Complex Tropical Currents

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Courants tropicaux complexes
(Complex Tropical currents)

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et de Andrea D'AGNOLO

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Complex Tropical Currents

Abstract

To a tropical p-cycle $V_{\mathbb{T}}$ in $\mathbb{R}^n$, we naturally associate a closed (p,p)-dimensional current of order zero on $(\mathbb{C}^*)^n$ denoted by $\mathscr{T}_n^p(V_{\mathbb{T}})$. Such a "tropical current" $\mathscr{T}_n^p(V_{\mathbb{T}})$ cannot be an integration current along any analytic set since its support has the form $\mathrm{Log}^{-1}(V_{\mathbb{T}}) \subset (\mathbb{C}^*)^n$, where Log is the coordinate-wise valuation with $\log(|.|)$. We provide sufficient (local) conditions on a tropical p-cycle such that its associated tropical current is "strongly extremal" in $\mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$. In particular, if these conditions hold for the effective cycles, then the associated currents are extremal in the cone of strongly positive closed currents of bidimension (p,p) on $(\mathbb{C}^*)^n$. Next we explain how to extend the currents and the extremality results to $\mathbb{CP}^n$. Further, we demonstrate how to use the intersection theory of currents to derive an intersection theory for the underlying tropical cycles. The explicit calculations will be established by using a formula for the real Monge-Ampère measure of a tropical polynomial. Finally, we explain certain relations between approximation problems of tropical cycles by amoebas of algebraic cycles and approximations of the associated currents by positive multiples of integration currents along analytic cycles. It will be discussed how these approximation problems are related to a stronger formulation of the Hodge conjecture.

Keywords: Theory of currents, tropical geometry, Monge-Ampére measures, approximation of currents.

Resumé

À tout p-cycle tropical $V_{\mathbb{T}}$ de $\mathbb{R}^n$, on attache naturellement un courant fermé (p,p) dimensionnel d'ordre 0 sur $(\mathbb{C}^*)^n$, noté $\mathscr{T}_n^p(V_{\mathbb{T}})$. Un tel "courant tropical" $\mathscr{T}_n^p(V_{\mathbb{T}})$ ne saurait être le courant d'intégration sur un quelconque sous-ensemble analytique de $(\mathbb{C}^*)^n$ du fait qu'il a pour support l'ensemble $\mathrm{Log}^{-1}(V_{\mathbb{T}}) \subset (\mathbb{C}^*)^n$, où l'application Log désigne la multivaluation $(z_1, ..., z_n) \mapsto (\log|z_1|, ..., \log|z_n|)$. On donne des conditions suffisantes (de nature locale) sur un p-cycle tropical $V_{\mathbb{T}}$ pour que le courant tropical $\mathscr{T}_n^p(V_{\mathbb{T}})$ qui lui est associé soit "fortement extrémal" dans $\mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$. En particulier, si une telle condition s'avère remplie pour un p-cycle tropical effectif, alors le courant tropical qui lui est attaché est extrémal dans le cône des courants fermés de bidimension (p,p) sur $(\mathbb{C}^*)^n$. On explique ensuite comment prolonger ces courants tropicaux et les propriétés d'extrémalité dont ils héritent à l'espace projectif $\mathbb{CP}^n$. On montre également comment définir le produit de tels courants tropicaux pour en déduire une théorie de l'intersection entre cycles tropicaux. Pour opérer ces calculs, on établit une formule pour la mesure de Monge Ampère réelle associée à un polynôme tropical. On explicite enfin certains liens entre les problèmes relevant de l'approximation (au sens ensembliste, pour la métrique de Hausdorff) des cycles tropicaux de $\mathbb{R}^n$ par les amibes de cycles algébriques de $(\mathbb{C}^*)^n$ et l'approximation (au sens faible) des courants tropicaux associés par des multiples positifs de courants d'intégration sur de tels cycles algébriques. On explique en quoi ces questions d'approximation se trouvent reliées à une formulation forte de la célèbre conjecture de Hodge.

Mots Clès: Théorie de courants, géometrie tropicale, mesure de Monge-Ampére, approximation de courants.

Compendio

Ad ogni p-ciclo tropicale $V_{\mathbb{T}}$ di $\mathbb{R}^n$ si associa in maniera naturale una corrente chiusa (p,p)-dimensionale di ordine 0 su $(\mathbb{C}^*)^n$, indicata con $\mathscr{T}_n^p(V_{\mathbb{T}})$. Una tale "corrente tropicale" $\mathscr{T}_n^p(V_{\mathbb{T}})$ non puo' essere la corrente d'integrazione associata ad un qualche sottoinsieme analitico di $(\mathbb{C}^*)^n$, avendo essa per supporto l'insieme $\mathrm{Log}^{-1}(V_{\mathbb{T}}) \subset (\mathbb{C}^*)^n$, dove Log denota la multi-valutazione $(z_1, ..., z_n) \mapsto (\log|z_1|, ..., \log|z_n|)$. Si danno delle condizioni sufficienti (di natura locale) su un p-ciclo tropicale $V_{\mathbb{T}}$ affinchè la sua corrente tropicale associata $\mathscr{T}_n^p(V_{\mathbb{T}})$ sia "fortemente estremale" in $\mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$. In particolare, se tale condizione è soddisfatta per un p-ciclo tropicale effettivo, allora la sua corrente tropicale associata è estremale nel cono delle correnti chiuse di dimensione (p,p) su $(\mathbb{C}^*)^n$. Si mostra in seguito come estendere la nozione di corrente tropicale e i risultati di effettività allo spazio proiettivo $\mathbb{CP}^n$. Si mostra inoltre come definire il prodotto di tali correnti tropicali e come dedurne una teoria dell'intersezione tra cicli tropicali. Per i calcoli espliciti, si fa uso di una formula per la misura di Monge Ampère reale associata ad un polinomio tropicale. Infine, si esplicitano alcuni legami tra problemi d'approssimazione (in senso insiemistico, per la metrica di Hausdorff) di cicli tropicali di $\mathbb{R}^n$ tramite amebe associate a cicli algebrici di $(\mathbb{C}^*)^n$ e l'approssimazione (nel senso debole) delle associate correnti tropicali tramite multipli positivi di correnti di integrazione associate a cicli algebrici. Si conclude con una discussione su come queste questioni di approssimazione siano legate ad una formulazione forte della celebre congettura di Hodge.

Parole chiave: teoria delle correnti, geometria tropicale, misure di Monge-Ampére, approssimazioni di correnti.

Contents

Acknowledgments

In my class of 8th grade, one of our teachers used to write a poem in beautiful Persian calligraphy for students who got good grades. The poem he wrote for me meant "life is the making of pearls in one's own shell," I believe that the same is true about mathematics, however writing this simple little text in my own "shell" would have been impossible if it was not for the support and love of my family, and the mathematicians I met along the way.

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Farhad Babaee

To my family

Introduction

In this thesis we introduce the notion of **complex tropical currents**, which are certain currents associated to tropical cycles in $\mathbb{R}^n$. Recall that in theory of partial differential equations, solutions of the equations are often distributions which are "generalized functions" and they are understood by their actions on smooth functions with compact supports (test functions). In the same manner, on a real or a complex smooth manifold, one defines the currents which are "generalized forms", that is forms with distribution coefficients. The currents act on smooth forms with compact supports (test forms). For a real smooth manifold X of real dimension m (resp. complex smooth manifold of complex dimension n) let us denote by $\mathcal{D}_k(X)$, $0 \leq k \leq m$, the space of k-test forms (resp. $\mathcal{D}_{p,q}(X)$, $0 \leq p, q \leq n$, the space (p,q)-test forms on X). Accordingly, denote by $\mathcal{D}'_k(X)$ the space of k-currents (resp. $\mathcal{D}'_{p,q}(X)$ the space of (p,q) currents) on X.

Tropical cycles are special polyhedral complexes which are weighted with non-zero integers. In this thesis we consider tropical cycles in $\mathbb{R}^n$, and will associate to each one a current on $(\mathbb{C}^*)^n$. We will also consider their extension by zero to $\mathbb{CP}^n$.

Some (positively weighted) tropical cycles can be obtained as logarithmic limit sets of algebraic subvarieties $\{Z_t\}_{t\in\mathbb{R}^+} \subset (\mathbb{C}^*)^n$, ([Spe02]), *i.e.* as the limit (with respect to the Hausdorff metric on compact sets of $\mathbb{R}^n$) of $\operatorname{Log}_t(Z_t)$, where

$$\operatorname{Log}_t : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (z_1, \dots, z_n) \mapsto (\frac{\log|z_1|}{\log t}, \dots, \frac{\log|z_n|}{\log t})$$

When $t = \exp(1)$, Gelfand, Kapranov and Zelevinski, [GKZ08], called the set $\operatorname{Log}(Z) \subset \mathbb{R}^n$ the **amoeba** of algebraic subvariety $Z \subset (\mathbb{C}^*)^n$. See Figure 0.1, where Log_t of algebraic subvariety associated to $z + w + 1$ in $(\mathbb{C}^*)^2$ is approximating a tropical line on the right hand side.

The usage of current theoretic ideas to analyze amoebas can be found for instance

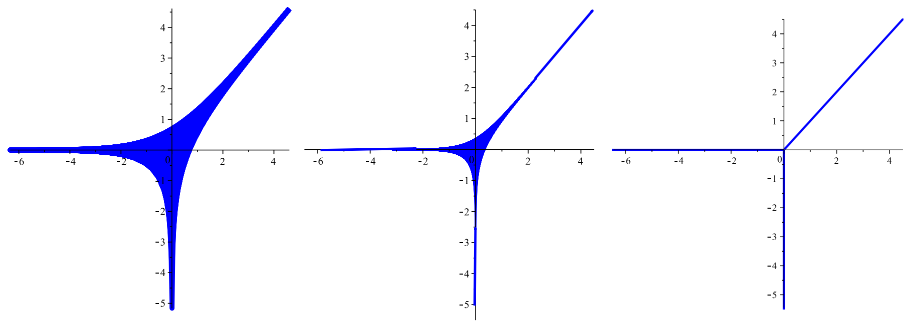

Figure 0.1: As t increases, $\operatorname{Log}_t(Z(z+w+1))$ converges to a "tropical line."

in [PR04], [Ras09a], [FGS07]. Later on, Lagerberg in [Lag12] introduced the notion of **super-currents** and as an application he analyzed the tropical hypersurfaces, and re-proved a few facts about intersections of tropical hypersurfaces. Afterwards, super-currents were generalized in the framework of Berkovich spaces by Chambert-Loir and Ducros ([CLD12]), in order to define Arakelov theory in the non-Archemidean setting. However, a notion of complex current associated to a tropical cycle was missing. In this thesis we suggest such a notion. In this way, we can benefit from strong tools in complex analysis and moreover ask whether these currents satisfy important properties such as extremality or approximability.

Let $p : \mathbb{R}^n \to \mathbb{R}$ be a tropical Laurent polynomial given by $x \mapsto \max_{\alpha\in A}\{c_\alpha + \langle \alpha, x\rangle\}$ where $A \subset \mathbb{Z}^n$ is a finite set, $\langle\, .\, ,\, .\,\rangle$ the usual inner product in $\mathbb{R}^n$ and $c_\alpha \in \mathbb{R}$. The tropical hypersurface associated to p, as a set, is the set of $x \in \mathbb{R}^n$ where $p(x)$ is not smooth. Let this set be denoted by $|V_{\mathbb{T}}(p)|$, and $\operatorname{Log} := \operatorname{Log}_{\exp(1)}$. Also assume that $d = \partial + \bar{\partial}$ is the decomposition of the usual de Rham differential operator and let $d^c = (\partial - \bar{\partial})/(2i\pi)$. The tropical currents associated to tropical hypersurface associated to p will have a simple representation

$$dd^c\ [p \circ \operatorname{Log}].$$

This current ignores the phases of the variables $(z_1, \ldots, z_n)$. The support of this current is given by $\operatorname{Log}^{-1}(|V_{\mathbb{T}}(p)|) \subset (\mathbb{C}^*)^n$ and thus, this current is not an integration current along any analytic set. Since p is a convex function, $p \circ \operatorname{Log}$ is a plurisubharmonic function (Definition 1.2.2), and therefore $dd^c\ [p \circ \operatorname{Log}]$ is a positive current (Definition 1.2.1). The analogue of this current in the setting of super-currents of Lagerberg is simply given by $dd^\sharp\ p$.

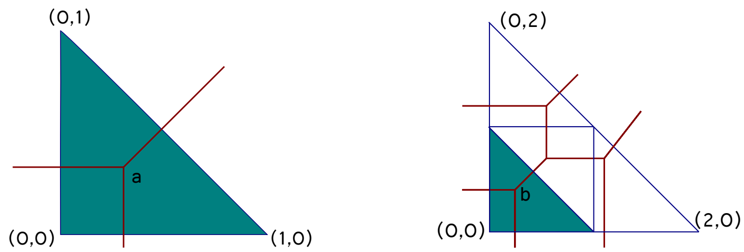

Figure 0.2: A tropical line and conic (red) with the dual subdivisions of their Newton polytopes.

The n-th power $(dd^c\ [p\circ \mathrm{Log}\,])^{\wedge n} = dd^c\ [p\circ \mathrm{Log}\,]\wedge\cdots\wedge dd^c\ [p\circ \mathrm{Log}\,]$ is a positive measure times the volume form in $(\mathbb{C}^*)^n$, and since the phases of the coordinates are ignored by the map Log , it is reasonable to have a real measure which describes $(dd^c\ [p\circ \mathrm{Log}\,])^{\wedge n}$. The corresponding real measure is in fact $n!$ times the real Monge-Ampère measure associated to p and denoted by $\mu[p]$. This is a consequence of Rashkovskii's formula ([Ras01]), which says that for any Borel subset $E\subset\mathbb{R}^n$,

$$\int_{\mathrm{Log}^{-1}(E)}(dd^c\,[p\circ \mathrm{Log}\,])^n = n!\int_E \mu[p]\ , \tag{0.0.1}$$

in fact this holds for any convex function u instead of p . The above measures on E with Lagerberg's notation is obtained by $\int_{E\times\mathbb{R}^n}(dd^{\sharp}\,p)^{\wedge n}$. The real Monge-Ampère measure associated to a convex function u can be defined independently as the Lebesgue measure of the generalized gradient set of u; more precisely

$$\int_E \mu[u] = \lambda(\{\xi\in\mathbb{R}^n :\ \exists\, x_0\in\ E,\ \text{with}\ \langle\xi,x_0\rangle - u(x_0)\geq\langle\xi,x\rangle - u(x)\ ,\forall x\in\mathbb{R}^n\}),$$

where λ is the standard Lebesgue measure on $\mathbb{R}^n$ (that is the measure of the set $\xi\in\mathbb{R}^n$ such that the maximum of $\langle\xi,x\rangle - u(x)$ is obtained at some $x_0\in E$).

There is a duality ([HS95]) between the tropical hypersurfaces associated to a tropical Laurent polynomial p and a triangulation of the Newton polytope of p, denoted by Δ_p. See Figure 0.2. On the left, a is the only vertex (or a zero cell) of the tropical curve, and the dual to this vertex is the whole triangle. On the right b is a vertex of the tropical conic (it is called a conic since the sides of the whole triangle have length 2), and the dual cell to b is given by the green area. Interestingly, the real Monge-Ampère measure associated to a tropical Laurent polynomial p is supported only in the vertices of $V_{\mathbb{T}}(p)$.

One finds the following formula :

$$\mu[p] = \sum_{a \in \mathcal{C}_0(V_T(p))} \mathrm{Vol}_n(\{a\}^*)\, \delta_a\,, \tag{0.0.2}$$

where $\mathcal{C}_0(V_T(p))$ is the set of zero-cells (or vertices) of $V_T(p)$ and $\{a\}^*$ denotes the dual to the vertex a . In Figure 0.2, $\mathrm{Vol}_n(\{a\}^*)$ and $\mathrm{Vol}_n(\{b\}^*)$ will be the volume of each green area.

For tropical polynomials $p_1, \dots, p_n$ the formula 0.0.1 can be polarized to obtain

$$\int_{\mathrm{Log}^{-1}(E)} dd^c\,[p_1 \circ \mathrm{Log}\,] \wedge \cdots \wedge dd^c\,[p_n \circ \mathrm{Log}\,] = n! \int_E \tilde{\mu}[p_1, \dots, p_n]\ . \tag{0.0.3}$$

The measure $\tilde{\mu}[p_1, \dots, p_n]$ which fits in the equation is called the mixed Monge-Ampére measure. Combining this with (0.0.2) gives (see section 4.2):

$$\int_{\mathrm{Log}^{-1}(E)} dd^c\,[p_1 \circ \mathrm{Log}\,] \wedge \cdots \wedge dd^c\,[p_n \circ \mathrm{Log}\,] = \sum_{\{a\} \in \mathcal{C}_0(V_{\mathbb{T}}(p_1) \cap \cdots \cap V_{\mathbb{T}}(p_n)) \cap E} \mathrm{Vol}_n(\{a\}^*) \delta_a\,, \tag{0.0.4}$$

which suggests considering $\mathrm{Vol}_n(\{a\}^*)$ as an intersection multiplicity at an intersection point $\{a\}$. These intersection multiplicities are compatible with [BB07], and [Mik06].

To understand the current $dd^c\,[p \circ \mathrm{Log}\,]$, we use a theorem of supports due to Demailly [Dem82]. This support theorem was used in [Dem82] to prove that the current $T_D = dd^c\,[\log\ \max\{1, |z_1|, |z_2|\}]$ is extremal in the cone of positive closed currents on $\mathbb{C}^2$. In any convex cone $\mathscr{C}$, an element $v \in \mathscr{C}$ is called extremal if any decomposition $v = v_1 + v_2$, for $v_1, v_2 \in \mathscr{C}$ implies that there are $\lambda_1, \lambda_2 > 0$ such that $v = \lambda_1 v_1 = \lambda_2 v_2$. Roughly said, by Choquet's theorem, every element of a closed convex cone can be approximated by positive linear combinations of the extremal elements of the cone, which reveals the importance of the set of extremal elements in any given cone.

For an analytic subset Z of a complex manifold X, which is of pure dimension p, the integration current along Z, denoted by $[Z]$, is defined by following action on (p,p)-forms (with compact support)

$$\langle [Z], \alpha \rangle = \int_Z \alpha\ .$$

Lelong in [Lel73] showed that the integration currents along (closed) irreducible analytic subsets are extremal in the cone of positive closed currents. He also asked whether positive multiples of integration currents along irreducible analytic cycles are the only extremal currents. Demailly's example was the first which is not an integration current along any analytic set, since its support has real dimension 3. Later on, Bedford noticed that many extremal currents naturally occur in dynamical systems on several complex variables whose supports are in general fractal sets, and therefore not analytic (see [Sib99], [DS05], [Gue05], [DS13] and references therein).

In this thesis, we extend the definition of extremality of positive currents to non-positive ones, which we call strong extremality. That is, a normal closed current T of bidimension (p,p) is called strongly extremal if, for any other normal closed current $\tilde{T}$ of bidimension (p,p) which has the same support as T, there exists a complex number ρ such that $\tilde{T} = \rho\, T$. The strong extremality is thus rather a property of supports.

As we mentioned before, to any tropical p-cycle $V_{\mathbb{T}} \subset \mathbb{R}^n$ (Definition 2.1.3) we will attach a normal closed current of bidimension (p,p) with support equal to $\mathrm{Log}^{-1}(V_{\mathbb{T}}) \subset (\mathbb{C}^*)^n$. We refer to such a current as the **tropical current** attached to the tropical p-cycle $V_{\mathbb{T}} \subset \mathbb{R}^n$, and we denote it by $\mathscr{T}_n^p(V_{\mathbb{T}})$. Such a construction could actually be carried out for any weighted rational polyhedral p-dimensional complex $\mathcal{P}$ and one will prove that if $\mathcal{P}$ is a tropical cycle, then the associated current is closed (Theorem 3.1.8). Moreover, we show that extremality of the tropical currents in the stronger sense is detectable from the structure of the corresponding tropical cycles ; suppose that a tropical p-cycle $V_{\mathbb{T}} \subset \mathbb{R}^n$ is connected in codimension 1, and that at each facet W, a set of primitive vectors $\{v_1, \dots, v_s\}$ which makes the balancing condition hold at W satisfies the following two conditions :

1. $\{h_W(v_1), \dots, h_W(v_s)\}$ spans the dual space $W^{\perp}$ as an $\mathbb{R}$-basis, where h_W is the projection along W;

2. every proper subset of $\{h_W(v_1), \dots, h_W(v_s)\}$ is a set of independent vectors ;

then the current $\mathscr{T}_n^p(V_{\mathbb{T}})$ stands as a strongly extremal element of (p,p)-dimensional normal closed currents on $(\mathbb{C}^*)^n$ (Theorem 3.1.8).

The strongly extremal currents on $(\mathbb{C}^*)^n$ can be extended by zero to obtain extremal currents in $\mathbb{CP}^n$. In the last chapter we come to the important problem of approximability of currents. We will ask whether all the (positive) (p,p)-dimensional extremal tropical currents on $X := (\mathbb{C}^*)^n$ are in the closure (in the sense of currents) of

$$\mathcal{I}^p(X) = \big\{\lambda[Z] : \ Z \ \ p\text{-dim. irreducible analytic subset in } X,\, \lambda \geq 0\big\}.$$

We will prove that if for a tropical p-cycle $V_{\mathbb{T}}$, there exists a family of algebraic subvarieties $\{X_t\}_{t\in\mathbb{R}^+} \subset (\mathbb{C}^*)^n$ such that

$$\operatorname{Log}_t(X_t) \to V_{\mathbb{T}} \qquad \text{as } t \to +\infty \ ,$$

in Hausdorff metric on compact sets of $\mathbb{R}^n$, then

$$\mathscr{T}_n^p(V_{\mathbb{T}}) \in \overline{\mathcal{I}^p((\mathbb{C}^*)^n)}.$$

This result easily extends on $\mathbb{CP}^n$. Hence, to find a possible candidate of an extremal current which is not approximable by integration currents along irreducible analytic sets, one can start from a tropical cycle which is not a Hausdorff limit of the amoebas $\operatorname{Log}_t(X_t)$. We do not know whether such a tropical cycle exists. In the theory of currents the approximability problem of a positive extremal current by integration currents along irreducible analytic cycles is related to a stronger formulation of the **Hodge conjecture** which we will explain in Chapter 5.

Most of the results of this thesis have already been announced in [Bab14]. This thesis is structured as follows. In Chapter 1, we state the preliminaries of the theory of currents. Chapter 2 is a brief discussion of tropical geometry, and Monge-Ampère measures. In Chapter 3, we define the tropical current $\mathscr{T}_n^p(\mathcal{P})$ attached to a weighted p-dimensional polyhedral complex $\mathcal{P}$. In this section, we state and prove the main result of this thesis about extremality (mentioned above) ; this is done step by step, first in the case $p = 1$, then in the case $p > 1$ assuming $\mathcal{P}$ has a single facet and then finally in the general case of $p > 1$ tropical cycles. We also illustrate how these constructions may be used in order to produce extremal currents on complex projective planes. Chapter 4 is a brief explanation of intersection of tropical currents. In Chapter 5, we discuss the problems of the approximability of tropical cycles by amoebas and will explain how it could be related to the problem of approximating the tropical currents by analytic cycles. We end this thesis with some open problems.

Chapter 1

Currents

1.1 Basic definitions

Let X be an analytic complex manifold of dimension n. If k, p, q are non-negative integers, possibly $k = \infty$, we denote by $\mathcal{C}^k_{p,q}(X)$ (resp. $\mathcal{D}^k_{p,q}(X)$) the space of differential forms of bidegree (p,q) and of class C^k (resp. with compact support) on X. The elements of $\mathcal{D}^k_{p,q}(X)$ are called test forms.

The space of currents of **order** k and of bidimension (p,q), or equivalently of bidegree $(n-p, n-q)$, is by definition the topological dual space $[\mathcal{D}^k_{p,q}(X)]'$, where $\mathcal{D}^k_{p,q}(X)$ is endowed with the inductive limit topology. $\mathcal{D}^\infty_{p,q}(X)$ (resp. $[\mathcal{D}^\infty_{p,q}(X)]'$) is usually denoted instead by $\mathcal{D}_{p,q}(X)$ (resp. by $\mathcal{D}'_{p,q}(X)$). A current $T \in \mathcal{D}'_{p,q}(X)$ is called **closed** if for every $\alpha \in \mathcal{D}_{p-1,q}(X)$,

$$\langle dT, \alpha \rangle := (-1)^{p+q-1} \langle T, d\alpha \rangle \tag{1.1.1}$$

vanishes.

Therefore the currents are defined by their actions on forms. Given a current $T \in \mathcal{D}'_{p,q}(X)$ and a form $\alpha \in \mathcal{D}_{p,q}(X)$, the action is defined by

$$\langle T, \alpha \rangle := \int_X T \wedge \alpha \quad \in \mathbb{C} \ .$$

It is convenient to understand currents as forms with distribution coefficients. Recall that the distributions are just topological dual to smooth functions with compact supports, the test functions. The currents of lower order are somehow "nicer", since they can act on forms of lower regularity. A current of order 0 can act on forms which coefficients are only continuous test functions, *i.e* the currents of order zero can be understood as forms with measure coefficients, which is a nice property since one can restrict the measures (and thus the order zero currents) to submanifolds of X.

An important family of currents are **currents of integration** associated to any p-dimensional submanifold $[Z]$ of X ; such a current is also denoted by $[Z]$ and acts on any $\alpha \in \mathcal{D}_{p,p}(X)$ by

$$\langle [Z], \alpha \rangle = \int_Z \alpha_{|Z} \qquad \in \mathbb{C} \ .$$

By Stoke's theorem one has $d[Z] = -[\partial Z]$. We will see in the following sections that such a current can be also defined for any analytic subset of X as well.

Let us finish this section by recalling the definition of **push-forward** of a current. Assume that $\tau : X \to X'$ is a holomorphic map between the complex manifolds X and X'. Let T be a current in $\mathcal{D}'_{p,q}(X)$ such that τ is proper on the support of T. Then the push-forward of T by τ, denoted by $\tau_*(T)$, is naturally given by

$$\langle \tau_*(T), \alpha \rangle = \langle T, \tau^*(\alpha) \rangle, \quad \alpha \in \mathcal{D}_{p,q}(X').$$

1.2 Positivity

An important concept in this theory is **positivity** which is due to Lelong and Oka.

Definition 1.2.1. A form $\psi \in \mathcal{C}^0_{p,p}(X)$ is called

- strongly positive, if for all $z \in X$, $\psi(z)$ is in the convex cone generated by (p,p) forms of the type
$$(i\psi_1 \wedge \bar{\psi}_1) \wedge \cdots \wedge (i\psi_p \wedge \bar{\psi}_p),$$
where $\psi_j \in \bigwedge^{1,0} T^*_z X$;
- positive, if at every point $z \in X$ and all p-planes F of the tangent space $T_z X$, the restriction $\psi(z)_{|F}$ is a strongly positive (p,p)-form.

A current $T \in \mathcal{D}'_{p,p}(X)$ is called strongly positive (resp. positive) if

$$\langle T, \psi \rangle \geq 0$$

for every positive (resp. strongly positive) test form $\psi \in \mathcal{D}_{p,p}(X)$. We denote the set of positive (resp. strongly positive) closed currents of bidimension (p,p) by

$$PC^p(X) \quad (\text{resp. } SPC^p(X)).$$

If T is (strongly) positive current then the positivity of its action on positive smooth test forms extends to continuous positive test forms, *i.e.* has order zero.

1.2.1 Examples of positive currents

Let $d = \partial + \bar{\partial}$ the usual decomposition of the de Rham (exterior) derivative and $d^c = (\partial - \bar{\partial})/(2i\pi)$, so that $dd^c = (1/i\pi)\,\partial\bar{\partial}$. An important family of $(n-1, n-1)$-dimensional positive currents are those of the form

$$dd^c\ \varphi\ ,$$

where φ is a **plurisubharmonic** function as defined below.

Definition 1.2.2. A function $u : \Omega \to [-\infty, \infty)$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic if

1. u is upper semicontinuous;

2. for every complex line $L \subset \mathbb{C}^n$, $u_{|\Omega \cap L}$ is subharmonic on $\Omega \cap L$.

The set of plurisubharmonic functions on Ω is denoted by $\mathrm{Psh}(\Omega)$.

Example 1.2.3 (See also [Dem92a])**.**

1. One can see that any combination of plurisubharmonic functions with positive coefficients is indeed plurisubharmonic ; this is also the case for decreasing limits of sequences of plurisubharmonic functions or upper envelopes of families of such functions.

2. Let $F : X \to Y$ be a holomorphic change of variables and $\varphi \in \mathrm{Psh}(Y)$, then one has $\varphi\ \circ\ F \in \mathrm{Psh}(X)$.

3. Let $u : \mathbb{R}^p \to \mathbb{R}$ convex and increasing on each variable then

$$\varphi_1, \ldots, \varphi_p \in \mathrm{Psh}(X) \implies u(\varphi_1, \ldots, \varphi_p) \in \mathrm{Psh}(X).$$

An important example of plurisubharmonic to be often considered in this text is

$$(z_1, \ldots, z_n) \in \mathbb{C}^n \mapsto u(\log|z_1|, \ldots, \log|z_n|) \in [-\infty, \infty)$$

where u is a convex function, and might be abridged to $u \circ \mathrm{Log}\,(z_1, \ldots, z_n)$. When $\varphi \in \mathrm{Psh}(X) \cap L^1_{\mathrm{loc}}(X)$ is a plurisubharmonic function, one has

$$dd^c\ \varphi = \frac{i}{\pi} \sum_{1 \le j,k \le n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k\ ,$$

which is a closed $(n-1, n-1)$-dimensional (or bidegree $(1,1)$) positive current. As a consequence for a convex (hence continuous and locally integrable) function u, the current

$$dd^c\left[u \circ \mathrm{Log}\,(z_1, \ldots, z_n)\right] = dd^c\left[u(\log|z_1|, \ldots, \log|z_n|)\right] \tag{1.2.2}$$

is a positive closed current of bidimension $(n-1, n-1)$. We refer the reader to [Ras12] for a study of plurisubharmonic singularities and the combinatorics of the extremal elements.

It is also easy to see that for any complex submanifold Z of X, the current of integration associated to Z is indeed a positive current, *i.e.* if $\alpha \in \mathcal{D}_{p,p}(X)$ is a positive form

$$\langle [Z], \alpha\rangle = \int_Z \alpha_{|Z} \qquad \in \mathbb{R}^{\geq 0} \ .$$

The **Lelong-Poincaré equation** (see (2.15) in [Dem]) relates some elements of these two families (plurisubharmonic functions on one side and integration currents on the other side).

Lemma 1.2.4 (Lelong-Poincaré equation)**.** *Let f be a non-zero meromorphic (resp. holomorphic) function on X, and let $\sum m_j Z_j$ be the divisor of f . Then the function $\log|f|$ is locally integrable on X and*

$$dd^c \log|f| = \sum m_j [Z_j],$$

in the spaces of currents (resp. positive currents) of bidimension $(n-1, n-1)$.

1.3 Extremality

In this paper we are mainly concerned with **extremal** currents. Recall that the **support** of a current is the smallest closed set in the ambient space X such that on its complement the current vanishes, and a current T is called **normal** if T and $d\,T$ are of order zero. We have mentioned that every positive current has order zero, thus every closed positive current is indeed normal.

Definition 1.3.1. A current $T \in PC^p(X)$ (resp. $\in SPC^p(X)$) is called extremal in $PC^p(X)$ (resp. in $SPC^p(X)$) if for any decomposition $T = T_1 + T_2$ with $T_1, T_2 \in PC^p(X)$ (resp. $\in SPC^p(X)$), there exist $\lambda_1, \lambda_2 \geq 0$ such that $T = \lambda_1 T_1$ and $T = \lambda_2 T_2$.

Let us introduce a stronger notion of extremality which does not assume positivity anymore.

Definition 1.3.2. *A closed current $T \in \mathcal{D}'_{p,p}(X)$ of order zero is called strongly extremal if, for any closed current $\tilde{T} \in \mathcal{D}'_{p,p}(X)$ of order zero which has the same support as T, there exists $\rho \in \mathbb{C}$ such that $T = \rho\tilde{T}$.*

Remark 1.3.3. Note that the extremality properties are invariant under invertible affine linear transformations. Furthermore, strong extremality of a positive (resp. strongly positive) closed current $T \in \mathcal{D}'_{p,p}(X)$ implies its extremality in $PC^p(X)$ (resp. in $SPC^p(X)$) . In addition, strong extremality can be considered as a property of supports (see also [DS13]). Also note that there are positive extremal currents which are not strongly extremal [Slo99].

Let us denote

$$\mathcal{I}^p(X) = \left\{\lambda[Z] : \ \lambda \geq 0,\, Z \subset X \text{ be a } p\text{-dimensional closed irreducible analytic subset}\right\}$$

and by $\mathcal{E}^p(X)$ the set of extremal elements of $SPC^p(X)$. Using the support theorems which are recalled in the next section, it is not hard to see that ([Lel73], [Dem])

$$\mathcal{I}^p(X) \subset \mathcal{E}^p(X).$$

1.4 Support theorems

We need to quote two important structure theorems for supports of currents which are due to Demailly [Dem82]. For a thorough treatment see [Dem].

Let $S \subset X$ be a closed C^1 real submanifold of X. The complex dimension

$$\dim_{\mathbb{C}} \left(T_x S \cap iT_x S\right),$$

is called the **Cauchy-Riemann** dimension of S at x. The maximal dimension

$$\max_{x\in S} \dim_{\mathbb{C}} \left(T_x S \cap iT_x S\right)$$

is called the Cauchy-Riemann dimension of S, denoted by $\mathrm{CRdim}\, S$. If this dimension is constant for all $x \in S$, then S is called a Cauchy-Riemann submanifold of X.

The following theorem implies that a complex structure of dimension at least p is needed on the support of a normal current in order to accommodate (p, p) test forms.

Theorem 1.4.1 ([Dem82])**.** *Suppose that $T \in \mathcal{D}'_{p,p}(X)$ is a normal current. If the support of T is contained in a real submanifold S of Cauchy-Riemann dimension less than p, then $T = 0$.*

The next theorem about supports permits us to streamline a current if its support is a fiber space.

Theorem 1.4.2 ([Dem82]). *Let $S \subset X$ be a Cauchy-Riemann submanifold with Cauchy-Riemann dimension p such that there is a submersion $\sigma : S \to Y$ of class C^1 whose fibers $\sigma^{-1}(y)$ are connected and that for all the points $z \in S$ we have*

$$T_zS \cap i\,T_zS = T_zF_z,$$

where $F_z = \sigma^{-1}(\sigma(z))$ is the fiber of the point z and T_zS, T_zF_z are the tangent spaces at z corresponding to S and F_z. Then, for every closed current T of bidimension (p,p) and of order 0 (resp. positive) with support in S, there exists a unique (resp. positive) Radon measure μ on Y such that

$$T = \int_{y\in Y} [\sigma^{-1}(y)] d\mu(y),$$

i.e.

$$\langle T, \psi\rangle = \int_{y\in Y} \big(\int_{\sigma^{-1}(y)} \psi\big) d\mu(y)\,,$$

for $\psi \in \mathcal{D}_{p,p}(X)$.

1.5 Theorems of Skoda-El Mir and Crofton

Skoda-El Mir theorem ([Dem, Theorem 2.3]) provides a sufficient condition for extension by zero of a current. To state the theorem we need to recall the notion of **mass**.

Definition 1.5.1. Let (X,ω) be a p-dimensional Hermitian complex manifold *i.e.* the Hermitian norm is induced by the form ω . The mass of a positive (p,p)-dimensional current T on a Borel set K is defined by

$$\mathrm{Mass}_{\omega,K}(T) := \int_K T \wedge \omega^p \ .$$

If K is contained in a fixed compact subset of X , then by changing the Hermitian metrics on X an equivalent norm is induced (see [DS13]).

For instance, for $X = \mathbb{C}^n$, the standard Euclidean metric is induced by the following Hermitian form

$$dd^c||z|| = \frac{1}{2\pi}\sum_{i,j} i dz_j \wedge d\bar{z}_j \ .$$

Also, the form $dd^c \log(||z||)$ (on $\mathbb{C}^{n+1}$) induces the **Fubini-Study metric**, denoted by ω_{FS}, on $\mathbb{CP}^n$ given in each chart $z_j \neq 0$, by $\frac{1}{2} dd^c \log(1 + |\zeta_0|^2 + \cdots + |\widehat{\zeta_j}|^2 + \dots |\zeta_n|^2)$, where $\zeta_j = z_0/z_j$ are the induced non-homogeneous coordinates. If T is a (p,p) current in some open subset of $\mathbb{C}^n$ or $\mathbb{CP}^n$, T is said to have finite local mass in some relatively compact neighborhood K of a point x_0 if

$$\int_K T \wedge (dd^c \log \|z\|)^p < +\infty.$$

Theorem 1.5.2 (Skoda-El Mir)**.** *Let $E \subset X$ be an analytic set and $T \in \mathcal{D}'_{p,p}(X \setminus E)$ be a closed positive current. Assume that T has a finite mass in a neighborhood of every point of E. Then the extension by zero of T to $\bar{T} \in \mathcal{D}'_{p,p}(X)$ is also closed and positive.*

Remark 1.5.3. The theorem of Skoda-El Mir is still true for a complete pluripolar set E . Moreover, one notes that the support of $\bar{T}$ is obtained as the closure of the support of T in $X \setminus E$ in the new ambient space X.

The theorem of Skoda-El Mir allows to extend the definition of currents of integration associated with (eventually singular) analytic subsets of X. Let $Z \subset X$ be an analytic subset, and $\alpha \in \mathcal{D}_{p,p}(X)$ a test form. Since for an analytic set the ramification points are finitely sheeted, the current $[Z_{\mathrm{reg}}]$ has finite mass near the analytic set $E = Z_{\mathrm{sing}}$ (the singular analytic subset of Z) ; therefore, by Skoda-El Mir theorem, $[Z_{reg}]$ certainly extends by zero from $X \setminus Z_{\mathrm{sing}}$ to X.

As the following fact shows one can also restrict the closed positive currents to analytic subsets.

Corollary 1.5.4 ([Dem, Corollary 2.4])**.** *If $T \in \mathcal{D}'_{p,p}(X)$ is a positive closed current and $E \subset X$ is a closed analytic subset, and $\mathbb{1}_E$ is its characteristic function, then $\mathbb{1}_E T$ and $\mathbb{1}_{X \setminus E} T$ are closed and positive.*

Next we recall the **Crofton's formula**, [Dem, Corollary 7.11].

Theorem 1.5.5 (Crofton's formula)**.** *Let dv be the unique measure which is invariant under the action of unitary group $U(n)$ and has mass 1 on the Grassmannian $G(p,n)$ of p-dimensional subspaces in $\mathbb{C}^n$. Then*

$$\int_{S \in G(p,n)} [S] dv(S) = (dd^c \log ||z||)^{n-p} \ .$$

We finish this chapter with a simple calculation that we will use in Chapter 5.

Example 1.5.6. Let D_P be the divisor of the homogeneous polynomial $P(z) = 0$ in $\mathbb{CP}^n$ and as before ω_{FS} the Fubini-Study metric. Crofton's formula allows us to interpret the total mass

$$\mathrm{Mass}_{\omega_{\mathrm{FS}}}(D_P) = \int_{\mathbb{CP}^n} [D_P] \wedge \omega_{\mathrm{FS}}^{n-1}$$

as the average of intersection numbers of D_P with affine hyperplanes in $\mathbb{CP}^n$. Since the degree of D_P is the total intersection number of intersection of D_P with generic hyperplanes, the total mass is exactly equal to the degree of D_P (see [Dem92a, Page 12] for another explanation of this fact).

Chapter 2

Tropical cycles and amoebas

The goal of this chapter is to introduce tropical cycles in $\mathbb{R}^n$ and to calculate Monge-Ampère measures associated to tropical polynomials. In order to do that we briefly explain amoebas and Ronkin functions and a duality between tropical hypersurfaces associated to a tropical polynomial and a subdivision of the Newton polytope of it. For a beautiful introduction to tropical geometry see [BS14a].

2.1 Tropical cycles

We start off by recalling the definition of tropical curves. Throughout this thesis a **rational graph** is a finite union of rays and segments in $\mathbb{R}^n$ whose directions have rational slopes. We refer to these rays and segments as edges or 1-cells and to the endpoints as vertices or 0-cells. Hence a graph Γ is the data $(\mathcal{C}_0(\Gamma), \mathcal{C}_1(\Gamma))$ of the 0-cells and 1-cells. A **primitive vector** is an integral vector such that the greatest common divisor of its components is 1. For each edge e incident to a vertex a there exists a primitive vector v_e which has a representative with support on e pointing away from a. Assume that every edge e of Γ is weighted by a non-zero integer m_e. We say that Γ satisfies the **balancing condition** at a vertex a if

$$\sum_{\{e \in \mathcal{C}_1(\Gamma); \{a\} \prec e\}} m_e v_e = 0, \tag{2.1.1}$$

where the sum is taken over all the edges incident to the vertex a. See Figure 2.1.

Definition 2.1.1 ([RGST05]). A tropical curve in $\mathbb{R}^n$ is a weighted rational graph $\Gamma = (\mathcal{C}_0(\Gamma), \mathcal{C}_1(\Gamma))$ which satisfies the balancing condition (2.1.1) at every vertex $a \in \mathcal{C}_0(\Gamma)$.

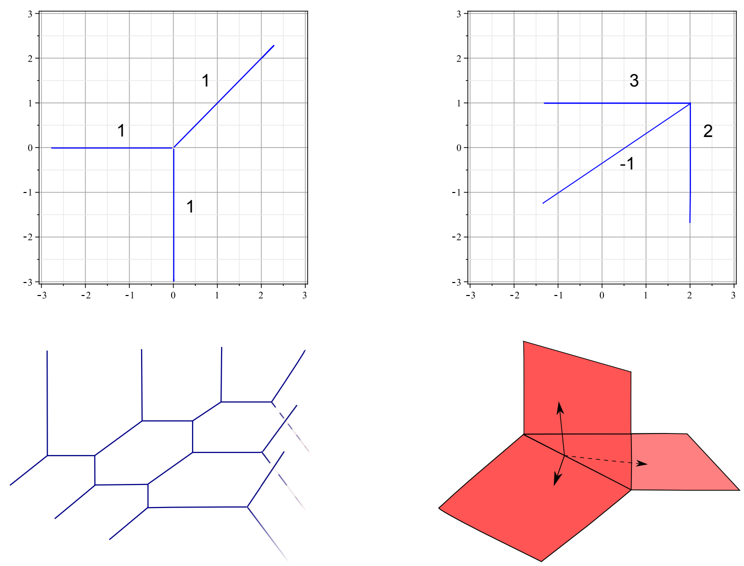

Figure 2.1: Balancing condition; traditionally weights equal to 1 are omitted in figures.

In the same spirit, one can define the **tropical p-cycles** in $\mathbb{R}^n$. First, a p-dimensional **polyhedral complex** is a finite set of p-dimensional polyhedra which are joined to each other along common faces. Such a p-dimensional polyhedral complex is called **rational** if each polyhedron is the intersection of rational half spaces, *i.e.* the half spaces which are given by the inequalities of the form

$$\langle \nu, x \rangle \geq a \quad (\text{with given constants} \quad \nu \in \mathbb{Z}^n\,,\ a \in \mathbb{R})\,.$$

Such a complex is said to be **weighted** if a non-zero integral weight is assigned to each of its p-dimensional cells. Let $V_1, \ldots, V_s$ be the p dimensional cells containing a $(p-1)$-dimensional cell W, $s \geq 2$, which have respective non-zero integer weights $m_1, \ldots, m_s$. Assume that W lies in an affine $(p-1)$-plane H_W and that each V_j lies in an affine p-plane H_{V_j}. One can find a $\mathbb{Z}$-basis $\{w_1, \ldots, w_{p-1}\}$ for $H_W \cap \mathbb{Z}^n$ (the initial point for these vectors is considered to be a point in W) and the unique inward (primitive) vector v_j in V_j such that $\{w_1, \ldots, w_{p-1}, v_j\}$ is a $\mathbb{Z}$-basis for $H_{V_j} \cap \mathbb{Z}^n$ [1]. One defines the balancing condition (corresponding to such a given $(p-1)$-dimensional cell W of the p-dimensional complex) as

[1]In such case, v_j is called a primitive vector with respect to the lattice $H_W \cap \mathbb{Z}^n$.

the sum of the vectors:

$$\sum_{j=1}^{s} m_j\, v_j \quad \text{lies in} \quad H_W\,. \tag{2.1.2}$$

Remark 2.1.2. The balancing condition simply implies that every $p \times p$ minor of the $n \times p$ matrix with column vectors $\big(w_1, \ldots, w_{p-1}, \sum_{j=1}^{s} m_j\, v_j\big)$ vanishes.

Definition 2.1.3 ([Mik06, AR10])**.** A weighted rational polyhedral complex of pure dimension p is called a tropical p-cycle if the balancing condition (2.1.2) is satisfied at every codimension 1 face. Such a cycle is called effective if every weight is a positive integer.

Therefore, tropical 1-cycles are the tropical graphs. Also, a tropical $(n-1)$-cycle in $\mathbb{R}^n$ is called a tropical hypersurface. To define the effective tropical cycles of codimension 1, one might use **tropical polynomials** which are defined as follows.

Definition 2.1.4. A tropical Laurent polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is a function of the form

$$(x_1, \ldots, x_n) \mapsto \max\nolimits_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n} \big\{ c_{(\alpha_1,\ldots,\alpha_n)} + \alpha_1 x_1 + \cdots + \alpha_n x_n \big\}, \tag{2.1.3}$$

over a finite set of indices, in which α_i, $i = 1, \ldots, n$ are integer numbers and $c_{\alpha_1,\ldots,\alpha_n}$ are real numbers; we might abbreviate the notation as

$$x \mapsto \max_{\alpha} \big\{ c_\alpha + \langle \alpha, x \rangle \big\}, \tag{2.1.4}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\langle\ ,\ \rangle$ is the usual inner product in $\mathbb{R}^n$.

To justify the preceding definition one considers the **tropical semi-field** $(\mathbb{T}, \oplus, \odot)$, where $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$, with the operations $a \oplus b = \max\{a, b\}$ and $a \odot b = a + b$ for $a, b \in \mathbb{T}$. Then the usual definition of a Laurent polynomial carried with tropical operations instead of the usual ones leads to that of a tropical Laurent polynomial in which the c_α's are coefficients and α_i's the respective exponents. If all $\alpha_i \in \mathbb{Z}^{\geq 0}$, the tropical Laurent polynomial p is said to be a tropical polynomial. The **tropical hypersurface** corresponding to a given tropical Laurent polynomial p is denoted by $V_{\mathbb{T}}(p)$ and defined as the set below

$$V_{\mathbb{T}}(p) = \big\{ x \in \mathbb{R}^n : \text{values of at least two monomials in } p \text{ coincide and maximize at } x \big\},$$

which is basically the corner locus of $x \mapsto p(x)$: the set of points over which the graph of the piece-wise linear convex function p is broken. This set has a rational polyhedral complex structure of pure dimension $n-1$. However, we still need to assign the weights to each of

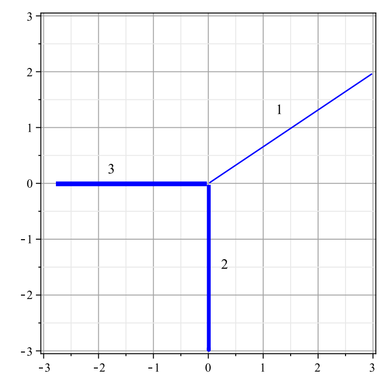

Figure 2.2: $V_{\mathbb{T}}(\max\{0, 2x, 3y\})$

the polyhedra to make it an honest tropical cycle : suppose F is a $(n-1)$-dimensional cell where the monomials $c_{\alpha_j} + \langle \alpha_j, x\rangle$, $\alpha_j \in \mathbb{Z}^n$, $j = 1, \dots, s$ are equal and maximized, then for dimensional reasons, the slopes α_j lie in a line in $\mathbb{Z}^n$; the weight $w(F)$ assigned to F is the maximal lattice length of this line segment connecting the points representing these slopes (the lattice length of a line segment being the number of lattice points on this line minus 1) ; one can check that with such weights $V_{\mathbb{T}}(p)$ satisfies the balancing condition at each facet, see Figure 2.2.

One also defines the **tropical projective space** in the following way.

Definition 2.1.5. The tropical projective space is the quotient

$$\mathbb{TP}^n = \mathbb{T}^{n+1} \backslash \{(-\infty)^{n+1}\} / \sim$$

where the equivalence relation $\sim$ is defined by

$$(x_0, \dots, x_n) \sim (\lambda \odot x_0, \dots, \lambda \odot x_n), \quad \lambda \in \mathbb{T}^* = \mathbb{R}\,.$$

One now considers the tropical cycles in $\mathbb{TP}^n$, which are locally tropical cycles in the $n+1$ local charts homeomorphic to $\mathbb{R}^n$; accordingly, the **homogeneous** tropical polynomials of degree d are of the form

$$(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mapsto \max_{\substack{(\alpha_0,\dots,\alpha_n)\in A \\ \alpha_0+\cdots+\alpha_n=d}} \left\{\alpha_0 x_0 + \cdots + \alpha_n x_n\right\}$$

with $A \subset \mathbb{Z}^{n+1}$ a finite subset. Finally one can consider the tropical hypersurfaces associated to tropical polynomials in $\mathbb{TP}^n$. The theorem below is now classic in tropical geometry.

Theorem 2.1.6 ([Mik04b], see also [RGST05] for $n = 2$)**.** *Every effective tropical hypersurface in $\mathbb{R}^n$ (resp. $\mathbb{TP}^n$) is of the form $V_{\mathbb{T}}(p)$ for some tropical (resp. homogeneous tropical) polynomial p.*

Another important notion to be recalled here is the notion of **amoeba**. Consider

$$\mathrm{Log}_t : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (z_1, \ldots, z_n) \mapsto (\frac{\log|z_1|}{\log t}, \ldots, \frac{\log|z_n|}{\log t}) \tag{2.1.5}$$

(when $t = \exp(1)$ we drop the subscript).

Definition 2.1.7 ([GKZ08])**.** The amoeba of an algebraic subvariety $V \subset (\mathbb{C}^*)^n$, denoted by $\mathcal{A}_V$, is the set $\mathrm{Log}\,(V) \subset \mathbb{R}^n$.

Given a family $(Z_t)_{t\in\mathbb{R}_+}$ of algebraic subvarieties of $(\mathbb{C}^*)^n$, one considers the family of amoebas $\mathrm{Log}_t(Z_t) \subset \mathbb{R}^n$, where $t \mapsto Z_t$ is analytic. Assume that $\mathrm{Log}_t(Z_t)$, as t goes to infinity, converges (with respect to the Hausdorff metrics on compact sets of $\mathbb{R}^n$) to a limit set V ; then V inherits a structure of a tropical cycle, *i.e.* as a set it is a rational polyhedral complex [BG84], which can moreover be equipped with positive integer weights to become balanced, (see [Spe02]). All the tropical hypersurfaces can be obtained in this way, which is not the case for higher codimensions. We explain in the next section the main properties of amoebas of hypersurfaces.

2.2 Amoebas of hypersurfaces

Let $F \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ be a Laurent polynomial. We denote by

$$\mathcal{A}_F := \mathrm{Log}\,(F^{-1}(0))$$

the amoeba of F which is a closed subset of $\mathbb{R}^n$. A convenient tool in order to study the amoeba of a Laurent polynomial and its deformation towards a tropical object is the **Ronkin function**.

Definition 2.2.1 ([Ron01])**.** Let F be a Laurent polynomial. The real function R_F defined in $\mathbb{R}^n$ by

$$R_F(x) = \frac{1}{(2\pi i)^n} \int_{z\in\mathrm{Log}^{-1}(x)} \log|F(z)| \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \ldots z_n},$$

is called the Ronkin function of F.

Indeed one needs to check that the above Lebesgue integral is convergent in $\mathcal{A}_F$. This is the case, thanks to the fact that logarithmic singularities remain integrable. But there is more to say :

Theorem 2.2.2 ([Ron01]). *Let F be a Laurent polynomial. The following statements hold.*

1. *The function R_F is convex.*
2. *R_F is affine linear in every open subset of any connected component of $\mathcal{A}_F^c = \mathbb{R}^n \backslash \mathcal{A}_F$.*
3. *If E is a connected component of $\mathcal{A}_F^c$ then the gradient $\nabla R_{F|_E}$ is an integral vector.*

Let E be a connected component of $\mathcal{A}_F^c$ then by items 2 and 3 of the above theorem, there exist $c_{\nu(E)} \in \mathbb{R}$ and $\nu(E) \in \mathbb{Z}^n$ such that

$$\nabla R_{F|_E} = c_{\nu(E)} + \langle \nu(E), x \rangle .$$

In this case the vector $\nu(E)$ is called the order of E see [FPT00], [Ron01], [Rul01]. It is not hard to see that every such E is in fact a convex set and that different connected components of $\mathcal{A}_F^c$ have different orders. Moreover, for every $x \in \mathbb{R}^n$

$$R_F(x) \geq c_{\nu(E)} + \langle \nu(E), x \rangle,$$

with equality exactly on the closure of E. Therefore it is reasonable to consider the tropical Laurent polynomial

$$p_F := \max_E \{c_{\nu(E)} + \langle \nu(E), x \rangle\}, \tag{2.2.6}$$

where E runs over the connected components of $\mathcal{A}_F^c$. The tropical hypersurface $V_{\mathbb{T}}(p_F)$ is called the **spine** of $\mathcal{A}_F$, see [PR04] and Figure 2.3.

It is in general very hard to calculate the spine of an amoeba, since positioning the components of $\mathcal{A}_F^c$ and the calculation of the constants $c_{\nu(E)}$ are both very hard. However, certain information can be deduced from the **Newton polytope** of F. Recall that the Newton polytope of a complex Laurent polynomial F (resp. tropical Laurent polynomial p) in n variables is the convex hull of the exponents of monomials in F (resp. p) considered as points in $\mathbb{R}^n$. Such polytopes in $\mathbb{R}^n$ are denoted by Δ_F (resp. Δ_p). It is not hard to see that whenever ν is a vertex of Δ_F then $\mathcal{A}_F^c$ has exactly one connected component with order ν . Thus $\Delta_F = \Delta_{p(F)}$, and the general duality (to be explained in a moment) between the natural polytopal subdivision of $\mathbb{R}^n$ induced by the tropical hypersurface of a tropical

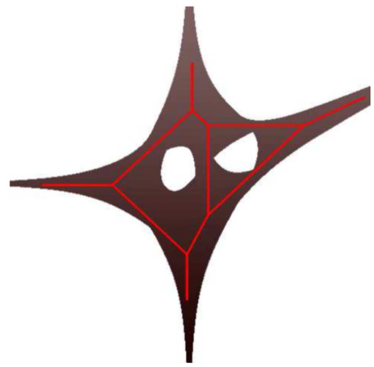

Figure 2.3: Amoeba of $1 + 5zw + w^2 - z^3 + 3z^2w - z^2w^2$ with its spine.

polynomial p and some polytopal subdivision of its Newton polytope Δ_p, when applied to the spine of $\mathcal{A}_F$, can be used to settle a duality between $\mathcal{A}_F$ and Δ_F. The spine of an amoeba $\mathcal{A}_F$ inherits the topological properties of the $\mathcal{A}_F$, namely the spine is a strong deformation retraction of the amoeba, see [PR04], [Rul01]. On the other hand, for any (effective) tropical hypersurface $V_{\mathbb{T}}(p)$ one can find a complex polynomial F such that spine of $\mathcal{A}_F$ coincides with $V_{\mathbb{T}}(p)$. Therefore, any tropical hypersurface is indeed approximable by a family of amoebas, see [Mik04b],[Rul01]. However this is not the case for higher codimension. We will explain the higher codimensional deformations in Chapter 5.

2.3 Polyhedral subdivisions

We recall a few definitions and theorems from Section 3 of [PR04] in order to perform a computation on Monge-Ampère measures in the next section.

Definition 2.3.1 ([PR04, Definition 1])**.** *Let K be a convex set in $\mathbb{R}^n$. A collection T of nonempty closed convex subsets of K is called a convex subdivision if it satisfies the following conditions :*

1. the union of all sets in T is equal to K ;
2. if $\sigma, \tau \in T$ and $\sigma \cap \tau$ is non empty, then $\sigma \cap \tau \in T$;
3. if $\sigma \in T$ and τ is any subset of σ then $\tau \in T$ if and only if τ is a face of σ ($\tau \prec \sigma$).

We say that T is locally finite if every compact set in K intersects only a finite number of $\sigma \in T$ and that T is polytopal if every $\sigma \in T$ is a polytope.

Given a convex cone C in $\mathbb{R}^n$, its dual cone is defined as

$$C^\vee = \{\xi \in \mathbb{R}^n; \langle \xi, x \rangle \le 0 \ \forall x \in C\}.$$

It is not hard to see that when C is closed, one has $C^{\vee\vee} = C$. If $\tau \subset \sigma$ are convex sets, define the convex cone

$$\operatorname{cone}(\tau, \sigma) = \big\{t(x-y); x \in \sigma, y \in \tau, t \ge 0\big\}.$$

Definition 2.3.2 ([PR04, Definition 2])**.** Let K, K' be convex sets in $\mathbb{R}^n$, and let T, T' be convex subdivisions of K, K'. We say that T and T' are dual to each other if there exists a bijective map $T \to T'$, denoted $\sigma \mapsto \sigma^*$, satisfying the following conditions :

1. for $\sigma, \tau \in T$, $\tau \subset \sigma$ if and only if $\sigma^* \subset \tau^*$;
2. if $\tau \subset \sigma$, then $\operatorname{cone}(\tau, \sigma)$ is dual to $\operatorname{cone}(\sigma^*, \tau^*)$.

Now let p be a tropical Laurent polynomial. The **Legendre transform** of $p(x)$ is defined as

$$\tilde{p}(\xi) = \sup_{x \in \mathbb{R}^n} \big(\langle \xi, x \rangle - p(x)\big) \in (-\infty, +\infty] \ ;$$

this is a piecewise linear convex function which takes finite values in the Newton polytope Δ_p (the closed convex envelope of the support of the Laurent polynomial p). Define $D_p : \Delta_p \times \mathbb{R}^n \to \mathbb{R}$ by

$$D_p(\xi, x) = p(x) + \tilde{p}(x) - \langle \xi, x \rangle.$$

One can see that $D_p(\xi, x) \ge 0$ and that $D_p(\xi, x)$ is convex in each argument when the other is fixed. Moreover

Lemma 2.3.3 ([PR04, Lemma 1])**.** *For every $x \in \mathbb{R}^n$ there is a $\xi \in K$ such that $D_p(\xi, x) = 0$, and for every $\xi \in K$ there exists an $x \in \mathbb{R}^n$ such that $D_p(\xi, x) = 0$.*

Define T to be the collection of all sets $\sigma_\xi = \{x \in \mathbb{R}^n \ : D_p(\xi, x) = 0\}$ for $\xi \in K$. Similarly let T' be the collection of all sets $\sigma'_x = \{\xi \in \Delta_p \ : D_p(\xi, x) = 0\}$. Then

Lemma 2.3.4 ([PR04, Proposition 1]). *With the notation of preceding paragraph, T and T' are dual convex subdivisions of $\mathbb{R}^n$ and Δ_p, where the correspondence between T and T' is given by*

$$\begin{aligned}\sigma^* &= \bigcap_{x\in\sigma} \sigma_x = \{\xi \in \Delta_p \ : D_p(\xi, x) = 0 \ \forall x \in \sigma\} \\ {\sigma'}^* &= \bigcap_{\xi\in\sigma'} \sigma_\xi = \{x \in \mathbb{R}^n \ : D_p(\xi, x) = 0 \ \forall \xi \in \sigma'\}\end{aligned}$$

for all $\sigma \in T$, $\sigma; \in T'$. Moreover, T is locally finite and T' is polytopal.

Example 2.3.5. Let us briefly analyze these functions. If $p(x) = \max_{\alpha\in A}\{c_\alpha + \langle\alpha, x\rangle\}$ for some finite set $A \subset \mathbb{Z}^n$, then for each $\alpha \in A$,

$$\tilde{p}(\alpha) = -c_\alpha\,. \tag{2.3.7}$$

To see this, for $\alpha \in A$ define

$$X_\alpha = \{x \in \mathbb{R}^n : c_\alpha + \langle\alpha, x\rangle = p(x)\}.$$

Obviously, $\langle\alpha, x_0\rangle - p(x_0) = -c_\alpha$ for some $x_0 \in X_\alpha$. However, when $x_1 \notin X_\alpha$, there exists $\beta \in A$, $\beta \neq \alpha$, such that $p(x_1) = c_\beta + \langle\beta, x_1\rangle \geq c_\alpha + \langle\alpha, x_1\rangle$. Thus $-c_\alpha \geq \langle\alpha, x_1\rangle - c_\beta - \langle\beta, x_1\rangle$, and therefore $\tilde{p}(\alpha) = -c_\alpha$ and $D_p(\alpha, x_0) = 0$. Hence $X_\alpha \subset \sigma_\alpha$.

On the other hand, for $x_0 \in \mathbb{R}^n$, consider

$$\sigma'_{x_0} = \{\xi \in \Delta_p : D_p(\xi, x_0) = 0\},$$

which is non-empty by Lemma 2.3.3. The condition $D_p(\xi, x_0) = 0$ or equivalently

$$\sup_{x\in\mathbb{R}^n} \big(\langle\xi, x\rangle - p(x)\big) = \langle\xi, x_0\rangle - p(x_0)$$

simply means that the maximum of this function is achieved at x_0. Therefore by the above argument $\alpha \in \sigma'_{x_0}$ whenever $x_0 \in X_\alpha$. Moreover, it is easy to see any convex combination of elements of σ'_{x_0} also lies in σ'_{x_0}. If moreover $x_0 \in V_{\mathbb{T}}(p)$, then there are $\alpha_i \in A$, $i = 1, \ldots, s$, such that $p(x_0) = c_{\alpha_i} + \langle\alpha_i, x_0\rangle$. It follows then

$$(\alpha_1\widehat{, \ldots,} \alpha_s) \subset \sigma'_{x_0}\,, \tag{2.3.8}$$

where $\widehat{}$ means the closed convex envelope.

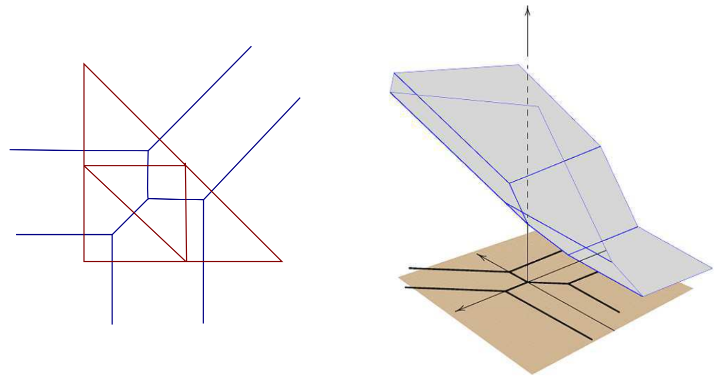

Figure 2.4: Privileged subdivision.

If the vectors $(\alpha_1, \ldots, \alpha_s)$ lie in a hyperplane of minimal dimension l then

$$\mathrm{Vol}_l(\sigma'_{x_1}) \geq \mathrm{Vol}_l(\widehat{\alpha_1, \ldots, \alpha_s})\,. \tag{2.3.9}$$

Therefore if we choose a vertex $a \in \mathcal{C}_0(V_{\mathbb{T}}(p))$, then

$$\mathrm{Vol}_n\big(\xi \in \Delta : \langle \xi, x\rangle - p(x) \text{ attains its global maximum at } a\big) \;\geq \mathrm{Vol}_n(\{a\}^*)\,. \tag{2.3.10}$$

On the other hand

$$\sum_{a \in \mathcal{C}_0(V_{\mathbb{T}}(p))} \mathrm{Vol}_n(\{a\}^*) = \mathrm{Vol}_n(\Delta).$$

Given that for different $a, b \in \mathcal{C}_0(V_{\mathbb{T}}(p))$, $\mathrm{Vol}_n(\{a\}^* \cup \{b\}^*) = 0$, we find

$$\mathrm{Vol}_n\big(\xi \in \Delta : \langle \xi, x\rangle - p(x) \text{ attains its global maximum at } a\big) \;= \mathrm{Vol}_n(\{a\}^*)\,. \tag{2.3.11}$$

This quantity is related to Monge-Ampère measure, which was first used in this context in [PR04].

Following 2.3.7 the above duality between a tropical hypersurface $V_{\mathbb{T}}(p)$ and the resulting subdivision of the Newton polytope of p can be understood in the following way : given $p(x) = \max_{\alpha \in A}\{c_\alpha + \langle \alpha, x\rangle\}$, consider $\tilde{\Delta}_p = \big\{(\alpha, -c_\alpha) : \alpha \in A \subset \Delta_p\big\} \subset \Delta_p \times \mathbb{R}$. Consider the faces of $\tilde{\Delta}_p$ which have an inward normal vector with negative last coordinate, project these faces down to Δ_p in order to obtain a **privileged subdivision** of Δ_p (see Figure 2.4, and [HS95]).

Let $\Delta^n \subset \mathbb{R}^n$ be the standard simplex given as the convex hull of $\{0, e_1, \ldots, e_n\}$, where e_j denote the standard basis. For any non-negative integer d, denote by $d\ \Delta^n$ the dilation of Δ^n , one has

$$\mathrm{Vol}_n(d\ \Delta^n) = \frac{d^n}{n!}\ .$$

The following definition will be generalized in Chapter 5.

Definition 2.3.6. A tropical hypersurface $V = V_{\mathbb{T}}(p) \subset \mathbb{R}^n$ is of degree d, if the Newton polytope Δ_p of p coincides with $d\ \Delta^n$.

2.4 Monge-Ampère measures

Let Ω be an open subset in $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a (possibly non-smooth) convex function. The generalized gradient of u at $x_0 \in \Omega$

$$\nabla\, u(x_0) = \{\xi \in \mathbb{R}^n\ : u(x) - u(x_0) \geq \langle \xi, x - x_0 \rangle\ \forall x \in \Omega\}\ . \tag{2.4.12}$$

For any subset $E \subset \Omega$, let

$$\nabla u(E) = \bigcup_{x \in E} \nabla u(x)\ . \tag{2.4.13}$$

The **Monge-Ampère measure** associated to u, denoted by $\mu[u]$, of any Borel subset $E \subset \Omega$ is defined as

$$\mu[u](E) := \lambda\left(\nabla u(E)\right)\ ,$$

where λ is the Lebesgue measure on $\mathbb{R}^n$. It is easy to see that when u is twice differentiable then

$$\mu[u] = \det\big(\big[\frac{\partial^2 u}{\partial x_i \partial x_j}\big]_{i,j}\big)\ \ \lambda\,.$$

Let now $\Omega = \mathbb{R}^n$ and $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function. One has $\mu[ku] = k^n \mu[u]$ for any positive integer k. Moreover, one can re-write (2.4.12) as

$$\nabla\, u(x_0) = \{\xi \in \mathbb{R}^n\ : \langle \xi, x_0 \rangle - u(x_0) \geq \langle \xi, x \rangle - u(x)\ \forall x \in \mathbb{R}^n\}\ .$$

That is, the set of $\xi \in \mathbb{R}^n$ where the function $x \in \Omega \mapsto \langle \xi, x \rangle - u(x)$ achieves its global maximum in $\mathbb{R}^n$ precisely at the point x_0. Accordingly, for a Borel set $E \subset \mathbb{R}^n$, $\mu[u](E) := \int_E \mu[u] = \lambda\left(\nabla u(E)\right)$ is the Lebesgue measure of the set of all points $\xi \in \mathbb{R}^n$ such that the global maximum of $x \in \mathbb{R}^n \mapsto \langle \xi, x \rangle - u(x)$ is achieved at some point x_0 in E.

An interesting fact is that taking $u = R_F$ the Ronkin function of a Laurent polynomial F, then the support of $\mu[u] = \mu[R_F]$ is exactly $\mathcal{A}_F$. This follows from the fact that R_F is affine linear outside $\mathcal{A}_F$. Also:

Proposition 2.4.1 ([PR04, Theorem 4])**.** *Let F be a Laurent polynomial and R_F be its associated Ronkin function ; then the total mass $\mu[R_F](\mathbb{R}^n)$ equals $\mathrm{Vol}_n(\Delta_F)$.*

Proof. We need to prove that $\xi \in \Delta_F$ if and only if $x \in \mathbb{R}^n \mapsto \langle \xi, x\rangle - R_F(x)$ is bounded from above. Let $\{\xi_1, \ldots, \xi_s\}$ be the vertices of Δ_F . We have mentioned before that for each of ξ_i there exists an unbounded component of $\mathcal{A}_F^c$ with order equal to ξ_i. Let p_F be the tropical Laurent polynomial corresponding to the spine of $\mathcal{A}_F$ defined in 2.2.6. If $\xi \in \Delta_F$ then there exist $0 \le \lambda_i \le 1$, $i = 1, \ldots, s$, with $\sum_i \lambda_i = 1$ such that $\xi = \sum \lambda_i\ \xi_i$. Thus, for any $x \in \mathbb{R}^n$,

$$\langle \xi, x\rangle - R_F(x) \ \le\ \langle \xi, x\rangle - p_F \ \le\ \max_i\{\langle \xi_i, x\rangle - p_F(x)\}\ ,$$

and the left hand side is obviously bounded from above. Conversely, if $\xi \notin \Delta_F$, one can find $x_0 \in \mathbb{R}^n$ such that $\langle \xi, x_0\rangle > \sup_{\eta\in\Delta_F}\langle \eta, x_0\rangle$. This supremum is achieved at some vertex of Δ_F, say ξ_j . If x_1 belongs to the connected component of $\mathbb{R}^n \setminus \mathcal{A}_F$ with order ξ_j, then so does $x_1 + tx_0$ for $t > 0$. Hence $\langle \xi, x_1 + t\, x_0\rangle - R_F(x_1 + tx_0) = \langle \xi - \xi_j, x_0 + tx_1\rangle - c_{\xi_j}$, which tends to $+\infty$ as $t \to +\infty$. $\square$

In the next example we give an explicit description of Monge-Ampère measures associated to tropical polynomials, see also [Yge13, Example 3.2]

Example 2.4.2. Assume that p is a tropical Laurent polynomial. The support of $\mu[p]$ lies inside $V_{\mathbb{T}}$. It is easy to see that $\xi \in \Delta_F$ if and only if $x \in \mathbb{R}^n \mapsto \langle \xi, x\rangle - p(x)$ is bounded from above. Hence

$$\mu[p](\mathbb{R}^n) = \mu[p](V_{\mathbb{T}}(p)) \le \mathrm{Vol}_n(\Delta_p)\,.$$

However, 2.3.11 can be interpreted as that for $a \in \mathcal{C}_0(V_T(p))$

$$\mu[p](\{a\}) = \mathrm{Vol}_n(\{a\}^*).$$

Since the dual cells $\{a\}^*$ provide a subdivision of Δ_p,

$$\sum_{a\in\mathcal{C}_0(V_T(p))} \mathrm{Vol}_n(\{a\}^*) = \mathrm{Vol}_n(\Delta_p)\ .$$

This implies that $\mu[p]$ has only atomic masses at 0-cells of $V_{\mathbb{T}}(p)$ and that

$$\mu[p] = \sum_{a \in \mathcal{C}_0(V_T(p))} \mathrm{Vol}_n(\{a\}^*)\, \delta_a\,, \tag{2.4.14}$$

where δ_a denotes the Dirac mass at a.

Now let $p'(x) = \max_{\beta \in B \subset \mathbb{Z}^n}\{c_\beta + \langle \beta, x\rangle\}$ be another tropical polynomial. Assume that $V_{\mathbb{T}}(p')$ does not pass a small neighborhood $E \subset \mathbb{R}^n$ of a fixed vertex $v \in \mathcal{C}_0(V_{\mathbb{T}}(p))$. This simply means that p' is affine linear near E *i.e.*

$$p'(x)_{|_E} = c_{\beta'} + \langle \beta', x\rangle_{|_E}, \quad \text{for a } \beta' \in B.$$

Consequently,

$$\nabla(p + p')(E) = \nabla(p)(E) + \beta\ . \tag{2.4.15}$$

And,

$$\mu[p + p'](E) = \mu[p](E) = \mu[p](v). \tag{2.4.16}$$

Notice that $p+p'$ as a new tropical polynomial has its own associated tropical hypersurface $V_{\mathbb{T}}(p+p')$ (which is in fact $V_T(p) \cup V_T(p')$). The Newton polytope of $p+p'$, $\Delta_{p+p'}$, also has its own subdivision dual to $V_{\mathbb{T}}(p+p')$. Therefore (2.4.15) and (2.4.16) imply that corresponding dual cells of v considering it as a vertex of either of $V_{\mathbb{T}}(p+p')$ or $V_{\mathbb{T}}(p)$ have the same shape and volume. Therefore

$$\mu[p + p'](E) = \mu[p](E) = \mu[p](v) = \mathrm{Vol}_n(\{v\}^*).$$

This means that for generic p and p',

$$\mu[p + p'] - \mu[p] - \mu[p']$$

does not have any mass on $\mathcal{C}_0(V_{\mathbb{T}}(p)) \cup \mathcal{C}_0(V_{\mathbb{T}}(p'))$, and only has masses on $\mathcal{C}_0(V_{\mathbb{T}}(p) \cap V_{\mathbb{T}}(p'))$. We will use this discussion for an intersection theory in Chapter 4.

Chapter 3

Tropical currents and extremality

3.1 Definition of tropical currents

Assume $V_{\mathbb{T}}$ is a tropical p-cycle. We define a current supported on $\text{Log}^{-1}(V_{\mathbb{T}})$ which inherits the respective weights of $V_{\mathbb{T}}$ and then determine whether this current is strongly extremal. We introduce the following abridged notations.

Notation 3.1.1. *For a complex number ζ and an integral vector $\nu = (\nu_1, \ldots, \nu_m)$ ($m \in \mathbb{N}^*$) we set*

$$\zeta^{\nu} = (\zeta^{\nu_1}, \ldots, \zeta^{\nu_m}).$$

Moreover for two vectors $\nu = (\nu_1, \ldots, \nu_m)$, $\nu' = (\nu'_1, \ldots, \nu'_m)$

$$\nu \star \nu' := (\nu_1 \, \nu'_1, \ldots, \nu_m \, \nu'_m).$$

Recall that a rational p-plane in $\mathbb{R}^n$ is given by equations of the form

$$\langle \nu_i, x \rangle = 0, \quad \nu_i \in \mathbb{Z}^n \, , i \, = \, 1, \ldots, n - p \, .$$

Lemma 3.1.2. *Suppose H is a rational p-plane in $\mathbb{R}^n$ (which passes the origin), ($1 \leq p \leq n$). Let $B = \big(w_1, \ldots, w_p\big)$ and $B' = \big(w'_1, \ldots, w'_p\big)$ be two $\mathbb{Z}$-basis for $H \cap \mathbb{Z}^n$. Define for any $\gamma \in (\mathbb{S}^1)^n$, the two subsets of $(\mathbb{C}^*)^n$:*

$$Z^{\gamma}_{B} := \{\tau_1^{w_1} \star \cdots \star \tau_p^{w_p} \star \gamma = \iota^{w}_{\gamma}(\tau) \; ; \tau_1, \ldots, \tau_p \in \mathbb{C}^*\}$$

and

$$Z^{\gamma}_{B'} = \{\tau_1^{w'_1} \star \cdots \star \tau_p^{w'_p} \star \gamma = \iota^{w'}_{\gamma}(\tau) \; ; \tau_1, \ldots, \tau_p \in \mathbb{C}^*\}.$$

Then, the integration currents

$$T = [Z_B^\gamma] := (\iota_\gamma^w)_*([(\mathbb{C}^*)^p]), \quad T' = [Z_{B'}^\gamma] := (\iota_\gamma^{w'})_*[(\mathbb{C}^*)^p]$$

coincide.

Proof. The analytic sets Z_B^γ and $Z_{B'}^\gamma$ are equal. We prove that they are analytically isomorphic. Consider B, B' as matrices with the given vectors as columns. There exists $C \in GL(p, \mathbb{Z})$, such that $B\,C = B'$. Set

$$(\tau_1', \ldots, \tau_p') = (\tau_1^{c_1} \star \cdots \star \tau_p^{c_p})$$

where $c_1, \ldots, c_p$ are the columns of C. This is an invertible monoidal change of coordinates, and it is easy to see that

$$(\tau_1')^{w_1} \star \cdots \star (\tau_p')^{w_p} = \tau_1^{w_1'} \star \cdots \star \tau_p^{w_p'},$$

which concludes the proof. □

Remark 3.1.3. The sets of the form Z_B^γ, when $\gamma = 1$, are referred to as **toric** sets [Stu96a]. They can be understood as zero locus of binomial ideals in $\mathbb{C}^n$. In fact, if $\xi_1, ..., \xi_M$ is a set of primitive generators for $\operatorname{Ker} B^t \cap \mathbb{Z}^n$, such that each ξ_ℓ splits into $\xi_\ell^+ - \xi_\ell^-$, with $\xi_\ell^+ = (\xi_{\ell,1}^+, \ldots, \xi_{\ell,n}^+)$ and $\xi^- = (\xi_{\ell,1}^-, \ldots, \xi_{\ell,n}^-)$ having non-negative components in $\mathbb{Z}^n$ and disjoint supports, then the current $[Z_B^\gamma]$ is given by

$$\mathbb{1}_{Z_B^\gamma} \cdot \Big[\frac{1}{2} dd^c \log\Big(\sum_{\ell=1}^M \Big|\prod_{j=1}^n \zeta_j^{\xi_{\ell,j}^+} - \prod_{j=1}^n \gamma_j^{-\xi_{\ell,j}} \zeta_j^{\xi_{\ell,j}^-}\Big|^2\Big)\Big]^{n-p}$$

by King's formula (see [Stu96b, Lemma 4.1] for the fact about toric sets and [Dem, Page 181]) for the King's formula).

As before let $H = H_0$ be a rational p-plane (passing through 0). One can find a $\mathbb{Z}$-basis for the lattice $L_H := H \cap \mathbb{Z}^n$, $B = \big(w_1, \ldots, w_p\big)$. Moreover, B can be completed as $D = \big(w_1, \ldots, w_p, u_1, \ldots, u_{n-p}\big)$ which stands, as a set, as a $\mathbb{Z}$-basis for $\mathbb{Z}^n$ (if D denotes the matrix of such vectors as columns, one has $\det D = \pm 1$). Note that, if D and D' are two such completions of B, one has

$$D' = D \cdot \begin{bmatrix} \mathrm{Id}_p & 0 \\ K & \widetilde{C} \end{bmatrix}$$

where K and $\widetilde{C}$ are respectively $(n-p,p)$ and $(n-p,n-p)$ matrices with integer coefficients and $\det \widetilde{C} = (\det D)^{-1} \times \det D' = \pm 1$. Fix for the moment a basis B and consider such a completion $D_B = D$ of B. Consider, for each $(\theta_{p+1}, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}$, the set

$$\Delta_{H,D}(\theta) := \{\tau_1^{w_1} \star \cdots \star \tau_p^{w_p} \star e^{2i\pi\theta_{p+1}u_1} \star \cdots \star e^{2i\pi\theta_n u_{n-p}}\,;\, \tau \in (\mathbb{C}^*)^p\}. \tag{3.1.1}$$

This is a p-dimensional analytic subset of $(\mathbb{C}^*)^n$ which is a toric set of the form $Z_B^{\gamma_u}$. In addition, one can parametrize $S_H := \mathrm{Log}^{-1}(H)$ in the following way :

$$S_H = \Big\{\tau_1^{w_1} \star \cdots \star \tau_p^{w_p} \star e^{2i\pi\theta_{p+1}u_1} \star \cdots \star e^{2i\pi\theta_n u_{n-p}}\ ; \tau \in (\mathbb{C}^*)^p,\ (\theta_{p+1}, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}\Big\}.$$

Therefore each $\Delta_{H,D}(\theta_{p+1}, \ldots, \theta_n)$ can be considered as the fiber over $(\theta_{p+1}, ..., \theta_n)$ of the submersion $\sigma_{H,D}$:

$$\begin{array}{c} \tau_1^{w_1} \star \cdots \star \tau_p^{w_p} \star e^{2i\pi\theta_{p+1}u_1} \star \cdots \star e^{2i\pi\theta_n u_{n-p}} \in S_H \\ \Big\downarrow \sigma_{H,D} \\ (\theta_{p+1}, \ldots, \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}\,. \end{array}$$

We define the positive (p,p) current $T_{H,D}$

$$T_{H,D} = \int_{(\theta_{p+1},\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} \big[\Delta_{H,D}(\theta_{p+1}, \ldots, \theta_n)\big]\, d\theta_{p+1} \ldots d\theta_n\,. \tag{3.1.2}$$

If one considers two completions $D = (B,U)$ and $D' = (B,U')$ of B, though the fibers $\Delta_{H,D}$ do vary when D is changed into D' (as well as the integration currents $[\Delta_{H,D}]$)), the sum $T_{H,D}$ does not since $U' = U \cdot \widetilde{C}$ (where $\widetilde{C} \in GL(n-p,\mathbb{Z})$) and the Lebesgue measure on $(\mathbb{R}/\mathbb{Z})^{n-p}$ is preserved under the action of monoidal automorphisms of the torus $(\mathbb{S}^1)^{n-p}$ whose matrix $\widetilde{C}$ of exponents belongs to $GL(n-p,\mathbb{Z})$. As a result, the current $T_{H,D}$ depends only on B and one can write $T_{H,D} = T_{H,D_B} = T_H^{[B]}$ for any completion D_B of B. On the other hand, if U is fixed, it follows from Lemma 3.1.2 that, if one considers $D = (B,U)$ and $D' = (B',U)$, where B and B' are two lattice basis of L_H, then $[\Delta_{H,D}(\theta)] = [\Delta_{H,D'}(\theta)]$ for any $\theta = (\theta_{p+1}, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}$, hence $T_{H,D} = T_{H,D'}$. Accordingly, $T_{H,D_B} = T_H^{[B]}$ is in fact independent of B, and one defines in such a way a positive current

$$T_H = T_H^{[B]} = T_{H,\{B,U_B\}} = T_{H,D_B}$$

which is independent of the choice of the lattice basis B for $L_H = H \cap \mathbb{Z}^n$ as well as that of its completion $D = D_B = (B, U_B)$ as a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. The support of T_H (considered as a (p,p)-dimensional positive current in $(\mathbb{C}^*)^n$ is clearly $\operatorname{Log}^{-1}(H) = S_H$.

Now assume that $H_a \subset \mathbb{R}^n$ is a rational affine p-plane obtained by translation of a rational p-plane $H = H_0$ via $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. Define the linear map

$$
\begin{aligned}
L_a : (\mathbb{C}*)^n &\to (\mathbb{C}^*)^n, \\
z = (z_1, \dots, z_n) &\mapsto \exp(-a) \star z = (\exp(-a_1)z_1, \dots, \exp(-a_n)z_n).
\end{aligned}
$$

Set

$$T_{H_a} := L_a^*(T_H) = \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [L_a^{-1}(\Delta_{H,D}(\theta_{p+1}, \dots, \theta_n))] d\theta_{p+1} \dots d\theta_n.$$

Accordingly,

$$S_{H_a} = \exp(a) \star S_H,$$

and

$$\Delta_{H_a,D} = \exp(a) \star \Delta_{H,D} \,.$$

It is easily seen that the definition of T_{H_a} is independent of the choice of the base point $a \in H_a$, which makes us ready to propose the following definition.

Definition 3.1.4. Assume $\mathcal{P}$ is a weighted rational polyhedral complex of pure dimension p. Let $\mathcal{C}_p(\mathcal{P})$ be the family of all p dimensional cells of $\mathcal{P}$. Each $P \in \mathcal{C}_p(\mathcal{P})$ is equipped with a non-zero integral weight m_P and lies in an affine p-plane H_{a_P} which passes through a chosen base point $a_P \in P$. Let

$$\mathscr{T}_P = \mathbb{1}_{\operatorname{Log}^{-1}(\operatorname{int} P)} \, T_{H_{a_P}}$$

be the restriction of the positive (p,p)-dimensional current $T_{H_{a_P}}$ (supported by $\operatorname{Log}^{-1}(H_{a_P})$) to $\operatorname{Log}^{-1}(\operatorname{int} P) \subset \operatorname{Log}^{-1}(H_{a_P}) \subset (\mathbb{C}^*)^n$. Here $\operatorname{int}(P)$ denotes the relative interior of P in the affine p-plane H_{a_P} . This definition is independent of the chosen base point a_P. We define

$$\mathscr{T}_n^p(\mathcal{P}) = \sum_{P \in \mathcal{C}_p(\mathcal{P})} m_P \, \mathscr{T}_P \,.$$

Obviously, if $\mathcal{P}$ is positively weighted, then $\mathscr{T}_n^p(\mathcal{P})$ is a positive current. In this thesis we are interested in the case where $\mathcal{P}$ is a tropical cycle $V_{\mathbb{T}}$. In such case, we call $\mathscr{T}_n^p(V_{\mathbb{T}})$ the **tropical current** associated to $\mathcal{P} = V_{\mathbb{T}}$.

Before stating the main theorem of this thesis, we introduce the following terminology.

Definition 3.1.5. A set of vectors is said to be linearly sub-independent over a field $\mathbb{K}$ if each proper subset of this set is a set of linearly independent vectors.

Remark 3.1.6. Suppose that the set of vectors $\{v_1, \ldots, v_s\}$ is linearly sub-independent over $\mathbb{R}$ and there exist $a_j, b_j \in \mathbb{C}$, $j = 1, \ldots, s$ such that $\sum_{j=0}^{s} a_j v_j = \sum_{j=0}^{s} b_j v_j = 0$. Then there exists a $\rho \in \mathbb{C}$ such that $a_j = \rho\, b_j$ for $j = 1, \ldots, s$.

Definition 3.1.7. A tropical p-cycle $V_{\mathbb{T}} \subset \mathbb{R}^n$ is said to be strongly extremal if

1. $V_{\mathbb{T}}$ is connected in codimension 1 ;
2. each $p-1$ dimensional face (facet) W of $V_{\mathbb{T}}$ is a common facet of exactly $n-p+2$ polyhedra (cells) of dimension p ;
3. for each facet of W of $V_{\mathbb{T}}$, let $\{v_1, \ldots, v_{n-p+2}\}$ be the primitive vectors, one in each of the $n-p+2$ polyhedra above, that make the balancing condition hold. Then, the set of their projections along W, $\{h_W(v_1), \ldots, h_W(v_{n-p+2})\}$, forms a sub-independent set.

For instance, when $V_{\mathbb{T}} \subset \mathbb{R}^n$ is a tropical 1-cycle, then the strong extremality conditions means that the graph is $(n+1)$-valent at every vertex and the corresponding $(n+1)$- primitive vectors span $\mathbb{R}^n$. It is also clear that for tropical hypersurfaces in the number $n-p+2$ is exactly $n-(n-1)+2 = 3$.

Theorem 3.1.8. *If $V_{\mathbb{T}} \subset \mathbb{R}^n$ is a tropical p-cycle, then the normal and (p,p)-dimensional tropical current $\mathscr{T}_n^p(V_{\mathbb{T}})$ is closed. If moreover $V_{\mathbb{T}}$ is strongly extremal, then $\mathscr{T}_n^p(V_{\mathbb{T}})$ is strongly extremal in $\mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$.*

In order to make our understanding progressive, we first explore the case of tropical curves ($p = 1$), then that of p-dimensional tropical cycles with a single codimension 1 face, a facet.

3.2 Tropical $(1,1)$-dimensional currents

In this section we study $\mathscr{T}_n^1(\Gamma)$, where Γ is a weighted rational graph. We prove Theorem 3.1.8 in this case. Suppose an edge e of Γ of weight m_e (spanning the affine line

$E \subset \mathbb{R}^n$) is parameterized by

$$t \mapsto t\, v_e + a,$$

where $\{a\}$ ($a \in \mathbb{R}^n$) is one of the vertices of e, $v_e = v_e^{[a\to]} \in \mathbb{R}^n$ is the corresponding (inward) primitive vector for e from the vertex $\{a\}$, and $t \in [0, t_0] \subset \mathbb{R}$ is a real parameter, $t_0 \in [0, +\infty]$; when $t = \infty$ the edge is a ray. We complete $\{v_e\}$ to a basis D_e of the lattice $\mathbb{Z}^n$, say $D_e = (v_e, U_e) = \big\{v_e, u_1^e, \dots, u_{n-1}^e\big\}$, that is, if one denotes also D_e as the matrix with columns $v_e^t, (u_1^e)^t, \dots, (u_{n-1}^e)^t$, one has $\det(D_e) = \pm 1$, *i.e.* $D_e \in GL(n, \mathbb{Z})$. We can now define an open subset $S_{e,D_e,a} \subset S_E := \mathrm{Log}^{-1}(E)$ as :

$$\begin{aligned} S_{e,D_e,a} := \Big\{ & \exp(a) \star \tau^{v_e} \star \exp(2i\pi\theta_2 u_1^e) \star \cdots \star \exp(2i\pi\theta_n u_{n-1}^e) \ ; \\ & \tau \in \mathbb{C}^*,\ 1 < |\tau| < \exp(t_0),\ \theta = (\theta_2, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-1} \Big\} . \end{aligned} \tag{3.2.3}$$

Such an open set $S_{e,D_e,a} \subset S_E$ (considered here as a submanifold with boundary of the manifold S_E with real dimension $n+1$) is injectively foliated over the Cartesian product $\big(\mathbb{R}/\mathbb{Z}\big)^{n-1}$ through the submersion

$$\begin{array}{c} \exp(a) \star \tau^{v_e} \star \exp(2i\pi\theta_2 u_1^e) \star \cdots \star \exp(2i\pi\theta_n u_{n-1}^e) \in S_{e,D,a} \\ \Big\downarrow{\scriptstyle \sigma_{e,D_e,a}} \\ (\theta_2\,, \dots, \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-1} . \end{array}$$

One also denotes as $\boldsymbol{\tau}_{e,D_e,a}$ the parameterization map from $(\mathbb{C}^*)^n$ into itself which is used to get (through its inverse) the submersion $\sigma_{e,D_e,a}$, that is the monoidal map :

$$\boldsymbol{\tau}_{e,D_e,a} \ : (\tau_1, \lambda_2, \dots, \lambda_n) \in (\mathbb{C}^*)^n \mapsto \exp(a) \star \tau_1^{v_e} \star \lambda_2^{u_1^e} \star \cdots \star \lambda_n^{u_{n-1}^e} \in (\mathbb{C}^*)^n.$$

Denote as $\Sigma_{e,D_e,a}$ the cycle

$$\begin{aligned} & \Sigma_{e,D_e,a} := \partial S_{e,D_e,a} \ : \\ & (\theta_1, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^n \mapsto \exp(a) \star \exp(2i\pi\theta_1\, v_e) \star \exp(2i\pi\theta_2\, u_1^e) \star \cdots \star \exp(2i\pi\theta_n u_{n-1}^e) \ . \end{aligned}$$

The support of the cycle $\Sigma_{e,D_e,a}$ equals $\mathrm{Log}^{-1}(\{a\})$. For each $(\theta_2, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-1}$, denote as $\Delta_{e,D_e,a}$ the fiber $\sigma^{-1}_{e,D_e,a}(\{(\theta_2, ..., \theta_n)\})$ of the submersion $\sigma_{e,D_e,a}$ over $(\theta_2, ..., \theta_n)$ and consider the $(1,1)$-dimensional positive current in $(\mathbb{C}^*)^n$ defined as

$$T_{e,D_e,a} := \int_{(\theta_2,...,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}} [\Delta_{e,D_e,a}(\theta_2, ..., \theta_n)]\, d\theta_2 \dots d\theta_n.$$

The current $T_{e,D_e,a}$ is obviously not closed ; nevertheless, its support is the set $\mathrm{Log}^{-1}(e)$. As we have explained in the beginning of this section, the current $T_{e,D_e,a}$ is independent of the choice of the completion D_e for $\{v_e\}$ because of the invariance of the Lebesgue measure on $(\mathbb{R}/\mathbb{Z})^{n-1}$ under the action of the linear group $GL(n-1,\mathbb{Z})$ (considered in the multiplicative sense). In fact $T_{e,D_e,a}$ depends only on e and stands as the current $\mathscr{T}_e$ obtained as the restriction to the edge e of the positive $(1,1)$-dimensional current T_E (in order to check this point, one can easily reduce the situation up to translation to the case $a=0$). We however keep track of the averaged representation

$$\mathscr{T}_e = T_{e,D_e,a} := \int_{(\theta_2,\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}} [\Delta_{e,D_e,a}(\theta_2,\dots,\theta_n)]\, d\theta_2\dots d\theta_n, \tag{3.2.4}$$

where the average of integration currents $[\Delta_{e,D_e,a}]$ indeed depend on the specified vertex a of e and on the completion D_e of the set $\{v_e\}$, where $v_e = v_e^{[a\to]}$ denotes the primitive (inward) vector spanning E and emanating from its specified vertex a.

Lemma 3.2.1. *Let ω be a 1-test form on $(\mathbb{C}^*)^n$, with support in a neighborhood of $\mathrm{Log}^{-1}(\{a\}) \subset (\mathbb{C}^*)^n$, with the restriction*

$$\omega_{|\mathrm{Log}^{-1}(\{a\})} = \sum_{j=1}^n \omega_j(t_1,\dots,t_n)\, dt_j \ .$$

Then

$$\langle d\mathscr{T}_e,\omega\rangle = \sum_{j=1}^n v_{e,j} \int_{\theta\in(\mathbb{R}/\mathbb{Z})^n} \omega_j\Big(v_{e,1}\theta_1+\sum_{\ell=1}^{n-1} u^e_{\ell,1}\theta_{\ell+1},\ \dots,\ v_{e,n}\theta_1+\sum_{\ell=1}^{n-1} u^e_{\ell,n}\,\theta_{\ell+1}\Big) d\theta_1\cdots d\theta_n\,. \tag{3.2.5}$$

Proof. By definition of differentiation of currents and Stokes' formula, it follows that, for such ω,

$$\begin{aligned}\langle d\mathscr{T}_e,\omega\rangle &:= -\int_{(\theta_2,\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}} \big\langle [\Delta_{e,D_e,a}(\theta_2,\dots,\theta_n)], d\omega\big\rangle\, d\theta_2\dots d\theta_n \\ &= \int_{(\theta_2,\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}} \big\langle [\partial\Delta_{e,D_e,a}(\theta_2,\dots,\theta_n)],\omega\big\rangle\, d\theta_2\dots d\theta_n,\end{aligned} \tag{3.2.6}$$

Note that the induced orientation on boundary of each fiber $\partial\Delta_{e,D_e,a}(\theta_2,\dots,\theta_n)$ is given by $-d\theta_1$, since this boundary is obtained by letting $\tau_1 = 1$ in (3.2.3). Moreover, for each fixed $(\theta_2,\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}$, $\partial\Delta_{e,D_e,a}(\theta_2,\dots,\theta_n)$ can be understood as the image of

$$\boldsymbol{\tau}_{e,D_e,a}^{(\theta_2,\dots,\theta_n)}(\mathbb{R}/\mathbb{Z}) := \boldsymbol{\tau}_{e,D_e,a}\big((\mathbb{R}/\mathbb{Z}),\theta_2,\dots,\theta_n\big), \tag{3.2.7}$$

where

$$\boldsymbol{\tau}_{e,D_e,a}^{(\theta_2,\ldots,\theta_n)}(\theta_1) := \boldsymbol{\tau}_{e,D_e,a}\big(\theta_1,\theta_2,\ldots,\theta_n\big).$$

Therefore,

$$\langle d\mathscr{T}_e,\omega\rangle = \int_{(\theta_2,\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}}\int_{\theta_1\in(\mathbb{R}/\mathbb{Z})}\big(\boldsymbol{\tau}_{e,D_e,a}^{(\theta_2,\ldots,\theta_n)}\big)^*(\omega)\,.$$

It is clear that

$$\big(\boldsymbol{\tau}_{e,D_e,a}^{(\theta_2,\ldots,\theta_n)}\big)^*(t_j) = t_j\circ\big(\boldsymbol{\tau}_{e,D_e,a}^{(\theta_2,\ldots,\theta_n)}\big) = v_{e,j}\theta_j+\sum_{\ell=1}^{n-1}u_{\ell,j}^e\theta_{\ell+1}, \tag{3.2.8}$$

and

$$\big(\boldsymbol{\tau}_{e,D_e,a}^{(\theta_2,\ldots,\theta_n)}\big)^*(dt_j) = d\Big(t_j\circ\big(\boldsymbol{\tau}_{e,D_e,a}^{(\theta_2,\ldots,\theta_n)}\big)\Big) = d(v_{e,j}\theta_1+\sum_{\ell=1}^{n-1}u_{\ell,j}^e\theta_{\ell+1}) = v_{e,j}\,d\theta_1\,,$$

which easily give the result. □

The next lemma relates the balancing condition to closedness of the corresponding currents. Suppose every edge e of Γ is weighted by a non-zero integer m_e. Then, one has the following lemma.

Lemma 3.2.2. *Let $\mathcal{P}$ a weighted rational 1-polyedral complex in $\mathbb{R}^n$, $\{a\}$ be one of its vertices and ω be a 1-test form in $(\mathbb{C}^*)^n$ supported in an open neighborhood of $\operatorname{Log}^{-1}(\{a\})$. One has*

$$\langle d\mathscr{T}_n^1(\mathcal{P}),\omega\rangle = \sum_{\{e\in\mathcal{C}_1(\mathcal{P})\,;\,\{a\}\prec e\}} m_e\,\langle d\mathscr{T}_e\,,\omega\rangle = 0 \quad\Longleftrightarrow\quad \sum_{\{e\in\mathcal{C}_1(\mathcal{P})\,;\,\{a\}\prec e\}} m_e v_e^{[a\to]} = 0\,, \tag{3.2.9}$$

where $\{a\}\prec e$ means that $\{a\}$ is a vertex of the edge e and $v_e^{[a\to]}$ denotes then the inward primitive vector contained in the edge e and pointing away from a ; In particular, the tropical current $\mathscr{T}_n^1(V_{\mathbb{T}})$ attached to a tropical curve $V_{\mathbb{T}}$ is closed.

Proof. To prove the lemma it is enough to check the result for any 1-test form ω in a neighborhood of $\operatorname{Log}^{-1}(\{a\})$ in $(\mathbb{C}^*)^n$ such that $\omega = e^{2i\pi\langle\nu,\theta\rangle}\,d\theta_j$ for some $j\in\{1,...,n\}$ and $\nu\in\mathbb{Z}^n$. This follows from the fact that the characters $\theta\mapsto\chi_{n,\nu}(\theta) := e^{2i\pi\langle\nu,\theta\rangle}$ ($\nu\in\mathbb{Z}^n$) form an orthonormal basis for the Hilbert space $L^2_{\mathbb{C}}((\mathbb{R}/\mathbb{Z})^n,d\theta)$. Then the equivalence stated here follows from the formula (3.2.5) established in Lemma 3.2.1. The second claim follows from the fact that the balancing condition is fulfilled at any vertex $\{a\}$ of any tropical curve $V_{\mathbb{T}}$. □

Recall that for a tropical curve $\Gamma \subset \mathbb{R}^n$ strong extremality means $(n+1)$-valency for any vertex $\{a\}$ and sub-independency of the set whose elements are the $(n+1)$ primitive vectors $v_e^{[a\to]}$ ($e \in \mathcal{C}_1(\Gamma)$ such that $\{a\} \prec e$).

Theorem 3.2.3. *Let $\Gamma \subset \mathbb{R}^n$ be a strongly extremal tropical curve. Then the $(1,1)$-dimensional closed current normal $\mathscr{T}_n^1(\Gamma)$ is strongly extremal in $\mathcal{D}'_{1,1}(\mathbb{C}^*)^n$.*

We first prove Theorem 3.2.3 for a tropical curve Γ which has only one vertex.

Lemma 3.2.4. *Suppose that $\Gamma \in \mathbb{R}^n$ is a tropical curve with only one vertex at the origin. Then $\mathscr{T}_n^1(\Gamma)$ is strongly extremal in $\mathcal{D}'_{1,1}(\mathbb{C}^*)^n$ if and only if Γ is strongly extremal*

Proof. The proof of the lemma is divided into five steps.

Each edge $e \in \mathcal{C}_1(\Gamma)$ is contained in an affine line E. For such a E consider $w = v_e$ the inward primitive vector $w = v_e^{[0\to]}$ initiated from the vertex $\{0\}$, lying in E. We fix an arbitrary completion D_e of $\{v_e\}$ with vectors $u_1^e, ..., u_{n-1}^e$ in $\mathbb{Z}^n$. One has

$$\mathscr{T}_1^n(\Gamma) = \sum_{e \in \mathcal{C}_1(\Gamma)} m_e\, T_{e,D_e,\{0\}}$$

as seen in Section 3.2 above.

We assume from now on that $\widetilde{\mathscr{T}}$ is a $(1,1)$-dimensional normal closed current in $(\mathbb{C}^*)^n$ with support equal to that of $\mathscr{T}_1^n(\Gamma)$, *i.e.* $\operatorname{Supp}(\widetilde{\mathscr{T}}) = \operatorname{Log}^{-1}(\Gamma)$.

Step 1. For any $e \in \mathcal{C}_1(\Gamma)$, let $\mathcal{U}_e$ be the open subset of $(\mathbb{C}^*)^n$ defined as

$$\mathcal{U}_e := \operatorname{Log}^{-1}\big(\mathbb{R}^n \setminus \bigcup_{\substack{e' \in \mathcal{C}_1(\Gamma)\\ e' \neq e}} |e'|\big).$$

It follows from Theorem 1.4.2 that, for each $e \in \mathcal{C}_1(\Gamma)$, there is a Radon measure $d\mu_e$ on $(\mathbb{R}/\mathbb{Z})^{n-1}$ such that (as currents in the open subset $\mathcal{U}_e$ of $(\mathbb{C}^*)^n$) :

$$\widetilde{\mathscr{T}}_{|\mathcal{U}_e} = \int\limits_{(\theta_2,\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}} [\Delta_{e,D_e,0}(\theta_2, ..., \theta_n)]\, d\mu_e(\theta_2, ..., \theta_n).$$

Since the normal current $\widetilde{\mathscr{T}}_{|\mathcal{U}_e}$ extends globally as the $(1,1)$-dimensional normal closed current $\widetilde{\mathscr{T}}$ in the whole ambient manifold $(\mathbb{C}^*)^n$, one can certainly define $(1,1)$-dimensional normal current $\widetilde{\mathscr{T}_e}$ in $(\mathbb{C}^*)^n$ as

$$\widetilde{\mathscr{T}_e} := \int\limits_{(\theta_2,\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}} [\Delta_{e,D_e,0}(\theta_2, ..., \theta_n)]\, d\mu_e(\theta_2, ..., \theta_n). \tag{3.2.10}$$

The support of $\widetilde{\mathscr{T}_e}$ equals $\text{Log}^{-1}(e)$, which implies that all currents $\widetilde{\mathscr{T}_{e'}}$ (for $e' \in \mathcal{C}_1(\Gamma)$ such that e' is distinct from e) vanish in U_e. Hence $\widetilde{\mathscr{T}} = \sum_{e\in\mathcal{C}_1(\Gamma)} \widetilde{\mathscr{T}_e}$ in each U_e. Hence the normal current $\widetilde{\mathscr{T}} - \sum_{e\in\mathcal{C}_1(\Gamma)} \widetilde{\mathscr{T}_e}$ is supported by $\text{Log}^{-1}(\{0\}) \simeq (\mathbb{R}/\mathbb{Z})^n$ with Cauchy-Riemann dimension 0. It follows then from Theorem 1.4.1 that one has the decomposition :

$$\widetilde{\mathscr{T}} = \sum_{e\in\mathcal{C}_1(\Gamma)} \widetilde{\mathscr{T}_e}$$

(as currents this time in the whole ambient space $(\mathbb{C}^*)^n$).

Remark 3.2.5. Although the current $\mathscr{T}_n^1(\Gamma)$ is not dependent on completions of v_e to lattice bases D_e, the representation in (3.2.10) is. The representation, indeed depends on the chosen foliation which comes from the completions D_e of v_e to a $\mathbb{Z}$-basis for every edge e of Γ. Therefore as mentioned before, at this point we need to fix a lattice basis for each of the edges of the tropical graph.

Step 2. One can repeat the proof of Lemma 3.2.1 for each edge $e \in \mathcal{C}_1(\Gamma)$ and use the expression (3.2.10) of $\widetilde{\mathscr{T}_e}$, in order to get the following result.

Lemma 3.2.6. *Let ω be a 1-test form on $(\mathbb{C}^*)^n$, with support in a neighborhood of $\text{Log}^{-1}(\{0\})$ with restriction given by*

$$\omega_{|\text{Log}^{-1}(\{0\})} = \sum_{j=1}^{n} \omega_j(t_1, ..., t_n)\, dt_j.$$

Then

$$\langle d\widetilde{\mathscr{T}_e}, \omega\rangle = \sum_{j=1}^{n} v_{e,j} \times$$
$$\int_{\theta\in(\mathbb{R}/\mathbb{Z})^n} \omega_j\Big(v_{e,j}\theta_1 + \sum_{\ell=1}^{n-1} u^e_{\ell,1}\theta_{\ell+1},\, \dots, v_{e,n}\theta_1 + \sum_{\ell=1}^{n-1} u^e_{\ell,n}\,\theta_{\ell+1}\Big)\, d\theta_1 \otimes d\mu_e(\theta_2, ..., \theta_n)\,. \tag{3.2.11}$$

Step 3. The current $\widetilde{\mathscr{T}} = \sum_{e\in\mathcal{C}_1(\Gamma)} \widetilde{\mathscr{T}_e}$ is closed by hypothesis. We try to fully exploit this property in order to derive information on the measures μ_e, $e \in \mathcal{C}_1(\Gamma)$. To do that, we use the fact that a Radon measure $d\mu$ on the group $(\mathbb{R}/\mathbb{Z})^{n-1}$ is characterized[1] by the complete

[1] These Fourier coefficients for measures are obtained by action of the measures on a Fourier basis of the Hilbert space $L^2_{\mathbb{C}}((\mathbb{R}/\mathbb{Z})^{n-1}, d\theta)$. Thus, two measures on a torus coincide if and only if their actions on each element of this basis coincide. In consequence, the list of Fourier coefficients characterizes the measures on tori.

list of its Fourier coefficients

$$\widehat{\mu}(\nu) = \int_{[0,1]^{n-1}} \chi_{n-1,\nu}(\theta_2, ..., \theta_n)\, d\theta_2 \dots d\theta_n := \int_{[0,1]^{n-1}} \exp(-i\langle \nu, \theta\rangle)\, d\theta_2 \dots d\theta_n$$
$$(\nu \in \mathbb{Z}^{n-1}).$$

Fix $e \in \mathcal{C}_1(\Gamma)$. Let $\omega_\nu^{[1]}$ be a 1-test form on $(\mathbb{C}^*)^n$, with support in a neighborhood of $\mathrm{Log}^{-1}(\{0\})$ such that its restriction is given by

$$(\omega_\nu^{[1]})_{|\mathrm{Log}^{-1}(\{0\})} = \chi_{n,\nu}(t_1, ..., t_n)\, dt_1.$$

After simplifications, (3.2.11) reduces to the scalar equation :

$$\langle d\, \widetilde{\mathscr{T}_e}, \omega_\nu^{[1]}\rangle = \delta^0_{\langle \nu, v_e\rangle}\, \widehat{\mu}_e\big(-\langle \nu, u_1^e\rangle, \dots, -\langle \nu, u_{n-1}^e\rangle\big)\, v_{e,1} \tag{3.2.12}$$

(δ^η_α denotes here the Kronecker's symbol). Since $\widetilde{\mathscr{T}}$ is closed we conclude, after performing the same computations for all e in $\mathcal{C}_1(\Gamma)$, that

$$0 = \langle d\, \widetilde{\mathscr{T}}, \omega_\nu^{[1]}\rangle = \sum_{e\in\mathcal{C}_1(\Gamma)} \langle d\, \widetilde{\mathscr{T}_e}, \omega_\nu^{[1]}\rangle = \sum_{e\in\mathcal{C}_1(\Gamma)} \delta^0_{\langle \nu, v_e\rangle}\, \widehat{\mu}_e\big(-\langle \nu, u_1^e\rangle, \dots, -\langle \nu, u_{n-1}^e\rangle\big)\, v_{e,1}\,. \tag{3.2.13}$$

If one performs the same operations when $\omega_\nu^{[1]}$ is replaced by $\omega_\nu^{[j]}$ $(1 \le j \le n)$ such that

$$(\omega_\nu^{[j]})_{|\mathrm{Log}^{-1}(\{0\})} = \chi_{n,\nu}(t_1, ..., t_n)\, dt_j,$$

one gets the vectorial equation

$$\sum_{e\in\mathcal{C}_1(\Gamma)} \delta^0_{\langle \nu, v_e\rangle} \widehat{\mu}_e\big(-\langle \nu, u_1^e\rangle, \dots, -\langle \nu, u_{n-1}^e\rangle\big)\, v_e = 0. \tag{3.2.14}$$

Step 4. Now assume that Γ is strongly extremal. Equation (3.2.14) implies the two following facts :

- Taking $\nu = (0, \dots, 0)$ leads to

$$\sum_{e\in\mathcal{C}_1(\Gamma)} \widehat{\mu}_e(0, \dots, 0)\, v_e = 0.$$

  Recall that the balancing condition $\sum_{e\in\mathcal{C}^1(\Gamma)} m_e\, v_e = 0$ is also satisfied, it follows from the sub-independency hypothesis (see Remark 3.1.6) that there exists a complex number ρ such that

$$\widehat{\mu}_e(0, \dots, 0) = \rho\, m_e \quad \forall\, e \in \mathcal{C}_1(\Gamma).$$

- Let $\ell = (\ell_2, \dots, \ell_n) \neq (0, \dots, 0)$ be an arbitrary non-zero integral vector. Fix $e \in \mathcal{C}_1(\Gamma)$. There exists a unique $\nu_e \in \mathbb{Z}^n$ such that at the same time $\langle \nu_e, v_e \rangle = 0$ and $\langle \nu_e, u_j^e \rangle = -\ell_{j+1}$ for $j = 1, ..., n-1$, since $D_e = \{v_e, u_1^e, ..., u_{n-1}^e\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Since the graph Γ is $(n+1)$-valent and the $(n+1)$-primitive vectors $v_{e'}$ $(e' \in \mathcal{C}_1(\Gamma))$ affinely span the whole $\mathbb{R}^n$, there exists at least one edge $e'[e]$ (distinct from e) of Γ such that $\langle \nu_e, v_{e'[e]} \rangle \neq 0$, thus $\delta^0_{\langle \nu_e, v_{e'[e]} \rangle} = 0$. Therefore, in view of Remark 3.1.6, all of the coefficients involved in the vectorial equation (3.2.14) must vanish, as well as $\delta^0_{\langle \nu_e, v_e \rangle} \widehat{\mu}_e(\ell) = \widehat{\mu}_e(\ell)$, that is $\widehat{\mu}_e(\ell) = 0$. Consequently, for every $0 \neq \ell \in \mathbb{Z}^{n-1}$, we have $\widehat{\mu}_e(\ell) = 0$.

It means that every $d\mu_e$ $(e \in \mathcal{C}_1(\Gamma))$ is a Lebesgue measure given by $d\mu_e(\theta_2, ..., \theta_n) = \rho\, m_e\, d\theta_2 \dots d\theta_n$, and therefore, $\widetilde{\mathscr{T}} = \rho \mathscr{T}_n^1(\Gamma)$. This concludes to the strong extremality of $\mathscr{T}_n^1(\Gamma)$.

Step 5. Assume that Γ is not strongly extremal. Then following Step 4, there are $\widehat{\mu}_e(\ell_2, \dots, \ell_n) \neq 0$ with $(\ell_2, \dots, \ell_n) \neq 0$ satisfying 3.2.14. For such Fourier coefficients, the current

$$\widetilde{\mathscr{T}} = \sum_e \int_{(\theta_2, \dots, \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-1}} [\Delta_{e, D_e, 0}(\theta_2, \dots, \theta_n)] d\mu_e(\theta_2, \dots, \theta_n),$$

cannot be a multiple of $\mathscr{T}_n^1(\Gamma)$.

□

Now it is easy to prove the Theorem 3.2.3.

Proof of Theorem 3.2.3. Let $\widetilde{\mathscr{T}}$ be a closed $(1,1)$-dimensional normal current with support $\mathrm{Log}^{-1}(|\Gamma|)$. For any vertex a of Γ there is an open neighborhood $\mathcal{V}_a$ of a in $\mathbb{R}^n$ which does not contain any other vertex of the tropical curve Γ. We are thus reduced to the situation of a tropical curve with just one vertex. It follows then from the Lemma 3.2.4 (the reasoning may be applied locally, in the open set $\mathrm{Log}^{-1}(\mathcal{V}_a)$ instead as in $\mathrm{Log}^{-1}(\mathbb{R}^n) = (\mathbb{C}^*)^n$) that for each vertex $\{a\}$ of Γ there is a complex number ρ_a such that

$$\widetilde{\mathscr{T}}_{|_{\mathrm{Log}^{-1}(\mathcal{V}_a)}} = \rho_a\, \mathscr{T}_n^1(\Gamma)_{|_{\mathrm{Log}^{-1}(\mathcal{V}_a)}}.$$

Similarly for an adjacent vertex $\{b\}$, we can write for some complex number ρ_b

$$\widetilde{\mathscr{T}}_{|_{\mathrm{Log}^{-1}(\mathcal{V}_b)}} = \rho_b\, \mathscr{T}_n^1(\Gamma)_{|_{\mathrm{Log}^{-1}(\mathcal{V}_b)}}.$$

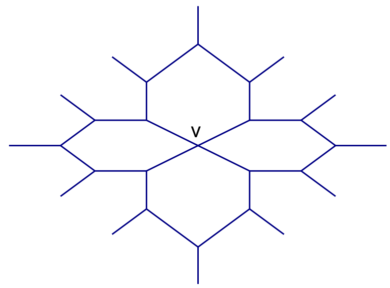

Figure 3.1: $\Gamma \subset \mathbb{R}^2$ is not strongly extremal, however $\mathscr{T}_2^1(\Gamma)$ is.

On the other hand, if $\{a\}$ and $\{b\}$ are connected via the edge e, then we have, using the notations from the previous lemma, that

$$\widetilde{\mathscr{T}_e} = \rho_a \, m_e \, \mathscr{T}_e = \rho_b \, m_e \, \mathscr{T}_e$$

(as currents in some open neighborhood of $\operatorname{Log}^{-1}(e \setminus \{a, b\})$ in $(\mathbb{C}^*)^n$). Hence $\rho_a = \rho_b$. Since Γ is strongly extremal and thus connected, one can show that all ρ_a are indeed equal by taking a chain of successive adjacent vertices from $\{a\}$ to an arbitrary other vertex. □

Remark 3.2.7. There are tropical graphs Γ which are not strongly extremal but the associated tropical current is strongly extremal. For instance in Figure 3.1 the graph $\Gamma \subset \mathbb{R}^2$ has a 4-valent, so it is not a strongly extremal graph. However, any edge incident to v is connected also to a 3-valent vertex, and all 3-valent are connected via a path of 3-valent vertices. Thus, (with the notation of the preceding proof) for all $e \in \mathcal{C}_1(\Gamma)$, the ρ_e coincide.

3.3 Tropical (p, p)-dimensional currents

In this section we prove the Theorem 3.1.8. We start by treating the simplest case, namely when $V_{\mathbb{T}} \subset \mathbb{R}^n$ is a tropical p-cycle with only one facet W. Note that such a hypothesis implies that this facet is in fact an affine $(p-1)$-plane in $\mathbb{R}^n$ and that all p-cells are of the form $[0, \infty[\times v_P + W$ for some primitive inward vector $v_P = v_P^{[W \to]}$. Let us analyze the current $\mathscr{T}_n^p(V_{\mathbb{T}})$ in that particular case. Assume that W (which is here assumed to be the sole facet of $V_{\mathbb{T}}$) passes through the origin and is the common facet of the p-dimensional polyhedra $P_1, \ldots, P_s$, $s \geq 3$, with corresponding weights m_P. We have already shown in the beginning of Section 3 that in the definition of $\mathscr{T}_n^p(V_{\mathbb{T}})$

$$\mathscr{T}_n^p(V_{\mathbb{T}}) = \sum_{P\in\{P_1,\dots,P_s\}} m_P\,\mathscr{T}_P,$$

is independent of the choice of the base point, and $\mathbb{Z}$-bases for $P\cap\mathbb{Z}^n$ as well as their completions to $\mathbb{Z}$-bases of $\mathbb{Z}^n$. Accordingly, we choose $\{w_1,\dots,w_{p-1}\}$ a $\mathbb{Z}$-basis for $W\cap\mathbb{Z}^n$ and for each $P\in\{P_1,\dots,P_s\}$, we choose the inward primitive vector $v_P = v_P^{[W\to]}\in\mathbb{Z}^n$ pointing inward P from the origin such that $\{w_1,\dots,w_{p-1},v_P\}$ is a $\mathbb{Z}$-basis for $H_P\cap\mathbb{Z}^n$ where H_P is the p-plane containing P. Also, the balancing condition means (see Remark 2.1.2) that every $p\times p$ minor of the $n\times p$ matrix of columns $\big(w_1,\dots,w_{p-1},\sum_P m_P v_P\big)$ vanishes. Equivalently, this implies that under the projection along W, $h_W:\mathbb{R}^n\to\mathbb{R}^{n-p+1}$, we have

$$\sum_P m_p\,h_W(v_P) = 0. \tag{3.3.15}$$

Furthermore, we extend each $\{w_1,\dots,w_{p-1},v_P\}$ to

$$D_P = \{w_1,\dots,w_{p-1},v_P,u_1^P,\dots,u_{n-p}^P\},$$

a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Correspondingly, we define the open subset S_{D_P} of $\operatorname{Log}^{-1}(H_P)$ by

$$\begin{aligned} S_{D_P} = \Big\{&(\tau_1^{w_1}\star\cdots\star\tau_{p-1}^{w_{p-1}}\star\tau^{v_P}\star\exp(2i\pi\theta_{p+1}u_1^P)\star\cdots\star\exp(2i\pi\theta_n u_{n-p}^P)\ ;\\ &(\tau_1,\dots,\tau_{p-1},\tau)\in(\mathbb{C}^*)^p,\ |\tau|>1,\ (\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}\Big\}. \end{aligned} \tag{3.3.16}$$

Each S_{D_P} (which is a $(n+p)$-dimensional real manifold) is injectively foliated over $(\mathbb{R}/\mathbb{Z})^{n-p}$ through the submersion

$$\begin{array}{c} \tau_1^{w_1}\star\cdots\star\tau^{w_{p-1}}\star\tau^{v_P}\star\exp(2i\pi\theta_{p+1}u_1^P)\star\cdots\star\exp(2i\pi\theta_n u_{n-p}^P)\in S_{D_P} \\ \Big\downarrow{\scriptstyle\sigma_P} \\ (\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}. \end{array}$$

We again denote the fiber over $(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}$, as $\Delta_P(\theta_{p+1},\dots,\theta_n)$. Note that the complex dimension of each $\Delta_P(\theta_{p+1},\dots,\theta_n)$ is p and that the boundary of such a fiber is a real analytic $(2p-1)$-cycle. Denote by $\boldsymbol{\tau}_P$ the parameterization map

$$\begin{aligned} &\boldsymbol{\tau}_P\ :(\tau,\theta_{p+1},\dots,\theta_n)\in(\mathbb{C}^*)^p\times(\mathbb{R}/\mathbb{Z})^{n-p}\mapsto\\ &\tau_1^{w_1}\star\dots\tau_{p-1}^{w_{p-1}}\star\tau_p^{v_P}\star\exp(2i\pi\theta_{p+1}u_1^P)\star\cdots\star\exp(2i\pi\theta_n u_{n-p}^P)\in(\mathbb{C}^*)^n. \end{aligned}$$

Identifying $\mathbb{C}^{p-1}\times(\mathbb{R}/\mathbb{Z})\simeq(\mathbb{R}^+)^{p-1}\times(\mathbb{R}/\mathbb{Z})^p$, $\partial\Delta_P(\theta_{p+1},\dots,\theta_n)$ (with orientation induced by $-d\theta_p$) can be therefore understood as the image

$$\begin{aligned}\boldsymbol{\tau}_P\big((\mathbb{R}^+)^{p-1}\times(\mathbb{R}/\mathbb{Z})^p\times\{(\theta_{p+1},\dots,\theta_n)\}\big)&\\ =: \boldsymbol{\tau}_P^{(\theta_{p+1},\dots,\theta_n)}\big((\mathbb{R}^+)^{p-1}\times(\mathbb{R}/\mathbb{Z})^p\big).&\end{aligned} \quad (3.3.17)$$

By definition of the tropical current associated to $V_{\mathbb{T}}$,

$$\mathscr{T}_n^p(V_{\mathbb{T}}) \;=\sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P\,\mathscr{T}_P \;=\sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P\int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}}[\Delta_P(\theta_{p+1},...,\theta_n)]\,d\theta_{p+1}\dots d\theta_n\,.$$

Lemma 3.3.1. *Let $V_{\mathbb{T}}\subset\mathbb{R}^n$ be a p-tropical cycle such that $\mathcal{C}_{p-1}(V_{\mathbb{T}})=\{W\}\ni\{0\}$. The current $\mathscr{T}_n^p(V_{\mathbb{T}})$, which can then be decomposed as*

$$\mathscr{T}_n^p(V_{\mathbb{T}}) = \sum_{P\in C_p(V_{\mathbb{T}})} m_P\int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}}[\Delta_P(\theta_{p+1},...,\theta_n)]\,d\theta_{p+1}\dots d\theta_n$$

is a closed (p,p)-dimensional current. Moreover, assume $\mathrm{card}(C_p(V_{\mathbb{T}}))=n-p+2$*, with inward vectors v_P in P, that make the balancing condition hold. If the set of vectors $\{h_W(v_P)\,;\,P\in C_p(V_{\mathbb{T}})\}$, h_W being the projection along W, is linearly sub-independent, then $\mathscr{T}_n^p(V_{\mathbb{T}})$ is strongly extremal.*

Proof. The proof is similar to that of Lemmas 3.2.2 and 3.2.4. Consider a $(2p-1)$-test form $\omega_{\eta,\nu}^{[K,J]}$ in $\mathbb{C}^*$ such that in a neighborhood of the $\mathrm{Log}^{-1}(0)\subset\mathrm{Log}^{-1}(W)$ is expressed in polar coordinates $z_j=r_j\,e^{2i\pi t_j}$, $j=1,...,n$, by

$$\omega_{\eta,\nu}^{[K,J]}(r_1,...,r_n,t_1,\dots,t_n) = \eta(r_1,...,r_n)\,\chi_{n,\nu}(t_1,\dots,t_n)\bigwedge_{k\in K}dr_k\wedge\bigwedge_{j\in J}dt_j,$$

where $K\subset J\subset\{1,...,n\}$ with $|K|=p-1\,,|J|=p\,,$ η a test function in $r=(r_1,\dots,r_n)$. Also $\nu\in\mathbb{Z}^n$ and $\chi_{n,\nu}$ denotes as before the character $t=(t_1,\dots,t_n)\mapsto\exp(2i\pi\langle\nu,t\rangle)$ on the torus $(\mathbb{R}/\mathbb{Z})^n$. Thanks to Fourier analysis looking at the application of $\mathscr{T}_n^p(V_{\mathbb{T}})$ on these forms one can extract all information needed in order to verify closedness as well as extremality of $\mathscr{T}_n^p(V_{\mathbb{T}})$.

By definition of the exterior derivative of a current and the Stokes' formula, taking into account the orientation induced on the boundary of each $\Delta_P(\theta_{p+1},\dots,\theta_n)\,,$

$$\begin{aligned}&\big\langle d\mathscr{T}_n^p(V_{\mathbb{T}}),\omega_{\eta,\nu}^{[K,J]}\big\rangle = \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P\big\langle\, d\mathscr{T}_P,\omega_{\eta,\nu}^{[K,J]}\big\rangle =\\ &\int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}}\sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P\big\langle\partial\Delta_P(\theta_{p+1},\dots,\theta_n),\omega_{\eta,\nu}^{[K,J]}\big\rangle\,.\end{aligned} \quad (3.3.18)$$

Therefore by (3.3.17),

$$\big\langle d\mathscr{T}_n^p(V_{\mathbb{T}}), \omega_{\eta,\nu}^{[K,J]}\big\rangle = \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P\Big(\int_{(\mathbb{R}^+)^{p-1}\times(\mathbb{R}/\mathbb{Z})^p} (\boldsymbol{\tau}_P^{(\theta_{p+1},\dots,\theta_n)})^*\big(\eta(r)\,dr_K\wedge\chi_{n,\nu}(t)\,dt_J\big)\Big)\,. \tag{3.3.19}$$

A computation similar to Lemma 3.2.1 and (3.2.13) gives the scalar equation

$$\begin{aligned}\big\langle d\mathscr{T}_n^p(V_{\mathbb{T}}), \omega_{\eta,\nu}^{[K,J]}\big\rangle &= \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P\,\times\, M_{\eta,K,W}\times\\ &\times\,\Big(\prod_{\ell=1}^{p-1}\delta^0_{\langle\nu,w_\ell\rangle}\Big)\delta^0_{\langle\nu,v_P\rangle}\,\delta^0_{\langle\nu,u_1^P\rangle}\cdots\delta^0_{\langle\nu,u_{n-p}^P\rangle}\,\mathrm{Det}_J\big(w_1,\dots,w_{p-1},v_P\big).\end{aligned} \tag{3.3.20}$$

In the above $M_{\eta,K,W}$ is a constant coming from integration of $\big(\boldsymbol{\tau}_P^{(\theta_{p+1},\dots,\theta_n)})^*\big(\eta(r)\,dr_K\big)$ depending on η, K, W which can be chosen to be 1. $\mathrm{Det}_J\big(w_1,\dots,w_{p-1},v_P\big)$ denotes the $p\times p$ minor of the $n\times p$ matrix $\big(w_1,\dots,w_{p-1},v_P\big)$ corresponding to the rows with indices $j\in J$, this term appears from $\big(\boldsymbol{\tau}_P^{(\theta_{p+1},\dots,\theta_n)})^*\big(dt_J\big)$ in (3.3.19) (compare to (3.2.8)). For any $0\neq\nu\in\mathbb{Z}^n$, (3.3.20) becomes zero. Assuming $\nu=0$, yields

$$\big\langle d\mathscr{T}_n^p(V_{\mathbb{T}}), \omega_{\eta,\nu}^{[K,J]}\big\rangle = 0 \quad \text{if and only if}\quad \mathrm{Det}_J\big(w_1,\dots,w_{p-1},\sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P\,v_P\big)=0,\quad \forall\,J\subset\{1,\dots,n\}, |J|=p\,.$$

The latter equation thus implies the equivalence of d-closedness of $\mathscr{T}_n^p(V_{\mathbb{T}})$ and the balancing condition.

For any $P\in\mathcal{C}_p(V_{\mathbb{T}})$, let $\mathcal{U}_P$ be the open subset of $(\mathbb{C}^*)^n$ defined as

$$\mathcal{U}_P := \mathrm{Log}^{-1}\Big(\mathbb{R}^n\setminus\bigcup_{\substack{P'\in\mathcal{C}_p(V_{\mathbb{T}})\\ P'\neq P}}|P'|\Big)$$

Suppose that $\widetilde{\mathscr{T}}$ is a (p,p)-dimensional normal current in $(\mathbb{C}^*)^n$ with support exactly $\mathrm{Log}^{-1}(V_{\mathbb{T}})$. As in the 1-dimensional case, by Theorem 1.4.2, for any $P\in\mathcal{C}_p(V_{\mathbb{T}})$, there exists a unique Radon measure $d\mu_P$ on $(\mathbb{R}/\mathbb{Z})^{n-p}$ such as (as currents in the open subset $\mathcal{U}_P\subset(\mathbb{C}^*)^n$) one has

$$\widetilde{\mathscr{T}}_{|\mathcal{U}_P} = \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_{P,D_P,W}(\theta_{p+1},\dots,\theta_n)]\,d\mu_P(\theta_{p+1},\dots,\theta_n).$$

Since $\widetilde{\mathscr{T}}_{|\mathcal{U}_P}$ extends globally as the normal closed current to the whole of $(\mathbb{C}^*)^n$, one defines normal currents $\widetilde{\mathscr{T}_P}$ on $(\mathbb{C}^*)^n$ by setting

$$\widetilde{\mathscr{T}_P} := \int_{(\theta_{p+1},\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_{P,D_P,W}(\theta_{p+1},...,\theta_n)]\, d\mu_P(\theta_{p+1},...,\theta_n).$$

The normal (p,p)-dimensional current $\widetilde{\mathscr{T}} - \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} \widetilde{\mathscr{T}_P}$, which is supported by $\mathrm{Log}^{-1}(W)$, equals zero for dimension reasons thanks to theorem 1.4.1, so that one has (as currents in $(\mathbb{C}^*)^n$ this time) the representation (which indeed depends on the chosen foliation):

$$\widetilde{\mathscr{T}} = \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} \int_{(\theta_{p+1},\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_{P,D_P,W}(\theta_{p+1},...,\theta_n)]\, d\mu_P(\theta_{p+1},...,\theta_n).$$

Using the fact that $\langle d\widetilde{\mathscr{T}}, \omega_{\eta,\nu}^{[K,J]}\rangle = 0$ for any $\nu \in \mathbb{Z}^n$, any $K \subset J \subset \{1,\ldots,n\}$ with $|J| = |K| + 1 = p$, any test function η in r with non-zero integral leads to

$$\sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} \Big(\prod_{\ell=1}^{p-1} \delta^0_{\langle\nu,w_\ell\rangle}\Big)\delta^0_{\langle\nu,v_P\rangle}\, \widehat{\mu}_P\big(-\langle\nu,u_1^P\rangle,...,-\langle\nu,u_{n-p}^P\rangle\big)\,\mathrm{Det}_J(w_1,...,w_{p-1},v_P) = 0\,,$$
$$J\subset\{1,\ldots,n\},\ |J| = p\,. \quad (3.3.21)$$

Recall that by hypothesis the set $\{h_W(v_P)\,;\, P\in\mathcal{C}_p(V_{\mathbb{T}})\}$ is linearly sub-independent and spans $\mathbb{R}^n/W$ as an $\mathbb{R}$-basis, where h_W is the projection along W. The balancing condition also gives $\sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P h_W(v_P) = 0\,$.

Similar to the bottom of the proof of Lemma 3.2.4 we deduce :

- Choosing $\nu = 0$, together with sub-independency implies that there exists a complex number ρ such that $\widehat{\mu}_P(0,...,0) = \rho\, m_P$ by the Remarks 3.1.6 and 3.3.15 for every P.
- Now assume $(\ell_{p+1},...,\ell_n)\in\mathbb{Z}^{n-p}$ is any non-zero vector. Since for any P, the set $\{w_1,...,w_{p-1},v_P,u_1^P,...,u_{n-p}^P\}$ is a lattice basis of $\mathbb{Z}^n$, there exists a unique $\nu_P\in\mathbb{Z}^n\cap W^{\perp}$ such that $\langle\nu_P,v_P\rangle = 0$, $\langle\nu_P,u_j^P\rangle = -\ell_{p+j}$ for $j = 1,...,n-p$. However, for at least one $P'\neq P$, $\langle\nu_P,v_{P'}\rangle\neq 0$, and therefore $\delta^0_{\langle\nu_P,v_{P'}\rangle} = 0$ in (3.3.21). The sub-independency thus implies that $\widehat{\mu}_P(\ell_{p+1},...,\ell_n) = 0$ for every P.

Therefore $d\mu_P(\theta_{p+1}\ldots\theta_n) = \rho\, m_P\, d\theta_{p+1}\ldots d\theta_n$ for any $P\in\mathcal{C}_p(V_{\mathbb{T}})$. This proves the strong extremality of $\mathscr{T}_n^p(V_{\mathbb{T}})$ and ends the proof of the lemma. □

Proof of Theorem 3.1.8. Let $V_{\mathbb{T}}$ be a strongly extremal tropical p-cycle. Let P be a p-dimensional cell of the tropical p-cycle $V_{\mathbb{T}}$. The current $\mathscr{T}_P$ defined (in the preliminaries of Section 3) as

$$\mathscr{T}_P := (T_{H_P})_{|\mathrm{Log}^{-1}(\mathrm{int(P)})}$$

coincides with the current

$$\int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_{P,D_P,W}(\theta_{p+1},\dots,\theta_n)]\, d\theta_{p+1}\dots d\theta_n$$

about any point (in $(\mathbb{C}^*)^n$) which belongs to the $(n+p)$-dimensional real submanifold $\mathrm{Log}^{-1}(\mathrm{int}(P))$, where $\mathrm{int}(P)$ denotes the relative interior of P in the affine p-plane H_P (the argument is again the same as the one which has been invoked in the discussion preceding the Lemma 3.2.1). About any point $\mathrm{Log}^{-1}(a)$, where a lies in the relative interior (in the affine $(p-1)$-plane H_W) of a given facet W of P, the normal current $\sum_{W\prec P'} \mathscr{T}_{P'}$ coincides with the current

$$\sum_{W\prec P'} \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_{P',D_{P'},W}(\theta_{p+1},\dots,\theta_n)]\, d\theta_{p+1}\dots d\theta_n,$$

By Theorem 1.4.1. Since closedness of a current can be tested locally, it follows from the argument developed in the proof of Lemma 3.3.1 that the current

$$\mathscr{T}_n^p(V_{\mathbb{T}}) = \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P \mathscr{T}_P$$

is closed in a any compact neighborhood of any point $\mathrm{Log}^{-1}(a)$ in $\mathrm{Log}^{-1}(\mathrm{int}(W))$, W being an arbitrary facet of P, which in turn implies the closedness of $\mathscr{T}_n^p(V_{\mathbb{T}})$, noting that in light of Theorem 1.4.1 we need not to check the closedness for faces of codimension higher than 1. Suppose now that for each facet $W\in\mathcal{C}_{p-1}(V_{\mathbb{T}})$, for each $P\in\mathcal{C}_p(V_{\mathbb{T}})$ that shares W as a facet, the projection along W of primitive vectors $v_P^{[\to W]}$ form a linearly sub-independent set with cardinality $n-p+2$. Let $\widetilde{\mathscr{T}} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$ be a normal closed current with support $\mathrm{Log}^{-1}(V_{\mathbb{T}})$. If $W\in\mathcal{C}_{p-1}(V_{\mathbb{T}})$ and a is a point in the relative interior of W in H_W, the argument used in the proof of Lemma 3.3.1 also shows that there exists some complex number $\rho_{W,a}$ such that, in neighborhood of $\mathrm{Log}^{-1}(a)$ in $(\mathbb{C}^*)^n$, one has $\widetilde{\mathscr{T}} = \rho_{W,a}\mathscr{T}_n^p(V_{\mathbb{T}})$. Obviously, all $\rho_{W,a}$ (for a in the relative interior of an arbitrary facet W of $V_{\mathbb{T}}$) are equal to some complex number ρ_W. This implies that $\mathbb{1}_{\mathrm{Log}^{-1}(\mathrm{int}P)}\,\widetilde{\mathscr{T}} = \rho_W\, m_P\, \mathscr{T}_P$. If $W'\neq W$ is another facet of P we find a complex number $\rho_{W'}$ such that

$$\mathbb{1}_{\mathrm{Log}^{-1}(\mathrm{int}P)}\,\widetilde{\mathscr{T}} = \rho_{W'}\, m_P\, \mathscr{T}_P\,,$$

and $\rho_W = \rho_{W'}$ is imposed. Connectivity of $V_{\mathbb{T}}$ in codimension 1 (as in the final step in the proof of Theorem 3.2.3) shows that all numbers ρ_W ($W \in \mathcal{C}_{p-1}(V_{\mathbb{T}})$) coincide (note that higher codimensional connectivity is not sufficient). This concludes the proof of the strong extremality of the current $\mathscr{T}_n^p(V_{\mathbb{T}})$. $\square$

3.4 Tropical currents in $\mathcal{D}'_{p,p}(\mathbb{CP}^n)$

We first show that for a given effective tropical p-cycle $V_{\mathbb{T}}$ in $\mathbb{R}^n$ the closed positive (p,p)-dimensional current $\mathscr{T}_n^p(V_{\mathbb{T}})$ (considered as a current in $(\mathbb{C}^*)^n$) can be extended by zero to a closed positive (p,p)-dimensional current in $\mathbb{CP}^n$.

Lemma 3.4.1. *For any effective tropical p-cycle $V_{\mathbb{T}}$ in $\mathbb{R}^n$, the positive tropical current $\mathscr{T}_n^p(V_{\mathbb{T}}) \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$ can be extended by zero to $\mathbb{CP}^n$ as a current $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}})$ in $\mathcal{D}'_{p,p}(\mathbb{CP}^n)$. Moreover, if $\mathscr{T}_n^p(V_{\mathbb{T}}) \in \mathcal{E}^p((\mathbb{C}^*)^n)$, then $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}}) \in \mathcal{E}^p(\mathbb{CP}^n)$.*

Proof. Let $(\zeta_1, \dots, \zeta_n)$ be the coordinates on the complex torus $(\mathbb{C}^*)^n$. Assume $P \in \mathcal{C}_p(V_{\mathbb{T}})$, and without loss of generality that $0 \in \mathrm{int}P$. The current T_{H_P} is expressed as the average

$$T_{H_P} = T_{H_P,D_P} = \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_{H_P,D_P}(\theta_{p+1},...,\theta_n)]\, d\theta_{p+1}\dots d\theta_n$$

(see formula (3.1.2)). For each $(\theta_{p+1},...,\theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}$, the complex p-dimensional analytic variety $\Delta_{H_P,D_P}(\theta_{p+1},...,\theta_n)$ is included in the toric subset of $(\mathbb{C}^*)^n$ defined in the coordinates $(\zeta_1,...,\zeta_n)$ by the set of binomial equations

$$\prod_{j=1}^n \zeta_j^{\xi^+_{\ell,j}} - \prod_{j=1}^n (\gamma_j(\theta,U_P,a))^{-\xi_{\ell,j}}\,\zeta_j^{\xi^-_{\ell,j}} = 0, \quad \ell = 1,...,n-p,$$

where the $\xi_\ell = \xi_\ell^+ - \xi_\ell^-$ form a set of generators for $\mathrm{Ker}_{B_P^t} \cap \mathbb{Z}^n$ and

$$\gamma_j(\theta,U_P,a) = \exp\left(2i\pi(\theta_{p+1}u^P_{1,j} + \cdots + \theta_n u^P_{n-p,j})\right) \in \{\zeta \in \mathbb{C}^*\,;\, |\zeta| = 1\}, \quad j = 1,...,n.$$

Each integration current $\Delta_{H_P,D_P}(\theta_{p+1},...,\theta_n)$ can then be extended to $\mathbb{CP}^n$ as the integration of the Zariski closure (in $\mathbb{CP}^n$) of the toric subset

$$\Big\{(\zeta_1,...,\zeta_n) \in (\mathbb{C}^*)^n\,;\, \prod_{j=1}^n \zeta_j^{\xi^+_{\ell,j}} - \prod_{j=1}^n (\gamma_j(\theta,U_P,a))^{-\xi_{\ell,j}}\,\zeta_j^{\xi^-_{\ell,j}} = 0\Big\}.$$

Since the degree of this projective algebraic variety is bounded independently of $(\theta_{p+1},...,\theta_n)$, the current $\mathscr{T}_n^p(V_{\mathbb{T}})$ has finite mass about any point in $\mathbb{CP}^n \setminus (\mathbb{C}^*)^n$. By the extension theorem of Skoda-El Mir (Theorem 1.5.2), this current can then be trivially extended by 0 as a positive (p,p)-dimensional closed current on $\mathbb{CP}^n$. The last assertion follows from the fact that $\mathscr{T}_n^p(V_{\mathbb{T}})$ and $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}})$ have the same support in the dense open subset $(\mathbb{C}^*)^n \subset \mathbb{CP}^n$ and support of $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}})$ is the closure of support of $\mathscr{T}_n^p(V_{\mathbb{T}})$ in $\mathbb{CP}^n$. □

The following theorem gives a simpler representation for tropical currents of bidimension $(n-1,n-1)$ (equivalently of bidegree $(1,1)$).

Theorem 3.4.2. *Positive tropical currents of bidimension $(n-1,n-1)$ in $(\mathbb{C}^*)^n$ (resp. their extension by zero to $\mathbb{CP}^n$) are exactly the currents of the form $dd^c[p\circ \mathrm{Log}\,]$, where p is a tropical polynomial on $\mathbb{R}^n$ (resp. p is a homogeneous tropical polynomial on $\mathbb{TP}^n$).*

Proof. Observe that for the given tropical polynomial $p:\mathbb{R}^n\to\mathbb{R}$, the two positive closed $(n-1,n-1)$-dimensional currents $dd^c[p\circ \mathrm{Log}\,]$ and $\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p))$ share the same support, $\mathrm{Log}^{-1}(V_{\mathbb{T}}(p))$, in $(\mathbb{C}^*)^n$. In order to show they coincide in $(\mathbb{C}^*)^n$, it is enough to prove they coincide in the open subset $\mathrm{Log}^{-1}\big(\mathbb{R}^n\setminus\bigcup_{\tau\in\mathcal{C}_{n-2}(V_{\mathbb{T}})}|\tau|\big)$ (then they coincide in the whole $(\mathbb{C}^*)^n$ for dimensional reasons thanks to theorem 1.4.1). Since equality of currents can be tested locally, it is even enough to test such an equality in a neighborhood of $\mathrm{Log}^{-1}(a)$, where a is an arbitrary point in the relative interior of a $(n-1)$-dimensional cell P of the tropical hypersurface $V_{\mathbb{T}}(p)$. By a translation in $\mathbb{R}^n$, one can then assume that $p(x)=\max\{\langle\alpha,x\rangle\,,0\}$, where $\alpha\in\mathbb{Z}^n\setminus\{(0,...,0)\}$. Let $\alpha=m_\xi\,\xi$, where ξ is a primitive vector in $\mathbb{Z}^n$ and $m_\xi\in\mathbb{N}^*$. Let $B:=\{w_1,...,w_{n-1}\}$ be a $\mathbb{Z}$-basis for $H_P\cap\mathbb{Z}^n$ and consider a completion $D_B:=\{w_1,...,w_{n-1},u\}$ of B as in the preliminaries of Section 3. For each $\theta\in(\mathbb{R}/\mathbb{Z})$, the toric set $\Delta_{H_P,D_B}(\theta)$ is the $(n-1)$-dimensional (reduced) toric hypersurface in $(\mathbb{C}^*)^n$ defined by the irreducible binomial $\prod_{j=1}^n\zeta_j^{\xi_j^+}-\gamma_u(\theta)\prod_{j=1}^n\zeta_j^{\xi_j^-}$ for some $\gamma_u(\theta)\in\mathbb{S}^1$ (see Remark 3.1.3).

Let $\overline{\Delta_{H_P,D_B}(\theta)}$ $(\theta\in\mathbb{R}/\mathbb{Z})$ be the Zariski closure of the hypersurface $\Delta_{H_P,D_B}(\theta)$ in $\mathbb{CP}^n$, which is in fact the zero set in $\mathbb{CP}^n$ of homogenization of the above equation. It follows that (see Example 1.5.6)

$$\deg\big(\overline{\Delta_{H_P,D_B}(\theta)}\big)=\max\Big\{\sum_{j=1}^n\xi_j^+,\sum_{j=1}^n\xi_j^-\Big\}=\int_{\mathbb{CP}^n}\big[\overline{\Delta_{H_P,D_B}(\theta)}\big]\wedge\omega^{n-1}$$

where ω denotes the Kähler form $\omega=dd^c\log\|\ \|$ in $\mathbb{CP}^n$. On the other hand, it is easy to see that, in the weak sense of currents in $(\mathbb{C}^*)^n$,

$$\lim_{m\to\infty}\frac{m_\xi}{m}\log\Big|\prod_{j=1}^n\zeta_j^{m\xi_j}+1\Big|=p\circ\mathrm{Log}\,,$$

which implies, taking dd^c,

$$\lim_{m\to\infty} \frac{m_\xi}{m}\, dd^c\big[\log\big|\prod_{j=1}^n \zeta_j^{m\xi_j}+1\big|\big] = dd^c\,[p\circ \mathrm{Log}\,].$$

It follows that, if one denotes as $\overline{dd^c[p\circ \mathrm{Log}\,]}$ the trivial extension by 0 of the positive closed $(n-1,n-1)$-dimensional current $dd^c[p\circ \mathrm{Log}\,]$ from $(\mathbb{C}^*)^n$ to $\mathbb{CP}^n$, one has

$$\int_{\mathbb{CP}^n} \overline{dd^c[p\circ \mathrm{Log}\,]}\wedge\omega^{n-1} = \\ \int_{\mathbb{CP}^n}\Big[\int_{\mathbb{R}/\mathbb{Z}} \overline{[\Delta_{H_P,D_B}(\theta)]}\,d\theta\Big]\wedge\omega^{n-1} = \max\big\{\sum_{j=1}^n \xi_j^+, \sum_{j=1}^n \xi_j^-\big\}. \quad (3.4.22)$$

Chose now $\xi'\in\mathbb{Z}^n\backslash\{(0,...,0)\}$ and a strictly increasing sequence $(N_k)_{k\geq 1}$ of positive integers such that all tropical $(n-1,n-1)$-hypersurfaces $V_{\mathbb{T}}(p_k)$, where

$$p_k\ : x\in\mathbb{R}^n \mapsto \max\big\{p(x),\langle\xi',x\rangle - N_k\big\} = \max\big\{\langle\xi,x\rangle, 0, \langle\xi',x\rangle - N_k\big\},\quad k\in\mathbb{N}^*,$$

are trivalent. For any relatively compact open subset $\mathcal{V}\subset\mathbb{R}^n$, $p\equiv p_k$ in $\mathcal{V}$ and the currents $dd^c[p\circ\mathrm{Log}\,]$ and $dd^c[p_k\circ\mathrm{Log}\,]$ coincide in $\mathrm{Log}^{-1}(\mathcal{V})$ provided k is large enough (depending on $\mathcal{V}$). Since the current $\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p_k))$ is extremal in $(\mathbb{C}^*)^n$ thanks to Theorem 3.1.8 ($p=n-1$), there exists, for each such $\mathcal{V}\subset\mathbb{R}^n$ and for any $k>>1$ large enough (depending on $\mathcal{V}$), a strictly positive constant $\rho_{\mathcal{V},k}$ such that one has

$$\begin{aligned}&\big(\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p))\big)_{|\mathrm{Log}^{-1}(\mathcal{V})} = \big(\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p_k))\big)_{|\mathrm{Log}^{-1}(\mathcal{V})} = \\ &= \rho_{\mathcal{V},k}\,\big(dd^c[p_k\circ\mathrm{Log}\,]\big)_{|\mathrm{Log}^{-1}(\mathcal{V})} = \rho_{\mathcal{V},k}\,\big(dd^c[p\circ\mathrm{Log}\,]\big)_{|\mathrm{Log}^{-1}(\mathcal{V})}.\end{aligned} \quad (3.4.23)$$

Taking an exhaustion of $\mathbb{R}^n$ with relatively open subsets $\mathcal{V}_\ell$, $\ell=1,2,...$, such that $\mathcal{V}_\ell\subset\mathcal{V}_{\ell+1}$ for any $\ell\in\mathbb{N}^*$, it follows that all $\rho_{\mathcal{V},k}$ are equal, so that there exists some strictly positive constant ρ such that

$$\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p)) = \rho\, dd^c\,[p\circ\mathrm{Log}\,]$$

(as currents in $(\mathbb{C}^*)^n$). The fact that the normalization constant ρ equals 1 follows from (3.4.22) since

$$\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p)) = \int_{\theta\in(\mathbb{R}/\mathbb{Z})}\big[\Delta_{H_P,D_B}(\theta)\big]\,d\theta$$

(so that the trivial extensions of $\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p))$ and $dd^c\,[p\circ\mathrm{Log}\,]$ to $\mathbb{CP}^n$ share the same total mass as currents in the projective space $\mathbb{CP}^n$ equipped with its Fubini-Study Kähler form).

Regarding the statement for the homogeneous tropical polynomials, observe that extension by zero of $dd^c\,[p \circ \mathrm{Log}\,]$ to $\mathbb{CP}^n$, in the sense of currents, is exactly $dd^c\,[\tilde{p}\ \circ\ \mathrm{Log}\,]$, where $\tilde{p}$ is the homogenization of p.

For the converse statement, just note that by Theorem 2.1.6, every tropical hypersurface $V_{\mathbb{T}}$ can be understood as $V_{\mathbb{T}}(p)$ for a tropical polynomial p, with equality of respective weights. □

By previous theorem and Theorem 3.1.8 we can finally generalize the extremal example of Demailly.

Corollary 3.4.3. *Let p be a homogeneous tropical polynomial defining a tropical hypersurface in $\mathbb{TP}^n$. Then, the positive current $dd^c\,[p \circ \mathrm{Log}\,]$ is in $\mathcal{E}^{n-1}(\mathbb{CP}^n)$ if every facet of the tropical hypersurface associated to p is the common intersection of exactly 3 polyhedra.*

Chapter 4

Intersections

Intersection theory for currents can be performed in several ways. One way is using their cohomology classes. As it will be explained in the Chapter 5, for any closed current T on a smooth manifold with class $\{T\}$ there is a smooth form α (of the same degree), such that the cohomology class of α, $\{\alpha\} = \{T\}$. In this way, even when for two currents T_1 and T_2, the product $T_1 \wedge T_2$ is not defined one can define $\{T_1 \wedge T_2\} := \{\alpha_1\} \smile \{\alpha_2\} := \{\alpha_1 \wedge \alpha_2\}$. However we will try to describe the intersection theory of tropical currents using the intersections of integration currents themselves. In this way (since we are working over $(\mathbb{C}^*)^n$ or $\mathbb{CP}^n$ that have simple cohomology groups) the information about intersection multiplicities will not be lost. The general theory for non-proper intersection theory with the current theoretic approach is still under development [ASWY12]. However, the situation is easier to understand for the tropical currents.

4.1 Wedge products of tropical currents

We first explain some intersection theory for integration currents in proper cases. Let $f_1, \dots, f_q$ be non-zero holomorphic functions on $X = \mathbb{C}^n$ or $\mathbb{CP}^n$ with the corresponding zero divisors $Z_1, \dots, Z_q$. Assume the supports of these divisors satisfy the proper intersection condition, that is, for every $1 \leq m \leq q$ the intersection $|Z_{j_1}| \cap \cdots \cap |Z_{j_m}|$ of the supports has pure codimension m. Then one has

Theorem 4.1.1 ([Dem, Proposition 4.12])**.** *With the above notations and codimension condition, let $(C_k)_{k \geq 1}$ be the irreducible components of the $|Z_1| \cap \cdots \cap |Z_q|$. There exist* *integers $m_k > 0$ such that*

$$[Z_1] \wedge \cdots \wedge [Z_q] = \sum m_k [C_k] \ . \tag{4.1.1}$$

The integer m_k is called the multiplicity of intersection of $Z_1, \ldots, Z_q$ along the component C_k.

Let us analyze the above theorem for two smooth hypersurfaces in $Z_1, Z_2 \subset \mathbb{C}^n$ given respectively as zero divisors of f_1 and f_2. By definition[1]

$$[Z_1] \wedge [Z_2] = dd^c \log |f_1| \wedge dd^c \log |f_2| = dd^c(\log |f_1| [Z_2]) \ .$$

Applying the Lelong-Poincaré equation (Lemma 5.2.6) on Z_2 as an ambient space, shows that

$$dd^c(\log |f_1| [Z_2]) = \sum m_k [C_k] \ ,$$

where m_k is the **vanishing order** of f_1 along C_k in Z_2 .

• **Multiplication of n currents.** Let $Y_1, \ldots, Y_n$ be n toric hypersurfaces given respectively as zero divisors of the binomials

$$Y_\ell \ = Y_\ell(\xi_\ell^+, \xi_\ell^-) \ : \ \prod_{j=1}^n \zeta_j^{\xi_{\ell,j}^+} - \prod_{j=1}^n \zeta_j^{\xi_{\ell,j}^-}, \quad \zeta_j \in \mathbb{C}^*, \ell = 1, \ldots, n \ ,$$

where $\xi_\ell^+ = (\xi_{\ell,1}^+, \ldots, \xi_{\ell,n}^+)$ and $\xi_\ell^- = (\xi_{\ell,1}^-, \ldots, \xi_{\ell,n}^-)$ are vectors with non-negative components in $\mathbb{Z}^n$ and disjoint supports (see also Remark 3.1.3). Assume that all ξ_ℓ are primitive. If $\{\xi_1, \ldots, \xi_n\}$ is an $\mathbb{R}$-basis for $\mathbb{R}^n$, then the intersection $|Y_1| \cap \cdots \cap |Y_n|$ at $(\zeta_1, \ldots, \zeta_n) = (1, \ldots, 1)$ is transversal, simply because the Jacobian at this point is the matrix with given rows $\xi_1, \ldots, \xi_n$ and thus is invertible. Therefore, the intersection multiplicity at each intersection point is exactly 1.

Let us fix, for any $\ell = 1, ..., n$, a $\mathbb{Z}$-basis for $\mathrm{Ker}\, \langle \xi_\ell, \cdot \rangle \cap \mathbb{Z}^n$, $B_\ell = \{w_1^\ell, \ldots, w_{n-1}^\ell\}$, and complete it with primitive vectors u_ℓ to $D_\ell = \{w_1^\ell, \ldots, w_{n-1}^\ell, u_\ell\}$ to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

[1]A priori the product of currents does not make sense. However, the equality $dd^c(uT) = dd^c\ u \wedge T$ holds for a smooth plurisubharmonic u, and a (p, p) closed positive current T. The equality can be extended to a more general plurisubharmonic function u if

1. u is locally integrable with respect to the measure $T \wedge (dd^c \log ||z||)^{n-p}$;
2. u can be approximated by a decreasing sequence (u_n) of plurisubharmonic functions;

then $dd^c u_n \wedge T$ converges weakly to $dd^c u \wedge T$. Therefore, for integration currents along hypersurfaces, the product is defined using Lelong-Poincaré equation. See also [DS13, Proposition 2.17].

Recall the notation for S_{H_ℓ} (where $H_\ell = \mathrm{Ker}\,\langle \xi_\ell, \cdot\rangle$), which is fibered over $(\mathbb{R}/\mathbb{Z})$ by σ_{H_ℓ} with fibers $\Delta_\ell(\theta)$. Assume that in a neighborhood of $\mathrm{Log}^{-1}(0)$, the intersection

$$\Delta_1(\theta_1) \cap \cdots \cap \Delta_n(\theta_n)$$

is a set of points $\{x_1, \ldots, x_\kappa\}$, each an with intersection multiplicity one. Because each fiber $\Delta_\ell(\theta_\ell)$ changes by an invertible linear transformation with respect to θ, the intersection of any set of different fibers

$$\Delta_1(\theta_1') \cap \cdots \cap \Delta_n(\theta_n')$$

has also κ points. Let $E \subset \mathbb{R}^n$ be a neighborhood of the origin. One has

$$\int_{(\theta_1,\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^n} [\Delta_1(\theta_1)] \wedge \cdots \wedge [\Delta_n(\theta_n)]_{|\mathrm{Log}^{-1}(E)} = \kappa \int_{\varphi\ \in\ (\mathbb{R}/\mathbb{Z})^n} \delta_{e^{2i\pi\varphi}}\ , \tag{4.1.2}$$

since as $(\theta_1, \ldots, \theta_n)$ vary, the κ intersection points uniformly cover the torus $(\mathbb{R}/\mathbb{Z})^n$, each with velocity 1, and each one covers equal portion. Note that the left hand side of the above equation can be written as

$$\mathscr{T}_n^{n-1}(H_1) \wedge \cdots \wedge \mathscr{T}_n^{n-1}(H_n).$$

In order to compute κ (which can be indeed calculated directly), we will use the following formula due to A. Rashkovskii in [Ras01] that relates the real Monge-Ampère measure to the complex Monge-Ampère operator (see [Rul01] for a proof).

Theorem 4.1.2. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and $E \subset \mathbb{R}^n$ be a Borel set ; then*

$$\int_{\mathrm{Log}^{-1}(E)} (dd^c\,[u \circ \mathrm{Log}\,])^n = n! \times \int_E \mu[u]\ . \tag{4.1.3}$$

The above formula can be **polarized** in the following way:

Lemma 4.1.3. *Let $u_1, \ldots, u_n : \mathbb{R}^n \to \mathbb{R}$ be n convex functions and $E \subset \mathbb{R}^n$ be a Borel set ; then*

$$\int_{\mathrm{Log}^{-1}(E)} dd^c\,[u_1 \circ \mathrm{Log}\,] \wedge \cdots \wedge dd^c\,[u_n \circ \mathrm{Log}\,] = n! \times \int_E \tilde{\mu}[u_1, \ldots, u_n], \tag{4.1.4}$$

where $\tilde{\mu}$ is the polarization of μ.

The above formula was also used in [PR04] to re-prove the complex Bernstein theorem. In the next section we explain the polarization formulas, and we will return to the calculation of the intersection multiplicity κ afterwards. We mention that Theorem 4.1.2 and the formula

$$\mu[p] = \sum_{a \in \mathcal{C}_0(V_T(p))} \mathrm{Vol}_n(\{a\}^*)\, \delta_a \,, \tag{4.1.5}$$

for a tropical polynomial p (see Example 2.4.2) imply following corollary.

Corollary 4.1.4. *The support of the current* $(dd^c\ [p \circ \mathrm{Log}\,])^{\wedge n}$ *for a tropical polynomial* p *is the set* $\bigcup_{a \in \mathcal{C}_0(V_{\mathbb{T}}(p))} \mathrm{Log}^{-1}(\{a\})$.

4.2 Polarizations

In this section we try to understand a few well-known results using a calculation of Monge-Ampère measure. These multiplicities were already employed in [BB07].

For two convex sets $K_1, K_2 \subset \mathbb{R}^n$, the **Minkowski sum** of K_1 and K_2, defined as

$$K_1 + K_2 = \{a + b\ : a \in K_1, b \in K_2\}\ ,$$

is also a convex set, and similarly for n objects. In this case the quantity

$$\frac{1}{2!}(\mathrm{Vol}_n(K_1 + K_2) - \mathrm{Vol}_n(K_1) - \mathrm{Vol}_n(K_2))$$

is called the **mixed volume** of K_1 and K_2. This can be generalized for n convex sets $K_1, \ldots, K_n$ using the following polarization formula :

$$\text{Mixed Vol}_n(K_1, \ldots, K_n) = \frac{1}{n!} \sum_{k=1}^{n} \sum_{1 \le j_1 < \cdots < j_k \le n} (-1)^{n-k}\ \mathrm{Vol}_n(K_{j_1} + \cdots + K_{j_k})\,.$$

Similarly one defines the **mixed Monge-Ampère measure** associated to n convex functions $u_1, \ldots, u_n$ as

$$\tilde{\mu}[u_1, \ldots, u_n] = \frac{1}{n!} \sum_{k=1}^{n} \sum_{1 \le j_1 < \cdots < j_k \le n} (-1)^{n-k}\ \mu[u_{j_1} + \cdots + u_{j_k}]\,. \tag{4.2.6}$$

Note that with this definition (4.1.3) easily gives (4.1.4). For the mixed Monge-Ampère measure, one has $\tilde{\mu}[u, \ldots, u] = \mu[u]$ for a convex function u . Consider now the tropical Laurent polynomials $p, p_1, \ldots, p_n$. Using the fact that the total mass of $\mu[p]$ equals

$$\tilde{\mu}[p, \ldots, p](\mathbb{R}^n) = \mu[p](\mathbb{R}^n) = \mathrm{Vol}_n(\Delta_p)\ ,$$

and comparing the mixed formulas above, one gets

$$\tilde{\mu}[p_1,\ldots,p_n](\mathbb{R}^n) := \int_{\mathbb{R}^n} \tilde{\mu}[p_1,\ldots,p_n] = \text{Mixed Vol}_n(\Delta_{p_1},\ldots,\Delta_{p_n}) \ . \tag{4.2.7}$$

It is not hard to see that

$$V_{\mathbb{T}}(p_1+\cdots+p_n) = \bigcup_{i=1}^{n} V_{\mathbb{T}}(p_i).$$

For generic choices of $p_1,\ldots,p_n$, the 0-cells of $V_{\mathbb{T}}(p_1+\cdots+p_n)$ are the union of zero cells of each of $V_{\mathbb{T}}(p_i)$, $i=1,\ldots,n$ and the new zero cells occure from intersection of the $V_{\mathbb{T}}(p_i)$'s. The discussion in Example 2.4.2 with the inclusion-exclusion principle imply

$$n! \times \tilde{\mu}[p_1,\ldots,p_n] \quad = \sum_{\{a\}\in\mathcal{C}_0(V_{\mathbb{T}}(p_1)\cap\cdots\cap V_{\mathbb{T}}(p_n))} \text{Vol}_n(\{a\}^*)\delta_a\,. \tag{4.2.8}$$

This, together with 4.1.4 suggest to use $\text{Vol}_n(\{a\}^*)$ as an **intersection multiplicity** at the intersection point a. Such intersection multiplicities were already used by Bertrand and Bihan in [BB07]. The interesting fact (see [HS95]) is that such a dual n-cell $\{a\}^*$ in the privileged subdivision of Δ_p is in fact a mixed cell of the form $C = C_1+\cdots+C_n$ where $C_i \in \mathcal{C}_1(V_{\mathbb{T}}(p_i))$, which is reasonable since a is in the intersection of n cells of dimension $(n-1)$, one in each $V_{\mathbb{T}}(p_i)$, see Figure 4.1, see also [ST10] and [BS14a]. With such multiplicities, and generic $p_1,\ldots,p_n$ the quantity $n!\times\tilde{\mu}[p_1,\ldots,p_n](\mathbb{R}^n)$ is just the total intersection number of the hypersurfaces $V_{\mathbb{T}}(p_1),\ldots,V_{\mathbb{T}}(p_n)$, taking the multiplicities into account. Therefore the Equation (4.2.7) simply gives

$$n! \times \tilde{\mu}[p_1,\ldots,p_n](\mathbb{R}^n) = n! \times \text{Mixed Vol}_n(\Delta_{p_1},\ldots,\Delta_{p_n})$$

which is considered in [BB07] as the tropical version of **Bernstein's theorem**. Note that the genericity condition insures that the intersections have the right dimensions. Let us also deduce the tropical versions of Bézout's theorem (as in [RGST05]) from the above formulas (see also [Lag12]).

Corollary 4.2.1. *The total intersection number of two generic tropical curves of degree d_1 and d_2 in $\mathbb{R}^2$ is exactly d_1d_2.*

Proof. Assume that the given curves $C_1 = V_{\mathbb{T}}(p_1)$, $C_2 = V_{\mathbb{T}}(p_2)$ have the respective degrees d_1 and d_2 . By Definition 2.3.6 this assumption means that the Newton polytopes of p_1

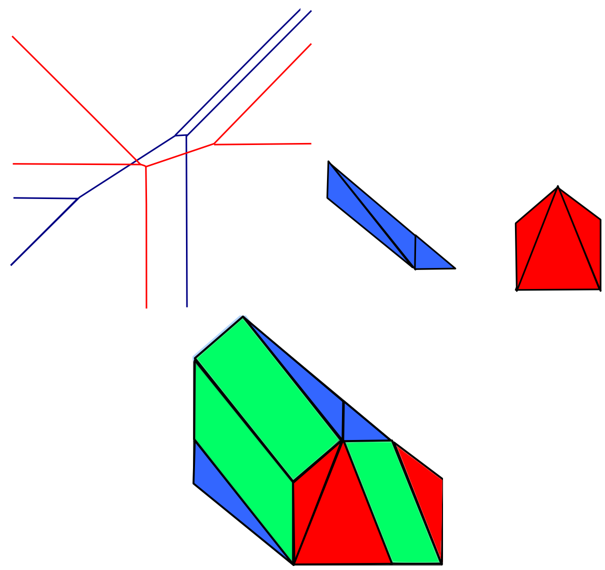

Figure 4.1: On top there are two graphs with the dual subdivision of their corresponding Newton polytopes, underneath the is Minkowskii sum of the two Newton polytopes which is the dual to the union of the graphs, and the green cells correspond to the intersections points of the graphs. The volume of each green cell is used as the intersection multiplicity.

and p_2 are respectively dilated standard simplexes $d_1\ \Delta^2$ and $d_2\ \Delta^2$. The total intersection number is thus given by $2! \times \tilde{\mu}[p_1, p_2]$. By (4.2.7) one has

$$2! \times \tilde{\mu}[p_1, p_2] = 2! \times \text{Mixed Vol}_2(d_1\ \Delta^2, d_2\ \Delta^2)$$
$$= \text{Vol}_2(d_1\ \Delta^2 + d_2\ \Delta^2) - \text{Vol}_2(d_1\ \Delta^2) - \text{Vol}_2(d_2\ \Delta^2)\ .$$

And the right hand side is $\frac{1}{2}(d_1 + d_2)^2 - \frac{1}{2}(d_1)^2 - \frac{1}{2}(d_2)^2 = d_1 d_2$. □

If in the above theorems $p_1, \ldots, p_n$ do not intersect properly, one can perturb the tropical polynomials to get a proper intersection for the perturbations $p_1^\epsilon, \ldots, p_n^\epsilon$ and find a intersection number corresponding to this perturbations. However, the continuous dependence of $\tilde{\mu}[p_1, \ldots, p_n]$ on $p_1, \ldots, p_n$ (with respect to point-wise convergence of functions), implies that the limit when $\epsilon_1, \ldots, \epsilon_n \to 0$ exists. Hence follows the notion of **stable intersection** for tropical hypersurfaces introduced in [RGST05, Mik06].

4.3 Intersection multiplicities

We will use the results of the previous section to calculate the intersection multiplicity $\kappa = \kappa(\xi_1, \ldots, \xi_n)$ of Section 4.1. Let E be a connected open neighborhood of $0 \in \mathbb{R}^n$. As in the proof of Theorem 3.4.2 for each $\ell = 1, ..., n$, let $p_\ell = \max\{-1, \langle \xi_\ell^+, \cdot \rangle, \langle \xi_\ell^-, \cdot \rangle\}$. Then $V_{\mathbb{T}}(p_\ell) \cap E = H_\ell \cap E$. Recall that the H_ℓ $(1 \leq \ell \leq n)$ only intersect at the origin in $\mathbb{R}^n$. This, together with Theorem 3.4.2, gives

$$\int_{(\theta_1,\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^n} [\Delta_1(\theta_1)] \wedge \cdots \wedge [\Delta_n(\theta_n)]_{\;|\mathrm{Log}^{-1}(E)} = \int_{\mathrm{Log}^{-1}(E)} dd^c\,[p_1 \circ \mathrm{Log}\,] \wedge \cdots \wedge dd^c\,[p_n \circ \mathrm{Log}\,].$$

Hence, by 4.1.4,

$$\int_{(\theta_1,\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^n} [\Delta_1(\theta_1)] \wedge \cdots \wedge [\Delta_n(\theta_n)]_{\;|\mathrm{Log}^{-1}(E)} = n! \times \tilde{\mu}[p_1, \ldots, p_n](E)\,.$$

By 4.2.8, the right hand side of this identity is exactly $\mathrm{Vol}_n(\xi_1, \ldots, \xi_n)\, \delta_0$, where $\xi_\ell = \xi_\ell^+ - \xi_\ell^-$ is now identified with the edge connecting ξ_ℓ^+ and ξ_ℓ^- considered as points in $\mathbb{Z}^n$. Comparing with 4.1.2 gives

$$\kappa = n! \times \text{Mixed Vol}_n(\xi_1, \ldots, \xi_n) = \mathrm{Vol}_n(\xi_1 + \cdots + \xi_n)\ .$$

The above discussion also implies that if $\xi_1, \ldots, \xi_n$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$, the intersection multiplicity

$$\kappa(\xi_1, \ldots, \xi_n) = n! \times \text{Mixed Vol}_n(\xi_1, \ldots, \xi_n) = 1, \tag{4.3.9}$$

since this volume is the volume of the parallelotope $\xi_1 + \cdots + \xi_n$. Let $H_{\xi_{\ell_1},\ldots,\xi_{\ell_p}}$ be a p-plane generated by $\langle \xi_{\ell_1}, \ldots, \xi_{\ell_p} \rangle$ over $\mathbb{R}$, then (4.3.9) implies that,

$$\mathscr{T}_n^p(H_{\xi_1,\ldots,\xi_p})_{|\mathrm{Log}^{-1}(E)} = \big(\mathscr{T}_n^{n-1}(H_{\xi_{p+1}}) \wedge \cdots \wedge \mathscr{T}_n^{n-1}(H_{\xi_n})\big)_{|\mathrm{Log}^{-1}(E)}. \tag{4.3.10}$$

since the intersection multiplicity in the right hand side has to be one, as it cannot increase when the number of intersecting toric sets decreases, and the left hand side is obtained by an injective foliation. Similarly if we have a $\mathbb{Z}$-basis $\{\xi'_1, \ldots, \xi'_n\}$ and define $H'_{\xi'_{\ell_1},\ldots,\xi'_{\ell_p}}$ similarly then

$$\mathscr{T}_n^p(H_{\xi_1,\ldots,\xi_p}) \wedge \mathscr{T}_n^k(H'_{\xi'_1,\ldots,\xi'_{n-p}}) = n! \times \text{Mixed Vol}_n(\xi'_1, \ldots, \xi'_p, \xi_1, \ldots, \xi_{n-p})[\mathrm{Log}^{-1}(\{0\})].$$

For two rational polytopes P and Q with complementary dimensions in $\mathbb{R}^n$ which intersect transversally at 0, assume that $P \subset H_{\xi_1,\ldots,\xi_p}$ and $Q \subset H'_{\xi'_1,\ldots,\xi'_{n-p}}$, then $\mathscr{T}_n^p(P)$ and $\mathscr{T}_n^{n-p}(Q)$

coincide with $\mathscr{T}_n^p(H_{\xi_1,\dots,\xi_p})$ and $\mathscr{T}_n^k(H'_{\xi'_1,\dots,\xi'_{n-p}})$ respectively, in the neighborhood E. And the intersection multiplicity $\kappa_{P,Q} := n!$ Mixed $\mathrm{Vol}_n(\xi'_1,\dots,\xi'_p,\xi_1,\dots,\xi_{n-p})$ only depends on P and Q. Therefore for the tropical p-cycle $V_{\mathbb{T}}$ and q-cycle $V'_{\mathbb{T}}$, that intersect transversally with $p+q=n$, one has:

$$\mathscr{T}_n^p(V_{\mathbb{T}}) \wedge \mathscr{T}_n^q(W_{\mathbb{T}}) = \sum_{(P,Q)\in\mathcal{C}_p(V_{\mathbb{T}})\times\mathcal{C}_q(W_{\mathbb{T}})}$$
$$m_P \times m_Q \times \int_{(\theta_{p+1},\dots,\theta_n,\theta'_{q+1},\dots,\theta'_n)\in(\mathbb{R}/\mathbb{Z})^{2n-p-q}} \big[\Delta_P(\theta_{p+1},\dots,\theta_n)\big] \wedge \big[\Delta'_Q(\theta'_{q+1},\dots,\theta'_n)\big],$$

which gives

$$\mathscr{T}_n^p(V_{\mathbb{T}}) \wedge \mathscr{T}_n^q(W_{\mathbb{T}}) = \sum_{(P,Q)\in\mathcal{C}_p(V_{\mathbb{T}})\times\mathcal{C}_q(W_{\mathbb{T}})} m_P \times m_Q \times \kappa_{P,Q} \times \big[\ \mathrm{Log}^{-1}(\{P\cap Q\})\big],$$

in $\mathrm{Log}^{-1}(E)$.

Chapter 5

Approximations of tropical currents

The goal of this chapter is to prove that if an effective strongly extremal tropical cycle $V_{\mathbb{T}}$ in $\mathbb{R}^n$ can be approximated by amoebas of algebraic cycles in Hausdorff metric, then the associated strongly positive extremal current $\mathscr{T}_n^p(V_{\mathbb{T}})$ inherits an approximation by integration currents along irreducible analytic cycles. The proof can easily be extended to the projective spaces. A stronger formulation of the Hodge conjecture [Dem82] due to Jean-Pierre Demailly (see (5.1.4) in Section 5.1) implies that, on a projective manifold, any positive extremal current of bidimension (p, p) with (real) Hodge cohomology class, can be approximated with positive multiples of integration currents along analytic p-cycles. However, this is not known even for $\mathbb{CP}^n$ when $p \neq 0, n-1, n$. There is some hope to find a counter-example for this conjecture with tropical extremal currents since the construction of such global extremal currents is achieved by gluing together "locally extremal" ones.

In tropical geometry, the approximability problems of tropical cycles are being fervently perused ([BBM14], [BS14b], [Mik06], [Spe02]). However, the approximation problem with which we are concerned is rather more flexible, since we are interested in approximating the (effective) tropical cycles with Hausdorff metrics as sets. Therefore, we do not require keeping the degrees of the algebraic cycles whose amoebas approximate the tropical cycle. For instance, Mikhalkin's example of a spatial tropical cubic (see Example 5.2.4) of genus 1 is not approximable by amoebas of cubic curves in $(\mathbb{C}^*)^3$ but, as a set, it is approximable by a family of sextic curves, the resulting sextic tropical curve being the cubic tropical curve with doubled weights. This discussion suggests that a candidate for an extremal current which is not approximable by integration currents along irreducible analytic cycles, is an extremal current associated to a non-approximable tropical cycle. We will make the discussion precise in the following sections.

5.1 Currents with cohomology classes in Hodge groups

Let (X, ω) be a connected compact Hermitian manifold. Denote by $X_{\mathbb{R}}$ the underlying real differentiable manifold of X. One can consider the real test k-forms on $X_{\mathbb{R}}$, accordingly defines by duality the "real" currents on the real manifold $X_{\mathbb{R}}$, denoted by ${\mathcal{D}'}_{2n-k}(X_{\mathbb{R}}) = {\mathcal{D}'}^{k}(X_{\mathbb{R}})$, which are currents of dimension $2n-k$ or degree k . One has

$$ {\mathcal{D}'}_{k}(X_{\mathbb{R}}) \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathcal{D}'^{p,q}(X) , $$

where $\mathcal{D}'^{p,q}(X) = {\mathcal{D}'}_{n-p,n-q}(X)$, are the currents of bidegree (p, q) or bidimension $(n-p, n-q)$.

For the de Rham differential operator d, the equation $d^2 = 0$ allows us to form a quotient of d-closed currents of degree k over d-exact currents of degree k and have the de Rham cohomology $H^k_{dR}(X, \mathbb{R})$. Moreover, a basic observation shows that the Poincaré lemma ([Dem, Chapter 1, Lemma 2.24]) also holds for current. The Poincaré lemma in the language of sheaves just says that the following sequence of sheaves is exact.

$$ 0 \to \mathbb{R}_X \xrightarrow{d} {\mathcal{D}'}^{0}(X_{\mathbb{R}}) \xrightarrow{d} {\mathcal{D}'}^{1}(X_{\mathbb{R}}) \xrightarrow{d} {\mathcal{D}'}^{2}(X_{\mathbb{R}}) \xrightarrow{d} \dots , \tag{5.1.1} $$

where $\mathbb{R}_X$ is the sheaf of locally constant real functions. Taking the global sections of this sequence, one has the sheaf cohomology groups

$$ H^k_{dR}(X, \mathbb{R}_X) = \frac{\operatorname{Ker}\Big(d : \Gamma\big(X, {\mathcal{D}'}^{k}(X_{\mathbb{R}})\big) \to \Gamma\big(X, {\mathcal{D}'}^{k+1}(X_{\mathbb{R}})\big)\Big)}{\operatorname{Im}\Big(d : \Gamma\big(X, {\mathcal{D}'}^{k-1}(X_{\mathbb{R}})\big) \to \Gamma\big(X, {\mathcal{D}'}^{k}(X_{\mathbb{R}})\big)\Big)} . $$

is exactly as the usual de Rham cohomology groups $H^k_{dR}(X, \mathbb{R})$. Now let us denote by $\check{H}^k(X; \mathcal{F})$ the **Čech cohomology** with values in a sheaf $\mathcal{F}$ of abelian groups. The de Rham-Weil isomorphism ([Wel08, Chapter II, Theorem 3.13]) tells us that

$$ \check{H}^k(X, \mathbb{R}_X) \simeq H^k_{dR}(X; \mathbb{R}_X) \simeq H^k_{dR}(X; \mathbb{R}). $$

One indeed has the above isomorphism derived from the exact sequence of sheaves of smooth form $0 \stackrel{d}{\to} \mathbb{R}_X \stackrel{d}{\to} \Omega^{\bullet}_X$, which in turn implies that for every current $T \in {\mathcal{D}'}^k(X_{\mathbb{R}}) = {\mathcal{D}'}_{2n-k}(X_{\mathbb{R}})$ with de-Rham cohomology class in $\{T\} \in H^k_{dR}(X;\mathbb{R})$ there exists a real smooth k-form such that its class $\{\alpha\}$ equals $\{T\}$.

We mention that in the same manner one defines the Dolbeault cohomology for currents, using the fact that the Dolbeault-Grothendieck lemma also holds for currents ([Dem, Chapter 1, Lemma 3.29]), and as a consequence the following sequence is an exact sequence of sheaves :

$$0 \to \Omega^p_X \stackrel{\bar\partial}{\to} \mathcal{D}'^{\ p,0}(X) \stackrel{\bar\partial}{\to} \mathcal{D}'^{\ p,1}(X) \stackrel{\bar\partial}{\to} \mathcal{D}'^{\ p,2}(X) \stackrel{\bar\partial}{\to} \dots \ . \tag{5.1.2}$$

The de Rham-Weil isomorphism again gives

$$\check{H}^q(X,\Omega^p_X) \simeq H^{p,q}(X) = \frac{\operatorname{Ker}\Big(\bar\partial : \Gamma\big(X,\mathcal{D}'^{p,q}(X)\big) \to \Gamma\big(X,\mathcal{D}'^{p,q+1}(X)\big)\Big)}{\operatorname{Im}\Big(\bar\partial : \Gamma\big(X,\mathcal{D}'^{p,q-1}(X)\big) \to \Gamma\big(X,\mathcal{D}'^{p,q}(X)\big)\Big)} \ .$$

Now suppose that (X,ω) is a Kähler manifold (*i.e.* the Hermitian form ω is closed), one has the **Hodge decomposition**.

$$H^k_{dR}(X,\mathbb{C}) = H^k_{dR}(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X).$$

Denote by $H^k_{dR}(X,\mathbb{Z})$ the image of natural morphism $\check{H}^k(X;\mathbb{Z}_X) \to \check{H}^k(X;\mathbb{R}_X) \simeq H^k_{dR}(X;\mathbb{R}) \to H^k_{dR}(X,\mathbb{C})$.

Therefore, the intersection of free groups $H^{2p}_{dR}(X,\mathbb{Z})/\text{tors} \cap H^{p,p}(X,\mathbb{C})$ does make sense. Accordingly, one defines the group of **Hodge classes** with coefficient in $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$.

$$\operatorname{Hdg}^p_{\mathbb{K}}(X) = \mathbb{K} \otimes_{\mathbb{Z}} \big(H^{2p}_{dR}(X,\mathbb{Z})/\text{tors} \ \cap \ H^{p,p}_{dR}(X,\mathbb{C})\big).$$

For any analytic subset $Z \subset X$ of (complex) dimension $n-p$ the existence of triangulation together with Poincaré duality implies that the cohomology class $\{[Z]\} \in H^{2p}_{dR}(X,\mathbb{Z})$. Therefore, for $q_j \in \mathbb{K}$ and $n-p$ dimensional analytic sets Z_j the sum $\sum_j q_j[Z_j]$ has a cohomology class in $\operatorname{Hdg}^p_{\mathbb{K}}(X)$. Moreover, for a closed current $T \in \mathcal{D}'^{p,p}(X)$ the map

$$T \to \{T\} \in H^{p,p}(X)$$

is continuous with respect to the weak topology of $\mathcal{D}'^{p,p}(X)$. Therefore if for a sequence $T_j \to T$, with $\{T_j\} \in \operatorname{Hdg}^p_{\mathbb{K}}(X)$ then also $\{T_j\} \in \operatorname{Hdg}^p_{\mathbb{K}}(X)$. This implies that if a current T

is approximable by sum of integration currents with coefficients in $\mathbb{K}$ then a-priori $\{T\}$ has to be in $\mathrm{Hdg}^p_{\mathbb{K}}(X)$.

The **Hodge conjecture** asserts the following

> Let X be a projective variety, then every cohomology class in $\mathrm{Hdg}^p_{\mathbb{K}}(X)$ is generated by cohomology classes of elements of the form
>
> $$\sum_j q_j[Z_j], \quad q_j \in \mathbb{K}\ ,\ Z_j : (n-p)\text{-cycle} \tag{5.1.3}$$

Since $\mathbb{Q} \subset \mathbb{R}$ is dense, the two versions (over $\mathbb{Q}$ or $\mathbb{R}$) of the conjecture are equivalent. Jean-Pierre Demailly in [Dem12] proves the following result.

Theorem 5.1.1 ([Dem12, Theorem 13.41])**.** *Let X be a projective n-dimensional manifold. The following statements are equivalent :*

1. *The Hodge conjecture is true in codimension p, i.e. $\mathrm{Hdg}^p_{\mathbb{R}}(X)$ is generated by classes of codimension p algebraic cycles with real coefficients.*
2. *Every real*[1] *closed current $T \in \mathcal{D}'_{n-p,n-p}(X)$ such that $\{T\} \in \mathrm{Hdg}^p_{\mathbb{R}}(X)$ is a weak limit of algebraic cycles $\sum \lambda_j[Z_j]$ of codimension p with real coefficients.*

Let us now denote by $SPC^p_{\mathbb{Z}}(X)$ the cone of currents $T \in \mathcal{D}'_{p,p}(X) = \mathcal{D}'^{n-p,n-p}(X)$ with cohomology class $\{T\} \in \mathrm{Hdg}^{n-p}_{\mathbb{R}}(X)$. Consider the following conjecture

$$SPC^p_{\mathbb{Z}}(X) = \widehat{\mathcal{I}^p(X)}. \tag{5.1.4}$$

i.e. every strongly positive current $T \in \mathcal{D}'_{p,p}(X)$ with cohomology class $\{T\} \in \mathrm{Hdg}^{n-p}_{\mathbb{R}}(X)$ is approximable by sum of $\sum_j \lambda^+_j[Z_j]$ where Z_j are irreducible analytic (=algebraic) cycles of dimension p and λ^+_j are positive real numbers. In other words, it asks whether in the second statement in Theorem 5.1.1 for a positive current T, the coefficients λ_j in $\sum \lambda_j[Z_j]$ can be chosen to be positive numbers.

By continuity of the map $\{\ \}$ on currents one obviously has $SPC^p_{\mathbb{Z}}(X) \supset \widehat{\mathcal{I}^p(X)}$. Demailly in [Dem82] proves that (5.1.4) implies the Hodge conjecture, however the equivalence is not known. He also proves that (5.1.4) holds true when $p = n-1$.

[1] Recall that a current T is called real, if it is equal to its conjugation, where the conjugation, $\overline{T}$, is defined by

$$\langle \overline{T}, \alpha \rangle := \overline{\langle T, \overline{\alpha} \rangle}.$$

Finally consider

$$\mathcal{E}^p(X) \cap SPC^p_{\mathbb{Z}}(X) \subset \overline{\mathcal{I}^p(X)} \ . \tag{5.1.5}$$

i.e the extremal currents with Hodge classes on a projective variety are approximable by integration currents along irreducible analytic cycles with positive coefficients. From the proof of Proposition 5.2 in [Dem82] it easily follows that one has the implication

$$\text{If} \quad SPC^p_{\mathbb{Z}}(X) = \widehat{\mathcal{I}^p(X)} \quad \text{then} \quad \mathcal{E}^p(X) \cap SPC^p_{\mathbb{Z}}(X) \subset \overline{\mathcal{I}^p(X)} \ .$$

This highlights the importance of approximability problem of extremal currents.

Example 5.1.2. Take $X = \mathbb{CP}^n$ the projective space. It is known that

$$H^{p,q}(X) = \begin{cases} \mathbb{C} & p = q \\ 0 & p \neq q \end{cases}$$

In fact $H^{p,p}(\mathbb{CP}^n)$ and $\mathrm{Hdg}^p_{\mathbb{R}}(\mathbb{CP}^n)$ are generated only by one element ω^p_{FS}. Hence a class of every positive current $T \in \mathcal{D}'_{p,p}(\mathbb{CP}^n)$ is given by Mass $_{\omega_{\mathrm{FS}}}(T)\{\omega^{n-p}_{\mathrm{FS}}\}$. Hence, in view of the Crofton's formula (1.5.5) for a closed analytic subset of dimension p, $V \subset \mathbb{CP}^n$ (which is also algebraic by Chow's theorem) the cohomology class $\{[V]\}$ is given by $\deg(V)$ $\{\omega^{n-p}_{\mathrm{FS}}\}$. This means that Hodge conjecture is trivially true for $\mathbb{CP}^n$. Thus, in particular, Theorem 5.1.1 implies that every tropical current $\bar{\mathscr{T}}^p_n(V_{\mathbb{T}}) \in \mathcal{D}'_{p,p}(\mathbb{CP}^n)$ is approximable by currents of the form $\sum_j \lambda_j [Z_j]$, where $[Z_j]$ are irreducible algebraic p-cycles in $\mathbb{CP}^n$ and $\lambda_j \in \mathbb{R}$. However it will remain as a very hard open problem whether every positive extremal tropical current is approximable by positive multiples of integration currents along irreducible analytic sets.

5.2 Set-wise approximability by amoebas

Let us recall the definition of the **degree** of a tropical p-cycle which generalizes the Definition 2.3.6. To define the degree of a given cycle $V_{\mathbb{T}} \subset \mathbb{R}^n$, we first consider the closure $\overline{V}_{\mathbb{T}} \subset \mathbb{TP}^n$. The degree of $\overline{V}_{\mathbb{T}}$ is the sum of tropical intersection numbers with any of the $n+1$ divisors at infinity ($\mathbb{TP}^{n-1}$) of $\mathbb{TP}^n$ ([Mik07], [BBM14]). The balancing condition insures that this number is independent of the choice of the divisors at infinity. This in fact implies that this number is a sum of intersection multiplicities of $V_{\mathbb{T}}$ with a generic hyperplane, and has the following consequence for the tropical currents.

Proposition 5.2.1. *Let $V_{\mathbb{T}} \subset \mathbb{R}^n$ be an effective p-cycle with closure $\overline{V}_{\mathbb{T}} \subset \mathbb{TP}^n$. Consider the associated positive current $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}}) \in \mathcal{D}'_{p,p}(\mathbb{CP}^n)$. Then*

$$\mathrm{Mass}_{\omega_{FS}}(\bar{\mathscr{T}}_n^p(V_{\mathbb{T}})) = \int_{\mathbb{CP}^n} \bar{\mathscr{T}}_n^p(V_{\mathbb{T}}) \wedge \omega_{FS}^p = \deg\ (V_{\mathbb{T}}).$$

In particular the cohomology class $\{\bar{\mathscr{T}}_n^p(V_{\mathbb{T}})\}$ is given by $\deg(V_{\mathbb{T}})\{\omega_{\mathrm{FS}}^{n-p}\}$.

Proof. By the above discussion, as a consequence of the balancing conditions, the degree of $V_{\mathbb{T}}$ is the sum of intersection multiplicities of $V_{\mathbb{T}}$ with a generic tropical $(n-p)$-plane. These (tropical) local intersection multiplicities are the same as the toric intersection multiplicities of $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}})$ and generic complex $(n-p)$-planes, as it was noted in in Subsection 4. Hence the total intersection number of $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}})$ with a generic complex $(n-p)$-plane is also the constant number $\deg(V_{\mathbb{T}})$. This number is therefore the average of total intersection numbers of $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}})$ with generic $(n-p)$-planes in the Grassmannian $G(n-p,n)$, which is equal to the total mass of $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}})$ by Crofton's formula.

The second assertion is followed by the discussion in Example 5.1.2. □

When $\lim_{t\to\infty} \mathrm{Log}\,(Z_t) = V_{\mathbb{T}}$ for family of algebraic cycles $Z_t \subset (\mathbb{C}^*)^n$ of the same degree d, depending analytically on t, then $V_{\mathbb{T}}$, equipped with the induced weights, will also have degree d. The approximation problem considered in [BS14b],[Kat12a],[Spe02],[BBM14] deals with approximation of a tropical cycle or a tropical curve by algebraic varieties Z_t with equal total degrees. However we are interested here in the problem of approximating the strongly extremal tropical cycles only from the set-wise point of view :

Definition 5.2.2. We call a tropical cycle set-wise approximable if its underlying set is approximable in Hausdorff metric by amoebas of algebraic varieties of any fixed degree.

Remark 5.2.3. The set-wise approximability for strongly extremal tropical cycles is equivalent to having a multiple (obtained by multiplying the weights) which is approximable by amoebas of algebraic subvarieties of $(\mathbb{C}^*)^n$ with equal degrees.

Let us look at the following spatial curve of Mikhalkin [Mik05].

Example 5.2.4. Consider the balanced green cubic curve Γ in Figure 5.1 embedded in $\mathbb{R}^3$. Assume that every edge at left of the points P, Q and R lie in the XY-plane and the edges incident to P, Q and R from the right side, do not. This curve has genus equal to

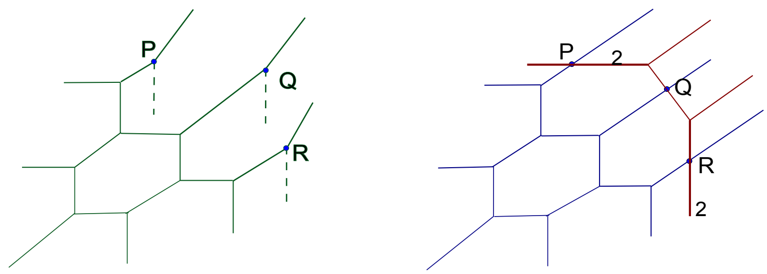

Figure 5.1: Mikhalkin's cubic is set-wise approximable by sextics.

1. If P, Q and R, do not lie in a tropical line, then the green curve cannot be obtained as a continuation of logarithmic limit of any planar family of algebraic curves. Hence it is a "spatial tropical elliptic curve." Now suppose that $\{Z_t\} \subset \mathbb{CP}^3$ is a family of algebraic curves such that

$$\operatorname{Log}_t(Z_t \cap (\mathbb{C}^*)^3) \to \Gamma, \quad \text{as } t \to \infty \ .$$

Therefore the genus of each Z_t is at least one ([Mik05]). The Riemann-Roch theorem implies that the projective cubics of genus one always lie in $\mathbb{CP}^2$. Thus $\{Z_t\}$ cannot be a family of cubics (see also [Spe02]). However, if one doubles the weights of Γ (that is $\Gamma' = 2 \cdot \Gamma$) in order to obtain a tropical sextic of genus one, the author learned from Erwan Brugallé that the 3-valent graph Γ' is approximable by sextic curves based on the following reasoning. In Figure 5.1 assume that blue curve and red curve are in the XY-plane, respectively given as the corner set of the cubic tropical polynomial $\max\{F(x,y),0\}$ and the conic tropical polynomial $\max\{G(x,y),0\}$. Moreover consider the tropical surface given by the equation $\max\{F(x,y),0\}$ in $\mathbb{R}^3$, *i.e.* the cylinder over the plane cubic, and the tropical surface given by the tropical conic equation $\max\{2z, G(x,y),0\}$. The tropical intersection of these two tropical surfaces is exactly the green spatial tropical cubic, with all edges having weight 2. It is approximable by amoebas since it is a tropical intersection of tropical hypersurfaces, and as we discussed in Section 2.2, all the tropical hypersurfaces are indeed approximable.

The following example is about the tropical currents in codimension one. Usually in codimension one all the approximability statements (in tropical geometry or for theory of currents) hold true.

Example 5.2.5. Consider the tropical polynomial

$$p : \mathbb{R}^n \to \mathbb{R}, \quad x = (x_1, \dots, x_n) \mapsto \max_{\alpha} \left\{ c_\alpha + \alpha_1 x_1 + \dots + \alpha_n x_n \right\}$$

attached to a finite set of indices $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$. Now, for each $m, l \in \mathbb{N}^*$, consider the polynomial map :

$$f_{l,m} : \mathbb{C}^n \to \mathbb{C}, \quad z = (z_1, \dots, z_n) \mapsto \sum_{\alpha} \exp(l\, c_\alpha) z_1^{m\alpha_1} \dots z_n^{m\alpha_n}.$$

It is not hard to see that for two non-zero complex numbers z, w with $|z| \neq |w|$, one has

$$\lim_{m\to\infty} \frac{1}{m} \log |z^m + w^m| = \max \ \{\log |z|, \log |w|\},$$

which implies that, in the sense of distributions,

$$\lim_{m\to\infty} \frac{1}{m} \ \log |f_{m,m}(z)| = p \circ \mathrm{Log}\,(z).$$

Poincaré-Lelong equation (Lemma 5.2.6) yields (in the sense of currents)

$$\lim_{m\to\infty} \frac{1}{m} [Z_{f_{m,m}}] = dd^c[p \circ \mathrm{Log}\,(z)], \tag{5.2.6}$$

where $Z_{f_{m,m}}$ denotes the divisor of $f_{m,m}$ with multiplicities taken into account. Moreover

$$\lim_{m\to\infty} \mathrm{Log}\,(\mathrm{Supp}\ [Z_{f_{m,m}}]) = \lim_{m\to\infty} \mathcal{A}_{f_{m,m}} = \lim_{m\to\infty} \frac{1}{m} \mathcal{A}_{f_{m,1}} = V_{\mathbb{T}}(p),$$

where in the third equation "multiplying" the amoeba $\mathcal{A}_{f_{m,1}}$ by $1/m$ means dilating this amoeba by this factor. Therefore, the support of the currents on the left hand side of (5.2.6) approximates the support of the current on the right hand side and the coefficient $\frac{1}{m}$ makes the total masses equal. Combining this with Theorem 3.4.2 one has

$$\lim_{m\to\infty} \frac{1}{m} [Z_{f_{m,m}}] = dd^c[p \circ \mathrm{Log}\,(z)] = \mathscr{T}_n^{n-1}(V_{\mathbb{T}})\,. \tag{5.2.7}$$

The first equality is the degeneration of integration currents of algebraic cycles to a tropical current, the second equality is a **tropical Lelong-Poincaré equation**. $dd^c[p \circ \mathrm{Log}\,(z)]$ is somehow the degenerated "Chern form" which was used in the intersection theory in the previous chapter.

Remark 5.2.6. With the notations of preceding example, one can also see that for the sequence of the Ronkin functions

$$\lim_{m\to\infty} \frac{1}{m} R_{f_{m,m}}(x) = p(x) \quad \text{ in } L^2_{loc}.$$

It is also interesting to note that the Monge-Ampère measure $\mu[\frac{1}{m} R_{f_{m,m}}]$ is invariant under this deformation and its total mass equals Δ_p .

Assume now that $V_{\mathbb{T}} \subset \mathbb{R}^n$ is an effective strongly extremal tropical p-cycle. Suppose next that there is a family of algebraic p-cycles $(Z_t)_{t>1}$ in $(\mathbb{C}^*)^n$ such that we have the set-wise approximation

$$\lim_{t\to\infty} \operatorname{Log}_t(Z_t) = V_{\mathbb{T}}\,, \tag{5.2.8}$$

where $\operatorname{Log}_t(z_1, \ldots, z_n) := (\log|z_1|^{1/\log t}, \ldots, \log|z_n|^{1/\log t})$ for $t > 1$. Starting with such a set-wise approximation, we intend to find a sequence of integration currents $\mathcal{I}^p((\mathbb{C}^*)^n)$ that converges to a multiple of $\mathscr{T}_n^p(V_{\mathbb{T}})$.

For every positive integer m, define the proper smooth map

$$\Phi_m : \mathbb{C}^n \to \mathbb{C}^n\ ,\ (z_1, \ldots, z_n) \mapsto (z_1^m, \ldots, z_n^m) \tag{5.2.9}$$

and consider the current integration current $\Phi_m^*[Z_t] := [\Phi_m^{-1}(Z_t)]$. The support of this current is obviously the set

$$\begin{aligned}
\Phi_m^{-1}(Z_t) &= \big\{(w_1, \ldots, w_n) \in (\mathbb{C}^*)^n\,;\,(w_1^m, \ldots, w_n^m) \in Z_t\big\} \\
&= \quad \Big\{\Big(\exp\big(\frac{2\pi i k_1 + \arg(z_1)}{m}\big)|z_1|^{1/m}, \ldots, \exp\big(\frac{2\pi i k_n + \arg(z_n)}{m}\big)|z_n|^{1/m}\Big)\ , \\
&\qquad\qquad (z_1, \ldots, z_n) \in Z_t\ , 0 \le k_j \le m-1\Big\}.
\end{aligned} \tag{5.2.10}$$

Note that as m increases, the set $\big\{e^{2\pi i k/m}, k = 0, \ldots, m-1\big\}$ tends to a dense set in the unit circle $\mathbb{S}^1$. Let $m\ : [1,\infty[\to \mathbb{N}$ be an increasing function tending to infinity when t tends to infinity. Therefore the support of a limit current for any convergent sequence of the form $\big(\lambda_{m(t_k)}[\Phi_{m(t_k)}^{-1}(Z_{t_k})/\deg Z_t]\big)_k$ such that $(t_k)_k$ tends to $+\infty$, is necessarily of the form $\operatorname{Log}^{-1}(V)$ for some closed set $V \subset \mathbb{R}^n$.

On the other hand, if $x = (x_1, \ldots, x_n) \in V_{\mathbb{T}}$, then there exists a sequence of points

$$\Big(\zeta_{t_{\nu_k}} = (\zeta_{t_{\nu_k},1}, \ldots, \zeta_{t_{\nu_k},n}) \in Z_{t_{\nu_k}}\Big)_k$$

such that

$$\operatorname{Log}_{t_{\nu_k}}(\zeta_{t_{\nu_k}}) \to x,$$

or

$$(|\zeta_{t_{\nu_k},1}|^{1/\log t_{\nu_k}}, \ldots, |\zeta_{t_{\nu_k},n}|^{1/\log t_{\nu_k}}) \to (e^{x_1}, \ldots, e^{x_n})$$

as the sub-sequence $(\nu_k)_k = (\nu_k(x))_k$ tends to $+\infty$. Comparing this with (5.2.10), if one takes $m\ : t \in [1,+\infty[\mapsto [\log t]$, the integer part of $\log t$, then the support of a limit current for any convergent sequence of the form $\big(\lambda_{m(t_k)}[\Phi^{-1}_{m(t_k)}(Z_{t_k})]\big)_k$ such that $(t_k)_k$ tends to $+\infty$ equals necessarily to $V_{\mathbb{T}}$. If one takes $\lambda_m = m^{n-p}$ the family of currents

$$\frac{1}{(m(t))^{n-p}}\,\frac{1}{\deg Z_t}\,\big[\Phi^*_{m(t)}[Z_t]\big], \qquad t>1$$

is normalized (with degrees all equal to 1). Thanks to Theorem 3.2.3, any subsequence of it converges towards the same multiple $\lambda\mathscr{T}^p_n(V_{\mathbb{T}})$ $(\lambda>0)$ of the extremal current $\mathscr{T}^p_n(V_{\mathbb{T}})$. Note that the convergence of the supports in Hausdorff metric, and finite masses of each current in this sequence, implies that the sequence itself is a Cauchy sequence, and therefore it is convergent. So we have proved the following.

Theorem 5.2.7. *Assume that the tropical cycle $V_{\mathbb{T}}$ is strongly extremal and set-wise approximable as $\lim_{t\to+\infty}\operatorname{Log}_t(Z_t)$ by amoebas of irreducible algebraic p-cycles $(Z_t)_{t>1}$ of $(\mathbb{C}^*)^n$. Then there exists $\lambda>0$ such that*

$$\mathscr{T}^p_n(V_{\mathbb{T}}) = \lambda \lim_{m\to\infty}\frac{1}{m^{n-p}}\Phi^*_m[Z_{e^m}].$$

In particular, $\mathscr{T}^p_n(V_{\mathbb{T}}) \in \overline{\mathcal{I}^p\big((\mathbb{C}^*)^n\big)}$.

Remark 5.2.8. Let $V_{\mathbb{T}}$ be an effective tropical p-cycle. By Theorem 3.4.1, the current $\mathscr{T}^p_n(V_{\mathbb{T}}) \in SPC^p((\mathbb{C}^*)^n)$ can be extended by zero to $\bar{\mathscr{T}}^p_n(V_{\mathbb{T}}) \in SPC^p_{\mathbb{Z}}(\mathbb{CP}^n)$. As a result, if in the above theorem one approximates $V_{\mathbb{T}}$ by amoebas of irreducible algebraic cycles (= analytic cycles by Chow's theorem) of $\mathbb{CP}^n$ which do not lie entirely in $\{z_0\cdots z_n=0\}$, then the theorem also gives $\bar{\mathscr{T}}^p_n(V_{\mathbb{T}}) \in \overline{\mathcal{I}^p\big(\mathbb{CP}^n\big)}$.

5.3 Open problems

The above discussion highlights the following important questions.

Problem 5.3.1. Are there strongly extremal tropical cycles which are not set-wise approximable?

The above problem is closely related to a question raised by June Huh, which asks: "Does the Bergman fan of every matroid has a multiple which is approximable?" ([Huh], [HK12]). This is also still open. Anyhow, if one can approximate every positive tropical current with integration currents along algebraic cycles such that their supports are also convergent in Hausdorff metric on compact sets of $\mathbb{C}^n$, then the above problem will be solved just by taking Log of support of such sequences. Furthermore note that, since the tropical hypersurfaces are all approximable, a stronger formulation of Problem 5.3.1 is to ask whether all (strongly extremal) tropical cycles are set-theoretical complete intersection of tropical hypersurfaces. Recall that in this way we showed in Example 5.2.4 that a spatial cubic of genus one is set-wise approximable.

Problem 5.3.2. [Converse of Theorem 5.2.7] Assume $V_{\mathbb{T}}$ is a tropical p-cycle such that $\mathscr{T}_n^p(V_{\mathbb{T}})$ is extremal. Does $\mathscr{T}_n^p(V_{\mathbb{T}}) \in \overline{\mathcal{I}^p\big((\mathbb{C}^*)^n\big)}$ imply that $V_{\mathbb{T}}$ is set-wise approximable by amoebas of algebraic varieties in $(\mathbb{C}^*)^n$?

The above problem seems to be very hard. However, in codimension 1, Duval and Sibony in [DS95] proved that, given a closed positive $(1,1)$ current $T = dd^c\phi \in \mathcal{D}'_{n-1,n-1}(\mathbb{C}^n)$ (for $\phi \in \operatorname{Psh}(\mathbb{C}^n)$), then there exists a sequence $\lambda_j[H_j]$ of integration currents along hypersurfaces $[H_j]$ such that $(H_j)_j$ approximates the support of T with Hausdorff metric on compact subsets of $\mathbb{C}^n$. If the statement raised here is true, then the question whether every extremal tropical current in $\mathbb{CP}^n$ is approximable by algebraic cycles become equivalent to the statement of the purely tropical Problem 5.3.1.

Problem 5.3.3. Generalize these constructions to "infinite" tropical cycles?

Some of the extremal currents which were already generated in dynamical systems obtained as invariant currents for polynomial endomorphisms, have structure of tropical currents. However, many of them have generally fractal supports. One might ask though, to what extent these extremal currents can be constructed tropically, if we allow the tropical cycles to have infinite number of cells.

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