2051901

① a) In the lecture notes, as a consequence of Nullstellensatz, we have the correspondence $\{\text{maximal ideals in } \mathbb{C}[x]\} \leftrightarrow \{\text{points in } \mathbb{A}^1\}$.

This can be seen by $(x-a) \subseteq \mathbb{C}[x] \leftrightarrow a \in \mathbb{A}^1$.

$\therefore \operatorname{maxSpec}(\mathbb{C}[x]) = \{(x-a) \mid a \in \mathbb{C}\}$.

We initially notice that $\mathbb{C}[x, 1/x] \cong \mathbb{C}[x,y]/(xy-1)$ as $(xy-1)$ is the kernel of $\varphi: \mathbb{C}[x,y] \to \mathbb{C}[x, 1/x]$, $\varphi(x) = x$ and $\varphi(y) = 1/x$. We want to find the maximal ideals of the RHS. These are of the form $(x-a, y-a^{-1})$ for $a \in \mathbb{C}^*$. Since we still have that $xy-1=0$, we deduce that $(x-a, y-a^{-1}) = (x-a)$.

$\therefore \operatorname{maxSpec}(\mathbb{C}[x, 1/x]) = \{(x-a) \mid a \in \mathbb{C}^*\}$.

b) $\varphi^*(1/x)(t) = 1/x(\varphi(t)) = 1/x(1/t) = t$ for any $t \neq 0 \in \mathbb{A}^1$.

$\therefore \varphi^*(1/x) = y \in \mathbb{C}[y, 1/y]$.

as φ^* is a morphism of $\mathbb{C}$-algebras

$\varphi^*(x) \cdot y = \varphi^*(x)\varphi^*(1/x) = \varphi^*(x \cdot 1/x) = \varphi^*(1) = 1$, so we deduce $\varphi^*(x) = 1/y$.

$\therefore \varphi^*(2-x) = \varphi^*(2) - \varphi^*(x) = 2 - 1/y$.

Similarly, $\varphi^*\left(2x^2 - \frac{2x^3-4x}{x^5}\right) = \varphi^*\left(2x^2 - \frac{2x^2-4}{x^4}\right) = \varphi^*(2x^2 - 1/x^4(2x^2+4))$

$= 2\varphi^*(x^2) - [(2\cdot\varphi^*(x^2)+4)\cdot\varphi^*(1/x^4)]$

$= 2\cdot 1/y^2 - [(2\cdot 1/y^2 + 4)\, y^4] = 2/y^2 - (2y^2+4y^4) = 2/y^2 - 2y^2 - 4y^4$.

② a) $(x,y,u) \mapsto (x,u)$ restricts to an isomorphism.

$y = ux$ so $(x,y,u) = (x, ux, u) \mapsto (x,u)$

If $\varphi(a, ab, b) = \varphi(a', a'b', b')$ then $(a,b) = (a',b') \Rightarrow (a, ab, b) = (a', a'b', b')$ so is injective.

By the nature of the map, surjectivity is clear to see. For every $\underline{a}' \in \mathbb{A}^2$ we can always find $\underline{a} \in \mathbb{A}^3$ such that $\varphi(\underline{a}) = \underline{a}'$ we simply do this by picking the necessary coordinates.

$\therefore$ we have an isomorphism.

b) $(x,y,u) \mapsto (x,y)$ doesn't restrict to an isomorphism.

$(x, ux, u) \mapsto (x, ux)$.

This is immediately clear when we see the surjectivity fails.

Due to $y = ux$, we will never be able to reach $(0,y) \in \mathbb{A}^2$ for any $y \neq 0$.

$\therefore$ this cannot be an isomorphism.

c) $\mathcal{O}_V(D(u))$ = the set of regular functions on $D(u) \subseteq V$.

$D(u) = \mathbb{A}^3 \setminus \mathbb{V}(u)$ → the open subset

$\mathcal{O}_V(\mathbb{A}^3 \setminus \mathbb{V}(u))$ = the set of regular functions where $u \neq 0$.

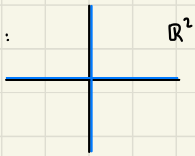
$= \mathbb{C}[V] \setminus \mathbb{V}(u)$?

(3) Assume $\bar{V}$ is irreducible. Let $V = A \cup B$ where A, B are two closed sets that are irreducible. We want either $V = A$ or $V = B$. Since $V = A \cup B$, it follows that $\bar{V} = \bar{A} \cup \bar{B}$. Since $\bar{V}$ is irreducible, let's say we have $\bar{V} = \bar{A}$. The closure of A as a subset of V is $\bar{A} \cap V = \bar{V} \cap V = V$. But, A is already closed in V, so $V = A$.

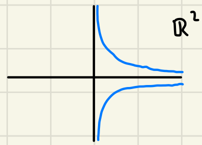
(4) Algebraic sets are either finite, or the whole space.
The set of roots for $y = \sin(x)$ is infinite, and therefore this analytic set is not algebraic.
This is not projective or compact, so Chow's lemma still holds.

(5) $V_0 = \mathbb{V}(xy^2)$
$V_1 = \mathbb{V}(xy^2 - 1)$
$V_2 = \mathbb{V}(xy^2 - 2)$

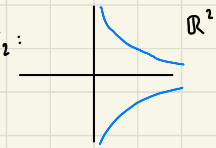
V_0 :



V_1 :



V_2 :



smooth : $\dim T_x V = \dim V$

$\dim V_0 = 2$, $\dim V_1 = 1$, $\dim V_2 = 1$ $\rightarrow$ because $\dim \mathbb{A}^2 - 1 = 1$.

$\nabla f(a,b)$:

$\nabla f(a,b) = (y^2, 2xy)\big|_{(a,b)} = (b^2, 2ab)$ $\rightarrow$ same for all 3.

Kernel of this $\uparrow$ is dimension 2, so V_0 is smooth, whereas V_1 and V_2 aren't.

irreducible: cannot be written as a union of two proper subsets.
V_0 is irreducible.