CONTINUITY OF THE SUPERPOTENTIALS AND SLICES OF TROPICAL CURRENTS

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Abstract.

Contents

1. Introduction

Let X be a complex manifold of dimension d, and p, q non-negative integers with $n = p + q$. We denote by $\mathcal{C}^q(X) = \mathcal{C}_p(X)$ the cone of positive closed bidegree (q, q), or bidimension $(p, p)-$ currents on X. We also consider $\mathcal{D}^q(X) = \mathcal{D}_p(X)$, the $\mathbb{R}$-vector space spanned by $\mathcal{C}^q(X)$. It is well-known that the intersection of two positive closed currents is not always defined. The main initial progress was due to the works of Federer [Fed69] and Bedford and Taylor in [BT82]. Federer defined the generic slicing theory of currents: for a dominant holomorphic map $f : X \longrightarrow Y$, and a positive closed current $\mathcal{T} \in \mathcal{C}_p(X)$, or more generally, a *flat current*, a *slice*

$$\mathcal{T} \wedge [f^{-1}(y)]$$

is well-defined for a generic $y \in Y$. Following Bedford and Taylor, Demailly [Dem12], and Fornaess–Sibony [FS95] if $\mathcal{S} = dd^c u$ is a bidegree $(1, 1)$-current, then

$$\mathcal{S} \wedge \mathcal{T} := dd^c(u\mathcal{T}),$$

can be defined or it is *admissible*, when

- The potential, u, is bounded
- u is unbounded, but its unbounded locus has a small intersection with $\mathrm{supp}(\mathcal{T})$.

For instance, when $\mathcal{S} = dd^c \log |f|$ and $\mathcal{T}$ are two integration currents, such that their supports intersect in the expected dimension, then

$$\mathcal{S} \wedge \mathcal{T} = \sum c_i[C_i],$$

where each C_i is an irreducible component of the intersection, and c_i is the corresponding vanishing number. This intersection coincides with the slicing of integration currents.

Demailly in [Dem92] asked the question of generalising the intersection theory to the case where $\mathcal{T}$ is of a higher bidegree. In several works, the second-named author and Sibony introduced *superpotential theory* and *density* of currents to answer this question. In this article, we adopt the approach of the second-named author and Sibony for our intersection theory. See also the works of Anderson, Eriksson, Kalm, Wulcan and Yger [AESK+21], and [ASKW22] a non-proper intersection theory.

The authors in [DS09] completely discussed the situation where X is a homogeneous space, and in [DS10] investigated the intersection theory for currents with continuous superpotentials, which is a generalisation of the case of bounded potentials in bidegree $(1,1)$. Once the intersections are defined, one can ask the following *continuity problem*:

Problem 1.1. Let $\mathcal{T}_k$ be a sequence of positive closed currents on X converging to $\mathcal{T}$. Let $\mathcal{S}$ be also a positive closed current on X. Find sufficient conditions such that

$$\lim_{k\to\infty}(\mathcal{S}\wedge\mathcal{T}_k)=\mathcal{S}\wedge(\lim_{k\to\infty}\mathcal{T}_k).$$

Roughly speaking, if $\mathcal{S}$ is a current on a compact Kähler manifold with a *continuous superpotential*, then for a current $\mathcal{T}$, the wedge product

$$\mathcal{S}\wedge\mathcal{T}:=\lim_{n\to\infty}(\mathcal{S}\wedge\mathcal{T}_n),$$

is independent of the choice of smooth approximation $\mathcal{T}_n\longrightarrow\mathcal{T}$. Consequently, by the Regularisation Theorem for any bidegree we can partially answer Problem 1.1.

Proposition 1.2. Let X be a compact Kähler manifold, $\mathcal{T}_k\longrightarrow\mathcal{T}$ be a convergent sequence in $\mathcal{D}^p(X)$. If a current $\mathcal{S}$ has a continuous superpotential, then

$$\mathcal{S}\wedge\mathcal{T}_n\longrightarrow\mathcal{S}\wedge\mathcal{T}.$$

Proof. The main result of [DS04] implies that any current $\mathcal{D}^q(X)$ can be weakly approximated by a difference of smooth closed positive of bidegree (p,p)-forms of bounded mass. The result then follows from the definition of continuity of super-potentials. $\square$

Problem 1.1 becomes more difficult when one considers continuity for slices, and the current $\mathcal{S}$ is an integration current. Borrowing ideas in tropical geometry, we discuss this problem for the very specific case where $\lim_{k\to\infty}\mathcal{T}_k$ is a *complex tropical current* [Bab14, BH17]. (Complex) tropical currents are closed currents on complex tori $(\mathbb{C}^*)^d$ or on a toric variety associated to a *tropical cycle*. Recall that a tropical cycle is a weighted polyhedral complex satisfying the *balancing condition* (see Definition 3.1). For a tropical cycle $\mathcal{C}\subseteq\mathbb{R}^d$, of dimension p, the associated tropical current $\mathcal{T}_{\mathcal{C}}\in\mathcal{D}_p((\mathbb{C}^*)^d)$, is a closed current with support $\mathrm{Log}^{-1}(\mathcal{C})$, where

$$\mathrm{Log}:(\mathbb{C}^*)^d\longrightarrow\mathbb{R}^d,\quad (z_1,\dots,z_d)\longmapsto(-\log|z_1|,\dots,-\log|z_d|).$$

The tropical current $\mathcal{T}_{\mathcal{C}}$ can be naturally presented as a locally fibration of $\mathrm{Log}^{-1}(\mathcal{C})$, and we say $\mathcal{C}$ is compatible with the fan Σ, if the fibres of $\overline{\mathcal{T}}_{\mathcal{C}}$ intersect the toric invariant divisors of the toric variety X_Σ transversely. Here $\overline{\mathcal{T}}_{\mathcal{C}}$ denotes the extension by zero of $\mathcal{T}_{\mathcal{C}}$ to the toric variety X_Σ.

Theorem 1.3. Let X_Σ be a smooth projective toric variety, and let $\mathcal{C}$ be a tropical cycle compatible with Σ. Then $\overline{\mathcal{T}}_{\mathcal{C}}$ has a continuous super-potential.

The preceding theorem allows for defining the intersection product of a tropical current with any current on a compatible toric variety, we can then restrict the intersection product to the complex torus $T_N \subseteq X_\Sigma$ and use the isomorphism $T_N \simeq (\mathbb{C}^*)^d$, to define the intersection product of two tropical currents in $(\mathbb{C}^*)^d$. On the tropical geometry side, there exists a *stable intersection theory* of tropical cycles. The word stable here precisely corresponds to the continuity of the definition with respect to generic translations of tropical cycles. With the stable intersection and natural addition of tropical cycles, we have the ring of tropical cycles.

Theorem 1.4. The assignment $\mathcal{C} \longmapsto \mathcal{T}_{\mathcal{C}}$ induces a $\mathbb{Z}$-algebra homomorphism between

(a) The $\mathbb{Z}$-algebra of tropical cycles in $\mathbb{R}^d$ with the natural addition (Definition 3.5) and stable intersection (Definition 3.3) as the multiplication.

(b) The $\mathbb{Z}$-algebra of tropical currents on $(\mathbb{C}^*)^d$ with the usual addition of currents and the wedge product of currents.

We also address Problem 1.1 in a very particular case of slicing of currents converging to a tropical current. The theorem is inspired by works in [BJS+07], [OP13] and [Jon16]. Add more theorems.

Theorem 1.5. Let $D, W \subseteq (\mathbb{C}^*)^d$ be an algebraic subtorus, and an algebraic subvariety respectively. Assume that $\mathrm{Log}(D)$ intersects $\mathrm{trop}(W)$ properly. Then,

$$\lim_{m\to\infty} \Big(\frac{1}{m^{d-p}}\Phi_m^*[W] \wedge [D]\Big) = \Big(\lim_{m\to\infty} \frac{1}{m^{d-p}}\Phi_m^*[W]\Big) \wedge [D],$$

where $\Phi_m : (\mathbb{C}^*)^d \longrightarrow (\mathbb{C}^*)^d$ is the m-th power map $(z_1, \dots, z_d) \longmapsto (z_1^m, \dots, z_d^m)$.

The proof relies on a theorem of Berteloot and the second-named author [BD20] that the limit of slices satisfies a certain continuity of harmonic functions, and we can use Fourier analysis to prove the theorems about tropical currents.

2. Tools from Superpotential Theory

Let (X, ω) be a compact Kähler manifold of dimension n. Assume that $\mathcal{S}$ is either a positive or a negative current of bidegree (q, q) on X. The quantity

$$|\langle \mathcal{S}, \omega^{n-q}\rangle|$$

is referred to as the *total mass* of $\mathcal{S}$. For $0 \le r \le n$, we consider the de Rham cohomology groups $H^r(X,\mathbb{C}) = H^r(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ with coefficients in $\mathbb{C}$. Recall that Hodge theory provides the following decomposition of the de Rham cohomology group into Dolbeault cohomology groups:

$$H^r(X,\mathbb{C}) \simeq \bigoplus_{p+q=r} H^{p,q}(X,\mathbb{C}).$$

We denote by $\mathcal{C}^q(X)$ the cone of positive closed bidegree (q, q)-currents or bidimension $(n-q, n-q)$ in X. We denote by $\mathcal{D}^q(X) = \mathcal{D}_{n-q}(X)$ the $\mathbb{R}$-vector space spanned by $\mathcal{C}^q(X)$, which is the space of closed real currents of bidegree (q, q). Every current $\mathcal{T} \in \mathcal{D}^q(X)$ has a cohomology class:

$$\{\mathcal{T}\} \in H^{q,q}(X,\mathbb{R}) = H^{q,q}(X,\mathbb{C}) \cap H^{2q}(X,\mathbb{R}).$$

We define $\mathcal{D}^{q,0}(X) = \mathcal{D}^0_{n-q}(X)$ to be the subspace of $\mathcal{D}^q(X)$, consisting of currents with vanishing cohomology. The $*$-topology on $\mathcal{D}^q(X)$ is defined by the norm:

$$\|\mathcal{S}\|_* := \min(\|\mathcal{S}^+\| + \|\mathcal{S}^-\|),$$

where the minimum is taken over positive currents $\mathcal{S}^+$ and $\mathcal{S}^-$ in $\mathcal{C}^q(X)$ that satisfy $\mathcal{S} = \mathcal{S}^+ - \mathcal{S}^-$. We say that $\mathcal{S}_n$ converges to $\mathcal{S}$ in $\mathcal{D}^q(X)$ if $\mathcal{S}_n$ converges weakly to $\mathcal{S}$ and moreover, $\|\mathcal{S}_n\|_*$ is bounded by a constant independent of n.

Let $h := \dim H^{q,q}(X, \mathbb{R})$, and fix a set of smooth forms $\alpha = (\alpha_1, \dots, \alpha_h)$ such that their cohomology classes $\{\alpha\} = (\{\alpha_1\}, \dots, \{\alpha_h\})$ form a basis for $H^{q,q}(X, \mathbb{R})$. By Poincaré duality, there exists a set of smooth forms $\alpha^\vee = (\alpha_1^\vee, \dots, \alpha_h^\vee)$ such that their cohomology classes $\{\alpha^\vee\}$ form the dual basis of $\{\alpha\}$, with respect to the cup-product. By adding $U_\mathcal{S}$ to a suitable combination of $\alpha_i^\vee$, we can assume that $\langle U_\mathcal{S}, \alpha_i\rangle = 0$, for all $i = 1, \dots, h$. In this case, we say that $U_\mathcal{S}$ is *α-normalised*.

Definition 2.1. Let $\mathcal{T} \in \mathcal{D}^q(X)$ and $\mathcal{S}$ be a smooth form in $\mathcal{D}^{n-q+1,0}(X)$.

(i) The *α-normalised super-potential* $\mathcal{U}_\mathcal{T}$ of $\mathcal{T}$ is given by the function

$$\begin{aligned} \mathcal{U}_\mathcal{T} : \{\mathcal{S} \in \mathcal{D}^{n-q+1,0}(X) : \text{smooth}\} &\longrightarrow \mathbb{R} \\ \mathcal{S} &\longmapsto \langle \mathcal{T}, U_\mathcal{S}\rangle, \end{aligned}$$

where $U_\mathcal{S}$ is the α-normalised potential of $\mathcal{S}$.

(ii) We say $\mathcal{T}$ has a *continuous super-potential*, if $\mathcal{U}_\mathcal{T}$ can be extended to a function on $\mathcal{D}^{n-q+1,0}(X)$ which is continuous with respect to the $*$-topology.

In general, consider $\mathcal{T} \in \mathcal{D}^q(X)$ and $\mathcal{S} \in \mathcal{D}^r(X)$. Assume that $q + r \leq n$ and $\mathcal{T}$ has a continuous super-potential. Let $\mathcal{U}_\mathcal{T}$ be the α-normalised super-potential of $\mathcal{T}$. Let $\beta \in \mathrm{Span}_\mathbb{R}\{\alpha\}$ such that $\{\beta\} = \{\mathcal{T}\}$. For any compactly supported smooth form φ of bidegree $(n-q-r, n-q-r)$, we define

$$\langle \mathcal{T} \wedge \mathcal{S}, \varphi\rangle := \mathcal{U}_\mathcal{T}(\mathcal{S} \wedge dd^c\varphi) + \langle \beta \wedge \mathcal{S}, \varphi\rangle. \tag{1}$$

Now assume that if $f : X \longrightarrow Y$, is a biholomorphism between smooth compact Kähler manifolds, then we have

$$f_*\mathcal{U}_{\mathcal{R}_1} = \mathcal{U}_{f_*\mathcal{R}_1}, \quad f^*\mathcal{U}_{\mathcal{R}_2} = \mathcal{U}_{f^*\mathcal{R}_2},$$

for $\mathcal{R}_1 \in \mathcal{D}^q(X)$ and $\mathcal{R}_2 \in \mathcal{D}^q(Y)$. I saw your comment, but I'm really sorry, I don't know if I need to add something here or not! Would you be able to re-write this part in latex?

Definition 2.2. Let $(\mathcal{T}_n)$ be a sequence of currents in $\mathcal{D}^q(X)$ weakly converging to $\mathcal{T}$. Let $\mathcal{U}_\mathcal{T}$ and $\mathcal{U}_{\mathcal{T}_n}$ be their α-normalised super-potentials. If $\mathcal{U}_{\mathcal{T}_n}$ converges to $\mathcal{U}_\mathcal{T}$ uniformly on any $*$-bounded sets of smooth forms in $\mathcal{D}^{n-q+1,0}(X)$, then the convergence is called *SP-uniform*.

It is shown in [DS10, Proposition 3.2.8] that any current with continuous super-potentials can be SP-uniformly approximated by smooth forms. Moreover, currents with continuous super-potentials have other nice properties:

Theorem 2.3 ([DNV18, Theorem 1.1])**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two positive currents in $\mathcal{D}_q(X)$, such that $\mathcal{T} \leq \mathcal{T}'$, *i.e.*, $\mathcal{T}' - \mathcal{T}$ is a positive current. Then, if $\mathcal{T}'$ has a continuous super-potential, then so does $\mathcal{T}$.

Theorem 2.4. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two positive closed currents, and $\mathcal{T}_1$ has a continuous superpotentials, then $\mathcal{T}_1 \wedge \mathcal{T}_2$ is well-defined. Moreover, if $\mathcal{T}_2$ has also a continuous superpotential, then

(a) [DS10, Proposition 3.3.3] $\mathcal{T}_1 \wedge \mathcal{T}_2$ has a continuous superpotential;
(b) [DS10, Proposition 3.3.3] This wedge product is continuous with respect to the SP-uniform convergence.
(c) [DS09, Theorem 4.2.4] $\mathrm{supp}(\mathcal{T}_1 \wedge \mathcal{T}_2) \subseteq \mathrm{supp}(\mathcal{T}_1) \cap \mathrm{supp}(\mathcal{T}_2)$.

Theorem 2.5 ([DS10, Proposition 3.3.4])**.** Assume that $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are closed positive currents, and $\mathcal{T}_1$ and $\mathcal{T}_2$ have continuous superpotentials. Then,

$$\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_2 \wedge \mathcal{T}_1 \quad \text{and} \quad (\mathcal{T}_1 \wedge \mathcal{T}_2) \wedge \mathcal{T}_3 = \mathcal{T}_1 \wedge (\mathcal{T}_2 \wedge \mathcal{T}_3).$$

Proposition 2.6. Let X be a compact Kähler manifold, $S_n \longrightarrow S$ be a convergent sequence in $\mathcal{D}^q(X)$. If a current $\mathcal{T}$ has a continuous superpotential, then

$$\mathcal{T} \wedge \mathcal{S}_n \longrightarrow \mathcal{T} \wedge \mathcal{S}.$$

Proof. The main result of [DS04] implies any current $\mathcal{T} \in \mathcal{D}^p(X)$ can be weakly approximated by a difference of smooth closed positive of bidegree (p,p)-forms. The result then follows from the definition of continuity of super-potentials and Equation (1). □

Lemma 2.7. Let $\mathcal{T}$, $\mathcal{T}'$ be positive closed currents such that $\mathcal{T}_{|\Omega} = \mathcal{T}'_{|\Omega}$ in an open subset $\Omega \subseteq X$, and both $\mathcal{T}$ and $\mathcal{T}'$ have continuous super-potentials. Then, for any $\mathcal{S} \in \mathcal{D}^r(X)$,

$$\big(\mathcal{T} \wedge \mathcal{S}\big)_{|\Omega} = \big(\mathcal{T}' \wedge \mathcal{S}\big)_{|\Omega}.$$

Proof. In [DS10], for any current $\mathcal{S}$ with continuous super-potential, a family $\{\mathcal{T}_\theta\}_{\theta \in \mathbb{C}^*}$ is constructed that $\mathcal{T}_\theta$ converges SP-uniformly to $\mathcal{S}$ as $|\theta| \to 0$. Let $\epsilon := |\theta| > 0$, be a small positive number, and $V \subseteq U$ be any open set such that V_ϵ, the ϵ-neighbourhood of V, is contained entirely in Ω. Therefore, by the hypothesis of the lemma, we can construct families of smooth forms $\mathcal{T}_n$ and $\mathcal{T}'_n$ converging SP-uniformly to $\mathcal{T}$ and $\mathcal{T}'$ respectively. Moreover,

$$\mathcal{T}_{n|_{V_\epsilon}} = \mathcal{T}'_{n|_{V_\epsilon}}.$$

Now, for any $(n-q-r, n-q-r)$ smooth form φ with compact support on Ω, we can cover the support of φ with an open set of the form V_ϵ and deduce

$$\big(\mathcal{T}_n \wedge \mathcal{S}\big) \wedge \varphi = \big(\mathcal{T}'_n \wedge \mathcal{S}\big) \wedge \varphi.$$

This, together with Theorem 2.4(b) implies the assertion.

□

We also have a very useful local version of Theorem 2.3.

Lemma 2.8. If $\mathcal{T}$ is a positive closed current on a compact Kähler manifold X, which is locally bounded by a product of positive closed bidegree $(1,1)$-currents of continuous potentials, resp. Hölder continuous potentials, then $\mathcal{T}$ has continuous superpotentials, respectively Hölder continuous potentials in X.

Proof. Fix a point a in X. In an open neighbourhood of a that we identify with the ball $B(0,2)$ in $\mathbb{C}^d$, we have

$$\mathcal{T} \leq dd^c u_1 \wedge \cdots \wedge dd^c u_p$$

with u_i continuous or Hölder continuous. Without loss of generality, we can assume that these functions are strictly negative. On $B(0,1)$, define

$$u_i' := \max(u_i, A\log\|z\|)$$

with A sufficiently large so that $u_i' = u_i$ on $B(0,1/2)$. Observe that $u_i' = A\log\|z\|$ near $\partial B(0,1)$. Hence we can extend it to a function which is smooth in a neighbourhood of $X \setminus B(0,1)$. Thus, this function is quasi-plurisubharmonic. We have

$$\mathcal{T} \leq (B\omega + dd^c u_1') \wedge \cdots \wedge (B\omega + dd^c u_p')$$

in a neighbourhood W_a of a if B is large enough.

Since we can cover X using a finite number of open sets W_{a_k}, we can add up all obtained quasi-plurisubharmonic functions together and obtain a quasi-plurisubharmonic function u. It is clear that

$$\mathcal{T} \leq (C\omega + u)^p$$

if C is large enough. The function u is continuous or Hölder continuous, and we deduce by Theorem 2.3. □

Theorem 2.9. Let $q : \widehat{X} \longrightarrow X$, be the blowing up of the compact Kähler manifold X along a submanifold. Assume that $\mathcal{T} \in \mathcal{D}_p(\widehat{X})$ is such that the support of $\mathcal{T}$ does not intersect the exceptional divisors of $\widehat{X}$. If the current $\mathcal{T}$ has a continuous superpotential then $q_*\mathcal{T}$ has the same property.

Proof. Assume that $\mathcal{S} \in \mathcal{D}^{d-p+1}(X)$. We need to show that the super-potential of $q_*\mathcal{T}$, which is a function defined on smooth forms in $\mathcal{D}^{n-q+1,0}(X)$, can be extended to a continuous function on $\mathcal{D}^{n-q+1,0}(X)$. Let α be a smooth closed (p,p)-form cohomologous to $q_*\mathcal{T}$ and define $\beta := q^*\alpha$. By hypothesis, we have $q^*q_*\mathcal{T} = \mathcal{T}$. It follows that $\mathcal{T}$ is cohomologous to β. Fix a potential U of $q_*\mathcal{T} - \alpha$ which is smooth near the blowup locus. We have $dd^cU = q_*\mathcal{T} - \alpha$ and therefore $dd^c q^*U = \mathcal{T} - \beta$. The smoothness of U near the blowup locus implies that $q_*(q^*U) = U$.

Since α is smooth, it is enough to show that the superpotential $\mathcal{U}$ of $q_*\mathcal{T} - \alpha$ can be extended to a continuous function on $\mathcal{D}^{n-q+1,0}(X)$. Since the last current has a vanishing cohomology class, its superpotential doesn't depend on the normalization.

Claim 1. Let $(\mathcal{S}_n)$ be a bounded sequence of smooth forms in $\mathcal{D}^{d-p+1}(X)$. Then the sequence $\mathcal{U}(\mathcal{S}_n)$ is bounded.

Proof of Claim 1. Since $\mathcal{S}_n$ is smooth, we have

$$\mathcal{U}(\mathcal{S}_n) = \langle U, \mathcal{S}_n\rangle = \langle q^*(U), q^*\mathcal{S}_n\rangle = \widehat{\mathcal{U}}(q^*\mathcal{S}_n),$$

where $\widehat{\mathcal{U}}$ denotes the superpotential of $\mathcal{T} - \beta$. Since the action of q^* on cohomology is bounded and the mass of a positive closed current only depends on its cohomology class, we see that $q^*\mathcal{S}_n$ is bounded in $\mathcal{D}^{d-p+1,0}(\widehat{X})$. Since $\mathcal{T}$ has a continuous superpotential, we deduce that $\widehat{\mathcal{U}}(q^*\mathcal{S}_n)$ is bounded and hence $\mathcal{U}(\mathcal{S}_n)$ is bounded as claimed. □

Claim 2. Let $(\mathcal{S}_n)$ be a bounded sequence of smooth forms in $\mathcal{D}^{d-p+1}(X)$ converging to 0. Then $\mathcal{U}(\mathcal{S}_n)$ tends to 0.

Proof of Claim 2. By extracting a subsequence, we can assume that $q^*\mathcal{S}_n$ converges to some current $\mathcal{R}$ supported by the exceptional divisor. By hypothesis, we have $q_*(\mathcal{R}) = 0$. Using the computation in Claim 1, and the fact that $\widehat{\mathcal{U}}$ is continuous, we get

$$\lim \mathcal{U}(\mathcal{S}_n) = \widehat{\mathcal{U}}(\mathcal{R}).$$

Now, since q^*U is smooth near the exceptional divisor which supports $\mathcal{R}$, we deduce that

$$\widehat{\mathcal{U}}(\mathcal{R}) = \langle q^*U, \mathcal{R}\rangle = \langle U, q_*\mathcal{R}\rangle = 0.$$

This proved the claim. □

To finish the proof, let $\mathcal{S}$ be any current in $\mathcal{D}^{d-p+1,0}(X)$. Choose a bounded sequence $\mathcal{R}_n$ of smooth forms in $\mathcal{D}^{d-p+1,0}(X)$ converging to $\mathcal{S}$. By Claim 1, extracting a subsequence allows to assume that $\mathcal{U}(\mathcal{R}_n)$ converges to some real number l. Consider now an arbitrary bounded sequence of $(\mathcal{S}_n)$ of smooth forms in $\mathcal{D}^{d-p+1,0}(X)$ converging to $\mathcal{S}$. To show that $\mathcal{U}$ extends to a continuous function at $\mathcal{S}$, it is enough to check that $\mathcal{U}(\mathcal{S}_n)$ converges to l. This is a consequence of Claim 2 applied to the sequence $\mathcal{S}_n - \mathcal{R}_n$. □

Theorem 2.10. For two complex manifolds X and Y, consider two convergent sequences of currents $\mathcal{T}_n \longrightarrow \mathcal{T}$ in $\mathcal{D}^q(X)$ and $\mathcal{S}_n \longrightarrow \mathcal{S}$ in $\mathcal{D}^r(Y)$. We have that

$$\mathcal{T}_n \otimes \mathcal{S}_n \longrightarrow \mathcal{T} \otimes \mathcal{S},$$

weakly in $\mathcal{D}^{q+r}(X \times Y)$.

Sketch of the proof. Let us denote by (x, y) the coordinates on $X \times Y$. Using local coordinates and a partition of unity and Weierstrass theorem we can approximate any smooth forms on $X \times Y$ with forms with polynomial coefficients in (x, y). The approximation is in C^∞. As a result, the convergence, we only need test forms with monomial coefficients. Thus, the variables x, y are separated and the convergence of the tensor products becomes the convergence of each factor. □

2.1. **Semi-continuity of slices.** Let $f : X \longrightarrow Y$ be a dominant holomorphic map between complex manifolds, not necessarily compact, of dimension n and m respectively. Let $\mathcal{T}$ be a positive closed current on X of bi-dimension (p, p) with $p \geq m$. Then a slice

$$\mathcal{T}_y = \langle \mathcal{T}|f|y\rangle$$

obtained by restricting $\mathcal{T}$ to $f^{-1}(y)$ exists for almost every $y \in Y$; see [Dem, Page 171]. This is a positive closed current of bi-dimension $(p-m, p-m)$ on X supported by $f^{-1}(y)$. If Ω is a smooth form of maximal bi-degree on Y and α a smooth $(q-m, q-m)$-form with compact support in X, then we have

$$\langle \mathcal{T}, \alpha \wedge f^*(\Omega)\rangle = \int_{y\in Y} \langle \mathcal{T}_y, \alpha\rangle \Omega(y).$$

In general, if $\mathcal{T}$ and $\mathcal{T}'$ are such that $\mathcal{T}_y = \mathcal{T}'_y$ for almost every y, we do not necessarily have $\mathcal{T} = \mathcal{T}'$. However, the following is true: let $f_1, \dots, f_k$ be dominant holomorphic maps from X to $Y_1, \dots, Y_k$. Consider the vector space spanned by all the differential forms of type $\alpha \wedge f_i^*(\Omega_i)$ for some α as above and for some smooth form Ω_i on Y_i of maximal degree. Assume this space is equal to space of all (q, q)-forms of compact support in X. Then if $\langle \mathcal{T}|f_i|y_i\rangle = \langle \mathcal{T}'|f_i|y_i\rangle$ for every i and almost every $y_i \in Y_i$, we have $\mathcal{T} = \mathcal{T}'$. The proof is a consequence of the above discussion. An important special case is the following:

Lemma 2.11. Assume that $\mathcal{T}$ and $\mathcal{T}'$ are two positive closed currents of bidimension (p,p) on $\mathbb{C}^d$ and

$$\mathcal{T} \wedge [L] = \mathcal{T}' \wedge [L],$$

for any generic $(d-p)$-dimensional affine plane L. Then $\mathcal{T} = \mathcal{T}'$.

Proof. The above equality means that for any smooth function α with compact support and generic L we have

$$\langle \mathcal{T} \wedge [L], \alpha \rangle = \langle \mathcal{T}' \wedge [L], \alpha \rangle.$$

For each L, we can find a projection $f : \mathbb{C}^d \longrightarrow \mathbb{C}^p$, such that $f^{-1}(0) = L$, and the above equality implies that

$$\langle \mathcal{T} | f | 0 \rangle = \langle \mathcal{T}' | f | 0 \rangle.$$

Let $z \in \operatorname{supp}(\mathcal{T})$, and consider the local coordinates centred at z. By [Dem, Lemma 1.4], the set of strongly positive forms

$$\beta_s = i\beta_{s,1} \wedge \bar{\beta}_{s,1} \wedge \cdots \wedge i\beta_{s,p} \wedge \bar{\beta}_{s,p}, \quad 1 \leq s \leq \binom{n}{p}$$

forms a basis for $\Lambda^{p,p}(\mathbb{C}^d)$, where $\beta_{s,i}$ are of the type $dz_j \pm dz_k$ or $dz_j \pm i dz_k$. These forms can be obtained by the pullback of the standard volume forms by the projections $f : \mathbb{C}^d \longrightarrow \mathbb{C}^p$, where $(z_1, \ldots, z_d) \longmapsto (\beta_{s,1}, \ldots, \beta_{s,p})$, for all different $\beta_{s,i}$. By an action of $\mathrm{GL}(n, \mathbb{C})$, we can assume that for different choices of these projections $L = f^{-1}(0)$, the slices are well-defined and the equality holds

$$\langle \mathcal{T} \wedge [L], \alpha \rangle = \langle \mathcal{T}' \wedge [L], \alpha \rangle,$$

and we conclude by the above discussion. □

Now let $U \subseteq \mathbb{C}^m$ and $V \subseteq \mathbb{C}^d$ be two bounded open sets. Assume that $\pi_1 : U \times V \longrightarrow U$ and $\pi_2 : U \times V \longrightarrow V$ are the canonical projections. Consider two closed positive currents $\mathcal{T}$ and $\mathcal{S}$ on $U \times V$ of bi-dimension (m,m) and (n,n) respectively. We say that $\mathcal{T}$ horizontal-like if $\pi_2(\operatorname{supp}(\mathcal{T}))$ is relatively compact in V. Similarly, if $\pi_1(\operatorname{supp}(\mathcal{S}))$ is relatively compact in U, $\mathcal{S}$ is called vertical-like.

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Theorem 2.12. Let X be a compact complex manifold of dimension n. Assume that V and W are two analytic sets of dimension p and q with $p + q \geq n$ with a proper intersection. Then $[V] \wedge [W]$ is well-defined and

$$[V] \wedge [W] = \sum c_i [C_i],$$

and the multiplicities coincide with the multiplicities in classical algebraic geometry. if the following Condition (C) is satisfied.

Condition 1 (C)**.** For any generic perturbation V^ϵ of V, each irreducible component Z^ϵ of the intersection $V^\epsilon \cap W$ contains a generic point that lies in the smooth loci of both V^ϵ and W, and at such a point, the intersection is transverse.

Proof. Let $z \in V \cap W$. By a partition of unity, and a local diffeomorphism $\varphi : N \subset X \longrightarrow \mathbb{C}^n$, we can reduce the problem to the local case of a small open set in $\varphi(N) \subseteq \mathbb{C}^d$. We define $([V] \wedge [W])|_N$ to be the unique current satisfying

$$(\varphi_*[V] \wedge \varphi_*[W]) \wedge [L] := \varphi_*[V] \wedge (\varphi_*[W] \wedge [L]).$$

for generic linear spaces L containing $\varphi(z)$ of dimension $p + q$. $\varphi_*[W] \wedge [L]$ is well-defined since L is a complete intersection of hypersurfaces, and $\varphi_*[V] \wedge (\varphi_*[W] \wedge [L])$ is well-defined, since $\operatorname{supp}(\varphi_*[W] \wedge [L])$ is a horizontal-like and we deduce by (D-Nguyen-Sibony, Bianchi-D-Rakhimov). Is this alright? Which theorems I should precisely cite?

Note that Condition (C) ensures that the intersection in the classical intersection theory is 1 and therefore the limit in the sense of currents yields the same multiplicities. □

Theorem 2.13 ([BD20, Lemma 3.7])**.** Let $(\mathcal{T}_n) \longrightarrow \mathcal{T}$ be a convergent sequence of horizontal-like positive closed currents to a horizontal-like current $\mathcal{T}$ in $U \times V$. Let $a \in U$ and assume that the sequence of measures $(\langle \mathcal{T}_n, \pi_1, a\rangle)_n$ is also convergent. Then,

$$\lim_{n\to\infty} \langle \mathcal{T}_n|\pi_1|a\rangle(\phi) \leq \langle \mathcal{T}|\pi_1|a\rangle(\phi)$$

for every plurisubharmonic function ϕ on $\mathbb{C}^d$.

There is an simple version of the above theorem for supports which will be useful later.

Lemma 2.14. Assume that $\mathcal{T}_n$'s, $\mathcal{S}$ and $\mathcal{T}$ are all closed positive currents, and $\mathcal{T}_i \wedge \mathcal{S}$ and $\mathcal{T} \wedge \mathcal{S}$ are well-defined. If we have the following weak convergence, together with the convergence of supports in the Hausdorff metric, that is,

$$\mathcal{T}_n \longrightarrow \mathcal{T}, \quad \operatorname{supp}(\mathcal{T}_n) \longrightarrow \operatorname{supp}(\mathcal{T}).$$

Then, $\operatorname{supp}(\lim(\mathcal{T}_n \wedge \mathcal{S})) \subseteq \operatorname{supp}(\mathcal{T}) \cap \operatorname{supp}(\mathcal{S})$.

Proof. For a point x outside the support of $\mathcal{T}$, there exists a sufficiently small radius ϵ, such that for a sufficiently large n, $\mathcal{T}_n$ vanishes on the ball $B_\epsilon(x)$ centred at x. It follows that any limit of $\mathcal{T}_n \wedge \mathcal{S}$ vanishes on $B_\epsilon(x)$. So its support does not contain a. Moreover, its support does not contain any point outside $\operatorname{supp}(\mathcal{S})$. □

3. Tropical Varieties, Tori, Tropical Currents

In this section, we recall the definition of tropical cycles and note that with the natural addition of tropical cycles and their *stable intersection*, the tropical cycles form a ring.

3.1. Tropical varieties.

A linear subspace $H \subseteq \mathbb{R}^d$ is said to be *rational* if there exists a subset of $\mathbb{Z}^d$ that spans H. A *rational polyhedron* is the intersection of finitely many rational half-spaces defined by

$$\{x \in \mathbb{R}^d : \langle m, x\rangle \geq c, \text{ for some } m \in \mathbb{Z}^d,\ c \in \mathbb{R}\}.$$

A *rational polyhedral complex* is a polyhedral complex consisting solely of rational polyhedra. The polyhedra in a polyhedral complex are also referred to as *cells*. A *fan* is a polyhedral complex whose cells are all cones. If every cone in a fan Σ is contained in another fan Σ', then Σ is called a *subfan* of Σ. The one-dimensional cones of a fan are often called *rays*. Throughout this article, all fans and polyhedral complexes are assumed to be *rational*.

For a given polyhedron σ, and a finitely generated abelian group N, we denote by

$$\begin{aligned} \mathrm{aff}(\sigma) &:= \text{affine span of } \sigma, \\ H_\sigma &:= \text{translation of } \mathrm{aff}(\sigma) \text{ to the origin}, \\ N_\sigma &:= N \cap H_\sigma, \\ N(\sigma) &:= N/N_\sigma. \end{aligned}$$

Consider τ, a codimension one face of a p-dimensional polyhedron σ, and let $u_{\sigma/\tau}$ be the unique outward generator of the one-dimensional lattice $(\mathbb{Z}^d \cap H_\sigma)/(\mathbb{Z}^d \cap H_\tau)$.

Definition 3.1 (Balancing Condition and Tropical Cycles)**.** Let $\mathcal{C}$ be a p-dimensional polyhedral complex whose p-dimensional cones are equipped with integer weights w_σ. We say that $\mathcal{C}$ satisfies the *balancing condition* at τ if

$$\sum_{\sigma \supset \tau} w(\sigma)\ u_{\sigma/\tau} = 0, \quad \text{in } \mathbb{Z}^d/(\mathbb{Z}^d \cap H_\tau),$$

where the sum is over all p-dimensional cells σ in $\mathcal{C}$ containing τ as a face. A *tropical cycle* or a *tropical variety* in $\mathbb{R}^d$ is a weighted complex with finitely many cells that satisfies the balancing condition at every cone of dimension $p-1$.

polytop algebra.

3.2. **The $\mathbb{Z}$-algebra of Tropical Cycles.** Recall that, generally speaking, the star of a cone in a complex is the extension of the local p-dimensional fan surrounding it. More precisely:

Definition 3.2. Given a polyhedral complex $\Sigma \subseteq \mathbb{R}^d$ and a cell $\tau \in \Sigma$, define the star of τ in Σ, denoted by $\mathrm{star}_\Sigma(\tau)$, is a fan in $\mathbb{R}^d$. The cones of $\mathrm{star}_\Sigma(\tau)$ are the *extensions* of cells σ that include τ as a face. Here, by extension, we mean

$$\bar{\sigma} = \{\lambda(x-y) : \lambda \geq 0, x \in \sigma,\, y \in \tau\}.$$

Definition 3.3 (Stable Intersection)**.**

(a) Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{R}^d$ be two tropical cycles of dimension p and q, intersecting transversely. That is, the top dimensional cells $\sigma_1 \in \mathcal{C}_1$ and $\sigma_2 \in \mathcal{C}_2$ intersect in dimension $p+q-n$. Then the stable intersection of $\mathcal{C}_1 \cdot \mathcal{C}_2$ is the tropical cycles supported on finitely many cells $\mathcal{C}_1 \cap \mathcal{C}_2$. In this case, the weight of a cell $\sigma_1 \cap \sigma_2$ if defined by

$$w_{\mathcal{C}_1 \cdot \mathcal{C}_2}(\sigma_1 \cap \sigma_2) = w_{\sigma_1} w_{\sigma_2} [N : N_{\sigma_1} + N_{\sigma_2}],$$

where $N = \mathbb{Z}^d$ here.

(b) When $\mathcal{C}_1$ and $\mathcal{C}_2$ do not intersect transversely, then $\mathcal{C}_1 \cdot \mathcal{C}_2$ as a set is the Hausdorff limit of

$$\mathcal{C}_1 \cap (\epsilon b + \mathcal{C}_2), \quad \text{as } \epsilon \to 0,$$

for a fixed generic $b \in \mathbb{R}^d$, and the weights are the sum of all the tropical multiplicities of the cells in the transversal intersection $\mathcal{C}_1 \cap (\epsilon b + \mathcal{C}_2)$ which converge to the same $p+q-n$-dimensional cell in the Hausdorff metric. Equivalently, for top dimensional cones $\sigma_1 \in \mathcal{C}_1$ and $\sigma_2 \in \mathcal{C}_2$

$$w_{\mathcal{C}_1 \cdot \mathcal{C}_2}(\sigma_1 \cap \sigma_2) = \sum_{\tau_1, \tau_2} w_{\tau_1} w_{\tau_2} [N : N_{\tau_1} + N_{\tau_2}],$$

where the sum is taken over all $\tau_1 \in \mathrm{star}_{\mathcal{C}_1}(\sigma_1 \cap \sigma_2), \tau_2 \in \mathrm{star}_{\mathcal{C}_2}(\sigma_1 \cap \sigma_2)$ with $\tau_1 \cap (\epsilon b + \tau_2) \neq \varnothing$, for some fixed generic vector $b \in \mathbb{R}^d$.

(c) When $p+q < n$, then the stable intersection of C_1 and C_2 is the empty set.

The following result is proved in tropical geometry; see [MS15, Lemmas 3.6.4 and 3.6.9]. We will revisit its proof later through the lens of superpotential theory.

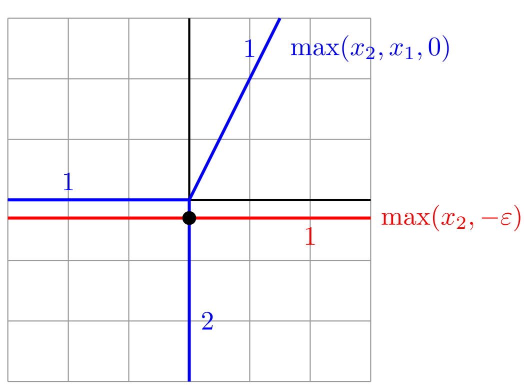
Theorem 3.4. When $p+q \geq n$, the stable intersection, defined above, yields a balanced polyhedral complex of dimension $p+q-n$.

We also need the following for turning the set of tropical cycles into a $\mathbb{Z}$-algebra.

Definition 3.5 (Addition of Tropical Cycles)**.** For two p-dimensional tropical cycles $\mathcal{C}_1, \mathcal{C}_2$ in $\mathbb{R}^d$, the addition $\mathcal{C}_1 + \mathcal{C}_2$ is the tropical cycle obtained by the common refinement of the support $|\mathcal{C}_1| \cup |\mathcal{C}_2|$ where the weights of a cone σ in the refinement are determined by $w_{\mathcal{C}_1+\mathcal{C}_2}(\sigma) = w_{\mathcal{C}_1}(\sigma) + w_{\mathcal{C}_2}(\sigma)$.

Let us end this section with an example of the stable intersection.

Example 3.6. Consider the tropical cycles below. We can easily check that both cycles with the given weights satisfy the balancing condition. The line $\{x_2 = 0\}$ with weight 1 can be considered a tropical cycle that does not properly intersect the blue tropical variety. We can therefore translate by the vector $(0, -\epsilon)$ it to obtain $x_2 = -\epsilon$, and compute the stable intersection by taking the limit as $\epsilon \to 0$. Calculating the multiplicities and Hausdorff limit gives the stable intersection equal to the origin $(0,0)$ with multiplicity 2. Note that we can choose any $\epsilon b \in \mathbb{R}^2$ generic for this translation, and thanks to Theorem 3.4 we obtain the same result.



4. Tropical Currents

Let us briefly recall the definition of tropical currents from [Bab14, BH17]. To fix the notation,

$$
\begin{aligned}
T_N &:= \text{ the complex algebraic torus } \mathbb{C}^* \otimes_{\mathbb{Z}} N,\\
S_N &:= \text{ the compact real torus } S^1 \otimes_{\mathbb{Z}} N,\\
N_{\mathbb{R}} &:= \text{ the real vector space } \mathbb{R} \otimes_{\mathbb{Z}} N.
\end{aligned}
$$

Let $\mathbb{C}^*$ be the group of nonzero complex numbers. As before, the logarithm map is the homomorphism

$$\operatorname{Log} : (\mathbb{C}^*)^d \longrightarrow \mathbb{R}^d, \qquad (z_1, \ldots, z_d) \longmapsto (-\log|z_1|, \ldots, -\log|z_d|),$$

and the *argument map* is

$$\operatorname{Arg} : (\mathbb{C}^*)^d \longrightarrow (S^1)^d, \qquad (z_1, \ldots, z_d) \longmapsto (z_1/|z_1|, \ldots, z_d/|z_d|).$$

For a rational linear subspace $H \subseteq \mathbb{R}^d$ we have the following exact sequences:

$$0 \longrightarrow H \cap \mathbb{Z}^d \longrightarrow \mathbb{Z}^d \longrightarrow \mathbb{Z}^d/(H \cap \mathbb{Z}^d) \longrightarrow 0.$$

Moreover,

$$0 \longrightarrow S_{H\cap\mathbb{Z}^d} \longrightarrow (S^1)^d = S^1 \otimes_{\mathbb{Z}} \mathbb{Z}^d \longrightarrow S_{\mathbb{Z}^d/(H\cap\mathbb{Z}^d)} \longrightarrow 0.$$

Define

$$\pi_H : \operatorname{Log}^{-1}(H) \xrightarrow{\operatorname{Arg}} (S^1)^d \longrightarrow S_{\mathbb{Z}^d/(H\cap\mathbb{Z}^d)} .$$

Similarly,

$$0 \longrightarrow T_{H\cap\mathbb{Z}^d} \longrightarrow (\mathbb{C}^*)^d = \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Z}^d \longrightarrow T_{\mathbb{Z}^d/(H\cap\mathbb{Z}^d)} \longrightarrow 0.$$

We define

$$\Pi_H : (\mathbb{C}^*)^d \simeq \mathbb{C}^* \otimes \big((H \cap \mathbb{Z}^d) \oplus \mathbb{Z}^d/(H \cap \mathbb{Z}^d)\big) \longrightarrow T_{\mathbb{Z}^d/(H\cap\mathbb{Z}^d)} .$$

One has

$$\ker(\Pi_H) = \ker(\pi_H) = T_{H\cap\mathbb{Z}^d} \subseteq (\mathbb{C}^*)^d.$$

As a result, when H is of dimension p, the set $\operatorname{Log}^{-1}(H)$ is naturally foliated by the $\pi_H^{-1}(x) = T_{H\cap\mathbb{Z}^d} \cdot x \simeq (\mathbb{C}^*)^p$ for $x \in S_{\mathbb{Z}^d/(H\cap\mathbb{Z}^d)}$. For a lattice basis $u_1, \ldots, u_p$, of $H \cap \mathbb{Z}^d$, the tori $T_{H\cap\mathbb{Z}^d}.x$ can be parametrised by the monomial map

$$(\mathbb{C}^*)^p \longrightarrow (\mathbb{C}^*)^d, \quad z \longmapsto x.z^{[u_1,\ldots,u_p]^t}$$

where $U = [u_1, \ldots, u_p]$ is the matrix with column vectors $u_1, \ldots, u_p$, and z^{U^t} denotes that $z \in (\mathbb{C}^*)^p$ is taken to have the exponents with rows of the matrix U. Accordingly, one can easily check that

$$T_{H\cap\mathbb{Z}^d} \cdot x = \{z \in (\mathbb{C}^*)^d : z^{m_i} = x^{m_i},\, i = 1, \ldots, d-p\}.$$

for any choice of a $\mathbb{Z}$-basis $\{m_1, \ldots, m_{d-p}\}$ of $\mathbb{Z}^d/(H \cap \mathbb{Z}^d)$.

Definition 4.1. Let H be a rational subspace of dimension p, and μ be the Haar measure of mass 1 on $S_{\mathbb{Z}^d/(H\cap\mathbb{Z}^d)}$. We define a (p,p)-dimensional closed current $\mathcal{T}_H$ on $(\mathbb{C}^*)^d$ by

$$\mathcal{T}_H := \int_{x\in S_{\mathbb{Z}^d/(H\cap\mathbb{Z}^d)}} \left[\pi_H^{-1}(x)\right]\, d\mu(x).$$

When A is a rational affine subspace of $\mathbb{R}^d$ parallel to the linear subspace $H = A - a$ for $a \in A$, we define $\mathcal{T}_A$ by translation of $\mathcal{T}_H$. Namely, we define the submersion π_A as the composition

$$\pi_A :\ \mathrm{Log}^{-1}(A) \xrightarrow{e^a} \mathrm{Log}^{-1}(H) \xrightarrow{\pi_H} S_{\mathbb{Z}^d/(H\cap\mathbb{Z}^d)}.$$

We will call $T^A := \pi_A^{-1}(1) = \ker \pi_A = e^{-a}T_{H\cap\mathbb{Z}^d}$, the *distinguished fibre* of $\mathcal{T}_A$.

Definition 4.2. Let $\mathcal{C}$ be a weighted polyhedral complex of dimension p. The tropical current $\mathcal{T}_\mathcal{C}$ associated to $\mathcal{C}$ is given by

$$\mathcal{T}_\mathcal{C} = \sum_\sigma w_\sigma\ \mathbb{1}_{\mathrm{Log}^{-1}(\sigma)}\mathcal{T}_{\mathrm{aff}(\sigma)},$$

where the sum runs over all p-dimensional cells σ of $\mathcal{C}$.

Theorem 4.3 ([Bab14])**.** A weighted complex $\mathcal{C}$ is balanced, if and only if, $\mathcal{T}_\mathcal{C}$ is closed.

Theorem 4.4 ([Bab14])**.** Any tropical current $\mathcal{T}_\mathcal{C} \in \mathcal{D}'_{n-1,n-1}((\mathbb{C}^*)^d)$ is of the form $dd^c[\mathfrak{q}\circ \mathrm{Log}]$, where $\mathfrak{q} : \mathbb{R}^d \longrightarrow \mathbb{R}$, is a tropical Laurent polynomial, that is $\mathfrak{q}(x) = \max_{\alpha\in A}\{c_\alpha + \langle \alpha, x\rangle\}$, for $A \subseteq \mathbb{Z}^d$ a finite subset and $c_\alpha \in \mathbb{R}$.

Remark 4.5. Note that the support of $dd^c[\mathfrak{q}\circ\mathrm{Log}]$, is given by $\mathrm{Log}^{-1}(\mathrm{Trop}(\mathfrak{q}))$, where $\mathrm{Trop}(\mathfrak{q})$ is the set of points $x \in \mathbb{R}^d$ where $\mathfrak{q}$ is not smooth at x. This set can be balanced with natural weights which coincides with the weights of the closed current $dd^c[\mathfrak{q}\circ\mathrm{Log}]$ and it is called the tropical variety associated to $\mathfrak{q}$.

Proposition 4.6 ([Bab23, Proposition 4.6])**.** Assume that $\mathcal{T} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^d)$ is a closed positive $(S^1)^d$-invariant current whose support is given by $\mathrm{Log}^{-1}(|\mathcal{C}|)$, for a polyhedral complex $\mathcal{C} \subseteq \mathbb{R}^d$ of pure dimension p. Then $\mathcal{T}$ is a tropical current.

5. Continuity of superpotentials

Let $\mathfrak{q} : \mathbb{R}^d \longrightarrow \mathbb{R}$, be a tropical polynomial function, and $\mathrm{Log} : (\mathbb{C}^*)^d \longrightarrow \mathbb{R}^d$, as before. The current $dd^c[\mathfrak{q}\circ\mathrm{Log}] \in \mathcal{D}'_{n-1,n-1}((\mathbb{C}^*)^d)$ has a bounded potential on any relatively compact open set, and by Bedford–Taylor theory, for any positive closed current $\mathcal{T} \in \mathcal{D}'_{p,p}(\mathbb{C}^*)^d$, the product

$$dd^c[\mathfrak{q}\circ\mathrm{Log}] \wedge \mathcal{T} = dd^c([\mathfrak{q}\circ\mathrm{Log}]\,\mathcal{T}),$$

is well-defined. See [Dem, Section III.3]. In higher codimensions though, to prove that any two tropical currents have a well-defined wedge product, we utilise the superpotential theory [DS09] on a compact Kähler manifold, and as a result, we extend the tropical currents to smooth compact toric varieties.

5.1. **Tropical Currents on Toric Varieties.** In a toric variety X_Σ, for a cone $\sigma \in \Sigma$, we denote by $\mathcal{O}_\sigma$, the toric orbit associated with σ. We have

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{O}_\sigma.$$

We also set D_σ to be the closure of $\mathcal{O}_\sigma$ in the X_Σ, and $\Sigma(p)$ the p-dimensional skeleton of Σ, that is, the union of p-dimensional cells of Σ. Fibres of tropical currents are algebraic varieties with finite degrees and can be extended by zero to any toric variety, in consequence, any tropical current can be extended by zero to toric varieties. Moreover, with the following compatibility condition, we can ask for the extension of the fibres to intersect the toric invariant divisors transversally.

Definition 5.1. (i) For a polyhedron σ, its *recession cone* is the convex polyhedral cone

$$\operatorname{rec}(\sigma) = \{b \in \mathbb{R}^d : \sigma + b \subseteq \sigma\} \subseteq H_\sigma.$$

(ii) Let $\mathcal{C}$ be a p-dimensional balanced weighted complex in $\mathbb{R}^d$, and Σ a p-dimensional fan. We say that $\mathcal{C}$ is *compatible* with Σ, if $\operatorname{rec}(\sigma) \in \Sigma$ for all $\sigma \in \mathcal{C}$.

(iii) We say the tropical current $\mathcal{T}_\mathcal{C}$ is *compatible* with X_Σ, if all the closures of the fibers $\pi^{-1}_{\operatorname{aff}(\sigma)}(x)$ in X_Σ of $\mathcal{T}_\mathcal{C}$ intersect the torus invariant divisors of X_Σ transversely.

Theorem 5.2 ([BH17, Lemma 4.10])**.** Let $\mathcal{C}$ be a p-dimensional tropical cycle and Σ be a fan. Assume that $\sigma \in \mathcal{C}$ is a p-dimensional polyhedron and $\rho \in \Sigma$ is a one-dimensional cone. Then

(a) The intersection $D_\rho \cap \overline{\pi^{-1}_{\operatorname{aff}(\sigma)}(x)}$ is non-empty and transverse, if and only if, $\rho \in \operatorname{rec}(\sigma)$. Here $\overline{\pi^{-1}_{\operatorname{aff}(\sigma)}(x)}$ corresponds the closure of a fiber of $\mathcal{T}_{\operatorname{aff}(\sigma)}$ in the toric variety X_Σ.

(b) In particular, $\mathcal{C}$ is compatible with Σ, if and only if, $\mathcal{T}_\mathcal{C}$ is compatible with X_Σ.

For a tropical current $\mathcal{T}_\mathcal{C} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^d)$, and given a toric variety X_Σ we denote its extension by zero $\overline{\mathcal{T}}_\mathcal{C} \in \mathcal{D}'_{p,p}(X_\Sigma)$.

Proposition 5.3. For every tropical variety $\mathcal{C}$, there exists a smooth projective toric fan Σ compatible with a subdivision of $\mathcal{C}$.

Proof. By [BS11], for $\mathcal{C}$ there is a refinement $\mathcal{C}'$, and a complete fan $\Sigma_1 \subseteq \mathbb{R}^d$ such that $\mathcal{C}'$ is compatible with Σ_1. Applying the toric Chow lemma [CLS11, Theorem 6.1.18] and the toric resolution of singularities [CLS11, Theorem 11.1.9] we can find a fan Σ which is a refinement of Σ_1 that defines a smooth projective variety X_Σ. The tropical variety $\mathcal{C}''$ which is the refinement of $\mathcal{C}'$ induced by Σ, satisfies the statement. □

Remark 5.4. When $\mathcal{C}'$ is a refinement of a tropical variety $\mathcal{C}$, then $\mathcal{C}'$ is a tropical variety with natural induced weights. It is also easy to check that we have the equality of currents $\mathcal{T}_\mathcal{C} = \mathcal{T}_{\mathcal{C}'}$ in $(\mathbb{C}^*)^d$; see [BH17, Section 2.6].

Lemma 5.5. Let $\mathfrak{q} : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a tropical Laurent polynomial and X_Σ be a smooth projective toric variety compatible with a subdivision of $\operatorname{Trop}(\mathfrak{q})$. Let $\rho \in \Sigma(1)$. Assume that $\zeta_0 \in D_\rho \cap \operatorname{supp}(\overline{dd^c[\mathfrak{q} \circ \operatorname{Log}]})$, and Ω is a sufficiently small neighbourhood of ζ_0.

Then, $\mathfrak{q} \circ \mathrm{Log} \in \mathrm{PSH}(\Omega \setminus D_\rho) \cap \mathcal{C}^0(\Omega \setminus D_\rho)$ can be extended to a function $u : \Omega \longrightarrow \mathbb{R} \cup \{+\infty\}$, such that

(a) In Ω, $u = g + \kappa \log |f|$, where g is a continuous function, f is the local equation for D_ρ, and κ is a negative integer.

(b) Restricted to Ω, we have $dd^c u = \overline{\mathcal{T}}_{\mathrm{Trop}(\mathfrak{q})} + \kappa[D_\rho]$.

(c) In Ω, we have $\overline{\mathcal{T}}_{\mathcal{C}} = dd^c g$. In particular, $\overline{\mathcal{T}}_{\mathrm{Trop}(\mathfrak{q})}$ has a continuous superpotential.

Proof. Assume that $\mathfrak{q} = \max_{\alpha \in A}\{c_\alpha + \langle \alpha, x\rangle\}$. Recall that

$$\mathrm{Log} = (-\log|\cdot|, \dots, -\log|\cdot|).$$

We write

$$\mathfrak{q} \circ \mathrm{Log} = \log \max_{\alpha}\{|e^{c_\alpha} z^{-\alpha}|\}.$$

Assume that near ζ_0, $\mathfrak{q} \circ \mathrm{Log}$ is given by $\max\{|e^{c_\beta} z^{-\beta}, |e^{c_\gamma} z^{-\gamma}|\}$. This implies that in $\mathrm{Log}(\Omega \setminus D_\rho)$, $\mathfrak{q}$ is given by $\max\{c_\beta + \langle \beta, x\rangle, c_\gamma + \langle \gamma, x\rangle\}$. For $\mathfrak{q} = \max_{\alpha \in A}\{c_\alpha + \langle \alpha, x\rangle\}$ we set $\mathrm{rec}(\mathfrak{q}) = \max_{\alpha \in A}\{\langle \alpha, x\rangle\}$. It is not hard to check that

$$\mathrm{rec}\big(\mathrm{Trop}(\mathfrak{q})\big) = \mathrm{Trop}(\mathrm{rec}(\mathfrak{q}));$$

see [MS15, Page 132].

We now show that by extending each $z^{-\alpha}$ as a rational function to X_Σ, the compatibility condition implies that $\mathfrak{q} \circ \mathrm{Log}$ extends to X_Σ. By [CLS11, Proposition 4.1.2] the divisor of the extension of a character z^α in X_Σ is given by

$$\mathrm{Div}(z^\alpha) = \sum_{\rho \in \Sigma(1)} \langle \alpha, n_\rho\rangle D_\rho, \tag{2}$$

where n_ρ the is the minimal generator of ρ. By assumption,

$$D_\rho \cap \mathrm{supp}(\overline{dd^c[\mathfrak{q} \circ \mathrm{Log}]}) \neq \varnothing.$$

Theorem 5.2 implies that

$$n_\rho \in \mathrm{rec}(V_{\mathrm{trop}(\mathfrak{q})}).$$

Moreover, if $\zeta_1 \in D_\rho \cap \mathrm{supp}(\overline{dd^c[\mathrm{rec}(\mathfrak{q}) \circ \mathrm{Log}]})$, then in a small neighbourhood of $\mathrm{Log}(\zeta_1)$, $\mathrm{rec}(\mathfrak{q})(x) = \max\{\langle \beta, x\rangle, \langle \gamma, x\rangle\}$. By definition

$$n_\rho \in \mathrm{rec}(\mathrm{Trop}(\mathfrak{q})) \quad \text{if and only if} \quad \kappa := \langle \beta, n_\rho\rangle = \langle \gamma, n_\rho\rangle.$$

This, together with Equation (2) implies that the extension of $z^{-\beta}$ and $z^{-\gamma}$ as rational functions to X_Σ have the same vanishing order along D_ρ, and we write $z^{-\beta} = f^\kappa \frac{g_1}{h_1}$ and $z^{-\gamma} = f^\kappa \frac{g_2}{h_2}$. Now note that in $\Omega \setminus D_\rho$,

$$\mathfrak{q} \circ \mathrm{Log} = \max \log\{|e^{c_\beta} z^{-\beta}|, |e^{c_\gamma} z^{-\gamma}|\} = \kappa \log|f| + \max\{|e^{c_\beta} \frac{g_1}{h_1}|, |e^{c_\gamma} \frac{g_2}{h_2}|\},$$

we must have $\kappa < 0$, otherwise $q \circ \mathrm{Log} = -\infty$ in $\Omega \setminus D_\rho$. Consequently, $\mathfrak{q} \circ \mathrm{Log} : \Omega \setminus D_\rho \longrightarrow \mathbb{R}$, can be extended to

$$u := \kappa \log|f| + \max\{|e^{c_\beta} \frac{g_1}{h_1}|, |e^{c_\gamma} \frac{g_2}{h_2}|\}$$

on Ω. Setting

$$g = \max\{|e^{-c_\beta} \frac{g_1}{h_1}|, |e^{-c_\gamma} \frac{g_2}{h_2}|\},$$

implies (a).

We have

$$dd^c[\mathfrak{q}\circ \mathrm{Log}]_{|\Omega\setminus D_\rho} = \big(dd^c \log|f|^\kappa\ dd^c \log|g|\big)_{|\Omega\setminus D_\rho} = dd^c \log|g|_{|\Omega\setminus D_\rho},$$

since $dd^c \log|f|^\kappa$ is holomorphic in $\Omega \setminus D_\rho$. As a result of compatibility with X_Σ, $\overline{dd^c[\mathfrak{q}\circ \mathrm{Log}]}$ does not charge any mass in D_ρ, and we obtain

$$\overline{dd^c[\mathfrak{q}\circ \mathrm{Log}]} = dd^c \log|g|.$$

This together with Theorem 4.4 implies (c) and (b). □

Lemma 5.6. Assume that σ is p-dimensional and $\mathrm{aff}(\sigma) = H_1 \cap \cdots \cap H_{d-p}$, is given as the transversal intersection hyperplanes $H_i \subseteq \mathbb{R}^d$. If Σ is a smooth projective fan compatible with $\bigcup_i H_i$, then

$$\overline{\mathcal{T}}_{\mathrm{aff}(\sigma)} \le \overline{\mathcal{T}_{H_1}\wedge\cdots\wedge\mathcal{T}_{H_{d-p}}} = \overline{\mathcal{T}}_{H_1}\wedge\cdots\wedge\overline{\mathcal{T}}_{H_{d-p}}.$$

Proof. By the definition of tropical currents, we have the inequality

$$\mathcal{T}_{\mathrm{aff}(\sigma)} \le \mathcal{T}_{H_1}\wedge\cdots\wedge\mathcal{T}_{H_{d-p}},$$

as currents in $(\mathbb{C}^*)^d$, since the right-hand side might have multiplicities but the currents have the same support. Now, the wedge products in X_Σ are well-defined by Lemma 5.5 and Theorem 2.4. As both currents on both sides of the equation coincide on $(\mathbb{C}^*)^d$, the support of the current on the right-hand side contains the closure of the support of $\mathcal{T}_{\mathrm{aff}(\sigma)}$ in X_Σ. For the equality, note that compatibility with Σ, implies that $\overline{\mathcal{T}}_{H_i}$ has a zero mass in $X_\Sigma \setminus T_N$. □

Theorem 5.7. Let $\mathcal{C}$ be a positively weighted tropical cycle of dimension p compatible with a smooth, projective fan Σ, then $\overline{\mathcal{T}}_{\mathcal{C}}$ has a continuous superpotential in X_Σ.

We need the following definition.

Definition 5.8. We define the affine extension p-dimensional a tropical cycle $\mathcal{C}$, by as the addition of tropical cycles

$$\widehat{\mathcal{C}} := \sum_{\sigma\in\mathcal{C}} w_\sigma \mathrm{aff}(\sigma).$$

It is clear that if $\mathcal{C}$ is a positively weighted tropical cycle, then $\mathcal{T}_{\widehat{\mathcal{C}}} - \mathcal{T}_{\mathcal{C}} \ge 0$.

Proof of 5.7 . Let $\widehat{\mathcal{C}}$ be the affine extension of $\mathcal{C}$, and $\widehat{\Sigma}$ be a smooth projective fan which is a refinement of Σ and compatible with $\widehat{\mathcal{C}}$. By the preceding lemma and repeated application of Theorem 2.4 for any $\sigma\in\mathcal{C}$, $\overline{\mathcal{T}}_{\mathrm{aff}(\sigma)}$ has a bounded superpotential, which implies this property for $\overline{\mathcal{T}}_{\widehat{\mathcal{C}}}$. Now, since $\mathcal{T}_{\widehat{\mathcal{C}}} - \mathcal{T}_{\mathcal{C}}$ is a positive closed tropical current in $(\mathbb{C}^*)^d$,

$$\overline{\mathcal{T}_{\widehat{\mathcal{C}}} - \mathcal{T}_{\mathcal{C}}} = \overline{\mathcal{T}}_{\widehat{\mathcal{C}}} - \overline{\mathcal{T}}_{\mathcal{C}} \ge 0$$

in $X_{\widehat{\Sigma}}$. Continuity of the superpotential of $\overline{\mathcal{T}}_{\mathcal{C}}$ in $X_{\widehat{\Sigma}}$ follows from Theorem 2.3.

We now show that $\overline{\mathcal{T}}_{\mathcal{C}}$ has also a continuous super-potential on X_Σ as well. We consider the proper map $f : X_{\widehat{\Sigma}} \longrightarrow X_\Sigma$, which can be understood as a composition of multiple blow-ups along toric points with exceptional divisors D_ρ for any ray $\rho \in \widehat{\Sigma}\setminus\Sigma$. These divisors satisfy $D_\rho \cap \mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}}) = \varnothing$. We conclude the proof by Theorem 2.9. □

Remark 5.9. When $\mathcal{C}$ is a tropical cycle which is not positively weighted, then there exist positively weighted tropical cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ such that

$$\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2.$$

Therefore, if Σ is compatible with both $\mathcal{C}_1$ and $\mathcal{C}_2$, then by preceding lemma, $\overline{\mathcal{T}}_{\mathcal{C}}$ has a continuous superpotential in X_Σ.

Proposition 5.10. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two positively weighted tropical cycles. If X_Σ is compatible $\mathcal{C}_1 + \mathcal{C}_2$, then

$$\overline{\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}} = \overline{\mathcal{T}}_{\mathcal{C}_1} \wedge \overline{\mathcal{T}}_{\mathcal{C}_2}.$$

Proof. The proof is clear since both $\overline{\mathcal{T}}_{\mathcal{C}_1}$ and $\overline{\mathcal{T}}_{\mathcal{C}_2}$ have continuous superpotentials with no mass on the boundary divisors $X_\Sigma \setminus T_N$. ☐

Proposition 5.11. For any two tropical currents $\mathcal{C}_1$ and $\mathcal{C}_2$, the intersection product

$$\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2} := \overline{\mathcal{T}}_{\mathcal{C}_1} \wedge \overline{\mathcal{T}}_{\mathcal{C}_2}|_{(\mathbb{C}^*)^d},$$

does not depend on the choice of a smooth projective toric variety of the fan Σ compatible with $\mathcal{C}_1 + \mathcal{C}_2$, where $(\mathbb{C}^*)^d$ is identified with $T_N \subseteq X_\Sigma$. Moreover, this product coincides with the definition of wedge products with bi-degree $(1,1)$ tropical currents in Bedford–Taylor Theory in $(\mathbb{C}^*)^d$.

Proof. This is a consequence of Lemma 2.7, and the fact that intersection product with a bidegree $(1,1)$ current in super-potential theory, in an open set of compact Kähler manifold, coincides with the Bedford–Taylor theory. ☐

Proposition 5.12. Stable intersection of tropical cycles is associative and commutative

Proof. This is the application of Theorem 5.7 and Theorem 2.5. ☐

5.2. **Proof of $\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2} = \mathcal{T}_{\mathcal{C}_1 \cdot \mathcal{C}_2}$.** We prove of our main intersection theorems here, and it is not hard to visit the proof of Theorem 3.4 using tools from superpotential theory.

Theorem 5.13. For two tropical varieties $\mathcal{C}$ and $\mathcal{C}'$ of dimension p and q, respectively, we have

$$\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2} = \mathcal{T}_{\mathcal{C}_1 \cdot \mathcal{C}_2}$$

where the $\mathcal{C} \cdot \mathcal{C}'$ is the stable intersection $\mathcal{C}_1$ and $\mathcal{C}_2$, defined in Definition 3.3. Moreover, the $\mathcal{C}_1 \cdot \mathcal{C}_2$ is a balanced polyhedral complex of dimension $p + q - n$.

Proposition 5.14 ([Kat09, Propositions 6.1])**.** Let $H_1, H_2 \subseteq \mathbb{R}^d$ be two rational planes of dimension p and q with $p + q = n$ that intersect transversely. Then, the complex tori $T_{H_1 \cap \mathbb{Z}^d}$ and $T_{H_2 \cap \mathbb{Z}^d}$ intersect at $[N : N_{H_1} + N_{H_2}]$ distinct points.

Proof of Theorem 5.13. Note that $\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}$ is well-defined by Proposition 5.11. Assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two tropical cycles of dimension p and q, respectively. Note that when $p+q < n$, both sides of the equality are zero. Therefore, we assume that $p+q \geq n$. We proceed with the following steps:

(a) In the transversal case $\mathrm{supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) = \mathrm{Log}^{-1}(\mathcal{C}_1 \cdot \mathcal{C}_2) = \mathrm{Log}^{-1}(\mathcal{C}_1 \cap \mathcal{C}_2)$.
(b) When $p + q = n$, $\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2} = \mathcal{T}_{\mathcal{C}_1 \cdot \mathcal{C}_2}$, in the transversal case.
(c) When $p + q > n$, $\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2} = \mathcal{T}_{\mathcal{C}_1 \cdot \mathcal{C}_2}$, in the transversal case.
(d) $\mathrm{Log}(\mathrm{supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2})) = \mathcal{C}_1.\mathcal{C}_2$ when $p + q = n$, also in the non-transverse case.

(e) $\mathrm{Log}(\mathrm{supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2})) = \mathcal{C}_1.\mathcal{C}_2$ when $p+q>n$, also in the non-transverse case.

To see (a), note the by Theorem 2.4(c), $\mathrm{supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) \subseteq \mathrm{Log}^{-1}(\mathcal{C}_1 \cdot \mathcal{C}_2) = \mathrm{Log}^{-1}(\mathcal{C}_1 \cap \mathcal{C}_2)$. Moreover, on $\mathcal{C}_1 \cap \mathcal{C}_2$, the fibres of $\mathcal{T}_{\mathcal{C}_1}$ and $\mathcal{T}_{\mathcal{C}_2}$ have a non-zero intersection.

To prove (b), let $a \in \mathcal{C}_1 \cap \mathcal{C}_2$ be an isolated point of intersection. We can choose a small ball $B_\epsilon(a) \subseteq \mathbb{R}^d$ such that a is an isolated point of intersection $\sigma_1 \cap \sigma_2 \cap B$, where σ_1 and σ_2 are cells of dimension p and q in $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. For any rational polyhedron σ, let

$$\begin{aligned} N_\sigma &:= N \cap \mathrm{aff}(\sigma), \\ S^1(\sigma) &:= S^1 \otimes_Z \big(\mathbb{Z}^d/(\mathbb{Z}^d \cap \mathrm{aff}(\sigma))\big), \\ \pi_\sigma &:= \pi_{\mathrm{aff}(\sigma)}, \end{aligned}$$

where $\pi_{\mathrm{aff}(\sigma)}$ was defined after Definition 4.1. By Lemma 2.7

$$\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}|_{\mathrm{Log}^{-1}(B)} = w_{\sigma_1} w_{\sigma_2} \mathbb{1}_{\mathrm{Log}^{-1}(B)} \int_{(x_1,x_2)\in S^1(\sigma_1)\times S^1(\sigma_2)} [\pi_{\sigma_1}^{-1}(x_1)] \wedge [\pi_{\sigma_2}^{-1}(x_2)]\, d\mu_{\sigma_1}(x_1) \otimes d\mu_{\sigma_2}(x_2).$$

Transversality of the fibres implies

$$[\pi_{\sigma_1}^{-1}(x_1)] \wedge [\pi_{\sigma_2}^{-1}(x_2)] = [\pi_{\sigma_1}^{-1}(x_1) \cap \pi_{\sigma_2}^{-1}(x_2)].$$

By Proposition 5.14, we have $\kappa = [\mathbb{Z}^d : \mathbb{Z}^d_{\mathrm{aff}(\sigma)} + \mathbb{Z}^d_{\mathrm{aff}(\sigma')}]$ distinct intersection points covering $\mathrm{Log}^{-1}(a) \simeq S^1(\sigma_1) \times S^1(\sigma_2)$. When $(x_1, x_2) \in \mathrm{Log}^{-1}(a) \simeq$ vary with respect to the normalised Haar measure, these κ points cover $S^1(\sigma_1) \times S^1(\sigma_2) \simeq (S^1)^d$ with speed κ. Here I don't mean a κ-covering of $(S^1)^d$, as κ is the Jacobian rather than the degree of the map. With (x,x') we cover $(S^1)^d \simeq S^1(\sigma_1) \times S^1(\sigma_2)$ once, but with speed κ. As a result,

$$\int_{(x_1,x_2)\in S^1(\sigma_1)\times S^1(\sigma_2)} [\pi_{\sigma_1}^{-1}(x)] \wedge [\pi_{\sigma_2}^{-1}(x')]\, d\mu_{\sigma_1}(x) \otimes d\mu_{\sigma_2}(x') = \int_{y\in(S^1)^d} \kappa [\pi_{\sigma_1\cap\sigma_2}^{-1}(y)] d\mu_{\sigma_1\cap\sigma_2}(y).$$

This proves (b).

To deduce (c), let $\sigma = \sigma_1 \cap \sigma_2$ be a $(p+q-n)$-dimensional cell in the intersection. Assume that $0 \in \sigma$, by a translation, and $L := \mathrm{aff}(\sigma)^\perp$. By Proposition 5.12,

$$(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) \wedge \mathcal{T}_L = \mathcal{T}_{\mathcal{C}_1} \wedge (\mathcal{T}_{\mathcal{C}_2} \wedge \mathcal{T}_L).$$

Note that $[N : N_\sigma + N_L] = 1$. Assume that $w_{\sigma_1} = w_{\sigma_2} = 1$. As a result, if the multiplicity of $\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}$ at σ equals κ, we have that the multiplicity of $(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) \wedge \mathcal{T}_L$ at the origin is also κ. We have that $[N : N_{\sigma_1} + N_{\sigma_2 \cap L}] = [N/N_\sigma : \big(N_{\sigma_1} + N_{\sigma_2\cap L}\big)/N_\sigma]$, which equals $[N/N_\sigma : N_{\sigma_1\cap L} + N_{\sigma_2\cap L}] = [N : N_{\sigma_1} + N_{\sigma_2}]$. As a consequence, the intersection multiplicity induced on σ by $\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}$ equals the intersection multiplicity in Definition 3.3.

To prove (d) note that if $\mathcal{C}_1 + \epsilon b$ is the translation of the the tropical variety, where $b \in \mathbb{R}^d$ and $\epsilon \in \mathbb{R}_{\geq 0}$, then $(e^{\epsilon b})^* \mathcal{T}_{\mathcal{C}_1} = \mathcal{T}_{\mathcal{C}_1 + \epsilon b}$. Moreover, we have the SP-convergence of currents with continuous superpotentials.

$$(e^{\epsilon b})^* \mathcal{T}_{\mathcal{C}_1} \longrightarrow \mathcal{T}_{\mathcal{C}_1}, \qquad \text{as } \epsilon \to 0.$$

Therefore, by Theorem 2.4,

$$(e^{\epsilon b})^* \mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2} = \mathcal{T}_{\mathcal{C}_1+\epsilon b} \wedge \mathcal{T}_{\mathcal{C}_2} \longrightarrow \mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}, \qquad \text{as } \epsilon \to 0. \tag{3}$$

Considering the support, we obtain the Hausdorff limit

$$\lim \operatorname{supp}((e^{\epsilon b})^* \mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) \supseteq \operatorname{supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}).$$

We now note that for all ϵ, the number of intersection points in $(\mathcal{C}_1+\epsilon b)\cap\mathcal{C}_2$ is uniformly bounded by the number of p-dimensional cells in $\mathcal{C}_1$ and q-dimensional cells in $\mathcal{C}_2$, the Hausdorff limit of $(\mathcal{C}_1 + \epsilon b) \cap \mathcal{C}_2$ is also zero dimensional. Now, by definition of $\mathcal{C}_1 \cdot \mathcal{C}_2$ it suffices to show that

$$\lim \operatorname{supp}((e^{\epsilon b})^* \mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) = \operatorname{supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}),$$

for any fixed generic b. This is also easy. Let $a_\epsilon \in (\mathcal{C}_1+\epsilon b)\cap\mathcal{C}_2$. Since the translation by ϵb does not change the slopes of the cells, as $a_\epsilon \to a$, the multiplicity for all a_ϵ remains constant for $\epsilon > 0$, therefore $\lim(e^{\epsilon b})^* \mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}$ has a non-zero mass at $\operatorname{Log}^{-1}(a)$.

For Part (e), first observe that any $\lim(\mathcal{C}_1 + \epsilon b) \cap \mathcal{C}_2$ is obtained by a translation ϵb, as $\epsilon \to 0$, of finitely many $(p + q - n)$-dimensional cells. Therefore, $\mathcal{C}_1 \cdot \mathcal{C}_2$ is also of dimension $p+q-n$, and the SP-convergence readily implies that the limit is independent of generic b. Now, it only remains to show that

$$\lim \operatorname{supp}((e^{\epsilon b})^* \mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) = \operatorname{supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}).$$

Let σ be a $p + q - n$ dimensional cell in $\mathcal{C}_1 \cdot \mathcal{C}_2$, and by translation, assume that $0 \in \sigma$. Let $L = \operatorname{aff}(\sigma)^\perp$. By Proposition 2.6,

$$\big((e^{\epsilon b})^* \mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}\big) \wedge \mathcal{T}_L \longrightarrow \big(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}\big) \wedge \mathcal{T}_L, \quad \text{as } \epsilon \to 0.$$

By Part (d), the left-hand-side has mass at the origin. This show (e).

To deduce Part (f), recall that in Equation (3) since we can choose b generically and the previous discussion. The balancing condition is also deduced by the fact that $\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}$ is closed and Theorem 4.3. ☐

Proposition 5.12 and Theorem 5.7 imply the following:

Theorem 5.15. (a)

(b)

5.2.1. *Calculating Intersection Multiplicities Using Monge-Ampère Measures.* Using the equality of the supports in the previous section, we only need to prove the intersection multiplicities in the transversal case locally.

5.2.2. *Real Monge–Ampére Measures.* Let $\Omega \subseteq \mathbb{R}^d$ be an open subset and $u : \Omega \longrightarrow \mathbb{R}$ be a convex (hence continuous) function. The *generalised gradient* of u at $x_0 \in \Omega$ is defined by

$$\nabla u(x_0) = \big\{\xi \in (\mathbb{R}^d)^* : u(x) - u(x_0) \geq \langle \xi, x - x_0\rangle, \text{ for all } x \in \Omega\big\}.$$

In the above, $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{R}^d$, and $(\mathbb{R}^d)^*$ is the dual. The real Monge-Ampère measure associated to a convex function u on a Borel set $E \subseteq \Omega$, is given by

$$\mathrm{MA}[u](E) = \mu\big(\bigcup_{y\in E} \nabla u(y)\big),$$

where μ is the Lebesgue measure on $(\mathbb{R}^d)^*$.

It is interesting that for the tropical polynomials, one can compute the associate real Monge–Ampère measures explicitly. Recall that, for any tropical polynomial, there is a natural subdivision of its Newton polytope which is dual to the tropical variety of it. See Figure for an example and [BS14, MS15] for details.

Lemma 5.16 ([Yge13, Page 59],[BGPS14, Proposition 2.7.4])**.** Let $\mathfrak{q} : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a tropical polynomial associated tropical variety $\mathcal{C} = V_{\text{trop}}(\mathfrak{q})$, one has

$$\mathrm{MA}[\mathfrak{q}] = \sum_{a\in\mathcal{C}(0)} \mathrm{Vol}\big(\{a\}^*\big)\delta_a,$$

where $\mathcal{C}(0)$ is the 0-dimensional skeleton of $\mathcal{C}$, and $\{a\}^*$ is the dual of the vertex $a \in \mathcal{C}(0)$.

A detailed discussion of the preceding theorem can be also found in [Bab14].

5.3. **Polarisation.** For n convex functions $u_1, \dots, u_d : \mathbb{R}^d \longrightarrow \mathbb{R}$, their *mixed Monge-Ampère measure* is defined by

$$\widetilde{\mathrm{MA}}[u_1, \dots, u_d] = \frac{1}{n!}\sum_{k=1}^{n} \sum_{1\le j_1<\cdots<j_k\le n} (-1)^{n-k}\,\mathrm{MA}[u_{j_1} + \cdots + u_{j_k}].$$

Recall that this is how the *mixed volume* of n convex bodies can be defined from the n-dimensional volume. Moreover, it is easy to check that for a convex function $u : \mathbb{R}^d \longrightarrow \mathbb{R}$, $\mathrm{MA}[u] = \widetilde{\mathrm{MA}}[u, \dots, u]$.

The following statements are clear from Proposition 5.16 by taking the total mass.

Proposition 5.17. Let $\mathfrak{q}, \mathfrak{q}_1, \dots, \mathfrak{q}_d : \mathbb{R}^d \longrightarrow \mathbb{R}$ be tropical polynomials. We have the following facts:

(a) $\mathrm{MA}[\mathfrak{q}](\mathbb{R}^d) = \mathrm{Vol}_d(\Delta_{\mathfrak{q}})$, where $\Delta_{\mathfrak{q}}$ is the Newton polytope of $\mathfrak{q}$.

(b) (Tropical Bernstein–Khovanskii–Kushnirenko Theorem) $\widetilde{\mathrm{MA}}[\mathfrak{q}_1, \dots, \mathfrak{q}_d](\mathbb{R}^d) = \widetilde{\mathrm{Vol}}(\Delta_{\mathfrak{q}_1}, \dots, \Delta_{\mathfrak{q}_d})$, where $\widetilde{\mathrm{Vol}}$ is the mixed volume.

Corollary 5.18. Assume that $\alpha_i, \beta_i \in \mathbb{Z}^d$ for $i = 1, \dots, n$. Let $\mathfrak{q}_i = \max\{\langle\alpha_i, x\rangle, \langle\beta_i, x\rangle\}$ be n tropical polynomials. Then,

$$n!\,\widetilde{\mathrm{MA}}[\mathfrak{q}_1, \dots, \mathfrak{q}_d] = \kappa\,\delta_0,$$

where κ is given by the volume *zonotope* of the Minkowski sum of the vectors $\sum_{i=1}^d[\alpha_i - \beta_i]$.

Proof. Note that $\Delta_{\mathfrak{q}_i}$ is the line segment between α_i and β_i. Moreover, in the definition of $\widetilde{\mathrm{MA}}[\mathfrak{q}_1, \dots, \mathfrak{q}_d]$ only $\mathrm{Vol}(\sum_{i=1}^d[\alpha_i - \beta_i])$ possibly has a non-zero n-dimensional volume. Finally, the origin is the only 0-dimensional cell of the tropical variety of polynomial

$\mathfrak{q}_1 + \cdots + \mathfrak{q}_d$, if and only if, $\{\alpha_1 - \beta_1, \cdots, \alpha_d - \beta_d\}$ forms a linearly independent set. Therefore, $n!\,\mathrm{MA}[\mathfrak{q}_1 + \cdots + \mathfrak{q}_d] = \kappa\,\delta_0$. □

6. Slicing tropical currents

Proposition 6.1. Let $\mathcal{C}$ be a p-dimensional positively weighted tropical cycle in $\mathbb{R}^d$ with $p \geq 1$. Assume that $S \subseteq (\mathbb{C}^*)^d$ is an algebraic hypersurface with transversal intersection with $\mathcal{T}_{\mathcal{C}}$. Then, $[S] \wedge \mathcal{T}_{\mathcal{C}}$ is admissible and it is a closed positive current of bidimension $(p-1, p-1)$ given by

$$[S] \wedge \mathcal{T}_{\mathcal{C}} = \sum_{\sigma \in \mathcal{C}} w_\sigma \int_{x \in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[S \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\big]\ d\mu(x).$$

Proof. The idea of the proof is similar to that of [BH17, Proposition 4.11]. Let f be a polynomial with vanishing S in $(\mathbb{C}^*)^d$. Assume that $\mathrm{Log}^{-1}(\sigma^\circ) \cap S \neq \varnothing$, for a p-dimensional cell $\sigma \in \mathcal{C}$, then for each fibre, $\pi_\sigma^{-1}(x) := \pi^{-1}_{\mathrm{aff}(\sigma)}(x)$, the transversality assumption allows for the application of the Lelong–Poincaré formula to deduce

$$dd^c\big(\log|f|\mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\big[\pi_\sigma^{-1}(x)\big]\big) = \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\big[S \cap \pi_\sigma^{-1}(x)\big] + \mathcal{R}_\sigma(x),$$

where $\mathcal{R}_\sigma(x)$ is a $(p-1, p-1)$-bidimensional current. The support of $\mathcal{R}_\sigma(x)$ lies in the boundary of $\mathrm{Log}^{-1}(\sigma)$, as $\mathcal{R}_\sigma(x)$ is the difference of two currents that coincide in any set of form $\mathrm{Log}^{-1}(B)$, where $B \subseteq \mathbb{R}^d$ is a small ball with

$$B \cap \sigma' = \varnothing, \quad \text{for a } p\text{-dimensional cell } \sigma' \in \mathcal{C}, \sigma' \neq \sigma\ ,$$

and both vanish outside $\mathrm{Log}^{-1}(\sigma)$. Integrating along the fibers, and adding for all p-dimensional cells $\sigma \in \mathcal{C}$, we obtain

$$[S] \wedge \mathcal{T}_{\mathcal{C}} = \sum_{\sigma \in \mathcal{C}} w_\sigma \int_{x \in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[S \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\big]\ d\mu(x) + \mathcal{R}_{\mathcal{C}},$$

where $\mathcal{R}_{\mathcal{C}}$ is $(p-1, p-1)$-dimensional current. We claim that $\mathcal{R}_{\mathcal{C}}$ is *normal*, *i.e.* $\mathcal{R}_{\mathcal{C}}$ and $d\mathcal{R}_{\mathcal{C}}$ have measure coefficients; $\mathcal{R}_{\mathcal{C}}$ is a difference of two normal currents, where the first current $[S] \wedge \mathcal{T}_{\mathcal{C}}$ is a positive closed current, and the second current is an addition of normal pieces. Moreover, the support of $\mathcal{R}_{\mathcal{C}}$ is a subset of S as it is a difference of two currents that both vanish outside S. As a result, the current $\mathcal{R}_{\mathcal{C}}$ is supported on $S \cap \bigcup_\sigma \partial\mathrm{Log}(\sigma)$. This set is a real manifold of Cauchy–Riemann dimension less than $p-1$, therefore by Demailly's first theorem of support the normal current $\mathcal{R}_{\mathcal{C}}$ vanishes; see also the discussion following [BH17, Proposition 4.11]. □

Corollary 6.2. Let $H \subseteq \mathbb{R}^d$ be a rational plane of dimension $r \geq d-p$ and $A := a+H$, a translation of H for $a \in \mathbb{R}^d$. Assume also that $\mathcal{C} \subseteq \mathbb{R}^d$ is a tropical variety of dimension p that intersects A transversely. Then

$$[(e^{-a})T_{H\cap\mathbb{Z}^d}] \wedge \mathcal{T}_{\mathcal{C}}$$

can be viewed as a tropical current of dimension $p-(n-r)$ in the complex subtori $T^A := (e^{-a})\,T_{H\cap Z^d} \subseteq (\mathbb{C}^*)^d$.

Proof. Note that the hypothesis implies that the intersection $T^A \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ is transversal for any $x \in S_{N(\sigma)}$. By translation, it is sufficient to prove the statement for $a = 0$. By the preceding theorem,

$$[T^A] \wedge \mathfrak{T}_{\mathcal{C}} = \sum_{\sigma \in \mathcal{C}} w_\sigma \int_{x \in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \left[T^A \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x) \right] \, d\mu(x).$$

The sets $T^A \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ can be understood as the toric sets in T^A. We can conclude since each every $d\mu$ associated to each σ is a Haar measure. □

Theorem 6.3. Let $M \subseteq (\mathbb{C}^*)^{d-p}$ and $N \subseteq (\mathbb{C}^*)^p$ be two bounded open subsets such that N contains the real torus $(S^1)^p$. Let $\pi : M \times N \longrightarrow M$ be the canonical projection. Let $\mathfrak{T}_n$ be a sequence of positive closed (p,p)-bidimensional currents on $M \times N$ such that $\overline{\mathrm{supp}(\mathfrak{T}_n)} \cap (M \times \partial \overline{N}) = \varnothing$. Assume that $\mathfrak{T}_n \longrightarrow \mathfrak{T}$ and $\mathrm{supp}(\mathfrak{T}) \subseteq M \times (S^1)^p$. Then we have the following convergence of slices

$$\langle \mathfrak{T}_n | \pi | x \rangle \longrightarrow \langle \mathfrak{T} | \pi | x \rangle \quad \text{for every } x \in M.$$

Note that all the above slices are well-defined for all $x \in M$.

Proof. Since all the currents $\mathfrak{T}_n$ and $\mathfrak{T}$ are horizontal-like, the slices are well-defined, and we prove that the slices have the same cluster value. Let $\mathcal{S}$ be any cluster value of $\langle \mathfrak{T}_n | \pi | x \rangle$. Note that such $\mathcal{S}$ always exists by Banach–Alaoglu theorem. As both measures $\mathcal{S}$ and $\langle \mathfrak{T} | \pi | x \rangle$, are supported $\{x\} \times (S^1)^p$ to prove their equality, it suffices to prove that they have the same Fourier coefficients. By Theorem 2.13, we have

$$\langle \mathcal{S}, \phi \rangle \leq \langle \mathfrak{T} | \pi | x \rangle (\phi),$$

for every plurisubharmonic function ϕ on $\mathbb{C}^d$, and the mass of $\mathcal{S}$ coincides with the mass of $\langle \mathfrak{T} | \pi | x \rangle$. Now, note that if ϕ is pluriharmonic, then $-\phi$ and ϕ are plurisubharmonic. As a result,

$$\langle \mathcal{S}, \phi \rangle = \langle \mathfrak{T} | \pi | x \rangle (\phi),$$

for every pluriharmonic function. Recall that if f is a holomorphic function, then $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are pluriharmonic. We now consider the elements of the Fourier basis $f(\theta) = \exp 2\pi i \langle \nu, \theta \rangle$ for $\nu \in \mathbb{Z}^d$. Then we have the equality

$$\langle \mathcal{S}, f \rangle = \langle \mathfrak{T} | \pi | x \rangle (f).$$

This implies that the Fourier measure coefficients of both $\mathcal{S}$ and $\langle \mathfrak{T} | \pi | x \rangle$ coincide. □

Lemma 6.4. Let $\mathcal{C} \subseteq \mathbb{R}^d$ be a tropical variety of dimension p, and L be a rational $(d-p)$-dimensional plane such that L is transversal to all the affine extensions $\mathrm{aff}(\sigma)$ for $\sigma \in \mathcal{C}$. Assume that $\mathfrak{T}$ is a positive closed current of bidimension (p,p) on a smooth projective toric variety X_Σ compatible with $\mathcal{C} + L$ such that $\mathrm{supp}(\mathfrak{T}) \subseteq \mathrm{supp}(\mathfrak{T}_{\mathcal{C}})$. Further, for all $a \in \mathbb{R}^d$,

$$\overline{\mathfrak{T}}_{L+a} \wedge \mathfrak{T} = \overline{\mathfrak{T}}_{L+a} \wedge \overline{\mathfrak{T}}_{\mathcal{C}}.$$

Then $\mathfrak{T} = \mathfrak{T}_{\mathcal{C}}$ in T_N.

Proof. Let us first remark that $\mathrm{rec}(L + a) = \mathrm{rec}(L)$ for all $a \in \mathbb{R}^d$ and therefore, all $\mathfrak{T}_{a+L}$ are compatible with X_Σ and have a continuous super-potenrial in X_Σ and as a result, all the above wedge products are well-defined. By Demailly's second theorem of support [Dem, III.2.13], there are measures $\mu_\sigma^{\mathcal{T}}$ such that

$$\mathcal{T} = \sum_\sigma \int_{x\in S(\mathbb{Z}^d\cap H_\sigma)} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\big[\pi_\sigma^{-1}(x)\big]\,d\mu_\sigma^{\mathcal{T}}(x).$$

By repeated application of Proposition 6.1,

$$\mathcal{T}_L \wedge \mathcal{T} = \sum_\sigma \int_{(x,y)\in S(\mathbb{Z}^d\cap H_L)\times S(\mathbb{Z}^d\cap H_\sigma)} \big[\pi_H^{-1}(x)\cap \pi_\sigma^{-1}(y)\big]\,d\mu_L(x)\otimes \mu_\sigma^{\mathcal{T}}(y).$$

Applying both sides of the equality $\mathcal{T}_L\wedge\mathcal{T} = \mathcal{T}_L\wedge\mathcal{T}_{\mathcal{C}}$ on test-functions of the form

$$\omega_\nu = \exp(-i\langle \nu,\theta\rangle)\rho(r)$$

where $\rho:\mathbb{R}^d\to\mathbb{R}$ is a smooth function with compact support of $r\in\mathbb{R}^d$ and $\theta\in[0,2\pi)^d$, and $\nu\in\mathbb{Z}^d$, completely determines the Fourier coefficients of $\mu_\sigma^{\mathcal{T}}$ which have to coincide with the normalised Haar measures multiplied by the weight of σ, *i.e.*, $\mu_\sigma^{\mathcal{T}} = w_\sigma\mu_\sigma$. ☐

Note that any subtorus of $(\mathbb{C}^*)^d$, can be understood as a fibre of a tropical current. We have the following slicing theorem.

Proposition 6.5. Let $\mathcal{C}\subseteq\mathbb{R}^d$ be a tropical variety and $A\subseteq\mathbb{R}^d$ be a rational hyperplane intersecting $\mathcal{C}$ transversely. Let Σ be a fan compatible with $\mathcal{C}+A$. Assume that $\overline{\mathcal{S}}_n$ is a sequence of positive closed currents on X_Σ, and denote by $\mathcal{S}_n$ the restriction to T_N. Further, assume that

(a) $\overline{\mathcal{S}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}}$;
(b) $\mathrm{supp}(\overline{\mathcal{S}}_n) \longrightarrow \mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}})$,

then

$$\lim_{n\to\infty}\big(\mathcal{S}_n\wedge[T^A]\big) = \mathcal{T}_{\mathcal{C}}\wedge[T^A],$$

as currents on $T_N\subseteq X_\Sigma$.

Proof. Assume that $L\subseteq\mathbb{R}^d$ is an $(d-p-1)$-dimensional affine plane intersecting all $\mathrm{aff}(\sigma)$ for all $\sigma\in\mathcal{C}\cap A$ transversely. Then, on a projective smooth toric variety $X_{\Sigma'}$ compatible with $\mathcal{C}+L+A$ the tropical currents $\overline{\mathcal{T}}_{a+L}$, $a\in\mathbb{R}^d$ have continuous super-potentials. Therefore, by Proposition 2.6, we have

$$\lim_{n\to\infty}\big(\overline{\mathcal{S}}_n\wedge\overline{\mathcal{T}}_{a+L}\big) = \overline{\mathcal{T}}_{\mathcal{C}}\wedge\overline{\mathcal{T}}_{a+L}.$$

Now, for any $x\in\mathcal{C}\cap L\cap A$, let $B\subseteq\mathbb{R}^d$ containing x be a bounded open set containing only x as an isolated point of the intersection. By a translation we can assume that $x=0$. Consider the isomorphism

$$\xi : (\mathbb{C}^*)^d \xrightarrow{\ \sim\ } T_{\mathbb{Z}^d/(\mathbb{Z}^d\cap A)}\times T_{\mathbb{Z}^d\cap A},$$

and let π_1 and π_2 be the respective projections. Note that $\pi_1^{-1}(1) = T^A$. We now set

$$\begin{aligned} U &:= \pi_1\circ\xi\big(\mathrm{Log}^{-1}(U)\cap\mathrm{supp}(\mathcal{T}_{\mathcal{C}}\wedge\mathcal{T}_{a+L})\big),\\ V &:= \pi_2\circ\xi\big(\mathrm{Log}^{-1}(U)\cap T^A\big),\\ \mathcal{T}_n &:= \xi_*\big(\mathcal{S}_n\wedge\mathcal{T}_{a+L}\big),\ \text{in } T_N,\\ \mathcal{T} &:= \xi_*\big(\mathcal{T}_{\mathcal{C}}\wedge\mathcal{T}_{a+L}\big). \end{aligned}$$

Note that $\mathcal{T}_{\mathcal{C}}$ are horizontal-like as in the setting of Theorem 6.3. Assumption (b) now implies that $\mathcal{T}_n$ for a large n, is also horizontal-like. Thus we obtain

$$\lim_{n\to\infty} \big(\mathcal{S}_n \wedge [T^A]\big) \wedge \mathcal{T}_{a+L} = \mathcal{T}_{\mathcal{C}} \wedge [T^A] \wedge \mathcal{T}_{a+L},$$

for every a. We now deduce the convergence on T_N by Lemma 6.4. $\square$

Theorem 6.6. Let $\mathcal{C} \subseteq \mathbb{R}^d$ be a tropical variety, and let $B = A_1 \cap \cdots \cap A_k \subseteq \mathbb{R}^d$ be a complete intersection of rational hyperplanes $A_1, \ldots, A_k$. Assume that $\mathcal{C}$ and B intersect transversely. Let Σ be a smooth, projective fan compatible with $\mathcal{C} + B$, and let $(\overline{\mathcal{S}}_n)$ be a sequence of positive closed currents on X_Σ. Suppose that:

(a) $\overline{\mathcal{S}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}}$;
(b) $\operatorname{supp}(\overline{\mathcal{S}}_n) \longrightarrow \operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}})$.

Then, the following limit holds in the smooth projective toric variety X_Σ:

$$\lim_{n\to\infty} \Big(\overline{\mathcal{S}}_n \wedge [\overline{T}^B]\Big) = \overline{\mathcal{T}}_{\mathcal{C}} \wedge [\overline{T}^B].$$

We need a simple observation:

Lemma 6.7. Let $U \subseteq \mathbb{C}^d$ be an open subset and D be an analytic subset of $\mathbb{C}^d$. Assume that we have the convergence of closed positive currents $\mathcal{V}_n \longrightarrow \mathcal{V}$ in $U \setminus D$, and $\mathcal{V}_n$'s and $\mathcal{V}$ have a uniformly bounded local masses near D. Further, assume that for any cluster value $\mathcal{W}$ of the sequence $\{\overline{\mathcal{V}}_n\}_n$, we have

(a) $\operatorname{supp}(\mathcal{W}) \subseteq \operatorname{supp}(\overline{\mathcal{V}})$,
(b) $\operatorname{supp}(\overline{\mathcal{V}}) \cap D$ has the expected Cauchy–Riemann dimension.

Then,

$$\overline{\mathcal{V}}_n \longrightarrow \overline{\mathcal{V}}.$$

Proof. $\mathcal{W} - \overline{\mathcal{V}}$ is a positive closed current with the Cauchy–Riemann dimension less than or equal to p, therefore, it must be zero by Demailly's first theorem of support [Dem, Theorem III.2.10]. $\square$

Proof of Theorem 6.6. Applying Theorem 5.2 (or [?Ossermad-payne, Proposition 3.3.2] to each fibre of $\overline{\mathcal{T}}_{\mathcal{C}}$ separately), we obtain $\operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}}) \cap \overline{T}^B \cap [D_\rho]$ has the expected Cauchy–Riemann dimension $p-k-1$. By Demailly's first theorem of support [Dem, Theorem III.2.10],

$$\overline{\mathcal{S}}_{\mathcal{C}} \wedge [\overline{T}^B] = \overline{\mathcal{T}_{\mathcal{C}} \wedge [T^B]}.$$

By assumption $\overline{\mathcal{S}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}}$ and $\operatorname{supp}(\mathcal{T}_n) \longrightarrow \operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}})$. The observation in Lemma 2.14,

$$\lim_{n\to\infty} \operatorname{supp}(\overline{\mathcal{S}}_n \wedge [\overline{T}^B]) \subseteq \operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}} \wedge [\overline{T}^B]).$$

Therefore, any cluster value of $\overline{\mathcal{S}_n \wedge [T^B]} \leq \overline{\mathcal{S}}_n \wedge [\overline{T}^B]$ has a support in $\operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}} \wedge [\overline{T}^B])$. Now, we set

(a) $\mathcal{V}_n := \mathcal{S}_n \wedge [T^B]$,
(b) $\mathcal{V} := \mathcal{T}_{\mathcal{C}} \wedge [T^B]$,
(c) $\mathcal{W}$ a cluster value of $\overline{\mathcal{T}_n \wedge [T^B]}$.

A repeated application of Proposition 6.5 for $B = A_1 \cap \cdots \cap A_k$, we are in the situation of Lemma 6.7, and conclude. $\square$

Lemma 6.8. Let X_Σ be a smooth projective toric variety, and $\bar{\Delta} \subseteq X_\Sigma \times X_\Sigma$ be the diagonal. Let $\mathcal{S}$ and $\mathcal{T}$ be two positive currents on X. Then, for any ray $\rho \in \Sigma$,

$$\operatorname{supp}(\mathcal{S}) \cap \operatorname{supp}(\mathcal{T}) \cap D_\rho \subseteq X_\Sigma$$

has a Cauchy–Riemann dimension ℓ, if and only if,

$$\operatorname{supp}(\mathcal{S} \otimes \mathcal{T}) \cap \bar{\Delta} \cap D_{(0,\rho)} \subseteq X_\Sigma \times X_\Sigma,$$

has a Cauchy–Riemann dimension ℓ, where $D_{(0,\rho)}$ is the toric invariant divisor corresponding to the ray $(0,\rho)$ in $\Sigma \times \Sigma$.

Proof. The fan of $X_\Sigma \times X_\Sigma$ is $\Sigma \times \Sigma$, we have that $D_{(0,\rho)} \simeq X_\Sigma \times D_\rho$ and the assertion follows. $\square$

Theorem 6.9. Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{R}^d$ be two tropical cycles intersecting properly. Assume that X_Σ is a smooth toric projective variety compatible with $\mathcal{C}_1 + \mathcal{C}_2$. If moreover, for two sequence of positive closed currents $\overline{\mathcal{V}}_n$ and $\overline{\mathcal{W}}_n$ we have

(a) $\overline{\mathcal{W}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}_1}$ and $\overline{\mathcal{V}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}_2}$,
(b) $\operatorname{supp}(\overline{\mathcal{W}}_n) \longrightarrow \operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}_1})$ and $\operatorname{supp}(\overline{\mathcal{V}}_n) \longrightarrow \operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}_2})$.
(c) For any large n, $\overline{\mathcal{W}}_n \wedge \overline{\mathcal{V}}_n$ is well-defined.

Then

$$\overline{\mathcal{W}}_n \wedge \overline{\mathcal{V}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}_1} \wedge \overline{\mathcal{T}}_{\mathcal{C}_2}, \quad \text{as } m \to \infty.$$

Proof. For two closed currents $\mathcal{S}$ and $\mathcal{T}$ on X_Σ we naturally identify

$$\mathcal{S} \wedge \mathcal{T} = \pi_*\big(\mathcal{S} \otimes \mathcal{T} \wedge [\bar{\Delta}]\big),$$

where $\pi : X_\Sigma \times X_\Sigma \longrightarrow X_\Sigma$ is the projection onto the first factor. Let $\mathcal{W}_n$ and $\mathcal{V}_n$ be the restriction of $\overline{\mathcal{W}}_n$ and $\overline{\mathcal{V}}_n$ to $T_N \subseteq X_\Sigma$. We can define the tropical current $\mathcal{T}_\mathcal{C} := \mathcal{T}_{\mathcal{C}_1} \otimes \mathcal{T}_{\mathcal{C}_2}$. By Theorem 4.3, $\mathcal{C}_1 \times \mathcal{C}_2 := \mathcal{C}$ is a balanced tropical variety. Let us denote the diagonal $\Delta_\mathbb{R} \subseteq \mathbb{R}^d \times \mathbb{R}^n$. It is not hard to see that the transversality of $\mathcal{C}_1$ and $\mathcal{C}_2$ is equivalent to the transversality of the intersection

$$(\mathcal{C}_1 \times \mathcal{C}_2) \cap \Delta_\mathbb{R} \subseteq \mathbb{R}^d \times \mathbb{R}^n.$$

Now, fixing the coordinates $\big((z_1, \dots, z_d), (z'_1, \dots, z'_d)\big) \in (\mathbb{C}^*)^d \times (\mathbb{C}^*)^d$, the diagonal of $(\mathbb{C}^*)^d \times (\mathbb{C}^*)^d$ is given by the complete intersection of the tori $z_i = z'_i$, $i = 1, \dots, n$. Therefore, by refining $\Sigma \times \Sigma$ to a fan compatible also with $\Delta_\mathbb{R}$, and setting $B = \Delta$ in Proposition 6.5, we obtain

$$\mathcal{T}_n \wedge [\Delta] \longrightarrow \mathcal{T}_\mathcal{C} \wedge [\Delta], \quad \textit{i.e.}, \ \mathcal{W}_n \wedge \mathcal{V}_n \longrightarrow \mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2},$$

in the open torus $T_N \times T_N$. Now, we can employ the compatibility of $\mathcal{C}_1 + \mathcal{C}_2$ and X_Σ together with Lemma 6.8, imply that for any ray $\rho \in \Sigma$,

$$\operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}_1} \otimes \overline{\mathcal{T}}_{\mathcal{C}_2}) \cap [\bar{\Delta}] \cap D_\rho$$

has the expected Cauchy–Riemann dimension, and finally conclude by Lemma 6.7. $\square$

7. Dynamical Tropicalisation with non-Trivial Valuations

7.1. **Dynamical tropicalisation with a non-trivial valuation.** Recall that for a field $\mathbb{K}$, $\nu : \mathbb{K} \longrightarrow \mathbb{R} \cup \{\infty\}$, is called a valuation if it satisfies the following properties for every $a, b \in \mathbb{K}$:

(a) $\nu(a) = \infty$ if and only if $a = 0$;
(b) $\nu(ab) = \nu(a) + \nu(b)$;
(c) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$.

A valuation is called *trivial*, if the valuation of any non-zero element is 0. For an element $a \in \mathbb{K}$, we denote by $\bar{a}$ its image in the residue field. We are interested in the case where $\mathbb{K} = \mathbb{C}((t))$, is the field of *formal Laurent series* with the variable t, with the usual valuation. That is, for $g(t) = \sum_{j \geq k} a_j t^j$, with $a_k \neq 0$, the valuation equals the minimal exponent $\nu(g) = k \in \mathbb{Z}$.

Definition 7.1. (a) Let $f = \sum_{\alpha \in A} c_\alpha z^\alpha \in \mathbb{K}[z^{\pm 1}]$, be a Laurent polynomial in n variables, where $A \subseteq \mathbb{Z}^d$ is a finite subset. The tropicalisation of f with respect to ν,

$$\begin{aligned} \operatorname{trop}_\nu(f) : \mathbb{R}^d &\longrightarrow \mathbb{R}, \\ x &\mapsto \max\{-\nu(c_\alpha) + \langle x, \alpha \rangle\}. \end{aligned}$$

(b) Let $I \subseteq \mathbb{K}[z^{\pm 1}]$ be an ideal. The tropical variety associated to I, as a set, is defined as

$$\operatorname{Trop}_\nu(I) := \bigcap_{f \in I} \operatorname{Trop}(\operatorname{trop}_\nu(f)),$$

where $\operatorname{Trop}(\operatorname{trop}_\nu(f))$ is the set of points where $\operatorname{trop}_\nu(f)$ is not differentiable; see Remark 4.5.

(c) For an algebraic subvariety of the torus $Z \subseteq (\mathbb{K}^*)^d$, with the associated ideal $\mathbb{I}(Z)$, the tropicalisation of Z, as a set, is $\operatorname{Trop}_\nu(Z) := \operatorname{Trop}_\nu(\mathbb{I}(Z))$.

(d) In all the situations above, trop_0 denotes the tropicalisation with respect to the trivial valuation.

We need to relate a non-trivial valuation to the trivial valuation. Compare to [BJS+07, Lemma 1.1].

Lemma 7.2. Consider the ideal $I \subseteq \mathbb{C}[t^{\pm 1}, z^{\pm 1}] \overset{\iota}{\hookrightarrow} \mathbb{C}((t))[z^{\pm 1}]$. Assume that (u, x) are the coordinates in $\mathbb{R} \times \mathbb{R}^d$. Then, we have the following equality of sets

$$\operatorname{Trop}_0(I) \cap \{u = -1\} = \operatorname{Trop}_\nu(\iota(I)),$$

where ν is the the usual valuation in $\mathbb{C}((t))$. In other words, the tropicalisation of I as an ideal in $\mathbb{C}[t^{\pm 1}, z^{\pm 1}]$ with respect to the trivial valuation intersected with $\{u = -1\}$ coincides with the tropicalisation of $I = \iota(I)$ with respect to the usual valuation in $\mathbb{C}((t))$.

The proof of the lemma becomes clear with the following example.

Example 7.3. Let

$$f(x, t) = 4(t^3 + t^{-1}) z_1 z_2 + (1 + t + t^2) z_1.$$

Then, the tropicalisation of $f \in \mathbb{C}[t^{\pm 1}, z^{\pm 1}]$, with respect to the trivial valuation equals:

$$\mathrm{trop}_0(f) = \max\big\{ \max\{3u + x_1 + x_2, -u + x_1 + x_2\}, \max\{x_1, u + x_1, 2u + x_1\}\big\}.$$

Letting $u := -1$, $\mathrm{trop}_0(f)(-1, x) = \max\{1 + x_1 + x_2, x_1\}$. The latter equals $\mathrm{trop}_\nu(f)$ as an element of $\mathbb{C}((t))[z^{\pm 1}]$.

Proof of Lemma 7.2. If f is a monomial in $\mathbb{C}[t][z]$, then it is clear that

$$\mathrm{trop}_0(f)(-1, x) = \mathrm{trop}_\nu(\iota(f)).$$

Therefore, we have the equality for any polynomial in $f \in \mathbb{C}[t, z]$. To prove the main statement, note that

$$\begin{aligned}\mathrm{Trop}_\nu(\iota(I)) &= \bigcap_{f \in \iota(I)} \mathrm{Trop}(\mathrm{trop}_\nu(f)) \\ &= \bigcap_{f \in I} \big(\mathrm{Trop}(\mathrm{trop}_0(f)) \cap \{u = -1\}\big) \\ &= \mathrm{Trop}_0(I) \cap \{u = -1\}.\end{aligned}$$

□

Remark 7.4. Bergman in [Ber71], shows that for an algebraic subvariety $Z \subseteq (\mathbb{C}^*)^d$, one has

$$\lim_{t\to\infty} \mathrm{Log}_t(Z) \subseteq \mathrm{Trop}_0(\mathbb{I}(Z)),$$

and he conjectured the equality. This conjecture was later proved by Bieri and Groves in [BG84]. More precisely, Bieri and Grove prove that $\lim \mathrm{Log}_t(Z) \cap (S^1)^d$ is a polyhedral sphere of real dimension equal to (the complex dimension) $\dim(Z) - 1$. Therefore, the fan $\lim \mathrm{Log}_t(Z)$ is a cone over their spherical complex. See also [MS15, Theorem 1.4.2].

Remark 7.5. The above lemma is related to the results of Markwig and Ren in [MR20]. They considered the tropicalisation of an ideal $J \subseteq R[[t]][z]$, where R is the ring of integers of a discrete valuation ring $\mathbb{K}$, which is non-trivially valued. To obtain finiteness properties, however, the authors consider the associated tropical variety in the half-space $\mathbb{R}_{\leq 0} \times \mathbb{R}^d$. Note that such a variety is almost never balanced. The authors also prove that for an ideal $I \subseteq \mathbb{K}[z]$, the tropicalisation of the natural inverse image $\pi^{-1}I \subseteq R[[t]][z]$ with respect to trivial valuation, intersected with $\{u = -1\}$ equals $\mathrm{trop}_\nu(I)$; [MR20, Theorem 4].

Let us also recall the main result of [Bab23].

Theorem 7.6. Let $Z \subseteq (\mathbb{C}^*)^d$ be an irreducible subvariety of dimension p, and $\overline{Z}$ be the closure of Z in a compatible smooth projective toric variety X_Σ. Define $\Phi_m : X_\Sigma \longrightarrow X_\Sigma$ to be the unique continuous extension of

$$\begin{aligned}(\mathbb{C}^*)^d &\longrightarrow (\mathbb{C}^*)^d, \\ z &\longmapsto z^m.\end{aligned}$$

Then,

$$\frac{1}{m^{d-p}} \Phi_m^*[\overline{Z}] \longrightarrow \overline{\mathcal{T}}_{\mathrm{Trop}_0(Z)}, \quad \text{as } m \to \infty,$$

where $\overline{\mathcal{T}}_{\mathrm{Trop}_0(Z)}$ is the extension by zero of $\mathcal{T}_{\mathrm{Trop}_0(Z)}$ to X_Σ. Moreover, the supports also converge in Hausdorff metric.

Note that since the limit of a sequence of closed currents is closed, the above theorem implies that $\mathrm{trop}_0(Z)$ can be equipped with weights to become balanced. Note that the compatibility is in the following sense of Tevelev and Sturmfels:

Theorem 7.7. (a) The closure $\overline{Z}$ of Z in X_Σ is complete, if and only if, $\mathrm{Trop}_0(Z) \subseteq |\Sigma|$; see [Tev07].

(b) We have $|\Sigma| = \mathrm{Trop}_0(Z)$, if and only if, for every $\sigma \in \Sigma$ the intersection $\mathcal{O}_\sigma \cap \overline{Z}$ is non-empty and of pure dimension $p - \dim(\sigma)$; see [ST08].

Theorem 7.8. Let $I \subseteq \mathbb{C}[t^{\pm 1}, z^{\pm 1}]$ be an ideal with the associated $(p+1)$-dimensional algebraic variety $W = \mathbb{V}(I) \subseteq (\mathbb{C}^*)^{d+1}$. Assume that the projection onto the first coordinate $\pi_1 : W \longrightarrow \mathbb{C}^*$ is surjective and Zariski closed. We denote the fibers as $W_t := \pi_1^{-1}(t)$. We have that

(a)
$$\frac{1}{m^{d-p}} \Phi_m^* [W_{e^m}] \longrightarrow \mathcal{T}_{\mathrm{Trop}_\nu(I)}, \quad \text{as } m \to \infty \ ,$$
in the sense of currents in $\mathcal{D}_p((\mathbb{C}^*)^d)$, and we have identified $\iota(I) = I$.

(b) $\mathrm{Trop}_\nu(I)$ can be equipped with weights to become balanced.

(c) $\lim \mathrm{supp}(\frac{1}{m^{d-p}} \Phi_m^* [W_{e^m}]) = \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_\nu(I)})$.

(d) On a toric variety X_Σ compatible with $\mathrm{trop}_0(W) + \{u = -1\}$,
$$\frac{1}{m^{d-p}} \Phi_m^* [\overline{W}_{e^m}] \longrightarrow \overline{\mathcal{T}}_{\mathrm{Trop}_\nu(I)}, \quad \text{as } m \to \infty.$$

For the proof, we need the following: (I've simplified and fixed this:)

Lemma 7.9. Let $W \subseteq (\mathbb{C}^*)^{n+1}$ be a $(p+1)$-dimensional subvariety, such that the projection onto the first factor, $\pi_1 : (\mathbb{C}^*)^{n+1} \longrightarrow \mathbb{C}^*$ is surjective and a Zariski closed morphism, and the projection of the singular points $\pi_1(W_{\mathrm{sing}}) \subsetneq \mathbb{C}^*$. Then for a sufficiently large $|t_0| >> 0$
$$[W_{t_0}] = [\pi_1^{-1}(t_0)] = [\{t = t_0\}] \wedge [W].$$

Proof. Let us first fix an ideal associated to $I = \mathbb{I}(W) = \langle f_1, \dots, f_k \rangle \subseteq \mathbb{C}[t^{\pm 1}, z^{\pm 1}]$. At any regular point $w \in W_{\mathrm{reg}}$, $T_w W = \ker J(f)(w) = \big(\frac{\partial f_i}{\partial z_j}(w)\big)_{k \times (n+1)}$ is of dimension $p+1$. We consider the *critical set*
$$C = \Big\{ w \in W : \dim \ker \begin{pmatrix} \nabla_w t \\ Jf(w) \end{pmatrix} \geq p+1 \Big\},$$
where $\nabla_w t$ is the gradient of t at w. We have that $\nabla_w t = e_1$, the first element of the standard basis of the $\mathbb{C}$-vector space $\mathbb{C}^{n+1}$. The set C contains all the points such that either

- $w \in W$ is regular and the intersection $W \cap \pi_1^{-1}(\pi_1(w))$ is not transversal. Note that in this case, $\mathrm{Image}(Jf(w)) \subseteq \mathrm{span}(e_1)$.
- $w \in W$ is singular, and therefore $\ker(Jf(w)) > p+1$.

As a result, $C \subseteq W$, is a closed algebraic subvariety given by the equations of the $(d-p+1) \times (d-p+1)$ minors of $\begin{pmatrix} e_1 \\ Jf(w) \end{pmatrix}$. Since π_1 is a closed map, if we observe that $\pi_1(C) \subsetneq \mathbb{C}^*$ then $\pi_1(C)$ is a finite set and we conclude. Note that $\pi_1(W_{\mathrm{reg}} \cap C) \subseteq \mathbb{C}^*$, implies that W is contained in $\{t = t_0\}$ for some $t_0 \in \mathbb{C}^*$, which is not the case as $\pi_1 : W \longrightarrow \mathbb{C}^*$ is surjective. Further. $\pi(W_{\text{sing}}) \subsetneq \mathbb{C}^*$ by assumption of the lemma. □

Proof of Theorem 7.8. By the preceding lemma, and the fact that Φ_m preserves transversal intersection, we have

$$\frac{1}{m^{d-p}}\Phi_m^*[W_{e^m}] = \frac{1}{m^{n-(p+1)}}\Phi_m^*[W] \wedge \frac{1}{m}\Phi_m^*[\{t = e^m\}],$$

for a large m. Since $\text{Trop}_0(W)$ is a fan, it is transversal to the plane $\{u = -1\} \subset \mathbb{R}^{n+1}$. In consequence, we can use Proposition 6.5 to write

$$\lim \frac{1}{m^{d-p}}\Phi_m^*[W_{e^m}] = \big(\lim \frac{1}{m^{n-(p+1)}}\Phi_m^*[W]\big) \wedge \big(\lim \frac{1}{m}\Phi_m^*[\{t = e^m\}]\big)$$

By Theorem 7.6, restricted to $(\mathbb{C}^*)^{n+1}$, and the fact that we used $\text{Log} = (-\log|\cdot|, \dots, -\log|\cdot|)$ in the definition of tropical currents, the above limits yield

$$\lim \frac{1}{m^{d-p}}\Phi_m^*[W_{e^m}] = \mathcal{T}_{\text{Trop}_0(W)} \wedge \mathcal{T}_{\{u=-1\}}.$$

Applying Theorem 5.13 and Lemma 7.2, we obtain the equality.

For the assertion (b), note that the limit $\mathcal{T}_{\text{Trop}_\nu(I)}$ is a closed current and Theorem 4.3 implies that $\text{Trop}_\nu(I)$ is naturally balanced. To observe (c), note that (a) implies

$$\lim \text{supp}(\frac{1}{m^{d-p}}\Phi_m^*[W_{e^m}]) \supseteq \text{supp}(\mathcal{T}_{\text{Trop}_\nu(I)}).$$

However, because of transversality, $\text{supp}(\mathcal{T}_{\text{Trop}_\nu(I)}) = \text{supp}(\mathcal{T}_{\text{Trop}_0(W)}) \cap \text{supp}(\mathcal{T}_{\{u=-1\}})$. At the same time,

$$\lim \text{supp}(\Phi_m^*[W_{e^m}]) = \lim \text{supp}(\Phi_m^*[W]) \cap \text{supp}(\Phi_m^*[\{t = e^m\}]).$$

Moreover, for the Hausdorff limit of sets $\lim(A_i \cap B_i) \subseteq (\lim A_i) \cap (\lim B_i)$. This implies

$$\lim \text{supp}(\Phi_m^*[W_{e^m}]) \subseteq \text{supp}(\mathcal{T}_{\text{Trop}_0(W)}) \cap \text{supp}(\mathcal{T}_{\{u=-1\}}),$$

which implies (c). Now, (d) is implied by Theorem 6.6. □

In the setting of the previous theorem, a generalisation of Bergman's theorem (see Remark 7.4) asserts that

$$\text{Log}_t(W_t) \longrightarrow \text{Trop}(I), \quad \text{as } t \longrightarrow \infty,$$

where Log_t is the logarithm with base t. This theorem can be understood as a counterpart of Lemma 7.2 for tropicalisation with Log. This generalisation was finally proved by Jonsson in [Jon16], and we can now deduce a sequential analogue:

Corollary 7.10. In the setting of the previous theorem

$$\frac{1}{m}\text{Log}(W_{e^m}) \longrightarrow \text{Trop}_\nu(I), \quad \text{as } m \to \infty,$$

in the Hausdorff metric in compact subsets of $\mathbb{R}^d$.

Proof. Note that for any variety $Z \subseteq (\mathbb{C}^*)^{n+1}$, as a set $\mathrm{Log}(\Phi_m^{-1}(Z)) = \frac{1}{m}\mathrm{Log}(Z)$. Therefore,

$$\mathrm{Log}\ \mathrm{supp}\big(\frac{1}{m^{d-p}}\Phi_m^*[W_{e^m}]\big) = \frac{1}{m}\mathrm{Log}(W_{e^m}).$$

The assertion now follows from Theorem 7.8(c) and continuity of Log with respect to the Hausdorff metric on compact sets. □

Let us first prove the analogous to [BJS+07, Lemma 3.2] and [OP13, Theorem 1.2].

Theorem 7.11. Assume that W and V are two algebraic subvarieties of $(\mathbb{C}^*)^d$ with respective dimensions p and q, with $p+q \geq n$. Assume that $\mathrm{Trop}_0(V)$ and $\mathrm{Trop}_0(W)$ intersect properly, then

$$\frac{1}{m^{2n-(p+q)}}\Phi_m^*\big([W]\wedge[V]\big) \longrightarrow \mathcal{T}_{\mathrm{Trop}_0(W)\cdot\mathrm{Trop}_0(V)}, \quad \text{as } m \to \infty.$$

Moreover, $\mathcal{T}_{\mathrm{Trop}_0(W\cap V)} \leq \mathcal{T}_{\mathrm{Trop}_0(W)\cdot\mathrm{Trop}_0(V)}$. In particular, the corresponding induced weights for $\tau \in \mathrm{Trop}_0(W\cap V)$, is less than or equal to the weight of τ induced by $\mathrm{Trop}_0(W)\cdot\mathrm{Trop}_0(V)$.

Example 7.12. To see that the inequality in the previous theorem can be strict, let us consider the subvarieties of $(\mathbb{C}^*)^2$, $W = \mathbb{V}(z_2 - 1)$ and $V = \mathbb{V}(z_2 - z_1^2 - 1)$. We have that

$$\mathrm{Trop}_0(W) = \mathrm{Trop}(\max\{x_2, 2x_1, 0\}),$$
$$\mathrm{Trop}_0(V) = \mathrm{Trop}(\max\{x_2, 0\}).$$

In Example 3.6, we discussed that the stable intersection of these two cycles is the origin $(0,0) \in \mathbb{R}^2$ with multiplicity 2. Note that the (set-theoretic) intersection $W\cap V = \{(0,1)\}$, thus $\mathrm{Trop}_0(W\cap V) = (0,0)$ with multiplicity 1, whereas $\mathrm{Trop}_0(W)\cdot\mathrm{Trop}_0(V) = (0,0)$ with multiplicity 2. We obtain

$$2\,\mathcal{T}_{\mathrm{Trop}_0(W\cap V)} = \mathcal{T}_{\mathrm{Trop}_0(W)\cdot\mathrm{Trop}_0(V)}.$$

Note that this inequality, gladly, does not contradict [OP13, 3.3.1] (a consequence of [Ful98, 8.2]), which the scheme-theoretic intersection $W\cap V$ is considered.

Proof of Theorem 7.11. Assume that X_Σ is a toric variety compatible with $\mathrm{Trop}_0(W)+\mathrm{Trop}_0(V)$. We need to show that the hypothesis (a), ..., (d) of Theorem 6.9 for $\overline{\mathcal{W}}_m = m^{p-n}\Phi_m^*[\overline{W}]$ and $\overline{\mathcal{V}}_m = m^{q-n}\Phi_m^*[\overline{V}]$ is satisfied. Note that hypotheses (a) and (b) are implied by Theorem 7.6. The hypotheses (c) and (d) for $\overline{\mathcal{W}}_1$ and $\overline{\mathcal{V}}_1$ are a result of [OP13, Proposition 3.3.2]. The hypotheses (c) and (d) for any m are implied by the fact that $\Phi_m^*\big([\overline{W}]\wedge[\overline{V}]\big) = \Phi_m^*[\overline{W}]\wedge\Phi_m^*[\overline{V}]$. Any quick explanation for this?. Therefore,

$$\overline{\mathcal{W}}_m \wedge \overline{\mathcal{V}}_m \longrightarrow \overline{\mathcal{T}}_{\mathrm{Trop}_0(W)} \wedge \overline{\mathcal{T}}_{\mathrm{Trop}_0(V)}, \quad \text{as } m \to \infty,$$

and the first assertion is obtained by restricting to $T_N \subseteq X_\Sigma$ and Theorem 5.13. To see the second assertion, note that by Theorem 7.6,

$$\frac{1}{m^{2n-(p+q)}}\Phi_m^*[W\cap V] \longrightarrow \mathcal{T}_{\mathrm{Trop}_0(W\cap V)}, \quad \text{as } m \to \infty.$$

However, as currents $[W \cap V] \leq [W] \wedge [V]$ since the right-hand side might produce multiplicities on the intersection. Applying Φ_m^* to both sides preserves this inequality. Therefore, for every m,

$$\Phi_m^*[W \cap V] \leq \Phi_m^*[W] \wedge \Phi_m^*[V].$$

We can conclude by taking the limit $m \longrightarrow \infty$.

□

The tropical version of the following was observed in various places [OP13, MS15], which does not need to assume the proper intersection of tropicalisations.

Lemma 7.13. Let W and V be two algebraic subvarieties of $(\mathbb{C}^*)^d$, and for $0 < \epsilon < 1$, let $U_\epsilon((S^1)^d)$ be an ϵ-neighbourhood of $(S^1)^d$. Then,

$$\frac{1}{m^{2n-(p+q)}} \int_{(t_1,t_2)\in U_\epsilon((S^1)^{2n})} \Phi_m^*[t_1V \cap t_2W]\, d\nu(t_1) \otimes d\nu(t_2) \longrightarrow \mathcal{T}_{\mathrm{Trop}_0(V)} \wedge \mathcal{T}_{\mathrm{Trop}_0(W)},$$

as $m \to \infty$. Here, the $d\nu$ are the normalised Lebesgue measures on $U_\epsilon((S^1)^d)$.

Proof. Note that $[t_1V \cap t_2W]$ is transversal for generic t_1 and t_2, and since Φ_m^* preserves transversality, we can separate the above integrand into $\Phi_m^*[t_1V] \wedge \Phi_m^*[t_2W]$. Using polar coordinates (r_i, θ_i) for t_i, we have

$$\int_{t_1\in U_\epsilon((S^1)^d)} \Phi_m^*[t_1V]\, d\nu(t_1) = \int_{t_1\in U_\epsilon((S^1)^d)} t_1^{1/m}\, \Phi_m^*[V]\, d\nu(t_1).$$

Here, with abuse of notation, we choose the m-th root $t_1^{1/m}$ to be the first root of $t^{1/m}$. We can also obtain a similar equation for t_2W. Further,

$$\big(\Phi_m^*[t_1V] \otimes \Phi_m^*[t_2W]\big) \wedge [\Delta] = \big(\Phi_m^*[V] \otimes \Phi_m^*[W]\big) \wedge (t_1)^{-1/m}(t_2)^{-1/m}[\Delta].$$

$$\int_{U_\epsilon(S^1)^{2n}} (t_1)^{-1/m}(t_2)^{-1/m}[\Delta] = \int_{[(1-\epsilon),1+\epsilon]} \mathcal{T}_\Delta^{R_m}\, dR,$$

where $R_m = (|t_1|^{-1/m}, |t_2|^{-1/m})$, and $\mathcal{T}_\Delta^{R_m}$ is the tropical current associated with the diagonal $\Delta \subseteq \mathbb{R}^d \times \mathbb{R}^d$, where the compact torus is rescaled to $R_m(S^1)^{2n}$. In a toric variety compatible with $\Delta \subseteq \mathbb{R}^d \times \mathbb{R}^d$, for any R, $\overline{\mathcal{T}_\Delta^{R_m}}$ has a continuous superpotential, and $\overline{\mathcal{T}_\Delta^{R_m}}$ is SP-convergent to $\overline{\mathcal{T}}_\Delta$. Using Proposition 2.6, Theorem 2.10, and restricting yields:

$$\frac{1}{m^{2n-(p+q)}} \big(\Phi_m^*[t_1V] \otimes \Phi_m^*[t_2W]\big) \wedge \int_{[(1-\epsilon),1+\epsilon]} \mathcal{T}_\Delta^{R_m}\, dR \longrightarrow \big(\mathcal{T}_{\mathrm{Trop}_0(V)} \otimes \mathcal{T}_{\mathrm{Trop}_0(W)}\big) \wedge \mathcal{T}_\Delta,$$

as $m \to \infty$. The latter equals

$$\mathcal{T}_{\mathrm{Trop}_0(V)\times \mathrm{Trop}_0(W).\Delta},$$

which can be identified with $\mathcal{T}_{\mathrm{Trop}_0(V)} \wedge \mathcal{T}_{\mathrm{Trop}_0(W)}$. □

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