1. (a)

(1) $\mathbb{V}(f+g)$

(2) $\mathbb{V}((f)+(g)) = \mathbb{V}((f)) \cap \mathbb{V}((g))$

(3) $\mathbb{V}((f)\cap(g)) = \mathbb{V}((f)) \cup \mathbb{V}((g))$

(4) $\mathbb{V}(f) \cap \mathbb{V}(g)$

(5) $\mathbb{V}(fg) = \{x\in\mathbb{C}^2 : f(x)=0 \text{ OR } g(x)=0\}$
$= \{x\in\mathbb{C}^2 : f(x)=0\} \cup \{x\in\mathbb{C}^2 : g(x)=0\}$
$= \mathbb{V}(f) \cup \mathbb{V}(g)$

- $\mathbb{V}((f)) = \{x\in\mathbb{C}^2 : \forall h\in\mathbb{C}^2[x_1,x_2] \;\; h(x)f(x)=0\}$
$= \{x\in\mathbb{C}^2 : f(x)=0\} = \mathbb{V}(f)$, since $h(x)f(x)=0$ must hold for uncountably many $h\in\mathbb{C}^2[x_1,x_2]$, and this is solved only if $f(x)=0$.

So $(3) = \mathbb{V}(f)\cup\mathbb{V}(g) = (5)$, and $(2) = \mathbb{V}(f)\cap\mathbb{V}(g) = (4)$.
since union of two distinct sets contains more elements than their intersection, $(3)\geq(2)$.

Now, $(1) = \{x\in\mathbb{C}^2 : [f(x)=0 \text{ AND } g(x)=0] \text{ OR } [f(x)=-g(x)]\}$
$= [\mathbb{V}(f)\cap\mathbb{V}(g)] \cup \underbrace{\{x\in\mathbb{C}^2 : f(x)=-g(x)\}}_{=:B}$
$= (2) \cup B \geq (2)$.

So we have $(1)\geq(2)$,
$(3)\geq(2)$.

1 ...

$f = y - x$, Then $(2,4) \in \mathbb{V}(f) \cup \mathbb{V}(g)$,
$g = y - x^2$. but $(2,4) \notin \mathbb{V}(f+g)$
So $(3) \not\subseteq (1)$. "$\not\supseteq$"? On the other hand

$$f + g = 2y - x^2 - x = 0$$
$$\Leftrightarrow y = \frac{x^2 + x}{2} = \frac{x(x+1)}{2} \rightsquigarrow \begin{matrix} x = -1 \\ y = 0 \end{matrix}$$

So $(-1,0) \in \mathbb{V}(f+g)$,

but $f(-1,0) = 1$, $g(-1,0) = 1$,

so $(-1,0) \notin \mathbb{V}(f) \cup \mathbb{V}(g)$.

So $(3) \not\supseteq (1)$

2. (c) Let $B = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$, $C = \mathbb{C}$.

Then $B, C \subseteq \mathbb{A}^1$, and $B \subsetneq C$.

Note that B is neither open nor closed in $\mathbb{A}^1$, and it is infinite.

The only closed set in $\mathbb{A}^1$ containing B is $\mathbb{C} = \mathbb{V}(0)$, so $C = \mathbb{C}$ is the smallest closed set containing B, hence $\overline{B} = C$.

By part (b), $\mathbb{V}(\mathbb{I}(B)) = \overline{B} = \mathbb{C} = \overline{C} = \mathbb{V}(\mathbb{I}(C))$.

(d)

$$\varphi(x, xy) = x$$

3. (a) A subset U in topological space (X,τ) is compact if every open cover of U has a finite subcover, i.e., if

$$U = \bigcup_{S\in C} S, \quad \text{then} \quad U = \bigcup_{S\in F} S,$$

where C is some collection of open sets, and F is finite collection of open sets.

(b) Let $\bigcup_{i\in J} U_i$ be some open cover for $\mathbb{V}(x^2-y)$.

That is, $\bigcup_{i\in J} U_i \supseteq \mathbb{V}(x^2-y)$.

Since $\bigcup_{i\in J} U_i$ is open, each $U_i \in \mathbb{A}^2$ is open, that is, $U_i = \mathbb{A}^2 \setminus \mathbb{V}(I_i)$, for some ideal I.

Then $\bigcup_{i\in J} \mathbb{A}^2 \setminus \mathbb{V}(I_i) \supseteq \mathbb{V}(x^2-y)$

$$\Leftrightarrow \mathbb{A}^2 \setminus \bigcap_{i\in J} \mathbb{V}(I_i) \supseteq \mathbb{V}(x^2-y)$$

$$\Leftrightarrow \mathbb{A}^2 \setminus \mathbb{V}\Big(\sum_{i\in J} I_i\Big) \supseteq \mathbb{V}(x^2-y).$$

If the index set J is uncoutable, take its countable subset $C \subset J$.
Note that $\mathbb{C}^2[x_1,x_2]$ is Noetherian, so its' ideals satisfy an ascending chain condition

$$I_0 \subset I_1 \subset \dots \subset I_N = I_{N+1} = \dots, \quad N \in \mathbb{N}.$$

Note also that $\underbrace{I_0}_{K_0} \subset \underbrace{I_0+I_1}_{K_1} \subset \dots \subset \underbrace{I_0+\dots+I_N}_{K_N}$. ideals

So $\sum_{i\in C} I_i = K_N$, for some ideal K_N, as ideals are closed under addition.

$\square$

3. (b) Since in Euclidean topology every compact set is closed and bounded, and $\mathbb{V}(x^2-y) = \{(t,t^2) : t\in\mathbb{R}\}$ is unbounded, it is not compact in Euclidean top°.

4. (a) A field K is alg. closed if for every $\alpha\in K$ there exists polynomial $f\in K[t]$ such that $f(\alpha)=0$.

(b) "$\Longrightarrow$" Suppose $(1)\neq I \subseteq K[x_1,\dots,x_n]$

Let us show that

$\mathbb{V}(I)\neq\emptyset$ in $\overline{K}^n$

Suppose by contradiction that

$$\mathbb{V}(I) = \emptyset \subseteq K[x_1,\dots,x_n] \subseteq \overline{K}[x_1,\dots,x_n]$$

since every ideal in $K[x_1,\dots,x_n]$ is finitely generated, $K[x_1,\dots,x_n]$ is Noetherian.

Let $J\subseteq \overline{K}[x_1,\dots,x_n]$ be an ideal generated by I, then $I\subseteq J \subsetneq \overline{K}[x_1,\dots,x_n]=(1)$, and $\mathbb{V}(J)=\{x\in\overline{K} \mid f(x)=0 \ \forall f\in I\}$.

- If $\mathbb{V}(J)=\emptyset$, then $\mathbb{V}(J)=\emptyset=\mathbb{V}(I)$ $\iff$ $J=(1)=\mathbb{I}(\mathbb{V}(I))$
  $\iff$ $J=(1)=\sqrt{I}$
  $\iff$ $I=(1)$, contradiction,
  as $\mathbb{I}(\emptyset)=\overline{K}[x_1,\dots,x_n] = (1)$.
- If $\mathbb{V}(J)\neq\emptyset$, then as $I\subseteq J$, $\emptyset=\mathbb{V}(I)\supseteq\mathbb{V}(J)$, so $\mathbb{V}(J)=\emptyset$.

Therefore $\mathbb{V}(I)\neq\emptyset$ in $\overline{K}$.

"$\Leftarrow$" Suppose $\emptyset \neq \mathbb{V}(I) \subseteq \mathbb{K}^n$. Let us show that $(1) \neq I$.

If $I = (1) = \mathbb{K}[x_1, \ldots, x_n]$, then since $1 \in (1)$, and $\mathbb{V}(I)$ is the common zero-set of all polys in I, we must have $\mathbb{V}(I) = \emptyset$, as there are no roots for constant polynomial 1. Therefore $I \neq (1)$.

5. (b) Suppose $\varphi : V \to U \subseteq W$ is iso^m.

By thm 2.37, the map $\varphi^* : \mathbb{C}[U] \longrightarrow \mathbb{C}[V]$ is iso^m.

Define restriction $i : W \to U$. Note it is surjective.

Then $i^* : \mathbb{C}[U] \longrightarrow \mathbb{C}[W]$ is also surjective.

For any $f \in \mathbb{C}[W]$ we have $f|_U = f$, so

$$i^*(f|_U) = f|_U \circ i = f.$$

Altogether, $i^* \circ \varphi^*$ is surjective map

from $\mathbb{C}[W]$ to $\mathbb{C}[V]$.