HW2

Wednesday, April 17, 2024 11:45 PM

1.a) $\mathbb{C}[x] = \mathbb{C}[x]/(0) = \mathbb{C}[x]/\mathbb{I}(\mathbb{A}^1) = \mathbb{C}[\mathbb{A}^1]$, And we know $\operatorname{Max Spec} \mathbb{C}[V]$ has a 1-to-1 correspondence with V for any closed affine algebraic variety V, by

$$\operatorname{MaxSpec} \mathbb{C}[V] \longleftrightarrow V.$$
$$m_{\vec{p}} \longleftrightarrow \vec{p} \in \mathbb{A}^n$$

Where $m_{\vec{p}}$ is the ideal $(x_1 - p_1, x_2 - p_2, \dots x_n - p_n)$ with $\mathbb{C}[V] = \mathbb{C}[x_1, \dots x_n]/\mathbb{I}(V)$ and $\vec{p} = (p_1, p_2 \dots p_n)$.

So for $V = \mathbb{A}^1$, $\operatorname{Max Spec} \mathbb{C}[V] = \operatorname{Max Spec} \mathbb{C}[x] = \{ (x-p) \mid p \in \mathbb{C} \}$. ideal generated by x-p.

For $\mathbb{C}[x, 1/x] = \mathbb{C}[\mathbb{V}(xy-1)]$

So $\operatorname{Max spec} \mathbb{C}[x, 1/x] \longleftrightarrow \mathbb{V}(xy-1)$ is 1-to-1 correspondence.

So $\operatorname{Max Spec} \mathbb{C}[x, 1/x] = \{ (x - a_0, 1/x - a_1) : a_0 a_1 - 1 = 0, a_0, a_1 \in \mathbb{C} \}$.

b) The pull back is sending x to $x \circ \varphi$ and $x \circ \varphi(y) = x \circ 1/y = 1/y$

(As a function: $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$)

sending $1/x$ to $1/x \circ \varphi$ and $1/x \circ \varphi(y) = 1/x \circ 1/y = \frac{1}{(1/y)} = y$

So that determines the $\mathbb{C}$-algebra homomorphism.

For example
$$\varphi^*\left(2x^2 + \frac{2x^3+4x}{x^5} \right) = \varphi^*\left(\frac{2x^7 + 2x^3 + 4x}{x^5} \right)$$
$$= \varphi^*(1/x^5)\, \varphi^*(2x^7 + 2x^3 + 4x)$$
$$= [\varphi^*(1/x)]^5 [2\varphi^*(x)^7 + 2\varphi^*(x)^3 + 4\varphi^*(x)]$$
$$= y^5 [2(1/y)^7 + 2(1/y)^3 + 4\, 1/y]$$
$$= \frac{2}{y^2} + 2y^2 + 4y^4$$

Similarly $\varphi^*(2-x) = 2\varphi^*(1) - \varphi^*(x) = 2 - 1/y$.

(φ^* is a $\mathbb{C}$-algebra homomorphism, so it always send 1 to 1.)

2. a) let $\varphi: V \to \mathbb{A}^2 \quad (x, y, u) \xmapsto{\varphi} (x, u)$

Define $\psi: \mathbb{A}^2 \to V \quad (w, z) \xmapsto{\psi} (w, wz, z)$

It is clear that ψ is well defined and is a morphism of varieties, (as $w \cdot z - (wz) = 0$, so $\psi(w,z) \in V$) (Only polynomials inside the brackets)

And one can check $\psi \circ \varphi = id_V$ and $\varphi \circ \psi = id_{\mathbb{A}^2}$

$$(x, y, u) \xmapsto{\varphi} (x, u) \xmapsto{\psi} (x, xu, u) = (x, y, u) \quad (xu = y \text{ since } (x,y,u) \in V)$$

$$(w, z) \xmapsto{\psi} (w, wz, z) \xmapsto{\varphi} (w, z)$$

So φ is an isomorphism as required.

b) let $\tilde{\varphi}: V \to \mathbb{A}^2 \quad (x, y, u) \mapsto (x, y)$

Suppose by contradiction there is morphism $\chi: \mathbb{A}^2 \to V$ such that

$(w, z) \mapsto (\gamma_1(w,z), \gamma_2(w,z), \gamma_3(w,z))$, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}[t_1, t_2]$

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$(w, z) \mapsto (\psi_1(w, z), \psi_2(w, z), \psi_3(w, z))$, $\psi_1, \psi_2, \psi_3 \in \mathbb{C}[t_1, t_2]$
such that $\psi\tilde{\varphi} = id_V$ and $\tilde{\varphi}\psi = id_{\mathbb{A}^2}$.
Since $\psi\tilde{\varphi} = id_V$ $(\psi_1(x, y), \psi_2(x, y), \psi_3(x, y)) = (x, y, u)$
for all $x, y \in \mathbb{C}$,
This implies $\psi_1(x, y) = x$, $\psi_2(x, y) = y$ (as polynomials).
Now since $(x, y, u) \in V$, $x \cdot \psi_3(x, y) = y$ for all $x, y \in \mathbb{C}$
But this can't hold for $x = 0$, $y = 1$ Contradiction.
So our assumption that there is an isomorphism ψ must be wrong.

c) Given $(x, y, u) \in \mathbb{A}^3$, note that there is an isomorphism
$V \cong \mathbb{A}^2$, so there are isomorphisms
$V \setminus \mathbb{V}(u) \cong \mathbb{A}^2 \setminus \mathbb{V}(u) \cong \mathbb{V}(zu - 1) \subseteq \mathbb{A}^3$:
$(x, y, u) \mapsto (x, u) \mapsto (x, u, 1/u)$
$(x, xu, u) \leftarrow (x, u) \leftarrow (x, u, z)$
$$\therefore \mathbb{C}[V \setminus \mathbb{V}(u)] \cong \mathbb{C}[\mathbb{V}(zu-1)] = \frac{\mathbb{C}[x, u, z]}{(zu-1)} \cong \mathbb{C}[x, u, 1/u].$$

Since $\mathcal{O}_V(D(u))$ is the set of regular function on $V \cap D(u)$
$$\text{so } \mathcal{O}_{V \setminus \mathbb{V}(u)}(V \setminus \mathbb{V}(u)) = \mathbb{C}[V \setminus \mathbb{V}(u)] \cong \mathbb{C}[x, u, 1/u].$$

Q.3. Without loss of generality, suppose $V \subseteq \mathbb{A}^n$ is identified with $\varphi(V)$
where $\varphi : \mathbb{A}^n \to \mathbb{P}^n$, $\varphi(x_1, \dots x_n) = [1 : x_1 : x_2 \dots : x_n]$.
Note that φ is a homeomorphism to its image $\{[x_0 : x_1 \dots : x_n] \in \mathbb{P}^n \mid x_0 \neq 0\}$.
Suppose the contrary that $\overline{V} = W_1 \cup W_2$, where W_1, W_2 are closed sets in $\mathbb{P}^n$. Then consider $\varphi^{-1}(W_1)$, since W_1 is closed, φ is continuous, so $\varphi^{-1}(W_1)$ is closed in $\mathbb{A}^n$. Similarly $\varphi^{-1}(W_2)$ are closed,
but since $\varphi^{-1}(W_1) \cup \varphi^{-1}(W_2) = \varphi^{-1}(W_1 \cup W_2) = \varphi^{-1}(\overline{V}) = V$.
and V is irreducible (in $\mathbb{A}^n$), $\varphi^{-1}(W_1) = V$ or $\varphi^{-1}(W_2) = V$.
Say (without loss of generality) $\varphi^{-1}(W_1) = \overline{V}$, so $\varphi(V) \subseteq W_1$,
since W_1 is closed by assumption, $W_1 = \overline{V}$, so similarly if $\varphi^{-1}(W_2) = V$,
$W_2 = \overline{V}$, so $\overline{V}$ is irreducible as required.

Q4. We first compute the closure of $\mathbb{V}(y - \sin x)$ in $\mathbb{A}^2$, then compute the projective closure. <u>Claim</u>. Closure of $\mathbb{V}(y - \sin x)$ in $\mathbb{A}^2$ is the whole $\mathbb{A}^2$.
<u>Proof</u>. Suppose $\mathbb{V}(y - \sin x) \subseteq \mathbb{V}(f)$ for some polynomial $f \in \mathbb{C}[x, y]$.
Now we use the fact that $\sin$ is surjective on $\mathbb{C}$ with period 2π,
So for any $\beta \in \mathbb{C}$ there exist $\alpha \in \mathbb{C}$ such that $\beta = \sin(\alpha + 2\pi n)$ for all $n \in \mathbb{Z}$.

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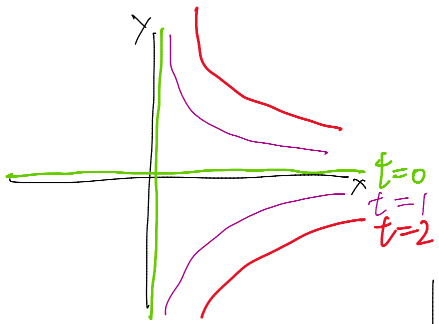
So for any $\beta \in \mathbb{C}$ there exist $\alpha \in \mathbb{C}$ such that $\beta = \sin(\alpha + 2\pi n)$ for all $n \in \mathbb{Z}$.
So $f(\alpha + 2\pi n, \beta) = 0$ for all $n \in \mathbb{Z}$. Now if I write.

$$f(x, y) = g_n(y)x^n + g_{n-1}(y)x^{n-1} + \cdots \quad g_0(y) \quad \text{for } g_n, \ldots g_0 \in \mathbb{C}[y].$$

Define $F(x) := f(x, \beta) = g_n(\beta)x^n + g_{n-1}(\beta)x^{n-1} + \cdots \quad g_0(x) \quad \in \mathbb{C}[x]$
and $F(\alpha + 2\pi n) = f(\alpha + 2\pi n, \beta) = 0$ for all $n \in \mathbb{Z}$, so $F \equiv 0$
(since a single-variable non-constant polynomial had only finitely many roots.)

So $g_n(\beta), g_{n-1}(\beta) \ldots g_0(\beta) = 0$ for all β by comparing coefficient.
Hence $g_n, g_{n-1} \ldots g_0 \equiv 0$ so $f(x, y) \equiv 0$. So $\mathbb{V}(f) = \mathbb{V}(0) = \mathbb{A}^2$ ※
So the projective closure is $\overline{\mathbb{A}^2} = \mathbb{P}^2$.
This does not contradict the Chow lemma, as $\mathbb{V}(y - \sin x)$ is an analytic <u>affine</u> subvariety of $\mathbb{A}^2$, but not an analytic projective subvariety of $\mathbb{P}^2$.

5.



Also, since V_t is a hyperplane, (for all $t \in \mathbb{C}$), it has dimension $\dim(\mathbb{A}^2) - 1 = 2 - 1 = 1$.

To compute smoothness we fix $\vec{p} \in V_t$.

$$T_p V_t := \left\{ \vec{v} \in \mathbb{A}^n \;\middle|\; \left(\frac{\partial(xy^2 - t)}{\partial x}, \frac{\partial(xy^2 - t)}{\partial y} \right)\Big|_{\vec{p}} \vec{v} = 0 \right\}$$

$$:= \left\{ \vec{v} \in \mathbb{A}^n \;\middle|\; (y^2, 2y)\big|_{\vec{p}} \, \vec{v} = 0 \right\}$$

If $\vec{p} = (x_0, y_0) \in V_t$,

$$:= \{ \vec{v} \in \mathbb{A}^n \mid (y_0^2, 2y_0)\, \vec{v} = 0 \}$$

Now, as long as the vector $(y_0^2, 2y_0)$ is non-zero, this is a rank 1 matrix hence $\dim T_p V_t = 2 - 1 = 1$ in this case.
Else, when $(y_0^2, 2y_0) = 0$, $\dim T_p V_t = 2$.

Now notice $(y_0, 2y_0)$ is the zero vector if and only if $y_0 = 0$
and $(x_0, 0) \in V_t$ if and only if $t = 0$.
So indeed V_0 is not smooth (at $\vec{p} = (0, 0)$), while V_1, V_2 are smooth.

V_0 is <u>NOT</u> irreducible since $xy^2 = 0 \iff x = 0$ or $y = 0$
so $V_0 = \mathbb{V}(x) \cup \mathbb{V}(y)$ and $\mathbb{V}(x) \subsetneq V_0$, $\mathbb{V}(y) \subsetneq V_0$.
We claim that the ideal $(xy^2 - t)$ when $t = 1, 2$ is prime, (then it is radical), so $\mathbb{I}(V_t) = \mathbb{I}(\mathbb{V}(xy^2 - t)) = (xy^2 - t)$ (by Nullstellensatz) is prime, hence V_t is irreducible.
To show the claim, since $\mathbb{C}[x_1, x_2]$ is a UFD. It suffice to check

that xy^2-t is irreducible. Suppost $xy^2-t = F(x,y)\,G(x,y)$, since x has degree 1, one of the polynomial (say G), must solely depend on y.

$$xy^2 - t = (G_1(y)\,x + G_2(y)) \cdot G(y)$$
$$= G_1(y)G(y)\,x + G_2(y)\,G(y)$$

By comparing coefficient and the fact that $t \neq 0$, both polynomial G_2, G must be non-zero constant functions.

So $xy^2 - t = F(x,y)\,G(x,y) \Rightarrow G$ must be constant

i.e. xy^2-t is irreducible for $t \neq 0$, hence V_1, V_2 are irreducible as required.