1

a) $\operatorname{maxSpec}(\mathbb{C}[x]) = \{ m \subset \mathbb{C}[x] : m \text{ a maximal ideal} \}$

ideal generated by ~~$\{(x-a)$~~

$= \{ (x-a) : a \in \mathbb{C} \} = m_a$

This is as ~~this set~~ m_a is the kernel of the surjective homomorphism

~~f~~ $\delta_a : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}$

$\delta_a(f) = f(a)$

Since $\mathbb{C}$ is a ring ~~ka~~ and $\mathbb{C}[x_1, \dots, x_n] / m_a = \mathbb{C}$, m_a is maximal

$\operatorname{maxSpec}(\mathbb{C}[x, 1/x]) = \{ (x-a) : a \in \mathbb{C} \setminus \{0\} \}$

$\operatorname{maxSpec}(\mathbb{C}[x, 1/x, y]) = \{ ((x-a),(y-b)) : a \in \mathbb{C} \setminus \{0\},\ b \in \mathbb{C} \}$

b) $\varphi^*\left(\frac{1}{x}\right) = \frac{1}{(1/x)} = x$

$\varphi^*\left(2x^2 + \frac{2x^3 + 4x}{x^5}\right) = 2\left(\frac{1}{x^2}\right) + \frac{2\left(\frac{1}{x}\right)^3 + 4\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)^5}$

$= 2x^{-2} + 2x^2 + 4x^4$

$\varphi^*(2 - x) = 2 - \frac{1}{x}$

2

a)

$$\varphi : V \to \mathbb{A}^2, \quad \varphi(x, y, u) = (x, u)$$

As stated in the question, φ is the restriction of a projection, and therefore polynomial mapping, from $\mathbb{A}^3$ to $\mathbb{A}^2$. It is therefore a morphism.

Let $\varphi^{-1} : \mathbb{A}^2 \to V$ be defined by $\varphi^{-1}(x, u) = (x, ux, u)$. This is clearly a well defined function and a polynomial mapping. It is therefore a morphism (Note that $\varphi^{-1}(A) = V$ as $y = ux$)

~~To $\varphi^{-1} \circ \varphi$~~

To check $\varphi^{-1} \circ \varphi = id_V$, we note that any $(x, y, u) \in V$ can be written as (x, ux, u). This is as (x, y, u) must be a solution to the equation $y - ux = 0$, so $y = ux$.

Thus, $\varphi^{-1} \circ \varphi(x, y, u) = \varphi^{-1}(x, ux, u) = (x, y, u)$

Clearly $\varphi \circ \varphi^{-1} = id_{\mathbb{A}^2}$ as $\varphi^{-1}(x, u) = (x, ux, u)$ and $\varphi(x, ux, u) = (x, u)$

So φ is an isomorphism

b) Let $\psi : V \to \mathbb{A}^2$ be the restriction of the projection to V. So $\psi(x, y, u) = (x, y)$

Since ψ is not injective, it cannot be a morphism. To see that it is not injective, consider $(0,0,1) \in V$ and $(0,0,0) \in V$. These are both in V as $0 - 0\times 0 = 0$ and $0 - 0 \times 1 = 0$.

$\Psi(0,0,0) = (0,0)$ and $\Psi(0,0,1) = (0,0)$

Ψ not injective

3

b) ~~$g \in \mathbb{C}[x,y]$~~

f_1 doesn't pass through any of the points. Neither does f_2.

f_3 and f_4 pass through $[0:1:0]$ ~~&~~ only

c)

For $\tilde{g}$ to not pass through any of the points in b we must be able to write $g(x,y) = ax^n + by^n + c$

4)

a) We show that $\mathbb{P}^n$ is compact by showing that ~~there is a~~ $\mathbb{P}^n$ is the image of a continuous function from $S^n = \{ z \in \mathbb{C}^{n+1} : \|z\| = 1\}$ (an obviously compact space) to $\mathbb{P}^n$

~~Choose~~ ~~[illegible]~~ Let $\varphi : \mathbb{C}^{n+1} \to \mathbb{P}^n$ be defined by $\varphi(z_0, \dots, z_n) = [z_0 : \dots : z_n]$

$\varphi|_{S^n}$ gives us a surjective function from S^n to $\mathbb{P}^n$. So $\varphi|_{S^n}(S^n) = \mathbb{P}^n$.

Now we show that $\varphi|_{S^n}$ is continuous. Let U be an open set in $\mathbb{P}^n$.

We can write $U = \{[x_0 : \dots : x_n] : (x_0, \dots, x_n) \in \bigcup_{i=0}^{n} W \cap U_i\}$

where W is open in S^n and U_i is the set $\{(x_0, \dots, x_n) : x_0 \neq 0\}$

Now $\varphi^{-1}|_{S^n}(U) = \{(x_0, \dots, x_n) : \bigcup_{i=1}^{n} W \cap U_i\}$

Since W is open in S^n and U_i is an open set of $\mathbb{C}^{n+1}$ (clearly $\{0\}$ is closed) $\varphi^{-1}|_{S^n}(U)$ is open so $\varphi^{-1}|_{S^n}(U)$ is continuous

In general, the continuous image of a compact set is compact. To see this, let $\{U_i\}_{i \in I}$ be an open cover of $f(A)$. Then $\{f^{-1}(U_i)\}_{i \in I}$ is a cover of A. Since A is compact we can ~~pick finitely many $i \in I$ such~~ I' finite such that $\{f^{-1}(U_i)\}_{i \in I'}$ covers A. But now $\{U_i = f(f^{-1}(U_i))\}_{i \in I'}$ covers $f(A)$. So $f(A)$ compact

The projective closure of $\mathbb{V}(\cancel{x}y - \sin x)$, $\overline{\mathbb{V}(y-\sin x)}$ is $\mathbb{P}^2$. ~~This doesn't cause any problems~~ ~~for with the~~

Since $\overline{\mathbb{V}(y-\sin x)} = \mathbb{P}^2$ is compact, Chow's lemma tells us that

$$\mathbb{V}(y-\sin x) \neq \overline{\mathbb{V}(y-\sin x)} = \mathbb{P}^2$$

(Otherwise $\mathbb{V}(y-\sin x) = \mathbb{P}^2$ compact implies $\mathbb{V}(y-\sin x)$ algebraic #).

5

a)

Let $L_1 = \mathbb{V}(a_1x + b_1y + c_1z)$ and
$L_2 = \mathbb{V}(a_2x + b_2y + c_2z)$

So we need $[x:y:z]$ to satisfy

$$a_1x + b_1y + c_1z = 0$$
and
$$a_2x + b_2y + c_2z = 0$$

So

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Since L_1 and L_2 are distinct in $\mathbb{P}^2$, they are linearly independent. So the rank of our matrix, or the dimension of the image of the linear map, is 2.

Now by the rank-nullity theorem,

$$\dim(\mathbb{C}^3) - \operatorname{rank}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = \operatorname{null}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

$$3 - 2 = 1$$

Since the nullity of $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$ is 1, there is a 1-dimensional subspace of $\mathbb{C}^3$ with solutions to

$$a_1 x + b_1 y + c_1 z = 0$$
$$\text{and} \quad a_2 x + b_2 y + c_2 z = 0$$

Equivalently there is a unique line through the origin, or an element of $\mathbb{P}^2$, that provides solutions to these equations.

b

Identify $\mathbb{A}^n$ with U_0 and consider $C_1, C_2 \subset U_0$

i) Since $C_1 \subset \overline{C_1}$ and $C_2 \subset \overline{C_2}$

$C_1 \cap C_2 \subset \overline{C_1} \cap \overline{C_2}$. Since $\overline{C_1} \cap \overline{C_2}$ is the intersection of closed sets, it is closed.

Since $\overline{C_1} \cap \overline{C_2}$ is a closed set containing $C_1 \cap C_2$

$$\overline{C_1 \cap C_2} \subset \overline{C_1} \cap \overline{C_2}$$

ii) $\mathbb{V}(x+y+z) = C_1$ and $\mathbb{V}(x+y+2z) = C_2$

Since the polynomials are both already homogenised $C_1 = \overline{C_1}$ and $C_2 = \overline{C_2}$ (see Q3(a)). So $\overline{C_1} \cap \overline{C_2} = C_1 \cap C_2$
$= \overline{C_1 \cap C_2}$

6 b)

i) $\mathcal{O}_X(V) = \{$ regular functions ~~from~~ on $V \subseteq X \}$

$f \in \mathcal{O}_X(V)$ means that, ~~there~~ ~~exist~~ for any open $U \subseteq V$ and for all $p \in U$, there is an open neighbourhood U' of p ~~such that~~ and polynomials $g, h \in \mathbb{C}[x_1, \dots, x_n]$ such that $h(p) \neq 0$, and $f|_{U'} = (\frac{g}{h})|_{U'}$

So for any open $W \subseteq U$ there is an open neighbourhood $U' \cap W$ and polynomial $g, h \in \mathbb{C}[x_1, \dots, x_n]$ such that $h(p) \neq 0$ and $f|_{U'} = \frac{g}{h}|_{U'}$

ii) As stated in part (a) $\mathcal{O}_X(U)$ is a $\mathbb{C}$-algebra and therefore a ring.

For each inclusion of open sets $U \subseteq V$ $\quad res_{V,U} : \mathcal{O}_{X}(V) \to \mathcal{O}_{X}(U)$

$$f \to f|_U$$

~~Clearly~~ $res_{U,U} = id_{\mathcal{O}_X(U)}$ as ~~$f|_U = f|_U$~~ if $f : U \to \mathbb{C}$ $\quad f|_U = f$

$res_{V,U} \circ res_{W,V} = (f|_V)|_U = f|_U$ since

$U \subseteq V \subseteq W$

3

a) ~~The ideal generated by $\tilde{g}$ is clearly homogeneous as it is generated by 1 (finite) homogeneous polynomial. So $\mathbb{V}(\tilde{g})$ is closed in $\mathbb{P}^n$~~

~~Suppose $W \subsetneq \mathbb{V}(\tilde{g})$ closed with $\mathbb{V}(g) \subset W$~~

By Theorem 2.8 $\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{I})$

where $\tilde{I} = \{\tilde{f} \in \mathbb{C}[x_0, \dots, x_n] \text{ homogeneous} : f \in (g)\}$

$= \{f \in \mathbb{C}[x_0, \dots, x_n] : f \in (\tilde{g})\}$

$= \mathbb{V}(\tilde{g})$