

Algebraic Geometry

1a)

The closure of a subset $A \subset \mathbb{A}^n$, in Zariski Topology,
 Notated by \bar{A} , is the smallest set containing A .

$$\bar{A} = \bigcap \{X \subset \mathbb{A}^n : X \text{ is closed}, A \subset X\}.$$

b) $\mathbb{I}(A)$ is the vanishing Ideal of A , $= \{f \in \mathbb{C}[x_1, \dots, x_n] : f(a) = 0, \forall a \in A\}$

$$\text{We want: } \bar{A} = V(\mathbb{I}(A))$$

In Zariski Topology, any set of the form $V(J)$, for some ideal J , is closed.

As, every $f \in \mathbb{I}(A)$ vanishes on A , $A \subset V(\mathbb{I}(A))$

Let X be any closed subset in \mathbb{A}^n , with $A \subset X$, we can say $X = V(J)$, for some ideal J of $\mathbb{C}[x_1, \dots, x_n]$.

$\forall f \in J$, at A , as $a \in X$, $f(a) = 0$, so $f \in \mathbb{I}(A)$
 and so $J \subset \mathbb{I}(A)$

$$\text{Taking zero sets, } V(\mathbb{I}(A)) \subset V(J) = X.$$

Hence, $V(\mathbb{I}(A))$ is contained in every closed set containing A , and is also closed and contains A .

$V(\mathbb{I}(A))$ = intersection of all closed sets containing A , and so is the smallest closed set containing A .

$$\text{And so, } \bar{A} = \bigcap \{X \subset \mathbb{A}^n : X \text{ closed}, A \subset X\} = V(\mathbb{I}(A))$$

c) If we let $B = \mathbb{Z} \subset \mathbb{C}$, ℓ = an affine line $A^1(\mathbb{C})$

In Zariski Topology on $A^1(\mathbb{C})$, closed sets are the zero sets of polynomials in one variable, but any non-zero polynomial in $\mathbb{C}[t]$ has finite roots, so the only infinite closed set is \mathbb{C} .

Any other infinite subset in \mathbb{C} would be dense in Zariski Topology, so $\bar{B} = \mathbb{C}$

In Euclidean Topology, discrete sets are closed if they have no limit, and taking $B = \mathbb{Z}$, as this is a discrete subset with no limit, B is closed in Euclidean Topology, so $\bar{B} = \mathbb{Z}$.

2a) A subset, K , of a Topological space is compact if every open cover of K has a finite subcover.

Taking $\{U_i\}_{i \in I}$ as a collection of open sets in X , where X is our topological space, and K is a subset of X , with: $K \subseteq \bigcup_{i \in I} U_i$, then there is a finite set of indices, $i_1, \dots, i_n \in I$, with:

$$K \subseteq U_{i_1} \cup \dots \cup U_{i_n}$$

b) $V(x^2 - y^3) \subseteq \mathbb{C}$. Let $X = V(x^2 - y^3)$
 $= \{(x, y) \in \mathbb{C}^2 : x^2 - y^3 = 0\}$

The Zariski Topology is sound with the zero sets of the collection of polynomials.

As the polynomial ring, $([x_1, \dots, x_n])$ is Noetherian, and the Zariski Topology on \mathbb{C}^n is Noetherian, then ~~any~~ every subset of \mathbb{C}^n will be compact.

As X is a subset of \mathbb{C}^2 , it is compact in Zariski topology.

The Heine-Borel Theorem tells us that a subset is compact if and only if it is closed and bounded.

$V(x^2 - y^3)$ is the zero set of a polynomial, and is closed in Euclidean Topology,

But, as $y \rightarrow \infty$, and $x = \pm y^{3/2}$, both $|x|$ and $|y|$ become large, so $\|(x, y)\| = \sqrt{|x|^2 + |y|^2}$ grows without bound. Hence, X is unbounded, and not compact in Euclidean Topology.

3a) Let $W = \bigcup_{\substack{Y+X \\ \in A^2}} (Y+X)$, and $\varphi: A^2 \rightarrow A^2$
 $(x, y) \mapsto (x^2, y^2)$

W is irreducible as it is just the line $\{a, -a : a \in \mathbb{C}\}$, defined by the irreducible polynomial $Y+X$.

$\varphi^{-1}(W)$ here is $\bigcup_{\substack{x^2+y^2 \text{ has factor} \\ (n+iy)(n-iy) \text{ over } \mathbb{C}}} (n+iy, n-iy)$ over \mathbb{C} ,

so $\varphi^{-1}(W) = \bigcup_{n \in \mathbb{C}} (n+iy) \cup \bigcup_{n \in \mathbb{C}} (n-iy)$, a union of two irreducible curves, hence $\varphi^{-1}(W)$ is reducible.

b) We want to show that if X is irreducible, then \bar{X} is irreducible.

We can do this by showing that if \bar{X} is reducible, then X must be reducible.

Let $\bar{X} = C_1 \cup \dots \cup C_n$ be reducible where C_i is a closed set in \bar{Y} for all i , and $\bar{X} \supseteq X$,

then, $X = \bar{X} \cap X = \underbrace{(C_1 \cap X) \cup \dots \cup (C_n \cap X)}_{\text{is closed by subspace topology}}$, $C_i \cap X$ is closed by subspace topology

Hence, If \bar{X} is reducible $\Rightarrow X$ is reducible.

3(c) Showing V is irreducible is equivalent to showing that $\mathbb{I}(V)$ is a prime Ideal, and this taking the coordinate ring:

$$\mathbb{C}[V] := \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbb{I}(V)} \text{ is an Integral Domain as } \mathbb{I}(V) \text{ is prime.}$$

So we have that $\mathbb{C}[V]$ is an integral domain.

An Isomorphism of $V \cong W$, gives us $\mathbb{C}[W] \cong \mathbb{C}[V]$

Taking $\varphi: R \cong S$, R an ID, then for $a, b \in S$, $ab = 0$,

$$\varphi^{-1}(ab) = \varphi^{-1}(0) = 0$$

$\Rightarrow \varphi^{-1}(a)\varphi^{-1}(b) = 0$, and as R is an ID, one of $\varphi^{-1}(a)$, or $\varphi^{-1}(b) = 0$.

So the Isomorphism gives: $\varphi(ab) = 0$, $\varphi(a) = 0$ or $\varphi(b) = 0$
 \Rightarrow Hence, S is an Integral Domain.

So, As $\mathbb{C}[V]$ is an Integral Domain, so is $\mathbb{C}[W]$

$\Rightarrow W$ is irreducible

Hence, the Isomorphism preserves irreducibility.

If $\dim(V) = n$, $\Rightarrow V = V_n \supseteq V_{n-1} \supseteq \dots \supseteq V_0 = \{pt\}$

Where $V_i \subseteq V$ are algebraic, irreducible subvarieties of V .

And so, $\mathbb{I}(V) = P_1 \cap \dots \cap P_n$ of monomial chain length,
 where $P_i \subseteq \mathbb{C}[x_1, \dots, x_n]$ are prime ideals.

The Correspondence Theorem says that this is equivalent to:

(o) $CQ_1 \cap \dots \cap Q_n$ of monomial chain length,
 with $Q_i \subseteq \mathbb{C}[V]$ prime ideals.

Applying $(\psi^*)^{-1}$ as an isomorphism,

(o) $C(V_1, \dots, C(V_n)$ of monic length, is equivalent to W having dimension n .

Hence, invertible projective transformations preserve irreducibility and Dimension.

3d) * Let $V = \mathbb{V}(zx-y, y^2-x^2(n+1)) \subseteq \mathbb{A}^3$.

If we let $y=zx$, then $y^2-x^2(n+1) \Rightarrow$
 $\Rightarrow (zx)^2 - x^2(n+1)$
 $= -x^2(x+(1-z^2))$
 $= -x(x)(x)(x+1-z^2)$

So, $V = (\mathbb{V}(x) \cup \mathbb{V}(x+1-z^2)) \cap \mathbb{V}(zx-y)$

$= \underbrace{\mathbb{V}(x, zx-y)}_{\text{Plane}} \cup \mathbb{V}(x+1-z^2, zx-y)$

Because $x=0 \Rightarrow zx=0$, y must also $=0$,

$= \mathbb{V}(x, y) \cup \mathbb{V}(x+1-z^2, zx-y)$

$\mathbb{V}(x, y)$ is just the z axis, and is isomorphic to \mathbb{A}^1 , which is irreducible in Zariski topology.

Let $W = \mathbb{V}(x+1-z^2, zx-y)$. To show this is irreducible is equivalent to showing $\mathbb{C}[w]$ is an integral domain.

For $\alpha: \mathbb{C}[x, y, z] \rightarrow \mathbb{C}(t)$, $\ker(\alpha) = \mathbb{I}(w)$

If $z \mapsto t$, then $x = t^2 - 1$, $y = t(t^2 - 1)$, so

$$\begin{aligned} x &\mapsto t^2 - 1 \\ y &\mapsto t^3 - t \\ z &\mapsto t \end{aligned}$$

which is a surjective map, with kernel,
 $\ker(\alpha) = (zx - y, x + 1 - z^2)$,

so, $\mathbb{C}[t] \cong \frac{\mathbb{C}[x, y, z]}{\ker(\alpha)} = \mathbb{C}(w)$, with ~~closed~~ and
 $\mathbb{C}(w)$ an integral domain,

so ~~(~~ $(zx - y, x + 1 - z^2)$ is prime ideal,
and so ~~V~~ $V(zx - y, x + 1 - z^2)$ is irreducible.

So The irreducible components of $V(zx - y, y^2 - x^2(x+1))$
 $= V(x, y) \cup V(x+1 - z^2, z = x - y)$

4a) For $a \notin V$, then $a \notin W(\mathbb{I}(v))$
 $= W(g_1, g_m)$

Hence, There exists some j , such that
 $g_j(a) \neq 0$

If we say that $X = g_j(a) \in \mathbb{C}^X$,
then $f = X^{-1} g_j = 1$ for $f(a)$, as

$$g_j(a) = X, X^{-1}X = 1.$$

$f \in \mathbb{I}(V)$ as $g_j \in \mathbb{I}(V)$, and ~~other members~~
Ideals are closed under multiplication,

and $(f(a) = X^{-1}g_j = 1)$

4bi) We have that $V(g) \supseteq V(f_1, \dots, f_k)$
 $= \bigcap_{i=1}^k V(f_i)$

And, $V(f_1, \dots, f_k, x_{n+1}, g-1) = V(f_1, \dots, f_k) \cap V(x_{n+1}, g-1)$

Let's call $f_1, \dots, f_k = I$, $x_{n+1}, g-1 = m$,
and

For $y \in V(I) \cap V(m)$, then $y \in V(I) \text{ and } y \in V(m)$.

Then, $f_j(y) = 0$ for all j , or $y_{n+1} g(y)^{k-1} = 0$

If $y \in V(f_1, \dots, f_k) \in V(g)$,

then, $y_{n+1} g(y)^{k-1} = 0 - 1 = -1 \neq 0$. A contradiction

So, $V(f_1, \dots, f_k) \cap V(x_{n+1}, g-1) = \emptyset$

By Nullstellensatz, if $J = (f_1, \dots, f_k, x_{n+1}, g-1)$

$\sqrt{J} = \mathbb{F}(V(J)) = \mathbb{F}(\emptyset) = \mathbb{F}(x_1, \dots, x_{n+1})$

And, as $1 \in \sqrt{J}$, $1^m \in J$ for some m .

$1^m = 1$ for all m , so

$I \in J$.

If $(x_1, \dots, x_{n+1}) = R$, and so, $R = R \cdot 1 \subseteq RJ \subseteq J$
 $\Rightarrow R \subseteq J$, $R = J$.

So, $(f_1, \dots, f_k, x_{n+1}, g-1) = (x_1, \dots, x_{n+1})$

4b ii) We know that $(f_1, \dots, f_k, x_{n+1}, g-1) = I[x_1, \dots, x_{n+1}]$,
and $1 \in (f_1, \dots, f_k, x_{n+1}, g-1)$, so

$$r_1 f_1 + \dots + r_k f_k + r_{n+1} (x_{n+1}, g-1) = 1$$

For any $x = (x_1, \dots, x_{n+1}) \in A^{n+1}$.

We have that $x_{n+1} = \frac{1}{g}$ as a result, so, letting point x

$$y = (x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n, \frac{1}{g(x_1, x_n)})$$

then we have that

$$1 = \sum_{j=1}^k r_j(x_1, \dots, x_n) f_j(x_1, \dots, x_n).$$

If we take r_j' as polynomials in $I[x_1, \dots, x_n]$,
and let n be the maximal degree of x_{n+1} for each
 $r_j f_j$, then: $\underline{x} = (\underline{x}_1, \dots, \underline{x}_n)$

$$1 = \frac{\sum_{j=1}^k r_j'(\underline{x}) f_j(\underline{x})}{g(\underline{x})^m},$$

$$\text{so, } g(\underline{x})^m = \sum_{j=1}^k r_j'(\underline{x}) f_j(\underline{x}), \forall \underline{x}.$$

And, as $\sum_{j=1}^k r_j'(\underline{x}) f_j(\underline{x}) \in (f_1, \dots, f_k) = I$,

$$g(\underline{x})^m \in I, \text{ so } g^m \in I$$

5a) For $V \subseteq \mathbb{A}^n$, $\mathcal{C}[x_1, \dots, x_n] = R$
 $W \subseteq \mathbb{A}^m$, $\mathcal{C}[x_1, \dots, x_m] = S$

And a morphism $\varphi: V \rightarrow W$

Then $\varphi^*: \mathcal{C}[W] \rightarrow \mathcal{C}[V]$ is injective if and only if $\ker(\varphi^*)$ is trivial.

$\ker(\varphi^*)$ comes from the map: $\psi: S \rightarrow \mathcal{C}[V]$
 $f \mapsto f \circ \varphi$

when it has ~~not~~ kernel $\mathbb{I}(W)$

$\ker(\psi)(\varphi)$ comes from the inverse of the image of $S \rightarrow R$, call this map ϕ .

So, φ^* is injective if and only if $\phi^{-1}(\mathbb{I}(V)) = \mathbb{I}(W)$

$$\begin{aligned}\text{And, } \phi^{-1}(\mathbb{I}(V)) &= \{ f \in S \mid f \circ \varphi(x) = 0 \ \forall x \in V \} \\ &= \{ f \in S \mid f(y) = 0 \ \forall y \in \varphi(V) \} \\ &= \mathbb{I}(\varphi(V)).\end{aligned}$$

Hence, φ^* is injective if and only if $\mathbb{I}(\varphi(V)) = \mathbb{I}(W)$.

And so we can write this as:

$$\begin{aligned}\mathbb{V}(\mathbb{V}(\mathbb{I}(\varphi(V)))) &= \mathbb{V}(\mathbb{I}(W)) \\ \Rightarrow \mathbb{V}(\mathbb{I}(\varphi(V))) &= W\end{aligned}$$

And, by 1 b, $\mathbb{V}(\mathbb{I}(\varphi(V))) = \overline{\varphi(V)}$, so:

$$\begin{aligned}\varphi^* \text{ injective} &\Leftrightarrow \ker(\varphi^*) \text{ is trivial} \\ &\Leftrightarrow \mathbb{I}(\varphi(V)) = \mathbb{I}(W) \\ &\Leftrightarrow \overline{\varphi(V)} = W\end{aligned}$$

But if the image of $\varphi(V) = W$, then this means that $\varphi(V)$ is dense in W , so φ^* is injective $\Leftrightarrow \varphi$ is dominant.

9b) $\varphi^*: \mathcal{C}(W) \rightarrow \mathcal{C}(V)$ surjective \Leftrightarrow φ defines an $I\!\!K$ -isomorphism between $I\!\!K$ -subalgebra subvarieties of W .

By Homomorphism theorem:

$$\frac{\mathcal{C}(W)}{\text{Ker } (\varphi^*)} \cong \text{Im } (\varphi^*)$$

And, the correspondence Theorem tells us that for some $J \subseteq S$, contains $\mathcal{I}(W)$, that the preimage of J is the preimage of $\text{Ker } (\varphi^*)$ under the map $S \rightarrow \mathcal{C}(W)$,

$$\text{So, } \frac{S}{J} \cong \frac{\mathcal{C}(W)}{\text{Ker } (\varphi^*)} \cong \text{Im } (\varphi^*)$$

As $\mathcal{I}(W) \subseteq J$, then $W = V(\mathcal{I}(W)) \supseteq V(J)$.

If we say that $V(J) = U$, and U is the algebraic subvariety of W , and as J is the preimage of $\mathcal{I}(V)$ under:

$$S \xrightarrow{f} R, \quad \text{then } J \text{ is radical, so } \sqrt{J} = J$$

We use Nullstellensatz: $\sqrt{J} = J = \mathcal{I}(V(J)) = \mathcal{I}(U)$

And so, $\mathcal{C}(U) = \frac{S}{J}$, and so φ^* is surjective.

$$\text{So, } \mathcal{C}(U) \cong \frac{\mathcal{C}(W)}{\text{Ker } (\varphi^*)} \cong \text{Im } (\varphi^*) = \mathcal{C}(V)$$

$$\therefore \mathcal{C}(U) \cong \mathcal{C}(V),$$

and so an $I\!\!K$ -isomorphism between V and U , the algebraic subvarieties of W .