TROPICAL DYNAMICS AND INTERSECTIONS

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Abstract.

Contents

1. Introduction

Assume that $\mathcal{T}_n \longrightarrow \mathcal{T}$ is a convergent sequence of positive closed currents. For $\mathcal{T}$, one can ask when we have

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Theorem 1.1. The assignment $\mathcal{C} \longmapsto \mathcal{T}_{\mathcal{C}}$ induces a $\mathbb{Z}$-algebra isomorphism.

2. Tools from Superpotential Theory

Let (X, ω) be a compact Kähler manifold of dimension n. Assume that $\mathcal{S}$ is a positive or negative current of bidegree (q, q) on X. The quantity $\langle \mathcal{S}, \omega^{n-q} \rangle$ is called the *total mass* of $\mathcal{S}$. For $0 \leq r \leq n$, we consider the de Rham cohomology group $H^r(X, \mathbb{C}) = H^r(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ with coefficients in $\mathbb{C}$. Recall that Hodge theory implies the following decomposition of de Rham cohomology group into the Dolbeault cohomology groups

$$H^r(X, \mathbb{C}) = \bigoplus_{r=p+q} H^{p,q}(X, \mathbb{C}).$$

Following [DS10] we denote by $\mathcal{C}_q$ the cone of positive closed bidegree (q, q)-currents in X. We denote by $\mathcal{D}_q$ the $\mathbb{R}$-vector space spanned by $\mathcal{C}_q$, that is the space of closed real currents of bidegree (q, q). Every current $\mathcal{T} \in \mathcal{D}_q$ has a cohomology class

$$\{\mathcal{T}\} \in H^{q,q}(X, \mathbb{R}) = H^{q,q}(X, \mathbb{C}) \cap H^{2q}(X, \mathbb{R}).$$

We also define $\mathcal{D}_q^0$ as the subspace of $\mathcal{D}_q$, consisting of currents with vanishing cohomology. One defines the $*$-topology on $\mathcal{D}_q$ by the norm

$$\|\mathcal{S}\|_* := \min\|\mathcal{S}^+\| + \|\mathcal{S}^-\|,$$

where the minimum is taken over positive $\mathcal{S}^+$ and negative $\mathcal{S}^-$ in $\mathcal{C}_p$ satisfying $\mathcal{S} = \mathcal{S}^+ + \mathcal{S}^-$.

2.1. **Super-potentials of currents.** Super-potential theory of currents was introduced by Dinh and Sibony in several papers, see [DS10, DS09] and we recall basic notions from this theory. For a current $\mathcal{T} \in \mathcal{D}_q$ the super-potential of $\mathcal{T}$ is defined as a function on smooth forms in $\mathcal{D}^0_{n-q+1}$. Recall that for a smooth form $\mathcal{S} \in \mathcal{D}^0_{n-q+1}$, the $\partial\bar{\partial}$-lemma or dd^c-lemma [Huy05, Corollary 3.2.10.iv] implies the existence of the smooth potential $U_\mathcal{S} \in \mathcal{D}_{n-q}$ such that $dd^cU_\mathcal{S} = \mathcal{S}$. (do we have this for currents by continuity?) Now let $h := \dim H^{q,q}(X,\mathbb{R})$, and also $\alpha = (\alpha_1, \ldots, \alpha_{\beta_h})$ a fixed set of smooth forms such that their cohomology classes $\{\alpha\} = (\{\alpha_1\}, \ldots, \{\alpha_h\})$ form a basis for $H^{q,q}(X,\mathbb{R})$. By Poincaré duality, there exists a set of smooth forms $\alpha^\vee = (\alpha_1^\vee, \ldots, \alpha_h^\vee)$ such that their cohomology classes $\{\alpha^\vee\}$ form the dual basis of $\{\alpha\}$, with respect to the cup-product $\smile$. Note that by adding $U_\mathcal{S}$ to a suitable combination of $\alpha_i^\vee$, we can assume that $\langle U_\mathcal{S}, \alpha_i\rangle = 0$, for all $i = 1, \ldots, h$. In this case, we say that $U_\mathcal{S}$ is *α-normalised.*

Definition 2.1. Let $\mathcal{T} \in \mathcal{D}_q$ and $\mathcal{S}$ be a smooth form in $\mathcal{D}^0_{n-q+1}$.

(i) The *α-normalised super-potential* $\mathcal{U}_\mathcal{T}$ of $\mathcal{T}$ is given by the function

$$\begin{aligned} \mathcal{U}_\mathcal{T} : \{\mathcal{S} \in \mathcal{D}^0_{n-q+1} : \text{smooth}\} &\longrightarrow \mathbb{R} \\ \mathcal{S} &\longmapsto \langle \mathcal{T}, U_\mathcal{S}\rangle, \end{aligned}$$

where $U_\mathcal{S}$ is the α-normalised potential of $\mathcal{S}$.

(ii) We say $\mathcal{T}$ has a *continuous super-potential*, if $\mathcal{U}_\mathcal{T}$ can be extended to a function on $\mathcal{D}^0_{n-q+1}$ which is continuous with respect to the $*$-topology.

In general, consider $\mathcal{T} \in \mathcal{D}_q$ and $\mathcal{T} \in \mathcal{D}_r$. Assume that $q+r \leq n$ and $\mathcal{T}$ has a continuous super-potential. Let $\mathcal{U}_\mathcal{T}$ be the α-normalised super-potential of $\mathcal{T}$. Let $\beta \in \mathrm{Span}_\mathbb{R}\{\alpha\}$ such that $\{\beta\} = \{\mathcal{T}\}$. We define

$$\langle \mathcal{T} \wedge \mathcal{S}, \varphi\rangle := \mathcal{U}_\mathcal{T}(\mathcal{S} \wedge dd^c\varphi) + \langle \mathcal{S} \wedge \varphi, \beta\rangle. \tag{1}$$

Now assume that if $f : X \longrightarrow Y$, is a biholomorphism between smooth compact Kähler manifolds, then we have

$$f_*\mathcal{U}_{\mathcal{R}_1} = \mathcal{U}_{f_*\mathcal{R}_1}, \quad f^*\mathcal{U}_{\mathcal{R}_2} = \mathcal{U}_{f^*\mathcal{R}_2},$$

for $\mathcal{R}_1 \in \mathcal{D}_q(X)$ and $\mathcal{R}_2 \in \mathcal{D}_q(Y)$.

Definition 2.2. Let $(\mathcal{T}_n)$ be a sequence of currents in $\mathcal{D}_q$ weakly converging to $\mathcal{T}$. Let $\mathcal{U}_\mathcal{T}$ and $\mathcal{U}_{\mathcal{T}_n}$ be their α-normalised super-potentials. If $\mathcal{U}_{\mathcal{T}_n}$ converges to $\mathcal{U}_\mathcal{T}$ uniformly on any $*$-bounded set of smooth form in $\mathcal{D}^0_{n-q+1}$, then the convergence is called *SP-uniform.*

It is shown in [DS10, Proposition 3.2.8] that any current with continuous super-potentials can be SP-uniformly approximated by smooth forms. Moreover, currents with continuous super-potentials have other nice properties:

Theorem 2.3 ([DNV18, Theorem 1.1])**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two positive currents in $\mathcal{D}_q$, such that $\mathcal{T} \leq \mathcal{T}'$, *i.e.*, $\mathcal{T}' - \mathcal{T}$ is a positive current. Then, if $\mathcal{T}'$ has a continuous super-potential, then so does $\mathcal{T}$.

Theorem 2.4 ([DS10, Proposition 3.3.3])**.** If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two positive closed currents, and $\mathcal{T}_1$ has a continuous superpotentials, then $\mathcal{T}_1 \wedge \mathcal{T}_2$ is well-defined. Moreover, if $\mathcal{T}_2$ has also a continuous superpotential, then

(a) $\mathcal{T}_1 \wedge \mathcal{T}_2$ has a continuous superpotential;
(b) This wedge product is continuous with respect to the SP-uniform convergence.

Theorem 2.5 ([DS10, Proposition 3.3.4])**.** Assume that $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are closed positive currents, and $\mathcal{T}_1$ and $\mathcal{T}_2$ have continuous superpotentials. Then,

$$\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_2 \wedge \mathcal{T}_1 \quad \text{and} \quad (\mathcal{T}_1 \wedge \mathcal{T}_2) \wedge \mathcal{T}_3 = \mathcal{T}_1 \wedge (\mathcal{T}_2 \wedge \mathcal{T}_3).$$

Lemma 2.6. Let $\mathcal{T}$, $\mathcal{T}'$ be positive closed currents such that $\mathcal{T}_{|U} = \mathcal{T}'_{|U}$ in an open subset $U \subseteq X$, and both $\mathcal{T}$ and $\mathcal{T}'$ have continuous super-potentials. Then, for any $\mathcal{S} \in \mathcal{D}_r$, with continuous super-potential

$$\big(\mathcal{T} \wedge \mathcal{S}\big)_{|U} = \big(\mathcal{T}' \wedge \mathcal{S}\big)_{|U}.$$

Proof. In [DS10], for any current $\mathcal{S}$ with continuous super-potential, a family $\{\mathcal{T}_\theta\}_{\theta \in \mathbb{C}^*}$ is constructed that $\mathcal{T}_\theta$ converges SP-uniformly to $\mathcal{S}$ as $|\theta| \to 0$. Therefore, by the hypothesis of the lemma, we can construct families of smooth forms $\mathcal{T}_n, \mathcal{T}'_n, \mathcal{S}_n$, converging *SP*-uniformly to $\mathcal{T}, \mathcal{T}', \mathcal{S}$ respectively. Moreover,

$$\mathcal{T}_{n|U_\epsilon} = \mathcal{T}'_{n|U_\epsilon},$$

where U_ϵ is an ϵ-neighbourhood of U. Now, for a $(n-q-r, n-q-r)$ smooth form φ with compact support on U

$$\big(\mathcal{T}_n \wedge \mathcal{S}_n\big) \wedge \varphi = \big(\mathcal{T}'_n \wedge \mathcal{S}_n\big) \wedge \varphi,$$

together with Theorem 2.4(b), implies the assertion.

□

Lemma 2.7. If $\mathcal{T}$ is a positive closed current on a compact Kähler manifold X, which is locally bounded by a product of positive closed $(1,1)$-currents of continuous potentials, resp. Hölder continuous potentials, then $\mathcal{T}$ has continuous superpotentials, resp. Hölder continuous potentials in X.

Proof. Fix a point a in X. In an open neighbourhood of a that we identify with the ball $B(0,2)$ in $\mathbb{C}^n$, we have

$$\mathcal{T} \leq dd^c u_1 \wedge \cdots \wedge dd^c u_p$$

with u_i continuous or Hölder continuous. Without loss of generality, we can assume that these functions are strictly negative. Define on $B(0,1)$

$$u'_i := \max(u_i, A \log \|z\|)$$

with A big enough so that $u'_i = u_i$ on $B(0,1/2)$. Observe that $u'_i = A \log \|z\|$ near $\partial B(0,1)$. Hence we can extend it to a function which is smooth in a neighbourhood of $X \setminus B(0,1)$. Thus, this function is quasi-plurisubharmonic. We have

$$\mathcal{T} \leq (B\omega + dd^c u'_1) \wedge \cdots \wedge (B\omega + dd^c u'_p)$$

in a neighbourhood W_a of a if B is large enough.

Since we can cover X using a finite number of open sets W_{a_k}, we can add up all obtained quasi-plurisubharmonic functions together and obtain a quasi-plurisubharmonic function u. It is clear that

$$\mathcal{T} \leq (C\omega + u)^p$$

if C is large enough. The function u is continuous or Hölder continuous, and we deduce by Theorem 2.3.

□

Corollary 2.8. Let $q : \widehat{X} \longrightarrow X$, be the blowing up of the compact Kähler manifold X. Assume that $\mathcal{T} \in \mathcal{D}_p(\widehat{X})$ is such that the support of $\mathcal{T}$ does not intersect the exceptional divisors of $\widehat{X}$. If the current on $\mathcal{T}$ has a continuous superpotential then $q_*\mathcal{T}$ has the same property.

Lemma 2.9 (to be removed)**.** Assume that $\mathcal{T}_n$ and $\mathcal{T}$ are closed, positive (p,p)-dimensional currents in $\mathbb{C}^n$, and

$$\mathcal{T}_n \wedge [H] \longrightarrow \mathcal{T} \wedge [H],$$

for any generic affine linear $(n-p)$-dimenstional subspace $H \subseteq \mathbb{C}^n$. Then,

$$\mathcal{T}_n \longrightarrow \mathcal{T}.$$

Let $U \subseteq \mathbb{C}^m$ and $V \subseteq \mathbb{C}^n$ be two bounded open sets. Assume that $\pi_1 : U \times V \longrightarrow U$ and $\pi_2 : U \times V \longrightarrow V$ are the canonical projections. Consider two closed positive currents $\mathcal{T}$ and $\mathcal{S}$ on $U \times V$ of bi-dimension (m,m) and (n,n) respectively. We say that $\mathcal{T}$ horizontal-like if $\pi_2(\mathrm{supp}(\mathcal{T}))$ is relatively compact in V. Similarly, if vertical-like if $\pi_1(\mathrm{supp}(\mathcal{S}))$ is relatively compact in U, $\mathcal{S}$ is called vertical-line.

Theorem 2.10 ([BD20, Lemma 3.7])**.** Let $(\mathcal{T}_n) \longrightarrow \mathcal{T}$ be a convergent sequence of horizontal-like positive closed currents in $\mathcal{D}_p(\mathbb{C}^n)$ and L be an $(n-p)$-dimensional plane. Assume that the currents $\mathcal{T}_n \wedge [L]$ and $\mathcal{T} \wedge [L]$ are well-defined, have compact support, and the sequence $\mathcal{T}_n \wedge [L]$ is convergent. Then,

$$\lim_{n\to\infty} \langle \mathcal{T}_n \wedge [L], \phi \rangle \leq \langle \mathcal{T} \wedge [L], \phi \rangle,$$

for every p.s.h function ϕ on $\mathbb{C}^n$.

Theorem 2.11. For two complex manifolds X and Y, consider two convergent sequences of currents $\mathcal{T}_n \longrightarrow \mathcal{T}$ in $\mathcal{D}_q(X)$ and $\mathcal{S}_n \longrightarrow \mathcal{S}$ in $\mathcal{D}_r(Y)$. We have that

$$\mathcal{T}_n \otimes \mathcal{S}_n \longrightarrow \mathcal{T} \otimes \mathcal{S},$$

weakly in $\mathcal{D}_{q+r}$.

Sketch of the proof. Let us denote by (x,y) the coordinates on $X \times Y$. Using local coordinates and a partition of unity and Weierstrass theorem we can approximate any smooth forms on $X \times Y$ with forms with polynomial coefficients in (x,y). The approximation is in C^∞. As a result, the convergence, we only need test forms with monomial coefficients. Thus, the variables x, y are separated and the convergence of the tensor products becomes the convergence of each factor.

□

2.2. **A slicing lemma.** Let $f : X \longrightarrow Y$ be a dominant holomorphic map between complex manifolds, not necessarily compact, of dimension n and m respectively. Let $\mathscr{T}$ be a positive closed current on X of bi-dimension (p,p) with $p \geq m$. Then the slice

$$\mathscr{T}_y = \langle \mathscr{T} | f | y \rangle$$

exists for almost every $y \in Y$. This is a positive closed current of bi-dimension $(p - m, p - m)$ on X supported by $f^{-1}(y)$. If Ω is a smooth form of maximal bi-degree on Y and α a smooth $(q-m, q-m)$-form with compact support in X, then we have

$$\langle \mathscr{T}, \alpha \wedge f^*(\Omega) \rangle = \int_{y \in Y} \langle \mathscr{T}_y, \alpha \rangle \Omega(y).$$

In general, if $\mathscr{T}$ and $\mathscr{T}'$ are such that $\mathscr{T}_y = \mathscr{T}'_y$ for almost every y, we do not necessarily have $\mathscr{T} = \mathscr{T}'$. However, the following is true: Let $f_1, \ldots, f_k$ be dominant holomorphic maps from X to $Y_1, \ldots, Y_k$. Consider the vector space spanned by all the differential forms of type $\alpha \wedge f_i^*(\Omega_i)$ for some α as above and for some smooth form Ω_i on Y_i of maximal degree. Assume this space is equal to space of all (q,q)-forms of compact support in X. Then if $\langle \mathscr{T} | f_i | y_i \rangle = \langle \mathscr{T}' | f_i | y_i \rangle$ for every i and almost every $y_i \in Y_i$, we have $\mathscr{T} = \mathscr{T}'$. The proof is a consequence of the above discussion.

Lemma 2.12. Assume that $\mathscr{T}$ and $\mathscr{T}'$ are two positive closed currents of bidimension (p,p) on $\mathbb{C}^n$ and

$$\mathscr{T} \wedge [L] = \mathscr{T}' \wedge [L],$$

for any generic $(n-p)$-dimensional affine plane L. Then $\mathscr{T} = \mathscr{T}'$.

Proof. The above equality means that for any smooth function α with compact support and generic L we have

$$\langle \mathscr{T} \wedge [L], \alpha \rangle = \langle \mathscr{T}' \wedge [L], \alpha \rangle.$$

For each L, we can find a projection $f : \mathbb{C}^n \longrightarrow \mathbb{C}^p$, such that $f^{-1}(0) = L$, and the above equality implies that

$$\langle \mathscr{T} | f | 0 \rangle = \langle \mathscr{T}' | f | 0 \rangle.$$

Let $z \in \mathrm{supp}(\mathscr{T})$, and consider the local coordinates centred at z. By [Dem, Lemma 1.4], the set of strongly positive forms

$$\beta_s = i\beta_{s,1} \wedge \bar{\beta}_{s,1} \wedge \cdots \wedge i\beta_{s,p} \wedge \bar{\beta}_{s,p}, \quad 1 \leq s \leq \binom{n}{p}$$

forms a basis for $\Lambda^{p,p}(\mathbb{C}^n)^*$, where $\beta_{s,i}$ are of the type $dz_j \pm dz_k$ or $dz_j \pm i dz_k$. These forms can be obtained by the pullback of the standard volume forms by the projections $f : \mathbb{C}^n \longrightarrow \mathbb{C}^p$, where $(z_1, \ldots, z_n) \longmapsto (\beta_{s,1}, \ldots, \beta_{s,p})$, for all different $\beta_{s,i}$. By an action of $\mathrm{GL}(n, \mathbb{C})$, we can assume that for different choices of these projections $L = f^{-1}(0)$, the slices are well-defined and the equality holds

$$\langle \mathscr{T} \wedge [L], \alpha \rangle = \langle \mathscr{T}' \wedge [L], \alpha \rangle,$$

and we conclude by the above discussion.

□

3. Tropical Varieties, Tori, Tropical Currents

In this section, we recall the definition of tropical cycles and note that with the natural addition of tropical cycles and their *stable intersection* the tropical cycles form a $\mathbb{Z}$-algebra.

3.1. **Tropical varieties.** We say that a linear subspace $H \subseteq \mathbb{R}^n$ is called *rational* if there is a subset of $\mathbb{Z}^n$ that spans H. An intersection of finitely many rational half-spaces which are defined by

$$\{x \in \mathbb{R}^n : \langle m, x\rangle \geq c, \text{ for some } m \in \mathbb{Z}^n, \ c \in \mathbb{R}\}.$$

is called a *rational polyhedron.* A rational polyhedral complex is a polyhedral complex with only rational polyhedrons. The polyhedrons in a polyhedral complex are also called *cells.* A *fan* is a polyhedral complex whose cells are all cones. If any cone of a fan Σ is contained in another fan Σ', then Σ is a *subfan* of Σ. One dimensional cones of a fan are often referred to as *rays.* All fans and polyhedral complexes considered in this article are *rational.*

For a given polyhedron σ, and a finitely generated abelian group N, we denote by

$$\begin{aligned}
\operatorname{aff}(\sigma) &:= \text{affine span of } \sigma,\\
H_\sigma &:= \text{translation of } \operatorname{aff}(\sigma) \text{ to the origin},\\
N_\sigma &:= N \cap H_\sigma,\\
N(\sigma) &:= N/N_\sigma.
\end{aligned}$$

Consider τ, a codimension one face of a p-dimensional polyhedron σ, and let $u_{\sigma/\tau}$ be the unique outward generator of the one-dimensional lattice $(\mathbb{Z}^n \cap H_\sigma)/(\mathbb{Z}^n \cap H_\tau)$.

Definition 3.1 (Balancing Condition and Tropical Cycles)**.** Let $\mathcal{C}$ be a p-dimensional polyhedral complex whose p-dimensional cones are equipped with integer weights. We say that $\mathcal{C}$ satisfies the *balancing condition* at τ if

$$\sum_{\sigma \supset \tau} w(\sigma)\ u_{\sigma/\tau} = 0, \quad \text{in } \mathbb{Z}^n/(\mathbb{Z}^n \cap H_\tau),$$

where the sum is over all p-dimensional cells σ in $\mathcal{C}$ containing τ as a face. A *tropical variety* in $\mathbb{R}^n$ is a weighted complex with finitely many cells that satisfies the balancing condition at every cone of dimension $p-1$.

3.2. **Addition of tropical cycles.**

3.2.1. *Stable intersection.* To state what intersection multiplicities are assigned in the stable intersection in tropical geometry, we need to recall the *star* of a $(p-1)$-dimensional cell in a polyhedral complex, which is, roughly speaking, the extension of the local p-dimensional fan around it.

Definition 3.2. Let $\Sigma \subseteq \mathbb{R}^n$ be a polyhedral complex and $\tau \in \Sigma$ be a $(p-1)$-dimensional cell. The star of σ in Σ, denoted by $\operatorname{star}_\Sigma(\sigma)$ is the union of the extension $\bar{\sigma}$ of p-dimensional cells σ containing τ as a face. Here, by the extension we mean

$$\bar{\sigma} = \{\lambda(x-y) : \lambda \geq 0, x \in \sigma, y \in \tau\}.$$

Definition 3.3 (Stable intersection)**.** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two tropical varieties in $\mathbb{R}^n$. The *stable intersection* $\mathcal{C}_1 \cdot \mathcal{C}_2$ is the polyhedral complex

$$\mathcal{C}_1 \cdot \mathcal{C}_2 = \bigcup_{\substack{\sigma_1 \in \mathcal{C}_1, \sigma_2 \in \mathcal{C}_2 \\ \dim(\sigma_1 + \sigma_2) = n}} \sigma_1 \cap \sigma_2.$$

For a top dimensional cell $\sigma_1 \cap \sigma_2$ in $\mathcal{C}_1 \cdot \mathcal{C}_2$, the weights are obtained by

$$w_{\mathcal{C}_1 \cdot \mathcal{C}_2}(\sigma_1 \cap \sigma_2) = \sum_{\tau_1, \tau_2} w_{\tau_1} w_{\tau_2} [N : N_{\tau_1} + N_{\tau_2}],$$

where the sum is over all $\tau_1 \in \mathrm{star}_{\mathcal{C}_1}(\sigma_1 \cap \sigma_2), \tau_2 \in \mathrm{star}_{\mathcal{C}_2}(\sigma_1 \cap \sigma_2)$ with $\tau_1 \cap (v + \tau_2) \neq \varnothing$, for a generic fixed $v \in \mathbb{R}^n$.

In tropical geometry it is shown that

Theorem 3.4 ([MS15, Chapter 3])**.** In the above definition, the stable intersection does not depend on the choice of generic $v \in \mathbb{R}^n$ and the stable intersection of two tropical varieties is indeed a tropical variety.

3.3. **The Complex Tori.** For a finitely generated abelian group N, we set

$$T_N := \mathbb{C}^* \otimes_Z N.$$

Thus, a rational p dimensional plane $H \subseteq \mathbb{R}^n$ defines the subtori $T_{H \cap \mathbb{Z}^n} \subseteq (\mathbb{C}^*)^n$ of complex dimension p. Its cosets are of the form $T_{H \cap N} \cdot y$, for $y \in S_{\mathbb{Z}^n / (\mathbb{Z}^n \cap H)}$.

Proposition 3.5 ([Kat09, Propositions 6.1])**.** Let $H_1, H_2 \subseteq \mathbb{R}^n$ be two rational planes of dimension p and q with $p + q = n$ that intersect transversely. Then, the complex tori $T_{H_1 \cap \mathbb{Z}^n}$ and $T_{H_2 \cap \mathbb{Z}^n}$ intersect at $[N : N_{H_1} + N_{H_2}]$ distinct points.

4. Tropical Currents

Let us briefly recall the definition of tropical varieties from [MS15] and tropical currents from [Bab14, BH17]. To fix the notation,

$$\begin{aligned} T_N &:= \text{the complex algebraic torus } \mathbb{C}^* \otimes_{\mathbb{Z}} N, \\ S_N &:= \text{the compact real torus } S^1 \otimes_{\mathbb{Z}} N, \\ N_{\mathbb{R}} &:= \text{the real vector space } \mathbb{R} \otimes_{\mathbb{Z}} N. \end{aligned}$$

Definition 4.1. We define the affine extension p-dimensional tropical variety $\mathcal{C}$, denoted by $\widehat{\mathcal{C}}$, as the tropical variety with support

$$|\widehat{\mathcal{C}}| = \bigcup_{\sigma \in \mathcal{C},\ \dim(\sigma) = p} \mathrm{aff}(\sigma)$$

and corresponding weights

$$w_{\mathrm{aff}(\sigma)} = \sum_{\sigma' \subseteq \mathrm{aff}(\sigma), \dim(\sigma') = p} w_{\sigma}.$$

4.1. **Toric sets and intersection numbers.** Let $\mathbb{C}^*$ be the group of nonzero complex numbers. As before, the logarithm map is the homomorphism

$$\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \qquad (z_1, \dots, z_n) \longmapsto (-\log|z_1|, \dots, -\log|z_n|),$$

and the *argument map* is

$$\mathrm{Arg} : (\mathbb{C}^*)^n \longrightarrow (S^1)^n, \qquad (z_1, \dots, z_n) \longmapsto (z_1/|z_1|, \dots, z_n/|z_n|).$$

For a rational linear subspace $H \subseteq \mathbb{R}^n$ we have the following exact sequences:

$$0 \longrightarrow H \cap \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n/(H \cap \mathbb{Z}^n) \longrightarrow 0,$$

Moreover,

$$0 \longrightarrow S_{H\cap\mathbb{Z}^n} \longrightarrow (S^1)^n = S^1 \otimes_{\mathbb{Z}} \mathbb{Z}^n \longrightarrow S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)} \longrightarrow 0.$$

Define

$$\pi_H : \ \mathrm{Log}^{-1}(H) \xrightarrow{\mathrm{Arg}} (S^1)^n \longrightarrow S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)} \ .$$

One has

$$\ker(\pi_H) = T_{H\cap\mathbb{Z}^n} \subseteq (\mathbb{C}^*)^n.$$

As a result, when H is of dimension p, the set $\mathrm{Log}^{-1}(H)$ is naturally foliated by the $\pi_H^{-1}(x) = T_{H\cap\mathbb{Z}^n} \cdot x \simeq (\mathbb{C}^*)^p$ for $x \in S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}$. For a lattice basis $u_1, \dots, u_p$, of $H \cap \mathbb{Z}^n$, the tori $T_{H\cap\mathbb{Z}^n}.x$ can be parametrised by the monomial map

$$(\mathbb{C}^*)^p \longrightarrow (\mathbb{C}^*)^n, \quad z \longmapsto x.z^{[u_1,\dots,u_p]^t}$$

where $U = [u_1, \dots, u_p]$ is the matrix with column vectors $u_1, \dots, u_p$, and z^{U^t} denotes that $z \in (\mathbb{C}^*)^p$ is taken to have the exponents with rows of the matrix U. Accordingly, one can easily check that

$$T_{H\cap\mathbb{Z}^n} \cdot x = \mathbb{V}\big(\{z \in (\mathbb{C}^*)^n : z^{m_i} = x^{m_i},\ i = 1, \dots, m-p\}\big).$$

for any choice of a $\mathbb{Z}$-basis $\{m_1, \dots, m_{n-p}\}$ of $\mathbb{Z}^n/(H \cap \mathbb{Z}^n)$.

Remark 4.2. When strictly convex cone $\sigma = \mathrm{cone}(u_1, \dots, u_n)$ where $\{u_1, \dots, u_n\}$ forms a $\mathbb{Z}$-basis for $\mathbb{Z}^n$, then the associated toric variety is isomorphic to $\mathbb{C}^n$, which can be obtained by z

Definition 4.3. Let H be a rational subspace of dimension p, and μ be the Haar measure of mass 1 on $S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}$. We define a (p,p)-dimensional closed current $\mathcal{T}_H$ on $(\mathbb{C}^*)^n$ by

$$\mathcal{T}_H := \int_{x\in S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}} \left[\pi_H^{-1}(x)\right] \, d\mu(x).$$

When A is an affine subspace of $\mathbb{R}^n$ parallel to the linear subspace $H = A - a$ for $a \in A$, we define $\mathcal{T}_A$ by translation of $\mathcal{T}_H$. Namely, we define the submersion π_A as the composition

$$\pi_A : \ \mathrm{Log}^{-1}(A) \xrightarrow{e^a} \mathrm{Log}^{-1}(H) \xrightarrow{\pi_H} S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}.$$

For σ a p-dimensional (rational) polyhedron in $\mathbb{R}^n$, we denote

$$\begin{aligned} \operatorname{aff}(\sigma) &:= \text{the affine span of } \sigma, \\ \sigma^\circ &:= \text{the interior of } \sigma \text{ in } \operatorname{aff}(\sigma), \\ H_\sigma &:= \text{the linear subspace parallel to } \operatorname{aff}(\sigma), \\ N(\sigma) &:= \mathbb{Z}^n/(H_\sigma \cap \mathbb{Z}^n). \end{aligned}$$

Definition 4.4. Let $\mathcal{C}$, be a weighted polyhedral complex of dimension p. The tropical current $\mathcal{T}_\mathcal{C}$ associated to $\mathcal{C}$ is given by

$$\mathcal{T}_\mathcal{C} = \sum_\sigma w_\sigma \; \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \mathcal{T}_{\operatorname{aff}(\sigma)},$$

where the sum runs over all p-dimensional cells σ of $\mathcal{C}$.

Theorem 4.5 ([Bab14])**.** A weighted complex $\mathcal{C}$ is balanced, if and only if, $\mathcal{T}_\mathcal{C}$ is closed.

Theorem 4.6 ([Bab14])**.** Any tropical current $\mathcal{T}_\mathcal{C} \in \mathcal{D}'_{n-1,n-1}((\mathbb{C}^*)^n)$ is of the form $dd^c[\mathfrak{q} \circ \mathrm{Log}]$, where $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$, is a tropical Laurent polynomial.

Proposition 4.7 ([Bab23, Proposition 4.6])**.** Assume that $\mathcal{T} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$ is a closed positive $(S^1)^n$-invariant current whose support is given by $\mathrm{Log}^{-1}(|\mathcal{C}|)$, for a polyhedral complex $\mathcal{C} \subseteq \mathbb{R}^n$ of pure dimension p. Then $\mathcal{T}$ is a tropical current.

5. Continuity of superpotentials

Let $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$, be a tropical polynomial function, and $\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$, as before. The current $dd^c[\mathfrak{q} \circ \mathrm{Log}] \in \mathcal{D}'_{n-1,n-1}(\mathbb{C}^*)^n$ has a bounded potential, and by Bedford–Taylor theory, for any positive closed current $\mathcal{T} \in \mathcal{D}'_{p,p}(\mathbb{C}^*)^n$, the product

$$dd^c[\mathfrak{q} \circ \mathrm{Log}] \wedge \mathcal{T} = dd^c([\mathfrak{q} \circ \mathrm{Log}]\, \mathcal{T}),$$

is well-defined. See [Dem, Section III.3]. In higher codimensions though, to prove that any two tropical currents have a well-defined wedge product, we utilise Dinh and Sibony's superpotential theory [DS09] on a compact Kähler manifold, and as a result, we extend the tropical currents to smooth compact toric varieties.

5.1. Tropical Currents on Toric Varieties.

In a toric variety X_Σ, for a cone $\sigma \in \Sigma$, we denote by $\mathcal{O}_\sigma$, the toric orbit associated with σ. We have

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{O}_\sigma.$$

We also set D_σ to be the closure of $\mathcal{O}_\sigma$ in the X_Σ. $\Sigma(p)$ p-dimensional skeleton.

Fibers of tropical currents are algebraic varieties with finite degrees and can be extended by zero to any toric variety, in consequence, any tropical current can be extended by zero to toric varieties. Moreover, with the following compatibility condition, we can ask for the extension of the fibres to intersect the toric invariant divisors transversally.

Definition 5.1. (i) For a polyhedron σ, its *recession cone* is the convex polyhedral cone

$$\operatorname{rec}(\sigma) = \{b \in \mathbb{R}^n : \sigma + b \subseteq \sigma\} \subseteq H_\sigma.$$

(ii) Let $\mathcal{C}$ be a p-dimensional balanced weighted complex in $\mathbb{R}^n$, and Σ a p-dimensional fan. We say that $\mathcal{C}$ is *compatible* with Σ, if $\operatorname{rec}(\sigma) \in \Sigma$ for all $\sigma \in \mathcal{C}$.
(iii) We say the tropical current $\mathcal{T}_{\mathcal{C}}$ is *compatible* with X_Σ, if all the closures of the fibers $\pi^{-1}_{\operatorname{aff}(\sigma)}(x)$ in X_Σ of $\mathcal{T}_{\mathcal{C}}$ intersect the torus invariant divisors of X_Σ transversely.

Theorem 5.2. Let $\mathcal{C}$ be a p-dimensional tropical cycle Σ be a fan. Assume that $\sigma \in \mathcal{C}$ is a p-dimensional polyhedron and $\rho \in \Sigma$ is a one-dimensional cone. Then

(a) The intersection $D_\rho \cap \overline{\pi^{-1}_{\operatorname{aff}(\sigma)}(x)}$ is non-empty and transverse, if and only if, $\rho \in \operatorname{rec}(\sigma)$. Here $\overline{\pi^{-1}_{\operatorname{aff}(\sigma)}(x)}$ corresponds the closure of a fiber of $\mathcal{T}_{\operatorname{aff}(\sigma)}$ in the toric variety X_Σ.
(b) In particular, if $\mathcal{C}$ is compatible with Σ, if and only if, $\mathcal{T}_{\mathcal{C}}$ is compatible with X_Σ.

Proof. See Lemma [BH17, Lemma 4.10]. □

For a tropical current $\mathcal{T}_{\mathcal{C}} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$, and given a toric variety X_Σ we denote its extension by zero $\overline{\mathcal{T}}_{\mathcal{C}} \in \mathcal{D}'_{p,p}(X_\Sigma)$.

Proposition 5.3. For every tropical variety $\mathcal{C}$, a smooth projective toric fan Σ compatible with a subdivision of $\mathcal{C}$.

Proof. By [BS11], for $\mathcal{C}$ there is a refinement $\mathcal{C}'$, and a complete fan $\Sigma_1 \subseteq \mathbb{R}^n$ such that $\mathcal{C}'$ is compatible with Σ_1. Applying the toric Chow lemma [CLS11, Theorem 6.1.18] and the toric resolution of singularities [CLS11, Theorem 11.1.9] we can find a fan Σ which is a refinement of Σ_1 that defines a smooth projective variety X_Σ. The tropical variety $\mathcal{C}''$ which is the refinement of $\mathcal{C}'$ induced by Σ, satisfies the statement.

□

Remark 5.4. When $\mathcal{C}'$ is a refinement of a tropical variety $\mathcal{C}$, then $\mathcal{C}'$ is a tropical variety with natural induced weights. It is also easy to check that we have the equality of currents $\mathcal{T}_{\mathcal{C}} = \mathcal{T}_{\mathcal{C}'}$ in $(\mathbb{C}^*)^n$; see [BH17, Section 2.6].

Lemma 5.5. Let $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tropical Laurent polynomial and X_Σ be a smooth projective toric variety compatible with a subdivision of $V_{\text{trop}}(\mathfrak{q})$. Let $\rho \in \Sigma(1)$. Assume that $\zeta_0 \in D_\rho \cap \operatorname{Supp}(\overline{dd^c[\mathfrak{q} \circ \operatorname{Log}]})$, and Ω is a sufficiently small neighbourhood of ζ_0. Then, $\mathfrak{q} \circ \operatorname{Log} \in \operatorname{PSH}(\Omega \setminus D_\rho) \cap \mathcal{C}^0(\Omega \setminus D_\rho)$ can be extended to a function $u : \Omega \longrightarrow \mathbb{R}$, such that

(a) In Ω, $u = g + \kappa \log|f|$, where g is a continuous function, f is the local equation for D_ρ, and κ is a negative integer.
(b) Restricted to Ω, we have $dd^c u = \overline{\mathcal{T}}_{V_{\text{trop}(\mathfrak{q})}} + c[D_\rho]$.
(c) In Ω, we have $\overline{\mathcal{T}}_{\mathcal{C}} = dd^c g$. In particular, $\overline{\mathcal{T}}_{V_{\text{trop}(\mathfrak{q})}}$ has a continuous superpotential.

Proof. Assume that $\mathfrak{q} = \max_{\alpha \in A}\{c_\alpha + \langle \alpha, x\rangle\}$. Recall that

$$\operatorname{Log} = (-\log|\cdot|, \dots, -\log|\cdot|).$$

We write

$$\mathfrak{q} \circ \operatorname{Log} = \log \max_\alpha \{|e^{c_\alpha} z^{-\alpha}|\}.$$

Assume that near ζ_0, $\mathfrak{q}\circ\mathrm{Log}$ is given by $\max\{|e^{c_\beta}z^{-\beta}, |e^{c_\gamma}z^{-\gamma}|\}$. This implies that in $\mathrm{Log}(\Omega\setminus D_\rho)$, $\mathfrak{q}$ is given by $\max\{c_\beta+\langle\beta,x\rangle, c_\gamma+\langle\gamma,x\rangle\}$. For $\mathfrak{q}=\max_{\alpha\in A}\{c_\alpha+\langle\alpha,x\rangle\}$ we set $\mathrm{rec}(\mathfrak{q})=\max_{\alpha\in A}\{\langle\alpha,x\rangle\}$. It is not hard to check that

$$\mathrm{rec}\big(V_{\mathrm{trop}}(\mathfrak{q})\big)=V_{\mathrm{trop}}(\mathrm{rec}(\mathfrak{q}));$$

see [MS15, Page 132].

We now show that by extending each $z^{-\alpha}$ as a rational function to X_Σ, the compatibility condition implies that $\mathfrak{q}\circ\mathrm{Log}$ extends to X_Σ. By [CLS11, Proposition 4.1.2] the divisor of the extension of a character z^α in X_Σ is given by

$$\mathrm{Div}(z^\alpha)=\sum_{\rho\in\Sigma(1)}\langle\alpha,n_\rho\rangle D_\rho, \tag{2}$$

where n_ρ the is the minimal generator of ρ. By assumption,

$$D_\rho\cap\mathrm{Supp}(\overline{dd^c[\mathfrak{q}\circ\mathrm{Log}]})\neq\varnothing.$$

Theorem 5.2 implies that

$$n_\rho\in\mathrm{rec}(V_{\mathrm{trop}(\mathfrak{q})}).$$

Moreover, if $\zeta_1\in D_\rho\cap\mathrm{Supp}(\overline{dd^c[\mathrm{rec}(\mathfrak{q})\circ\mathrm{Log}]})$, then in a small neighbourhood of $\mathrm{Log}(\zeta_1)$, $\mathrm{rec}(\mathfrak{q})(x)=\max\{\langle\beta,x\rangle,\langle\gamma,x\rangle\}$. By definition

$$n_\rho\in\mathrm{rec}(V_{\mathrm{trop}(\mathfrak{q})})\quad\text{ if and only if }\quad\kappa:=\langle\beta,n_\rho\rangle=\langle\gamma,n_\rho\rangle.$$

This, together with Equation 2 implies that the extension of $z^{-\beta}$ and $z^{-\gamma}$ as rational functions to X_Σ have the same vanishing order along D_ρ, and we write $z^{-\beta}=f^\kappa\frac{g_1}{h_1}$ and $z^{-\gamma}=f^\kappa\frac{g_2}{h_2}$. Now note that in $\Omega\setminus D_\rho$

$$\mathfrak{q}\circ\mathrm{Log}=\max\log\{|e^{c_\beta}z^{-\beta}|,|e^{c_\gamma}z^{-\gamma}|\}=\kappa\log|f|+\max\{|e^{c_\beta}\frac{g_1}{h_1}|,|e^{c_\gamma}\frac{g_2}{h_2}|\},$$

we must have $\kappa<0$, otherwise $q\circ\mathrm{Log}=-\infty$ in $\Omega\setminus D_\rho$. Consequently, $\mathfrak{q}\circ\mathrm{Log}:\Omega\setminus D_\rho\longrightarrow\mathbb{R}$, can be extended to

$$u:=\kappa\log|f|+\max\{|e^{c_\beta}\frac{g_1}{h_1}|,|e^{c_\gamma}\frac{g_2}{h_2}|\}$$

on Ω. Setting

$$g=\max\{|e^{-c_\beta}\frac{g_1}{h_1}|,|e^{-c_\gamma}\frac{g_2}{h_2}|\},$$

implies (a).

We have

$$dd^c[\mathfrak{q}\circ\mathrm{Log}]_{|_{\Omega\setminus D_\rho}}=\big(dd^c\log|f|^\kappa\ dd^c\log|g|\big)_{|_{\Omega\setminus D_\rho}}=dd^c\log|g|_{|_{\Omega\setminus D_\rho}},$$

since $dd^c\log|f|^\kappa$ is holomorphic in $\Omega\setminus D_\rho$. As a result of compatibility with X_Σ, $\overline{dd^c[\mathfrak{q}\circ\mathrm{Log}]}$ does not charge any mass in D_ρ, and we obtain

$$\overline{dd^c[\mathfrak{q}\circ\mathrm{Log}]}=dd^c\log|g|.$$

This together with Theorem 4.6 implies (c) and (b).

□

Lemma 5.6. Assume that $\mathrm{aff}(\sigma) = H_1 \cap \cdots \cap H_{n-p}$, where H_i are hyperplanes in $\mathbb{R}^n$. If Σ is a smooth projective fan compatible with $\bigcup_i H_i$, then

$$\overline{\mathcal{T}}_{\mathrm{aff}(\sigma)} = \overline{\mathcal{T}_{H_1} \wedge \cdots \wedge \mathcal{T}_{H_{n-p}}} \leq \overline{\mathcal{T}}_{H_1} \wedge \cdots \wedge \overline{\mathcal{T}}_{H_{n-p}}.$$

Proof. The wedge products in question are well-defined by Lemma 5.5 and Theorem 2.4. As both currents on both sides of the equation coincide on $(\mathbb{C}^*)^n$, the support of the current on the right-hand side contains the closure of the support of $\mathcal{T}_{\mathcal{C}}$ in X_Σ. □

Theorem 5.7. Let $\mathcal{C}$ be a tropical cycle of dimension p compatible with a smooth, projective fan Σ, then $\overline{\mathcal{T}}_{\mathcal{C}}$ has a continuous superpotential in X_Σ.

Proof. Let $\widehat{\mathcal{C}}$ be the affine extension of $\mathcal{C}$, and $\widehat{\Sigma}$ be a smooth projective fan, compatible with $\widehat{\mathcal{C}}$. By the preceding lemma and repeated application of Theorem 2.4 for any $\sigma \in \mathcal{C}$, $\overline{\mathcal{T}}_{\mathrm{aff}(\sigma)}$ has a bounded superpotential, which implies this property for $\overline{\mathcal{T}}_{\widehat{\mathcal{C}}}$. Now, since $\mathcal{T}_{\widehat{\mathcal{C}}} - \mathcal{T}_{\mathcal{C}}$ is a positive closed tropical current in $(\mathbb{C}^*)^n$,

$$\overline{\mathcal{T}_{\widehat{\mathcal{C}}} - \mathcal{T}_{\mathcal{C}}} = \overline{\mathcal{T}}_{\widehat{\mathcal{C}}} - \overline{\mathcal{T}}_{\mathcal{C}} \geq 0$$

in $X_{\widehat{\Sigma}}$. Continuity of the superpotential of $\overline{\mathcal{T}}_{\mathcal{C}}$ in $X_{\widehat{\Sigma}}$ follows from Theorem 2.3.

We now show that $\overline{\mathcal{T}}_{\mathcal{C}}$ has also a continuous super-potential on X_Σ as well. We consider the proper map $f : X_{\widehat{\Sigma}} \longrightarrow X_\Sigma$, which can be understood as a composition of multiple blow-ups along toric points with exceptional divisors D_ρ for any ray $\rho \in \widehat{\Sigma} \setminus \Sigma$. These divisors satisfy $D_\rho \cap \mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}}) = \varnothing$. We deduce by Corollary 2.8.

□

Proof. This is an easy corollary of previous lemma, since q is biholomorphic near $\mathrm{supp}(\mathcal{T})$. □

Proposition 5.8. For any two tropical currents $\mathcal{C}_1$ and $\mathcal{C}_2$, the intersection product

$$\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2} := \overline{\mathcal{T}}_{\mathcal{C}_1} \wedge \overline{\mathcal{T}}_{\mathcal{C}_2}|_{(\mathbb{C}^*)^n},$$

does not depend on the choice of a smooth projective toric variety compatible with $\widehat{\mathcal{C}}_1 + \widehat{\mathcal{C}}_2$. Moreover, this product coincides with the definition of wedge products with bi-degree $(1,1)$ tropical currents in Bedford–Taylor Theory in $(\mathbb{C}^*)^n$.

To be added. □

6. The Support of the Intersection

Let us recall the notion of the stable intersection, which is proven to be well-defined.

Definition 6.1. Support of the *stable intersection* two tropical cycles $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{R}^n$, is the Gromov–Hausdorff limit (in compact subsets of $\mathbb{R}^n$) of

$$(\mathcal{C}_1 + \varepsilon b) \cap \mathcal{C}_2$$

as the real number $\varepsilon \to 0$, where $b \in \mathbb{R}^n$ is a generically chosen.

In this section, we show that

Proposition 6.2. $\mathrm{Supp}(\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}) = \mathrm{Log}^{-1}(|\mathcal{C} \wedge \mathcal{C}'|)$.

Lemma 6.3.

Lemma 6.4. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two tropical cycles in $\mathbb{R}^n$, then

$$\mathrm{Supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) \subseteq \mathrm{Log}^{-1}(|\mathcal{C}_1 \wedge \mathcal{C}_2|)$$

Proof. Fist assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect transversely, then the fibres of $\mathcal{T}_{\mathcal{C}_1}$ and $\mathcal{T}_{\mathcal{C}_2}$ also intersect transversely. In this case, we have

$$\mathrm{Supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) = \mathrm{Log}^{-1}(|\mathcal{C}_1| \cap |\mathcal{C}_2|) = \mathrm{Log}^{-1}(|\mathcal{C}_1 \wedge \mathcal{C}_2|)$$

For the general case, note that if $\mathcal{C} + b$ is the translation of the the tropical variety by $b \in \mathbb{R}^n$, then $(e^b)^* \mathcal{T}_{\mathcal{C}} = \mathcal{T}_{\mathcal{C}+b}$. Moreover, we have the SP-convergence of currents with continuous superpotentials.

$$(e^{\varepsilon b})^* \mathcal{T}_{\mathcal{C}} \longrightarrow \mathcal{T}_{\mathcal{C}}\,, \qquad \text{as } \varepsilon \to 0.$$

Therefore, by Theorem 2.4,

$$(e^{\varepsilon b})^* \mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'} = \mathcal{T}_{\mathcal{C}+\varepsilon b} \wedge \mathcal{T}_{\mathcal{C}'} \longrightarrow \mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}\,, \qquad \text{as } \varepsilon \to 0.$$

Since for a generic choice of b, $\mathcal{C} + \varepsilon b$ and $\mathcal{C}$ are transversal, by Remark **??**, we deduce that

$$\mathrm{Supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) \subseteq \mathrm{Log}^{-1}(|\mathcal{C}_1 \wedge \mathcal{C}_2|).$$

□

Lemma 6.5. When $p + q = n$, $\mathrm{Supp}(\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}) = \mathrm{Log}^{-1}(|\mathcal{C} \wedge \mathcal{C}'|)$.

Proof. Assume that $x \in (\mathcal{C} + \varepsilon b) \cap \mathcal{C}'$ and the intersection is transversal at x. Then, $\mathrm{Log}(x) \subseteq \mathrm{supp}(\mathcal{T}_{\mathcal{C}+\varepsilon b} \wedge \mathcal{T}_{\mathcal{C}'})$. Since the mass remains constant in the ball containing $x + \varepsilon b$, ... For the general case, when $p + q > n$, we can find... □

7. Calculating Intersection Multiplicities

7.0.1. *Real Monge–Ampere Measures.* Let $\Omega \subseteq \mathbb{R}^n$ be an open subset and $u : \Omega \longrightarrow \mathbb{R}$ be a convex (hence continuous) function. The *generalised gradient* of u at $x_0 \in \Omega$ is defined by

$$\nabla u(x_0) = \big\{\xi \in (\mathbb{R}^n)^* : u(x) - u(x_0) \geq \langle \xi, x - x_0 \rangle, \text{ for all } x \in \Omega\big\}.$$

In the above, $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{R}^n$, and $(\mathbb{R}^n)^*$ is the dual. The real Monge-Ampére measure associated to a convex polynomial of a Borel set $E \subseteq \Omega$, is given by

$$\mathrm{MA}[u](E) = \mu\big(\bigcup_{y \in E} \nabla u(y)\big),$$

where μ is the Lebesgue measure on $\mathbb{R}^n$.

It is interesting that for the tropical polynomials, one can compute the associate real Monge–Ampére measures explicitly. Recall that, for any tropical polynomial, there is a natural subdivision of its Newton polytope which is dual to the tropical variety of it. See Figure for an example and [BS14, MS15] for details.

Lemma 7.1 ([Yge13, Page 59],[BGPS14, Proposition 2.7.4])**.** Let $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tropical polynomial $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$ with associated tropical variety $\mathcal{C} = V_{\mathrm{trop}}(\mathfrak{q})$, one has

$$\mathrm{MA}[\mathfrak{q}] = \sum_{a \in \mathcal{C}(0)} \mathrm{Vol}\big(\{a\}^*\big)\delta_a,$$

where $\mathcal{C}(0)$ is the 0-dimensional skeleton of $\mathcal{C}$, and $\{a\}^*$ is the dual of the vertex $a \in \mathcal{C}(0)$.

A detailed discussion of the preceding theorem can be also found in [Bab14].

7.1. **Polarisation.** For n convex functions $u_1, \dots, u_n : \mathbb{R}^n \longrightarrow \mathbb{R}$, their *mixed Monge-Ampére measure* is defined by

$$\widetilde{\mathrm{MA}}[u_1, \dots, u_n] = \frac{1}{n!} \sum_{k=1}^{n} \sum_{1 \le j_1 < \dots < j_k \le n} (-1)^{n-k} \, \mathrm{MA}[u_{j_1} + \dots + u_{j_k}].$$

Recall that this is how the *mixed volume* of n convex bodies can be defined from the n-dimensional volume. Moreover, it is easy to check that for a convex function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathrm{MA}[u] = \widetilde{\mathrm{MA}}[u, \dots, u]$.

The following statements are clear from 7.1 by taking the total mass.

Proposition 7.2. Let $\mathfrak{q}, \mathfrak{q}_1, \dots, \mathfrak{q}_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ be tropical polynomials. We have the following facts:

(a) $\mathrm{MA}[\mathfrak{q}](\mathbb{R}^n) = \mathrm{Vol}_n(\Delta_{\mathfrak{q}})$, where $\Delta_{\mathfrak{q}}$ is the Newton polytope of $\mathfrak{q}$.

(b) (Tropical Bernstein Theorem) $\widetilde{\mathrm{MA}}[\mathfrak{q}_1, \dots, \mathfrak{q}_n](\mathbb{R}^n) = \widetilde{\mathrm{Vol}}(\Delta_{\mathfrak{q}_1}, \dots, \Delta_{\mathfrak{q}_n})$, where $\widetilde{\mathrm{Vol}}$ is the mixed volume.

Corollary 7.3. Assume that $\alpha_i, \beta_i \in \mathbb{Z}^n$ for $i = 1, \dots, n$. Let $\mathfrak{q}_i = \max\{\langle \alpha_i, x \rangle, \langle \beta_i, x \rangle\}$ be n tropical polynomials. Then,

$$n! \, \widetilde{\mathrm{MA}}[\mathfrak{q}_1, \dots, \mathfrak{q}_n] = \kappa \, \delta_0,$$

where κ is given by the volume *zonotope* of the Minkowski sum of the vectors $\sum_{i=1}^n [\alpha_i - \beta_i]$.

Proof. Note that $\Delta_{\mathfrak{q}_i}$ is the line segment between α_i and β_i. Moreover, in the definition of $\widetilde{\mathrm{MA}}[\mathfrak{q}_1, \dots, \mathfrak{q}_n]$ only $\mathrm{Vol}(\sum_{i=1}^n [\alpha_i - \beta_i])$ possibly has a non-zero n-dimensional volume. Finally, the origin is the only 0-dimensional cell of the tropical variety of polynomial $\mathfrak{q}_1 + \dots + \mathfrak{q}_n$, if and only if, $\{\alpha_1 - \beta_1, \cdots, \alpha_n - \beta_n\}$ forms a linearly independent set. Therefore, $n! \, \mathrm{MA}[\mathfrak{q}_1 + \dots + \mathfrak{q}_n] = \kappa \, \delta_0$. $\square$

7.1.1. *Direct Calculations.* Therefore, $\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'} = \mathcal{T}_{\mathcal{C}\cdot\mathcal{C}'}$.

8. Slicing tropical currents

Proposition 8.1. Let $\mathcal{C}$ be a p-dimensional tropical cycle in $\mathbb{R}^n$, and $S \subseteq (\mathbb{C}^*)^n$ be an algebraic hypersurface with transversal intersection with $\mathcal{T}_{\mathcal{C}}$. Then, $[S]\wedge\mathcal{T}_{\mathcal{C}}$ is admissible and it is a closed positive current of bidimension $(p-1, p-1)$ given by

$$[S] \wedge \mathcal{T}_{\mathcal{C}} = \sum_{\sigma\in\mathcal{C}} w_\sigma \int_{x\in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[S \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\big]\ d\mu(x).$$

Proof. The idea of the proof is similar to that of [BH17, Proposition 4.11]. Let f be the equation of S in $(\mathbb{C}^*)^n$. Assume that $\mathrm{Log}^{-1}(\sigma^\circ) \cap S \neq \varnothing$, for a p-dimensional cone $\sigma \in \mathcal{C}$, then for each fiber, $\pi_\sigma^{-1}(x)$ the transversality assumption allows for application of the Lelong–Poincaré formula to deduce

$$\begin{aligned} &dd^c\big(\log|f|\mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\big[\pi_\sigma^{-1}(x)\big]\big) \\ &\qquad\qquad = \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\big[S\cap\pi_\sigma^{-1}(x)\big] + \mathcal{R}_\sigma(x). \end{aligned}$$

where $\mathcal{R}_\sigma(x)$ is a $(p-1,p-1)$-bidimensional current. The support of $\mathcal{R}_\sigma(x)$ lies in the boundary of $\mathrm{Log}^{-1}(\sigma)$, as $\mathcal{R}_\sigma(x)$ is the difference of two currents that coincide in any set of form $\mathrm{Log}^{-1}(B)$, where $B \subseteq \mathbb{R}^n$ is a small ball with

$$B \cap \sigma^\circ \neq \varnothing, \quad B \cap \partial\sigma = \varnothing,$$

and both vanish outside $\mathrm{Log}^{-1}(\sigma)$. Integrating along the fibers, and adding for all p-dimensional cones $\sigma \in \mathcal{C}$, we obtain

$$[S] \wedge \mathcal{T}_{\mathcal{C}} = \sum_{\sigma\in\mathcal{C}} w_\sigma \int_{x\in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[S \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\big]\ d\mu(x) + \mathcal{R}_{\mathcal{C}},$$

where $\mathcal{R}_{\mathcal{C}}$ is $(p-1,p-1)$-dimensional current. We claim that $\mathcal{R}_{\mathcal{C}}$ is *normal*, *i.e.* $\mathcal{R}_{\mathcal{C}}$ and $d\mathcal{R}_{\mathcal{C}}$ have measure coefficients; $\mathcal{R}_{\mathcal{C}}$ is a difference of two normal currents, where the first current $[S]\wedge\mathcal{T}_{\mathcal{C}}$ is a positive closed current, and the second current is an addition of normal pieces. Moreover, the support of $\mathcal{R}_{\mathcal{C}}$ is a subset of S as it is a difference of two currents that both vanish outside S. As a result, the current $\mathcal{R}_{\mathcal{C}}$ is supported on $S \cap \bigcup_\sigma \partial\mathrm{Log}(\sigma)$. This set is a real manifold of Cauchy–Riemann dimension less than $p-1$, therefore by Demailly's first theorem of support the normal current $\mathcal{R}_{\mathcal{C}}$ vanishes; see also the discussion following [BH17, Proposition 4.11]. ☐

Corollary 8.2. Let $H \subseteq \mathbb{R}^n$ be a rational plane of dimension r and $A := a + H$, a translation of H for $a \in \mathbb{R}^n$. Assume also that $\mathcal{C} \subseteq \mathbb{R}^n$ is a tropical variety of dimension p that intersects A transversely. Then

$$[(e^{-a})T_{H\cap\mathbb{Z}^n}] \wedge \mathcal{T}_{\mathcal{C}}$$

can be viewed as a tropical current of dimension $p-(n-r)$ in $T_A := (e^{-a})\,T_{H\cap Z^n}$.

Proof. Note that the hypothesis implies that the intersection $T_A \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ is transversal for any $x \in S_{N(\sigma)}$. By translation, it is sufficient to prove the statement for $a = 0$. By preceding theorem,

$$[T_A] \wedge \mathcal{T}_{\mathcal{C}} = \sum_{\sigma\in\mathcal{C}} w_\sigma \int_{x\in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[T_A \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\big]\ d\mu_\sigma(x).$$

The sets $T_A \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ can be understood as a translation toric sets in T_A and $d\mu_\sigma(x)$ are Haar measures, which imply the assertion. □

Theorem 8.3. Let $M \subseteq (\mathbb{C}^*)^{n-p}$ and $N \subseteq (\mathbb{C}^*)^p$ be two bounded open subsets such that N contains the real torus $(S^1)^p$. Let $\pi : M \times N \longrightarrow M$ be the canonical projection. Let $\mathcal{T}_n$ be a sequence of positive closed (p,p)-bidimensional currents on $M \times N$ such that $\overline{\mathrm{supp}(\mathcal{T}_n)} \cap (M \times \partial\overline{N}) = \varnothing$. Assume that $\mathcal{T}_n \longrightarrow \mathcal{T}$ and $\mathrm{supp}(\mathcal{T}) \subseteq M \times (S^1)^p$. Then we have the following convergence of slices

$$\langle \mathcal{T}_n|\pi|x\rangle \longrightarrow \langle \mathcal{T}|\pi|x\rangle \quad \text{for every } x \in M.$$

Note that all the above slices are well-defined for all $x \in M$.

Proof. Let $\mathcal{S}$ be any cluster value of $\langle \mathcal{T}_n|\pi|x\rangle$. Note that such $\mathcal{S}$ always exists by Banach–Alaoglu (check.) theorem. As both measures $\mathcal{S}$ and $\langle \mathcal{T}|\pi|x\rangle$, are supported $\{x\} \times (S^1)^p$ to prove their equality, it suffices to prove that they have the same Fourier coefficients. By Theorem 2.10, we have

$$\langle \mathcal{S}, \phi\rangle \leq \langle \mathcal{T}|\pi|x\rangle(\phi),$$

for every plurisubharmonic function ϕ on $\mathbb{C}^n$, and the mass of $\mathcal{S}$ coincides with the mass of $\langle \mathcal{T}|\pi|x\rangle$. Now, note that if ϕ is pluriharmonic, then $-\phi$ and ϕ are plurisubharmonic. As a result,

$$\langle \mathcal{S}, \phi\rangle = \langle \mathcal{T}|\pi|x\rangle(\phi),$$

for every pluriharmonic function. Recall that if f is a holomorphic function, then $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are pluriharmonic. We now consider the elements of the Fourier basis $f(\theta) = \exp i\langle \nu, \theta\rangle$ for $\nu \in \mathbb{Z}^n$. then we have the equality

$$\langle \mathcal{S}, f\rangle = \langle \mathcal{T}|\pi|x\rangle(f)$$

This implies that the Fourier measure coefficients of both $\mathcal{S}$ and $\langle \mathcal{T}|\pi|x\rangle$ coincide.

Why are all the slices well-defined?

□

Lemma 8.4. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a tropical variety of dimension p, and L be a rational $(n-p)$-dimensional plane such that L is transveral to all the affine extensions $\mathrm{aff}(\sigma)$ for $\sigma \in \mathcal{C}$. Assume that $\mathcal{T}$ be a positive closed current of bidimension (p,p) on a smooth projective toric variety (X_Σ compatible with $\mathcal{C} + L$ such that $\mathrm{supp}(\mathcal{T}) \subseteq \mathrm{supp}(\mathcal{T}_{\mathcal{C}})$. Further, for all $a \in \mathbb{R}^n$,

$$\overline{\mathcal{T}}_{L+a} \wedge \mathcal{T} = \overline{\mathcal{T}}_{L+a} \wedge \overline{\mathcal{T}}_{\mathcal{C}}\,,$$

then $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$ in $(\mathbb{C}^*)^n$.

Proof. Let us first remark that $\mathrm{rec}(L+a) = \mathrm{rec}(L)$ for all $a \in \mathbb{R}^n$ and therefore, all $\mathcal{T}_{a+L}$ are compatible with X_Σ and have a continuous super-potenrial in X_Σ and as a result, all the above wedge products are well-defined.

Third, by Demailly's second theorem of support [Dem, III.2.13], there are measures μ_σ such that

$$\mathcal{T} = \sum_\sigma \int_{x \in S(Z^n \cap H_\sigma)} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\big[\pi_\sigma^{-1}(x)\big]\, d\mu_\sigma^{\mathcal{T}}(x).$$

By repeated application of Proposition 8.1,

$$\mathcal{T}_L \wedge \mathcal{T} = \sum_{\sigma} \int_{(x,y)\in S(\mathbb{Z}^n\cap H_L)\times S(\mathbb{Z}^n\cap H_\sigma)} \left[\pi_H^{-1}(x) \cap \pi_\sigma^{-1}(y)\right] d\mu_L(x) \otimes \mu_\sigma^{\mathcal{T}}(y).$$

Applying both sides of the equality $\mathcal{T}_L \wedge \mathcal{T} = \mathcal{T}_L \wedge \mathcal{T}_{\mathcal{C}}$ on test-functions of the form

$$\omega_\nu = \exp(-i\langle \nu, \theta\rangle)\rho(r)$$

where $\rho : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support and $\theta \in [0, 2\pi)^n$, and $\nu \in \mathbb{Z}^n$, completely determines the Fourier coefficients of $\mu_\sigma^{\mathcal{T}}$ which have to coincide with the normalised Haar measures multiplied by the weight of σ, *i.e.*, $\mu_\sigma^{\mathcal{T}} = w_\sigma \mu_\sigma$.

□

As in Lemma 8.4, we denote $T_A := (e^{-a})\, T_{H\cap Z^n}$, for a rational affine plane A and a linear subspace H such that $A = a + H$.

Corollary 8.5. Let $W \subseteq (\mathbb{C}^*)^n$ be a closed algebraic variety of dimension p and $A \subseteq \mathbb{R}^n$ be a rational hyperplane such that A intersects $\mathrm{trop}(W)$ transversely. Assume that X_Σ is a projective smooth toric variety compatible with $A \cup \mathrm{trop}(W)$. Then,

$$\lim_{m\to\infty} \big(\frac{1}{m^{n-p}} \Phi_m^*[W] \wedge [T_A]\big) = \big(\lim_{m\to\infty} \frac{1}{m^{n-p}} \Phi_m^*[W]\big) \wedge [T_A].$$

Proof. Denote by $\mathcal{W}_n := \frac{1}{m^{n-p}}\Phi_m^*[W]$. Assume that $L \subseteq \mathbb{R}^n$ is an $(n-p-1)$-dimensional affine plane intersecting all $\mathrm{aff}(\sigma)$ for all $\sigma \in \mathrm{trop}(W)$ transversely. Then, on a projective smooth toric variety compatible with $\mathrm{trop}(W) + L$ the tropical currents $\overline{\mathcal{T}}_{a+L}$, $a \in \mathbb{R}^n$ have continuous super-potentials. Therefore, by a standard smooth approximation,

$$\lim_{m\to\infty} \big(\mathcal{W}_n \wedge \mathcal{T}_{a+L}\big) = \mathcal{T}_{\mathrm{trop}(W)} \wedge \mathcal{T}_{a+L}.$$

Now, for any $x \in \mathrm{trop}(W) \cap L$ let $U \subseteq \mathbb{R}^n$ containing x be a bounded open set. By a translation and monomial change of coordinate $\xi : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$, and setting

$$\begin{aligned}
M &:= \xi\big(\mathrm{Log}^{-1}(U) \cap \mathrm{supp}(\mathcal{T}_{\mathrm{trop}(W)} \wedge \mathcal{T}_{a+L})\big)\\
N &:= \xi\big(\mathrm{Log}^{-1}(U) \cap T_A\big),\\
\mathcal{T}_n &:= \xi_*\big(\mathcal{W}_n \wedge \mathcal{T}_{a+L}\big),\\
\mathcal{T} &:= \xi_*\big(\mathcal{T}_{\mathrm{trop}(W)} \wedge \mathcal{T}_{a+L}\big),
\end{aligned}$$

we are in situation of Theorem 8.3. Using the fact that slicing behaves well under pullback and pushforward of proper maps we conclude by Lemma 8.4.

explain more!!

□

9. Applications

Proposition 9.1.
- Commutative, Associative. Kazaronovski (?)
- Projection Formula
- Bernstein
- Comparison to other Gubler etc.
- Analogous result of Jonsson [Jon16]
- Analogous results of Payne–Cartwright, Maclagan–Surmfels etc.

  $W_n \otimes S_n \wedge \Delta$ small perturbation of Δ.

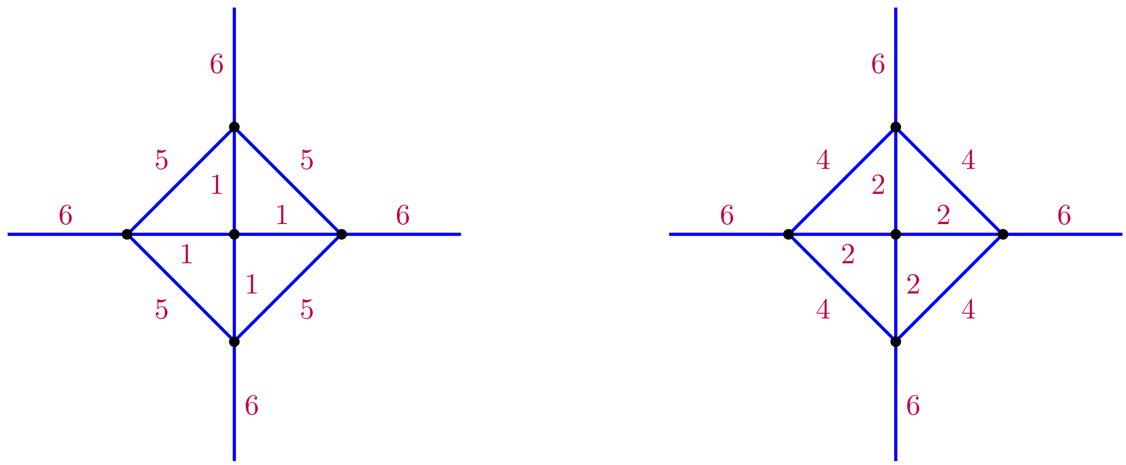

FIGURE 1. Two different tropical varieties with the same recession fan and same support.

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