UNIVERSITY OF BRISTOL

School of Mathematics

Algebraic Geometry
MATHM0036
(Paper code MATHM0036)

April/May 2025   2 hour(s) 30 minutes

The exam contains FOUR questions
All Four answers will be used for assessment.

Calculators of an approved type (permissible for A-Level examinations) are permitted.

Candidates may bring ONE hand-written sheet of A4 notes, written double-sided into the examination. Candidates must insert this sheet into their answer booklet(s) for collection at the end of the examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

Q1. (a) (**5 marks**) Show that any polynomial $f \in \mathbb{C}[x, y, z]$ can be expressed as

$$f = r_1(x^2 - y) + r_2(x^3 - z) + g,$$

for $r_1, r_2 \in \mathbb{C}[x, y, z]$ and $g \in \mathbb{C}[x]$.

(b) (**5 marks**) Define the *twisted cubic* $V = \mathbb{V}(x^2 - z, x^3 - y)$, and consider the parametrisation:

$$\begin{aligned} \varphi : \mathbb{A}^1 &\to \mathbb{A}^3, \\ t &\mapsto (t, t^2, t^3). \end{aligned}$$

Prove that the pullback map

$$\varphi^* : \mathbb{C}[x, y, z] \to \mathbb{C}[t]$$

induces an isomorphism of $\mathbb{C}$-algebras $\mathbb{C}[V] \simeq \mathbb{C}[t]$.

(c) (**5 marks**) Explain why the result from part (b) implies that V is irreducible.

(d) (**5 marks**) We know that the closure of V in $\mathbb{P}^3$, is given by $\overline{V} = \Phi(\mathbb{P}^1)$ where

$$\begin{aligned} \Phi : \mathbb{P}^1 &\to \mathbb{P}^3 \\ [t : s] &\mapsto [s^3 : ts^2 : t^2s : t^3]. \end{aligned}$$

Prove that $\overline{V} = \mathbb{V}(xz - y^2, yw - z^2, xw - yz) \subseteq \mathbb{P}^3$.

(e) (**5 marks**) Explain why the irreducibility of V implies that $\overline{V}$ is also irreducible.

Q2. (a) (**15 marks**) Recall the following definition:

Let X, Y be two algebraic varieties (*i.e.,* affine, quasi-affine, projective or quasi-projective). A morphism $\varphi : X \longrightarrow Y$, is a map such that

- φ is continuous;
- For any for every open set $V \subseteq Y$, and for every regular function $f \in \mathcal{O}_Y(V)$, $\varphi^*(f) = f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$.

Prove the following theorem:

Let X be an algebraic variety, $Y \subseteq \mathbb{A}^n$ a closed affine algebraic variety, and $\varphi : X \longrightarrow Y$ a map of sets. Then, $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a morphism, if and only if, for all i, coordinate function $\varphi_i \in \mathcal{O}_X(X)$.

(b) (**10 marks**) Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be two closed affine algebraic varieties and

$$\varphi : V \longrightarrow W$$

a morphism. Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W.

Q3. Let $V = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2$.

(a) (**5 marks**) Sketch $V \cap \mathbb{R}^2$ in $\mathbb{R}^2$.

(b) (**5 marks**) Find all the singular point of V.

(c) (**10 marks**) Identify the irreducible components of $\mathbb{V}(y^2 - x^3, xz - y) \subseteq \mathbb{A}^3$.

(d) (**5 marks**) Show that $\mathbb{V}(xz - y) \subseteq \mathbb{A}^3$ is isomorphic to $\mathbb{A}^2$.

Q4. Consider the cone $\sigma = \text{cone}(\{e_1, e_1 + 3e_2\}) \subseteq \mathbb{R}^2$.

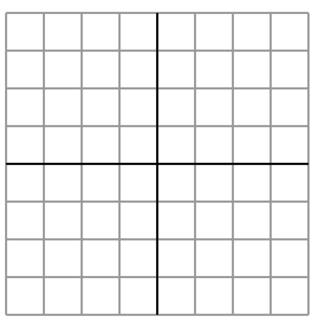
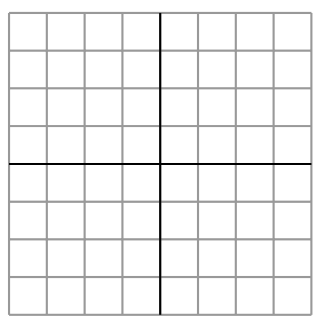
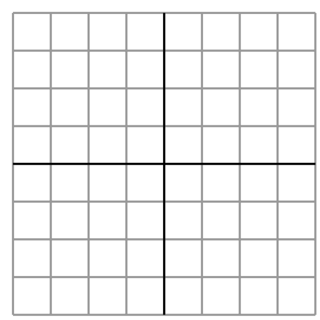
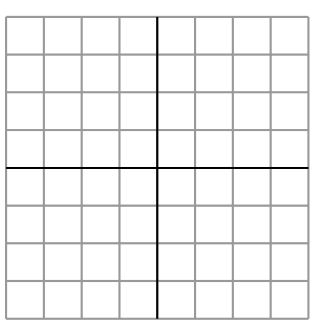
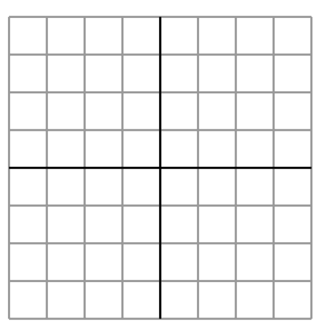
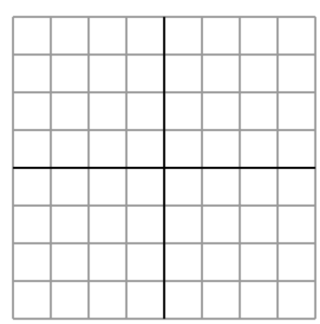
(a) (**5 marks**) Explain why the affine toric variety X_σ is not smooth. Subdivide σ into a union of smooth two-dimensional cones.

(b) (**10 marks**) Select two of the two-dimensional cones from your subdivision and denote them by σ_1 and σ_2. Let $\tau = \sigma_1 \cap \sigma_2$. Describe the the toric varieties X_{σ_1}, X_{σ_2}, and X_τ and their coordinate rings.

(c) (**5 marks**) Justify why we have the inclusions

$$\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau], \quad \mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau].$$

(d) (**5 marks**) Explain why X_{σ_1} and X_{σ_2} contain X_τ as an open set and describe the glueing of X_{σ_1} and X_{σ_2} along X_τ.

End of examination.