Linear Algebra: Sheet 8

Present all your answers in complete sentences. There is also a Numbas quiz.

Hand-in question

Submit your solution on Blackboard by **1pm on Wednesday (Week 10)** for feedback from your tutor.

1. Recall, from Sheet 7, the basis $\mathcal{A} = \{v_1, v_2\}$ of $\mathbb{R}^2$ with $v_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $v_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
   a) What property does the basis $\mathcal{A}$ have? Describe, in words, the conditions to check for this property.
   b) Using the property identified in (a), find $M_{\mathcal{A}\mathcal{A}}(f)$ where $f : (x, y) \mapsto (x, -y)$.
   c) Now let $g : (x, y) \mapsto (3x, 2y)$ and find $M_{\mathcal{A}\mathcal{A}}(g)$.
   d) Confirm your answers to (b) and (c) by somehow using the image of the basis vectors under f and g.

   Solution:

   We work with $\mathcal{A} = \{v_1, v_2\}$ of $\mathbb{R}^2$ where $v_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $v_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

   a) The set $\mathcal{A}$ is an orthonormal basis. In words, this says that the inner product of any given vector in the set with itself is 1 and the inner product of distinct elements in the set is zero.
   b) We use the example from lectures to find $M_{\mathcal{A}\mathcal{A}}(f)$ where $f : (x, y) \mapsto (x, -y)$. Recall that this matrix has entries given by $v_i \cdot f(v_j)$, since $\mathcal{A}$ is an ONB. Thus
   $$M_{\mathcal{A}\mathcal{A}}(f) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$
   c) For $g : (x, y) \mapsto (3x, 2y)$, the same idea above gives us that
   $$M_{\mathcal{A}\mathcal{A}}(g) = \begin{pmatrix} 5/2 & -1/2 \\ -1/2 & 5/2 \end{pmatrix}.$$
   d) We know that the ith column of $M_{\mathcal{A}\mathcal{A}}(f)$ tells us the linear combination, in $\mathcal{A}$, obtained from computing $f(v_i)$. This is indeed the case: $f(v_1) = -v_2$ and $f(v_2) = -v_1$. Similarly we have that $g(v_1) = \frac{5}{2}v_1 - \frac{1}{2}v_2$ and $g(v_2) = -\frac{1}{2}v_1 + \frac{5}{2}v_2$.

Additional questions

Try these questions and look at the solutions for feedback. They might also be discussed in your tutorial.

2. Let $A \in M_n(\mathbb{R})$ be a matrix which is symmetric ($A^t = A$) and positive definite ($\mathbf{x} \cdot A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \backslash \{0\}$). Our aim in this question is to show that we obtain an inner product on $V = \mathbb{R}^n$ by defining

$$\langle \mathbf{x}, \mathbf{y} \rangle_A := \mathbf{x} \cdot A\mathbf{y} = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} y_j.$$

In each part, a specific property of A may be helpful. There are hints[1] below.

a) Show that $\langle v, v \rangle_A \geq 0$ and $\langle v, v \rangle_A = 0$ if and only if $v = 0$.

b) Write out $\sum_{i=1}^n x_i \sum_{j=1}^n a_{ij} y_j$, and so justify this equals $\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i y_j$.

c) Show that $\langle v, w \rangle_A = \langle w, v \rangle_A$ for any $v, w \in \mathbb{R}^n$.

d) Show that $\langle v, u + w \rangle_A = \langle v, u \rangle_A + \langle v, w \rangle_A$ and $\langle v, \lambda w \rangle_A = \lambda \langle v, w \rangle_A$.

e) Pick a matrix A that is not symmetric. Which property of $\langle \cdot, \cdot \rangle_A$ do we expect to fail? Find a specific matrix $A \in M_2(\mathbb{R})$ where this property does fail.

f) What about if A is not positive definite? Again, find a specific $A \in M_2(\mathbb{R})$.

Solution:

a) Given $v \neq 0$, we note that $\langle v, v \rangle_A > 0$ from our assumption that A is positive definite. If $v = 0$, then $\langle v, v \rangle_A = v \cdot Av = 0 \cdot 0 = 0$.

b) Directly computing, we see that

$$\begin{aligned} \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} y_j &= x_1 \sum_{j=1}^{n} a_{1j} y_j + \cdots + x_n \sum_{j=1}^{n} a_{nj} y_j \\ &= \sum_{j=1}^{n} a_{1j} x_1 y_j + \cdots + \sum_{j=1}^{n} a_{nj} x_n y_j = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i y_j \end{aligned}$$

c) We first apply our observation from (b),

$$\langle v, w \rangle_A = v \cdot Aw = \sum_{i=1}^{n} v_i \sum_{j=1}^{n} a_{ij} w_j = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} v_i w_j.$$

Similarly we can apply this to see that

$$\langle w, v \rangle_A = w \cdot Av = \sum_{k=1}^{n} w_k \sum_{l=1}^{n} a_{kl} v_l = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} w_k v_l.$$

Using that a double summation commutes, we see that

$$\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} w_k v_l = \sum_{l=1}^{n} \sum_{k=1}^{n} a_{kl} w_k v_l = \sum_{l=1}^{n} \sum_{k=1}^{n} a_{kl} v_l w_k.$$

Replacing the indices l and k with i and j respectively means the previous expression is $\sum_{i=1}^n \sum_{j=1}^n a_{ji} v_i w_j$. At this point we recall our assumption that $A = A^t$, and so $\langle w, v \rangle_A = \langle v, w \rangle_A$.

d) There are two possible approaches. For one, we could apply properties of matrix multiplication. Then $\langle v, u + w \rangle_A$ becomes

$$v \cdot (A(u + w)) = v \cdot (Au + Aw) = v \cdot Au + v \cdot Aw = \langle v, u \rangle_A + \langle v, w \rangle_A.$$

[1] For (a), replacing v with $\mathbf{x}$ may help.
For (c), use (b) together with $\sum_i \sum_j a_i b_j = \sum_j \sum_i a_i b_j$...now, what property does A have?
For (d), write out the LHS and RHS in each case; try and show that they are equal.

Alternatively, for a more algebraic approach, we could observe that

$$\langle v, u+w\rangle_A = v \cdot A(u+w) = \sum_{i=1}^{n} v_i \sum_{j=1}^{n} a_{ij}(u+w)_j = \sum_{i=1}^{n} v_i \sum_{j=1}^{n} a_{ij}(u_j + w_j).$$

We are merely working with real numbers, and so this can be manipulated to get

$$\begin{aligned}\sum_{i=1}^{n} v_i \sum_{j=1}^{n} a_{ij}(u_j + w_j) &= \sum_{i=1}^{n} v_i \sum_{j=1}^{n} (a_{ij}u_j + a_{ij}w_j) \\ &= \sum_{i=1}^{n} v_i \sum_{j=1}^{n} a_{ij}u_j + \sum_{i=1}^{n} v_i \sum_{j=1}^{n} a_{ij}w_j = \langle v, u\rangle_A + \langle v, w\rangle_A.\end{aligned}$$

With a similar approach, we see that

$$\langle v, \lambda w\rangle_A = v \cdot A(\lambda w) = v \cdot (\lambda A w) = \lambda v \cdot Aw = \lambda \langle v, w\rangle_A.$$

e) It was only in part (c) that we used the assumption that A should be symmetric. It may appear clear that such an assumption is necessary for that argument to work, but a concrete example provides us with a proper justification. (In fact, any non-symmetric real matrix will provide a counter-example, but then finding suitable v and w may be more difficult.) We use

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for which one can check that $\langle e_1, e_1 + e_2\rangle_A \neq \langle e_1 + e_2, e_1\rangle_A$.

f) We only used the positive definite assumption in part (a). The matrix we happened to choose in (e) fails to be positive definite in a strong way: $\langle v, v\rangle_A = 0$ for every $v \in \mathbb{R}^2$. This therefore does not give rise to an inner product, since it fails the first condition given in our definition in the lecture notes.

3. Let V be a complex inner product space with orthonormal basis $\mathcal{A} = \{u_1, u_2, u_3\}$.

a) Define the function $P(v) := \langle u_1, v \rangle u_1$.

(i) Show that P is a linear map.

(ii) Show that $P^2 = P$.

(iii) Show that $P = P^*$, where P^* denotes the adjoint of P.

(iv) Calculate $P(u_1), P(u_2), P(u_3)$ and hence find the matrix $M_{\mathcal{A}\mathcal{A}}(P)$.

(v) Consider what P does to a general vector $v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$.

(vi) From the above, find $\operatorname{Im} P$ and express your answer as a span.

b) Let $Q(v) := \langle u_1, v \rangle u_1 + \langle u_2, v \rangle u_2$. Answer (a)(i)-(vi) for Q. It is easier to first show (i) and then (iv) for Q, in order to deduce answers for Q.

c) Finally, let $R(v) = \langle u_1, v \rangle u_1 + \langle u_2, v \rangle u_2 + \langle u_3, v \rangle u_3$. What is the function R? (It may be helpful to do (i) and (iv) for R.)

Solution:

Recall that V is a complex inner product space and $\mathcal{A} = \{u_1, u_2, u_3\}$ an orthonormal basis. We have first defined the function $P(v) := \langle u_1, v \rangle u_1$.

a) (i) To show that P is a linear map, we check each condition in turn.

$$\begin{aligned} P(v + w) &= (\langle u_1, v + w \rangle) u_1 = (\langle u_1, v \rangle + \langle u_1, w \rangle) u_1 \\ &= \langle u_1, v \rangle u_1 + \langle u_1, w \rangle u_1 = P(v) + P(w) \\ P(\lambda v) &= (\langle u_1, \lambda v \rangle) u_1 = (\lambda \langle u_1, v \rangle) u_1 = \lambda \langle u_1, v \rangle u_1 = \lambda P(v) \end{aligned}$$

(ii) Again, we compute directly. Let $v \in V$. Then

$$P^2(v) = P(\langle u_1, v \rangle u_1) = \langle u_1, \langle u_1, v \rangle u_1 \rangle u_1 = \langle u_1, v \rangle (\langle u_1, u_1 \rangle u_1) = \langle u_1, v \rangle$$

because $\langle u_1, u_1 \rangle = 1$ from our assumption that $\mathcal{A}$ is an ONB.

(iii) Recall that the adjoint of P is the function such that

$$\langle P^*(v), w \rangle = \langle v, P(w) \rangle \text{ for every } v, w \in V.$$

We will show that $\langle P(v), w \rangle = \langle v, P(w) \rangle$ so to show that $P = P^*$, using properties of an inner product covered in lectures. Note that

$$\begin{aligned} \langle v, P(w) \rangle &= \langle v, \langle u_1, w \rangle u_1 \rangle = \langle u_1, w \rangle \langle v, u_1 \rangle \text{ and similarly} \\ \langle P(v), w \rangle &= \langle \langle u_1, v \rangle u_1, w \rangle = \overline{\langle u_1, v \rangle} \langle u_1, w \rangle = \langle v, u_1 \rangle \langle u_1, w \rangle. \end{aligned}$$

(iv) To calculate $P(u_1), P(u_2), P(u_3)$, we rely heavily on $\mathcal{A}$ being an ONB, and find that

$$\begin{aligned} P(u_1) &= \langle u_1, u_1 \rangle u_1 = u_1; \\ P(u_2) &= \langle u_1, u_2 \rangle u_1 = 0; \text{ and} \\ P(u_3) &= \langle u_1, u_3 \rangle u_1 = 0. \end{aligned}$$

These are already naturally linear combinations in $\mathcal{A}$. Hence

$$M_{\mathcal{A}\mathcal{A}}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(v) Take $v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in V$. Using linearity,

$$P(v) = P(\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3) = \sum_{i=1}^{3} \lambda_i P(u_i) = \lambda_1 u_1.$$

(vi) From our linearity calculation above, we need only find the image of u_1, u_2, and u_3 in order to find $\operatorname{Im} P$. We did this above. Hence $\operatorname{Im} P$ contains $\lambda_1 u_1$ for every $\lambda_1 \in \mathbb{C}$. Furthermore, given $v \in V$ we have that $v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$ and so $P(v) = \lambda u_1$ for some $\lambda \in \mathbb{C}$. Hence $\operatorname{Im} P = \operatorname{span}\{u_1\}$. Another approach is to use that the image of P is related to the span of the columns of $M_{\mathcal{A}\mathcal{A}}(P)$ with respect to $\mathcal{A}$.

b) We have that $Q(v) := \langle u_1, v\rangle u_1 + \langle u_2, v\rangle u_2$. As suggested by the question, we first show (i) and then (iv) for Q.

(i) We show that Q is a linear map in the same way that we did for P.

$$\begin{aligned}
Q(v+w) &= (\langle u_1, v+w\rangle)u_1 + (\langle u_2, v+w\rangle)u_2 \\
&= (\langle u_1, v\rangle + \langle u_1, w\rangle)u_1 + (\langle u_2, v\rangle + \langle u_2, w\rangle)u_2 \\
&= \langle u_1, v\rangle u_1 + \langle u_1, w\rangle u_1 + \langle u_2, v\rangle u_2 + \langle u_2, w\rangle u_2 = Q(v) + Q(w) \\
Q(\lambda v) &= (\langle u_1, \lambda v\rangle)u_1 + (\langle u_2, \lambda v\rangle)u_2 \\
&= (\lambda\langle u_1, v\rangle)u_1 + (\lambda\langle u_2, v\rangle)u_2 \\
&= \lambda(\langle u_1, v\rangle u_1 + \langle u_2, v\rangle u_2) = \lambda Q(v)
\end{aligned}$$

(iv) We now wish to find $M_{\mathcal{A}\mathcal{A}}(Q)$. We note, from $\mathcal{A}$ an ONB, that

$$\begin{aligned}
P(u_1) &= \langle u_1, u_1\rangle u_1 + \langle u_2, u_1\rangle u_2 = u_1; \\
P(u_2) &= \langle u_1, u_2\rangle u_1 + \langle u_2, u_2\rangle u_2 = u_2; \text{ and} \\
P(u_3) &= \langle u_1, u_3\rangle u_1 + \langle u_2, u_3\rangle u_2 = 0.
\end{aligned}$$

Again, these are already naturally linear combinations in $\mathcal{A}$. Hence

$$M_{\mathcal{A}\mathcal{A}}(Q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(ii) We confirm that $Q^2 = Q$ by composing our matrix with itself.

(iii) Similarly $Q = Q^*$ from our matrix form.

(v) Again the linearity of Q can allow us to find $\operatorname{Im} Q$, or alternatively we can find the image of a vector under $M_{\mathcal{A}\mathcal{A}}(Q)$ to see that $(\lambda_1, \lambda_2, \lambda_3)_{\mathcal{A}}$ is sent to $(\lambda_1, \lambda_2, 0)_{\mathcal{A}}$.

(vi) From (v), we see that $\operatorname{Im} Q = \operatorname{span}\{u_1, u_2\}$.

c) In the case of $R(v) = \langle u_1, v\rangle u_1 + \langle u_2, v\rangle u_2 + \langle u_3, v\rangle u_3$, we see that $M_{\mathcal{A}\mathcal{A}}(R)$ is the identity matrix, and so R sends $(\lambda_1, \lambda_2, \lambda_3)_{\mathcal{A}}$ to $(\lambda_1, \lambda_2, \lambda_3)_{\mathcal{A}}$. Hence R is the identity function.

4. A function satisfying 3(a)(i)-(iii) is called an orthogonal projection. Let V be a vector space, $P : V \to V$ an orthogonal projection, and I be the identity map.
   a) Show that $I - P$ is also an orthogonal projection.
   b) Show that $\mathrm{Im}(I - P) = \ker P$.[2]

   Solution:

   We have that P is an orthogonal projection, and I is the identity function.

   a) We check each of the three conditions in turn.
      (i) The sum of two linear maps is a linear map.
      (ii) By considering the sum and composition of functions, we have

$$
\begin{aligned}
(I-P)^2 &= (I-P)(I-P) \\
&= I \circ I - P \circ I - I \circ P + P \circ P \\
&= I - P - P + P^2 \\
&= I - P - P + P = I - P
\end{aligned}
$$

      (iii) Finally we note that $(I - P)^* = I^* - P^* = I - P^* = I - P$.

   b) As mentioned in the hint, there are two natural approaches. First, we show each is included in the other.

      If $v \in \ker P$, then $(I - P)v = Iv - Pv = v$. Thus $v \in \mathrm{Im}(I - P)$ and $\ker P \subset \mathrm{Im}(I - P)$. On the other hand, $v \in \mathrm{Im}(I - P)$ means there exists a $w \in V$ with $v = (I - P)w$. Then $Pv = P(I - P)w = (P - P^2)w = 0$, and so $\mathrm{Im}(I - P) \subset \ker P$.

      Alternatively, we could use the example from the lecture notes. This states that $V = \mathrm{Im}(P) \oplus \ker(P)$. Thus, given $v \in V$, we have that $x = x_1 + x_2$ where $x_1 \in \mathrm{Im}(P)$ and $x_2 \in \ker(P)$. Now,

      - $P(x_1) = x_1$ and so $x_1 - P(x_1) = x_1 - x_1$, i.e. $x_1 \in \ker(I - P)$.
      - $P(x_2) = x_2 - P(x_2) = x_2$, i.e. $x_2 \in \mathrm{Im}(I - P)$.

      Thus $x \in \mathrm{Im}(I - P)$ if and only if $x \in \ker(P)$.

5. Determine which of the following matrices are hermitian.

$$
\text{(a) } A = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} \qquad \text{(b) } B = \begin{pmatrix} \mathrm{i} & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(c) } C = \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}
$$

   Solution:

   We compute the adjoint matrix, and compare this with the given matrix.

$$
\text{(a) } A^* = A \qquad \text{(b) } B^* \neq B \qquad \text{(c) } C^* = C
$$

   Therefore A and C are hermitian, whereas B is not.

[2] There are two approaches. Either show each are subsets of one another, or look at the example on orthogonal projections in the lecture notes.

6. a) Find the complex conjugate of each of the following expressions
   (i) $(a+bi)^{-1}$, where $a, b \in \mathbb{R}$. Compare this to $(a-bi)^{-1}$.
   (ii) e^{ai}, where $a \in \mathbb{R}$. Can you express this as e^{bi} for some $b \in \mathbb{R}$?[3]
   b) Based on your answers above, decide whether D is hermitian, where

$$D = \begin{pmatrix} 1 & 2+\mathrm{i} & \mathrm{e}^{\mathrm{i}} \\ 2-\mathrm{i} & 1 & \frac{1}{1-\mathrm{i}} \\ \mathrm{e}^{-\mathrm{i}} & \frac{1}{1+\mathrm{i}} & 0 \end{pmatrix}.$$

Solution:

a) We compute in each case.
   (i) Rationalising the denominator, we see that

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} \Rightarrow \overline{\left(\frac{1}{a+bi}\right)} = \frac{a+bi}{a^2+b^2}.$$

   Similarly, $(a-bi)^{-1} = (a+bi)^{-1}$
   (ii) We use Euler's formula, which says that $e^{ai} = \cos a + i \sin a$. Hence $\overline{e^{ai}} = \cos a - i \sin a = \cos(-a) + i \sin(-a) = e^{-ai}$.

b) From our computations in (a), the matrix D is hermitian.

7. Which of the following matrices are unitary?

$$\text{(a) } A = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \text{(b) } B = \begin{pmatrix} \mathrm{i} & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(c) } C = \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$$

Solution:

Computing we see that $AA^* = I_2$ but BB^*, CC^* are not I_2. Therefore only A is unitary.

8. Recall the complex inner product on $\mathbb{C}^n$ defined by $\langle v, w \rangle := \overline{v} \cdot w$. Take $v_1, v_2, v_3 \in \mathbb{C}^3$ to be an ONB with respect to this inner product.
   a) Let $U = (v_1\ v_2\ v_3)$. Explain why $U^*U = I_3$.
   b) Possibly by using the idea from (a), show that D is unitary, where

$$D = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{\mathrm{i}}{\sqrt{3}} & \frac{-\mathrm{i}}{\sqrt{3}} \\ \frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\mathrm{i}}{\sqrt{6}} & \frac{2\mathrm{i}}{\sqrt{6}} \end{pmatrix}.$$

Solution:

a) To help with the visualisation, the entries of U and U^* could be written out. Note that U^* has rows given by $\overline{v_1}^t$, $\overline{v_2}^t$, and $\overline{v_3}^t$. Thus the ijth entry of U^*U is given by $\overline{v_i} \cdot v_j$. From our assumption that v_1, v_2, v_3 form an ONB, the resulting matrix has entries $a_{ij} = \delta_{ij}$, which defines the identity matrix.

b) The columns of D form an ONB, and so D is unitary. Note that some ONBs are easy to spot, and the computations here seem simpler than finding D^*D: we computed $\langle v_i, v_i \rangle$ for $i = 1, 2, 3$ and $\langle v_1, v_2 \rangle$, $\langle v_1, v_3 \rangle$, and $\langle v_2, v_3 \rangle$.

[3] Euler's formula involving e^{ix} is helpful here.

9. Let V be an inner product space and $v_1, v_2, \dots, v_k \in V \setminus \{0\}$ be mutually orthogonal, i.e, $\langle v_i, v_j \rangle = 0$ if $i \neq j$. Show $v_1, v_2, \dots, v_k$ are linearly independent.

   Solution:

   Let $\sum_{j=1}^{k} \lambda_j v_j = 0$, for some real or complex λ_i (depending on whether V is over $\mathbb{R}$ or $\mathbb{C}$). Fix an $i \in \{1, \dots, k\}$ and take the inner product of both sides of $\sum_{j=1}^{k} \lambda_j v_j = 0$ with v_i. If $i \neq j$, then $\langle v_i, v_j \rangle = 0$, and so what remains is $\lambda_i \langle v_i, v_i \rangle = 0$. Since $\langle v_i, v_i \rangle > 0$, this means $\lambda_i = 0$. Our choice of i was arbitrary, and so the vectors $\{v_1, \dots, v_k\}$ are linearly independent.

10. Let $A \in M_{n\times n}(\mathbb{C})$ satisfy $A^* = -A$ and let λ be an eigenvalue of A. Show that $\bar{\lambda} = -\lambda$. Deduce that all eigenvalues of any such A are imaginary numbers.[4]

    Solution:

    By assumption, $T^* = -T$. We let $Tv = \lambda v$ for a nonzero vector v. Thus

    $$\langle v, Tv \rangle = \lambda \langle v, v \rangle = \lambda \|v\|^2.$$

    For every $v \neq 0$, $\langle v, v \rangle$ is a positive real number. Our assumption on T now states that

    $$\lambda \|v\|^2 = \langle v, Tv \rangle = \langle T^* v, v \rangle = -\langle Tv, v \rangle = -\overline{\langle v, Tv \rangle}.$$

    Thus $\mu = \langle v, Tv \rangle$ is a complex number equal to $-\overline{\langle v, Tv \rangle} = -\overline{\mu}$. More precisely, if $\mu = a + bi$, then $-\overline{\mu} = -a + bi$ implying that $a = 0$. Hence μ (and so also λ) are pure imaginary.

[4] A good starting point is to take $v \in E(\lambda)$ and try to compute $\langle v, Av \rangle$ in two different ways.

11. Let $T_A : \mathbb{C}^3 \to \mathbb{C}^3$, $x \to Ax$ where

$$A := \begin{pmatrix} 1 & 0 & \mathrm{i} \\ 0 & 2 & 0 \\ -\mathrm{i} & 0 & 1 \end{pmatrix}.$$

a) Show that A is hermitian.

b) Compute the eigenvalues and a set of orthonormal eigenvectors of A.

c) Find a unitary matrix U such that U^*AU is diagonal.

Solution:

Recall that we are working with the function $T_A : \mathbb{C}^3 \to \mathbb{C}^3$, $x \to Ax$ where

$$A := \begin{pmatrix} 1 & 0 & \mathrm{i} \\ 0 & 2 & 0 \\ -\mathrm{i} & 0 & 1 \end{pmatrix}.$$

a) Direct computation shows that $A^* = A$.

b) The characteristic polynomial of A is

$$p_A(\lambda) = (2-\lambda)((1-\lambda)^2 - 1) = -\lambda(\lambda-2)^2,$$

and so we have two eigenvalues, $\lambda_1 = 0$ and $\lambda_2 = 2$. Let us next find the eigenvectors.

$\lambda_1 = 0$. We have to solve $A\mathbf{x} = 0$, which gives

$$x + \mathrm{i}z = 0, \quad 2y = 0, \quad -\mathrm{i}x + z = 0,$$

and so any vector of the form $\mathbf{x} = (x, 0, \mathrm{i}x)$ with $x \neq 0$ is an eigenvector. In order to be normalised, we choose $x = \frac{1}{\sqrt{2}}$ so that $\mathbf{x}_1 = \frac{1}{\sqrt{2}}(1, 0, \mathrm{i})$.

$\lambda_2 = 2$. We have to solve $(A - 2I)\mathbf{x} = 0$, which gives

$$-x + \mathrm{i}z = 0 \text{ and } -\mathrm{i}x - z = 0.$$

Hence any vector of the form $\mathbf{x} = (\mathrm{i}z, y, z)$ with $y, z \neq 0$ is an eigenvector. Since we have two free parameters, the eigenspace is actually two-dimensional and we have to choose two orthogonal and normalised eigenvectors. One possible choice is $\mathbf{x}_2 = (0, 1, 0)$ and $\mathbf{x}_3 = \frac{1}{\sqrt{2}}(\mathrm{i}, 0, 1)$.

c) The columns of U are given by any orthonormal basis of eigenvectors of A, and in our case we find that

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & \mathrm{i} \\ 0 & \sqrt{2} & 0 \\ \mathrm{i} & 0 & 1 \end{pmatrix}.$$

12. Let $T_B : \mathbb{R}^3 \to \mathbb{R}^3$, $x \to Bx$ where

$$B = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}.$$

a) Is B a special kind of matrix? Compute the eigenvalues of B.

b) Find an orthonormal set of eigenvectors of B.

c) Hence find an orthogonal matrix O such that $O^t BO$ is diagonal.

Solution:

We have $T_B : \mathbb{R}^3 \to \mathbb{R}^3$, $x \to Bx$ where

$$B = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}.$$

a) Clearly B is symmetric. The characteristic polynomial is

$$p_B(\lambda) = -(\lambda + 5)(\lambda^2 - \lambda - 2)$$

which has roots $\lambda_1 = -5$, $\lambda_2 = 2$ and $\lambda_3 = -1$.

b) There are three distinct roots, and so the eigenvectors will be orthogonal. We now find the eigenvectors.

$\lambda_1 = -5$. We have $(A + 5I)\mathbf{x} = 0$, which gives us

$$6x + \sqrt{2}\,y = 0 \qquad \sqrt{2}\,x + 5y = 0$$

implying that $x = y = 0$. Hence $\mathbf{x}_1 = (0, 0, 1)$ is a normalized eigenvector.

$\lambda_2 = 2$. Here $(A - 2I)\mathbf{x} = 0$ gives

$$-x + \sqrt{2}\,y = 0\ , \quad \sqrt{2}\,x - 2y = 0\ , \quad -7z = 0$$

and so a normalised solution is $\mathbf{x}_2 = \frac{1}{\sqrt{3}}(\sqrt{2}, 1, 0)$.

$\lambda_3 = -1$ Finally, $(A + I)\mathbf{x} = 0$ gives

$$2x + \sqrt{2}\,y = 0\ , \quad \sqrt{2}\,x + y = 0\ , \quad -4z = 0$$

and so a normalised solution is $\mathbf{x}_3 = \frac{1}{\sqrt{3}}(1, -\sqrt{2}, 0)$.

c) Putting the normalised eigenvectors as the columns of the transition matrix we find

$$O = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & \sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \\ \sqrt{3} & 0 & 0 \end{pmatrix}.$$