Intersection of positive closed currents

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Conférence à la mémoire de Jean-Pierre Demailly

Outline

Problem (Demailly)

Study the existence of the wedge-product (intersection) $T \wedge S$ of positive closed currents T and S of higher bi-degrees (we can consider more than 2 currents).

Jean-Pierre Demailly, Courants positifs et théorie de l'intersection. *Gaz. Math.* **53** (1992), 131-159.

COURANTS POSITIFS ET THÉORIE DE L'INTERSECTION

Jean-Pierre DEMAILLY (Institut Fourier, Grenoble 1)

1. Introduction

La notion de multiplicité locale d'intersection des cycles algébriques ou analytiques est maintenant bien comprise d'un point de vue algébrique depuis plusieurs décennies (travaux de Samuel [Sa51], Serre [Se57]), voire depuis le XIXème siècle. Nous allons dans la suite adopter un point de vue assez différent, mais il est sans doute utile de rappeler quelques notions fondamentales pour situer le contexte.

Rappelons qu'un cycle algébrique de codimension p dans une variété algébrique X est une combinaison linéaire formelle $A = \sum \lambda_j A_j$ dans le groupe abélien libre engendré par les ensembles algébriques irréductibles de codimension p: les A_j sont donc de tels ensembles et $\lambda_j \in \mathbb{Z}$; le cycle est dit effectif si $\lambda_j \geq 0$. On s'intéressera en fait aussi aux cycles réels ($\lambda_j \in \mathbb{R}$). Le support de A est l'ensemble $|A| = \bigcup_{\lambda_j \neq 0} A_j$.

Ideal situation (quite restrictive)

For suitable classes of currents:

- We can define $T \wedge S$ locally.
- The definition is independent of the choice of (local) coordinates.
- The definition is compatible with the wedge-product of forms (smooth case), the intersection of cycles (geometric case), cohomology, or other known cases.
- The product is (semi)-continuous with respect to T and/or S.
- $\cdots$

Case of bi-degree $(1, 1)$: Bedford-Taylor, Demailly, Oka-Fornaess-Sibony...

- Assume that T is of bi-degree $(1, 1)$.
- Write locally $T = dd^c u$ where u is a psh function, unique modulo a pluriharmonic function.
- If u is integrable with respect to S, then define

$$T \wedge S = dd^c u \wedge S = dd^c(uS).$$

We use here that S is closed and psh functions are defined at every point.

Examples

- u is continuous or locally bounded.
- u is bounded outside a compact subset of a Stein open set.
- u is bounded outside a closed set which is "pseudoconvex enough"...

Remark (non-pluripolar product of currents)

Bedford-Taylor, Boucksom-Eyssidieux-Guedj-Zeriahi, Darvas-Di Nezza-Lu, Vu...

Particular case: pullback operator

- Let $f : X \to Y$ be a surjective meromorphic map and Γ its graph in $X \times Y$.
- Let π_X, π_Y be the projections from $X \times Y$ to X and Y.
- Let T be a positive closed current on T. Define (when meaningful)

$$f^*(T) = (\pi_X)_*\big((\pi_Y)^*(T) \wedge [\Gamma]\big).$$

Remarks

- $(\pi_Y)^*(T)$ is always well-defined.
- $(\pi_X)_*$ is well-defined when π_X is proper on $(\pi_Y)^{-1}(\mathrm{supp}\,(T)) \cap \Gamma$, e.g. when Y is compact.

Examples (ideal situation)

- T is of bi-degree $(1,1)$ (Méo).
- Every fiber of $\pi_Y : \Gamma \to Y$ is either empty or of dimension $\dim X - \dim Y$ (D-Sibony).

Particular case: Federer slicing theory

- Let $f : X \to Y$ be a holomorphic submersion. Let T be a positive closed current on X with dimension $\geqslant \dim Y$.
- For almost every $y \in Y$, the slice $\langle T \,|\, f \,|\, y \rangle$ is well-defined. This is similar to Lebesgue points of integrable functions.
- This can be seen as the wedge product $T \wedge [f^{-1}(y)]$.

Particular case: Federer slicing theory

Example (D-Nguyen-Sibony, Bianchi-D-Rakhimov)

- Let $M \subset \mathbb{C}^{n-p}$ and $N \subset \mathbb{C}^p$ be open sets. Let T be positive closed of bi-degree (p,p) on $M \times N$ such that $\pi_M : \text{supp}\,(T) \to M$ is proper.
- Then $\langle T \,|\, \pi_M \,|\, y \rangle$ is well-defined for every y.
- If ϕ is psh on $M \times N$, then $y \mapsto \langle T \,|\, \pi_M \,|\, y \rangle(\phi)$ is either $-\infty$ or psh.
- More generally, if S is positive closed of bi-degree $(n-p, n-p)$ on $M \times N$ with vertical support, then $T \wedge S$ is well-defined.

Outline

Super-potentials of positive closed currents (D-Sibony, Nguyen, Vu)

- Assume $X = \mathbb{P}^n$ for simplicity. Let ω_{FS} be the Fubini-Study form on $\mathbb{P}^n$.
- Let T be a positive closed (p, p)-current on X.
- Let $\mathcal{C}_{n-p+1}$ be the set of all positive closed $(n-p+1, n-p+1)$-currents of mass 1 on $\mathbb{P}^n$. This is a compact convex metric space.
- Wasserstein distance:

$$\mathrm{dist}(R, R') := \sup \left\{ |\langle R - R', \alpha \rangle| \text{ with } \|\alpha\|_{\mathcal{C}^1} \leqslant 1 \right\}.$$

- We can define a function $\mathcal{U}_T : \mathcal{C}_{n-p+1} \to \mathbb{R} \cup \{-\infty\}$ by

$$\mathcal{U}_T(R) := \langle T, V_R \rangle,$$

where V_R is a $(n-p, n-p)$-current such that

$$dd^c V_R = R - \omega_{FS}^{n-p+1} \qquad \text{and} \qquad \langle V_R, \omega_{FS}^p \rangle = 0.$$

- The definition is independent of the choice of V_R.
- $\mathcal{U}_T$ can be seen as a quasi-psh function on $\mathcal{C}_{n-p+1}$.
- The general case of Kähler manifolds X is more complicated.
- Skoda's type estimate: $|\mathcal{U}_T(R)| \lesssim \log \|R\|_\infty$.

Remark (similarity with the case of $(1,1)$-currents)

- If T is of bi-degree $(1,1)$ of mass 1, we can write $T = \omega_{FS} + dd^c u_T$ with $u_T : \mathbb{P}^n \to \mathbb{R} \cup \{-\infty\}$ quasi-psh and $\langle \omega_{FS}^n, u_T \rangle = 0$.
- Skoda estimate: $\langle \omega_{FS}^n, e^{|u_T|} \rangle \leqslant \text{const}$.

Intersection of currents: first answer to Demailly's problem

- Let S be a positive closed (q,q)-current with $q \leqslant n-p$.
- Assume that $\mathcal{U}_T(S \wedge \omega_{FS}^{n-p-q+1}) \neq -\infty$. Then, we can define $T \wedge S$.
- More precisely, for α a real smooth $(n-p-q, n-p-q)$-form, write $dd^c\alpha = \beta^+ - \beta^-$ with $\beta^\pm$ positive closed. We have

$$\langle T \wedge S, \alpha \rangle = \mathcal{U}_T(\beta^+ \wedge S) - \mathcal{U}_T(\beta^- \wedge S) + \text{some correction.}$$

- The definition is independent of the choice of $\beta^\pm$ ...
- $(T, S) \mapsto T \wedge S$ is continuous with respect to the "standard" regularization of T and S ...
- If T has a bounded super-potential, then $T \wedge S$ is defined for every S.

Remark (for $p = 1$, Bedford-Taylor, Demailly, Oka-Fornaess-Sibony...)

- Write $T = dd^c v_T + \omega_{FS}$. Assume that v_T is integrable wrt $S \wedge \omega_{FS}^{n-s}$.
- Then, we can define $T \wedge S = dd^c(v_T S) + \omega_{FS} \wedge S$ or equivalently

$$\langle T \wedge S, \alpha \rangle = \langle v_T S, dd^c\alpha \rangle + \langle \omega_{FS} \wedge S, \alpha \rangle = \langle v_T S, \beta^+ \rangle - \langle v_T S, \beta^- \rangle + \langle \omega_{FS} \wedge S, \alpha \rangle.$$

Application in dynamics

Theorem (Cantat for projective K3 surfaces, D-Sibony)
Let $f : X \to X$ be a holomorphic automorphism on a compact Kähler manifold X of dimension n. Suppose the action of f on Hodge cohomology is simple: there is only one eigenvalue of maximal modulus and it is simple. Denote this eigenvalue by d and let $1 \leqslant p \leqslant n-1$ be such that the corresponding eigenvectors are in $H^{p,p}(X, \mathbb{R})$. Then

- *There are unique positive closed (p,p)-current T^+ and $(n-p, n-p)$-current T^- of mass 1 such that*

$$f^*(T^+) = dT^+ \qquad \text{and} \qquad f_*(T^+) = dT^-.$$

- *If S is a positive closed (p,p)-current, then $d^{-n}(f^n)^*(S)$ converges exponentially fast to a multiple of T_+. A similar property holds for $(f^n)_*$.*
- *T_+ and T_- have Hölder continuous super-potentials.*
- *$T^+ \wedge T^-$ is the unique invariant measure of maximal entropy...*

Remark (other applications in dynamics)
D-Nguyen-Sibony, de Thélin-Vigny, Ahn, Bianchi-D-Rakhimov...

Application to complex Monge-Ampère equation

Theorem (D-Nguyen)

Let μ be a probability measure on a compact Kähler manifold (X, ω). Then μ is the Monge-Ampère measure with Hölder potential:

$$\mu = (dd^c u + \omega)^n \quad \textit{with } u \textit{ Hölder continuous } \omega\textit{-psh}$$

if and only if μ has a Hölder continuous super-potential (linear condition):

$$\mathcal{U}_\mu : \mathcal{C}_1 \to \mathbb{R} \cup \{-\infty\}.$$

Example (Yau, Kolodziej, Hiep, Vu)

Lebesgue measures on generic real Cauchy-Riemann submanifolds of real dimension $\geqslant n$ of X.

Remarks

- Demailly's technique of regularization of quasi-psh functions plays a crucial role in the proof.
- Related results: Kolodziej, Hiep, D-Nguyen-Sibony, Demailly-Dinew-Guedj-Hiep-Kolodziej-Zeriahi, Vu, D-Kolodziej-Nguyen ...

Outline

Density of positive closed currents (D-Sibony, Nguyen, Vu)

- Consider the case $S = [V]$ with V a submanifold of dimension $n - q$ of X.
- To define $T \wedge [V]$, we "dilate" a neighbourhood W of V in the normal directions to V.

Density of positive closed currents (D-Sibony, Nguyen, Vu)

- To do this, we identify W with a neighbourhood W' of the zero section of the normal vector bundle $\mathrm{N}_{V|X}$ using a suitable diffeomorphism τ.
- Use $A_\lambda : \mathrm{N}_{V|X} \to \mathrm{N}_{V|X}$ which is the multiplication by $\lambda \in \mathbb{C}$ on the fibers.
- Consider the following currents and limits of subsequences

$$T_\lambda = (A_\lambda)_* \tau_*(T).$$

- If $V = 1$ point, we have Lelong's number theory (Harvey's viewpoint).
- Kiselman: in general, the limit doesn't exist when $\lambda \to \infty$; we need to take subsequences.

Density of positive closed currents

Theorem (D-Sibony)

- *Any limit* T_∞ *of a subsequence of* (T_{λ_j}) *is a positive closed* (p,p)*-current of* $\overline{N_{V|X}}$ *which is invariant by* A_λ*.*
- *The cohomology class* $\{T_\infty\}$ *in* $H^{p,p}(\overline{N_{V|X}}, \mathbb{R})$ *doesn't depend on the subsequence.*
- *Let* $\pi : \overline{N_{V|X}} \to V$ *be the canonical projection. We can define the dimension of "*$T \wedge [V]$*" as the maximal integer* s *such that*

$$T_\infty \wedge \pi^*(\omega_V^s) \neq 0.$$

  It is independent of the subsequence and the Kähler form ω_V*.*
- *If* $T_\infty \neq 0$*, we have* $s \geqslant \max(n-p-q, 0)$*.*
- *We can define the shadow* Θ *of* T_∞ *to* V*. This is a positive closed current of bi-dimension* (s,s) *on* V*. Its cohomology class doesn't depend on the subsequence.*

Intersection with(out) dimension excess: 2nd answer to Demailly's problem

- The cohomology class $\{T_\infty\}$ measures the size of the intersection "$T \wedge [V]$".
- When s is larger than the expected dimension, we say that "$T \wedge [V]$" has a dimension excess.
- When there is no dimension excess and T_∞ is unique, we define $T \wedge [V]$ as the shadow Θ of T_∞ to V. In this case, we have $T_\infty = \pi^*(\Theta)$.
- For general currents T, S, the intersection $T \wedge S$ is defined as the intersection of $T \otimes S$ with the diagonal $[\Delta]$ of $X \times X$.
- This definition is consistent with other known definitions, where applicable.

Theorem (Bedford-Lyubich-Smillie for $n = 2$, D-Sibony)
Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a Hénon-Sibony polynomial automorphism. Then the (saddle) periodic points are equidistributed with respect to the invariant probability measure μ of maximal entropy:

$$\lim_{k\to\infty} \sum_{f^k(a)=a} \delta_a = \mu.$$

Remarks

- The measure μ is obtained as the intersection of two canonical invariant positive closed currents: $\mu = T^+ \wedge T^-$.
- In the proof, using the density theory, we show that this intersection is transversal in a suitable sense.

Applications in complex dynamics

Theorem (D-Nguyen-Truong)

Let $f : X \to X$ be a dominant meromorphic map on a compact Kähler manifold. Then it is an Artin-Mazur map: if $P_k(f)$ is the number of its isolated periodic points of period k (counted with multiplicity), then $P_k(f)$ grows at most exponentially fast as $k \to \infty$

$$P_k(f) \lesssim \|(f^k)^* : H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})\|.$$

Remarks

- The dynamical ζ-function is always analytic near 0, where

$$\zeta_f(z) := \sum_{k \geqslant 1} \frac{1}{k} P_k(f) z^k.$$

  Is it rational ?

- Kaloshin: for smooth real diffeomorphisms, this number can grows as fast as we want.

Application in foliation theory: unique ergodicity

Theorem (Fornaess-Sibony and D-Sibony for $\mathbb{P}^2$, D-Nguyen-Sibony)
Let $\mathcal{F}$ be a foliation by Riemann surfaces on a compact Kähler surface X. Assume that $\mathcal{F}$ is generic: all singularities are hyperbolic and there are no invariant positive closed currents. Then

- *there is a unique positive dd^c-closed current T of mass 1 directed by $\mathcal{F}$.*
- *its cohomology class $\{T\}$ in $H^{1,1}(X, \mathbb{R})$ is big and nef.*
- *all leaves of $\mathcal{F}$ are equidistributed wrt T.*

Remarks

- When $\mathcal{F}$ admits invariant positive closed currents, we can classify them.
- The proof uses the density of dd^c-closed currents to show $T \wedge T = 0$.
- Then a version of Hodge-Riemann theorem implies the uniqueness of T.

Outline

Intersection and limit

Problem 1

Let T_k be a sequence of positive closed currents on X converging to a current T. Let S be a positive closed current on X. Find sufficient conditions such that

$$\lim_{k\to\infty}(T_k \wedge S) = T \wedge S = (\lim_{k\to\infty} T_k) \wedge S.$$

Problem 2

Assume that T_k converges to T exponentially fast. Find sufficient conditions such that $T_k \wedge S$ converges to $T \wedge S$ exponentially fast.

Example

T_k and S are currents of integration on complex submanifolds of X, but not the current T.

Main motivation: complex dynamics

Problem 3

Let $f : X \to X$ be a suitable meromorphic map or correspondence. Prove that its isolated periodic points are equidistributed with respect to a canonical invariant measure (with an exponential speed).

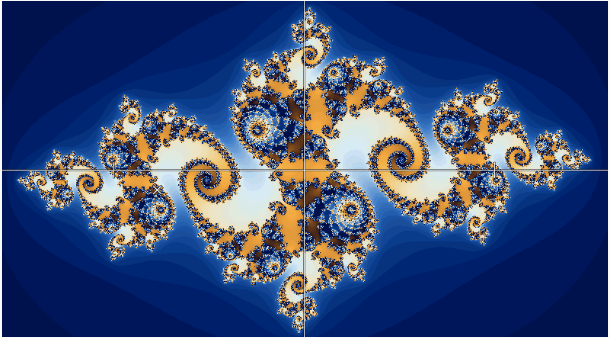
Strategy

- Let Γ_k denote the graph of f^k in $X \times X$ and $[\![\Gamma_k]\!]$ be the normalized current of integration on Γ_k.
- Show that $[\![\Gamma_k]\!]$ converges (exponentially fast) to some positive closed current Γ_∞ in $X \times X$.
- The periodic points of period k are identified with the intersection $\Gamma_k \cap \Delta$, where Δ is the diagonal of $X \times X$.
- Show that $[\![\Gamma_k]\!] \wedge [\Delta]$ converges (exponentially fast) to $\Gamma_\infty \wedge [\Delta]$.

Main motivation: complex dynamics

Theorem (Brolin, Lyubich, D-Kaufmann)
Let $f : \mathbb{C} \to \mathbb{C}$ *be a polynomial of degree* $d \geqslant 2$*. Let* μ *be the invariant probability measure of maximal entropy of* f*. Then the periodic points of* f *equidistribute towards* μ *exponentially fast:*

$$d^{-k} \sum_{f^k(a)=a} \delta_a \to \mu \quad \textit{exponentially fast.}$$



source: internet.

Another problem

Problem (Demailly)

Let T *be a positive closed current of mass* 1 *on a projective manifold* X*. Can we write* $T = T^+ - T^-$ *with* $T^\pm$ *positive closed approximable by effective cycles and* $\|T^\pm\| \leqslant \mathrm{const}$ *? We assume here that there is no cohomological obstruction.*

Remark

Babaee-Huh: we cannot remove T^-.