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to I. R. Shafarevich on his 80*th birthday*

c-fans and Newton polyhedra of algebraic varieties

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Abstract. To every algebraic subvariety of a complex torus there corresponds a Euclidean geometric object called a c-fan. This correspondence determines an intersection theory for algebraic varieties. c-fans form a graded commutative algebra with visually defined operations. The c-fans of algebraic varieties lie in the subring of rational c-fans. It seems that other subrings may be used to construct an intersection theory for other categories of analytic varieties. We discover a relation between an old problem in the theory of convex bodies (the so-called Minkowski problem) and the ring of c-fans. This enables us to define a correspondence that sends any algebraic curve to a convex polyhedron in the space of characters of the torus.

§ 1. Introduction

1.1. The rings of rational and Euclidean c-fans. Consider the algebraic subvarieties of the complex torus $(\mathbb{C} \setminus 0)^n$ (from now on denoted by $\mathbb{T}$) given by sets of common zeros of non-degenerate systems of equations. The invariants of such subvarieties can be described in terms of the geometry of Newton polyhedra [1]–[7]. If we take another non-degenerate system of equations of the variety, then the change of Newton polyhedra is practically uncontrollable (except the case of a hypersurface). However, the product of the Newton polyhedra in the ring of c-fans (defined below) turns out to be invariant (c is the first letter of the word "current").

For every d-dimensional algebraic subvariety M of the torus we define a Euclidean geometric object Σ_M, which is called a d-*dimensional* c-*fan* (see § 3). A c-fan is something very close (see Definition 1 in § 2) to a d-dimensional fan of rational cones (see Remark 1 in § 1.5) whose cones of dimension d are taken with integer multiplicities. These multiplicities must satisfy certain linear conditions. The c-fan Σ_M has positive multiplicities, which imposes strong restrictions on the geometry of the fan. The c-fan of a hypersurface is something very close (see § 1.4) to its Newton polyhedron. c-fans form a commutative graded $\mathbb{Z}$-algebra with visually defined operations. If M is given by the set of common zeros of a non-degenerate system with Newton polyhedra, then Σ_M is the product of these polyhedra. The map Σ_M determines an intersection theory for algebraic varieties with values in the ring of c-fans (see § 3).

The c-fans described above are said to be *rational*. They lie in a space with an integer lattice and consist of rational cones with an integer multiplicities.

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One can construct the ring of c-fans in an abstract real vector space E. (In terms of Newton polyhedra, this corresponds to the passage from polynomials to exponential sums with real exponents [15]–[19].) Then the multiplicity of a cone K must be defined as an odd volume form on $E/E(K)$, where $E(K)$ is the subspace of E generated by the cone (see § 7).

If E is endowed with an integer lattice and K is a rational cone, then determining an odd volume form on $E/E(K)$ is equivalent to determining the multiplicity of K as the integral of this form over the fundamental cell of the lattice in $E/E(K)$. Abstract c-fans supported on rational fans of cones with integer multiplicities form the $\mathbb{Z}$-algebra of rational c-fans. The Euclidean metric on E also enables one to replace the odd form on $E/E(K)$ by a real-valued multiplicity, which equals the integral of the form over the unit cube of $E/E(K)$. This makes the description of operations on c-fans more simple and geometric. Such c-fans are said to be *Euclidean*. Precise definitions of c-fans and operations under them are given in §§ 2, 4–6. The properties of rational and Euclidean c-fans are described in a parallel way. To distinguish them, we use the expressions "in the rational case" or "in the Euclidean case".

Rings of c-fans are close in meaning and structure to the cohomology rings of toric varieties [7]. In particular, the $\mathbb{Z}$-algebra of rational c-fans was constructed in [27] in the context of toric varieties as a realization of the toric Chow ring, that is, the inductive limit of the cohomology rings of non-singular toric varieties. (The multiplicities of cones are called *Minkowski weights* in [27].)

The correspondence $M \to \Sigma_M$ may be constructed either directly from an ideal of the variety or as an asymptotic density of the variety (see § 3). For a description of Σ_M in terms of toric varieties see § 1.5.

There are now various realizations of the toric Chow ring and its Euclidean analogues [7], [8], [11] [26], [27]. Using c-fans and their description as differential forms (see § 1.3 below), we get another realization and another Euclidean analogue of the toric Chow ring. Some possible applications of c-fans are described in §§ 1.2, 1.3.

1.2. The Minkowski problem and Newton polyhedra of algebraic varieties. The cohomology rings of toric varieties and their Euclidean analogues turned out to be closely related to McMullen's conjecture on the Dehn–Sommerville numbers of simple polyhedra and played an important role in its proof [10]–[13]. The notion of a c-fan reveals the relation of this problem to another problem on the geometry of convex bodies. The so-called "Minkowski problem for intermediate curvatures" in the piecewise-linear set-up ([22]–[24]) is directly stated as a question on the structure of the ring of c-fans (see § 8). We now make an assertion about this ring which is a restatement of the results of [20], [21].

Let $\mathcal{R}^k_+$ be the set of homogeneous $(n-k)$-dimensional abstract c-fans whose multiplicities are positive and whose supports do not lie in a proper subspace of E. (As explained in § 1.4 and § 7, $\mathcal{R}^1_+$ consists of c-fans corresponding to convex polyhedra of full dimension in the space E^*.)

Assertion 1. *The kth power map $d_k\colon \mathcal{R}^1_+ \to \mathcal{R}^k_+$ is injective for $1 \leqslant k \leqslant n-1$ and bijective for $k = 1, n-1$.*

If $k = n-1$, this assertion is just Minkowski's theorem on convex polyhedra (see § 8). This theorem asserts that sending each convex polyhedron in $\mathbb{R}^n$ $(n > 1)$ to the set of exterior normal vectors to $(n-1)$-dimensional faces with the length of each vector being equal to the area of the face, gives a one-to-one correspondence between convex polyhedra of full dimension (up to translation) and non-degenerate (that is, not lying in a proper subspace) sets of mutually non-codirected vectors with zero sum.

The Minkowski problem for intermediate curvatures (in the piecewise-linear set-up) asks for a description of the images of the maps d_k. In other words, one must describe the powers (see § 1.4 below) of convex polyhedra.

A convex polyhedron Δ is called a *Newton polyhedron* of an $(n-k)$-dimensional algebraic variety M if the c-fan Σ_M is the kth power of Δ. One can state the Minkowski problem in the language of algebraic geometry as follows. *Which varieties have a Newton polyhedron, and which polyhedra, other than those with vertices at integer points, can be Newton polyhedra of algebraic varieties*? A Newton polyhedron of an $(n-k)$-dimensional algebraic variety (see § 1.4) has rational slopes of all faces and integer-valued areas of k-dimensional faces when measured in terms of the corresponding k-dimensional integer lattices.

Minkowski's theorem implies that an algebraic curve has a Newton polyhedron provided that the curve does not lie in a translate of a subtorus of smaller dimension (see § 8). A non-degenerate intersection of k hypersurfaces with the same Newton polyhedron Δ also has a Newton polyhedron, which coincides with Δ.

Besides the geometric (old) and algebraic-geometric versions, we also have the following arithmetic version of the Minkowski problem, which arises in the context of c-fans and occupies an intermediate position. What can be said about a convex polyhedron if we know that its kth power $(1 < k < n)$ is a rational c-fan? For example, take a polyhedron with vertices at integer points and multiply it by a kth root of an integer.

1.3. The ring generated by convex bodies. One can regard c-fans as currents (that is, functionals on the space of smooth compactly supported differential forms) on the complexification of E (see § 7). The construction of the c-fan Σ_M as a current is described in § 3 (Theorem 1).

Each polyhedron generates the current dd^cH, where H is its supporting function extended to $E+\sqrt{-1}E$ by the identity on the imaginary part. In Theorem 3 we shall see that abstract c-fans (regarded as currents) form a ring generated by finite linear combinations of the monomials $dd^cH_1 \wedge \cdots \wedge dd^cH_k$, where the H_i are continuous piecewise-linear functions on E extended to $E + \sqrt{-1}E$ by the identity on the imaginary part. This ring is appropriate for constructing an intersection theory for varieties defined by the zeros of finite systems of exponential sums with real exponents [15]–[19].

The $\mathbb{Z}$-algebra of rational c-fans is generated by the monomials whose H_i take integer values on the integer lattice of E. This can be proved by using the description of the cohomology ring of a toric variety (see § 1.5 below). We shall only prove in § 7 that c-fans belong to the $\mathbb{Q}$-algebra generated by the monomials whose H_i take rational values on the integer lattice.

If we assume that the H_i are the supporting functions of arbitrary convex bodies (extended by the identity on the imaginary part), we get the ring generated by convex bodies in the dual space E^*. This ring may be appropriate for constructing an analogue of intersection theory for varieties that are common roots of systems of Fourier transforms of compactly supported distributions. Here the role of the "Newton polygon of a distribution" is played by the convex hull of its support.

If we assume that the H_i are arbitrary piecewise-linear functions on a complex vector space, we get a ring that seems to be appropriate for constructing an intersection theory for varieties that are common roots of systems of exponential sums with complex exponents ([17], [18]).

1.4. Examples. Here we describe the correspondence $M \to \Sigma_M$ for varieties of dimension 0, 1, $n-1$, n, state the relation between $(n-1)$-dimensional c-fans and convex polyhedra, and describe an algorithm for taking powers of polyhedra.

Example 1 (dimensions 0, n). If M consists of finitely many points with multiplicities, then Σ_M is the zero point with multiplicity equal to the number of points in M. If M consists of m copies of $\mathbb{T}$, then Σ_M is the whole space with multiplicity m.

Example 2 (curves). If M is a curve, then Σ_M consists of finitely many rational rays and the zero point. The curve tends to infinity in the directions of these rays. The multiplicity of a ray is the number of points of M at infinity in the corresponding direction.

A rational ray with multiplicity m can be identified with its minimal integer vector multiplied by m. The sum of all vectors corresponding to a curve is zero (compare the multiplicity restrictions in § 2). Thus the c-fan of a curve is a finite set of integer vectors with zero sum.

For planar curves, these vectors are sides of a convex polygon (the Newton polygon of the curve), which is determined up to a translation. The product of two c-fans is the zero point with multiplicity equal to twice the mixed area of the corresponding polygons. Hence the algorithm of taking the square of such a c-fan is just the algorithm of computing the area of a convex polygon in terms of its sides.

Example 3 (hypersurfaces and powers of polyhedra). Given a convex polyhedron Λ with vertices at integer points (or any convex polyhedron in the Euclidean case), we have an $(n-1)$-dimensional c-fan Σ_Λ. The c-fan of a hypersurface is constructed from its Newton polyhedron.

The support of the c-fan Σ_Λ is the fan formed by cones that are dual to positive-dimensional faces of Λ. Every cone of highest dimension is dual to an edge of Λ. The multiplicity of this cone equals the number of integer points on the dual edge (or the length of the edge in the Euclidean case) minus one. In § 2 we prove that Σ_Λ with these multiplicities is indeed a c-fan (Proposition 3).

Let Σ_Λ^k be the $(n-k)$-dimensional fan of cones that are dual to faces of Λ of dimension at least k, with the multiplicity of an $(n-k)$-dimensional cone being equal to $k!$ times the area of the dual k-dimensional face. (In the rational case, the area is measured in terms of the k-dimensional integer lattice.) By definition, $\Sigma_\Lambda = \Sigma_\Lambda^1$ and $\Sigma_\Lambda^k = 0$ for $k > n$. In § 7 we prove that Σ_Λ^k is the kth power of Σ_Λ in the ring of c-fans (Proposition 12). We shall sometimes call the c-fan Σ_Λ^k the *kth power of* Λ.

It follows that the n-th power Σ_Λ^n of a polyhedron Λ is the point 0 with multiplicity equal to $n!$ times the volume of Λ. Hence the product of n polyhedra is the zero point with multiplicity equal to $n!$ times their mixed volume (compare [3]).

Example 4 (c-fans of polyhedra and piecewise-linear functions). Let E be an n-dimensional vector space with an integer lattice or a metric. A continuous function on E is said to be *piecewise-linear* if it is linear on every cone of some complete fan of cones. If there is an integer lattice, we also assume that the function takes integer values on it.

Let H be a piecewise-linear function and $\mathcal{K}$ the corresponding fan of cones. Assigning multiplicities by the rule described below, we get the structure of a c-fan on the $(n-1)$-skeleton of $\mathcal{K}$. The function H is the supporting function of a convex polyhedron if and only if these multiplicities are non-negative. Conversely, any $(n-1)$-dimensional c-fan can be obtained in this way. The function is uniquely determined by the c-fan up to adding a linear function. (These assertions are proved in § 7.)

Here is the rule for assigning multiplicities. Consider an $(n-1)$-dimensional cone K and the adjacent n-dimensional cones A_1, A_2 of the fan $\mathcal{K}$. Let h_1, h_2 be the points of E^* that determine the function H on the cones A_1, A_2. The modulus of the multiplicity of K is equal to the (integer or Euclidean) length of the vector $h_2 - h_1$. The hyperplane of K carries two co-orientations. The first is given by the ordered pair of half-spaces containing A_1 and A_2. The second is given by the vector $h_2 - h_1$. The sign of the multiplicity of K is $+$ if these co-orientation coincide and $-$ otherwise.

1.5. The connection with toric varieties.

Remark 1. We recall that a finite set of polyhedral convex cones of dimensions from 0 to d in a real vector space is called a *d-dimensional fan of cones* (in a space with a fixed integer lattice, the cones are assumed to be *rational*) if

1) all cones are open subsets of the subspaces they generate,
2) each face of every cone is also a cone of the fan,
3) the intersection of the closures of any two cones is their common face.

The cone of lowest dimension in a fan is always a subspace. A fan is said to be *sharp* if this subspace is zero-dimensional. The support of a fan is the union of its cones. A fan is said to be *complete* if its support is the whole space. A cone K of a sharp rational fan is said to be *simple* if the primitive integer vectors lying on the edges form a basis for the integer lattice in the space $E(K)$. A fan of simple cones is said to be *simple*.

A sharp rational fan of cones determines a toric variety [1], [7]. If the fan is complete (resp. simple), then the variety is compact (resp. non-singular).

The algebraic-geometric part of the results of this paper is interpreted in the language of toric geometry: we consider the toric cohomology of a variety M, express it in terms of c-fans, and compute it via Newton polyhedra.

The closure of M in a non-singular toric variety determines a cycle, and intersection with this cycle defines a cohomology class. These classes together form an element $\sigma(M)$ in the inductive limit of the cohomology rings of a projective system of non-singular toric varieties, that is, an element of the toric Chow ring [7].

If M, N are varieties given by the common zero sets of two non-degenerate systems of k equations with the same sets of Newton polyhedra [4], then $\sigma(M) = \sigma(N)$. Hence the set of convex polyhedra with vertices at integer points carries a symmetric polyadditive k-form $\sigma(\Delta_1, \dots, \Delta_k)$ with values in the homogeneous component of the toric Chow ring. This form is equal to the product of polyhedra in the ring of c-fans defined below (which is isomorphic to the Chow ring). If $k=n$, then $\sigma(M)$ is an integer equal to the number of points of the finite set M, and $\sigma(\Delta_1, \dots, \Delta_n)$ is $n!$ times the mixed volume of the polyhedra [4].

The c-fan Σ_M may be defined in the language of toric varieties as follows. A sharp fan is said to be *sufficient* for a d-dimensional variety M if the closure of $M \subset \mathbb{T}$ in the corresponding toric variety is compact. A fan is sufficient for M if and only if its support contains the set $\Re_M$ (defined in § 3).

We consider a sharp fan with support $\Re_M$ and the corresponding toric variety. (As remarked in § 3, such a fan exists, and each of its cones is contained in one of the truncation chambers.) We assume that this fan is simple (every fan can be refined to a simple one).

Let K be a d-dimensional cone of the fan and F_K the closure of the $(n-d)$-dimensional pasted-in torus corresponding to K. We put the multiplicity of K equal to the intersection number of the closure of M and F_K. One can verify that cones lying in one truncation chamber have equal multiplicities, and the multiplicity restrictions in § 2 hold. If M is equidimensional, the resulting c-fan coincides with the c-fan Σ_M (which is constructed in § 3). Moreover, one can obtain Σ_M by a direct geometric construction as the "asymptotic density" of the variety (Theorem 1 in § 3).

We now explain the relation between the ring of c-fans and the cohomology ring of toric varieties. Let $\mathcal{K}$ be the fan of dual cones for a simple polyhedron Δ, and let X be the corresponding projective non-singular toric variety. For every k-dimensional cone $K \in \mathcal{K}$ we have a $2(n-k)$-dimensional cycle F_K, which is defined to be the closure of the pasted-in torus corresponding to K. Hence to every element $\sigma \in H^{2(n-k)}(X, \mathbb{Z})$ there corresponds a set of multiplicities $\{m(K) = \sigma(F_K)\}$ assigned to the highest-dimensional cones of the k-skeleton of $\mathcal{K}$. The relation between the ring of c-fans and the cohomology of toric varieties explains the following fact (see, for example, [27]). *The sets of multiplicities distinguished in this way are the sets that satisfy the multiplicity restrictions of* § 2.

Thus one can use the cohomology of toric varieties to verify that the operations are well defined and the ring axioms hold for rational c-fans. However this indirect approach leads to geometric losses.

The layout of the paper is as follows. In § 2 we define c-fans. In § 3 we construct the map $M \to \Sigma_M$ and give a preliminary justification of the intersection theory. In §§ 4–6 we consider the sum, projections, and product of c-fans. § 7 contains the representation of Euclidean c-fans by currents and some properties of the ring generated by convex bodies. In § 8 we consider abstract c-fans and the Minkowski problem.

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§ 2. Definition of a c-fan

Let $\mathcal{K}$ be a d-dimensional fan of cones (with real-valued multiplicities assigned to the d-dimensional cones) that lies in the n-dimensional real vector space E endowed with either a Euclidean metric or an integer lattice. In the rational case, all subspaces of E appearing below are assumed to be rational (that is, generated by vectors of the integer lattice), and the multiplicities of cones are assumed to be integers.

We call $\mathcal{K}$ a *d-dimensional pre-*c*-fan* if

1) each of its cones is a face of some d-dimensional cone (this property is referred to as the *homogeneity condition*);

2) the multiplicities of the cones satisfy the *multiplicity restrictions* stated below.

The *support* of a d-dimensional pre-c-fan is the closure of the union of those d-dimensional cones whose multiplicities are *non-zero*.

To state the multiplicity restrictions, we define a function $\xi(A, B)$ on pairs of subspaces of E to be the area distortion coefficient for the space $A/(A \cap B)$ under the projection $\pi\colon E/(A \cap B) \to E/B$. This means that, given any domain $U \subset A/(A \cap B)$, the area of $\pi(U)$ equals the area of U multiplied by $\xi(A, B)$ (areas in quotient spaces are measured using either quotient metrics or quotient lattices). If $A \cup B$ belongs to a proper subspace of E, then $\xi(A, B) = 0$ (in this case $\dim A/(A \cap B) < \dim E/B$).

We list without proof some simple properties of ξ.

Lemma 1. 1) $\xi(A, B) = \xi(B, A)$.

2) *If* $\dim A + \dim B = \dim E$, *then* $\xi(A, B)$ *equals the volume of the parallelepiped defined by the fundamental cells of the lattices or* (*in the Euclidean case*) *the unit vectors of these subspaces.*

3) *The function* $\Xi(A, B, C) = \xi(A, B)\xi(A \cap B, C)$ *is invariant under permutations of the arguments.*

We extend ξ to shifted subspaces by putting $\xi(a + A, b + B) = \xi(A, B)$. Given a d-dimensional fan $\mathcal{K}$ whose cones of dimension d are endowed with multiplicities $m(K)$, we define a function on the set of $(n-d)$-dimensional affine subspaces of E by setting

$$\xi_{\mathcal{K}}(A) = \sum \xi(A, E(K))m(K),$$

where the sum is taken over all highest-dimensional cones of the fan that intersect A, and $E(K)$ is the d-dimensional subspace of E generated by the cone K.

The multiplicity restrictions. *For every* $(n-d)$*-dimensional subspace* A *the function* $\xi_{\mathcal{K}}$ *is constant on the set of all affine subspaces parallel to* A *which do not intersect the* $(d-1)$*-skeleton of* $\mathcal{K}$.

Two d-dimensional pre-c-fans with the same support are said to be *equivalent* if any two cones with d-dimensional intersection have equal multiplicities.

Definition 1. A *d-dimensional* c*-fan* (or a *homogeneous* c*-fan of degree* $n-d$) is an equivalence class of d-dimensional pre-c-fans. A c*-fan* is a formal sum of homogeneous c-fans of different degrees.

Remark 2. Let $\mathbb{T}_1$, $\mathbb{T}_2$ be subtori in $\mathbb{T}_0$. In general, $\mathbb{T}_1 \cap \mathbb{T}_2$ is the union of finitely many shifts of some subtorus of codimension $\operatorname{codim}\mathbb{T}_1 + \operatorname{codim}\mathbb{T}_2$. The reason for introducing the function ξ is that the number of these shifts is equal to $\xi(\tau_1, \tau_2)$, where τ_i is the imaginary subspace of the Lie algebra of $\mathbb{T}_i$, and the ambient space E with the fixed integer lattice coincides with τ_0.

Remark 3. The multiplicity restrictions are closely related to the Minkowski conditions for convex polyhedra (see below).

Remark 4. The multiplicity restrictions are crucial in giving the correct definitions of operations with c-fans such as projections or the product (see §§ 5, 6). These restrictions themselves coincide with the conditions under which the product of a c-fan and a subspace of complementary dimension (regarded as a c-fan) is well defined.

Remark 5. The associative property of the product of c-fans is based on the fact that ξ satisfies an "associativity identity" (assertion 3 of Lemma 1), which itself expresses the associativity of the product of subspaces regarded as c-fans (Corollary 2).

Proposition 1. *Let $v_1, \dots, v_N$ be primitive integer vectors or unit vectors* (*in the Euclidean case*) *on the rays of a one-dimensional fan of cones with multiplicities $m_1, \dots, m_N$. Then the multiplicity restriction is equivalent to the equation $m_1v_1 + \dots + m_Nv_N = 0$. Hence one-dimensional* c*-fans with positive multiplicities are just finite sets of mutually non-codirected vectors with zero sum.*

Proof. Indeed, if A is the zero subspace of a functional $\varphi \in E^*$, then it is easy to verify that the multiplicity restriction for the shifts of A is equivalent to the equation $m_1\varphi(v_1) + \dots + m_N\varphi(v_N) = 0$.

The verification of the multiplicity restrictions can be reduced to the case of one-dimensional fans. Indeed, let φ be a vector in a $(d-1)$-dimensional cone of the d-dimensional fan $\mathcal{K}$ with multiplicities $\{m_i\}$. The localization $\mathcal{K}^\varphi$ (see Remark 6 below) is a fan like a *d-dimensional book*, that is, it consists of d-dimensional half-spaces $K_1, \dots, K_N$ (with multiplicities $m_1, \dots, m_N$) whose intersection is the common $(d-1)$-dimensional subspace H. The multiplicity restrictions for such a fan are equivalent to those for the quotient fan (see Remark 6 below), that is, for a one-dimensional fan.

Proposition 2. *The multiplicity restrictions hold for a d-dimensional fan $\mathcal{K}$ if and only if they hold for all localizations of $\mathcal{K}$ that are like d-dimensional books or, equivalently, for all one-dimensional fans that are quotients of these localizations.*

Proof. It suffices to note that

1) the set of affine subspaces that have a fixed $(n-d)$-dimensional base and are disjoint from the $(d-2)$-skeleton of the d-dimensional fan is connected;

2) when the subspace passes through a $(d-1)$-dimensional cone, the value of the function $\xi_{\mathcal{K}}$ remains the same because of the multiplicity restrictions for the corresponding d-dimensional book.

Let $\Lambda \in E^*$ be a full-dimensional convex polyhedron, $v_\Delta \in E$ the unit exterior normal vector, and m_Δ the area of an $(n-1)$-dimensional face Δ. In the rational case we assume that the $(n-1)$-dimensional faces of Λ lie on shifts of rational subspaces.

The areas of the k-dimensional faces of such a polyhedron can be measured in terms of integer lattices in the rational subspaces generated by the faces, and the *normal vector* to a face Δ is understood to be the primitive vector of the integer lattice in E that assumes its maximal value (as a functional restricted to the polyhedron) at all points of Δ. It is easy to verify the equation $\Sigma_\Delta m_\Delta v_\Delta = 0$. We call it the *Minkowski condition* for the polyhedron Λ.

Let us verify the multiplicity restrictions for the c-fan Σ_Λ^k defined in § 1.4.

Proposition 3. Σ_Λ^k *is a* c*-fan.*

Indeed, Proposition 2 implies that the multiplicity restrictions for Σ_Λ^k are just the Minkowski conditions for the $(k+1)$-dimensional faces of Λ (regarded as full-dimensional convex polyhedra in the subspaces generated by them).

Remark 6. We recall two simple constructions that are related to fans and transfer to c-fans. Let φ be a vector in the closure of a cone K endowed with multiplicity m. The cone $K^\varphi = \{k + \mu\varphi \colon k \in K,\ \mu \in \mathbb{R}\}$ with the same multiplicity is called the *φ-localization* of K. If φ does not belong to the closure of K, we put $K^\varphi = \varnothing$ with multiplicity zero. If $\mathcal{K}$ is a fan of cones with multiplicities, then its φ-localization $\mathcal{K}^\varphi$ is the set of cones $\{K^\varphi \colon K \in \mathcal{K}\}$ (this set forms a fan which is not sharp) with the multiplicities described above.

It is easily verified that the φ-localizations of equivalent pre-c-fans are equivalent, and the multiplicity restriction is preserved. We thus get a well-defined operation of φ-localization for a homogeneous c-fan.

Another useful operation is that of taking the quotient of a non-sharp fan. If $E_1 \subset E$ is a subspace (rational for rational c-fans) in the minimal cone of the fan, then the image of the fan under the factorization map $E \to E/E_1$ is again a fan of cones. We say that a c-fan is *non-sharp* if it can be represented as a non-sharp pre-c-fan. It is easily verified that the quotient of a non-sharp c-fan is well defined.

§ 3. The c-fan of a polynomial ideal

If we put aside the justification, then the simplest definition of the object Σ_M (corresponding to an equidimensional algebraic variety $M \subset \mathbb{T}$) is in terms of currents on the Lie algebra of $\mathbb{T}$. (The interpretation of c-fans as currents is contained in § 7.) Let $\mathcal{M}_k$ be the current of integration over the exponential pre-image of the algebraic variety formed by the kth roots of the elements of the torus that belong to M.

Theorem 1. *Suppose that* $\dim M = d$. *Then* $\lim_{k\to\infty} \mathcal{M}_k/k^{n-d} = \Sigma_M$.

We do not give a proof here. The proof uses Theorem 2 and the interpretation of c-fans as currents.

The construction of the c-fan corresponding to an ideal I of the ring of Laurent polynomials (that is, the ring of regular functions on the torus) was given (without taking account of multiplicities) and justified in [14] (and announced in [15]). Here we recall this construction (without justification) along with some necessary explanations.

Let A be the ring of Laurent polynomials. We denote by E the n-dimensional real space dual to the space containing the character lattice of the torus $\mathbb{T}$ and endow

E with the dual integer lattice. (There are convenient identifications $\mathbb{T} = (\mathbb{C} \setminus 0)^n$ and $E = \operatorname{Re} \mathbb{C}^n$.) The set of characters occurring in a Laurent polynomial f with non-zero coefficients is called the *support of the polynomial.* Given $\varphi \in E$, we define the *φ-degree* of the polynomial as the maximal value of φ on the support of f. The highest φ-homogeneous component of f is called its *φ-truncation* and is denoted by f_φ.

Definition 2. The *φ-truncation of an ideal I in the ring A* is the ideal I_φ generated by the polynomials $\{f_\varphi,\ f \in I\}$. The *φ-truncation of the algebraic variety M which is the zero set of I* is the variety M_φ of zeros of the truncated ideal I_φ.

In Definition 2 and throughout this section we assume that the correspondence $\{\text{ideals}\} \leftrightarrow \{\text{varieties}\}$ takes multiplicities of components into account.

We denote by $\Re_I$ or $\Re_M$ the set of points $\varphi \in E$ such that $I_\varphi \neq A$. The formula $\varphi \sim \psi \Leftrightarrow I_\varphi = I_\psi$ defines an equivalence relation on $\Re_I$. Connected components of the equivalence classes are called the *truncation chambers* of the ideal (or variety).

The set of truncation chambers is finite and their union is closed. Each chamber is an open conical subset of a rational subspace of E. The maximal dimension of a truncation chamber is equal to $\dim M$, and the corresponding truncated variety is the union of finitely many shifts of a subtorus of dimension $\dim M$. This enables one to assign positive integer multiplicities to the highest-dimensional chambers.

There are d-dimensional ($d = \dim M$) fans of cones with support $\Re_M$. Such fans are constructed from certain special systems of generators of I. The proof of the existence of such systems is the main technical result of [14]. It is also true that each truncation chamber consists entirely of the cones of any fan with support $\Re_M$ [14]. Hence the d-dimensional cones of any such fan can be endowed with non-negative integer multiplicities. If M is equidimensional, then every truncation chamber belongs to the closure of a highest-dimensional chamber, and this guarantees the validity of the homogeneity condition for such fans. (See the definition of a pre-c-fan at the beginning of § 2.)

Theorem 2. *Suppose that the variety M is equidimensional and d-dimensional. Then the construction described above determines a d-dimensional* c*-fan Σ_M with support $\Re_M$.*

Apart from the multiplicity restriction, the proof of this theorem is given in [14]. The multiplicity restriction follows from Theorems 3 and 4 of [14] and is a geometric reflection of the fact that each fibre of some branched covering over a connected base contains the same number of points (counting multiplicities). This covering arises when we restrict to M the factorization map of $\mathbb{T}$ over the $(n-d)$-dimensional subtorus whose Lie algebra is generated by the subspace A that occurs in the multiplicity restrictions. (If A is transversal to all d-dimensional chambers, then this map is proper and surjective.)

Remark 7. If $\dim M = 1$ or $n-1$, then the truncation chambers are cones and form a fan of cones. This remains valid for *irreducible* two-dimensional varieties [14].

Remark 8. The truncation chambers are not always convex. For example, if M is the union of two 2-dimensional subtori of a 4-dimensional torus, then there are

3 chambers: the zero point and two 2-dimensional planes without zero, all of multiplicity 1.

Theorem 2 of [14] yields the following proposition.

Proposition 4. *For every $\varphi \in \Re_M$, the φ-localization Σ_M^{φ} of the* c*-fan Σ_M coincides with the* c*-fan Σ_{M_φ} of the truncated variety.*

We list the assertions of intersection theory with values in the ring of c-fans.

1) $\Sigma_{M \bigcup N} = \Sigma_M + \Sigma_N$.

2) $\Sigma_{M \bigcap gN} = \Sigma_M \Sigma_N$ for every $g \in \mathbb{T} \setminus D(M,N)$, where $D(M,N)$ is a proper algebraic subvariety of $\mathbb{T}$ that depends on M and N and becomes empty for pairs (M,N) in general position (see Proposition 5).

3) For every subtorus $\mathbb{T}_1 \subset \mathbb{T}$ of codimension at least d there is a homomorphism of the additive group of homogeneous d-dimensional c-fans to the additive group of homogeneous d-dimensional c-fans that are constructed from the torus $\mathbb{T}/\mathbb{T}_1$ such that the homomorphism is compatible with projections of algebraic varieties.

These assertions follow from the results of [14] and the definitions of operations with c-fans given in §§ 4–6. The properties of the c-fan of an intersection of varieties are briefly described as follows.

Proposition 5. *For any equidimensional algebraic subvarieties M, N of the torus there is a discriminant subvariety $D(M,N) \subset \mathbb{T}$ such that the* c*-fans of the varieties $M \cap (\tau N)$ coincide for all $\tau \notin D(M,N)$. If the truncation chambers of M, N are transversal, then $D(M,N) = \varnothing$.*

Proof. This is obvious when $\dim M + \dim N \leqslant n$. Let $\mathbb{T}_\varphi^M = \{t \in \mathbb{T}\colon tM_\varphi = M_\varphi\}$ be the stabilizer of the truncated variety M_φ. The Lie algebra of the subtorus $\mathbb{T}_\varphi^M$ is the subspace generated by the vectors of the truncation chamber that contains φ. The truncated variety M_φ is the pre-image of some equidimensional subvariety of the quotient torus $\mathbb{T}/\mathbb{T}_\varphi^M$ whose codimension equals the codimension of M [14].

If $\varphi \in \Re_M \cap \Re_N$ is non-zero, then the subtorus $\mathbb{T}_\varphi = \mathbb{T}_\varphi^M \cap \mathbb{T}_\varphi^N$ has the same dimension as the intersection of the truncation chambers of M and N that contain φ. We denote by M^φ, N^φ the images of the truncated varieties under the projection $\mathbb{T} \to \mathbb{T}/\mathbb{T}_\varphi$, and let $D_\varphi(M,N)$ be the pre-image of the discriminant variety $D(M^\varphi, N^\varphi)$. The results of [14] imply that we may use induction on the dimension of the torus to put $D(M,N) = \cup_\varphi D_\varphi(M,N)$, where the sum is taken over a finite set of non-zero vectors φ that has exactly one element in each non-empty pairwise intersection of the truncation chambers of M, N.

For large *admissible* (see § 6) values of $\operatorname{Re}\log\tau$, one easily verifies (using the definitions of § 6) that the c-fan $\Sigma_{M\cap\tau N}$ equals the product $\Sigma_M\Sigma_N$ of the c-fans.

§ 4. The sum of c-fans

Let $\mathcal{K}$ and $\mathcal{L}$ be d-dimensional fans with multiplicities k and l (respectively) that make them into pre-c-fans. The union of their supports decomposes into disjoint subsets of three types:

$$\{K \cap L\colon K \in \mathcal{K},\ L \in \mathcal{L}\}, \qquad \{K \setminus \operatorname{supp}\mathcal{L}\colon K \in \mathcal{K}\}, \qquad \{L \setminus \operatorname{supp}\mathcal{K}\colon L \in \mathcal{L}\}.$$

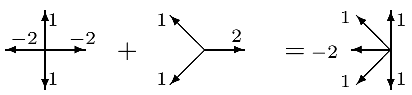
The subsets of the first (resp. second and third) type are cones (resp. open domains of cones). We endow the d-dimensional subsets of these three types with multiplicities $k(K) + l(L)$, $k(K)$, and $l(L)$ respectively.

Let $\mathcal{P}$ be a fan of cones supported on the union of the supports of $\mathcal{K}$ and $\mathcal{L}$ and such that each of the subsets indicated above is a union of cones of $\mathcal{P}$. (Such fans clearly exist.) We endow the d-dimensional cones of $\mathcal{P}$ with the multiplicities of the subsets that contain them. It is easy to see that we get a pre-c-fan $\widetilde{\mathcal{P}}$. If we take another fan $\mathcal{Q}$ with the same properties, then the pre-c-fans $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{Q}}$ are equivalent. Moreover, if we replace $\mathcal{K}$, $\mathcal{L}$ by equivalent pre-c-fans, then the equivalence class of $\widetilde{\mathcal{P}}$ does not change.

We have thus defined the sum of homogeneous c-fans of equal degree. This operation is clearly associative and commutative. If we pass to the inverse element, then the cones are preserved and the multiplicities are multiplied by -1.

We recall that the sum of c-fans of different degrees is defined formally.

Here is an example of adding 1-dimensional c-fans on the plane:



§ 5. Projections of c-fans

Let $\mathcal{K}$ be a d-dimensional pre-c-fan, $\{m\}$ the set of multiplicities of its d-dimensional cones, and $\pi\colon E \to E/H$ the projection, where H is a subspace in E of dimension at most $n-d$. Vectors in $\pi(\operatorname{supp}\mathcal{K})$ are said to be *equivalent* if their pre-images have non-empty intersection with the same cones of the fan $\mathcal{K}$. It is clear that there is a fan $\mathcal{K}_\pi$ with support $\pi(\operatorname{supp}\mathcal{K})$ such that each cone of $\mathcal{K}_\pi$ lies in its equivalence class. We now describe the multiplicities m_π of d-dimensional cones of this fan.

Let K_π be a d-dimensional cone of $\mathcal{K}_\pi$. We put $m_\pi(K_\pi) = \sum \mu(E(K), H)m(K)$, where the sum is taken over all cones K of $\mathcal{K}$ that intersect $\pi^{-1}(K_\pi)$, and μ is the area distortion coefficient of $E(K)$ under π.

If $\dim H = n-d$, then $\mu(E(K), H) = \xi(E(K), H)$, and the multiplicity restrictions for $\{m_\pi\}$ (stating that all multiplicities are equal) are equivalent to the multiplicity restrictions for the initial c-fan $\mathcal{K}$ applied to shifts of the subspace H. Hence we get a c-fan of degree d in the space E/H, that is, a cone which coincides with the whole space E/H, along with its multiplicity. If the initial c-fan has positive multiplicities and the projection of at least one of its d-dimensional cones has no kernel, then this multiplicity is non-zero. If $\dim H < n-d$, then the multiplicity restrictions for $\{m_\pi\}$ follow from those for the pre-c-fan $\mathcal{K}$ applied to shifts of subspaces containing H.

The fan-projection may not satisfy the homogeneity condition since projections of algebraic varieties do not preserve equidimensionality. To guarantee this condition, one must exclude from the fan-projection all cones that do not lie in the closures of d-dimensional cones.

§ 6. The product of c-fans

The product operation is obtained by transferring an operation with fans (called their *stable intersection*) to c-fans. The stable intersection of fans is generally not associative.

Let $\mathcal{A}, \mathcal{B}$ be subsets of the vector space E. A point of $\mathcal{A} \cap \mathcal{B}$ is said to be *stable* if each of its neighbourhoods has non-empty intersection with $\mathcal{A} \cap (\mathcal{B} + e)$ for all sufficiently small $e \in E$. We denote the set of stable intersection points by $\mathcal{A} \cap^s \mathcal{B}$. For example, if $\mathcal{A}, \mathcal{B}$ are subspaces of E, then $\mathcal{A} \cap^s \mathcal{B}$ either coincides with $\mathcal{A} \cap \mathcal{B}$ (if the intersection is transversal) or is empty.

Let $\mathcal{K}$, $\mathcal{N}$ be fans of cones. Their intersection $\mathcal{K} \cap \mathcal{N}$ can be defined as the fan formed by the cones $\{K \cap N \colon K \in \mathcal{K},\, N \in \mathcal{N}\}$. We define a fan $\mathcal{K} \cap^s \mathcal{N}$, which is called the *stable intersection of fans*, as follows:

1) the support of $\mathcal{K} \cap^s \mathcal{N}$ is the stable intersection of the supports $\operatorname{supp} \mathcal{K} \cap^s \operatorname{supp} \mathcal{N}$;

2) the cones of $\mathcal{K} \cap^s \mathcal{N}$ are elements of the set $\{K \wedge^s N = K \cap N \cap (\operatorname{supp}(\mathcal{K} \cap^s \mathcal{N}))\colon K \in \mathcal{K},\, N \in \mathcal{N}\}$.

We easily see that the set $K \wedge^s N$ is either empty or coincides with $K \cap N$. Hence it is a cone of the fan-intersection $\mathcal{K} \cap \mathcal{N}$. The stable intersection of closed sets is closed, whence the union of the cones $K \wedge^s N$ is closed. Therefore $\mathcal{K} \cap^s \mathcal{N}$ is indeed a fan of cones.

The following lemma contains some simple properties of the stable intersection of fans that will be used below.

Lemma 2. 1) *Stable intersection of fans commutes with localization. More precisely, if $K \cap N$ contains a non-zero vector φ, then $\mathcal{K}^\varphi \cap^s \mathcal{N}^\varphi = (\mathcal{K} \cap^s \mathcal{N})^\varphi$, and $K \wedge^s N = (K \cap N) \cap (K^\varphi \wedge^s N^\varphi)$.*

2) *Stable intersection commutes with factorization. More precisely, let K, N be minimal cones of the fans $\mathcal{K}, \mathcal{N}$, and let $\mathcal{T}$ be the stable intersection of the quotients of these fans by $K \cap N$. Then $K \wedge^s N$ is the pre-image of the minimal cone of $\mathcal{T}$ under the factorization map $E \to E/(K \cap N)$.*

3) *If the sum of the dimensions of the fans equals $n + k$, then the dimension of their stable intersection does not exceed k.*

Let $\mathcal{P}$ be a k-dimensional fan, $\mathcal{Q}$ an l-dimensional fan, p and q the multiplicities that make them into pre-c-fans, and $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{Q}}$ the corresponding c-fans. A point $e \in E$ is said to be *admissible* for this pair of fans if the set $D(e) = \operatorname{supp} \mathcal{P} \cap (e + \operatorname{supp} \mathcal{Q})$ consists only of transversal intersections of cones.

When $k + l < n$ we put $\widetilde{\mathcal{P}}\widetilde{\mathcal{Q}} = 0$.

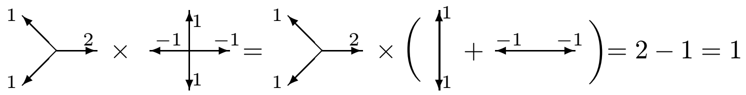
When $k + l = n$ the product $\widetilde{\mathcal{P}}\widetilde{\mathcal{Q}}$ is a 0-dimensional c-fan, that is, the zero point with integer multiplicity $m(0)$. The multiplicities when $k + l = n$ are defined as follows. We take an admissible point $e \in E$ and put

$$m(0) = \sum_{x \in D(e),\, x = P \cap (Q + e)} \xi(E(P), E(Q)) p(P) q(Q).$$

We recall that $E(K)$ is the subspace of E generated by the cone K, and the function ξ is defined in § 2.

Corollary 1. *Let E_1, E_2 be subspaces of complementary dimensions in E. We regard them as* c*-fans. Then their product is the zero point with multiplicity $\xi(E_1, E_2)$.*

Here is an example of the product of 1-dimensional c-fans on the plane:



Proposition 6. *The multiplicity $m(0)$ is independent of the choice of an admissible point e.*

Remark 9. This fact may be interpreted as the existence of an intersection number for c-fans of complementary dimensions.

The proof of Proposition 6 makes use of the following arguments.

1. Consider the set $\mathcal{D}$ of all points $e \in E$ such that $\operatorname{supp}\mathcal{P} \cap (e + \operatorname{supp}\mathcal{Q})$ contains (apart from the intersection points of the highest-dimensional cones) only points of a discrete intersection of cones of total dimension $n-1$.

2. The number $m(0)$ is preserved by any continuous variation of e in the domain of admissible points. Any two admissible points can be connected by a curve that lies in $\mathcal{D}$ and contains only finitely many non-admissible points.

3. The multiplicity restrictions (in the local form; see § 2) guarantee that passing through a non-admissible point of the curve does not change $m(0)$ as a function on the curve.

If $k + l = n + d$ with $d > 0$, then the construction of the product of pre-c-fans consists of

1) *an algorithm determining the multiplicities* of d-dimensional cones of the fan $\mathcal{P} \cap^s \mathcal{Q}$ (the stable intersection of fans as defined above);

2) the verification of the multiplicity restrictions.

Determining the multiplicities for $k + l = n + d$, $d > 0$, reduces to the case $k + l = n$ by means of localization and factorization. Suppose that K is a d-dimensional cone of the fan $\mathcal{P} \cap^s \mathcal{Q}$, $\varphi \in K$, and $\mathcal{P}^\varphi$, $\mathcal{Q}^\varphi$ are the localizations of the fans $\mathcal{P}$, $\mathcal{Q}$. The minimal cones of these fans contain the subspace $E(K)$. We denote by $\mathcal{P}^\varphi_f$, $\mathcal{Q}^\varphi_f$ the quotients of the localized fans by the subspace $E(K)$. According to § 2, they are pre-c-fans in $E/E(K)$ if we regard them as fans with multiplicities. Since the fan $(\mathcal{P} \cap^s \mathcal{Q})^\varphi_f$ is the zero point, we have $\mathcal{P}^\varphi_f \cap^s \mathcal{Q}^\varphi_f = (0)$ by Lemma 2. The multiplicity $m(0)$ in this case was constructed above. We assign this multiplicity to K.

We now verify the multiplicity restrictions for $k + l = n + d$, $d > 0$, in the local form (Proposition 2) or, in other words, in a neighbourhood of any $(d-1)$-dimensional cone H of the fan $\mathcal{P} \cap^s \mathcal{Q}$. We choose $\varphi \in H$ and replace $\mathcal{P}$, $\mathcal{Q}$ first by their φ-localizations, and then by the quotients by the space $E(H)$. This reduces the verification of the restrictions to the case $k + l = n + 1$, when the fan $\mathcal{P} \cap^s \mathcal{Q}$ is one-dimensional.

In this case it suffices (compare Proposition 1) to verify the multiplicity restrictions for shifts of a hyperplane A which is transversal to every non-zero cone of the fans $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{P} \cap^s \mathcal{Q}$. Let $h \in E^*$ be an equation of the hyperplane A. If e is an admissible point, then the set $D(e) = \operatorname{supp}\mathcal{P} \cap (e + \operatorname{supp}\mathcal{Q})$ consists of transversal intersections of cones with sum of dimensions either n or $n+1$, whence $D(e)$ is a polygonal arc with finitely many edges. Each edge is the intersection of cones with sum of dimensions $n+1$. We endow the edge $\lambda = P \cap Q$ with multiplicity $m(\lambda) = \xi(E(P), E(Q))$, where ξ is the function on pairs of subspaces defined in § 2.

Lemma 3. *Let γ be a vertex of the polygonal arc $D(e)$. We denote by $\{\alpha_-\}$ (resp. $\{\alpha_+\}$) the set of edges of $D(e)$ that connect γ with the domain of larger (resp. smaller) values of h. Then*

$$\sum_{\alpha_-} m(\alpha_-)\xi(E(\alpha_-), A) = \sum_{\alpha_+} m(\alpha_+)\xi(E(\alpha_+), A).$$

Proof. Suppose that γ is the intersection of cones $P \in \mathcal{P}$, $Q \in \mathcal{Q}$ of dimensions $k-1$, l. Let $\{P_+\}$ (resp. $\{P_-\}$) be the set of k-dimensional cones of the fan $\mathcal{P}$ containing those edges of $D(e)$ that connect γ with the domain of larger (resp. smaller) values of h.

Using assertion 3 of Lemma 1, we can rewrite the terms on both sides as $m(\alpha_\pm)\xi(E(\alpha_\pm), A) = \xi(A, E(Q))\xi(A \cap E(Q), E(P_\pm))$. Dividing by $\xi(A, E(Q))$, we get the local multiplicity restrictions for the fan $\mathcal{P}$ (Proposition 2):

$$\sum_{P_+} \xi(A \cap E(Q), E(P_+)) = \sum_{P_-} \xi(A \cap E(Q), E(P_-))$$

in a neighbourhood of the cone P for the subspace $A \cap E(Q)$.

The intersection of $D(e)$ with the domain of sufficiently large positive values of h consists of finitely many rays $\{\kappa_+\}$. (These rays are parts of edges of the polygonal arc.) They are parallel to one-dimensional cones of the fan $\mathcal{P} \cap^s \mathcal{Q}$ that lie in the half-space $h > 0$. We put $m_+ = \sum_{\kappa_+} m(\kappa_+)\xi(E(\kappa_+), A)$.

The algorithm for determining the multiplicities in a product of c-fans implies that the multiplicity restrictions to be verified reduce to the equation $m_+ = m_-$, which follows from Lemma 3. Here m_- is defined similarly to m_+, using the domain of sufficiently large *negative* values of h.

It is easy to see that if the initial pre-c-fans are replaced by equivalent ones, then the result lies in the same equivalence class of pre-c-fans.

Here are some remarks on the associativity of the product. Let $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$ be c-fans of dimensions k_0, k_1, k_2. Arguing as in the verification of the multiplicity restrictions above, we reduce the verification of the property $\mathcal{P}_0(\mathcal{P}_1\mathcal{P}_2) = (\mathcal{P}_0\mathcal{P}_1)\mathcal{P}_2$ to the case $k_0 + k_1 + k_2 = 2n$. A pair of points e_1, e_2 is said to be *admissible* if the triple intersections of cones $K_0 \cap (e_1 + K_1) \cap (e_2 + K_2)$ are transversal for all $K_i \in \mathcal{P}_i$. We endow each point of such a triple intersection with multiplicity $\Xi(E(K_0), E(K_1), E(K_2))$ (see Lemma 1) and denote the sum of all multiplicities by μ. As in the proof of Proposition 6, we see that μ is independent of the choice of an admissible pair e_1, e_2.

We claim that the triple product of these fans is the zero point with multiplicity μ, independently of the order of brackets. The proof of this claim is based on the symmetry property of the function Ξ of three arguments (see Lemma 1). We omit the details (which are similar to those in the proof of Lemma 3) since the associativity of the product of c-fans follows from the interpretation of this product as a product of differential forms (§§ 7, 8).

This completes the construction of the ring of c-fans. For later use, we state two obvious properties of this ring.

Proposition 7. 1) *If $\varphi \in E$, then the map of φ-localization is an endomorphism of the ring of* c*-fans.*

2) *Let $\mathcal{A}^{\psi}$ be the subring of all* c*-fans representable as pre-*c*-fans whose minimal cone contains the line ψ. Then the factorization map determines a homomorphism of the ring $\mathcal{A}^{\psi}$ to the ring of* c*-fans of the space E/ψ.*

§ 7. c-fans as differential forms

We identify the n-dimensional Euclidean vector space E with the space $\operatorname{Re}\mathbb{C}^n$. Let $f, g, \dots$ be continuous piecewise-linear functions on $\mathbb{C}^n$ that are constant along the imaginary space $\operatorname{Im}\mathbb{C}^n$. We denote by $\mathcal{F}$ the ring of differential forms on $\mathbb{C}^n$ generated (over $\mathbb{R}$) by the 2-forms $dd^c f$. (We recall that d^c is the difference of the holomorphic and antiholomorphic parts of the differential.)

Since each $dd^c f$ is a current, that is, a form whose coefficients are distributions, we must verify that $\mathcal{F}$ is well defined. It is known [25] that $dd^c f_1 \wedge \dots \wedge dd^c f_k$ is a well-defined current if $f_1, \dots, f_k$ are continuous plurisubharmonic (in particular, convex) functions on $\mathbb{C}^n$. This means that if a sequence of k-tuples of smooth plurisubharmonic functions converges to $\{f_1, \dots, f_k\}$ uniformly on compacta, then the corresponding sequence of differential forms converges weakly to a current which is independent of the choice of the sequence. It is known that the limiting current is defined (as a functional) on compactly supported forms with locally integrable coefficients and assumes non-negative values on positive forms. Hence it is a non-negative current of measure type. Since every piecewise-linear function is represented as the difference of two convex functions, we now see that $\mathcal{F}$ is well defined.

As described below, for every k-dimensional c-fan $\mathcal{K}$ we have a homogeneous current $W(\mathcal{K})$ of bidegree (k, k). (This means that $W(\mathcal{K})$ vanishes on all bihomogeneous forms whose number of holomorphic or antiholomorphic differential is different from $n - k$.)

Theorem 3. *The map W establishes an isomorphism between the ring $\mathcal{A}$ of Euclidean* c*-fans and the ring $\mathcal{F}$.*

Remark 10. The proof of the theorem does not use the associativity of the product of c-fans. Hence this associativity is a corollary of Theorem 3.

Remark 11. If we change the definition of rational c-fans by admitting rational multiplicities (not only integer ones), then all the arguments below remain valid and give the following result. The map W is an isomorphism of this ring onto the ring generated (over $\mathbb{Q}$) by the forms $dd^c H$, where H is the supporting function of a convex polyhedron whose vertices are points with rational coordinates.

The current $W(\mathcal{K})$ corresponding to a k-dimensional c-fan $\mathcal{K}$ is defined by

$$W(\mathcal{K})(\varphi) = \sum_K \int_{(K+\operatorname{Im}\mathbb{C}^n)} \omega_K \wedge \varphi, \tag{1}$$

where φ is a compactly supported differential form of degree $2k$ in $\mathbb{C}^n$, the sum is taken over all k-dimensional cones of the c-fan, and ω_K is an odd $(n-k)$-form on the space $E(K) + \operatorname{Im}\mathbb{C}^n$ to be defined below.

We recall that an odd form is an exterior form (on a vector space) that changes sign if we reverse the orientation. (Hence it is an odd function on the set of orientations.) The exterior product of an odd form and an even form is an odd form. An odd volume form on a vector space is uniquely determined by the volume of a fixed bounded domain in this space.

Given an m-dimensional cone $K \in \mathcal{K}$ with multiplicity m_K, we construct an odd volume form u_K on $E/E(K)$ by declaring the volume of the unit cube in the Euclidean metric on the space $E/E(K)$ to be equal to m_K. For every cone K in E we define a map $\rho_K\colon (E(K)+\operatorname{Im}\mathbb{C}^n) \to E/E(K)$ as multiplication by $\sqrt{-1}$ followed by taking the quotient by the subspace $E(K)+\sqrt{-1}E(K)$. We put $\omega_K = \rho_K^*(u_K)$, where the relation between the orientations of $E(K)+\operatorname{Im}\mathbb{C}^n$ and $E/E(K)$ (which is necessary for the definition of the pre-image of an odd form) is established by orienting the kernel of ρ_K as a *complex* vector space.

To prove Theorem 3 we must restate the multiplicity restrictions of § 2 in terms of the forms u_K defined above. Let the orientations of a k-dimensional cone K and a $(k-1)$-dimensional face L of K be related by means of the "interior normal vector". This relation enables us to consider an odd $(n-k)$-form u_K^L which is the pre-image of u_K under the projection map $E/E(L) \to E/E(K)$. We restate the local multiplicity restrictions (Proposition 2) as follows.

Proposition 8. *For every $(k-1)$-dimensional cone L of a k-dimensional pre-*c*-fan $\mathcal{K}$ we have $\sum_K u_K^L = 0$, where the sum is taken over all k-dimensional cones K of $\mathcal{K}$ having L as a face.*

Let $\Delta \subset E^*$ be a convex polyhedron, Γ a face of Δ, K_Γ the dual cone of Γ, u_Γ the odd form on $E/E(K_\Gamma)$ constructed as above with multiplicity equal to the area of Γ, and u_Γ^Λ the pre-image of the odd form u_Γ on the space $E/E(K_\Lambda)$, where Λ is a face of Δ such that Λ contains Γ and has dimension exceeding that of Γ by 1. Applying Proposition 8 to the pre-c-fan Σ_Δ^k defined in § 1.4, we obtain a restatement of the Minkowski conditions for the $(n-k+1)$-dimensional faces of Δ (as explained at the end of § 2).

Corollary 2. *For every $(n-k+1)$-dimensional face Λ of the polyhedron Δ we have $\sum_\Gamma u_\Gamma^\Lambda = 0$, where the sum is taken over all $(n-k)$-dimensional faces of Λ.*

We state (without proof) some more identities that concern odd forms u_Γ^Λ and are used in the proof of Theorem 3. These identities are equivalent to the formula expressing the volume of a polyhedron as the sum of the areas of its faces multiplied by the heights to these faces. (This formula is applied to an $(n-k+1)$-dimensional face Λ of Δ.)

Lemma 4. *Let $h_D \in E^*$ be a point of the affine subspace generated by a face D of the polyhedron Δ. Then for every $(n-k+1)$-dimensional face Λ of Δ we have $u_\Lambda = \sum_\Gamma h_\Gamma \wedge u_\Gamma^\Lambda$ (in the notation of Corollary 2).*

The following assertions divide the assertion of Theorem 3 into parts.

Proposition 9. *The map W establishes a ring isomorphism between $\mathcal{A}_0$ and $\mathcal{F}$, where $\mathcal{A}_0$ is the subring of $\mathcal{A}$ generated by the* c*-fans of degree* 1.

Proposition 10. *The ring of* c*-fans is generated by elements of degree* 1, *that is,* $\mathcal{A} = \mathcal{A}_0$.

Since W is obviously monomorphic and preserves sums, Proposition 9 follows from Proposition 11 below and the algorithm for taking powers of polyhedra described in § 1.4 (its justification is given in Proposition 12 below). Indeed, since every piecewise-linear function is a difference of convex functions, we can apply the polarization formula to represent any element $f \in \mathcal{F}$ of degree m in the form $\sum_\Delta \pm(dd^c H_\Delta)^m$, where H_Δ is the supporting function of Δ. Therefore $f = \sum_\Delta \pm W(\Sigma_\Delta^m)$ (see Proposition 11). Then Proposition 12 (see below) implies that $f = \sum_\Delta \pm W(\Sigma_\Delta)^m$.

Proposition 11. *We have $(dd^c H_\Delta)^m = W(\Sigma_\Delta^m)$, where Σ_Δ^m is the* c*-fan defined in* § 1.4.

Proof. We use the identity

$$(dd^c H)^m(\varphi) = (dd^c H)^{m-1}(H dd^c \varphi),$$

which holds for any function H and any smooth compactly supported function φ, along with the inductive assumption (Σ^0 is the c-fan consisting of one cone E with multiplicity 1) to conclude that $(dd^c H)^m(\varphi) = W(\Sigma_\Delta^{m-1})(H dd^c \varphi)$. Hence, according to (1),

$$(dd^c H)^m(\varphi) = \sum_{L \in \Sigma_\Delta^{m-1},\, \dim L = n-m+1} \int_{(L+\operatorname{Im}\mathbb{C}^n)} \omega_L \wedge (H dd^c \varphi).$$

We represent the integrand as the sum of two terms: $d(H_L \omega_L \wedge d^c\varphi)$ and $-\omega_L \wedge dH_L \wedge d^c\varphi$. Here H_L is any point of the face of Δ dual to the cone L, and we regard H_L as a real linear function on $\mathbb{C}^n$. (This function coincides with H on the domain $L + \operatorname{Im}\mathbb{C}^n$ of integration.)

Lemma 5. 1) *The sum of the integrals of the first term is zero.*

2) *The restriction of the second term to the domain $L + \operatorname{Im}\mathbb{C}^n$ of integration is equal to $d(\omega_L \wedge d^c H_L \wedge \varphi)$.*

The first assertion of the lemma follows from the multiplicity restrictions for the c-fan Σ_Δ^{m-1} (see above). The second is a corollary of the following facts:

a) the component of bidegree $(n-m+1, n-m+1)$ of the form $dH_L \wedge d^c\varphi + d^c H_L \wedge d\varphi$ vanishes on the subspace $E(L) + \sqrt{-1}E(L)$, which lies in the kernel of ω_L;

b) the form $d^c H_L$ is closed.

Applying Stokes' formula to the integrals of the second term, we express $(dd^c H)^m(\varphi)$ as a sum of integrals over the domains $L + \operatorname{Im}\mathbb{C}^n$, where the sum is taken over the $(n-m)$-dimensional cones L of the fan Σ_Δ^{m-1}. The form to be integrated over $L + \operatorname{Im}\mathbb{C}^n$ is given by $\left(\sum_K \omega_K \wedge d^c H_K\right) \wedge \varphi$, where the sum is taken over the $(n-m+1)$-dimensional cones K of the fan Σ_Δ^{m-1} having L as a face. Then Lemma 4 (see also the definition of the form ω_L above) implies that this form equals ω_K, as required. This proves the proposition.

Proposition 12. $\Sigma^k_\Lambda \Sigma^p_\Lambda = \Sigma^{k+p}_\Lambda$.

Proof. The algorithm of § 6 for determining multiplicities implies that it suffices to verify the equation $\Sigma^l_\Delta \Sigma^m_\Delta = \Sigma^{l+m}_\Delta$ in the case when $l+m=n$. Let H be a supporting function of Δ and put $H_e(x) = H(x+e)$. Using Proposition 11 and Lemma 6 below, we get

$$W(\Sigma^l_\Delta \Sigma^m_\Delta) = \lim_{e\to 0}(dd^cH)^l \wedge (dd^cH_e)^m = (dd^cH)^n = W(\Sigma^n_\Delta).$$

Lemma 6. *Let $\mathcal{P}$, $\mathcal{Q}$ be* c*-fans of complementary dimensions. Then* $W(\mathcal{P}\mathcal{Q}) = \lim_{e\to 0} W(\mathcal{P})W_e(\mathcal{Q})$, *where $W_e(\mathcal{Q})$ is the shift of the current $W(\mathcal{Q})$ by an admissible* (*see* § 6) *point e.*

The lemma is easily proved by a continuity argument or by direct calculation. We omit the details.

Let $\mathcal{K}$ be the fan of cones that are dual to the faces of a simple polyhedron Δ (this means that all cones of $\mathcal{K}$ are simplicial), $\mathcal{B}$ the ring of all c-fans with cones in $\mathcal{K}$, and $\mathcal{B}^k$ the set of homogeneous c-fans in $\mathcal{B}$ of degree k (equivalently, $(n-k)$-dimensional c-fans).

Theorem 4. 1) *The ring $\mathcal{B}$ is generated by elements of degree* 1.

2) *The product of* c*-fans determines a non-degenerate pairing $\mathcal{B}^k \oplus \mathcal{B}^{n-k} \to \mathbb{R}$.*

Every c-fan is contained in a ring of this form. Therefore Proposition 10 (and hence Theorem 3) follows from Theorem 4. We combine the two assertions of Theorem 4 into one as follows.

Lemma 7. *Let $\mathcal{B}^k_0$ be the part of $\mathcal{B}^k$ generated by $\mathcal{B}^1$. Then the pairing $\mathcal{B}^k \oplus \mathcal{B}^{n-k}_0 \to \mathbb{R}$ is non-degenerate for $0 \leqslant k \leqslant n$.*

Proof. Suppose that $\mathcal{P} \in \mathcal{B}^k$ lies in the kernel of the pairing, that is, the product $\mathcal{P}\mathcal{Q}$ is zero for all $\mathcal{Q} \in \mathcal{B}^{n-k}_0$. We take k to be as large as possible, that is, the pairing in $\mathcal{B}^m$ has zero kernel for $m > k$. (Clearly, $k < n$.) Then $\mathcal{P}\mathcal{R} = 0$ for every $\mathcal{R} \in \mathcal{B}^1$. Let Λ be an $(n-1)$-dimensional face of the polyhedron Δ with dual cone $\varphi \subset \operatorname{supp} \mathcal{P}$.

If $k < n-1$, then we use localization and factorization of fans (Proposition 7) to reduce the assertion to the case of a fan of smaller dimension. (The quotient fan is the dual fan of the simple polyhedron Λ.)

If $k = n-1$, then we denote by Γ the polyhedron obtained from Δ by a small parallel translation of the face Λ. Since Δ is simple, we see that $(\Sigma_\Delta - \Sigma_\Gamma) \in \mathcal{B}^1$. It is now easy to conclude from the equation $(\Sigma_\Delta - \Sigma_\Gamma)\mathcal{P} = 0$ that the multiplicity of the ray φ in the c-fan $\mathcal{P}$ is equal to zero.

§ 8. Abstract c-fans and the Minkowski problem

Unlike the ring of Euclidean c-fans, the ring $\mathcal{F}$ of currents is defined independently of the choice of a metric in $\operatorname{Re}\mathbb{C}^n$. To give an invariant (independent of the metric) definition of c-fans in a real vector space E, one must consider the multiplicity of a cone K as an odd volume form u_K on $E/E(K)$, where $E(K)$ is the subspace of E generated by K. Using Definition 1 and taking Proposition 8 as the definition of the multiplicity restrictions, we get the definition of an abstract c-fan.

The product of abstract c-fans can be defined by the algorithm of § 6. (The sum is defined in the obvious way.) Indeed, as in the Euclidean and rational cases, it suffices to construct the product for subspaces E_1, E_2 (regarded as c-fans) with odd forms u_1, u_2 on the quotient spaces E/E_1, E/E_2. (These forms are replaced by numerical multiplicities in the Euclidean and rational cases.) Hence we must construct the "product" of u_1 and u_2, that is, an odd volume form $u_{1,2}$ on the space $E/(E_1 \cap^s E_2)$, where $E_1 \cap^s E_2$ is the stable intersection of subspaces (see § 6).

If E_1, E_2 are not transversal, we put $u_{1,2} = 0$. If they are transversal, then we choose orientations of the spaces E/E_i, take the compatible orientation of $E/(E_1 \cap E_2)$, and put $u_{1,2} = u_1^* \wedge u_2^*$, where u_i^* is the pre-image of u_i under the projection $E/(E_1 \cap E_2) \to E/E_i$. (We have $E_1 \cap^s E_2 = E_1 \cap E_2$ in the transversal case.) The form $u_{1,2}$ is indeed odd, that is, it changes sign under reversal of orientation. (We note that u_1^*, u_2^* are not odd forms.)

We introduce the symbols C_{e}, C_{r}, C_{a} for the rings of Euclidean, rational, abstract c-fans. The isomorphism W was constructed in § 7 as the composite of the maps $C_{\mathrm{e}} \to C_{\mathrm{a}} \to \mathcal{F}$. Hence the rings of Euclidean and abstract c-fans are isomorphic. For every rational cone with multiplicity $m(K)$ we take the odd volume form on $E/E(K)$ such that the volume of the fundamental cell of the integer lattice is equal to $m(K)$. This yields an embedding $C_{\mathrm{r}} \to C_{\mathrm{a}}$.

The map sending a polyhedron $\Delta \subset E^*$ to the c-fan Σ_Δ^k is independent of the choice of metric and integer lattice. The abstract c-fan Σ_Δ^k is constructed from the convex polyhedron Δ in the following way.

Given a bounded domain A in a vector space H, one can take an odd volume form ψ_A on the dual space H^* such that the ψ_A-volume of any domain $U \subset H^*$ is equal to the Liouville volume of the domain $A \oplus U \subset H \oplus H^*$. The abstract c-fan Σ_Δ^k is the $(n-k)$-skeleton of the fan of dual cones with odd volume forms ψ_Λ on the spaces $E/E(K_\Lambda)$, where K_Λ is the cone dual to the k-dimensional face Λ. Thus results on powers of polyhedra (in particular, Minkowski's theorem) may be stated without using the Euclidean metric. It is easily verified that the assertion of § 1.2 on the powers of polyhedra is indeed a restatement of the results of [20], [21].

Let v_i be primitive integer vectors in a space E with a lattice and let m_i be positive integers such that $m_1 v_1 + \dots + m_N v_N = 0$. Then the set $\{v_i, m_i\}$ is called *Minkowski data*. The theorem of Minkowski on convex polyhedra implies that there is a unique (up to translation) convex polyhedron $\Delta \subset E^*$ such that

(i) the number of $(n-1)$-dimensional faces of Δ is equal to N;

(ii) these faces lie on shifts of rational subspaces;

(iii) v_i are their exterior unit normal vectors;

(iv) m_i are their areas.

(The notions of an exterior normal vector and the area of a face were defined at the end of § 2 for polyhedra in spaces with integer lattices.)

The c-fan of an algebraic curve (see § 1.4) determines Minkowski data. Hence every curve determines a convex polyhedron.

In all examples (known to the author) of 3-dimensional polyhedra with integer Minkowski data, the lengths of the edges (measured using the integral lattice) are sums of quadratic irrationalities.

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