INTERSECTION OF TROPICAL CURRENT AND TORIC SET

1. Notation

For any lattice N, denote

$$\begin{aligned} T_N &= \mathbb{C}^* \otimes_{\mathbb{Z}} N, \\ N_{\mathbb{R}} &= \mathbb{R} \otimes_{\mathbb{Z}} N, \\ S_N &= \mathbb{S}^1 \otimes_{\mathbb{Z}} N. \end{aligned}$$

We will also consider the maps

$$\begin{aligned} -\log : \mathbb{C}^* &\longrightarrow \mathbb{R} \\ z &\longmapsto -\log |z|, \end{aligned}$$

$$\begin{aligned} \arg : \mathbb{C}^* &\longrightarrow \mathbb{S}^1 \\ z &\longmapsto z/|z|, \end{aligned}$$

and, the maps

$$\begin{aligned} \mathrm{Log}_N &:= -\log \otimes 1_N : T_N \longrightarrow N_{\mathbb{R}} \\ \mathrm{Arg}_N &:= \arg \otimes 1_N : T_N \longrightarrow S_N. \end{aligned}$$

By taking $(\mathbb{C}^*)^n = \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Z}^n$, $\mathbb{R}^n = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}^n$ and $(\mathbb{S}^1)^n = \mathbb{S}^1 \otimes_{\mathbb{Z}} \mathbb{Z}^n$, we have in particular the maps

$$\begin{aligned} \mathrm{Log}_n &:= \mathrm{Log}_{\mathbb{Z}^n} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n \\ \mathrm{Arg}_n &:= \mathrm{Arg}_{\mathbb{Z}^n} : (\mathbb{C}^*)^n \longrightarrow (\mathbb{S}^1)^n. \end{aligned}$$

If σ is any rational polyhedron in $\mathbb{R}^n$, we will denote

$$\begin{aligned} \mathrm{aff}(\sigma) &:= \text{ the affine span of } \sigma, \\ \sigma^\circ &:= \text{ the relative interior of } \sigma, \\ H_\sigma &:= \text{ the linear subspace parallel to } \mathrm{aff}(\sigma), \\ N_\sigma &:= H_\sigma \cap \mathbb{Z}^n \\ N(\sigma) &:= \mathbb{Z}^n / N_\sigma. \end{aligned}$$

Finally, we will consider the quotient map

$$q_\sigma : \mathbb{Z}^n \longrightarrow N(\sigma),$$

and, given $a \in \mathrm{aff}(\sigma)$, the submersion

$$\pi_\sigma := (\arg \otimes q_\sigma) \circ e^{(-a)} : \mathrm{Log}_n^{-1}(\mathrm{aff}(\sigma)) \longrightarrow S_{N(\sigma)},$$

where $e^{(-a)} : \mathrm{Log}_n^{-1}(\mathrm{aff}(\sigma)) \longrightarrow \mathrm{Log}_n^{-1}(H_\sigma)$ is the pointwise product $z \mapsto e^{(-a)} \cdot z$, and with $\arg \otimes q_\sigma$ we mean its restriction to $\mathrm{Log}_n^{-1}(H_\sigma) \subseteq (\mathbb{C}^*)^n$.

Notice that π_σ doesn't depend on the choice of a and, in particular, $\pi_\sigma = \arg \otimes q_\sigma$ if $0 \in \sigma$.

2. Results

[1] Is this what we want?

Let $\mathcal{C}$ be a tropical cycle in $\mathbb{R}^n$ and $D = \{z^\alpha - z^\beta = 0\} \subset (\mathbb{C}^*)^n$ a toric set with $\alpha, \beta \in \mathbb{Z}^n$. Assume also that the hyperplane $V = \{x \in \mathbb{R}^n : (\alpha - \beta) \cdot x = 0\}$ satifies that for all $\sigma \in \mathcal{C}(p)$, either $\sigma \cap V = \emptyset$ or $\sigma^\circ \cap V \neq \emptyset$ with $\dim(\sigma \cap V) = p-1$. Without loss of generality, we can assume $\alpha - \beta$ is a primitive ray generator. Note that $D = T_{N_V}$. Finally, assume that for all $\sigma \in \Sigma$ and for all $x \in S_{N(\sigma)}$, $\dim(D \cap \pi_\sigma^{-1}(x)) = p-1$.

[2] Can this be deduced by the previous assumption, or viceversa? MAYBE: Lemma 8.6 of tropical toolkit can be somehow generalised? Note that $\mathrm{Trop}(\pi_\sigma^{-1}(x)) = (N_\sigma)_\mathbb{R}$ is transversal to V=Trop(D).

Proposition 2.1. *$[D] \wedge \mathcal{T}_\mathcal{C} = \mathcal{T}_{\mathcal{C} \cap V}$ as currents in D.*

Proof. We will prove the result locally. Let $\sigma \in \mathcal{C}(p)$ such that $\sigma^\circ \cap V \neq \emptyset$, and assume first that $0 \in \sigma$. Choose a $\mathbb{Z}$-basis $\{e_1, \ldots, e_{p-1}, e_n\}$ of N_σ that completes the $\mathbb{Z}$-basis $\{e_1, \ldots, e_{p-1}\}$ of $N_{\sigma \cap V} = H_{\sigma \cap V} \cap \mathbb{Z}^n = H_\sigma \cap V \cap \mathbb{Z}^n = N_\sigma \cap V$. Finally, complete $\{e_1, \ldots, e_{p-1}\}$ to a $\mathbb{Z}$-basis $\{e_1, \ldots, e_{n-1}\}$ of N_V. Notice that the lattice N_V contains the sublattices $N_{\sigma \cap V}$ and $N'(\sigma) = N_V / N_{\sigma \cap V}$. Call $i : N_V \hookrightarrow \mathbb{Z}^n$ the inclusion homomorphism.

Consider the isomorphism of lattices

$$\varphi : N_V \longrightarrow \mathbb{Z}^{n-1}$$

that takes $\{e_1, \ldots, \ldots, e_{n-1}\}$ to the canonical basis of $\mathbb{Z}^{n-1}$. This isomorphism induces isomorphisms of Lie groups

$$\begin{aligned} \varphi_\mathbb{R} := 1_\mathbb{R} \otimes \varphi : & V = (N_V)_\mathbb{R} \longrightarrow \mathbb{R}^{n-1}, \\ \varphi_S := 1_{\mathbb{S}^1} \otimes \varphi : & S_{N_V} \longrightarrow (\mathbb{S}^1)^{n-1}, \end{aligned}$$

and the isomorphism of algebraic tori

$$\varphi_\mathbb{C} := 1_{\mathbb{C}^*} \otimes \varphi : T_{N_V} \longrightarrow (\mathbb{C}^*)^{n-1}.$$

We have that

$$\begin{aligned} \mathrm{Log}_{n-1} &= -\log \otimes 1_{\mathbb{Z}^{n-1}} \\ &= (1_\mathbb{R} \circ (-\log) \circ 1_{\mathbb{C}^*}) \otimes (\varphi \circ 1_{N_V} \circ \varphi^{-1}) \\ &= \varphi_\mathbb{R} \circ \mathrm{Log}_{N_V} \circ \varphi_\mathbb{C}^{-1}, \end{aligned}$$

so that

$$\begin{aligned} \varphi_\mathbb{C}(\mathrm{Log}_{N_V}^{-1}(H_{\sigma \cap V})) = (\mathrm{Log}_{N_V} \circ \varphi_\mathbb{C}^{-1})^{-1}(H_{\sigma \cap V}) &= (\varphi_\mathbb{R}^{-1} \circ \mathrm{Log}_{n-1})^{-1}(H_{\sigma \cap V}) \\ = \mathrm{Log}_{n-1}^{-1}(\varphi_\mathbb{R}(H_{\sigma \cap V})) &= \mathrm{Log}_{n-1}^{-1}(\mathbb{R}^{p-1} \times \{0\}) \\ = (\mathbb{C}^*)^{p-1} \times (\mathbb{S}^1)^{n-p}. \end{aligned}$$

This, together with the fact that $\mathrm{Log}_{N_V} = \mathrm{Log}_n|_D$, implies

$$\mathrm{Log}_n^{-1}(H_\sigma) \cap D = \mathrm{Log}_{N_V}^{-1}(H_{\sigma \cap V}) = \varphi_\mathbb{C}^{-1}((\mathbb{C}^*)^{p-1} \times (\mathbb{S}^1)^{n-p}). \tag{1}$$

Consider the plane $H = \mathbb{R}^{p-1} \times \{0\} \subseteq \mathbb{R}^{n-1}$ and the map

$$\pi_H : \mathrm{Log}_{n-1}^{-1}(H) = (\mathbb{C}^*)^{p-1} \times (\mathbb{S}^1)^{n-p} \longrightarrow (\mathbb{S}^1)^{n-p}$$

given by $\pi_H = \arg \otimes q_{n-p}$, where $q_{n-p} : \mathbb{Z}^{n-1} \longrightarrow \mathbb{Z}^{n-p}$ is the projection to the last $n-p$ coordinates. In particular, π_H is given by the projection to the last $n-p$

coordinates and hence, for a given point $x \in (\mathbb{S}^1)^{n-p}$, $\pi_H^{-1}(x) = (\mathbb{C}^*)^{p-1} \times \{x\}$. By taking into account the commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^{n-1} \arrow[r, "\varphi^{-1}", "\cong"'] \arrow[d, "q_{n-p}"'] & N_V \arrow[r, hook, "i"] \arrow[d, "q'_\sigma"'] & \mathbb{Z}^n \arrow[d, "q_\sigma"] \\
\mathbb{Z}^{n-p} \arrow[r, "\cong", "\overline{\varphi}^{-1}"'] & N'(\sigma) \arrow[r, hook, "\overline{i}"'] & N(\sigma)
\end{tikzcd}$$

which induces the commutative diagram

$$\begin{tikzcd}
(\mathbb{S}^1)^{n-1} \arrow[r, "1_{\mathbb{S}^1}\otimes\varphi^{-1}", "\cong"'] \arrow[d, "1_{\mathbb{S}^1}\otimes q_{n-p}"'] & S_{N_V} \arrow[r, hook, "1_{\mathbb{S}^1}\otimes i"] \arrow[d, "q'_\sigma"'] & (\mathbb{S}^1)^n \arrow[d, "1_{\mathbb{S}^1}\otimes q_\sigma"] \\
(\mathbb{S}^1)^{n-p} \arrow[r, "\cong", "1_{\mathbb{S}^1}\otimes\overline{\varphi}^{-1}"'] & S_{N'(\sigma)} \arrow[r, hook, "1_{\mathbb{S}^1}\otimes\overline{i}"'] & S_{N(\sigma)}
\end{tikzcd}$$

we get that

$$\begin{aligned}
\pi_H^{-1}(x) &= \mathrm{Log}_{n-1}^{-1}(H) \cap ((1_{\mathbb{S}^1} \otimes q_{n-p}) \circ (\arg \otimes 1_{\mathbb{Z}_n}))^{-1}(x) \\
&= (\mathbb{C}^*)^{p-1} \times (\mathbb{S}^1)^{n-p} \cap (\arg \otimes 1_{\mathbb{Z}_n})^{-1}[(1_{\mathbb{S}^1} \otimes q_{n-p})^{-1}(x)] \\
&= (\mathbb{C}^*)^{p-1} \times (\mathbb{S}^1)^{n-p} \cap (\arg \otimes 1_{\mathbb{Z}_n})^{-1}[(1_{\mathbb{S}^1} \otimes (q_\sigma \circ i \circ \varphi^{-1}))^{-1}(1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x))] \\
&= (\mathbb{C}^*)^{p-1} \times (\mathbb{S}^1)^{n-p} \cap [(1_{\mathbb{S}^1} \otimes (q_\sigma \circ i \circ \varphi^{-1})) \circ \arg \otimes 1_{\mathbb{Z}_n}]^{-1}(1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x)) \\
&= (\mathbb{C}^*)^{p-1} \times (\mathbb{S}^1)^{n-p} \cap [\arg \otimes (q_\sigma \circ i \circ \varphi^{-1})]^{-1}(1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x)) \\
&= (\mathbb{C}^*)^{p-1} \times (\mathbb{S}^1)^{n-p} \cap [(\arg \otimes q_\sigma) \circ (1_{\mathbb{C}^*} \otimes i) \circ (1_{\mathbb{C}^*} \otimes \varphi^{-1})]^{-1}(1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x)) \\
&= (\mathbb{C}^*)^{p-1} \times (\mathbb{S}^1)^{n-p} \cap \varphi_{\mathbb{C}} \left([(\arg \otimes q_\sigma) \circ (1_{\mathbb{C}^*} \otimes i)]^{-1}(1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x))\right) \\
&= (\mathbb{C}^*)^{p-1} \times (\mathbb{S}^1)^{n-p} \cap \varphi_{\mathbb{C}} \left(D \cap [(\arg \otimes q_\sigma)^{-1}(1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x)]\right) \\
&=^{1} \varphi_{\mathbb{C}} \left(D \cap \mathrm{Log}_n^{-1}(H_\sigma) \cap [(\arg \otimes q_\sigma)^{-1}(1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x)]\right) \\
&= \varphi_{\mathbb{C}} \left(D \cap \pi_\sigma^{-1}((1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x))\right).
\end{aligned}$$

Calling $\sigma_V := \varphi_{\mathbb{R}}(\sigma \cap V)$ (so that $H_{\sigma_V} = H$), we have the equality of currents in D

$$\varphi_{\mathbb{C}}^*[\pi_{\sigma_V}^{-1}(x)] = \varphi_{\mathbb{C}}^*[\pi_H^{-1}(x)] = [\varphi_{\mathbb{C}}^{-1}(\pi_H^{-1}(x))] = [D \cap \pi_\sigma^{-1}((1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x))].$$

As $\pi_H^{-1}(x)$ is a torus and $\varphi_{\mathbb{C}}$ is biholomorphic, we have that the smooth analytic set $D \cap \pi_\sigma^{-1}((1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x)) \subseteq (\mathbb{C}^*)^n$ is irreducible and then the wedge product $[D] \wedge [\pi_\sigma^{-1}((1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x))]$ is a positive integral multiple of integration current $[D \cap \pi_\sigma^{-1}((1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x))]$. In detail,

$$[D] \wedge [\pi_\sigma^{-1}((1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x))] = m\varphi_{\mathbb{C}}^*[\pi_{\sigma_V}^{-1}(x)],$$

where m is the order of vanishing of $f = z^\alpha - z^\beta$ along $D \cap \pi_\sigma^{-1}((1_{\mathbb{S}^1} \otimes (\overline{i} \circ \overline{\varphi}^{-1})(x))$.
□