generated by d elements, $d = \dim A$, then $e_A(J) = l(A/J)$ precisely when A is Cohen-Macaulay (cf. Example 7.1.3). Nagata (1) also generalized Chevalley's criterion of multiplicity one: if $\hat{A}$ is unmixed, then $e_A(J) = 1$ if and only if A is regular and J is maximal. Nagata also gave an example to show that the criterion may fail for general local rings.

Lech (1) proved a remarkable asymptotic formula:

$$e_A(a_1, \ldots, a_d) = \lim_{\min(t_i) \to \infty} l(A/(a_1^{t_1}, \ldots, a_d^{t_d}))/t_1 \cdot \ldots \cdot t_d$$

which he used to prove the associativity formula for multiplicities (cf. Example 7.1.8).

In 1957 Serre (4) showed that $e_A(a_1, \ldots, a_d)$ is the alternating sum of the lengths of a Koszul complex (cf. Example 7.1.2); or an alternating sum of lengths of Tor modules, this definition, unlike previous algebraic definitions extends to intersections where neither factor is defined by a regular sequence. Several authors, beginning with Kleiman (5), have constructed other ideals J' in A so that $e_A(J) = l(A/J')$; one such is worked out and applied to Bézout's theorem by Vogel (1). Teissier (1), (2) has given some interesting new multiplicity formulas.

The definition of the intersection multiplicity $i(Z, X \cdot V)$ given in this chapter is also a length – the length of the local ring of the normal cone $C_{V \cap X} V$ at the component lying over Z. Since normal cones are constructed from associated graded rings $\oplus J^m/J^{m+1}$, it is not hard to see that this definition agrees with Samuel's (Example 7.1.1). This calculation of intersection multiplicities occurs implicitly in Verdier (5), and in Fulton-MacPherson (1), (2). Basic properties of intersection multiplicities, in this geometric context, follow from the properties proved for more general intersections in Chap. 6; other than the algebra in Appendix A, none of the previous multiplicity theory is required. The proof of the criterion of multiplicity one in § 7.2 is new, to our knowledge.

It should be emphasized that all of the above constructions of intersection multiplicities, with the notable exception of Serre's Tor definition, are valid only when one of the varieties being intersected is regularly imbedded in the ambient space. Intersection multiplicities for arbitrary varieties on a nonsingular variety are defined by reduction to the diagonal, as discussed in the next chapter.

Chapter 8. Intersections on Non-singular Varieties

Summary

If Y is a non-singular variety, the diagonal imbedding δ of Y in $Y \times Y$ is a regular imbedding. For $x, y \in A_* Y$, the product $x \cdot y \in A_* Y$ is defined by the formula

$$x \cdot y = \delta^*(x \times y) .$$

Setting $A^p Y = A_{n-p} Y$, $n = \dim Y$, this product makes $A^* Y$ into a commutative, graded, ring, with unit $[Y]$.

If $f: X \to Y$ is a morphism, with Y non-singular, the graph morphism γ_f from X to $X \times Y$ is a regular imbedding. For $x \in A_* X$, $y \in A^* Y$, define

$$x \cdot_f y = \gamma_f^* (x \times y) \qquad \in A_* X .$$

This product makes $A_* X$ into a graded module over $A^* Y$. If X is also non-singular, setting

$$f^*(y) = [X] \cdot_f y$$

defines a homomorphism $f^* : A^* Y \to A^* X$ of graded rings.

Using the refined operation $\gamma_f^!$ in place of γ_f^*, $x \cdot_f y$ has a canonical refinement in $A_*(|x| \cap f^{-1}(|y|))$. In particular, if V and W are subvarieties of a non-singular variety Y, the intersection class $V \cdot W$ is defined in $A_m(V \cap W)$, $m = \dim V + \dim W - \dim Y$. Any m-dimensional irreducible component Z of $V \cap W$ has a coefficient in $V \cdot W$, called the intersection multiplicity, and denoted $i(Z, V \cdot W; Y)$. The expected properties of these intersection products and multiplicities follow readily from the general properties proved in Chaps. 6 and 7.

Bézout's theorem, in its simplest form, states that $A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1})$, where h is the class of a hyperplane. A deeper analysis of intersections on projective space will be given in Chap. 12.

8.1 Refined Intersections

A variety Y will be called *non-singular* if it is smooth over the given ground field. For our purposes, the important point (Appendix B.7.3) is that the diagonal imbedding

$$\delta : Y \to Y \times Y$$

is a regular imbedding of codimension n, $n = \dim Y$.

The (global) *intersection product* is the composite

$$A_k Y \otimes A_l Y \xrightarrow{\times} A_{k+l}(Y \times Y) \xrightarrow{\delta^*} A_{k+l-n} Y,$$

where δ^* is the Gysin homomorphism (§ 6.2). We write $x \cdot y = \delta^*(x \times y)$ for $x, y \in A_*(Y)$.

More generally, if X is a scheme, Y a non-singular variety, $f: X \to Y$ a morphism, then the graph morphism

$$\gamma_f : X \to X \times Y,$$

$\gamma_f(P) = (P, f(P))$, is a regular imbedding of codimension n, $n = \dim(Y)$. Define a *cap product*

$$A_i Y \otimes A_j X \to A_{i+j-n} X,$$

denoted

$$y \otimes x \mapsto f^*(y) \cap x,$$

by the formula $f^*(y) \cap x = \gamma_f^*(x \times y)$. When f is the identity on Y, this is the previous product. This product is also denoted by $x \cdot_f y$. If X is also non-singular, we write $f^* y$ for $f^* y \cap [X]$.

By using the refined Gysin homomorphisms $\delta^!$ and $\gamma_f^!$ in place of δ^* and γ_f^*, these products can also be refined. If x and y are cycles on a non-singular variety Y, with supports $|x|$ and $|y|$, then $x \in A_*|x|$, $y \in A_*|y|$. Form the fibre square

$$\begin{array}{ccc} |x| \cap |y| & \to & |x| \times |y| \\ \downarrow & & \downarrow \\ Y & \to & Y \times Y \end{array} .$$

We have $\delta^!(x \times y) \in A_*(|x| \cap |y|)$. This product, also denoted $x \cdot y$, maps to the corresponding global product in $A_*(Y)$. If $x = [V]$, $y = [W]$, for V, W pure dimensional subschemes of Y of dimensions k, l, we write the refined product

$$V \cdot W = [V] \cdot [W] \in A_{k+l-n}(V \cap W).$$

Recalling the procedure of Chap. 6.1, this product $V \cdot W$ is constructed as follows. The normal bundle to the diagonal imbedding of Y in $Y \times Y$ is the tangent bundle T_Y to Y. Let T be the restriction of T_Y to $V \cap W$, s the zero section of T. The normal cone $C = C_{V \cap W}(V \times W)$ is a $(k+l)$-dimensional subscheme of T, and

$$V \cdot W = s^*[C],$$

i.e., $V \cdot W$ is the intersection of the cycle of C with the zero section of T.

Similarly, if $f: X \to Y$, Y non-singular, x a cycle on X, y a cycle on Y, the cap product $f^* y \cap x$ has a canonical refinement in $A_*(|x| \cap f^{-1}(|y|))$, which we denote by $x \cdot_f y$:

$$x \cdot_f y = \gamma_f^!(x \times y) \in A_*(|x| \cap f^{-1}(|y|)).$$

These products have the following common generalization.

Definition 8.1.1. Let $f: X \to Y$ be a morphism, with Y non-singular of dimension n. Let $p_X: X' \to X$, $p_Y: Y' \to Y$ be morphisms of schemes X', Y' to X and Y respectively, and let $x \in A_k X'$, $y \in A_l Y'$. Form the fibre square

$$\begin{array}{ccc} X' \times_Y Y' & \to & X' \times Y' \\ \downarrow & & \downarrow {\scriptstyle p_X \times p_Y} \\ X & \underset{\gamma_f}{\to} & X \times Y \,. \end{array}$$

Define

$$x \cdot_f y = \gamma_f^!(x \times y) \in A_{k+l-n}(X' \times_Y Y')\,,$$

where $x \times y \in A_{k+l}(X' \times Y')$ is the exterior product (§ 1.10) and $\gamma_f^!$ is the refined Gysin homomorphism (§ 6.2). When $X' = X$, $Y' = Y$, these are the global products; when $X' = |x|$, $Y' = |y|$, the preceding refinements.

The following proposition proves the expected formal properties of these refined products. In this proposition, it is assumed that each named variety X, Y, Z, Y_i, comes equipped with a morphism $p_X: X' \to X$, $p_Y: Y' \to Y$, etc. and a class $x \in A_* X'$, $y \in A_* Y'$, etc.

Proposition 8.1.1. (a) (Associativity). *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$, with Y and Z non-singular. Then*

$$x \cdot_f (y \cdot_g z) = (x \cdot_f y) \cdot_{gf} z$$

in $A_*(X' \times_Y Y' \times_Z Z')$.

(b) (Commutativity). *Let $f_i: X \to Y_i$, Y_i non-singular, $i = 1, 2$. Then*

$$(x \cdot_{f_1} y_1) \cdot_{f_2} y_2 = (x \cdot_{f_2} y_2) \cdot_{f_1} y_1$$

in $A_*(Y_1' \times_{Y_1} X' \times_{Y_2} Y_2')$.

(c) (Projection formula). *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$, with Z non-singular. Let $f': X' \to Y'$ be a proper morphism such that $p_Y f' = f\, p_X$; let $f'' = f' \times_Z 1_{Z'}$ be the base extension. Then*

$$f''_*(x \cdot_{gf} z) = f'_*(x) \cdot_g z$$

in $A_*(Y' \times_Z Z')$.

(d) (Compatibility). *Let $f: X \to Y$, with Y non-singular and let $g: V' \to Y'$ be a regular imbedding. Then*

$$g^!(x \cdot_f y) = x \cdot_f g^! y$$

in $A_*(X' \times_Y V')$.

Proof. For (a) consider the fibre square

$$\begin{array}{ccc} X & \xrightarrow{\gamma_{gf}} & X \times Z \\ {\scriptstyle \gamma_f}\downarrow & & \downarrow{\scriptstyle \gamma_f \times 1_Z} \\ X \times Y & \underset{1_X \times \gamma_g}{\longrightarrow} & X \times Y \times Z\,. \end{array}$$

The canonical map from $X' \times Y' \times Z'$ to $X \times Y \times Z$ induces a fibre cube lying over this square. Then

$$\begin{aligned} x \cdot_f (y \cdot_g z) &= \gamma_f^!(x \times \gamma_g^!(y \times z)) \\ &= \gamma_f^!(1_X \times \gamma_g)^!(x \times y \times z) = (\gamma_f \times 1_Z)^!(1_X \times \gamma_g)^!(x \times y \times z) \end{aligned}$$

by two applications of Theorem 6.2 (c). Now by Theorem 6.4,

$$\begin{aligned}(\gamma_f \times 1_Z)^!(1_X \times \gamma_g)^!(x \times y \times z) &= (1_X \times \gamma_g)^!(\gamma_f \times 1_Z)^!(x \times y \times z)\\ &= \gamma_{gf}^!((x \cdot_f y) \times z) = (x \cdot_f y) \cdot_{gf} z\,,\end{aligned}$$

using Theorem 6.2 (c) again.

Similarly (b) follows by applying the commutativity theorem (§ 6.4) to the fibre square

$$\begin{tikzcd} X \arrow[r, "\gamma_{f_2}"] \arrow[d, "\gamma_{f_1}"'] & X \times Y_2 \arrow[d, "\gamma_{f_1} \times 1_{Y_2}"] \\ Y_1 \times X \arrow[r, "1_{Y_1} \times \gamma_{f_2}"'] & Y_1 \times X \times Y_2 \end{tikzcd}$$

and the class $y_1 \times x \times y_2$ in $A_*(Y_1' \times X' \times Y_2')$.

For (c), apply Theorem 6.2 (a) to the diagram

$$\begin{tikzcd} X' \times_Z Z' \arrow[r] \arrow[d, "f''"'] & X' \times Z' \arrow[d, "f' \times 1_{Z'}"] \\ Y' \times_Z Z' \arrow[r] \arrow[d] & Y' \times Z' \arrow[d] \\ Y \arrow[r, "\gamma_g"'] & Y \times Z\,. \end{tikzcd}$$

This gives the formula $f_*'(x) \cdot_g z = f_*''(\gamma_g^!(x \times z))$. From the fibre square

$$\begin{tikzcd} X \arrow[r, "\gamma_{gf}"] \arrow[d, "f"'] & X \times Z \arrow[d, "f \times 1_Z"] \\ Y \arrow[r, "\gamma_g"'] & Y \times Z\,, \end{tikzcd}$$

Theorem 6.2 (c) gives

$$\gamma_g^!(x \times z) = \gamma_{gf}^!(x \times z) = x \cdot_{gf} z\,,$$

which concludes the proof of (c).

For (d), apply Theorem 6.4 to the diagram

$$\begin{tikzcd} X' \times_Y V' \arrow[r] \arrow[d] & X' \times V' \arrow[r] \arrow[d] & V' \arrow[d, "g"] \\ X' \times_Y Y' \arrow[r] \arrow[d] & X' \times Y' \arrow[r] \arrow[d] & Y' \\ X \arrow[r, "\gamma_f"'] & X \times Y & \end{tikzcd} \quad . \ \square$$

The following corollary follows from (d).

Corollary 8.1.1. *If Y is non-singular, and $j\colon V \to Y$ is a regular imbedding, and x is a cycle on Y, then*

$$x \cdot [V] = j^!(x)$$

in $A_*(|x| \cap V)$. $\square$

Corollary 8.1.2. *If $f\colon X \to Y$, with X and Y non-singular, and $\Gamma_f \subset X \times Y$ is the graph of f, then for cycles x on X, y on Y,*

$$x \cdot_f y = (x \times y) \cdot [\Gamma_f]$$

in $A_*(|x|)\cap f^{-1}(|y|))$. *In particular, for cycles* x, y *on a non-singular* Y, $x\cdot y$ *is the intersection product of* $x\times y$ *with the diagonal* Δ_Y *on* $Y\times Y$.

Proof. Apply Corollary 8.1.1 with $j=\gamma_f$ the imbedding of $X=\Gamma_f$ in $X\times Y$. □

Corollary 8.1.3. *Let* $f\colon X\to Y$, Y *non-singular,* x *a cycle on* X. *Then*

$$x\cdot_f[Y]=x\,.$$

Proof. By (c), one may assume $x=[X]$, X a variety. Then $x\cdot_f[Y]=\gamma_f^*[X\times Y]=[X]$ since γ_f is a regular imbedding. □

Definition 8.1.2. Let $f\colon X\to Y$ be a morphism from a purely m-dimensional scheme X to a non-singular n-dimensional variety Y. For any morphism $g: Y'\to Y$, define a refined Gysin homomorphism

$$f^!: A_k Y'\to A_{k+m-n}X'$$

with $X'=X\times_Y Y'$, by the formula

$$f^!(y)=[X]\cdot_f y\,.$$

Proposition 8.1.2. (a) *If* f *is also flat, then* $f^!(y)=f'^*(y)$, *where* f' *is the induced morphism from* X' *to* Y'.

(b) *If* f *is also a l.c.i. morphism, then* $f^!$ *agrees with the homomorphism constructed in § 6.6.*

Proof. (a) follows from Proposition 6.5 (a). For (*b*), if $i: X\to P$, $p: P\to Y$ factors f, with p smooth, i a regular imbedding, then we have a factorization

$$X\xrightarrow{\gamma_i}X\times P\xrightarrow{1\times p}X\times Y\xrightarrow{\mathrm{pr}}Y$$

of f, and the conclusion follows from Proposition 6.5 (b). □

The *functoriality* of these refined Gysin homomorphisms follows from Proposition 8.1.1 (a), the *projection formula* from Proposition 8.1.1 (c), and the *commutativity* of these Gysin homomorphisms with l.c.i. Gysin homomorphisms from Proposition 8.1.1 (d). Similarly all other formal properties of refined Gysin homomorphisms given for l.c.i. morphisms in Chap. 6 are valid for morphisms to non-singular varieties. These results will all be subsumed in Chap. 17, so we do not write them out here.

Example 8.1.1. Let V be a closed subscheme of a non-singular variety Y such that the imbedding i of V in Y is a regular imbedding. Then

$$[V]\cdot y=i_*i^*(y)\quad\in A_*Y$$

for all $y\in A_*(Y)$. (Use Corollary 8.1.1.)

Example 8.1.2. Both classes in part (a) of Proposition 8.1.1 (a) are equal to $x\cdot_{(f,gf)}(y\times z)$, as well as to $(x\cdot_{gf}z)\cdot_f y$. (The morphism (f,gf) from X to $X\times Y\times Z$ is the composite of morphisms given in the proof of (a); use Theorem 6.5.)

Both classes in (b) are equal to $x\cdot_{(f_1,f_2)}(y_1\times y_2)$.

Example 8.1.3. If in Proposition 8.1.1 (d) one assumes that g is flat instead of a regular imbedding, then

$$g'^*(x \cdot_f y) = x \cdot_f g^*(y)$$

where $g': X' \times_Y V' \to X' \times_Y Y'$ is induced by g.

On the other hand, if g is assumed to be proper, and $v \in A_* V'$, then

$$g'_*(x \cdot_f v) = x \cdot_f g_*(v)\,.$$

Formula (d) is also valid for $g: V' \to Y'$ an arbitrary l.c.i. morphism as in § 6.6.

Example 8.1.4. If $f_i: X_i \to Y_i$ are morphisms, with Y_i non-singular, $i = 1, \ldots, r$, then

$$(x_1 \times \ldots \times x_r) \cdot_{(f_1 \times \ldots \times f_r)} (y_1 \times \ldots \times y_r) = (x_1 \cdot_{f_1} y_1) \times \ldots \times (x_r \cdot_{f_r} y_r)$$

in $A_*((X_1' \times_{Y_1} Y_1') \times \ldots \times (X_r' \times_{Y_r} Y_r'))$. (Use Example 6.5.2.)

Example 8.1.5. Let Y be non-singular, $V_1, \ldots, V_r$ regularly imbedded subschemes. Then the intersection product of $V_1 \times \ldots \times V_r$ by the diagonal $\Delta = Y$ in $Y \times \ldots \times Y$ is the same as the intersection product of Δ by $V_1 \times \ldots \times V_r$. (Use Theorem 6.4.)

Example 8.1.6. Let $f: X \to Y$, Y non-singular, and let E be a vector bundle on Y. Then, for all i,

$$x \cdot_f (c_i(E) \cap y) = (c_i(f^*E) \cap x) \cdot_f y$$

in $A_*(X' \times_Y Y')$. (Use Proposition 6.3.)

Example 8.1.7. *Projection formula.* Let $f: X \to Y$ be a proper morphism of non-singular varieties, and let x (resp. y) be a cycle on X (resp. Y). Then

$$f'_*(x \cdot_f y) = f_*(x) \cdot y$$

in $A_*(f(|x|) \cap |y|)$, where f' is the induced map from $|x| \cap f^{-1}(|y|)$ to $f(|x|) \cap |y|$. In particular

$$f_*(f^*(y) \cap x) = y \cdot f_*(x)$$

in $A_*(Y)$. (Use Proposition 8.1.1 (c).)

Example 8.1.8. If $f: X \to Y$ is a morphism, with Y non-singular, there is a fibre square

$$\begin{array}{ccc} X & \xrightarrow{\gamma_f} & X \times Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times 1_Y} \\ Y & \xrightarrow[\delta]{} & Y \times Y \end{array},$$

with δ and γ_f regular imbeddings of the same codimension. With x and y cycles as in Definition 8.1, it follows (Theorem 6.2 (c)) that

$$x \cdot_f y = \gamma_f^!(x \times y) = \delta^!(x \times y)\,.$$

In this sense all the intersections of this section are intersections with a diagonal.

Example 8.1.9. Let Y be non-singular, and let δ_r be the r-fold diagonal imbedding of Y in $Y \times \ldots \times Y$. for cycles $y_1, \ldots, y_r$ on Y, define

$$y_1 \cdot \ldots \cdot y_r = \delta_r^!(y_1 \times \ldots \times y_r)$$

in $A_*(|y_1| \cap \ldots \cap |y_r|)$. Then

$$y_1 \cdot \ldots \cdot y_r = y_1 \cdot (y_2 \cdot \ldots \cdot y_r)\,.$$

(See the proof of Proposition 8.1.1 (a).)

Example 8.1.10. Let X be a non-singular closed subvariety of a non-singular variety, i the inclusion of X in Y. Then for any cycles y_1, y_2 on Y,

$$[X] \cdot y_1 \cdot y_2 = i^!(y_1 \cdot y_2) = i^*(y_1) \cdot i^*(y_2)$$

in $A_*(X \cap |y_1| \cap |y_2|)$. (The intersection products in the first two formulas are taken on Y, the third on X.) In particular, if X is a hypersurface on Y, and V_1, V_2 are subvarieties of Y not contained in X, then

$$[X] \cdot [V_1] \cdot [V_2] = [X \cap V_1] \cdot [X \cap V_2]$$

in $A_*(X \cap V_1 \cap V_2)$, the first intersection taken on Y, the second on X.

Example 8.1.11. Let V, W be subvarieties of a non-singular, n-dimensional variety Y. If the diagonal imbedding of the intersection scheme $V \cap W$ into $V \times W$ is a regular imbedding of codimension n, then

$$V \cdot W = [V] \cdot [W] = [V \cap W]$$

in $A_*(V \cap W)$. This equation holds if the imbedding of $V \cap W$ in $V \times W$ is regular of codimension n on an open set of each component of $V \cap W$ (cf. Remark 6.2.2). This is valid in particular if V and W are non-singular varieties meeting transversally at generic points of $V \cap W$.

Example 8.1.12. If X is an n-dimensional non-singular variety, and $\Delta \subset X \times X$ is the diagonal then (Corollary 8.1.1 and Corollary 6.3)

$$\Delta \cdot \Delta = c_n(T_X) \cap [X]\,.$$

In particular, the degree of $\Delta \cdot \Delta$ is the topological Euler characteristic $\int_X c_n(T_X)$. If $n = 1$, $\int \Delta \cdot \Delta = 2 - 2g$ (cf. Example 3.2.13).

Example 8.1.13. Consider a fibre square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

with Y and Y' non-singular, and g a closed imbedding of codimension d and normal bundle N. Then for $x \in A_* X'$, $y \in A_* Y'$,

$$h_*(x) \cdot_f g_*(y) = h_*(c_d(f'^* N) \cap (x \cdot_{f'} y))$$

in $A_* X$. (Use Theorem 6.3.) This formula corrects Lemma 2.2 (4) of Fulton (2).

8.2 Intersection Multiplicities

Let Y be an n-dimensional non-singular variety. Let V and W be closed subschemes of Y of pure dimensions k and l. From the fibre square

$$\begin{array}{ccc} V \cap W & \rightarrow & V \times W \\ \downarrow & & \downarrow \\ \Delta_Y = Y & \xrightarrow[\delta]{} & Y \times Y \end{array}$$

and Lemma 7.1, it follows that every irreducible component Z of $V \cap W$ has dimension at least $k + l - n$. One says that Z is a *proper component* of the intersection of V and W, or that V and W *meet properly at* Z, if

$$\dim Z = k + l - n .$$

If Z is proper, the coefficient of Z in the intersection class $V \cdot W \in A_{k+l-n}(V \cap W)$ is called the *intersection multiplicity* of Z in $V \cdot W$, and denoted $i(Z, V \cdot W; Y)$. In other words,

$$i(Z, V \cdot W; Y) = i(Z, \Delta_Y \cdot (V \times W); Y \times Y) ,$$

where the right side is defined in § 7.1. Setting $T = T_Y|_{V \cap W}$, and $C = C_{V \cap W}(V \times W)$, $i(Z, V \cdot W; Y)$ is the coefficient of $T|_Z$ in the cycle $[C]$.

If Y° is an open subscheme of Y which meets Z, and $V^\circ = V \cap Y^\circ$, $W^\circ = W \cap Y^\circ$, $Z^\circ = Z \cap Y^\circ$, then by Theorem 6.2 (c)

$$i(Z, V \cdot W; Y) = i(Z^\circ, V^\circ \cdot W^\circ; Y^\circ) .$$

If Y is a singular variety, but Z is not contained in the singular locus of Y, the preceding formula, with Y° the non-singular part of Y, defines $i(Z, V \cdot W; Y)$.

If every component of $V \cap W$ is proper, the intersection class $V \cdot W \in A_{k+l-n}(V \cap W)$ is a well-defined *cycle*

$$V \cdot W = \sum_Z i(Z, V \cdot W; Y) \cdot [Z] .$$

Proposition 8.2. *Assume Z is a proper component of $V \cap W$. Then*

(a) $1 \leqq i(Z, V \cdot W; Y) \leqq l(\mathcal{O}_{Z, V \cap W})$.

(b) *If the local ring of $V \times W$ along Z is Cohen-Macaulay, then*

$$i(Z, V \cdot W; Y) = l(\mathcal{O}_{Z, V \cap W}) .$$

(c) *If V and W are varieties, then $i(Z, V \cdot W; Y) = 1$ if and only if the maximal ideal of $\mathcal{O}_{Z,Y}$ is the sum of the prime ideals of V and W; in this case $\mathcal{O}_{Z,V}$ and $\mathcal{O}_{Z,W}$ are regular.*

Proof. (a) and (b) follow from Proposition 7.1. By Proposition 7.2 (and Example 7.2.1 if $V \times W$ is not a variety), the intersection number is 1 precisely when there is an open subset U of $Y \times Y$, meeting Z, such that the diagonal Δ_Y intersects $U \cap (V \times W)$ scheme-theoretically in the (reduced) variety Z. Since $\Delta_Y \cap (V \times W)$ is scheme-theoretically isomorphic to $V \cap W$, this implies the equivalence stated in (c). To see that $\mathcal{O}_{Z,V}$ and $\mathcal{O}_{Z,W}$ are regular, let $A = \mathcal{O}_{Z,Y}$, and let p, q and m be the prime ideals of V, W and Z in A, and let $K = A/m$. From the exact sequence

$$0 \to (p + m^2)/m^2 \to m/m^2 \to m/(p + m^2) \to 0$$

and the identification of $m/(p + m^2)$ with $M_{V,Z}/(M_{V,Z})^2$, it follows that

$$\dim_K((p + m^2)/m^2) \leqq \dim(Y) - \dim(V),$$

with equality if and only if $\mathscr{O}_{V,Z}$ is regular (Lemma A.6.2); similarly for W. Since $p + q = m$,

$$(p + m^2)/m^2 + (q + m^2)/m^2 = m/m^2.$$

Since $(\dim(Y) - \dim(V)) + (\dim(Y) - \dim(W)) = \dim(Y) - \dim(Z)$, the spaces on the left must have the maximum dimensions, and the conclusion follows. □

Remark 8.2. If the ground field is algebraically closed, and Z is a (closed) point, the last displayed equation decomposes the cotangent space of Y at Z into a direct sum of the cotangent spaces of V and W at Z. The intersection number is one precisely when V and W are non-singular and meet transversally at Z. When $\dim(Z) > 0$, the condition says that V and W are generically non-singular along Z, and generically meet transversally along Z.

Example 8.2.1. If $V_1, \ldots, V_r$ are pure-dimensional closed subschemes of a non-singular variety Y, an irreducible component Z of $\bigcap V_i$ is a *proper component* if $\dim(Z) = \sum \dim(V_i) - (r-1) \cdot \dim(Y)$. The *intersection multiplicity*

$$i(Z, V_1 \cdot \ldots \cdot V_r; Y)$$

is the coefficient of Z in the class $V_1 \cdot \ldots \cdot V_r = \delta_r^!(V_1 \times \ldots \times V_r)$, where δ is the r-fold diagonal imbedding of Y in $Y \times \ldots \times Y$. Proposition 8.2 is valid for r factors.

Example 8.2.2. Let $V_1, \ldots, V_n$ be hypersurfaces in an n-dimensional variety Y. Assume: P is a (closed) point of Y, rational over the ground field; P is a non-singular point of Y and of each V_i; P is a component of $\bigcap V_i$; all of the V_i are tangent to each other at P. Then

$$i(P, V_1 \cdot \ldots \cdot V_n; Y) \geqq 2^{n-1}.$$

(Use Example 8.1.10.)

Example 8.2.3 (cf. Samuel (2)). Let K be a field of characteristic $p \neq 0$, with an element a in K with no p^{th} root in K. Let Y be the affine plane over K with coordinates x, y, and let $V = V(y)$, $W = V(y^2 - x^p + a)$, $Z = V(y, x^p - a)$. Then

(a) Z is a proper component of the intersection of V with W, and $i(Z, V \cdot W; Y) = 1$.

(b) If the ground field is extended to a field containing a p^{th} root of a, the intersection still has one irreducible component, but the intersection number becomes p. (Note that $\mathscr{O}_{Z,W}$ is regular, but W is not smooth over $\operatorname{Spec}(K)$ at Z.)

Example 8.2.4. If Y, V, W, Z are as at the beginning of this section, and the imbedding of V in Y is a regular imbedding, then the intersection multiplicity $i(Z, V\cdot W; Y)$ defined in this section agrees with that defined in § 7.1. (Use Corollary 8.1.1, with $x=[W]$.) Similarly if $V_1, \ldots, V_r$ are regularly imbedded subschemes of a non-singular Y, and Z is a proper component of $\bigcap V_i$, then the intersection number $i(Z, V_1\cdot\ldots\cdot V_r; Y)$ obtained by intersecting $V_1\times\ldots\times V_r$ by the diagonal, is the same as that obtained by intersecting the diagonal by $V_1\times\ldots\times V_r$ (cf. Example 8.1.5). In particular, if $V_1, \ldots, V_r$ are hypersurfaces in Y, the intersection number defined here agrees with that defined in Example 7.1.10.

Example 8.2.5. Let $f\colon Y'\to Y$ be a finite surjective morphism of non-singular varieties. Let V', W' be irreducible subvarieties of Y', $V=f(V')$, $W=f(W')$. Let Z be an irreducible component of $V\cap W$, and assume there is only one irreducible component Z' of $f^{-1}(Z)$ which is contained in either V' or W'. Assume that Z is a proper component of $V\cap W$, and that f is étale at the generic point of Z'. Then V' and W' meet properly at Z', and

$$\deg(V'/V)\cdot\deg(W'/W)\cdot i(Z, V\cdot W; Y)=\deg(Z'/Z)\cdot i(Z', V'\cdot W'; Y').$$

(The assumptions imply that $(f\times f)^{-1}(\Delta_Y)\cap(V'\times W')$ is equal to $\Delta_{Y'}\cap(V'\times W')$ in a neighborhood of Z'. Therefore $i(Z', V'\cdot W'; Y')$ is the coefficient of Z' in the class $\delta^!(V'\times W')$ where δ is the diagonal imbedding of Δ_Y in $Y\times Y$ (cf. Theorem 6.2(c)). Then apply Theorem 6.2(a) to the diagram

$$\begin{array}{ccc} V'\times_Y W' & \to & V'\times W' \\ \downarrow & & \downarrow \\ V\cap W & \to & V\times W \\ \downarrow & & \downarrow \\ \Delta_Y & \xrightarrow[\delta]{} & Y\times Y) \end{array}$$

Example 8.2.6. Let Y be a subvariety of $\mathbb{P}^N$, V, W subvarieties of Y, Z a proper component of $V\cap W$ which is not in the singular locus of Y. Then for a generic projection $f\colon Y\to\mathbb{P}^n$, with $n=\dim Y$,

$$i(Z, V\cdot W; Y)=i(f(Z), f(V)\cdot f(W); \mathbb{P}^n).$$

(The hypotheses of Example 8.2.5 are satisfied by a generic projection.) This was one of Severi's methods for reducing intersection multiplicities on general varieties to intersections on projective space (Severi (9) p. 203). A similar construction was used by Chevalley (1) in his algebraic definition of intersection multiplicities.

Example 8.2.7. The intersection number is given by the length as in (b) whenever V and W are Cohen-Macaulay schemes, i.e., all their local rings are Cohen-Macaulay. Indeed, the Cartesian product $V\times W$ is then Cohen-Macaulay ([EGA]IV.6.7.3).

8.3 Intersection Ring

For an n-dimensional, non-singular variety Y, set

$$A^p Y = A_{n-p} Y .$$

With this indexing by codimension, the product $x \otimes y \to x \cdot y$ of § 8.1 reads

$$A^p Y \otimes A^q Y \to A^{p+q} Y ,$$

i.e., the degrees add. Similarly if $f: X \to Y$, the cap product $y \otimes x \to f^*(y) \cap x$ reads

$$A^p Y \otimes A_q X \xrightarrow{\cap} A_{q-p} X .$$

If X is also non-singular the pull-back $y \to f^*(y)$ preserves degrees:

$$f^* : A^p Y \to A^p X .$$

Let $1 \in A^0 Y$ denote the class corresponding to $[Y]$ in $A_n Y$. Set $A^* Y = \bigoplus A^p Y$. We sometimes write $A(Y)$ in place of $A^* Y$ or $A_* Y$ when the grading is unimportant.

Proposition 8.3. (a) *If Y is non-singular, the intersection product makes $A^* Y$ into a commutative, associative ring with unit 1. The assignment*

$$Y \rightsquigarrow A^* Y$$

is a contravariant functor from non-singular varieties to rings.

(b) *If $f: X \to Y$ is a morphism from a scheme X to a non-singular variety Y, the cap product*

$$A^p Y \otimes A_q X \xrightarrow{\cap} A_{q-p} X$$

makes $A_* X$ into an $A^* Y$-module.

(c) *If $f: X \to Y$ is a proper morphism of non-singular varieties, then*

$$f_*(f^* y \cdot x) = y \cdot f_*(x)$$

for all classes x on X, y on Y.

Proof. The associativity and commutativity of $A^* Y$ follow from Proposition 8.1.1 (a) and (b), with $f_i = f = g$ the identity map of Y; that 1 is a unit follows from Corollary 8.1.3. The functoriality of the pull-back follows from Proposition 8.1.1 (a), with $x = [X]$, $y = [Y]$. The projection formula (c) is a special case of Proposition 8.1.1 (c), with $Y = Z$. Proposition 8.1.1 (a), with g the identity on $Y = Z$, gives the formula

$$f^*(y \cdot z) \cap x = f^* z \cap (f^* y \cap x)$$

for $x \in A_* X$, $y, z \in A^* Y$. This formula shows that $A_* X$ is an $A^* Y$-module; setting $x = [X]$, it shows that f^* preserves products. □

Remark 8.3. If Y is non-singular, and $y_1, \ldots, y_r$ are cycles on Y, then

$$y_1 \cdot \ldots \cdot y_r = \delta_r^* (y_1 \times \ldots \times y_r)$$

where δ_r is the diagonal imbedding of Y in $Y \times \ldots \times Y$ (r factors). This follows by writing $\delta_r = (\delta_{r-1} \times 1_Y) \circ \delta_2$ and applying Theorem 6.5 (or see Example 8.1.9).

The ring A^*Y is often called the *Chow ring* of Y; the notation $CH^*(Y)$ is also common.

Example 8.3.1. Let $f\colon X \to Y$ be a morphism, with Y non-singular, X pure dimensional. If f is flat, or a regular imbedding, or a l.c.i. morphism, then the cap product $f^*y \cap [X]$ agrees with the Gysin pull-back f^*y constructed in § 1.7, § 6.2, or § 6.6, respectively. (Use Proposition 8.1.2.)

Example 8.3.2. If $X \xrightarrow{f} Y \xrightarrow{g} Z$, with Y, Z non-singular, then (Proposition 8.1.1 (a))

$$(gf)^*(z) \cap x = f^*(g^*z) \cap x$$

for all $x \in A_*X$, $z \in A^*(Z)$.

Example 8.3.3. Let $P = P(c_{i_1}(E_1), \ldots, c_{i_r}(E_r))$ be a polynomial in Chern classes of vector bundles on a non-singular variety Y. Suppose that $P \cap [Y] = 0$ in $A_*(Y)$. Then for all morphisms $f\colon X \to Y$, and all $x \in A_*X$, $f^*(P) \cap x = 0$ in A_*X. (By Example 8.1.6, $f^*(P) \cap x = x \cdot_f (P \cap [Y])$.) There is therefore no loss of generality in identifying P with its image in A_*Y. This aspect of Poincaré duality will be formalized in Chap. 17. If P is homogeneous of isobaric degree m, we regard $P \in A^m Y$.

Example 8.3.4. Let E be a vector bundle of rank r on a non-singular variety Y, let $X = P(E)$, with projection $p : X \to Y$. Then

$$A^*X \cong A^*Y[\zeta]/(\zeta^r + c_1(E)\,\zeta^{r-1} + \ldots + c_r(E))$$

as graded rings. Here ζ corresponds to $c_1(\mathcal{O}_E(1))$. (This follows from Theorem 3.3 (b).)

Example 8.3.5. If $f\colon X \to Y$ is an isomorphism of non-singular varieties, with inverse f^{-1}, then

$$f_* = (f^{-1})^* = (f^*)^{-1}\colon A(X) \to A(Y).$$

In particular, f_* preserves products. (By the projection formula, $f_*(f^*y) = y$.)

Example 8.3.6. Let Y be non-singular, $y_i \in A^*Y$, $U_i \subset Y$ open subschemes such that y_i restricts to 0 in A^*U_i, for $i = 1, \ldots, r$. Then $y_1 \cdot \ldots \cdot y_r \in A^*Y$ restricts to 0 in $A^*(U_1 \cup \ldots \cup U_r)$. In particular, if the U_i cover Y, then $y_1 \cdot \ldots \cdot y_r = 0$. (Let $Y_i' = Y - U_i$. By Proposition 1.8, y_i is represented by a cycle on Y_i'. The refined intersection product of § 8.1 then gives a class on $\bigcap Y_i'$ which represents $y_1 \cdot \ldots \cdot y_r$.)

Example 8.3.7. Let X and Y be non-singular varieties. Then the exterior product

$$A^*X \otimes A^*Y \xrightarrow{\times} A^*(X \times Y)$$

is a homomorphism of rings, preserving the grading. (Use Example 8.1.4.) If $Y = \mathbb{P}^n$, this homomorphism is an isomorphism.

Example 8.3.8. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, $i: X \to \mathbb{P}^3$ the Segre imbedding. The induced map from $A^1 \mathbb{P}^3 = \mathbb{Z}$ to $A^1(X) = \mathbb{Z} \oplus \mathbb{Z}$ takes 1 to $(1, 1)$. For any surface H in $\mathbb{P}^3$ of degree d, $i^*[H] = (d, d)$. Therefore no irreducible curve C on X, of bidegree (d, e) with $d \neq e$, can be – even set-theoretically – the intersection of X with any surface in $\mathbb{P}^3$.

Example 8.3.9. Let

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{j} & \tilde{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[i]{} & Y \end{array}$$

be a blow-up diagram as in § 6.7, with Y, X, and therefore $\tilde{Y}$, $\tilde{X}$ non-singular. The ring structure on $A(\tilde{Y})$ is determined by the following rules:

(i) $f^*y \cdot f^*y' = f^*(y \cdot y')$.

(ii) $j_*(\tilde{x}) \cdot j_*(\tilde{x}') = j_*(c_1(j^*\mathcal{O}_{\tilde{Y}}(\tilde{X})) \cdot \tilde{x} \cdot \tilde{x}')$.

(iii) $f^*y \cdot j_*(\tilde{x}) = j_*((g^*i^*y) \cdot \tilde{x})$.

Example 8.3.10. Let Y be a non-singular surface, and let $\tilde{Y} \xrightarrow{\pi} Y$ be the blow-up of Y at r points. Then

$$A_1 \tilde{Y} = A_1 Y \oplus \sum_{i=1}^{r} \mathbb{Z}[E_i]$$

with E_i the exceptional divisors, and $A_0 \tilde{Y} = A_0 Y$. The product is given by: $[E_i] \cdot [E_i] = -[P_i]$, $P_i \in E_i$; $[E_i] \cdot [E_j] = 0$ for $i \neq j$, $f^*[D] \cdot [E_i] = 0$, $f^*[D] \cdot f^*[D'] = f^*([D] \cdot [D'])$.

Example 8.3.11. Let V be a complete surface, $\pi: X \to V$ a resolution of singularities of V. Assume that $\pi^{-1}(P)$ is connected for all $P \in V$ (V normal, for example). For any irreducible curve A on V there is a unique one-cycle A^* supported on the exceptional locus of X, with rational coefficients, such that if $\tilde{A}$ is the proper transform of A, then

$$(\tilde{A} \cdot E_i)_X + (A^* \cdot E_i)_X = 0$$

for all irreducible components E_i of the exceptional locus (Example 7.1.16). Set $A' = \tilde{A} + A^*$. There is a unique homomorphism, denoted $\alpha \to \alpha'$, from $A_1 V$ to $A_1(X)_{\mathbb{Q}}$, which takes $[A]$ to $[A']$, for irreducible curves A. (Use Example 7.1.16 (i).). This determines a product

$$A_1 V \otimes A_1 V \to A_0(V)_{\mathbb{Q}}$$

by $\alpha \cdot \beta = \pi_*(\alpha' \cdot \beta')$, which is symmetric, bilinear, and independent of choice of X. For any Cartier divisor D on X, $[D] \cdot \beta$ is the image of the integral class $D \cdot \beta$ (defined in § 2.3) in $A_0 V$.

Example 8.3.12. If $X = Y/G$ is a quotient variety of a non-singular variety Y by a finite group G of automorphisms of Y, then $A_* X_{\mathbb{Q}}$ may also be made into a ring. Indeed, in this case one has an isomorphism (Example 1.7.6)

$$A_* X_{\mathbb{Q}} = (A_* Y_{\mathbb{Q}})^G,$$

so $A_* X_{\mathbb{Q}}$ is the ring of G-invariants of $A(Y)_{\mathbb{Q}}$. In fact, if V, W are subvarieties of X, one may construct a refined intersection class $V \cdot W$ in $A_m(V \cap W)_{\mathbb{Q}}$, $m = \dim V + \dim W - \dim X$; with the notation of Example 1.7.6,

$$V \cdot W = (1/\# G)\, \eta_*(\pi^*[V] \cdot \pi^*[W]),$$

where η is the projection from $\pi^{-1}(V \cap W)$ to $V \cap W$. In particular, for any m-dimensional component Z of $V \cap W$, one has a *rational* intersection number $i(Z, V \cdot W; X)$, the coefficient of Z in the class $V \cdot W$ (cf. Matsusaka (2), Briney (1)). Note that the product on X is determined so that $\pi^*(a \cdot b) = \pi^*(a) \cdot \pi^*(b)$, and $\pi_*((\pi^* a) \cdot c) = a \cdot \pi_*(c)$, for cycles a, b on X, c on Y. For a proof that these definitions are independent of the presentation of X as a quotient variety, see Examples 17.4.10 and 16.1.13.

Mumford (7) has constructed a ring structure on $A_* X_{\mathbb{Q}}$ when X is the moduli space of stable curves of genus g, in characteristic zero. He uses the fact that such X is locally (in the étale topology) a quotient of a non-singular variety by a finite group, and X is globally a quotient of a Cohen-Macaulay variety by a finite group, which dominates the local charts.

Example 8.3.13 (cf. Fulton (2) § 3). If X is a quasi-projective scheme, one may define a graded ring A^*X by

$$A^* X = \varinjlim A^* Y$$

where the limit is over all pairs (Y, f) with Y a non-singular quasi-projective scheme, f a morphism from X to Y. This is a contravariant functor from quasi-projective schemes to graded rings. There are (Example 8.3.2) cap products $A^p X \otimes A_q X \xrightarrow{\frown} A_{q-p} X$, with a projection formula $f_*(f^* x \cap x') = x \cap f_* x'$ for a proper morphism $f: X' \to X$.

Any vector bundle E on X has Chern classes $c_i(E)$ in $A^i X$, satisfying the formal properties of § 3.2, such that for any $f: X' \to X$, $x \in A_* X'$, $f^* c_i(E) \cap x'$ is the class $c_i(f^* E) \cap x'$ defined in § 3.2. (The essential points for this are the following facts (loc. cit. § 3.2): (i) There is a morphism $f: X \to Y$, Y non-singular, and a vector bundle $\tilde{E}$ on Y, with $f^* \tilde{E} \cong E$; (ii) If $f': X \to Y'$, $\tilde{E}'$ is another, there is a non-singular $Z, g: X \to Z$, $h: Z \to Y$, $h': Z \to Y$ with $hg = f$, $h'g = f'$, and $h^* \tilde{E} \cong h'^* \tilde{E}'$; (iii) Any exact sequence of vector bundles on X pulls back from an exact sequence on some non-singular Y.)

More generally, for any scheme X which admits a closed imbedding in a non-singular scheme, one may define A^*X to be $\varinjlim A^* Y$, the limit over all $X \to Y$, Y non-singular. One has the same properties as in the quasi-projective case. For (i), one uses Lemma 18.2.

Another "cohomology theory" to pair with A_* is discussed in Chap. 17.

Example 8.3.14. *Ruled varieties* (cf. B. Levi (1)). Let Y be a non-singular projective curve, let E be a vector bundle of rank d on Y, and let $X = P(E)$, $p: X \to Y$ the projection. Let V be a vector space of dimension $m+1$, $P(V) \cong \mathbb{P}^m$, and let $f: X \to P(V)$ be a morphism, with $f^* \mathcal{O}_V(1) = \mathcal{O}_E(1) \otimes p^* L$ for some line bundle L on Y; the fibres X_y of p are imbedded as linear subspaces of $P(V)$. Set $\zeta = c_1(\mathcal{O}_E(1))$, $h = c_1(f^* \mathcal{O}_V(1))$. For any subvariety W

of X of dimension w set

$$n(W) = \int_X h^w \cap [W] = \deg(W/f(W)) \cdot \deg(f(W)).$$

(In characteristic zero, $\deg(W/f(W))$ is the number of rulings, i.e., fibres X_y of p, which pass through a general point of $f(W)$.) Set

$$k(W) = \int_X \zeta^{w-1}[X_y] \cdot [W] = \int_X h^{w-1} \cdot p^*[y] \cdot [W].$$

($k(W)$ is the degree of $X_y \cap W$ as a subscheme of $X_y = \mathbb{P}^{d-1}$, for generic $y \in Y$.) Then

$$[W] = k(W)\, h^{d-w} + p^*(\alpha)\, h^{d-w-1},$$

where α is a zero-cycle on Y of degree $n(W) - k(W) \cdot N$, with $N = n(X)$. (Use Theorem 3.3(b).) If $W_1, \ldots, W_r$ are subvarieties of X with $\sum \operatorname{codim}(W_i) = d$, then

$$\deg(W_1 \cdot \ldots \cdot W_r) = \sum_{i=1}^{r} n_i \left(\prod_{j \neq i} k_j\right) - (r-1)\, N \prod_{i=1}^{r} k_i,$$

where $n_i = n(W_i)$, $k_i = k(W_i)$, $N = n(X)$.

Let Y be a non-singular curve of genus g in $P(V) = \mathbb{P}^m$, and let X be the ruled surface of tangent lines to Y, i.e. $X = P(E)$, where E is the "principal parts" bundle, constructed to make the following diagram have exact columns and rows:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to \mathcal{O} \to & & E & \to & T_Y & \to 0 \\
\| & & \downarrow & & \downarrow & & \\
0 \to \mathcal{O} \to & & V \otimes \mathcal{O}_Y(1) & \to & T_{\mathbb{P}^m}|_Y & \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & N_Y\mathbb{P}^m & = & N_Y\mathbb{P}^m & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

In this case $N = n(X) = 2g - 2 + 2\deg(Y)$. If $m = 2$, N is the degree of the dual curve.

For more on ruled surfaces, see Beauville (2) III.

8.4 Bézout's Theorem (Classical Version)

Intersections on projective space $\mathbb{P}^n$ are particularly simple, as well as important for applications.

We saw in § 3.3 that $A_k(\mathbb{P}^n) \cong \mathbb{Z}$, with generator $[L^k]$, L^k a k-dimensional linear subspace of $\mathbb{P}^n$, $k = 0, 1, \ldots, n$. For any k-cycle α on $\mathbb{P}^n$, the *degree* of α, $\deg(\alpha)$, is defined to be the integer such that

$$\alpha = \deg(\alpha) \cdot [L^k]$$

in $A_k\mathbb{P}^n$. Equivalently, $\deg(\alpha) = \int_{\mathbb{P}^n} c_1(\mathcal{O}(1))^k \cap \alpha$. This follows from the fact that $c_1(\mathcal{O}(1))^k \cap [L^k] = 1$ (Proposition 3.1 (a) (ii)).

Write $A^d\,\mathbb{P}^n = A_{n-d}\,\mathbb{P}^n$.

Proposition 8.4 (Bézout's Theorem). *Let* $\alpha_i \in A^{d_i}\mathbb{P}^n$, $i = 1, \ldots, r$. *If* $d_1 + \ldots + d_r \leqq n$, *then*

$$\deg(\alpha_1 \cdot \ldots \cdot \alpha_r) = \deg(\alpha_1) \cdot \ldots \cdot \deg(\alpha_r)\,.$$

Proof. From the preceding considerations, the ring homomorphism from $\mathbb{Z}[h]/(h^{n+1})$ to $A^*\mathbb{P}^n$ which takes h to $c_1(\mathcal{O}(1)) \cap [\mathbb{P}^n] = [L^{n-1}]$ is an isomorphism, from which the proposition follows. (For a more direct proof, see Example 6.2.6.) □

If $V_1, \ldots, V_r$ are pure-dimensional subschemes of $\mathbb{P}^n$ which meet properly, i.e. the irreducible components $Z_1, \ldots, Z_t$ of $\bigcap V_i$ all have codimension equal to $\sum \operatorname{codim}(V_i, \mathbb{P}^n)$, then

$$V_1 \cdot \ldots \cdot V_r = \sum_{j=1}^{t} i(Z_j, V_1 \cdot \ldots \cdot V_r; \mathbb{P}^n) \cdot [Z_j]\,,$$

where $i(Z_j, V_1 \cdot \ldots \cdot V_r; \mathbb{P}^n)$ is the intersection multiplicity (cf. Example 8.2.1). In this case Bézout's theorem says that

$$\sum_{j=1}^{t} i(Z_j, V_1 \cdot \ldots \cdot V_r; \mathbb{P}^n) \cdot \deg(Z_j) = \prod_{i=1}^{r} \deg(V_i)\,. \tag{1}$$

For example, if H is a hypersurface, and V is a subvariety of $\mathbb{P}^n$ not contained in H, then

$$H \cdot V = [H \cap V] = \sum_{j=1}^{t} m_j [Z_j]$$

with

$$\sum m_j \cdot \deg(Z_j) = \deg(H) \cdot \deg(V)\,. \tag{2}$$

Another important case is the one considered originally by Bézout: Let $H_1, \ldots, H_n$ be hypersurfaces in $\mathbb{P}^n_K$ with only a finite number of common points. For each $P \in H_i$, let $\mathcal{O}_P(\bigcap H_i) = \mathcal{O}_{P,\mathbb{P}^n}/(h_1, \ldots, h_n)$ be the local ring of $\mathbb{P}^n$ at P modulo the ideal generated by local equations h_i for H_i at P (i.e., the local ring of the scheme $\bigcap H_i$ at P). Then

$$\sum_P \dim_K(\mathcal{O}_P(\bigcap H_i)) = \prod_{j=1}^{n} \deg(H_j)\,. \tag{3}$$

Indeed, in this case each H_i is Cohen-Macaulay, so by Proposition 8.2 (b)

$$i(P, H_1 \cdot \ldots \cdot H_n; \mathbb{P}^n) = l(\mathcal{O}_P(\bigcap H_i)).$$

For a (closed) point P, $\deg P = [R(P) : K]$, and

$$\dim_K \mathcal{O}_P(\bigcap H_i) = l(\mathcal{O}_P(\bigcap H_i)) \cdot [R(P) : K]$$

by Lemma A.1.3. Thus (3) follows from (1).

Example 8.4.1. Let Y be an n-dimensional variety, smooth over a field K. If $V_1, \ldots, V_r$ are pure-dimensional subschemes of Y which meet properly in a finite set of points, then

$$\sum_P i(P, V_1 \cdot \ldots \cdot V_r; Y) \cdot [R(P):K] = \int_Y [V_1] \cdot \ldots \cdot [V_r].$$

if $H_1, \ldots, H_n$ are hypersurfaces meeting properly, then

$$\sum_P \dim_K(\mathcal{O}_P(\cap H_i)) = \int_Y [H_1] \cdot \ldots \cdot [H_r].$$

Example 8.4.2. (a) If s (resp. t) is the class of a hyperplane on $\mathbb{P}^n$ (resp. $\mathbb{P}^m$), then (Example 8.3.7)

$$A^*(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{Z}[s, t]/(s^{n+1}, t^{m+1}).$$

(b) If $H_1, \ldots, H_{n+m}$ are hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^m$, and H_i has bidegree (a_i, b_i) (i.e., $[H_i] = a_i \cdot s + b_i \cdot t$), then

$$\int [H_1] \cdot \ldots \cdot [H_{n+m}] = \sum a_{i_1} \cdot \ldots \cdot a_{i_n} b_{j_1} \cdot \ldots \cdot b_{j_m},$$

where the sum is over all permutations $(i_1, \ldots, i_n, j_1, \ldots, j_m)$ of $(1, \ldots, n+m)$ with $i_1 < i_2 < \ldots < i_n$ and $j_1 < \ldots < j_m$.

(c) if Δ is the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$, then

$$[\Delta] = \sum_{i=0}^{n} s^i t^{n-i}$$

in $A^n(\mathbb{P}^n \times \mathbb{P}^n)$. (Write $[\Delta] = \sum a_i s^i t^{n-i}$, and intersect both sides with $[L \times M]$, where L and M are linear spaces of complementary dimensions meeting transversally at a point. Alternatively, the composite of the canonical maps

$$\mathrm{pr}_1^* \mathcal{O}(-1) \hookrightarrow \mathcal{O}^{\oplus(n+1)} \hookrightarrow \mathrm{pr}_2^*(T_{\mathbb{P}^n}(-1))$$

corresponds to a section of $\mathrm{pr}_2^*(T_{\mathbb{P}^n}) \otimes \mathcal{O}(1, -1)$ whose zero-scheme is Δ; the top Chern class of this bundle is $\sum s^i t^{n-i}$.)

Example 8.4.3. (a) Let $\varphi: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ be the Segre imbedding, $N = nm + n + m$. If u is the hyperplane class on $\mathbb{P}^N$, then $\varphi^* u = s + t$. The degree of the image of φ is $\binom{n+m}{n}$.

(b) If $\psi: \mathbb{P}^n \to \mathbb{P}^N$ is the m-fold Veronese imbedding, $N = \binom{n+m}{n} - 1$, and s, u are hyperplane classes on $\mathbb{P}^n$ and $\mathbb{P}^N$, then $\psi^* u = m \cdot s$. If V is a k-dimensional subvariety of $\mathbb{P}^n$ of degree d, then $\psi(V)$ has degree $d \cdot m^k$.

Example 8.4.4. Let V be a subvariety of $\mathbb{P}^n \times \mathbb{P}^m$ of dimension k, so

$$[V] = \sum_{i+j=k} a_{ij} s^{n-i} t^{m-j};$$

the a_{ij} are the *bidegrees* of V. Define a variety $V' \subset \mathbb{P}^{n+m+1}$ as follows. Geometrically,

$$V' = \{(\lambda x_0 : \lambda x_1 : \ldots : \lambda x_n : \mu y_0 : \ldots : \mu y_m) \in \mathbb{P}^{n+m+1} \mid (x) \times (y) \in V, (\lambda : \mu) \in \mathbb{P}^1\}.$$

Algebraically, if I is the bihomogeneous ideal in $K[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ defining V, then the ideal of V' is generated by those elements in I which are homogeneous in all variables. Then V' is a variety of dimension $k+1$, and

$$\deg V' = \sum_{i+j=k} a_{ij}$$

(cf. Van der Waerden (7)).

Example 8.4.5. Let V, W be irreducible subvarieties of $\mathbb{P}^n$ of dimensions k, l respectively. Let $J(V, W) \subset \mathbb{P}^{2n+1}$ be the *ruled join* of V and W, i.e. let $\mathbb{P}_1^n$ (resp. $\mathbb{P}_2^n$) be the linear subspace of $\mathbb{P}^{2n+1}$ where the last (resp. the first) $n+1$ coordinates vanish; regard $V \subset \mathbb{P}_1^n$, $W \subset \mathbb{P}_2^n$, and define $J(V, W)$ be the union of all lines in $\mathbb{P}^{2n+1}$ from points of V to points of W. In the notation of the previous example, $J(V, W) = (V \times W)'$. If p, q are the ideals of V and W, then the ideal of $J(V, W)$ is the ideal in $K[X, Y]$ generated by all $f(X)$, $f \in p$ and $g(Y)$, $g \in q$.

Let L be the linear subspace of $\mathbb{P}^{2n+1}$ defined by

$$L = \{(x_0 : \ldots : x_n : y_0 : \ldots : y_n) \mid x_i = y_i \quad \text{for} \quad 0 \leqq i \leqq n\}.$$

The imbedding $i: \mathbb{P}^n \to \mathbb{P}^{2n+1}$, which takes $(x_0 : \ldots : x_n)$ to $(x_0 : \ldots : x_n : x_0 : \ldots : x_n)$, maps $\mathbb{P}^n$ isomorphically onto L, and determines an isomorphism of schemes:

$$V \cap W \cong L \cap J(V, W).$$

Moreover, if $\mathbb{P}_{\circ}^{2n+1} = \mathbb{P}^{2n+1} - (\mathbb{P}_1^n \cup \mathbb{P}_2^n)$, the canonical projection p from $\mathbb{P}_{\circ}^{2n+1}$ to $\mathbb{P}^n \times \mathbb{P}^n$ maps L isomorphically onto the diagonal Δ, and maps $L \cap J(V, W)$ to $\Delta \cap (V \times W)$. This projection p is smooth – in fact an $(\mathbb{A}^1 - \{0\})$-bundle. The claim is that

$$V \cdot W = L \cdot J(V, W)$$

in $A_{k+l-n}(V \cap W)$. To prove this consider the diagram

$$\begin{tikzcd}
 & \mathbb{P}_{\circ}^{2n+1} \arrow[dd, "p"] & J_0(V, W) \arrow[l] \arrow[dd, "p'"] \\
\mathbb{P}^n \arrow[ur, "i_{\circ}"] \arrow[dr, "\delta"'] & & \\
 & \mathbb{P}^n \times \mathbb{P}^n & V \times W \arrow[l]
\end{tikzcd}$$

where $J_{\circ}(V, W) = J(V, W) \cap \mathbb{P}_{\circ}^{2n+1}$. By the definition of $V \cdot W$, and Proposition 6.5(b),

$$V \cdot W = \delta^! [V \times W] = i_{\circ}^! \, p'^* [V \times W] = i_{\circ}^! [J_{\circ}(V, W)].$$

And $i_{\circ}^! [J_{\circ}(V, W)] = i^! [J(V, W)] = L \cdot J(V, W)$ by Theorem 6.2(b) and Corollary 8.1.1.

In fact, by Example 6.5.4, the canonical decomposition of the intersection class $\Delta \cdot (V \times W)$ is the same as the canonical decomposition of the intersection class $L \cdot J(V, W)$. Note in particular that $\deg J(V, W) = \deg(V) \cdot \deg(W)$, as stated in Example 8.4.4.

A similar discussion is valid for the intersection of r varieties in $\mathbb{P}^n$; the ruled join is a subvariety of $\mathbb{P}^{r(n+1)-1}$. In summary, all intersection products on projective space can be realized by an intersection of a subvariety by a linear space (cf. Gaeta (1)).

Example 8.4.6. Let $V_1, \ldots, V_r$ be subvarieties of $\mathbb{P}^n$. Let $Z_1, \ldots, Z_t$ be the irreducible components of $V_1 \cap \ldots \cap V_r$. Then

$$\sum_{i=1}^{t} \deg(Z_i) \leqq \prod_{j=1}^{r} \deg(V_j) .$$

In particular, the number of irreducible components of V_j is at most the product of the degrees. (By the preceding example, one may assume $r = 2$ and V_1 is a linear subspace. Write V_1 as an intersection of hyperplanes; by induction one may assume V_1 is a hyperplane. Then either $V_1 \supset V_2$, or $V_1 \cdot V_2 = \sum m_i [Z_i]$, from which the inequality is clear.) A refinement of this inequality will be discussed in § 12.3. For an application, see Bass-Connell-Wright (1) p. 293.

A typical classical application of Bézout's theorem is a proof that an irreducible subvariety $X \subset \mathbb{P}^n$ of degree d, not contained in a hyperplane, satisfies

$$\dim X + d \geqq n + 1 .$$

Indeed, taking generic linear sections, one is reduced to the case where X is a curve. There is a hyperplane through any n points, which contradicts Bézout's theorem if $n > d$ and the n points are on the curve.

Example 8.4.7. Let X be a projective scheme. Let $V_1, \ldots, V_r$ be subvarieties of X, and let d_i be the degree of V_i with respect to some imbedding of X in $\mathbb{P}^N$. If $V_1 \cap \ldots \cap V_r$ is finite, then

$$\operatorname{card}(V_1 \cap \ldots \cap V_r) \leqq d_1 \cdot \ldots \cdot d_r .$$

(Use Example 8.4.6.) Note that this upper bound depends only on the (numerical) equivalence classes of the $[V_i]$.

Example 8.4.8. Let L be a linear subspace of $\mathbb{P}^n$, V a subvariety of $\mathbb{P}^n$, Z a proper component of $L \cap V$. If $\dim Z = k$, then for a generic linear subspace $M \subset \mathbb{P}^n$ of codimension k, and a point P on $Z \cap M$, and $L' = L \cap M$, then P is a proper component of $L' \cap V$ and

$$i(Z, L \cdot V; \mathbb{P}^n) = i(P, L' \cdot V; \mathbb{P}^n) .$$

(Choose M transversal to Z at P and apply associativity of intersection products.) Together with Examples 8.4.5 and 8.2.6, this shows how intersection multiplicities on arbitrary non-singular varieties are determined by intersection multiplicities of varieties of complementary dimension in $\mathbb{P}^n$, with one factor a linear subspace.

Example 8.4.9. Let $V_1, \ldots, V_r$ be subvarieties of $\mathbb{P}^n$ with

$$m = \sum_{i=1}^{r} \dim(V_i) - (r-1)\, n \geqq 0 .$$

(a) The intersection $\bigcap_{i=1}^{r} V_i$ cannot be empty.

(b) Assume that each V_i has odd degree. Then $V_1 \cap \ldots \cap V_r$ must contain an m-dimensional variety of odd degree. (The intersection class $V_1 \cdot \ldots \cdot V_r$ is represented by a cycle of odd degree *on* $\cap V_i$.) In particular, for at least one irreducible component Z of $\cap V_i$, the restriction of $\mathcal{O}(1)$ to Z is not the square of any line bundle.

Example 8.4.10 (cf. Fulton-MacPherson (2)). Given $V, W \subset \mathbb{P}^n$, with $\dim V + \dim W = n$, there is no way to assign integers to the irreducible components of $V \cap W$ so that the sum is the Bézout number, at least if one requires the assignment to be preserved by automorphisms of $\mathbb{P}^n$. For example, let $n = 4$,

$$V = V(x_3^3 - x_1 x_2 (x_2 - 2x_1), x_4)$$
$$W = V(x_4^3 - x_2 x_1 (x_1 - 2x_2), x_3) .$$

Then $V \cap W$ is the union of two lines; the involution interchanging x_1 and x_2, and x_3 and x_4, takes V to W and interchanges the two lines; thus each would have to be assigned the same number, but the Bézout number 9 is odd. (In fact, the point of intersection of the two lines is a distinguished variety for the intersection of $V \times W$ by Δ.)

Example 8.4.11. Let V be a subvariety of $\mathbb{P}^n_K$ of dimension d, with K infinite. Then there is a linear space $L \subset \mathbb{P}^n$ of codimension d which meets V properly, and

$$\deg(V) = \sum_P i(P, L \cdot V; \mathbb{P}^n) \cdot [R(P):K] .$$

(Choose, inductively, d hyperplanes $H_1, \ldots, H_d$ so that H_i meets all components of $V \cap H_1 \cap \ldots \cap H_{i-1}$ properly.)

Example 8.4.12. *A Bertini theorem.* Let X be an n-dimensional subvariety of $\mathbb{P}^m$, over an algebraically closed field. Let $0 < k < m$, and let G be the Grassmannian of k-planes in $\mathbb{P}^m$ (cf. § 14.7).

(a) There is a non-empty open set $U \subset G$ such that for all L in U,

$$\dim(X \cap L) = n + k - m$$

if $n + k - m \geqq 0$, or $X \cap L = \emptyset$ if $n + k - m < 0$. (Set

$$I = \{(x, L) \in X \times G \mid x \in L\} .$$

The projection from I to X is smooth of relative dimension $k(m-k)$, so $\dim(I) - \dim(G) = n + k - m$. The generic fibre of $I \to G$ therefore has the indicated dimension.)

(b) Let X_0 be the smooth locus of X. For $x \in X_0$, let $T_x \subset \mathbb{P}^m$ be the projective n-plane tangent to X at x. If $n + k \geqq m$, there is a non-empty open set $U \subset G$ such that every L in U meets X_0 transversally, i.e.

$$\dim(T_x \cap L) = n + k - m$$

for all $x \in X_0 \cap L$. (Set

$$J_l = \{(x, L) \in X_0 \times G \mid x \in L, \dim(T_x \cap L) = n + k - m + l\} ,$$

$l = 1, 2, \ldots$. The projection from J_l to X_0 is smooth, and a dimension count shows that $\dim J_l < \dim G$.)

(c) Set $k = m - n$. There is an open set $U \subset G$ such that every $L \in U$ meets X transversally in $\deg(X)$ points. (Apply (b) to X_0, (a) to the components of $X - X_0$. More generally for X, Y subvarieties of $\mathbb{P}^m$ of complementary dimension, X meets $\sigma(Y)$ transversally, in $\deg(X)\deg(Y)$ points, for generic $\sigma \in \mathrm{Aut}(\mathbb{P}^m)$ (Appendix B.9).

Example 8.4.13. *Resultants.* Fix positive integers $d_1, \dots, d_n$. The hypersurfaces of degree d_i in $\mathbb{P}^n$ are parametrized by $\mathbb{P}^{m_i}$, $m_i = \binom{d_i + n}{n} - 1$. Let $x_0, \dots, x_n$ be homogeneous coordinates on $\mathbb{P}^n$, $t^i_{(j)}$ homogeneous coordinates on $\mathbb{P}^{m_i}$, so that $\mathscr{F}_i = \sum t^i_{(j)} x^{(j)}$ defines the universal hypersurface in $\mathbb{P}^n \times \mathbb{P}^{m_i}$. Set $T = \mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_n}$, and define $\mathscr{V} \subset \mathbb{P}^n \times T$ by

$$\mathscr{V} = \{(x, t^1, \dots, t^n) \mid \mathscr{F}_i(t^i, x) = 0 \quad \text{for} \quad i = 1, \dots, n\}.$$

By projecting to $\mathbb{P}^n$, one verifies that $\mathscr{V}$ is a smooth irreducible subvariety of $\mathbb{P}^n \times T$ of codimension n. The fibre $\mathscr{V}_t$ of $\mathscr{V}$ over t in T is the intersection of the corresponding hypersurfaces. Set

$$\begin{aligned} L &= \{(x) \in \mathbb{P}^n \mid x_0 = x_1 = 0\}, \\ T^\circ &= \{t \in T \mid \mathscr{V}_t \cap L = \emptyset\}, \\ \mathscr{V}^\circ &= \mathscr{V} \cap (\mathbb{P}^n \times T^\circ), \end{aligned}$$

so T° is a non-empty open subvariety of T, $\mathscr{V}^\circ$ is open in $\mathscr{V}$, and closed in $(\mathbb{P}^n - L) \times T^\circ$. Let p be the projection from $\mathbb{P}^n - L$ to $\mathbb{P}^1$, $p(x) = (x_0 : x_1)$. Let

$$\mathscr{R}^\circ = (p \times 1)(\mathscr{V}^\circ) \subset \mathbb{P}^1 \times T^\circ.$$

Since $\mathscr{V}^\circ$ is proper over T°, the induced morphism f from $\mathscr{V}^\circ$ to $\mathscr{R}^\circ$ is proper, so $\mathscr{R}^\circ$ is a closed subvariety of $\mathbb{P}^1 \times T^\circ$. In fact, f maps $\mathscr{V}^\circ$ birationally onto $\mathscr{R}^\circ$, i.e.

$$f_*[\mathscr{V}^\circ] = [\mathscr{R}^\circ].$$

Indeed, if $T^{\circ\circ}$ consists of those $t \in T^\circ$ for which the corresponding hypersurfaces meet in $d_1 \cdot \ldots \cdot d_n$ points with distinct projections to $\mathbb{P}^1$, f restricts to an isomorphism over $T^{\circ\circ}$. In particular $\mathscr{R}^\circ$ is a hypersurface in $\mathbb{P}^1 \times T^\circ$. Let

$$R = R(x_0, x_1; t^1, \dots, t^n)$$

be the equation, unique up to scalars, for the closure of $\mathscr{R}^\circ$ in $\mathbb{P}^1 \times T$; R is called the *resultant*. For any particular hypersurfaces $F_1, \dots, F_n$ of degrees $d_1, \dots, d_n$, the resultant $R(F_1, \dots, F_n)$ is obtained by substituting the corresponding coefficients $t^1, \dots, t^n$ into R. The resultant is characterized, up to multiplication by polynomials in $t^1, \dots, t^n$, by being homogeneous of degree $d_1 \cdot \ldots \cdot d_n$ in x_0, x_1 (cf. below) and vanishing at points $((x_0 : x_1), (t^1, \dots, t^n))$, in any algebraic closure of the ground field, where the $F_i(t^i, x)$ have common zeros not on L. (For some of the many classical methods for calculating resultants, see Salmon (2) pp. 66–98, and Van der Waerden (4) § XI. For a modern discussion see Jouanolou (4) and Lazard (1).)

Theorem. *Let $V_i \subset \mathbb{P}^n$ be the hypersurface defined by equation F_i, $i = 1, \dots, n$. Assume $\bigcap V_i \cap L = \emptyset$, and $\bigcap V_i$ is finite. Then, for any $Q \in \mathbb{P}^1$,*

$$(*) \qquad \mathrm{ord}_Q(R(F_1, \dots, F_n)) = \sum_{\substack{P \in \mathbb{P}^n - L \\ p(P) = Q}} i(P, V_1 \cdot \ldots \cdot V_n; \mathbb{P}^n).$$

To prove this, let t: Spec$(K) \to T$ be the point corresponding to $F_1, \ldots, F_n$. Let $\mathscr{V}_i^\circ$ be the subscheme of $(\mathbb{P}^n - L) \times T^\circ$ defined by $F_i(x, t^i)$. Form the fibre diagram:

$$\begin{array}{ccccccc}
\bigcap V_i & \to & V_1^\circ \times \ldots \times V_n^\circ & \to & \mathbb{P}^1 & \to & \operatorname{Spec}(K) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow t \\
\mathscr{V}^\circ & \to & \mathscr{V}_1^\circ \times \ldots \times \mathscr{V}_n^\circ & \to & \mathbb{P}^1 \times T^\circ & \to & T^\circ \\
\downarrow & & \downarrow & & & & \\
\mathbb{P}^n & \underset{\delta}{\to} & \mathbb{P}^n \times \ldots \times \mathbb{P}^n . & & & &
\end{array}$$

By Theorem 6.4, and Theorem 6.2 (b), (c),

$$\begin{aligned} t^![\mathscr{V}^\circ] &= t^!\,\delta^![\mathscr{V}_1^\circ \times \ldots \times \mathscr{V}_n^\circ] = \delta^!\,t^![\mathscr{V}_1^\circ \times \ldots \times \mathscr{V}_n^\circ] \\ &= \delta^![V_1^\circ \times \ldots \times V_n^\circ] = [V_1] \cdot \ldots \cdot [V_n] \end{aligned}$$

in $A_0(\bigcap V_i)$. Let f' be the morphism from $\bigcap V_i$ to $\mathbb{P}^1$ induced by p. By Theorem 6.2 (a),

$$f'_*([V_1] \cdot \ldots \cdot [V_n]) = f'_*\,t^![\mathscr{V}^\circ] = t^!\,f_*[\mathscr{V}^\circ] = t^![\mathscr{R}^\circ].$$

Since $R(F_1, \ldots, F_n) \neq 0$, $t^![\mathscr{R}^\circ]$ is the cycle determined by the divisor of $R(F_1, \ldots, F_n)$. The equality of these cycles in $A_0(f'(\bigcap V_i))$ gives (*).

If one does not assume $\bigcap V_i$ finite, but only that $\bigcap V_i$ is contained in a finite number of fibres of p, the same proof shows that $\operatorname{ord}_Q(R(F_1, \ldots, F_n))$ is the total contribution of the intersection class $V_1 \cdot \ldots \cdot V_n$ which is supported on $\bigcap V_i \cap p^{-1}(Q)$.

Adding formula (*) over $Q \in \mathbb{P}^1$, for any $V_1, \ldots, V_n$ which meet properly (off L) shows that R has degree $d_1 \cdot \ldots \cdot d_n$ in x_0, x_1.

Notes and References

The procedure of *reduction to the diagonal,* i.e., of intersecting two cycles on a non-singular variety by intersecting their exterior product with the diagonal, has played an important role in intersection theory. One may detect its presence in the nineteenth century theory of correspondences (cf. Pieri (1)). Apparently Weil (2) in 1946 was the first to use this principle in modern geometry, although Lefschetz (3) had made extensive use of cycles on product manifolds.

Following Lefschetz's model in topology, and ideas of Severi in algebraic geometry, the construction of an intersection ring A^*Y for a non-singular projective variety has usually proceeded in two separate steps: (1) a theory of intersection multiplicities was developed (see the notes to Chap. 7), so that properly intersecting cycles had a well-defined intersection cycle; (2) one showed that two cycle classes have representatives which meet properly, and that the resulting product is well defined up to rational equivalence. This is the approach followed by Samuel (3), Chow (1), and Chevalley (2). Earlier