UNIVERSITY OF BRISTOL

School of Mathematics

LINEAR ALGEBRA MOCK EXAM

MATH10015 MOCK

2 hours 30 minutes

This paper contains two sections: Section A and Section B.
Each section should be answered in a separate booklet.

Section A contains FIVE questions and Section B contains FOUR questions.
All NINE answers will be used for assessment.

Calculators are not permitted.

Do not turn over until instructed.

Section A: Short Questions

A1. (a) (**3 marks**) Write the following in the form $a + ib$, where a and b are real numbers:

(i) $z = \dfrac{7+3i}{4-2i}$.

(ii) $w = 5e^{-\frac{3\pi i}{4}}$.

(b) (i) (**2 marks**) Let $A = \begin{pmatrix} 1 & 2 & a \\ 3 & 7 & b \\ -4 & 0 & c \end{pmatrix}$. Give an example of non-zero values for a, b and c such that A is not invertible.

(ii) (**3 marks**) Let $B, C \in M_2(\mathbb{R})$. If $\det B = 2$ and $\det C = 5$ then must we have that $\det(B + C) = 7$? Give a proof or a counter-example.

A2. (a) (**3 marks**) Find the determinant of $\begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 3 \\ -1 & -1 & 0 \end{pmatrix}$.

(b) (**2 marks**) Consider $v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$. Is $\{v_1, v_2, v_3\}$ a basis for $\mathbb{R}^3$? Justify your answer.

(c) (**3 marks**) Consider a linear map $S : \mathbb{R}^3 \to \mathbb{R}^4$. Suppose that we know $S(v_1), S(v_2)$ and $S(v_3)$. Can we determine $S(w)$ for any $w \in \mathbb{R}^3$? Justify your answer.

A3. (a) (**3 marks**) Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

State whether each of A, B and C are in row echelon form, reduced row echelon form, or neither. You do not need to justify your answers.

(b) (**5 marks**) Now consider a real system of equations with coefficients represented by the matrix A above.

(i) What is $S(A, 0)$?

(ii) Suppose that $x = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ is a solution to the system $Ax = b$ for some $b \in \mathbb{R}^3$. State another solution to the system $Ax = b$.

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A4. Let $V = \mathbb{R}^2$, a vector space over $\mathbb{R}$. Let $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and define $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $f(v, w) = v \cdot Aw$, where '$\cdot$' denotes the usual dot product. For each of the following, carefully justify your answers.

(a) (**4 marks**) For $v = (a_1, a_2)$ and $w = (b_1, b_2) \in \mathbb{R}^2$, find an expression for $f(v, w)$. Hence decide if it is the case that $f(v, v) \geq 0$ for all $v \in \mathbb{R}^2$.

(b) (**2 marks**) Do we have that $f(v, v) = 0$ if and only if $v = 0$?

(c) (**2 marks**) Is it the case that $f(v, w) = f(w, v)$ for every $v, w \in \mathbb{R}^2$?

(d) (**2 marks**) Prove, for all $u, v, w \in V$, that $f(v, u + w) = f(v, u) + f(v, w)$.

A5. Let $f : \mathbb{R}^3 \to \mathbb{R}$ satisfy $f(e_1) = f(e_2) = f(e_3) = 2e_1$. Assume that f is $\mathbb{R}$-linear.

(a) (**3 marks**) With $\mathcal{E} = \{e_1, e_2, e_3\}$ and $\mathcal{E}' = \{e_1\}$, find $M_{\mathcal{E}'\mathcal{E}}(f)$.

(b) (**3 marks**) Construct an example of a non-linear function $g : \mathbb{R}^3 \to \mathbb{R}$ that satisfies $g(e_1) = g(e_2) = g(e_3) = 2e_1$. Make sure that you define $g(x)$ for every $x \in \mathbb{R}^3$.

(c) (**2 marks**) How many distinct $\mathbb{R}$-linear functions $h : \mathbb{R}^3 \to \mathbb{R}$ are there that satisfy $h(e_1) = h(e_2) = h(e_3) = 2e_1$? You should give a brief justification of your answer.

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Section B: Longer Questions

Please use a new booklet for this Part.

B1. (a) (i) (**3 marks**) Let $u, v, w \in \mathbb{R}^n$. Prove that the vectors $u - v, v - w$ and $w - u$ are linearly dependent over $\mathbb{R}$.

(ii) (**4 marks**) Let $A \in M_{m,n}(\mathbb{R})$ with $m < n$. Is it possible for the rows of A to be linearly independent? Is it possible for the columns of A to be linearly independent? Justify your answers.

(b) (i) (**3 marks**) Is the set $U = \{u \in \mathbb{R}^3 : u \text{ is a unit vector}\}$ a subspace of $\mathbb{R}^3$ over $\mathbb{R}$?

(ii) (**5 marks**)Let V and W be subspaces of $\mathbb{R}^3$ over $\mathbb{R}$. Let B_V be a basis for V and B_W be a basis for W. Show that $B_V \cap B_W$ is not necessarily a basis for $V \cap W$ by giving an example.

B2. (a) (**5 marks**) Find the inverse of the matrix

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 1 & -1 & 0 \end{pmatrix}.$$

(b) (**3 marks**) Let $A, B \in M_n(\mathbb{R})$. If A and AB are invertible, is B also invertible? Give a proof or a counter-example.

(c) (**4 marks**) Consider a system of linear equations represented by the matrix equation $Cx = b$. Let D be an invertible matrix. Show that the system $(DC)x = b$ has exactly one solution if and only if $Cx = b$ has exactly one solution.

(d) (**3 marks**) Let $A \in M_n(\mathbb{R})$ be an invertible matrix and suppose that B is a matrix that can be obtained from A by applying a finite number of row operations. Show that B is also invertible.

B3. Let $V = \mathbb{R}^3$ over $\mathbb{R}$, $f : V \to V$, $\mathcal{E} = \{e_1, e_2, e_3\}$ and

$$M_{\mathcal{E}\mathcal{E}}(f) := A = \begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) (**4 marks**) Find the characteristic polynomial of A.

(b) (**2 marks**) Find the eigenvalues of A. State their algebraic multiplicities.

(c) (**8 marks**) Find the corresponding eigenvectors for your eigenvalues in (b). State the geometric multiplicity of each eigenvalue.

(d) (**1 marks**) Find a basis $\mathcal{A}$ such that $M_{\mathcal{A}\mathcal{A}}(f)$ will be a diagonal matrix.

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B4. Define $\mathcal{S} := \{p_i : i \in \mathbb{N} \cup \{0\}\} \subseteq F(\mathbb{R}, \mathbb{R})$ by setting $p_0(x) := 1$ for all $x \in \mathbb{R}$ and for $i \in \{1, 2, \ldots\}$ let $p_i(x) := x^i$ for all $x \in \mathbb{R}$. Let $\mathbb{P} := \mathrm{span}_{\mathbb{R}}(\mathcal{S})$ and $V := \mathbb{R}^4$ be vector spaces over $\mathbb{R}$, and define the $\mathbb{R}$-linear function $\phi : V \to \mathbb{P}$ by

$$(a, b, c, d) \mapsto ap_0(x) + bp_1(x) + cp_2(x) + d(-p_0(x) - p_1(x) - p_2(x)).$$

(a) (**3 marks**) Find a basis for $\mathrm{Im}(\phi)$. Justify your answer.

(b) (**3 marks**) Find a basis for $\ker(\phi)$. Justify your answer.

(c) (**3 marks**) State the definition of an isomorphism. Is ϕ an isomorphism? Justify your answer.

(d) (**2 marks**) Show that the equation for the Rank-Nullity Theorem is satisfied by ϕ.

(e) (**4 marks**) Define $\psi : V \to \mathrm{Im}(\phi)$ so that $\psi(x) = \phi(x)$ for all $x \in \mathbb{R}^4$. You can assume that ψ is linear. Is ψ (i) injective and (ii) surjective? Justify your answer.

End of examination.