UNIVERSITY OF BRISTOL

School of Mathematics

Solutions to Algebraic Geometry

MATHM0036

(Paper code MATHM0036)

April/May 2025 2 hour(s) 30 minutes

The exam contains FOUR questions
All Four answers will be used for assessment.

Calculators of an approved type (permissible for A-Level examinations) are permitted.

Candidates may bring ONE hand-written sheet of A4 notes, written double sided into the examination. Candidates must insert this sheet into their answer booklet(s) for collection at the end of the examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

Q1. (a) (**5 marks**) Show that any polynomial $f \in \mathbb{C}[x, y, z]$ can be expressed as

$$f = r_1(x^2 - y) + r_2(x^3 - z) + g,$$

for $r_1, r_2 \in \mathbb{C}[x, y, z]$ and $g \in \mathbb{C}[x]$.

Solution: (Easy, bookwork.) Consider any $f \in \mathbb{C}[x, y, z]$. We can use the division algorithm, or replace any occurrence of y by $(y - x^2) + x^2$ and any occurrence of z by $(z - x^3) + x^3$. Re-arranging as

$$f = r_1(x^2 - y) + r_2(x^3 - z) + g,$$

where $r_1, r_2 \in \mathbb{C}[x, y, z]$, we obtain that $g \in \mathbb{C}[x, y, z]$ is a polynomial free of y or z.

(b) (**5 marks**)(Easy, bookwork.) Define the *twisted cubic* $V = \mathbb{V}(x^2 - z, x^3 - y)$, and and consider the parametrisation:

$$\begin{aligned} \varphi : \mathbb{A}^1 &\to \mathbb{A}^3, \\ t &\mapsto (t, t^2, t^3). \end{aligned}$$

Prove that the pullback map

$$\varphi^* : \mathbb{C}[x, y, z] \to \mathbb{C}[t]$$

induces an isomorphism of $\mathbb{C}$-algebras $\mathbb{C}[V] \simeq \mathbb{C}[t]$.

Solution: (Easy, unseen, bookwork.) We use Part (a) can write any $f \in \mathbb{C}[x, y, z]$ as $f = r_1(x^2 - y) + r_2(x^3 - z) + g$, where $g \in \mathbb{C}[x]$. Now, it is easy to see that $\varphi^*(f) = (f \circ \varphi)(t) = g(t)$, and $\ker(\varphi^*) = \mathbb{I}(V) = (x^2 - y, x^3 - z)$.

(c) (**5 marks**) Explain why the result from part (b) implies that V is irreducible.

Solution: (Easy, seen, bookwork.) The fact that $\mathbb{C}[V]$ is an integral domain implies that $\mathbb{I}(V)$ is prime and V is irreducible.

(d) (**5 marks**) We know that the closure of V in $\mathbb{P}^3$, is given by $\overline{V} = \Phi(\mathbb{P}^1)$ where

$$\begin{aligned} \Phi : \mathbb{P}^1 &\to \mathbb{P}^3 \\ [t : s] &\mapsto [s^3 : ts^2 : t^2 s : t^3]. \end{aligned}$$

Prove that $\overline{V} = \mathbb{V}(xz - y^2, yw - z^2, xw - yz) \subseteq \mathbb{P}^3$.

Solution: (Easy, unseen, bookwork.) We can show the equality for an affine cover of $\mathbb{P}^3$. For instance on $U_x = \{[x : y : z : w] \in \mathbb{P}^3 : x = 1\}$, we check that the equations become $\{z - y^2, yw - z^2, w - yz\}$. On the other hand, on this chart, $s^3 \neq 0$. Therefore, the image is understood by $\Phi([t : 1]) = [1 : t : t^2 : t^3] = [1 : x : y : z]$. Therefore, $\Phi([t : 1]) \subseteq \mathbb{V}(\{z - y^2, yw - z^2, w - yz\})$ is clear. It is also obvious that $z = y^2, w = yz = y^3$ includes all the points $[1 : y : y^2 : y^3]$.

(e) (**5 marks**) Explain why irreducibility of V implies that $\overline{V}$ is also irreducible.

Solution: (Easy, seen, bookwork.) If $\overline{V} = X_1 \cup X_2$, for X_1 and X_2 two Zariski-closed subsets of $\overline{V}$. Taking intersections with U_x gives $V = (X_1 \cap U_x) \cap (X_2 \cap U_x)$. Hence, $V \subseteq (X_1 \cap U_x)$ or $V \subseteq (X_2 \cap U_x)$. Therefore, $\overline{V} \subseteq X_1$ or $\overline{V} \subseteq X_2$, as X_i's are closed and contain the closure.

Q2. (a) (**15 marks**) Recall the following definition:

Let X, Y be two algebraic varieties (*i.e.*, affine, quasi-affine, projective or quasi-projective). A morphism $\varphi : X \longrightarrow Y$, is a map such that

- φ is continuous;
- For any for every open set $V \subseteq Y$, and for every regular function $f \in \mathcal{O}_Y(V)$, $\varphi^*(f) = f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$.

Prove the following theorem:

Let X be an algebraic variety, $Y \subseteq \mathbb{A}^n$ a closed affine algebraic variety, and $\varphi : X \longrightarrow Y$ a map of sets. Then, $\varphi = (\varphi_1, \dots, \varphi_n)$ is a morphism, if and only if, for all i, coordinate function $\varphi_i \in \mathcal{O}_X(X)$.

Solution: (Standard techniques, unseen)

'$\implies$' Take $x_i \in \mathbb{C}[Y]$. Then $\varphi^*(x_i) = \varphi_i \in \mathcal{O}_X(X)$.

'$\impliedby$' For $f \in \mathbb{C}[x_1, \dots, x_n]$, we define

$$\varphi^*(f) = (P \mapsto f(\varphi_1(P), \dots, \varphi_n(P))) = f(\varphi_1, \dots, \varphi_n) \in \mathcal{O}_X(X).$$

This follows because $\mathcal{O}_X(X)$ is a k-algebra and contains the φ_i. Hence, for all $f \in \mathbb{C}[x_1, \dots, x_n]$, we have

$$\varphi^{-1}(Z(f)) = \{P \in X \mid f(\varphi(P)) = 0\} = (\varphi^*(f))^{-1}(\{0\}).$$

Now, since $\varphi^*(f) \in \mathcal{O}_X(X)$, and because continuity is a local property, and regular functions are continuous, we obtain that φ is continuous.

To show that φ is a morphism, let $U \subseteq \mathbb{A}^n$ be open, and let $f \in \mathcal{O}_{\mathbb{A}^n}(U)$. We must show that $\varphi^*(f) : \varphi^{-1}(U) \to k$ is regular. This is a local condition, and we may reduce to the case where X is an affine variety, embedded as a closed subset in $\mathbb{A}^m$.

Let $P \in \varphi^{-1}(U)$. Write $f = g/h$ in a neighborhood of $\varphi(P)$, where $g, h \in k[x_1, \dots, x_n]$ and $h \neq 0$. Then

$$\varphi^*(f) = \frac{g(\varphi_1, \dots, \varphi_n)}{h(\varphi_1, \dots, \varphi_n)}.$$

Since the φ_i are given by polynomial functions on $\mathbb{A}^m$ (using a theorem in the notes that implies $\mathcal{O}_X(X) = \mathbb{C}[X]$ for X c.a.a.v.), it follows that $\varphi^*(f)$ is regular. Therefore, φ is a morphism.

(b) (**10 marks**) Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be two closed affine algebraic varieties and

$$\varphi : V \longrightarrow W$$

a morphism. Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W.

Solution: (Standard, seen)

"$\implies$". We claim that $Z := \mathbb{V}(\ker(\varphi^*))$ is a closed affine algebraic subvariety of W isomorphic to V. Note that $\ker(\varphi^*) = \{g \in \mathbb{C}[W] : g \circ \varphi \in \mathbb{I}(V)\} = \{g \in \mathbb{C}[W] : g \circ \varphi(x) = 0, \text{ for all } x \in V\}$ which includes $\mathbb{I}(W)$. Since φ^* is a homomorphism of $\mathbb{C}$-algebras $\ker(\varphi^*)$ is an ideal, and

$$\mathbb{C}[W]/\ker(\varphi^*) \simeq \mathbb{C}[Z] \simeq \mathbb{C}[V] \implies Z \simeq W.$$

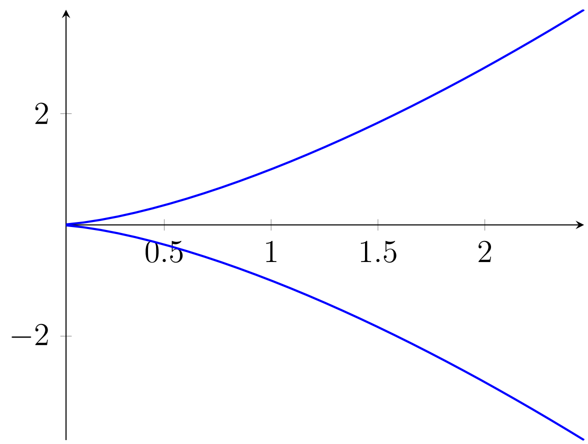
" $\Longleftarrow$ " Assume that φ induces an isomorphism $V \simeq \varphi(V)$. Note that isomorphism are closed maps, so $\varphi(V)$ is a closed affine algebraic variety. Therefore, φ^* is a $\mathbb{C}$-algebra isomorphism between $\mathbb{C}[\varphi(V)] \subseteq \mathbb{C}[W]$ and $\mathbb{C}[V]$.

(a) (**10 marks**) Let $V = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2$.

(i) Sketch $V \cap \mathbb{R}^2$ in $\mathbb{R}^2$.

(ii) Find all the singular point of V.

Solution: (Easy, unseen, bookwork.)

(i) 

(ii) Since V is given by one non-constant equation, by a theorem in the notes, it's of dimension 1. $\nabla(y^2 - x^3) = (-3x^2, 2y)$ which has nullity 2 if and only if $(x, y) = (0, 0)$ which is inside the curve. Therefore, $(0, 0)$ is the only singular point.

(b) (**10 marks**) (Standard, unseen.) Find the irreducible components of $\mathbb{V}(x^2 - y^3, xz - y) \subseteq \mathbb{A}^3$.

Solutions: Substituting $y = xz$ in $x^3 = y^2$ gives $x^3 = (xz)^2$. So $x^2(x - z^2) = 0$. Therefore, we obtain

- $x^2 = 0 \implies x = 0$. Since $x^3 = y^2$, $y = 0$ which gives the *exceptional divisor* $\{(0, 0, z) : z \in \mathbb{C}\}$, which is a line and smooth and connected.
- $x = z^2$ and $xz = y$ yield $y = x^3$. This gives the curve (x, x^3, x^2), which is our famous twisted cubic in Q1, from a different angle.

(c) (**5 marks**) (Standard, unseen.) Show that $\mathbb{V}(xz - y) \subseteq \mathbb{A}^3$ is isomorphic to $\mathbb{A}^2$.

Solution: Let $Y := V(xz - y)$. Consider the maps

$$\varphi : \mathbb{A}^2 \to \mathbb{A}^3, \quad (x, z) \mapsto (x, xz, z),$$

and

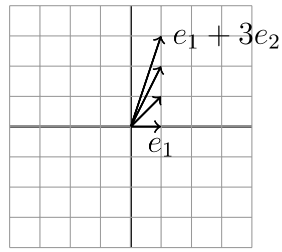
$$\psi : \mathbb{A}^3 \to \mathbb{A}^2, \quad (x, y, z) \mapsto (x, z).$$

Both φ and ψ are clearly morphisms. Moreover, we observe that

$$\psi \circ \varphi = \mathrm{id}_{\mathbb{A}^2} \quad \text{and} \quad \varphi \circ \psi = \mathrm{id}_Y\,.$$

Thus, φ and ψ establish an isomorphism between Y and $\mathbb{A}^2$.

Q4. Consider the cone $\sigma = \text{cone}(e_1, e_1 + 3e_2) \subseteq \mathbb{R}^2$.



(a) (**5 marks**) Explain why the affine toric variety X_σ is not smooth. Subdivide σ into a union of smooth two-dimensional cones.

Solution: (Bookwork, unseen.) $\det \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} = 3$. Therefore σ is not smooth. We can subdivide it into

$$\sigma_1 = \text{cone}(e_1, e_1 + e_2), \sigma_2 = \text{cone}(e_1 + e_2, e_1 + 2e_2), \sigma_3 = \text{cone}(e_1 + 2e_2, e_1 + 3e_2).$$

It's easy to check that the generators of all these cones form a matrix with determinant ± 1 and are smooth.

(b) (**10 marks**) Select two of the two-dimensional cones from your subdivision and denote them by σ_1 and σ_2. Let $\tau = \sigma_1 \cap \sigma_2$. Describe the toric varieties X_{σ_1}, X_{σ_2}, and X_τ and their coordinate rings.

Solution: (Bookwork, unseen.) I choose σ_1 and σ_2 with $\tau = \sigma_1 \cap \sigma_2$. Duals are given by $\sigma_1^\vee = \text{cone}(e_1 - e_2, e_2)$. $\sigma_2^\vee = \text{cone}(-e_1 + e_2, 2e_1 - e_2)$, and $\tau^\vee = \text{cone}(e_1 + e_2, e_1 - e_2, -e_1 + e_2)$. We have that $\mathbb{C}[X_{\sigma_1}] = \mathbb{C}[y, xy^{-1}]$, $\mathbb{C}[X_{\sigma_2}] = \mathbb{C}[x^{-1}y, x^2y^{-1}]$, $\mathbb{C}[X_\tau] = \mathbb{C}[xy, xy^{-1}, x^{-1}y, x, y]$. Taking maxSpec gives the associated toric varieties.

(c) (**2 marks**) Justify why we have the inclusions

$$\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau], \quad \mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau].$$

Solution: (Bookwork, unseen.) $\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau]$ is clear. For $\mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau]$ note that $x^2y^{-1} = (x)(xy^{-1})$, so the generators of $\mathbb{C}[X_{\sigma_2}]$ can be generated in $\mathbb{C}[X_\tau]$.

(d) (**8 marks**) Explain why X_{σ_1} and X_{σ_2} contain X_τ as an open set and describe the glueing of X_{σ_1} and X_{σ_2} along X_τ.

Solution: (Bookwork, unseen.) We have equalities $\mathbb{C}[X_{\sigma_1}]_{xy^{-1}} = \mathbb{C}[X_\tau] = \mathbb{C}[X_{\sigma_2}]_{yx^{-1}}$. These equalities give rise to the inclusions of open sets $X_\tau \subseteq X_{\sigma_1}$ and $X_\tau \subseteq X_{\sigma_2}$. We also have the isomorphisms of $\mathbb{C}$-algebras

$$\begin{aligned} \Phi : \mathbb{C}[X_{\sigma_1}] \supseteq \mathbb{C}[X_\tau] &\longrightarrow \mathbb{C}[X_\tau] \subseteq \mathbb{C}[X_{\sigma_2}] \\ x^{-1}y &\longmapsto xy^{-1} \\ xy^{-1} &\longmapsto x^{-1}y \\ y &\longmapsto x^2y^{-1}. \end{aligned}$$

The map Φ provides the information for glueing the coordinate rings, as well as the corresponding varieties $X_\tau \subseteq X_{\sigma_1}$ and $X_\tau \subseteq X_{\sigma_2}$.

End of examination.