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RESEARCH ARTICLE

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Structure of hyperbolic polynomial automorphisms of $\mathbb{C}^2$ with disconnected Julia sets

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Abstract

For a hyperbolic polynomial automorphism of $\mathbb{C}^2$ with a disconnected Julia set, and under a mild dissipativity condition, we give a topological description of the components of the Julia set. Namely, there are finitely many "quasi-solenoids" that govern the asymptotic behavior of the orbits of all nontrivial components. This can be viewed as a refined spectral decomposition for a hyperbolic map, as well as a two-dimensional version of the (generalized) Branner–Hubbard theory in one-dimensional polynomial dynamics. An important geometric ingredient of the theory is a John-like property of the Julia set in the unstable leaves.

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1 | INTRODUCTION

1.1 | Preamble on hyperbolic dynamics

The classical *spectral decomposition* of a hyperbolic (Axiom A) real diffeomorphism f of a compact manifold (developed by Smale, Anosov, Sinai, Bowen, and others) provides us with a rather complete topological picture of its dynamics. Namely, the nonwandering set $\Omega(f)$ is decomposed into finitely many *basic sets*, each of which modeled on an irreducible Markov chain. Among these basic sets, there are several *attractors* that govern the asymptotic behavior of generic points of the manifold. This picture has become an archetype for numerous other settings, including one-dimensional, noninvertible, holomorphic, partially, or nonuniformly hyperbolic dynamical systems.

In the context of complex polynomial automorphisms of $\mathbb{C}^2$, hyperbolic maps arise naturally as perturbations of one-dimensional hyperbolic polynomials. They were first studied in the late 1980s by Hubbard and Oberste-Vorth [24, 25] who showed that their topological structure can be fully described in terms of the original one-dimensional maps, whose Julia set and attracting cycles get perturbed to the basic sets of f (see also Fornæss–Sibony [19]).

Computer experiments indicate that, though hyperbolicity is not a prevalent phenomenon in dimension two, there should still exist plenty of nonperturbative examples. The first such candidate (a quadratic Hénon map with two coexisting attracting cycles) was proposed by Hubbard; it was further investigated by Oliva in his thesis [38]. However, it is a challenging problem, which requires computer assistance, to prove the hyperbolicity of a particular example, and this one still remains unconfirmed. Some time later, Ishii justified the hyperbolicity of several other nonperturbative Hénon maps: see [26–28] (of course, along with each such example comes an open set of hyperbolic parameters).

A systematic theory of hyperbolic polynomial automorphisms of $\mathbb{C}^2$ was launched by Bedford and Smillie in the early 1990s, relying notably on methods from pluripotential theory. In particular, they showed in [3] that any such a map has only one nontrivial basic set, its Julia set $J(f)$, while all others are just attracting cycles. Further combinatorial study of hyperbolic Hénon maps was carried out by Ishii and Smillie [29].

In this paper, we will reveal a finer structure of the Julia set, related to its connected components, which leads to a finer "spectral decomposition." Namely, under mild dissipativity assumptions, we will show that there are finitely many *quasi-solenoids* that govern the asymptotic behavior of all nontrivial components. Some of these quasi-solenoids are *tame* (i.e., lie on the boundary of the basins of some attracting cycles), while others might be *queer* (we do not know whether they actually exist).

Let us conclude this preamble by suggesting a potentially important role that hyperbolic maps may play in the Hénon story. They are not only interesting simple models for the general nonuniformly hyperbolic situation, but they may also be seen as "germs" for a renormalization theory that would lead to self-similarity features of the parameter spaces. In this respect, renormalizing hyperbolic Hénon maps around quasi-solenoids would be the beginning of this story.

1.2 | One-dimensional prototype

Understanding the topological structure of the Julia set is one of the most basic problems in holomorphic dynamics. For polynomials in one variable, Fatou and Julia proved that the connectivity properties of the Julia set are dictated by the dynamical behavior of critical points. When the critical points do not escape, the Julia set J is connected; on the contrary, if all critical points do escape, J is a Cantor set. If J is connected and locally connected, the theory of external rays of Douady and Hubbard [13] and the theory of geodesic laminations of Thurston [43] give a topological model for the Julia set as the quotient of the circle by an equivalence relation that records the landing pattern of external rays. When the Julia set of a polynomial is disconnected, it admits uncountably many components, and one challenge is to characterize when a component is nontrivial (i.e., not a point) in terms of the induced dynamics on the set of components. It turns out that this happens when and only when this component is preperiodic to a component containing a critical point: this is due to Branner and Hubbard [8] for cubic polynomials, and Qiu and Yin [40] in the general case (based upon the Kahn–Lyubich machinery [30, 31]). Then, one may describe nontrivial periodic components by realizing them as Julia sets of connected polynomial-like maps and using the Douady and Hubbard Straightening Theorem [14].

In the hyperbolic case, the above theory is much easier and had belonged to folklore of the field.

Theorem 1.1. *Let p be a hyperbolic polynomial in $\mathbb{C}$, with a disconnected Julia set. Then, the filled Julia set K has uncountably many components, and only countably many of them are nontrivial. Any nontrivial component is preperiodic, and there are finitely many periodic components, each of which containing an attracting periodic point.*

Note that this is really a statement about polynomials: there are examples of hyperbolic rational maps on $\mathbb{P}^1$ whose Julia sets are Cantor sets of circles [37].

1.3 | Main result

In this article, we address similar issues in the setting of polynomial automorphisms of $\mathbb{C}^2$. Let f be a polynomial automorphism of $\mathbb{C}^2$ with nontrivial dynamics: by this, we mean, for instance, that the algebraic degree of the iterates f^n tends to infinity (see below §2.1 for more details on this). Its Julia set $J = J_f$ is the set of points at which both $(f^n)_{n\geqslant 0}$ and $(f^{-n})_{n\geqslant 0}$ are not locally normal. We also classically denote by K^+ (resp. K^-), the set of points with bounded forward (resp. backward) orbits, $K = K^+ \cap K^-$ and $J^\pm = \partial K^\pm$, so that $J = J^+ \cap J^-$. The complex Jacobian $\operatorname{Jac} f$ is a nonzero constant. Thus, replacing f by f^{-1} if necessary, without loss of generality, we assume from now on that $|\operatorname{Jac} f| \leqslant 1$.

In this context, the connected versus disconnected dichotomy for the Julia set was studied by Bedford and Smillie [5], who proved that the connectedness of J, or equivalently of K, is equivalent to the nonexistence of "unstable critical points," which are defined as tangencies between certain dynamically defined foliations. (Recall that f has no critical point in the usual sense, but these unstable critical points play the same role as escaping critical points in dimension 1.) Bedford and Smillie also showed that when J is connected, there is a well-defined family of external rays along unstable manifolds, as well as a semiconjugacy between the dynamics on $J^- \setminus K^+$ and the "complex solenoid" that is the inverse limit of the dynamical system defined by $z \mapsto z^d$ on $\mathbb{C} \setminus \overline{\mathbb{D}}$ (in which the external rays become radial lines).

To proceed further and try to extend the Douady–Hubbard description of the Julia set in terms of the combinatorics of external rays, given our current state of knowledge, we need to assume that f is uniformly hyperbolic. Recall from [3] that f is said to be *hyperbolic* if J is a hyperbolic set, which must then be of saddle type. In this case, f satisfies Smale's Axiom A in $\mathbb{C}^2$, and the Fatou set is the union of finitely many basins of attraction. (See [27] for an introductory account to this topic, which also discusses some combinatorial/topological models for Julia sets.)

If f is hyperbolic and J is connected, it is shown in [6] that the semiconjugacy to the complex solenoid is, in fact, a conjugacy, and the convergence of unstable external rays ultimately implies that J can be described as a finite quotient of the solenoid at infinity. A nontrivial consequence of the results of [4, 5] and [6] is that in this case, f cannot be conservative, that is, $|\mathrm{Jac}\, f| < 1$ (see [6, Cor. A.3]; recall that we assume $|\mathrm{Jac}\, f| \leqslant 1$ here). An alternate argument for this fact was given by the first-named author in [15], where it is shown that a hyperbolic automorphism f with connected Julia set must possess an attracting periodic point, so, in particular, $|\mathrm{Jac}\, f| < 1$. Surprisingly enough, the existence of an attracting point does not seem to follow easily from the description of J as a quotient of the solenoid.

In this article, we focus on the disconnected case. A motivating question is the following conjecture from [15].

Conjecture 1.2. *Let f be a dissipative and hyperbolic automorphism of $\mathbb{C}^2$, without attracting points. Then, J is a Cantor set.*

Our main result is an essentially complete generalization of Theorem 1.1 in two dimensions, under a mild dissipativity assumption.

Main Theorem. *Let f be a hyperbolic polynomial automorphism of $\mathbb{C}^2$, with a disconnected Julia set, and such that $|\mathrm{Jac}\, f| \leqslant 1/(\deg f)^{1/2}$. Then there are uncountably many components of J, which can be of three (mutually exclusive) types:*

(1) *point;*
(2) *leafwise bounded;*
(3) *or quasi-solenoid.*

Quasi-solenoidal components are periodic and there are only finitely many of them. Any component of type (2) is wandering and converges to a quasi-solenoidal one under forward iteration. The components of K are classified accordingly.

Under an additional assumption (nondivergence of holonomy [NDH]) on the behavior of stable holonomy between components, any quasi-solenoidal component of K contains an attracting periodic point.

Here, $\deg f$ refers to the dynamical degree of f, which is the growth rate of the algebraic degree under iteration (see §2.1). By definition, a component of J is *leafwise bounded* if it is a relatively bounded subset of some unstable manifold; this implies that its topology is that of a full plane continuum, properly embedded in $\mathbb{C}^2$. A *quasi-solenoid* is a connected component with local product structure, which is totally disconnected in the stable direction and locally connected and leafwise unbounded in the unstable direction (see Definition 6.2). Components of type (2) are analogous to strictly preperiodic components in dimension 1; note, however, that by the local product structure of J, there are uncountably many of them. Countability is restored by saturating with semilocal stable manifolds (see Theorem 5.20). The meaning of the (NDH) assumption will be explained below.

1.4 | Outline

Let us discuss some of the main ideas of the proof, which occupies the most part of the paper. First, the assumption on the Jacobian is used to guarantee that *the slices of J (resp. K) by stable manifolds are totally disconnected*. It is reminiscent of the stronger *substantial dissipativity* assumption $|\mathrm{Jac}\, f| < 1/(\deg f)^2$ used in [17, 35, 36]. We could indeed use substantial dissipativity and Wiman's theorem in the style of these papers to achieve stable total disconnectivity. However, hyperbolicity allows for a Hausdorff dimension calculation that gives a better bound on the Jacobian (see Section 4).

The key step of the finiteness property in the main theorem is an analysis of the geometry of the unstable slices of J and K. Using external rays, we first show in Section 3 that the complement of K along unstable manifolds satisfies a weak version of the *John property*. This property implies that the components of $K \cap W^u$ are locally connected, and that locally there are only finitely many components of diameter bounded from below.

This finiteness is used to get a classification of *semilocal components* of J^+ and K^+. By this, we mean that we fix a large bidisk $\mathbb{B}$ (in adapted coordinates) in which J^+ and K^+ are vertical-like objects, and we look at components of $J^+ \cap \mathbb{B}$ (resp. $K^+ \cap \mathbb{B}$). We prove that *these semilocal components behave like components of J (resp. K) for one-dimensional polynomials*: only countably many of them are nontrivial, that is, not reduced to vertical submanifolds, and any nontrivial such component is preperiodic. Besides the finiteness induced by the John-like property, this relies on a key *homogeneity property* of such a semilocal component: either all its unstable slices are "thin," or all of them are "thick." To prove this *thin-thick dichotomy*, we show that if a semilocal component admits a thin unstable slice, then by a careful choice of $\mathbb{B}$, we can arrange that the stable foliation of this semilocal component is transverse to $\partial\mathbb{B}$. It follows that this component has a global product structure in $\mathbb{B}$; hence, it is thin everywhere (see Section 5 for details).

If C is a nontrivial component of J, it is easy to see that the ω-limit set of C must be contained in one of the finitely many thick semilocal components of J^+. We show that it must have local product structure, hence be a quasi-solenoidal component of J. The main step is the following: for large $m \neq n$, by the expansion in the unstable direction, the unstable slices of $f^m(C)$ and $f^n(C)$ have a diameter bounded from below, so if $x_n \in f^n(C)$ is close to $x_m \in f^m(C)$, by the finiteness given by the John-like property, $f^n(C)$ and $f^m(C)$ must correspond one to the other under local stable holonomy. Furthermore, such a quasi-solenoidal component must coincide with the limit set of its semilocal component in J^+, and the finiteness of the number of attractors follows (see Section 6).

To get a complete generalization of the one-dimensional situation, it remains to show that such a quasi-solenoidal component must "enclose" some attracting periodic point. Unfortunately, all our attempts toward this result stumbled over the following issue: if $x, y \in J$ are such that $y \in W^s(x)$, the stable holonomy induces a local homeomorphism $J \cap W^u_{\mathrm{loc}}(x) \to J \cap W^u_{\mathrm{loc}}(y)$. The point is that it might not be the case in general that this local homeomorphism can be continued along paths in $J \cap W^u(x)$, even when $J \cap W^u(x)$ is a relatively bounded subset of $W^u(x)$. (Compare with the Reeb phenomenon for foliations, illustrated in Figure 6.) This is a well-known difficulty in hyperbolic dynamics, which was encountered, for instance, in the classification of Anosov diffeomorphisms (see §8.1 for a short discussion). If this continuation property holds — this is the NDH property referred to in the main theorem — then we can indeed conclude that nontrivial periodic components of K contain attracting orbits (see Section 8, in particular, Theorem 8.5). This yields, in particular, a conditional proof of Conjecture 1.2. Let us also note that a simple instance where the NDH property holds is when the stable lamination of J^+ is transverse to $\partial\mathbb{B}$ (for some choice of $\mathbb{B}$), a property that can be checked in practice on specific examples (see Example 8.3 and the discussion following it).

In the course of the paper, we also establish a number of complementary facts, which do not enter into the proof of the main theorem: the existence of an external ray landing at every point of J (see Theorem 3.4); the structure of attracting basins (see § 7.2); a simple topological model for the dynamics on Julia components (see § 7.3); the topological transitivity of quasi-solenoids (see Theorem 8.8). In Appendix A, we sketch the construction of the *core* of a quasi-solenoidal component, which aims at describing its topological structure.

2 | PRELIMINARIES AND NOTATION

2.1 | Vocabulary of complex Hénon maps

If $\mathbb{B} = D \times D$ is a bidisk, we denote by $\partial^v B = \partial D \times D$ (resp. $\partial^h B = D \times \partial D$) the vertical (resp. horizontal) boundary. An object in $\mathbb{B}$ is horizontal if it intersects $\partial\mathbb{B}$ only in $\partial^v\mathbb{B}$, and likewise for vertical objects. A closed horizontal submanifold is a branched cover of finite degree over the first projection.

Let us collect some standard facts and notation (see [2, 3, 19, 21]). If f is a polynomial diffeomorphism of $\mathbb{C}^2$ with nontrivial dynamics, then by making a polynomial change of coordinates, we may assume that f is a composition of complex Hénon mappings $(z, w) \mapsto (p_i(z) + a_i w, a_i z)$. In particular, $\deg(f^n) = (\deg f)^n$ for every $n \geqslant 0$. We fix such coordinates from now on. As it is customary in this area of research, we will often abuse terminology and simply refer to f as a *complex Hénon map*. The degree of f is $d = \prod \deg(p_i) \geqslant 2$ and the relation $\deg(f^n) = d^n$ holds so that d coincides with the so-called *dynamical degree* of f.

In these adapted coordinates, there exists $R > 0$ such that for the bidisk $\mathbb{B} := D(0, R)^2$, we have that $f(\mathbb{B}) \cap \mathbb{B}$ (resp. $f^{-1}(\mathbb{B}) \cap \mathbb{B}$) is horizontally (resp. vertically) contained in $\mathbb{B}$ and the points of $\partial^v(\mathbb{B})$ (resp. $\partial^h(\mathbb{B})$ escape under forward (resp. backward) iteration.

- $K^\pm$ is the set of points with bounded forward orbits under $f^{\pm 1}$ and $K = K^+ \cap K^-$. Note that K^+ is vertical in $\mathbb{B}$ and $f(\mathbb{B} \cap K^+) \subset K^+$. Similarly, K^- is horizontal and $f^{-1}(\mathbb{B} \cap K^-) \subset K^-$.
- $J^\pm = \partial K^\pm$ are the forward and backward Julia sets. If f is dissipative, then $K^- = J^-$.
- $J = J^+ \cap J^-$ is the Julia set.

Following [5], we say that f is *unstably disconnected* if for some (and hence any) saddle periodic point p, $W^u(p) \cap K^+$ admits a compact component (relative to the topology induced by the biholomorphism $W^u(p) \simeq \mathbb{C}$), and unstably connected otherwise. If f is unstably disconnected, then it admits an *unstable transversal* Δ^u, that is, a relatively compact domain in $W^u(p)$ that is a horizontal submanifold in $\mathbb{B}$: indeed pick a bounded Jordan domain $U \subset W^u(p)$ containing a compact component of $W^u(p) \cap K^+$ such that $\partial U \cap K^+ = \emptyset$ and iterate it forward.

2.2 | Hyperbolicity and local product structure

Throughout the paper, we assume that f is hyperbolic on J (hence Axiom A on $\mathbb{C}^2$ by [3]), with hyperbolic splitting $T\mathbb{C}^2|_J = E^u \oplus E^s$. Then, there exists a continuous Riemannian metric $|\cdot|$ on J

and constants $s < 1 < u$ such that for any $x \in J$, and any $v \in E^u(x) \setminus \{0\}$, $|Df_x \cdot v| \geqslant u\,|v|$ (resp. for any $v \in E^s(x)$, $|Df_x \cdot v| \leqslant s\,|v|$). By [16], it is enough to assume that f is hyperbolic on $J^\star$, where $J^\star$ is the closure of saddle periodic points (and a posteriori one deduces that $J = J^\star$ by [3]).

In this situation, the local stable and unstable manifolds of points of J have local uniform geometry: there exists a uniform $r > 0$ such that for every $x \in J$, $W^u(x)$ (resp. $W^s(x)$ *is of size* r at x, in the sense that it contains a graph of slope at most 1 over a disk of radius r in $E^u(x)$ (resp. $E^s(x)$). The reader is referred to [1, 7] for a detailed study of this notion. We denote by $W^{s/u}_\delta(x)$ the local stable/unstable manifold of radius δ at x, which is by definition the component of $W^{s/u}(x)$ in $B(x,\delta)$. When the precise size does not matter, we simply denote them by $W^{s/u}_{\mathrm{loc}}$. Slightly reducing the expansion constant u if necessary, given two points z, z' in some local unstable manifold $W^u_\delta(x)$, there is a uniform constant C such that $d(f^{-n}(z), f^{-n}(z')) \leqslant Cu^{-n}$, for all $n \geqslant 0$.

There exists $\delta > 0$ and a neighborhood $\mathcal{N}$ of J such that the restriction to $\mathcal{N}$ of the family local stable/unstable manifolds of radius δ is a lamination, denoted by $\mathcal{W}^{u/s}$. The Julia set has local product structure, so there is a covering by topological bidisks Q (flow boxes) such that the laminations $\mathcal{W}^{u/s}$ are trivial in Q and

$$J \cap Q \simeq (W^s_Q(x) \cap J) \times (W^u_Q(x) \cap J) = (W^s_Q(x) \cap J^-) \times (W^u_Q(x) \cap J^+).$$

It is shown in [3] that the family of global stable and unstable manifolds of points of J also has a lamination structure, which will be denoted by $\mathcal{W}^{s/u}$. More precisely, in the dissipative case, $\mathcal{W}^s$ is a lamination of J^+ is laminated by stable manifolds and the other hand, $\mathcal{W}^u$ is a lamination of $J^- \setminus \{a_1, \dots, a_N\}$, where $\{a_1, \dots, a_N\}$ is the finite set of attracting periodic points of f. No unstable leaf extends across an attracting point, even as a singular analytic set: indeed an unstable leaf is biholomorphic to $\mathbb{C}$, therefore such an extension would yield a submanifold of $\mathbb{C}^2$ biholomorphic to a (possibly singular) copy of $\mathbb{P}^1$, which is impossible.

Under additional dissipativity assumptions, it was shown in [36] that the stable lamination $\mathcal{W}^s$ in $\mathbb{B}$ can be extended to a C^1 foliation in some neighborhood of J^+: see Lemma 5.7 below.

Let us conclude this paragraph with a useful elementary result.

Lemma 2.1. *If f is hyperbolic, every holomorphic disk contained in K^+ is either contained in the Fatou set or in the stable manifold of a point of J.*

Proof. Indeed, if Δ is a disk contained in K^+, then Δ is a Fatou disk, that is, $(f^n|_\Delta)_{n\geqslant 0}$ is a normal family. Now there are two possibilities: either Δ is contained in $\mathrm{Int}(K^+)$ hence in the Fatou set, or it intersects J^+. In the latter case, either Δ is contained in a stable leaf or by [2, Lem. 6.4], Δ must have a transversal intersection with some unstable manifold, so by the inclination lemma, it is not a Fatou disk, which is a contradiction. □

2.3 | Affine structure

Global stable and unstable manifolds are uniformized by $\mathbb{C}$, so they admit a natural affine structure. Since any automorphism of $\mathbb{C}$ is affine, f acts affinely on leaves. In particular, there is a well-defined notion of a round disk, which is f-invariant. Likewise, the Euclidean distance is well defined in the leaves, up to a multiplicative constant.

For any $x \in J$, we choose a uniformization $\psi_x^u : \mathbb{C} \xrightarrow{\sim} W^u(x)$ such that $\psi_x^u(0) = x$ and $|(\psi_x^u)'(0)| = 1$

Lemma 2.2. *The family of uniformizations $(\psi_x^u)_{x\in J}$ is continuous up to rotations, that is, if $x_n \to x$, then $(\psi_{x_n}^u)$ is a normal family and its cluster values are of the form $\psi_x^u(e^{i\theta}\cdot)$.*

Proof. The result follows from the continuity of the affine structure on the unstable leaves (see Theorem B.1). □

It is unclear whether the assignment $J \ni x \mapsto \psi_x^u$ can be chosen to be continuous, that is, if a consistent choice of rotation factor $e^{i\theta}$ can be made. This can be done locally but there might be topological obstructions to extend the continuity to J. Notice that the (ψ_x^u) provide a normalization for the leafwise Euclidean distance. The normalized Euclidean distance on $W^u(x)$ will be denoted by d_x^u.. If $C \subset W^u(x)$, its diameter with respect to d_x^u will be denoted by Diam_x. By Lemma 2.2, d_x^u varies continuously with x. For $R > 0$, we let $D^u(x, R) := \psi_x^u(D(0, R))$.

By construction, f is a uniformly expanding linear map in these affine coordinates, that is, $f \circ \psi_x^u = \psi_{f(x)}^u(\lambda_x^u \cdot)$, with $|\lambda_x^u| = \left\| df|_{E_x^u} \right\|$. By hyperbolicity, there is a positive constant C such that for every $x \in J$,

$$\left| \prod_{i=0}^{n-1} \lambda_{f^i(x)}^u \right| \geqslant C u^n, \tag{1}$$

where $u > 1$ was defined in §2.2.

By the Koebe Distortion Theorem, there exists a uniform $r > 0$ such that the $D^u(x, r)$ are contained in the flow boxes (see, e.g., [7, Lemma 3.7]). By the local bounded geometry of the leaves, the distance induced by the affine structure on the $D^u(x, r)$ is equivalent to that induced by the ambient Hermitian structure. Then, iterating finitely many times, we can promote this result on the $D^u(x, R)$ for every given $R > 0$.

All the above discussion holds for stable manifolds, with superscripts u replaced by s.

2.4 | Connected and semilocal components

For every $x \in J$ (or more generally $x \in K^+ \cap \mathbb{B}$), we denote by $K_{\mathbb{B}}^+(x)$ the connected component of $K^+ \cap \mathbb{B}$ containing x, which is a vertical subset of $\mathbb{B}$. It follows from the Hénon-like property that $f(K_{\mathbb{B}}^+(x)) \subset K_{\mathbb{B}}^+(f(x))$, thus f induces a (noninvertible) dynamical system on the set of connected components of $K^+ \cap \mathbb{B}$. The same discussion applies to components of $J^+ \cap \mathbb{B}$. More generally, for any closed connected subset $C \subset J$ (resp. $C \subset K$), we define $J_{\mathbb{B}}^+(C)$ (resp. $K_{\mathbb{B}}^+(C)$) to be the connected component of $J^+ \cap \mathbb{B}$ (resp. $K^+ \cap \mathbb{B}$) containing C. Of course, for $x \in C$, $J_{\mathbb{B}}^+(x) = J_{\mathbb{B}}^+(C)$ holds. A related concept is $W_{\mathbb{B}}^s(x)$, the component of $\mathbb{B} \cap W^s(x)$ containing x. If we set $W_{\mathbb{B}}^s(C) = \bigcup_{x\in C} W_{\mathbb{B}}^s(x)$, then $W_{\mathbb{B}}^s(C)$ is contained in $K_{\mathbb{B}}^+(C)$ but this inclusion may be strict. This phenomenon may happen when for some $x \in C$, $W_{\mathbb{B}}^s(x)$ is tangent to $\partial\mathbb{B}$ (see Figure 1).

For $x \in K$, we denote by $K^s(x)$ (resp. $K^u(x)$) the connected component of $K \cap W^s(x) = K^- \cap W^s(x)$ (resp. $K \cap W^u(x) = K^+ \cap W^u(x)$) containing x, and also $K(x)$ its connected component in K. For $x \in J$, we define $J^s(x)$, $J^u(x)$ and $J(x)$ similarly. More generally, if needed, we use the notation $\mathrm{Comp}_E(x)$ for the connected component of E containing x.

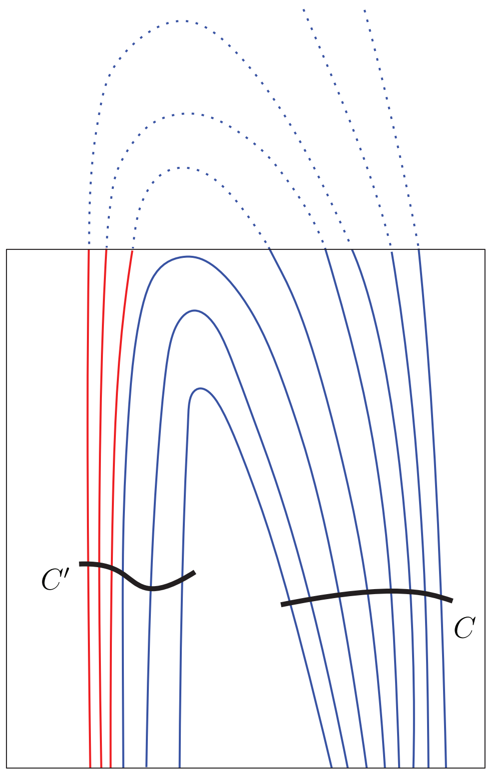

FIGURE 1 The red components on the left belong to $K^+_{\mathbb{B}}(C)$ but not to $W^s_{\mathbb{B}}(C)$.

If L is a complex line in $\mathbb{C}^2$ (or more generally a complex submanifold), we write $\partial_L E$ for the boundary of a set E for the induced topology on L. Likewise, we use the subscript "i" to denote topological operations (interior, closure, etc.) relative to the intrinsic topology in stable/unstable manifolds $W^\bullet$, which is the topology induced by the biholomorphism $W^\bullet \simeq \mathbb{C}$. Notice that since $W^\bullet$ (with $\bullet = u$ or s) is not an embedded submanifold, this does not coincide with the induced topology given by the inclusion $W^\bullet \subset \mathbb{C}^2$.

Lemma 2.3. *Assume that f is hyperbolic. Then, every connected component of $K^+ \cap \mathbb{B}$ has a connected boundary, which is a component of $J^+ \cap \mathbb{B}$.*

Proof. Observe that if p is an interior point of $K^+ \cap L$, where L is a horizontal line, then it belongs to a Fatou disk. Since L is not contained in J^+, by Lemma 2.1, we get that $p \in \mathrm{Int}(K^+)$. This implies that for every $x \in K^+ \cap \mathbb{B}$, $\partial K^+_{\mathbb{B}}(x) \subset \bigcup_{t\in\mathbb{D}} \partial_{L_t}(K^+_{\mathbb{B}}(x) \cap L_t)$, where $L_t = \mathbb{D} \times \{t\}$ and ∂_{L_t} refers to the boundary in L_t. The converse inclusion is obvious, so $\partial K^+_{\mathbb{B}}(x) \cap \mathbb{B} = \bigcup_{t\in\mathbb{D}} \partial_{L_t}(K^+_{\mathbb{B}}(x) \cap L_t)$. Since $K^+_{\mathbb{B}}(x) \cap L_t$ is compact and polynomially convex, and obviously $K^+_{\mathbb{B}}(x) = \bigcup_{t\in\mathbb{D}} K^+_{\mathbb{B}}(x) \cap L_t$, this means that $K^+_{\mathbb{B}}(x)$ is obtained from $\partial K^+_{\mathbb{B}}(x) \cap \mathbb{B}$ by filling the holes of all components of $\partial_L(K^+_{\mathbb{B}}(x) \cap L)$ in every horizontal line. Now assume $\partial K^+_{\mathbb{B}}(x) \cap \mathbb{B}$ is disconnected, so we can write it as $B_1 \cup B_2$, where each B_i is relatively open and $B_1 \cap B_2 = \emptyset$. In every horizontal slice L, $B_i \cap L$ must be a union of components of $\partial_L(K^+_{\mathbb{B}}(x) \cap L)$. For $i = 1, 2$, let $\widehat{B}_i$ be the set obtained by filling the holes of B_i in each horizontal line in $\mathbb{B}$. The previous discussion shows that

$K^+_{\mathbb{B}}(x) = \widehat{B}_1 \cup \widehat{B}_2$, where the $\widehat{B}_i$ are relatively open in $K^+_{\mathbb{B}}(x)$ and disjoint. This is a contradiction; therefore, $\partial K^+_{\mathbb{B}}(x) \cap \mathbb{B}$ is connected.

For the second statement, simply observe that if $D \subset J^+ \cap \mathbb{B}$ is a connected set such that $\partial K^+_{\mathbb{B}}(x) \cap \mathbb{B} \subset D$, then D is contained in $K^+_{\mathbb{B}}(x)$ and also in ∂K^+ so $D \subset \partial K^+_{\mathbb{B}}(x) \cap \mathbb{B}$ and we are done. □

2.5 | Basic properties of leafwise components

Here, we assume that f is a hyperbolic and dissipative complex Hénon map. The following result is well known.

Lemma 2.4. *For every $x \in K$, we have $\mathrm{Int}_{\mathrm{i}}(K^u(x)) \subset \mathrm{Int}(K^+)$ and $\partial_{\mathrm{i}}(K^u(x)) \subset J$. In particular, if $\mathrm{Int}_{\mathrm{i}}(K^u(x))$ is nonempty, each of its components is contained in an attracting basin. Likewise $\mathrm{Int}_{\mathrm{i}}\, K^s(x) = \emptyset$ and $J^s(x) = K^s(x)$.*

Proof. Indeed, since stable and unstable manifolds cannot coincide along some open set, if Δ is a disk contained in $K^u(x)$, it follows from Lemma 2.1 that $\Delta \subset \mathrm{Int}(K^+)$, and the remaining conclusions follow. □

For x in J, $K^u(x)$ may be bounded or unbounded for the intrinsic (leafwise) topology. By the maximum principle, $K^u(x)$ is polynomially convex, so if $K^u(x)$ (or equivalently $J^u(x)$) is leafwise bounded, then $K^u(x)$ is simply the polynomially convex hull of $J^u(x)$ (i.e., is obtained by filling in the leafwise bounded components of the complement).

Lemma 2.5. *Given $x \in K$, in the following properties, we have $(iv) \Leftrightarrow (iii) \Rightarrow (ii) \Leftrightarrow (i)$:*

(i) *$K^u(x)$ is leafwise bounded;*
(ii) *$J^u(x)$ is leafwise bounded;*
(iii) *$W^u_{\mathbb{B}}(x)$ is leafwise bounded;*
(iv) *$W^u_{\mathbb{B}}(x)$ is a closed horizontal submanifold of $\mathbb{B}$.*

Furthermore if (ii) holds, then (iii) holds for $f^n(x)$ for sufficiently large n.

Proof. The implication $(i) \Rightarrow (ii)$ follows directly from the fact that $J^u(x) = \partial_{\mathrm{i}} K^u(x)$. Now assume that $J^u(x)$ is leafwise bounded. Working in $W^u(x) \simeq \mathbb{C}$, we have that $K^u(x)$ is a closed connected polynomially convex set and $J^u(x)$ is a bounded connected component of $\partial_{\mathrm{i}} K^u(x)$. Since every point of $J^u(x)$ lies on the boundary of $W^u(x) \setminus K^+$ (for the intrinsic topology), the compact set obtained by filling the holes of $J^u(x)$ must be $K^u(x)$, so the converse implication holds.

Since $K^u(x) \subset W^u_{\mathbb{B}}(x)$, obviously (iii) implies (i). Conversely, $K^u(x)$ is the decreasing intersection of the sequence of components of x in $W^u(x) \cap f^{-n}(\mathbb{B})$. Hence, if $K^u(x)$ is leafwise bounded, it follows that $\mathrm{Comp}_{W^u(x)\cap f^{-n}(\mathbb{B})}(x)$ is leafwise bounded for large enough n, and so, is $W^u(f^n(x)) \cap \mathbb{B}$.

Recall that for every x, $W^u(x)$ is an injectively immersed copy of $\mathbb{C}$, whose image is a leaf of the lamination of $J^- \setminus \{a_1, \dots, a_N\}$. Here, the a_i are the attracting points, and a leaf never extends to a submanifold in the neighborhood of a_i (†). In particular, J^- is laminated near $\partial\mathbb{B}$. If $W^u_{\mathbb{B}}(x)$

† Indeed, otherwise this would induce a compactification of unstable manifolds, yielding an embedding of $\mathbb{P}^1$ into $\mathbb{C}^2$.

is leafwise bounded, then it is of the form $\psi^u_x(\Omega)$, where Ω is some bounded open set in $\mathbb{C}$. Since ψ^u extends to a neighborhood of $\overline{\Omega}$, $W^u_{\mathbb{B}}(x)$ it is a properly embedded submanifold of $\mathbb{B}$, which extends to a neighborhood of $\overline{\mathbb{B}}$. So (*iii*) implies (*iv*). Finally, if (*iv*) holds, since J^- is a lamination near $\partial\mathbb{B}$, we see that $W^u_{\mathbb{B}}(x)$ extends to a submanifold S in a neighborhood of $\overline{\mathbb{B}}$. Then, $W^u_{\mathbb{B}}(x)$ is relatively compact in $S \subset W^u(x)$ so if Ω is such that $\psi^u_x(\Omega) = W^u_{\mathbb{B}}(x)$, then Ω is relatively compact in $\mathbb{C}$, and (*iii*) follows. $\square$

3 | EXTERNAL RAYS

In this section, we study external rays along the unstable lamination (i.e., along J^-) for a hyperbolic complex Hénon map. The existence and convergence properties of external rays were studied in the unstably connected case in [5, 6]. Recall that when $|\mathrm{Jac}(f)| < 1$, unstable connectedness is equivalent to the connectedness of J. The results that we prove here do not rely on any unstable connectivity or dissipativity assumption, nevertheless what we have in mind is the case of a dissipative unstably disconnected map.

3.1 | Escaping from K^+ along an external ray

By definition, an *unstable external ray* (simply called "external rays" in the following) is a piecewise smooth continuous path contained in a leaf $W^u(x)$ of the unstable lamination, which is a union of gradient lines of $G^+|_{W^u(x)}$ outside the (leafwise locally finite) set of critical points of $G^+|_{W^u(x)}$. As usual, we assume that G^+ is strictly monotone along external rays (which will be considered as ascending or descending depending on the context). We do not prescribe rules for the behavior of rays hitting critical points, so, in particular, there is no attempt at defining a notion of "external map."

In the next proposition, the length of curves is relative to the ambient metric in $\mathbb{C}^2$. We show that external rays ascend fairly quickly.

Proposition 3.1. *Let f be a hyperbolic polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d > 1$. For every $r_1 < r_2$, there exists $\ell(r_1, r_2)$ such that for every $x \in J^- \setminus K^+$ such that if $G^+(x) = r_1$, any external ray through x reaches $\{G^+ = r_2\}$ along a path whose length is bounded by $\ell(r_1, r_2)$. In addition, $\ell(r_1, r_2)$ is bounded by a function $\overline{\ell}(r_2)$ depending only on r_2. Furthermore, $\ell(r_1, r_2) \to 0$ when $r_1 \to r_2$ and $\overline{\ell}(r_2) = O(r_2^\alpha)$ when $r_2 \to 0$, for some $\alpha > 0$.*

Remark 3.2. Notice that no dissipativity is assumed here so the result holds along stable leaves as well.

Proof. Start with $r_1 = 1$ and $r_2 = d$. In $J^- \cap \{1 \leqslant G^+ \leqslant d\}$, the leaves of $\mathcal{W}^u$ have uniform geometry and no leaf of $\mathcal{W}^u$ is contained in an equipotential hypersurface of the form $\{G^+ = C\}$, in particular unstable critical points have uniform order. Thus, by compactness and continuity of G^+, we infer the existence of uniform δ_0 and ℓ_0 such that for every $x \in J^- \cap \{1 \leqslant G^+ \leqslant d\}$, any external ray through x of length ℓ_0 reaches $\{G^+ = r\}$ with $r \geqslant G^+(x) + \delta_0$. By concatenating such pieces of rays, we deduce the conclusion of the proposition for $r_1 = 1$ and $r_2 = d$

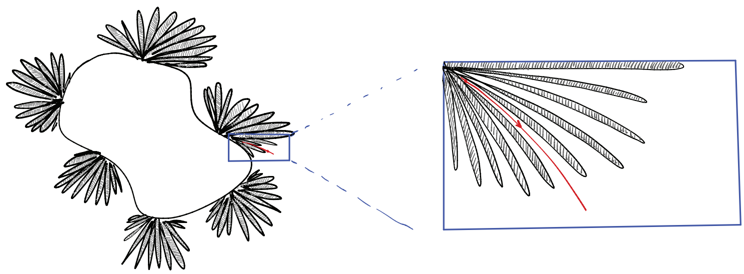

FIGURE 2 Comb-like structure prevents fast escaping.

(and $\ell(1,d) \leqslant (d-1)\ell_0/\delta_0$). Pulling back finitely many times and concatenating again, we get a similar conclusion for $\{r_0 \leqslant G^+ \leqslant d\}$ for any fixed r_0.

Let us now fix r_0 such that $\{0 < G^+ \leqslant dr_0\} \cap J^-$ is contained in $W^u_{\mathrm{loc}}(J)$. Any piece of external ray between the levels $\{G^+ = r_0/d^n\}$ and $\{G^+ = r_0/d^{n-1}\}$ is the pullback of a piece of external ray in $\{r_0 \leqslant G^+ \leqslant dr_0\}$. Thus, by concatenation, it follows that any external ray starting from $\{G^+ = r_0/d^n\}$ reaches $\{G^+ = r_0\}$ along a path of length bounded by $\leqslant C\ell(r_0, dr_0)\sum_{k=1}^n u^{-k}$, where u is the expansion constant introduced in §2.2. This proves the existence of the functions $\ell(r_1, r_2)$ and $\overline{\ell}(r_2)$

The same ideas imply immediately that $\ell(r_1, r_2) \to 0$ when $r_1 \to r_2$. For the last statement, simply note that for every $r_1 < r_2 \leqslant r_0$,

$$\ell(r_1, r_2) \leqslant C \sum_{k=k_0}^{\infty} u^{-k} = O(u^{-k_0}),$$

where k_0 is the greatest integer such that $r_0 d^{-k_0} \geqslant r_2$, therefore $\ell(r_1, r_2) = O(r_2^\alpha)$, with $\alpha = \frac{\log u}{\log d}$. □

It is easy to deduce from these ideas that all (descending) external rays land. However, since there is no well-defined external map, the characterization of the set of landing points does not seem to follow directly from this landing property.

Corollary 3.3 (John–Hölder property). *There exists a constant $\alpha > 0$ such that for any sufficiently small $\eta > 0$, for any $x \in J^- \setminus K^+$ sufficiently close to K^+, there exists a path of length at most $O(\eta^\alpha)$ in $W^u(x) \setminus K^+$ joining x to a point η-far from K^+.*

Proof. By the previous proposition, there exists a path of length $O(r^{\alpha_1})$ joining x to a point y such that $G^+(y) = r$. Now the Green function is Hölder continuous (see [19]) and that $K^+ = \{G^+ = 0\}$, so $d(x, K^+) \geqslant Cr^{\alpha_2}$. The result follows. □

This John–Hölder property has deep consequences for the topology of $K^+ \cap W^u(x)$, which will play an important role in the paper. Intuitively, it means that there cannot exist long "channels" between local components of $K^+ \cap W^u(x)$, nor comb-like structure at the boundary of components (see Figures 2 and 3).

This property is strongly reminiscent of the so-called John condition for plane domains, which have been much studied in one-dimensional dynamics, in relation with nonuniform

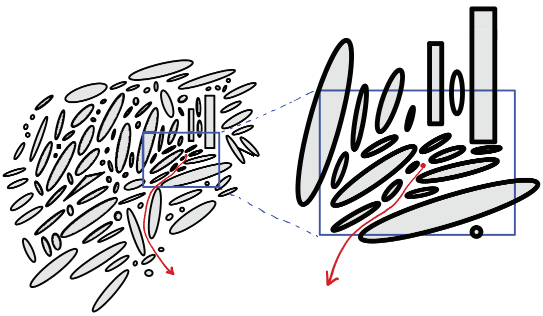

FIGURE 3 Too many large components prevents fast escaping.

hyperbolicity (see, e.g., [11, 23]). In the Hénon context, it was shown in [6] that for unstably connected hyperbolic maps, the components of $W^u(x) \setminus K^+$ satisfy the John property. It is very likely that using the continuity of affine structure along unstable leaves, their arguments can be adapted to the disconnected case as well: this would upgrade Corollary 3.3 to the actual John condition. One advantage of this weaker property is that it makes no reference to the affine structure of the leaves, so it is more flexible and may be adapted to semilocal situations (e.g., Hénon-like maps).

3.2 | Accesses and landing

Theorem 3.4. *Let f be a hyperbolic polynomial automorphism of $\mathbb{C}^2$ with dynamical degree $d > 1$.*

(1) *For every $x \in J$, $D^u(x, 1) \setminus K^+$ admits finitely many connected components, and at least one of these components contains x in its closure.*
(2) *For any component Ω of $D^u(x, 1) \setminus K^+$ such that $\overline{\Omega} \ni x$, there is an external ray landing at x through Ω.*

For the proof, it is convenient to work in the affine coordinates given by the unstable parameterizations. We work in the disks $D^u(x, 1)$ and measure path length relative to the normalized affine metric, which is equivalent to the ambient one.

Proof. The first observation is that $D^u(x, 1) \setminus K^+$ contains x in its closure: otherwise x would lie in the leafwise interior of K^+, thus contradicting Lemma 2.4. Furthermore, by the maximum principle, if $y \in D^u(x, 1) \setminus K^+$ is arbitrary, the component of y in $D^u(x, 1) \setminus K^+$ reaches the boundary of $D^u(x, 1)$.

We claim that there exists $\eta_1 > 0$ such that for any $x \in J$ and any component Ω of $D^u(x, 1) \setminus K^+$ such that $\Omega \cap D^u(x, 1/4) \neq \emptyset$, then:

$$\sup G^+|_{D^u(x,1/2)\cap\Omega} \geqslant \eta_1.$$

This follows directly from Proposition 3.1: indeed, there exists $\eta_1 > 0$ such that any point of $J^- \setminus K^+$ reaches $\{G^+ = \eta_1\}$ along a path of length $1/4$. By the Hölder continuity of G^+, we infer that any such component Ω contains a disk of radius $C\eta_1^\alpha$, so there are finitely many of them.

In particular, if (x_n) is a sequence in $D^u(x,1)\setminus K^+$ converging to x, infinitely many of them must belong to the same component Ω of $D^u(x,1)\setminus K^+$, which shows that $\overline{\Omega}$ contains x. This proves assertion (1) of the theorem.

Fix now a component Ω of $D^u(x,1)\setminus K^+$ such that $\overline{\Omega}\ni x$. Let η_1 be as above and fix ε such that $\varepsilon<\eta_1/d$ and $\ell(\varepsilon,d\varepsilon)<\min(1/2,(u-1)/2)$ where $\ell(\cdot)$ is as in Proposition 3.1 and the constant u was defined in §2.2. We do the following construction: for every point $y\in\{G^+=\varepsilon\}\cap\overline{D^u(x,1/2)}$, we consider all ascending external rays emanating from y until they reach $\{G^+=d\varepsilon\}$. The lengths of the corresponding rays is not larger than $\ell(\varepsilon,d\varepsilon)$. These are the rays of zeroth generation and we denote by E_0 the set of their endpoints (†), which by the assumption on $\ell(\varepsilon,d\varepsilon)$ is contained in $\{G^+=d\varepsilon\}\cap D^u(x,1)$. We note that E_0 is a closed set because it is the ending point set of a compact family of external rays. Since $\varepsilon<\eta_1/d$, E_0 has nonempty intersection with Ω.

Performing the same construction in $D^u(f(x),1)$, we obtain a set of rays of zeroth generation in that disk, which connect $\{G^+=\varepsilon\}\cap\overline{D^u(f(x),1/2)}$ to $\{G^+=d\varepsilon\}$, and their endpoints lie in

$$\{G^+=d\varepsilon\}\cap\overline{D^u\left(f(x),\tfrac{1}{2}+\ell(\varepsilon,d\varepsilon)\right)}.$$

The pullbacks of these rays by f have their endpoints in

$$\{G^+=\varepsilon\}\cap\overline{D^u\left(x,\tfrac{1}{u}\left(\tfrac{1}{2}+\ell(\varepsilon,d\varepsilon)\right)\right)}\subset\{G^+=\varepsilon\}\cap D^u\left(x,\tfrac{1}{2}\right),$$

by the assumption on $\ell(\varepsilon,d\varepsilon)$. These are the rays of first generation in $D^u(x,1)$. We define $E_1\subset E_0$ to be the closed set of points for which we can concatenate a ray of zeroth generation with a ray of first generation to descend all the way to $\{G^+=\varepsilon/d\}$. Notice that $f(\Omega)\cap D^u(f(x),1)$ is not necessarily connected, so it is a union of components of $D^u(f(x),1)\setminus K^+$, and since $\overline{f(\Omega)}\ni f(x)$, at least one of these components reaches $D^u(f(x),1/2)$, so it contains rays of zeroth generation. This shows that E_1 has nonempty intersection with Ω.

Continuing inductively this construction, we obtain a decreasing sequence (E_n) of closed subsets in $\{G^+=d\varepsilon\}\cap D^u(x,1)$, each of which intersecting Ω. If $e\in\bigcap_n E_n\cap\Omega$, then there is a ray through e (hence in Ω) converging to K^+, whose part in $\{\varepsilon d^{-n-1}\leqslant G^+\leqslant\varepsilon d^{-n}\}$ is the pullback under f^n of a piece of external ray in $D^u(f^n(x),1)$. Therefore, this ray lands at x, and the proof of assertion (2) is complete. □

Remark 3.5. The existence of a convergent external ray along any access to a saddle periodic point can be obtained exactly as in the one-dimensional case (see [18]), without assuming uniform hyperbolicity. In that case, the Denjoy–Carleman–Ahlfors Theorem is used instead of the John–Hölder property to guarantee the finiteness of the number of local components.

3.3 | Topology of $K^+\cap W^u$

In this section, we review the consequences of Corollary 3.3 for the topology of unstable components of K^+.

† Recall that since we do not prescribe the behavior of external rays at critical points of G^+, there is no reason that external rays fill up the whole unstable lamination, so E_0 could be smaller than $\{G^+=d\varepsilon\}$

Theorem 3.6. *Let f be a hyperbolic Hénon map. Then for every $x \in J$:*

(i) *Every component of $K^+ \cap W^u(x)$ (resp. $J^+ \cap W^u(x)$) is locally connected;*
(ii) *For any smoothly bounded domain $\Omega \subset W^u(x)$, for every $\delta > 0$, $K^+ \cap \Omega$ (resp. $J^+ \cap \Omega$) admits at most finitely many components of diameter larger than δ.*

As before, this follows from [6] when f is unstably connected (see Theorems 3.5 and 5.6 there), so we focus on the unstably disconnected case. In this case, it is known that $K^+ \cap W^u(x)$ has uncountably many point components (see [6, Thm 3.1]). Using (*ii*), we can be more precise.

Corollary 3.7. *Let f be hyperbolic and unstably disconnected. Then, for every $x \in J$, all but at most countably many components of $K^+ \cap W^u(x)$ are points.*

Let us stress that the conclusions of the theorem follow solely from Corollary 3.3 together with some elementary topological considerations. Remark also that the assumption that Ω has smooth boundary in Theorem 3.6(ii) is necessary to prevent artificial boundary effects: indeed, otherwise it might cut a component of K^+ in infinitely many parts of large diameter (think, e.g., of a domain with a comb-like boundary).

Part or all of Theorem 3.6 is presumably known to specialists; however, for completeness, we provide some details. Let us first define a notion of "fast escaping from a compact set."

Definition 3.8. Let Ω be a smoothly bounded domain in $\mathbb{C}$ and K be a closed subset in $\Omega \subset \mathbb{C}$. We say that K satisfies the *fast escaping property* in Ω if there exists an increasing continuous function ℓ with $\ell(0) = 0$ such that for any sufficiently small $\eta > 0$ and any $x \notin K$, there exists a path $\gamma : [0,1] \to \Omega \setminus K$ of length at most $\ell(\eta)$ such that $\gamma(0) = x$ and $d(\gamma(1), K) \geqslant \eta$.

Corollary 3.3 asserts that if f is hyperbolic, then for every $x \in J$, and any leafwise bounded domain $\Omega \subset W^u(x)$, $K^+ \cap W^u(x)$ satisfies the fast escaping property in Ω with $\ell(\eta) = c\eta^\alpha$. Note that both properties (*i*) and (*ii*) in Theorem 3.6 are local in $W^u(x)$ so the choice of ambient or leafwise topology or metric is harmless.

The following lemma takes care of item (*ii*) of the theorem (see Figure 3 for an illustration).

Lemma 3.9. *Let K be a closed subset of a smoothly bounded domain $\Omega \subset \mathbb{C}$, satisfying the fast escaping property. Then, for every $\delta > 0$, there are at most finitely many components of K (resp. of* $\mathrm{Int}(K)$*, of ∂K) of diameter greater than δ.*

Proof. We first prove the result for K and $\mathrm{Int}(K)$ and then explain how to modify the proof to deal with ∂K. Let us first assume that Ω is the unit square Q, and denote by π_1 and π_2 the coordinate projections of Q. Assume by contradiction that there are infinitely many components $(C_i)_{i \geqslant 0}$ of K with diameter $\geqslant \delta$. Then, there exists $\pi \in \{\pi_1, \pi_2\}$ such that infinitely many C_i satisfy $\mathrm{Diam}(\pi(C_i)) \geqslant \delta/2$. Therefore, there is an interval I of length $\delta/4$ such that for infinitely many i, C_i disconnects the strip $\pi^{-1}(I)$, and we conclude that $\pi^{-1}(I) \setminus \bigcup C_i$ has infinitely many connected components U_j going all the way across the strip. (Notice that the U_j may contain other points of K.) Let c be the center point of I. Since the C_i are distinct components of K, for each j, there exists a point x_j in $U_j \cap \pi^{-1}(c)$ that does not belong to K. If η is chosen such that $\ell(\eta) = \delta/20$, we infer from the fast escaping property that for every j, U_j contains a disk of radius η, which is the desired contradiction.

For Int(K), the argument is identical except that instead of c, we take a small open interval I' about c and argue that if the C_i are distinct components of Int(K), there exists $x_j \in U_j \cap \pi^{-1}(I')$ that does not belong to K.

In the general case, take a square Q such that $\Omega \Subset Q$ and replace K by $K' = \overline{K \cap \Omega}$. Let us check that K' satisfies the fast escaping property in Q. Indeed, if $x \in Q \setminus K'$, we have either $x \in \Omega$, $x \in \partial\Omega$ or $x \in Q \setminus \overline{\Omega}$. In the first case, we take the path γ given by the fast escaping property of K in Ω. In the second case, any small ball B about x intersects $\Omega \setminus K$, and we simply take a path starting from some $x' \in B \cap (\Omega \setminus K)$. Finally, in the last case, we use the fact that $\overline{\Omega}$ has the fast escaping property in Q.

By the first part of the proof, we conclude that K' has finitely many components of diameter $\geqslant \delta$. Since any component of K (resp. Int(K)) is contained in a component of K' (resp. Int(K')), we are done.

The proof that ∂K admits only finitely many components of diameter greater than δ goes exactly along the same lines. We assume that there are infinitely many components C_i of ∂K disconnecting the strip $\pi^{-1}(I)$, so that $\pi^{-1}(I) \setminus \bigcup C_i$ also has infinitely many components U_j. The difference with the previous case is that some of these components may be completely included in K. We modify the argument as follows. Denote by U_j' the components completely included in K and by U_j'' the remaining ones. We claim that there are infinitely many U_j'''s. Indeed, since the C_i are components of ∂K, two components of the form U_j' must be separated by a component of the form U_j''. So, there are infinitely many such components. Then, we take a small open interval $I' \subset I$ containing c and we repeat this argument, to obtain that there are infinitely many j's such that $U_j'' \cap \pi^{-1}(I')$ contains a point x_j that does not belong to K. Then, we proceed with the proof as in the previous case, by constructing infinitely many disjoint disks of radius η in Q to get a contradiction. □

Proof of (i) in Theorem 3.6. Since $J^+ \cap W^u(x) = \partial_{\mathrm{i}}(K^+ \cap W^u(x))$, general topology implies that local connectivity of $J^+ \cap W^u(x)$ implies that of $K^+ \cap W^u(x)$ (see [34, §49.III]), so it is enough to focus on J^+. For convenience, we plug in some dynamical information. Since f is unstably disconnected, it admits an unstable transversal Δ^u, that is a horizontal disk of finite degree in $\mathbb{B}$ contained in some unstable manifold (of a periodic saddle point, say). For every $x \in J$, $W^s(x)$ intersects Δ^u: this easily follows from the density of $W^s(x)$ in J^+ and the local product structure. Fix $y \in W^s(x) \cap \Delta^u$. By using the local holonomy along the stable lamination $W^u_{\mathrm{loc}}(x) \to W^u_{\mathrm{loc}}(y)$, we see that $J^+ \cap W^u(x)$ is locally connected at x if and only if $J^+ \cap W^u(y)$ is locally connected at y. Therefore, it is enough to show that $J^+ \cap \Delta^u$ is locally connected. Since $K^+ \cap \Delta^u$ is polynomially convex and compactly contained in Δ^u, it follows that $\Omega := \Delta^u \setminus K^+$ is connected and $J^+ \cap \Delta^u = \partial\Omega$. Likewise every component of $\partial\Omega$ is of the form ∂A, where A is a component of $\Delta^u \cap K^+$. For such a component, by Carathéodory's theorem local connectivity of ∂A is equivalent to that of A, which is, of course, equivalent to local connectivity of A at every point of its boundary. Let us fix $x_0 \in \partial A$: to complete the proof, we have to show that A is locally connected at x_0.

Assume by contradiction that A is not locally connected at x_0. Then, for small $\varepsilon > 0$ such that if C denotes the component of $A \cap \overline{B(x_0, \varepsilon)}$, then $x_0 = \lim x_n$, where x_n belongs to $A \setminus C$. Without loss of generality, we can assume that $x_n \in B(x_0, \varepsilon/2)$. Let $C_n = \mathrm{Comp}_{A \cap \overline{B(x_0,\varepsilon)}}(x_n)$, which by definition is disjoint from C. Passing to a subsequence if necessary, we may assume that the C_n are disjoint (the construction here is similar to that of convergence continua in [34, §49.VI]). Since C and the C_n intersect $\partial B(x_0, \varepsilon)$, their diameter is bounded from below by some $\delta > 0$. From this point, the proof is similar to that of Lemma 3.9: we can find an orthogonal projection π such that

C and the C_n cross the strip $\pi^{-1}(I)$ horizontally and $\pi^{-1}(I) \setminus (C \cup \bigcup C_n)$ admits infinitely many connected components U_j going all the way across the strip. If $\pi^{-1}(c)$ denotes the center line of the strip, for every j, $\pi^{-1}(c) \cap U_j$ has nontrivial intersection with Ω, and the fast escaping property of Ω gives a contradiction as before. ☐

3.4 | Complement: John–Hölder property in basins

We illustrate the comments from § 3.1 on the versatility of the John–Hölder property by sketching a proof of the following result.

Theorem 3.10. *Let f be a hyperbolic polynomial automorphism of $\mathbb{C}^2$, and $\mathcal{B}$ be an attracting basin. Then, the John–Hölder property holds in $\mathcal{B}$, that is, for any component Ω of $\mathcal{B} \cap W^u(x)$, there exists positive constants η and α depending only on Ω such that for any $y \in \Omega$ sufficiently close to J, there exists a path in of length $O(\eta^\alpha)$ in $W^u(x)$ joining y to a point η-far from J.*

Remark 3.11. A difference between this result and Corollary 3.3 is that in Corollary 3.3, the constant η is independent of the component of $W^u(x) \setminus K^+$, because G^+ reaches arbitrary large values in each component. Here, the situation is different because $\mathcal{B} \cap W^u(x)$ typically has (infinitely) many small components, so how far we can get from the boundary really depends on the component.

Proof. For convenience, we present a proof that is purposely close to that of Proposition 3.1 and Corollary 3.3. Replace f by some iterate so that $\mathcal{B}$ is the basin of attraction of a fixed point a with multipliers λ_1, λ_2, with $|\lambda_2| \leqslant |\lambda_1|$. There exists a biholomorphism $\phi : \mathcal{B} \to \mathbb{C}^2$ that conjugates the dynamics to that of the triangular map $(z_1, z_2) \mapsto (\lambda_1 z_1 + r(z_2), \lambda_2 z_2)$, where r is a polynomial that is nonzero only when there is a resonance $\lambda_2 = \lambda_1^j$ between the eigenvalues (see [42]). Introduce the function

$$\tilde{H}(z_1, z_2) = \left|z_1 - r(z_2/\lambda_2)\right|^2 + \left|z_2\right|^{2\gamma}, \text{ where } \gamma = \frac{\log \lambda_1}{\log \lambda_2} \geqslant 1$$

and put $H = \tilde{H} \circ \phi$. This is a smooth strictly psh function on $\mathcal{B}$ that satisfies $H \circ f = \left|\lambda_1\right|^2 H$. To get a better analogy with the previous case, we may consider H^{-1} that satisfies $H^{-1} \circ f = \left|\lambda_1\right|^{-2} H^{-1}$, and tends to zero when approaching J. The restriction of this function to any local unstable disk in $\mathcal{B} \setminus \{a\}$ is nonconstant and one easily checks that its set of critical points is discrete.

Arguing in Proposition 3.1, we define a family of rays in $\mathcal{B}$ by considering gradient lines of H (or equivalently H^{-1}) along $\mathcal{W}^u$, first in the fundamental domain $\left\{ \left|\lambda_1\right|^2 \leqslant H^{-1} \leqslant 1 \right\}$ and then in $\left\{0 < H^{-1} \leqslant 1\right\}$ by pulling back. It follows that for every component Ω of $\mathcal{B} \cap W^u(x)$, for every $0 < r_1 < r_2 < \max_\Omega \left|H^{-1}\right|$, and any $y \in \Omega$ such that $H^{-1}(y) = r_1$, there exists a ray of length $\ell(r_1, r_2) = O(r_2^{\alpha_1})$ joining y to a point of $\left\{H^{-1} = r_2\right\}$ for some $\alpha_1 > 0$.

To conclude the argument, we need to adapt the proof of Corollary 3.3, which relies on the Hölder continuity of the Green function. Instead, we use an argument based on uniform hyperbolicity. Indeed, let $x \in J$ and $y \in W^u_{\mathrm{loc}}(x)$ be such that $d^u(x, y) = \varepsilon$. We want to show that $H^{-1}(y) \lesssim \varepsilon^{\alpha_2}$ for some α_2. By the expansion along unstable manifolds and the local uniform geometry, it takes at most $N \leqslant C\,|\log \varepsilon|$ iterates to map y into a given compact subset of $\mathcal{B}$. Hence,

$$H^{-1}(y) = |\lambda_1|^{2N} H^{-1}(f^N(y)) \leqslant C|\lambda_1|^{2N} \leqslant C|\lambda_1|^{2C|\log \varepsilon|} = C\varepsilon^{-2C\log|\lambda_1|},$$

and we are done. □

4 STABLE TOTAL DISCONNECTEDNESS

We say that f (or J) is *stably totally disconnected* if for every $x \in J$, $W^s(x) \cap J^-$ is totally disconnected. Note that since $W^s(x) \subset J^+$, we have $W^s(x) \cap J = W^s(x) \cap J^-$.

Proposition 4.1. *Let f be a hyperbolic Hénon map. The following assertions are equivalent.*

(i) *Every leaf of the stable lamination in $\mathbb{B}$ is a vertical submanifold of finite degree.*
(ii) *The leaves of the stable lamination in $\mathbb{B}$ are vertical submanifolds of uniformly bounded degree.*
(iii) *For every x in J, $J^s(x) = K^s(x) = \{x\}$, that is, f is stably totally disconnected.*

Note that dissipativity is not required here, so this result holds in the unstable direction as well.

Proof. The implication $(ii) \Rightarrow (i)$ is obvious and its converse $(i) \Rightarrow (ii)$ follows from the semicontinuity properties of the degree and is identical to [36, Lemma 5.1]. To prove that $(iii) \Rightarrow (i)$, we use Lemma 2.5 for the stable lamination: indeed, if $J^s(x)$ is a point for every x, then all four conditions of Lemma 2.5 are equivalent, and the equivalence of properties *(ii)* and *(iii)* there yields the result. Finally, $(ii) \Rightarrow (iii)$ does not require hyperbolicity and was established in [15, Prop. 2.14]. For convenience, let us recall the argument: for every vertical disk D of degree $\leqslant k$, and every component D' of $D \cap f(\mathbb{B})$, the modulus of the annulus $D \setminus D'$ is bounded below by $m = m(k) > 0$, and for every $x \in J$, there is an infinite nest of such annuli surrounding the component of x in $W^s(x) \cap J$. So, $W^s(x) \cap J$ is totally disconnected and we are done. □

A way to ensure the boundedness of the degrees of semilocal stable manifolds originates in [17] and relies on Wiman's theorem for entire functions. The following result is contained in [36].

Proposition 4.2. *Let f be a hyperbolic Hénon map such that $|\mathrm{Jac}\, f| \leqslant d^{-2}$. Then, f is stably totally disconnected.*

Proof (sketch). Fix $x \in J$ and $v \in E^s(x)$. Uniform hyperbolicity together with the assumption on the Jacobian imply that $\|df^n_x(v)\| \leqslant Cs^n$, where $s < d^{-2}$. Denote as before ψ^s_y the normalized stable parameterization at some $y \in J$. It follows that $f^n \circ \psi^s_x(\cdot) = \psi^s_{f^n(x)}(\lambda_n \cdot)$, where $|\lambda_n| \leqslant Cs^n$. Then, from the relation

$$G^- \circ \psi^s_x(\lambda_n^{-1}\zeta) = d^n G^- \circ \psi^s_{f^n(x)}(\zeta),$$

we deduce that $G^- \circ \psi^s_x$ is a subharmonic function of order smaller than $1/2$ and Wiman's theorem implies that $\mathrm{Comp}_{(\psi^s_x)^{-1}(\mathbb{B})}(x)$ is a bounded domain in $\mathbb{C}$; thus, $W^s_{\mathbb{B}}(x)$ has bounded vertical degree and we are done. □

Another idea is to use a Hausdorff dimension argument to prove directly that stable slices of J are totally disconnected, following Wolf [47, Cor. 4.9]. Indeed the Hausdorff dimension of stable slices of J^- can be estimated using thermodynamic formalism for hyperbolic maps. This turns out to give a better bound on the Jacobian.[†]

Proposition 4.3 (Wolf [47]). *Let f be a hyperbolic Hénon map such that $|\mathrm{Jac}\, f| \leqslant d^{-1/2}$. Then, f is stably totally disconnected.*

Proof. Since J is a locally maximal hyperbolic set, the C^1 smoothness of the stable and unstable laminations (see Lemma 5.7 below) implies that the Hausdorff dimensions

$$\delta^s := \dim_H \left(J \cap W^s_{\mathrm{loc}}(x)\right) \text{ and } \delta^u := \dim_H \left(J \cap W^u_{\mathrm{loc}}(x)\right),$$

do not depend on $x \in J$. In addition, there are exact formulas for δ^u and δ^s.

A first result is that $\delta^u < 2$: indeed if not, it would follow that $J \cap W^u_{\mathrm{loc}}(x)$ has positive Lebesgue measure, and by the absolute continuity of the stable foliation, J^+ would have positive measure, contradicting a famous result of Bowen (see [46, Thm. 4.4] for details).

The Bowen–Ruelle formula asserts that δ^s (resp. δ^u) is the unique zero of the stable (resp. unstable) pressure function $t \mapsto P^{s/u}(f, t\phi^s)$ (resp. $t \mapsto P^{s/u}(f, -t\phi^u)$) where $\phi^{s/u}(x) = \log \left|df|_{E^{s/u}(x)}\right|$. Both functions P^s and P^u are convex and decreasing. The constancy of the Jacobian implies that for every $t \geqslant 0$, $P^u(t) = P^s(t) - t \log |\mathrm{Jac}\, f|$ (see [47, Prop. 4.6]). Since f is dissipative, for $t > 0$, $P^s(t) < P^u(t)$ and it follows that $\delta^s < \delta^u$. Furthermore, since $P^s(\delta^u) = \delta^u \log \left|\mathrm{Jac}(f)\right| < 0$, the graph of P^s lies below the line joining $(0, \log d)$ and $(\delta^u, \delta^u \log \left|\mathrm{Jac}(f)\right|)$. From this, it follows that

$$\delta^s \leqslant \frac{\delta^u \log d}{\log d - \delta^u \log \left|\mathrm{Jac}(f)\right|},$$

and plugging in $|\mathrm{Jac}\, f| \leqslant d^{-1/2}$ and $\delta^u < 2$ implies that $\delta^s < 1$, hence f is stably totally disconnected, as asserted. $\square$

Note that in the previous proposition, by the topological stability of J, we can further relax the assumption by only requiring that the hyperbolic component containing f contains some g with $|\mathrm{Jac}\, g| \leqslant d^{-1/2}$.

Question 4.4. Is a dissipative hyperbolic Hénon map always stably totally disconnected?

5 | CLASSIFICATION OF SEMI-LOCAL COMPONENTS OF K^+ AND J^+

Throughout this section, f is a dissipative and hyperbolic complex Hénon map of degree d with a disconnected Julia set (or equivalently, f is unstably disconnected). We assume moreover that *f is stably totally disconnected*. The results of §4 imply that this holds whenever $|\mathrm{Jac}\, f| \leqslant 1/\sqrt{d}$. We fix a large bidisk $\mathbb{B}$ as before, and our purpose is to classify the connected components of $J^+ \cap \mathbb{B}$ and to study the induced dynamics on this set of components.

[†] In a previous version of this paper, the bound $|\mathrm{Jac}\, f| < d^{-1}$ was obtained, the new bound $d^{-1/2}$ takes better advantage of the fact that the Jacobian is constant

5.1 Geometric preparations

We start with some general lemmas about vertical submanifolds in a bidisk. We define the angle $\angle(v, w)$ between two complex directions v and w at $x \in \mathbb{C}^2$ to be their distance in $\mathbb{P}(T_x\mathbb{C}^2) \simeq \mathbb{P}^1$ relative to the Fubini-Study metric induced by the standard Hermitian structure of $T_x\mathbb{C}^2 \simeq \mathbb{C}^2$.

Lemma 5.1. *Let M be a vertical submanifold in $\mathbb{D} \times \mathbb{D}$, and let $a \in \mathbb{D}$ and $0 < r < 1$ be such that M has no horizontal tangency in $\mathbb{D} \times D(a, 2r)$. Then, there exists a universal constant C_0 such that for any $x \in \mathbb{D} \times D(a, r)$, the angle between T_xM and the horizontal direction is bounded from below by $C_0 r$.*

Proof. If M has no horizontal tangency in $\mathbb{D} \times D(a, 2r)$, then $M \cap (\mathbb{D} \times D(a, 2r))$ is the union of $\deg(M)$ vertical graphs. Let Γ be one of these graphs. Then, $\varphi := \pi_1 \circ (\pi_2|_\Gamma)^{-1}$ maps $D(a, 2r)$ into $2\mathbb{D}$ and $\Gamma = \{(\varphi(w), w),\ w \in D(a, 2r)\}$. By the Cauchy estimate, we get that $|\varphi'| \leqslant 2/r$ on $D(a, r)$ and the result follows. ☐

A typical use of this result is by taking the contraposite: if a vertical submanifold M in $\mathbb{D} \times \mathbb{D}$ has a near horizontal tangency in $\mathbb{D} \times D(a, r)$, then it has an actual horizontal tangency in $\mathbb{D} \times D(a, 2r)$. Let us denote by $[e_1] \in \mathbb{P}(T\mathbb{C}^2)$ the horizontal direction.

Corollary 5.2. *Let M be a vertical submanifold in $\mathbb{D} \times \mathbb{D}$ that extends as a vertical submanifold to $\mathbb{D} \times (3/2)\mathbb{D}$. There exists constants C_1 and θ_0 such that if for some $\theta \leqslant \theta_0$ and $a \in \mathbb{D}$, there exists $x \in M \cap (\mathbb{D} \times \{a\})$ such that $\angle(T_xM, [e_1]) < \theta$, then there exists $a' \in (3/2)\mathbb{D}$ such that $|a - a'| < C_1\theta$ and M is tangent to $\mathbb{D} \times \left\{a'\right\}$.*

For the sake of completeness, let us also state a slightly stronger result:

Corollary 5.3. *Let M be a vertical submanifold in $\mathbb{D} \times \mathbb{D}$ of degree at most k that extends as a vertical submanifold to $\mathbb{D} \times r_0\mathbb{D}$ for some $r_0 > 1$ (say $r_0 = 3/2$). There exists a function $h = h_k$ such that $h(\theta) \to 0$ as $\theta \to 0$ with the following property: if $x \in M$ is such that the angle between T_xM and the horizontal direction is bounded by $\theta \ll 1$, then there exists $x' \in M$ with $d(x, x') \leqslant h(\theta)$ such that M has a horizontal tangency at x'.*

Proof. Indeed, letting $a = \pi_2(x)$, and applying Corollary 5.2, we see that the connected component of M containing x in $D(a, C_1\theta) \times \mathbb{D}$ cannot be a vertical graph, so it admits a horizontal tangency. Furthermore, an easy compactness argument shows that the diameter of a connected component of $M \cap D(a, r) \times \mathbb{D}$ is bounded by $h_k(r)$ with $h_k(r) \to 0$ as $r \to 0$. The result follows. ☐

Remark 5.4. It is likely that $h_k(r) = O\left(r^{1/k}\right)$.

The following result is a precise version of the Reeb Stability Theorem (see [10]) that is specialized to our setting.

Lemma 5.5. *Let $x_0 \in J$ be such that $W^s_{\mathbb{B}}(x_0)$ is transverse to $\partial\mathbb{B}$. Then there exists δ depending only on*

$$\min_{y \in W^s_{\mathbb{B}}(x_0) \cap \partial\mathbb{B}} \angle\left(T_y W^s_{\mathbb{B}}(x_0), [e_1]\right)$$

such that if $\tau \subset J^u(x_0)$ is a connected compact set containing x_0, of diameter less than δ, then for every $x \in \tau$, $W^s_{\mathbb{B}}(x)$ is transverse to $\partial\mathbb{B}$, $\deg W^s_{\mathbb{B}}(x) = \deg W^s_{\mathbb{B}}(x_0)$ and $\bigcup_{x\in\tau} W^s_{\mathbb{B}}(x)$ is homeomorphic to $\tau \times W^s_{\mathbb{B}}(x_0)$.

Note that it is slightly abusing to say that $W^s_{\mathbb{B}}(x)$ is transverse to $\partial(\mathbb{B})$ since $W^s_{\mathbb{B}}(x)$ precisely stops at $\partial\mathbb{B}$. Of course, $W^s_{\mathbb{B}}(x)$ extends to a neighborhood of $\overline{\mathbb{B}}$ and what we mean is transversality for this extension.

Remark 5.6. Later on, we will use this lemma with $r\mathbb{B}$ instead of $\mathbb{B}$ for $1 \leqslant r \leqslant 2$ (see Proposition 5.12). It will be important there that the constant δ is uniform for $r \in [1, 2]$, which easily follows from the proof.

Proof. We start with a qualitative argument. By compactness of $W^s_{\underline{\mathbb{B}}}(x_0) \cap \partial\mathbb{B}$, there exists $\eta_0 > 0$ such that $W^s_{(1+\eta)\mathbb{B}}(x_0)$ is transverse to $\partial((1+\eta)\mathbb{B})$ for any $|\eta| \leqslant \eta_0$. Since the stable leaves in $\mathbb{B}$ are simply connected, we can apply a local version of the Reeb Stability Theorem (see [10, Prop. 11.4.8]) that asserts that when $\tau \subset J \cap W^u(x_0)$ is sufficiently small, for $x \in \tau$, by the local triviality of the stable lamination, the domain $W^s_{(1+\eta)\mathbb{B}}(x_0) \subset W^s(x_0)$ can be lifted to a domain $D_x \subset W^s(x)$ such that $W^s_{\underline{\mathbb{B}}}(x) \Subset D_x$, and the collection $\{D_x,\ x \in \tau\}$ is topologically a product. Since $W^s_{\mathbb{B}}(x_0)$ is transverse to $\partial\mathbb{B}$, $W^s_{\mathbb{B}}(x_0) \subset W^s_{(1+\eta)\mathbb{B}}(x_0)$ is a smoothly bounded domain and, reducing τ if necessary, for $x \in \tau$, this transversality persists, $\mathrm{Comp}_{D_x\cap\mathbb{B}}(x) = W^s_{\mathbb{B}}(x)$ varies continuously and $\bigcup_{x\in\tau} W^s_{\mathbb{B}}(x)$ is a product. Finally, if we fix any horizontal line close to $\partial\mathbb{B}$ by transversality and continuity, its number of intersection points with $W^s_{\mathbb{B}}(x)$ is constant for $x \in \tau$, hence the statement on the degree.

What remains to be seen is why the size of the allowed transversal τ in the previous argument depends only on the minimal angle

$$\theta = \min_{y\in W^s_{\underline{\mathbb{B}}}(x_0)\cap\partial\mathbb{B}} \angle\Big(T_y W^s_{\underline{\mathbb{B}}}(x_0), [e_1]\Big).$$

First, we observe that the stable lamination in $(1+\eta_0)\mathbb{B}$ is covered by a finite number of flow boxes $(\mathcal{F}_i)_{i\in I}$. Recall that by definition, a *plaque* is the intersection between a leaf an a flow box. First concentrate on the subfamily $(\mathcal{F}_i)_{i\in I'}$ of flow boxes covering $W^s_{\underline{\mathbb{B}}}(x_0) \cap \partial\mathbb{B}$. Since the plaques of these flow boxes vary continuously for the C^1 topology, there exists $\delta' > 0$ depending only on θ such that if Δ' is a plaque of $\mathcal{F}_i$, $i \in I'$, that is δ'-close to a plaque Δ of $\mathcal{F}_i$ contained in $W^s_{(1+\eta_0)\mathbb{B}}(x_0)$ (measured in the C^1 topology or in some fixed family of transversals to $\mathcal{F}_i$), then the angle between Δ' and $\partial\mathbb{B}$ is greater than $\theta/2$.

Now, if $x \in \tau$ is close to x_0, we can follow x under holonomy along the stable foliation. More precisely, given a path γ in $W^s_{\mathbb{B}}(x_0)$ joining x_0 to some $y \in \partial\mathbb{B}$, there is a homeomorphism h_γ defined in a δ-neighborhood τ_δ of x_0 in τ, with values in a transversal to the stable lamination at y, and the diameter of its image depends only on δ and the number of flow boxes crossed by γ. The key point (which is the basic mechanism of the Reeb Stability Theorem) is that h_γ depends only on the homotopy class of γ with fixed extremities, so, since in our situation $W^s_{\mathbb{B}}(x_0)$ is simply connected, h_γ is independent of γ. Thus, since $W^s_{\mathbb{B}}(x_0)$ is properly embedded in $\mathbb{B}$ (or equivalently, it is a vertical submanifold of finite degree), we can fix once for all a compact family of such paths $(\gamma_y)_{y\in W^s_{\underline{\mathbb{B}}}(x_0)\cap\partial\mathbb{B}}$, respectively, joining x_0 to any point $y \in W^s_{\underline{\mathbb{B}}}(x_0) \cap \partial\mathbb{B}$, and the number of flow boxes crossed by these γ_y is uniformly bounded. Therefore, we can choose δ so that the diameter of $h_{\gamma_y}(\tau_\delta)$ is smaller than δ', and we conclude from the previous observations. $\square$

We will also need the following extension lemma.

Lemma 5.7 [36, Prop. 5.8]. *There exists a neighborhood $\mathcal{N}$ of $J^+ \cap \mathbb{B}$ such that the stable lamination $\mathcal{W}^s$ extends to a C^1 foliation of $\mathcal{N}$.*

Observe that in [36], it is assumed that $|\mathrm{Jac}\, f| < d^{-2}$ but what is really needed for extending the stable lamination is the boundedness of the vertical degree that holds in our setting (cf. Proposition 4.1). The C^1 regularity of the holonomy will not be used in the paper.

Using the above extension lemma, we can refine Lemma 5.5 to a statement about an open neighborhood of $W^s_{\mathbb{B}}(x_0)$ with exactly the same proof.

Lemma 5.8. *Let $x_0 \in J$ be such that $W^s_{\mathbb{B}}(x_0)$ is transverse to $\partial\mathbb{B}$. Then there exists δ depending only on*

$$\min_{y \in W^s_{\underline{r\mathbb{B}}}(x_0) \cap \partial\mathbb{B}} \angle\left(T_y W^s_{\underline{\mathbb{B}}}(x_0), [e_1]\right)$$

such that for every $x \in D^u(x_0, \delta)$, $W^s_{\mathbb{B}}(x)$ is transverse to $\partial\mathbb{B}$, $\deg W^s_{\mathbb{B}}(x) = \deg W^s_{\mathbb{B}}(x_0)$ and

$$\bigcup_{x \in D^u_{x_0}(x_0,\delta)} W^s_{\mathbb{B}}(x)$$

is homeomorphic to $D^u_{x_0}(x_0, \delta) \times W^s_{\mathbb{B}}(x_0)$.

5.2 | Thin and thick components

In this section, we study the geometry of the components of $J^+ \cap \mathbb{B}$. The arguments rely mostly on the geometry of the stable lamination, not on the dynamics of f. One main result is that thin components of $K^+ \cap \mathbb{B}$ have a simple leaf structure (Proposition 5.12). It follows that for a given component of $J^+ \cap \mathbb{B}$, either all its unstable slices are small, or all of them are large (Proposition 5.13). Together with the results of §3.3, this leads to a description and some regularity properties of components of $J^+ \cap \mathbb{B}$ and $K^+ \cap \mathbb{B}$.

We start with a simple case.

Proposition 5.9. *If $x \in J$ is such that $K^u(x) = J^u(x) = \{x\}$, then $K^+_{\mathbb{B}}(x) = J^+_{\mathbb{B}}(x) = W^s_{\mathbb{B}}(x)$.*

Proof. As observed above, the inclusion $W^s_{\mathbb{B}}(x) \subset K^+_{\mathbb{B}}(x)$ is obvious. For the converse inclusion, observe that for every $n \in \mathbb{Z}$, $K^u(f^n(x)) = \{f^n(x)\}$. For $n \geqslant 1$, consider a small loop $\gamma_n \subset W^u(f^n(x))$ around $f^n(x)$ that is disjoint from K^+. By the local product structure, we can extend it to a germ of 3-manifold $\widetilde{\gamma}_n$ transverse to $W^u(f^n(x))$, disjoint from K^+, and of size uniformly bounded from below in the stable direction. Since $W^s_{2\mathbb{B}}(x)$ has finite vertical degree in $2\mathbb{B}$, it admits finitely many horizontal tangencies, so we can fix $1 \leqslant r \leqslant 2$ such that $W^s_{r\mathbb{B}}$ is transverse to $\partial(r\mathbb{B})$. Then, by the Inclination Lemma, for large n, $f^{-n}(\widetilde{\gamma}_n)$ contains a small “tube” around $W^s_{r\mathbb{B}}(x)$ whose boundary is disjoint from K^+. It follows that $K^+_{r\mathbb{B}}(x) = W^s_{r\mathbb{B}}(x)$, hence $K^+_{\mathbb{B}}(x) \subset W^s_{r\mathbb{B}}(x) \cap \mathbb{B}$. Finally, $W^s_{r\mathbb{B}}(x) \cap \mathbb{B}$ has finitely many components, and one of them is $W^s_{\mathbb{B}}(x)$, so $K^+_{\mathbb{B}}(x) = W^s_{\mathbb{B}}(x)$. □

Here is a first interesting consequence.

Corollary 5.10. *All but countably many components of $K^+ \cap \mathbb{B}$ are vertical submanifolds.*

Proof. Fix a global unstable transversal Δ^u in $\mathbb{B}$. Then every component of $K^+ \cap \mathbb{B}$ intersects Δ^u. Indeed, for any such component C, ∂C is contained in J^+ so it contains stable manifolds. Stable manifolds in $\mathbb{B}$ are vertical and of finite degree, so they have nontrivial (transverse) intersection with Δ^u. Now if C is nontrivial, that is, not reduced to a vertical submanifold, then by Proposition 5.9, any component of $C \cap \Delta^u$ is nontrivial, and the result follows from Corollary 3.7. □

Another case where $J^+_{\mathbb{B}}(x)$ is easily understood is when stable leaves are transverse to $\partial\mathbb{B}$.

Proposition 5.11. *Assume that $J^u(x)$ is a leafwise bounded component such that for every $y \in J^u(x)$, $W^s_{\mathbb{B}}(y)$ is transverse to $\partial\mathbb{B}$. Then*

$$J^+_{\mathbb{B}}(x) = \bigcup_{y \in J^u(x)} W^s_{\mathbb{B}}(y). \tag{2}$$

Note that this result is not true if the transversality assumption is omitted (see Figure 1 for a visual explanation).

Proof. Let C be defined by the right-hand side of (2). Since the $W^s_{\mathbb{B}}(y)$, $y \in J^u(x)$, are transverse to $\partial\mathbb{B}$, they vary continuously with y. It follows that C is a closed connected set. To show that $C = J^+_{\mathbb{B}}(x)$, it is convenient to use the extension of the stable lamination to a neighborhood of $J^+ \cap \mathbb{B}$ (given in Lemma 5.7). Let (U_n) be a basis of open neighborhoods of $J^u(x)$ in $W^u(x)$ such that for every n, $\partial U_n \cap J = \emptyset$. For every $\delta > 0$, U_n is contained in the δ-neighborhood of $J^u(x)$ for large n. Thus, by Lemma 5.8, the leaves issued from U_n are transverse to $\partial\mathbb{B}$ and stay close to C. Let $\widetilde{U}_n$ be the saturation of U_n in the extended foliation. Then, $(\widetilde{U}_n)$ is a basis of neighborhoods of C in $\mathbb{B}$ such that $\partial\widetilde{U}_n$ is disjoint from J^+. We conclude that $C = J^+_{\mathbb{B}}(x)$. □

The structure of $J^+_{\mathbb{B}}(x)$ is not so easy to describe without this transversality assumption. Still, the argument can (almost) be salvaged if $J^u(x)$ is small enough. This will be a key property in the following.

Proposition 5.12. *There exists $\delta_1 > 0$ such that if $x \in J$ is such that $\mathrm{Diam}_x(J^u(x)) \leqslant \delta_1$, then there exists $1 \leqslant r \leqslant 2$ such that for every $y \in J^u(x)$, $W^s_{r\mathbb{B}}(y)$ is transverse to $\partial(r\mathbb{B})$ and $J^u(x)$ can be followed under holonomy along $W^s_{r\mathbb{B}}(x)$. In particular, $J^+_{r\mathbb{B}}(x)$ is homeomorphic to $J^u(x) \times W^s_{r\mathbb{B}}(x)$ and*

$$J^+_{\mathbb{B}}(x) \subset J^+_{r\mathbb{B}}(x) = W^s_{r\mathbb{B}}(J^u(x)) \subset W^s_{2\mathbb{B}}(J^u(x)) = \bigcup_{y \in J^u(x)} W^s_{2\mathbb{B}}(y). \tag{3}$$

Recall that Diam_x denotes the diameter relative to the normalized leafwise metric d^u_x induced by the affine structure. By polynomial convexity, if $K^u(x)$ is leafwise bounded, then $J^u(x) = \partial_{\mathrm{i}} K^u(x)$ so $\mathrm{Diam}_x(K^u(x)) = \mathrm{Diam}_x(J^u(x))$. Recall from §2.3 that by the Koebe Distortion Theorem, the ambient distance d and the leafwise Euclidean distance d^u_x are equivalent in a small neighborhood of x, with universal bounds, that is, in some neighborhood of x in $W^u(x)$, we have

$d/2 \leqslant d_x^u \leqslant 2d$. In particular, if $\mathrm{Diam}_x(J^u(x))$ is small enough then $\mathrm{Diam}(J^u(x))$ and $\mathrm{Diam}(K^u(x))$ are comparable to $\mathrm{Diam}_x(J^u(x))$ (where Diam denotes the ambient diameter).

Proof of Proposition 5.12. Recall that every leaf of the stable lamination in $3\mathbb{B}$ is a vertical disk of degree bounded by D, so by the Riemann–Hürwitz formula, it admits at most $D-1$ horizontal tangencies. For $k = 0, \dots, D$, let $r_k = 1 + \frac{k}{D}$, and fix $\theta < \frac{C_0}{8D}$, where C_0 is as in Lemma 5.1. Let $x \in J$ be arbitrary. By the pigeonhole principle, there exists $k \in \{0, \dots, D-1\}$ such that $W^s_{2\mathbb{B}}(x)$ has no horizontal tangency in $r_{k+1}\mathbb{B} \setminus r_k\mathbb{B}$. So, by Lemma 5.1 (scaled to $2\mathbb{B}$ and applied to any a such that $|a| = R(r_k + r_{k+1})/2$, where R is the radius of $\mathbb{B}$), we infer that

$$\min_{y \in \cap \partial(r'_k\mathbb{B})} \angle(T_y W^s_{\mathbb{B}}(x_0), [e_1]) \geqslant \theta, \text{ where } r'_k = \frac{r_k + r_{k+1}}{2}.$$

Therefore, by Lemma 5.5 and Remark 5.6, there exists δ_1 depending only on θ, hence ultimately only on D, hence on f, such that if $\mathrm{Diam}_x(J^u(x)) \leqslant \delta_1$, then for every $y \in J^u(x)$, $W^s_{r'_k\mathbb{B}}$ is transverse to $\partial(r'_k\mathbb{B})$ and $W^s_{r'_k\mathbb{B}}(J^u(x))$ is topologically a product. This completes the proof of the first part of the proposition. From this point, the description of $J^+_{2\mathbb{B}}(x)$ in (3) directly follows from Proposition 5.11. $\square$

It follows from this analysis that if C is a semilocal component of J^+, then either all its unstable slices are large or all of them are small.

Proposition 5.13. *There exists $0 < \delta_1 \leqslant \delta_2$ such that for every component C of $J^+ \cap \mathbb{B}$, the following alternative holds:*

(i) *either for every $x \in C \cap J$, $\mathrm{Diam}_x J^u(x) \leqslant \delta_2$;*
(ii) *or for every $x \in C \cap J$, $\mathrm{Diam}_x J^u(x) > \delta_1$.*

In addition if (i) holds, then C satisfies the conclusions of Proposition 5.12.

Referring to this dichotomy in the following, we will say that a component is *thin* (resp. *thick*) if it satisfies (*i*) (resp. (*ii*)). We stress that the proposition asserts that a component is thick as soon as *one* of its unstable slices has intrinsic diameter larger than δ_2. As seen before (see, e.g., Corollary 5.10), if Δ^u is an unstable transversal, every semilocal component of J^+ intersects Δ^u, so from Theorem 3.6, we immediately deduce the following.

Corollary 5.14. *There are only finitely many thick components of $J^+ \cap \mathbb{B}$.*

Proposition 5.13 is a direct consequence of the following lemma.

Lemma 5.15. *Let δ_1 be as in Proposition 5.12. There exists $\delta_2 \geqslant \delta_1$ such that if x is such that $\mathrm{Diam}_x(J^u(x)) \leqslant \delta_1$, then for every $y \in J^+_{\mathbb{B}}(x) \cap J$, $\mathrm{Diam}_y(J^u(y)) \leqslant \delta_2$.*

Proof. Indeed, by Proposition 5.12, if $\mathrm{Diam}_x(J^u(x)) \leqslant \delta_1$, then any point in $J^+_{\mathbb{B}}(x)$ can be joined to $y \in J^u(x)$ by a path contained in $W^s_{2\mathbb{B}}(y)$. Furthermore, as explained in the proof of Lemma 5.5, the plaque length of such a γ is uniformly bounded. The bound on $\mathrm{Diam}_y(J^u(y))$ then follows from the uniform continuity of holonomy along bounded paths in the stable lamination. $\square$

Remark 5.16. The argument of Propositions 5.12 and 5.13 makes no use of the fact that $J^u(x)$ is a component of $J \cap W^u(x)$. Thus, the same statements hold for the saturation by semilocal stable leaves of any (say closed) subset X of an unstable manifold: if its diameter of X is small enough, then, changing the bidisk $\mathbb{B}$ if necessary, the saturation $\hat{X}$ of X by semilocal stable manifolds is a product and all the stable slices of $\hat{X}$ have a small diameter.

Proposition 5.17. *Let Δ^u be an unstable transversal in $\mathbb{B}$. For every connected component C of $J^+ \cap \mathbb{B}$ (resp. $K^+ \cap \mathbb{B}$), $C \cap \Delta^u$ admits finitely many connected components.*

Proof. Let us first discuss the case of components of $J^+ \cap \mathbb{B}$. For thick components, the result follows immediately from Corollary 5.14, so we may assume that C is thin. As already seen, C intersects Δ^u. Pick $x \in C \cap \Delta^u$, in particular, $x \in J$. Since C is thin, for some $1 \leqslant r \leqslant 2$, $W^s_{r\mathbb{B}}(x)$ is transverse to $\partial(r\mathbb{B})$ and by Proposition 5.12, $J^u(x)$ can be followed under holonomy along $W^s_{r\mathbb{B}}(x)$. Since $W^s_{r\mathbb{B}}(x)$ and Δ^u have finitely many intersection points, we infer that $J^+_{r\mathbb{B}}(x) \cap \Delta^u$ has finitely many connected components. Finally, $J^+_{\mathbb{B}}(x) = C$ coincides with the component of $J^+_{r\mathbb{B}}(x) \cap \mathbb{B}$ containing x, so $C \cap \Delta^u$ is a union of connected components of $J^+_{r\mathbb{B}}(x) \cap \Delta^u$ and we conclude that there are finitely many of them.

We now discuss components of $K^+ \cap \mathbb{B}$. Recall from Lemma 2.3 that for such a component C, ∂C is a component of $J^+ \cap \mathbb{B}$. Assume first that such a component A is thin. Given $x \in A \cap \Delta^u$, $J^u(x)$ can be followed under holonomy along $W^s_{r\mathbb{B}}(x)$ for some $1 \leqslant r \leqslant 2$. If the polynomial hull of $J^u(x)$ is nonempty, then it has a small diameter and it can be followed by holonomy in $r\mathbb{B}$ along the extended foliation just as in Proposition 5.12 and it is topologically a product. It follows that $C \cap \Delta^u$ is the polynomial hull of $J^+_{\mathbb{B}}(x) \cap \Delta^u$ and it has finitely many components. On the other hand, if every component of ∂C is thick, then $\partial C \cap \Delta^u$ is contained in the finitely many components of $K^+ \cap \Delta^u$ of diameter greater than some δ, and so is $C \cap \Delta^u$. This concludes the proof. □

We conclude this subsection by giving a general description of components of $J^+ \cap \mathbb{B}$. Fix an unstable transversal Δ^u. Let $x \in J \cap \Delta^u$ and consider $W^s_{\mathbb{B}}(J^u(x)) = \bigcup_{y \in J^u(x)} W^s_{\mathbb{B}}(y)$. If every $W^s_{\mathbb{B}}(y)$ is transverse to $\partial\mathbb{B}$, then by Proposition 5.11, $W^s_{\mathbb{B}}(J^u(x)) = J^+_{\mathbb{B}}(x)$. In the general case, we define a relation between components of $J^+ \cap \Delta^u$ by declaring that $C_1 \leftrightarrow C_2$ if and only if there exists $x \in C_1$ such that $W^s_{\mathbb{B}}(x) \cap C_2 \neq \emptyset$ (or equivalently there exists $(x_1, x_2) \in C_1 \times C_2$ such that $W^s_{\mathbb{B}}(x_1) = W^s_{\mathbb{B}}(x_2)$). Then extend this relation to an equivalence relation (still denoted by $\leftrightarrow$) by allowing finite chains $C_1, \dots, C_n$. Finally, we define

$$\widehat{W^s_{\mathbb{B}}}(J^u(x)) := \bigcup_{C \leftrightarrow J^u(x)} \bigcup_{y \in C} W^s_{\mathbb{B}}(y).$$

Proposition 5.18. *For any $x \in J$, $J^+_{\mathbb{B}}(x)$ coincides with $\widehat{W^s_{\mathbb{B}}}(J^u(x))$.*

Proof. By Proposition 5.17, $J^+_{\mathbb{B}}(x) \cap \Delta^u$ admits finitely many connected components $(C_i)_{i \in I}$. Every point $z \in J^+_{\mathbb{B}}(x)$ belongs to some $W^s_{\mathbb{B}}(y)$, $y \in \Delta^u$, and necessarily y belongs to some C_i, say C_{i_0}. Furthermore, if $z' \in J^+_{\mathbb{B}}(x)$ is close to z, by the continuity of stable manifolds, there exists $y' \in \Delta^u$ close to y such that $z' \in W^s_{\mathbb{B}}(y')$. Since the C_i are at positive distance from each other, it follows that y' belongs to C_{i_0}. In other words, $W^s_{\mathbb{B}}(C_{i_0})$ is relatively open in $J^+_{\mathbb{B}}(x)$. Clearly, $W^s_{\mathbb{B}}(C_{i_0})$ is connected, and even arcwise connected since by Theorem 3.6 C_i is locally connected. Thus, the $W^s_{\mathbb{B}}(C_i)$ realize a finite cover of $J^+_{\mathbb{B}}(x)$ by connected open sets, which are contained in or disjoint from $J^+_{\mathbb{B}}(x)$. Define a nonoriented graph on I by joining i and j whenever $W^s_{\mathbb{B}}(C_i) \cap W^s_{\mathbb{B}}(C_j) \neq \emptyset$.

If we fix i_0 such that $W^s_{\mathbb{B}}(C_{i_0}) \subset J^+_{\mathbb{B}}(x)$, it follows that $J^+_{\mathbb{B}}(x) = \bigcup_{i\in I_0} W^s_{\mathbb{B}}(C_i)$ where I_0 is the component of i_0 in the graph. This is exactly the announced description. □

Let us point out the following interesting consequence of the proof.

Corollary 5.19. *Every connected component of $J^+ \cap \mathbb{B}$ (resp. $K^+ \cap \mathbb{B}$) is locally connected.*

Proof. Given a component $J^+_{\mathbb{B}}(x)$ of $J^+ \cap \mathbb{B}$, with notation as in the previous proof, $(W^s_{\mathbb{B}}(C_i))_{i\in I}$ is a finite cover of $J^+_{\mathbb{B}}(x)$ by locally connected and relatively open sets: local connectedness follows. If now C is a component of $K^+ \cap \mathbb{B}$, we saw in the proof of Proposition 5.17 that ∂C is a finite union of components of $J^+ \cap \mathbb{B}$; therefore, ∂C is locally connected. General topology then implies that C is locally connected and we are done. □

5.3 | Induced dynamics on the set of components of J^+

We still consider a uniformly hyperbolic dissipative Hénon map, with a disconnected and stably totally disconnected Julia set, and fix a large bidisk $\mathbb{B}$ as before. Since f maps $K^+ \cap \mathbb{B}$ (resp. $J^+ \cap \mathbb{B}$) into itself, it induces a dynamical system on the set of its connected components. Recall that a component is said *nontrivial* if it is not reduced to a vertical submanifold.

Theorem 5.20. *Let f be dissipative and hyperbolic with a disconnected and stably totally disconnected Julia set and $\mathbb{B} \subset \mathbb{C}^2$ be a large bidisk. Then $K^+ \cap \mathbb{B}$ (resp. $J^+ \cap \mathbb{B}$) admits uncountably many components, at most countably many of which being nontrivial. Any nontrivial connected component of $K^+ \cap \mathbb{B}$ (resp. $J^+ \cap \mathbb{B}$) is preperiodic, and there are finitely many nontrivial periodic components.*

Remark 5.21. Notice that a periodic component of $K^+ \cap \mathbb{B}$ can be trivial, that is, a vertical submanifold. Since it is mapped into itself by some f^N in this case, we conclude that it is of the form $W^s_{\mathbb{B}}(x)$ for some saddle periodic point x.

Lemma 5.22. *The function $y \mapsto \mathrm{Diam}_y(J^u(y))$ (resp. $y \mapsto \mathrm{Diam}_y(K^u(y))$) is upper semicontinuous on J. In particular, if $y_n \to y_\infty$ and $(\mathrm{Diam}_{y_n}(K^u(y_n)))$ is unbounded, then $K^u(y_\infty)$ is leafwise unbounded, and likewise for J^u.*

Proof. Recall that $\mathrm{Diam}_y(J^u(y)) = \mathrm{Diam}_y(K^u(y))$ for every $y \in J$ (including the case where it is infinite), so it is enough to deal with $K^u(y)$. Assume first that the y_n belong to the same local leaf and $y_n \to y_\infty$. If $K^u(y_\infty)$ is leafwise bounded, we can consider a closed loop γ enclosing it and disjoint from K^+. Then, for large enough n, γ also encloses $K^u(y_n)$, and any cluster value of this sequence for the Hausdorff topology is a continuum contained in K^+ and containing y_∞. It follows that

$$\limsup_{n\to\infty} \mathrm{Diam}_{y_\infty}(K^u(y_n)) \leqslant \mathrm{Diam}_{y_\infty}(K^u(y_\infty))$$

hence

$$\limsup_{n\to\infty} \mathrm{Diam}_{y_n}(K^u(y_n)) \leqslant \mathrm{Diam}_{y_\infty}(K^u(y_\infty)),$$

as desired. Of course, if $K^u(y_\infty)$ is leafwise unbounded, the inequality is obvious.

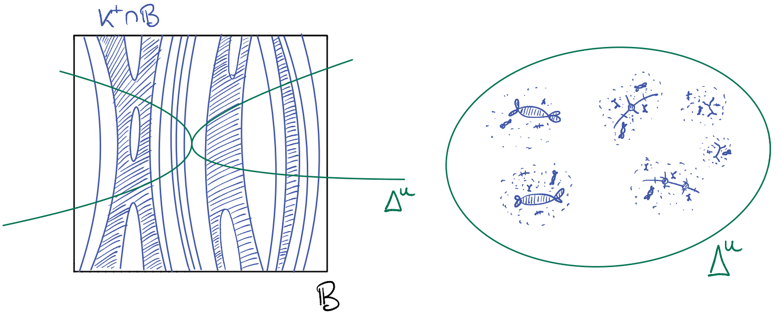

FIGURE 4 Illustration to Theorem 5.20.

Assume now that the y_n belong to different local leaves. As before, the case where $K^u(y_\infty)$ is leafwise unbounded is obvious. If $K^u(y_\infty)$ is leafwise bounded, again we consider a closed loop γ enclosing it and disjoint from K^+. In addition, we can assume that $\mathrm{Diam}_{y_\infty}(\gamma)$ is arbitrary close to $\mathrm{Diam}_{y_\infty}(K^u(y_\infty))$. When $y_n \to y_\infty$, γ can be lifted to a loop $\widetilde{\gamma}_n$ in $W^u(y_n)$, with roughly the same diameter (here we use the continuity of the leafwise distance d^u_y), and $K^u(y_n)$ is enclosed in $\widetilde{\gamma}_n$. The semicontinuity of the diameter follows. $\square$

Proof of Theorem 5.20. (See Figure 4 for a visual illustration of the proof.) Fix an unstable transversal Δ^u, and recall that any component of $K^+ \cap \mathbb{B}$ (resp. $J^+ \cap \mathbb{B}$) intersects Δ^u. By [5, Thm 7.1], $J^+ \cap \Delta^u$ admits uncountably many point components, thus the first assertion of the theorem follows from Proposition 5.9. Then Corollary 5.10 asserts that at most countably many components are nontrivial.

Let $x \in J^+ \cap \Delta^u$ and assume that $J^+_{\mathbb{B}}(x)$ (or equivalently $K^+_{\mathbb{B}}(x)$) is nontrivial. Since Δ^u is a global transversal, $J^u(x)$ is leafwise bounded. For $n \geqslant 0$, $J^u(x_n) = f^n(J^u(x))$ where $x_n = f^n(x)$, and by (1),

$$\mathrm{Diam}_{x_n}(J^u(x_n)) \geqslant Cu^n \,\mathrm{Diam}_x(J^u(x)) \underset{n\to\infty}{\longrightarrow} \infty. \tag{4}$$

Let x_∞ be any accumulation point of (x_n). By Lemma 5.22, $J^u(x_\infty)$ is leafwise unbounded, and so does $K^u(x_\infty)$.

By local product structure, for large n, the holonomy along the stable lamination defines a projection

$$D^u(x_n, 3/2) \cap J^+ \to D^u(x_\infty, 2) \cap J^+$$

which we simply denote by π^s. It is Lipschitz (see Lemma 5.7) and a homeomorphism onto its image. Notice that $\pi^s(D^u(x_n, 3/2) \cap J^+)$ contains $D^u(x_\infty, 1) \cap J^+$ for large n. For large n, $J^u(x_n)$ intersects the boundary of $D^u(x_n, 3/2)$, so the sets $J^u(\pi^s(x_n))$ define a sequence of components of $J^+ \cap D^u(x_\infty, 1)$ of diameter bounded from below. From Theorem 3.6, we infer that this sequence is finite. Let us denote by C_j, $j = 1, \dots, N$ these components. By the Pigeonhole Principle, there exist $n \neq n'$ such that $\pi_s(x_n)$ and $\pi_s(x_{n'})$ belong to the same C_j, thus x_n and $x_{n'}$ belong the local stable saturation of C_j. Therefore, the sequence $(J^+_{\mathbb{B}}(x_n))$ is eventually periodic, and so is $(K^+_{\mathbb{B}}(x_n))$.

Consider now a nontrivial periodic component C of $J^+ \cap \mathbb{B}$. Then it is of the form $J^+_{\mathbb{B}}(x)$ for some $x \in \Delta^u \cap J^+$. The previous argument shows that there are points $x' \in C \cap J$ such that $J^u(x')$

is leafwise unbounded. By Proposition 5.13, the components of the slices $J^+_{\mathbb{B}} \cap \Delta^u$ have diameter uniformly bounded from below (here we use the fact that for every $x \in \Delta^u \cap J^+$, the distance d^u_x is uniformly comparable to the ambient distance on Δ^u). Thus, by Theorem 3.6, only finitely many such components can arise and we conclude that C belongs to a finite set of components. The corresponding result for components of $K^+ \cap \mathbb{B}$ follows from Lemma 2.3. $\square$

Remark 5.23. Using techniques similar to those of §5.2, it is easily seen that any component of $K^+ \cap \mathbb{B}$ has finitely many preimages. In other words, the induced dynamical system on components of $K^+ \cap \mathbb{B}$ is finite-to-1. Indeed, assume by contradiction that C is a component such that $f^{-1}(C) \cap \mathbb{B}$ has infinitely many preimages C_i. Then by Theorem 3.6, for some i, $C_i \cap \Delta^u$ has a component of small diameter. Therefore, by pushing forward, there is some $x \in C \cap J$ such that $\mathrm{Diam}_x(J^u(x))$ is small, that is, $J^+_{\mathbb{B}}(x)$ (or equivalently $K^+_{\mathbb{B}}(x)$) is thin. But it is easy to show that a thin component admits finitely many preimages, and we arrive at the desired contradiction.

6 | COMPONENTS OF J AND K

We keep the same setting as before, that is, f is a uniformly hyperbolic dissipative Hénon map, with a disconnected and stably totally disconnected Julia set. In this section, we complete the proof of the main theorem by classifying the connected components of J and K.

We start with an easy fact. Recall the notation $E(x) = \mathrm{Comp}_E(x)$.

Proposition 6.1. *If $x \in J$ is such that $J^u(x)$ is leafwise bounded, then $J(x) = J^u(x)$.*

Proof. First, $J^u(x)$ is a connected set such that $x \in J^u(x) \subset J$, so it is contained in $J(x)$. To prove the converse statement, let (U_n) be a sequence of open neighborhoods of $J^u(x)$ in $W^u(x)$ decreasing to $J^u(x)$ and such that $\partial_{\mathrm{i}} U_n \cap J = \emptyset$. Since $J^s(x) = \{x\}$, for every n, any sufficiently small loop γ about x in $W^s(x)$ can be propagated along U_n to yield an open set $\widetilde{U}_n$ such that $\partial \widetilde{U}_n = \emptyset$. Note that we did not prove any extension result for the unstable lamination, so we cannot simply say that we propagate γ by using some "unstable holonomy." On the other hand, we can simply use the inclination lemma, by pushing forward a small thickening of $f^{-n}(\gamma)$ as a 3-manifold transverse to $W^s(f^{-n}(x))$. Finally, for every n, $\partial \widetilde{U}_n$ is relatively open and closed in J, so it contains $J(x)$ and we conclude that $J(x) = J^u(x)$. $\square$

To understand the structure of periodic components of J, let us introduce a definition.

Definition 6.2. A *quasi-solenoid* is a saddle hyperbolic set such that $f^k(\Lambda) = \Lambda$ for some k and:

- Λ is connected;
- Λ has local product structure;
- for every $x \in \Lambda$, $\Lambda \cap W^u(x)$ is leafwise unbounded and locally connected, and $\Lambda \cap W^s(x)$ is totally disconnected.

Observe that in this definition, we do not require that $\Lambda \cap W^s_{\mathrm{loc}}(x)$ is a Cantor set. In other words, we allow for isolated points in a stable transversal (this phenomenon will be ruled out later under appropriate hypotheses, see Theorem 8.8).

Theorem 6.3. *Let f be dissipative and hyperbolic with a disconnected and stably totally disconnected Julia set and $\mathbb{B}$ be as above. Let C be a periodic component of $J^+ \cap \mathbb{B}$ and k be its period. Then $\Lambda := \bigcap_{n\geqslant 0} f^{kn}(C)$ is a point or a quasi-solenoid, and it is a connected component of J.*

Proof. Replacing f by some iterate, we may assume that C is invariant, that is, $k = 1$. If C is a vertical manifold, it follows from Remark 5.21 that Λ is a point, and the other properties follow easily, so the interesting case is when C is nontrivial. Then, arguing in the proof of Theorem 5.20, by (4), C contains points such that $\mathrm{Diam}_x(J^u(x))$ is arbitrary large, so it is thick in the sense of Proposition 5.13. Define $\Lambda := \bigcap_{n\geqslant 0} f^n(C) = \bigcap_{n\geqslant 0} \overline{f^n(C)}$. Since by assumption $f(C) \subset C$, Λ is a decreasing intersection of compact connected sets. Hence, Λ is an invariant connected hyperbolic set contained in J, and $f(\Lambda) = \Lambda$. Let us show that it is a connected component of J. For this, let Λ' be the connected component of Λ in J. By definition, $\Lambda \subset \Lambda'$. Since Λ' is connected and contained in $J^+ \cap \mathbb{B}$, it must be contained in C. Furthermore, since $f(\Lambda) = \Lambda$, and f permutes the components of J, we have that $f(\Lambda') = \Lambda'$, hence for every $n \geqslant 1$, $f^{-n}(\Lambda') \subset C$, and we conclude that $\Lambda' \subset \bigcap_{n\geqslant 0} f^n(C) = \Lambda$, as was to be shown.

We claim that for every $x \in \Lambda$, $J^u(x)$ is leafwise unbounded. Indeed, for every $x \in \Lambda$, we have that $x = f^n(x_{-n})$ with $x_{-n} = f^{-n}(x) \in C$ and since C is thick, $\mathrm{Diam}_{x_{-n}}(J^u(x_{-n}))$ is uniformly bounded from below, and the result follows.

By Lemma 3.9, for every $x \in \Lambda$, there are only finitely many components of $J \cap D^u(x, 1)$ intersecting $\partial_{\mathrm{i}} D^u(x, 1)$ and $D^u(x, 1/2)$. A simple compactness argument using the holonomy invariance of J^+ shows that this number is uniformly bounded; therefore, there exists a uniform $\delta > 0$ such that leafwise unbounded components of J^+ intersecting $D^u(x, 1/2)$ are δ-separated in $D^u(x, 1)$ relative to the distance d^u_x (or equivalently, relative to the ambient one). From this, we deduce that for every $x \in \Lambda$, there exists $\delta > 0$ such that Λ coincides with $J^u(x)$ in $W^u_\delta(x)$, and it follows from Theorem 3.6 that Λ is locally connected in the unstable direction.

Let us show that Λ has local product structure. For this, let $y_1, y_2 \in \Lambda$ be close (i.e., $d(y_1, y_2) \ll \delta$), denote by $\pi^s : W^u_{\mathrm{loc}}(y_1) \to W^u_{\mathrm{loc}}(y_2)$ the projection along stable leaves, and let $z_2 = \pi^s(y_1)$. Since $J^u(y_1)$ and $J^u(y_2)$ are leafwise unbounded, if $d(y_1, y_2)$ is small enough, $J^u(z_2)$ intersects $\partial_{\mathrm{i}} D^u(y_2, 1)$, and so does $J^u(y_2)$. By definition of δ, it follows that $J^u(y_2) = J^u(z_2)$, hence y_2 and z_2 belong to the same connected component of J. In particular, z_2 belongs to C. Since f^{-1} contracts distances along unstable manifolds, and respects connected components of J, we can repeat this argument with $f^{-n}(y_2)$ and $f^{-n}(z_2)$ for any $n \geqslant 0$ and we conclude that $z_2 \in \Lambda$, as was to be shown. $\square$

Theorem 6.4. *Let f be dissipative and hyperbolic with a disconnected and stably totally disconnected Julia set. Then every component of J is either*

(1) *a point;*
(2) *or of the form $J^u(x)$ with $J^u(x)$ nontrivial and leafwise bounded;*
(3) *or a periodic quasi-solenoid.*

In addition:

(i) *There are finitely many quasi-solenoidal components*
(ii) *Every periodic component of J is either a point or a quasi-solenoid.*
(iii) *Every nontrivial component of J is attracted by a quasi-solenoid. More precisely, given a nontrivial component C for every $\delta > 0$, there exists n such that $f^{kn}(C) \subset W^s_\delta(\Lambda)$, where Λ is a quasi-solenoid of period k.*

Note that in assertion (*ii*), the uniformity of n as a function of δ is not a direct consequence of the fact that $\omega(C) \subset \Lambda$.

Proof. To establish the announced trichotomy, by Proposition 6.1, it is enough to show that if C is a component such that for some $x \in C$, $J^u(x)$ is leafwise unbounded, then C is a periodic quasi-solenoid. Note that for every $n \geqslant 1$, $J^u(f^{-n}(x))$ is leafwise unbounded. Therefore, the component of $f^{-n}(x)$ in $J^+ \cap \mathbb{B}$ is thick in the sense of Proposition 5.13, and by Corollary 5.14, $J^+_{\mathbb{B}}(f^{-n}(x))$ belongs to a finite set of semilocal components. Thus, there exists a component C^+ of $J^+ \cap \mathbb{B}$ and an infinite sequence n_i such that $f^{-n_i}(x) \in C^+$, hence C^+ is periodic of some period k and reversing time we get that $J^u(x)$ is included in $\Lambda := \bigcap_{n\geqslant 0} f^{kn}(C^+)$. By Theorem 6.3, Λ is a quasi-solenoid and $J(x) = C = \Lambda$.

Since there are only finitely many periodic semilocal components of J^+, this argument shows that J has only finitely many solenoidal components.

For assertion (*ii*), let C be a periodic component of J that is not reduced to a point, and let $x \in C$. Without loss of generality, we assume that C is fixed. Expansion in the unstable direction shows that if $J^u(x)$ is leafwise bounded, then $J^u(x) = \{x\}$, which is a contradiction. Thus, by the first part of the proof, C is a quasi-solenoid.

To prove (*iii*), let C be a nontrivial component of J, and for some large bidisk $\mathbb{B}$, let C^+ be the component of $J^+ \cap \mathbb{B}$ containing C. Then by Theorem 5.20, C^+ is ultimately periodic (with preperiod k), thus by Theorem 6.3, $\bigcap_{n\geqslant 0} f^{kn}(C^+)$ is a periodic quasi-solenoid Λ. This shows that C is attracted by Λ in the sense that for large n, $f^{kn}(C)$ is contained in a δ-neighborhood of Λ. To get the more precise statement that $f^{kn}(C) \subset W^s_\delta(\Lambda)$, we have to show that $W^s_\delta(\Lambda)$ is relatively open in $C^+ \cap J$. The argument is the same as for the local product structure: since large leafwise components of J are separated by some uniform distance and C is thick, if $x \in C \cap J$ is sufficiently close to $y \in \Lambda$, $W^s_{\mathrm{loc}}(x) \cap W^u_{\mathrm{loc}}(y)$ must belong to a large component of $W^u_{\mathrm{loc}}(y) \cap J$, therefore it belongs to $J^u(y)$, and we are done. □

Remark 6.5. Leafwise bounded components of J are locally connected, as follows from Theorem 3.6. On the other hand, a quasi-solenoid is not locally connected, since it locally has the structure of a Cantor set times a (locally) connected set.

The following result says that there is a 1-1 correspondence between components of K and J, so that the previous theorems yield a description of components of K as well.

Proposition 6.6. *Every component of K contains a unique component of J.*

For polynomials in one variable, the analogous statement is the fact that every component of K has a connected boundary, which follows from polynomial convexity. Here, components of K have empty interior so this has to be formulated differently.

Proof. Every component of K contains a point of J, for otherwise it would be contained in $\mathrm{Int}(K^+)$, so it is of the form $K(x)$ for some $x \in J$. If $J(x) = \{x\}$, the result is obvious. Now assume that $J^u(x)$ is leafwise bounded. By Lemma 2.4, $K^u(x)$ is obtained by filling the holes of $J^u(x)$ in $W^u(x) \simeq \mathbb{C}$, so $J^u(x)$ is equal to the intrinsic boundary of $K^u(x)$ and the result follows.

The most interesting case is when $J(x)$ is a quasi-solenoid. Replacing f by f^k for some $k \geqslant 1$, we may assume that $J(x)$ is fixed. We proved in Theorem 6.3 that $J(x) = \bigcap_{n\geqslant 0} f^n(J^+_{\mathbb{B}}(x))$. The very same proof shows that $K(x) = \bigcap_{n\geqslant 0} f^n(K^+_{\mathbb{B}}(x))$. By Lemma 2.3, $\partial K^+_{\mathbb{B}}(x)$ contains a unique component of $J^+_{\mathbb{B}}(x)$ (namely, its boundary), and we conclude by arguing that if $K(x)$ contained two distinct components $J(x)$ and $J(y)$ of J, then $K^+_{\mathbb{B}}(x)$ would contain $J^+_{\mathbb{B}}(x)$ and $J^+_{\mathbb{B}}(y)$, which must be distinct because $\bigcap_{n\geqslant 0} f^n(J^+_{\mathbb{B}}(x)) \neq \bigcap_{n\geqslant 0} f^n(J^+_{\mathbb{B}}(y))$, and this is impossible. $\square$

7 | COMPLEMENTS

We keep the setting as in Sections 5 and 6. Here, we prove a number of complementary facts that do not enter into the proof of the main theorem, so we sometimes allow the presentation to be a little sketchy.

7.1 | Transitivity

A desirable property of quasi-solenoids is transitivity, or chain transitivity. At this stage, we are not able to show that quasi-solenoidal components are transitive, but let us already explain a partial result in this direction. The full statement will be obtained in Theorem 8.8 under an additional assumption.

Proposition 7.1. *If Λ is a quasi-solenoidal component of J of period k, there exists a quasi-solenoid $\Lambda' \subset \Lambda$ of period $k\ell$, which is saturated by unstable components (i.e., if $x \in \Lambda'$, then $J^u(x) \subset \Lambda'$), with the property that $f^{k\ell}|_{\Lambda'}$ is topologically mixing. In addition, stable slices of Λ' are Cantor sets and for every periodic point $p \in \Lambda'$, $\Lambda' = \overline{J^u(p)}$.*

This proposition follows from general facts from hyperbolic dynamics. Let us recall some basics. Recall that if Λ is a compact hyperbolic set with local product structure, then by Smale's Spectral Decomposition Theorem (see, e.g., [48, §4.2]), the nonempty closed invariant subset

$$\Omega := C(f|_\Lambda) = \overline{\mathrm{Per}(f|_\Lambda)}$$

(where by definition $C(f|_\Lambda)$ is the chain recurrent set of $f|_\Lambda$) admits a decomposition of the form $\Omega = \Omega_1 \cup \cdots \cup \Omega_N$. The Ω_i are called the basic pieces. They are closed (and hence relatively open in Ω), f induces a permutation on the basic pieces and if q is the least integer such that $f^q(\Omega_i) = \Omega_i$, then $f^q|_{\Lambda_i}$ is topologically mixing. In addition, Ω and the Ω_i have local product structure.

Proof. For notational simplicity, replace f^k by f so that $k = 1$. Consider the ω-limit set $\omega(\Lambda) = \bigcup_{x\in\Lambda} \omega(x)$. Since a limit point is nonwandering, it is chain recurrent, so $\omega(\Lambda) \subset \Omega$. Conversely, since any periodic point is an ω-limit point, we see that $\mathrm{Per}(f|_\Lambda) \subset \omega(\Lambda)$, hence $\Omega \subset \omega(\Lambda)$ and $\omega(\Lambda) = \Omega$. Then, the Shadowing Lemma implies that $\Lambda \subset W^s(\Omega) = \bigcup_{x\in\Omega} W^s(x)$. Fix a small $\delta > 0$: then $W^s(\Omega) = \bigcup_{n\geqslant 0} f^{-n}\left(W^s_\delta(\Omega)\right)$. By Baire's theorem, there exists n such that $f^{-n}\left(W^s_\delta(\Omega)\right)$ has nonempty relative interior in Λ, hence so does $W^s_\delta(\Omega)$, and we conclude that for some i_0, $W^s_\delta(\Omega_{i_0})$ has relative nonempty interior in Λ. Let us show that $\Lambda' = \Omega_{i_0}$ satisfies the requirements of the proposition.

If ℓ is the least integer such that $f^\ell(\Lambda') = \Lambda'$, the fact that $f^\ell|_{\Lambda'}$ is topologically mixing follows from the spectral decomposition theorem. Since Λ' has local product structure and $W^s_\delta(\Lambda')$ has relative nonempty interior in Λ, we see that there exists a relatively open subset U in Λ' such that for any $x_0 \in U$, a neighborhood of x_0 in $J^u(x_0)$ in contained in Λ'. Since $f^\ell|_{\Lambda'}$ is topologically transitive, we may assume that x_0 has a dense orbit under f^ℓ. So, if $y \in \Lambda'$ is arbitrary, we can find a sequence (n_j) such that $f^{\ell n_j}(x_0) \to y$. By expansion in the unstable direction, there exists a uniform $\delta > 0$ such that for every j, $f^{\ell n_j}(\Lambda') = \Lambda'$ contains a δ-neighborhood of $f^{\ell n_j}(x_0)$ in $J^u(f^{\ell n_j}(x_0))$, so by local product structure, we conclude that a neighborhood of y in $J^u(y)$ is contained in Λ'. On the other hand, since Λ' is closed, it is also relatively closed in unstable manifolds. This shows that Λ' is saturated by unstable components.

Let us show that for every periodic point $p \in \Lambda'$, $\overline{J^u(p)} = \Lambda'$. Let $N = \ell m$ be the period of p. Since $f^\ell|_{\Lambda'}$ is topologically mixing, $f^{\ell m}|_{\Lambda'}$ is topologically transitive, so there exists y arbitrary close to p such that $(f^{\ell mn}(y))_{n\geqslant 0}$ is dense in Λ'. Let y' be the projection of y in $W^u_{\mathrm{loc}}(p)$ under stable holonomy. By local product structure, y' belongs to $J^u(p)$, and $y' \in W^s(y)$ so $(f^{\ell mn}(y'))$ is dense, too. Since all these points belong to $J^u(p)$, we conclude that $J^u(p)$ is dense in Λ', as asserted.

For p as above, since $J^u(p)$ is leafwise unbounded, it must accumulate nontrivially in Λ'. More precisely, there exists $x \in \Lambda'$ and a sequence of points $x_n \in J^u(p)$, with $x_n \notin W^u_{\mathrm{loc}}(x)$ and $x_n \to x$. Note that by local product structure, $W^u_{\mathrm{loc}}(x_n) \cap \Lambda'$ corresponds to $W^u_{\mathrm{loc}}(x) \cap \Lambda'$ under local stable holonomy. Now as before there exists $y' \in W^u_{\mathrm{loc}}(p) \cap \Lambda'$ whose orbit is dense in Λ'. Thus, any $z \in \Lambda'$ is the limit of $f^{n_j}(y')$ for some subsequence n_j. But $f^{n_j}(y')$ is an accumulation point of $W^s_{\mathrm{loc}}(f^{n_j}(y')) \cap \Lambda'$, so the same holds for z, and we conclude that Λ' is transversally perfect in the stable direction, hence it is transversally a Cantor set. $\square$

7.2 | Basins and solenoids

Assume that f has an attracting cycle $\{a_1, \dots, a_q\}$ of exact period q. We denote by $\mathcal{B}$ its basin of attraction, which is made up of q connected components $\mathcal{B}_i$ biholomorphic to $\mathbb{C}^2$. For every i, we can write $\mathcal{B}_i \cap \mathbb{B}$ as the (at most) countable union $(\mathcal{B}_{i,j})_{j\geqslant 0}$ of its components, with $a_i \in \mathcal{B}_{i,0}$. We refer to these open sets as basin components and to $\mathcal{B}_{i,0}$ as the *immediate basin* of a_i. Note that if we replace f by f^q, the basin of attraction of a_i is now made up of a single component, but $\mathcal{B}_{i,0}$ is unchanged.

By definition, a *Jordan star* in $U \subset \mathbb{C}$ is a finite union of simple Jordan arcs in U, intersecting at a single point.

Theorem 7.2. *Let f be dissipative and hyperbolic with a disconnected and stably totally disconnected Julia set. Suppose that f admits an attracting fixed point with immediate basin $\mathcal{B}_0$. Then:*

(i) *$\partial\mathcal{B}_0$ is a properly immersed topological submanifold of dimension 3, which intersects any global unstable transversal in finitely many Jordan domains.*
(ii) *$\bigcap_{n\geqslant 0} f^n(\partial\mathcal{B}_0)$ is a quasi-solenoid, whose unstable slices are Jordan stars. In particular, there is a (saddle) periodic point in $\partial\mathcal{B}_0$.*

We can be more precise about the structure of $\partial\mathcal{B}_0$: locally, it is homeomorphic to the product of a 2-disk by a Jordan star. The proof of the theorem shows that if the components of $\mathcal{B}_0 \cap \Delta^u$ have disjoint closures, then these stars are reduced to Jordan arcs, that is, $\partial\mathcal{B}_0$ is a topological submanifold.

The following basic fact is crucial for the proof.

Lemma 7.3. *The stable lamination $\mathcal{W}^s$ respects basin boundaries. That is, if $x \in J^+$ belongs to the boundary of an attracting basin $\mathcal{B}$, then so does its image under stable holonomy.*

Proof. This follows readily from the existence of a local extension of the stable lamination (Lemma 5.7): indeed, if a leaf of the extended foliation joined a point from $\mathrm{Int}(K^+)$ to a point of $(K^+)^\complement$, it would have to intersect J^+. (See also [16], Step 3 of the proof of the main theorem, for an alternate argument without extending the stable lamination.) □

Proof of Theorem 7.2. Fix a global unstable transversal Δ^u. Since every semilocal stable manifold intersects Δ^u, $\mathcal{B}_0 \cap \Delta^u$ is nonempty, and by the Maximum Principle, each of its connected components is a topological disk. Pick such a connected component Ω_0. By the John–Hölder property (Theorem 3.10), $\partial\Omega_0$ is locally connected, and by the Maximum Principle again, there is no cut point, and it follows that Ω_0 is a Jordan domain (see [39, Thm 2.6]).

If the diameter of Ω_0 is small, then, by Remark 5.16, enlarging $\mathbb{B}$ if necessary the saturation $\widehat{\partial\Omega_0}$ of $\partial\Omega_0$ by semilocal stable leaves is topologically a product and we infer that $\widehat{\partial\Omega_0} \cap \Delta^u$ has finitely many components. Otherwise the diameter is large and by the same remark, every component of $\widehat{\partial\Omega_0} \cap \Delta^u$ has a large diameter. Then, the finiteness of the number of such components follows from the John–Hölder property of $W^u(x) \setminus K^+$, Proposition 5.17, and the finiteness statement for interior components in Lemma 3.9.

By the Maximum Principle, if Ω_0 and Ω_1 are two components of $\mathcal{B}_0 \cap \Delta^u$ such that $\overline{\Omega_0} \cap \overline{\Omega_1} \neq \emptyset$, then $\overline{\Omega_0} \cap \overline{\Omega_1}$ is a single point. Indeed, if this set contained two distinct points z and z', by using crosscuts of Ω_0 and Ω_1 ending at z and z', we could construct a Jordan domain U with $\partial U \subset \overline{\Omega_0} \cup \overline{\Omega_1}$, and U would be contained in the Fatou set, a contradiction. Create a plane graph from $\mathcal{B}_0 \cap \Delta^u$ whose vertices are its components and edges are added when two components touch. The Maximum Principle again shows that this graph is a finite union of trees. Since the stable holonomy respects $\partial\mathcal{B}_0$ and $\partial\mathcal{B}_0$ is obtained from $\partial\mathcal{B}_0 \cap \Delta^u$ by saturating by stable manifolds, the description of $\partial\mathcal{B}_0$ as a properly immersed topological submanifold of dimension 3 follows.

The proof of the second item of the theorem is similar to that of Theorem 6.3. First, $\partial\mathcal{B}_0$ is connected: the argument is identical to that of Lemma 2.3. Then, for every $x \in \partial\mathcal{B}_0 \cap J^-$, there are only finitely many components of $\mathcal{B}_0 \cap D^u(x,1)$ (resp. $\partial\mathcal{B}_0 \cap D^u(x,1)$) intersecting $D^u(x,1/2)$. Indeed, observe first that it is enough to prove this in $D^u(x,r)$ for some uniform r. By the uniform boundedness of the degree of semilocal stable manifolds in $\mathbb{B}$, there is a uniform r such that $D^u(x,r)$ can be pushed to Δ^u by stable holonomy, and the applying item (i) of the theorem completes the argument. From this point, we proceed exactly as in Theorem 6.3. The existence of a periodic point in $\partial\mathcal{B}_0$ follows from general hyperbolic dynamics (see the comments after Proposition 7.1). □

Remark 7.4. It follows from this description that if $x \in \Lambda$ lies at the boundary of $\mathcal{B}_0$, then in $W^u(x)$, x belongs to the boundary of a component Ω of $\mathcal{B}_0 \cap W^u(x)$. In particular, Ω is a Fatou disk contained in $\mathrm{Comp}_K(x)$.

Remark 7.5. We do not know whether components of $\mathcal{B}_0 \cap \Delta^u$ can actually bump into each other, or equivalently if $\bigcap_{n\geqslant 0} f^n(\partial\mathcal{B}_0)$ does contain stars. If bumping occurs, let E be the finite set of points at which the closures of the components of $\mathcal{B}_0 \cap \Delta^u$ touch each other. Then, $W^s_{\mathbb{B}}(E)$ is a finite union of vertical submanifolds, and $f(W^s_{\mathbb{B}}(E)) \subset W^s_{\mathbb{B}}(E)$. It follows that $\bigcap_{n\geqslant 0} f^n(W^s_{\mathbb{B}}(E))$

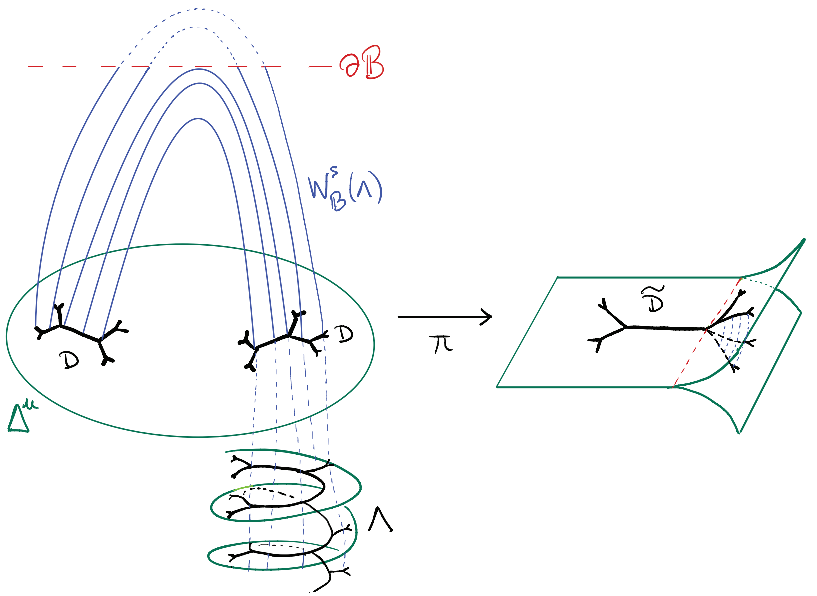

FIGURE 5 Branched Julia set model. Tangencies between $\partial\mathbb{B}$ and stable manifolds create branching.

is a finite set of periodic points, and for any other point x in the limiting quasi solenoid $\Lambda := \bigcap_{n\geqslant 0} f^n(\partial\mathcal{B}_0)$, $\Lambda \cap W^u_{\mathrm{loc}}(x)$ is a Jordan arc. Thus, roughly speaking, Λ *has the structure of finitely many solenoids attached at periodic "junction" points.*

7.3 | Branched Julia set model

Let Λ be a quasi-solenoidal component of J, and without loss of generality, assume that Λ is fixed. Let $J^+_{\mathbb{B}}(\Lambda)$ be its connected component in $J^+_{\mathbb{B}}$ and consider its intersection $D := J^+_{\mathbb{B}}(\Lambda) \cap \Delta^u$ with some unstable transversal, which is made up of finitely many thick components. Introduce a relation $\sim$ on D by $x \sim y$ if and only if $W^s_{\underline{\mathbb{B}}}(x) = W^s_{\underline{\mathbb{B}}}(y)$, where by definition $W^s_{\underline{\mathbb{B}}}(x) = \bigcap_{\varepsilon>0} W^s_{(1+\varepsilon)\mathbb{B}}(x)$. Equivalently, $x \sim y$ if and only if $\overline{W^s_{\mathbb{B}}(x)} \cap \overline{W^s_{\mathbb{B}}(y)} \neq \emptyset$: concretely, this means that x and y are related when they are connected by a stable manifold that is tangent to $\partial\mathbb{B}$. This defines a closed equivalence relation on D. We denote by $\tilde{D} := D/\sim$ the quotient topological space, which is compact (and Hausdorff) and by $\pi : D \to \tilde{D}$ the natural projection. Since $f(W^s_{\underline{\mathbb{B}}}(x)) \subset W^s_{\underline{\mathbb{B}}}(f(x))$, f descends to the quotient $\tilde{D} := D/\sim$ to a well-defined continuous map $\tilde{f}$. Geometrically $\tilde{D}$ has to be thought of as a *branched Julia set*, lying on the branched surface — in the sense of Williams [45] — obtained by collapsing the semilocal stable leaves of the extended stable lamination. See Figure 5 for a (tentative) visual explanation. Then, $\tilde{f}$ is expanding on the plaques of this branched manifold,[†] and its iterates are uniformly quasi-conformal wherever defined, since they are obtained by iterating f and projecting along the stable lamination. Observe that f is not necessarily surjective, since for every $x \in D$, $f^n(x)$ eventually belongs to $W^s_{\mathbb{B}}(\Lambda)$, which may be smaller than $J^+_{\mathbb{B}}(\Lambda)$ (cf. Figure 1). On the other hand, by the last assertion of Theorem 6.4, there exists a

[†] Here, by plaque, we mean one of the finitely many overlapping disks that make up a local chart of a branched manifold, see [45, Def. 1.0]

uniform N such that $f^N(J^+_{\mathbb{B}}(\Lambda)) \subset W^s_{\mathbb{B}}(\Lambda)$. It follows that the sequence $\bigcap_{0\leqslant k\leqslant n} \tilde{f}^k(\tilde{D})$ is stationary for $n \geqslant N$ and that $\tilde{D}' := \pi(W^s_{\mathbb{B}}(\Lambda) \cap \Delta^u)$, is an invariant, closed, and plaque-open subset of $\tilde{D}$ on which $\tilde{f}$ is surjective.

Proposition 7.6. *With the above definitions, the dynamical system (Λ, f) is topologically conjugate to the natural extension of $(\tilde{D}, \tilde{f})$ (or equivalently $(\tilde{D}', \tilde{f})$).*

Proof. Indeed, define $h : \varprojlim(\tilde{D}, \tilde{f}) \to \Lambda$ by $h((\tilde{x}_n)_{n\in\mathbb{Z}}) = \bigcap_{n\geqslant 0} f^n(W^s_{\underline{\mathbb{B}}}(x_{-n}))$, whose inverse is $y \mapsto h^{-1}(y) = (\pi(W^s_{\underline{\mathbb{B}}}(f^n(y)) \cap \Delta^u))_{n\in\mathbb{Z}}$. □

8 | NONDIVERGENCE OF HOLONOMY AND APPLICATIONS

8.1 | The NDH property

We say that the property of *Nondivergence of Holonomy* (NDH) holds if for every pair of points $x, y \in J$ such that y belongs to $W^s(x)$, the stable holonomy, which is locally defined from a neighborhood of x in $W^u(x)$ to a neighborhood of y in $W^u(y)$, can be continued along any path contained in $J^u(x)$.

Remark 8.1.

(1) The stable holonomy $h : W^u(x) \to W^u(y)$ is independent of the choice of a path c from x to y in $W^s(x)$ because $W^s(x)$ is simply connected.
(2) An unstable component $J^u(x)$ is typically *not* simply connected (since it may enclose the trace of an attracting basin on $W^u(x)$). So, even if the stable holonomy from x to y admits an extension along continuous paths, it does not generally yield a well-defined map from $J^u(x)$ to $J^u(y)$.

We do not know any example where the NDH property fails. See Figure 6 for a visual explanation of a mechanism leading to a breakdown of holonomy continuation. An analog of this property was studied in the context of the classification of Anosov diffeomorphisms, where it is expected to be a crucial step in the classification program. It was established in the two-dimensional case in [20] (see also [9, 32] for related results).

Back to automorphisms of $\mathbb{C}^2$, we have the following simple criterion.

Proposition 8.2. *A sufficient condition for the NDH property is that the stable lamination $\mathcal{W}^s$ of J^+ is transverse to $\partial\mathbb{B}$ (No Tangency condition, NT).*

Proof. Assume that the No Tangency condition holds and let $x, y \in J$ be such that y belongs to $W^u(x)$. Replacing x and y by $f^k(x)$ and $f^k(y)$ for some positive k, we may assume that $y \in W^s_{\mathbb{B}}(x)$. There is a germ of stable holonomy h sending a neighborhood of x in $J^u(x)$ to some neighborhood of $y \in J^u(y)$. Let $\gamma : [0, 1] \to J^u(x)$ be a continuous path: we have to show that h can be continued along γ. For this, introduce $E \subset [0, 1]$ the set of parameters t such that h can be continued along $\gamma|_{[0,t]}$ and $h(\gamma(t)) \in W^s_{\mathbb{B}}(\gamma(t))$. Obviously, E is a relatively open subinterval of [0,1] containing 0, and the proof will be complete if we show that E is closed. Thus, assume that $(t_n) \in E^{\mathbb{N}}$ is an

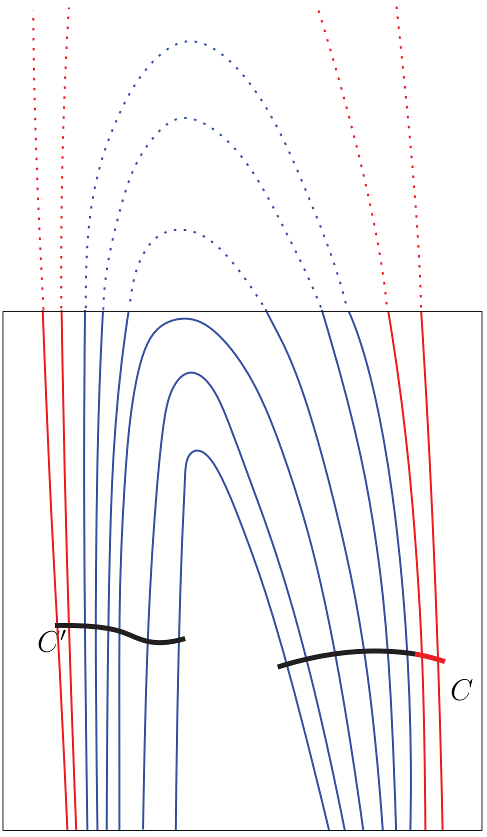

FIGURE 6 Divergence of holonomy: the holonomy between C and C' cannot be continued from the left part of C to its right part (in red).

increasing sequence converging to t_∞, and let y_∞ be any cluster value of the sequence $(h(\gamma(t_n)))$. The main observation is that since $\mathcal{W}^s$ is transverse to $\partial\mathbb{B}$, $W^s_{\mathbb{B}}(\gamma(t_n))$ converges to $W^s_{\mathbb{B}}(\gamma(t_\infty))$ in the Hausdorff topology, with multiplicity 1, or equivalently in the C^1 topology. Furthermore, by the uniform boundedness of the vertical degree, there is a uniform L such that for every n, there is a path of length at most L joining $\gamma(t_n)$ to $h(\gamma(t_n))$ in $W^s(\gamma(t_n))$. It follows that the assignment $\gamma(t_n) \mapsto h(\gamma(t_n))$ is equicontinuous. Let y_∞ be a cluster value of $(h(\gamma(t_n)))$. The equicontinuity property shows that $h(\gamma(t_n))$ actually converges to y_∞, and also that the points $h(\gamma(t_n))$ belong to the same local plaque of the unstable lamination, which must thus coincide with $W^u_{\mathrm{loc}}(y_\infty)$. From this, we conclude that h extends to a neighborhood of $\gamma(t_\infty)$, with $h(\gamma(t_\infty)) = y_\infty$, and we are done. $\square$

Example 8.3. *A small perturbation of a hyperbolic polynomial in the Hénon family satisfies the NT condition.*

This follows from the analysis of [25], let us recall the argument for convenience. Assume that $p : \mathbb{C} \to \mathbb{C}$ is a hyperbolic polynomial and let $f : (x, y) \mapsto (ay + p(x), x)$. Take a covering of J_p by a family of disks $(D_x)_{x\in J_p}$ with the property that p is univalent on D_x and $D_{p(x)} \supset \overline{D}_x$ for every $x \in J_p$. Then, if we denote by p_x^{-n} the branch of p^{-n} sending $p^n(x)$ to x, $p_x^{-n}(D_{p^n(x)})$ is a nested sequence of disks with $\bigcap_{n\geqslant 0} p_x^{-n}(D_{p^n(x)}) = \{x\}$. If R is as in § 2.1, for small enough $|a|$, for every

$x \in J_p$, f defines a crossed mapping of degree 1 from $D_x \times D(0,R)$ to $D_{p(x)} \times D(0,R)$, therefore

$$\Gamma_x := \bigcap_{n \geqslant 0} f^{-n}\left(D_{p^n(x)} \times D(0,R)\right)$$

defines a vertical graph in $D_x \times D(0,R)$, hence in $\mathbb{B}$. The collection of vertical graphs $(\Gamma_x)_{x \in J_p}$ form the stable lamination $W^s_{\mathbb{B}}(J)$, which is naturally fibered over J_p. In particular, there is no tangency with $\partial\mathbb{B}$, as asserted.

On the other hand, we do not know whether Ishii's examples from [26] satisfy the NT (nor the NDH) condition.

One may argue that the NT condition is not intrinsic since it depends on the choice of the bidisk $\mathbb{B}$. To get around this issue, we may consider the following variant:

(NT_G) there exists $R > 0$ such that the stable foliation admits no tangency with the hypersurface $\{G^- = R\}$.

Note that the level set $\{G^- = R\}$ is smooth near J^+ for every $R > 0$: indeed, by the local structure of G^- near infinity, this is the case when R is large, and then we use invariance to propagate this property to all $R > 0$. Arguing exactly as in the previous proposition shows that the NT_G property implies NDH.

Using this idea also enables us to understand more precisely how the NDH property may fail. If x and y are two points in J with $y \in W^s(x)$, define the *Green distance*

$$d_G(x,y) := \inf_{c : x \to y} \max(G^-|_c),$$

where the infimum runs over the set of continuous paths $c : [0,1] \to W^s(x)$ joining x to y. Since $W^s(x) \cap J$ is totally disconnected, this indeed defines an ultrametric on $W^s(x) \cap J$, which is uniformly contracted by f: $d_G(f(x), f(y)) = d^{-1} d_G(x,y)$. It provides an intrinsic way of measuring how far we need to go in $\mathbb{C}^2$ to connect two unstable components by stable manifolds. Arguing exactly as in Proposition 8.2 shows:

Proposition 8.4. *Let $x, y \in J$ with $y \in W^s(x)$ and denote by h the germ of stable holonomy $h : W^u_{\mathrm{loc}}(x) \to W^u_{\mathrm{loc}}(y)$. Let $\gamma : [0,1] \to J^u(x)$ be a continuous path and assume that h can be continued along $\gamma([0,t^\star))$. Then, h admits an extension to $t^\star$ if and only if $d_G(\gamma(t), h(\gamma(t)))$ is bounded as $t \to t^\star$.*

8.2 | No queer components

Theorem 8.5. *Let f be dissipative and hyperbolic, with a disconnected and stably totally disconnected Julia set. Assume further that the NDH property holds. Then any nontrivial periodic component of K contains an attracting point.*

Proof. We argue by contradiction: assume that Λ is a component of K that does not contain any attracting periodic point. Let C be the component of Λ in $K^+ \cap \mathbb{B}$. Our hypothesis implies that C has empty interior, so C is a component of $J^+ \cap \mathbb{B}$ (and Λ is a component of J. Fix an unstable transversal Δ^u and let E be a component of $C \cap \Delta^u$, which must have empty interior in Δ^u by Lemma 2.1. Thus, E is a locally connected continuum with empty interior, that is, a dendrite.

Lemma 8.6. *For every $x \in E$, $W^s(x) \cap E = \{x\}$.*

Assuming this lemma for the moment, let us complete the proof. By the expansion in the unstable direction, for every $x \in E$, there exists $\delta_1 > 0$ such that for every $n \geqslant 0$, $f^n(E)$ is not relatively compact in $D^u(f^n(x), \delta_1)$, and by the John–Hölder property, there exists $\delta_2 > 0$ such that any two components of $f^n(E)$ in $D^u(f^n(x), \delta_1)$ intersecting $D^u(f^n(x), \delta_1/2)$ are δ_2-separated. Fix a covering of J by unstable flow boxes. By the product structure of J, there exists $\varepsilon > 0$ such that if $y, z \in f^n(E)$ are ε-close in $\mathbb{C}^2$ but not on the same unstable plaque, then the components $\mathrm{Comp}_{f^n(E) \cap D^u(y,\delta_1)}(y)$ and $\mathrm{Comp}_{f^n(E) \cap D^u(z,\delta_1)}(z)$ are related by local stable holonomy. Finally, by expansion along the unstable direction and the previous separation property, $f^n(E)$ cannot be contained in boundedly many unstable plaques as $n \to \infty$. Thus, for sufficiently large n, we can find two points in $f^n(E)$ that are ε-close in $\mathbb{C}^2$ but not on the same unstable plaque, so there exists $y \in f^n(E)$ such that $W^s_{\mathrm{loc}}(y)$ intersects $f^n(E)$ in another point. This contradicts Lemma 8.6 and we are done. ☐

Proof of Lemma 8.6. Assume that $W^s(x) \cap E$ contains another point $y \neq x$. Then the stable holonomy defines a germ of homeomorphism $h : E \cap U_x \to E \cap U_y$, where U_x is some neighborhood of x (resp. y). By the NDH property, h can be continued along paths in E. Since E is simply connected, this extends to a globally defined map $h : E \to E$, sending x to y, which is a local homeomorphism, hence a covering, so again using the fact that E is simply connected, we conclude that h is a homeomorphism.

It is a classical fact that any continuous self-map of E admits a fixed point. For the reader's convenience, let us include the argument. View E as a subset of the plane. Then, by the Carathéodory theorem, the Riemann map $\mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus E$ extends to a continuous and surjective map $\partial\mathbb{D} \to \partial E = E$. From this, we can construct a topological disk $U \supset E$ and a retraction $r : \overline{U} \to E$: indeed, take the disk bounded by some equipotential and define r as collapsing each external ray to its endpoint. Now let $g = h \circ r$. Since g maps $\overline{U}$ into itself, by the Brouwer fixed point theorem, it admits a fixed point x_0. Finally, since $g(\overline{U}) \subset E$, x_0 belongs to E, so $g(x_0) = h(r(x_0)) = h(x_0) = x_0$.

To conclude the proof, we show that the existence of such a fixed point contradicts the hyperbolicity of f. For this, fix a continuous path $(x_t)_{t\in[0,1]}$ joining x_0 to $x_1 := x$ and let $t^\star = \max\{t \in [0,1],\ h(x_t) = x_t\}$, which satisfies $0 \leqslant t^\star < 1$. As $t > t^\star$ tends to $t^\star$, we see that the two point set $\{x_t, h(x_t)\}$ collapses to $\{x_{t^\star}\}$. This means that there is a tangency between the stable lamination and Δ^u at $x_{t^\star}$, which is the desired contradiction. ☐

Remark 8.7. With notation as in the proof of the theorem, it is not difficult to deduce from the proof that for every $\delta > 0$, for $n \geqslant n(\delta)$ there exists a nontrivial simple closed curve contained in $W^s_\delta(f^n(E))$. So, by the last assertion of Theorem 6.4, there is a nontrivial simple closed curve contained in $W^s_\delta(\Lambda)$. Without the NDH property, we cannot exclude a situation where these simple closed curves do not enclose an attracting basin. We may qualify these dendrites and their limit sets as *queer components* of J. So, Theorem 8.5 asserts that under the NDH property, *queer components of J do not exist.*

8.3 | Topological mixing

Theorem 8.8. *If the NDH property holds, if Λ is a quasi-solenoidal component of period k, then $f^k|_\Lambda$ is topologically mixing. In particular, Λ is transversally a Cantor set.*

Proof. Without loss of generality, we may assume $k = 1$. We resume Proposition 7.1 and its proof. Let Λ' be as in Proposition 7.1, and let us show that $\Lambda' = \Lambda$. Since Λ' is saturated in the unstable direction, $W^s(\Lambda')$ is relatively open in Λ. The NDH property shows that if $y \in W^s(\Lambda')$, then $J^u(y) \subset W^s(\Lambda')$: indeed, the set of points $z \in J^u(y)$ such that $z \in W^s(\Lambda')$ is open because $W^s(\Lambda')$ is relatively open, and since $J^u(y)$ is arcwise connected, the NDH property implies that it is closed as well. Thus, by the local product structure of Λ, we conclude that $W^s(\Lambda')$ is relatively closed in Λ, and by connectedness, we conclude that $W^s(\Lambda') = \Lambda$.

Fix a small $\delta > 0$. By Baire's theorem, we infer that $f^{-n}(W^s_\delta(\Lambda'))$ has nonempty relative interior in Λ for large n, hence so does $W^s_\delta(\Lambda')$ by invariance. Arguing as in Proposition 7.1, we see that by topological transitivity, $W^s_\delta(\Lambda')$ is actually relatively open in Λ. Therefore, $\bigcup_{n\geqslant 0} f^{-n}\left(W^s_\delta(\Lambda')\right)$ is an open cover of Λ and by compactness, we conclude that Λ is contained in $\bigcup_{0\leqslant n\leqslant n_0} f^{-n}\left(W^s_\delta(\Lambda')\right)$ for some n_0; and since $f^{n_0}(\Lambda) = \Lambda$, we finally deduce that $\Lambda \subset W^s_\delta(\Lambda')$. Since δ was arbitrary, $\Lambda \subset \Lambda'$, and we are done. □

Remark 8.9. A similar argument shows that under the NDH property, the quasi-solenoids obtained as limit sets of basin boundaries in Theorem 7.2 are transitive.

As a consequence of transitivity, we can be more precise about the topological structure of periodic components of K.

Proposition 8.10. *Let f be dissipative and hyperbolic, with a disconnected and stably totally disconnected Julia set. Assume further that the NDH property holds. Then for any nontrivial component D of K, $D \cap \mathrm{Int}(K^+)$ is dense in D. Equivalently, for any $x \in D$, $D \cap W^u(x)$ is the closure of its interior for the intrinsic topology.*

Proof. The equivalence between the two assertions follows from Lemma 2.1, Lemma 7.3, and the local product structure. Let D be as in the statement of the proposition and C be its component in $K^+ \cap \mathbb{B}$. Let also Λ the unique component of J contained in D (Proposition 6.6). Without loss of generality, we may assume that D (hence C and Λ) is fixed by f. By Theorem 8.5, D contains an attracting periodic point a, so the immediate basin $\mathcal{B}_0$ of a is contained in C. By Theorem 7.2, $\partial \mathcal{B}_0$ contains a saddle periodic point p, which must belong to Λ (indeed by Lemma 2.3 and Theorem 6.3, $\Lambda = \bigcap_{n\geqslant 0} f^n(\partial C)$). The topological mixing of $f|_\Lambda$ (Theorem 8.8) classically implies that $W^s(p) \cap \Lambda$ is dense in Λ. Indeed, let U be a product neighborhood of p in Λ, and V be an arbitrary open subset of Λ. Then, for sufficiently large $q \geqslant 0$, there exists $y_q \in V$ such that $f^q(y_q) \in U$. Since Λ has local product structure $[f^q(y_q), p] := W^u_{\mathrm{loc}}(f^q(y)) \cap W^s_{\mathrm{loc}}(p)$ belongs to Λ, hence increasing n again if needed, $z_q := f^{-q}([f^q(y_q), p])$ is a point in $W^s(p) \cap V$.

To conclude from this point, we observe that by Remark 7.4 (applied to $f^{-q}(\mathcal{B}_0)$), z_q belongs to the boundary of a component Ω of $W^u(z_q) \cap f^{-q}(\mathcal{B}_0)$ contained in D, and we are done. □

8.4 | A concluding remark

The problem of existence of queer components bears some similarity with another well-known problem that had remained open for some time, of the existence of Herman rings for complex Hénon maps. Examples were recently announced by Krikorian [33], in the almost conservative regime. Still, the question remains whether such examples can exist in the moderately dissipative regime.

APPENDIX A: THE CORE OF A QUASI-SOLENOID

In this appendix, we sketch the construction of the *core* of a quasi-solenoidal component, which should intuitively be understood as the space obtained from this component after removing all "bounded decorations" in unstable manifolds. Initially designed as a potential tool to prove the nonexistence of queer quasi-solenoids, it also gives interesting information on the combinatorial structure of tame ones. It would be interesting to compare it with other constructions such as Ishii's Hubbard trees (see [27]). We keep the setting as in the previous sections, that is, f is a uniformly hyperbolic dissipative Hénon map, with a disconnected and stably totally disconnected Julia set.

A.1 | Number of accesses

The discussion in this paragraph is reminiscent of [6, §7], which deals with the connected case. Pick $x \in J$. For any $R > 0$, define $N^u(x, R)$ to be the number of connected components Ω of $D^u(x, R) \setminus J$ such that $x \in \overline{\Omega}$ and $\overline{\Omega} \cap \partial D^u(x, R) \neq \emptyset$. Since $K \cap D^u(x, R)$ has the John–Hölder property, Corollary 3.3 implies that $N^u(x, R) < \infty$. Thus, $R \mapsto N^u(x, R)$ is a integer-valued nonincreasing function that drops when two components of $D^u(x, R) \setminus J$ merge. The limit

$$N^u_{\mathrm{loc}}(x) := \lim_{R \to 0} N^u(x, R)$$

is the number of local accesses to x, and

$$N^u(x) := \lim_{R \to \infty} N^u(x, R)$$

is the number of unbounded connected components of $W^u(x) \setminus J$ accumulating at x. Note that if $J^u(x)$ is bounded, then $N^u(x) = 1$, so this notion is interesting only when x belongs to a quasi-solenoidal component.

We can also restrict to counting accesses from infinity, that is, components of $D^u(x, R) \setminus K^+$, and we obtain corresponding numbers $N^u_\infty(x, R)$, $N^u_{\infty,\mathrm{loc}}(x)$ and $N^u_\infty(x)$. We have that $N^u_\infty(x) \leqslant N^u(x)$ (and similarly for the other quantities), and, since every point of J is accessible from infinity, $N^u_\infty(x) \geqslant 1$. (†)

Lemma A.1. *N^u (resp. N^u_∞) is upper semicontinuous on J, that is, for any $k \geqslant 1$, $\{x,\ N^u(x) \geqslant k\}$ is closed.*

Proof. We deal with N^u, the proof for N^u_∞ is similar. It is enough to assume that $k \geqslant 2$. By the local product structure of J, it is enough to study the semicontinuity of $x \mapsto N^u(x)$ separately along stable and unstable manifolds. Let us start by studying this semicontinuity along a local stable transversal. We have to prove that $\{x,\ N^u(x) < k\}$ is open. Indeed, assume that there are $j < k$ accesses to x in $W^u(x) \setminus J$. This means that for large R, $D^u(x, R) \setminus J$ has j connected components

† The John–Hölder property of the basin of infinity directly guarantees the finiteness of $N^u_{\infty,\mathrm{loc}}(x)$, but not that of $N^u_{\mathrm{loc}}(x)$ (see Remark 3.11). This property can actually be salvaged as follows: if for small R, $N^u(x, R)$ is large, then for some $k \gg 1$, $N^u(f^k(x), 1)$ is large, and projecting to some fixed transversal yields a contradiction.

accumulating at x. If $x' \in W^s(x)$, then the local stable holonomy between $W^u_{\mathrm{loc}}(x)$ and $W^u_{\mathrm{loc}}(x')$ is a homeomorphism, which locally preserves the number of components of $W^u_{\mathrm{loc}}(x) \setminus J$. In addition, if x' is sufficiently close to x, this holonomy is defined in $D^u(x, R)$. Indeed, for this, it is enough to iterate backward until $f^{-n}(D^u(x, R))$ is contained in the domain of the extended stable lamination. Therefore, there is a large domain D' in $W^u(x')$ such that $D' \setminus J$ has j connected components accumulating on x'. Since the number of components may drop when enlarging this disk further, we conclude that $N^u(x') \leqslant j$.

Now we work inside a given unstable manifold. Let R be such that $N^u(x, s) = N^u(x) = j$ for $s \geqslant R - 1$. By the Hölder–John property, for $R' < R$, $D^u(x, R) \setminus J$ admits finitely many components intersecting $D^u(x, R')$. So, if $N^u(x) = j$, there is some $0 < \varepsilon < 1$ such that only j of these components reach $D^u(x, \varepsilon)$, and we conclude that for $x' \in D^u(x, \varepsilon)$, $N^u(x', R - 1) \leqslant j$, hence $N^u(x') \leqslant j$, as asserted. □

Since f acts linearly on unstable parameterizations, $N^u(x, R) = N^u(f(x), \lambda^u R)$, and we obtain:

Corollary A.2. *If $N^u_{\mathrm{loc}}(x) \geqslant k$, then for any $y \in \omega(x)$, $N^u(y) \geqslant k$.*

An argument similar to that of the second part of Lemma A.1 implies (compare [6, pp. 490-491]):

Lemma A.3. *For any $R > 0$ and any $x \in \Lambda$, the set $\{y \in W^u(x),\ N^u(y, R) \geqslant 3\}$ is discrete for the intrinsic topology.*

Proposition A.4. *The set $\{x \in J,\ N^u(x) \geqslant 3\}$ is a finite set of saddle periodic points.*

Proof. By Lemma A.3, the set $\{x \in J,\ N^u(x, R) \geqslant 3\}$ is contained in a countable union of local stable manifolds. Since any point in J can be joined to a given unstable transversal Δ^u by a stable path of uniform length, by taking small enough R we infer that the projection of this set to Δ^u is actually finite. Therefore, the set $\{x \in \Lambda,\ N^u(x) \geqslant 3\}$ is a closed invariant set contained in a finite union of semi-local stable manifolds, so it is finite. □

A.2 | Definition(s) and properties of the core

Let Λ be a quasi-solenoidal component of J. We define the *core* of Λ to be

$$\mathrm{Core}(\Lambda) = \{x \in \Lambda,\ N^u(x) \geqslant 2\}.$$

It is obvious from the definition that $\mathrm{Core}(\Lambda)$ is invariant and Lemma A.1 implies that it is closed. Hence it is a closed hyperbolic set. Another natural open question is whether $\mathrm{Core}(\Lambda)$ is connected.

The core of the Julia set is the union of the cores of its finitely many quasi-solenoidal components. If $x \in J$ is any point such that $W^u(x) \setminus J$ has several local accesses, then $\omega(x) \subset \mathrm{Core}(J)$.

We say that $x \in \mathrm{Core}(\Lambda)$ is *regular* if $N^u(x) = 2$ and *singular* otherwise. Recall that the singular set is a finite set of periodic points. Note that if x belongs to the core, then $J^u(x)$ disconnects $W^u(x)$.

Conjecture A.5. $\mathrm{Core}(\Lambda)$ *has local product structure near any regular point, and is locally the product of a Jordan arc by a totally disconnected set.*

On the other hand, Core(Λ) does not have local product structure in the neighborhood of any of its singular points, unless it is locally contained in a single unstable manifold. So the structure of the core should be that of a union of solenoids joined at finitely many branch points. It seems that in the example described in [26, Thm 4.23], one quasisolenoidal component has a core made of two solenoids attached at a fixed saddle point.

Note that if Λ is not a queer component, that is the associated component of K contains an attracting periodic point, then the solenoid at the boundary of the immediate basin, constructed in § 7.2, is contained in the core. Indeed, it is obtained by taking limits of Jordan arcs locally separating an attracting basin from the basin of infinity. So, the topological structure of the core should give an account how these various basins are organized and attached to each other in Λ. (Comparing with a one-dimensional situation, it is instructive to take a look at he lift of the Hubbard tree to the natural extension.)

Finally, we may also define $\mathrm{Core}_\infty(\Lambda) = \left\{x \in \Lambda,\ N^u_\infty(x) \geqslant 2\right\}$. (If Λ is a queer component, then $\mathrm{Core}_\infty(\Lambda) = \mathrm{Core}(\Lambda)$.) We expect that $\mathrm{Core}_\infty(\Lambda)$ is a finite set. Indeed, if not, it should contain a Jordan arc such that every point is accessible from both sides by the basin of infinity, and such arcs should not exist. Indeed, iterating forward, and arguing as in Theorem 8.5, a large iterate of this arc must spiral and come close to itself, hence, projecting to an unstable transversal, this would cut out a Fatou disk, and we conclude that one side of the arc is contained in an attracting basin.

Remark A.6. These ideas admit a counterpart for polynomials in dimension 1. In this setting, the set of biaccessible points (from the basin of infinity) and its limit set can sometimes be completely characterized, and it is indeed tightly connected with the Hubbard tree (see, e.g., [12, 44, 49]). This is particularly accurate for a strictly critically finite quadratic polynomial (hence not hyperbolic), which has no bounded Fatou components. In this case, any biaccessible point eventually falls in the Hubbard tree, except possibly for the β fixed point (this follows from [44, Lemmas 2.1 and 4.6]), and the analog of $\mathrm{Core}_\infty(J)$ would be a subtree of the Hubbard tree. However, since we work in the hyperbolic case and consider both the accessibility from infinity and the bounded Fatou components, the analog of the core is rather related to "disked" version of the Hubbard tree decorated with attracting basins boundaries (see, e.g., [12]).

APPENDIX B: CONTINUITY OF THE AFFINE STRUCTURE

Here, we present the following mild generalization of a theorem by Étienne Ghys [22]. Recall that the ratio of a triple $(u, v, w) \in \mathbb{C}^3$ is $\frac{u-v}{u-w}$.

Theorem B.1. *Let $\psi : \mathbb{C} \to \mathbb{C}^2$ be an injective holomorphic immersion, and $L = \psi(\mathbb{C})$. Assume that (L_n) is a sequence of immersed complex submanifolds converging to L in the following sense: if $K \Subset L$ is any relatively compact subset (relative to the leafwise topology), then L_n contains a graph over a neighborhood of K for large n, that is there exists a neighborhood $N(K)$ of K in L and a sequence of injective holomorphic maps $\pi_n : N(K) \to L_n$ such that $\pi_n(x) \to x$ for every x. Assume further that for every n, L_n is biholomorphic to $\mathbb{C}$.*

Then, the affine structures on the L_n converge to that of L in the following sense: for any compact set $K \Subset L$ as above and any triple $(x, y, z) \in K^3$, if $(x_n, y_n, z_n) \in \pi_n(N(K))$ are close to $(\pi_n(x), \pi_n(y), \pi_n(z))$ and converge to (x, y, z), then the corresponding ratios converge as well.

The point of this statement is to emphasize that there is no need in Ghys' theorem to work with the leaves of a Riemann surface lamination. Also, compactness of the ambient space is not required. The theorem is certainly not written in its most general form: one might assume more generally that

- the π_n are $(1+\varepsilon_n)$ quasi-conformal for some $\varepsilon_n \to 0$;
- L and the L_n are parabolic Riemann surfaces instead of copies of $\mathbb{C}$.

The adaptation is left to the reader. Notice also that any submanifold V of a Stein manifold admits a neighborhood W endowed with a holomorphic retraction $W \to V$ (see [41, Cor. 1]). Therefore, our convergence assumption essentially means that L_n converges to L with multiplicity 1.

Proof. We follow [22, §4] closely. Pick a triple of distinct points (x, y, z) in L and R_0 such that $\psi(D(0, R_0))$ contains x, y, z. For $\alpha \in L$, let $\widetilde{\alpha} = \psi^{-1}(\alpha)$. Without loss of generality, we may assume $R_0 = 1$. Let R be a large positive number to be determined. For $n \geqslant n(R)$, π_n is well defined in $\psi(D(0, R))$. Let $(x_n, y_n, z_n) \in \pi_n(D(0,1))^3$ converging to (x, y, z), and fix $\varepsilon > 0$. Then by assumption $(\pi_n^{-1}(x_n), \pi_n^{-1}(y_n), \pi_n^{-1}(z_n))$ converges to (x, y, z) for the leafwise topology in L. Let $\psi_n : \mathbb{C} \to L_n$ be any parameterization, and let $\widetilde{x}_n = \psi_n^{-1}(x_n)$, $\widetilde{y}_n = \psi^{-1}(y_n)$ and $\widetilde{z}_n = \psi^{-1}(z_n)$. Without loss of generality, we may assume $\widetilde{x}_n = 0$. We have to show that for large n, the ratio of $(\widetilde{x}_n, \widetilde{y}_n, \widetilde{z}_n)$ is close to that of $(\widetilde{x}, \widetilde{y}, \widetilde{z})$.

By assumption $h_n := \psi_n^{-1} \circ \pi_n \circ \psi : D(0, R) \to \mathbb{C}$ is an injective holomorphic map. By renormalizing ψ_n, we may assume that $h_n'(0) = 1$ (we use $L_n \simeq \mathbb{C}$ precisely here). Then by the Koebe distortion theorem, h_n is almost affine in $D(0,1)$, that is, it distorts the ratios of points in $D(0,1)$ by some small amount $\varepsilon(R)$. Fix R so large that $\varepsilon(R) < \varepsilon$. In particular, for $n \geqslant n(R)$, we get that

$$\left| \frac{h_n(\widetilde{x}) - h_n(\widetilde{y})}{h_n(\widetilde{x}) - h_n(\widetilde{z})} - \frac{\widetilde{x} - \widetilde{y}}{\widetilde{x} - \widetilde{z}} \right| \leqslant \varepsilon.$$

Now for $\alpha \in K$, $h_n(\widetilde{\alpha})$ is the parameter in $\mathbb{C}$ corresponding to $\pi_n(\alpha) \in L_n$, so $\widetilde{\alpha}_n$ is close to $h_n(\widetilde{\alpha})$ in $\mathbb{C}$ and for large n, we also get that

$$\left| \frac{h_n(\widetilde{x}) - h_n(\widetilde{y})}{h_n(\widetilde{x}) - h_n(\widetilde{z})} - \frac{\widetilde{x}_n - \widetilde{y}_n}{\widetilde{x}_n - \widetilde{z}_n} \right| \leqslant \varepsilon,$$

and we are done. □

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