Slicing. Let $f : X \to Y$ be a dominant holomorphic map between complex manifolds, not compact in general, of dimension n and m respectively. Let T be a positive closed current on X of bi-dimension (q, q) with $q \geq m$. Then the slice

$$T_y = \langle T|f|y\rangle$$

exists for almost every $y \in Y$. This is a positive closed current of bi-dimension $(q-n, q-n)$ on X supported by $f^{-1}(y)$. If Ω is a smooth form of maximal bi-degree on Y and α a smooth $(q-n, q-n)$-form with compacts upport in X, then we have

$$\langle T, \alpha \wedge f^*(\Omega)\rangle = \int_{y\in Y} \langle T_y, \alpha\rangle \Omega(y).$$

In general, if T and T' are such that $T_y = T'_y$ for almost every y, we don't necessarily have $T = T'$. However, the following is true.

Let $f_1, \ldots, f_k$ be dominant holomorphic maps from X to $Y_1, \ldots, Y_k$. Consider the vector space spanned by all the differential forms of type $\alpha \wedge f_i^*(\Omega_i)$ for some α as above and for some smooth form Ω_i on Y_i of maximal degree. Assume this space is equal to space of all (q, q)-forms of compact support in X. Then if $\langle T|f_i|y_i\rangle = \langle T'|f_i|y_i\rangle$ for every i and almost every $y_i \in Y_i$, we have $T = T'$. The proof is a consequence of the above discussion.

As a basic example, consider $X = \mathbb{C}^2$, $Y = \mathbb{C}$ and $q = 1$. We can choose the maps $z \mapsto z_1$, $z \mapsto z_2$, $z \mapsto z_1 \pm z_2$ and $z \mapsto z_1 \pm iz_2$. It is not difficult to check that these maps satisfy the above condition or equivalently: every differential $(1, 1)$-form on $\mathbb{C}^2$ is a linear combination of the following forms (with functions as coefficients)

$$dz_1 \wedge d\overline{z}_1, \quad dz_1 \wedge d\overline{z}_1, \quad d(z_1 \pm z_2) \wedge d\overline{(z_1 \pm z_2)}, \quad d(z_1 \pm iz_2) \wedge d\overline{(z_1 \pm iz_2)}.$$

The higher dimension case is similar but we need much more projections.

Measure comparison. Let μ_1, μ_2 be two probability measures in $(\mathbb{C}^*)^n$, both supported by $(\mathbb{S}^1)^n$. Assume that

$$\langle \mu_1, \phi\rangle \leq \langle \mu_2, \phi\rangle$$

for every psh function ϕ on $(\mathbb{C}^*)^n$ (or a fixed neighbourhood of $(\mathbb{S}^1)^n$ but this is a particular case). Then $\mu_1 = \mu_2$.

This is a consequence of the fact that continuous psh functions are dense in the space of continuous functions on $(\mathbb{S}^1)^n$. We can even show that holomorphic functions are dense by using Fourier series.

On Theorem 8.3. I guess the theorem holds for $A = a + H$ with H rational but a is not necessarily rational (?). If I am not wrong, the map π_H in Section 4.1 extends to a holomorphic map from $(\mathbb{C}^*)^n$ to some $(\mathbb{C}^*)^{n-p}$ which contains the image of π_H as a torus (?).

We know that $W_n = m^{-n+p}\Phi_m^*[W]$ converges to $T = T_{trop(W)}$. Since T_A is foliated by complex varieties, we can replace T_A by such a variety, i.e. a fiber V of the above extension of π_H, for simplicity.

The first case is when $T\wedge[V]$ is a measure. Its support is a finite union of disjoint tori. In order to apply the above measure comparison, we need a more quantitative transversality in $W_m\cap V$, namely, a horizontal property of W_m near each torus. More precisely, consider a small neighbourhood in $(\mathbb{C}^*)^n)$ of such a torus which has a product form where V is vertical and all W_m (with m big) are horizontal. If so, in this neighbourhood, any limit of $W_n \wedge [V]$ is smaller than $T \wedge [V]$ when testing psh functions. Thus, we can apply the above measure comparison to conclude.

For the general case, we need the above slicing theory for a rich enough family of maps $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^q$. Here, the bi-dimension of $T \wedge [V]$ is (q, q). The choice of q reduces the problem to the case of measures in each slice. Do you think this is OK ?