lemma 6.7, and conclude. □

Lemma 6.8. Let X_Σ be a smooth projective toric variety, and $\bar{\Delta} \subseteq X_\Sigma$ be the diagonal. Let $\mathcal{S}$ and $\mathcal{T}$ be two positive currents on X. Then, for any ray $\rho \in \Sigma$,

$$\operatorname{supp}(\mathcal{S}) \cap \operatorname{supp}(\mathcal{T}) \cap D_\rho \subseteq X_\Sigma$$

has a Cauchy–Riemann dimension ℓ, if and only if,

$$\operatorname{supp}(\mathcal{S} \otimes \mathcal{T}) \cap \bar{\Delta} \cap D_{(0,\rho)} \subseteq X_\Sigma \times X_\Sigma,$$

has a Cauchy–Riemann dimension ℓ, where $D_{(0,\rho)}$ is the toric invariant divisor corresponding to the ray $(0, \rho)$ in $\Sigma \times \Sigma$.

Proof. The fan of $X_\Sigma \times X_\Sigma$ is $\Sigma \times \Sigma$, we have that $D_{(0,\rho)} \simeq X_\Sigma \times D_\rho$ and the assertion follows. □

Theorem 6.9. Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{R}^n$ be two tropical cycles intersecting properly. Assume that X_Σ is a smooth toric projective variety compatible with $\mathcal{C}_1 + \mathcal{C}_2$. If moreover, for two sequence of positive closed currents $\overline{\mathcal{V}}_n$ and $\overline{\mathcal{W}}_n$ we have

(a) $\overline{\mathcal{W}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}_1}$ and $\overline{\mathcal{V}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}_2}$,
(b) $\operatorname{supp}(\overline{\mathcal{W}}_n) \longrightarrow \operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}_1})$ and $\operatorname{supp}(\overline{\mathcal{V}}_n) \longrightarrow \operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}_2})$,
(c) For any n, $\operatorname{supp}(\overline{\mathcal{W}}_n) \cap \operatorname{supp}(\overline{\mathcal{V}}_n)$ has the expected dimension.
(d) For any n, and any ray $\rho \in \Sigma$, $\operatorname{supp}(\overline{\mathcal{W}}_n) \cap \operatorname{supp}(\overline{\mathcal{V}}_n) \cap D_\rho$ has the expected dimension.

Then

$$\overline{\mathcal{W}}_n \wedge \overline{\mathcal{V}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}} \wedge \overline{\mathcal{T}}_{\mathcal{C}'}.$$

Proof. For two closed currents $\mathcal{S}$ and $\mathcal{T}$ on X_Σ we naturally identify $\mathcal{S} \wedge \mathcal{T} = \pi_*\big(\mathcal{S} \otimes \mathcal{T} \wedge [\bar{\Delta}]\big)$, where $\pi : X_\Sigma \times X_\Sigma \longrightarrow X_\Sigma$ is the projection. In $T_N \times T_N \subseteq X_\Sigma \times X_\Sigma$ we $\mathcal{T}_n := \mathcal{W}_n \otimes \mathcal{V}_n$ and $\mathcal{T}_{\mathcal{C}} := \mathcal{T}_{\mathcal{C}_1} \otimes \mathcal{T}_{\mathcal{C}_2}$. Now note that the diagonal in the open torus is the complete intersection of the tori $x_i = y_i$, $i = 1, \dots, n$. This together with assumption (c) allows for a repeated application of Theorem 6.5 to obtain

$$\mathcal{W}_n \otimes \mathcal{V}_n \wedge [\Delta] \longrightarrow \mathcal{T}_{\mathcal{C}_1} \otimes \mathcal{T}_{\mathcal{C}_2} \wedge [\Delta].$$

By assumption (c), and Lemma 6.8, for large n and rays $\rho \in \Sigma$,

$$\operatorname{supp}(\overline{\mathcal{W}}_n \otimes \overline{\mathcal{V}}_n) \cap [\overline{\Delta}] \cap D_\rho$$

have the expected dimension. Lemma 6.8, and the compatibility assumption imply that $\operatorname{supp}(\mathcal{W}_n \otimes \mathcal{V}_n) \cap \bar{\Delta} \cap D_{(0,\rho)}$ and $\operatorname{supp}(\mathcal{T}_{\mathcal{C}} \otimes \mathcal{T}_{\mathcal{C}'}) \cap \bar{\Delta} \cap D_{(0,\rho)}$ have the expected Cauchy–Riemann dimension. Therefore, Lemma 2.12 brings us to the situation of Lemma 6.7 and we conclude.

Can we drop assumption (b)? □

7. Dynamical Tropicalisation with non-Trivial Valuations

7.1. **Dynamical tropicalisation with a non-trivial valuation.** Recall that for a field $\mathbb{K}$, $\nu : \mathbb{K} \longrightarrow \mathbb{R} \cup \{\infty\}$, is called a valuation if it satisfies the following properties for every $a, b \in \mathbb{K}$:

(a) $\nu(a) = \infty$ if and only if $a = 0$;
(b) $\nu(ab) = \nu(a) + \nu(b)$;
(c) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$.

A valuation is called *trivial*, if the valuation of any non-zero element is 0. For an element $a \in \mathbb{K}$, we denote by $\bar{a}$ its image in the residue field. We are interested in the case where $\mathbb{K} = \mathbb{C}((t))$, is the field of *formal Laurent series* with the parameter t. with the usual valuation. That is, for $g(t) = \sum_{j \geq k} a_j t^j$, with $a_k \neq 0$, the valuation equals the the minimal exponent $\nu(g) = k \in \mathbb{Z}$.

Definition 7.1. (a) Let $f = \sum_{\alpha \in \mathbb{N}} c_\alpha z^\alpha \in \mathbb{K}[z^{\pm 1}]$, be a Laurent polynomial in n variables. The tropicalisation of f with respect to ν,

$$\mathrm{trop}_\nu(f) : \mathbb{R}^n \longrightarrow \mathbb{R},$$
$$x \mapsto \max\{-\nu(c_\alpha) + \langle x, \alpha \rangle\}.$$

(b) Let $I \subseteq \mathbb{K}[z^{\pm 1}]$ be an ideal. The tropical variety associated to I, as a set, is defined as

$$\mathrm{Trop}_\nu(I) := \bigcap_{f \in I} \mathrm{Trop}(\mathrm{trop}_\nu(f)),$$

where $\mathrm{Trop}(\mathrm{trop}_\nu(f))$ is the set of points where $\mathrm{trop}_\nu(f)$ is not differentiable; see Remark 4.5.

(c) For an algebraic subvariety of the torus $Z \subseteq (\mathbb{K}^*)^n$, with the associated ideal $\mathbb{I}(Z)$, the tropicalisation of Z, as a set, is $\mathrm{Trop}_\nu(Z) := \mathrm{Trop}_\nu(\mathbb{I}(Z))$.

(d) In all the situations above, trop_0 denotes the tropicalisation with respect to the trivial valuation.

We need to relate a non-trivial valuation to the trivial valuation.

Lemma 7.2. Consider the ideal $I \subseteq \mathbb{C}[t^{\pm 1}, z^{\pm}1] \overset{\iota}{\hookrightarrow} \mathbb{C}((t))[z]$. Assume that (u, x) are the coordinates in $\mathbb{R} \times \mathbb{R}^n$. Then, we have the following equality of sets

$$\mathrm{Trop}_0(I) \cap \{u = -1\} = \mathrm{Trop}_\nu(\iota(I)).$$

That is, the tropicalisation of I as an ideal in $\mathbb{C}[t, x]$ with respect to the trivial valuation intersected with $\{u = -1\}$ coincides with the tropicalisation of $I = \iota(I)$ with respect to the usual valuation in $\mathbb{C}((t))$.

The proof of the lemma becomes clear with the following example.

Example 7.3. Let

$$f(x, t) = 4(t^3 + t^{-1})z_1 z_2 + (1 + t + t^2)z_1.$$

Then, the tropicalisation of $f \in \mathbb{C}[t, z]$, with respect to the trivial valuation equals:

$$\mathrm{trop}_0(f) = \max\big\{ \max\{3u + x_1 + x_2, -u + x_1 + x_2\}, \max\{x_1, u + x_1 + 2u + x_1\}\big\}$$

Letting $u := -1$, $\mathrm{trop}_0(f)(-1, x) = \max\{1 + x_1 + x_2, x_1\}$. The latter equals $\mathrm{trop}_\nu(f)$ as an element of $\mathbb{C}((t))[z]$.

Proof of Lemma 7.2. If f is a monomial in $\mathbb{C}[t][z]$, then it is clear that

$$\operatorname{trop}_0(f)(-1,x) = \operatorname{trop}_\nu(\iota(f)).$$

Therefore, we have the equality for any polynomial in $f \in \mathbb{C}[t,z]$. To prove the main statement, note that

$$\begin{aligned}\operatorname{Trop}_\nu(\iota(I)) &= \bigcap_{f\in\iota(I)} \operatorname{Trop}(\operatorname{trop}_\nu(f)) \\ &= \bigcap_{f\in I} \big(\operatorname{Trop}(\operatorname{trop}_0(f)) \cap \{u=-1\}\big) \\ &= \operatorname{Trop}_0(I) \cap \{u=-1\}.\end{aligned}$$

□

Remark 7.4. Bergman in [Ber71], shows that for an algebraic subvariety $Z \subseteq (\mathbb{C}^*)^n$, one has

$$\lim \operatorname{Log}_t(Z) \subseteq \operatorname{Trop}_0(\mathbb{I}(Z)),$$

and he conjectured the equality. This conjecture was later proved by Bieri and Groves in [BG84]. More precisely, Bieri and Grove prove that $\lim \operatorname{Log}_t(Z) \cap (S^1)^n$ is a polyhedral sphere of real dimension equal to (the complex dimension) $\dim(Z) - 1$. Therefore, the fan $\lim \operatorname{Log}_t(Z)$ is a cone over their spherical complex. See also [MS15, Theorem 1.4.2].

Remark 7.5. The above lemma is related to the results of Markwig and Ren in [MR20]. They considered the tropicalisation of an ideal $J \subseteq R[[t]][x]$, where R is the ring of integers of a discrete valuation ring $\mathbb{K}$, which is non-trivially valued. To obtain finiteness properties, however, the authors consider the associated tropical variety in the half-space $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$. Note that such a variety is almost never balanced. The authors also prove that for an ideal $I \subseteq \mathbb{K}[x]$, the tropicalisation of the natural inverse image $\pi^{-1}I \subseteq R[[t]][x]$ with respect to trivial valuation, intersected with $\{u=-1\}$ equals $\operatorname{trop}_\nu(I)$; [MR20, Theorem 4].

Let us also recall the main result of [Bab23].

Theorem 7.6. Let $Z \subseteq (\mathbb{C}^*)^n$ be an irreducible subvariety of dimension p, and $\overline{Z}$ the closure of Z in the compatible smooth projective toric variety X. Then,

$$\frac{1}{m^{n-p}} \Phi_m^*[\overline{Z}] \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}}, \quad \text{as } m \to \infty,$$

where $\Phi_m : X \longrightarrow X$ is the continuous extension of $\Phi_m : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$, and $\overline{\mathcal{T}}_{\operatorname{trop}_0(Z)}$ is the extension by zero of $\mathcal{T}_{\operatorname{trop}_0(Z)}$ to X. Moreover, the supports also converge in Hausdorff metric.

Note that since the limit of a sequence of closed currents is closed, the above theorem implies that $\operatorname{trop}_0(Z)$ can be equipped with weights to become balanced. Note that the compatibility is in the following sense of Tevelev and Sturmfels:

Theorem 7.7.
(a) The closure $\bar{Z}$ of Z in X_Σ is complete, if and only if, $\operatorname{trop}(Z) \subseteq |\Sigma|$; see [Tev07].
(b) We have $|\Sigma| = \operatorname{trop}(Z)$, if and only if, for every $\sigma \in \Sigma$ the intersection $\mathcal{O}_\sigma \cap \bar{Z}$ is non-empty and of pure dimension $p - \dim(\sigma)$; see [ST08].

Theorem 7.8. Let $I \subseteq \mathbb{C}[t^{\pm 1}, x^{\pm 1}]$ be an ideal with the associated $(p+1)$-dimensional algebraic variety $W = \mathbb{V}(I) \subseteq (\mathbb{C}^*)^{n+1}$. Assume that the projection onto the first coordinate $\pi_1 : W \longrightarrow \mathbb{C}^*$ is surjective and Zariski closed. We denote the fibers as $W_t := \pi_1^{-1}(t)$. We have that

(a)
$$\frac{1}{m^{n-p}} \Phi_m^* [W_{e^m}] \longrightarrow \mathcal{T}_{\mathrm{Trop}_\nu(I)}, \quad \text{as } m \to \infty \ ,$$
in the sense of currents in $\mathcal{D}_p((\mathbb{C}^*)^n)$.

(b) $\mathrm{Trop}_\nu(I)$ can be equipped with weights to become balanced.

(c) $\lim \mathrm{supp}(\frac{1}{m^{n-p}} \Phi_m^* [W_{e^m}]) = \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_\nu(I)})$.

(d) On a toric variety X_Σ compatible with $\mathrm{trop}_0(W) + \{u = -1\}$,
$$\frac{1}{m^{n-p}} \Phi_m^* \overline{[W_{e^m}]} \longrightarrow \overline{\mathcal{T}}_{\mathrm{Trop}_\nu(I)}, \quad \text{as } m \to \infty$$

We need the following:

Lemma 7.9. Let $W \subseteq (\mathbb{C}^*)^{n+1}$ be a $(p+1)$-dimensional smooth subvariety, such that the projection onto the first factor, $\pi_1 : (\mathbb{C}^*)^{n+1} \longrightarrow \mathbb{C}^*$ is surjective and a Zariski closed morphism. Assume that W Then for a sufficiently large $|t_0| >> 0$
$$[W_{t_0}] = [\pi_1^{-1}(t_0)] = [\{t = t_0\}] \wedge [W].$$

Proof. We first prove that the set of singular points of W, together with the set of points where $[\{t = t_0\}] \wedge [W]$ has a multiplicity greater than 1, is contained in a Zariski closed set in W. We define the *critical set*,
$$C = \big\{ w \in W_{\mathrm{reg}} : \dim \big(T_w W \cap \ker \ \nabla_w t \big) = p + 1 \big\},$$
which is the set of points where the tangent space of $T_w W_{reg}$ is included in the tangent space of $T_w\{t = t_0\}$, and this set contains the set of points $w \in W_{\mathrm{reg}}$ points the intersection multiplicity of $\{t = t_0\}$ and W exceeds 1. We fix an ideal associated to $I = \mathbb{I}(W) = \langle f_1, \dots, f_k \rangle \subseteq \mathbb{C}[t, x]$. At any regular point $w \in W_{\mathrm{reg}}$, $T_w W$ is of dimension $p+1$, and the rank of the Jacobian matrix $J(f)(w) = \big(\frac{\partial f_i}{\partial z_j}(w)\big)_{k \times (n+1)}$ equals codimension of W, $(n+1) - (p+1) = n - p$. We have that $\nabla_w t = e_1$, where e_1 is the first element of the standard basis for the $\mathbb{C}$-vector space $\mathbb{C}^{n+1}$. We have $w \in C$, if and only if,
$$\ker \begin{pmatrix} e_1 \\ Jf(w) \end{pmatrix} = \ker \big(Jf(w) \big) \, .$$
As a result, C is an algebraic variety given as the intersection of $W \setminus W_{\mathrm{sing}}$ with the intersection of zero loci of $(q+1) \times (q+1)$-minors of $\begin{pmatrix} e_1 \\ Jf(w) \end{pmatrix}$. Therefore, the closure of C in W, $\overline{C}$ union W_{sing} is a Zariski-closed subset of W. Since W is not contained in $\{t = t_0\}$, as π_1 is surjective, then $\pi_1(\overline{C} \cup W_{\mathrm{sing}})$ is a Zariski closed proper subset in $\mathbb{C}^* \subseteq \mathbb{C}$, and hence finite.

□

Proof of Theorem 7.8. By the preceding lemma, and the fact that Φ_m preserves transversal intersection, we have
$$\frac{1}{m^{n-p}} \Phi_m^* [W_{e^m}] = \frac{1}{m^{n-(p+1)}} \Phi_m^* [W] \wedge \frac{1}{m} \Phi_m^* [\{t = e^m\}],$$

for a large m. Since $\mathrm{trop}_0(W)$ is a fan and it is transversal to the plane $\{u=-1\} \subset \mathbb{R}^{n+1}$ are transversal, we can use Theorem 6.5 to write

$$\lim \frac{1}{m^{n-p}} \Phi_m^*[W_{e^m}] = \big(\lim \frac{1}{m^{n-(p+1)}} \Phi_m^*[W]\big) \wedge \big(\lim \frac{1}{m} \Phi_m^*[\{t=e^m\}]\big)$$

By Theorem 7.6, restricted to $(\mathbb{C}^*)^{n+1}$, and the fact that we used $\mathrm{Log} = (-\log|\cdot|, \dots, -\log|\cdot|)$ in the definition of tropical currents, the above limits yield

$$\lim \frac{1}{m^{n-p}} \Phi_m^*[W_{e^m}] = \mathcal{T}_{\mathrm{Trop}_0(W)} \wedge \mathcal{T}_{\{u=-1\}}.$$

Applying Theorems 5.11 and Lemma 7.2, we obtain the equality. For the assertion (b), note that the limit $\mathcal{T}_{\mathrm{Trop}_\nu(I)}$ is a closed current and Theorem 4.3 implies that $\mathrm{Trop}_\nu(I)$ is naturally balanced. To observe (c), note that (a) implies

$$\lim \mathrm{supp}(\frac{1}{m^{n-p}} \Phi_m^*[W_{e^m}]) \supseteq \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_\nu(I)}).$$

However, because of transversality, $\mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_\nu(I)}) = \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_0(W)}) \cap \mathrm{supp}(\mathcal{T}_{\{u=-1\}})$. At the same time,

$$\lim \mathrm{supp}(\Phi_m^*[W_{e^m}]) = \lim \mathrm{supp}(\Phi_m^*[W]) \cap \mathrm{supp}(\Phi_m^*[\{t=e^m\}]).$$

Moreover, for the Hausdorff limit of sets $\lim(A_i \cap B_i) \subseteq (\lim A_i) \cap (\lim B_i)$. This implies

$$\lim \mathrm{supp}(\Phi_m^*[W_{e^m}]) \subseteq \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_0(W)}) \cap \mathrm{supp}(\mathcal{T}_{\{u=-1\}}),$$

which implies (c). Now, (d) is implied by Theorem 6.6. □

Let us first prove the analogous result to the main result of Bogart Jensen, Speyer, Sturmfels, and Thomas in [BJS+07]. See also [OP13] for generalisation.

Theorem 7.10. Assume that W and Z
respectively. Further,

(a) the supports converge in the Hausdorff metric
(b) $\mathcal{C}$ and $\mathcal{C}'$ intersect properly.

Then, $\mathcal{W}_n \wedge \mathcal{V}_n$ converges to $\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}$.

Proof to be completed. When $\mathcal{C}$ and $\mathcal{C}'$ intersect properly, it implies that the fibres of $\mathcal{T}_{\mathcal{C}}$ and $\mathcal{T}_{\mathcal{C}'}$ intersect transversely. In this situation,

Let $I \subseteq \mathbb{C}[t^{\pm 1}, x^{\pm 1}]$ be an ideal with the associated $(p+1)$-dimensional algebraic variety $W = \mathbb{V}(I) \subseteq (\mathbb{C}^*)^{n+1}$. Assume that the projection onto the first coordinate $\pi_1 : W \longrightarrow \mathbb{C}^*$ is surjective and Zariski closed. We denote the fibres as $W_t := \pi_1^{-1}(t)$. We have that

$$\frac{1}{m^{n-p}} \Phi_m^*[W_{e^m}] \longrightarrow \mathcal{T}_{\mathrm{trop}_\nu(I)}, \quad \text{as } m \to \infty ,$$

in the sense of currents in $\mathcal{D}_p((\mathbb{C}^*)^n)$. In particular, $\mathrm{trop}_\nu(I)$ can be equipped with weights to become balanced. Moreover, if Σ is a toric variety compatible with $\mathrm{trop}_0(W)$ and $\{u=-1\}$, then on X_Σ,

$$\frac{1}{m^{n-p}} \Phi_m^*[\overline{W}_{e^m}] \longrightarrow \overline{\mathcal{T}}_{\mathrm{trop}_\nu(I)}, \quad \text{as } m \to \infty .$$

□

Add Tevelev's theorem.

Proposition 7.11.
- Commutative, Associative. (?)
- Projection Formula
- Comparison to other Gubler etc.
- Analogous result of Jonsson [Jon16]
- Analogous results of Payne–Cartwright, Maclagan–Surmfels etc.

  $W_n \otimes S_n \wedge \Delta$ small perturbation of Δ.

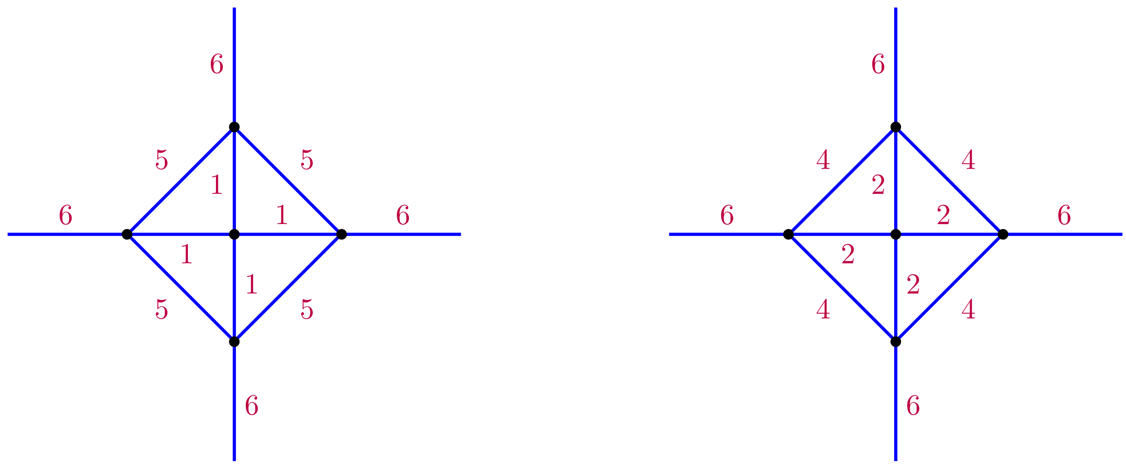

FIGURE 1. Two different tropical varieties with the same recession fan and same support.

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