UNIVERSITY OF BRISTOL

School of Mathematics

Algebraic Geometry Resit SOLUTIONS
MATHM0036
(Paper code MATHM0036R)

AUGUST 2025 2 hours 30 minutes

The exam contains FOUR questions
All Four answers will be used for assessment.

Calculators of an approved type (permissible for A-Level examinations) are permitted.

Candidates may bring four sheets of A4 notes written double-sided into the examination.
Candidates must insert these into their answer booklet(s) for collection at the end of the examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

Q1. (**25 marks**) Show that $\mathrm{GL}_n(\mathbb{C})$, the set of invertible $n \times n$ matrices with entries in $\mathbb{C}$ is isomorphic to an affine algebraic variety.

Solution. (a) The set of $n \times n$ matrices $M_{n\times n}(\mathbb{C})$ can be identified with $\mathbb{C}^{n\times n}$. In addition, equipped with the Zariski topology, we can regard $M_{n\times n}(\mathbb{C})$ as $\mathbb{A}^{n^2}$. Now, consider $\det : M_{n\times n}(\mathbb{C}) \longrightarrow \mathbb{C}$. $\mathrm{GL}_n(\mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) : \det(A) \neq 0\}$. Now, we can appeal the observations on Pages 41 and 42 in the notes that for any polynomial function f, the sets $D(f)$ are isomorphic to affine algebraic varieties or prove directly that

$$\mathrm{GL}_n(\mathbb{C}) = \mathbb{A}^{n^2} \setminus \{\det = 0\} \simeq \mathbb{V}(y \det -1).$$

Solution (b). In addition to what is asked in the question, note that the coordinate ring of $\mathrm{GL}_n(\mathbb{C}) \subseteq \mathbb{A}^{n^2}$ is

$$\mathbb{C}[\mathrm{GL}_n(\mathbb{C})] = \mathbb{C}[z_1, \dots, z_{n^2}, det^{-1}] \simeq \frac{\mathbb{C}[z_1, \dots, z_{n^2}]}{(y \det -1)}.$$

Q2. Consider the *Veronese* map

$$\begin{aligned} \varphi : \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ [s:t] &\longmapsto [s^3 : s^2t : st^2 : t^3] \end{aligned}$$

(a) (**15 marks**) Prove that φ is a morphism. (Hint. Describe the map φ in some affine charts.)

(b) (**10 marks**) Find the homogeneous ideal $\mathbb{I}\big(\varphi(\mathbb{P}^1)\big)$.

Solution. (a) Solution. With respect to Definition 3.24, this question is obvious, since if a map is given as a polynomial morphism globally it is also a polynomial morphism locally. For instance, for $U_0 = \{[s : t] : s \neq 0\} \subseteq \mathbb{P}^1$, $U_1 = \{[s : t] : t \neq 0\} \subseteq \mathbb{P}^1$. $\varphi_{|_{U_i}}[s : t] = [s^3 : s^2t : st^2 : t^3]$. Solution 2. We can use the more general Definition 4.12 and use Theorem 4.14. In U_0, we have that $\mathrm{image}(\varphi)(U_0) \subseteq \{[1 : t : t^2 : t^3]\} \subseteq \{[A : B : C : D] : A \neq 0\} \subseteq \mathbb{P}^3$. The coordinates of this map are obviously regular functions, so the map is a morphism. Similarly, for the other chart.

(b) Note that $\varphi(\mathbb{P}^1) \subseteq \mathbb{V}(AD - BC)$, since $(s^3)(t^3) - (s^2t)(st^2) = 0$. We expect to have the dimension of $\varphi(\mathbb{P}^1)$ to be one, so there must be more equations. With a little effort, we see that $B^2 - AC$ and $DB - C^2$ are also satisfied, since $(s^2t)^2 - (s^3)(st^2) = 0$, $(s^2t)(t^3) - (st^2)^2 = 0$. Let's write $f(A, B, C, D) = AD - BC$, $g(A, B, C, D) = B^2 - AC$ and $h(A, B, C, D) = DB - C^2$. We have shown that $\varphi(\mathbb{P}^1) \subseteq \mathbb{V}(f, g, h)$. To show the converse inclusion, assume that $p = [A : B : C : D] \in \mathbb{V}(f, g, h)$, and that p is in the chart $A \neq 0$. Then we can take $A = 1$. Then $D = BC$, $B^2 = C, DB = C^2 = B^4 \implies D = B^3$. I.e. $(A = 1, B, C, D) = (1, B, B^2, B^3) = \varphi[1 : B]$. Similarly, other points in $\mathbb{V}(f, g, h)$ are in $\varphi(\mathbb{P}^1)$.

To complete the solution we consider $\mathbb{I}(V(f, g, h)) = \sqrt{(f, g, h)}$, by Nullstellensatz.

Q3. (a) (**10 marks**) Consider the family of algebraic varieties, with parameter $t \in \mathbb{C}$, given by

$$V_t := \mathbb{V}(xy - t) \subseteq \mathbb{A}^2.$$

Sketch the variety of V_0, V_1, and V_2 in $\mathbb{R}^2$. Determine whether or not these varieties are smooth. Briefly justify your answers.

(b) (**15 marks**) Prove that the locus of singular points of a quasi-projective **hypersurface** V forms proper closed subset of V. Recall that a variety is called a hypersurface if it can be given with only one equation.

Solution. (a) For $t = 0$, we have the union of both axis $x = 0$ and $y = 0$, for values of $t = 1, t = 2$ we have two parabola. For $t = 0$, we have $f(x, y) = xy$. $\nabla f(x, y)|_{(a,b)} = (b, a)$. Clearly, when $(a, b) \neq (0, 0)$ this matrix is non-zero and it's kernel has rank $1 = 2 - 1$ which is the dimension of the curve. Therefore, for $(a, b) \neq (0, 0)$, $xy = 0$ is smooth. For $g(x, y) = xy - 1$, we have $\nabla g(x, y)_{|(a,b)} = (b, a)$. This matrix has rank zero if $(a, b) = (0, 0)$ but $(0, 0) \notin \mathbb{V}(xy - 1)$, therefore this curve is smooth everywhere. Similarly, $xy - 2$ is a smooth curve.

(b) I copy the solution from Page 93 of Karen Smith et al. book. Closeness is easy to see in any dimension: since a quasi-projective variety has a basis of affine open subvarieties, it suffices to prove that the singular locus of an affine variety is a proper closed subset. We have that As in the lectures, we can think of the tangent space as the kernel of linear map given by the Jacobian matrix

$$T_a V = \ker \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1}(a) & \cdots & \frac{\partial f_k}{\partial x_n}(a) \end{pmatrix}.$$

The rank of this matrix is less than $n - d$ if and only if all the $(n - d) \times (n - d)$ minors vanish. Therefore, the singular locus is defined by the zero loci of all these minors which give polynomial equations. This shows that the singular locus is a *closed* subvariety of V.

For properness: we now have to show that the singular locus is a proper subset of V, *i.e.*, there exist some points $p \in V$ such that the variety is non-singular. Note that the singular locus of $V = \mathbb{V}(F)$ is given by $\mathbb{V}(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}) \cap V$. If V is everywhere singular, then each $\frac{\partial F}{\partial x_i}$ must vanish everywhere on V. This means that $\frac{\partial F}{\partial x_i}$ is in the ideal $\mathbb{I}(V) = (F)$ defining V. But since the degree of $\frac{\partial F}{\partial x_i}$ is strictly less than the degree of F, this is impossible for all x_i.

Q4. Let Σ be the fan consisting of

- σ_1 cone spanned by $\{(-1, -1), (0, 1)\}$;
- σ_2 cone spanned by $\{(0, 1), (1, 0)\}$;
- τ cone spanned by $\{(0, 1)\}$.

(a) (**6 marks**) Determine whether or not the toric variety X_Σ has the following properties. Briefly justify your answer.

(i) smooth;

(ii) complete.

(b) (**9 marks**) Describe the coordinate rings of X_{σ_1}, X_{σ_2}, and X_τ.

(c) (i) (**5 marks**) Explain why we have the inclusions $\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau]$, $\mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau]$;

(ii) (**5 marks**) Describe the gluing of X_{σ_1} and X_{σ_2} along X_τ.

Solution. (a) Since the determinant of the generators of both cones is ± 1 the variety is smooth. Since σ_1 and σ_2 don't cover whole $\mathbb{R}^2$ the variety is not complete.

(b) We have that $\sigma_1^\vee = \text{cone}\{(-1,0),(-1,1)\}$, $\sigma_2^\vee = \text{cone}\{(1,0),(0,1)\}$, $\tau^\vee = \text{cone}\{(0,1),(-1,0),(1,0)\}$. By definition

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[S_{\sigma_1}] = \mathbb{C}[z_1^{-1}, z_1^{-1}z_2]$$

$$\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[S_{\sigma_2}] = \mathbb{C}[z_1, z_2]$$

$$\mathbb{C}[S_\tau] = \mathbb{C}[S_\tau] = \mathbb{C}[z_2, z_1^{-1}, z_1].$$

(c) (i) We have obviously the inclusions of $\mathbb{C}$-algebras $\mathbb{C}[S_{\sigma_1}] \subseteq \mathbb{C}[S_\tau]$, since z_1, z_2 are the generators of $\mathbb{C}[S_{\sigma_1}]$ and appear in the given representation of $\mathbb{C}[S_\tau]$. For the inclusion $\mathbb{C}[S_{\sigma_2}] \subseteq \mathbb{C}[S_\tau]$, we have that $z_1^{-1} \in \mathbb{C}[S_\tau]$ and $z_1^{-1}, z_2 \in \mathbb{C}[S_\tau]$ so $z_1^{-1}z_2 \in \mathbb{C}[S_\tau]$. So the we have the inclusion of $\mathbb{C}$-algebras generated by these generators.

(c) (ii)
Since maxSpec is a contravariant functor we obtain

$$X_\tau \subseteq X_{\sigma_1}$$

and

$$X_\tau \subseteq X_{\sigma_2}.$$

Moreover, from the given coordinate rings we see that we have

$$X_\tau = X_{\sigma_1} \setminus \{(z_1)^{-1} = 0\}$$

$$X_\tau = X_{\sigma_2} \setminus \{(z_1) = 0\}$$

Therefore we have the inclusion of the open subsets $X_\tau \subseteq X_{\sigma_1}$ and $X_\tau \subseteq X_{\sigma_2}$. The correct gluing to make X_Σ separated is the map induced by $\mathbb{C}$-algebra isomorphism which assigns

$$\begin{aligned} g*_{\sigma_2\sigma_1} : \mathbb{C}[S_\tau] &\longrightarrow \mathbb{C}[S_\tau] \\ z_1 &\longmapsto z_1^{-1} \\ z_2 &\longmapsto z_1^{-1}z_2. \end{aligned}$$

End of examination.