The excess intersection formula (§ 6.3) has many precedents. In the non-singular case, the self-intersection formula had been proved by Mumford in 1959, the key formula by Jouanolou (1) § 4.1; see also Lascu-Mumford-Scott (1). A topological version of the general formula was given by Quillen (1); J. King (2), (4) gave an analytic analogue. Illusie asked us if such a formula was known for rational equivalence. Such an excess intersection formula was given in Fulton-MacPherson (1) and by H. Gillet (unpublished).

The formula for intersection classes in terms of Segre classes of cones first appeared in Fulton-MacPherson (1). As mentioned in the notes to Chap. 4, such classes were constructed in many cases by B. Segre, who stressed the importance of blowing up to simplify problems in intersection theory.

The extension from regular imbeddings to l.c.i. morphisms follows the formalism of [SGA6]. Kleiman (12) has also developed and applied this extension.

In the case of smooth quasi-projective varieties, most of Proposition 6.7 was proved by Jouanoulou (1) § 9, by essentially the same calculations; the case of codimension 2 had been done by Samuel. See also Beauville (1) Prop. 0.1.3.

The blow-up formula of Theorem 6.7 is apparently new, even in the non-singular case.

Example 6.1.4 comes from R. Lazarsfeld.

Chapter 7. Intersection Multiplicities

Summary

As in Chap. 6, consider a fibre square

$$\begin{array}{ccc} W & \to & V \\ \downarrow & & \downarrow f \\ X & \xrightarrow[i]{} & Y \end{array}$$

with i a regular imbedding of codimension d, V a k-dimensional variety. If Z is an irreducible component of W of dimension $k-d$, the intersection multiplicity $i(Z, X \cdot V; Y)$ is defined to be the coefficient of Z in the intersection class $X \cdot V \in A_{k-d}(W)$. The intersection multiplicity is a positive integer, satisfying

$$i(Z, X \cdot V; Y) \leqq \text{length}\,(\mathcal{O}_{Z,W}).$$

Examples show that this inequality may be strict; equality holds, however, if $\mathcal{O}_{Z,V}$ is a Cohen-Macaulay ring.

On the other hand, the criterion of multiplicity one asserts that $i(Z, X \cdot V; Y)$ is one precisely when $\mathcal{O}_{Z,V}$ is a regular local ring with maximal ideal generated by the ideal of X in Y.

The standard properties of intersection multiplicities, worked out in the examples, follow from the basic properties of the general intersection product which were proved in Chapter 6.

7.1 Proper Intersections

Consider, as in § 6.1, a fibre square

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ g\downarrow & & \downarrow f \\ X & \xrightarrow[i]{} & Y \end{array}$$

with i a regular imbedding of codimension d, V a purely k-dimensional scheme. Let

$$C = C_W V, \qquad [C] = \sum_{i=1}^{r} m_i [C_i],$$

C_i the irreducible components of C, and let Z_i be the support of C_i; $Z_1, \ldots, Z_r$ are the distinguished varieties of the intersection.

Lemma 7.1. (a) *Every irreducible component of W is distinguished.*
(b) *For any distinguished variety Z,*

$$k-d \leqq \dim Z \leqq k\,.$$

Proof. (a) follows from the fact that the support of $C_W V$ is W, for any closed subscheme W of a scheme V. Since C_i is an irreducible subvariety of $g^* N_X Y$ which projects onto Z_i, if N_i is the restriction of $g^* N_X Y$ to Z_i, then

$$C_i \subset N_i\,.$$

Therefore

$$\dim Z_i \leqq \dim C_i \leqq \dim (N_i) = \dim Z_i + d\,.$$

Since C_i is k-dimensional (Appendix B.6.6), (b) follows. □

If $\dim Z_i = k-d$, the inclusion $C_i \subset N_i$ of irreducible k-dimensional varieties must be an isomorphism. In particular, the class α_i obtained by intersecting $[C_i]$ with the zero-section of N_i is just $[Z_i]$, and the equivalence of Z_i for the intersection is $m_i[Z_i]$.

Definition 7.1. An irreducible component Z of $W = f^{-1}(X)$ is a *proper* component of intersection of V by X if $\dim(Z) = k-d$. The *intersection multiplicity* of Z in $X \cdot V$, denoted

$$i(Z, X \cdot V; Y)$$

or simply $i(Z, X \cdot V)$, or $i(Z)$, is the coefficient of Z in the class $X \cdot V$ in $A_{k-d}(W)$. Equivalently, the equivalence of Z for the intersection class is

$$i(Z, X \cdot V; Y)\,[Z]$$

If N_Z is the pull-back of $N_X Y$ to Z, then $i(Z, X \cdot V; Y)$ is the coefficient of N_Z in the cycle $[C]$ of the cone $C = C_W V$.

Let $A = \mathcal{O}_{Z,V}$ be the local ring of V along Z, and let $J \subset A$ be the ideal of W; A/J has finite length when Z is an irreducible component of W.

Proposition 7.1. *Assume Z is a proper component of W. Then*

(a) $1 \leqq i(Z, X \cdot V; Y) \leqq l(A/J)$, *where $l(A/J)$ is the length of A/J.*

(b) *If J is generated by a regular sequence of length d, then*

$$i(Z, X \cdot V; Y) = l(A/J)\,.$$

If A is Cohen-Macaulay (e.g. regular) the local equations for X in Y give a regular sequence generating J, and the equality in (b) holds.

Proof. Let $N = g^* N_X Y$. The restriction N_Z of N to Z is an irreducible component of N. Since N is a vector bundle over W, the coefficient of N_Z in the cycle $[N]$ is the same as the coefficient of Z in the cycle $[W]$, which is $l(A/J)$. Since C is a closed subscheme of N, the coefficient of any irreducible component of N is no larger in $[C]$ than it is in $[N]$ (Lemma A.1.1). Since the coefficient of N_Z in $[C]$ is $i(Z, X \cdot V; Y)$, (a) follows.

If J is generated by a regular sequence of d elements, replacing V by an open subscheme which meets Z (which doesn't effect the intersection multiplicity, by Theorem 6.2(b)), we may assume the imbedding of W in V is regular of codimension d. Then C is a sub-bundle of N of rank d, so $C=N$, and the coefficients of N_Z in $[C]$ and in $[N]$ coincide.

The last assertion of the proposition follows from Lemma A.7.1. □

The inequalities in (a) may be strict, as shown by Macaulay (Example 7.1.5).

Example 7.1.1. Let $(e_W V)_Z$ be the multiplicity of V along W at Z, as defined in Example 4.3.4. Then

$$i(Z, X\cdot V;\, Y) = (e_W V)_Z$$

i.e., the intersection multiplicity defined here agrees with Samuel's.

Example 7.1.2. Let $a_1, \ldots, a_d$ be the images in A of a regular sequence of elements defining X in Y (locally, in an open set which meets $f(Z)$). Then

$$i(Z, X\cdot V;\, Y) = \chi_A(\mathbf{a}) = e_A(a_1, \ldots, a_d)$$

where $\chi_A(\mathbf{a}) = \sum_{i=0}^{d} l_A(H_i(K_*(\mathbf{a})))$, with $K_*(\mathbf{a})$ the Koszul complex defined by $a_1, \ldots, a_d$ (Appendix A.5). Serre (4) IV.A3 showed generally that $\chi_A(\mathbf{a})$ gives Samuel's multiplicity. We sketch an alternative proof, by induction on d. If $d=1$ it says

$$\sum l_{A_p}(A_p)\cdot l_A(A/p + aA) = e_A(a, A)\,,$$

where the sum is over the minimal primes p of A; this is a special case of Lemma A.2.7. For the inductive step, localize so that one has a fibre diagram

$$\begin{array}{ccccc} W & \subset & W' & \subset & V \\ \downarrow & & \downarrow & & \downarrow \\ X & \subset & X' & \subset & Y \end{array},$$

with $X'\subset Y$ and $X\subset X'$ regular imbeddings of codimension $d-1$ and 1, and local equations pulling back to $a_2, \ldots, a_d$ in A and $\bar{a}_1$ in $A/(a_2, \ldots, a_d)$ respectively; we may also assume the localization is sufficient so that Z is the only irreducible component of W, and all the irreducible components W_i' of W' contain Z, and therefore have dimension $k-d+1$. Let p_i be the prime ideal of A corresponding to W_i'. By induction

$$X'\cdot_Y V = \sum_i e_{A_{p_i}}(a_2, \ldots, a_d)\,[W_i'] \,.$$

By functoriality (§ 6.5), $X\cdot_Y V = X\cdot_{X'}(X'\cdot_Y V)$. Let $H_k = H_k(K_*(a_2, \ldots, a_d))$. Then

$$\begin{aligned} i(Z, X\cdot V;\, Y) &= \sum e_{A_{p_i}}(a_2, \ldots, a_d)\, l_A(A/p_i + a_1 A) \\ &= \sum_{i,k} (-1)^k l_{A_{p_i}}((H_k)_{p_i})\cdot l_A(A/p_i + a_1 A) \\ &= \sum_k (-1)^k e_A(a_1, H_k) = e_A(a_1, \ldots, a_d) \end{aligned}$$

by Lemma A.2.7 and Example A.5.1. (Note that each H_k has support in $V(a_2, \ldots, a_d)$, so H_k is an $\bar{A}$-module for $\bar{A} = A/(a_2^m, \ldots, a_d^m)$, some $m > 0$ – in fact $m = 1$ will do; Lemma A.2.7 may be applied over the one-dimensional ring $\bar{A}$.)

Example 7.1.3. With the notation of the preceding example, the following are equivalent:

(i) $i(Z, X \cdot V; Y) = l(A/J)$.
(ii) J is generated by a regular sequence of length d.
(iii) $a_1, \ldots, a_d$ is a regular sequence in A.
(iv) $H_k(K_*(\mathbf{a})) = 0$ for all $k > 0$.

In particular, $i = l$ if and only if A is Cohen-Macaulay. Algebraic proofs are given by Serre (4)IV. To prove directly that (i) implies (iii), the main point is to show that the equality of cycles $[C] = [N]$ implies that $C = N$, at least after replacing V by an open subset which meets Z. (Indeed, if A is an Artin local ring, and Q is a homogeneous ideal in $A[T_1, \ldots, T_d]$ whose localization at the minimal prime is zero, then $Q = 0$.)

Example 7.1.4. Let

$$Y = \mathbb{A}^4, \qquad X = V(x_1 - x_3, x_2 - x_4), \qquad V = V(x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4)\,.$$

Then V is purely 2-dimensional, $[V] = [V(x_1, x_2)] + [V(x_3, x_4)]$. The intersection number of the origin in $X \cdot V$ is 2, while $l(A/J) = 3$ (cf. Hartshorne (5) p. 428.)

Example 7.1.5. Let $Y = \mathbb{A}^4$, $X = V(x_1, x_4)$, and let $V \subset \mathbb{A}^4$ be the image of the finite morphism φ from $\mathbb{A}^2$ to $\mathbb{A}^4$ given by

$$\varphi(s, t) = (s^4, s^3 t, s t^3, t^4)\,.$$

The origin P is a proper component of the intersection of V by X.

(i) $I(V) = (x_1 x_4 - x_2 x_3,\ x_1^2 x_3 - x_2^3,\ x_2 x_4^2 - x_3^3,\ x_2^2 x_4 - x_3^2 x_1)$.
(ii) $[X \cap V] = 5[P]$, $l(A/J) = 5$.
(iii) $i(P, X \cdot V; Y) = 4$.

(For (iii), note that $\varphi_*[\mathbb{A}^2] = 4[V]$. Apply Theorem 6.2 (a) to the situation

$$\begin{tikzcd} V(s^4, t^4) \arrow[r] \arrow[d, "\psi"'] & \mathbb{A}^2 \arrow[d, "\varphi"] \\ X \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "i"'] & Y\,. \end{tikzcd}$$

giving that $\psi_*(i^![\mathbb{A}^2]) = i^!\,\varphi_*[\mathbb{A}^2] = 4\,X \cdot V$. Since $\mathbb{A}^2$ is regular, Proposition 7.1 (b) gives

$$i(Q, X \cdot \mathbb{A}^2; Y) = l(K[s, t]/(s^4, t^4)) = 16\,.$$

with Q the origin in $\mathbb{A}^2$. Therefore $16 = 4\,i(P, X \cdot V; Y)$, as required.) Note that the kernel of multiplication by x_4 on $A/x_1 A$ has length 1, which accounts for the difference between (ii) and (iii).

Example 7.1.6. Without regularity assumptions, irreducible components Z of $X \cap V$ may have dimension smaller than $\dim V - \operatorname{codim}(X, Y)$. The standard example is $Y = V(x_1 x_4 - x_2 x_3) \subset \mathbb{A}^4$, $X = V(x_1, x_2)$, $V = V(x_3, x_4)$.

Example 7.1.7 (Commutativity). If $V \to Y$ is also a regular imbedding, then by Theorem 6.4,

$$i(Z, X \cdot V; Y) = i(Z, V \cdot X; Y).$$

Example 7.1.8 (Associativity). Let $i: X \to Y$ factor into a composite $i': X \to X'$, $j: X' \to Y$ of regular imbeddings. Let $W_1', \ldots, W_r'$ be the irreducible components of $f^{-1}(X')$ which contain Z. If Z is a proper component of W, then Z is a proper component of the intersection of each W_h' by X on X', each W_h' is a proper component of the intersection of V by X' on Y, and by Theorem 6.5,

$$i(Z, X \cdot V; Y) = \sum_{h=1}^{r} i(Z, X \cdot W_h'; X') \cdot i(W_h', X' \cdot V; Y).$$

Example 7.1.9 (Projection formula). Let $g: V' \to V$ be a proper surjective morphism of k-dimensional varieties, and let $Z_1, \ldots, Z_r$ be the irreducible components of $g^{-1}(Z)$. If each Z_j and Z have dimension $k-d$, then by Proposition 6.2 (a),

$$\deg(V'/V) \cdot i(Z, X \cdot V; Y) = \sum_{j=1}^{r} \deg(Z_j/Z) \cdot i(Z_j, X \cdot V'; Y).$$

Example 7.1.10. (a) Let $D_1, \ldots, D_d$ be effective Cartier divisors in a k-dimensional variety V. An irreducible component Z of $\bigcap D_i$ of dimension $k-d$ is called a *proper* component, and the *intersection multiplicity*

$$i(Z, D_1 \cdot \ldots \cdot D_d; V)$$

is defined to be the intersection multiplicity of Z in the intersection of $V = \Delta_V$ by $D_1 \times \ldots \times D_d$:

$$\begin{array}{ccc} \cap D_i & \to & V \\ \downarrow & & \downarrow \delta \\ D_1 \times \ldots \times D_d & \to & V \times \ldots \times V. \end{array}$$

Equivalently (cf. Example 6.5.1), $i(Z, D_1 \cdot \ldots \cdot D_d; V)$ is the coefficient of Z in the intersection cycle $D_1 \cdot \ldots \cdot D_d$ in $A_{k-d}(\bigcap D_i)$ (Definition 2.4.2). If A is the local ring of V along Z, and a_i is a local equation for D_i in A, then

$$i(Z, D_1 \cdot \ldots \cdot D_d; V) = e_A(a_1, \ldots, a_d).$$

If A is Cohen-Macaulay, then

$$i(Z, D_1 \cdot \ldots \cdot D_d; V) = l_A(A/(a_1, \ldots, a_d)).$$

For example, if $d = k$, and Z is a simple point on V, the intersection multiplicity is given by the length.

(b) Let $D_1, \ldots, D_d$ be hypersurfaces in $\mathbb{A}_K^d$ defined by polynomials $f_1, \ldots, f_d$, and assume $P = (0, \ldots, 0)$ is an isolated (i.e. proper) point of intersection of $\bigcap D_i$. Then

$$i(P, D_1 \cdot \ldots \cdot D_d; \mathbb{A}^d) = \dim_K(\mathcal{O}_{P,\mathbb{A}^d}/(f_1, \ldots, f_d))$$
$$= \dim_K(K[\![x_1, \ldots, x_d]\!]/(f_1, \ldots, f_d)) .$$

If $K = \mathbb{C}$, one may replace formal power series by convergent power series in the last formula. (Note that modules of finite length are not altered by completion, cf. Zariski-Samuel (1) VIII.2.)

Example 7.1.11. Let V be an n-dimensional variety, P a simple point on V, $\pi: \tilde{V} \to V$ the blow-up of V at P, E the exceptional divisor. For an effective Cartier divisor D on V, let $\tilde{D}$ be the blow-up of D along P, i.e. the proper transform of D in $\tilde{V}$. If $D_1, \ldots, D_n$ are divisors such that $\bigcap_i \tilde{D}_i \cap E$ is finite, then

$$i(P, D_1 \cdot \ldots \cdot D_n; V) = \prod_{i=1}^{n} e_P(D_i) + \sum_{Q \in E} i(Q; \tilde{D}_1 \cdot \ldots \cdot \tilde{D}_n; \tilde{V}) .$$

If $n = 2$, and D_1 and D_2 meet properly at P, it follows that the intersection multiplicity $i(P, D_1 \cdot D_2; X)$ is the sum of the products of multiplicities of D_1 and D_2 at all infinitely near points, a result of M. Noether. (By Example 4.3.9, $\pi^* D_i = \tilde{D}_i + e_P(D_i) E$. Write out the product of the $\tilde{D}_i$, and push forward to V.) Generalizations will be given in Example 12.4.8.

Example 7.1.12. The equivalence of a distinguished variety Z of the minimal dimension $k - d$ is always a positive multiple of $[Z]$. If $\dim Z > k - d$, the equivalence of Z may be represented by negative cycles. For example, if Y is the blow-up of a surface at a simple point, and $X = V = E$ is the exceptional divisor, then $Z = E$ is the only distinguished variety, and its equivalence is $-[P]$, P a point on E.

Example 7.1.13. Let E be a vector bundle of rank r on a purely n-dimensional scheme X, s a section of E, $Z(s)$ the zero-scheme of s:

$$(*) \qquad \begin{array}{ccc} Z(s) & \to & X \\ \downarrow & & \downarrow s \\ X & \xrightarrow[s_E]{} & E \end{array}$$

where s_E is the zero section. If Z is a proper component of $Z(s)$, i.e. $\dim Z = n - r$, then the intersection construction from (*) determines an intersection multiplicity $i(Z)$. If A is the local ring of X at Z, the stalk of E at Z is a free A-module with an induced section s_A, which determines a Koszul complex $\Lambda^*(s_A)$ (Definition A.5). Then

$$i(Z) = \chi_A(s_A) .$$

Example 7.1.14. Let $f: X \to C$ be a morphism from a smooth n-dimensional variety to a smooth curve C. The tangent map $df: T_X \to f^* T_C$ corresponds to a section s of $T_X \otimes f^* \Omega_C^1$. If $x \in X$ is an isolated zero of this section, the intersection multiplicity of x in the intersection of $s(X)$ by the zero section (as in the preceding example) is called the *multiplicity of x as a critical point* of f, and denoted $\mu_x(f)$. If f is given in local coordinates by a function

$f(z_1, \ldots, z_n)$, then

$$\mu_x(f) = l(\mathcal{O}_{x,X}/(\partial f/\partial z_1, \ldots, \partial f/\partial z_n)) .$$

For a discussion of this multiplicity from an analytic and topological point of view, see Milnor (3) and Orlik (1), cf. Example 14.1.5.

Example 7.1.15. Let $f: Y' \to Y$ be a proper surjective morphism of varieties. Let X' be a subvariety of Y', $X = f(X')$; assume that $\dim X = \dim X'$, and X' is an irreducible component of $f^{-1}(X)$, and that X is regularly imbedded in Y. Then the ramification index of f at X' (Example 4.3.7) is given by an intersection multiplicity

$$e_{X'}(f) = i(X', X \cdot Y'; Y) .$$

This applies in particular if f is finite and X is a simple point of Y.

Example 7.1.16. *Fractional intersection numbers on normal surfaces,* (cf. Mumford (1)II(b), Reeve (2)). Let $\pi: X \to V$ be a resolution of a singular point P on a surface V, $\pi^{-1}(P) = E_1 \cup \ldots \cup E_r$ connected, as in Example 2.4.4. For an irreducible curve A on V, there are unique rational numbers $\lambda_1, \ldots, \lambda_r$ so that if $\tilde{A}$ is the proper transform of A on X,

$$(\tilde{A} \cdot E_i)_X + \sum_{j=1}^{r} \lambda_j (E_j \cdot E_i)_X = 0$$

for all i. Set $A' = \tilde{A} + \sum \lambda_i E_i \in Z_1 X_{\mathbb{Q}} = Z_1 X \otimes_{\mathbb{Z}} \mathbb{Q}$. This extends to a homomorphism $\alpha \to \alpha'$ from $Z_1 V$ to $Z_1 X_{\mathbb{Q}}$, satisfying

(i) $[D]' = [\pi^* D]$ for any Cartier divisor D on V.
(ii) If A is positive and contains P, then all the λ_i are positive.

(For (i), $(\pi^* D \cdot E_i)_X = (D \cdot \pi_* E_i)_V = 0$. For (ii), with D_i, Z as in Example 2.4.4, let $A' = \tilde{A} + \sum \mu_j D_j$, with μ_i minimal among the μ_j, $\mu_i \leqq 0$. Then $0 = A' \cdot D_i \geqq \sum_j \mu_j (D_j \cdot D_i)_X \geqq \mu_i (\sum_j (D_j \cdot D_i)_X = -\mu_i (Z \cdot D_i)_X \geqq 0$; the connectedness of E then implies that all μ_j are zero.)

For any two one-cycles A, B on V which meet only at P, set

$$j(P, A \cdot B) = (A' \cdot B')_X \in \mathbb{Q} .$$

(This is defined since $|A'| \cap |B'| \subset E$, which is complete.) This intersection number is symmetric and bilinear; it is non-negative if either A or B is positive, and positive if A and B are positive and pass through P. If $A = [D]$ is the Weil divisor of a Cartier divisor D, then

$$j(P, A \cdot B) = (D \cdot B)_V \in \mathbb{Z} .$$

This definition of multiplicity is independent of the resolution. (If $\varrho: \tilde{X} \to X$ blows up a point on X, $\varrho^* A'$ is perpendicular to all exceptional components, and $(\varrho^* A' \cdot \varrho^* B')_{\tilde{X}} = (A' \cdot B')_X$.)

If X is a quadric cone with vertex P, and A and B are generating lines of the cone, then $j(P, A \cdot B) = 1/2$.

Example 7.1.17. Let C be an irreducible curve on a scheme X, D an effective Cartier divisor on X, with C not contained in the support of D. Let

$f\colon C' \to X$ be a finite morphism which maps an irreducible curve C' birationally onto C. Then

$$i(P, D \cdot C; X) = \sum_{f(Q)=P} \operatorname{ord}_Q(f^* D).$$

(Use Theorem 6.2(a).) For example if $C' = \mathbb{P}^1$, f^*D is given by a polynomial, and the intersection multiplicities are given by multiplicities of roots of this polynomial.

7.2 Criterion for Multiplicity One

Let Z be a proper component of the intersection of V by X on Y. Let $A = \mathcal{O}_{Z,V}$, J the ideal in A generated by the ideal of X in Y, and let m be the maximal ideal of A.

Proposition 7.2. *Assume that V is a variety. The following are equivalent:*

(i) $i(Z, X \cdot V; Y) = 1$.
(ii) *A is a regular local ring, and $J = m$.*

Recall that a d-dimensional local ring is regular if its maximal ideal has d generators, which necessarily form a regular sequence (Lemma A.6.2). Since J always has d generators, the regularity of A follows from the assertion that $J = m$.

Proof. The implication (ii) $\Rightarrow$ (i) is a special case of Proposition 7.1, since $l(A/m) = 1$. We prove (i) $\Rightarrow$ (ii) by induction on d. The assertions are unchanged if V and Y are replaced by open subschemes which meet Z and $f(Z)$ respectively. Therefore, we may assume that Z is the only irreducible component of W, that Y is affine, and X is defined in Y by a regular sequence in the coordinate ring of Y.

If $d = 1$, then $X \cdot V = [W]$. The coefficient of Z in $[W]$ is $l_A(A/J)$, which can be 1 only if $J = m$.

Let $d > 1$, and assume (i) $\Rightarrow$ (ii) for smaller d. Assume first that A is a normal domain. Let X' be the divisor on Y defined by the first of the equations defining X. Form the fibre diagram

$$\begin{array}{ccccc} W & \to & W' & \to & V \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow[i']{} & X' & \to & Y . \end{array}$$

Since $\dim W = k - d$, $W' \neq V$, so W' is a Cartier divisor on V and $X' \cdot_Y V = [W']$. By functoriality (Theorem 6.5),

$$i'^{!}[W'] = i^{!}[V] = [Z].$$

In particular W' can have only one irreducible component which contains Z, and this component appears in the cycle $[W']$ with coefficient 1. In other words, $a_1 A$ has only one minimal prime ideal p containing it, and

$$l_{A_p}(A_p/a_1 A_p) = 1 \ .$$

Since A is a normal domain, p is the only prime ideal associated to $a_1 A$ (Lemma A.8.1), so $a_1 A$ is p-primary. Since $a_1 A_p = pA_p$, it follows that $a_1 A = p$. Therefore W' is a variety, and $A/a_1 A$ the local ring of Z on W'. By induction, the images of $a_2, \ldots, a_d$ in $A/a_1 A$ form a regular sequence generating $m/a_1 A$, so $a_1, \ldots, a_d$ form a regular sequence generating m.

Returning to the general case, it remains to show that A must be normal. Let $g: \tilde{V} \to V$ be the normalization of V in its function field. Let h be the induced morphism from $g^{-1}(W)$ to W. By Theorem 6.2(a), since g is proper and $g_*[\tilde{V}] = [V]$,

$$h_*(i^![\tilde{V}]) = i^![V] \ .$$

If $i^![\tilde{V}] = \sum_{i=1}^r m_i [\tilde{Z}_i]$, this gives

$$\sum_{i=1}^r m_i \deg(\tilde{Z}_i/Z) = 1 \ .$$

Therefore $r = m_1 = \deg(\tilde{Z}_1/Z) = 1$. The local ring A' of $\tilde{Z}_1$ in $\tilde{V}$ is the integral closure of A in its field of fractions. The case of (i) $\Rightarrow$ (ii) proved above applies to $\tilde{V}$, so J generates the maximal ideal m' of A'; in particular, $mA' = m'$. Since $\deg(\tilde{Z}_1/Z) = 1$, the canonical map from A/m to A'/m' is an isomorphism. Therefore $A' = A + mA'$; since A' is finite over A, Nakayama's lemma implies that $A = A'$, so A is normal, as required. □

Example 7.2.1. It is not enough to assume that V is a pure-dimensional scheme in Proposition 7.2. For example, let $Y = \mathbb{A}^2$, $X = V(y)$, $V = V(xy, x^2)$. Then X and V meet properly at the origin, and the intersection multiplicity is 1, but the local ring of V at the origin is not regular. However, if all associated primes p in A have $\dim A/p = d$, then the intersection multiplicity is one only if A is regular and J is the maximal ideal. (Since $X \cdot V = X \cdot [V]$ (Example 6.2.1), A can have only one minimal prime p, and $l(A_p) = 1$; i.e. $p_p = 0$; since elements outside p are assumed to be non-zero divisors, $p = 0$, and Proposition 7.2 applies.)

Nagata has extended this to general local rings A whose completion is unmixed (cf. Nagata (2)40.6 and Huneke (1)). Nagata has given an example of a local Noetherian domain whose multiplicity is one without being regular (cf. Nagata (2) Appendix A1).

Notes and References

The problem of assigning a multiplicity to an isolated solution of n polynomial equations in n variables can be traced back near the beginnings of algebraic geometry, although clear statements did not appear until relatively recently.

Two points of view, which remain vital, can be found in the work of Newton and his contemporaries:

(1) The *dynamic* approach, where the multiplicity of a solution is the number of solutions near the given solution when the equations are varied. For example, a point of tangency of a line with a curve is a limit of intersections of nearby secant lines.

(2) The *static* approach, where the multiplicity is obtained without varying the given equations. For $n = 2$, Newton and Leibnitz showed how to eliminate one of the variables, obtaining a polynomial equation whose roots give the abscissas where the equations have common solutions, the multiplicity question is likewise reduced to the multiplicity of a root of a polynomial in one variable.

In 1822, Poncelet (1) made the dynamic point of view quite explicit with his "principle of continuity". Rules of this type were given for calculating intersection multiplicities, e.g. by Cayley (1), Halphen (1), Schubert (1), and Zeuthen (3). A useful summary of this era is given by Zeuthen and Pieri (1). We will discuss these principles in Chap. 11.

Elimination theory and the calculation of resultants also received considerable attention; the names of Euler, Bézout, Cayley, Sylvester, Kronecker, and Hilbert should at least be mentioned. This is discussed by Salmon (2) and B. Segre (8); cf. Example 8.4.13.

In 1915 Macaulay (1) gave a static definition in terms of the length of a ring modulo an ideal, and proved Bézout's theorem for n hypersurfaces in $\mathbb{P}^n$.

The intersection of more general varieties than hypersurfaces in n-space was taken up, from the dynamic point of view, by Severi, Van der Waerden, and Weil in the 1930's. In 1928 Van der Waerden (2), borrowing an example from Macaulay (Example 7.1.5 above), showed that the naive definition using length would not always work. Van der Waerden (3) also pointed out in 1930 that the Poincaré-Lefschetz intersection theory in topology includes a notion of intersection multiplicity for complex varieties, since they can be triangulated. Severi's treatments (cf. Severi (7) for a summary) were almost entirely geometric. Van der Waerden (1), Weil (2), and Barsotti (1) developed algebraic notions of specialization to make such geometric ideas rigorous, not relying on geometric intuition, and valid over general ground fields.

Chevalley (1), in 1945, gave an important new definition of intersection multiplicity in terms of completions of the local rings; his theory was therefore equally valid in the analytic or formal case. He also gave a criterion for multiplicity one, which includes that given in § 7.2. Samuel (1) gave the first definition valid for a general Noetherian local ring A. As in Examples 4.3.1 and 4.3.4, he defined a multiplicity $e_A(J)$ for an ideal J primary to the maximal ideal. Samuel proved many basic properties for this multiplicity, including its agreement with Chevalley's.

We can only mention a few of the very many subsequent treatises on multiplicities in general local rings. The books of Nagata (2), Northcott (2), and Kunz (1) may be consulted for this literature. Nagata proved that when J is