# Algebraic Geometry Coursework 1

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1)

Q1. Let $A \subseteq \mathbb{A}^n$ be a subset.

(a) (**5 marks**) What is the definition of the closure of $A$ in $\mathbb{A}^n$?

(b) (**5 marks**) Prove that $\mathbb{V}(\mathbb{I}(A))$ equals the Zariski closure of $A$ in $\mathbb{A}^n$.

(c) (**5 marks**) Give an example of a subset in $B \subseteq \mathbb{C}$ whose closure in the Zariski topology does not coincide with its closure in the Euclidean topology.

a) If $A \subseteq \mathbb{A}^n$, then the closure of $A$ in $\mathbb{A}^n$ is the intersection of all closed sets in $\mathbb{A}^n$ containing $A$:

$$\overline{A} = \bigcap_{\substack{C \text{ closed} \\ C \supseteq A}} C$$

b) Let $x \in \mathbb{V}(\mathbb{I}(A))$, thus $f(x) = 0$ for all $f \in \mathbb{I}(A)$.
Let $C$ be a closed subset of $\mathbb{A}^n$ containing $A$. Then $C$ is a c.a.a.v., and so $C = \mathbb{V}(\{f_i\}_{i \in I})$.
Since $A \subseteq C$, $f_i(a) = 0$ for all $a \in A$, and so $f_i \in \mathbb{I}(A)$.
Thus $f_i(x) = 0$ and so $x \in C$.
Since $C$ was chosen arbitrarily, $x$ is in the intersection of all such $C$ and therefore $x \in \overline{A}$, and so $\mathbb{V}(\mathbb{I}(A)) \subseteq \overline{A}$.

Now let $x \in \overline{A}$. Then for any closed set $C \subseteq \mathbb{A}^n$ containing $A$, $x \in C$.
In particular, $\mathbb{V}(\mathbb{I}(A))$ is a closed set (as it is a c.a.a.v.), and $A \subseteq \mathbb{V}(\mathbb{I}(A))$ since if $a \in A$, then $f(a) = 0$ for all $f \in \mathbb{I}(A)$.
Hence $x \in \mathbb{V}(\mathbb{I}(A))$. Hence $\overline{A} \subseteq \mathbb{V}(\mathbb{I}(A))$.

c) Let $B = \{ z \in \mathbb{C} \mid |z| < 1 \} \subseteq \mathbb{C}$. Then since $B$ is an infinite subset of $\mathbb{C}$, $B$ is not a variety, and thus $\mathbb{I}(B) = \{0\}$.
So $\overline{B} = \mathbb{V}(\{0\}) = \mathbb{C}$ in the Zariski topology by (b).

However in the Euclidean topology, the limit points for $B$ are $\{ z \in \mathbb{C} \mid |z| = 1 \}$, since for any such $z = e^{i\theta}$ ($\theta \in [0, 2\pi)$) and $\delta > 0$, $B_\delta(z)$ contains a point in $B$, namely $w = (1 - \delta/2)e^{i\theta}$ since $|w| = 1 - \delta/2 < 1$, and $|z - w| = |\delta/2 \, e^{i\theta}| = \delta/2 < \delta$, and if $|z| > 1$, then $z = (1+\varepsilon)e^{i\varphi}$, $\varepsilon > 0$, $\varphi \in [0, 2\pi)$, then for all $w \in B_{\varepsilon/2}(z)$,

$$\begin{aligned} |w| &= |z - (z - w)| \\ &\geq \big| |z| - |z - w| \big| \quad \text{(Triangle inequality)} \\ &\geq | (1+\varepsilon) - \varepsilon/2 | \\ &= |1 + \varepsilon/2| > 1, \quad \text{so } w \notin B. \end{aligned}$$

So such a $z$ is not a limit point.

$$\begin{aligned} \Rightarrow \overline{B} &= B \cup \{ z \in \mathbb{C} \mid |z| = 1 \} \\ &= \{ z \in \mathbb{C} \mid |z| \leq 1 \} \\ &\neq \mathbb{C} \end{aligned}$$

Hence the closure of $B$ differs in the Euclidean and Zariski topologies.

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2)

Q2. (a) (**5 marks**) What is the definition of a compact subset of a topological space?

(b) (**10 marks**) Prove that $\mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}^2$ is compact in the Zariski topology but not in the Euclidean topology.

a) Let $(X, \mathcal{T})$ be a topological space and $X' \subseteq X$. Then $X'$ is a compact subset of $X$ if every open cover $\{U_i\}_{i \in I}$ of $X'$ (i.e., a collection of open sets such that $X' \subseteq \bigcup_{i \in I} U_i$) has a finite subcover (i.e., a finite collection $U_1, \dots, U_n \in \{U_i\}_{i \in I}$ such that $X' \subseteq \bigcup_{i=1}^n U_i$).

b) Suppose that $V = \mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}^2$ has an open cover $\{U_i\}_{i \in I}$. Then $V_i := U_i^c$ is a c.a.a.v. in $\mathbb{A}^2$.

If $I$ is finite, then we are done since $\{U_i\}_{i \in I}$ is a finite subcover, otherwise take countably many of the $V_i$: $V_1, V_2, \dots$, and define:

$$W_j := \bigcap_{i=1}^j V_i \qquad (j \in \mathbb{N}).$$

Then $W_j$ is a c.a.a.v. (as it is the finite intersection of closed sets in the Zariski topology) and $W_1 \supseteq W_2 \supseteq \dots$

$\Rightarrow \mathbb{I}(W_1) \subseteq \mathbb{I}(W_2) \subseteq \dots$ (Hilbert's Correspondence is inclusion reversing).

As $\mathbb{C}[x,y]$ is Noetherian, there exists $r \in \mathbb{N}$ such that $\mathbb{I}(W_r) = \mathbb{I}(W_{r+1}) = \dots$

$$\Rightarrow \begin{cases} \mathbb{I}(W_r) \subseteq \mathbb{I}(W_{r+1}) \subseteq \dots \\ \mathbb{I}(W_r) \supseteq \mathbb{I}(W_{r+1}) \supseteq \dots \end{cases}$$

$$\Rightarrow \begin{cases} W_r \supseteq W_{r+1} \supseteq \dots \\ W_r \subseteq W_{r+1} \subseteq \dots \end{cases}$$

$$\Rightarrow \quad W_r = W_{r+1} = \dots$$

In particular, $W_r = W_r \cap V_{r+1}$. Since $\{U_i\}_{i \in I}$ is an open cover for $V$, for each $x \in V$, there exists $j \in I$ such that $x \in U_j$, and so $x \notin V_j$.
If $V_j \in \{V_1, \dots, V_r\}$, then $x \notin W_r$, and if not take $V_j = V_{r+1}$, then $x \notin V_{r+1}$ and so $x \notin W_r \cap V_{r+1} = W_r$.

Hence for all $x \in V$, $x \in W_r^c = (\bigcap_{i=1}^r V_i)^c = \bigcup_{i=1}^r V_i^c = \bigcup_{i=1} U_i$. Hence $V \subseteq \bigcup_{i=1}^r U_i$ and so $\{U_1, \dots, U_r\}$ is a finite subcover for $V$. Hence $V$ is compact in the Zariski topology.

However, in the Euclidean topology, for each $z \in V$, let $U_z := B_1(z)$ (the open ball of radius $1$ centred at $z$). Then each $U_z$ is open in $\mathbb{C}^2$ as it is an open ball, and for each $z \in V$, $z \in U_z$, and so $z \in \bigcup_{z' \in V} U_{z'}$.
Thus $\{U_{z'}\}_{z' \in V}$ is an open cover for $V$.

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Suppose there exists a finite subcover $\{U_1, \dots, U_r\}$ for $V$.
Then $U_i = U_{z_i}$ for some $z_i \in V$.
Let $j$ be such that $|z_j| = \max\{|z_1|, \dots, |z_r|\}$, then for any $z \in \bigcup_{i=1}^{r} U_i$, $|z| \le |z_j| + 1 = M^3$, for some $M > 0$.

Then $w = (M^3, M^2) \in V$ but:

$$|w| = \sqrt{M^6 + M^4} > \sqrt{M^6} = M^3$$

So $w \notin \bigcup_{i=1}^{r} U_i$, and thus $\{U_1, \dots, U_r\}$ does not form a subcover of $V$. So $V$ is not compact in the Euclidean topology.

3)

Q3. (a) (**5 marks**) Find a curve $W \subseteq \mathbb{A}^2$ and a morphism $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$, such that $W$ is irreducible but $\varphi^{-1}(W)$ is not.

(b) (**5 marks**) Let $Y$ be a topological space and consider $X \subseteq Y$ with the subspace topology. Prove that if $X$ is irreducible then so is its closure.

(c) (**5 marks**) Prove that isomorphisms preserve irreducibility and dimension of closed affine algebraic varieties.

(d) (**10 marks**) Find the irreducible components of $\mathbb{V}(zx - y, y^2 - x^2(x+1)) \subseteq \mathbb{A}^3$. You need to justify why each component is irreducible.

a) Let $W = \{(0, y) \mid y \in \mathbb{C}\} \subseteq \mathbb{A}^2$.
Then $\mathbb{I}(W) = \{f \in \mathbb{C}[x,y] \mid f(0,y) = 0 \;\forall y \in \mathbb{C}\}$
$= (x)$

This is because if $p(x,y) \in (x)$, then $p(x,y) = x f(x,y)$ for some $f \in \mathbb{C}[x,y]$, and so $p(0,y) = 0$, thus $p \in \mathbb{I}(W)$, and if $p \notin (x)$, then $\exists m \in \mathbb{N}_0$ such that the coefficient of $y^m$ is non-zero, hence $p(0,y)$ is a non-zero polynomial in $y$, and therefore there exists some value of $y$ such that $p(0,y) \ne 0$. Thus $p \notin \mathbb{I}(W)$.

Moreover, $\mathbb{I}(W)$ is prime since if $p_1, p_2 \notin \mathbb{I}(W)$, then there exist maximal $m_1, m_2 \in \mathbb{N}_0$ such that the coefficients of $y^{m_1}$ and $y^{m_2}$ in $p_1$ and $p_2$ respectively are non-zero. Hence the coefficient of $y^{m_1+m_2}$ in $p_1 p_2$ is also non-zero and so $p_1 p_2 \notin \mathbb{I}(W)$.
Thus by theorem 2.16, $W$ is irreducible.

Now define $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ by $\varphi((x,y)) = (x^2 - y^2, y)$.
This is a morphism of varieties since $\varphi_1(x,y) = x^2 - y^2$, $\varphi_2(x,y) = y$ are polynomial maps. Then:

$$\begin{aligned}
\varphi^{-1}(W) &= \{(x,y) \in \mathbb{C}^2 \mid (x^2 - y^2, y) \in W\} \\
&= \{(x,y) \in \mathbb{C}^2 \mid (x^2 - y^2, y) = (0, y') \text{ for some } y' \in \mathbb{C}\} \\
&= \{(x,y) \in \mathbb{C}^2 \mid x^2 - y^2 = 0\} \\
&= \{(x,y) \in \mathbb{C}^2 \mid x - y = 0\} \cup \{(x,y) \in \mathbb{C}^2 \mid x + y = 0\} \\
&= \mathbb{V}(x - y) \cup \mathbb{V}(x + y)
\end{aligned}$$

is reducible.

b) Let $Y$ be a topological space, $X \subseteq Y$ with the subspace topology. Let $X$ be irreducible and suppose for a contradiction that $\overline{X}$ is reducible.

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Then $\overline{X} = C_1 \cup C_2$, where $C_1, C_2$ are closed in $\overline{X}$ and $\overline{X} \nsubseteq C_1$, $\overline{X} \nsubseteq C_2$.
Now $C_1 = \overline{X} \cap Y_1$, $C_2 = \overline{X} \cap Y_2$, where $Y_1, Y_2$ are closed in $Y$.

Since $\overline{X} \nsubseteq C_i$ (for $i=1,2$), $X \nsubseteq C_i = \overline{X} \cap Y_i$ and as $X \subseteq \overline{X}$, $X \nsubseteq Y_i$. Then:

$$\begin{aligned}
X &= X \cap \overline{X} \quad (\text{since } X \subseteq \overline{X}) \\
&= X \cap (C_1 \cup C_2) \\
&= (X \cap C_1) \cup (X \cap C_2) \\
&= (X \cap (\overline{X} \cap Y_1)) \cup (X \cap (\overline{X} \cap Y_2)) \\
&= ((X \cap \overline{X}) \cap Y_1) \cup ((X \cap \overline{X}) \cap Y_2) \\
&= (X \cap Y_1) \cup (X \cap Y_2).
\end{aligned}$$

As $Y_1, Y_2$ are closed in $Y$, $X \cap Y_1$, $X \cap Y_2$ are closed in $X$ with the subspace topology. However by the above result, we had $X \nsubseteq Y_i$, and so $X \nsubseteq X \cap Y_i$. Thus $X$ is reducible, giving a contradiction.

c) Let $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ and suppose that $V \simeq W$. Then there exists an isomorphism $\varphi: V \to W$.
Suppose that $V$ is irreducible, and let $W = W_1 \cup W_2$, where $W_1, W_2$ are closed in $\mathbb{A}^m$.

Since $\varphi$ is an isomorphism (and hence surjective), and $W_1, W_2 \subseteq W$, we have that $W_1 = \varphi(V_1)$, $W_2 = \varphi(V_2)$, for some $V_1, V_2 \subseteq \mathbb{A}^n$. These are closed affine algebraic varieties since if $W_1 = \mathbb{V}(\{f_i\}_{i \in I})$, $W_2 = \mathbb{V}(\{g_j\}_{j \in J})$, then:

$$\begin{aligned}
V_1 &= \varphi^{-1}(W_1) \\
&= \{ z \in \mathbb{C}^n \mid \varphi(z) \in W_1 \} \\
&= \{ z \in \mathbb{C}^n \mid f_i(\varphi(z)) = 0 \quad \forall i \in I \} \\
&= \{ z \in \mathbb{C}^n \mid (f_i \circ \varphi)(z) = 0 \quad \forall i \in I \} \\
&= \mathbb{V}(\{f_i \circ \varphi\}_{i \in I})
\end{aligned}$$

$$V_2 = \mathbb{V}(\{g_j \circ \varphi\}_{j \in J}) \quad (\text{by the same argument})$$

and $V = V_1 \cup V_2$ since if $z \in V$, then $z \in \varphi^{-1}(W)$

$$\begin{aligned}
&\Rightarrow \varphi(z) \in W = W_1 \cup W_2 \\
&\Rightarrow \varphi(z) \in W_1 \text{ or } \varphi(z) = W_2 \\
&\Rightarrow z \in \varphi^{-1}(W_1) = V_1 \text{ or } z \in \varphi^{-1}(W_2) = V_2 \\
&\Rightarrow z \in V_1 \cup V_2
\end{aligned}$$

and if $z \in V_1 \cup V_2$, then $z \in V_1$ or $z \in V_2$:

$$\begin{aligned}
&\Rightarrow z \in \varphi^{-1}(W_1) \text{ or } z \in \varphi^{-1}(W_2) \\
&\Rightarrow \varphi(z) \in W_1 \text{ or } \varphi(z) \in W_2 \\
&\Rightarrow \varphi(z) \in W_1 \cup W_2 = W \\
&\Rightarrow z \in \varphi^{-1}(W) = V
\end{aligned}$$

Since $V$ is irreducible, and $V = V_1 \cup V_2$, with $V_1, V_2 \subseteq \mathbb{A}^n$ closed, we must have that $V = V_1$ or $V = V_2$.

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If $W \neq W_1$, then $\varphi(V) \neq \varphi(V_1)$, and so $V \neq V_1$ (this can be seen easily by the contrapositive). Also if $W \neq W_2$, then $V \neq V_2$. However since $V$ is irreducible and thus $V = V_1$ or $V = V_2$, we must have $W = W_1$ or $W = W_2$, giving that $W$ is irreducible.

Moreover if $W$ is irreducible, then $\varphi(V)$ is irreducible, and since $\varphi^{-1}: W \to V$ is an isomorphism, by the previous part, $\varphi^{-1}(\varphi(V)) = V$ is irreducible.

So isomorphism preserves irreducibility.

Next suppose that $V$ has dimension $\dim(V) = d$. Then the maximal dimension of any irreducible variety $V' \subset V$ is $d$

Let $V' \subseteq V$ be such that there exists a chain $V' = V_d \supsetneq V_{d-1} \supsetneq \dots \supsetneq V_0 = \{*\}$, where $V_i \subseteq V$ are irreducible subvarieties of $V$.

For $i = 0, 1, \dots, d$, let $W_i = \varphi(V_i)$. Then by the previous part, $W_i$ is irreducible, and $W_0 = \varphi(\{*\}) = \{\varphi(*)\}$. Also since $V_{i+1} \supsetneq V_i$ for $i = 0, \dots, d-1$, $W_{i+1} \supsetneq W_i$ (this is because if $w \in W_i = \varphi(V_i)$, then $\varphi^{-1}(w) \in V_i$, so $\varphi^{-1}(w) \in V_{i+1}$, and thus $w \in \varphi(V_{i+1}) = W_{i+1}$; and $V_{i+1} \neq V_i \Rightarrow W_{i+1} \neq W_i$).
Now since $V_i$ is an algebraic subvariety of $V$, $V_i = V \cap Z$ where $Z \subseteq \mathbb{A}^n$ is a c.a.a.v.
Then $W_i = \varphi(V_i) = \varphi(V \cap Z)$ and we observe that:

$$\begin{aligned} x \in \varphi(V \cap Z) &\iff \varphi^{-1}(x) \in V \cap Z \\ &\iff \varphi^{-1}(x) \in V \text{ and } \varphi^{-1}(x) \in Z \\ &\iff x \in \varphi(V) \text{ and } x \in \varphi(Z) \\ &\iff x \in \varphi(V) \cap \varphi(Z) = W \cap \varphi(Z) \end{aligned}$$

So $W_i = W \cap \varphi(Z)$. We showed previously that if $Z$ is closed, then so is $\varphi(Z)$, and thus $W_i$ is an algebraic subvariety of $W$. Hence $W_d$ has dimension $d$, and so $\dim(W) \geq d$.

Now suppose there exists a chain $W_n \supsetneq W_{n-1} \supsetneq \dots \supsetneq W_0 = \{*\}$ as above for some $d > n$.
Then since $\varphi^{-1}$ is an isomorphism, and applying the above result with $\varphi^{-1}$ instead of $\varphi$, then if $V_i = \varphi^{-1}(W_i)$ $(i = 0, \dots, n)$, then $V_n \supsetneq V_{n-1} \supsetneq \dots \supsetneq V_0 = \{\varphi^{-1}(*)\}$ is a chain with the desired properties. So $\dim(V) \geq n > d$, which is a contradiction.
Thus $\dim(W) \leq d$, and so $\dim(W) = d$.

So if $V \simeq W$, then $\dim(V) = \dim(W)$.

d) Let $V = \mathbb{V}(zx - y,\ y^2 - x^2(x+1))$
$= \{(x,y,z) \in \mathbb{A}^3 \mid zx - y = 0,\ y^2 - x^2(x+1) = 0\}$

$$(x,y,z) \in V \iff \begin{cases} y = xz & (1) \\ y^2 = x^2(x+1) & (2) \end{cases}$$

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$$\Leftrightarrow \quad x^2 z^2 = x^2(x+1)$$
$$\Leftrightarrow \quad x = 0 \quad \text{or} \quad z^2 = x+1$$

If $x=0$, then $y=0$, and $z$ can be arbitrary.
If $z^2 = x+1$, then $y = (z^2-1)z = z^3 - z$.

So $(x,y,z) \in V \iff (x=0, y=0)$ or $(x = z^2-1, y = z^3-z)$

Thus $V = \underbrace{\mathbb{V}(x,y)}_{V_1} \cup \underbrace{\mathbb{V}(x-z^2+1, y-z^3+z)}_{V_2 = \{(t^2-1, t^3-t, t) \mid t \in \mathbb{C}\}}$.

Define $\varphi : \mathbb{A}^1 \to V_1$, $t \mapsto (0,0,t)$. This is clearly a morphism and is invertible with inverse $(0,0,t) \mapsto t$. So $\varphi$ is an isomorphism, and thus $\mathbb{A}^1 \simeq V_1$.
By 3(c), since $\mathbb{A}^1$ is irreducible (by Corollary 2.18), so is $V_1$.

Define $\psi : \mathbb{A}^1 \to V_2$, $t \mapsto (t^2-1, t^3-t, t)$. This is again a morphism (as the components are polynomial maps) and invertible with inverse $(t^2-1, t^3-t, t) \mapsto t$. So $\psi$ is an isomorphism, and thus $\mathbb{A}^1 \simeq V_2$, and hence $V_2$ is irreducible.

Thus $V_1$ and $V_2$ are the irreducible components of $V$.

4)

Q4. (a) (**10 marks**) Let $V \subseteq \mathbb{A}^n$ be a Zariski-closed subset and $a \in \mathbb{A}^n \setminus V$ be a point. Find a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that

$$f \in \mathbb{I}(V), \quad f(a) = 1.$$

(b) (**15 marks**) Let $I, (g) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be two ideals. Assume that $\mathbb{V}(g) \supseteq \mathbb{V}(I)$.

(i) Prove that if $I = (f_1, \ldots, f_k)$, then

$$(f_1, \ldots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \ldots, x_{n+1}]. \tag{1}$$

(ii) By only using Equation (1) and not the nullstellensatz, prove that there exists a positive integer $m$ such that $g^m \in I$.

a) Let $V \subseteq \mathbb{A}^n$ be Zariski closed and $a \in \mathbb{A}^n \setminus V$.
Then $V \neq V \cup \{a\}$.

$\Rightarrow \mathbb{I}(V) \neq \mathbb{I}(V \cup \{a\})$ (otherwise $\mathbb{V}(\mathbb{I}(V)) = \mathbb{V}(\mathbb{I}(V \cup \{a\}))$ and so $V = V \cup \{a\}$)

$\Rightarrow \mathbb{I}(V) \neq \mathbb{I}(V) \cap \mathbb{I}(\{a\})$
$\Rightarrow \exists f \in \mathbb{I}(V)$ such that $f \notin \mathbb{I}(\{a\})$
$\Rightarrow \exists f \in \mathbb{I}(V)$ such that $f(a) \neq 0$.

Then define $g(x) = \frac{f(x)}{f(a)} \in \mathbb{C}[x_1, \ldots, x_n]$. Then for any $z \in V$, $g(z) = f(z)/f(a) = 0$ since $f \in \mathbb{I}(V)$. Hence $g \in \mathbb{I}(V)$.

Also $g(a) = f(a)/f(a) = 1$.

b) Let $I$ and $(g)$ be ideals of $\mathbb{C}[x_1, \ldots, x_n]$ with $\mathbb{V}(g) \supseteq \mathbb{V}(I)$.
Let $I = (f_1, \ldots, f_k)$, then:

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$$\mathbb{V}(f_1,\dots,f_k, x_{n+1}g-1) = \mathbb{V}((f_1,\dots,f_k)+(x_{n+1}g-1))$$
$$= \mathbb{V}(f_1,\dots,f_k) \cap \mathbb{V}(x_{n+1}g-1).$$
$$= \mathbb{V}(I) \cap \mathbb{V}(x_{n+1}g-1).$$

Let $z \in \mathbb{V}(I)$, then $z \in \mathbb{V}(g)$ by assumption, and so $g(z)=0$. Hence $x_{n+1}g-1 = -1 \neq 0$, so $z \notin \mathbb{V}(x_{n+1}g-1)$.

Thus $\mathbb{V}((f_1,\dots,f_k, x_{n+1}g-1)) = \emptyset$.

If $(f_1,\dots,f_k, x_{n+1}g-1) \neq \mathbb{C}[x_1,\dots,x_{n+1}]$, then there exists a maximal ideal $\mathfrak{M}$ with $(f_1,\dots,f_k, x_{n+1}g-1) \subseteq \mathfrak{M} \subseteq \mathbb{C}[x_1,\dots,x_{n+1}]$.

So $\mathfrak{M} = (x_1-a_1,\dots, x_{n+1}-a_{n+1})$, and so $\mathbb{V}(\mathfrak{M}) = \{(a_1,\dots,a_{n+1})\}$.

Now $(f_1,\dots,f_k, x_{n+1}g-1) \subseteq \mathfrak{M} \Rightarrow \emptyset \supseteq \mathbb{V}(\mathfrak{M})$ (As $\mathbb{V}$ is inclusion reversing)
$\Rightarrow \mathbb{V}(\mathfrak{M}) = \emptyset$

but this is a contradiction of $\mathbb{V}(\mathfrak{M}) = \{(a_1,\dots,a_{n+1})\}$.
So $(f_1,\dots,f_k, x_{n+1}g-1) = \mathbb{C}[x_1,\dots,x_{n+1}]$.

$\square$

c) By (1), since $1 \in \mathbb{C}[x_1,\dots,x_{n+1}]$, there exist $r_1,\dots,r_k,r_{k+1} \in \mathbb{C}[x_1,\dots,x_n]$

$$1 = r_1 f_1 + \dots + r_k f_k + r_{k+1}(x_{n+1}g-1)$$

If we let $x_{n+1} = 1/g(x_1,\dots,x_n)$, then $1 = r_1(x_1,\dots,\frac{1}{g})f_1 + \dots + r_k(x_1,\dots,1/g)f_k$, (since $f_1,\dots,f_k,g$ have no $x_{n+1}$ term). Each of the $r_i$ will have some negative power of $g$ with maximal absolute value $m_i$. So $g^{m_i}\cdot r_i \in \mathbb{C}[x_1,\dots,x_n]$, and thus if $m = m_1\cdots m_k$, then $p_i := g^m r_i \in \mathbb{C}[x_1,\dots,x_n]$ and so:

$$g^m = g^m(r_1 f_1 + \dots + r_k f_k)$$
$$= p_1 f_1 + \dots + p_k f_k$$
$$\in (f_1,\dots,f_k) = I$$

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5)

Q5. Prove at least one implication from each of the following equivalences.

(a) (**10 marks**) Show that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is injective if and only if $\varphi$ is *dominant*. Recall that a map, $\varphi$, is called dominant if its image, $\varphi(V)$, is dense in $W$.

(b) (**10 marks**) Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if $\varphi$ defines an isomorphism between $V$ and some algebraic subvariety of $W$.

a) $(\Rightarrow)$: Let $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ be injective, then $\ker \varphi^* = \{0\}$.

Suppose for a contradiction that $\varphi(V)$ is not dense in $W$. Then there exists some $x \in W$ and open neighbourhood $U$ of $x$ such that $U \cap \varphi(V) = \emptyset$.

So for all $z \in \varphi(V)$, $z \notin U$ and so $z \in U^c$, which is a c.a.a.v. by definition of the Zariski topology. Thus $U = \mathbb{V}(\{f_i\}_{i \in I})$ for some family of polynomials $\{f_i\}_{i\in I}$.

For all $i \in I$, $z \in \varphi(V)$, $f_i(z) = 0$.

$\Rightarrow f_i(\varphi(v)) = 0$ for all $v \in V$
$\Rightarrow \varphi^*(f_i) = 0$
$\Rightarrow f_i \in \ker \varphi^*$ for all $i \in I$.

There must exist some $j \in I$ such that $f_j \neq 0$, otherwise $U^c = \mathbb{V}(0) = W$, and so $U = \emptyset$, which is a contradiction of $x \in U$. Thus there is some non-trivial polynomial in $\ker \varphi^*$, contradicting injectivity.

$(\Leftarrow)$: Suppose $\varphi^*$ is not injective, then there exists some $0 \neq f \in \ker \varphi^*$

$\Rightarrow \varphi^*(f) = f \circ \varphi = 0$
$\Rightarrow f(\varphi(x)) = 0$ for all $x \in V$

Let $V' = \mathbb{V}(f) \subseteq W$, then since $f(\varphi(x)) = 0$ for all $x \in V$, thus $f(z) = 0$ for all $z \in \varphi(V)$, we have that $\varphi(V) \subseteq V'$.
Note that since $f \neq 0$, $V' \neq W$.

Then $(V')^c$ is an open set (since $V'$ is a c.a.a.v.) and $(V')^c \neq \emptyset$, and we have that $(V')^c$ contains no elements of $\varphi(V)$, as $\varphi(V) \subseteq V'$.

Thus there exists some $v \in (V')^c \subseteq W$ and open neighborhood of $v$ (namely $(V')^c$) such that $(V')^c \cap \varphi(V) = \emptyset$

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b) $(\Rightarrow)$: Suppose that $\varphi$ defines an isomorphism between $V$ and some algebraic subvariety $W'$ of $W$. Then $V \simeq W'$, and so $\mathbb{C}[V] \cong \mathbb{C}[W']$ by Exercise 2.40.

(Note the first isomorphism is an isomorphism of varieties and the second is an isomorphism of $\mathbb{C}$-algebras, hence the different notation).

Indeed $\psi = \varphi^*|_{\mathbb{C}[W']} : \mathbb{C}[W'] \to \mathbb{C}[V]$ defines isomorphism (again by Exercise 2.40).

Moreover, if $i : W' \hookrightarrow W$ is the inclusion map (which is a morphism), then $\psi \circ i^* = \varphi^*$ since:

$$\begin{aligned}(\psi \circ i^*)(f) &= \psi(i^*(f)) \\ &= \psi(f \circ i) \\ &= \varphi^*(f \circ i) \\ &= (f \circ i) \circ \varphi\end{aligned}$$

So $(\psi \circ i^*)(f)(x) = f(i(\varphi(x)))$
$= f(\varphi(x))$ (as $\varphi$ is an isomorphism $\varphi(x) \in W'$)
$= (f \circ \varphi)(x)$
$= \varphi^*(f)(x)$

$\Rightarrow \psi \circ i^* = \varphi^*$

Now $i^*$ is surjective, since for any $f \in \mathbb{C}[W']$, we define $f' : W \to \mathbb{C}$ by extending $f$ from $W'$ to $W$ (with the same functional form, but different domain, this polynomial excists), and then:

$$\begin{aligned}f &= f' \circ i \\ &= i^*(f)\end{aligned}$$

and thus since $\psi$ and $i^*$ are surjective, so is the composition $\varphi^* = \psi \circ i^*$.

$(\Leftarrow)$: Suppose that $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective.

Then by the first isomorphism theorem $\mathbb{C}[W]/\ker \varphi^* \cong \mathbb{C}[V]$.

Since $\mathbb{C}[V]$ is reduced (Theorem 2.38), $\ker \varphi^*$ is radical (Exercise 2.3), so by the Nullstellensatz $\mathbb{I}(\mathbb{V}(\ker \varphi^*)) = \ker \varphi^*$.

$$\begin{aligned}\text{So } \mathbb{C}[\mathbb{V}(\ker \varphi^*)] &= \mathbb{C}[x_1, \dots, x_n] / \mathbb{I}(\mathbb{V}(\ker \varphi^*)) \\ &= \mathbb{C}[x_1, \dots, x_n] / \ker \varphi^* \\ &= \mathbb{C}[W] / \ker \varphi^*\end{aligned}$$

(Since $\mathbb{V}(\ker \varphi^*) \subseteq W$, so the polynomials restricted from $\mathbb{A}^n$ to $\mathbb{V}(\ker \varphi^*)$ are precisely the polynomials

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restricted from $W$ to $\mathbb{V}(\ker\varphi^*)$

$$\cong \mathbb{C}[V]$$

Then by Exercise 2.40, $V \simeq \mathbb{V}(\ker\varphi^*)$. Since $\mathbb{V}(\ker\varphi^*)$ is a c.a.a.v. in $\mathbb{A}^n$ and $\mathbb{V}(\ker\varphi^*) \subseteq W$, $\mathbb{V}(\ker\varphi^*) = \mathbb{V}(\ker\varphi^*) \cap W$, so $\mathbb{V}(\ker\varphi^*)$ is an algebraic subvariety of $W$.