The Reve's Puzzle

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Abstract

In 2014, it was proven by Thierry Bousch that N discs in The Reve's Puzzle could be transferred from an initial peg to destination peg in $2^{\nabla 0} + 2^{\nabla 1} + \cdots + 2^{\nabla (N-1)}$ moves. This paper will explore this proof in depth.

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Signed ___ALangley________________________

Date ___18th February 2025_________________

1 Introduction

As The Reve's Puzzle is a variant on the simpler Tower of Hanoi puzzle, we will first explore the Tower of Hanoi to gain a better understanding of the puzzle.

1.1 The origin of the Tower of Hanoi

The Tower of Hanoi was a puzzle invented by a French Mathematician, Édouard Lucas in 1883. It is so called due to its resemblance to a type of Vietnamese building, where Hanoi is the capital of Vietnam. The puzzle consists of three pegs and N discs starting on the first peg. The discs are stacked with the largest disc at the bottom, and decreasing in size as you move up, so the smallest disc is at the top. The goal is to move all discs to the last peg, whilst only moving one disc at a time, and not placing a larger disc on top of a smaller disc at any point. The starting point of the puzzle is illustrated in the figure below.

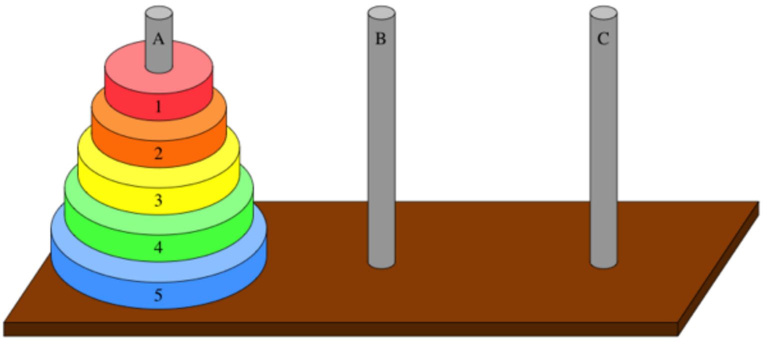

Figure 1: Graph of the Tower of Hanoi puzzle with 5 discs [1].

Of course, the question was then asked about what the most efficient way to complete the puzzle was, i.e., what is the minimum number of moves required to move all the discs from the first peg to the last peg whilst respecting the rules of the game. The answer turns out to be $2^N - 1$ moves, where N is the number of discs, and we will prove this is the optimum solution later in the paper.

1.2 Myths

In 1883, Professor N. Claus of the Collège Li-Sou-Stian (which was later discovered to be Lucas's pseudonym), reported about the origins of a game which he called LA TOUR D'HANOI [2]. A translation of his work creates an incredible story, which we will give a brief explanation of. The report states that there is a great temple at Benares which is a city in Uttar Pradesh, India, where there are three tall diamond needles. Upon creation of the needles, God placed 64 discs of gold, starting with the largest at the bottom and those stacked above it decreasing in size all the way to the top disc. This was the Tower of Bramah. It is said that priests work tirelessly to transfer these discs from the first needle to the last needle abiding by the same laws we set out above for the Tower of Hanoi. It was said that when all 64 discs have been moved to the last needle, the world would end.

It has been speculated that the creation of this myth was to signify the Hindu triad of Gods Brahma, Vishnu, and Shiva. As Brahma is known as the creator, it would seem that he was the one to place the discs onto the first peg, and when all discs reach the last peg, signifying Shiva, the God of destruction, then this would signify the worlds end.

Luckily, upon knowing that it takes $2^N - 1$ moves to complete the puzzle in the most efficient way, it appears that even if these discs were transferred at the rate of 1 move per second, then the world would end in $2^{64} - 1$ seconds, which is equivalent to 585 billion years. To put that into context, that is also 42 times greater than the current estimated age of the Universe [3].

1.3 Solving the Tower of Hanoi

Proving the solution for the minimum number of moves required to complete the Tower of Hanoi puzzle is actually relatively simple. The solution is given by the formula $2^N - 1$ where N is the number of discs that start on the first peg. We will prove that this is the minimum number of moves required to complete the puzzle by induction.

Let's start with the base case of $N = 1$. We know that if we were to start with 1 disc on the first peg it would only require one move, (i.e., moving this disc from the first to last peg) to complete the puzzle. Putting $N = 1$ into our formula we get $2^1 - 1 = 2 - 1 = 1$. Therefore, the formula is true for our base case.

Now we will assume that the formula is true for the case $N = K$. So our number of moves required is $2^K - 1$.

Finally, we will prove that the formula also holds true for the case $N = K + 1$. As such, we wish to prove that the number of moves needed is equal to $2^{K+1} - 1$. We will use the answer from $N = K$, and start out by assuming that instead of moving the first K discs to the last peg, we have moved them to the middle peg. It is clear to see that this will be the same number of moves as before, and will also allow us to move the $K + 1$ disc to the final peg. This of course, will add 1 move. As such, at this point we have had $2^K - 1 + 1 = 2^K$ moves.

Now, we will need to move all the discs in the middle to the last peg. We know we can do this in the same number of moves that was required to get the to the middle peg (as we are essentially reversing the process to get them there but swapping the first and last pegs). Therefore, the total number of moves required to get the final stacking is $2^K + 2^K - 1 = 2(2^K) - 1 = 2^{K+1} - 1$. This was equal to our aim result, and as such we have proved by induction that the formula holds true for all cases where $N \geq 1$.

The simplicity of the Tower of Hanoi comes from the fact that there are only three discs, and so in order to move a disc from one peg to another, all of the smaller discs are restricted to the peg that is not the initial or destination peg. We will see over the rest of this paper that removing this restriction causes a lot of problems in determining the path that some discs must take, and as such creates problems in determining the number of moves it takes to get from one stacking to another.

2 The Reve's Puzzle

2.1 The origin

As previously stated, the Reve's puzzle is a variation on the Towers of Hanoi puzzle, where instead of the typical three pegs, there are four pegs. This gives us significantly more freedom as to where we can place discs, in order to find the minimum number of moves for N number of discs.

The puzzle first appeared in a mathematical puzzle book published in 1907 by Henry Dudeney. The book is titled 'The Canterbury Puzzles and Other Curious Problems', and contains 114 puzzles, of which The Reve's puzzle is the first. A Reve can be an officer, steward, or governor, and we are not told which our Reve is in the book. It is only noted that he is something of a scholar. In the puzzle, the Reve is in a tavern, and he saw four stools and a few wheels of cheese. He placed the eight cheeses on the first stool, in decreasing size so the smallest was at the top. He then challenged the men in the tavern to move all the cheeses to the stool at the other end, moving one at a time, and without putting a larger cheese on top of a smaller cheese [4].

Of course, the stools act as the column and the cheeses act as the discs, and shall be referred to as columns and discs for the rest of the paper.

In the solutions of the Canterbury puzzles, we are given some answers for a number of moves required for 3, 4, and 5 stools, and varying numbers of cheeses. For example, we're told for 5 stools, that the number of moves to move 84 cheeses in this way, is 2815.

Figure 2: An illustration of The Reve's Puzzle [4].

It is stated the Dudeney found an equation that stated at least a possible solution for some values of N, say 3, 4, and 5. This will be (1) which we will see later in the paper. However, he did not prove, and may not have known, that he had actually found the optimal solution for the puzzle, and it was only proved to be optimal in 2014 by Bousch.

2.2 The differences

It is clear to see that adding an extra peg to the problem causes difficulties, and in this section we are going to talk about why. In the Tower of Hanoi, the restriction of only having three pegs gives a very clear path to reach the end of the puzzle, for example, we know that to move the largest disc to the last peg, we must move disc $N-1$ to the middle peg, so we must move disc $N-2$ to the last peg, and so on. This gives us very little freedom in completing the puzzle, as essentially all ways of completing the puzzle must follow the same path, with any moves over the optimum having to be reversed to continue along this path.

However, when we gain another peg to see The Reve's Puzzle, there is a lot more flexibility as to where the discs can move in the intermediate steps in the puzzle. The smallest disc now has 3 options of where to go on the first move, leaving the second disc with two options after that, and so on, meaning we can have multiple different paths to get us to the final stacking. Therefore, finding the optimum solution, and therefore optimum path becomes a lot more tricky. Also, it is interesting to note that upon finding the optimum path, there will be a symmetrical solution, where any intermediate moves in the middle two pegs can be swapped between these pegs to create a path requiring the same number of moves.

In the next section, the solution to the puzzle will be stated, and then further on in the paper, we will prove that this is the optimum.

2.3 The optimum solution

The optimum solution for The Reve's puzzle was found by Thierry Bousch, a French mathematician. He then published the result in a paper titled 'La quatrième tour de Hanoï" which appeared in the Bulletin of the Belgian Mathematical Society - Simon Stevin in 2014 [5]. He proved that the optimal number of moves required to complete the puzzle as given by

$$\Phi(N) = 2^{\nabla 0} + 2^{\nabla 1} + \cdots + 2^{\nabla (N-1)}, \tag{1}$$

where ∇n denotes the largest integer p such that $p(p+1)/2 \leq n$, which can also be called the triangular root of n.

It is possible to show that we can rewrite this equation in the form

$$\Phi(N) = \min_{0 \leq M < N} 2\Phi(M) + 2^{N-M} - 1 \quad (N \geq 1), \tag{2}$$

which can be reduced to

$$2\Phi(M) + 2^{N-M} - 1 = \Phi(N).$$

This can be done by choosing a suitable value of M, and we will look further into what the optimal value of M will be in the section which explores the Frame-Stewart algorithm. We will prove that we can rewrite the initial equation into the second equation in the next section, after introducing suitable notation. However it is interesting to note that in the paper by Thierry Bousch he states that Dudeney showed that the initial equation was possible at least for certain values of N. However it was not until the paper by Bousch in 2014 that it was proved this was the optimum solution.

For puzzles with 5 or more pegs, it appears that Frame and Stewart produced a recursive algorithm to find the minimum number of moves required to complete these puzzles. However, it has not been proven that this algorithm produces the optimal solution, and as such the Frame-Stewart conjecture is still open for 5 or more pegs. We will explore this conjecture later in section 6.1 which looks at the Frame-Stewart algorithm.

3 Preliminaries

In order to continue with the proof we will first set out some important notation. As Bousch is a French mathematician, the convention he introduced was to denote $\mathbb{N}$ as the natural numbers including 0. In this paper we will stick with this convention. Now for any natural number n, we will have $[n]$ to mean the set of the smallest n natural numbers. So $[n] = \{0, 1, ..., n-1\}$. We will also define $\Delta n = 1 + 2 + \cdots + n = n(n+1)/2$ for $n \in \mathbb{N}$, essentially the triangular numbers. Finally, we will stick with our definition of ∇n being the largest integer p such that $p(p+1)/2 \leq n$ which we introduced in the previous section. Therefore, by inspection we can see the following result:

$$\forall m, n \in \mathbb{N}, \quad \Delta m \leq n \Leftrightarrow m \leq \nabla n.$$

To help visualise the notation for ∇n and Δn, below is a table for these values for a few numbers

n	0	1	2	3	4	5	6
Δn	0	1	3	6	10	15	21
∇n	0	1	1	2	2	2	3

Also in order to prove that we can rewrite the first equation, we will need to prove a Lemma which is as follows.

Lemma 1. *If n and p are two natural numbers such that $p \leq n+1$, then we have*

$$\Phi(\Delta n + p) = 1 + (n+p-1) \cdot 2^n.$$

Proof. We see that we should write $N = \Delta n + p$, and we will use the fact from the original equation that

$$\Phi(N) = \sum_{M=0}^{N-1} 2^{\nabla M},$$

where this is of course just the original equation using summation notation. This was allow us to split the sum into multiple ranges with a constant value of ∇M. For each sum we can write $M = \Delta m + r$, where $0 \leq r < m$. If we group the terms with a common value of m, we can rewrite this as

$$\Phi(N) = \sum_{m=0}^{n-1} (m+1)2^m + p2^n.$$

To understand how we get here, we can see that for each value of M, there are $m+1$ terms, each of which will add 2^m to the sum. The last term then comes from the fact we may not have a full grouping of ΔM, and if this is the case there will be p terms which each add 2^n to the sum. Now we will focus on evaluating the sum, and we will do this by splitting it into two sums. So we have

$$\sum_{m=0}^{n-1} (m+1)2^m = \sum_{m=0}^{n-1} m2^m + \sum_{m=0}^{n-1} 2^m.$$

Now we can evaluate the two sums separately. We can see that the second one is simply a geometric series. We can use the fact that there are n terms with a common ratio 2 and an initial value one to find that it is equal to $2^n - 1$. The first sum, however, will take slightly more work. Let this sum be denoted S. We can then rewrite it in the following way

$$S = 1 \cdot 2 + 2 \cdot 2^2 + ... + (n-1) \cdot 2^{n-1}.$$

Now, we will find $2S$, and rewrite S in terms of $S = 2S - S$.

$$\begin{aligned}
2S &= 1 \cdot 2^2 + 2 \cdot 2^3 + \cdots + (n-1) \cdot 2^n, \\
2S - S &= -2(1 + 2 + 2^2 + \cdots + 2^{n-1}) + (n-1) \cdot 2^n, \\
2S - S &= n2^n - 2 \cdot 2^n + 2.
\end{aligned}$$

Now we have evaluated both sums and so we can add them both together and add back in the $p2^n$ term, to get that $\Phi(N) = 1 + p2^n + n2^n - 2^n$. We can now simplify this and substitute our $N = \Delta n + p$ to get

$$\Phi(\Delta n + p) = 1 + (n+p-1) \cdot 2^n,$$

thus proving the lemma. □

Now we have proved that and set out our notation, we can prove that we can rewrite our original equation in the form

$$\Phi(N) = \min_{0 \le M < N} 2\Phi(M) + 2^{N-M} - 1 \quad (N \ge 1).$$

We are going to start by fixing a value of N, and setting

$$f_N(M) = 2\Phi(M) + 2^{N-M} - 1 \quad (0 \le M < N).$$

Now, using our result from Lemma 1 and the fact we have $M = \Delta m + r$ as before, we can rewrite this as

$$f_N(M) = 2(1 + (m + r - 1)2^M) + 2^{N-M} - 1,$$
$$f_N(M+1) = 2(1 + (m + r)2^M) + 2^{N-M} - 1.$$

After expanding the brackets, we have

$$f_N(M+1) - f_N(M) = 2 + m2^{m+1} + r2^{m+1} + 2^{N-M-1} - 1 - 2 - m2^{m+1} - r2^{m+1} + 2^{m+1} - 2^{N-M} + 1.$$

This can be rewritten as

$$f_N(M+1) - f_N(M) = 2^{m+1} - 2^{N-M-1}.$$

Now we want to show that this expression changes sign at $M = N - n - 1$ as this will show us that this is the point where $\Phi(N)$ is at a minimum. We will start by writing $N - 1 = \nabla n + p$. Therefore, we have that $N - n - 1 = \nabla(n-1) + p$. Using that fact that $M = \nabla m + r$, we can check that $M < N - 1 - n$ implies $m \le n - 1$. This is due to the fact that ∇ preserves inequalities, as such $\nabla x \le \nabla y$ implies $x \le y$, so then $\nabla m + r \le N - 1 - n - 1 = \nabla(n-1) + p - 1$ implies $m \le n - 1$. Now, we can rewrite $M < N - 1 - n$ as $N - M - 1 > n$. Using this, and the fact that $m \le n - 1$, we have that the equation above gives a negative value. By the same method as above, we get that $M \ge N - 1 - n$ implies $m \ge n - 1$, and so $N - N - 1 \le n$. Therefore the equation above is greater than or equal to zero. Using these equalities, we therefore have that the equation must change sign when $M = N - n - 1$ as required.

We need to show that $f_N(N - n - 1) = \Phi(N)$. As we have just shown that $M = N - n - 1$ at the point where $\Phi(N)$ is at its minimum, the claim has been proven. Therefore, we have shown that we can rewrite the original equation from the previous section into the one that we claimed in this section. This is helpful as it allows us to put the overall solution to the problem in terms of the number of discs M and N, and we will later determine the optimum value or values of M in terms of N.

Throughout the paper, we will often use a weaker version of equation 2, which can be expressed as follows

$$\forall a, b \in \mathbb{N} \qquad \Phi(a + b) \le 2\Phi(a) + 2^b - 1. \tag{3}$$

This comes from (2), where we replace N with a + b to get

$$\Phi(a + b) = 2\Phi(M) + 2^{(a+b-M)} - 1.$$

Then, as we know we have $M < a + b$, we can see that $2\Phi(M) + 2^{a+b-M} \le 2\Phi(a) + 2^b$, and as such we can see the inequality from (3).

In order to help us prove the main theorem, we are first going to start by proving a few lemmas. Each of these lemmas are also explored in the 2014 paper by Bousch [5]. These won't have a clear link to the puzzle at first, but they will all be referenced in the main proof, and as such it is important to ensure that each are proved individually here.

We are going to start by introducing another function Ψ. Let E be a finite subset of $\mathbb{N}$, and for any natural number L we define

$$\Psi_L(E) = (1-L)2^L - 1 + \sum_{n\in E} 2^{\min(\nabla n, L)}.$$

As such, we can then also define

$$\Psi(E) = \sup_{L\in\mathbb{N}} \Psi_L(E).$$

We can see that as $\Phi_L(E) \to -\infty$ as $L \to \infty$, the supremum is actually a maximum. This is because the supremum is attained, so $\Psi(E) = \Psi_L(E)$ for some $L \in \mathbb{N}$.

Our next lemma is as follows:

Lemma 2. *For any natural number n,*

$$\Psi[n] = \frac{\Phi(n+1)-1}{2} = \frac{1}{2}(2^{\nabla 1} + 2^{\nabla 2} + \cdots + 2^{\nabla n}).$$

Proof. To prove this, we can start with the fact we know $\Phi[0] = 0$, as [n] is the smallest set of natural numbers $0, 1, \ldots, n-1$, and as such $[0] = \emptyset$. Therefore we can suppose that $n \geq 1$, and we can write $n = \Delta m + p$ with $m = \nabla n$, where $m \geq 1$ and $0 \leq p \leq m$. As per Lemma 1, we have the following two equations:

$$\Phi(\Delta m) = 1 + (m-1)2^m,$$
$$\Phi(n+1) = 1 + (m+p)2^m.$$

We can see this through direct substitution into Lemma 1 where for the first equation with Δm we can set $\Delta m = \Delta n$ and $p = 0$, and for the second equation with $n+1$ we can use that $n+1 = \Delta m + p + 1$. Now, for any natural number L, we can use the definition of $\Psi_L(E)$ that we have above to see that

$$\begin{aligned}\Psi_{L+1}[n] - \Psi_L[n] &= (1-L+1)2^{L+1} - 1 + \sum_{n\in E} 2^{\min(\nabla n, L+1)} - (1-L)2^L + 1 - \sum_{n\in E} 2^{\min(\nabla n, L)} \\ &= -(L+1)2^L + \sum_{k\in[n]} 2^{\min(\nabla k, L+1)} - 2^{\min(\nabla k, L)}.\end{aligned}$$

Now this can be written as

$$2^L[\#\{k \in [n] : k \geq \Delta(L+1)\} - (L+1)].$$

To see that we can write it in this form, we need to understand that # denoted the cardinality or size of the set. We can easily see that the last part of the rewritten equation corresponds to the first part of the first equation, and so we now need to show that the first part of the rewritten equation corresponds to the summation notation. If we have the size of the set is 0, then there are no values of k for which $k \geq \Delta(L+1)$. As such, the sum becomes $2^{(\nabla k)} - 2^{(\nabla k)} = 0$. Therefore the sum is equal to 0 which is equal to $2^L \cdot 0$. However, if the size is 1, that means that there is a value of $k \geq \Delta(L+1)$ and as such the sum becomes $2^{L+1} - 2^L = 2^L$. This is equal to $2^L \cdot 1$. Now if the size is greater than 1, then we have the previous sum multiplied by the size, as this means there are that many values of $k \geq \Delta(L+1)$ and so we must do the sum this many times. This means that the first half of the rewritten equation has been shown to be equal to the summation notation. Now we can write this as

$$2^L[[n - \Delta(L+1)]^+ - (L+1)],$$

where for a number z, z^+ is the positive part, which may also be expressed as $\max(z, 0)$. We can see that the last part of this equation corresponds to the last part of the previous equation. Therefore the first part corresponds to the first part of the previous equation. We can see this as $n - \Delta(L+1)$ is the number of values k in [n] where $k \geq \Delta(L+1)$. Also, if there are no values of k that satisfy this equation, then due to taking the positive part we get 0, and this is equal to when in the original definition of $\Phi_L(E)$ we have the size of the set being 0. Now we can say this expression is positive only when $n - \Delta(L+1) - (L+1) > 0$ and therefore when $n - \Delta(L+1) - L - 1 \geq 1$. Rearranging this, we have the equation is only positive when $n \geq \Delta(L+1) + L + 2 = \Delta(L+2)$. This is equivalent to $\nabla n \geq L+2$, or $L < m-1$, as at the start of this proof we wrote $m = \nabla n$. As a consequence of this, the sequence $\Psi_L[n]$ reaches a maximum when $L = m-1$. So, we have that

$$\Psi[n] = \Psi_{m-1}[n] = (2-m)2^{m-1} - 1 + \sum_{0 \leq k < \Delta m} 2^{\nabla k} + \sum_{\Delta m \leq k < n} 2^{m-1}.$$

This of course comes from the original definition of $\Psi_L(E)$, and the reason we have two separate sums in this equation is because we are breaking up the sum into the one where ∇k is the minimum, and the one where L, or in this case $m-1$ is the minimum. This is reflected in the inequalities below the sum, which show the different values of k which we are using within the sums. Now we are going to simplify this equation down. First, we have

$$\Phi[n] = (2-m)2^{m-1} - 1 + \Phi(\Delta m) + (n - \Delta m)2^{m-1}.$$

We can see that the first parts on the right hand side of each of the equations are equal. Now, we can see that the first sum is equal to $\Phi(\Delta m)$ from (1). Then, we can see that the second sum is equal to the last part of the equation as we have a sum for the values between Δm and n, and as such we have $n - \Delta m$ values, each which add 2^{m-1} to the sum. Therefore we get $(n - \Delta m)2^{m-1}$ as in the previous equation. This can be further simplified into

$$\Psi[n] = (2-m)2^{m-1} + (m-1)2^m + p2^{m-1}.$$

This simplification comes directly from the equation above where we found $\Phi(\Delta m)$, as well as the fact we have had $n - \Delta m = p$. Our final simplification then becomes

$$\Psi[n] = (m+p)2^{m-1}.$$

Where the right hand side comes from a rearrangement of the previous equation. Finally, we can see that

$$\frac{\Phi(n+1) - 1}{2} = \frac{1 + (m+p)2^m - 1}{2} = (m+p)2^{m-1}.$$

As such, we have found the first two equations in the Lemma are equivalent. To find the final equation in the lemma we can simply take our definition of $\Phi(N)$ from (1) to prove that these are equivalent, and that concludes the proof of this lemma. □

We can now find want to find an inequality similar to the one in (3), but in the form of Ψ rather than Φ. We do this by using the result from the lemma as well as the result from (3). From (3) we have

$$\Phi(a+b) \leq 2\Phi(a) + 2^b - 1,$$
$$\Phi(a+b) - 1 \leq 2\Phi(a) + 2^b - 2,$$
$$\frac{\Phi(a+b) - 1}{2} \leq \Phi(a) + 2^{b-1} - 1.$$

This then implies

$$\Psi[a+b-1] \leq 2\Psi[a-1] + 2^{b-1}.$$

So we reach our final inequality of

$$\forall a, b \in \mathbb{N} \qquad \Psi(a+b) \leq 2\Psi(a) + 2^{b-1}. \tag{4}$$

Next, we have the following lemma:

Lemma 3. *We have $\Psi[n+2] \geq 2^{(\nabla n)+1}$ for all natural numbers n.*

Proof. Let us set $s = \nabla n$. When $\Psi[\cdot]$ is increasing, we have $\Psi[\Delta s + 2] \geq 2^{s+1}$ from the result in our lemma. This inequality is satisfied for s = 0 and 1, and we have that $\Psi[2] = \frac{1}{2}(2^{\nabla 1} + 2^{\nabla 2}) = \frac{1}{2}(2+2) = 2$. We also have that $\Psi[3] = \frac{1}{2}(2^{\nabla 1} + 2^{\nabla 2} + 2^{\nabla 3}) = \frac{1}{2}(2+2+4) = 4$. Now for $s \geq 2$, we can use the equations in Lemma 2 and Lemma 1 to show $\Psi[\Delta s + 2] = (s+2)2^{s-1}$. From Lemma 2 we get

$$\Psi[\Delta s + 2] = \frac{\Phi(\Delta s + 3) - 1}{2}.$$

We can show that by rewriting $\Delta n + p$ as $\Delta s + 3$ to substitute into Lemma 1, we get $\Phi(\Delta s + 3) = 1 + (s+2)2^s$. As such, we have

$$\Psi[\Delta s + 2] = \frac{1 + (s+2)2^s - 1}{2} = (s+2)2^{s-1}.$$

Therefore we have that $\Psi[\Delta s + 2] \geq 2^{s+1}$ for $s + 2 \geq 4$, as $2^{s+1} = 4 \cdot 2^{s-1}$, and so substituting back our $s = \nabla n$, we see that the Lemma is proved. □

Now, we would like to prove the following lemma:

Lemma 4. *For any finite subset E of $\mathbb{N}$, we have*

$$n \leq \Psi[n] \leq \Psi(E) \leq 2^n - 1,$$

where n is the cardinality of E.

Proof. To prove this we will take each inequality individually. The first inequality, $n \leq \Psi[n]$, we can prove through saying that $n = \Psi_0[n]$ and due to the definition of $\Psi_L(E)$ from before, this means

$$\Psi_0[n] = (1-0)2^0 - 1 + \sum_{n \in E} 2^{\min(\nabla n, 0)} = n \cdot 2^0 = n.$$

We know $\Psi[n] \geq \Psi_0[n]$ as the supremum will increase as L increases, due to the sum in the definition increasing to the power of 2 for each L increase by 1.

Therefore the first inequality has been proven. Now, we will look at the second inequality $\Psi[n] \leq \Psi(E)$. Let's start by naming $e_0 < e_1 < \ldots < e_{n-1}$ as the n elements of E. Since $e_k \geq k$ for all k, due to E being a subset of $\mathbb{N}$ then

$$\begin{aligned}\Psi_L(E) &= (1-L)2^L - 1 + \sum_{0 \leq k < n} 2^{\min(\nabla e_k, L)} \\ &\geq (1-L)2^L - 1 + \sum_{0 \leq k < n} 2^{\min(k, L)} = \Psi_L[n].\end{aligned}$$

As such, we have that $\Psi_L(E) \geq \Psi_L[n]$ and this implies $\Psi(E) \geq \Psi[n]$, and we have proven the second inequality.

Now, we will look at the last inequality, $\Psi(E) \leq 2^n - 1$. For any natural number L, we have

$$\begin{aligned}\Psi_L(E) &= (1-L)2^L - 1 + \sum_{k\in E} 2^{\min(\nabla k, L)} \\ &\leq (1-L)2^L - 1 + \sum_{k\in E} 2^L = (1+n-L)2^L - 1.\end{aligned}$$

However, we know that $2^s \geq 1+s$ for all of the relevant integers s, which gives $2^n \geq (1+n-L)2^L$ by taking $s = n-L$, and therefore $\Psi_L(E) \leq 2^n - 1$. This proves the third and final inequality and therefore proves the lemma as a whole. We would also like to consider cases of equality in this lemma, for example where $\Psi(E) = 2^n - 1$, we have the cases where $2^L = 2^{\min(\nabla k, L)}$, and as such equality occurs when $L \leq \nabla k$. We can see that equality only occurs in the first inequality when $L = 0$, and similarly it is easy to see that for the second inequality, equality only exists when $\nabla e_k = k$. □

Next we have the following lemma,

Lemma 5. *Let A and B be two finite subsets of N. We have:*

$$\Psi(A) - \Psi(B) \leq \sum_{k\in A-B} 2^{\nabla k}.$$

Proof. To prove this we start with letting L be a natural number such that $\Psi(A) = \Psi_L(A)$. Therefore

$$\Psi(A) - \Psi(B) \leq \Psi_L(A) - \Psi_L(B) \leq \Psi_L(A) - \Psi_L(A\cap B).$$

The second inequality comes from the fact we know $\Psi_L(B) \geq \Psi_L(A\cap B)$, as the intersection of A and B can only reduce the number we have in the sum from the number we get from just having B. Then we can use the definitions we were given previously to rewrite this as

$$\Psi_L(A) - \Psi_L(A\cap B) = \sum_{k\in A-B} 2^{min(\nabla k, L)} \leq \sum_{k\in A-B} 2^{\nabla k}.$$

This is what we were required to prove and thus we have completed the proof of this lemma. □

Next we have the following lemma:

Lemma 6. *For A, a finite subset of $\mathbb{N}$, there is a natural number such that the set $A - [\Delta s]$ contains at most s elements. Then*

$$\Psi(A) - \Psi(A - \{a\}) \leq 2^{s-1},$$

for any A.

Proof. We can assume A is non-empty, and as such we have that $s \geq 1$ Then, for all $L \geq s-1$ we have

$$\begin{aligned}\Psi_{L+1}(A) - \Psi_L(A) &= 2^L[\#\{n \in A : n \geq \Delta(L+1)\} - (L+1)] \\ &\leq 2^L[\#\{n \in A : n \geq \Delta s\} - s] \leq 0.\end{aligned}$$

The first line occurs in exactly the same way as in the proof of Lemma 2, and the second line is a logical follow on from the fact that $L \geq s-1$. We also know this is less than or equal to 0 as by definition, the number of elements $n \in A$ such that $n \geq \Delta s$ is less than s. This then shows that $\Psi_L(A)$ is decreasing from $L = s-1$. Therefore, there exists $L \leq s-1$ such that $\Psi(A) = \Psi_L(A)$, and

$$\Psi(A) - \Psi(A - \{a\}) \leq \Psi_L(A) - \Psi_L(A - \{a\}) = 2^{\min(\nabla a, L)} \leq 2^L \leq 2^{s-1},$$

where this comes from the proof of Lemma 5, replacing B by (a - {a}). Therefore we have completed the proof of this lemma. □

Lemma 7. *Let n and s be two natural numbers such that $s \geq 1$, and $n \geq \Delta(s-1)$ and let A be a subset of $[n]$. Then,*

$$\Psi(A \cup \{b_1, \ldots, b_s\}) - \Psi(A) \leq \Psi[n+s] - \Psi[n],$$

for any $b_1, \ldots, b_s$ (not necessarily distinct) in N.

Proof. To prove this lemma, we will let $A_t = A \cup \{b_1, \ldots, b_t\}$, for $0 \leq t \leq s$. It is sufficient to show that:

$$\Psi(A_t) - \Psi(A_{t-1}) \leq \Psi[n+t] - \Psi[n+t-1],$$

for all t such that $1 \leq t \leq s$. Where it can be seen that $\Psi(A_{t-1}) = A_t \setminus \{b_t\}$. Of course, we can assume the inclusion of A_{t-1} in A is strict. On the right hand side of this equation we use Lemma 2 to show that this is equal to $\frac{1}{2}(2^{\nabla(n+t)}) = 2^{\nabla(n+t-1)} = 2^{\sigma-1}$, where $\sigma = \nabla(n+t)$, and the inequality will result from Lemma 4, provided we show the set $A_t - [\Delta\sigma]$ has a cardinality less than or equal to σ. First we note that $\Delta(\sigma+1) > n+t$ from the definition of sigma, which is equivalent to saying $\Delta\sigma + \sigma \geq n+t$. Then the proof continues,

$$\begin{aligned}\#(A_t - [\Delta\sigma]) &\leq t + \#([n] - [\Delta\sigma]) = t + (n - \Delta\sigma)^+ \\ &= \max(t, t+n-\Delta\sigma) \leq \max(t, \sigma).\end{aligned}$$

The right hand side of the above inequality comes form the inequality $\Delta\sigma + \sigma \geq n+t$ and now we need to verify that $t \leq \sigma$. To do this, we can observe that

$$\Delta t - t = \Delta(t-1) \leq \Delta(s-1) \leq n,$$

which comes from the idea that $t \leq \nabla(n+t) = \sigma$ and then we have $\Delta t \leq n+t$, which is what we needed to show. □

We have one final lemma left to prove before we can move onto the main theorem and it is as follows:

Lemma 8. *Let A and B be two finite subsets of $\mathbb{N}$. Then we have*

$$\Psi(A) + \Psi(B) \geq \frac{\Phi(n+3) - 5}{4} = \frac{1}{2}\Psi[n+2] - 1$$

$$= \frac{1}{4}(2^{\nabla 3} + 2^{\nabla 4} + ... + 2\nabla(n+2)),$$

where n is the cardinality of $A \cup B$.

Proof. To prove this, let $E = A \cup B$. Then let L be any natural number. As seen in the proof of Lemma 4, we have $\Psi_L(E) \geq \Psi_L[n]$. Then we can use the definition of $\Phi_L(E)$ from above to produce the following inequalities

$$\begin{aligned}\Psi(A) + \Psi(B) \geq \Psi_L(A) + \Psi_L(B) &= \Psi_L(A \cap B) + \Psi_L(A \cup B) \\ &\geq \Psi_L(\phi) + \Psi_L(E) \geq \Psi_L[0] + \Psi_L[n],\end{aligned}$$

where $A \cup B$ means A and B are disjoint, and we get the second line from Lemma 4. Now we can write $n + 3 = \Delta m + p$, with $m = \nabla(n + 3)$, and therefore $m \geq 2$ and $0 \leq p \leq m$. Then we have $n \geq \Delta(m - 2)$ as $n = \Delta m + p - 3$, and

$$\begin{aligned}\Phi(n + 3) &= 1 + (m + p - 1)2^m, \\ \Phi(\Delta(m - 2)) &= 1 + (m - 3)2^{m-2},\end{aligned}$$

which is the result we get from using Lemma 1 which we proved previously. Now let $L = m - 2$. Then

$$\Psi_L[0] + \Psi_L[n] = (1 - L)2^{L+1} - 2 \sum_{0 \leq k < n} 2^{min(\nabla k, L)},$$

and we can simplify this by substituting $L = m - 2$ in, and splitting up the sums into a term when ∇k is the minimum and when $m - 2$ is the minimum. So the right han side of the equation gives us

$$(3 - m)2^{m-1} - 2 + \sum_{0 \leq k < \Delta(m-2)} 2^{\nabla k} + \sum_{\Delta(m-2) \leq k < n} 2^{m-2}.$$

We continue simplifying this down, remembering that we can express p as $n + 3 - \Delta m$,

$$\begin{aligned}&(3 - m)2^{m-1} - 2 + \Phi(\Delta(m - 2)) + (n + \Delta(m - 2))2^{m-2} \\ &= (3 - m)2^{m-1} - 1 + (m - 3)2^{m-2} + (p + 2m - 4)2^{m-2} \\ &= (m + p - 1)2^{m-2} - 1 = \frac{\Phi(n + 3) - 5}{4}.\end{aligned}$$

As such we have proven this lemma. We cans show this final result is equal to $\frac{1}{2}\Psi[n + 2] - 1$ by the result from Lemma 2. □

3.1 The main theorem

Now we have proved all the prerequisite lemmas, we can revisit the Dudeney problem to attempt to find the minimum number of moves required to complete The Reve's Puzzle. We will now denote the pegs within the puzzle as a set $\mathcal{C}$. This is a set of 4 elements which we will label 0, 1, 2, 3. We can also represent the discs in the puzzle of which we will have N discs, by the integers 0 to $N - 1$. The largest integers will of course correspond to the largest discs. We will also introduce the concept of stackings, which will be states that the puzzle can be in. For example, we will have $\mathbf{u}$ as the initial stacking, so state which all discs are in at the start of the puzzle. We will also denote the distance between any two stackings $\mathbf{a}$ and $\mathbf{b}$ as $d(\mathbf{a}, \mathbf{b})$, which will be the minimum number of moves required to get from stacking $\mathbf{a}$ to $\mathbf{b}$, whilst respecting the rules of the game that we set out in the introduction.

So, we have that the main Theorem of the paper by Bousch in 2014 is:

Theorem 1. *Let $\mathcal{C} = \{0, 1, 2, 3\}$. Let N be a natural number, and let $\mathbf{u}, \mathbf{v} : [N] \to \mathcal{C}$ be two stackings of N discs. Suppose that in stacking $\mathbf{v}$, columns $\{0\}$ and $\{1\}$ are empty, i.e., $\mathbf{v}[N] \subseteq \{2, 3\}$. Then:*

$$d(\mathbf{u}, \mathbf{v}) \geq \Psi\{k \in [N] : \mathbf{u}(k) = 0\}. \tag{5}$$

We will now prove this theorem in the next section. The reason this theorem is so important to our understanding and completion of the puzzle, is that by Lemma 2, we must have 2 empty pegs at what will essentially be the halfway point of our problem. This is because, if we ignore the largest disc, we know that all the smaller discs must be on two columns so that the largest disc can move from its initial to final point. Then we can repeat the set of movements but in reverse for all the smaller discs onto the final stacking on the final peg.

In the theorem it states that we are assuming discs 0 and 1 are empty. Of course in the problem we need the first and final discs to be empty, but we can simply relabel the pegs and the number of moves will be the same regardless of this labelling. So the next section will essentially be looking to find the minimum number of moves required to get to this 'halfway point', and we can then use this to solve the problem as a whole. Therefore, we can 'forget' about the largest disc N, because it is irrelevant in this smaller breakdown of the puzzle.

We also want to be convinced that in the most efficient case, that the largest disc will move only once. This is a fact used in the overall puzzle, as well as in the smaller, simpler puzzle in the next section. The proof of this fact follows.

Let m be the minimum number of moves required to move n discs from one peg to two other pegs. A tower of size $n+1$ can be moved from one peg to another one in at most $2m+1$ moves. Any solution of this problem needs at least m moves before the first move of the disc $n+1$ and at least m moves after the last one. Therefore, it would need at least $2m+2$ moves if the disc $n+1$ were to move more than once, so this cannot be optimal. Therefore the largest disc will only move once in the optimal solution.

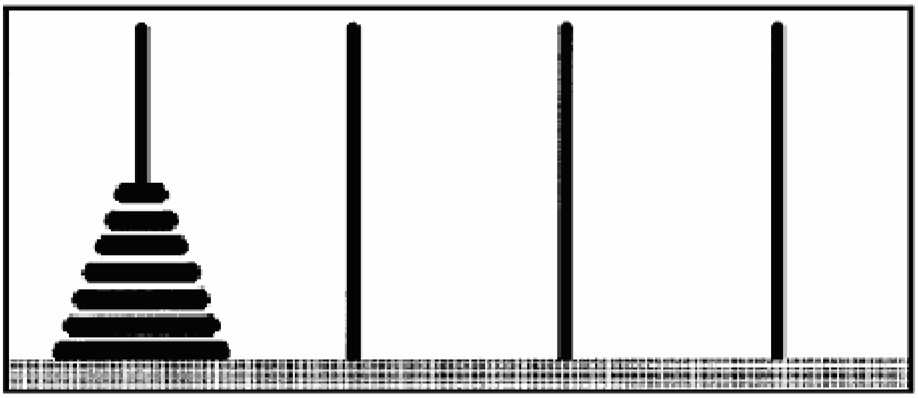

Figure 3: An illustration the 4 peg puzzle [6].

In the figure above which provides a visual of The Reve's Puzzle, we can label the columns 0, 1, 2, and 3 from left to right.

4 Proving the theorem

4.1 The idea

The general idea to prove the previous theorem, is to find an exhaustive list of cases, where for each we can find a lower bound for $d(\mathbf{u}, \mathbf{v})$. Each of these cases will take a different kind of path to get from the initial stacking to the final stacking, depending on the properties of the case, and we will find the lower bound for the overall path for each case. This will be done by finding 'checkpoints' which will essentially be stackings we know will appear in each case, and finding the required number of moves to get to each checkpoint, to find the number of moves required overall. Of course, the optimum case is where the fewest number of moves are required to get from the initial stacking to the final stacking. The optimal number of moves we will later show to be $\Psi(E)$, and as such when looking at each case we will prove that the other lower bounds are greater than this value.

An exhaustive list of the cases we will be looking at is provided below, and we will define all the notation within this list later on in the paper:

- $E' = \emptyset$
- $\Delta K > T$
- $K = 0(E'' = \emptyset)$
- $T \geq \Delta K > 0$ and $t_1 > t_3 + 1$
- $T \geq \Delta K > 0$ and $t_1 < t_3 + 1$

It should also be noted that the last three cases are sub-cases of when $T \geq \Delta K$ and so will have some properties in common.

For each case, we can fix the number of discs N, and we can assume that the result holds true for any number of discs strictly smaller than N. Let $\mathbf{u}, \mathbf{v}$ be two stackings of N discs, with $\mathbf{v}[N] \subseteq \{2, 3\}$. So in this stacking, all the discs are on pegs 2 and 3, such as in Theorem 1. As noted before, this is essentially the 'halfway point' of the overall problem. So, we will be looking to find the minimum number of moves to get from the point where all of the discs are in column 0 to where all of the discs are in column 2 and 3, leaving column 0 and 1 empty.

We define $E = \{k \in [N] : \mathbf{u}(k) = 0\}$, which represents the discs of column 0 in the stacking $\mathbf{u}$, which essentially represents the start of the puzzle. We can assume E is non-empty; otherwise, the result is trivial, as this means there are no discs to move. Therefore, in particular, we have that $N \geq 1$.

Now we will let $\mathbf{u'}$ and $\mathbf{v'}$ be the stackings $\mathbf{u}$ and $\mathbf{v}$, but where we are ignoring the disc $N - 1$. This can be done to simplify the puzzle, and therefore these particular stackings will contain $N - 2$ discs, and we can apply the induction hypothesis to them:

$$\begin{aligned} d(\mathbf{u}, \mathbf{v}) \geq d(\mathbf{u'}, \mathbf{v'}) &\geq \Psi\{k \in [N-1] : \mathbf{u'}(k) = 0\} \\ &= \Psi\{k \in [N-1] : \mathbf{u}(k) = 0\} \\ &= \Psi(E - \{N-1\}). \end{aligned}$$

The first inequality simply comes from Theorem 1, and then we simplify the last two through definitions. For example, we have $\Psi\{k \in [N-1] : \mathbf{u'}(k) = 0\} = \Psi\{k \in [N-1] : \mathbf{u}(k) = 0\}$ as $[N-1]$ is the group of discs smaller than $N-1$, and so $N-1$ is irrelevant in either case, so it makes no difference if it is ignored in the stacking or not. The we have $\Psi\{k \in [N-1] : \mathbf{u}(k) = 0\} = \Psi(E - \{N-1\})$ from the definition of E, where clearly we are replacing N with $N-1$.

Now, if E does not contain $N-1$, then we have $d(\mathbf{u}, \mathbf{v}) \geq \Psi E$ which is the result we are seeking. This trivial case is resolved, so we will assume throughout the rest of the discussion that E does contain $N-1$, meaning that the largest disc in the stacking $\mathbf{u}$ is in column 0. This same disc, in the stacking $\mathbf{v}$, will be in column 2 or 3. We can assume without loss of generality that it is in column 2, i.e., $\mathbf{u}(N-1) = 0$ and $\mathbf{v}(N-1) = 2$.

Let D be the distance between $\mathbf{u}$ and $\mathbf{v}$, and let $\gamma : [D+1] \to \mathcal{C}^{[N]}$ be the shortest path from $\mathbf{u}$ to $\mathbf{v}$ in the graph $\mathcal{C}^{[N]}$, meaning that $\gamma(0) = \mathbf{u}$, which is the start of our puzzle, and $\gamma(D) = \mathbf{v}$ which becomes the end of our puzzle, and $d(\gamma(i), \gamma(j)) = |i - j|$, so essentially the number of moves between i and j for all i, j in $[D+1]$. We will sometimes denote $\gamma(i)$ as γ_i.

Now define $E' = \{k \in E : \exists t \in [D+1] \text{ such that } \gamma_t(k) = 3\}$, which is the set of discs that are initially in column 0 and pass through column 3 at least once.

4.2 The case $E' = \emptyset$

First, consider the case where E' is empty. Essentially, this means that the discs in E (i.e., all the discs that start on peg 0) never pass through peg 3. In particular, we will say that they end up on peg 2. Due to all the discs being restricted to only using three pegs out of the four available, their movement requires at least $2^{\#E} - 1$ moves, (where of course $\#E$ denoted the size of the set E), which we know from the 3-peg Tower of Hanoi problem, which implies $D \geq 2^{\#E} - 1$. Therefore by the result we proved in Lemma 4, where we showed $2^{n-1} \geq \Psi[E]$, we see the inequality $D \geq \Psi E$ is clearly satisfied in this case. Therefore, we have shown that this is not the most efficient path, and is not the one we will use to solve the overall problem.

Henceforth, we will assume that E' is non-empty, so at least some discs will pass through peg 3, and let T be its largest element. Therefore T is the largest disc which passes through column 3. We also define $E'' = \{k \in E : k > T\}$, so this is the set of discs larger than T, i.e., large discs that don't pass through column 3. We can denote the cardinality of this set by K (which may be zero), with $b_1 < b_2 < \ldots < b_k$ being the elements in the set. We see that

$$E \subseteq [T] \sqcup \{T\} \sqcup \{b_1, \ldots, b_k\} \subseteq [N]. \tag{6}$$

In this equation $\sqcup$ denotes a disjoint union as of course we have seen that none of these groups of discs share a disc in common, and therefore we can have that

$$T + K + 1 \leq N. \tag{7}$$

Of course T is the cardinality of $[T]$, 1 is the cardinality of $\{T\}$, and K is the cardinality of $\{b_1, \ldots, b_k\}$.

Now in all of the rest cases, we can let the following notation hold. Let t_0 be the first moment when the disc T is no longer in column 0, i.e., the smallest element of $[D+1]$ for which $\gamma_{t_0}(T) \neq 0$. Since this is the smallest, we have $\gamma_{t_0 - 1}(T) = 0$, which means that between times $t_0 - 1$ and t_0, it is the disc T that is moved from column 0 to another column. Such a move is only possible if column 0 and the destination column contain no discs smaller than T. Let $\mathbf{x}_0 = \gamma(t_0 - 1)$, which is the stacking just before the movement of disc T. In this stacking, column 0 contains disc T and no discs smaller than T, and another column (the destination) contains no discs less than or equal to T.

Let t_1 be the first moment when disc T is in column 3, i.e., the smallest integer for which $\gamma_{t_1}(T) = 3$, and let $\mathbf{x}_3 = \gamma(t_1)$ be the corresponding stacking. By similar arguments, in this stacking, column 3 contains disc T and no discs smaller than T, and another column contains no discs smaller than or equal to T. The times t_0 and t_1 satisfy $1 \leq t_0 \leq t_1 \leq D$, as it requires at least one move to move T off of column 0, and it will be moved to column 3 either at or after this point, and either at or after this point is when we have traversed the entire path.

Let t_2 be the first moment when disc $N-1$ is no longer in column 0, i.e.,$\gamma_{t_2}(N-1) \neq 0$, and let $\mathbf{z}_0 = \gamma(t_2 - 1)$. In this stacking, column 0 contains only disc $N-1$, and another column is empty. Finally, let t_3 be the last moment when disc $N-1$ is not in column 2, i.e., $\gamma_{t_3}(N-1) \neq 2$, and let $\mathbf{z}_2 = \gamma(t_3 + 1)$. In this stacking, column 2 contains only disc $N-1$, and another column is empty. The times t_2 and t_3 satisfy $1 \leq t_2 \leq t_3 + 1 \leq D$.

As with $\mathbf{u}$ and $\mathbf{v}$, the notations $\mathbf{x}'_{a'}$ and $\mathbf{z}'_{b'}$ will denote the restrictions of x_a and z_b to $[N-1]$, i.e., ignoring the largest disc. Additionally, we will denote $\mathbf{u}'', \mathbf{v}'', \mathbf{x}''_a, \mathbf{z}''_b$ as the stackings restricted to $[T]$, i.e., ignoring all discs greater than or equal to T.

We observe that in the stacking $\mathbf{z}'_0$, which is just before $N-1$ leaves column 0, whilst ignoring $N-1$. Column 0 is empty as well as another (the destination) column, as they must have no discs smaller than $N-1$ in order to move $N-1$ between them, and this allows us to apply the induction hypothesis:

$$\begin{aligned} d(\mathbf{u}, \mathbf{z}_0) \geq d(\mathbf{u}', \mathbf{z}'_0) \geq \Psi\{k \in [N-1] : \mathbf{u}(k) = 0\} \\ = \Psi(E - \{N-1\}). \end{aligned} \tag{8}$$

In a similar way,

$$d(\mathbf{u}, \mathbf{x}_0) \geq d(\mathbf{u}'', \mathbf{x}''_0) \geq \Psi(E \cap [T]). \tag{9}$$

This of course means the number of moves from the initial stacking to when T first leaves column 0, is greater or equal to than the number of moves for the same stackings whilst ignoring the discs bigger than or equal to T, which of course is the number of moves to move any discs in the intersection of E (so starting on column 0), and discs being smaller than T. In fact, when $T = N-1$, this is exactly the same inequality.

4.3 The case $\Delta K > T$

So let's consider the case where $\Delta K > T$. This essentially means that there will be some discs that pass through column 3, but there will also be some large discs (a set of cardinality K) which will never pass through column 3. We will continue to ignore the largest disc N.

First, the fact that $\Delta K > T$ implies $K \geq 1$ as $T > 0$, and $T < b_k = N-1$ which is the largest disc that never passes through column 3. Furthermore, the set $\{x \in E : x \geq \Delta K\}$ is contained within $\{x \in E : x > T\} = E''$ and therefore has at most K elements due to the definition of E''. By Lemma 6, this leads to

$$\Psi E - \Psi(E - \{N-1\}) \leq 2^{K-1}. \tag{10}$$

Rearranging this we get

$$\Psi E - 2^{K-1} \leq \Phi(E - \{N-1\}).$$

Thus, from (3.3), as $d(\mathbf{u}, \mathbf{z_0}) \geq \Phi(E - \{N-1\})$, we get the result:

$$d(\mathbf{u}, \mathbf{z}_0) \geq \Psi E - 2^{K-1}. \tag{11}$$

As $t_2 \leq t_3 + 1$, (essentially the first moment $N-1$ is in column two is earlier than the last instant $N-1$ is in column 3 by at least one move) the path γ traverses the configurations $\mathbf{u}, \mathbf{z}_0, \mathbf{z}_2$, and $\mathbf{v}$ in the following order (broadly speaking, some stackings might coincide):

$$\mathbf{u} \to \mathbf{z}_0 \to \mathbf{z}_2 \to \mathbf{v}.$$

This essentially means that the stackings will follow the path of being in the initial stacking, to the point where $N-1$ is no longer in column 0, to the last moment $N-1$ is not in column 2, to the final stacking.

In the stacking $\mathbf{z_2}$, column 2 and column $c = \gamma_{t_3}(N-1)$, which is either 0 or 1, do not contain any discs smaller than $N-1$. Thus, the discs $b_1, \ldots, b_{K-1}$ are all in column $1-c$. In the final stacking $\mathbf{v}$, all these discs must end up in column 2. Moving these discs, which are not allowed to pass through column 3, requires at least $2^{K-1}-1$ moves (the fact that these discs are only moving in 3 columns allows us to use the equation from the towers of Hanoi puzzle), and the reason there are $K-1$ discs in this stacking is because $N-1$ itself is a large disc that never passes through peg 3, but we know that it is not in this stack. Therefore:

$$d(\mathbf{z}_2, \mathbf{v}) \geq 2^{K-1} - 1.$$

So it requires this many moves to get from the last moment $N-1$ is not in column two to the final stacking. Finally, $d(\mathbf{z}_0, \mathbf{z}_2) \geq 1$, as disc $N-1$ must move from column 0 to column 2, so we have found the number of moves required to complete each step and an overview is shown in the table below.

Step	Moves required
$d(\mathbf{u}, \mathbf{z_0})$	$\Psi E - 2^{K-1}$
$d(\mathbf{z_0}, \mathbf{z_2})$	1
$d(\mathbf{z_2}, \mathbf{v})$	$2^{K-1} - 1$

We can then add the number of moves required to complete each step together to get the overall lower bound for this path;

$$\begin{aligned} D = d(\mathbf{u}, \mathbf{v}) &= d(\mathbf{u}, \mathbf{z}_0) + d(\mathbf{z}_0, \mathbf{z}_2) + d(\mathbf{z}_2, \mathbf{v}) \\ &\geq (\Psi E - 2^{K-1}) + 1 + (2^{K-1} - 1) = \Psi E, \end{aligned}$$

which was to be demonstrated.

4.4 The case $\Delta K \leq T$

Now we will look at the case where $\Delta K \leq T$. This case has discs that pass through peg 3, however there is now a larger proportion of discs that do pass through peg 3.

This can actually be subdivided into three smaller cases, one where $K = 0$, one where $t_1 > t_3 + 1$ and one where $t_1 > t_3 + 1$, but first we will look at what holds for all of these cases.

We can no longer maintain the inequality from before where $\Psi E - \Psi(E - N - 1) \leq 2^{K-1}$, but there is a similar inequality that we can use:

$$\Psi E - \Psi(E - N - 1) \leq 2^{\nabla(T+K+1)-1}. \tag{12}$$

We can deduce this by Lemma 6, showing that $E - [\Delta s]$ contains more than s elements or $s = \nabla(T + K + 1)$. When E includes $[T + 1] \cup E''$, the difference $E - [\Delta s]$ contains more than $(T + 1 - \Delta s)^+ + K$ elements, and it is sufficient to note that the number is $\leq s$. Then we end up with the following inequalities:

$$\begin{aligned} K &\leq s, \\ T + 1 - \Delta s + K &\leq s. \end{aligned}$$

The second inequality is equivalent to $T + K + 1 \leq s + \Delta s < \Delta(s + 1)$. Then with our first inequality then we have $k \leq s$ which comes from the fact that $\nabla T \geq K$ by the properties of this case, and $\nabla T \leq s$ by $s = \nabla(T + K + 1)$. This therefore shows that the inequality in (12) holds.

From

$$d(\mathbf{u}, \mathbf{z_0}) \geq d(\mathbf{u}', \mathbf{z_0}') \geq \Psi(E - \{N - 1\}),$$

in (8), we get the result

$$d(\mathbf{u}, \mathbf{z_0}) \geq d(\mathbf{u}', \mathbf{z}'_{\mathbf{0}}) \geq \Psi E - 2^{\nabla(T+K+1)-1}. \tag{13}$$

4.4.1 The case $K = 0$

We can have $K = 0$, which is equivalent to $T = N - 1$, as for $K = 0$ all discs must pass through column 3 and so $N - 1$ is the largest disc that does so. We will first examine this particular case. This means that the largest disc, which is initially in column 0 and eventually in column 2, must pass through column 3 at least once. Here T and $N - 1$ are the same disc, so $t_1 \leq t_3$, as of course the first moment T is in column 3 is before the last moment T is not in column 2, and the path γ will therefore have a configuration of:

$$\mathbf{u} \to \mathbf{z_0} = \mathbf{x_0} \to \mathbf{x_3} \to \mathbf{z_2} \to \mathbf{v}. \tag{14}$$

Of course $\mathbf{z_0} = \mathbf{x_0}$ in this case as the stacking just before $N - 1$ leaves column 0 is the same as the stacking just before T leaves column 0. This path therefore means that we have the initial stacking, followed by $N - 1$ leaving column 0, followed by $N - 1$ arriving in column 3, followed by $N - 1$ arriving in column 2, followed by the final stacking.

We have $c = \gamma_{t_3}(N - 1)$ which is not equal to 2, as at the point just before $N - 1$ arrives in column 2 it could be in any other column, and c denotes this column. In the stacking $\mathbf{z}_2'$, so where we ignore the disc $N - 1$ columns 2 and c are empty, to allow disc $N - 1$ to move, and all the discs are in the other two columns. Therefore we have

$$\{0, 1, 2, 3\} - \{2, c\} = \{a, b\}, \tag{15}$$

where $a \in \{0, 3\}$ and $b \in \{0, 1\}$. So our possible stackings can be shown in the following table:

c	0	1	3
a	3	3	0
b	1	0	1

So if $N - 1$ is in column 0 there are discs in column 3 and 1, and so forth. Of course the column 2 remains empty so that $N - 1$ can move to it. Now define

$$\begin{aligned} A &= \{k \in [N - 1] : \mathbf{z}_2(k) = a\}, \\ B &= \{k \in [N - 1] : \mathbf{z}_2(k) = b\}. \end{aligned} \tag{16}$$

So A is the set of discs in column a and B is the set of discs in column b. These sets satisfy $A \sqcup B = [N - 1]$, as all discs smaller than $N - 1$ are in either of these sets, so by Lemma 8 as A and B are two finite subsets of $\mathbb{N}$,

$$\begin{aligned} \Psi A + \Psi B &\geq \frac{1}{2}\Psi[N + 1] - 1 \\ &= \frac{1}{4}(2^{\nabla(N+1)} + 2^{\nabla N}) + \frac{1}{2}\Psi[N - 1] - 1 \\ &\geq 2^{\nabla(T+K+1)-1} + \frac{1}{2}\Psi[N - 1] - 1. \end{aligned} \tag{17}$$

Of course from Lemma 8 we have that $n = [N - 1]$ so we substitute $[N + 1]$ in for $[n + 2]$ for the first equation. We can then break this up still using the result from Lemma 8 into the second line. Finally, we gain the third equation through the understanding that $\frac{1}{4}(2^{\nabla(N+1)} + 2^{\nabla N}) \geq 2^{\nabla(T+K+1)-1}$. We can see how this follows in the below inequalities, remembering that

$N = T - 1$, $K = 0$, and that ∇ and Δ preserve inequalities;

$$\begin{aligned}
&\frac{1}{4}(2^{\nabla(N+1)} + 2^{\nabla N}) \geq 2^{\nabla(T+K+1)-1} \\
\iff &\frac{1}{4}(2^{\nabla(N+1)} + 2^{\nabla N}) \geq 2^{\nabla(N+K)-1} \\
\iff &2^{\nabla(N+1)} + 2^{\nabla N} \geq 2^{\nabla(N+K)+1} \\
\iff &2^{N+1} + 2^{N} \geq 2^{N+K+1} \\
\iff &2^{N+1} + 2^{N} \geq 2^{N+1}.
\end{aligned}$$

So this inequality clearly holds.

In the stacking $\mathbf{x}'_a$, which is the state just before $N-1$ goes to column a, whilst also ignoring the disc $N-1$ itself, column a is empty as well as another column (from where $N-1$ is coming from), which allows us to apply the recurrence hypothesis:

$$d(\mathbf{z}'_{2'}, \mathbf{x}'_a) \geq \Psi A.$$

So the number of moves required to get from when $N-1$ arrives in column 2 to when $N-1$ arrives in column a (0 or 3), will be at least the number of moves that it takes to move all the discs off of column a.

Similarly, in the stacking $\mathbf{v}'$ column b is empty, as well as the column $1-b$, as such both the columns 0 and 1 are empty, and so:

$$d(\mathbf{z}_2, \mathbf{v}) \geq d(\mathbf{z}'_{2'}, \mathbf{v}') \geq \Psi B.$$

So the number of moves required to get from when $N-1$ arrives in column 2 to the final stacking when column 0 and 1 are empty, will be at least the number of moves that it takes to move all the discs off of column b as this is either column 0 or 1.

Between the stackings $\mathbf{z_0}$ and $\mathbf{z_2}$, so when $N-1$ leaves column 0 to when it arrives at column 2, discs smaller than $N-1$ make at least $d(\mathbf{x}'_{a'}, \mathbf{z}'_2)$ moves as they need to move away from the initial and destination pegs of the disc $N-1$, and the disc $N-1$ makes at least two moves (to pass from column 0, to column 3, and then to column 2), so:

$$d(\mathbf{z}_0, \mathbf{z}_2) \geq d(\mathbf{x}'_{a'}, \mathbf{z}'_2) + 2 \geq \Psi A + 2.$$

Therefore we have found the number of moves required to complete each step and an overview is seen in the table below.

Step	Moves required
$d(\mathbf{u}, \mathbf{z_0})$	$\Psi E - 2^{\nabla(T+K+1)-1}$
$d(\mathbf{z_0}, \mathbf{z_2})$	$\Psi A + 2$
$d(\mathbf{z_2}, \mathbf{v})$	ΨB

Then we can compute the overall number of moves by adding the number of moves between each step, to get:

$$\begin{aligned}
D = d(\mathbf{u}, \mathbf{v}) &= d(\mathbf{u}, \mathbf{z_0}) + d(\mathbf{z_0}, \mathbf{z_2}) + d(\mathbf{z_2}, \mathbf{v}) \\
&\geq \Psi E - 2^{\nabla(T+K+1)-1} + \Psi A + 2 + \Psi B \\
&\geq \Psi E + 1 + \frac{1}{2}\Psi[N-1] \geq \Psi E,
\end{aligned}$$

where the rearrangement from the second to the third line comes from (17). This concludes the proof, as we have found that the number of moves required to complete this path is more than in the case $\Delta K > T$. In this particular case it should also be relatively easy to understand that it is not the most efficient. As we said before, the largest disc N should only move once in the most efficient case, and $N-1$, should be able to do the same due to having multiple intermediate pegs.

We will now assume $K \geq 1$ (and still $\Delta K \leq T$), so $T < b_K = N-1$ (the largest disc never passes through column 3). At this point, there is no direct way to compare t_1 and t_3+1, the times at which the stackings $\mathbf{x_3}$ and $\mathbf{z_2}$ appear, respectively. We can only say that these two moments are different because the last disc moved is not the same in both cases. This means that we must break down this specific case of $0 < \Delta K \leq T$ into two smaller sub-cases which will have different paths. These two cases include the one where $t_1 > t_3+1$ and the one where $t_1 < t_3+1$.

4.4.2 The case $t_1 > t_3 + 1$

Now suppose that $t_1 > t_3+1$. This essentially means that the first moment T is in column 3 is after the time when $N-1$ moves to column 2. Then the path γ has the configuration:

$$\mathbf{u} \to \mathbf{z_0} \to \mathbf{z_2} \to \mathbf{x_3} \to \mathbf{v}.$$

Essentially, we have when all discs are in column 0, then after this $N-1$ leaves column 0, then after this it will arrive in column 2. Then T will arrive in column 3, then after this we will have the final stacking when all discs are in column 2 or 3.

In the stacking $\mathbf{x_3}$, column 3 contains no disc smaller than T as this is the last moment before T moves into this column and the same applies to column $d = \gamma_{t_1-1}(T)$, so essentially the column that T is coming from. The discs smaller than T are then distributed across the other two columns. Then we have $c = \gamma_{t_3}(N-1) \in \{0,1\}$ as before as in the last moment before $N-1$ is in column 2 it must be in either 0 or 1 (as it is larger than T in this case it can never be in column 3), and let

$$\{0,1,2,3\} - \{3,d\} = \{a,b\},$$

with $a \in \{2,c\}$, (so either $\{0,2\}$ or $\{1,2\}$) and $b \in \{0,1\}$. Therefore we can have the table below, which shows what these columns must be in different cases.

d	0 or 1	2
a	2	c
b	$1-d$	$1-c$

This essentially shows that at the moment just before T moves into column 3, all discs smaller than this are will be in any combination of columns 0 and 1, 0 and 2, or 1 and 2 depending on where T is moving from.

Then we can define:

$$A = \{k \in [T] : \mathbf{x}_3(k) = a\},$$
$$B = \{k \in [T] : \mathbf{x}_3(k) = b\}.$$

This of course means that A is the set of discs which are smaller than T and are in column a, and B is the set of discs that are smaller than T and in column b.

These sets are complementary in $[T]$, as of course all discs smaller than T have to be in column a or b. In the stacking $\mathbf{z_2}''$, which is when we're ignoring all the discs larger than and including T, column a is empty along with another column, as these are the initial and destination pegs for when we are just about to move disc $N-1$ so:

$$d(\mathbf{x_3}, \mathbf{z_2}) \geq d(\mathbf{x_3}'', \mathbf{z_2}'') \geq \Psi A.$$

This inequality occurs as we have to move all the discs in set A off of column a, which will take ΨA moves, between these two stackings.

Similarly in $\mathbf{v}''$, columns b and $1-b$ are empty, essentially columns 0 and 1, so

$$d(\mathbf{x_3}, \mathbf{v}) \geq d(\mathbf{x_3}'', \mathbf{v}'') \geq \Psi B.$$

This inequality occurs as we have to move all the discs in set B off of column b, which will take ΨB moves, between these two stackings.

Moreover, $d(\mathbf{z_0}, \mathbf{z_2}) \geq 1$, as it will take at least one move for the disc $N-1$ to move from column 0 to column 2. Therefore we have found the minimum number of moves required to complete each step, and there is an overview in the table below.

Step	Moves required
$d(\mathbf{u}, \mathbf{z_0})$	$\Psi E - 2^{\nabla(T+K+1)-1}$
$d(\mathbf{z_0}, \mathbf{z_2})$	1
$d(\mathbf{z_2}, \mathbf{x_3})$	ΨA
$d(\mathbf{x_3}, \mathbf{v})$	ΨB

Therefore we can add the number of moves required to complete each step to determine a lower bound for the number of moves required to complete the overall puzzle:

$$\begin{aligned} D = d(\mathbf{u}, \mathbf{v}) &= d(\mathbf{u}, \mathbf{z_0}) + d(\mathbf{z_0}, \mathbf{z_2}) + d(\mathbf{z_2}, \mathbf{x_3}) + d(\mathbf{x_3}, \mathbf{v}) \\ &\geq \Psi E - 2^{\nabla(T+K+1)-1} + 1 + \Psi A + \Psi B \\ &\geq \Psi E - 2^{\nabla(T+K+1)-1} + \frac{1}{2}\Psi[T+2]. \end{aligned}$$

Now let $s = \nabla(T+K+1)$. By definition of our case, $T \geq \Delta K$ which is equivalent to $T+K+1 \geq \Delta(K+1)$, and so $s \geq K+1$. Therefore, it follows that $T = (T+K+1)-(K+1) \geq \Delta s - s = \Delta(s-1)$, meaning $\nabla T \geq s-1$.

We can now use Lemma 3, which gives us the below inequalities:

$$\begin{aligned} &\Psi[n+2] \geq 2^{(\nabla n)+1} \\ \iff &\Psi[T+2] \geq 2^{(\nabla T)+1} \\ \iff &\frac{1}{2}\Psi[T+2] \geq 2^{\nabla T} \\ \iff &\frac{1}{2}\Psi[T+2] \geq 2^{s-1} \\ \iff &\frac{1}{2}\Psi[T+2] \geq 2^{\nabla(T+K+1)-1}. \end{aligned}$$

Therefore $D \geq \Psi E$, which is what we needed to prove. This is because we have proof that this case is less efficient then the case we had before where $\Delta K > T$. Therefore, we can now move on to check the final case.

4.4.3 The case $t_1 < t_3 + 1$

The final case is where $t_1 < t_3+1$. Essentially when the final moment T is in column 3 is earlier than the last moment $N-1$ is not in 2. The path γ then has the configuration:

$$\mathbf{u} \to \mathbf{x_0} \to \mathbf{x_3}, \mathbf{z_0}, \to \mathbf{z_2} \to \mathbf{v}.$$

As we can see, there is an overlap of the stackings. However, in the overlap we are looking at different sets of discs, so this will not affect our result. The reason this overlap occurs is because in this case, the disc T can go to column 3 either before or after the disc $N-1$ arrives in column 2.

In the stacking $\mathbf{z_2}'$ columns 2 and $c = \gamma_{t_3}(N-1) \in \{0, 1\}$ are empty to allow the disc $N-1$ to move, and all the other discs are in columns 3 and $b = 1 - c$, where b is ether 0 or 1. In particular, the discs $b_1, \dots, b_{K-1}$ are in column b, as T has gone to column 3, and we cannot stack any larger discs on top of it. We also note that $b_K = N-1$ which is why it is not included in the above statement.

In the stacking $\mathbf{x_3}''$, so where we ignore the disc T column 3 is empty along with another column (where T is coming from) so $d(\mathbf{z_2}'', \mathbf{x+3}'') \geq \Psi A$, where,

$$A = \{k \in [T] : \mathbf{z_2}(k) = 3\},$$

as all the discs in set A are smaller than T, and are on column 3, so we must move these out of the way of T which takes ΨA moves. In the final stacking $\mathbf{v}'$, the columns b and $b-1$ are empty (so columns 0 and 1 are empty), so by the backward induction hypothesis we have

$$d(\mathbf{z_2}, \mathbf{v}) \geq d(\mathbf{z_2}', \mathbf{v}') \geq \Psi B.$$

In this equation we have

$$\begin{aligned} B &= \{k \in [N-1] : \mathbf{z_2}(k) = b\} \\ &\supseteq \{k \in [T] : \mathbf{z_2}(k) = b\} \sqcup \{b_1, ..., b_{K-1}\}, \end{aligned}$$

as all the discs in set B are smaller than $N-1$, and are on the destination column, so we must move these out of the way of $N-1$ which takes ΨB moves.

The set $A \cup B$ contains $[T] \sqcup \{b_1, ..., b_{K-1}\}$, as of course neither contain $N-1$ or T, so this union has at least $T + K - 1$ elements, and:

$$\Psi A + \Psi B \geq \frac{1}{2}\Psi[T+K+1] - 1,$$

which holds by Lemma 8, where $n+2$ is replaced by $T+K-1$.

Between $\mathbf{x_3}$ and $\mathbf{z_2}$, the discs smaller than T make at least ΨA movements, and between $\mathbf{u}$ and $\mathbf{x_0}$ there are at least $\Psi(E \cap [T])$ movements, as of course we have to move all discs smaller than T that are on column 0 before we can move T itself off of column 0. In $\mathbf{u}$ the discs $b_1, \dots, b_{K-1}$ are all in column 0, while in $\mathbf{z_0}$ they are all in another column (which can be 1 or 2). Since these discs avoid column 3, they must make at least $2^{K-1} - 1$ movements between $\mathbf{u}$ and $\mathbf{z_0}$, by the Tower of Hanoi.

Finally, disc T makes at least one move between $\mathbf{x_0}$ and $\mathbf{x_3}$, as it must take one move to get from column 0 to column 3, and the disc $N-1$ makes at least one move between $\mathbf{z_0}$ and $\mathbf{z_2}$, as it takes at least one move for it to get from column 0 to column 2. Therefore the total number of movements between $\mathbf{u}$ and $\mathbf{z_2}$ satisfies the lower bound:

Step	Moves required
$d(\mathbf{u}, \mathbf{x_0})$	$\Psi(E \cup [T])$
$d(\mathbf{z_2}'', \mathbf{x_3}'')$	ΨA
$d(\mathbf{u}, \mathbf{z_0})$	$2^{K-1} - 1$
$d(\mathbf{z_0}, \mathbf{z_2})$	1
$d(\mathbf{x_0}, \mathbf{x_3})$	1

Therefore we find the minimum number of moves required to get from $\mathbf{u}$ to $\mathbf{z_2}$ is found by adding these all together:

$$\begin{aligned} d(\mathbf{u}, \mathbf{z_2}) \geq d(\mathbf{u}, \mathbf{x_0}) + d(\mathbf{z_2}'', \mathbf{x_3}'') + d(\mathbf{u}, \mathbf{z_0}) + d(\mathbf{z_0}, \mathbf{x_0}) + d(\mathbf{z_2}, \mathbf{x_3}) \\ d(\mathbf{u}, \mathbf{z_2}) \geq \Psi(E \cap [T]) + \Psi A + 2^{K-1} + 1. \end{aligned}$$

From this we can find the overall number of moves to get from $\mathbf{u}$ to $\mathbf{v}$ by adding this result to the number of moves required to get from $\mathbf{z_2}$ to $\mathbf{v}$:

$$\begin{aligned} D = d(\mathbf{u}, \mathbf{v}) = d(\mathbf{u}, \mathbf{z_2}) + d(\mathbf{z_2}, \mathbf{v}) \\ \geq \Psi(E \cap [T]) + \Psi A + 2^{K-1} + 1 + \Psi B \\ \geq \Psi(E \cap [T]) + \frac{1}{2}\Psi[T + K + 1] + 2^{K-1}. \end{aligned}$$

Then, we have $E = (E \cap [T]) \sqcup \{T, b_1, ..., b_K\}$ and $T \geq \Delta K$, so:

$$\Psi E - \Psi(E \cap [T]) \leq \Psi[T + K + 1] - \Psi[T],$$

by Lemma 7. As such, we have that

$$D - \Psi E \geq \Psi[T] + 2^{K-1} - \frac{1}{2}\Psi[T + K + 1].$$

We know that this expression is positive by the result we had in equation 4 which states that $\Psi[a + b] \leq 2\Psi[a] + 2^{b-1}$. We substitute $a = T$ and $b = K + 1$. It follows that

$$\begin{aligned} 2\Psi[a] + 2^{b-1} \geq \Psi[a + b] \\ \iff 2\Psi[T] + 2^K \geq \Psi[T + K + 1] \\ \iff 2\Psi[T] + 2^K + \Psi[T + K + 1] \geq 0 \\ \iff \Psi[T] + 2^{K-1} + \frac{1}{2}\Psi[T + K + 1] \geq 0. \end{aligned}$$

So again, we have proven that in this case the minimum number of moves required to complete the puzzle by traversing this path is greater, and therefore this path less efficient than in the case $\Delta K > T$.

This concludes the proof of the final case and as such the theorem is established.

5 Finalising the proof

Below there is a table of each of the cases that have been looked at, and also the lower bound of moves required to complete the puzzle for each of these cases, so we can find the optimum one.

Case	Lower bound
$E' = \emptyset$	$2^{\#E} - 1$
$\Delta K > T$	ΨE
$K = 0 \quad (E'' = \emptyset)$	$\Psi E + 1 + \frac{1}{2}\Psi[N-1]$
$T \geq \Delta K > 0$ and $t_1 > t_3 + 1$	$\Psi E - 2^{\nabla(T+K+1)-2} + \frac{1}{2}\Psi[T+2]$
$T \geq \Delta K > 0$ and $t_1 < t_3 + 1$	$\Psi(E \cap T) - 2^{K-1} + \frac{1}{2}\Psi[T+K+1]$

Whilst exploring the lower bound for each case, we proved that none were smaller than ΨE, therefore the case $\Delta K > T$ is clearly the optimum case, and as such is the one we will continue with to complete the paper. So, this is of course the minimum number of moves required to get to the point where all discs $N-1$ and smaller are stacked on the two columns, which are neither the starting, nor destination column for the largest disc.

From Theorem 1, we can then deduce the the result which summaries the above section.

Lemma 9. *In the four-column variant of the Towers of Hanoi, at least $\Phi(N)$ movements are required to move N discs from one column to another.*

At this point, we want to add the largest disc N back into the puzzle. To prove the above lemma, we assume $N \geq 1$. Let $\mathbf{u} : [N] \to \mathcal{C}$ be the stacking where all discs are in column 0, and $\mathbf{v}$ be the stacking where all discs are in column 2, so all are stacked on the destination column. Define the path γ as before and the intermediate stackings $\mathbf{z_0}, \mathbf{z_2}$ as before however with N now being the largest disc rather $N-1$, and these stackings are traversed in the following order:

$$\mathbf{u} \to \mathbf{z_0} \to \mathbf{z_2} \to \mathbf{v}.$$

In the stacking $\mathbf{z_0}'$, so where we are ignoring the disc N the column 0 is empty as well as the destination column. Therefore, by Theorem 1 as well as the case $\Delta K > T$ we have that all discs $N-1$ and smaller are in the other two columns, in this case 1 and 3, and we have that

$$d(\mathbf{u}, \mathbf{z_0}) \geq d(\mathbf{u}', \mathbf{z_0}') \geq \Psi\{k \in [N-1] : \mathbf{u}(k) = 0\} = \Psi[N-1].$$

This essentially is what we showed in the previous section in the case $\Delta K = T$. Of course, as we are continuing to ignore the largest disc, we know that $\Psi[E] = \Psi[N-1]$, as there will start $N-1$ discs in the starting column. Similarly, in stacking $\mathbf{z_2}'$, so just before the disc N moves to column 2, but ignoring the disc N, column 2 is empty as well as one other (the starting column), so by the same theorem

$$d(\mathbf{v}, \mathbf{z_2}) \geq d(\mathbf{v}', \mathbf{z_2}') \geq \Psi\{k \in [N-1] : \mathbf{v}(k) = 2\} = \Psi[N-1].$$

This makes sense, as at this point it takes the same number of moves to restack all discs smaller than N as it did to unstack them. Again, $d(\mathbf{z_0}, \mathbf{z_2}) \geq 1$, because disc the N must move from the starting to destination columns. It follows that:

$$d(\mathbf{u}, \mathbf{v}) = d(\mathbf{u}, \mathbf{z_0}) + d(\mathbf{z_0}, \mathbf{z_2}) + d(\mathbf{z_2}, \mathbf{v}) \geq 1 + 2\Psi[N-1] = \Phi(N).$$

In this equation we can of course write Φ in terms of Ψ by Lemma 2. Of course, the above inequality turns into equality when we only move the largest disc N once, which we will do in the most efficient case. As such, we have shown that $\Phi(N)$ is the optimal solution to The Reve's Puzzle, and we can conclude the proof. One thing, however, that was not proven was the optimal value for T, or M as it's noted in (2). Whilst it isn't needed to find the value of Φ from (1), it is something we will look at in the section concerning the Frame-Stewart algorithm.

6 Other variants

At the time of writing this paper, a similar theorem to determine the minimum number of moves required to complete a five peg variant of this puzzle has not been found. However the Frame-Stewart algorithm provides us with a good starting point.

6.1 The Frame-Stewart algorithm

I previously noted the Frame-Stewart algorithm, which is a recursive algorithm that is assumed to define the optimum number of moves for Tower of Hanoi variants which have four or more pegs. Whilst the algorithm has been assumed to provide the best solution in these cases, it wasn't until 2014 that Thierry Bousch proved that it was optimum for the case when the number of pegs was equal to 4, and for any greater number of pegs, any similar proofs still have not been presented.

However, whilst in popular literature the Frame-Stewart algorithm is presented a single algorithm to find solutions, they are actually two separate algorithms that were produced in 1941, one by Frame and one by Stewart, which can be proved to be 'essentially the same'. Originally, it was noted that the two algorithms outputted the same value for all tested number of discs and pegs, but a paper published in 1999 titled 'On the frame–stewart algorithm for the multi-peg tower of hanoi problem' proved that whilst written differently, the two algorithms were the same [7].

For Frame's algorithm, let $p \geq 4$ and $n \geq p$, where n is the number of discs and p is the number of pegs.

$$\begin{aligned} F(n,p) = \min\{&2F(n_1,p) + 2F(n_2,p-1) + \cdots + 2F(n_{p-2},3) + 1| \\ &n_1 + \cdots + n_{p_2} + 1 = n, n_1 \geq n_2 \geq \cdots \geq n_{p-2}\}. \end{aligned} \tag{18}$$

For Stewart's algorithm, again let $p \geq 4$ and $n \geq p$.

$$S(n,p) = \min\{2S(n_1,p) + S(n_2,p-1)|n_1 + n_2 = n\}. \tag{19}$$

For all $n_1, n_2 \in \mathbb{Z}^+$. As the paper published in 1999 proved these were the same, they where then combined to create the Frame-Stewart algorithm to credit both mathematicians. As such, we can define the overall algorithm below.

Let n be the number of discs and r the number of pegs. Then define $T(N,r)$ to be the fewest number of moves needed to move N discs using r pegs. Then, we can define a recursive algorithm in three steps.

First, chose some k with $1 \leq k < n$, and transfer the top k discs to one peg which is neither the original or destination peg. This will take $T(k,r)$ moves.

In the next step, we will ignore the peg that the discs k and smaller have been moved to, and now move the remaining $N-k$ discs to the destination peg. As we are ignoring one peg, this will require $r-1$ pegs. As such, this step will take $T(N-k, r-1)$ moves.

Finally, we now move the top k discs to the destination peg, which will take $T(k,r)$ moves.

Therefore, the overall process will take $2T(k,r)+T(N-k,r-1)$ moves. Of course, we have to choose k to be the value which provides the minimum number of moves in each case, like we did when solving The Reve's Puzzle.

Now, we want to show that this recursive algorithm works for 4 pegs. Let $k = M$, and of course $r = 4$ as there are 4 pegs. Then, we get the number of moves is $\min\{2T(M,4) + T(N-M,3)\}$. Of course, we know that $T(N-M,3) = 2^{N-M} - 1$ from the Tower of Hanoi.

Then, we have the number of moves to complete the Reve's puzzle is $\min_{1 \geq k < N}\{2T(k,4) + 2^{N-M} - 1$. Which, you should see is equivalent to (2).

At the time of writing this paper, a the only output from the Frame-Stewart algorithm proven to be optimal is the four peg solution. As such, proving this result is optimal for five pegs or more remains an open problem. There is also a general computational problem in determining the optimum k, which changes depending on both the number of pegs, and the number of discs in any given problem.

We can however, define the Frame-Stewart numbers as follows:

Definition 1. *For all n in $\mathbb{N}$, including 0,* $FS_3^n = 2^n - 1$
$FS_4^0 = 0 \quad \forall n \in \mathbb{N}_0 \quad FS_4^n = \min\{2FS_4^m + FS_3^{n-m} | m \in [n]_0\}$.

These of course are the numbers used to solve the 3 and 4 peg problems.

We can also find the frame numbers, denoted by $\overline{F}_4^n$. To show the number of moves required to get to the 'halfway point' of the problem, as found in section 3. These are defined below.

Definition 2. $F_4^0 = 0$; *Then for all n in $\mathbb{N}$, including 0:* $\quad \overline{F}_4^n = \min\{F_4^m + F_3^{n-m} | m \in [n+1]_0, 2m \geq n\}, \quad F_4^{n+1} = 2\overline{F}_4^n + 1$.

Then F_p^n is the overall number of moves to complete the puzzle, and as such we have the following argument.

Proposition 1. *For all n in $\mathbb{N}$, including 0:* $\quad F_4^n = FS_4^n$.

The Frame-Stewart conjecture, again produced in 1941, is shown below, where we again write d as the minimal distance required to get from one state to another.

Conjecture 2. *The Frame-Stewart Conjecture:* $\quad \forall n \in \mathbb{N}_0 : \quad d(0^n, 2^n) = FS_4^n$.

This conjecture can be verified by hand for a small number of discs, as such a small n. As of 2013, it had also been tested to be true for up to 30 discs using a search algorithm on a computer, developed by R. E. Korf and A. Felner [8]. Of course it was then proven to be true for all n by Thierry Bousch in 2014.

As we can see, Frame and Stewart made some massive contributions to the solving of the puzzle, and so it is still important to credit and examine their work.

In the Towers of Hanoi, Myths and maths by Hinz [2], there was an analysis of for which m, that $f(m) = 2FS_4^m + FS_3^{n-m}$ reaches its minimum. In other words, what is the value of M in (2) where the puzzle takes the fewest number of moves to complete? Recall $\Delta p = \frac{p(p+1)}{2}$. We have that every $n \in \mathbb{N}_0$ can be uniquely written as $\Delta p + x$, where

$$p = \lfloor \frac{\sqrt{8n+1} - 1}{2} \rfloor \in \mathbb{N}_0,$$

with an excess $x \in [p+1]_0$. This property comes from the fact that $\Delta p \leq n < \Delta(p+1)$, and so p is the largest integer such that $p(p+1) \leq 2n$. From there, we can complete the square to get our answer.

$$\begin{aligned} p(p+1) &\leq 2n \\ \iff 4p(p+1) &\leq 8n \\ \iff 4p(p+1) + 1 &\leq 8n + 1 \\ \iff (2v+1)^2 &\leq 8n + 1 \\ \iff v &\leq \frac{\sqrt{8n+1} - 1}{2}. \end{aligned}$$

Then as we take the largest integer v, this becomes an equality with the excess described before. From this, it was determined we can write $f(m_n) = (p-1+x)2^p + 1$, and this is proven in The Towers of Hanoi, Myths and Maths [2]. the minimum of the function $f(m)$ was found to be $m_n = \Delta p + x$, as well as there being another minimum at $m_n - 1$. There being two minimums is as to be expected due to the fact that there two intermediate pegs in the puzzle, you can have two symmetrical optimal solutions. This optimal M is what we will use in the code in Appendix B.

6.2 The Double Decker

A simple variation on the Tower of Hanoi is known as the Double Decker, which is explored in a 2019 paper called 'Simple Variations on The Tower of Hanoi: A Study of Recurrences and Proofs by Induction' [9]. In this variation, we have $2n$ discs on the first peg, and we achieve this many discs by duplicating each disc from the original problem. As such, we have two discs of each size. Other than that, the rules of the Tower of Hanoi puzzle are maintained.

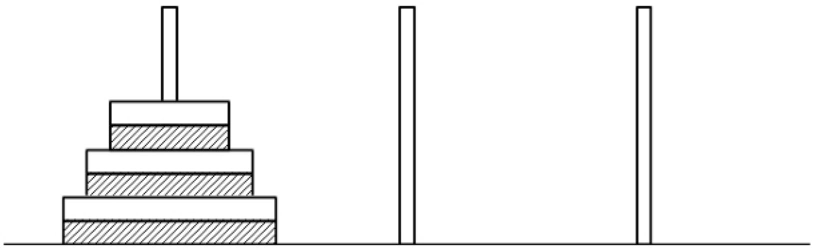

Figure 4: Graph of the Double Decker problem for $n = 3$ [9].

A simple way to complete the puzzle would be to use $2^{2n} - 1 = 4^n - 1$ moves, as we would in the Tower of Hanoi. However, this is likely not the most efficient way to complete the puzzle as it doesn't benefit from having discs of the same size. So, we are going to explore other ways to complete this variation. Now, since we are looking at duplicate discs, we can ask the question as to whether we want to preserve the order of the discs or not, so, in the figure above, can we switch the shaded and non-shaded discs in our final stacking? We will look at both of these cases, to understand the differences between them.

6.2.1 Not preserving the order

In the case where we don't preserve the order of the discs in the final stacking, we can use a recurrence relation in order to find our optimal solution. We can label the amount of moves required to move n groups of discs by a_n, so for example a_1 will be the number of moves required to move both of the disc of smallest size to the final peg. As such, we can find $a_0 = 0$, $a_1 = 2$, $a_3 = 6$, and so on. This produces a recurrence relation $a_n = 2a_{n-1} + 2$. We will now try to find an expression for a_n not in terms of a. To do this, we can expand our first few terms:

$$\begin{aligned} a_1 &= 2a_0 + 2, \\ a_2 &= 2a_1 + 2, \\ a_3 &= 2a_2 + 2. \end{aligned}$$

We can then use these to put a_2 and a_3 in terms of a_0.

$$\begin{aligned} a_2 &= 2(2a_0 + 2) + 2 = 4a_0 + 6, \\ a_3 &= 2(4a_0 + 6) + 2 = 8a_0 + 14. \end{aligned}$$

From this, we can see the general form of a_n in terms of a_0. We have $a_n = 2^n a_0 + (2 + 2^2 + 2^3 + \ldots + 2^n)$. We can now see that the terms inside the brackets form a geometric series, with first term 2 and common ratio 2. Therefore, we can find out what this sums to.

$$s_n = a(\frac{1-r^n}{1-r}) = 2(\frac{1-2^n}{-1}) = 2(2^n - 1).$$

As we have already determined that $a_0 = 0$, this therefore means that $a_n = 2(2^n - 1)$, and this is the optimal solution in this case.

6.2.2 Preserving the order

Now we want to look at the case where we do preserve the order. So the discs in figure 9 end up on the destination peg in the same order that they start at on the initial peg. Essentially, the shaded disc is below the non-shaded disc of the same size once it reaches the final peg, but the order does not matter for discs of the same size in intermediate steps.

Interestingly, in the previous case, despite not having to preserve the order of the discs in the final stacking, the only ones that are switched are the largest two. This is because they both only move once to get to the final peg, and the non-shaded one is free to move first, so ends up on the bottom. So they make an odd number of moves. However, every other smaller disc makes an even number of moves, maintaining their order. So, to get to a solution, we can make the two largest discs move an even number of times.

This means, that for the case where we want to preserve the order, we can have a solution where we simply double the number of moves it takes to get to the final stacking to maintain the order of the larger two discs. This gives us $2 \cdot 2(2^n - 1) = 4(2^n - 1)$ moves. However, this is not actually the optimal solution. In the optimal case, we can move the largest shaded disc only once, as this will maintain the order. Therefore the optimal number of moves to complete the puzzle whilst preserving the order of the discs becomes $4(2^n - 1) - 1$ moves.

6.3 The Tower of London

There are many other variants of the Tower of Hanoi which don't just include increasing the number of pegs. There are some which change the number of discs of each size, some that change how many discs each peg can hold, and some which introduce other rules that make the puzzle more complex. One such variant of the Tower of Hanoi is called the Tower of London. It was invented in 1982 by Shallice, and seemingly has applications in psychology. He proposed the puzzle as an alternative to the classic Tower of Hanoi task, and it is now used to study problem-solving and planning in both clinical and non-clinical environments [2].

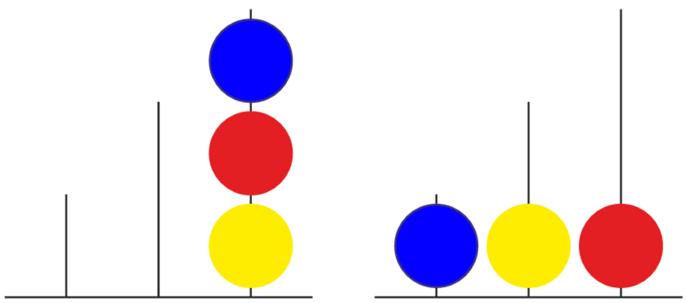

Figure 5: An illustration of The Tower of London [2].

The classical version of the problem consists of three different coloured balls, (in place of discs), of equal size. We will designated these to be blue, red, and yellow. There are three pegs, which can hold up to 1, 2, and 3 balls respectively.

The aim in this puzzle is to reach a destination 'state' (i.e., an ordering of the balls) from a given state in the minimum number of moves. In order to help us in the mathematical sense, we will label the balls with numbers, so let the blue ball = 1, the red ball = 2, and the yellow ball = 3.

We are able to determine that there exist 36 possible states in this puzzle. This is due to the fact we have 6 types of states, which can be listed as follows:

- All balls on the last peg
- 2 balls on the last peg, 1 on the middle
- 2 balls on the last peg, 1 on the first
- 1 ball on the last peg, 2 in the middle
- 1 ball on each peg
- 1 ball on the first, 2 in the middle

Each of these types of states has $3! = 6$ different possibilities, which comes from a reordering of the colours. As such, we have $6 \cdot 6 = 36$ total states.

Clearly the optimum number of moves to reach one state from another depends on how different the starting states were. It has been found that maximum optimum number of moves is 8, whilst of course the minimum optimum number of moves is 1 (as we assume the initial and final states are different), and this was explored in the Journal of clinical and experimental neuropsychology in 2002 [10].

The finding of the optimum number of moves between two states was done by creating a graph with all states listed, and then drawing a line between each set of states where one could become the other in only one move. From this graph, we can see by inspection that the shortest path between some states is 8 moves, and the shortest path between all other pairs is less than 8.

7 Conclusion

This paper has explored the origin and optimal solution for both the Tower of Hanoi and The Reve's puzzle, highlighting how even a small variation in the rules of a simple problem can lead to a much more complex problem which requires years of mathematical research.

Through looking into the Reve's puzzle we examined all different possible paths to complete the puzzle and from there determined which was the most efficient path and therefore the optimal solution. Additionally, we explored the Frame-Stewart algorithm which remains the assumed optimal solution for variations on the puzzle with 4 or more pegs.

Beyond the simple variations of adding more pegs, have also explored a different variant of the Tower of Hanoi problem, called the Tower of London, and looked into how this variant has practical uses, though of course there are many more variants of the puzzle each with their own mathematical challenges.

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A Notation

Below I have included multiple tables from my supervisor that state the definition and meaning for some notation which is included throughout the paper. The first table includes notation that describes the configuration or 'state' at any time during the puzzle.

Notation	Definition	Meaning
$\mathbf{u}$	.	Initial configuration
$\mathbf{v}$	.	Final configuration
$\gamma(t) = \gamma_t$	.	Configuation at time t
$\mathbf{x_0}$	$\gamma(t_0 - 1)$	Configuration just before T first leaves column 0
$\mathbf{x_3}$	$\gamma(t_1)$	Configuration when T first arrives in column 3
$\mathbf{z_0}$	$\gamma(t_2 - 1)$	Configuration just before $N-1$ first leaves column 0
$\mathbf{z_3}$	$\gamma(t_3 + 1)$	Configuration just after $N-1$ arrives in column 2 for the last time
$\mathbf{x}'$	$\mathbf{x}\|_{[N-1]}$	State $\mathbf{x}$, ignoring disc $N-1$
$\mathbf{x}''$	$\mathbf{x}\|_{[T]}$	State $\mathbf{x}$, ignoring all discs $\geq T$

The next table is one that has notation which describes moments or times in the puzzle.

Notation	Definition	Meaning
t_0	$\min\{t : \gamma_t(T) \neq 0\}$	First moment disc T is not in column 0
t_1	$\min\{t : \gamma_t(T) = 3\}$	First moment disc T is in column 3
t_2	$\min\{t : \gamma_t(N-1) \neq 0\}$	First moment disc $N-1$ is not in column 0
t_3	$\max\{t : \gamma_t(N-1) \neq 2\}$	Last moment disc $N-1$ is not in column 2

Finally, there is a table which has the notation that describes groups of discs.

Notation	Definition	Meaning
E	$\{k \in [N] : \mathbf{u}(k) = 0\}$	Discs initially in column 0
E'	$\{k \in E : \exists t \leq D : \gamma_t(k) = 3\}$	Discs that start in column 0 and pass through column 3 at least once
T	$\max E'$	Largest disc that starts in column 0 and passes through column 3
E" $= \{k \in E : k > T\}$	$\max\{t : \gamma_t(N-1) \neq 2\}$	Large discs that never go through column 3

B The use of computers

For the Tower of Hanoi, The Reve's Puzzle, and any other variations on these puzzles, computers can be extremely useful tools. Below, we have a code which, uses the optimum value for M in the section on the Frame-Stewart numbers, to produce an output describing how to move all the discs from the initial to final peg in The Reve's Puzzle. This code can be run in Python. The idea is to create a recursive function which moves the discs in the path described by the case $\Delta K > T$. As such, we move all the discs k and smaller onto an intermediate peg, before moving the larger discs onto the destination column, and finally move all the smaller discs onto this destination column as well.

```
 import math

#Calculate p (rounded)
def calculate_p(n):
    p = (math.sqrt(8 * n + 1) - 1) / 2
    return round(p)

#Calculate m_n
def calculate_m_n(p):
    return (p * (p - 1)) // 2

#Tower of Hanoi
def hanoi(n, k, from_col, temp_col, to_col):
    if n == 0:
        return
    hanoi(n - 1, k, from_col, to_col, temp_col)
    print(f"Move disc {n + k} from column {from_col} to column {to_col}")
    hanoi(n - 1, k, temp_col, from_col, to_col)
```

```python
#Reve's puzzle
def reve(n, from_col, temp_col, to_col, temp_col2):
    #Base case
    if n == 1:
        print(f"Move disc {n} from column {from_col} to column {to_col}")
        return

    #Calculate p
    p = calculate_p(n)

    #Calculate m_n
    k = calculate_m_n(p)

    #Case if k is 0
    if k == 0:
        print(f"Move disc {n} from column {from_col} to column {to_col}")
        return

    #Moving k discs from 'from_col' to 'temp_col2' using 'to_col'
    reve(k, from_col, to_col, temp_col, temp_col2)

    #Moving the remaining n-k discs to 'to_col' using Tower of Hanoi
    hanoi(n - k, k, from_col, temp_col2, to_col)

    #Moving the k discs from 'temp_col2' to 'to_col' using 'from_col'
    reve(k, temp_col, from_col, to_col, temp_col2)

#Function to start the recursion, using 5 discs as our example
def run_reve():
    n = 5
    reve(n, "A", "B", "D", "C")

run_reve()
```

We have let n = 5 in this case, which produces an output:
Move disc 1 from column A to column D
Move disc 2 from column A to column C
Move disc 3 from column A to column B
Move disc 2 from column C to column B
Move disc 1 from column D to column B
Move disc 4 from column A to column C
Move disc 5 from column A to column D
Move disc 4 from column C to column D
Move disc 1 from column B to column A
Move disc 2 from column B to column C
Move disc 3 from column B to column D
Move disc 2 from column C to column D
Move disc 1 from column A to column D

We can check that the number of moves adds up to what we expect. In this case there are 13 moves required as per the output of the code. Now, we will check this is what is expected using the Φ function.

$$\begin{aligned}\Phi(5) &= 2^{\nabla 0} + 2^{\nabla 1} + 2^{\nabla 2} + 2^{\nabla 3} + 2^{\nabla 4},\\ &= 1 + 2 + 2 + 4 + 4 = 13.\end{aligned}$$

Therefore, the number of moves that the code produced is the number of moves we expect to see in an optimal solution of The Reve's Puzzle. A code like this then allows us to not only see the minimum number of moves required to complete the puzzle, but also the actual moves that can be taken to complete the puzzle in the most efficient way. In order to check other values of n, we can simply adjust this in the line of the code which currently has $n = 5$ to be a different number.