TROPICAL DYNAMICS AND INTERSECTIONS

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Abstract.

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Let (X, ω) be a compact Kähler manifold of dimension n. Assume that $\mathcal{S}$ is a positive or negative current of bidegree (q, q) on X. The quantity $\langle \mathcal{S}, \omega^{n-q} \rangle$ is called the *total mass* of $\mathcal{S}$. For $0 \leq r \leq n$, we consider the de Rham cohomology groups $H^r(X, \mathbb{C}) = H^r(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ with coefficients in $\mathbb{C}$. Recall that Hodge theory implies the following decomposition of de Rham cohomology group into the Dolbeault cohomology groups

$$H^r(X, \mathbb{C}) = \bigoplus_{r=p+q} H^{p,q}(X, \mathbb{C}).$$

We denote by $\mathcal{C}^q(X)$ the cone of positive closed bidegree (q, q)-currents or bidimention $(n-q, n-q)$ in X. We denote by $\mathcal{D}^q(X) = \mathcal{D}_{n-q}(X)$ the $\mathbb{R}$-vector space spanned by $\mathcal{C}^q(X)$, that is the space of closed real currents of bidegree (q, q). Every current $\mathcal{T} \in \mathcal{D}^q(X)$ has a cohomology class

$$\{\mathcal{T}\} \in H^{q,q}(X, \mathbb{R}) = H^{q,q}(X, \mathbb{C}) \cap H^{2q}(X, \mathbb{R}).$$

We define $\mathcal{D}^{q,0}(X) = \mathcal{D}^0_{n-q}(X)$ to be the subspace of $\mathcal{D}^q(X)$, consisting of currents with vanishing cohomology. One defines the $*$-topology on $\mathcal{D}^q(X)$ by the norm

$$\|\mathcal{S}\|_* := \min \|\mathcal{S}^+\| + \|\mathcal{S}^-\|,$$

where the minimum is taken over positive currents $\mathcal{S}^+$ and $\mathcal{S}^-$ in $\mathcal{C}^q(X)$ satisfying $\mathcal{S} = \mathcal{S}^+ - \mathcal{S}^-$.

2.1. **Super-potentials of currents.** The super-potential theory of currents on a compact Kähler manifold was introduced by Dinh and Sibony in several papers, see for instance [DS10, DS09]. Let us recall basic notions from this theory. For a current $\mathcal{T} \in \mathcal{D}^q(X)$, on a compact Kähler manifold X, the super-potential is a function defined on $\mathcal{D}^{n-q+1,0}(X)$. The classical $\partial\bar{\partial}$-lemma or dd^c-lemma from Hodge theory implies that if $\mathcal{S} \in \mathcal{D}^{n-q+1,0}(X) = \mathcal{D}^0_{q-1}(X)$ is smooth, there exist a smooth potential $U_\mathcal{S} \in \mathcal{D}^{n-q}(X)$ such that $dd^c U_\mathcal{S} = \mathcal{S}$. Gillet and Soulé generalised this lemma for currents, that is, if $\mathcal{S} \in \mathcal{D}^{n-q+1,0}(X)$, then a potential, not necessarily smooth, exists such that $dd^c U_\mathcal{S} = \mathcal{S}$; see [GS90, Theorem 1.2.1].

Let $h := \dim H^{q,q}(X, \mathbb{R})$, and fix a set of smooth forms $\alpha = (\alpha_1, \dots, \alpha_{\beta_h})$ such that their cohomology classes $\{\alpha\} = (\{\alpha_1\}, \dots, \{\alpha_h\})$ form a basis for $H^{q,q}(X, \mathbb{R})$. By Poincaré duality, there exists a set of smooth forms $\alpha^\vee = (\alpha_1^\vee, \dots, \alpha_h^\vee)$ such that their cohomology classes $\{\alpha^\vee\}$ form the dual basis of $\{\alpha\}$, with respect to the cup-product. By adding $U_\mathcal{S}$ to a suitable combination of $\alpha_i^\vee$, we can assume that $\langle U_\mathcal{S}, \alpha_i \rangle = 0$, for all $i = 1, \dots, h$. In this case, we say that $U_\mathcal{S}$ is *α-normalised*.

Definition 2.1. Let $\mathcal{T} \in \mathcal{D}^q(X)$ and $\mathcal{S}$ be a smooth form in $\mathcal{D}^{n-q+1,0}(X)$.

(i) The *α-normalised super-potential* $\mathcal{U}_\mathcal{T}$ of $\mathcal{T}$ is given by the function

$$\begin{aligned} \mathcal{U}_\mathcal{T} : \{\mathcal{S} \in \mathcal{D}^{n-q+1,0}(X) : \text{smooth}\} &\longrightarrow \mathbb{R} \\ \mathcal{S} &\longmapsto \langle \mathcal{T}, U_\mathcal{S} \rangle, \end{aligned}$$

where $U_\mathcal{S}$ is the α-normalised potential of $\mathcal{S}$.

(ii) We say $\mathcal{T}$ has a *continuous super-potential*, if $\mathcal{U}_\mathcal{T}$ can be extended to a function on $\mathcal{D}^{n-q+1,0}$ which is continuous with respect to the $*$-topology.

In general, consider $\mathcal{T} \in \mathcal{D}^q(X)$ and $\mathcal{T} \in \mathcal{D}^r(X)$. Assume that $q + r \leq n$ and $\mathcal{T}$ has a continuous super-potential. Let $\mathcal{U}_\mathcal{T}$ be the α-normalised super-potential of $\mathcal{T}$. Let $\beta \in \mathrm{Span}_\mathbb{R}\{\alpha\}$ such that $\{\beta\} = \{\mathcal{T}\}$. We define

$$\langle \mathcal{T} \wedge \mathcal{S}, \varphi \rangle := \mathcal{U}_\mathcal{T}(\mathcal{S} \wedge dd^c \varphi) + \langle \beta \wedge \mathcal{S}, \varphi \rangle. \tag{1}$$

Now assume that if $f : X \longrightarrow Y$, is a biholomorphism between smooth compact Kähler manifolds, then we have

$$f_* \mathcal{U}_{\mathcal{R}_1} = \mathcal{U}_{f_* \mathcal{R}_1}, \quad f^* \mathcal{U}_{\mathcal{R}_2} = \mathcal{U}_{f^* \mathcal{R}_2},$$

for $\mathcal{R}_1 \in \mathcal{D}^q(X)$ and $\mathcal{R}_2 \in \mathcal{D}^q(Y)$.

Definition 2.2. Let $(\mathcal{T}_n)$ be a sequence of currents in $\mathcal{D}^q(X)$ weakly converging to $\mathcal{T}$. Let $\mathcal{U}_\mathcal{T}$ and $\mathcal{U}_{\mathcal{T}_n}$ be their α-normalised super-potentials. If $\mathcal{U}_{\mathcal{T}_n}$ converges to $\mathcal{U}_\mathcal{T}$ uniformly on any $*$-bounded sets of smooth form in $\mathcal{D}^{n-q+1,0}(X)$, then the convergence is called *SP-uniform*.

It is shown in [DS10, Proposition 3.2.8] that any current with continuous super-potentials can be SP-uniformly approximated by smooth forms. Moreover, currents with continuous super-potentials have other nice properties:

Theorem 2.3 ([DNV18, Theorem 1.1])**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two positive currents in $\mathcal{D}_q(X)$, such that $\mathcal{T} \leq \mathcal{T}'$, *i.e.*, $\mathcal{T}' - \mathcal{T}$ is a positive current. Then, if $\mathcal{T}'$ has a continuous super-potential, then so does $\mathcal{T}$.

Theorem 2.4. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two positive closed currents, and $\mathcal{T}_1$ has a continuous superpotentials, then $\mathcal{T}_1 \wedge \mathcal{T}_2$ is well-defined. Moreover, if $\mathcal{T}_2$ has also a continuous superpotential, then

(a) [DS10, Proposition 3.3.3] $\mathcal{T}_1 \wedge \mathcal{T}_2$ has a continuous superpotential;
(b) [DS10, Proposition 3.3.3] This wedge product is continuous with respect to the SP-uniform convergence.
(c) [DS09, Theorem 4.2.4] $\operatorname{supp}(\mathcal{T}_1 \wedge \mathcal{T}_2) \subseteq \operatorname{supp}(\mathcal{T}_1) \cap \operatorname{supp}(\mathcal{T}_2)$.

Theorem 2.5 ([DS10, Proposition 3.3.4])**.** Assume that $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are closed positive currents, and $\mathcal{T}_1$ and $\mathcal{T}_2$ have continuous superpotentials. Then,

$$\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_2 \wedge \mathcal{T}_1 \quad \text{and} \quad (\mathcal{T}_1 \wedge \mathcal{T}_2) \wedge \mathcal{T}_3 = \mathcal{T}_1 \wedge (\mathcal{T}_2 \wedge \mathcal{T}_3).$$

Proposition 2.6. Let X be a compact Kähler manifold, $S_n \longrightarrow S$ be a convergent sequence in $\mathcal{D}^q(X)$. If a current $\mathcal{T}$ has a continuous superpotential, then

$$\mathcal{T} \wedge \mathcal{S}_n \longrightarrow \mathcal{T} \wedge \mathcal{S}.$$

Proof. The main result of Dinh and Sibony in [DS04] implies any current $\mathcal{T} \in \mathcal{D}^p(X)$ can be weakly approximated by a difference of smooth closed positive of bidegree (p,p)-forms. The result then follows from the definition of continuity of super-potentials. $\square$

Lemma 2.7. Let $\mathcal{T}$, $\mathcal{T}'$ be positive closed currents such that $\mathcal{T}_{|_U} = \mathcal{T}'_{|_U}$ in an open subset $U \subseteq X$, and both $\mathcal{T}$ and $\mathcal{T}'$ have continuous super-potentials. Then, for any $\mathcal{S} \in \mathcal{D}^r(X)$,

$$\big(\mathcal{T} \wedge \mathcal{S}\big)_{|_U} = \big(\mathcal{T}' \wedge \mathcal{S}\big)_{|_U}.$$

Proof. In [DS10], for any current $\mathcal{S}$ with continuous super-potential, a family $\{\mathcal{T}_\theta\}_{\theta \in \mathbb{C}^*}$ is constructed that $\mathcal{T}_\theta$ converges SP-uniformly to $\mathcal{S}$ as $|\theta| \to 0$. Therefore, by the hypothesis of the lemma, we can construct families of smooth forms $\mathcal{T}_n, \mathcal{T}'_n$ converging SP-uniformly to $\mathcal{T}, \mathcal{T}'$ respectively. Moreover,

$$\mathcal{T}_{n|_{U_\epsilon}} = \mathcal{T}'_{n|_{U_\epsilon}},$$

where U_ϵ is an ϵ-neighbourhood of U. Now, for a $(n-q-r, n-q-r)$ smooth form φ with compact support on U

$$\big(\mathcal{T}_n \wedge \mathcal{S}\big) \wedge \varphi = \big(\mathcal{T}'_n \wedge \mathcal{S}\big) \wedge \varphi,$$

together with Theorem 2.4(b), implies the assertion.

$\square$

We also have a very useful local version of Theorem 2.3.

Corollary 2.8. Let $q : \widehat{X} \longrightarrow X$, be the blowing up of the compact Kähler manifold X. Assume that $\mathcal{T} \in \mathcal{D}_p(\widehat{X})$ is such that the support of $\mathcal{T}$ does not intersect the exceptional divisors of $\widehat{X}$. If the current on $\mathcal{T}$ has a continuous superpotential then $q_*\mathcal{T}$ has the same property.

Proof. This is an easy corollary of previous lemma, since q is biholomorphic near $\operatorname{supp}(\mathcal{T})$. $\square$

Theorem 2.9. For two complex manifolds X and Y, consider two convergent sequences of currents $\mathcal{T}_n \longrightarrow \mathcal{T}$ in $\mathcal{D}^q(X)$ and $\mathcal{S}_n \longrightarrow \mathcal{S}$ in $\mathcal{D}^r(Y)$. We have that

$$\mathcal{T}_n \otimes \mathcal{S}_n \longrightarrow \mathcal{T} \otimes \mathcal{S},$$

weakly in $\mathcal{D}^{q+r}(X \times Y)$.

Sketch of the proof. Let us denote by (x, y) the coordinates on $X \times Y$. Using local coordinates and a partition of unity and Weierstrass theorem we can approximate any smooth forms on $X \times Y$ with forms with polynomial coefficients in (x, y). The approximation is in C^∞. As a result, the convergence, we only need test forms with monomial coefficients. Thus, the variables x, y are separated and the convergence of the tensor products becomes the convergence of each factor.

□

2.2. **Semi-continuity of slices.** Let $f : X \longrightarrow Y$ be a dominant holomorphic map between complex manifolds, not necessarily compact, of dimension n and m respectively. Let $\mathcal{T}$ be a positive closed current on X of bi-dimension (p, p) with $p \geq m$. Then the slice

Let $U \subseteq \mathbb{C}^m$ and $V \subseteq \mathbb{C}^n$ be two bounded open sets. Assume that $\pi_1 : U \times V \longrightarrow U$ and $\pi_2 : U \times V \longrightarrow V$ are the canonical projections. Consider two closed positive currents $\mathcal{T}$ and $\mathcal{S}$ on $U \times V$ of bi-dimension (m, m) and (n, n) respectively. We say that $\mathcal{T}$ horizontal-like if $\pi_2(\mathrm{supp}(\mathcal{T}))$ is relatively compact in V. Similarly, if $\pi_1(\mathrm{supp}(\mathcal{S}))$ is relatively compact in U, $\mathcal{S}$ is called vertical-like.

Theorem 2.10 ([BD20, Lemma 3.7])**.** Let $(\mathcal{T}_n) \longrightarrow \mathcal{T}$ be a convergent sequence of horizontal-like positive closed currents to a horizontal-like current $\mathcal{T}$ in $U \times V$. Let $a \in U$ and assume that the sequence of measures $(\langle \mathcal{T}_n, \pi_1, a\rangle)_n$ is also convergent. Then,

$$\langle \lim_{n\to\infty} \mathcal{T}_n | \pi_1 | a\rangle(\phi) \leq \langle \mathcal{T} | \pi_1 | a\rangle(\phi)$$

for every plurisubharmonic function ϕ on $\mathbb{C}^n$.

There is an simple version of the above theorem for supports which will be useful later.

Lemma 2.11. Assume that $\mathcal{T}_i$'s, $\mathcal{S}$ and $\mathcal{T}$ are all closed positive currents, and $\mathcal{T}_i \wedge \mathcal{S}$ and $\mathcal{T} \wedge \mathcal{S}$ are well-defined. If,

$$\mathcal{T}_i \longrightarrow \mathcal{T}, \quad \mathrm{supp}(\mathcal{T}_i) \longrightarrow \mathrm{supp}(\mathcal{T})$$

Then,

$$\mathrm{supp}(\lim(\mathcal{T}_i \wedge \mathcal{S})) \subseteq \mathrm{supp}(\mathcal{T}) \cap \mathrm{supp}(\mathcal{S}).$$

If moreover, $\mathcal{T}_i \wedge \mathcal{S} \longrightarrow \mathcal{T} \wedge \mathcal{S}$, then

$$\mathrm{supp}(\lim(\mathcal{T}_i \wedge \mathcal{S})) = \mathrm{supp}(\mathcal{T}) \cap \mathrm{supp}(\mathcal{S}).$$

Proof. For a point x outside the support of $\mathcal{T}$, There is a sufficiently small radius ϵ, such that for a sufficiently large i, $\mathcal{T}_i$ vanishes on the ball $B_\epsilon(x)$ centred at x. It follows that any limit of $\mathcal{T}_i \wedge \mathcal{S}$ vanishes on $B_\epsilon(x)$. So its support does not contain a. Moreover, its support does not contain any point outside $\mathrm{supp}(S)$. The second statement is now clear.

□

3. Tropical Varieties, Tori, Tropical Currents

In this section, we recall the definition of tropical cycles and note that with the natural addition of tropical cycles and their *stable intersection*, the tropical cycles form a ring.

3.1. **Tropical varieties.** A linear subspace $H \subseteq \mathbb{R}^n$ is said to be *rational* if there exists a subset of $\mathbb{Z}^n$ that spans H. A *rational polyhedron* is the intersection of finitely many rational half-spaces defined by

$$\{x \in \mathbb{R}^n : \langle m, x\rangle \geq c, \text{ for some } m \in \mathbb{Z}^n, \ c \in \mathbb{R}\}.$$

A *rational polyhedral complex* is a polyhedral complex consisting solely of rational polyhedra. The polyhedra in a polyhedral complex are also referred to as *cells*. A *fan* is a polyhedral complex whose cells are all cones. If every cone in a fan Σ is contained in another fan Σ', then Σ is called a *subfan* of Σ. The one-dimensional cones of a fan are often called *rays*. Throughout this article, all fans and polyhedral complexes are assumed to be *rational*.

For a given polyhedron σ, and a finitely generated abelian group N, we denote by

$$\begin{aligned} \mathrm{aff}(\sigma) &:= \text{affine span of } \sigma, \\ H_\sigma &:= \text{translation of } \mathrm{aff}(\sigma) \text{ to the origin}, \\ N_\sigma &:= N \cap H_\sigma, \\ N(\sigma) &:= N/N_\sigma. \end{aligned}$$

Consider τ, a codimension one face of a p-dimensional polyhedron σ, and let $u_{\sigma/\tau}$ be the unique outward generator of the one-dimensional lattice $(\mathbb{Z}^n \cap H_\sigma)/(\mathbb{Z}^n \cap H_\tau)$.

Definition 3.1 (Balancing Condition and Tropical Cycles)**.** Let $\mathcal{C}$ be a p-dimensional polyhedral complex whose p-dimensional cones are equipped with integer weights. We say that $\mathcal{C}$ satisfies the *balancing condition* at τ if

$$\sum_{\sigma \supset \tau} w(\sigma)\ u_{\sigma/\tau} = 0, \quad \text{in } \mathbb{Z}^n/(\mathbb{Z}^n \cap H_\tau),$$

where the sum is over all p-dimensional cells σ in $\mathcal{C}$ containing τ as a face. A *tropical variety* in $\mathbb{R}^n$ is a weighted complex with finitely many cells that satisfies the balancing condition at every cone of dimension $p-1$.

3.2. **Stable intersection.** Recall that, roughly speaking, the extension of the local p-dimensional fan around it. More precisely:

Definition 3.2. Let $\Sigma \subseteq \mathbb{R}^n$ be a polyhedral complex and $\tau \in \Sigma$ be a $(p-1)$-dimensional cell. The star of σ in Σ, denoted by $\mathrm{star}_\Sigma(\sigma)$ is the union of the extension $\bar{\sigma}$ of p-dimensional cells σ containing τ as a face. Here, by the extension we mean

$$\bar{\sigma} = \{\lambda(x-y) : \lambda \geq 0, x \in \sigma, y \in \tau\}.$$

Definition 3.3 (Stable intersection)**.** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two tropical varieties in $\mathbb{R}^n$. The *stable intersection* $\mathcal{C}_1 \cdot \mathcal{C}_2$ is the polyhedral complex

$$\mathcal{C}_1 \cdot \mathcal{C}_2 = \bigcup_{\substack{\sigma_1 \in \mathcal{C}_1, \sigma_2 \in \mathcal{C}_2 \\ \dim(\sigma_1+\sigma_2)=n}} \sigma_1 \cap \sigma_2.$$

For a top dimensional cell $\sigma_1 \cap \sigma_2$ in $\mathcal{C}_1 \cdot \mathcal{C}_2$, the weights are obtained by

$$w_{\mathcal{C}_1 \cdot \mathcal{C}_2}(\sigma_1 \cap \sigma_2) = \sum_{\tau_1, \tau_2} w_{\tau_1} w_{\tau_2} [N : N_{\tau_1} + N_{\tau_2}],$$

where the sum is over all $\tau_1 \in \mathrm{star}_{\mathcal{C}_1}(\sigma_1 \cap \sigma_2), \tau_2 \in \mathrm{star}_{\mathcal{C}_2}(\sigma_1 \cap \sigma_2)$ with $\tau_1 \cap (v + \tau_2) \neq \varnothing$, for a generic fixed $v \in \mathbb{R}^n$.

In tropical geometry, it is shown that:

Theorem 3.4 ([MS15, Chapter 3])**.** In the above definition, the stable intersection does not depend on the choice of generic $v \in \mathbb{R}^n$ and the stable intersection of two tropical varieties is indeed a tropical variety. Moreover, the support of the stable intersection of two tropical cycles $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{R}^n$, is the Gromov–Hausdorff limit (in compact subsets of $\mathbb{R}^n$) of

$$(\mathcal{C}_1 + \varepsilon v) \cap \mathcal{C}_2$$

as the real number $\varepsilon \to 0$.

4. Tropical Currents

Let us briefly recall the definition of tropical currents from [Bab14, BH17]. To fix the notation,

$$\begin{array}{rcl} T_N & := & \text{the complex algebraic torus } \mathbb{C}^* \otimes_{\mathbb{Z}} N, \\ S_N & := & \text{the compact real torus } S^1 \otimes_{\mathbb{Z}} N, \\ N_{\mathbb{R}} & := & \text{the real vector space } \mathbb{R} \otimes_{\mathbb{Z}} N. \end{array}$$

Let $\mathbb{C}^*$ be the group of nonzero complex numbers. As before, the logarithm map is the homomorphism

$$\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \qquad (z_1, \dots, z_n) \longmapsto (-\log|z_1|, \dots, -\log|z_n|),$$

and the *argument map* is

$$\mathrm{Arg} : (\mathbb{C}^*)^n \longrightarrow (S^1)^n, \qquad (z_1, \dots, z_n) \longmapsto (z_1/|z_1|, \dots, z_n/|z_n|).$$

For a rational linear subspace $H \subseteq \mathbb{R}^n$ we have the following exact sequences:

$$0 \longrightarrow H \cap \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n/(H \cap \mathbb{Z}^n) \longrightarrow 0,$$

Moreover,

$$0 \longrightarrow S_{H \cap \mathbb{Z}^n} \longrightarrow (S^1)^n = S^1 \otimes_{\mathbb{Z}} \mathbb{Z}^n \longrightarrow S_{\mathbb{Z}^n/(H \cap \mathbb{Z}^n)} \longrightarrow 0.$$

Define

$$\pi_H : \ \mathrm{Log}^{-1}(H) \xrightarrow{\mathrm{Arg}} (S^1)^n \longrightarrow S_{\mathbb{Z}^n/(H \cap \mathbb{Z}^n)} \ .$$

Similarly,

$$0 \longrightarrow T_{H \cap \mathbb{Z}^n} \longrightarrow (\mathbb{C}^*)^n = \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Z}^n \longrightarrow T_{\mathbb{Z}^n/(H \cap \mathbb{Z}^n)} \longrightarrow 0,$$

We define

$$\Pi_H : (\mathbb{C}^*)^n \simeq \mathbb{C}^* \otimes \big((H \cap \mathbb{Z}^n) \oplus \mathbb{Z}^n/(\mathbb{Z}^n \cap H)\big) \longrightarrow T_{\mathbb{Z}^n/(H \cap \mathbb{Z}^n)} \ .$$

One has

$$\ker(\Pi_H) = \ker(\pi_H) = T_{H\cap\mathbb{Z}^n} \subseteq (\mathbb{C}^*)^n.$$

As a result, when H is of dimension p, the set $\mathrm{Log}^{-1}(H)$ is naturally foliated by the $\pi_H^{-1}(x) = T_{H\cap\mathbb{Z}^n}\cdot x \simeq (\mathbb{C}^*)^p$ for $x \in S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}$. For a lattice basis $u_1,\ldots,u_p$, of $H\cap\mathbb{Z}^n$, the tori $T_{H\cap\mathbb{Z}^n}.x$ can be parametrised by the monomial map

$$(\mathbb{C}^*)^p \longrightarrow (\mathbb{C}^*)^n, \quad z \longmapsto x.z^{[u_1,\ldots,u_p]^t}$$

where $U = [u_1,\ldots,u_p]$ is the matrix with column vectors $u_1,\ldots,u_p$, and z^{U^t} denotes that $z \in (\mathbb{C}^*)^p$ is taken to have the exponents with rows of the matrix U. Accordingly, one can easily check that

$$T_{H\cap\mathbb{Z}^n}\cdot x = \mathbb{V}\big(\{z \in (\mathbb{C}^*)^n : z^{m_i} = x^{m_i},\ i = 1,\ldots,m-p\}\big).$$

for any choice of a $\mathbb{Z}$-basis $\{m_1,\ldots,m_{n-p}\}$ of $\mathbb{Z}^n/(H\cap\mathbb{Z}^n)$.

Definition 4.1. Let H be a rational subspace of dimension p, and μ be the Haar measure of mass 1 on $S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}$. We define a (p,p)-dimensional closed current $\mathcal{T}_H$ on $(\mathbb{C}^*)^n$ by

$$\mathcal{T}_H := \int_{x\in S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}} \big[\pi_H^{-1}(x)\big]\ d\mu(x).$$

When A is an affine subspace of $\mathbb{R}^n$ parallel to the linear subspace $H = A - a$ for $a \in A$, we define $\mathcal{T}_A$ by translation of $\mathcal{T}_H$. Namely, we define the submersion π_A as the composition

$$\pi_A:\ \mathrm{Log}^{-1}(A) \xrightarrow{e^a} \mathrm{Log}^{-1}(H) \xrightarrow{\pi_H} S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}.$$

We will call $T^A := \pi_A^{-1}(1) = \ker\pi_A = e^{-a}T_{H\cap\mathbb{Z}^n}$, the *distinguished fibre* of $\mathcal{T}_A$.

Definition 4.2. Let $\mathcal{C}$, be a weighted polyhedral complex of dimension p. The tropical current $\mathcal{T}_\mathcal{C}$ associated to $\mathcal{C}$ is given by

$$\mathcal{T}_\mathcal{C} = \sum_\sigma w_\sigma\ \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\mathcal{T}_{\mathrm{aff}(\sigma)},$$

where the sum runs over all p-dimensional cells σ of $\mathcal{C}$.

Theorem 4.3 ([Bab14])**.** A weighted complex $\mathcal{C}$ is balanced, if and only if, $\mathcal{T}_\mathcal{C}$ is closed.

Theorem 4.4 ([Bab14])**.** Any tropical current $\mathcal{T}_\mathcal{C} \in \mathcal{D}'_{n-1,n-1}((\mathbb{C}^*)^n)$ is of the form $dd^c[\mathfrak{q}\circ\mathrm{Log}]$, where $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$, is a tropical Laurent polynomial, that is $\mathfrak{q}(x) = \max_{\alpha\in A}\{c_\alpha + \langle\alpha, x\rangle\}$, for $A \subseteq \mathbb{Z}^n$ a finite subset and $c_\alpha \in \mathbb{R}$.

Remark 4.5. Note that the support of $dd^c[\mathfrak{q}\circ\mathrm{Log}]$, is given by $\mathrm{Log}^{-1}(\mathcal{V}(\mathfrak{q}))$, where $\mathcal{V}(\mathfrak{q})$ is the set of points $x \in \mathbb{R}^n$ where $\mathfrak{q}$ is not smooth at x. This set can be balanced with natural weights which coincides with the weights of the closed current $dd^c[\mathfrak{q}\circ\mathrm{Log}]$ and it is called the tropical variety associated to $\mathfrak{q}$.

Proposition 4.6 ([Bab23, Proposition 4.6])**.** Assume that $\mathcal{T} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$ is a closed positive $(S^1)^n$-invariant current whose support is given by $\mathrm{Log}^{-1}(|\mathcal{C}|)$, for a polyhedral complex $\mathcal{C} \subseteq \mathbb{R}^n$ of pure dimension p. Then $\mathcal{T}$ is a tropical current.

5. Continuity of superpotentials

Let $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$, be a tropical polynomial function, and $\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$, as before. The current $dd^c[\mathfrak{q} \circ \mathrm{Log}] \in \mathcal{D}'_{n-1,n-1}(\mathbb{C}^*)^n$ has a bounded potential, and by Bedford–Taylor theory, for any positive closed current $\mathcal{T} \in \mathcal{D}'_{p,p}(\mathbb{C}^*)^n$, the product

$$dd^c[\mathfrak{q} \circ \mathrm{Log}] \wedge \mathcal{T} = dd^c([\mathfrak{q} \circ \mathrm{Log}]\, \mathcal{T}),$$

is well-defined. See [Dem, Section III.3]. In higher codimensions though, to prove that any two tropical currents have a well-defined wedge product, we utilise Dinh and Sibony's superpotential theory [DS09] on a compact Kähler manifold, and as a result, we extend the tropical currents to smooth compact toric varieties.

5.1. **Tropical Currents on Toric Varieties.** In a toric variety X_Σ, for a cone $\sigma \in \Sigma$, we denote by $\mathcal{O}_\sigma$, the toric orbit associated with σ. We have

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{O}_\sigma.$$

We also set D_σ to be the closure of $\mathcal{O}_\sigma$ in the X_Σ. $\Sigma(p)$ p-dimensional skeleton.

Fibers of tropical currents are algebraic varieties with finite degrees and can be extended by zero to any toric variety, in consequence, any tropical current can be extended by zero to toric varieties. Moreover, with the following compatibility condition, we can ask for the extension of the fibres to intersect the toric invariant divisors transversally.

Definition 5.1. (i) For a polyhedron σ, its *recession cone* is the convex polyhedral cone

$$\mathrm{rec}(\sigma) = \{b \in \mathbb{R}^n : \sigma + b \subseteq \sigma\} \subseteq H_\sigma.$$

(ii) Let $\mathcal{C}$ be a p-dimensional balanced weighted complex in $\mathbb{R}^n$, and Σ a p-dimensional fan. We say that $\mathcal{C}$ is *compatible* with Σ, if $\mathrm{rec}(\sigma) \in \Sigma$ for all $\sigma \in \mathcal{C}$.

(iii) We say the tropical current $\mathcal{T}_\mathcal{C}$ is *compatible* with X_Σ, if all the closures of the fibers $\pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ in X_Σ of $\mathcal{T}_\mathcal{C}$ intersect the torus invariant divisors of X_Σ transversely.

Theorem 5.2. Let $\mathcal{C}$ be a p-dimensional tropical cycle Σ be a fan. Assume that $\sigma \in \mathcal{C}$ is a p-dimensional polyhedron and $\rho \in \Sigma$ is a one-dimensional cone. Then

(a) The intersection $D_\rho \cap \overline{\pi^{-1}_{\mathrm{aff}(\sigma)}(x)}$ is non-empty and transverse, if and only if, $\rho \in \mathrm{rec}(\sigma)$. Here $\overline{\pi^{-1}_{\mathrm{aff}(\sigma)}(x)}$ corresponds the closure of a fiber of $\mathcal{T}_{\mathrm{aff}(\sigma)}$ in the toric variety X_Σ.

(b) In particular, if $\mathcal{C}$ is compatible with Σ, if and only if, $\mathcal{T}_\mathcal{C}$ is compatible with X_Σ.

Proof. See Lemma [BH17, Lemma 4.10]. □

For a tropical current $\mathcal{T}_\mathcal{C} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$, and given a toric variety X_Σ we denote its extension by zero $\overline{\mathcal{T}}_\mathcal{C} \in \mathcal{D}'_{p,p}(X_\Sigma)$.

Proposition 5.3. For every tropical variety $\mathcal{C}$, a smooth projective toric fan Σ compatible with a subdivision of $\mathcal{C}$.

Proof. By [BS11], for $\mathcal{C}$ there is a refinement $\mathcal{C}'$, and a complete fan $\Sigma_1 \subseteq \mathbb{R}^n$ such that $\mathcal{C}'$ is compatible with Σ_1. Applying the toric Chow lemma [CLS11, Theorem 6.1.18] and the toric resolution of singularities [CLS11, Theorem 11.1.9] we can find a fan Σ which is a refinement of Σ_1 that defines a smooth projective variety X_Σ. The tropical variety $\mathcal{C}''$ which is the refinement of $\mathcal{C}'$ induced by Σ, satisfies the statement. □

Remark 5.4. When $\mathcal{C}'$ is a refinement of a tropical variety $\mathcal{C}$, then $\mathcal{C}'$ is a tropical variety with natural induced weights. It is also easy to check that we have the equality of currents $\mathcal{T}_\mathcal{C} = \mathcal{T}_{\mathcal{C}'}$ in $(\mathbb{C}^*)^n$; see [BH17, Section 2.6].

6. Slicing tropical currents

Proposition 6.1. Let $\mathcal{C}$ be a p-dimensional tropical cycle in $\mathbb{R}^n$, and $S \subseteq (\mathbb{C}^*)^n$ be an algebraic hypersurface with transversal intersection with $\mathcal{T}_\mathcal{C}$. Then, $[S]\wedge\mathcal{T}_\mathcal{C}$ is admissible and it is a closed positive current of bidimension $(p-1, p-1)$ given by

$$[S] \wedge \mathcal{T}_\mathcal{C} = \sum_{\sigma\in\mathcal{C}} w_\sigma \int_{x\in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[S \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\big] \; d\mu(x).$$

Proof. The idea of the proof is similar to that of [BH17, Proposition 4.11]. Let f be the equation of S in $(\mathbb{C}^*)^n$. Assume that $\mathrm{Log}^{-1}(\sigma^\circ) \cap S \neq \varnothing$, for a p-dimensional cone $\sigma \in \mathcal{C}$, then for each fiber, $\pi_\sigma^{-1}(x)$ the transversality assumption allows for application of the Lelong–Poincaré formula to deduce

$$dd^c\big(\log|f|\mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\big[\pi_\sigma^{-1}(x)\big]\big) \\ = \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\big[S \cap \pi_\sigma^{-1}(x)\big] + \mathcal{R}_\sigma(x).$$

where $\mathcal{R}_\sigma(x)$ is a $(p-1, p-1)$-bidimensional current. The support of $\mathcal{R}_\sigma(x)$ lies in the boundary of $\mathrm{Log}^{-1}(\sigma)$, as $\mathcal{R}_\sigma(x)$ is the difference of two currents that coincide in any set of form $\mathrm{Log}^{-1}(B)$, where $B \subseteq \mathbb{R}^n$ is a small ball with

$$B \cap \sigma^\circ \neq \varnothing, \quad B \cap \partial\sigma = \varnothing,$$

and both vanish outside $\mathrm{Log}^{-1}(\sigma)$. Integrating along the fibers, and adding for all p-dimensional cones $\sigma \in \mathcal{C}$, we obtain

$$[S] \wedge \mathcal{T}_\mathcal{C} = \sum_{\sigma\in\mathcal{C}} w_\sigma \int_{x\in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[S \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\big] \; d\mu(x) + \mathcal{R}_\mathcal{C},$$

where $\mathcal{R}_\mathcal{C}$ is $(p-1, p-1)$-dimensional current. We claim that $\mathcal{R}_\mathcal{C}$ is *normal*, *i.e.* $\mathcal{R}_\mathcal{C}$ and $d\mathcal{R}_\mathcal{C}$ have measure coefficients; $\mathcal{R}_\mathcal{C}$ is a difference of two normal currents, where the first current $[S] \wedge \mathcal{T}_\mathcal{C}$ is a positive closed current, and the second current is an addition of normal pieces. Moreover, the support of $\mathcal{R}_\mathcal{C}$ is a subset of S as it is a difference of two currents that both vanish outside S. As a result, the current $\mathcal{R}_\mathcal{C}$ is supported on $S \cap \bigcup_\sigma \partial\mathrm{Log}(\sigma)$. This set is a real manifold of Cauchy–Riemann dimension less than $p-1$, therefore by Demailly's first theorem of support the normal current $\mathcal{R}_\mathcal{C}$ vanishes; see also the discussion following [BH17, Proposition 4.11]. □

Corollary 6.2. Let $H \subseteq \mathbb{R}^n$ be a rational plane of dimension r and $A := a + H$, a translation of H for $a \in \mathbb{R}^n$. Assume also that $\mathcal{C} \subseteq \mathbb{R}^n$ is a tropical variety of dimension p that intersects A transversely. Then

$$[(e^{-a})T_{H\cap\mathbb{Z}^n}] \wedge \mathcal{T}_{\mathcal{C}}$$

can be viewed as a tropical current of dimension $p - (n - r)$ in the complex subtori $T^A := (e^{-a})\, T_{H\cap Z^n} \subseteq (\mathbb{C}^*)^n$.

Proof. Note that the hypothesis implies that the intersection $T^A \cap \pi^{-1}_{\text{aff}(\sigma)}(x)$ is transversal for any $x \in S_{N(\sigma)}$. By translation, it is sufficient to prove the statement for $a = 0$. By preceding theorem,

$$[T^A] \wedge \mathcal{T}_{\mathcal{C}} = \sum_{\sigma\in\mathcal{C}} w_\sigma \int_{x\in S_{N(\sigma)}} \mathbb{1}_{\text{Log}^{-1}(\sigma^\circ)} \left[T^A \cap \pi^{-1}_{\text{aff}(\sigma)}(x)\right] \, d\mu(x).$$

The sets $T^A \cap \pi^{-1}_{\text{aff}(\sigma)}(x)$ can be understood as a translation toric sets in T^A and $d\mu_\sigma(x)$ are Haar measures, which imply the assertion. $\square$

Theorem 6.3. Let $M \subseteq (\mathbb{C}^*)^{n-p}$ and $N \subseteq (\mathbb{C}^*)^p$ be two bounded open subsets such that N contains the real torus $(S^1)^p$. Let $\pi : M \times N \longrightarrow M$ be the canonical projection. Let $\mathcal{T}_n$ be a sequence of positive closed (p, p)-bidimensional currents on $M \times N$ such that $\overline{\text{supp}(\mathcal{T}_n)} \cap (M \times \partial\overline{N}) = \varnothing$. Assume that $\mathcal{T}_n \longrightarrow \mathcal{T}$ and $\text{supp}(\mathcal{T}) \subseteq M \times (S^1)^p$. Then we have the following convergence of slices

$$\langle \mathcal{T}_n|\pi|x\rangle \longrightarrow \langle \mathcal{T}|\pi|x\rangle \quad \text{for every } x \in M.$$

Note that all the above slices are well-defined for all $x \in M$.

Note that any subtorus of $(\mathbb{C}^*)^n$, can be understood as a fibre of a tropical current. We have the following slicing theorem.

Theorem 6.4. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a tropical variety and $A \subseteq \mathbb{R}^n$ a rational hyperplane intersecting $\mathcal{C}$ transversely. Let Σ be a fan compatible with $\mathcal{C} + A$. Assume that $\overline{\mathcal{S}}_n$ is a sequence of positive closed currents on X_Σ, and denote by $\mathcal{S}_n$ the restriction to T_N. Further,

- $\overline{\mathcal{S}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}}$;
- $\text{supp}(\overline{\mathcal{S}}_n) \longrightarrow \text{supp}(\overline{\mathcal{T}}_{\mathcal{C}})$.

We have that

$$\lim_{n\to\infty} \left(\mathcal{S}_n \wedge [T^A]\right) = \mathcal{T}_{\mathcal{C}} \wedge [T^A],$$

as currents on $T_N \subseteq X_\Sigma$.

Proof. Assume that $L \subseteq \mathbb{R}^n$ is an $(n - p - 1)$-dimensional affine plane intersecting all $\text{aff}(\sigma)$ for all $\sigma \in \mathcal{C} \cap A$ transversely. Then, on a projective smooth toric variety $X_{\Sigma'}$ compatible with $\mathcal{C} + L + A$ the tropical currents $\overline{\mathcal{T}}_{a+L}$, $a \in \mathbb{R}^n$ have continuous super-potentials. Therefore, by Proposition 2.6, we have

$$\lim_{m\to\infty} \left(\overline{\mathcal{S}}_n \wedge \overline{\mathcal{T}}_{a+L}\right) = \overline{\mathcal{T}}_{\mathcal{C}} \wedge \overline{\mathcal{T}}_{a+L}.$$

Now, for any $x \in \mathcal{C} \cap L \cap A$, let $B \subseteq \mathbb{R}^n$ containing x be a bounded open set containing only x as an isolated point of the intersection. By a translation we can assume that $x = 0$. Let H be the linear space parallel to A, and

$$\xi : (\mathbb{C}^*)^n \xrightarrow{\sim} T_{\mathbb{Z}^n/(Z^n \cap H)} \times T_{\mathbb{Z}^n \cap H}$$

be the isomorphism, and π_1 and π_2 be the respective projections. Note that for $x \in S^1_{\mathbb{Z}^n/(\mathbb{Z}^n \cap H)}$, we have $\pi_1^{-1}(1) = T^A$. We now set

$$\begin{aligned} U &:= \pi_1 \circ \xi\big(\mathrm{Log}^{-1}(U) \cap \mathrm{supp}(\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{a+L})\big) \\ V &:= \pi_2 \circ \xi\big(\mathrm{Log}^{-1}(U) \cap T^A\big), \\ \mathcal{T}_n &:= \xi_*\big(\mathcal{S}_n \wedge \mathcal{T}_{a+L}\big), \text{ in } T_N, \\ \mathcal{T} &:= \xi_*\big(\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{a+L}\big), \end{aligned}$$

Therefore, for large n, $\mathcal{T}_n$ and $\mathcal{T}_{\mathcal{C}}$ are horizontal-like. By Theorem 6.4, we obtain

$$\lim_{n\to\infty} \big(\mathcal{S}_n \wedge [T^A]\big) \wedge \mathcal{T}_{a+L} = \mathcal{T}_{\mathcal{C}} \wedge [T^A] \wedge \mathcal{T}_{a+L},$$

for every a. We now deduce the convergence on $X_{\Sigma'}$ by Lemma **??**. Finally the convergence on $(\mathbb{C}^*)^n \simeq T_N$ follows from restriction.

□

Theorem 6.5. In the situation of Theorem 6.4,

$$\lim_{n\to\infty} \big(\mathcal{S}_n \wedge [\overline{T}^A]\big) = \overline{\mathcal{T}}_{\mathcal{C}} \wedge [\overline{T}^A],$$

where the extension is considered in a smooth projective toric variety X_Σ compatible with $\mathrm{trop}(W) + A$.

Corollary 6.6. Let $W \subseteq (\mathbb{C}^*)^n$ be an algebraic subvariety. Let $\mathcal{C} = \mathrm{trop}(W)$ be the tropicalisation of W with respect to the trivial valuation. Assume that A is a rational affine hyperplane in $\mathbb{R}^n$ intersecting $\mathcal{C}$ properly. Then,

$$\lim_{m\to\infty} \big(\frac{1}{m^{n-p}} \Phi_m^*[W] \wedge [T^A]\big) = \big(\lim_{m\to\infty} \frac{1}{m^{n-p}} \Phi_m^*[W]\big) \wedge [T^A].$$

Proof. Let $\mathcal{T}_m : \frac{1}{m^{n-p}} \Phi_m^*[\overline{W}]$, where closure is taken on a toric variety X_Σ compatible with $\mathrm{trop}(W) + A$. Note that when $\mathrm{trop}(W) \cap A$ intersect transversely, $\mathrm{supp}(\mathcal{T}_{\mathcal{C}}) \cap T^A$ is compact. Moreover, since $\mathrm{supp}(T_m)$ in Hausdorff metric converges to $\mathcal{T}_{\mathcal{C}}$ then for any large m, $\mathrm{supp}(\mathcal{T}_m) \cap T^A$ is also compact. Now the statement follows from Theorems 7.6 and 6.4. □

Lemma 6.7. Let X_Σ be a smooth projective toric variety, and $\bar{\Delta} \subseteq X_\Sigma$ be the diagonal. Let $\mathcal{S}$ and $\mathcal{T}$ be two positive currents on X. Then, for any ray $\rho \in \Sigma$,

$$\mathrm{supp}(\mathcal{S}) \cap \mathrm{supp}(\mathcal{T}) \cap D_\rho \subseteq X_\Sigma$$

has a Cauchy–Riemann dimension ℓ, if and only if,

$$\mathrm{supp}(\mathcal{S} \otimes \mathcal{T}) \cap \bar{\Delta} \cap D_{(0,\rho)} \subseteq X_\Sigma \times X_\Sigma,$$

has a Cauchy–Riemann dimension ℓ, where $D_{(0,\rho)}$ is the toric invariant divisor corresponding to the ray $(0, \rho)$ in $\Sigma \times \Sigma$.

Proof. The fan of $X_\Sigma \times X_\Sigma$ is $\Sigma \times \Sigma$, we have that $D_{(0,\rho)} \simeq X_\Sigma \times D_\rho$ and the assertion follows. ◻

Theorem 6.8. Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{R}^n$ be two tropical cycles intersecting properly. Assume that X_Σ is a smooth toric projective variety compatible with $\mathcal{C}_1 + \mathcal{C}_2$. If moreover, for two sequence of positive closed currents $\overline{\mathcal{V}}_n$ and $\overline{\mathcal{W}_n}$ we have

(a) $\overline{\mathcal{W}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}_1}$ and $\overline{\mathcal{V}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}_2}$,
(b) $\operatorname{supp}(\overline{\mathcal{W}}_n) \longrightarrow \operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}_1})$ and $\operatorname{supp}(\overline{\mathcal{V}}_n) \longrightarrow \operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}_2})$,
(c) For any n, $\operatorname{supp}(\overline{\mathcal{W}}_n) \cap \operatorname{supp}(\overline{\mathcal{V}}_n)$ has the expected dimension.
(d) For any n, and any ray $\rho \in \Sigma$, $\operatorname{supp}(\overline{\mathcal{W}}_n) \cap \operatorname{supp}(\overline{\mathcal{V}}_n) \cap D_\rho$ has the expected dimension.

Then

$$\overline{\mathcal{W}}_n \wedge \overline{\mathcal{V}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}} \wedge \overline{\mathcal{T}}_{\mathcal{C}'}.$$

Proof. For two closed currents $\mathcal{S}$ and $\mathcal{T}$ on X_Σ we naturally identify $\mathcal{S} \wedge \mathcal{T} = \pi_*\big(\mathcal{S} \otimes \mathcal{T} \wedge [\bar{\Delta}]\big)$, where $\pi : X_\Sigma \times X_\Sigma \longrightarrow X_\Sigma$ is the projection. In $T_N \times T_N \subseteq X_\Sigma \times X_\Sigma$ we $\mathcal{T}_n := \mathcal{W}_n \otimes \mathcal{V}_n$ and $\mathcal{T}_\mathcal{C} := \mathcal{T}_{\mathcal{C}_1} \otimes \mathcal{T}_{\mathcal{C}_2}$. Now note that the diagonal in the open torus is the complete intersection of the tori $x_i = y_i$, $i = 1, \dots, n$. This together with assumption (c) allows for a repeated application of Theorem 6.4 to obtain

$$\mathcal{W}_n \otimes \mathcal{V}_n \wedge [\Delta] \longrightarrow \mathcal{T}_{\mathcal{C}_1} \otimes \mathcal{T}_{\mathcal{C}_2} \wedge [\Delta].$$

By assumption (c), and Lemma 6.7, for large n and rays $\rho \in \Sigma$,

$$\operatorname{supp}(\overline{\mathcal{W}}_n \otimes \overline{\mathcal{V}}_n) \cap [\overline{\Delta}] \cap D_\rho$$

have the expected dimension. Lemma 6.7, and the compatibility assumption imply that $\operatorname{supp}(\mathcal{W}_n \otimes \mathcal{V}_n) \cap \bar{\Delta} \cap D_{(0,\rho)}$ and $\operatorname{supp}(\mathcal{T}_\mathcal{C} \otimes \mathcal{T}_{\mathcal{C}'}) \cap \bar{\Delta} \cap D_{(0,\rho)}$ have the expected Cauchy–Riemann dimension. Therefore, Lemma 2.11 brings us to the situation of Lemma **??** and we conclude. ◻

7. Applications

7.1. **Dynamical tropicalisation with a non-trivial valuation.** Recall that for a field $\mathbb{K}$, $\nu : \mathbb{K} \longrightarrow \mathbb{R} \cup \{\infty\}$, is called a valuation if it satisfies the following properties for every $a, b \in \mathbb{K}$:

(a) $\nu(a) = \infty$ if and only if $a = 0$;
(b) $\nu(ab) = \nu(a) + \nu(b)$;
(c) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$.

A valuation is called *trivial*, if the valuation of any non-zero element is 0. For an element $a \in \mathbb{K}$, we denote by $\bar{a}$ its image in the residue field. We are interested in the case where $\mathbb{K} = \mathbb{C}((t))$, is the field of *formal Laurent series* with the parameter t. with the usual valuation. That is, for $g(t) = \sum_{j \geq k} a_j t^j$, with $a_k \neq 0$, the valuation equals the the minimal exponent $\nu(g) = k \in \mathbb{Z}$.

Definition 7.1. (a) Let $f = \sum_{\alpha\in\mathbb{N}} c_\alpha z^\alpha \in \mathbb{K}[z^{\pm 1}]$, be a Laurent polynomial in n variables. The tropicalisation of f with respect to ν,

$$\operatorname{trop}_\nu(f) : \mathbb{R}^n \longrightarrow \mathbb{R},$$
$$x \mapsto \max\{-\nu(c_\alpha) + \langle x, \alpha\rangle\}.$$

(b) Let $I \subseteq \mathbb{K}[z^{\pm 1}]$ be an ideal. The tropical variety associated to I, as a set, is defined as

$$\operatorname{trop}_\nu(I) := \bigcap_{f\in I} \mathcal{V}(\operatorname{trop}_\nu(f)),$$

where $\mathcal{V}(\operatorname{trop}_\nu(f))$ is the tropical variety associated to $\operatorname{trop}_\nu(f)$; see Remark 4.5.

(c) For an algebraic subvariety of the torus $Z \subseteq (\mathbb{K}^*)^n$, with the associated ideal $\mathbb{I}(Z)$, the tropicalisation of Z, as a set, is $\operatorname{trop}_\nu(Z) := \operatorname{trop}_\nu(\mathbb{I}(Z))$.

(d) In all the situations above, trop_0 denotes the tropicalisation with respect to the trivial valuation.

We need to relate non-trivial valuation to the trivial valuation.

Lemma 7.2. Consider the ideal $I \subseteq \mathbb{C}[t^{\pm 1}, z^{\pm}1] \overset{\iota}{\hookrightarrow} \mathbb{C}((t))[z]$. Assume that (u, x) are the coordinates in $\mathbb{R}\times\mathbb{R}^n$. Then, we have the following equality of sets

$$\operatorname{trop}_0(I) \cap \{u = -1\} = \operatorname{trop}_\nu(\iota(I)).$$

That is, the tropicalisation of I as an ideal in $\mathbb{C}[t, x]$ with respect to the trivial valuation intersected with $\{u = -1\}$ coincides with the tropicalisation of $I = \iota(I)$ with respect to the usual valuation in $\mathbb{C}((t))$.

The proof of the lemma becomes clear with the following example.

Example 7.3. Let

$$f(x, t) = 4(t^3 + t^{-1})z_1z_2 + (1 + t + t^2)z_1.$$

Then, the tropicalisation of $f \in \mathbb{C}[t, z]$, with respect to the trivial valuation equals:

$$\operatorname{trop}(f) = \max\big\{\max\{3u + x_1 + x_2, -u + x_1 + x_2\}, \max\{x_1, u + x_1 + 2u + x_1\}\big\}$$

Letting $u := -1$, $\operatorname{trop}(f)(-1, x) = \max\{1 + x_1 + x_2, x_1\}$. The latter equals $\operatorname{trop}_\nu(f)$ as an element of $\mathbb{C}((t))[z]$.

Proof of Lemma 7.2. If f is a monomial in $\mathbb{C}[t][z]$, then it is clear that

$$\operatorname{trop}_0(f)(-1, x) = \operatorname{trop}_\nu(\iota(f)).$$

Therefore, we have the equality for any polynomial in $f \in \mathbb{C}[t, z]$. To prove the main statement, note that

$$\begin{aligned}\operatorname{trop}_\nu(\iota(I)) &= \bigcap_{f\in\iota(I)} \mathcal{V}(\operatorname{trop}_\nu(f)) \\ &= \bigcap_{f\in I} \big(\mathcal{V}(\operatorname{trop}_0(f)) \cap \{u = -1\}\big) \\ &= \operatorname{trop}_0(I) \cap \{u = -1\}.\end{aligned}$$

□

Remark 7.4. Bergman in [Ber71], shows that for an algebraic subvariety $Z \subseteq (\mathbb{C}^*)^n$, one has

$$\lim \operatorname{Log}_t(Z) \subseteq \operatorname{trop}_0(\mathbb{I}(Z)),$$

and he conjectured the equality. This conjecture was later proved by Bieri and Groves in [BG84].

Remark 7.5. The above lemma is related to the results of Markwig and Ren in [MR20]. They considered the tropicalisation of an ideal $J \subseteq R[[t]][x]$, where R is the ring of integers of a discrete valuation ring $\mathbb{K}$, which is non-trivially valued. To obtain finiteness properties, however, the authors consider the associated tropical variety in the half-space $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$. Note that such a variety is almost never balanced. The authors also prove that for an ideal $I \subseteq \mathbb{K}[x]$, the tropicalisation of the natural inverse image $\pi^{-1}I \subseteq R[[t]][x]$ with respect to trivial valuation, intersected with $\{u = -1\}$ equals $\operatorname{trop}_\nu(I)$; [MR20, Theorem 4].

Let us also recall the main result of [Bab23].

Theorem 7.6. *Let $Z \subseteq (\mathbb{C}^*)^n$ be an irreducible subvariety of dimension p, and $\overline{Z}$ the closure of Z in the compatible smooth projective toric variety X. Then,*

$$\frac{1}{m^{n-p}} \Phi_m^*[\overline{Z}] \longrightarrow \overline{\mathcal{T}}_{\operatorname{trop}_0(Z)}, \quad \text{as } m \to \infty,$$

where $\Phi_m : X \longrightarrow X$ is the continuous extension of $\Phi_m : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$, and $\overline{\mathcal{T}}_{\operatorname{trop}_0(Z)}$ is the extension by zero of $\mathcal{T}_{\operatorname{trop}_0(Z)}$ to X. Moreover, the supports also converge in Hausdorff metric.

Note that since the limit of a sequence of closed currents is closed, the above theorem implies that $\operatorname{trop}_0(Z)$ can be equipped with weights to become balanced.

Theorem 7.7. *Let $I \subseteq \mathbb{C}[t^{\pm 1}, x^{\pm 1}]$ be an ideal with the associated $(p+1)$-dimensional algebraic variety $W = \mathbb{V}(I) \subseteq (\mathbb{C}^*)^{n+1}$. Assume that the projection onto the first coordinate $\pi_1 : W \longrightarrow \mathbb{C}^*$ is surjective and Zariski closed. We denote the fibres as $W_t := \pi_1^{-1}(t)$. We have that*

$$\frac{1}{m^{n-p}} \Phi_m^*[W_{e^m}] \longrightarrow \mathcal{T}_{\operatorname{trop}_\nu(I)}, \quad \text{as } m \to \infty \ ,$$

in the sense of currents in $\mathcal{D}_p((\mathbb{C}^*)^n)$. In particular, $\operatorname{trop}_\nu(I)$ can be equipped with weights to become balanced. Moreover, if Σ is a toric variety compatible with $\operatorname{trop}_0(W)$ and $\{u = -1\}$, then on X_Σ,

$$\frac{1}{m^{n-p}} \Phi_m^*[\overline{W}_{e^m}] \longrightarrow \overline{\mathcal{T}}_{\operatorname{trop}_\nu(I)}, \quad \text{as } m \to \infty \ .$$

We need the following:

Lemma 7.8. *Let $W \subseteq (\mathbb{C}^*)^{n+1}$ be a $(p+1)$-dimensional smooth subvariety, such that the projection onto the first factor, $\pi_1 : (\mathbb{C}^*)^{n+1} \longrightarrow \mathbb{C}^*$ is surjective and a Zariski closed morphism. Assume that W Then for a sufficiently large $|t_0| >> 0$*

$$[W_{t_0}] = [\pi_1^{-1}(t_0)] = [\{t = t_0\}] \wedge [W].$$

Proof. We first prove that the set of singular points of W together with the set of points where $[\{t=t_0\}]\wedge[W]$ has a multiplicity greater than 1, is contained in a Zariski closed set in W. We define the *critical set*,

$$C=\big\{w\in W_{\text{reg}}:\dim\big(T_wW\cap\ker\ \nabla_w t\big)=p+1\big\},$$

which is the set of points where the tangent space of T_wW_{reg} is included in the tangent space of $T_w\{t=t_0\}$, and this set contains the set of points $w\in W_{\text{reg}}$ points the intersection multiplicity of $\{t=t_0\}$ and W exceeds 1. We fix an ideal associated to $I=\mathbb{I}(W)=\langle f_1,\dots,f_k\rangle\subseteq\mathbb{C}[t,x]$. At any regular point $w\in W_{\text{reg}}$, T_wW is of dimension $p+1$, and the rank of the Jacobian matrix $J(f)(w)=\big(\frac{\partial f_i}{\partial z_j}(w)\big)_{k\times(n+1)}$ equals codimension of W, $(n+1)-(p+1)=n-p$. We have that $\nabla_w t=e_1$, where e_1 is the first element of the standard basis for the $\mathbb{C}$-vector space $\mathbb{C}^{n+1}$. We have $w\in C$, if and only if,

$$\ker\begin{pmatrix}e_1\\ Jf(w)\end{pmatrix}=\ker\big(Jf(w)\big).$$

As a result, C is an algebraic variety given as the intersection of $W\setminus W_{\text{sing}}$ with the intersection of zero loci of $(q+1)\times(q+1)$-minors of $\begin{pmatrix}e_1\\ Jf(w)\end{pmatrix}$. Therefore, the closure of C in W, $\overline{C}$ union W_{sing} is a Zariski-closed subset of W. Since W is not contained in $\{t=t_0\}$, as π_1 is surjective, then $\pi_1(\overline{C}\cup W_{\text{sing}})$ is a Zariski closed proper subset in $\mathbb{C}^*\subseteq\mathbb{C}$, and hence finite.

□

Proof of Theorem 7.7. By preceding lemma, and the fact that Φ_m preserves transversal intersection, we have

$$\frac{1}{m^{n-p}}\Phi_m^*[W_{e^m}]=\frac{1}{m^{n-(p+1)}}\Phi_m^*[W]\wedge\frac{1}{m}\Phi_m^*[\{t=e^m\}],$$

for a large m. Since $\operatorname{trop}_0(W)$ is a fan and it is transversal to the plane $\{u=-1\}\subset\mathbb{R}^{n+1}$ are transversal, we can use Theorem 6.4 to write

$$\lim\frac{1}{m^{n-p}}\Phi_m^*[W_{e^m}]=\Big(\lim\frac{1}{m^{n-(p+1)}}\Phi_m^*[W]\Big)\wedge\Big(\lim\frac{1}{m}\Phi_m^*[\{t=e^m\}]\Big)$$

By Theorem 7.6, restricted to $(\mathbb{C}^*)^{n+1}$, and the fact that we used $\operatorname{Log}=(-\log|\cdot|,\dots,-\log|\cdot|)$ in the definition of tropical currents, the above limits yield

$$\lim\frac{1}{m^{n-p}}\Phi_m^*[W_{e^m}]=\mathcal{T}_{\operatorname{trop}_0(W)}\wedge\mathcal{T}_{\{u=-1\}}.$$

Applying Theorems **??** and Lemma 7.2, we obtain the equality. To see the final assertion, note that the limit $\mathcal{T}_{\operatorname{trop}_\nu(I)}$ is a closed current and Theorem 4.3 implies that $\operatorname{trop}_\nu(I)$ is naturally balanced.

For the extension of the theorem on X_Σ, we apply Theorem 6.5. □

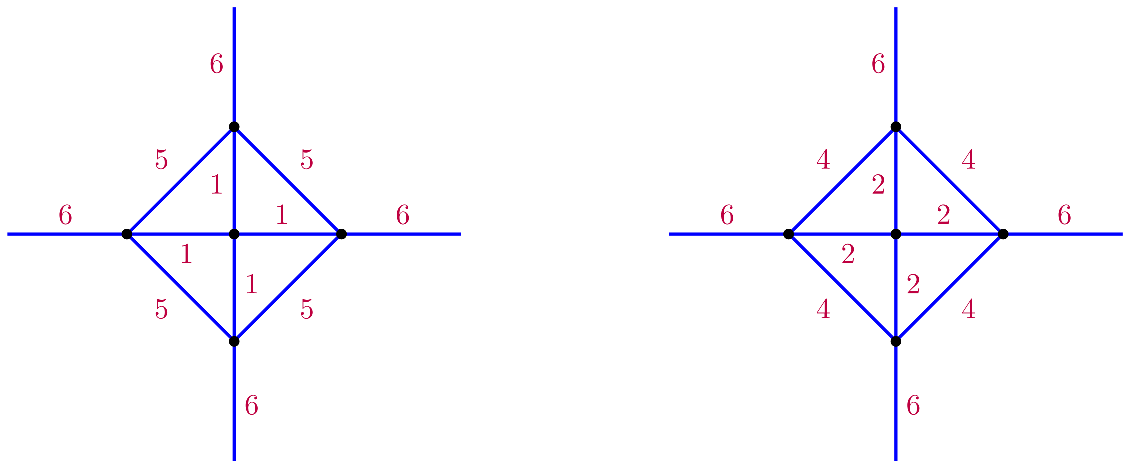

FIGURE 1. Two different tropical varieties with the same recession fan and same support.

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