Non-archimedean dynamical tropicalisation and Monge–Ampère Equations

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Tropicalisation is a certain degeneration of algebraic varieties of complex torus into polyhedral complexes called *tropical varieties*. Tropical geometry is the study of tropical varieties. A crucial property of tropicalization is its capacity to capture a significant amount of information from the original algebraic varieties, even as it transforms algebro-geometric objects into simplified combinatorial and polyhedral structures. This characteristic and the naturality of tropical varieties have been the source of applications of tropical geometry in the past twenty years, leading to many groundbreaking works.

Relations of tropical geometry with analytic geometry were been made possible in the real setting by works of Lagerberg [Lag12], which were later generalised to the setting of *Berkovich analytic spaces* in [CD12] and [GK17]. The theory of Berkovich spaces provides an analogue of complex analytic geometry in the case of non-archimedean valued fields, constructing spaces that have nice topological properties, and therefore are particularly suitable for dynamics.

The bridge between tropical geometry and complex analytic geometry was started by introducing *complex tropical currents* by the author in [Bab14]. A notable application of the complex tropical currents was to disprove a generalised Hodge conjecture for the positive current in the work of the author and June Huh (Fields Medal 2022) in [BH17]. Further, understanding tropicalisation in terms of holomorphic dynamics appeared in [Bab23] and will be generalised to the non-trivial valuation case in the ongoing works of the author and Tien-Cuong Dinh.

The **first objective** of this Focused Research Group is to gather experts in Berkovich spaces and dynamical systems to generalise the main result of [Bab23] to the setting of *Berkovich Spaces*. This work is significant as there are no known general results about dynamics on the Berkovich spaces in higher dimensions. We are hopeful that we can settle the special case of *dynamical tropicalisation map* in any dimension and codimension. Moreover, the main result of [Bab23] is reminiscent of a series of strong results in arithmetic geometry, known as *Equidistribution Theorems*, which have culminated with the paper [Yua08]. Our first objective will also have certain applications in analytic geometry. The **second objective** of this Focused Research Group would be to explore the relations of the dynamical tropicalisation and applications to the *non-archimedean Monge–Ampère equations* and relations to *mirror symmetry*. Specifically, we wish to explore whether a specific degeneration of the Calabi–Yau metric structures can be understood using dynamical tropicalisation. More specifically, let $f \in \mathbb{C}[z_0, \dots, z_{d+1}]$ be a *generic* homogeneous polynomial of degree $d+2$. Consider the family of algebraic varieties $(X_s, \omega_{\mathrm{CY}}) = \mathbb{V}(z_0 \cdots z_{d+1} - sf) \subseteq \mathbb{CP}^{d+1}$, for $s \in \mathbb{C}\setminus\{0\}$. Predicting how the Calabi–Yau metrics structure change as $s \to 0$, is the content of two foundational conjectures in algebraic geometry. On the one hand, the celebrated Strominger–Yau–Zaslov conjecture [SYZ96], predicts that for small s, X_s admit a *special Lagrangian torus fibration*. On the other hand, the conjecture by Kontsevich–Soibelman predicts that $(X_s, \omega_{\mathrm{CY}})$ converges in Gromov–Hausdorff distance to a metric of Monge–Ampère type on the base which has an integral affine structure. Recently, in [HJMM24], Mazzon and others have resolved a non-archimedean Monge–Ampère equation on the boundary of the unit simplex, showing some special cases of the above conjectures, and making some connections of Monge–Amprère equations and mirror symmetry more explicit. We wish to investigate whether these results can be understood using dynamical tropicalisation.

Proposed Participants. I have invited several experts who are enthusiastic about these projects. Dr Daniele Turchetti (Durham) is an expert in Berkovich spaces and their tropicalisations. Dr Roberto Gualdi (Barcelona) has contributed to the equidistribution theorems in arithmetic geometry. Dr Nguyen-Bac Dang (Paris-Saclay) is an expert in both holomorphic dynamics and Berkovich analytification. Dr Enrica Mazzon (Regensburg) is one of the authors of [HJMM24] and having her insight for the second part of the project is crucial. Finally, Prof Charles Favre (École Polytechnique) is a world-leading expert in the fields of complex dynamics and non-archimedean geometry. From Bristol, the Heilbronn research fellows Dr James Maxwell and Dr Sean Dewar are interested in these projects, and my PhD student Daniel Green Tripp will also benefit from the participation and discussion with the collaborators.

Proposed Budget. We expect the non-local participants to stay for 1 - 2 weeks each. We are requesting £6500, comprising of the following:

- Accommodation - 42 nights at £100 per night - £4200.
- UK travel - 1 return train from Durham - £200.
- EU travel - 4 return flights - £1600 (average £400 per person).
- Provision of refreshments and a group dinner - £500 (10 people).

References

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