1. (a) By Nullstellensatz, max. ideals of $\mathbb{C}[x]$ correspond to points in $\mathbb{A}^1$ by $a \in \mathbb{A}^1 = \mathbb{C}[x] \longleftrightarrow (x-a) \subseteq \mathbb{C}[x]$.

So $\max\operatorname{Spec}(\mathbb{C}[x]) = \{(x-a) \mid a \in \mathbb{C}\}$.

For $\max\operatorname{Spec}(\mathbb{C}[x, 1/x])$, consider a morphism

$\varphi : \mathbb{C}[x,y] \to \mathbb{C}[x, 1/x]$, $\varphi(x) = (x)$, $\varphi(y) = \frac{1}{x}$.

Then $\ker(\varphi) = (xy-1)$, so $\mathbb{C}[x, 1/x] \cong \mathbb{C}[x,y]/(xy-1)$.

Let us find the maximal ideals of $\mathbb{C}[x,y]/(xy-1) = \mathbb{C}[V]$, where $V = \mathbb{V}(xy-1) = \{(a, a^{-1}) \mid a \in \mathbb{C}^*\}$.

The maximal ideals are of the form $(\bar{x}-a, \bar{y}-\frac{1}{a})$, and since $\overline{xy}-1=0$ we have

$$-\frac{1}{a}\bar{y}(\bar{x}-a) = -\frac{1}{a}\overline{xy} + \bar{y} = \bar{y} - \frac{1}{a}, \text{ so}$$

$(\bar{x}-a, \bar{y}-\frac{1}{a}) = (\bar{x}-a)$. Hence $\max\operatorname{Spec}(\mathbb{C}[V]) = \{(\bar{x}-a) : a \in \mathbb{C}^*\}$.

Therefore $\max\operatorname{Spec}(\mathbb{C}[x, 1/x]) = \{(x-a) : a \in \mathbb{C}^*\}$ by iso^m.

1. (b).

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, "f\circ\varphi = \varphi^*(f)"'] & W \arrow[d, "f"] \\ & \mathbb{C} \end{tikzcd}$$

(i) $\varphi^*\left(\frac{1}{x}\right) = \frac{1}{x} \circ \varphi(a)$

$= \frac{1}{\varphi(a)} = \frac{1}{\frac{1}{a}} = a.$

So $\varphi^*\left(\frac{1}{x}\right) = x.$

(ii) Let $f(x) = 2x^2 + \frac{2x^3+4x}{x^5}$. Then

$\varphi^*(f(x)) = f\circ\varphi(x) = \frac{2}{x^2} + \frac{\frac{2}{x^3}+\frac{4}{x}}{\frac{1}{x^5}} = \frac{2}{x^2} + x^5\left(\frac{2}{x^3}+\frac{4}{x}\right)$

$= \frac{2}{x^2} + 2x^2 + 4x^4$

(iii) Let $f(x) = 2-x$. Then $\varphi^*(f) = f\circ\varphi(x) = 2 - \frac{1}{x}.$

2. $V = \mathbb{V}(y - ux) = \{ (x,y,u) \mid x = \frac{y}{u},\ u = \frac{y}{x} \} \subseteq \mathbb{A}^3.$

(a) Let $\Phi : \mathbb{A}^3 \to \mathbb{A}^2$ be a projection, which is a poly^n map.
$(x,y,u) \mapsto (x,u)$

Define a morphism

$$\varphi := \Phi|_V : V \to \mathbb{A}^2$$
$$(x,y,u) \mapsto = (x,u) = \left(\frac{y}{u}, \frac{y}{x}\right).$$

Note that φ is well-defined, as it is a restriction of a poly^n map.

Let us show that φ is an isom^m. For this we need to find an inverse $\psi : \mathbb{A}^2 \to V$ s.t. $\psi \circ \varphi = id_V$, $\varphi \circ \psi = id_{\mathbb{A}^2}$.

Define $\psi : (x,u) \mapsto (x, ux, u)$. Then

(i) $\psi \circ \varphi\,(x,y,u) = \psi\left(\frac{y}{u}, \frac{y}{x}\right) = \left(\frac{y}{u}, y, \frac{y}{x}\right) = (x,y,u)$, in V.

(ii) $\varphi \circ \psi\,(x,u) = \varphi(x, ux, u) = (x,u)$ in $\mathbb{A}^2$.

Since $(0,0,0) \in V$, (i) and (ii) need to be well defined at this point. Indeed, for (i) we can take $(1,0,1) = (x,y,u)$, and $(0,0) = (x,u)$ for (ii).

2.(b) $\Phi : \mathbb{A}^3 \to \mathbb{A}^2$ a poly^n map.
$(x,y,u) \mapsto (x,y)$

Define morphism as a restriction of Φ by V as follows

$$\varphi := \Phi|_V \; : \; (x,y,u) \mapsto (x,y) = \left(\frac{y}{u}, ux\right).$$

Suppose by contradiction φ is an iso^m. Then there exists an inverse $\psi : \mathbb{A}^2 \to V$ s.t.

$$\psi \circ \varphi = id_V, \qquad \varphi \circ \psi = id_{\mathbb{A}^2}.$$

we want: $\psi \circ \varphi(x,y,u) = \psi(x,y) = \left(x, y, \frac{y}{x}\right) = \left(\frac{y}{u}, ux, \frac{y}{x}\right).$

But since $(0,0,0) \in V$ we need the inverse ψ to be well defined at this point, and this means we need to choose $y=0$ and either $u=0$ or $x=0$, since

$$\psi(x,y) = \left(\frac{y}{u}, ux, \frac{y}{x}\right).$$

But choosing $u=0$ or $x=0$, we get division by zero, which is undefined. Hence the inverse ψ does not exist. Therefore φ is not isomorphism. $\square$

2.(c) $V = \mathbb{V}(y-ux) \subseteq \mathbb{A}^3$.

$$\mathcal{O}_V(D(u)) = \mathcal{O}_V(V \setminus \mathbb{V}(u)) = \mathcal{O}_V(\mathbb{V}(y-ux) \setminus \mathbb{V}(u)) = \mathcal{O}_V(\mathbb{V}(y-ux) \setminus \{0\})$$

$$\mathcal{O}_V(D(u)) = \frac{\mathbb{C}[x,y,u]}{(y-ux)}, \quad \text{since} \quad D(u) \cong \mathbb{V}(y-ux) = V,$$

and $\mathcal{O}_V(V) \cong \mathcal{O}_{D(x)}(D(x))$

3. Let V be an irreducible alg. variety. Then for some alg. varieties T, S if $V = S \cup T$ then $V = S$ or $V = T$.

Let us show that the closure $\overline{V}$ is irreducible.

Suppose $\overline{V} = \mathbb{V}(I) \cup \mathbb{V}(J)$.

We have $V \subseteq \overline{V}$.

So $V = V \cap \overline{V} = V \cap (\mathbb{V}(I) \cup \mathbb{V}(J))$

$$= \left(V \cap \mathbb{V}(I)\right) \cup \left(V \cap \mathbb{V}(J)\right).$$

But since V is irreducible, WLOG we have

$$V = V \cap \mathbb{V}(I) = V \cap \overline{V}.$$

So $\overline{V} = \mathbb{V}(I)$, for some homogenised ideal I.

4. Note that $\mathbb{V}(y-\sin(x))$ is not an affine algebraic variety, as it intersects with $y=\frac{1}{2}$ at infinitely many points.

Let us write $V=\mathbb{V}(y-\sin(x))$ as a union of affine charts.
We have

$$V = U_x \cup U_y, \quad \text{where}$$

$$U_x = \{[1: \tfrac{\sin(x)}{x}] : x \neq 0\},$$

$$U_y = \{[\tfrac{x}{\sin(x)} : 1] : \sin(x) \neq 0\} = \{[\tfrac{x}{\sin(x)} : 1] : x \neq \pi k,\ k \in \mathbb{Z}\}$$

$$= \{[\tfrac{x}{\sin(x)} : 1] : x \neq 0\} = U_x.$$

So $V = U_x$.
Note that V contains $[1:0]$, when $x=\pi$, and $[(2n+1)\pi : 2]$ when $x = \frac{n\pi}{2}$, $n \in \mathbb{Z}$.
$= [1 : \frac{2}{(2+n)\pi}]$

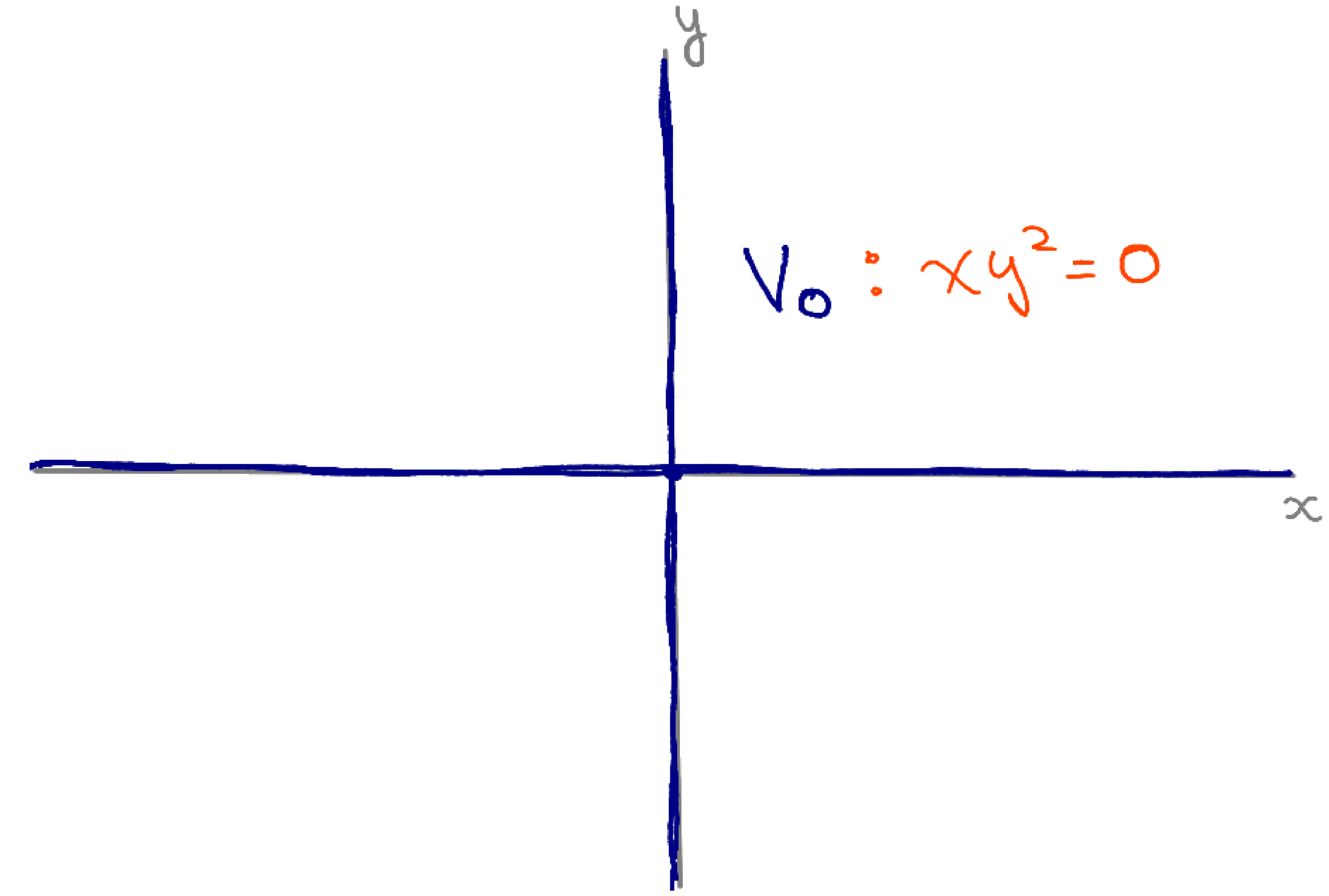
This does not contradict Chow's Lemma, as V is affine analytic in $\mathbb{P}^2$.

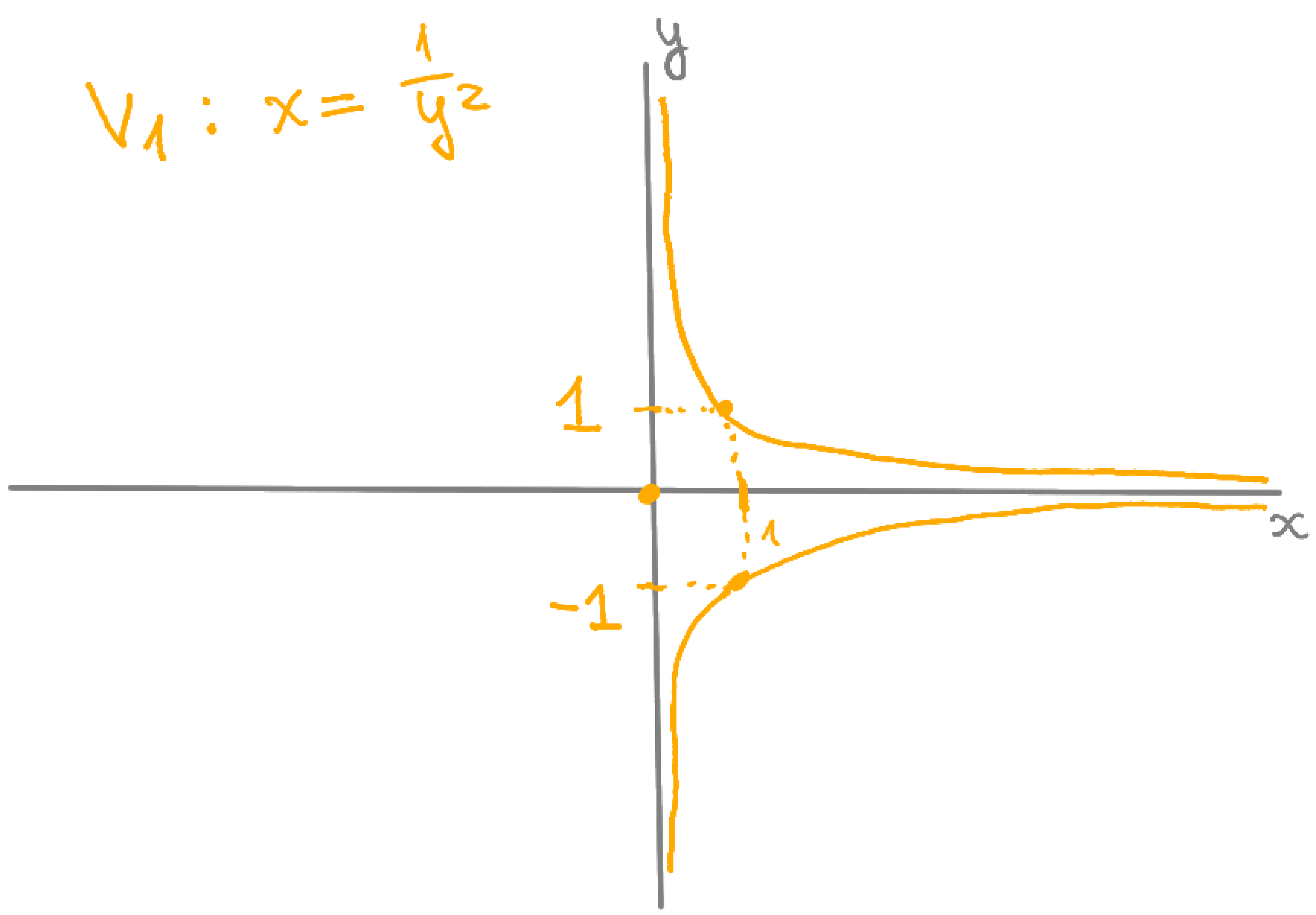
5. $V_t := \mathbb{V}(xy^2 - t) \subseteq \mathbb{A}^2, \quad t \in \mathbb{C}.$

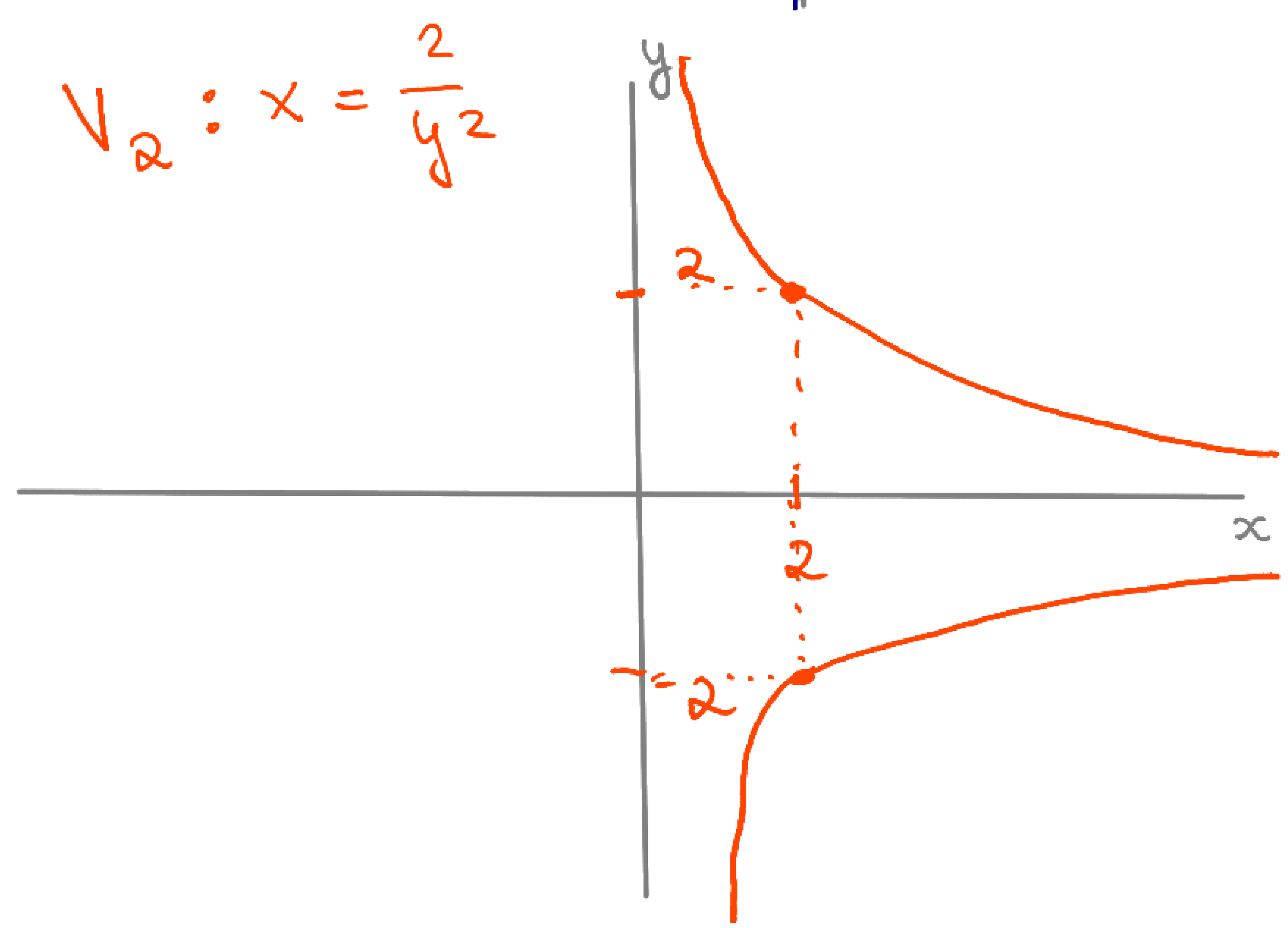
$V_0 = \mathbb{V}(xy^2)$

$V_1 = \mathbb{V}(xy^2 - 1)$

$V_2 = \mathbb{V}(xy^2 - 2)$







V_0 is irreducible, V_1 and V_2 are not.

5.

$$\dim V_1 = \dim V_2 = \dim(\mathbb{A}^2) - \dim(V) = 2 - 1 = 1$$

$$V_1 = (xy^2 - 1) \rightsquigarrow x = \frac{1}{y^2}$$

$$\dim(\ker T_{(\frac{1}{a^2},a)}(V_1)) = \dim(\ker \nabla(xy^2-1)|_{(\frac{1}{a^2},a)} = \dim(\ker(y^2, 2xy)|_{(\frac{1}{a^2},a)})$$

$$= \dim(\ker(a^2, \tfrac{2}{a})) =$$

$$\ker(a^2, \tfrac{2}{a}) = \left\{ \begin{pmatrix} b \\ c \end{pmatrix} \in \mathbb{C}^2 : \begin{pmatrix} a^2 \\ \frac{2}{a} \end{pmatrix} \cdot \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$\Leftrightarrow \underline{0} = \begin{pmatrix} a^2 \\ \frac{2}{a} \end{pmatrix} \cdot \begin{pmatrix} b \\ c \end{pmatrix} = a^2 b + \frac{2c}{a}$$

$$\Leftrightarrow a^3 b + 2c = 0$$

$$\Leftrightarrow c = -\frac{a^3 b}{2}$$

So $\ker(a^2, \frac{2}{a})$ not linearly independent, and has one variable.
Therefore $\dim(\ker(a^2, \frac{2}{a})) = 1 = \dim V_1$,

so V_1 is smooth.

Moreover, $\nabla V_1 = \nabla V_2$, so $\dim(\ker \nabla V_2) = \dim(\ker \nabla V_1) = \dim V_2$
Hence V_2 is also smooth.