1

a) The closure of a set $A \subseteq \mathbb{A}^n$, denoted $\bar{A}$ is the smallest closed set such that $A \subset \bar{A}$

Equivalently $\bar{A} = \bigcap_{\substack{C_\alpha \supset A \\ C_\alpha \text{ closed}}} C_\alpha$ (is the intersection of all the closed sets containing A.

b)

$$\mathbb{I}(A) = \{ f \in \mathbb{C}[x_1, \dots, x_n] : f(x) = 0 \quad \forall x \in A \}$$

$$\mathbb{V}(\mathbb{I}(A)) = \bigcap_{f_i \in \mathbb{I}(A)} \mathbb{V}(\{f_i\})$$

Now suppose C_α is a ~~closed~~ Zariski closed set $C_\alpha \subset \mathbb{A}^n$ with $A \subset C_\alpha$

We have that $C_\alpha = \bigcap_{j \in J} \mathbb{V}(\{f_j\})$.
Since $A \subset C_\alpha$, we have that $f_j(x) = 0$ $\forall x \in A$, $\forall j \in J$.

Remembering that $\bar{A}$ is the intersection of all such C_α we get:

$$\bar{A} = \bigcap_{\substack{C_\alpha \supset A \\ C_\alpha \text{ closed}}} \bigcap_{j \in J} \mathbb{V}\{f_i\} = \bigcap_{\beta \in B} \mathbb{V}\{f_\beta\}$$, where

$f_\beta(0)$, $\forall x \in A$ $\forall \beta \in B$

So $\bar{A} = \bigcap_{\beta \in B} \mathbb{V}(\{f_\beta\}) = \bigcap_{f_i \in \mathbb{I}(A)} \mathbb{V}(\{f_i\}) = \mathbb{V}(\mathbb{I}(A))$

c)

$B = \{ x \in \mathbb{C} : |x| \leq 1 \}$

Given that $\mathbb{A}^1$ has the cofinite topology and B is an infinite set, the only Zariski closed set containing B is $\mathbb{C}$ itself. So the closure of B in the Zariski topology is $\mathbb{C}$

The closure of B in the Euclidean topology is B itself, as B is closed with respect to the Euclidean topology

2

a)

A collection of sets $\{U_\alpha\}_{\alpha \in A}$ is said to be a cover of a set $E \subset X$ if $\bigcup_{\alpha \in A} U_\alpha \supset E$. It is said to be open if each U_α is open.

A subset $E \subset X$ of a topological space is said to be compact if, for every open cover, there exists a finite subcover.

(i.e. there exists a finite set $A' \subset A$ such that $\bigcup_{\alpha \in A'} U_\alpha \supset E$)

b) First we show that $\mathbb{V}(x^2 - y^3)$ is not compact in the Euclidean topology. To do this, we note that it is not bounded. In fact, the subset of $\mathbb{V}(x^2-y^3)$ with imaginary part 0 is unbounded. So now, since $\mathbb{V}(x^2-y^3)$ must be closed and bounded to be compact (in the Euclidean topology), it is not compact.

Now we show that, since $\mathbb{V}(x^2-y^3)$ is a closed affine algebraic variety, it is compact.

Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of $\mathbb{V}(x^2-y^3)$. Since each U_α is open we can write each $U_\alpha = X\setminus V_\alpha$ with V_α closed.

Now $$\bigcup_{\alpha\in A} U_\alpha = \bigcup_{\alpha\in A}(X\setminus V_\alpha) = X\setminus\bigcap_{\alpha\in A} V_\alpha$$

Since $\bigcap_{\alpha\in A} V_\alpha$ is an intersection of closed sets, it is closed and is a closed affine algebraic variety. By a result in the lecture notes, it can be written as the intersection of finitely many hypersurfaces

So $$\bigcap_{\alpha\in A} V_\alpha = \bigcap_{i=1}^{n} H_i$$

(Before I thought I could pick n members of A such that $V_{\alpha_i}\subset H_i$. If so we would be done as

$$\bigcap_{\alpha\in A} V_\alpha \subseteq \bigcap_{i=1}^{n} V_{\alpha_i} \subseteq \bigcap_{i=1}^{n} H_i \quad \text{and} \quad \bigcap_{\alpha\in A} V_\alpha = \bigcap_{i=1}^{n} H_i$$

So we would have equality and $\{X\setminus V_{\alpha_i}\}_{i=1}^{n}$ would be our subcover

This doesn't seem obvious now. So here is a different proof

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover. Suppose there doesn't exist a finite subcover

Now (maybe using axiom of choice) choose a countable collection $\{U_i\}_{i=1}^{\infty} \subset \{U_\alpha\}_{\alpha \in A}$ and let

$U^k = \bigcup_{i=1}^{k} U_i$. Since each U_i is open, each U^k is open.

Now the sets $X \setminus U^k$ form a decreasing sequence of closed sets

$X \setminus U^1 \supseteq X \setminus U^2 \supseteq \dots$

This corresponds to an increasing chain of ideals

$\mathbb{I}(X \setminus U^1) \subseteq \mathbb{I}(X \setminus U^2) \subseteq \dots$

But since $\mathbb{C}[x, y, \ldots]$ is Noetherian, there exists $K \in \mathbb{N}$ such that $\mathbb{I}(X \setminus U^k) = \mathbb{I}(X \setminus U^K)$ for all $k > K$

And so if the functions defining these Zariski closed sets are the same, the sets are the same. So $\exists$ $K \in \mathbb{N}$ such that $X \setminus U^k = X \setminus U^K$ for all $k > K$

Since this is true for any collection $\{U_i\}_{i=1}^{\infty} \subset \{U_\alpha\}_{\alpha \in A}$

$\bigcup_{\alpha \in A} U_\alpha = \bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{K} U_i$

3

a)

$W = \{(0,0)\} \subseteq \mathbb{A}^2$

$\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ defined by $\varphi(x,y) = ((x-1)(x-2), y)$

Clearly, as $\mathbb{A}^2$ is a singleton it is irreducible. But $\varphi^{-1}(W) = \{(1,0)\} \cup \{(2,0)\}$ and it is therefore reducible

b)

We will prove this by showing that if $\bar{X}$ is reducible, then so is X

So suppose $\bar{X}$ is reducible with $\bar{X} = X_1 \cup X_2$ such that $X_1 \neq \bar{X}$, $X_2 \neq \bar{X}$ and both X_1, X_2 closed in $\bar{X}$. Since X_1, X_2 are closed in $\bar{X}$ and $\bar{X}$ is closed, they are closed in Y. So $X_1 \cap X$ and $X_2 \cap X$ are closed in X. Now suppose that $(X_1 \cap X) = X$. This would give a contradiction as we would have $X \subsetneq X_1 \subsetneq \bar{X}$. And X_1 would be a closed set $X_1 \subsetneq X$ with $X \subset X_1$. So $(X \cap X_1) \neq X$. The same argument can be made for $(X \cap X_2)$ and so X is reducible.

c)

Let $\varphi : V \to W$ be a morphism and let V be irreducible

Now let Y_1, Y_2 be closed sets of W

such that $Y_1 \cup Y_2 = W$. Since morphisms are continuous, we get $\varphi^{-1}(Y_1)$ and $\varphi^{-1}(Y_2)$ are closed. Also if $Y_1 = W$ then $\varphi^{-1}(Y_1) = V$ and $Y_2 = W \Rightarrow \varphi^{-1}(Y_2) = V$.
So if W is reducible V must be too.
Since V is irreducible, W is as well.

~~def~~

Now let V have dimension n and $\varphi : V \to W$ be an isomorphism.
~~We shall prove that for any chain~~

Let there be a chain

$$V \supsetneq V_{n-1} \dots \supsetneq V_1 \supsetneq \{pt\}$$

Since φ is an isomorphism we have a $\psi : W \to V$ such that $\psi \circ \varphi = id_V$
Now, for any V_i in this chain, and since ψ is a morphism (and continuous) $\psi^{-1}(V_i)$ is closed. But since $\psi \circ \varphi = id_V$ $\psi^{-1}(V_i) = \varphi(V_i)$.
Finally since φ and ψ are well defined the sets $\varphi(V) = W$, $\varphi(V_{n-1}), \dots, \varphi(V_1), \varphi(\{pt\})$ form a chain such that

$$W \supsetneq \varphi(V_{n-1}) \supsetneq \dots \supsetneq \varphi(V_1) \supsetneq \{pt\}$$

So W has dimension at least n. Since ψ is an isomorphism from W to V we can make the same argument in the opposite direction and V has dimension at least $\dim(W)$. So $\dim(W) = n = \dim(V)$

d)

$$\mathbb{V}(zx-y,\ y^2-x^2(x+1))$$
$$= \mathbb{V}(zx-y) \cap \mathbb{V}(y^2-x^2(x+1))$$

Note that if $(x,y,z) \in \mathbb{V}(zx-y)$, then $y = zx$

So
$$\mathbb{V}(zx-y) \cap \mathbb{V}(y^2-x^2(x+1))$$
$$= \mathbb{V}(zx-y) \cap \mathbb{V}((zx)^2-x^2(x+1))$$
$$= \mathbb{V}(zx-y) \cap \mathbb{V}(x^2(z^2-x-1))$$
$$= \mathbb{V}(zx-y) \cap (\mathbb{V}(x^2) \cup \mathbb{V}(z^2-x-1))$$
$$= (\mathbb{V}(zx-y) \cap \mathbb{V}(x)) \cup (\mathbb{V}(zx-y) \cap \mathbb{V}(z^2-x-1))$$

Now note that the polynomials $zx-y$, x and z^2-x-1 are irreducible and therefore generate prime ideals in $\mathbb{C}[x,y,z]$ so $\mathbb{V}(zx-y)$, $\mathbb{V}(x)$ and $\mathbb{V}(z^2-x-1)$ are irreducible

Giving us that

$\mathbb{V}(zx-y) \cap \mathbb{V}(x)$ and $\mathbb{V}(zx-y) \cap \mathbb{V}(z^2-x-1)$ are the irreducible components of V.

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4

a) Choose $h \in \mathbb{I}(V)$. Now define f by $f(x) = h(x)\left(x_1 - a_1 + \frac{1}{h(a)}\right)$. Since $h \in \mathbb{I}(V)$ f must be 600. Also, $f(a) = h(a)\left(0 + \frac{1}{h(a)}\right)$
$= 1$

b)

i)

~~Aclli~~

We aim to show that we can write
$$1 = h_{n+1}(x_1, \dots, x_{n+1})(x_{n+1}g(x_1, \dots, x_n) - 1) + \sum_{i=1}^{n} h_i(x_1, \dots, x_{n+1}) f_i(x_1, \dots, x_n)$$

for some $h_1, \dots, h_{n+1} \in \mathbb{C}[x_1, \dots, x_{n+1}]$

Since then for any $p \in \mathbb{C}[x_1, \dots, x_{n+1}]$
$p = p \cdot 1 \in (f_1, \dots, f_n, x_{n+1}g - 1)$ and

$$(f_1, \dots, f_n, x_{n+1}g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}]$$

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ii)

Since $(f_1, \dots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}]$

we can write

(*) $$1 = h_{n+1}(x_1, \dots, x_{n+1})(x_{n+1}g(x_1, \dots, x_n) - 1) + \sum_{i=1}^{k} h_i(x_1, \dots, x_{n+1}) f_i(x_1, \dots, x_n)$$

~~At this point~~ (I'm writing this before part (i) so I may have already proved this)

~~Now if we consider these polynomials as elements of the field of fractions of $\mathbb{C}[x_1, \dots, x_{n+1}]$ this equation (*) still holds.~~

Note that if we let $x_{n+1} = \frac{1}{g(x_1, \dots, x_n)}$ and consider equation (*) with polynomials in the field of fractions of $\mathbb{C}[x_1, \dots, x_n]$ we get

$$1 = 0 + \sum_{i=1}^{k} h_i\left(x_1, \dots, x_n, \frac{1}{g(x_1, \dots, x_n)}\right) f_i(x_1, \dots, x_n)$$

Now let a_i be the lowest common multiple of the powers that $\frac{1}{g(x_1, \dots, x_n)}$ is raised to by h_i and let $m = LCM(a_1, \dots, a_n)$.

Then we can write

$$1 = \left(\sum_{i=1}^{k} p_i(x_1, \dots, x_n) f_i(x_1, \dots, x_n)\right) / g(x_1, \dots, x_n)^m$$

for some polynomials p_i in $\mathbb{C}[x_1, \dots, x_n]$

So $$g^m = \left(\sum_{i=1}^{k} p_i(x_1, \dots, x_n) f_i(x_1, \dots, x_n)\right)$$

and $g^m \in I$

5

a)

First ~~suppose~~ φ^* is injective. This now tells us that for all $f \in \mathbb{C}[W]$ $\varphi^*(f) = 0$ if and only if $f = 0$.
So now the set $\mathbb{I}(\varphi(V))$ of polynomial f such that $f \circ \varphi$ is 0 is just the set $\mathbb{I}(\varphi(V)) = \{0\}$.

~~Let~~ By Q1 (b) $\mathbb{V}(\mathbb{I}(\varphi(V))) = \overline{\varphi(V)}$
~~But~~ So $\overline{\varphi(V)} = \mathbb{V}(\{0\}) = W$. φ is dominant

Now suppose that $\varphi(V)$ is dense in W
Note that if ~~$f\circ\varphi$~~ f $f(y) = 0$ $\forall y \in \varphi(V)$ then $f(y) = 0$ $\forall y \in \overline{\varphi(V)}$.
Otherwise $\overline{\varphi(V)} \cap \mathbb{V}(f)$ is a closed set strictly smaller than $\overline{\varphi(V)}$ containing $\varphi(V)$ giving us a contradiction. So $\ker(\varphi^*) = \{0\}$ and φ^* is injective.

b)

Let φ define an isomorphism to some subvariety $D \subset W$. So $\exists\ \varphi^{-1} : D \to V$ such that $\varphi \circ \varphi^{-1} = id_D$ and $\varphi^{-1} \circ \varphi = id_V$

Choose $f \in \mathbb{C}[V]$ and notice that there exists $f \circ \varphi^{-1} \in \mathbb{C}[D]$ such that $\varphi^*(f \circ \varphi^{-1}) = f \circ \varphi^{-1} \circ \varphi = f \circ id_V = f$. Since φ^{-1} is a morphism, it is a restriction of a polynomial mapping Φ to D. Let Ψ be the restriction of Φ to W and $\exists\ f \circ \Psi \in \mathbb{C}[W]$ such that $\varphi^*(f \circ \Psi) = f$.
φ^* is surjective