in another fan Σ', then Σ is called a *subfan* of Σ. The one-dimensional cones of a fan are often called *rays*. Throughout this article, all fans and polyhedral complexes are assumed to be *rational*.

For a given polyhedron σ, and a finitely generated abelian group N, we denote by

$$\begin{aligned} \mathrm{aff}(\sigma) &:= \text{affine span of } \sigma, \\ H_\sigma &:= \text{translation of aff}(\sigma) \text{ to the origin}, \\ N_\sigma &:= N \cap H_\sigma, \\ N(\sigma) &:= N/N_\sigma. \end{aligned}$$

Consider τ, a codimension one face of a p-dimensional polyhedron σ, and let $u_{\sigma/\tau}$ be the unique outward generator of the one-dimensional lattice $(\mathbb{Z}^n \cap H_\sigma)/(\mathbb{Z}^n \cap H_\tau)$.

Definition 3.1 (Balancing Condition and Tropical Cycles). Let $\mathcal{C}$ be a p-dimensional polyhedral complex whose p-dimensional cones are equipped with integer weights. We say that C satisfies the *balancing condition* at τ if

$$\sum_{\sigma \supset \tau} w(\sigma)\, u_{\sigma/\tau} = 0, \quad \text{in } \mathbb{Z}^n/(\mathbb{Z}^n \cap H_\tau),$$

w

where the sum is over all p-dimensional cells σ in C containing τ as a face. A *tropical variety* in $\mathbb{R}^n$ is a weighted complex with finitely many cells that satisfies the balancing condition at every cone of dimension $p-1$.

cycle = variety ?

dim = p ?

3.2. **The $\mathbb{Z}$-algebra of Tropical Cycles.** Recall that, generally speaking, the star of a cone in a complex is the extension of the local p-dimensional fan surrounding it. More precisely:

Definition 3.2 (Star of a Cone). Given a polyhedral complex $C \subset \mathbb{R}^n$ and a cell τ within C, define the star of σ in Σ, denoted by $\mathrm{star}_\Sigma(\tau)$, as a fan in $\mathbb{R}^n$. The cones of $\mathrm{star}_\Sigma(\tau)$ are the *extensions* of cones σ that include τ as a face. Here, by extension, we mean

$$\bar{\sigma} = \{\lambda(x-y) : \lambda \geq 0, x \in \sigma, y \in \tau\}.$$

if σ and τ are cones we don't need λ

Definition 3.3 (Stable Intersection). (a) Let C_1 and C_2 be two tropical cycles intersecting transversely, then the stable intersection of $C_1 \cdot C_2$ is the tropical cycles supported on finitely many zero dimensional cells $C_1 \cap C_2$. In this case, the weight of a cell $\sigma_1 \cap \sigma_2$, where $\sigma_1 \in C_1$ and $\sigma_2 \in C_2$ are top dimensional cells, we define the weights by

$$w_{C_1 \cdot C_2}(\sigma_1 \cap \sigma_2) = w_{\sigma_1} w_{\sigma_2} [N : N_{\sigma_1} + N_{\sigma_2}].$$

you only consider the dimension 0 case ?

complementary dimension ?

N ?

some confusion about $\mathbb{Z}^n$, N

(b) When C_1 and C_2 do not intersect transverse, then $C_1 \cdot C_2$ as a set is the Hausdorff limit of

$$C_1 \cap (\epsilon b + C_2), \quad \text{as } \epsilon \longrightarrow 0,$$

for a fixed generic $b \in \mathbb{R}^n$, and the weights are the sum of all the tropical multiplicities of the cells in the transversal intersection $C_1 \cap (\epsilon b + C_2)$ which converge to the same zero-dimensional cell in the Hausdorff metric. Equivalently, for top dimensional cones $\sigma_1 in C_1$ and $\sigma_2 \in C_2$

?

/

$$w_{C_1 \cdot C_2}(\sigma_1 \cap \sigma_2) = \sum_{\tau_1, \tau_2} w_{\tau_1} w_{\tau_2} [N : N_{\tau_1} + N_{\tau_2}],$$

it is better to ~~do~~ use C_1, C_2 (or C, C')

but the same in Th 5.11

where the sum is taken over all $\tau_1 \in \operatorname{star}_{\mathcal{C}_1}(\sigma_1 \cap \sigma_2), \tau_2 \in \operatorname{star}_{\mathcal{C}_2}(\sigma_1 \cap \sigma_2)$ with $\tau_1 \cap (v + \tau_2) \neq \varnothing$, for some fixed generic vector $b \in \mathbb{R}^n$.

In tropical geometry, the following theorem is shown which we reprove using superpotential theory.

Theorem 3.4 (Stable Intersection Invariance). *The stable intersection, as defined above, does not depend on the choice of a generic vector $b \in \mathbb{R}^n$ and the induced weights satisfy the balancing condition on the support.*

We also need the following for turning the set of tropical cycles into a $\mathbb{Z}$-algebra.

Definition 3.5 (Addition of Tropical Cycles). For two p-dimensional tropical cycles $\mathcal{C}_1, \mathcal{C}_2$ in $\mathbb{R}^n$, the addition $\mathcal{C}_1 + \mathcal{C}_2$ is the tropical cycle obtained by the common refinement of the support $|\mathcal{C}_1| \cup |\mathcal{C}_2|$ where the weights of a cone σ in the refinement are determined by $w_{\mathcal{C}_1+\mathcal{C}_2}(\sigma) = w_{\mathcal{C}_1}(\sigma) + w_{\mathcal{C}_2}(\sigma)$.

4. Tropical Currents

Let us briefly recall the definition of tropical currents from [Bab14, BH17]. To fix the notation,

$$\begin{aligned} T_N &:= \text{the complex algebraic torus } \mathbb{C}^* \otimes_{\mathbb{Z}} N, \\ S_N &:= \text{the compact real torus } S^1 \otimes_{\mathbb{Z}} N, \\ N_{\mathbb{R}} &:= \text{the real vector space } \mathbb{R} \otimes_{\mathbb{Z}} N. \end{aligned}$$

Let $\mathbb{C}^*$ be the group of nonzero complex numbers. As before, the logarithm map is the homomorphism

$$\operatorname{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \qquad (z_1, \ldots, z_n) \longmapsto (-\log|z_1|, \ldots, -\log|z_n|),$$

and the *argument map* is

$$\operatorname{Arg} : (\mathbb{C}^*)^n \longrightarrow (S^1)^n, \qquad (z_1, \ldots, z_n) \longmapsto (z_1/|z_1|, \ldots, z_n/|z_n|).$$

For a rational linear subspace $H \subseteq \mathbb{R}^n$ we have the following exact sequences:

$$0 \longrightarrow H \cap \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n/(H \cap \mathbb{Z}^n) \longrightarrow 0,$$

Moreover,

$$0 \longrightarrow S_{H \cap \mathbb{Z}^n} \longrightarrow (S^1)^n = S^1 \otimes_{\mathbb{Z}} \mathbb{Z}^n \longrightarrow S_{\mathbb{Z}^n/(H \cap \mathbb{Z}^n)} \longrightarrow 0.$$

Define

$$\pi_H : \operatorname{Log}^{-1}(H) \xrightarrow{\operatorname{Arg}} (S^1)^n \longrightarrow S_{\mathbb{Z}^n/(H \cap \mathbb{Z}^n)} .$$

Similarly,

$$0 \longrightarrow T_{H \cap \mathbb{Z}^n} \longrightarrow (\mathbb{C}^*)^n = \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Z}^n \longrightarrow T_{\mathbb{Z}^n/(H \cap \mathbb{Z}^n)} \longrightarrow 0,$$

We define

$$\Pi_H : (\mathbb{C}^*)^n \simeq \mathbb{C}^* \otimes \big((H \cap \mathbb{Z}^n) \oplus \mathbb{Z}^n/(\mathbb{Z}^n \cap H)\big) \longrightarrow T_{\mathbb{Z}^n/(H \cap \mathbb{Z}^n)} .$$

One has

$$\ker(\Pi_H) = \ker(\pi_H) = T_{H \cap \mathbb{Z}^n} \subseteq (\mathbb{C}^*)^n.$$

As a result, when H is of dimension p, the set $\mathrm{Log}^{-1}(H)$ is naturally foliated by the $\pi_H^{-1}(x) = T_{H\cap\mathbb{Z}^n}\cdot x \simeq (\mathbb{C}^*)^p$ for $x \in S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}$. For a lattice basis $u_1,\ldots,u_p$, of $H\cap\mathbb{Z}^n$, the tori $T_{H\cap\mathbb{Z}^n}.x$ can be parametrised by the monomial map

$$(\mathbb{C}^*)^p \longrightarrow (\mathbb{C}^*)^n, \quad z \longmapsto x.z^{[u_1,\ldots,u_p]^t}$$

where $U = [u_1,\ldots,u_p]$ is the matrix with column vectors $u_1,\ldots,u_p$, and z^{U^t} denotes that $z \in (\mathbb{C}^*)^p$ is taken to have the exponents with rows of the matrix U. Accordingly, one can easily check that

$$T_{H\cap\mathbb{Z}^n}\cdot x = \{z \in (\mathbb{C}^*)^n : z^{m_i} = x^{m_i},\ i = 1,\ldots,\cancel{m} - p\}.$$ n

check it seems that equal modulo $T_{H\cap\mathbb{Z}^n}$

for any choice of a $\mathbb{Z}$-basis $\{m_1,\ldots,m_{n-p}\}$ of $\mathbb{Z}^n/(H\cap\mathbb{Z}^n)$.

Definition 4.1. Let H be a rational subspace of dimension p, and μ be the Haar measure of mass 1 on $S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}$. We define a (p,p)-dimensional closed current $\mathcal{T}_H$ on $(\mathbb{C}^*)^n$ by

$$\mathcal{T}_H := \int_{x\in S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}} [\pi_H^{-1}(x)]\ d\mu(x).$$

rational

When A is an affine subspace of $\mathbb{R}^n$ parallel to the linear subspace $H = A - a$ for $a \in A$, we define $\mathcal{T}_A$ by translation of $\mathcal{T}_H$. Namely, we define the submersion π_A as the composition

$$\pi_A : \mathrm{Log}^{-1}(A) \xrightarrow{e^a} \mathrm{Log}^{-1}(H) \xrightarrow{\pi_H} S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}.$$

We will call $T^A := \pi_A^{-1}(1) = \ker \pi_A = e^{-a}T_{H\cap\mathbb{Z}^n}$, the *distinguished fibre* of $\mathcal{T}_A$.

Definition 4.2. Let $\mathcal{C}$, be a weighted polyhedral complex of dimension p. The tropical current $\mathcal{T}_\mathcal{C}$ associated to $\mathcal{C}$ is given by

$$\mathcal{T}_\mathcal{C} = \sum_\sigma w_\sigma\ \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\mathcal{T}_{\mathrm{aff}(\sigma)},$$ ~~no~~ it seems that σ is also OK ?

where the sum runs over all p-dimensional cells σ of $\mathcal{C}$.

Theorem 4.3 ([Bab14]). A weighted complex $\mathcal{C}$ is balanced, if and only if, $\mathcal{T}_\mathcal{C}$ is closed.

Theorem 4.4 ([Bab14]). Any tropical current $\mathcal{T}_\mathcal{C} \in \mathcal{D}'_{n-1,n-1}((\mathbb{C}^*)^n)$ is of the form $dd^c[\mathrm{q}\circ\mathrm{Log}]$, where $\mathrm{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$, is a tropical Laurent polynomial, that is $\mathrm{q}(x) = \max_{\alpha\in A}\{c_\alpha + \langle\alpha, x\rangle\}$, for $A \subseteq \mathbb{Z}^n$ a finite subset and $c_\alpha \in \mathbb{R}$.

Remark 4.5. Note that the support of $dd^c[\mathrm{q}\circ\mathrm{Log}]$, is given by $\mathrm{Log}^{-1}(\mathcal{V}(\mathrm{q}))$, where $\mathcal{V}(\mathrm{q})$ is the set of points $x \in \mathbb{R}^n$ where q is not smooth at x. This set can be balanced with natural weights which coincides with the weights of the closed current $dd^c[\mathrm{q}\circ\mathrm{Log}]$ and it is called the tropical variety associated to q.

Proposition 4.6 ([Bab23, Proposition 4.6]). Assume that $\mathcal{T} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$ is a closed positive $(S^1)^n$-invariant current whose support is given by $\mathrm{Log}^{-1}(|\mathcal{C}|)$, for a polyhedral complex $\mathcal{C} \subseteq \mathbb{R}^n$ of pure dimension p. Then $\mathcal{T}$ is a tropical current.

5. Continuity of superpotentials

Let $\mathrm{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$, be a tropical polynomial function, and $\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$, as before. The current $dd^c[\mathrm{q} \circ \mathrm{Log}] \in \mathcal{D}'_{n-1,n-1}(\mathbb{C}^*)^n$ has a bounded potential, and by Bedford–Taylor theory, for any positive closed current $\mathcal{T} \in \mathcal{D}'_{p,p}(\mathbb{C}^*)^n$, the product

locally

$$dd^c[\mathrm{q} \circ \mathrm{Log}] \wedge \mathcal{T} = dd^c([\mathrm{q} \circ \mathrm{Log}]\, \mathcal{T}),$$

is well-defined. See [Dem, Section III.3]. In higher codimensions though, to prove that any two tropical currents have a well-defined wedge product, we utilise ~~Dinh and Sibony's~~ superpotential theory [DS09] on a compact Kähler manifold, and as a result, we extend the tropical currents to smooth compact toric varieties.

5.1. Tropical Currents on Toric Varieties. In a toric variety X_Σ, for a cone $\sigma \in \Sigma$, we denote by $\mathcal{O}_\sigma$, the toric orbit associated with σ. We have

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{O}_\sigma.$$

We also set D_σ to be the closure of $\mathcal{O}_\sigma$ in the X_Σ. $\Sigma(p)$ p-dimensional skeleton.

Fibers of tropical currents are algebraic varieties with finite degrees and can be extended by zero to any toric variety, in consequence, any tropical current can be extended by zero to toric varieties. Moreover, with the following compatibility condition, we can ask for the extension of the fibres to intersect the toric invariant divisors transversally.

Definition 5.1. (i) For a polyhedron σ, its *recession cone* is the convex polyhedral cone

$$\mathrm{rec}(\sigma) = \{b \in \mathbb{R}^n : \sigma + b \subseteq \sigma\} \subseteq H_\sigma.$$

(ii) Let $\mathcal{C}$ be a p-dimensional balanced weighted complex in $\mathbb{R}^n$, and Σ a p-dimensional fan. We say that $\mathcal{C}$ is *compatible* with Σ, if $\mathrm{rec}(\sigma) \in \Sigma$ for all $\sigma \in \mathcal{C}$.

(iii) We say the tropical current $\mathcal{T}_\mathcal{C}$ is *compatible* with X_Σ, if all the closures of the fibers $\pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ in X_Σ of $\mathcal{T}_\mathcal{C}$ intersect the torus invariant divisors of X_Σ transversely.

Theorem 5.2. Let $\mathcal{C}$ be a p-dimensional tropical cycle, and Σ be a fan. Assume that $\sigma \in \mathcal{C}$ is a p-dimensional polyhedron and $\rho \in \Sigma$ is a one-dimensional cone. Then

(a) The intersection $D_\rho \cap \overline{\pi^{-1}_{\mathrm{aff}(\sigma)}(x)}$ is non-empty and transverse, if and only if, $\rho \in \mathrm{rec}(\sigma)$. Here $\overline{\pi^{-1}_{\mathrm{aff}(\sigma)}(x)}$ corresponds the closure of a fiber of $\mathcal{T}_{\mathrm{aff}(\sigma)}$ in the toric variety X_Σ.

(b) In particular, ~~if~~ $\mathcal{C}$ is compatible with Σ, if and only if, $\mathcal{T}_\mathcal{C}$ is compatible with X_Σ.

Proof. See Lemma [BH17, Lemma 4.10]. □

For a tropical current $\mathcal{T}_\mathcal{C} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$, and given a toric variety X_Σ we denote its extension by zero $\overline{\mathcal{T}}_\mathcal{C} \in \mathcal{D}'_{p,p}(X_\Sigma)$.

Proposition 5.3. For every tropical variety $\mathcal{C}$, there exists a smooth projective toric fan Σ compatible with a subdivision of $\mathcal{C}$.

Proof. By [BS11], for C there is a refinement C', and a complete fan $\Sigma_1 \subseteq \mathbb{R}^n$ such that C' is compatible with Σ_1. Applying the toric Chow lemma [CLS11, Theorem 6.1.18] and the toric resolution of singularities [CLS11, Theorem 11.1.9] we can find a fan Σ which is a refinement of Σ_1 that defines a smooth projective variety X_Σ. The tropical variety C'' which is the refinement of C' induced by Σ, satisfies the statement. □

Remark 5.4. When C' is a refinement of a tropical variety C, then C' is a tropical variety with natural induced weights. It is also easy to check that we have the equality of currents $\mathcal{T}_C = \mathcal{T}_{C'}$ in $(\mathbb{C}^*)^n$; see [BH17, Section 2.6].

Lemma 5.5. Let $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tropical Laurent polynomial and X_Σ be a smooth projective toric variety compatible with a subdivision of $V_{\text{trop}}(\mathfrak{q})$. Let $\rho \in \Sigma(1)$. Assume that $\zeta_0 \in D_\rho \cap \text{supp}(\overline{dd^c[\mathfrak{q} \circ \text{Log}]})$, and Ω is a sufficiently small neighbourhood of ζ_0. Then, $\mathfrak{q} \circ \text{Log} \in \text{PSH}(\Omega \setminus D_\rho) \cap \mathcal{C}^0(\Omega \setminus D_\rho)$ can be extended to a function $u : \Omega \longrightarrow \mathbb{R}$, $\cup \{+\infty\}$
such that

(a) In Ω, $u = g + \kappa \log |f|$, where g is a continuous function, f is the local equation for D_ρ, and κ is a negative integer.

(b) Restricted to Ω, we have $dd^c u = \overline{\mathcal{T}}_{V_{\text{trop}(\mathfrak{q})}} + c[D_\rho]$. $c = \kappa$

(c) In Ω, we have $\overline{\mathcal{T}}_C = dd^c g$. In particular, $\overline{\mathcal{T}}_{V_{\text{trop}(\mathfrak{q})}}$ has a continuous superpotential.

Proof. Assume that $\mathfrak{q} = \max_{\alpha \in A}\{c_\alpha + \langle \alpha, x\rangle\}$. Recall that

$$\text{Log} = (-\log|\cdot|, \ldots, -\log|\cdot|).$$

We write

$$\mathfrak{q} \circ \text{Log} = \log \max_{\alpha}\{|e^{c_\alpha} z^{-\alpha}|\}.$$

$\beta, \gamma \ldots$

Assume that near ζ_0, $\mathfrak{q} \circ \text{Log}$ is given by $\max\{|e^{c_\beta} z^{-\beta}, |e^{c_\gamma} z^{-\gamma}|\}$. This implies that in $\text{Log}(\Omega \setminus D_\rho)$, $\mathfrak{q}$ is given by $\max\{c_\beta + \langle \beta, x\rangle, c_\gamma + \langle \gamma, x\rangle\}$. For $\mathfrak{q} = \max_{\alpha \in A}\{c_\alpha + \langle \alpha, x\rangle\}$ we set $\text{rec}(\mathfrak{q}) = \max_{\alpha \in A}\{\langle \alpha, x\rangle\}$. It is not hard to check that

$$\text{rec}(V_{\text{trop}}(\mathfrak{q})) = V_{\text{trop}}(\text{rec}(\mathfrak{q}));$$

see [MS15, Page 132].

We now show that by extending each $z^{-\alpha}$ as a rational function to X_Σ, the compatibility condition implies that $\mathfrak{q} \circ \text{Log}$ extends to X_Σ. By [CLS11, Proposition 4.1.2] the divisor of the extension of a character z^α in X_Σ is given by

$$\text{Div}(z^\alpha) = \sum_{\rho \in \Sigma(1)} \langle \alpha, n_\rho\rangle D_\rho, \tag{2}$$

where n_ρ the is the minimal generator of ρ. By assumption,

$$D_\rho \cap \text{supp}(\overline{dd^c[\mathfrak{q} \circ \text{Log}]}) \neq \varnothing.$$

Theorem 5.2 implies that

$$n_\rho \in \text{rec}(V_{\text{trop}(\mathfrak{q})}).$$

Moreover, if $\zeta_1 \in D_\rho \cap \text{supp}(\overline{dd^c[\text{rec}(\mathfrak{q}) \circ \text{Log}]})$, then in a small neighbourhood of $\text{Log}(\zeta_1)$, $\text{rec}(\mathfrak{q})(x) = \max\{\langle \beta, x\rangle, \langle \gamma, x\rangle\}$. By definition

$$n_\rho \in \text{rec}(V_{\text{trop}(\mathfrak{q})}) \quad \text{if and only if} \quad \kappa := \langle \beta, n_\rho\rangle = \langle \gamma, n_\rho\rangle.$$

This, together with Equation 2 implies that the extension of $z^{-\beta}$ and $z^{-\gamma}$ as rational functions to X_Σ have the same vanishing order along D_ρ, and we write $z^{-\beta} = f^\kappa \frac{g_1}{h_1}$ and $z^{-\gamma} = f^\kappa \frac{g_2}{h_2}$. Now note that in $\Omega \setminus D_\rho$

$$\mathrm{q} \circ \mathrm{Log} = \max \log\{|e^{c_\beta} z^{-\beta}|, |e^{c_\gamma} z^{-\gamma}|\} = \kappa \log|f| + \max\{|e^{c_\beta}\frac{g_1}{h_1}|, |e^{c_\gamma}\frac{g_2}{h_2}|\},$$

we must have $\kappa < 0$, otherwise $q \circ \mathrm{Log} = -\infty$ in $\Omega \setminus D_\rho$. Consequently, $\mathrm{q} \circ \mathrm{Log} : \Omega \setminus D_\rho \longrightarrow \mathbb{R}$, can be extended to

$$u := \kappa \log|f| + \max\{|e^{c_\beta}\frac{g_1}{h_1}|, |e^{c_\gamma}\frac{g_2}{h_2}|\}$$

on Ω. Setting

$$g = \max\{|e^{-c_\beta}\frac{g_1}{h_1}|, |e^{-c_\gamma}\frac{g_2}{h_2}|\},$$

implies (a).

We have

$$dd^c[\mathrm{q} \circ \mathrm{Log}]_{|\Omega \setminus D_\rho} = \left(dd^c \log|f|^\kappa \, dd^c \log|g|\right)_{|\Omega \setminus D_\rho} = dd^c \log|g|_{|\Omega \setminus D_\rho},$$

since $dd^c \log|f|^\kappa$ is holomorphic in $\Omega \setminus D_\rho$. As a result of compatibility with X_Σ, $\overline{dd^c[\mathrm{q} \circ \mathrm{Log}]}$ does not charge any mass in D_ρ, and we obtain

$$\overline{dd^c[\mathrm{q} \circ \mathrm{Log}]} = dd^c \log|g|.$$

This together with Theorem 4.4 implies (c) and (b). □

Lemma 5.6. Assume that σ is p-dimensional and $\mathrm{aff}(\sigma) = H_1 \cap \cdots \cap H_{n-p}$, is given as the transveral intersection hyperplanes $H_i \subseteq \mathbb{R}^n$. If Σ is a smooth projective fan compatible with $\bigcup_i H_i$, then

$$\overline{\mathcal{T}}_{\mathrm{aff}(\sigma)} \leq \overline{\mathcal{T}_{H_1} \wedge \cdots \wedge \mathcal{T}_{H_{n-p}}} \leq \overline{\mathcal{T}}_{H_1} \wedge \cdots \wedge \overline{\mathcal{T}}_{H_{n-p}}.$$

= no ? there is no mass on analy sets.

Proof. By the definition of tropical currents we have the inequality

$$\mathcal{T}_{\mathrm{aff}(\sigma)} \leq \mathcal{T}_{H_1} \wedge \cdots \wedge \mathcal{T}_{H_{n-p}},$$

as currents in $(\mathbb{C}^*)^n$, since the right hand side might have multiplicities but the currents have the same support. Now, the wedge products in X_Σ are well-defined by Lemma 5.5 and Theorem 2.4. As both currents on both sides of the equation coincide on $(\mathbb{C}^*)^n$, the support of the current on the right-hand side contains the closure of the support of $\mathcal{T}_C$ in X_Σ. □

seems to be useful Yes

Should I modify this for non-positive tropical cycles too? Since they can be written as a difference of two positive cycles, this is easy.

Theorem 5.7. Let C be a positively weighted tropical cycle of dimension p compatible with a smooth, projective fan Σ, then $\overline{\mathcal{T}}_C$ has a continuous superpotential in X_Σ.

We need the following definition.

Definition 5.8. We define the affine extension p-dimensional a tropical cycle $\mathcal{C}$, by as the addition of tropical cycles

$$\widehat{\mathcal{C}} := \sum_{\sigma \in \mathcal{C}} w_\sigma \mathrm{aff}(\sigma).$$

It is clear that if $\mathcal{C}$ is a positively weighted tropical cycle, then $\mathscr{T}_{\widehat{\mathcal{C}}} - \mathscr{T}_{\mathcal{C}} \geq 0$.

Theorem

Proof of 5.7 . Let $\widehat{\mathcal{C}}$ be the affine extension of $\mathcal{C}$, and $\widehat{\Sigma}$ be a smooth projective fan which is a refinement of Σ and compatible with $\widehat{\mathcal{C}}$. By the preceding lemma and repeated application of Theorem 2.4 for any $\sigma \in \mathcal{C}$, $\overline{\mathscr{T}}_{\mathrm{aff}(\sigma)}$ has a bounded superpotential, which implies this property for $\overline{\mathscr{T}}_{\widehat{\mathcal{C}}}$. Now, since $\mathscr{T}_{\widehat{\mathcal{C}}} - \mathscr{T}_{\mathcal{C}}$ is a positive closed tropical current in $(\mathbb{C}^*)^n$,

$$\overline{\mathscr{T}_{\widehat{\mathcal{C}}} - \mathscr{T}_{\mathcal{C}}} = \overline{\mathscr{T}}_{\widehat{\mathcal{C}}} - \overline{\mathscr{T}}_{\mathcal{C}} \geq 0$$

in $X_{\widehat{\Sigma}}$. Continuity of the superpotential of $\overline{\mathscr{T}}_{\mathcal{C}}$ in $X_{\widehat{\Sigma}}$ follows from Theorem 2.3.

We now show that $\overline{\mathscr{T}}_{\mathcal{C}}$ has also a continuous super-potential on X_Σ as well. We consider the proper map $f : X_{\widehat{\Sigma}} \longrightarrow X_\Sigma$, which can be understood as a composition of multiple blow-ups along toric points with exceptional divisors D_ρ for any ray $\rho \in \widehat{\Sigma} \setminus \Sigma$. These divisors satisfy $D_\rho \cap \mathrm{supp}(\overline{\mathscr{T}}_{\mathcal{C}}) = \varnothing$. We deduce by Corollary 2.9. □

check

5

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Proposition 5.9. In a toric variety X_Σ compatible with the tropical cylce $\mathcal{C}_1 + \mathcal{C}_2$,

$$\overline{\mathscr{T}_{\mathcal{C}_1} \wedge \mathscr{T}_{\mathcal{C}_2}} = \overline{\mathscr{T}}_{\mathcal{C}_1} \wedge \overline{\mathscr{T}}_{\mathcal{C}_2}.$$

Proof. The proof is clear since both $\overline{\mathscr{T}}_{\mathcal{C}_1}$ and $\overline{\mathscr{T}}_{\mathcal{C}_2}$ have continuous superpotentials with no mass on the boundary divisors $X_\Sigma \setminus T_N$. □

the role of N seems to be different in different places

Proposition 5.10. For any two tropical currents $\mathcal{C}_1$ and $\mathcal{C}_2$, the intersection product

$$\mathscr{T}_{\mathcal{C}_1} \wedge \mathscr{T}_{\mathcal{C}_2} := \overline{\mathscr{T}}_{\mathcal{C}_1} \wedge \overline{\mathscr{T}}_{\mathcal{C}_2}|_{(\mathbb{C}^*)^n},$$

does not depend on the choice of a smooth projective toric variety of the fan Σ compatible with $\mathcal{C}_1 + \mathcal{C}_2$, where $(\mathbb{C}^*)^n$ is identified with $T_N \subseteq X_\Sigma$. Moreover, this product coincides with the definition of wedge products with bi-degree $(1,1)$ tropical currents in Bedford–Taylor Theory in $(\mathbb{C}^*)^n$.

$\mathbb{C}$

Proof. This is a consequence of Lemma 2.7, and the fact that intersection product with a bidegree $(1,1)$ current in super-potential theory, in an open set of compact Kähler manifold, coincides with the Bedford–Taylor theory. □

5.2. **Proof of $\mathscr{T}_{\mathcal{C}} \wedge \mathscr{T}_{\mathcal{C}'} = \mathscr{T}_{\mathcal{C}\cdot\mathcal{C}'}$.**

Theorem 5.11. For two tropical varieties $\mathcal{C}$ and $\mathcal{C}'$ with complementary dimensions the notion of stable intersection is well-defined and we have

necessary?

$$\mathscr{T}_{\mathcal{C}} \wedge \mathscr{T}_{\mathcal{C}'} = \mathscr{T}_{\mathcal{C}\cdot\mathcal{C}'}.$$

Proposition 5.12 ([Kat09, Propositions 6.1]). Let $H_1, H_2 \subseteq \mathbb{R}^n$ be two rational planes of dimension p and q with $p+q=n$ that intersect transversely. Then, the complex tori $T_{H_1 \cap \mathbb{Z}^n}$ and $T_{H_2 \cap \mathbb{Z}^n}$ intersect at $[N : N_{H_1} + N_{H_2}]$ distinct points.

Proof of Theorem 5.11. Note that $\mathcal{T}_C \wedge \mathcal{T}_{C'}$ is well-defined by Proposition 5.10. We proceed with the following steps:

(a) $\mathcal{T}_C \wedge \mathcal{T}_{C'} = \mathcal{T}_{C.C'}$ in the transversal case.
(b) $\mathrm{Log}(\mathrm{supp}(\mathcal{T}_C \wedge \mathcal{T}_{C'}))$ is 0-dimensional in the general case.
(c) $\mathrm{supp}(\mathcal{T}_C \wedge \mathcal{T}_{C'}) = \mathrm{Log}^{-1}(C.C')$.
(d) Proof of Theorem 3.4.

may be change the notation, a?

$\mathcal{T}_C \wedge \mathcal{T}_{C'}$ is invariant under the action of $(S^1)^n$, and therefore it is a tropical current. To see (a), when $C \cap C'$ is transverse. Assume that $x \in C \cap C'$ and the intersection is transverse. Since C and C' are of complementary dimensions, we can choose a small ball $B =$ $B_\epsilon(x) \in \mathbb{R}^n$ such that x is an isolated point of intersection in B. Now, by Lemma 2.7

not the same x above

$$\mathcal{T}_C \wedge \mathcal{T}_{C'}|_{\mathrm{Log}^{-1}(B)} = w_\sigma w_{\sigma'} \mathbb{1}_{\mathrm{Log}^{-1}(B)}$$

$$\int_{(x,x')\in(S^1)^n} [\pi_\sigma^{-1}(x)] \wedge [\pi_{\sigma'}^{-1}(x')]\, d\mu_\sigma(x) \otimes d\mu_{\sigma'}(x').$$

σ, σ' not introduced as well as the notation π_σ, $\pi_{\sigma'}$

Transversality of the fibers implies

conjuring; normally we have a product of real tori and a map onto $(S^1)^n$, of degree K.

$$[\pi_\sigma^{-1}(x)] \wedge [\pi_{\sigma'}^{-1}(x')] = [\pi_\sigma^{-1}(x) \cap \pi_{\sigma'}^{-1}(x')].$$

By Proposition 5.12, we have $\kappa = [N : N_{\mathrm{aff}(\sigma)} + N_{\mathrm{aff}(\sigma')}]$ distinct intersection points covering $\mathrm{Log}^{-1}(x)$. When $(x,x') \in (S^1)^n$ vary with respect to normalised Haar measure, these κ points cover $(S^1)^n$ with speed κ. As a result,

do you mean "k times"?

$$\int_{(x,x')\in(S^1)^n} [\pi_\sigma^{-1}(x)] \wedge [\pi_{\sigma'}^{-1}(x')]\, d\mu_\sigma(x) \otimes d\mu_{\sigma'}(x') = \int_{y\in(S^1)^n} \kappa[\pi_{\sigma\cap\sigma'}^{-1}(y)] d\mu_{\sigma\cap\sigma'}(y).$$

new line

This proves (a). To prove (b) note that if $C + b$ is the translation of the the tropical variety by $b \in \mathbb{R}^n$, then $(e^b)^*\mathcal{T}_C = \mathcal{T}_{C+b}$. Moreover, we have the SP-convergence of currents with continuous superpotentials.

you can change b by εb

$$(e^{\varepsilon b})^*\mathcal{T}_C \longrightarrow \mathcal{T}_C, \qquad \text{as } \varepsilon \to 0.$$

Therefore, by Theorem 2.4,

$$(e^{\varepsilon b})^*\mathcal{T}_C \wedge \mathcal{T}_{C'} = \mathcal{T}_{C+\epsilon b} \wedge \mathcal{T}_{C'} \longrightarrow \mathcal{T}_C \wedge \mathcal{T}_{C'}, \qquad \text{as } \varepsilon \to 0. \tag{3}$$

Considering the support, we obtain the Hausdorff limit

$$\lim \mathrm{supp}((e^{\varepsilon b})^*\mathcal{T}_C \wedge \mathcal{T}_{C'}) \supseteq \mathrm{supp}(\mathcal{T}_C \wedge \mathcal{T}_{C'}).$$

We now note that for all ϵ, the number of intersection points in $(C+\epsilon b)\cap C'$ is uniformly bounded by the number of p-dimensional cells in C and q-dimensional cells in C, the Hausdorff limit of $(C + b\epsilon) \cap C'$ is also zero dimensional. To prove (c), by definition of $C \cdot C'$ it suffices to show that

εb

C'

P.19 not introduced

$$\lim \mathrm{supp}((e^{\varepsilon b})^*\mathcal{T}_C \wedge \mathcal{T}_{C'}) = \mathrm{supp}(\mathcal{T}_C \wedge \mathcal{T}_{C'}),$$

for any fixed generic b. This is also easy. Let $x_\epsilon \in (C + \epsilon b) \cap C'$. Since the translation by ϵb does not change slopes of the cells, as $x_\epsilon \to x$, the multiplicity for all x_ϵ remains constant for $\epsilon > 0$, therefore ~~the mass~~ $\lim (e^{\epsilon b})^*\mathcal{T}_C \wedge \mathcal{T}_{C'}$ has a non-zero mass at $\mathrm{Log}^{-1}(x)$. Now Part (d) is deduced from Equation (3) since we can choose b generically and Parts (a),(b),(c). □

5

here you only consider the case of complementary dimensions slicing is needed?

Theorem 5.13. Stable intersection of tropical cycles is associative and commutative, and $\mathcal{C} \mapsto \mathcal{T}_{\mathcal{C}}$ induces an isomorphism of $\mathbb{Z}$-algebras between effective tropical cycles and positive tropical currents on $(\mathbb{C}^*)^n$.

Proof. This is the application of Theorem 5.7 and Theorem 2.5, and Theorem 5.11. □

5.2.1. *Calculating Intersection Multiplicities Using Monge-Ampère Measures.* In this section, we explain how to calculate intersection multiplicities in two different ways. Note that by the equality of the supports in the previous section, we only need to prove the intersection multiplicities in the transversal case locally.

5.2.2. *Real Monge–Ampère Measures.* Let $\Omega \subseteq \mathbb{R}^n$ be an open subset and $u : \Omega \longrightarrow \mathbb{R}$ be a convex (hence continuous) function. The *generalised gradient* of u at $x_0 \in \Omega$ is defined by

$$\nabla u(x_0) = \{\xi \in (\mathbb{R}^n)^* : u(x) - u(x_0) \geq \langle \xi, x - x_0\rangle, \text{ for all } x \in \Omega\}.$$

In the above, $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{R}^n$, and $(\mathbb{R}^n)^*$ is the dual. The real Monge-Ampére measure associated to a convex polynomial u of a Borel set $E \subseteq \Omega$, is given by

$$\mathrm{MA}[u](E) = \mu\Big(\bigcup_{y\in E} \nabla u(y)\Big),$$

where μ is the Lebesgue measure on $(\mathbb{R}^n)^*$.

It is interesting that for the tropical polynomials, one can compute the associate real Monge–Ampére measures explicitly. Recall that, for any tropical polynomial, there is a natural subdivision of its Newton polytope which is dual to the tropical variety of it. See Figure for an example and [BS14, MS15] for details.

Lemma 5.14 ([Yge13, Page 59],[BGPS14, Proposition 2.7.4]). Let $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tropical polynomial ~~$\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$~~ with associated tropical variety $\mathcal{C} = V_{\text{trop}}(\mathfrak{q})$, one has

$$\mathrm{MA}[\mathfrak{q}] = \sum_{a\in\mathcal{C}(0)} \mathrm{Vol}(\{a\}^*)\delta_a,$$

where $\mathcal{C}(0)$ is the 0-dimensional skeleton of $\mathcal{C}$, and $\{a\}^*$ is the dual of the vertex $a \in \mathcal{C}(0)$.

A detailed discussion of the preceding theorem can be also found in [Bab14].

5.3. **Polarisation.** For n convex functions $u_1, \ldots, u_n : \mathbb{R}^n \longrightarrow \mathbb{R}$, their *mixed Monge-Ampére measure* is defined by

$$\widetilde{\mathrm{MA}}[u_1, \ldots, u_n] = \frac{1}{n!}\sum_{k=1}^{n} \sum_{1\leq j_1<\cdots<j_k\leq n} (-1)^{n-k}\,\mathrm{MA}[u_{j_1} + \cdots + u_{j_k}].$$

Recall that this is how the *mixed volume* of n convex bodies can be defined from the n-dimensional volume. Moreover, it is easy to check that for a convex function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathrm{MA}[u] = \widetilde{\mathrm{MA}}[u, \ldots, u]$.

The following statements are clear from 5.14 by taking the total mass.

Proposition 5.15. Let $\mathfrak{q}, \mathfrak{q}_1, \ldots, \mathfrak{q}_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ be tropical polynomials. We have the following facts: