Coursework 1

15 January 2025 13:48

Q1. Let $A \subseteq \mathbb{A}^n$ be a subset.

(a) (**5 marks**) What is the definition of the closure of A in $\mathbb{A}^n$?

(b) (**5 marks**) Prove that $\mathbb{V}(\mathbb{I}(A))$ equals the Zariski closure of A in $\mathbb{A}^n$.

(c) (**5 marks**) Give an example of a subset in $B \subseteq \mathbb{C}$ whose closure in the Zariski topology does not coincide with its closure in the Euclidean topology.

(a) For a topological space X, the closure, $\overline{Y}$, of $Y \subseteq X$, is the smallest closed set in X such that $Y \subseteq \overline{Y}$, i.e. $\overline{Y} = \bigcap_{B \text{ closed}} B$ s.t. $Y \subseteq B$.

In particular, for $\mathbb{A}^n$, the closure of $A \subseteq \mathbb{A}^n$ is defined to be

$$\bar{A} = \bigcap_{i \in I} X_i$$

s.t. each X_i is closed & $A \subseteq X_i$ for all i.

(b) By definition, we have that,

$\mathbb{I}(A) = \{ f \in \mathbb{C}[x_1, \ldots, x_n] : f(a) = 0 \ \forall \ a \in A \}$.

Hence,

$$\mathbb{V}(\mathbb{I}(A)) = \mathbb{V}(\{f_j \in \mathbb{C}[x_1, \ldots, x_n] : f_j(a) = 0 \ \forall \ a \in A\}_{j \in J})$$

$$= \bigcap_{j \in J} (\mathbb{V}\{f_j \in \mathbb{C}[x_1, \ldots, x_n] : f_j(a) = 0 \ \forall \ a \in A\})$$

On the other hand, $\bar{A} = \bigcap_{i \in I} X_i$ s.t. X_i are closed & $A \in X_i \ \forall \ i \in I$. Thus since $A_i \subseteq X_i \ \forall \ i$, we have that

$$\bigcap_{i \in I} X_i = \bigcap_{j \in J} (\mathbb{V}(\{f_j \in \mathbb{C}[x_1, \ldots, x_n] : f_j(a) = 0 \ \forall \ a \in A\})$$

$$= \mathbb{V}(\{f_j \in \mathbb{C}[x_1, \ldots, x_n] : f_j(a) = 0 \ \forall \ a \in A\})$$
$$= \mathbb{V}(\mathbb{I}(A)).$$

(c) Let $A = \{(x, 0) : x \in \mathbb{C}\} \subseteq \mathbb{A}^2$. Then, $\bar{A}$ in $\mathbb{A}^2$ can be described by the polynomial $y = 0$, i.e. $\bar{A} = \mathbb{V}(\{y\})$. However, $\bar{A} = \mathbb{V}(\{y\}) = \{(x,0) : x \in \mathbb{C}\} \cup \{(x,y) : x, y \in \mathbb{C}\} = \mathbb{A}^2$. Thus, $\bar{A} = \mathbb{A}^2$.

On the other hand, for $\bar{A}$ in $\mathbb{C}^2$ under the Euclidean topology, we have that $\bar{A} = A$ since A is closed under the Euclidean topology. A is closed since any open neighbourhood of some point $(x_0, 0) \in A$ is wholly contained in A.

Q2. (a) (**5 marks**) What is the definition of a compact subset of a topological space?

(b) (**10 marks**) Prove that $\mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}^2$ is compact in the Zariski topology but not in the Euclidean topology.

(a) In a topological space, X, a subset $U \subseteq X$ is compact if for all open coverings, there is a finite subcovering.

Wherein, a collection of open sets, $\{O_i\}_{i \in I} \subseteq X$, each O_i is open, is an open covering if $\bigcup_{i \in I} O_i = X$. For $J \subseteq I$, $\{O_i\}_{i \in J}$ is a subcovering if it is still an open covering.

(b) Suppose $\{O_i\}_{i \in I}$ is an open covering for $V = \mathbb{V}(x^2 - y^2) \subseteq \mathbb{C}^2$. Then, let

$$C = \{U : U = \bigcup_{i=1}^{n} O_i\}$$

i.e. C is a set of open sets that can be found in our open cover for V. If $V \notin C$, then there exists an infinite ascending chain of open sets namely $U_1 \subseteq U_2 \subseteq \cdots$ for $U_i \in C$. However, this implies that $V_1 \supseteq V_2 \supseteq \cdots$ is an infinite descending chain of closed sets & then by Hilbert's correspondence this descending chain corresponds to $\mathbb{I}(V_1) \subseteq \mathbb{I}(V_2) \subseteq \cdots$ being an infinite ascending chain, but this is a contradiction since polynomial rings are Noetherian by Hilbert's Basis Theorem. Therefore, every ascending chain of open sets stabilises & so $V \in C$, & in particular there exists a finite subcovering of V given by $\{U_i\}_{i \in I}$. Thus, V is compact in the Zariski topology.

On the other hand, if $\{O_i\}_{i \in I}$ is an open covering for V in the Euclidean topology, then as before let

$$C = \{U \mid U = \bigcup_{i=1}^{n} O_i\}.$$

Suppose that $V \in C$, then $\exists \ U \in C$ s.t. $V \subseteq U$. However, V is unbounded since $V = \mathbb{V}(x^2 - y^2) = \{(x,y) : x^2 = y^2\}$ & so for any $x \in \mathbb{C} \ \exists \ y \in \mathbb{C}$ s.t. $(x,y) \in V$, in particular we can choose infinitely many $(x,y) \in V$ & so there does not exist an open ball which wholly contains V. Thus, $\exists$ infinitely many $(x,y) \in V$ s.t. $(x,y) \notin C$, & so $V \subseteq C$, i.e. V is not compact in the Euclidean topology.

(C) $\Rightarrow \varphi$ cannot be an isomorphism

Q3. (a) (**5 marks**) Find a curve $W \subseteq \mathbb{A}^2$ and a morphism $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$, such that W is irreducible but $\varphi^{-1}(W)$ is not. $W = \mathbb{V}(y - x^2)$ also works?

(b) (**5 marks**) Let Y be a topological space and consider $X \subseteq Y$ with the subspace topology. Prove that if X is irreducible then so is its closure.

(c) (**5 marks**) Prove that isomorphisms preserve irreducibility and dimension of closed affine algebraic varieties.

(d) (**10 marks**) Find the irreducible components of $\mathbb{V}(zx - y, y^2 - x^2(x+1)) \subseteq \mathbb{A}^3$. You need to justify why each component is irreducible.

(a) Let $W = \mathbb{V}(x - y)$ & let $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ be given by $(x,y) \longmapsto (x^2, y^2)$. Then, W is clearly irreducible but $\varphi^{-1}(W)$ is not, since φ is not injective, & thus,

$$\varphi^{-1}(\mathbb{V}(x-y)) = \mathbb{V}(x^2 - y^2) = \mathbb{V}(x+y) \cup \mathbb{V}(x-y).$$

Therefore, $\varphi^{-1}(W)$ is irreducible.

(b) Let Y be a topological space & $X \subseteq Y$ be endowed with the subspace topology. Let X be irreducible. Assume for contradiction that $\bar{X}$ is reducible, i.e. $\bar{X} = X_1 \cup X_2$ for $X_1, X_2 \subseteq Y$ closed subsets. However, we have that

$$X = X \cap \bar{X}$$
$$= X \cap (X_1 \cup X_2)$$
$$= (X \cap X_1) \cup (X \cap X_2)$$

equality. Let X be irreducible. Assume for contradiction that $\bar{X}$ is reducible, i.e. $\bar{X} = X_1 \cup X_2$ for $X_1, X_2 \subseteq Y$ closed subsets. However, we have that

$$X = X \cap \bar{X}$$
$$= X \cap (X_1 \cup X_2)$$
$$= (X \cap X_1) \cup (X \cap X_2)$$

which contradicts X being irreducible unless $X_1 = X_2 = X$ & in this case $\bar{X} = X$ which is another contradiction. Therefore, if X is irreducible then so is $\bar{X}$.

(c) Let $\varphi : \mathbb{A}^n \longrightarrow \mathbb{A}^m$ be an isomorphism & suppose that $V \subseteq \mathbb{A}^n$ is an irreducible closed affine algebraic variety. Suppose for contradiction that $\varphi(V)$ is reducible i.e. $\varphi(V) = W_1 \cup W_2$ for W_1, W_2 closed proper subsets of $\mathbb{A}^m$. But since φ is an isomorphism, we have that $\varphi^{-1}(W_1 \cup W_2) = \varphi^{-1}(W_1) \cup \varphi^{-1}(W_2) \Rightarrow V = \varphi^{-1}(W_1) \cup \varphi^{-1}(W_2)$ for $\varphi^{-1}(W_1), \varphi^{-1}(W_2)$ closed proper subsets of $\mathbb{A}^n$, which is a contradiction. Thus, $\varphi(V)$ is irreducible, & hence isomorphisms preserve irreducibility.

Let $V \subseteq \mathbb{A}^n$ be a c.a.a.v s.t. $\dim(V) = n$, i.e. $V \supseteq V_n \supseteq \dots \supseteq V_0$. Then, since φ (as described above) is an isomorphism, $\varphi(W)$ is a c.a.a.v for any c.a.a.v $W \subseteq \mathbb{A}^n$. So, if $V_i \subseteq V_j$ then $\varphi(V_i) \subseteq \varphi(V_j)$, since if $x \in V_i$, then $x \in V_j$ & so $\varphi(x) \in \varphi(V_j)$. Therefore, isomorphisms preserves chains of varieties, i.e. if $V \supseteq V_n \supseteq \dots \supseteq V_0$, then $\varphi(V) \supseteq \varphi(V_n) \supseteq \dots \supseteq \varphi(V_0)$, & in particular if $\dim(V) = n$, then $\dim(\varphi(V)) = n$.

(d) Let $V = \mathbb{V}(zx - y, y^2 - x^2(x+1))$, then $y = zx$ for all points V.
So, we have that

$$\mathbb{V}(zx-y, y^2 - x^2(x+1)) = \mathbb{V}(zx - y, (zx)^2 - x^2(x+1))$$
$$= \mathbb{V}(zx - y, x^2(z^2 - (x+1)))$$
$$= \mathbb{V}(zx - y, x^2) \cup \mathbb{V}(zx - y, z^2 - (x+1))$$

Thus, the irreducible components of V are $V_1 = \mathbb{V}(zx - y, x^2)$ & $V_2 = \mathbb{V}(zx - y, z^2 - (x+1))$.
To see that those components are irreducible, we construct the following isomorphisms:

$$\varphi_1 : \mathbb{A}^1 \longrightarrow V_1 \quad , \quad \varphi_2 : \mathbb{A}^1 \longrightarrow V_2$$
$$s \longmapsto (0,0,s) \qquad t \longmapsto (t, t\sqrt{t+1}, \sqrt{t+1}).$$

Since, by (c), isomorphisms preserves irreducibility & $\mathbb{A}^1$ is irreducible, it follows that V_1 & V_2 are irreducible.
Note that φ_1 & φ_2 are indeed isomorphisms since there exists morphisms

$$\varphi_1^{-1} : V_1 \longrightarrow \mathbb{A}^1 , \quad \varphi_2^{-1} : V_2 \longrightarrow \mathbb{A}^1$$
$$(0,0,s) \longmapsto s \qquad (t, t\sqrt{t+1}, \sqrt{t+1}) \longrightarrow t$$

s.t. $\varphi_1^{-1} \circ \varphi_1 = \varphi_1 \circ \varphi_1^{-1}$ & $\varphi_2^{-1} \circ \varphi_2 = \varphi_2 \circ \varphi_2^{-1}$ are both identity maps on φ.

Q4. (a) (**10 marks**) Let $V \subseteq \mathbb{A}^n$ be a Zariski-closed subset and $a \in \mathbb{A}^n \setminus V$ be a point. Find a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ such that

$$f \in \mathbb{I}(V), \quad f(a) = 1.$$

(b) (**15 marks**) Let $I, (g) \subseteq \mathbb{C}[x_1, \dots, x_n]$ be two ideals. Assume that $\mathbb{V}(g) \supseteq \mathbb{V}(I)$.

(i) Prove that if $I = (f_1, \dots, f_k)$, then

$$(f_1, \dots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}]. \tag{1}$$

(ii) By only using Equation (1) and not the nullstellensatz, prove that there exists a positive integer m such that $g^m \in I$.

(a) To find $f \in \mathbb{I}(V)$ s.t. $f(a) = 1$, we want $f(v_i) = 0 \ \forall \ v_i \in V$, so we can take $f(x) = \frac{(x - v_1)(x - v_2) \cdots (x - v_k)}{(a - v_1)(a - v_2) \cdots (a - v_k)} = \prod_{i \in I} \frac{(x - v_i)}{(a - v_i)}$, for $x = (x_1, x_2, \dots, x_n)$.

This holds since $a \notin V$ & so $a - v_i \neq 0$ for any $v_i \in V$. & clearly $f(a) = \prod_{i \in I} \frac{(a - v_i)}{(a - v_i)} = 1$.

(b) Let $I, (g) \subseteq \mathbb{C}[x_1, \dots, x_n]$ be two ideals such that $\mathbb{V}(I) \subseteq \mathbb{V}(g)$. Suppose $I = (f_1, \dots, f_k)$.

(i) Let $J = (f_1, \dots, f_k, x_{n+1}g - 1)$. Then clearly $J \subseteq \mathbb{C}[x_1, \dots, x_{n+1}]$.
To show that $\mathbb{C}[x_1, \dots, x_n] \subseteq J$, we want to write any $q \in \mathbb{C}[x_1, \dots, x_{n+1}]$ in terms of the generators of J, that is for $p_i \in \mathbb{C}[x_1, \dots, x_{n+1}]$

$$q = p_1 f_1 + \dots + p_k f_k + p_{k+1}(x_{n+1}g - 1).$$

Since $\mathbb{V}(I) \subseteq \mathbb{V}(g)$, we have that $f_1, \dots, f_k$ & g all vanish at the same time. Therefore, in the case they all vanish, we have $q = -p_{k+1}$, & thus $q \in J$. Since any polynomial that vanishes on $\mathbb{V}(I)$ can be written as a linear combination of $f_1, \dots, f_k$, we can always find a polynomial that makes $p_1 f_1 + \dots + p_k f_k$ vanish, & when all f_i vanish so does g. Therefore, we can always find $q = -p_{k+1} \in J$, i.e. any polynomial $q \in \mathbb{C}[x_1, \dots, x_n]$ is contained in J, & thus $\mathbb{C}[x_1, \dots, x_{n+1}] \subseteq J$ & so $(f_1, \dots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}]$.

(ii) Since $(f_1, \dots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}]$
$$= (1)$$
we can write

$$p_1 f_1 + \dots + p_k f_k + p_{k+1}(x_{n+1}g - 1) = 1$$

for any $p_i \in \mathbb{C}[x_1, \dots, x_{n+1}]$. If we take $p_{k+1} = -1$, then we get

$$p_1 f_1 + \dots + p_k f_k - x_{n+1}g + 1 = 1$$
$$\Rightarrow p_1 f_1 + \dots + p_k f_k = x_{n+1}g.$$

Then, we can choose $p_1, \dots, p_k$ such that they all have a common factor of x_{n+1}, say $p_i = s_i x_{n+1}$, for $s_i \in \mathbb{C}$. then we have that

$$s_1 x_{n+1} f_1 + \dots + s_k x_{n+1} f_k = x_{n+1} g$$
$$\Rightarrow s_1 f_1 + \dots + s_k f_k = g.$$

Thus, g can be written as a linear combination of f_i & hence $g \in I$.

(Attempted both directions for (a) & the 'only if' for (b).

Q5. Prove at least one implication from each of the following equivalences.

(a) (**10 marks**) Show that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is injective if and only if φ is *dominant*. Recall that a map, φ, is called dominant if its image, $\varphi(V)$, is dense in W.

(b) (**10 marks**) Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W.

Suppose that $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ are two c.a.a.v, & let $\varphi : V \to W$ be a morphism.

(a) First, we note that if $\varphi(V)$ is dense in W, then it has nonempty intersection with all open sets of W.
Suppose that $\varphi(V) \subseteq W$ is dense, then for any open set $U \subseteq W$, $\varphi(V) \cap U \neq \emptyset$. In particular, the open set $U = \{x : f(x) \neq 0\}$ where $f \in \mathbb{C}[W]$ has nonempty intersection with $\varphi(V)$. Then, for $x \in \varphi(V) \cap U$, we have that $(\varphi^*(f))(x) \neq 0$, & thus $\varphi^*(f) \neq 0$. But this in turn implies that $\ker \varphi^* = 0$, since f was arbitrarily taken from $\mathbb{C}[W]$, & hence φ^* is injective.

Suppose that $\varphi(V) \subseteq W$ is not dense. Then there exists some open subset U such that $\varphi(V) \cap U = \emptyset$, in particular we have that $\varphi(V) \subset U^c$. Since $U^c \subseteq W$, it follows that there exists some nonzero $g \in \mathbb{C}[W]$ such that U^c is contained in some set of the form $V(g)$. Therefore, we must have that $\varphi^*(g) = 0$, & so φ^* must not be injective.
Thus we have proved the contrapositive & so we have that if φ^* is injective, then $\varphi(V)$ is dense in W.

(b) Suppose that $\varphi: V \to X$ is an isomorphism, where $X \subseteq W$ is a subvariety of W. Then, there exists an injective map $\Phi: X \to W$, & so we have the map $\Phi \circ \varphi : V \to X \to W$. This composition of maps has pullback $(\Phi \circ \varphi)^* = \varphi^* \circ \Phi^* : \mathbb{C}[W] \to \mathbb{C}[X] \to \mathbb{C}[V]$, & since φ is an isomorphism, φ^* is an isomorphism. Thus if $\Phi^* : \mathbb{C}[W] \to \mathbb{C}[X]$ is surjective, then the pullback $\varphi^* \circ \Phi^*$ is surjective. But Φ^* is clearly surjective since Φ is injective & $X \subseteq W$. Therefore, the pullback $\varphi^* \circ \Phi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective.