LATTICE POLYTOPES AND SEMIGROUP ALGEBRAS: GENERIC LEFSCHETZ PROPERTIES AND PARSEVAL-RAYLEIGH IDENTITIES

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ABSTRACT. We study semigroup algebras associated to lattice polytopes.

We begin by generalizing and refining work of Hochster, and describe their volume maps, that is, their fundamental classes, in terms of Parseval-Rayleigh identitites and differential equations, which we prove to be equivalent.

We use these descriptions to establish strong Lefschetz properties.

A consequence is the resolution of several conjectures concerning unimodality properties of the h^*-polynomial of lattice polytopes arising within the theory of lattice polytopes.

1. LATTICE POLYTOPES AND SEMIGROUP ALGEBRAS

The main player of this paper is a convex polytope P all whose vertices lie in a lattice $\mathbb{Z}^d$. These polytopes are called lattice polytopes, and are of tremendous importance throughout mathematics, see [BR15, BB96, AK00, PK92, KKMS73, Bar97, CLS11]. Not getting distracted with an endless list, let us simply note this: One of the main players and focuspoints in this context is often the function

$$E_P(i) \ := \ \#\{iP \cap \mathbb{Z}^d\},$$

firstly at nonnegative integers i. It is one of the fundamental facts that this function is in fact a polynomial.

It is often convenient to this polynomial into a generating function

$$\mathrm{Ehr}_P(t) \ := \ \sum_{i=0}^{\infty} E_P(i) t^i.$$

It is convenient to write it into this way because another polynomial appears: We can write

$$\mathrm{Ehr}_P(t) \ = \ \frac{h^*_P(t)}{(1-t)^{d+1}},$$

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where d is the dimension of P and $h^*_P(t) = h^*_0 + h^*_1 t + \ldots + h^*_d t^d$ is a polynomial of degree at most d. Now, if we were only a combinatorialist, we would try to simply say that we try to understand the coefficients of these polynomials. But that is only a distraction. Because there is more at play here.

This is because $\mathrm{Ehr}_P(t)$ and $h^*_P(t)$ have an algebraic interpretation, and it is really this algebraic structure that is a greater mystery, and that has attracted almost as much attention as the simple numerical question about understanding the aforementioned functions. The story goes like this:

Embed the polytope P in $\mathbb{R}^d \times \mathbb{R}$ at height 1, that is, in the affine hyperplane $\mathbb{R}^d \times \{1\}$. Consider the cone over $P \times \{1\}$.

$$\mathrm{cone}(P) \coloneqq \mathbb{R}_{\geq 0}(P \times \{1\}).$$

As such, it generates a semigroup algebra

$$\Bbbk^*[P] \coloneqq \Bbbk^*[\mathrm{cone}(P) \cap (\mathbb{Z}^d \times \mathbb{Z})],$$

graded by the last coordinate which we will refer to as the **height**[1].

We refer to [BG02] for an introduction into the subject.

Here $\Bbbk$ is any field, though we generally assume the field to be infinite to ensure the existence of an Artinian reduction. In this case $\Bbbk[P]$ is Cohen-Macaulay [Hoc72] with Hilbert series $\mathrm{Ehr}_P(t)$. For a choice of linear system of parameters $\theta_1, \ldots, \theta_{d+1} \in \mathscr{A}^1(P)$, the Artinian reduction

$$\mathscr{A}^*(P) \coloneqq \Bbbk^*[P] \Big/ \langle \theta_1, \ldots, \theta_{d+1} \rangle \, \Bbbk^*[P]$$

has $\dim \mathscr{A}^k(P) = h^*_k$. It follows that the h^*_k are nonnegative.

It has been a central question in the theory of lattice polytopes to determine additional properties for the coefficients of the h^* polynomial. In particular, Hibi and Ohsugi conjectured that under two special conditions, the coefficients form a unimodal sequence [OH06], see also [Bra16, **?**].

The first of these conditions is that P has the **integer decomposition property**, short **IDP**: every lattice point of $\mathrm{cone}(P)$ is a nonnegative integral combination of lattice points in $P \times \{1\}$, or equivalently that $\Bbbk^*[P]$ is generated in degree one.

[1] We apologize to the reader, but using notions from different areas of mathematics that have different traditions of naming and notation will force us to use some terms in a different way than you, dear reader, might be used to. We will try to be as clear about every aspect of this as possible.

The second property is the **reflexive** property: there is a lattice point p in $\mathbb{Z}^d \times \{1\}$ such that

$$\mathrm{cone}^\circ(P) \cap \mathbb{Z}^{d+1} = \ p + \mathrm{cone}(P) \cap \mathbb{Z}^{d+1},$$

where $\mathrm{cone}^\circ(P)$ is the interior of $\mathrm{cone}(P)$.

This is equivalent to $\Bbbk[P]$ being algebraically Gorenstein. In other words, the Artinian reduction is a Poincaré duality algebra with socle degree d.

Hence, it also implies, and is in fact known to be equivalent, to a palindromicity condition analogous to the Dehn-Sommerville relations for polytopes:

$$h^*_k = h^*_{d-k} \text{ for all } k \leq d/2.$$

Numerically on the level of the h^*-vector, the restriction to reflexive polytopes rather than all Gorenstein polytopes is without loss of generality. Bruns and Herzog [BR07] showed that for every Gorenstein polytope there is a reflexive polytope with the same h^*-vector.

We resolve the following conjecture of Hibi and Ohsugi.

Conjecture 1.1 (Hibi-Ohsugi)**.** *For any IDP reflexive lattice polytope $P \subset \mathbb{R}^d$, the coefficients of the h^*-polynomial are unimodal:*

$$h^*_0 \leq h^*_1 \leq \ldots \leq h^*_{\lfloor d/2 \rfloor} = h^*_{\lceil d/2 \rceil} \geq \ldots \geq h^*_d$$

This is the updated form of a conjecture of Hibi [Hib92], after Mustaţa and Payne gave an example showing the necessity of the IDP assumption [MP05]. These conjectures in turn go back to a more general one of Stanley [Sta89], who proposed that the unimodality may hold for a general Gorenstein standard graded integral domain, no doubt motivated by the g-conjecture.

In fact, we shall prove statements that are more powerful than this, and apply to more general cases. For instance, in the case of polytopes that have only the integer decomposition property, we still obtain monotone decreasing coefficients in the second half, i.e.

$$h^*_{\lfloor d/2 \rfloor} \geq \ldots \geq h^*_d.$$

More importantly, we actually prove algebraic theorems on $\Bbbk[P]$ that illuminate why and how the integer decomposition property enters. And some that in particular applies, to some degree, apply to all lattice polytopes.

The first (to be stated, yet not to be proven) algebraic result is this: we prove Conjecture 1.1 by proving a Lefschetz property for a generic Artinian reduction of the associated semigroup algebra.

Theorem 1.2. *If P is an IDP reflexive polytope, and the characteristic of $\mathbb{k}$ is 2 or 0, then there is an Artinian reduction $\mathscr{A}^*(P)$ of $\widetilde{\mathbb{k}}^*[P]$ has the Lefschetz property, i.e., there is a linear element $\ell \in \mathscr{A}^1(P)$ such that for any $k \leq {}^{d}/_{2}$, the map*

$$\mathscr{A}^k(P) \xrightarrow{\cdot \ell^{d-2k}} \mathscr{A}^{d-k}(P)$$

is an isomorphism.

Again: The integer decomposition property enters subtly, and we will attempt to make clear where and when it happens.

We will provide two proofs of this theorem, distributed over two papers. The first relies on the idea of generic Lefschetz theory introduced by the first three authors [Adi18, PP20, APP21].

In [APP25], we provide a second proof that combinates "classical" combinatorial Hodge theory, as in, we establish the Hodge-Riemann bilinear relations for a deformation of $\mathscr{A}^*(P)$, with generic Lefschetz theory using nonstandard analysis.

Back to the topic at hand: *generic* shall mean that the Artinian reduction is taken by linear forms

$$\theta_1, \ldots, \theta_{d+1}, \quad \text{where} \quad \theta_i = \sum_{p \in P \cap \mathbb{Z}^d} \theta_{i,p},$$

with transcedentally independent coefficients $\theta_{i,p}$, necessitating passing to a transcendental field extension $\widetilde{\mathbb{k}}$ of $\mathbb{k}$.

This is a rather extreme choice of linear system of parameters, necessitated by the proof via anisotropy. We want to emphasize the importance of the choice of l.s.o.p., as it makes a crucial difference for the ring we end up working with.

For a specific, and somewhat canonical, choice of l.s.o.p., $\mathscr{A}^*(P)$ is isomorphic to the orbifold Chow ring of the associated toric Deligne-Mumford stack [BCS05]. In contrast to this choice, we make of a generic Artinian reduction here. The special choice of linear system, as well as consequences particular to that specific linear system, will be discussed in [APP25].

As graded vector spaces, the results are isomorphic and the inequalities on the dimensions of the graded pieces remain unaffected.

Yet, as observed in [BD16], whether an Artinian reduction admits a Lefschetz element depends on the Artinian reduction, and not only on $\Bbbk^*[P]$. Braun and Davis gave an example of an IDP reflexive simplex and an Artinian reduction of the associated semigroup algebra which does not even admit a weak Lefschetz element.

The linear system of parameters chosen there is however not the canonical system for the orbifold Chow ring, nor is it generic.

As the composition of individual multiplications with ℓ, the Lefschetz isomorphism gives us an injection in the first half and a surjection in the second half, and thus the desired inequalities on the dimensions of the graded pieces:

Theorem 1.3 (Adiprasito-Papadakis-Petrotou-Steinmeyer, [APPS22])**.** *The h^*-polynomial of a reflexive IDP lattice polytope of dimension d has a unimodal sequence of coefficients. Moreover, we have that $(h^*_i - h^*_{i-1})_{1\leq i\leq d/2}$ is an M-vector in the sense of Macaulay* [Mac27]*: It is the Hilbert polynomial of a commutative graded algebra generated in degree one.*

Here the M-vector property too immediately follows from the Lefschetz property in the usual fashion [Sta87]: The vector of differences $\max\{h^*_i - h^*_{i-1}, 0\}$ is the Hilbert vector of a standard graded algebra, namely

$$\mathscr{A}^*(P)\Big/\ell\mathscr{A}^*(P).$$

Idea and setup. The overall idea is based on the recent works of Adiprasito [Adi18], Papadakis and Petrotou [PP20, APP21], in that we reduce the Lefschetz property to a property of pairings, introduced as biased pairings in [Adi18] and anisotropy in [PP20].

However, our work requires a critical new ingredient: The aforementioned works are much simplified because we are gifted detailed knowledge of the rings involved, including their fundamental class (also called the **volume map** in this setting) based on a wealth of previous works that describe the Chow rings of toric varieties, first from the perspective of algebraic geometry, then using combinatorics [Tuc, Bri97].

In the case of lattice polytopes, we have little to work with. The case of simplicial spheres (and then (pseudo)manifolds and cycles) made use of explicitly combinatorial techniques to reach the desired goal, and the algebra we investigate here is not immediately as governed by a combinatorial struture as the previous one, being cut out by binomials rather than monomials.

And so while Hochster [Hoc72] studied the canonical module for the semigroup algebras associated to lattice polytopes, he did not provide enough for us to proceed. In particular, we need a sufficiently explicit description of the fundamental class, that is, the isomorphism between the top nontrivial cohomology and the ground field. We know that such a description exists in principle, of course, see [BH93] and [CLS11, Theorem 13.4.7]. That said, there is no direct description of this polynomial. Worse, it is not canonically defined; two definitions may differ up to a unit of the base field. We build on Hochster's work and provide a sufficiently explicit description in two ways.

First, we recall the Kustin-Miller normalization of [APP24]. We proposed there a way to define fundamental class uniquely (and not just up to a scalar) at least within a certain field extension that parametrizes Artinian reductions.

Second, we give two descriptions, which we prove to be equivalent: The first is an identity of Parseval-Rayleigh's type. The second is a system of differential equations.

The paper is organized as follows. We start in Section 2 by generalizing the setup in order to state our main theorem and deduce the individual numerical corollaries within Erhart theory. For sake of completeness, we then recall in Section 3 the necessary parts of the machinery of [Adi18, PP20, APP21] in order to prove the Lefschetz statements by way of anisotropy.

Following this setup, we give our new contributions to the theory. Section 4 contains the normalization of the fundamental class and an auxiliary identity, while Section 8 contains the key identity of Parseval-Rayleigh type, and a differential equation for the volume map we prove to be equivalent. We finish with a discussion of open questions in Section 11. The appendix contains additional material, such as alternative proofs and illuminating facts that are not necessary on the way to the proof of the main results, but can be helpful independently.

2. The algebraic results, reviewed

For P reflexive, the semigroup algebra $\mathbb{k}^*[P]$ is Gorenstein of Krull dimension equal to the dimension of the polytope plus one [BH93]: After an Artinian reduction using a linear system of parameters of length equal to the Krull dimension, we arrive at a Poincaré duality algebra of socle degree d, that is, a graded ring whose top nontrivial degree is d, is one-dimensional as a vectorspace and so that the ring becomes a Poincaré duality algebra with respect to this copy of $\mathbb{k}$.

For general IDP polytopes, the situation is a little more delicate. One can force Poincaré duality however, using the usual trick: we allow for relative objects. The $\mathbb{k}^*[P]$-module defined by

$$\mathbb{k}^*[P, \partial P] \coloneqq \mathbb{k}^*[\mathrm{cone}^\circ(P)], \quad \mathrm{cone}^\circ(P) \ = \ (\mathrm{cone}(P) \setminus \partial \,\mathrm{cone}(P)) \cap (\mathbb{Z}^d \times \mathbb{Z}),$$

is the canonical module of the Cohen-Macaulay ring $\Bbbk^*[P]$, see [Hoc72], or more explicitly [BH93, Chapter 6].

After Artinian reduction, we are left with a perfect bilinear pairing

$$\mathscr{A}^k(P) \times \mathscr{A}^{d+1-k}(P, \partial P) \longrightarrow \mathscr{A}^{d+1}(P, \partial P) \cong \Bbbk.$$

Let us state the first key result:

Theorem 2.1. *If P is an IDP polytope of dimension d, and the characteristic of $\Bbbk$ is 2 or 0, then a generic Artinian reduction $\mathscr{A}^*(P)$ of $\widetilde{\Bbbk}^*[P]$ has the relative Lefschetz property, i.e., there exists a linear element $\ell \in \mathscr{A}^1(P)$ such that for all $k \leq {}^{d+1}/_2$,*

$$\mathscr{A}^k(P, \partial P) \xrightarrow{\cdot \ell^{d+1-2k}} \mathscr{A}^{d+1-k}(P)$$

is an isomorphism.

For general IDP polytopes, Theorem 2.1 gives surjections

$$\mathscr{A}^{d-k}(P) \xrightarrow{\cdot \ell} \mathscr{A}^{d+1-k}(P) \quad \text{and} \quad \mathscr{A}^k(P) \xrightarrow{\cdot \ell^{d+1-2k}} \mathscr{A}^{d+1-k}(P)$$

for $k \leq \lfloor {}^{d}/_2 \rfloor$ and thus we obtain the following result.

Corollary 2.2. *The h^*-polynomial of an IDP lattice polytope P of dimension d has monotone decreasing coefficients in the second half, i.e.,*

$$h^*_{\lfloor {}^{d+1}/_2 \rfloor} \geq \ldots \geq h^*_d \geq\geq h^*_{d+1} = 0.$$

Moreover, for all $k \leq {}^{d+1}/_2$, we have

$$h^*_k \geq h^*_{d+1-k}.$$

A corollary then concerns Stapledon's a-polynomial [Sta09].

Corollary 2.3. *For any IDP lattice polytope, the a-polynomial has unimodal coefficients.*

The last part also follows from Theorem 2.5, by observing that the a-polynomial corresponds exactly to the h^*-polynomial of ∂P as a **lattice complex**. Here, a lattice complex is a polyhedral complex built out of lattice polytopes, such that the lattice structure agrees on intersections of faces [BG09]. We will discuss this object a bit later again.

In addition, we obtain some interesting consequences if we know at what height the interior of $\mathrm{cone}^\circ(P)$ is generated: Suppose all minimal lattice points of $\mathrm{cone}^\circ(P)$, with respect to the order induced by the semigroup $\mathrm{cone}(P) \cap (\mathbb{Z}^d \times \mathbb{Z})$, are of height at most j, or in other words, $\Bbbk^*[P, \partial P]$, the ideal generated by the interior lattice points of $\mathrm{cone}(P)$, is generated in degree $\leq j$ as an ideal over $\Bbbk^*[P]$.

Theorem 2.4. *The h^*-polynomial of an IDP lattice polytope P of dimension d with $\mathrm{cone}^\circ(P)$ generated at height $\leq j$ has monotone increasing coefficients in the initial part, i.e.,*

$$h_0^* \leq \ldots \leq h^*_{\lceil \frac{d-j}{2} \rceil}.$$

Moreover, for $k \leq \lceil \frac{d-j}{2} \rceil$ we have

$$h_k^* \leq h^*_{d+1-j-k}.$$

This is not an immediate consequence of Theorem 2.1, but instead follows from an analogous Lefschetz Theorem 6.3. This in particular includes the result on Gorenstein IDP lattice polytopes Theorem 1.2, which is the case when $\mathscr{A}^*(P, \partial P)$ is generated by a single element.

These results extend to sheafified versions, and in fact, these form an important step:

Consider an abstract polytopal complex X (that is, a strongly regular CW complex [Hat02]) whose cells are lattice polytopes, with the property that the lattices agree in common intersections. This is what we call a **lattice complex**. We obtain an analogous ring $\mathbb{k}^*[X]$, that naturally generalizes the face ring (or Stanley-Reisner ring):

It is defined quite simply as the disjoint union of the individual semigroup rings of the cells, identified at the common intersections of faces, with the additional constraint that two monomials x_a and x_b that do not lie in a common face multiply to 0. This is a good definition: For instance, we have the short exact sequence

$$0 \longrightarrow \mathbb{k}^*[P, \partial P] \longrightarrow \mathbb{k}^*[P] \longrightarrow \mathbb{k}^*[\partial P] \longrightarrow 0.$$

We summarize basic results of [BBR07]: This ring is Cohen-Macaulay if X is topologically Cohen-Macaulay, and in particular if X is a sphere Σ (in which case the ring is also algebraically Gorenstein) or a ball Δ (in which case the Poincaré pairing applies to the pair of spaces $\mathscr{A}^*(\Delta, \partial\Delta)$ and $\mathscr{A}^*(\Delta)$).

Motivated by this, we put special focus on lattice balls and spheres, that is, lattice complexes homeomorphic, or $\mathbb{Z}_2$ homology equivalent, to a ball resp. sphere with lattice structure in which every element has the property. We call such complexes IDP if every single of its cells is IDP.

In analogy with the proofs of the g-theorem of [AHK18, PP20, APP21], we have:

Theorem 2.5. *If X is an IDP lattice sphere or ball of dimension d, and the characteristic of $\mathbb{k}$ is 2 or 0, then a generic Artinian reduction $\mathscr{A}^*(X)$ of $\widetilde{\mathbb{k}}^*[X]$ has the Lefschetz property*

$$\mathscr{A}^k(X, \partial X) \xrightarrow{\ \cdot \ell^{d+1-2k}\ } \mathscr{A}^{d+1-k}(X).$$

Similar results hold for manifolds and cycles (again in direct analogy to [APP21]) but seem less immediately relevant here. They can be worked out easily using the methods we provide here, however.

3. From anisotropy to the Lefschetz property

To deduce the Lefschetz property from anisotropy, we employ a reduction found in [Adi18], see also [PP20, APP21]: It is enough to demonstrate a nondegeneracy property of the Poincaré pairing at certain ideals. It is useful to introduce an intermediate property, which we call the **Hall-Laman relations**, which describe anisotropy of the Hodge-Riemann bilinear form.

The overall strategy is to show that (suitable) anisotropy implies the (suitable) Hall-Laman relations which imply the generic Lefschetz property. The final implication is rather easy, but the first takes a thought: we use the lifting trick of [Adi18] to describe Hodge-Riemann pairings of arbitrary semigroup algebras of lattice balls in terms of the middle Poincaré pairing in the semigroup algebra of a higher dimensional lattice ball, see Lemma 3.9.

We note that we give here the derivation for Theorem 2.1. For Theorem 2.4 we need a different form of the Lefschetz property, and consequently a different form of anisotropy. Nevertheless, since the basic building blocks are the same, we focus on the derivation of the former here.

3.1. **Definition of various Ideals.**

CASE 1. Assume $P \subset \mathbb{R}^d$ is an IDP lattice polytope. We do not assume that $\dim P = d$. We denote the lattice point set of P by $V(P)$. We consider the polynomial ring

$$R = \Bbbk[x_u : u \in V(P)].$$

We consider the convex cone $C(P)$ over P defined by

$$C(P) = \{ \sum_{1 \le t \le r} \lambda_t(u_t, 1) : r \ge 1, u_t \in P, \lambda_t \in \mathbb{R}_{\ge 0} \} \subset \mathbb{R}^{d+1}.$$

Then, under addition $C(P) \cap \mathbb{Z}^{d+1}$ is a submonoid of $\mathbb{Z}^{d+1}$ and we set

$$\Bbbk[P] = \Bbbk[C(P) \cap \mathbb{Z}^{d+1}].$$

In other words, $\Bbbk[P]$ is the $\Bbbk$-algebra associated to the monoid $C(P) \cap \mathbb{Z}^{d+1}$. We remark that since P is IDP, we have by ??? that

$$C(P) \cap \mathbb{Z}^{d+1} = \{ \sum_{1 \le t \le r} \rho_t(u_t, 1) : r \ge 1, u_t \in P, \rho_t \in \mathbb{Z}_{\ge 0} \} \}.$$

We consider the unique $\Bbbk$-algebra homomorphism

$$R \to \Bbbk[P]$$

with the property that $x_v \mapsto v$ for all $v \in V(P)$ and denote by I_P its kernel. Since P is IDP the homomorphism is surjective, hence it induces a $\Bbbk$-algebra isomorphism

$$R/I_P \to \Bbbk[P].$$

CASE 2. We keep the assumptions and notation of the above CASE 1. We denote by $F_1, \dots, F_r$ the components of the boundary of P. We denote by J the ideal of R generated by all elements

$$x_{v_1} x_{v_2} \dots x_{v_s}$$

such that $s \geq 1$, $v_i \in V(P)$ for all $1 \leq i \leq s$, and $\{v_1, \dots, v_s\}$ is not a subset of F_j for all $1 \leq j \leq r$. The ideal $I_{\partial P}$ of the boundary of P is by definition the ideal of R generated by the subset $I_P \cup J$.

CASE 3. Assume $X \subset \mathbb{R}^d$ is a lattice polytopal complex , with facets the lattice polytopes $P_1, \dots, P_r$. We assume that X is IDP, which means that each P_i is IDP. We denote by $V(P_i)$ the lattice point set of the polytope P_i and define the set $V(X)$ of lattice points of X by

$$V(X) = \cup_{i=1}^r V(P_i).$$

We denote by R the polynomial ring

$$R = \Bbbk[x_u : u \in V(X)].$$

We denote by J the ideal of R generated by all elements

$$x_{v_1} x_{v_2} \dots x_{v_s}$$

such that $s \geq 1$, $v_j \in V(X)$ for all $1 \leq j \leq s$ and $\{v_1, \dots, v_s\}$ is not a subset of P_i for all $1 \leq i \leq r$. Recall that, for $1 \leq i \leq r$, we defined in CASE 1, an ideal I_{P_i} of $\Bbbk[x_u : u \in V(P_i)]$. We denote by I_X the ideal of R generated by the union

$$(\cup_{i=1}^r I_{P_i}) \cup J.$$

CASE 4. We keep the assumptions and notation of the above CASE 3. We denote by $F(P_i)$ the set of facets of P_i, and we set

$$F(X) = \cup_{i=1}^r F(P_i).$$

We denote by Q the ideal of R generated by all elements

$$x_{v_1}x_{v_2}\dots x_{v_s}$$

such that $s \geq 1$, $v_i \in V(X)$ for all $1 \leq i \leq s$, and $\{v_1, \dots, v_s\}$ is not a subset of Z for all $Z \in F(X)$. The ideal $I_{\partial X}$ of the boundary of X is by definition the ideal of R generated by the subset $I_X \cup Q$.

3.2. **Natural maps.** Assume Δ is IDP lattice polytopal ball or sphere of dimension d.

1. The inclusion map

$$\widetilde{\mathbb{k}}^*[\Delta, \partial\Delta] = I_{\partial\Delta}/I_\Delta \to \widetilde{\mathbb{k}}^*[\Delta] \tag{1}$$

induces, for all $k \geq 0$, natural maps

$$\mathscr{A}^k(\Delta, \partial\Delta) \ \to \mathscr{A}^k(\Delta) \tag{2}$$

which, in general, are not injective. For example, for $k = d+1$, we have $\mathscr{A}^k(\Delta) = 0$ while $\dim_{\widetilde{\mathbb{k}}} \mathscr{A}^k(\Delta, \partial\Delta) = 1$.

2. The multiplication map

$$\widetilde{\mathbb{k}}^*[\Delta] \times \widetilde{\mathbb{k}}^*[\Delta, \partial\Delta] \to \widetilde{\mathbb{k}}^*[\Delta, \partial\Delta] \tag{3}$$

induces, for all k, l, multiplication maps

$$\widetilde{\mathbb{k}}^k[\Delta] \times \widetilde{\mathbb{k}}^l[\Delta, \partial\Delta] \to \widetilde{\mathbb{k}}^{k+l}[\Delta, \partial\Delta]. \tag{4}$$

There is an induced multiplication map

$$\mathscr{A}^*(\Delta) \times \mathscr{A}^*(\Delta, \partial\Delta) \to \mathscr{A}^*(\Delta, \partial\Delta) \tag{5}$$

giving $\mathscr{A}^*(\Delta, \partial\Delta)$ the structure of $\mathscr{A}^*(\Delta)$-module. This map induces, for all k, l, multiplication maps

$$\mathscr{A}^k(\Delta) \times \mathscr{A}^l(\Delta, \partial\Delta) \to \mathscr{A}^{k+l}(\Delta, \partial\Delta). \tag{6}$$

We remark that if $k + l = d + 1$, the pairing

$$\mathscr{A}^k(\Delta) \times \mathscr{A}^{d+1-k}(\Delta, \partial\Delta) \to \mathscr{A}^{d+1}(\Delta, \partial\Delta) \cong \widetilde{\mathbb{k}} \tag{7}$$

is perfect.

3. Composing the maps in Equation (1) and (3) we get a multiplication map

$$\widetilde{\mathbb{k}}^*[\Delta, \partial\Delta] \times \widetilde{\mathbb{k}}^*[\Delta, \partial\Delta] \to \widetilde{\mathbb{k}}^*[\Delta, \partial\Delta] \tag{8}$$

which induces a multiplication map

$$\mathscr{A}^*(\Delta, \partial\Delta) \times \mathscr{A}^*(\Delta, \partial\Delta) \to \mathscr{A}^*(\Delta, \partial\Delta). \tag{9}$$

4. Assume p is a homogeneous polynomial of degree s with $0 \le s \le d+1$. For $k \ge 0$ we set

$$p\mathscr{A}^k(\Delta, \partial\Delta) = \{pu : u \in \mathscr{A}^k(\Delta, \partial\Delta)\} \subset \mathscr{A}^{k+s}(\Delta, \partial\Delta) \tag{10}$$

and

$$p\mathscr{A}^k(\Delta) = \{pu : u \in \mathscr{A}^k(\Delta)\} \subset \mathscr{A}^{k+s}(\Delta). \tag{11}$$

We claim that for all k there exists a well-defined perfect pairing

$$\phi_p : p\mathscr{A}^k(\Delta) \times p\mathscr{A}^{d+1-k-s}(\Delta, \partial\Delta) \to \mathscr{A}^{d+1}(\Delta, \partial\Delta) \cong \widetilde{\mathbb{k}} \tag{12}$$

defined by $\phi_p(pa, pb) = \mathrm{vol}(pab)$ for all $a \in \mathscr{A}^k(\Delta)$ and $b \in \mathscr{A}^{d+1-k-s}(\Delta, \partial\Delta)$.

We first prove that it is well defined. Assume first that $c \in \mathscr{A}^k(\Delta)$ has the property that $pc = 0$ in $\mathscr{A}^{k+s}(\Delta)$. Then for all $b \in \mathscr{A}^k(\Delta, \partial\Delta)$ we get that $\mathrm{vol}(pcb) = 0$. Assume now that $c \in \mathscr{A}^{d+1-k-s}(\Delta, \partial\Delta)$ has the property that $pc = 0$ in $\mathscr{A}^{d+1-k}(\Delta, \partial\Delta)$. Then for all $a \in \mathscr{A}^k(\Delta)$ we get that $\mathrm{vol}(pac) = \mathrm{vol}(apc) = 0$.

We now prove it is a perfect pairing. Assume first $a \in \mathscr{A}^k(\Delta)$ has the property that $pa \neq 0$ in $\mathscr{A}^{k+s}(\Delta)$. Then, since the pairing in Equation (7) is perfect, there exists $b \in \mathscr{A}^{d+1-k-s}(\Delta, \partial\Delta)$ such that $\mathrm{vol}(pab) \neq 0$. Hence $\phi_p(pa, pb) \neq 0$. Assume now $b \in \mathscr{A}^{d+1-k-s}(\Delta, \partial\Delta)$ has the property that $pb \neq 0$ in $\mathscr{A}^{d+1-k}(\Delta, \partial\Delta)$. Then, since the pairing in Equation (7) is perfect, there exists $a \in \mathscr{A}^k(\Delta)$ such that $\mathrm{vol}(apb) \neq 0$. Hence $\phi_p(pa, pb) \neq 0$.

Remark 3.1. Assume $\mathscr{I}^*$ is a nonzero graded $\mathscr{A}^*(\Delta)$-submodule of $\mathscr{A}^*(\Delta, \partial\Delta)$. Then, there exists $k \ge 0$ and nonzero $u \in \mathscr{I}^k$. Using the perfect pairing

$$\mathscr{A}^k(\Delta) \times \mathscr{A}^{d+1-k}(\Delta, \partial\Delta) \to \mathscr{A}^{d+1}(\Delta, \partial\Delta) \cong \widetilde{\mathbb{k}}$$

of Equation (7), there exists $w \in \mathscr{A}^{d+1-k}(\Delta)$ such that wu is a nonzero element of $\mathscr{A}^{d+1}(\Delta, \partial\Delta)$. Hence, wu is a nonzero element of $\mathscr{I}^{d+1}$. Since $\mathscr{A}^{d+1}(\Delta, \partial\Delta)$ is a 1-dimensional vector space over $\widetilde{\mathbb{k}}$, it follows that $\mathscr{I}^{d+1} = \mathscr{A}^{d+1}(\Delta, \partial\Delta) \cong \widetilde{\mathbb{k}}$.

3.3. **Anisotropy.** The prototype of anisotropy is the following.

Theorem 3.2. *If X is an IDP lattice ball or sphere of dimension d, and the characteristic of $\mathbb{k}$ is 2 or 0, then the generic Artinian reduction $\mathscr{A}^*(X, \partial X)$ of $\widetilde{\mathbb{k}}^*[X, \partial X]$ has the anisotropy* *property. This means that for every nonzero $u \in \mathscr{A}^k(X, \partial X)$ of degree $k \leq {}^{(d+1)}/_{2}$, we have*

$$u^2 \neq 0,$$

in other words u^2 is a nonzero element of $\mathscr{A}^{2k}(X, \partial X)$. Moreover, if m is a monomial of degree $\leq d+1-2k$ such that mu is nonzero in $\mathscr{A}^{k+\deg(m)}(X, \partial X)$, then

$$mu^2 \neq 0,$$

in other words mu^2 is a nonzero element of $\mathscr{A}^{2k+\deg(m)}(X, \partial X)$.

Most of the remainder of this paper is devoted to proving this theorem (see Section 5), and related results, see Section 6. Let us note that it is enough to prove this theorem in characteristic 2, see [KLS24, Section 3.2]. Before we do that, however, we follow the derivation of the Lefschetz property from it, based on the ideas of [Adi18, PP20, APP21].

3.4. **The Hall-Laman relations.** Consider a lattice ball Δ of dimension d. Assume that $k \leq \frac{d+1}{2}$, $\ell \in \mathscr{A}^1(\Delta)$ and $\mathscr{I}^* \subset \mathscr{A}^*(\Delta, \partial\Delta)$ is a nonzero graded submodule. We say that $\mathscr{A}^*(\Delta, \partial\Delta)$ satisfies the **Hall-Laman relations** for the triple $(k, \ell, \mathscr{I}^*)$ if the pairing

$$\begin{array}{ccccc} \mathscr{I}^k & \times & \mathscr{I}^k & \longrightarrow & \mathscr{A}^{d+1}(\Delta, \partial\Delta) \cong \widetilde{\Bbbk} \\ a & & b & \longmapsto & \mathrm{vol}(ab\ell^{d+1-2k}) \end{array} \tag{13}$$

is nondegenerate. We say that $\mathscr{A}^*(\Delta, \partial\Delta)$ satisfies the **absolute Hall-Laman relations** with respect to the pair (k, ℓ) if for *all* nonzero graded submodules $\mathscr{I}^* \subset \mathscr{A}^*(\Delta, \partial\Delta)$ the Hall-Laman relations are true for the triple $(k, \ell, \mathscr{I}^*)$.

Proposition 3.3. *The absolute Hall-Laman relations are true for the pair (k, ℓ) if and only if the Hodge-Riemann bilinear form*

$$\begin{array}{cccccc} Q_{\ell,k}: & \mathscr{A}^k(\Delta, \partial\Delta) & \times & \mathscr{A}^k(\Delta, \partial\Delta) & \longrightarrow & \mathscr{A}^{d+1}(\Delta, \partial\Delta) \cong \widetilde{\Bbbk} \\ & a & & b & \longmapsto & \mathrm{vol}(ab\ell^{d+1-2k}) \end{array}$$

is anisotropic in the following sense: if $u \in \mathscr{A}^k(\Delta, \partial\Delta)$ is not zero, then $Q_{\ell,k}(u, u) \neq 0$.

Proof. The absolute Hall-Laman relations state that the Hodge-Riemann bilinear form does not degenerate at any nonzero graded submodule $\mathscr{I}^*$ of $\mathscr{A}^*(\Delta, \partial\Delta)$. But it clearly suffices to verify this fact at principal ideals generated by single elements u in $\mathscr{A}^k(\Delta, \partial\Delta)$, which in turn is the property of anisotropy. The other direction is clear. $\square$

Hence, since every anisotropic symmetric bilinear form is nondegenerate, we get from Proposition 3.3 that the absolute Hall-Laman relations for the pair (k, ℓ) imply a Lefschetz type property for $\mathscr{A}^*(\Delta, \partial\Delta)$ at degree k.

We introduce refinements of these properties: Assume p is a homogeneous polynomial of degree s in $\mathbb{k}[\Delta]$, k is an integer with $k \leq \frac{d+1-s}{2}$, $\ell \in \mathscr{A}^1(\Delta)$ and $\mathscr{I}^* \subset \mathscr{A}^*(\Delta, \partial\Delta)$ is a nonzero graded submodule. We say that $\mathscr{A}^*(\Delta, \partial\Delta)$ satisfies the **Hall-Laman relations with respect to the quadruple** $(k, p, \ell, \mathscr{I}^*)$ if the pairing

$$\begin{array}{ccccc} p\mathscr{A}^k(\Delta, \partial\Delta) & \times & p\mathscr{A}^k(\Delta, \partial\Delta) & \longrightarrow & \mathscr{A}^{d+1}(\Delta, \partial\Delta) \cong \mathbb{k} \\ pa & & pb & \longmapsto & \mathrm{vol}(pab\ell^{d+1-s-2k}) \end{array}$$

(which is well-defined by the obvious argument) is perfect when restricted to $p\mathscr{I}^k \times p\mathscr{I}^k$. If this is true for every nonzero graded submodule $\mathscr{I}^* \subset \mathscr{A}^*(\Delta, \partial\Delta)$ we say $\mathscr{A}^*(\Delta, \partial\Delta)$ satisfies the **absolute Hall-Laman relations with respect to the triple** (k, p, ℓ). Arguing similarly as in the proof of Proposition 3.3 this is equivalent to saying that for all $u \in \mathscr{A}^k(\Delta, \partial\Delta)$ such that $pu \in \mathscr{A}^{k+s}(\Delta, \partial\Delta)$ is nonzero it holds that $pu^2\ell^{d+1-s-2k}$ is a nonzero element of $\mathscr{A}^{d+1}(\Delta, \partial\Delta)$.

Recall the definition of $p\mathscr{A}^k(\Delta, \partial\Delta)$ in Equation (10), the definition of $p\mathscr{A}^k(\Delta)$ in Equation (11) and the perfect pairing ϕ_p in Equation (12). For a vector subspace U of $\mathscr{A}^k(\Delta, \partial\Delta)$ we set

$$pU = \{pu : u \in U\} \subset p\mathscr{A}^k(\Delta, \partial\Delta) \tag{14}$$

and

$$\mathrm{ann}(pU) = \{pb : b \in \mathscr{A}^{d+1-k-s}(\Delta) \ \text{ and } \ \mathrm{vol}(pab) = 0 \ \text{ for all } \ a \in U\}. \tag{15}$$

We have that pU is a vector subspace of $p\mathscr{A}^k(\Delta, \partial\Delta)$ and $\mathrm{ann}(pU)$ is a vector subspace of $p\mathscr{A}^{d+1-k-s}(\Delta)$. Moreover,

$$\mathrm{ann}(pU) = \{x \in p\mathscr{A}^{d+1-k-s}(\Delta) \ : \ \phi_p(x, z) = 0 \ \text{ for all } \ z \in pU\}. \tag{16}$$

Lemma 3.4. *The following are equivalent:*

(1) $\mathscr{A}^*(\Delta, \partial\Delta)$ satisfies the Hall-Laman relations with respect to the quadruple $(k, p, \ell, \mathscr{I}^*)$.
(2) The map

$$p\mathscr{I}^k \xrightarrow{\cdot \ell^{d+1-s-2k}} p\mathscr{A}^{d+1-k-s}(\Delta) \big/ \mathrm{ann}(p\mathscr{I}^k) \tag{17}$$

is an isomorphism.

(We set $t = d + 1 - k - s$. The map in Equation (17) sends pa to the class of $pa\ell^{t-k}$ in $p\mathscr{A}^t(\Delta) \big/ \mathrm{ann}(p\mathscr{I}^k)$ for all $a \in \mathscr{I}^k$. Moreover, it is the composition of the inclusion map $p\mathscr{I}^k \to p\mathscr{A}^k(\Delta, \partial\Delta)$ map with the multiplication by ℓ^{t-s} map $p\mathscr{A}^k(\Delta, \partial\Delta) \to p\mathscr{A}^t(\Delta, \partial\Delta)$ (compare Equation (5)), with the natural map $p\mathscr{A}^t(\Delta, \partial\Delta) \to p\mathscr{A}^t(\Delta)$ (compare Equation (2)), with the natural quotient map $p\mathscr{A}^t(\Delta) \to p\mathscr{A}^t(\Delta) \big/ \mathrm{ann}(p\mathscr{I}^k)$.)

Proof. We define

$$\rho_1 : p\mathscr{I}^k \times p\mathscr{I}^k \longrightarrow \mathscr{A}^{d+1}(\Delta, \partial\Delta) \cong \mathbb{k}$$

with $\rho_1(pa, pb) = \mathrm{vol}(pab\ell^{d+1-s-2k})$ for all $a, b \in \mathscr{I}^k$. As observed before, ρ_1 is well-defined.

Recall that if $\rho : V \times W \to \widetilde{\mathbb{k}}$ is a perfect pairing of finite dimensional $\widetilde{\mathbb{k}}$-vector spaces, X is a vector subspace of V, and we set

$$\mathrm{ann}(X) = \{w \in W : \rho(x, w) = 0 \text{ for all } x \in X\},$$

then it follows that the pairing ρ induces a perfect pairing $\rho' : X \times (W/\mathrm{ann}(X)) \to \widetilde{\mathbb{k}}$ such that $\rho'(x, [w]) = \rho(x, w)$ for all $x \in X, w \in W$. Applying that to the perfect pairing ϕ_p defined in Equation (12) we get an induced perfect pairing

$$\rho_2 : p\mathscr{I}^k \times p\mathscr{A}^{d+1-k-s}(\Delta)\big/\mathrm{ann}(p\mathscr{I}^k) \longrightarrow \mathscr{A}^{d+1}(\Delta, \partial\Delta) \cong \mathbb{k}.$$

We define

$$h : p\mathscr{I}^k \times p\mathscr{I}^k \to p\mathscr{I}^k \times p\mathscr{A}^{d+1-k-s}(\Delta)\big/\mathrm{ann}(p\mathscr{I}^k)$$

by $h(pa, pb) = (pa, pb\ell^{d+1-s-2k})$ for all $a, b \in \mathscr{I}^k$. It easy to see that h is well-defined and that

$$\rho_1 = \rho_2 \circ h.$$

Since ρ_2 is a perfect pairing, we get that ρ_1 is a perfect pairing if and only if h is an isomorphism. The result follows. $\square$

3.5. **Pyramids.** To prove the Hall-Laman relations, let us introduce an auxiliary construction:

Definition 3.5 (Pyramids)**.** Given a lattice polytope P with ambient lattice $\mathbb{Z}^d$, the **pyramid** over **base** P is constructed as the convex hull of $P \times \{0\}$ and $\mathbf{a} = (0, \cdots, 0, 1)$, the **apex**, in $\mathbb{Z}^d \times \mathbb{Z}$. We also denote this as $\mathrm{pyr}_{\mathbf{a}} P$. The pyramid over a lattice complex is the collection of pyramids over its elements.

If X is a lattice complex, then $\mathrm{pyr}_{\mathbf{a}} X$ is the lattice complex consisting of cones $\mathrm{pyr}_{\mathbf{a}} P$, $P \in X$ together with X itself. The pyramid over a lattice ball is also a lattice ball. The following lemma is immediate from the definitions.

Lemma 3.6. *The pyramid over a lattice polytope is IDP if and only if the base is IDP. The pyramid over a lattice polytopal complex is IDP if and only if the basis is IDP.*

3.6. **Reduction by lifting.** On the level of semigroup algebras, a pyramid corresponds to the introduction of a new indeterminate, corresponding to the apex. Consider a lattice complex X of dimension d. If X is a lattice ball then $\mathrm{pyr}_{\mathbf{a}} X$ is a lattice ball of dimension $d + 1$.

We use the linear system of parameters consisting of the rows of the $(d+2) \times 1$ matrix $\Theta = (\theta_{i,j})\mathbf{x}$, where i ranges from 1 to $d+2$, j ranges over the lattice points of $\mathrm{pyr}_{\mathbf{a}} X$ and $\mathbf{x}$ is the column matrix with $(1,j)$-entry equal to x_j.

Without loss of generality we assume that the last column of the matrix $(\theta_{i,j})$ corresponds to the apex $\mathbf{a}$ and is equal to the transpose of the matrix $(0, 0, \cdots, 0, 1)$.

We consider the Artinian reduction $\mathscr{A}^*(\mathrm{pyr}_{\mathbf{a}} X)$ of $\Bbbk[\mathrm{pyr}_{\mathbf{a}} X]$ with respect to this linear system of parameters, and the Artinian reduction $\mathscr{A}^*(X)$ of $\Bbbk[X]$ with respect to the linear system system of paramaters specified by the first $d+1$ rows of Θ.

We denote by $\overline{h}$ the $(d+2)$-th row of the matrix Θ. With respect to the above linear systems of parameters, the following lemma is a straightforward computation, which will be given in Subsection 3.7.

Lemma 3.7 (Pyramid lemma)**.** *(1) The inclusion $\Bbbk[X] \hookrightarrow \Bbbk[\mathrm{pyr}_{\mathbf{a}} X]$ induces, for all m, an isomorphism*

$$\mathscr{A}^m(X) \to \mathscr{A}^m(\mathrm{pyr}_{\mathbf{a}} X).$$

(2) The multiplication

$$\Bbbk[\mathrm{pyr}_{\mathbf{a}} X] \xrightarrow{\cdot \mathbf{x_a}} \Bbbk[\mathrm{pyr}_{\mathbf{a}} X]$$

induces, for all m, an isomorphism

$$\Bbbk^m[\mathrm{pyr}_{\mathbf{a}} X] \;\cong\; \Bbbk^{m+1}[\mathrm{pyr}_{\mathbf{a}} X, X]$$

and therefore an isomorphism

$$\mathscr{A}^m(\mathrm{pyr}_{\mathbf{a}} X) \;\cong\; \mathscr{A}^{m+1}(\mathrm{pyr}_{\mathbf{a}} X, X).$$

(3) The multiplication

$$\Bbbk[\mathrm{pyr}_{\mathbf{a}} X] \xrightarrow{\cdot \mathbf{x_a}} \Bbbk[\mathrm{pyr}_{\mathbf{a}} X]$$

also induces, for all m, an isomorphism

$$\mathscr{A}^m(X, \partial X) \to \mathscr{A}^{m+1}(\mathrm{pyr}_{\mathbf{a}} X, \partial \mathrm{pyr}_{\mathbf{a}} X).$$

Example 3.8. We consider the 1-dimensional lattice polytope

$$X = [1, 4] = \{x \in \mathbb{R} : 1 \le x \le 4\}$$

and the pyramid $Z = \mathrm{pyr}_{\mathbf{a}} X$ over it. We denote the apex of the pyramid by 5. We set $R_{sm} = \Bbbk[x_1, \ldots, x_4]$ and $R = \Bbbk[x_1, \ldots, x_5]$. We have

$$I_X = (x_2^2 - x_1x_3, x_3^2 - x_2x_4, x_1x_4 - x_2x_3) \subset R_{sm},$$

$$I_{\partial X} = (x_2, x_3) + I_X \subset R_{sm}$$

$$I_Z = (x_2^2 - x_1x_3, x_3^2 - x_2x_4, x_1x_4 - x_2x_3) \subset R,$$

hence I_Z is the ideal of R generated by I_X and

$$I_{\partial Z} = (x_5 x_2, x_5 x_3) + I_Z \subset R.$$

Since I_Z is the ideal of R generated by I_X and the Krull dimension of R/I_Z is one more than the Krull dimension of R/I_X, we get that the inclusion $\Bbbk[X] \hookrightarrow \Bbbk[\mathrm{pyr}_{\mathbf{a}} X]$ induces, for all m, an isomorphism

$$\mathscr{A}^m(X) \to \mathscr{A}^m(\mathrm{pyr}_{\mathbf{a}} X).$$

By the above descriptions of $I_X, I_{\partial X}, I_Z, I_{\partial Z}$, there exists a well-defined multiplication by x_5 map of R_{sm}-modules

$$\psi_1 : \Bbbk[X, \partial X] = I_{\partial X}/I_X \to \Bbbk[Z, \partial Z] = I_{\partial Z}/I_Z.$$

The map ψ_1 is not surjective, since, for example, the class of $x_5^2 x_2$ in $\Bbbk[Z, \partial Z]$ is not in its image. However, the induced map

$$\psi_1' : \mathscr{A}(X, \partial X) \to \mathscr{A}(Z, \partial Z)$$

is bijective.

3.7. **Proof of Pyramid Lemma 3.7.** We assume X is an IDP lattice ball and denote by $Z = \mathrm{pyr}_{\mathbf{a}} X$ the pyramid over X with apex $\mathbf{a}$. We consider the polynomial ring

$$R_{sm} = \Bbbk[\mathbf{x_v} : v \text{ lattice point of } X]$$

which is a subring of the polynomial ring

$$R = \Bbbk[\mathbf{x_v} : v \text{ lattice point of } Z] = R_{sm}[\mathbf{x_a}].$$

Consider the ideals $I_X, I_{\partial X}$ of R_{sm} and the ideals $I_Z, I_{\partial Z}$ of R, see Subsection 3.1. We define three ideals $I_1, \dots, I_3$ of R as follows: I_1 is the ideal generated by the subset I_X, I_2 is the ideal generated by the subset $I_{\partial X}$ and I_3 is the ideal of the subcomplex X of Z.

From the definitions in Subsection 3.1 and using that $Z = \mathrm{pyr}_{\mathbf{a}} X$ it follows that $\Bbbk[Z, X]$, which by definition is I_3/I_Z, is the ideal of R/I_Z generated by the class of $\mathbf{x_a}$. In other words,

$$I_3 = I_Z + (\mathbf{x_a}). \tag{18}$$

Moreover, it follows that

$$I_Z = I_1 \tag{19}$$

and

$$I_{\partial Z} = \mathbf{x_a} I_2 + I_Z. \tag{20}$$

We now prove Part (1) of the Pyramid Lemma 3.7. Using Equation (19) we get that $\Bbbk[Z]$ is the polynomial ring in variable $\mathbf{x_a}$ of $\Bbbk[X]$. Taking into account the linear system of parameters we use for the Artinian reductions, Part (1) of the Pyramid Lemma 3.7 follows.

Part (2) of the Pyramid Lemma 3.7 follows from Equation (18).

We now prove Part (3) of the Pyramid Lemma 3.7. We first remark that by elementary properties of tensor product and flat modules (see for example, [Rotman, An Introduction to homological algebra, Springer 2nd Edition] we have that R is a faithfully flat R_{sm}-module (see [Rotman, p. 152, Exerc. 3.35]) and hence the natural extension of scalars maps $I_X \otimes_{R_{sm}} R \to I_1$, and $I_{\partial X} \otimes_{R_{sm}} R \to I_2$ are isomorphism (see [Rotman, p. 139, Corollary 3.59]).

Consider the short exact sequence

$$0 \to I_X \to I_{\partial X} \to I_{\partial X / I_X} \to 0$$

Since R is a flat R_{sm}-module, we get an exact sequence

$$0 \to I_X \otimes_{R_{sm}} R \to I_{\partial X} \otimes_{R_{sm}} R \to (I_{\partial X}/I_X) \otimes_{R_{sm}} R \to 0$$

hence, an exact sequence

$$0 \to I_1 \to I_2 \to (I_{\partial X}/I_X) \otimes_{R_{sm}} R \to 0$$

Therefore, the R-module $(I_{\partial X}/I_X) \otimes_{R_{sm}} R$ is naturally isomorphic to the R-module I_2/I_1. Since $R_{sm} = R[\mathbf{x_a}]$, by the choice of θ_i we get that there exists a unique isomorphism

$$\mathscr{A}(X, \partial X) = (I_{\partial X}/I_X) \otimes_{R_{sm}} R_{sm}/(\theta_1, \dots, \theta_{d+1}) \to (I_2/I_1) \otimes_R R/(\theta_1, \dots, \theta_{d+2})$$

such that $[u] \otimes 1 \mapsto [u] \otimes 1$ for all $u \in I_{\partial X}$.

Combining this isomorphism with the fact that due to Equations (19) and Equations (20) the multiplication by $\mathbf{x_a}$ map induces an isomorphism

$$I_2/I_1 \to \Bbbk[Z, \partial Z] = I_{\partial Z}/I_Z$$

Part (3) of the Pyramid Lemma 3.7 follows.

3.8. **A crucial Lemma.** Consider now the case when $X = \Delta$, where Δ is a lattice ball of dimension d. The crucial lemma is the following.

Lemma 3.9 (compare [Adi18, Lemma 7.5])**.** *Let $k < \frac{d+1}{2}$ and $\mathscr{I}^*$ be a nonzero graded submodule of $\mathscr{A}^*(\Delta, \partial\Delta)$. We consider the induced graded submodule $\mathbf{x_a}\mathscr{I}^*$ of $\mathscr{A}^*(\mathrm{pyr}_{\mathbf{a}}\Delta, \partial \mathrm{pyr}_{\mathbf{a}}\Delta)$. We also set $h = \mathbf{x_a} - \overline{h}$. Then the following two are equivalent:*

1. *The Hall-Laman relations for the triple $(k+1, \mathbf{x_a}, \mathbf{x_a}\mathscr{I}^*)$.*
2. *The Hall-Laman relations for the triple $(k, h, \mathscr{I}^*)$.*

This extends naturally to the Hall-Laman relations relative to a homogeneous polynomial p. We denote the degree of p by s, and assume $k < \frac{d+1-s}{2}$. Then the following two are equivalent:

(3) The Hall-Laman relations for the quadruple $(k+1, p, \mathbf{x_a}, \mathbf{x_a}\mathscr{I}^*)$.

(4) The Hall-Laman relations for the quadruple $(k, p, h, \mathscr{I}^*)$.

Let us note that we actually do only need the case when $d - 2k - s = 1$. But the lemma deserves to be stated fully nevertheless.

Proof. In the present proof all vertical maps are coming from the pyramid Lemma 3.7. Moreover, we remark that $h = \mathbf{x_a}$ in $\mathscr{A}^*(\mathrm{pyr}_{\mathbf{a}}\Delta)$.

CASE 1. We assume that $p = 1$ and $\mathscr{I}^* = \mathscr{A}^*(\Delta, \partial\Delta)$. We consider the commutative diagram

$$\begin{tikzcd}
\mathscr{A}^k(\Delta, \partial\Delta) \arrow[r, "\cdot h^{d+1-2k}"] \arrow[d, "\sim"] & \mathscr{A}^{d+1-k}(\Delta) \arrow[d, "\sim"] \\
\mathscr{A}^{k+1}(\mathrm{pyr}_{\mathbf{a}}\Delta, \partial \mathrm{pyr}_{\mathbf{a}}\Delta) \arrow[r, "\cdot \mathbf{x}_{\mathbf{a}}^{d-2k}"] & \mathscr{A}^{d+1-k}(\mathrm{pyr}_{\mathbf{a}}\Delta)
\end{tikzcd}$$

The horizontal map on the top being an isomorphism is equivalent to the horizontal map on the bottom being an isomorphism.

CASE 2. We assume that $p = 1$ and $\mathscr{I}^*$ is a nonzero graded submodule of $\mathscr{A}^*(\Delta, \partial\Delta)$. For U a vector subspace of $\mathscr{A}^k(\Delta, \partial\Delta)$ we define

$$\mathrm{ann}_1(U) = \{x \in \mathscr{A}^{d+1-k}(\Delta) \ : \ \mathrm{vol}(xz) = 0 \ \text{ for all } \ z \in U\} \tag{21}$$

and for W a vector subspace of $\mathscr{A}^{k+1}(\mathrm{pyr}_{\mathbf{a}}\Delta, \partial \mathrm{pyr}_{\mathbf{a}}\Delta)$ we define

$$\mathrm{ann}_1(W) = \{x \in \mathscr{A}^{d+1-k}(\mathrm{pyr}_{\mathbf{a}}\Delta) \ : \ \mathrm{vol}(xz) = 0 \ \text{ for all } \ z \in W\}. \tag{22}$$

We consider the isomorphism

$$\tau_1 : \mathscr{A}^{d+1-k}(\Delta) \to \mathscr{A}^{d+1-k}(\mathrm{pyr}_{\mathbf{a}}\Delta)$$

of Part (1) of Lemma 3.7. It has the property that for any element $u \in \Bbbk^{d+1-k}[\Delta]$ it holds $\tau_1[u] = [u]$.

We claim that

$$\tau_1(\mathrm{ann}_1(\mathscr{I}^k)) = \mathrm{ann}_1(\mathbf{x_a}\mathscr{I}^k). \tag{23}$$

The inclusion $\tau_1(\mathrm{ann}_1(\mathscr{I}^k)) \subset \mathrm{ann}_1(\mathbf{x_a}\mathscr{I}^k)$ is obvious. We assume that $u \in \Bbbk^{d+1-k}[\Delta]$ has the property that u is not an element of $\mathrm{ann}_1(\mathscr{I}^k)$. By the perfect pairing of Equation (7) there exists $w \in \mathscr{A}^k(\Delta, \partial\Delta)$ such that uw is a nonzero element of $\mathscr{A}^{d+1}(\Delta, \partial\Delta)$, Using the isomorphism of Part (3) of Lemma 3.7 we get that $\mathbf{x}_{\mathbf{a}}uw$ is a nonzero element of $\mathscr{A}^{d+1}(\mathrm{pyr}_{\mathbf{a}}\Delta), \partial\mathrm{pyr}_{\mathbf{a}}\Delta)$. Hence, $\tau_1(u)$ is not an element of $\mathrm{ann}_1(\mathbf{x}_{\mathbf{a}}\mathscr{I}^k)$.

Using Equation (23), we have, similarly to Case 1, a commutative diagram

$$\begin{array}{ccc} \mathscr{I}^k & \xrightarrow{\cdot h^{d+1-2k}} & \mathscr{A}^{d+1-k}(\Delta)/\mathrm{ann}_1(\mathscr{I}^k) \\ \downarrow\wr & & \downarrow\wr \\ \mathbf{x}_{\mathbf{a}}\mathscr{I}^k & \xrightarrow{\cdot \mathbf{x}_{\mathbf{a}}^{d-2k}} & \mathscr{A}^{d+1-k}(\mathrm{pyr}_{\mathbf{a}}\Delta)/\mathrm{ann}_1(\mathbf{x}_{\mathbf{a}}\mathscr{I}^k) \end{array}$$

The horizontal map on the top being an isomorphism is equivalent to the horizontal map on the bottom being an isomorphism.

CASE 3. We assume that p is a homogeneous polynomial of degree s and $\mathscr{I}^* = \mathscr{A}^*(\Delta, \partial\Delta)$. We consider the commutative diagram

$$\begin{array}{ccc} p\mathscr{A}^k(\Delta, \partial\Delta) & \xrightarrow{\cdot h^{d+1-2k-s}} & p\mathscr{A}^{d+1-k-s}(\Delta) \\ \downarrow\wr & & \downarrow\wr \\ p\mathscr{A}^{k+1}(\mathrm{pyr}_{\mathbf{a}}\Delta, \partial\mathrm{pyr}_{\mathbf{a}}\Delta) & \xrightarrow{\cdot \mathbf{x}_{\mathbf{a}}^{d-2k-s}} & p\mathscr{A}^{d+1-k-s}(\mathrm{pyr}_{\mathbf{a}}\Delta) \end{array}$$

The horizontal map on the top being an isomorphism is equivalent to the horizontal map on the bottom being an isomorphism.

CASE 4. We assume that p is a homogeneous polynomial of degree s and $\mathscr{I}^*$ is a nonzero graded submodule of $\mathscr{A}^*(\Delta, \partial\Delta)$. We set $t = d + 1 - k - s$. Arguing similarly as in Case 2, we have that under the natural map

$$p\mathscr{A}^t(\Delta) \to p\mathscr{A}^t(\mathrm{pyr}_{\mathbf{a}}\Delta)$$

of Part (1) of the pyramid Lemma 3.7 the submodule $\mathrm{ann}(p\mathscr{I}^k)$ of $p\mathscr{A}^t(\Delta)$ maps isomorphically onto the submodule $\mathrm{ann}(\mathbf{x}_{\mathbf{a}}p\mathscr{I}^k)$ of $p\mathscr{A}^t(\mathrm{pyr}_{\mathbf{a}}\Delta)$, where ann was defined in Equation (16). Hence, similarly to Case 3 we have a commutative diagram

$$\begin{array}{ccc} p\mathscr{I}^k & \xrightarrow{\cdot h^{d+1-2k-s}} & p\mathscr{A}^{d+1-k-s}(\Delta)/\mathrm{ann}(p\mathscr{I}^k) \\ \downarrow\wr & & \downarrow\wr \\ \mathbf{x}_{\mathbf{a}}p\mathscr{I}^k & \xrightarrow{\cdot \mathbf{x}_{\mathbf{a}}^{d-2k-s}} & p\mathscr{A}^{d+1-k-s}(\mathrm{pyr}_{\mathbf{a}}\Delta)/\mathrm{ann}(\mathbf{x}_{\mathbf{a}}p\mathscr{I}^k) \end{array}$$

The horizontal map on the top being an isomorphism is equivalent to the horizontal map on the bottom being an isomorphism. □

3.9. **Consequences of anisotropy.** With this, we are ready to conclude the following consequence of Theorem 3.2:

Theorem 3.10. *We assume that the characteristic of $\Bbbk$ is 2 or 0. Assume Δ is an IDP lattice disk of dimension d, m is a monomial of degree s in $\Bbbk[\Delta]$, and $k \leq {(d+1-s)}/{2}$. Then there exists a linear element $\ell \in \mathscr{A}^1(\Delta)$ such that $\mathscr{A}^*(\Delta, \partial\Delta)$ satisfies the absolute Hall-Laman relations with respect to the triple (k, m, ℓ).*

Proof. The characteristic 0 case follows from the characteristic 2 by arguing as in Subsection 3.2 in the paper [Karu-Larson-Stapledon, Differential Operators, Anisotropy, and Simplicial Spheres, arXiv v1]. Hence, in the rest of the proof we assume that the field $\Bbbk$ has characteristic 2.

We also remark that since the field $\Bbbk$ is infinite, it is well-known that the existence of a linear element $\ell \in \mathscr{A}^1(\Delta)$ such that $\mathscr{A}^*(\Delta, \partial\Delta)$ satisfies the absolute Hall-Laman relations with respect to the triple (k, m, ℓ) is equivalent to the existance of a nonempty Zariski open subset $W' \subset \mathscr{A}^1(\Delta)$ such that for all $\ell' \in W'$ it holds that $\mathscr{A}^*(\Delta, \partial\Delta)$ satisfies the absolute Hall-Laman relations with respect to the triple (k, m, ℓ').

We set $t = d + 1 - s - 2k$ and do induction on $t \geq 0$.

STEP 1. We assume $t = 0$. We assume ℓ is a nonzero element of $\mathscr{A}^1(\Delta)$. As noted above, the absolute Hall-Laman relations with respect to the triple (k, m, ℓ) are equivalent to proving that all $u \in \mathscr{A}^k(\Delta, \partial\Delta)$ such that $mu \in \mathscr{A}^{k+s}(\Delta, \partial\Delta)$ is nonzero it holds that $mu^2\ell^{d+1-s-2k}$ is a nonzero element of $\mathscr{A}^{d+1}(\Delta, \partial\Delta)$. Since $d + 1 - s - 2k = t = 0$, it is enough to prove that mu^2 is a nonzero element of $\mathscr{A}^{d+1}(\Delta, \partial\Delta)$, which is true by Theorem 3.2.

STEP 2. We assume $t = 1$. We set $Z = \mathrm{pyr}_{\mathbf{a}}\Delta$. By STEP 1, if ℓ is a nonzero element of $\mathscr{A}^1(Z)$ we have that $\mathscr{A}^*(Z, \partial Z)$ satisfies the absolute Hall-Laman relations with respect to the triple $(k+1, m, \ell)$. Hence, $\mathscr{A}^*(Z, \partial Z)$ satisfies the absolute Hall-Laman relations with respect to the triple $(k+1, m, \mathbf{x_a})$. Using the equivalence of Part (3) and Part (4) of Lemma 3.9 it follows that $\mathscr{A}^*(\Delta, \partial\Delta)$ satisfies the absolute Hall-Laman relations with respect to the triple (k, m, h).

STEP 3. We assume $t \geq 2$ and, by inductive hypothesis, that that the statement of the Theorem is true for the values $t - 2$ and $t - 1$. We first prove that for a Zariski general $\ell \in \mathscr{A}^1(\Delta)$ and for any u in $\mathscr{A}^k(\Delta, \partial\Delta)$ such that $mu \neq 0$ we have

$$mu \cdot \ \ell \neq 0.$$

For this, notice that because Δ is IDP and due to Poincaré duality (see Equation (7)) there exists a lattice point v in Δ such that

$$\mathbf{x}_v mu \neq 0.$$

It follows from the inductive case for $t-1$ that the absolute Hall-Laman relations are true relative to the triple $(k, \mathbf{x}_v m, \ell)$, and hence

$$(\mathbf{x}_v m)u^2\ell^{d+1-2k-s-1} \neq 0.$$

In particular,

$$mu \cdot \ell \neq 0$$

since $d+1-2k-s-1 = t-1 \geq 1$.

We set $u' = u\ell$. By the inductive case for $t-2$ we have that $\mathscr{A}^*(\Delta, \partial\Delta)$ satisfies the absolute Hall-Laman relations with respect to the triple $(k-1, m, \ell)$, where ℓ is any Zariski general element of $\mathscr{A}^1(\Delta)$. Hence, $mu' \neq 0$ implies that

$$m(u')^2\ell^{d-s-2k-1} \neq 0.$$

Since

$$mu^2\ell^{d+1-s-2k} = m(u')^2\ell^{d-s-2k-1}$$

it follows that

$$mu^2\ell^{d+1-s-2k} \neq 0. \qquad \square$$

Corollary 3.11. *If Δ is an IDP lattice ball of dimension d, and the characteristic of $\Bbbk$ is 2 or 0, then a generic Artinian reduction $\mathscr{A}^*(\Delta)$ of $\widetilde{\Bbbk}^*[\Delta]$ has the relative Lefschetz property, i.e., there exists a linear element $\ell \in \mathscr{A}^1(\Delta)$ such that for all $k \leq {}^{d+1}/_2$,*

$$\mathscr{A}^k(\Delta, \partial\Delta) \xrightarrow{\cdot\ell^{d+1-2k}} \mathscr{A}^{d+1-k}(\Delta)$$

is an isomorphism.

Proof. The characteristic 0 case follows from the characteristic 2 case by arguing as in Subsection 3.2 in the paper [Karu-Larson-Stapledon, Differential Operators, Anisotropy, and Simplicial Spheres, arXiv v1]. Hence, in the rest of the proof we assume that the field $\Bbbk$ has characteristic 2.

Using Theorem 3.10 for the special case $m = 1$ there exists a linear element $\ell \in \mathscr{A}^1(\Delta)$ such that that $\mathscr{A}^*(\Delta, \partial\Delta)$ satisfies the absolute Hall-Laman relations with respect to the triple $(k, 1, \ell)$. The result follows by applying Lemma 3.4 for $p = 1$ and $\mathscr{I}^* = \mathscr{A}^*(\Delta, \partial\Delta)$. $\square$

Specializing to $\Delta = P$ gives Theorem 2.1. We can conclude that we are left with the task of proving Theorem 3.2.

4. Kustin-Miller normalization of the volume map

Our next goal is to to prove anisotropy for a general IDP lattice disk. Key to this is to understand the fundamental class, that is, the map $\mathrm{vol} : \mathscr{A}^{d+1}(P, \partial P) \to \tilde{\Bbbk}$ for lattice polytopes of dimension d, specifically when we evaluate it at squares: we want to understand $\mathrm{vol}(u^2)$ as a function of u and the linear system of parameters $\theta_{i,j}$.

The map vol is usually called the degree map, but to avoid confusion with the degree of a polynomial, we shall instead call this identification the *volume map* (alluding to the fact that what we are aiming to understand is actually the volume map [Tuc]).

One curiosity of lattice polytopes is that even though we know the canonical module thanks to Hochster (it is, as you will recall, simply the ideal generated by the interior of the cone), we do not actually know of an identification of the top degree with the base field, that is, we do not have a canonical choice of volume map. The proof of Theorem 3.2 is reliant on understanding this, however. In the situation of classical algebraic geometry, there is a canonical such identification, which leads to a classical combinatorial formula in toric geometry [Bri97]. In our case, no such canonical identification seems to have been explored (though we note that in several ways, the one we rely on is canonical). Let us recall it now.

4.1. **Normalizing the volume map.** In [APP24], we provided a convention that, for general Gorenstein rings, provides such a normalization that is canonical in the group quotient $\tilde{\Bbbk}^\times/\Bbbk^\times$. We discuss it again here, with an alternative proof, simplified somewhat because we work in characteristic 2 and do not have to care about signs. It is called the **Kustin-Miller normalization**.

To determine the volume map uniquely, of course, is easy: The vectorspace $\mathscr{A}^{d+1}(P, \partial P)$ is of dimension one, it suffices to give one nontrivial affine condition. A good definition, of course, should come with desirable properties, and we will discuss it here, but leave some more algebraic justifications for the original [APP24].

The Kustin-Miller normalization: We start with the simplest case: Recall that a lattice simplex is called unimodular if the associated semigroup algebra is isomorphic to a polynomial ring. In other words, affine integral combinations of the vertices of the simplex generate the lattice.

Now, if P itself is a unimodular simplex, the situation is classical, see [Bri97].

Prototype. If P is a unimodular simplex, then we obtain the desired normalization by defining

$$1 \;=\; \mathrm{vol}(\mathbf{x}_P)\det(\Theta_{|P}) \tag{24}$$

where $\Theta = (\theta_{i,j})$ is the matrix of coefficients in the linear system of parameters.

This ends up being a good definition. For instance, if Σ is a lattice sphere whose facets are simply unimodular lattice simplices, then the ring obtained is simply the face ring (or Hochster-Reisner-Stanley ring), and the volume map defined in this way coincides with the canonical map from toric geometry.

Simple case. If P has a unimodular boundary facet τ in ∂P, then we obtain the desired normalization by matching the face ring picture and setting

$$1 \;=\; \sum_{p \in P \cap \mathbb{Z}^d} \mathrm{vol}(\mathbf{x}_p \mathbf{x}_\tau) \det(\Theta_{|\tau,p}) \tag{25}$$

where $\Theta = (\theta_{i,j})$ is the matrix of coefficients in the linear system of parameters.

General case. In general, consider a flag (τ_i) of faces of P such that $\tau_d = P$ and such that τ_i is a facet of τ_{i+1}. We say a set $\sigma = \{\sigma_0, \dots, \sigma_d\} \subset (P \cap \mathbb{Z}^d)^{d+1}$ without repetitions is **coherent** with (τ_i) if it intersects τ_i in a set of cardinality $i+1$. We then normalize by setting

$$1 \;=\; \sum_{\sigma \text{ coherent with } (\tau_i)} \mathrm{vol}(\mathbf{x}_\sigma) \det(\Theta_{|\sigma}) \tag{26}$$

The question remains, of course, to establish that this actually a good definition.

4.2. **The normalization is well-defined.**

Theorem 4.1. *The normalization is independent of the flag (τ_i) chosen, and it agrees with the Kustin-Miller normalization of* [APP24].

We shall only recall the first part here, to convince the reader we are justified in choosing the normalization this way. More detail, as well as the second part, is provided in [APP24]. The proof we follow here is a little different from the one provided in the aforementioned paper.

Let us begin with a simple observation:

Lemma 4.2. *If P is a d-dimensional lattice polytope, then the boundary of the pyramid over P is a lattice sphere that contains P as a facet.*

We begin with a *construction*: Given the lattice polytope P of dimension d, we consider the d**th generation porcpupine**:

We start by considering the pyramid over P. Let us call the apex point α_0. It is also the 1st generation porcupine.

The boundary of P has several facets, that is, maximal faces that are of the form $\mathrm{pyr}_{\alpha_0} F$, where is a facet of ∂P.

Consider the pyramids over those facets, each with its own apex $\alpha_{1,F}$. We obtain pyramids $\mathrm{pyr}_{\alpha_{1,F}} \mathrm{pyr}_{\alpha_0} F$, each of which naturally attach to $\mathrm{pyr}_{\alpha_0} P$ along their base, resulting in a lattice ball consisting of $\mathrm{pyr}_{\alpha_0} P$, and the $\mathrm{pyr}_{\alpha_{1,F}} \mathrm{pyr}_{\alpha_0} F$.

The boundary of this ball has several facets of the form $\mathrm{pyr}_{\alpha_{1,F}} \mathrm{pyr}_{\alpha_0} G$, where G ranges over facets of F. This is the 2nd generation porcupine.

It should be apparent to the reader now how porcupines of this kind and nature procreate, so we arrived at the nature of the d-th generation porcupine. Let us denote it by $\mathrm{porc}_d P$.

We need a further small lemma that is immediate from the definition of Artinian reductions:

Lemma 4.3. *Consider any $1 \leq s \leq d+1$, and $\mathbf{x}_I$ a monomial of degree d of $\mathscr{A}(X)$, where X is some d-dimensional lattice complex. Then in $\mathscr{A}(X)$ we have*

$$\sum_{p \text{ latticepoint in } X} \theta_{s,p} \mathrm{vol}(\mathbf{x}_I \mathbf{x}_p) \ = \ 0.$$

Indeed, this last identity arises simply because we quotient by the linear elements

$$\theta_j = \sum_{p \in P \cap \mathbb{Z}^d} \theta_{j,p} \mathbf{x}_p$$

when constructing the Artinian reduction.

We now return to Theorem 4.1. A direct numerical proof is rather uninformative, but we can use an indirect argument without getting our fingers dirty, and that informs what is really happening and how it connects to the naturality of the definition. However, for the reader preferring a down to earth and explicit proof, we refer them to the Appendix B.

An intermediate word is useful: We are working to establish the volume map on $\mathscr{A}^*(P, \partial P)$. The idea now is to think of P as a facet in a larger sphere Σ of the same dimension, and use properties of that larger space. Let us call this a locality principle: The volume is locally defined on P, but is consistent with the algebraic structure of the larger space. It is justified by the following Lemma.

Lemma 4.4. *[Locality principle] Consider Δ a lattice disk of dimension d, and $X \supset \Delta$ a lattice sphere or ball of the same dimension that contains Δ as subcomplex. Then*

$$\mathscr{A}^*(\Delta, \partial\Delta) \ \longhookrightarrow \ \mathscr{A}^*(X, \partial X)$$

Proof. We begin by defining the lattice complex $X - \Delta$ as induced by those facets of X not in Δ. We have a short exact sequence

$$0 \longrightarrow \Bbbk^*[\Delta, \partial\Delta] \longrightarrow \Bbbk^*[X, \partial X] \longrightarrow \Bbbk^*[(X - \Delta) \cup \partial X, \partial X] \longrightarrow 0.$$

All three modules in this sequence are Cohen-Macaulay, hence the sequence remains exact after Artinian reduction. The claim follows. □

Proof of Theorem 4.1. To see how to apply this, let us start by considering a lattice d-sphere Σ. Given a maximal flag $T = (\tau_i)$ of faces of Σ, we denote the sum

$$\mathscr{N}[T] := \sum_{\sigma \text{ coherent with } (\tau_i)} \mathrm{vol}(\mathbf{x}_\sigma) \det(\Theta_{|\sigma}). \tag{27}$$

We now consider the boundary $\partial \mathrm{porc}_d P$ of the d-th generation porcupine. It is a lattice sphere, and therefore the resulting algebra is Gorenstein [BBR07].

Notice that it has several unimodular facets, in particular those of the form

$$\mathrm{pyr}_{\alpha_{d-1,S}} \cdots \mathrm{pyr}_{\alpha_{1,F}} \mathrm{pyr}_{\alpha_0} v,$$

where v is a vertex of P and $S \subset \ldots \subset G \subset F$ is a full flag. We already know the natural normalization on these unimodular faces, and it states

$$1 \;=\; \mathscr{N}[(v, \mathrm{pyr}_{\alpha_0} v, \mathrm{pyr}_{\alpha_{1,F}} \mathrm{pyr}_{\alpha_0} v, \cdots, \mathrm{pyr}_{\alpha_{d-1,S}} \cdots \mathrm{pyr}_{\alpha_{1,F}} \mathrm{pyr}_{\alpha_0} v)].$$

It suffices to prove that the normalization of Equation (26) is consistent with the normalization of the unimodular simplex.

Hence, we want to prove that for an arbitrary maximal flag (τ_i) ending with $\tau_d = P$, the sum $\mathscr{N}[T]$ equals such a term. The proof is simple and involves an alternating application of Lemma 4.3 and a simple rewrite of the sum (27): We have

$$\begin{aligned}
\mathscr{N}[T] \;&=\; \mathscr{N}[(\tau_0, \tau_1, \cdots, \tau_{d-1}, \tau_d)] \\
&\overset{2}{=}\; \mathscr{N}[(\tau_0, \tau_1, \cdots, \tau_{d-1}, \mathrm{pyr}_{\alpha_{1,\tau_{d-1}}} \tau_{d-1})] \\
&\overset{3}{=}\; \mathscr{N}[(\tau_0, \tau_1, \cdots, \mathrm{pyr}_{\alpha_{1,\tau_{d-1}}} \tau_{d-2}, \mathrm{pyr}_{\alpha_{1,\tau_{d-1}}} \tau_{d-1})] \\
&=\; \cdots \\
&=\; \mathscr{N}[(\tau_0, \mathrm{pyr}_{\alpha_0} \tau_0, \mathrm{pyr}_{\alpha_{1,\tau_{d-1}}} \mathrm{pyr}_{\alpha_0} \tau_0, \cdots, \mathrm{pyr}_{\alpha_{d-1,\tau_1}} \cdots \mathrm{pyr}_{\alpha_{1,\tau_{d-1}}} \mathrm{pyr}_{\alpha_0} \tau_0)]
\end{aligned}$$

Hence $\mathscr{N}[T]$ equals the canonical normalization on a unimodular simplex, which is indepentent of the simplex. Hence the $\mathscr{N}[T]$ coincide. □

[2] This is Lemma 4.3.
[3] This is trivial.

5. Differential equations for the volume map

This definition of the volume as well as the anisotropy we wish to prove is dependent of our choice of linear system of parameters θ_i, so we will consider vol as a rational function in the variables $\theta_{i,j}$. This allows us to formulate the auxiliary lemma on the way to anisotropy. For a family of lattice points A, we denote by $|A|$ their sum.

Lemma 5.1. *Consider P a $(2k-1+j)$-dimensional polytope, the volume normalized, and $F = (f_1, \dots, f_{2k+j}) \subset (P \cap \mathbb{Z}^d)^{2k+j}$ a collection of lattice points, and $\sigma = (\sigma_1, \dots, \sigma_j) \subset (P \cap \mathbb{Z}^d)^j$ is another family of j lattice points. Assume that*

$$|F| - |\sigma| = \sum f_i - \sum \sigma_j = 2|G|$$

for some family of lattice points G. Then, in $\mathscr{A}(P)$ considered over $\Bbbk(\theta_{i,j})$, where $\Bbbk$ is of characteristic two, we have

$$\partial_F \mathrm{vol}(u^2) \ = \ \mathrm{vol}_\sigma(u \cdot \mathbf{x}_G)^2.$$

Here, ∂_F is the differential operator obtained as the composition of the composition of differentials after the varialbles $\theta_{1,f_1}, \theta_{2,f_2}, \theta_{3,f_3}, \theta_{4,f_4}, \theta_{5,f_5}, \dots$, that is,

$$\frac{\partial}{\partial\theta_{1,f_1}} \frac{\partial}{\partial\theta_{2,f_2}} \frac{\partial}{\partial\theta_{3,f_3}} \frac{\partial}{\partial\theta_{4,f_4}} \frac{\partial}{\partial\theta_{5,f_5}} \cdots,$$

applied to the volume as a rational function in variables $(\theta_{i,j})$, and vol_σ is the volume map with respect to the pullback to $\mathscr{A}(P) \big/ \mathrm{ann}\, \mathbf{x}_\sigma$.

We postpone the proof of this lemma to the end of the next section, where we will derive it from the key identity of Parseval-Rayleigh type.

From this differential identity, anisotropy follows at once.

Proof of Theorem 3.2. Consider u of degree $k \leq \frac{d+1}{2}$ in $\mathscr{A}^k(\Delta, \partial\Delta)$. The pairing

$$\mathscr{A}^k(\Delta, \partial\Delta) \times \mathscr{A}^{d+1-k}(\Delta)$$

is nondegenerate. Since Δ is IDP, each $P \in \Delta$ is IDP, there exist $P \in \Delta$, and families of lattice points $\sigma = \{\sigma_1, \dots, \sigma_{d+1-2k}\} \subset (P \cap \mathbb{Z}^d)^{d+1-2k}$ and $G = \{g_1, \dots, g_k\} \subset (P \cap \mathbb{Z}^d)^k$ so that $\mathbf{x}_\sigma \cdot \mathbf{x}_G \cdot u$ is not 0.

Following the differential identity, and using $G + G + \sigma$ to denote the concatenation of two copies of the family G and one copy of σ, $\mathrm{vol}_\sigma(u^2)$ is not zero as

$$\partial_{G+G+\sigma} \mathrm{vol}(u^2) \ = \ \mathrm{vol}_\sigma(u \cdot \mathbf{x}_G)^2.$$

Hence u^2 is not 0. $\square$

6. LEVEL PROPERTIES AND ANOTHER LEFSCHETZ/ANISOTROPY THEOREM

But we also obtain other anisotropy theorems from here. Consider Theorem 2.4. Its Lefschetz variant is the following:

Theorem 6.1. *The semigroup algebra of an IDP lattice polytope P of dimension d with $\mathrm{cone}^\circ(P)$ generated at height $\leq j$ satisfies an almost Lefschetz theorem: We have that*

$$\mathscr{A}(P)^k \xrightarrow{\cdot \ell^{d+1-j-2k}} \mathscr{A}(P)^{d+1-j-k}$$

is an injection for some ℓ in $\mathscr{A}(P)^1$ and every $k \leq \lceil \frac{d-j}{2} \rceil$.

It is the consequence of two lemmas. We need the concept of an **interior simplex**, that is, a collection of lattice points that do not lie in a common face of the polytope. The first is a version of partition of unity in the given situation [AY20]:

Lemma 6.2. *In the semigroup algebra of an IDP lattice polytope P of dimension d with $\mathrm{cone}^\circ(P)$ generated at height $\leq j$ we have an injection*

$$\mathscr{A}(P) \longhookrightarrow \bigoplus_{\sigma \text{ interior cardinality } j \text{ simplex}} \mathscr{A}(P)\big/_{\mathrm{ann}\, \mathbf{x}_\sigma}$$

Proof. $\mathscr{A}(P)$ is Poincaré dual to $\mathscr{A}(P, \partial P)$, which in turn is generated by elements of degree $\leq j$ by assumption. $\square$

The second is a Lefschetz type fact:

Lemma 6.3. *If σ is a j simplex of an IDP lattice d-polytope P satisfies the Lefschetz property: We have that*

$$\mathscr{A}^k(P)\big/_{\mathrm{ann}\, \mathbf{x}_\sigma} \xrightarrow{\cdot \ell^{d+1-j-2k}} \mathscr{A}^{d+1-j-k}(P, \partial P)\big/_{\mathrm{ann}\, \mathbf{x}_\sigma}$$

is an injection for some ℓ in $\mathscr{A}(P)^1$ and every $k \leq \lceil \frac{d-j}{2} \rceil$.

The underlying anisotropy lemma goes as follows.

Lemma 6.4. *The semigroup algebra of an IDP lattice complex of dimension d with $\mathrm{cone}^\circ(P)$ generated at height $\leq j$ satisfies anisotropy in every pullback to an interior simplex σ of cardinality at most j, every nonzero element u in $\mathscr{A}(P)\big/_{\mathrm{ann}\, \mathbf{x}_\sigma}$ of degree $k \leq \lceil \frac{d-j}{2} \rceil$ has nonzero square.*

That the latter implies the former follows by the same lifting trick we used to prove the Lefschetz property earlier. And that the latter is true, just like Theorem 3.2, follows at once from Lemma 5.1: Because of Poincaré duality, we have that $\mathrm{vol}_\sigma(u \cdot \mathbf{x}_G)$ is nonzero for some G. Hence $\partial_F \mathrm{vol}(u^2)$, and in particular, $\mathrm{vol}_\sigma(u^2)$ is nonzero for $F = G + G + \sigma$.

7. Locality in lattice sheaves

The locality Lemma 4.3 and Theorem 4.1 imply that for a facet P of a lattice sphere or ball X, the volume map on $(P, \partial P)$ coincides with the volume map on $(X, \partial X)$, restricted to P.

It behoves us to ask more questions. For instance, consider the following case:

Consider for instance a lattice polytope P of dimension d, and a lattice polytope Q of the same dimension inside it, and of the same dimension. Consider a monomial m of $\mathbb{k}^{d+1}[Q, \partial Q]$. What can be said of the relation of $\mathrm{vol}_Q(m)$ and $\mathrm{vol}_P(m)$ in $\mathscr{A}^*(Q, \partial Q)$ resp. $\mathscr{A}^*(P, \partial P)$? Unlike in the previous case, they do not coincide.

We will ask such questions here, and prove two theorems. Let us for this purpose introduce another parameter t. Given a set of lattice points V in a polytope P, we wish to study the following variation of the generic linear system of parameters: Instead of using the linear system of parameters $\theta_{i,j}$, we use a modified system of parameters

$$\theta_{i,j}^V[t] = \begin{cases} t\theta_{i,j} & \text{if } j \in V \\ \theta_{i,j} & \text{otherwise} \end{cases}$$

We denote the corresponding Artinian reduction by $\mathscr{A}^*(P, \partial P)[\theta_{i,j}^V[t]]$

7.1. Finer lattices.

Consider the following situation: P is a d-dimensional lattice polytope with lattice $\mathbb{Z}^d$. And Λ is some finer lattice: it contains $\mathbb{Z}^d$ as a strict subset. Of course, we could consider $\mathscr{A}^*(P, \partial P)[\mathbb{Z}^d]$, that is, P and the semigroup algebra with respect to the lattice $\mathbb{Z}^d$. But we could equally consider $\mathscr{A}^*(P, \partial P)[\Lambda]$.

A prototype of such a situation is to consider P, and a positive dilate nP.

In either case, how do they relate? The answer is actually easy:

Theorem 7.1. *If V consists of those lattice points of $\Lambda \setminus \mathbb{Z}^d$ that lie in P, we have*

$$\mathscr{A}^*(P, \partial P)[\Lambda][\theta_{i,j}^V[0]] \;=\; \mathscr{A}^*(P, \partial P)[\mathbb{Z}^d].$$

Moreover, if m is a monomial of degree $d+1$ in the $\mathbb{k}^*[P, \partial P]$, then, marking down the obvious dependencies, we have

$$\mathrm{vol}_{P,\Lambda,[\theta_{i,j}^V[t]]}(m) - \mathrm{vol}_{P,\mathbb{Z}^d,[\theta_{i,j}]}(m)$$

is a rational function that vanishes at $t = 0$.

Both of these facts are obvious. We come to a slightly more intericate case.

7.2. Bigger polytopes.

We now go back to the original situation:

Theorem 7.2. *Consider a lattice polytope P of dimension d, and a lattice polytope Q of the same dimension inside it. Consider a monomial m of $\Bbbk^{d+1}[Q, \partial Q]$, and assume additionally that Q is obtained from P by cutting the latter with a halfspace delimited by hyperplane H. Let V denote the lattice points of P not in Q. Then*

$$\mathrm{vol}_{P,\Lambda,[\theta^V_{i,j}[t]]}(m) - \mathrm{vol}_{Q,\mathbb{Z}^d,[\theta_{i,j}]}(m)$$

is a rational function that vanishes at $t = 0$.

Remark 7.3. In fact, the additional assumption is not necessary, but that requires some further elaboration that we will only be able to discuss in the next section. We contend ourselves with this version, which is enough for our purposes.

Proof. Consider the polytopes P, Q, and the complement Q' of Q in P. On each of these individually, the linear relations, combined with the affine Kustin-Miller normalization, determines the volume map uniquely.

Let us see how this works, first on the smaller polytope Q: The volume map $\mathrm{vol}_{Q,\mathbb{Z}^d,[\theta_{i,j}]}$ is determined by an invertible square matrix C^Q and an equation

$$\mathrm{C}^Q \mathbf{m}_Q = e_Q$$

where $\mathbf{m}$ is the vector of degree $d+1$ monomials in $\Bbbk[Q]$, and C^Q is the matrix of linear relations arising from the linear system of parameters and the Kustin-Miller normalization, and $e = (0, \cdots, 0, 1)$ is a vector of appropriate length.

Similarly, we play the game with P and $\mathrm{vol}_{P,\Lambda,[\theta^V_{i,j}[t]]}$ and get analogously an equation matrix

$$\mathrm{C}^P \mathbf{m}_P = e_P.$$

We can do the same for Q', Kustin-Miller normalized at $Q' \cap H$.

Let us examine the relation of the two matrices: They are of the form

$$\mathrm{C}^P = \begin{pmatrix} \mathrm{C}^{Q'} & B \\ A & \mathrm{C}^Q \end{pmatrix}$$

Perform row operations using the fact that $\mathrm{C}^{Q'}$ is of full rank, and we can transform to

$$\mathrm{C}'_P = \begin{pmatrix} \mathrm{C}^{Q'} & B \\ 0 & \mathrm{C}^Q + tA' \end{pmatrix}$$

where is A' is a matrix without pole at $t = 0$. The equation determining $\mathrm{vol}_{P,\Lambda,[\theta^V_{i,j}[t]]}$ becomes

$$\widetilde{\mathrm{C}}^P \mathbf{m}_P = e_P.$$

To summarize, the entries of $\mathrm{vol}_{P,\Lambda,[\theta^V_{i,j}[t]]}(m)$ are determined by the entries of the matrix

$$(\mathrm{C}^Q + tA')^{-1}e_Q,$$

the entries of $\mathrm{vol}_{Q,\mathbb{Z}^d,[\theta_{i,j}]}(m)$ are determined by the entries of the matrix

$$(\mathrm{C}^Q)^{-1}e_Q,$$

Hence, we need to understand the difference $\left((\mathrm{C}^Q + tA')^{-1} - (\mathrm{C}^Q)^{-1}\right) e_Q$.

The rest is easy. Using, for instance, Cramer's rule, it is easy to see that the entries of $(\mathrm{C}^Q + tA')^{-1} - (\mathrm{C}^Q)^{-1}$ are rational functions in t that vanish at 0. The claim follows. $\square$

8. Parseval-Rayleigh identities

While the differential equation of Lemma 5.1 is superficially similar to identities proven in the case of simplicial cycles in [APP21, PP20], where they follow immediately from the known formulas for the volume map in toric varieties, the case of lattice polytopes is much harder: we understand the volume map only indirectly, using a nonhomogeneous equation that takes the form of an identity of the Parseval-Rayleigh type. We consider lattice polytopes of dimension d in $\mathbb{Z}^d$.

Assume $v = (v_1, \ldots, v_{d+1}) \in (P \cap \mathbb{Z}^d)^{d+1}$. We set as usual

$$\mathbf{x}_v \ = \prod_{1\le i\le d+1} \mathbf{x}_{v_i}.$$

Note that since we are working in the semigroup algebra, this only depends on

$$|v| = v_1 + v_2 + \cdots + v_{d+1},$$

the sum over the entries of v within the semigroup $\mathrm{cone}(P) \cap (\mathbb{Z}^d \times \mathbb{Z})$.

Theorem 8.1. *For a lattice d-polytope P, and α is a lattice point of $\mathrm{cone}(P) \cap (\mathbb{Z}^d \times \{d+1\})$ we have in $\mathscr{A}^*(P, \partial P)$ over characteristic 2*

$$\mathrm{vol}(\mathbf{x}_\alpha) \ = \sum_{\beta \in (P\cap\mathbb{Z}^d)^{d+1}} \mathrm{vol}(\mathbf{x}_{\frac{\alpha+\beta}{2}})^2 \theta^\beta \tag{28}$$

Remark 8.2. Note that we do not require α to be a lattice point in the interior of $\mathrm{cone}(P) \cap (\mathbb{Z}^d \times \{d+1\})$. If α is a boundary point

Here, we follow the convention $\theta^\beta := \prod_{1\le i\le d+1} \theta_{i,\beta_i}$. Moreover, $\mathrm{vol}(\mathbf{x}_{\frac{\alpha+\beta}{2}})$ is defined to be $\mathrm{vol}(\mathbf{x}_\gamma)$ if there is an $\mathbf{x}_\gamma \in \Bbbk[P]$ such that $\mathbf{x}_\alpha \mathbf{x}_\beta = \mathbf{x}_\gamma^2$, and 0 otherwise.

This specializes to the following identity for $\alpha = \sigma + 2\alpha'$, which explains the naming of this identity:

Lemma 8.3. *For a lattice d-polytope P, in $\mathscr{A}^*(P, \partial P)$ over characteristic 2, and σ a family of lattice points of P and for $d+1+\#\sigma$ even, we have*

$$\mathrm{vol}(\mathbf{x}_\sigma \mathbf{x}_{\alpha'}^2) \ = \sum_{\beta \in (P \cap \mathbb{Z}^d)^{d+1}} \mathrm{vol}(\mathbf{x}_{\alpha'} \cdot \mathbf{x}_{\frac{\sigma+\beta}{2}})^2 \theta^\beta \tag{29}$$

where $\sigma + \beta$ denotes the concatenation of the families σ and β.

From here we conclude identities deserving their name:

Lemma 8.4 (The Parseval-Rayleigh identity)**.** *For a lattice d-polytope P, in $\mathscr{A}^*(P, \partial P)$ over characteristic 2, and σ an interior simplex and for $d+1+\#\sigma$ even, we have*

$$\mathrm{vol}(\mathbf{x}_\sigma u^2) \ = \sum_{\beta \in (P \cap \mathbb{Z}^d)^{d+1}} \mathrm{vol}(u \cdot \mathbf{x}_{\frac{\sigma+\beta}{2}})^2 \theta^\beta. \tag{30}$$

This extends to polynomials u defined over $\Bbbk^*[P]$.

Here for a non-monomial $\mathbf{x}_\sigma u^2$ we get the result by a quirk of working over characteristic 2:

$$\begin{aligned}
\mathrm{vol}(\mathbf{x}_\sigma u^2) \ &= \ \mathrm{vol}\left(\mathbf{x}_\sigma \left(\sum_a \lambda_a \mathbf{x}_a\right)^2\right) \ = \ \sum_a \lambda_a^2 \, \mathrm{vol}(\mathbf{x}_\sigma \mathbf{x}_a^2) \\
&= \sum_a \lambda_a^2 \sum_{\beta \in (P \cap \mathbb{Z}^d)^{d+1}} \mathrm{vol}(\mathbf{x}_a \cdot \mathbf{x}_{\frac{\sigma+\beta}{2}})^2 \theta^\beta \\
&= \sum_{\beta \in (P \cap \mathbb{Z}^d)^{d+1}} \mathrm{vol}\left(\sum_a \lambda_a \, \mathbf{x}_a \cdot \mathbf{x}_{\frac{\sigma+\beta}{2}}\right)^2 \theta^\beta \\
&= \sum_{\beta \in (P \cap \mathbb{Z}^d)^{d+1}} \mathrm{vol}(u \cdot \mathbf{x}_{\frac{\sigma+\beta}{2}})^2 \theta^\beta
\end{aligned}$$

In order to prove the Parseval-Rayleigh identity, we note a basic identity for the volume:

Lemma 8.5. *Consider any two elements $I, J \in (P \cap \mathbb{Z}^d)^d$, where at least one point of I lies in the interior of P, and an index $\mu \in [d+1]$. Then we have*

$$\mathbf{R}[J, \mu, |I|] \ := \sum_{p \in P \cap \mathbb{Z}^d} \theta^{J *_\mu p} \mathrm{vol}(\mathbf{x}_I \mathbf{x}_p) \ = \ 0.$$

Here, $J *_\mu p$ is the vector family J to which p is inserted at place μ.

This is an immediate consequence of Lemma 4.3.

8.1. **The case of the simplex.** To simplify the proof of the Theorem 8.1, we first prove a variant:

Lemma 8.6. *Consider a lattice d-dimplex S that is a dilation of a unimodular simplex, and let α denote the sum $\sum v$, where v ranges over the vertices of $S \times \{1\}$. Then*

$$\mathrm{vol}(\mathbf{x}_\alpha) \ = \sum_{\beta \in (S \cap \mathbb{Z}^d)^{d+1}} \mathrm{vol}(\mathbf{x}_{\frac{\alpha+\beta}{2}})^2 \theta^\beta \tag{31}$$

Proof. First, we require some notation: As before, for a family of lattice points $v = (v_1, \dots, v_q) \in (S \cap \mathbb{Z}^d)^q$, we denote by $|v|$ their sum $v_1 + v_2 + \cdots + v_q$.

Next, given a $d+1$-tuple γ in $(S \cap \mathbb{Z}^d)^{[d+1] \setminus \{\mu\}}$ and $\iota \in \mathrm{cone}(S) \cap (\mathbb{Z}^d \times \{d\})$, and α as in the statement of the lemma, we denote by $t(\gamma, \iota)$ the lattice point

$$t(\gamma, \iota) = \alpha + |\gamma| - 2\iota$$

in $\mathbb{Z}^d \times \{1\}$ and note that it does not necessarily lie in $S \times \{1\}$.

Now, let us discuss the trick of the proof: We want to find linear combinations of $\mathbf{R}[J, \mu, |I|]$ such that

$$\sum_{\beta \in (S \cap \mathbb{Z}^d)^{d+1}} \mathrm{vol}(\mathbf{x}_{\frac{\alpha+\beta}{2}})^2 \theta^\beta - \sum \lambda_{J,\mu,|I|} \mathbf{R}[J, \mu, |I|]$$

equals $\mathrm{vol}(\alpha)$ times the volume normalization. Before we take things further, we note that the coefficient $\lambda_{J,\mu,|I|}$ will take a simple form: it is

$$\lambda_{J,\mu,|I|} \ = \ \delta_{J,\mu,|I|} \mathrm{vol}(\mathbf{x}_{\alpha+|J|-|I|})$$

where $\delta_{J,\mu,|I|}$ is 0 or 1.

Given γ and ι, we have the simplex $\gamma *_\mu t(\gamma, \iota)$ of cardinality $d+1$. We now want to define an order on d-simplices that says that $\delta_{J,\mu,|I|}$ is nonzero if J is the least cardinality d subset of $J *_\mu t(J, I)$. To achieve this, we define a perturbation vector: These are simply $d+1$ generic and small enough vectors $\rho = (\vec{\rho}_b)$ in $\mathbb{R}^d$ that sum to 0.

Let us now illustrate what happens: We start with $d = 1$. Then S is a lattice segment, and α is its center. Let h be a generic linear functional $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and let us distinguish two cases: if $0 \geq h(|j| - |i|)$, we simply determine that $\delta_{j,\mu,i} = 1$ if μ is the maximal column of $\{j\} *_\mu t(j,i) + \rho$ under h, or in other words, the column corresponding to j is the minimal one. If $0 < h(|j| - |i|)$, we set $\delta_{j,\mu,i} = 1$ if μ corresponds to the minimal column.

Now, let us consider the sum

$$\sum \delta_{j,\mu,|i|} \mathrm{vol}(\mathbf{x}_{\alpha+|j|-|i|}) \mathbf{R}[j, \mu, |i|] \tag{32}$$

blindly, without second thought. Then we immediately find the resulting sum to be precisely $\sum_{\beta \in (S \cap \mathbb{Z}^d)^{d+1}} \mathrm{vol}(\mathbf{x}_{\frac{\alpha+\beta}{2}})^2 \theta^\beta$. Alas, there is an issue that in our blind joy we overlooked: $\mathbf{R}[j,\mu,|i|]$ is only defined if i is in the interior of S. That, however, can fail in our rule, and δ can be nonzero specifically if i a boundary vertex. This arises only if i is the minimal vertex m of S under h, and $j=i$.

In other words, we cannot use those terms in the sum. Hence, if we actually only sum over admissible j and i, we are left with an error of exactly those terms:

$$\sum \mathrm{vol}(\mathbf{x}_\alpha)\mathrm{vol}(\mathbf{x}_{m+t})\det(\Theta_{|\{m,p\}})$$

Now, for general d, we are a bit more explicit with the choice of h. Or rather, we need several functions.

First, let us consider S, a lattice simplex of dimension d. It is subdivided canonically into $d+1$ parallelohedra along the barycenters of faces: We consider the barycentric subdivision of S, whose central vertex, the barycenter of S, is $b = {}^{\alpha}/_{d+1}$.

Now, group up all those simplices of the subdivision incident to a single vertex v of S. Call their union C_v, they form a parallelohedron.

Finally, order the vertices of S from 1 to $d+1$, that, $v_1, \cdots, v_{d+1}$. Finally, as before, we have the

Now, given a triple (J,μ,I), we first associate it to a unique parallelohedron. Choose the minimal i so that C_{v_i} contains $\alpha+|J|-|I|$.

Next, consider the vector $h_i = b - v_i$.

We set $\delta_{J,\mu,|I|}=1$ if $\mu\in\{1,2\}$ and out of the first two column vectors of $J *_\mu t(J,|I|)+\rho$, the column corresponding to μ is the maximal one under interior product with h_i.

We now sum

$$\sum \delta_{J,\mu,|I|}\mathrm{vol}(\mathbf{x}_{\alpha+|J|-|I|})\mathbf{R}[J,\mu,|I|] \tag{33}$$

keeping in mind that I needs to be interior of the cone over S. As before, these boundary issues prevent the sum from simply equalling the right hand side of the Parseval formula, and it is easy to see that after some basic cancellations, the error is given by $\mathrm{vol}(\mathbf{x}_\alpha)$ times normalization. $\square$

8.2. **The general case.** We now obtain the proof of the general Parseval-Rayleigh identities. For this, we only need to use Theorems 7.2 and Theorem 7.1, as well as Lemma 8.6 of course.

Let us start with a pair of a rational polytope Q, and an interior point α. Consider a polytope P containing Q.

We say that the triple $(P,Q;\alpha)$ tight if **tight** if:

- $Q = P \cap H$, where H is some halfspace, and both P and Q are of the same dimension, and
- For every point v in P, the point ${v+\alpha}/{2}$ lies in the interior of Q.

The two crucial lemmata now are the following:

Lemma 8.7. *Given any rational polytope Q and a rational interior point α, there exists a finite sequence (P_i) $i = 0, \cdots, n$ of rational polytopes such that*

- *P_n is a dilation of a unimodal simplex, and α is its barycenter.*
- *$P_0 = Q$.*
- *$(P_{i+1}, P_i; \alpha)$ is tight.*

Proof. This is clear by gradually moving out the hyperplanes defining Q until we are left with a simplex. Moving them out further assures we can make α the barycenter. Doing this in discrete, small enough steps gives the finite sequence. □

We now return to lattice polytopes:

Lemma 8.8. *Consider Q a lattice polytope of dimensionn d, and α a point in $\mathrm{cone}^\circ(Q) \cap (\mathbb{Z}^d \times \{d+1\})$, and P a lattice polytope so that the triple $(P, Q; {\alpha}/{d+1})$ is tight. Assume that the Parseval-Rayleigh identity of Theorem 8.1 holds for P and α. Then it holds for Q and α.*

Proof. This is an immediate consequence of Theorem 7.2. □

Let us similarly note a curious consequence of Theorem 7.1.

Lemma 8.9. *Consider Q a lattice polytope of dimensionn d with respect to a lattice Λ, let $\Lambda' \supset \Lambda$ denote some finer lattice, and α a point in $\mathrm{cone}^\circ(Q) \cap (\Lambda' \times \{d+1\})$. Assume that the Parseval-Rayleigh identity of Theorem 8.1 holds for Q and α with respect to Λ'. Then*

(1) Consider the set V of lattice points of $Q \cap \Lambda'$ not in Λ. Then, if $\alpha \notin \mathrm{cone}^\circ(Q) \cap (\Lambda \times \{d+1\})$, we have

$$\mathrm{vol}_{Q,\Lambda',[\theta^V_{i,j}[0]]}(\mathbf{x}_\alpha) \;=\; 0$$

(2) The Parseval-Rayleigh identities hold for Q and α with respect to Λ.

Remark 8.10. It is in fact not hard to see that Lemma 8.9 can be proven directly, and holds independently of the characteristic of the underlying field.

Example 8.11. We give an example to demonstrate Lemma 8.9. Assume $\mathbb{k}$ is a field of characteristic 2, $\Lambda' = \mathbb{Z}^2$, $\Lambda = 2\Lambda'$ and $P \subset \mathbb{R}^2$ is the convex hull of the set of points $\{(0,0), (0,2), (2,0)\}$. The lattice point set of P with respect to the lattice Λ' is the set

$$\{q_1 = (0,0),\ q_2 = (0,1),\ q_3 = (0,2),\ q_4 = (1,0),\ q_5 = (1,1),\ q_6 = (2,0)\},$$

while the lattice point set of P with respect to the lattice Λ is the set $\{q_1, q_3, q_6\}$. For $1 \leq i \leq 3$, we denote by θ_i a general linear combination of the variables x_1, x_3, x_6. Then $\Theta = (\theta_1, \ldots, \theta_3)$ is a linear system of parameters for both $\Bbbk_\Lambda[P]$ and $\Bbbk_{\Lambda'}[P]$ and the following holds: Assume $m = \prod_{i=1}^6 x_i^{a_i}$ is a monomial in $\Bbbk[x_1, \ldots, x_6]$ of degree 3 such that $m \in I_{\partial P}$. If $\sum_{i=1}^6 a_i q_i \notin \Lambda$, then the class of m in $\mathscr{A}^3_{\Lambda',\Theta}(P, \partial P)$ is zero. Finally, we mention that by Remark 8.10 the same results holds in any characteristic.

Proof of Theorem 8.1. Consider a given lattice polytope Q and α a point in $\mathrm{cone}^\circ(Q) \cap (\mathbb{Z}^d \times \{d+1\})$. We may assume that $\alpha = (0, \cdots, 0, d+1)$.

We want to prove Theorem 8.1 for Q with respect to α. Consider a sequence P_i as given by Lemma 8.7. Since the polytopes involved are rational, we can find a sufficiently large dilation NP_i such that all involved polytopes are lattice polytopes.

We conclude from Lemma 8.8 and Lemma 8.6 that the Parseval-Rayleigh identities hold for NQ with respect to α. It follows that the Parseval-Rayleigh identities hold for Q by Lemma 8.9. $\square$

9. Differential equations via Parseval-Rayleigh identities

We now prove the differential equations.

Proof of Lemma 5.1 using Lemma 8.4. We give the argument for σ empty; the case for nonempty σ follows by pullback (or, in fact, an analogous calculation, using Theorem 8.1 applied to $\alpha = \sigma + 2\alpha'$).

For P a $(2k-1)$-dimensional polytope, the volume normalized and F a family of $2k = d+1$ lattice points in P, let $u \in \mathscr{A}^k(P, \partial P)$. We have the Parseval-Rayleigh identity

$$\mathrm{vol}(u^2) \;=\; \sum_{\beta \in (P \cap \mathbb{Z}^d)^{d+1}} \mathrm{vol}(u \cdot \mathbf{x}_{\frac{\beta}{2}})^2 \cdot \theta^\beta$$

Now differentiation by F yields

$$\begin{aligned} \partial_F \mathrm{vol}(u^2) \;&=\; \sum_{\beta \in (P \cap \mathbb{Z}^d)^{d+1}} \partial_F \left(\mathrm{vol}(u \cdot \mathbf{x}_{\frac{\beta}{2}})^2 \cdot \theta^\beta \right) \\ &=\; \sum_{\beta \in (P \cap \mathbb{Z}^d)^{d+1}} \mathrm{vol}(u \cdot \mathbf{x}_{\frac{\beta}{2}})^2 \cdot \partial_F \theta^\beta, \end{aligned}$$

in characteristic two, as differentiating $\mathrm{vol}(u \cdot \mathbf{x}_{\frac{\beta}{2}})^2$ introduces a factor of 2.

Now, $\partial_F \theta^\beta = 1$ if and only if $F = \beta$, and $\partial_F \theta^\beta = 0$ otherwise, so we get the desired identity

$$\partial_F \mathrm{vol}(u^2) \ = \ \mathrm{vol}(u \cdot \mathbf{x}_F)^2. \qquad \square$$

10. Beyond the integer decomposition property

Before we discuss open questions, let us point out again that the integer decomposition property was used once, and once only: when concluding the anisotropy property from the differential relations/Parseval-Rayleigh identities. Let us return therefore to Theorem 3.2 as an example. As a reminder: it states that if X is an IDP lattice ball or sphere of dimension d, and the characteristic of $\Bbbk$ is 2, then a generic Artinian reduction $\mathscr{A}^*(X, \partial X)$ of $\widetilde{\Bbbk}^*[X, \partial X]$ has the anisotropy property for $\mathscr{A}$ over $\Bbbk(\theta_{i,j})$: for every nontrivial $u \in \mathscr{A}^k(X, \partial X)$ of degree $k \leq {}^{d+1}/_2$, we have

$$u^2 \ \neq \ 0.$$

Now, if we want to remove the integer decomposition property, we have to consider the subalgebra of $\mathscr{A}^*(X)$ generated by elements of degree 1. Let us denote it by $\mathscr{S}^*(X)$, that is, the standard subalgebra of $\mathscr{A}^*(X)$. We then have the following:

Proposition 10.1 (Anisotropy in lattice polytopes)**.** *If X is an lattice ball or sphere of dimension d, and the characteristic of $\Bbbk$ is 2, then a generic Artinian reduction $\mathscr{A}^*(X, \partial X)$ of $\widetilde{\Bbbk}^*[X, \partial X]$ has a weak anisotropy property for $\mathscr{A}$ over $\Bbbk(\theta_{i,j})$, that is, for every nontrivial $u \in \mathscr{A}^k(X, \partial X)$ of degree $k = {}^{d+1}/_2$ that pairs with some element of $\mathscr{S}^*(X)$, we have*

$$u^2 \ \neq \ 0.$$

11. Outlook and open questions

The non-lattice cases of Stanley's conjecture remain. Even for IDP lattice polytopes or stronger yet, for lattice polytopes with a regular unimodular triangulation, we are left with a gap in the inequalities restricting h^* if the interior of the cone is generated in higher degree. We conjecture that the Lefschetz property, and in fact the unimodality of h^* fails in general.

The intuition here is that lattice polytopes behave like triangulated disks, which can have non-unimodal h-vectors. The idea here could rely on constructing appropriate connected sums: as we saw above, Gorenstein polytopes have h^*-polynomials that peak at half of their socle degree (which is $d + 1 - s$, s being the minimal dilation constant so that the polytope has an interior vertex). By connected sum of polytopes with different socle degree, one could hope to turn a dromedary into a camel (though a mythical beast with more humps is not beyond our imagination, alas such a creature has to be high-dimensional).

A word of caution, however, lies in an inequality for the h^*-polynomial arising from work of Eisenbud and Harris [Sta91]: we have that for any nonnegative k, and s the degree of the h^*-polynomial, we have

$$h^*_0 + \ldots + h^*_k \leq h^*_s + \ldots + h^*_{s-k}.$$

This inequality is special to domains, and prevents us from introducing a hump below half the socle degree easily; it remains to understand the impact of this inequality in general. The most promising approach is then to look among polytopes whose interior is generated in high degree, and look for non-unimodality between half the degree of the h^*-polynomial and half the dimension of the polytope. Another direction that we shall investigate in [APP25] is the impact of restricted systems of parameters. Here, we proved the Lefschetz property for linear systems of parameters that correspond to orbifold Chow rings; this leads to unimodality results for the local h^*-vector.

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APPENDIX A. DIFFERENTIAL EQUATIONS AND THE EULER FORMULA

We have seen that the Parseval-Rayleigh identities imply the differential equations that we needed to prove our anisotropy theorem. Of course, as both differential equations and Parseval-Rayleigh identities are nontrivial nonhomogeneous relations that uniquely determine the fundamental class, they are equivalent. However, their connection is more direct: We have seen that the Parseval-Rayleigh identities imply the differential equations. We now also provide the other direction, even if it is unnecessary for our purposes.

Assume that $d \geq 2$, $m \geq 3$, p is a prime number and k is a field of characteristic p. Consider the polynomial ring

$$R = \Bbbk[\theta_{(i,j)} \ : \ 1 \leq i \leq d, \ 1 \leq j \leq m].$$

Assume $f, g \in R \setminus \{0\}$ such that, for all i with $1 \leq i \leq d$, the polynomials f, g are homogeneous with respect to the set of variables

$$\theta_{(i,1)}, \theta_{(i,2)}, \ldots, \theta_{(i,m)}$$

of degrees $\deg_i f$, $\deg_i g$, respectively and

$$\deg_i f - \deg_i g = -1.$$

We denote by $\mathscr{A} \subseteq R$ the following set of monomials:

$$\mathscr{A} = \{z = \prod_{1\leq i\leq d, 1\leq j\leq m} \theta_{(i,j)}^{e_{i,j}} \colon e_{i,j} \geq 0 \text{ and for all i, } \sum_{1\leq j\leq m} e_{i,j} = p-1\}.$$

Remark A.1. It is clear that each $z \in \mathscr{A}$ is a homogeneous element of R of degree $d(p-1)$. Moreover, for each i with $1 \leq i \leq d$, z is homogeneous with respect to the variables

$$\theta_{(i,1)}, \theta_{(i,2)}, \dots, \theta_{(i,d)}$$

of degree $p-1$. Each $z \in \mathscr{A}$ defines the differential operator

$$\partial_z = \frac{\partial^{d(p-1)}}{\prod_{1\leq i\leq d, 1\leq j\leq m} (\partial\theta_{(i,j)}^{e_{i,j}})}.$$

Example A.2. Assume that $d=2, m=6$, the characteristic of the field is equal to 3 and $z = \theta_{(1,1)}\theta_{(1,5)}\theta_{(2,1)}^2 \in \mathscr{A}$. Then,

$$\partial_z = \frac{\partial^4}{\partial\theta_{(1,1)}\partial\theta_{(1,5)}\partial\theta_{(2,1)}\partial\theta_{(2,1)}}$$

Proposition A.3. *We have*

$$\frac{f}{g} = (-1)^d \sum_{z\in\mathscr{A}} z\partial_z\left(\frac{f}{g}\right)$$

Proof. For fixed i, with $1 \leq i \leq m$, the rational function f/g is homogeneous with respect to the variables

$$\theta_{(i,1)}, \theta_{(i,2)}, \dots, \theta_{(i,m)}$$

of degree equal to -1. Then, from the Euler formula for homogeneous rational functions ([Hof, Theorem 1])

$$[(p-1)!\,]^d \frac{f}{g} = \sum_{z\in\mathscr{A}} z\partial_z\left(\frac{f}{g}\right).$$

By Wilson's theorem p divides $(p-1)!+1$. The proposition follows. ☐

Remark A.4. We refer the reader to the previous Section 8 for the setting of Parseval-Rayleigh Identities. Assume D is a simplicial sphere of dimension $d-1$ or P is an IDP lattice polytope of dimension $d-1$ and m is a monomial in the $\mathbf{x}_i$ of degree d. We have that $\mathrm{vol}(m)$ is an element of the field of fractions $Q(R)$ of R. It is clear that $\mathrm{vol}(m)$ is, for fixed m, homogeneous with respect to the set of variables

$$\theta_{(i,1)}, \theta_{(i,2)}, \dots, \theta_{(i,m)}$$

of degree -1. Hence, Proposition A.3 implies that

$$\mathrm{vol}(m) = (-1)^d \sum_{z\in\mathscr{A}} z\partial_z(\mathrm{vol}(m)).$$

Hence the differential identities imply the Parseval-Rayleigh Identities.

APPENDIX B. THE KUSTIN-MILLER NORMALIZATION, REVISITED

In this section we give a second proof that the volume normalization of a lattice polytope P induced by a boundary flag is independent, up to sign, of the choice of the flag. As a bonus, we work in arbitrary characteristic, paying attention to signs.

In the following $P \subset \mathbb{R}^N$ denotes a d-dimensional lattice polytope with lattice point set $\{1, \dots, m\}$. We denote by $L(P)$ the set of lattice points of τ. For a nonempty subset Z of $\mathbb{R}^N$ we denote by $\operatorname{aff}(Z)$ the smallest affine subset of $\mathbb{R}^N$ containing Z.

Proposition B.1. *Assume*

$$\tau = (\tau_0, \tau_1, \dots, \tau_d = P)$$

is a boundary flag of P, in the sense that for all $1 \le i \le d$ we have that τ_{i-1} is a facet of the boundary of τ_i. Suppose $a_0 \in L(\tau_0)$ and for all $1 \le i \le d$, $a_i \in L(\tau_i) \setminus L(\tau_{i-1})$. Then, for all $0 \le i \le d$, it holds that

$$\dim(\operatorname{aff}(a_0, \dots, a_i)) = i$$

and

$$\operatorname{aff}(a_0, \dots, a_i) = \operatorname{aff}(\tau_i).$$

Proof. It is well-known that $\dim(\operatorname{aff}(\tau_i)) = i$. We use induction on i. For $i = 0$, we have $\tau_0 = \{a_0\}$ and the two claims are true. Assume that $1 \le i \le d-1$ and the two claims are true for i. Hence $\dim(\operatorname{aff}(a_0, \dots, a_i)) = i$ and $(\operatorname{aff}(a_0, \dots, a_i)) = (\operatorname{aff}(\tau_i))$. Since τ_i is a facet of τ_{i+1}, it holds that $\tau_i = \tau_{i+1} \cap H$ where $H = \operatorname{aff}(\tau_i)$. Since $a_i \in L(\tau_i) \setminus L(\tau_{i-1})$ we get that $a_i \notin H$. Consequently, H is a proper subset of the $\operatorname{aff}(H \cup \{a_{i+1}\})$. This implies that $\dim(\operatorname{aff}(H \cup \{a_{i+1}\})) = i+1$. Since by the inductive hypothesis $(\operatorname{aff}(a_0, \dots, a_i)) = (\operatorname{aff}(\tau_i))$ we get that $\dim(\operatorname{aff}(a_0, \dots, a_{i+1})) = i+1$. Taking into account that $(\operatorname{aff}(a_0, \dots, a_{i+1})) \subset (\operatorname{aff}(\tau_{i+1}))$ and that $\dim(\operatorname{aff}(\tau_{i+1})) = i+1$, we get that $(\operatorname{aff}(a_0, \dots, a_{i+1})) = (\operatorname{aff}(\tau_{i+1}))$, which finishes the proof. $\square$

We consider the $(d+1) \times m$ matrix $M_\theta = [\theta_{i,j}]$, where $\theta_i = \sum_{j=1}^m \theta_{i,j} \mathbf{x}_j$ are the linear polynomials we use for the Artinian reduction. We use the notations

$$(a_0, \dots, a_d) = [a_0, \dots, a_d] \mathbf{x}_{a_0} \mathbf{x}_{a_1} \dots \mathbf{x}_{a_d}$$

and

$$\mathbf{R}_{a_0, \dots, a_{d-1}} = \sum_{i=1}^m (a_0, \dots a_{d-1}, i) = \sum_{i=1}^m [a_0, \dots a_{d-1}, i] \mathbf{x}_{a_0} \mathbf{x}_{a_1} \dots \mathbf{x}_{a_{d-1}} \mathbf{x}_i,$$

where $[a_0, \dots, a_d]$ denotes the determinant of the $(d+1) \times (d+1)$ submatrix of M_θ specified by the columns $a_0, \dots, a_d$ (see also Lemma 8.5).

Proposition B.2. *(i) We have that*

$$\mathbf{R}_{a_0,\ldots,a_{d-1}} = \operatorname{sign}(\sigma)\mathbf{R}_{b_0,\ldots,b_{d-1}}$$

if $(b_0, \ldots, b_{d-1})$ is obtained from $(a_0, \ldots, a_{d-1})$ by a permutation σ and $\operatorname{sign}(\sigma)$ denotes the sign of the permutation. Moreover,

$$\mathbf{R}_{a_0,\ldots,a_{d-1}} = 0$$

if there exists $i \neq j$ with $a_i = a_j$.

(ii) Given $a_0, \ldots, a_{d-2}$, we have that

$$\sum_{i=1}^{m} \mathbf{R}_{a_0,a_1,\ldots,a_{d-2},i} = 0.$$

Proof. Since $[a_0, \ldots, a_{d-1}, i]$ is a determinant, (i) follows immediately.

We now prove (ii). We have

$$\sum_{i=1}^{m} \mathbf{R}_{a_0,a_1,\ldots,a_{d-2},i} = \sum_{i=1}^{m}\sum_{j=1}^{m}(a_0, a_1, \ldots, a_{d-2}, i, j)$$

Since for all i

$$(a_0, a_1, \ldots, a_{d-2}, i, i) = 0$$

and when $i \neq j$ we have

$$(a_0, a_1, \ldots, a_{d-2}, i, j) + (a_0, a_1, \ldots, a_{d-2}, j, i) = 0$$

the result follows. □

We keep assuming that P is a d-dimensional lattice polytope with lattice point set $\{1, \ldots, m\}$. Assume $T = (\tau_0, \tau_1, \ldots, \tau_d = P)$ is a boundary flag of P in the above sense. We define the expression H_T as follows

$$H_T = \sum_{a_0,\ldots,a_d} (a_0, \ldots, a_d)$$

with the sum for all $a_i \in L(\tau_i)$.

Proposition B.3. *We have that*

$$H_T = \sum_{a_0,\ldots,a_d} (a_0, \ldots, a_d)$$

with the sum for $a_0 \in L(\tau_0)$ and $a_i \in L(\tau_i) \setminus L(\tau_{i-1})$ for $i > 0$.

Proof. Assume $a_i \in L(\tau_i)$, for $i = 0 \dots d$. If $a_1 \in L(\tau_0)$ we get that $a_1 = a_0$. Hence, $(a_0, \dots, a_d) = 0$. Consequently,

$$H_T = \sum_{a_0,\dots,a_d} (a_0, \dots, a_d)$$

with the sum for $a_0 \in L(\tau_0)$, $a_1 \in L(\tau_1) \setminus L(\tau_0)$ and $a_i \in L(\tau_i)$ for $i \geq 2$. Assume now that $a_2 \in L(\tau_1)$. If $a_2 = a_0$ then $(a_0, \dots, a_d) = 0$. Similarly if $a_2 = a_1$. Otherwise, both terms $(a_0, a_1, a_2, \dots, a_d)$ and $(a_0, a_2, a_1, \dots, a_d)$ appear in the sum defining H_T and they cancel each other. Consequently,

$$H_T = \sum_{a_0,\dots,a_d} (a_0, \dots, a_d)$$

with the sum for $a_0 \in L(\tau_0)$, $a_1 \in L(\tau_1) \setminus L(\tau_0)$, $a_2 \in L(\tau_2) \setminus L(\tau_1)$ and $a_i \in L(\tau_i)$ for $i \geq 3$. Continuing on the same way the result follows. □

Proposition B.4. *We have that*

$$H_T = \sum_{a_0,\dots,a_{d-1}} \mathbf{R}_{a_0,\dots,a_{d-1}}$$

with the sum for $a_i \in L(\tau_i)$.

Proof. It is clear from the definitions. □

Proposition B.5. *We have that*

$$H_T = \sum_{a_0,\dots,a_{d-1}} \mathbf{R}_{a_0,\dots,a_{d-1}}$$

with the sum for $a_0 \in L(\tau_0)$ and $a_i \in L(\tau_i) \setminus L(\tau_{i-1})$ for $i > 0$.

Proof. It follows immediately from Proposition B.3. □

Assume $t < d$ and we have two boundary flags of P:

$$T_1 = (\sigma_0, \sigma_1, \dots, \sigma_d = P)$$

$$T_2 = (\rho_0, \rho_1, \dots, \rho_d = P)$$

with the property that they only differ on the t-position, in the sense that $\sigma_i = \rho_i$ when i is different from t and $\sigma_t \neq \rho_t$.

We set for $j > 0$ and $j \notin \{t, t+1\}$

$$S_j = L(\sigma_j) \setminus L(\sigma_{j-1})$$

$$E_{sp} = L(\sigma_{t+1}) \setminus (L(\sigma_t) \cup L(\rho_t))$$

$$U_1 = S_1 \times S_2 \times \ldots \times S_{t-1}$$

$$U_2 = S_{t+2} \times S_{t+3} \times \ldots \times S_d$$

We set if

$$t = 0 \textit{ and } d = 1,\ U = E_{sp}.$$

$$t = 0 \textit{ and } d \geq 2,\ U = E_{sp} \times U_2.$$

$$t = 1 \textit{ and } d < 3,\ U = L(\sigma_0) \times E_{sp}.$$

$$t = 1 \textit{ and } d \geq 3,\ U = L(\sigma_0) \times E_{sp} \times U_2.$$

$$t \geq 2 \textit{ and } d \geq t+2,\ U = L(\sigma_0) \times U_1 \times E_{sp} \times U_2.$$

$$t \geq 2 \textit{ and } d < t+2,\ ,U = L(\sigma_0) \times U_1 \times E_{sp}.$$

where $\times$ denotes the Cartesian product of sets.

For $z = (z_1, z_2, \ldots, z_d) \in U$, we set

$$\mathbf{R}_z = \sum_{i=1}^{m} (z_1, z_2, \ldots, z_d, i)$$

Hence,

$$\mathbf{R}_z = \sum_{i=1}^{m} [z_1, z_2, \ldots, z_d, i] \mathbf{x}_{z_1} \ldots \mathbf{x}_{z_d} \mathbf{x}_i.$$

Theorem B.6. *(i) We have the following equality of polynomials*

$$H_{T_1} + H_{T_2} = (-1)^{t+d} \sum_{z \in U} \mathbf{R}_z.$$

(ii) Assume $z = (z_1, z_2, \ldots, z_d) \in U$. Then the set $\{z_1, z_2, \ldots, z_d\}$ is not contained in a facet of P.

Proof. We prove (i) only for the case $t \geq 2$ and $d \geq t+2$. All the other cases can be proven in a similar way.

We assume first that $t \geq 2$ and $d > t+2$ and that we have two boundary flags

$$T_1 = (\sigma_0, \dots, \sigma_{t-1}, \sigma_t, \sigma_{t+1}, \dots, \sigma_d = P)$$

$$T_2 = (\sigma_0, \dots, \sigma_{t-1}, \rho_t, \sigma_{t+1}, \dots, \sigma_d = P)$$

differing only on the t-th position.

We set $L(\sigma_0) = \{a\}$ and

$$M_1 = \{a\} \times (L(\sigma_1) \setminus L(\{a\})) \times \ldots \times (L(\sigma_{t-1}) \setminus L(\sigma_{t-2}))$$

$$M_2 = (L(\sigma_{t+2}) \setminus L(\sigma_{t+1}) \ldots \times (L(\sigma_{d-1}) \setminus L(\sigma_{d-2}).$$

We have

$$U = M_1 \times (L(\sigma_{t+1}) \setminus (L(\sigma_t) \cup L(\rho_t))) \times M_2 \times (L(\sigma_d) \setminus (L(\sigma_{d-1}))$$

By Proposition B.5,

$$H_{T_1} = \sum_{(a_0,a_1,\dots,a_{d-1}) \in W_1} \mathbf{R}_{a_0,a_1,\dots,a_{d-1}}$$

$$H_{T_2} = \sum_{(b_0,b_1,\dots b_{d-1}) \in W_2} \mathbf{R}_{b_0,b_1\dots,b_{d-1}},$$

where

$$W_1 = M_1 \times (L(\sigma_t) \setminus L(\sigma_{t-1})) \times (L(\sigma_{t+1}) \setminus L(\sigma_t)) \times M_2$$

and

$$W_2 = M_1 \times (L(\rho_t) \setminus L(\sigma_{t-1})) \times (L(\sigma_{t+1}) \setminus L(\rho_t)) \times M_2.$$

We set

$$V_1 = M_1 \times (L(\sigma_t) \setminus L(\sigma_{t-1})) \times (L(\rho_t) \setminus L(\sigma_{t-1})) \times M_2$$

$$V_2 = M_1 \times (L(\sigma_t) \setminus L(\sigma_{t-1})) \times (L(\sigma_{t+1}) \setminus (L(\sigma_t) \cup L(\rho_t)) \times M_2$$

Since $L(\sigma_{t+1}) \setminus L(\sigma_t)$ is the disjoint union of

$$L(\rho_t) \setminus L(\sigma_{t-1}) \textit{ and } L(\sigma_{t+1}) \setminus (L(\sigma_t) \cup L(\rho_t))$$

it follows that W_1 is the disjoint union of V_1 and V_2.

We also set

$$V_3 = M_1 \times (L(\rho_t) \setminus L(\sigma_{t-1})) \times (L(\sigma_t) \setminus L(\sigma_{t-1})) \times M_2,$$

$$V_4 = M_1 \times (L(\rho_t) \setminus L(\sigma_{t-1})) \times (L(\sigma_{t+1}) \setminus (L(\sigma_t) \cup L(\rho_t)) \times M_2$$

Since $L(\sigma_{t+1}) \setminus L(\rho_t)$ is the disjoint union of

$$L(\sigma_t) \setminus L(\sigma_{t-1}) \textit{ and } L(\sigma_{t+1}) \setminus (L(\sigma_t) \cup L(\rho_t)),$$

it follows that W_2 is the disjoint union of V_3 and V_4.

Therefore, H_{T_1} becomes a sum of two expressions, one for V_1 and one for V_2. Similarly H_{T_2} becomes a sum of two expressions, one for V_3 and one for V_4. Moreover, the exchange of the t and $t+1$ positions give a bijection between V_1 and V_3, and using the first part of Proposition B.2, the corresponding terms in the sum $H_{T_1} + H_{T_2}$ add to zero. As a consequence,

$$H_{T_1} + H_{T_2} = \sum_{u \in V_1} \mathbf{R}_u + \sum_{u \in V_2} \mathbf{R}_u + \sum_{u \in V_3} \mathbf{R}_u + \sum_{u \in V_4} \mathbf{R}_u =$$

$$= \sum_{u \in V_2} \mathbf{R}_u + \sum_{u \in V_4} \mathbf{R}_u$$

Hence,

$$H_{T_1} + H_{T_2} - (-1)^{(t+d)} \sum_{z \in U} \mathbf{R}_z = \sum_{(a_1,\dots,a_{t-1},a_t,a_{t+1},a_{t+2},\dots,a_{d-1}) \in V_2} \mathbf{R}_{a,a_1,a_2,\dots,a_{d-1}} +$$

$$\sum_{(b_1,\dots,b_{t-1},b_t,b_{t+1},b_{t+2},\dots,b_{d-1}) \in V_4} \mathbf{R}_{a,b_1,\dots,b_2,\dots,b_{d-1}} - (-1)^{(t+d)} \sum_{z \in U} \mathbf{R}_z =$$

$$= (-1)^{d-1-t} \sum_{(a_1,\ldots,a_{t-1},a_t,a_{t+1},a_{t+2},\ldots,a_{d-1})\in V_2} \mathbf{R}_{a,a_1,\ldots,a_{t-1},a_{t+1},a_{t+2},\ldots,a_{d-1},a_t} +$$

$$(-1)^{d-1-t} \sum_{(b_1,\ldots,b_{t-1},b_t,b_{t+1},b_{t+2},\ldots,b_{d-1})\in V_4} \mathbf{R}_{a,b_1,\ldots,b_{t-1},b_{t+1},b_{t+2},\ldots,b_{d-1},b_t}$$

$$-(-1)^{(t+d)} \sum_{z\in U} \mathbf{R}_z = (-1)(-1)^{t+d} \sum_{(z_1,\ldots,z_{d-1}\in U} \sum_{i=1}^{m} \mathbf{R}_{z_1,z_2,\ldots,z_{d-1},i} = 0$$

where for the final equality we used that

$$(-1)^{d-1-t} = (-1)(-1)^d(-1)^{-t} = (-1)(-1)^d(-1)^t = (-1)(-1)^{t+d}$$

and Proposition B.2.

We now assume that $t \geq 2$ and $d = t+2$ and that we have two boundary flags

$$T_1 = (\sigma_0, \ldots, \sigma_{t-1}, \sigma_t, \sigma_{t+1}, \sigma_{t+2} = P)$$

$$T_2 = (\sigma_0, \ldots, \sigma_{t-1}, \rho_t, \sigma_{t+1}, \sigma_{t+2} = P)$$

We set $L(\sigma_0) = \{a\}$ and

$$M_1 = \{a\} \times (L(\sigma_1) \setminus \{a\}) \times \ldots \times (L(\sigma_{t-1}) \setminus L(\sigma_{t-2}))$$

We observe that in this case M_2 defined above does not exist.

We have

$$U = M_1 \times (L(\sigma_{t+1}) \setminus (L(\sigma_t) \cup L(\rho_t))) \times (L(\sigma_{t+2}) \setminus L(\sigma_{t+1}))$$

By Proposition B.5,

$$H_{T_1} = \sum_{(a_0,a_1,\ldots,a_{t+1})\in W_1} \mathbf{R}_{a_0,a_1,\ldots,a_{t+1}}$$

$$H_{T_2} = \sum_{(b_0,b_1,\ldots b_{t+1})\in W_2} \mathbf{R}_{b_0,b_1\ldots,b_{t+1}}$$

where,

$$W_1 = M_1 \times (L(\sigma_t) \setminus L(\sigma_{t-1})) \times (L(\sigma_{t+1}) \setminus L(\sigma_t))$$

and

$$W_2 = M_1 \times (L(\rho_t) \setminus L(\sigma_{t-1})) \times (L(\sigma_{t+1}) \setminus L(\rho_t))$$

We set

$$V_1 = M_1 \times (L(\sigma_t) \setminus L(\sigma_{t-1})) \times (L(\rho_t) \setminus L(\sigma_{t-1}))$$

$$V_2 = M_1 \times (L(\sigma_t) \setminus L(\sigma_{t-1})) \times (L(\sigma_{t+1}) \setminus (L(\sigma_t) \cup L(\rho_t)))$$

Since $L(\sigma_{t+1}) \setminus L(\sigma_t)$ is the disjoint union of

$$L(\rho_t) \setminus L(\sigma_{t-1}) \text{ and } L(\sigma_{t+1}) \setminus (L(\sigma_t) \cup L(\rho_t))$$

it follows that W_1 is the disjoint union of V_1 and V_2.

We also set

$$V_3 = M_1 \times (L(\rho_t) \setminus L(\sigma_{t-1})) \times (L(\sigma_t) \setminus L(\sigma_{t-1})),$$

$$V_4 = M_1 \times (L(\rho_t) \setminus L(\sigma_{t-1})) \times (L(\sigma_{t+1}) \setminus (L(\sigma_t) \cup L(\rho_t)))$$

Since $L(\sigma_{t+1}) \setminus L(\rho_t)$ is the disjoint union of

$$L(\sigma_t) \setminus L(\sigma_{t-1}) \text{ and } L(\sigma_{t+1}) \setminus (L(\sigma_t) \cup L(\rho_t)),$$

it follows that W_2 is the disjoint union of V_3 and V_4.

Therefore, H_{T_1} becomes a sum of two expressions, one for V_1 and one for V_2. Similarly H_{T_2} becomes a sum of two expressions, one for V_3 and one for V_4. Moreover, the exchange of the t and $t+1$ positions give a bijection between V_1 and V_3, and using the first part of Proposition B.2, the corresponding terms in the sum $H_{T_1} + H_{T_2}$ add to zero. As a consequence,

$$H_{T_1} + H_{T_2} = \sum_{u \in V_1} \mathbf{R}_u + \sum_{u \in V_2} \mathbf{R}_u + \sum_{u \in V_3} \mathbf{R}_u + \sum_{u \in V_4} \mathbf{R}_u =$$

$$= \sum_{u \in V_2} \mathbf{R}_u + \sum_{u \in V_4} \mathbf{R}_u$$

Hence,

$$H_{T_1} + H_{T_2} - (-1)^{(2t+2)} \sum_{z \in U} \mathbf{R}_z = \sum_{(a_1,\dots,a_{t-1},a_t,a_{t+1}) \in V_2} \mathbf{R}_{a,a_1,a_2,\dots,a_t,a_{t+1}} +$$

$$\sum_{(b_1,\dots,b_{t-1},b_t,b_{t+1}) \in V_4} \mathbf{R}_{a,b_1,\dots,b_2,\dots,b_t,b_{t+1}} - (-1)^{(2t+2)} \sum_{z \in U} \mathbf{R}_z =$$

$$= (-1) \sum_{(a_1,\dots,a_{t-1},a_t,a_{t+1}) \in V_2} \mathbf{R}_{a,a_1,\dots,a_{t-1},a_{t+1},a_t} + (-1) \sum_{(b_1,\dots,b_{t-1},b_t,b_{t+1},b_{t+2},\dots,b_{d-1}) \in V_4} \mathbf{R}_{a,b_1,\dots,b_{t-1},b_{t+1},b_t}$$

$$-(-1)^{(2t+2)} \sum_{z \in U} \mathbf{R}_z = (-1) \sum_{(z_1,\dots,z_{t+1}) \in U} \sum_{i=1}^{m} \mathbf{R}_{z_1,z_2,\dots,z_{t+1},i} = 0$$

where for the final equality we used the second part of Proposition B.2.

We now prove (ii).

We recall that every facet of a d-dimensional lattice polytope P has dimension $d-1$. Assume that $\{z_1, z_2, \dots, z_d\}$n is contained in a facet of P. Then $\{z_1, z_2, \dots, z_d\}$ should have dimension less or equal to $d-1$. By Proposition B.1, we get a contradiction since the affine span of $\{z_1, z_2, \dots, z_d\}$ has dimension d. □

Denote by $\mathscr{A}(P, \partial P)$ the generic Artinian reduction of $k[P, \partial P]$. That is,

$$\mathscr{A}(P, \partial P) = \mathscr{I}_{\partial P} / (\mathscr{I}_{\partial P} \mathscr{I}_{lins} + \mathscr{I}_P),$$

where we denote by $\mathscr{I}_P$ and $\mathscr{I}_{\partial P}$ the ideals of P and of the boundary of P respectively and by $\mathscr{I}_{lins} = (\theta_1, \dots, \theta_{d+1})$ the ideal of the linear relations.

Theorem B.7. *We have*

$$H_{T_1} + H_{T_2} = 0$$

in $\mathscr{A}(P, \partial P)$.

Proof. Recall that by the first part of Theorem B.6,

$$H_{T_1} + H_{T_2} = (-1)^{t+d} \sum_{z \in U} \mathbf{R}_z$$

where

$$\mathbf{R}_z = \sum_{i=1}^{m} [z_1, z_2, \ldots, z_d, i] \mathbf{x}_{z_1} \mathbf{x}_{z_2} \ldots \mathbf{x}_{z_d} \mathbf{x}_i$$

for $z = (z_1, z_2, \ldots, z_d) \in U$.

It is enough to prove that $\sum_{z \in U} \mathbf{R}_z$ belongs to the ideal $\mathscr{I}_{\partial P} \mathscr{I}_{lins} + \mathscr{I}_P$.

By the second part of Theorem B.6,

$$\mathbf{x}_{z_1} \mathbf{x}_{z_2} \ldots \mathbf{x}_{z_d}$$

is an element of the ideal $\mathscr{I}_{\partial P}$ of the boundary of the P. By Lemma 4.3. The result follows. □

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