Lecture Notes
Geometry of Manifolds
MATHM0037 [1]

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Contents

Chapter 0

Preliminaries: Point set Topology.

Here we give a very brief introduction to point set topology. This should only be seen as a glossary of terms that will be used later, and you are referred to other texts for a more comprehensive introduction to the subject. Any introduction level to topology textbook will cover everything we mention here, for example "*Topology*" by Munkres.

0.1 Topological spaces

In your Metric Spaces course you learned what a metric space is: a non-empty set set X together with a *metric* d, i.e. a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$ we have $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$, and $d(x, y) \leq d(x, z) + d(z, x)$. Using the metric one can define many notions about subsets of X: what it means for a set to be open, compact, connected etc, as well as what it means for a map between two metric spaces to be continuous. In fact, all these notions can be defined in terms of open sets. A *topological space* is a set X together with a topology T describing which sets are open. In particular, any metric space is a topological space, however in general we have no notion of distance between points in a topological space. Nonetheless, any notion defined in terms of open sets can be defined for these more general spaces too. The topology T has to satisfy some basic properties, mimicking the behavior of open sets in a metric space:

Definition 0.1 (Topology) Let X be a set. A *topology* T on X is a collection of subsets of X such that

- $\emptyset, X \in T$
- An arbitrary union of sets in T also belongs to T, I.e. if $U_\alpha \in T$ for all α in some index set A, then $\bigcup_{\alpha \in A} U_\alpha \in T$.
- A finite intersection of sets in T also belongs to T. I.e. if $U_1, U_2, \ldots, U_n \in T$ then $\bigcap_{i=1}^n U_i \in T$.

The elements of T are called *open* sets. If T is a topology on X we say (X, T) (or just X) is a *topological space*.

Exercise 0.2 Let (X, d) be a metric space. Recall that $U \subset X$ is said to be an *open* set in (X, d) if every point in U is an interior point; that is, for all $x \in U$ there exists $\epsilon > 0$ such that $d(x, \epsilon) \subset U$. Show that $T = \{U \,|\, U \text{ open in } (X, d)\}$ is a topology on X. (Hence every metric space is a topological space).

To help illustrate the definition, we give some toy examples: Let X be a set containing just 4 elements, say $X = \{a, b, c, d\}$. Then

- $T = \{\emptyset, \{a, b, c, d\}\}$ is a topology on X.
- $T = \{\emptyset, \{a, b, c, d\}, \{a\}, \{b, c\}, \{a, b, c\}\}$ is a topology on X.
- $T = \{\emptyset, \{a, b, c, d\}, \{a\}, \{b, c\}, \{a, b, c\}, \{a, b\}\}$ is *not* a topology on X (since $\{a, b\}$ and $\{b, c\}$ are open sets but their intersection is not.
- $T = \{\emptyset, \{a, b, c, d\}, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ is a topology on X.

Note that for any set X, $T = \{\emptyset, X\}$ is always a topology on X (as in the first example above). On the other extreme, the collection of all subsets of X (as in the last example above) always determines a topology on X. This is called the *discrete* topology.

Exercise 0.3 Let $T = \{U \subset \mathbb{R} \,|\, U = \emptyset \text{ or } \mathbb{R} \setminus U \text{ is finite }\}$. Prove that T is a topology on $\mathbb{R}$. (This topology is called the finite complement topology).

The next several examples will be more useful for the purpose of this course:

(1) (Metric topology) As mentioned in Exercise 0.2 above, any metric space is a topological space with topology $T = \{U \,|\, U \text{ is open in } (X, d)\}$. In particular $\mathbb{R}^n$ with the Euclidean metric is a topological space, which we call the *Euclidean space of dimension* n, or Euclidean n-space.

If (X, T) is a topological space and if there exists a metric d on X such that the topology given by d is the same as T we say that X is *metrizable*.

Exercise 0.4 Show that the finite complement topology on $\mathbb{R}$ (see Exercise 0.3) is *not* metrizable.

(2) Suppose (X, T) is a topological space and let $A \subset X$ be any subset. Then the *subspace* (A, T') where $T' = \{U \cap A \,|\, A \in T\}$ is a topological space. We say T' is the *subspace topology* on A. That is, if X is a topological space and A is a subspace, then V is open in A if and only if $V = U \cap A$ for some open set A in X. Unless otherwise stated, we always assume a subset of a topological space is equipped with the corresponding subspace topology. As an example, consider $\mathbb{R}$ with the usual topology and consider $[0, 1]$ with the subspace topology. Then $[0, 1/2)$, $(1/3, 1/2)$, and $(1/2, 1]$ are all examples of open sets in $[0, 1]$ (while only the second one is open in $\mathbb{R}$).

(3) Suppose (X_α, T_α) is a family of topological spaces, where α is in some index set $\mathcal{A}$ (which could be finite, infinite, countable, or uncountable). Then the *product space* is (X, T) where X is the Cartisian product $X = \Pi_{\alpha \in \mathcal{A}} X_\alpha$ and $U \in T$ (i.e. U is open in this topology) if and only if it can be written as a union $U = \Pi_{\alpha \in \mathcal{A}} U_\alpha$ where *all but finitely many* $U_\alpha = X_\alpha$. For each $\beta \in \mathcal{A}$ define the *projection map* to be $\pi_\beta : X \to X_\beta$ defined by $(x_\alpha)_{\alpha \in \mathcal{A}} \mapsto x_\beta$. This map is continuous for each β. In fact, the product topology is the coarsest (i.e. the one with the least amount of open sets) topology on the set X for which the projection maps are continuous.

In this course we will only be concerned with the product topology for finitely many factors. In this case the situation is simplier (although, of course, is the same as the above): if $X_1, X_2, \ldots X_n$ are topological spaces, then the product space is $X_1 \times X_2 \times \cdots X_n$ where a set U is open in the product if it can be written as $U_1 \times U_2 \times \cdots \times U_n$ for open sets $U_i \in x^i$.

Remark 0.5 In the case where we allow infinitely many factors in the product, we could instead allowed any product of open sets to be open in X but we would get a different topology called the *box topology.* Note that, clearly, if $\mathcal{A}$ is finite, the box topology and product topology agree. However they are not the same if the index set is infinite. When we work with infinite index set we always, unless otherwise stated, work with the product topology. To see why this is the "right" topology, consider the following examples. Let $X_i = \mathbb{R}$ for each $i \in \mathbb{N}$ and let $X = \Pi_{i \in \mathbb{N}} X_i$. Let (x_n) be a sequence in X, say $x_n = (x_{1,n}, x_{2,n}, x_{3,n}, \ldots)$ for each n. In the product topology (x_n) converges as $n \to \infty$ to some point $x = (x_1, x_2, x_3, \ldots) \in X$ if and only if $(x_{i,n})$ converges as $n \to \infty$ to x_i for each i. This is not true in the box topology. Another motivation: The map $f : \mathbb{R} \to X$ defined by $f(x) = (x, x, x, \ldots)$ is continuous in the product topology but not in the box topology. We leave it as an exercise to verify these claims.

Often it is hard to list, or even describe, all open sets. For example, in the case of a metric space we know how to define an open set but in general there is no clean way to explain what they look like. However, we can easily describe an open ball in a metric space. It turns out that this is enough in order to describe all open sets; the open balls *generate* the topology given by the metric in the sense that they are a *basis* for the topology.

Definition 0.6 (Basis of a topology) Let X be a set. A *basis* for a topology on X is a set $\mathcal{B}$ consisting of subsets of X (called *basis elements*) such that

- for all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$
- if $x \in B_1 \cap B_2$, where B_1, B_2 are two basis elements, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

  If $\mathcal{B}$ is a basis (i.e. satisfies the conditions above) then *the topology T generated by $\mathcal{B}$* is defined as

$$T = \{U \subset X \,|\, \text{ for all } x \in U \text{ there exists } B \in \mathcal{B} \text{ such that } x \in B \subset U\}.$$

In the above definition we use the word "topology" for T without justifying why it is a topology. However, it is and we leave it as an exercise to prove it:

Exercise 0.7 Let X be a set and $\mathcal{B}$ be a basis. Show that the topology generated by T is indeed a topology on X.

The main example to keep in mind is, as always, Euclidean space $\mathbb{R}$ or $\mathbb{R}^n$. In this case one can take as a basis the collection of all open metric balls.

Exercise 0.8 Consider $\mathbb{R}$ and let $\mathcal{B} = \{(a, b) \,|\, a, b \in \mathbb{R}, a < b\}$ (i.e. the collection of all open intervals). Show that $\mathcal{B}$ is a basis and that it generates the usual (Euclidean) topology on $\mathbb{R}$. More generally, show that the set of all open balls in $\mathbb{R}^n$ is a basis for the Euclidean topology on $\mathbb{R}^n$. (Note that this is true for any metric space: any open set in a metric space (X, d) can be written as a union of open balls and hence the set of all open sets or the set of all open balls generate the same topology, namely the metric topology, on X).

0.2 Topological Properties

Let (X, T) be a topological space. Hence we have a notion for a subset of X to be open, and hence we can extend all the notions we learned in Metric Spaces defined in terms of open sets to the topological space (X, T). (However, we remark again that we may no longer have a notion of distance between points in X.) We list some of these properties below, and also give some new, related, definitions and terminology. The below can be used as a glossary to come back to while reading the rest of the lecture notes.

Throughout the below definitions we let X be a topological space and T its topology. Whenever we consider a subset of X we allow for the possibility of the subset to be all of X. (I.e. $A \subset X$ always includes the possibility of $A = X$).

(Open) neighborhood. Let $x \in X$. An open neighborhood (often just called a neighborhood) U of x is any open set U in X which contains x.

Continuous maps. Let X and Y be topological spaces and $f : X \to Y$ be a map. We say f is *continuous at a point* $x \in X$ if for every neighborhood U of $f(x)$ there exists a neighborhood V of x such that $f(V) \subset U$. We say that f is *continuous* (on X) if it is continuous at every x in X.

Equivalently, we say that f is continuous (on X) if $f^{-1}(U)$ is open in X for every open set $U \subset Y$. (In other words: if T is the topology on X and T' is the topology on Y, we say f

is continuous if $f^{-1}(U) \in T$ for all $U \in T'$). (You should verify that the two definitions are equivalent!).

Homeomorphisms. A map $f : X \to Y$ is a homeomorphism if it is continuous, bijective, and its inverse $f^{-1} : Y \to X$ is also continuous.

Interior point Let $A \subset X$. A point $x \in A$ is an interior point of A if there exists an open neighborhood U of x such that $U \subset A$. Note that A is an open set if and only if every point of A is an interior point.

Closed set. A subset $A \subset X$ is closed if its complement $X \setminus A$ is open (that is, $X \setminus A \in T$.) Note that in particular X is always closed (and open!).

Limit point Let $A \subset X$. We say $x \in X$ is a limit point of A if every open neighborhood of x intersects A; that is x is a limit point of A if for any open $U \subset X$ with $x \in U$ we have $(U \setminus \{x\}) \cap A \neq \emptyset$. It is a fact that A is a closed set if and only if it contains all its limit points.

Closure of a set. Let $A \subset X$. The closure of A in X, denoted $\bar{A}$, is defined as $\bar{A} = A \cup \{\text{all limit points of } A\}$.

Connectedness. A set $A \subset X$ is connected if there do *not* exists nonempty, open, disjoint sets U, V in X that cover A and such that $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$.

Path connectedness. A set $A \subset X$ is said to be path-connected if any two points $x, y \in A$ can be connected be a (continuous) path contained in A. That is, for all $x, y \in A$ there exists a continuous function $\gamma : [0, 1] \to A$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

(Connected) Component Let $A \subset X$. A connected component (sometimes just called a component) is a set $C \subset A$ such that C is connected and if there exists a connected set D such that $C \subset D \subset A$, then either $C = D$. That is, C is a "largest" connected subset of A. (Note that if A is connected, then it only has one component, namely A itself).

Open cover. Let $A \subset X$. A collection (which could be finite or infinite, countable or uncountable) of sets $\{U_\alpha\}$ is an open cover for A if each U_α is an open set in X and A is contained in the union of all U_α that is, $A \subset \bigcup U_\alpha$. (We say that A can be *covered by* these sets).

Locally finite. A collection $\{A_\alpha\}$ of subsets of X is said to be locally finite if every point in x has a neighborhood which intersects at most finitely many of the sets. That is, for all $x \in X$ there exists a neighborhood U if x such that $U \cap A_\alpha = \emptyset$ for all but finitely many α.

Compactness. Let $K \subset X$. We say that K is compact if *every* open cover of K has a *finite subcover*. That is, if $\{U_\alpha\}$ is an open cover of K then K can in fact be covered by only finitely many of these sets, i.e. there exists $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n} \in \{U_\alpha\}$ such that $K = \cup_{i=1}^n U_{\alpha_i}$.

Sequentially compactness. Let $K \subset X$. We say that K is sequentially compact if every sequence (x_n) in K has a subsequence that converges in K (i.e. it converges and the limit is in K).

Remark 0.9 (Compactness vs. Sequential Compactness) The two notions of compactness are in general different (there are topological spaces that are compact but not sequentially compact and vice versa). However, for any metrizable space they are equivalent. In fact, for any "nice enough" space (those that are Hausdorff, second countable, and regular) the two notions are equivalent. The spaces we will consider in this course–smooth manifolds–all have the required properties needed for the two notions to be equivalent. Hence from now on you can take "compact" to mean either of the two definitions given.

Locally compact We say that $K \subset X$ is locally compact if every point $x \in K$ has an open neighborhood U and a compact set K' such that $U \subset K'$. Equivalently (at least in all cases we will encounter), X is locally compact if every point has a neighborhood whose closure is compact. For example, $\mathbb{R}$ is not compact, but is locally compact.

Precompactness. We say $K \subset X$ is precompact if its closure $\bar{K}$ is compact. For example, in $\mathbb{R}$ an open interval (a, b) is precompact.

Paracompactness. Given an open cover $\mathcal{U} = \{U_\alpha\}$ of X we say $\mathcal{V} = \{V_\beta\}$ is a *refinement* of $\mathcal{U}$ if $\mathcal{V}$ is also an open cover of X and each V_β is a subset of some U_α. We say $K \subset X$ is *paracompact* if every open cover of X has a refinement that is locally finite.

Hausdorff. We say X is Hausdorff (also knows as T2) if for any two distinct points $x, y \in X$ there exists *disjoint* non-empty open sets U, V such that $x \in U$ and $y \in V$. (We say that x and y can be *separated* by (non-empty) open sets.

Second Countability. We say that X is second countable if its topology T has a countable basis (i.e. a basis consisting of countably many sets).

The last three notions above might be new to you. We note that $\mathbb{R}$, and in fact $\mathbb{R}^n$ for any $n \geq 1$ (with the usual topology defined by its Euclidean metric) is Hausdorff, second countable, and paracompact. In the next section (and throughout the course) we will look at a particular class of topological spaces called manifolds. We will require them to be Hausdorff, paracompact, and second countable in an attempt to mimic $\mathbb{R}^n$ as much as possible. These assumption guarantees that some notions that feel intuitively true to us are indeed true. For example, a space X being Hausdorff implies that a singleton subset $\{x\}$ of X is a closed set, and that a sequence can have at most one limit.

Exercise 0.10 Prove that $\mathbb{R}$ (and more generally $\mathbb{R}^n$) is Hausdorff, second countable, and paracompact, but *not* compact.

Exercise 0.11 Prove that Hausdorff implies that singletons are closed and that limits are unique.

Lastly, we comment on the importance of homeomorphisms. If X and Y are topological spaces that are homeomorphic, then they share all the same topological properties. For example, X is compact if and only if Y is; X is connected if and only if Y is; X is Hausdorff if and only if Y is, and so on. Hence, we think of X and Y as being *the same* topological space. I.e. we view topological spaces "up to homeomorphism".

Exercise 0.12 Show that the following spaces are homeomorphic:

- Any open interval (a, b) and $\mathbb{R}$.
- The open interval $(0, 1)$ and $\mathbb{S} \setminus \{(1, 0)\}$ where $\mathbb{S}$ is the unit circle (viewed as a subset of $\mathbb{R}^2$).
- $\mathbb{R}$ and $\mathbb{S} \setminus \{(1, 0)\}$. (Is $\mathbb{S}$ and $\mathbb{R}$ homeomorphic?)

Remark 0.13 We will only be considering spaces that are Hausdorff and paracompact (and in fact, also second countable) but, for the interested reader, we give her a couple of examples of topological spaces that do not have all of these properties and leave it as an exercise to verify the claims. We will never encounter any of these spaces again in the course.

- The so-called *line with two origins* is second countable and paracompact but *not* Hausdorff.
- Let X be an uncountable set and equip it with the discrete topology. Then X is Hausdorff and paracompact, but *not* second countable.
- The so-called *long line* is Hausdorff but *not* second countable and *not* paracompact.

Chapter 1

Manifolds and smooth maps.

1.1 Topological and Smooth Manifolds

Intuitively, a (topological) manifold is a topological space which locally looks like Euclidean space $\mathbb{R}^n$ for some n. For technical reasons we will always require our manifolds to be Hausdorff, paracompact, and second countable. More precisely:

> **Definition 1.1 (Topological manifold)** A *topological manifold of dimension* n (or topological n-manifold) is a topological space M which is Hausdorff, paracompact, and second countable such that every point $p \in M$ has an open neighborhood which is homeomorphic to $\mathbb{R}^n$.
>
> That is, for all $p \in M$ there exists an open neighborhood $U \subset M$ of p and a homeomorphism $\varphi : U \to \tilde{U}$ for some open set $\tilde{U} \subset \mathbb{R}^n$.
>
> A *manifold with boundary* is a topological space satisfying all the properties above except that $\mathbb{R}^n$ is replaced with the closed half-space of $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ (equipped with the subspace topology).

We will see some examples of manifolds below, but a trivial example is $\mathbb{R}^n$ itself. The n-sphere or a torus are other familiar examples. In this course we will (almost) only consider manifolds without boundary, but most of the theory generalizes directly to manifolds with boundary. A simple example of a manifold with boundary is a cylinder, another is a closed ball in $\mathbb{R}^n$.

Remark 1.2 If M is a topological n-manifold with boundary, then its *interior*

$$M^\circ = \{p \in M \text{ such that } p \text{ has a neighborhood homeomorphic to to an open set in } \mathbb{R}^n\}$$

is a topological n-manifold (without boundary) and its *boundary* $\partial M = M \setminus M^0$ is a topological $(n-1)$-manifold (without boundary).

We do not require in the definition that M is connected, i.e. it could have several components; this is for example sometimes the case when we consider the boundary of a manifold with boundary, for example the boundary of a (closed) cylinder is the 1-manifold given by the union of two circles. However, with the exception of when we consider the boundary of a manifold, unless we say otherwise our manifolds will be connected.

Remark 1.3 (Second countable manifolds) Sometimes a manifold is defined to be just a Hausdorff, second countable space which is locally Euclidean, i.e. the paracompact assumption is left out. However, as long as the space has at most countably many components, a locally Euclidean, Hausdorff space is second countable if and only if it is paracompact. In this course we will never consider manifolds having uncountably many components (in fact, they will almost always just have one) and you could define then using either second countable or paracompact. We add both words in the assumption since we will use both facts frequently.

Remark 1.4 (A note on the dimension) It is not immediate that the dimension n in the definition of an n-manifold is well-defined. That is, how do we know that an n-manifold M is not also a k-manifold for some $k \neq m$? This follows from the *Invariance of Domain Theorem* which says that if two non-empty open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^k$ are homeomorphic, then we must have $n = k$. Hence the dimension is indeed well-defined and we will write $\dim M = n$ if the M is an n-manifold. The statement of the Invariance Domain Theorem might sound obvious; however it is actually a non-trivial result in topology and requires some real work to prove (and we will not do so in this course). When it comes to smooth manifolds, which we will define soon and is what we will consider in this course, it is much easier to see that the dimension is well-defined.

Moreover, in our definition we assume that every component of M has the same dimension and this will be enough for our purposes. Some authors define a manifold to be a locally Euclidean, Hausdorff, paracompact space in which case it follows that each connected component has a well-defined dimension (by the Invariance of Domain Theorem) but two components might not have the same dimension; for example the union of a sphere and a circle would be a manifold in this definition. Since we will mostly consider connected (or in the case of the boundary, a union of manifolds of the same dimension) the definition we give works well.

We note that although a manifold can be *locally* identified with Euclidean space it cannot, in general, be identified with one globally. Meaning, locally, around each point, the manifold is homeomorphic to a subset of a Euclidean space but there might not exists a homeomorphism between M and $\mathbb{R}^n$. For example, the 2-sphere $\mathbb{S}^2$ is locally homeomorphic to $\mathbb{R}^2$ as we will see below but there exists no homeomorphism from $\mathbb{S}^2$ to $\mathbb{R}^2$. To see this, note that the former is compact while the latter is not.

The pair (U, φ) in the definition above is called a (coordinate) *chart* of M, φ is a coordinate map and U is a coordinate domain (although we often abuse notation and refer to φ, or even U, as the chart). The collection of all charts is called an *atlas* for M. That is, $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ (for some index set A) is an atlas for a topological manifold if $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M and for each α, $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism onto its image. Given a chart (U, φ) for a n-dimensional manifold, we usually *identify* U with $\tilde{U} = \varphi(U) \subset \mathbb{R}^n$, i.e. view it as a subset of $\mathbb{R}^n$ and hence identify each point $p \in M$ with a point in $\mathbb{R}^n$ (i.e. an n-tuple of real numbers). Writing φ in its coordinate functions, $\varphi = (x^1, x^2, \dots, x^n)$, we have $\varphi(p) = (x^1(p), x^2(p), \dots, x^n(p))$; we call $(x^1, x^2, \dots, x^n)$ *local coordinates* of U.

Now, suppose (U, φ) and (V, ψ) are two charts. If $U \cap V \neq \emptyset$ then the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called a *transition map*, or a *change of coordinates map*. It is clearly a homeomorphism.

Next we will define a *smooth manifold*, one which locally inherits not only topological properties from Euclidean space, but also properties which allow us to do calculus on $\mathbb{R}^n$. The main idea for this is to require the transition maps to be *smooth maps*.

Recall that a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if all the partial derivatives of each coordinate function exists. I.e. if $f = (f_1, \dots, f_m)$ and we use coordinates $x = (x^1, \dots, x^n)$ for $\mathbb{R}^n$ and $y = (y^1, \dots, y^m)$ for $\mathbb{R}^m$, f is differentiable at x if

$$\frac{\partial f_i}{\partial y^j}(x)$$

exists for all $i, j \in \{1, \dots m\}$ (and, as always, we say it's differentiable on $U \subset \mathbb{R}^n$ if it is differentiable at every $x \in U$). We say f is *smooth*, or f is C^∞, if it is infinitely differentiable, i.e. the partial derivatives above exists of all orders.

For all intent and purposes, a smooth manifold is a topological manifold for which the transition maps are smooth:

Definition 1.5 (Working definition of smooth manifold) A *smooth manifold* (of dimension n) is a topological manifold M (of dimension n) together with an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ with the property that for all U_α, U_β with $U_\alpha \cap U_\beta \neq \emptyset$ the transition map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth.

Although the above definition is exactly how we will think about and use the term smooth manifolds, for technical reasons we need one more assumption for the definition to be well-defined. The issue is that we want a well-defined way to determine whether two smooth manifolds are the same or not and we want two diffeomorphic manifolds (those that are related by a smooth bijective map whose inverse is also smooth; see Definition 1.18) to be considered the same, even if the smooth atlas in the definition might look different. For this reason we introduce the notion of *smooth structure*, as follows. First, let M be a topological manifold and $\mathcal{A}$ an atlas for M. We say two charts (U, φ), (V, ψ) are *smoothly compatible* if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is smooth. We say the atlas $\mathcal{A}$ is a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible. Lastly, we say a smooth atlas is *maximal* if it is not properly contained in a larger smooth atlas, i.e. $\mathcal{A}$ is a maximal smooth atlas if whenever $(U, \varphi) \in \mathcal{A}$ and there exists open $V \subset M$ and a map $\psi : V \to \mathbb{R}^n$, homeomorphism onto its image, which is smoothly compatible with (U, φ), then it must be the case that (V, ψ) already belonged to $\mathcal{A}$. We call a chart that belongs to a smooth atlas a *smooth chart*.

Definition 1.6 (Technical definition of smooth manifold) Let M be a topological manifold. A *smooth structure* on M is a maximal smooth atlas. If $\mathcal{A}$ is a maximal smooth atlas on M we say that $(M, \mathcal{A})$ (or usually just M) is a *smooth manifold*.

As one can imagine, finding a maximal atlas is in general very difficult. Fortunately, we will never have to do so thanks to the following lemma (which also justifies our "working definition" of a smooth manifold):

Lemma 1.7 *Let M be a topological manifold. Every smooth atlas for M is contained in a unique maximal smooth atlas. Moreover, two smooth atlases for M determine the same maximal smooth atlas if and only if their union is a smooth atlas.*

We give some examples of smooth manifolds:

Example 1.8 (Smooth manifolds)

1. (Euclidean space) $\mathbb{R}^n$ is a smooth n-manifold. It is not hard to verify that any Euclidean space is Hausdorff and paracompact. The atlas containing a single (global) chart $\mathcal{A} = \{(\mathbb{R}^n, id)\}$ is a smooth atlas. Another smooth atlas is $\mathcal{A}' = \{(B_x, id|_{B_x})\}_{x \in \mathbb{R}}$ where B_x is the open unit ball centered at x. Since $\mathcal{A} \cup \mathcal{A}'$ is also a smooth atlas, these two determine the same smooth structure on $\mathbb{R}^n$, which we usually call the *standard* structure.

2. (A different structure on $\mathbb{R}$). Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(x) = x^3$. Then $\{(\mathbb{R}, \phi)\}$ is a smooth atlas on $\mathbb{R}$. However, it does not give the same smooth structure as the standard one, since $id \circ \phi^{-1} = x^{1/3}$ is not a smooth function (it is not differentiable at 0).

3. (Spheres) For any n the n-sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \,|\, |x| = 1\}$ is a smooth n-manifold. Hausdorff and paracompactness follow from being a subset of Euclidean space. For each $i \in \{1, 2, \ldots, n+1\}$ let $U_i^+ = \{(x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \,|\, x^i > 0\}$ and $U_i^- = \{(x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \,|\, x^i < 0\}$. Then $\{U_i^+, U_i^-\}$ is an open cover of $\mathbb{S}^n$. Now let $\varphi_i^+ : U_i^+ \to \mathbb{B}^n \subset \mathbb{R}^n$ (where $\mathbb{B}^n$ is the open unit ball) be defined by $\varphi_i^+(x^1, \ldots, x^{n+1}) = (x^1, \ldots x^{i-1}, x^{i+1}, \ldots, x_{n+1})$ and define φ_i^- similarly. Then each φ_i^+, φ_i^- is a homeomorphisms, and the transition maps are smooth.

4. (Product Manifolds) Suppose M is a smooth n-manifold and N is a smooth k-manifold. Then $M \times N$ (with the product topology) is a smooth $(n + k)$-manifold. One easily verifies that the product of two Hausdorff, paracompact spaces is Hausdorff and paracompact. If $\mathcal{A}$ is a smooth atlas for N and $\mathcal{B}$ a smooth atlas for N, then $\{(U \times V, \varphi \times \psi) \,|\, (U, \varphi) \in \mathcal{A}, (V, \psi) \in \mathcal{B}\}$ is a smooth atlas for $M \times N$, where $\varphi \times \psi : U \times V \to \mathbb{R}^{n+m}$ is defined by $(u, v) \mapsto (\varphi(u), \psi(v))$. Since φ and ψ are homeomorphisms onto their images, this defines a homeomorphism onto its image. Moreover, a transition map is of the form $(\varphi_1 \times \psi_1) \times (\varphi_2 \circ \psi_2)^{-1} = (\varphi_1 \circ \varphi_2^{-1}) \times (\psi_1 \times \psi_2^{-1})$ and hence smooth. This construction extends directly to a product of finitely many manifolds $M_1 \times \cdots \times M_p$.

5. (Tori) The n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ is a smooth n-manifold by the previous two examples.

Exercise 1.9 Find an explicit smooth atlas for the 2-torus T^2.

6. (Matrices) Let $M(n, \mathbb{R})$ be the set of all $n \times n$-matrices with real entries. Then $M(n, \mathbb{R})$ is an n^2-manifold. In fact, it is homeomorphic to $\mathbb{R}^{n^2}$ by $\psi : M(n, \mathbb{R}) \to \mathbb{R}^{n^2}$,
$$(a_{ij}) \mapsto (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots a_{nn}).$$

7. (Open submanifolds) Let M be a smooth n-manifold and $U \subset M$ any open non-empty subset. Then U is also a smooth n-manifold. If $\mathcal{A} = \{(V, \varphi)\}$ is a smooth atlas for M, then $\mathcal{A}_U = \{(V, \varphi) \in \mathcal{A} \,|\, V \subset U\}$ is a smooth atlas for U.

   In particular, any open set $U \subset \mathbb{R}^n$ is a smooth manifold.

8. (General Linear Group) Let $GL(n, \mathbb{R})$ denote the set of all $n \times n$ matrices with real entries and with non-zero determinant (i.e. invertible). This is a smooth n^2-manifold, since it is an open subset of $G(n, \mathbb{R})$ (it is open since determinant is a continuous function).

In the example of $M(n, \mathbb{R})$ above we cheated a bit since we didn't define a topology on this set (or rather, it inherited the topology from $\mathbb{R}^{n^2}$ by identifying it with it, giving us a circular argument for why it is a manifold. However, one can construct manifolds by giving the charts using the following lemma:

Proposition 1.10 (Manifold Construction Lemma) *Let M be a set and suppose we are given a collection $\{U_\alpha\}_{\alpha\in A}$ of subsets of M and a collection of injective maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ for each α such that the following hold:*

- *for each α the set $\tilde{U}_\alpha = \varphi_\alpha(U_\alpha)$ is an open subset of $\mathbb{R}^n$,*
- *for all α, β the set $\varphi_\beta(U_\alpha \cap U_\beta)$ is an open subset of $\mathbb{R}^n$,*
- *for all α, β the map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a homeomorphism,*
- *whenever $x, y \in M$ are two distinct points then either there exists α with $x, y \in U_\alpha$ or there exists α, β with $x \in U_\alpha$, $y \in U_\beta$ and $U_\alpha \cap U_\beta = \emptyset$.*
- *there is a countable $I \subset A$ such that the collection $\{U_\alpha\}_{\alpha\in I}$ covers M.*

Then M has a unique topology with respect to which the collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ is an atlas for M.

Moreover, if we require the maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ in the third condition above to be diffeomorphisms *instead, then M has a unique* smooth *structure with respect to which $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas (i.e. M with this atlas is a smooth manifold).*

Exercise 1.11 Use the Manifold Construction Lemma to prove that $M(n, \mathbb{R})$ is a smooth manifold.

Exercise 1.12 Let V be an n-dimensional (real) vector space. Prove that it is a smooth n-manifold.

1.2 Smooth maps between manifolds

Our next goal is to define what it means for a map between (smooth) manifolds to be a smooth map. We start with the case when the target manifold is a Euclidean space.

Definition 1.13 (Smooth function into $\mathbb{R}^k$) Let M be a smooth n-manifold. A function $f : M \to \mathbb{R}^k$ is said to be *smooth* if for every $p \in M$ there exists a smooth chart (U, φ) for M with $p \in U$ such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is a smooth map.

It follows from the definition of smooth structures that if $F : M \to \mathbb{R}^k$ is smooth, then $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ for *every* smooth chart (U, φ), which we ask you to check for your self:

Exercise 1.14 If $F : M \to \mathbb{R}^k$ is smooth, then $F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth for *every* smooth chart (U, φ).

The definition above can be generalized to maps between arbitrary smooth manifolds:

Definition 1.15 (Smooth map) Let M, N be smooth manifolds and $F : M \to N$ a continuous map. We say that F is *smooth* if for every $p \in M$ there exists a smooth chart (U, φ) of M containing p and a smooth chart (V, ψ) of N containing $F(p)$ such that $F(U) \subset V$ and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth.

Remark 1.16 The assumption that $F(U) \subset V$ is a technicality and by enlarging to a maximal atlas will always be satisfied. The assumption is needed in order for $\psi \circ F \circ \varphi^{-1}$ to be well-defined. However, given smooth charts (U, φ) at p and (V, ψ) at $F(p)$ one can always replace U by $U \cap F^{-1}(V)$, which is open since F is continuous, and φ by its restriction to this set. This chart will be smoothly compatible with all the other charts in the atlas of M since this is true of (U, φ) and hence adding it in will not change the smooth structure on M—alternatively, assume M is equipped with its maximal atlas in which case this chart is already in it.

By a very similar argument to what is used for the above exercise we have the following:

Remark 1.17 If $F : M \to N$ is a smooth map, then $\psi \circ F \circ \varphi^{-1}$ is smooth on $\varphi(U)$ for *all* smooth charts (U, φ) of M.

Equipped with the notion of smooth maps between manifolds we can talk about manifolds being diffeomorphic:

Definition 1.18 (Diffeomorphism) A map $f : M \to N$ is a *diffeomorphism* if it is a smooth bijective map such that its inverse is also smooth. If such a map exists we say that M and N are *diffeomorphic*.

Being diffeomorphic is an equivalence relation as is easily checked, and we think of diffeomorphic manifolds as being equivalent, i.e. the same.

Exercise 1.19 Let M be a smooth manifold and let $\mathrm{Diffeo}(M)$ denote the set of all diffeomorphisms of M onto itself. Prove that $\mathrm{Diffeo}(M)$ is a group (with composition as the group operation).

Exercise 1.20 Let M be a smooth manifold with smooth atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$. Show that each $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is smooth, in fact, that it is a diffeomorphism. (Hence, every point on a smooth manifold has a neighborhood which is *diffeomorphic* to an open set in Euclidean space).

Exercise 1.21 Let M be a (smooth) n-manifold. Show that every point on M has a neighborhood which is homeomorphic (indeed, diffeomorphic if M is smooth) to an open ball in $\mathbb{R}^n$.

Exercise 1.22 Let M be a (smooth) n-manifold. Show that every point on M has a neighborhood which is homeomorphic (indeed, diffeomorphic if M is smooth) to all of $\mathbb{R}^n$. *Hint:* Use the above exercise together with the map $f : B_r \to \mathbb{R}^n$ (where B_r is the open ball of radius r centered at the origin) defined by $x \mapsto \frac{rx}{\sqrt{r^2 - |x|^2}}$.

Remark 1.23 (A note on the dimension of a smooth manifold) Unlike in the case of topological manifolds, it is easy to see that the dimension of a smooth manifold is well-defined. If $p \in M$ has both a neighborhood homeomorphic to an open set in $\mathbb{R}^n$ and a neighborhood homeomorphic to an open set in $\mathbb{R}^k$ this results in a diffeomorhism between two non-empty open sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^k$ (the two images of the intersection of the neighborhoods). However A and B can only be diffeomorphic if $n = k$; this just follows from the fact that the Jacobian of the inverse of a bijective map is the inverse of Jacobian of the map, and so the Jacobian has to be invertible and in particular be a square matrix.

1.3 Lie Groups

We can now define another class of examples of smooth manifolds, which we will come back to several times throughout the course:

Definition 1.24 (Lie Group) A *Lie Group* is a group G which is a smooth manifold and such that the group operations (multiplication and taking inverses)

$$m : G \times G \to G, \quad (g, h) \mapsto gh$$

and

$$i : G \to G, \quad g \mapsto g^{-1}$$

are smooth maps.

As a slight shortcut to checking whether the group operations are smooth, we note that it is enough to check that the operation $(g, h) \mapsto gh^{-1}$ is smooth for all $g, h \in G$.

Some of the standard examples of Lie groups are given below (we will see a couple more in a later section):

Example 1.25 (Lie Groups) We leave it as an exercise to verify the claims below.

- For each n, $(\mathbb{R}^n, +)$ is a Lie group. We saw above that $\mathbb{R}^n$ (with the usual topology) is a smooth manifold, and addition and subtraction are smooth operations.
- Non-zero real numbers under multiplication $(\mathbb{R}^*, \cdot)$ is a Lie group.
- The unit circle S^1 with addition of angle (or multiplication using complex numbers)
- For any Lie groups G, H, their product $G \times H$ is a Lie group.
- The matrix group of invertible $n \times n$ matrices with real entries, $GL(n, \mathbb{R})$ is a Lie group. We showed above that this is a smooth manifold, and matrix multiplication and taking inverse of a matrix are smooth operations.

In particular, Lie groups are *topological groups*, which are groups together with a topology on the group with respect to which the group operations are continuous functions. For matrix groups (like $GL(n, \mathbb{R})$ above) we always equip the group with the topology it inherits by viewing it as a subset of $\mathbb{R}^{n^2}$.

Exercise 1.26 Verify that the above examples are indeed Lie groups.

1.4 Partition of Unity

The goal of this section is to define *partition of unities*. These will be crucial tools in the second half of the course when defining Riemannian manifolds, but they also have other nice consequences. It will allow us to patch-up smooth maps, defined on parts of a manifold, to a global smooth map defined on the entire manifold. Moreover, here we will use them to prove (a version of) Whitney's Embedding Theorem which implies that we can view a manifold as a subset of a Euclidean space. We start with the definition.

First, recall that the *support* of a function $f : M \to \mathbb{R}$ is the closure of the set of points in M for which f takes non-zero values:

$$\operatorname{supp}(f) = \overline{\{x \in M \mid f(x) \neq 0\}}.$$

Definition 1.27 (Partition of unity) Let $\mathcal{A} = \{X_\alpha\}_{\alpha \in A}$ be an open cover of M. A *partition of unity subordinate to* $\mathcal{A}$ is a family $\{\psi_\alpha\}_{\alpha \in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ such that

(i) $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and $x \in M$

(ii) $\operatorname{supp}(\psi_\alpha) \subset X_\alpha$ for each $\alpha \in A$

(iii) The collection of sets $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha \in A}$ is *locally finite*, i.e. for all $x \in M$ there exists an open neighborhood V of x such that $V \cap \operatorname{supp}(\psi_\alpha) = \emptyset$ for all but finitely many $\alpha \in A$.

(iv) $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for each $x \in M$.

Moreover, if M is a smooth manifold, a *smooth partition of unity* is one for which each function ψ_α is smooth.

Note that condition (iii) implies that the sum in (iv) has finitely many non-zero terms and hence there is no issue of convergence.

Theorem 1.28 ((Smooth) Partitions of unity exist) *Suppose M is a smooth manifold and $\mathcal{A} = \{X_\alpha\}_{\alpha \in A}$ is an open cover. Then there exists a (smooth) partition of unity subordinate to $\mathcal{A}$.*

Proof. We will just prove the continuous version and leave the smooth version as an exercise (alternatively refer to Lee's *An introduction to smooth manifolds* for a proof). Let $\{(V_\beta, \varphi_\beta)\}_{\beta \in B}$ be a (smooth) atlas for M. Consider $\{X_\alpha \cap V_\beta \mid \alpha \in A, \beta \in B\}$. This is also an open cover of M. Take a locally finite refinement of this cover and call the sets in this cover U_α and let φ_α be the corresponding charts (technically, the restriction to U_α of the original maps). Note that each U_α is contained in some X_γ. Moreover, we can assume that the *closure* of each U_α is contained in some X_γ (see the Lemma below).

For each α let $\tilde{U}_\alpha = \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$. Without loss of generality we can assume that each $\tilde{U}_\alpha$ is bounded, by applying another homeomorphism if necessary.

For each α let $\tilde{f}_\alpha : \mathbb{R}^n \to \mathbb{R}$ be defined by $\tilde{f}_\alpha(x) = d(x, \tilde{U}_\alpha^c)$ where d is Euclidean distance. Then $\tilde{f}_\alpha$ is continuous, $f(x) > 0$ for all $x \in \tilde{U}_\alpha$ and $f(x) = 0$ for all $x \notin \tilde{U}_\alpha$. Define $f_\alpha : M \to \mathbb{R}$ by $f(x) = \tilde{f}_\alpha \circ \varphi_\alpha(x)$ if $x \in U_\alpha$ and $f(x) = 0$ otherwise. Note that the support of of f_α is contained in $\overline{U}_\alpha$.

Finally, for each $x \in M$ set $F(x) = \sum_\alpha f_\alpha(x)$. Since $\{U_\alpha\}$ is locally finite, $F(x) < \infty$ for each x. Moreover, $F(x) > 0$ for all x since for each x we have that $x \in U_\alpha$ for some α. Now let $g_\alpha : M \to [0, 1]$ be defined by $g_\alpha(x) = f_\alpha(x)/F(x)$. This is the desired (continuous) partition of unity.

□

In the above proof we used the following fact from topology:

Lemma 1.29 *Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas for M. There exists a locally finite refinement $\{V_\alpha\}$ of the cover $\{U_\alpha\}$ such that for each α we have $\overline{V_\alpha} \subset U_\beta$ for some β.*

Proof. Let $\mathcal{A} = \{V \subset M \text{ open } | \overline{V} \subset U_\alpha \text{ for some } \alpha\}$. We claim that $\mathcal{A}$ is an open cover of M. Note that once we have shown this we are done since M is paracompact. So, let $x \in M$. Then $x \in U_\alpha$ for some α. Let $\tilde{x} = \varphi_\alpha(x)$ and $\tilde{U}_\alpha = \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$. Since $\tilde{U}_\alpha$ is an open set in Euclidean space, there exists a ball B_x around x such that $\overline{B}_x \subset \tilde{U}_\alpha$. Let $V = \varphi^{-1}(B_x)$. Then $x \in V$ and $V \in \mathcal{A}$.

□

We now give some applications to the existence of partitions of unity. Let M be a smooth manifold, $A \subset M$ a closed subset, and $U \subset M$ any open set containing A. A continuous function $f : M \to \mathbb{R}$ is called a *bump function for A supported in U* if $f(x) \in [0, 1]$ for all $x \in M$, $f(x) = 1$ for all $x \in A$ and $\operatorname{supp}(f) \subset U$. An immediate consequence of the existence of smooth partitions of unity is that smooth bump function always exists:

Proposition 1.30 (Bump functions exist) *Let M be a smooth manifold, $A \subset M$ any closed subset, and $U \subset M$ any open set containing A. Then there exists a smooth bump function for A supported in U.*

Proof. Let $X_1 = U$ and $X_2 = M \setminus A$. Then $\mathcal{A} = \{X_1, X_2\}$ is an open cover of M. Let $\{\psi_1, \psi_2\}$ be a smooth partition of unity subordinate to $\mathcal{A}$. We claim that ψ_1 is the desired bump function. First note that by definition $0 \le \psi_1 \le 1$ and $\operatorname{supp}(\psi_1) \subset U$. Now, since $\operatorname{supp}(\psi_2) \subset M \setminus A$ we have $\psi_2(x) = 0$ for all $x \in A$ and hence since for each $x \in M$ $\psi_1(x) + \psi_2(x) = 1$ we have $\psi_1(x) = 1$ for all $x \in A$.

□

Partition of unity can also be used to show that a smooth function $f : A \to \mathbb{R}$ defined on a closed subset $A \subset M$ can be extended to a smooth function defined on the entire manifold. First we need to define what we mean for a function defined on a closed set to be smooth: We say that f is smooth on A if it has a smooth extension in a neighborhood of each point, that is, for each $p \in A$ there is an open neighborhood $W_p \subset M$ of p and a smooth map $\tilde{f}_p$ whose domain is $W_p \to \mathbb{R}$ and such that its restriction to $W_p \cap A$ agrees with f.

Theorem 1.31 (Extension Lemma) *Let M be a smooth manifold, $A \subset M$ a closed subset, and $f : A \to \mathbb{R}$ be a smooth function. For any open set $U \subset M$ such that $A \subset U$ there exists a smooth function $\tilde{f} : M \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for all $x \in A$ and $f(x) = 0$ for all $x \notin U$.*

Proof. Suppose M is a smooth manifold, $A \subset M$ is closed, $f : A \to \mathbb{R}$ is smooth, and $U \subset M$ any open set such that $A \subset U$. We want to prove that we can extend f to a function $\tilde{f} : M \to \mathbb{R}$ which is supported in U and agrees with f on A.

Let $p \in A$. By the comment just preceding the proof, f being smooth on A means that we can choose a neighborhood W_p of p and a smooth map $\tilde{f}_p : W_p \to \mathbb{R}$ which agrees with f on $W_p \cap A$. By replacing W_P by $W_p \cap U$ if necessary, we can assume that $W_p \subset U$. Note that the sets W_p (as p varies over all points in A) cover A. Hence $\mathcal{A} = \{W_p \,|\, p \in A\} \cup \{M \setminus A\}$ is an open cover of M. Let $\{\psi_p \,|\, p \in A\} \cup \{\psi_0\}$ be a smooth partition of unity subordinate to $\mathcal{A}$ with $\operatorname{supp}(\psi_p) \subset W_p$ and $\operatorname{supp}(\psi_0) \subset M \setminus A$.

For each $p \in A$, consider the function $\psi_p \tilde{f}_p$. It is smooth on W_p and can be extended to a function on all of M by defining $\psi_p \tilde{f}_p(x) = 0$ for all x in the open set $M \setminus \operatorname{supp} \psi_p$. Note that the extended function is also smooth, since it is smooth on each of the two open sets in the open cover $\{W_p, M \setminus \operatorname{supp} \psi_p\}$ and the two definitions agree on the overlapping domain $W_p \setminus \operatorname{supp}(\psi_p)$.

Now define $\tilde{f} : M \to \mathbb{R}$ by

$$\tilde{f} = \sum_{p \in A} \psi_p(x) \tilde{f}_p(x).$$

We claim that $\tilde{f}$ is the desired extension. First, to see that it is smooth, note that since the collection of supports $\{\operatorname{supp}(\psi_p)\}|_{p \in A}$ is locally finite, the sum has only finitely many non-zero terms in a neighborhood of each x and since each term in a smooth function, $\tilde{f}$ is thus smooth. Also, $\tilde{f}$ agrees with f on A because we have $\psi_0(x) = 0$ for $x \in A$ and whenever $\psi_p(x) \neq 0$ we have $\tilde{f}_p(x) = f(x)$ so we have,

$$\tilde{f}(x) = \sum_{p \in A} \psi_p(x) \tilde{f}_p(x) = \left(\psi_0(x) + \sum_{p \in A} \psi_p(x) \right) f(x) = f(x)$$

for all $x \in A$. (In the last equality we used the fact that we have a partition of unity, i.e. that the sum is 1). Finally, we need to prove that the support of $\tilde{f}$ is contained in U. For this we use the following general fact (which you should verify is true): If $\{A_\alpha\}$ is any locally finite collection of subsets of some topological space X, then $\overline{\cup A_\alpha} = \cup \overline{A}_\alpha$. Hence, since the collections of supports is a locally finite collection, each set is closed, and $\operatorname{supp}(\psi_p) \subset W_p \subset U$ we have:

$$\operatorname{supp} \tilde{f} = \overline{\bigcup \operatorname{supp}(\psi_p)} = \bigcup \operatorname{supp}(\psi_p) \subset U.$$

□

Another important consequence of the existence of smooth partitions of unity is the Whitney Embedding Theorem.

Definition 1.32 ((topological) Embedding) We say a map $f : M \to N$ is an *(topological) embedding* if f is a homeomorphism onto its image $f(X)$. That is, f is continuous, injective, and $f^{-1}\big|_{f(X)} : f(X) \to X$ is continuous.

Up until this course, you have probably only encountered manifolds as subsets of $\mathbb{R}^3$ (or higher dimensional Euclidean spaces). Now we have defined them intrinsically, without an ambient space that they sit inside. However, even with the abstract definition, it turns out we can still view them as subsets of Euclidean space (although we might have to go up in dimension):

Theorem 1.33 ((compact) Whitney Embedding Theorem) *Let M be a compact n-manifold. Then there exists an embedding $f : M \to \mathbb{R}^k$ for some $k \geq n$. (That is, any manifold can be embedded into Euclidean space).*

Remark 1.34 Whitney Embedding Theorem is actually stronger: it also holds also for non-compact manifolds, and k can be taken to be $2n + 1$. Moreover, when M is a smooth manifold, the embedding can be taken to be smooth; in fact, it can be taken to be a *smooth embedding* (which is stronger notion than a topological embedding which is a smooth map—we will define it in a later section).

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas for M. Since M is compact, there exists a finite subcover $\{U_1, U_2, \dots U_m\}$. Let $\varphi_1, \varphi_2, \dots \varphi_m$ be the corresponding coordinate charts.

Now, let $\{g_1, g_2, \dots g_m\}$ be a partition of unity subordinate to $\{U_1, U_2, \dots U_m\}$. In particular, each $g_i : M \to \mathbb{R}$ is continuous, and $g_i(p) = 0$ for all $p \notin U_i$.

Now, define, for each $i = 1, \dots, m$, a map $h_i : M \to \mathbb{R}^n$ by $h_i(p) = g_i(p)\varphi_i(p)$ if $p \in U_i$ and $h_i(p) = g_i(p) = 0$ for all $p \notin U_1$. Since g_i is continuous on M and supported on U_i and φ_i is continuous on U_i, we have that h_i is continuous.

Finally, define $f : M \to \mathbb{R}^{m(n+1)}$ by $f(p) = (g_1(p), \dots, g_m(p), h_1(p), \dots, h_m(p))$. We claim that f is the desired embedding. Since each component of f is continuous, f is continuous. Next we show it is injective. Suppose $p, q \in M$ such that $f(p) = f(q)$. Then $g_i(p) = g_i(q)$ and $h_i(p) = h_i(q)$ for all $i = 1, \dots m$. Now, since the $\{g_1, \dots, g_m\}$ is a partition of unity, there exists a $j \in \{1, 2, \dots, m\}$ such that $g_j(p) \neq 0$. Thus also $g_j(q) \neq 0$ and so $p, q \in U_j$. Also, since $h_j(p) = h_j(q)$ we have $g_j(p)\varphi_j(p) = g_j(q)\varphi_j(q)$ (by definition) and so $\varphi_j(p) = \varphi_j(q)$. But $\varphi_j : U_j \to \hat{U}_j$ is a homeomorphism and in particular injective, so $p = q$. Hence f is injective. We then have that $f : M \to f(M)$ is a continuous bijection. But since M is compact it follows that $f^{-1} : f(M) \to M$ is also continuous (*you should verify this!*). Hence f is an embedding as desired.

□

Chapter 2

Tangent space and differentials

2.1 Tangent vectors as derivations

Recall that given a smooth map between Euculidean spaces, $F : \mathbb{R}^n \to \mathbb{R}^m$, its (total) derivative at a point $p \in \mathbb{R}^n$ (which we will call the *differential at* p) is a linear map between the tangent spaces $T_p\mathbb{R}^n \simeq \mathbb{R}^n$ and $T_{F(p)}\mathbb{R}^k \simeq \mathbb{R}^k$ given by the Jacobian, i.e. the matrix

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}.$$

In this chapter we will define the differential of a smooth map between manifolds, but before we can do that we need to define what the tangent space at a point of a manifold is.

Let M be a smooth n-manifold. When M is Euclidean space we know how to make sense of the tangent space at at p: we take all vectors based at p. However, for an abstract smooth manifold we have to define it using only what we know about abstract smooth manifolds. All we really have to work with are smooth maps. Accordingly, denote by $C^\infty(M)$ the set of all smooth functions $f : M \to \mathbb{R}$, i.e. the set of all smooth real-valued functions on M. We define the tangent space at a point in terms of *derivations at that point*:

Definition 2.1 (Derivation at p) Let $p \in M$. We say a map $v : C^\infty(M) \to \mathbb{R}$ is a *derivation at p* if it is a *linear map* and it satisfies

$$v(fg) = f(p)vg + g(p)vf. \tag{2.1}$$

Equation 2.1 should remind you of the product rule, and we will refer to it as such. Here v being linear means that $v(f + g) = vf + vg$ and $v(\lambda f) = \lambda vf$ for all $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$. We define the tangent space in terms of derivations and will show below that this agrees with our notion of tangent space in Euclidean space.

Definition 2.2 (Tangent space) The *tangent space of M at p*, denoted by T_pM is the set of all derivations at p. An element of $v \in T_pM$ will be referred to both as a derivation at p and as a *tangent vector at p*.

Note that T_pM is a vector space over $\mathbb{R}$ where, for v, w two tangent vectors (derivations) at p, their sum is a derivation defined by $(v + w)f = vf + wf$ for all $f \in C^\infty(M)$ and for $\lambda \in \mathbb{R}$, λv is defined by $(\lambda v)f = v(\lambda f)$. We give some properties of derivations (tangent vectors) at a point:

Lemma 2.3 *Let $v \in T_pM$ and $f, g \in C^\infty(M)$.*

(a) If f is a constant function, then $vf = 0$.

(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Proof. The assertions follow by (2.1). We first prove (a) for the constant function $f_1 \equiv 1$. Then $vf = v(ff) = f(p)vf + f(p)vf = 2vf$ and hence $vf = 0$. Now let f be an arbitrary constant function, say $f \equiv \lambda$. Then $f = \lambda f_1$ and so $vf = v(\lambda f_1) = \lambda v f_1 = 0$.

Part (b) follows immediately by applying (2.1).

□

Now, to give some motivation for the above definition, we consider the case when $M = \mathbb{R}^n$. Let $p \in \mathbb{R}^n$. Define $\mathbb{R}^n_p$ to be the set of all vectors in $\mathbb{R}^n$ based at p. Each vector can be written as an n-tuple of real numbers and hence be identified with a point in $\mathbb{R}^n$. If v is a vector based at p we use the notation (p, v) (or sometimes v_p or $v|_p$) to denote it. Accordingly, $\mathbb{R}^n_p = \{(p, v) \mid v \in \mathbb{R}^n\}$. Note that $\mathbb{R}^n_p$ is isomorphic to $\mathbb{R}^n$ by the natural isomorphism $(p, v) \mapsto v$. In particular it is n-dimensional. This is probably how you think of the tangent space of $\mathbb{R}^n$ at p. We will see below that it agrees with our definition above, i.e. that $T_p\mathbb{R}^n$ (as defined above) and $\mathbb{R}^n_p$ are the same spaces (i.e. they are isomorphic).

Next, we recall the definition of *directional derivative* of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ at a point. Let $p \in \mathbb{R}^n$ and v a vector (direction) based at p. Then the directional derivative at p (in direction v) is a function $D_v|_p : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$D_v|_p f = D_v f(p) = \frac{d}{dt}\Big|_{t=0} f(p + tv) = \lim_{t\to 0} \frac{f(p + tv) - f(p)}{t}$$

for $f \in C^\infty(\mathbb{R}^n)$. If we let $e_1|_p, \dots e_n|_p$ denote the standard basis vectors (based at p) and we write $v|_p = \sum v^i e_i|_p$, then we can equivalently (and maybe more familiarly) write

$$D_v|_p f = \sum v^i \frac{\partial f}{\partial x^i}(p) \tag{2.2}$$

using the chain rule.

[To see that the two expressions are the same: Note that fixing p and v, letting $g(t) = f(p + tv)$ we have $g'(0) = D_v|_p f$. On the other hand, writing $g(t) = f(x^1, x^2, \dots, x^n)$ where $x^i = p_i + tv^i$ we get by the chain rule that $g'(t) = \frac{dg}{dt} = \sum \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} = \sum v^i \frac{\partial f}{\partial x^i}$ and $g'(0) = \sum v^i \frac{\partial f}{\partial x^i}(p_1, \dots, p_n)$.]

Note that $D_v|_p$ is linear and satisfied the product rule (2.1), i.e. it is a derivation. As we see next, every derivation in $T_p\mathbb{R}^n$ is of this form.

Proposition 2.4 *For any $p \in \mathbb{R}^n$, the spaces $\mathbb{R}^n_p$ and $T_p\mathbb{R}^n$ are isomorphic. In fact, the map $D : \mathbb{R}^n_p \to T_p\mathbb{R}^n$ defined by $D : (p, v) \mapsto D_v|_p$ is an isomorphism.*

Proof. To see that D is a homomorphism, we need to show that for any $f \in C^\infty(\mathbb{R}^n)$ we have $D_{v+w}|_p f = D_v|_p f + D_w|_p f$. This is true since, by (2.2) we have

$$D_{v+w}|_p f = \sum (v^i + w^i) \frac{\partial f}{\partial x^i}(p) = \sum v^i \frac{\partial f}{\partial x^i}(p) + \sum w^i \frac{\partial f}{\partial x^i}(p) = D_v|_p f + D_w|_p f$$

for any $f \in C^\infty(\mathbb{R}^n)$.

To see that it is injective, suppose that $D(p, v) = 0$ i.e. $D_v|_p f = 0$ for all $f \in C^\infty(\mathbb{R}^n)$. We need to show that v is the 0 vector. For each $K = 1, \ldots, n$ consider the function $x^k : \mathbb{R}^n \to \mathbb{R}$ such that $x^k(p_1, \ldots, p_n) = p_k$ (i.e. the k^{th}-coordinate function, which is a smooth function). Then we have

$$0 = D_v|_p x^k = \sum v^i \frac{\partial x^k}{\partial x^i}(p) = v^k.$$

Since this holds for all k we get that $v = (0, 0, \ldots 0)$ as desired.

Lastly, we show why D is surjective. Let $v \in T_p\mathbb{R}^n$ be a derivation at p and $x^k : \mathbb{R}^n \to \mathbb{R}$ be the coordinate functions as above. Since $v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$, we have $v(x^k) \in \mathbb{R}$ for each k, say $v(x^k) = v^k$. Set $\bar{v} = \sum v^i e_i$, that is $\bar{v} = (v^1, \ldots, v^n)$. Then $(p, \bar{v}) \in \mathbb{R}^n_p$ and one can show that $D(p, \bar{v}) = v$, i.e. that $D_{\bar{v}}|_p$ is exactly the derivation v. To see this we use Taylor's Theorem. Let $f \in C^\infty(\mathbb{R}^n)$ and $a = (a^1, \ldots, a^n) \in \mathbb{R}^n$. There exist smooth functions $g_i : \mathbb{R}^n \to \mathbb{R}$ with $g_i(a) = 0$ and such that

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + \sum_{i=1}^{n} g_i(x)(x^i - a^i).$$

We will apply v to f. By linearity of v this is the same as applying v to each summand in the right hand side above. Note that by Lemma 2.3 $v(f(a)) = 0$ and $v(g_i(x)(x^i - a^i)) = 0$. Hence

$$vf = \sum_{i=1}^{n} v\left(\frac{\partial f}{\partial x^i}(a)(x^i - a^i)\right) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)(vx^i - va^i) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)v^i = D_{\bar{v}}|_p f$$

as desired. (Here we have used that $vx^i = v^i$ and that $va_i = 0$ for each i.)

□

It follows from the above proposition that $T_p\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ (since $\mathbb{R}^n_p$ is) and so is n-dimensional. In fact, we have an explicit basis. For each $i \in \{1, \ldots, n\}$ consider the derivation

$$\left.\frac{\partial}{\partial x^i}\right|_p \text{ defined by } \left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p).$$

Note that $\frac{\partial}{\partial x^i}|_p = D_{e_i}|_p$. Since $e_1|_p, \ldots e_n|_p$ is a basis for $\mathbb{R}^n_p$, we have that $\frac{\partial}{\partial x^i}|_p$, $i = 1, \ldots, n$ form a basis for $T_p\mathbb{R}^n$ by the isomorphism given in the above proposition. We record this observation for future reference:

Corollary 2.5 (Basis for tangent space) *For any $p \in \mathbb{R}^n$ the n derivations*

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$$

is a basis for $T_p\mathbb{R}^n$.

□

In fact, for any smooth n-dimensional manifold M the tangent space T_pM is n-dimensional, as we will see in the next section.

2.2 Differentials

Having defined the tangent space, we can now talk about the differential at a point of a smooth map. As we recalled in the beginning of the chapter, given a smooth map $F : \mathbb{R}^n \to \mathbb{R}^k$ between Euclidean spaces, its (total) derivative is given by its Jacobian matrix and is hence a linear map (in fact, it is the best linear approximation to the map at a given point). We want to generalize this to smooth maps between abstract smooth manifolds.

Definition 2.6 (Differential at p) Let M, N be smooth manifolds and $F : M \to N$ a smooth map. For $p \in M$ the *differential of F at p* is the map $dF_p : T_pM \to T_{F(p)}N$ that maps $v \in T_pM$ to the derivation $dF_p(v)$ defined by

$$dF_p(v)(f) = v(f \circ F)$$

for all $f \in C^\infty(N)$.

To see that the definition above is well-defined, first note that if $f \in C^\infty(N)$ then $f \circ F \in C^\infty(M)$. We leave it as an exercise to show that, for each $v \in T_pM$, $dF_p(v) \in T_{F(p)}N$, i.e. is a derivation at $F(p)$ on N:

Exercise 2.7 Show that $dF_p(v)$ is a derivation at $F(p)$, i.e. is linear and satisfies the product rule.

We record some important facts about the differential at a point:

Lemma 2.8 *Let M, N be smooth manifolds, $F : M \to N$ a smooth map, and $p \in M$. Then*

(1) $dF_p : T_pM \to T_{F(p)}N$ is a linear map.

(2) If F is a diffeomorphism, then dF_p is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

(3) (Chain Rule) If K is another smooth manifold and $G : N \to K$ is a smooth map, then $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}K$.

Proof. To see that it is a linear map, let $v, w \in T_pM$. Then, for any $f \in C^\infty(M)$ we have $dF_p(v+w)(f) = (v+w)(f \circ F) = v(f \circ F) + w(f \circ F) = dF_p(v)(f) + dF_p(w)(f)$. Similarly one shows that $dF_p(\lambda v)(f) = \lambda dF_p(v)(f)$ for all $\lambda \in \mathbb{R}$.

To show the second claim, we need to also prove that dF_p is invertible (i.e. bijective) when F is a diffeomorphism. So suppose F is a diffeomorphism. Then F^{-1} exists and is smooth. We claim that $d(F^{-1})_{F(p)}$ is the inverse of dF_p, i.e. that $(dF_p)^{-1} = d(F^{-1})_{F(p)}$. To see this, let $v \in T_pM$ and note that $(d(F^{-1})_{F(p)} \circ dF_p)(v)(f) = d(F^{-1})_{F(p)}(dF_p(v)(f)) = d(F^{-1})(v(f \circ F)) = v(f \circ F \circ F^{-1}) = v(f)$ for all $f \in C^\infty(M)$. Similarly $(dF_p \circ d(F^{-1})_{F(p)})(w)(f) = w(f)$ for all $w \in T_{F(p)}$ and $f \in C^\infty(N)$.

For the last claim, let $H = G \circ F : M \to K$. By definition $dH_p : T_pM \to T_{H(p)}K$ is defined by $dH_p(v)(f) = v(f \circ H)$ for all $f \in C^\infty(K)$. So $d(F \circ G)(v)(f) = v(f \circ G \circ F)$. On the other hand, for $v \in T_pM$, $dG_{F(p)} \circ dF_p(v) = dG_{F(p)}(dF_p(v))$ and so for $f \in C^\infty(K)$ we have $dG_{F(p)}(dF_p(v))(f) = dF_p(v)(f \circ G) = v(f \circ G \circ F)$.

□

Now, we want to use the second claim above to show that the tangent space of an n-manifold at each point is n-dimensional. The idea is that around each $p \in M$ there is an open set U which is diffeomorphic to an open set in $\mathbb{R}^n$ (through the coordinate map φ) and hence the tangent space at p should be isomorphic to the tangent space at $\varphi(p)$ in $\mathbb{R}^n$, which is isomorphic to $\mathbb{R}^n$ (in particular, in local coordinates we get a basis). There is a technicality we have to take care of first though, namely we need to make sure that this does not depend on the open set U we choose. This follows from the fact that tangent vectors act locally in the following sense:

Proposition 2.9 *Let M be a smooth manifold, $p \in M$, and $v \in T_pM$. If $f, g \in C^\infty(M)$ agree in a neighborhood of p, then $vf = vg$.*

Proof. Let V be an open neighborhood such that $f(x) = g(x)$ for all $x \in V$. Let $h = f - g$. Then h is smooth and $h(x) = 0$ for all $x \in V$. Let $A = \operatorname{supp}(h)$ be the support of h, and set $U = M \setminus \{p\}$. Note that A is closed, U is open, and $A \subset U$. Hence we can choose, by Proposition 1.30, a smooth bump function $\psi : M \to \mathbb{R}$ such that $\psi(x) = 1$ for all $x \in A$ and $\operatorname{supp}(\psi) \subset U$. Note that $\psi h = h$. Also, $h(p) = \psi(p) = 0$ so by Lemma 2.3 we have $vh = v(\psi h) = 0$. But $vh = v(f - g) = vf - vg$ so $vf = vg$.

□

Using this fact one can show that the tangent space to an (open) subset of a manifold can be identified with the tangent space to the entire manifold:

Proposition 2.10 *Let M be a smooth manifold and $U \subset M$ an open subset. Let $\iota : U \hookrightarrow M$ be the inclusion map. For every $p \in M$ the differential $d\iota_p : T_pU \to T_pM$ is an isomorphism. In particular, T_pU and T_pM are isomorphic (and we will identify them as being the same).*

Proof. Let $p \in M$. Since $d\iota_p$ is a linear map, we only need to show it is bijective.

For injectivity: Let $v \in T_pU$ and suppose $d\iota_p(v) = 0$, i.e. $d\iota_p(v)\tilde{f} = 0$ for all $\tilde{f} \in C^\infty(M)$. We want to show that $v = 0$, i.e. that $vf = 0$ for all $f \in C^\infty(U)$. Let $f \in C^\infty(U)$. Let B be an open neighborhood of p in U such that $\bar{B} \subset U$. Using the Extension Lemma (Theorem 1.31) pick a function $\tilde{f} \in C^\infty(M)$ such that $\tilde{f}(x) = f(x)$ for all $x \in \bar{B}$. Then f and $\tilde{f}|_U$ are both smooth functions on U and agree on the neighborhood B of p and hence by Proposition 2.9 we have $vf = v(\tilde{f}|_U)$. But $v(\tilde{f}|_U) = v(\tilde{f} \circ \iota) = d\iota_p(v)\tilde{f} = 0$ by assumption. Hence $vf = 0$. Since f was arbitrary we have $v = 0$ as desired.

For surjectivity: Let $w \in T_pM$. Define $v : C^\infty(U) \to \mathbb{R}$ as follows. For each $f \in C^\infty(U)$ choose any $\tilde{f} \in C^\infty(M)$ such that $\tilde{f}(x) = f(x)$ for all $x \in \bar{B}$ (as above) and define $vf = w\tilde{f}$. By Proposition 2.9 vf is independent of the choice of $\tilde{f}$ and hence v is well-defined. One can check that v is a derivation at p (i.e. linear and satisfies the product rule). We claim that $d\iota_p(v) = w$, proving surjectivity. To see this, let $h \in C^\infty(M)$ and note that $d\iota_p(v)h = v(h \circ \iota) = w(\widetilde{h \circ \iota}) = wh$, where the last equality is due to the fact that $\widetilde{h \circ \iota}$ and h agree on B.

□

We are now ready to prove the claim we made earlier: the tangent space to a manifold is of the same dimension as the manifold:

Theorem 2.11 *If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional vector space.*

Proof. Let $p \in M$ and (U, φ) a smooth chart containing p. Then $\varphi : U \to \hat{U}$ (for some open $\hat{U} \in \mathbb{R}^n$) is a diffeomorphism. Hence $d\varphi_p : T_pU \to T_{\varphi(p)}\hat{U}$ is an isomorphism by Proposition 2.8. Also, by Proposition 2.10 we have that T_pM and T_pU are isomorphic and $T_{\varphi(p)}\hat{U}$ and $T_{\varphi(p)}\mathbb{R}^n$ are isomorphic. Hence T_pM and $T_{\varphi(p)}\mathbb{R}^n$ must be isomorphic. Since the latter is n-dimensional, so is T_pM.

□

2.3 Tangent space and differentials in coordinates

The construction above might seem very abstract, but using the fact that the tangent space is defined locally and that a manifold is locally Euclidean, we will see that we can treat tangent vectors (derivations) in T_pM as vectors in Euclidean space and that locally the differential is nothing but the Jacobian. First we construct a basis for T_pM. Let (U, φ) be a chart containing p. Then $\varphi = (x^1, \ldots, x^n)$ is a diffeomorphism from U to an open set $\tilde{U}$ in $\mathbb{R}^n$ and hence, by Proposition 2.8 and after identifying T_pU with T_pM and $T_{\varphi(p)}\tilde{U}$ with $T_{\varphi(p)}\mathbb{R}^n$ by Proposition 2.10, we have $d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism. We know, see Corollary 2.5, that the derivations $\partial/\partial x^i|_{\varphi(p)}$, for $i = 1, \ldots n$, is a basis for $T_{\varphi(p)}\mathbb{R}^n$. Hence the preimages of these form a basis for T_pM. For simplicity, we use the same notation for these, i.e. we use the convention

$$\left.\frac{\partial}{\partial x^i}\right|_p := (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = d(\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right). \tag{2.3}$$

That is, for each $f \in C^\infty(U)$ (in particular, for any $f \in C^\infty(M)$)

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} (f \circ \varphi^{-1}) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)). \tag{2.4}$$

In a sense, what we have done is that we have *pulled back* the basis elements of $T_{\varphi(p)}\mathbb{R}^n$ to basis elements on T_pM using diffeomorphism φ (although, for simplicity we just identify them and denote them by the same symbols). We will generalize this notion of pullbacks later, see Definition 4.11 and compare the above equations with equation (4.3).

With the conventions above, we see that each $\left.\frac{\partial}{\partial x^i}\right|_p$ is a derivation at p, and so is an element of T_pM. By the above discussion above they, in these local coordinates, make up a basis. We have shown the following:

Theorem 2.12 (Tangent vector in local coordinates) *Let M be a smooth n-manifold and $p \in M$. Let (U, φ) be a smooth chart containing p, where we write $\varphi = (x^1, \ldots x^n)$. Then any $v \in T_pM$ can be written uniquely as a linear combination*

$$v = \sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x^i}\right|_p$$

where $\frac{\partial}{\partial x^i}$ is defined by (2.3) and (2.4).

□

The basis

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

is called a *coordinate basis* for T_pM and the numbers in $(v^1, \ldots, v^n)$ are called the *components* of v. Once we have fixed a chart $(U, \varphi) = (U, (x^i))$ we can thus identify $v \in T_pM$ with a vector $(v^1, \ldots v^n)$ in $\mathbb{R}^n$, again giving our more familiar notion of tangent vectors.

Remark 2.13 The above is an example of a theme we will see throughout the course—and in a sense the main idea of differential geometry—although smooth manifolds and associated notions such as tangent vectors are defined very abstractly, we will continuously take advantage of the *locally Euclidean* property and, after fixing a chart, translate everything to Euclidean space where we know how to do calculus. We saw this earlier too when defining smooth maps: being smooth is a local property and hence we use local coordinates to translate everything to Euclidean spaces and defined the property of being smooth as a property of the resulting map of Euclidean spaces being smooth.

Next we express the differential in coordinates. As always, we first define it when the smooth manifolds are (open subsets of) Euclidean spaces, and then explain how to do it in the general situation using local coordinates. So suppose first that $F : U \to V$ is a smooth map between open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^k$. Let $p \in U$. We will show that the differential at p, $dF_p : T_pU \to T_{F(p)}V$, which we know is a linear map, is given by the Jacobian matrix of F at p. Let us denote points in U by $(x^1, \ldots x^n)$ and points in V by $(y^1, \ldots y^k)$. Then we get, for all $f \in C^\infty(V)$:

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f = \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) = \sum_{j=1}^k \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) = \sum_{j=1}^k \left(\frac{\partial F^j}{\partial x^i}(p)\right)\left.\frac{\partial}{\partial y^j}\right|_{F(p)} f$$

where we used the definition of dF_p in the first equality, the chain rule for the second equality, and linearity in the last. In other words,

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \sum_{j=1}^k \frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}. \tag{2.5}$$

Recall that given any $v \in T_p\mathbb{R}^n$ we can identify it with the vector $(v^1, \ldots, v^n)$ where v^i are the components with respect to the basis $\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$. Similarly any $w \in T_{F(p)}\mathbb{R}^k$ can be identified with $w = (w^1, \ldots, w^k)$ with respect to the basis $\left\{ \left.\frac{\partial}{\partial y^1}\right|_{F(p)}, \ldots, \left.\frac{\partial}{\partial y^k}\right|_{F(p)} \right\}$. The above computation shows that the basis element $\frac{\partial}{\partial x^i}|_p$ gets mapped to $w = (w^1, \ldots w^k)$ whose components are given by $w_j = \frac{\partial F^j}{\partial x^i}(p)$.

That is, in the coordinate basis, the linear map dF_p is given by the matrix

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^k}{\partial x^1}(p) & \cdots & \frac{\partial F^k}{\partial x^n}(p) \end{pmatrix}$$

i.e., the Jacobian of F at p.

To clarify: fixing the bases for $T_p\mathbb{R}^n$ and $T_{F(p)}\mathbb{R}^k$ as above and identify the tangent vectors with their component vectors with respect to these, we have that (the component vector for) $dF_p(v) \in T_{F(p)}\mathbb{R}^k$ is given by

$$dF_p(v) = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^k}{\partial x^1}(p) & \cdots & \frac{\partial F^k}{\partial x^n}(p) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Now we want to do the same for a smooth map $F : M \to N$ where M is a smooth n-manifold and N is a smooth k-manifold. Let $p \in M$ and choose smooth charts (U, φ) around p and (V, ψ) around $F(p)$ and consider

$$\tilde{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$$

which is a smooth map between open sets in $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively (i.e. we are in the previous setting!). Let $\tilde{p} = \varphi(p) \in \mathbb{R}^n$. Then, by (2.5) above, $d\tilde{F}_{\tilde{p}}$ is given by the Jacobian of $\tilde{F}$ at $\tilde{p}$. We get

$$\begin{aligned} dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) &= dF_p\left(d(\varphi^{-1})_{\tilde{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\tilde{p}}\right)\right) = d(\psi^{-1})_{\tilde{F}(\tilde{p})}\left(d\tilde{F}_{\tilde{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\tilde{p}}\right)\right) \\ &= d(\psi^{-1})_{\tilde{F}(\tilde{p})}\left(\sum_{j=1}^m \frac{\partial \tilde{F}^j}{\partial x^i}(\tilde{p}) \left.\frac{\partial}{\partial y^j}\right|_{\tilde{F}(\tilde{p})}\right) = \sum_{j=1}^m \frac{\partial \tilde{F}^j}{\partial x^i}(\tilde{p}) \left.\frac{\partial}{\partial y^j}\right|_{F(p)}. \end{aligned} \tag{2.6}$$

The first and last equalities follows by definition (see (2.3)), the second because $F \circ \varphi^{-1} = \psi^{-1} \circ \tilde{F}$ and hence, using the chain rule from Lemma 2.8,

$$dF_p \circ d(\varphi^{-1})_{\tilde{p}} = d(F \circ \varphi^{-1})_{\tilde{p}} = d(\psi^{-1} \circ \tilde{F})_{\tilde{p}} = d(\psi^{-1})_{\tilde{F}(\tilde{p})} \circ d\tilde{F}_{\tilde{p}},$$

and the third follows by (2.5). Hence, dF_p is represented (in local coordinates) by the Jacobian of $\tilde{F}$ at $\tilde{p}$. We have shown:

Proposition 2.14 (Differential in local coordinates) *Let $F : M \to N$ be a smooth map. For $p \in M$ let $\varphi = (x^i)$ be local coordinates at x and $\psi = (y^j)$ local coordinates at $F(p)$. Then the differential of F at p, $dF_p : T_pM \to T_{F(p)}N$, is given by the Jacobian of F at p with these coordinates. In particular, the action of dF_p on the basis elements is given by*

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \sum_{j=1}^m \frac{\partial \tilde{F}^j}{\partial x^i}(\varphi(p)) \left.\frac{\partial}{\partial y^j}\right|_{F(p)}$$

and if $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}|_p$ then $dF_p(v) = \sum_{i=1}^k w^i \frac{\partial}{\partial y^i}|_{F(p)}$ where $(w_1, \ldots, w_k)$ is given by the column vector:

$$\begin{pmatrix} \frac{\partial \tilde{F}^1}{\partial x^1}(\tilde{p}) & \cdots & \frac{\partial \tilde{F}^1}{\partial x^n}(\tilde{p}) \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{F}^k}{\partial x^1}(\tilde{p}) & \cdots & \frac{\partial \tilde{F}^k}{\partial x^n}(\tilde{p}) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Here, as always, $\tilde{F} = \psi \circ F \circ \varphi$, $\tilde{p} = \varphi(p)$, and $\left.\frac{\partial}{\partial x^i}\right|_p$ and $\left.\frac{\partial}{\partial y^j}\right|_{F(p)}$ are as in (2.3).

□

What to take away from the discussion in this section is the following: *after fixing local coordinates, the differential is nothing but the Jacobian!*

However, whenever we define something in local coordinates, we have a choice in what charts to pick and we need to know how things change with this choice. We end by getting a change of coordinate formula: given a basis for T_pM in terms of local coordinates given by (U, φ) how does it differ if instead use a chart (V, ψ) also containing p?

Let M be a smooth manifold, $(U, \varphi), (V, \psi)$ two smooth charts such that $p \in U \cap V$ and write $\varphi = (x^1, \dots, x^n)$ and $\psi = (\tilde{x}^1, \dots \tilde{x}^n)$. Let $v \in T_pM$. Then we can write $v = \sum v^i \frac{\partial}{\partial x^i}|_p$ and $v = \sum \tilde{v}^i \frac{\partial}{\partial \tilde{x}_i}|_p$ for some $(v^1, \dots, v^n), (\tilde{v}^1, \dots, \tilde{v}^n) \in \mathbb{R}^n$.

Lemma 2.15 (Change of coordinates) *If $(U, \varphi), (V, \psi)$ are two smooth charts of M, $\varphi = (x^i), \psi = (\tilde{x}^i)$, such that $p \in U \cap V$ and $v \in T_pM$ as above, then*

$$\tilde{v}^j = \sum_{i=1}^{n} \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) v^i.$$

Proof. Consider $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$. For $x \in \varphi(U \cap V)$ we use slight abuse of notation to write

$$\psi \circ \varphi^{-1}(x) = (\tilde{x}^1(x), \dots, \tilde{x}^n(x)).$$

By (2.6) we have

$$d(\psi \circ \varphi^{-1})|_{\varphi(p)} \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \sum_{j=1}^{m} \frac{\partial (\psi \circ \varphi^{-1})^j}{\partial x^i} (\varphi(p)) \left. \frac{\partial}{\partial \tilde{x}^j} \right|_{\psi \circ \varphi^{-1}(\varphi(p))} = \sum_{j=1}^{m} \frac{\partial \tilde{x}^j}{\partial x^i} (\varphi(p)) \left. \frac{\partial}{\partial \tilde{x}^j} \right|_{\psi(p)}.$$

Now, by the above and the definition of the coordinate vectors, we have

$$\begin{aligned}
\left. \frac{\partial}{\partial x^i} \right|_p &= d(\varphi^{-1})_{\varphi(p)} \left(\left. \frac{\partial}{\partial x^i} \right|_{\varphi(p)} \right) = d(\psi^{-1} \circ \psi \circ \varphi^{-1})_{\varphi(p)} \left(\left. \frac{\partial}{\partial x^i} \right|_{\varphi(p)} \right) \\
&= d(\psi^{-1})_{\psi(p)} \circ d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\left. \frac{\partial}{\partial x^i} \right|_{\varphi(p)} \right) = d(\psi^{-1})_{\psi(p)} \sum_{j=1}^{m} \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \left. \frac{\partial}{\partial \tilde{x}^j} \right|_{\psi(p)} \\
&= \sum_{j=1}^{m} \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) d(\psi^{-1})_{\psi(p)} \left. \frac{\partial}{\partial \tilde{x}^j} \right|_{\psi(p)} = \sum_{j=1}^{m} \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \left. \frac{\partial}{\partial \tilde{x}^j} \right|_p .
\end{aligned}$$

(The first and last equalities follow by definition of coordinate vectors (2.3), the second is trivial, the third is the chain rule for differentials (Lemma 2.8), the fourth by the above equation, and the second to last by linearity.)

We have proved the claim.

□

Orientability

We comment briefly on the orientability of a smooth manifold. In fact, orientability makes sense also for topological manifolds. Intuitively a topological manifold is orientable if one can choose an orientation in a consistent way—going around a loop on the manifold you always return to your starting point oriented the same way (for (non)-example, the projective plane or the Klein bottle are non-orientable manifolds, and the Möbius band is a non-roentable manifold with boundary.). However, for smooth manifolds it is easier to state the definition precisely. Let $\mathcal{A}$ be a smooth atlas on M. We say that a transition map is *orientation preserving* if the Jacobian of its differential at every point has positive determinant. We say that $\mathcal{A}$ is an *oriented atlas* if every transition map is orientation preserving. Finally, we say that M is *orientable* if it admits an oriented atlas. We say that M together with an oriented atlas is an *oriented manifold*.

Although most results discussed here hold whether or not the manifold is orientable (unless otherwise specified) we will only consider orientable manifolds in this course.

Chapter 3

Submanifolds

3.1 Local diffeomorphisms, immersions, submersions, and smooth embeddings

We start by discussing some special classes of smooth maps between manifolds.

Let M, N be smooth manifolds of dimension n and k, respectively (we sometimes write this as M^n and N^k; when this notation is used the superscript is to be interpreted as the dimension of the manifold, it is not a power). Let $F : M \to N$ be a smooth map and consider, for each $p \in M$, the differential $dF_p : T_pM \to T_{F(p)}N$, which is a linear map of vector spaces (of dimension n and k, respectively). Recall that the rank of a linear map L is equal to the dimension of its image, that is $rank(L) = \dim Im(L)$.

Definition 3.1 (Immersion, submersion, smooth embedding) Let M be a smooth n-manifold, N a smooth k-manifold and $F : M \to N$ be a smooth map. Then:

- F is an *immersion* if dF_p is injective for all $p \in M$. That is, $rank(dF_p) = n$ for all $p \in M$.
- F is a *submersion* if dF_p is surjective for all $p \in M$. That is, $rank(dF_p) = k$ for all $p \in M$.
- F is a *smooth embedding* if it is an immersion which is also a topological embedding.

We note that being a smooth embedding is stronger than being a topological embedding which is smooth (see Exercise 3.2 below).

If $F : M \to N$ is an immersion then we must have that $n \leq k$, while if it is a submersion we must have $n \geq k$. A standard example of an immersion is

$$F : \mathbb{R}^n \to \mathbb{R}^k$$

for $n \leq k$, defined by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, 0, \dots, 0).$$

Note that this is in fact also a smooth embedding. A standard example of a submersion is

$$F : \mathbb{R}^n \to \mathbb{R}^k$$

for $k \geq n$, defined by

$$(x_1, \ldots, x_n, \ldots, x_k) \mapsto (x_1, \ldots x_n).$$

For an example of an immersion that is not a smooth embedding, consider mapping the circle to a figure 8: this map is not injective so is not a topological embedding and hence cannot be a smooth embedding (but note that the differential is injective at all points, so it is indeed an immersion).

Exercise 3.2 Find and example of a map which is smooth and a topological embedding, but is not a smooth immersion.

We make one last definition of a class of maps between smooth manifolds. As usual in topology, we say something is "*locally* X" if it has property X in a neighborhood of every point; doing this to the property of being a diffeomorphism we have the following:

Definition 3.3 (Local diffeomorphism) A map $F : M \to N$ is a *local diffeomorphism* if for all $p \in M$ there is a neighborhood U of p such that $F(U)$ is open in N and $F|_U : U \to F(U)$ is a diffeomorphism.

Exercise 3.4 Show that a bijective local diffeomorphism is a diffeomorphism.

We recall an important theorem from Calculus, here stated in the more general setting of manifolds:

Theorem 3.5 (Inverse Function Theorem) *Suppose $F : M \to N$ is smooth. Let $p \in M$ and suppose that $dF_p : T_pM \to T_{F(p)}N$ is bijective. Then there exists a neighborhood $U \subset M$ of p such that $F : U \to F(U)$ is a diffeomorphism.*

With the definition of a local diffeomorphism in mind, we have the immediate corollary:

Corollary 3.6 *Let $F : M \to N$ be a smooth map where both M and N are n-manifolds. Suppose for every $p \in M$, dF_p is bijective (i.e. rank n). Then F is a local diffeomorphism.*

Putting the above discussions together we have the following facts:

- $F : M \to N$ is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.

- Suppose both M and N are of dimension n and $F : M \to N$ is smooth. If F is bijective and either a smooth immersion or a smooth submersion, then F is a diffeomorphism.

Exercise 3.7 Justify the two bullet points above.

3.2 Embedded submanifolds

Previously we saw that an open subset of a smooth manifold is a smooth manifold, which is sometimes referred to as an open submanifold. In this chapter we discuss other submanifolds, i.e. subsets of a smooth manifold that are smooth manifolds in its own right. The most common kind, and the ones we will be concerned with here, are *embedded submanifold*.

Definition 3.8 (Embedded Submanifold) Let M be a smooth n-manifold and let $m \leq n$. $S \subset M$ is an m-dimensional *embedded submanifold* if for all $p \in S$ there exists a chart (U, φ) of M with $p \in U$ such that $\varphi(S \cap U) = \varphi(U) \cap P$ where P is an m-dimensional plane in $\mathbb{R}^n$. Note that by composing φ with another homeomorphism, we can assume that $P = \mathbb{R}^m \times \{0\} = \{(x_1, \ldots, x_m, 0, \ldots, 0)\} \subset \mathbb{R}^n$.

All submanifolds we will consider are embedded submanifolds and we will often drop the adjective embedded and only say submanifold. If S is a submanifold of dimension m in a manifold M of dimension n we say that N is of *codimension* $k = n - m$ in M. We note that the only open submanifolds are those of codimension 0.

An embedded submanifold S of M is indeed a smooth manifold, and in fact the inclusion map $\iota : S \to M$ is a smooth embedding (and hence the name) as the following lemma shows:

Lemma 3.9 *Let S be a k-dimensional embedded submanifold of M. Then with the subspace topology, S admits a smooth structure so that*

(a) S is a smooth k-manifold

(b) The inclusion map $\iota : S \hookrightarrow M$ is a smooth embedding.

Proof. Since M is Hausdorff and paracompact, so is S (with the subspace topology). Let $\mathcal{A}$ be a smooth atlas for M. It is not hard to check that $\mathcal{A}' = \{(S \cap U, \varphi|_{S\cap U}) \,|\, (U, \varphi) \in \mathcal{A}\}$ is a smooth atlas for S.

To see that $\iota : S \to M$ is an embedding, first note that it is clearly a topological embedding (since the identity map is a homeomorphism). So we need to prove that ι is an immersion, i.e. ι is a smooth map and that for all $p \in S$ the map $d\iota_p : T_pS \to T_pM$ is injective. Consider the standard immersion

$$j : \mathbb{R}^k \to \mathbb{R}^n, \quad (x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, 0, \ldots, 0).$$

Let $p \in S \subset M$ and choose a chart (U, φ) of M containing p. We can assume that $\varphi(U \cap S) = \mathbb{R}^k \subset \mathbb{R}^n$ (or an open subset thereof) where we have identified $\mathbb{R}^k$ with the first factor of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. Let $V = U \cap S$ and $\psi = \varphi|_{U\cap S}$. Then (V, ψ) is a smooth chart of the smooth manifold S around of p. To see that ι is smooth we need to show that $\varphi \circ \iota \circ \psi^{-1}$ is smooth. But note that $\varphi \circ \iota \circ \psi^{-1} = j$ which is smooth. Moreover, since $\iota = \varphi^{-1} \circ j \circ \psi$, we have by the product rule that $d\iota_p = d(\varphi^{-1})_{\varphi^{-1}(\psi(p))} \circ dj_{\psi(p)} \circ d\psi_p$ and all three maps on the right are injective ($d\psi_p$ and $d\phi_p$ are isomorphism and j an immersion) and hence so is $d\iota_p$. Therefor ι is an immersion and so a smooth embedding.

$\square$

In particular, any embedded submanifold of M is the image of a smooth embedding into M. In fact the converse is also true: the image $f(N)$ of a smooth embedding $f : N \to M$ is an embedded submanifold of M. Hence embedded submanifolds are exactly the images of smooth embedding. However, we will not need this fact here and we skip the proof of the other direction.

Taking submanifolds of a manifold is a good way to come up with examples of manifolds, and hence it is useful to have a way to do this, and the goal of the rest of this chapter is to prove the Regular Value Theorem (sometimes called the Inverse Image Theorem) that does just that. First we need to define a few terms.

Definition 3.10 (Regular and critical points/values) Let M be a smooth n-manifold, N a smooth k-manifolds and $F : M \to N$ a smooth map.

- A point $p \in M$ is called a *regular point* if $dF_p : T_pM \to T_pN$ is surjective (i.e. has rank k).
- A point $p \in M$ that is not a regular point is is called a *critical point*.
- A point $q \in N$ is called a *regular value* if *every* point in $F^{-1}(q)$ is a regular point, and it is called a *critical value* otherwise.

Note that if $n < k$ then every point is critical. Also, every point of M is a regular point if and only if F is a submersion.

We can now state the main result of this chapter:

Theorem 3.11 (Regular Value Theorem) *Let M be a smooth n-manifold and N a smooth k-manifold. Suppose $F : M \to N$ is a smooth map and $q \in N$ is a regular value. Then $F^{-1}(q)$ is an embedded submanifold of dimension $n - k$ (i.e. of codimension k).*

In order to prove this theorem we need to recall the Implicit Function Theorem:

Theorem 3.12 (Implicit Function Theorem) *Let $U \subset \mathbb{R}^{m+k}$ be open and $F : U \to \mathbb{R}^k$ be a smooth map. Identify $\mathbb{R}^{m+k}$ with the set of points $\{(x, y) \,|\, x \in \mathbb{R}^m, y \in \mathbb{R}^k\}$. Suppose there is a point $(a, b) \in \mathbb{R}^{m+k}$ such that $F(a, b) = 0$ and that the $k \times k$ matrix*

$$\left(\frac{\partial F^i}{\partial y^j}\bigg|_{(a,b)} \right)$$

is invertible. Then there exists open sets $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^k$ with $(a, b) \in A \times B$ and a unique smooth map $g : A \to B$ such that $F(x, g(x)) = 0$.

In other words, for all $x \in A$ there exists a unique $y \in B$ such that $F(x, y) = 0$. This means that, locally in $A \times B$, y can be represented as a function of x. That is, locally we can construct (implicitly) a function g whose graph $(x, g(x))$ is precisely the set of points (x, y) for which $F(x, y) = 0$.

We proved versions of the Implicit Function Theorem, in cases such as $F : \mathbb{R}^n \to \mathbb{R}^m$, in Metric Spaces and Intro to Geometry; the general case is similar.

We are now ready to prove the Regular Value Theorem:

Proof. [Proof of Theorem 3.11] Let $F : M \to N$ be smooth and $q \in N$ a regular value. Let $S = F^{-1}(q)$. We want to show that $S \subset M$ is an embedded submanifold of codimension k.

Let $p \in S$ and (U, φ) a chart of M with $p \in U$ and (V, ψ) a chart in N with $F(p) = q \in V$. Without loss of generality (by composing with a translation if necessary) we can assume that $\varphi(p) = 0 \in \mathbb{R}^n$ and $\psi(q) = 0 \in \mathbb{R}^k$.

Consider $\tilde{F} : \psi \circ F \circ \varphi^{-1} : \tilde{U} \to \mathbb{R}^k$, where $\tilde{U} = \varphi(U)$. Set $m = n - k$ and write $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k = \{(x_1, \dots, x_m, y^1, \dots, y^k)\}$. Note that $\tilde{F}(0,0) = 0$ and by assumption, that $d\tilde{F}_0$ has rank k. We can, by applying a rotation if necessary, assume that it is the right hand side $k \times k$-matrix in the Jacobian of $\tilde{F}$ at $(0,0)$ that span the image, i.e. that

$$\left(\frac{\partial F^i}{\partial y^j}\bigg|_{(0,0)} \right)$$

is invertible, where $i, j \in \{1, \dots, k\}$.

We apply the Implicit Function Theorem to $\tilde{F}$ with $(a,b) = (0,0)$. We get that there exists open sets $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^k$, and a smooth map $g : A \to \mathbb{R}^k$ such that $F(x,y) = 0$, with $(x,y) \in A \times B$, if and only if $y = g(x)$.

By making A and B smaller if necessary, we can assume that $A \times B \subset \tilde{U} = \varphi(U)$. Also note that $\tilde{F}^{-1}(0) = \varphi(S \cap U)$.

Anyways, let $(x,y) \in \tilde{F}^{-1}(0) \cap (A \times B)$. Then $F(x,y) = 0$ and $y = g(x)$. Define $\tilde{\varphi} : \mathbb{R}^n \to \mathbb{R}^n$ by $(x,y) \mapsto (x, y - g(x))$. Then $\tilde{\varphi}(x,y) = (x,0)$ for all $(x,y) \in \tilde{F}^{-1}(0) \cap (A \times B)$. That is, $\tilde{\varphi}(\tilde{F}^{-1}(0) \cap (A \times B)) = \tilde{\varphi}(A \times B) \cap \mathbb{R}^m$.

Now, let $U' = \varphi^{-1}(A \times B) \subset U$. Then U' is an open set in M containing p. Let $\varphi' = \tilde{\varphi} \circ \varphi : U' \to \mathbb{R}^n$. Then (U', φ') is a chart at p and $\varphi'(U' \cap S) = \tilde{\varphi} \circ \varphi(U' \cap S) = \tilde{\varphi}((A \times B) \cap \tilde{F}^{-1}) = \tilde{\varphi}(A \times B) \cap \mathbb{R}^m$. Since $p \in S$ was arbitrary, this shows that S is an embedded submanifold of dimension $m = n - k$, i.e. of codimension k.

□

Using this theorem, we can for example prove that the n-sphere is an embedded submanifold of $\mathbb{R}^{n+1}$ (and in particular give a simple proof that it's a smooth manifold):

Example 3.13 (Spheres) For each n, the sphere $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$ (of dimension n, i.e. codimension 1). To see this, define $F : \mathbb{R}^{n+1} \to \mathbb{R}$ by $F(x) = ||x||^2 = x_1^2 + x_2^2 + \dots x_{n+1}^2$. It is straightforward to check that F is smooth. Note that $F^{-1}(1) = \mathbb{S}^n$. Hence, if we show that 1 is a regular value of F we are done. Note that, for each $x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ and $v = (v^1, \dots v^{n+1}) \in T_x\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$, we have $dF_x(v) = 2\sum x^i v^i$ which defines a surjective map $T_x\mathbb{R}^{n+1} \to T_{F(x)}\mathbb{R} = \mathbb{R}$ for all $x \neq 0$. In particular it is surjective for all $x \in F^{-1}(1)$, since $0 \notin F^{-1}(1)$.

Example 3.14 (More Lie Groups) Applying the Implicit Function Theorem also gives us more examples of Lie groups:

(1) The special linear group $SL(n, \mathbb{R})$ consisting of matrices with determinant 1 is an embedded submanifold of $M(n, \mathbb{R})$. Note that this implies that $SL(n, \mathbb{R})$ is a Lie group since it means that it is a smooth manifold and since it is a subgroup of the Lie group $GL(n, \mathbb{R})$ and hence the group operations are smooth.

    To see that it is an embedded submanifold, consider the map

$$F : M(n, \mathbb{R}) \to \mathbb{R}, \quad F(A) = \det(A).$$

    We identify $M(n, \mathbb{R})$ with $\mathbb{R}^{n^2}$ and hence its tangent space at a point with $\mathbb{R}^{n^2}$. Note that F is a polynomial map and hence smooth. By definition $SL(n, \mathbb{R})$ is the preimage of 1 under

F and hence we need to prove that 1 is a regular value of F. Let $A \in F^{-1}(1) = SL(n,\mathbb{R})$ and consider $dF_A : T_A\mathbb{R}^{n^2} \to T_1\mathbb{R}$. Since $T_1\mathbb{R}$ is 1-dimensional, the linear map is surjective if there exists a v such that $dF_A(v) \neq 0$. Recall that $dF_A(v) \in T_1\mathbb{R}$ i.e. is a derivation at 1 on the manifold $\mathbb{R}$. So to show that this is not the 0 derivation, we need to show there is $f \in C^\infty(\mathbb{R})$ such that $dF_A(v)f = v(f \circ F) \neq 0$. Let f be the identity function, so $v(f \circ F) = v(F)$. Recall that this is nothing but the directional derivative of F in direction v (since v is a derivation on Euclidean space $\mathbb{R}^{n^2}$). Hence

$$v(F) = D_v|_A F(A) = \frac{d}{dt}\bigg|_{t=0} F(A + tv).$$

Taking $v = A$ (as we can identify both v and A with points in $\mathbb{R}^{n^2}$) we get

$$\frac{d}{dt}\bigg|_{t=0} F(A + tA) = \frac{d}{dt}\bigg|_{t=0} \det(A + tA) = \frac{d}{dt}\bigg|_{t=0} (1+t)^n \det(A) = n\det(A) = n$$

since $A \in SL(n,\mathbb{R})$. Hence 1 is a regular value and we are done.

(2) The orthogonal group $O(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \,|\, AA^T = I\}$ is an embedded submanifold of $M(n,\mathbb{R})$.

(3) The special orthogonal group $SO(n,\mathbb{R}) = O(n,\mathbb{R}) \cap SL(n,\mathbb{R})$ is an embedded submanifold of $M(n,\mathbb{R})$.

In particular the groups in the example above are smooth manifolds and the group operations are smooth (since this is true for $GL(n,\mathbb{R})$ of which they are subgroups) and hence they are all Lie groups.

Exercise 3.15 Prove the last two assertions in the examples above, i.e. that $O(n,\mathbb{R})$ and $SO(n,\mathbb{R})$ are embedded submanifolds. *Hint:* Consider the map $F : M(n,\mathbb{R}) \to S(n,\mathbb{R})$, where $S(n,\mathbb{R})$ denotes symmetric matrices, defined by $F(A) = AA^T$ and apply the Regular Value Theorem.

We end by describing the tangent space to a submanifold:

Proposition 3.16 (Tangent space of submanifold) *Let $F : M \to N$ be a smooth map, q a regular value of F, and consider the submanifold $S = F^{-1}(q) \subset M$. Then, for all $p \in S$, the tangent space at p is the subspace of T_pM given by the kernel of the differential $dF_p : T_pM \to T_pN$, that is*

$$T_pS = \ker(dF_p).$$

Proof. Suppose the dimension of M is n and dimension of N is k (so S is of dimension $n-k$). Let $F|_S$ be the restriction of F to $S \subset M$, that is $F|_S : S \to N$ with $F|_S(p) = F(p)$ for all $p \in S$. Note that this is the constant map $F|_S(p) = q$ for all $p \in S$. Let $p \in S \subset M$ and choose charts $(U, \varphi = (x^i))$ and $(V, \psi = (y^i))$ of p and $F(p) = q$, respectively. Let $\tilde{F} = \psi \circ F \circ \varphi^{-1}$ and $\tilde{F}|_S = \psi \circ F|_S \circ \varphi^{-1}$.

We can assume $\varphi(U \cap S) = \varphi(U) \cap \mathbb{R}^{n-k}$ where we identify $\mathbb{R}^{n-k}$ with

$$\{(x^1, \ldots, x^{n-k}, 0, \ldots, 0)\} \subset \mathbb{R}^n$$

i.e. as the first factor of $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$. We have $d\tilde{F}_p : \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^k$ (where we've done the usual identifications) and $d(\tilde{F}|_S)_p : \mathbb{R}^{n-k} \to \mathbb{R}^k$ where $d\tilde{F}_p(v,0) = d(\tilde{F}|_S)_p(v)$ for all $v = (v', 0) \in \mathbb{R}^{n-k} \times \{0\} \subset \mathbb{R}^n$.

Now, since $\tilde{F}|_S$ is constant, $d(\tilde{F}|_S)_p$ is the 0-map. Hence $d\tilde{F}_p(v,0) = d(\tilde{F}|_S)_p(v) = 0$ for all $v = (v', 0) \in \mathbb{R}^{n-k} \times \{0\}$. That is $d(F|_S)_p(v) = 0$ for all $v \in T_pS$. We have shown that $T_pS \subset \ker(dF_p)$. However, since both these vector spaces has dimension $n-k$, they must then be equal.

Chapter 4

Tangent bundle and vector fields

4.1 Tangent Bundle

Now that we are familiar with tangent spaces, our goal is to define the *tangent bundle*. The tangent bundle is an example of a *vector bundle* which, intuitively, is a disjoint union of vector spaces (one for each point of a manifold M) fitting together in a nice way (in particular so that it is a manifold itself). Here we will take a goal oriented approach and define the tangent bundle directly since this will be enough for our purposes. However, vector bundle is an important concept in differential topology and we refer the interested reader to Appendix A were we cover vector bundles in general.

Definition 4.1 (Tangent Bundle (working definition)) Let M be a smooth manifold. The tangent bundle of M, denoted by TM, is the disjoint union

$$TM = \bigsqcup_{p \in M} T_pM.$$

The surjective map

$$\pi : TM \to M, \quad v_p \mapsto p \quad \text{for all } v_p \in T_pM$$

is the associated *projection* of TM to M.

Technically, the tangent bundle is the space TM together with the projection π and is hence a pair (TM, π); for simplicity we mostly write just TM. (Even more technically, and what is missing from the definition above, is that the tangent bundle is this pair together with "local trivilalizations", see remark below and the appendix).

We will next prove that TM is a smooth manifold. Note that we can identify TM with the set $\{(p, v) \,|\, p \in M, v \in T_pM\}$ and in particular, if M is n-dimensional, then TM is of dimension $2n$.

Proposition 4.2 *Let M be a smooth n-manifold. Then TM is a smooth manifold of dimension $2n$.*

Proof. We use the Manifold Construction Lemma. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a maximal smooth atlas for M. Writing $\varphi_\alpha = (x^1_\alpha, \ldots, x^n_\alpha)$ we have (for $p \in U_\alpha$) the basis $\{\frac{\partial}{\partial x^1_\alpha}|_p, \ldots, \frac{\partial}{\partial x^n_\alpha}|_p\}$ for T_pM. Accordingly for $v \in T_pM$ we write $v = (v^1, v^2, \ldots, v^n)$ if $v = \sum v^i \frac{\partial}{\partial x^i_\alpha}|_p$.

For each α let

$$V_\alpha = \pi^{-1}(U_\alpha) = \bigsqcup_{p \in U_\alpha} T_pM = \{(p, v) \,|\, p \in U_\alpha, v \in T_pM\}$$

and

$$\Psi_\alpha : V_\alpha \to \mathbb{R}^{2n}, \quad (p, v) \mapsto (\varphi_\alpha(p), v^1, \ldots, v^n)$$

where v_i are the coordinates of v in the fixed basis for T_pM as above. Note that Ψ_α is injective on V_α since φ_α is and once p is fixed the coordinates for v are unique.

Now, for each α, we have that $\varphi_\alpha(U_\alpha)$ is open in $\mathbb{R}^n$ and hence that $\Psi_\alpha(V_\alpha) = \varphi_\alpha(U_\alpha) \times \mathbb{R}^n$ is an open set in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

Moreover, for each α, β, we have $\Psi_\beta(V_\alpha \cap V_\beta) = \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n$ which is open in $\mathbb{R}^{2n}$.

Also, for each α, β, note that $\Psi_\alpha(V_\alpha \cap V_\beta) = \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$ and $\Psi_\beta(V_\alpha \cap U_\beta) = \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n$. Letting $\{x_i\}$ and $\{\tilde{x}_i\}$ be the local coordinates for ϕ_α and ϕ_β respectively and so $\{\frac{\partial}{\partial x^i}\}$ and $\{\frac{\partial}{\partial \tilde{x}^i}\}$ be the corresponding basis for the tangent spaces. Then in each basis v can be expressed as $(v_1, \ldots, v_n)$ and $(\tilde{v}_1, \ldots, \tilde{v}_n)$ where $\tilde{v}_j$ is given in terms of the v_i according to the the change of coordinates formula in Lemma 2.15. Note that this map is a diffeomorphism. Now we have $\Psi_\beta \circ \Psi_\alpha^{-1}(x, v_1, \ldots, v_n) = (\varphi_\beta \circ \varphi_\alpha^{-1}(x), \tilde{v}_1, \ldots, \tilde{v}_n)$ for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$. Since also $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a diffeomorphism it follows that so is $\Psi_\beta \circ \Psi_\alpha^{-1} : \Psi_\alpha(V_\alpha \cap V_\beta) \to \Psi_\beta(V_\alpha \cap U_\beta)$.

Now let $(p, v), (q, w) \in TM$ (that is, $p, q \in M$, $v \in T_pM$, and $w \in T_qM$). If $p \neq q$ then, since M is Hausdorff there are open sets A and B in M such that $p \in A$, $q \in B$ and $A \cap B = \emptyset$. We can in fact assume that $A = U_\alpha$ and $B = U_\beta$ for some α, β (i.e. that these open sets are in our cover of M). To see this, with A and B being any open sets separating p, q as above, take $U_{\alpha'}$, $U_{\beta'}$ any charts at p and q, respectively, and note that $A \cap U_\alpha$ is an open set containing p and hence, since our atlas is maximal, this intersection is the domain of a chart. Similarly we can find U_β. Now, $(p, v) \in V_\alpha$, $(q, w) \in V_\beta$ and $V_\alpha \cap V_\beta = \emptyset$.

Lastly, note that for each chart $(U_\alpha, \varphi_\alpha)$ in $\mathcal{A}$, U_α can be written as a union of basis elements, and for each such basis element B we have that $(B, \varphi_\alpha|_B)$ also belongs to M (it is smoothly compatible with all other charts since $(U_\alpha, \varphi_\alpha)$ is). Note that the domains of these charts cover M and since M is second countable, there are countably many of them, say $\{B_i\}$. Since $\{B_i\}$ cover M we have that those V_α corresponding to preimages $\pi^{-1}(B_i)$ of these chart domains cover TM, and in particular we have a countable subcover.

We can now conclude by the Manifold Construction Lemma that TM is a smooth $2n$-manifold.

□

Definition 4.3 (Trivial vector bundle) We say M has a *trivial* vector bundle if TM is homeomorphic to $M \times \mathbb{R}^k$.

Remark 4.4 As mentioned above, the tangent bundle is an example of a vector bundle (see Definition A.1) and as such has more structure than we describe here. Essentially, this extra structure is that locally (for each open set U of M) the vector bundle TM looks like (is diffeomorphic to) $U \times \mathbb{R}^k$. More precisely, for each $p \in M$ there is an open set U and a diffeomorphism $\Psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that for all $q \in U$ we have that $\pi \circ \Psi^{-1}(q, t) = q$ and Ψ restricted to T_qM is an isomorphism between T_qM and $\mathbb{R}^k$. This is called a "local trivialization"; see the appendix for more details. In particular, if U can be chosen to be M (and so Ψ is a global map) then TM is trivial.

4.2 Vector Fields

Our next goal is to define vector fields, which are *smooth sections* of the tangent bundle.

Definition 4.5 (Vector Field) Let M be a smooth manifold. A (smooth) *vector field* X on M is a (smooth) section of the tangent bundle (TM, π), i.e. a (smooth) map

$$X : M \to TM$$

such that

$$\pi(X(p)) = p$$

for all $p \in M$. We denote the set of all smooth vector fields on M by $\Gamma(TM)$ (which is in fact a vector space over $\mathbb{R}$ under point-wise addition and scalar multiplication).

We often write X_p for $X(p)$. Note that X_p is a point in T_pM. Letting (U, φ) be a smooth chart of M where $\varphi = (x^1, \ldots, x^n)$, recall that

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

is a (coordinate) basis for T_pM for all $p \in U$. Hence we can, for each $p \in U$, write X_p in terms of this basis as

$$X_p = \sum_{i=1}^{n} X^i(p) \frac{\partial}{\partial x^i}\bigg|_p \tag{4.1}$$

for some (unique) real numbers $X^i(p)$. The functions $X^i : U \to \mathbb{R}$ (for $i = 1, \ldots, n$) are called the *component functions* of X. Equivalently, we can write

$$X|_U : U \to TM \text{ given by } X|_U = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} \tag{4.2}$$

by which we mean that for each $p \in U$ we have that $X(p) = X_p$ is obtained as in (4.1) with $X^i : U \to \mathbb{R}$ being the component functions. Note that X is *smooth* vector field if and only if its component functions are smooth.

Example 4.6 Let M be any smooth n-manifold and (U, φ) a smooth chart with local coordinates $\varphi = (x^1, \ldots, x^n)$. Note that for each i,

$$\frac{\partial}{\partial x^i} : U \to TM$$

defined, for points in $U \subset M$, by

$$\frac{\partial}{\partial x^i}(p) = \left.\frac{\partial}{\partial x^i}\right|_p$$

is a vector field. That is, $\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM$ is the derivation defined by $\left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p)$ using (as always) the notation of (2.3) and (2.4).

Example 4.7 Consider $\mathbb{R}^2$ and use coordinates (x, y) to denote its (global) coordinates. For any smooth functions $F, G : \mathbb{R}^2 \to \mathbb{R}$

$$X = F\frac{\partial}{\partial x} + G\frac{\partial}{\partial y}$$

is a vector field, where

$$X_p = F(p)\left.\frac{\partial}{\partial x}\right|_p + G(p)\left.\frac{\partial}{\partial y}\right|_p$$

for each $p \in \mathbb{R}^2$.

Since it will be used in the second half of the course, we introduce some more terminology. As in the example above, $X = \frac{\partial}{\partial x^i}$ is a smooth vector field for each $i = 1, \ldots, n$, The tuple $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$ is called a (local) *frame*. This can be defined more generally:

Definition 4.8 Let $U \subset M$ be an open subset. Suppose $(E_1, \ldots, E_n)$ is an n-tuple of vector fields on U such that for all $p \in U$, $\{E_1|_p, \ldots, E_n|_p\}$ is a basis for T_pM. Then $(E_1, \ldots, E_n)$ is a *local frame* for M. If $U = M$ we call it a *global frame*.

We say a vector field X is a *non-zero vector field* if it is not identically 0, i.e. if there exists at least one $p \in M$ such that $X_p \neq 0$. That is, there is at least one p such that $X_p \in T_pM$ is not the the 0 derivation at p (so there exists at least one $f \in C^\infty(M)$ with $X_p f \neq 0$). Given a manifold M, is there always a non-zero smooth vector field on M? In the special case that $M = \mathbb{R}^n$ it is clear that the answer is yes. Intuitively in $\mathbb{R}^n$ a vector field can be viewed as something that at each point $p \in \mathbb{R}^n$ assigns a vector v_p based at p and the vector field is smooth if these vectors change in a smooth way as p changes. The vector field is non-zero if not all of these vectors are the 0-vector. Hence, one can create a smooth non-zero vector field by choosing a constant one: choose one non-zero vector and "attached" this to every point. This can be generalized to any manifold for which the tangent bundle is trivial as we explain next.

Suppose $TM = M \times \mathbb{R}^n$, i.e. the tangent bundle is trivial. Then there is a non-zero vector field on M, i.e. one for which $X_p \neq 0$ for all p, as such is easy to construct: for each $p \in M$ define $X_p = (p, 1, 1, \ldots, 1)$.

However, not every manifold has a non-zero vector field. The famous "Hairy Ball Theorem" which you might have heard of states that the sphere $\mathbb{S}^2$ has no non-zero vector field (one cannot comb a hairy ball such that all hairs lie in the same direction, i.e. you create a cowlick). More precisely it states that for any continuous map X which to each point in $\mathbb{S}^2$ assigns a vector in $\mathbb{R}^3$ (in other words, any vector field $X : \mathbb{S}^2 \to T\mathbb{S}^2$) there exists at least one $p \in \mathbb{S}^2$ such that $X(p) = 0$. As a consequence:

Fact 4.9 The tangent bundle to $\mathbb{S}^2$ is non-trivial.

□

However, there are non-trivial example of manifolds for which the tangent bundle is trivial—in fact all Lie groups have trivial tangent bundle.

Lemma 4.10 *Let G be a Lie group. Then the tangent bundle TG is trivial. In particular, TG is homeomorphic (in fact, diffeomorphic) to $G \times T_eG$.*

Proof. Suppose G is of dimension n. Recall that T_eG is isomorphic to $\mathbb{R}^n$ and hence if we prove the last claim it follows that the tangent bundle is trivial.

Let $g \in G$ and consider the smooth map $L_g : G \to G$ given by left multiplication by g, i.e. $L_g(h) = gh$ for each $h \in G$. This is clearly bijective, and hence a diffeomorphism (since the group operations are smooth). Therefore the differential at the identity e,

$$dL_g := d(L_g)_e : T_eG \to T_gG$$

is an isomorphism.

Now, define $\Phi : G \times T_eG \to TG$ by $\Phi(g, v) = dL_g(v)$. Since dL_g is an isomorphism, Φ is bijective. We leave it as an exercise to verify that Φ is a homeomorphism (in fact, a diffeomorphism).

□

If two manifolds M and N are diffeomorphic, one can *pullback* a vector field on N to one of M. That is, a diffeomorphism $F : M \to N$ induces a map $F^* : \Gamma(TN) \to \Gamma(TM)$ defined as follows. Since F is a diffeomorphism, for all $p \in M$ the differential $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and so $dF^{-1}_{F(p)} : T_{F(p)}N \to T_pM$ is well-defined. Now, given a vector field Y on N we want to associate a vector field F^*Y on M. For each $p \in M$ take the tangent vector $Y_{F_p} \in T_{F(p)}N$ and *pull it back* using $dF^{-1}_{F(p)}$ to the tangent vector $dF^{-1}_{F(p)}(Y_{F(p)}) \in T_pM$. This is the tangent vector we define F^*Y_p to be. We call F^*Y the *pullback* of Y.

Similarly, we can *push forward* a vector field on M to one on N. The diffeomorphism $F : M \to N$ gives rise to a map $F_* : \Gamma(TM) \to \Gamma(TN)$ defined as follows: given a vector field X on M define F_*X to be the vector field on N given by $F_*(X)(q) = dF_{F^{-1}(q)}(X_{F^{-1}(q)})$ for each $q \in N$. To see that this makes sense: for each $q \in N$ we have the differential $dF_{F^{-1}(q)} : T_{F^{-1}(q)}M \to T_qN$. Since $Y_{F^{-1}(q)} \in T_{F^{-1}(q)}N$ we have that $dF_{F^{-1}(q)}(Y_{F^{-1}(q)})$ is indeed a tangent vector in T_qN.

We summarize this discussion below for future reference:

Definition 4.11 (Pullback and pushforward of vector fields) Let M and N be smooth manifolds and $F : M \to N$ a diffeomorphism.
Let Y a vector field on N. The pullback of Y is the vector field $F^*Y : M \to TM$ on M defined by

$$F^*(Y)_p = dF^{-1}_{F(p)}(Y_{F(p)})$$

for each $p \in M$.
Let X be a vector field on M. The *pushforward* of X is the vector field $F_*X : N \to TN$ on N defined by

$$F_*(X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)})$$

for each $q \in N$.

For example, with $F : M \to N$ diffeomorphism as above, let $\left\{ \frac{\partial}{\partial y^1}\Big|_q, \dots, \frac{\partial}{\partial y^n}\Big|_q \right\}$ be a local coordinate basis for T_qN. For each i, consider the vector field

$$\frac{\partial}{\partial y^i} : U \to TN$$

as in Example 4.6. The pullback of this vector space is given by

$$F^*\left(\frac{\partial}{\partial x^i}\right)_p = dF^{-1}_{F(p)}\left(\frac{\partial}{\partial y^i}\Big|_{F(p)}\right). \tag{4.3}$$

We ask the reader to compare the above equation with Equation (2.3).

In particular, if Y is any vector field on N recall from (4.1) that it can be written in local coordinates as

$$Y_q = \sum_{i=1}^n Y^i(q)\frac{\partial}{\partial y^i}\Big|_q$$

for some smooth functions Y^i. Then the pullback of Y can be written as

$$F^*(Y)_p = \sum Y^i(F(p))dF^{-1}_{F(p)}\left(\frac{\partial}{\partial y^i}\Big|_{F(p)}\right).$$

Similar expressions can be obtained for the pushforward, which we leave as an exercise.

Lie groups and left invariant vector fields*

**This section will not be covered in the course and reading it is optional.*

Using similar methods as in the proof of Lemma 4.10 above we can show that the tangent space at e is in bijection with left-invariant vector spaces on G, as we will explain next. For each $g \in G$ define $L_g : G \to G$ as in the mentioned proof, i.e $L_g(h) = gh$. For each $h \in G$ this induces the map $d(L_g)_h : T_hG \to T_{gh}G$. With this notation, we define what it means for a vector field to be left-invariant:

Definition 4.12 (Left Invariant Vector Fields) Let G be a Lie groups and $X : G \to TG$ a smooth vector field on G. We say that X is *left invariant* if $d(L_g)_h(X_h) = X_{gh}$ for all $g, h \in G$. We sometimes abbreviate this notation by writing $(L_g)_*(X) = X$.

Note that if $X : G \to TG$ is a left invariant vector field then it is determined by how it acts on one element, in particular it is determined by $X(e) = X_e$ since for any $g \in G$, we have $X(g) = X_g = d(L_g)_e(X_e)$.

Also note that the set of all left invariant smooth vector fields on a Lie group is a vector space.

We encourage you to work out a concrete example:

Exercise 4.13 Consider the Lie group $(\mathbb{R}^2, +)$. Show that there is a bijection between the set of left invariant vector fields on $\mathbb{R}^2$ and $\mathbb{R}^2$ itself.

In fact, for any Lie group the space of left invariant vector fields is in bijection with the tangent space at the identity:

Lemma 4.14 *Let G be a Lie group. Then the space of all left-invariant smooth vector fields is isomorphic to T_eG.*

We leave the proof to the reader.

We will come back to left-invariant vector fields below when we define the Lie algebra of a Lie group.

4.3 Lie Brackets

The goal of this section is to define the Lie bracket of two vector fields which is a way to combine two vector fields into a new vector field. Hence taking Lie brackets is a binary operation on the set of all vector fields and makes this set into an algebraic object, which is an example of a so-called Lie algebra. There are other Lie algebras; we comment on some in the last subsection on this chapter but this will not be an (examinable) part of this course. For the interested reader who read the optional section above on left-invariant vector fields on Lie groups: the set of such vector fields is closed under taking Lie brackets and that set together with this operation is called its Lie algebra.

First we take a slightly different view of vector fields, viewing them as *derivations*.

Lemma 4.15 *Let M be a smooth manifold and X a smooth vector field on M. Then X gives rise to a linear map (which by abuse of notation we also call X)*

$$X : C^\infty(M) \to C^\infty(M), \quad f \mapsto Xf$$

defined by $Xf(p) = X_pf$.

Equivalently, if in local coordinates (i.e. in a chart $(U, \phi = (x^1, \dots, x^n))$ the vector field is given by $X = \sum X_i \frac{\partial}{\partial x^i}$ then

$$Xf = \sum X_i \frac{\partial f}{\partial x^i}$$

and

$$Xf(p) = X_pf = \sum X_i(p) \frac{\partial f}{\partial x^i}(p).$$

Proof. [Proof of Lemma 4.15] Since X is a smooth vector field Xf defined by $Xf(p) = X_pf$ is indeed a smooth function on M (since each component function X_i as well as each $\frac{\partial f}{\partial x^i}$ are smooth) so X is well-defined. Recall that X_p is a derivation at p, and in particular is linear. Hence $X(f+g)(p) = X_p(f+g) = X_pf+X_pg = Xf(p)+Xg(p)$ for all p and similarly $X(\lambda f)(p) = X_p(\lambda f) = \lambda X_pf = \lambda Xf(p)$ for all p. That is, $X(f+g) = Xf+Xg$ and $X(\lambda f) = \lambda Xf$, in other words, X is a linear map.

Since X_p is a derivation at p for each p also means that X_p satisfies the product rule at p, from which one can prove the following exercise:

Exercise 4.16 Prove that the linear map X induced by a vector field satisfies the *product rule*:

$$X(fg) = fXg + gXf$$

.

In fact, any linear map from $C^\infty(M)$ to itself that satisfies the product rule defined in the exercise above is called a *derivation*:

Definition 4.17 (Derivations) Let M be a smooth manifold. A derivation is a linear map

$$X : C^\infty(M) \to C^\infty(M)$$

satisfying the product rule

$$X(fg) = fXg + gXf$$

for all $f, g \in C^\infty(M)$.

Compare the definition of derivation with the earlier definition of *derivations at a point p*: its relationship to the above definition is motivated by Lemma 4.15 where the derivation X is defined in terms of X_p which is a derivation at p.

Anyways, Lemma 4.15 together with the exercise following it shows that every vector field is (can be viewed as) a derivation. The converse is also true:

Proposition 4.18 (Derivations are vector fields) *Let M be a smooth manifold. A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some smooth vector field X on M.*

Proof. As mentioned, Lemma 4.15 together with the exercise following it shows that every vector field is a derivation.

For the converse, suppose $D : C^\infty(M) \to C^\infty(M)$ is a derivation, i.e. is linear and satisfies the product rule. For each $p \in M$ define

$$X_p : C^\infty(M) \to \mathbb{R} \text{ by } X_p f = (Df)(p).$$

Since D is a derivation (i.e. is linear and satisfies the product rule) it follows that X_p is a derivation at p, and so $X_p \in T_pM$ for each p. Hence we can define

$$X : M \to TM \text{ by } X(p) = X_p.$$

By construction $X(f) = D(f)$ for all $f \in C^\infty(M)$. That X is a *smooth* vector field follows from the fact that Xf is smooth for all smooth functions f. To see this, consider $f_k = x^k$, the k^th-coordinate function, then Xf_k is the k^{th}-component function X^k of X is smooth. Since this is true for all k, X is smooth. □

Example 4.19 (Vector fields and derivations are the same) Let M be any smooth n-manifold and (U, φ) a smooth chart with local coordinates $\varphi = (x^1, \ldots, x^n)$. Recall from Example 4.6 that for each i we have the vector field

$$\frac{\partial}{\partial x^i} : M \to TM$$

defined by

$$\frac{\partial}{\partial x^i}(p) = \frac{\partial}{\partial x^i}\bigg|_p$$

for each $p \in U$. That is, $\frac{\partial}{\partial x^i}\big|_p \in T_pM$ is the derivation defined by

$$\frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial f}{\partial x^i}(p).$$

On the other hand, we can view it as a map

$$\frac{\partial}{\partial x^i} : C^\infty(M) \to C^\infty(M)$$

defined by

$$\frac{\partial}{\partial x^i} f = \frac{\partial f}{\partial x^i}.$$

This is a linear map which satisfies the product rule, i.e. is a derivation. Also, for each $p \in U$ we have

$$\left(\frac{\partial}{\partial x^i} f\right)(p) = \frac{\partial f}{\partial x^i}(p)$$

so is the same as the vector field above.

From now on we will think of derivations and smooth vector fields as being the same.

Next we will define a binary operation on the set of smooth vector fields, i.e. a way of combining two vector fields into a new one. Note that if X and Y are smooth vector fields (derivations) then

$$XY : C^\infty(M) \to C^\infty(M) \text{ defined by } (XY)(f) = X(Yf)$$

is a linear map from $C^\infty(M)$ to itself. Similarly,

$$YX : C^\infty(M) \to C^\infty(M) \text{ defined by } (YX)(f) = Y(Xf)$$

is a linear map from $C^\infty(M)$ to itself. *However, these maps might in general not satisfy the product rule and hence are not derivations, i.e. vector fields.* Instead, we take their difference to ensure we get a vector field, and define the *Lie bracket* as follows:

Definition 4.20 (Lie Bracket) Let X, Y be two smooth vector fields on M. The *Lie bracket* of X and Y is the operation

$$[X, Y] : C^\infty(M) \to C^\infty(M)$$

defined by

$$[X, Y]f = XYf - YXf.$$

The Lie bracket of two vector fields is indeed a vector field:

Proposition 4.21 *Let X, Y be two smooth vector fields on M. Then $[X, Y]$ is a smooth vector field on M.*

Proof. We need to check that $[X, Y]$ is a derivation. Linearity follows from linearity of X and Y so we need to check that the product rule is satisfied. Let $f, g \in C^\infty(M)$. Then

$$\begin{aligned}
[X,Y](fg) &= XY(fg) - YX(fg)\\
&= X(Y(fg)) - Y(X(fg))\\
&= X(fYg + gYf) - Y(fXg + gXf) \quad \text{(since } X \text{ and } Y \text{ satisfy the product rule)}\\
&= X(fYg) + X(gYf) - Y(fXg) - Y(gXf) \quad \text{(by linearity of } X \text{ and } Y\text{)}\\
&= X(f(Yg)) + X(g(Yf)) - Y(f(Xg)) - Y(g(Xf))\\
&= fXYg + YgXf + gXYf + YfXg - fYXg - XgYf - gYXf - XfYg \quad \text{(by product rule)}\\
&= fXYg + gXYf - fYXg - gYXf \quad \text{(by cancelation; multiplication is abelian)}\\
&= f(XYg - YXg) + g(XYf - YXf) \quad \text{(by linearity)}\\
&= f[X,Y]g + g[X,Y]f.
\end{aligned}$$

□

We can also write the Lie bracket in local coordinates; i.e. using component functions. Let (U, φ) be a smooth chart of M where φ gives the local coordinates $(x^1, \ldots, x^n)$. Then, as always,

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^1}\right|_p \right\}$$

is a coordinate basis for T_pM and restricting the domain of X and Y to U we have

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} \text{ and } Y = \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j}$$

for points in U where $X^i : U \to \mathbb{R}$ and $Y^j : U \to \mathbb{R}$ are the (smooth) component functions; see (4.1) and (4.2). In these coordinates we get that

$$\begin{aligned}
[X,Y]f =& X(Yf) - Y(Xf)\\
=& X\left(\sum_j Y^j \frac{\partial f}{\partial x^j}\right) - Y\left(\sum_i X^i \frac{\partial f}{\partial x^i}\right)\\
=& \sum_i X^i \frac{\partial}{\partial x^i}\left(\sum_j Y^j \frac{\partial f}{\partial x^j}\right) - \sum_j Y^j \frac{\partial}{\partial x^j}\left(\sum_i X^i \frac{\partial f}{\partial x^i}\right)\\
=& \sum_i\sum_j X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + \sum_i\sum_j X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_i\sum_j Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i} - \sum_i\sum_j Y^j X^i \frac{\partial^2 f}{\partial x^j \partial x^i}\\
=& \sum_i\sum_j \left(X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i}\right)
\end{aligned}$$

for all smooth $f : U \to \mathbb{R}$. Here we used linearity and product rule for the partial derivatives to obtain the fourth equality and the fact that

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$$

to obtain the last equality. Now since a tangent vector is determined by its action on functions locally by Proposition 2.9, this is true for all $f : M \to \mathbb{R}$, i.e. for all $f \in C^\infty(M)$. Hence we have, for any choice of local coordinates, that

$$[X,Y] = \sum_i \sum_j \left(X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i} \right). \tag{4.4}$$

That is,

$$[X,Y] : C^\infty(M) \to C^\infty(M)$$

is the derivation (vector field) defined by

$$[X,Y](f) = \sum_i \sum_j \left(X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} \right).$$

Equivalently,

$$[X,Y] : M \to TM$$

is the vector field defined by

$$[X,Y]_p = \sum_i \sum_j \left(X^i_p \frac{\partial Y^j}{\partial x^i}(p) \frac{\partial}{\partial x^j}\bigg|_p - Y^j_p \frac{\partial X^i}{\partial x^j}(p) \frac{\partial}{\partial x^i}\bigg|_p \right).$$

Example 4.22 Consider $\mathbb{R}^2$; let us use coordinates (x,y) for points in $\mathbb{R}^2$ and hence for each $p \in \mathbb{R}^2$ we have that $\left\{ \frac{\partial}{\partial x}|_p, \frac{\partial}{\partial y}|_p \right\}$ is a (global) basis for $T_p \in \mathbb{R}^2$. Then the following are two vector fields on $\mathbb{R}^2$:

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$
$$Y = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

The component functions of X are $X^1 : \mathbb{R}^2 \to \mathbb{R}$ given by $X^1(x,y) = -y$ and $X^2 : \mathbb{R}^2 \to \mathbb{R}$ given by $X^2(x,y) = x$. Similarly, the component functions of Y are Y^1 and Y^2 defined by $Y^1(x,y) = y$ and $Y^2(x,y) = x$.

Let us compute the Lie Bracket $[X,Y]$. Using (4.4) we get:

$$\begin{aligned}
[X,Y] &= \left(X^1 \frac{\partial Y^1}{\partial x} - Y^1 \frac{\partial X^1}{\partial x} \right) \frac{\partial}{\partial x} + \left(X^2 \frac{\partial Y^1}{\partial y} - Y^2 \frac{\partial X^1}{\partial y} \right) \frac{\partial}{\partial x} \\
&\quad + \left(X^1 \frac{\partial Y^2}{\partial x} - Y^1 \frac{\partial X^2}{\partial x} \right) \frac{\partial}{\partial y} + \left(X^2 \frac{\partial Y^2}{\partial y} - Y^2 \frac{\partial X^2}{\partial y} \right) \frac{\partial}{\partial y} \\
&= \left(-y \frac{\partial y}{\partial x} - y \frac{\partial(-y)}{\partial x} \right) \frac{\partial}{\partial x} + \left(x \frac{\partial y}{\partial y} - x \frac{\partial(-y)}{\partial y} \right) \frac{\partial}{\partial x} \\
&\quad + \left(-y \frac{\partial x}{\partial x} - y \frac{\partial x}{\partial x} \right) \frac{\partial}{\partial y} + \left(x \frac{\partial x}{\partial y} - x \frac{\partial x}{\partial y} \right) \frac{\partial}{\partial y} \\
&= 2x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y}
\end{aligned}$$

So $[X,Y]$ is the vector field on $\mathbb{R}^2$ given by $2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$.

$\square$

Example 4.23 As in Example 4.19, let M be any smooth n-manifold and (U, φ) a smooth chart with local coordinates $\varphi = (x^1, \ldots, x^n)$ and consider the vector fields (derivations)

$$\frac{\partial}{\partial x^i} : M \to TM$$

or, equivalently,

$$\frac{\partial}{\partial x^i} : C^\infty(M) \to C^\infty(M)$$

for $i = 1, \ldots, n$. Note that for all i, j we have

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0.$$

That is this Lie bracket is the 0-vector field given by

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] f = 0$$

for all $f \in C^\infty(M)$. We leave it as an exercise to verify this.

□

The Lie bracket satisfy the following properties; we leave the proof as an exercise (each follow either from the definition or by a computation):

Lemma 4.24 (Properties of the Lie bracket) *Let X, Y, Z be smooth vector fields on M. The Lie bracket satisfy the following identities:*

(1) (Bilinear) For all $a, b \in \mathbb{R}$, $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ and $[Z, aX + bY] = a[Z, X] + b[Z, Y]$

(2) (Antisymmetric) $[X, Y] = -[Y, X]$

(3) (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Lie Algebra*

**This section will not be covered in the course and reading it is optional.*

We comment briefly on *Lie algebras*, since it is the natural thing to do with the above section in mind. However, this will not be used again in the course and we do not prove the assertions made here and instead refer to Lee's Introduction to Smooth Manifolds (section 8.3 and 8.4) for these.

A *Lie algebra* is a vector space $\mathfrak{g}$ endowed with any bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the 3 properties in Lemma 4.24.

Example 4.25 (Lie algebras) The following are examples of Lie algebras:

- The vector space consisting of all vector fields on a manifold M together with the Lie bracket

- Any vector space V with the trivial bracket $[X, Y] = 0$ for all $X, Y \in V$.
- The vector space of all $n \times n$ matrices $M(n, \mathbb{R})$ together with the bracket $[\cdot, \cdot] : M(n, \mathbb{R}) \times M(n, \mathbb{R}) \to M(n, \mathbb{R})$ defined by $[A, B] = AB - BA$. (You should verify that this bracket satisfies the 3 properties above!). We denote this Lie algebra by $\mathfrak{gl}(n, \mathbb{R})$.

If we look at vector fields on a Lie group and restrict to subspace of the left-invariant ones, the Lie bracket maps this space to itself (as the below lemma shows) and hence it is a Lie algebra under the Lie bracket.

Lemma 4.26 *The vector space $\mathcal{L}$ of left invariant vector fields on a Lie group is closed under taking Lie brackets. That is, if X, Y are two left invariant vector fields on a Lie group G, then $[X, Y]$ is left invariant.*

Hence the vector space of all left-invariant vector fields (which by Lemma 4.14 is isomorphic to T_eG) on a Lie group G together with the Lie bracket for a Lie algebra, which we define to be the *Lie algebra of a Lie group* and denote by $\mathfrak{g} = Lie(G)$.

Example 4.27 (Lie algebras of Lie groups)

(1) The Lie algebra of the Lie group $(\mathbb{R}^n, +)$ is $\mathbb{R}^n$ itself together with the trivial bracket $[\cdot, \cdot] \equiv 0$. Recall that we already proved above that the set of left invariant vector fields can be identified with $\mathbb{R}^n$ itself, since every such vector field is constant. We will show that $[X, Y] = 0$ for all such vector spaces, For ease of notation assume $n = 2$, the general case is a direct generalization. Let X, Y be two vector fields on $\mathbb{R}^2$ that are left invariant. We know they must be of the form $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ and $Y = c\frac{\partial}{\partial x} + c\frac{\partial}{\partial y}$ for some $a, b, c, d \in \mathbb{R}$. It follows by the properties of the Lie bracket that

$$\begin{aligned}[X, Y] &= ac\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right] + ad\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] + bc\left[\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] + bd\left[\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right] \\ &= ad\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] + bc\left[\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right]\end{aligned}$$

where we have used the fact that $[A, A] = 0$ (since $[A, A] = -[A, A]$). Now, for any f,

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right] f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y} = 0.$$

Hence $[\frac{\partial}{\partial x}, \frac{\partial}{\partial x}] = 0$ and similarly $[\frac{\partial}{\partial y}, \frac{\partial}{\partial y}] = 0$. It follows that $[X, Y] = 0$ as desired.

(2) The Lie algebra of $GL(n, \mathbb{R})$ is $\mathfrak{gl}(n, \mathbb{R})$.

Exercise 4.28 Verify part (2) in the example above, i.e. find the Lie algebra for the Lie group $SL(n, \mathbb{R})$. (This is non-trivial: See Proposition 8.41 in Lee's *Introduction to Smooth Manifolds, 2^{nd} ed.*).

Chapter 5

Differential forms

5.1 Linear Algebra of Alternating Forms

Let V be an n-dimensional vector space (over $\mathbb{R}$). We say $\varphi : V \times V \times \cdots \times V \to \mathbb{R}$ is an *alternating form of degree* k, or *alternating* k*-form* (where k is the number of factors of V in $V \times V \times \cdots V$) if it is

(i) multi-linear (i.e. linear in each coordinate)

(ii) alternating, meaning that $\varphi(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\varphi(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$.

The set of all alternating k-forms form a vector space (over $\mathbb{R}$) under pointwise addition and scalar multiplication. That is, if φ, ψ are two alternating k-forms, then $\varphi + \psi$ is the alternating k-form defined by

$$(\varphi + \psi)(v_1, \ldots, v_k) = \varphi(v_1, \ldots, v_k) + \psi(v_1, \ldots, v_k)$$

and if $\lambda \in \mathbb{R}$, $\lambda\varphi$ is the alternating form

$$(\lambda\varphi)(v_1, \ldots, v_k) = \lambda\varphi(v_1, \ldots, v_k).$$

We denote the vector space of alternating k-forms on V by $\mathrm{Alt}^k(V)$. Note that when $k = 1$ this is exactly the dual V^* of V (i.e. the vector space of all linear maps $\varphi : V \to \mathbb{R}$) and when $k = 2$ it is the space of all anti-symmetric bilinear maps.

We note that if V is n-dimensional and $k > n$, then any alternating k-form must be identical 0, that is $\mathrm{Alt}^k(V) = \{0\}$ is the trivial vector space whenever $k > \dim(V)$. We leave this as an exercise to check:

Exercise 5.1 Let V be an n-dimensional vector space and φ an alternating k-form on V for some $k > n$. Prove that $\varphi = 0$.

We can combine two alternating forms and get a new alternating form by taking their *wedge product*:

Definition 5.2 (Wedge product of alternating forms) Let $\varphi \in \mathrm{Alt}^k(V)$ and $\psi \in \mathrm{Alt}^l(V)$. Their *wedge product* $\varphi \wedge \psi \in \mathrm{Alt}^{k+l}(V)$ is the alternating $(k+l)$-form defined by

$$\varphi \wedge \psi(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\varphi(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\psi(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

where S_n is the group of permutations on n elements.

Recall that the sign of a permutation σ, denoted $\mathrm{sgn}(\sigma)$ is 1 if σ is even (i.e. can be written as a product of an even number of transpositions) and is -1 if it is odd.

We leave it as an exercise to check the following properties of the wedge product:

Lemma 5.3 (Properties of the wedge-product) *Let* $\varphi \in \mathrm{Alt}^k(V), \psi \in \mathrm{Alt}^l(V), \eta \in \mathrm{Alt}^s(V)$. *Then*

- *(Bilinearity)* $(a\varphi + b\psi) \wedge \eta = a(\varphi \wedge \eta) + b(\psi \wedge \eta)$ *and* $\eta \wedge (a\varphi + b\psi) = a(\eta \wedge \varphi) + b(\eta \wedge \psi)$ *for all* $a, b \in \mathbb{R}$
- *(Associativity)* $(\varphi \wedge \psi) \wedge \eta = \varphi \wedge (\psi \wedge \eta)$
- $\varphi \wedge \psi = (-1)^{kl}\varphi \wedge \psi$.

Note that in particular if φ and ψ are both 1-forms (i.e. elements of V^*) then $\varphi \wedge \psi = -\psi \wedge \varphi$ and $\varphi \wedge \varphi = 0$. This is a useful observation and we record it for future reference:

Corollary 5.4 *Let* $\varphi, \psi \in \mathrm{Alt}^1(V) = V^*$. *Then*

$$\varphi \wedge \psi = -\psi \wedge \varphi$$

and

$$\varphi \wedge \varphi = 0.$$

As noted above, the set $\mathrm{Alt}^k(V)$ of all alternating k-forms on V is a vector space over $\mathbb{R}$ and next we will show that it is of dimension $\binom{n}{k}$ where n is the dimension of V. In fact, we will find an explicit basis for $\mathrm{Alt}^k(V)$.

Fix a basis $\{e_1, \ldots, e_n\}$ for V. Let $\{e_1^*, \ldots, e_n^*\}$ be the dual basis which is defined by

$$e_i^* : V \to \mathbb{R}, \quad e_i^*\left(\sum a_j e_j\right) = a_i.$$

We will see below that the dual basis is in fact a basis for the dual space V^*. For now, note that each e_i^* is an alternating 1-form and hence a wedge product of k (distinct) of them, $e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$ is an alternating k-form. Also, by the anti-commutative property in Lemma 5.3 we have that up to changing the sign, a wedge product of k of them can be written as $e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$ with $i_1 < i_2 < \cdots < i_k$. We claim that taking all possible such products gives a basis:

Lemma 5.5 *The set* $\{e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* \,|\, i_1 < i_2 < \cdots < i_k\}$ *is a basis for* $\mathrm{Alt}^k(V)$.

Proof. We prove it in the special case when $k = 1$, i.e. we will show that $\{e_1^*, \ldots, e_n^*\}$ is a basis for $\mathrm{Alt}^1(V) = V^*$. The general case follow using same logic only with more bookkeeping.

First we show that $\{e_1^*, \ldots, e_n^*\}$ is a linearly independent set. So suppose there exists $a_1, \ldots, a_n \in \mathbb{R}$ such that $\sum_{i=1}^n a_i e_i^* = 0$. That is, for all $v \in V$ we have $\sum_{i=1}^n a_i e_i^*(v) = 0(v) = 0$. In particular, this holds for each basis element of V giving us

$$0 = \sum_{i=1}^n a_i e_i^*(e_j) = a_j$$

for each $j = 1, \ldots, n$. That is, $a_j = 0$ for all j, and we can conclude that $\{e_1^*, \ldots, e_n^*\}$ is a linearly independent set.

Next we must show that the set spans $\mathrm{Alt}^1(V)$. So suppose $\varphi \in \mathrm{Alt}^1(V)$. Set $\lambda_i = \varphi(e_i)$ for each $i = 1, \ldots, n$. Let $\psi = \lambda_1 e_1^* + \cdots \lambda_n e_n^*$ and note that we have $\psi \in span\{e_1^*, \ldots, e_n^*\}$; we claim that $\psi = \varphi$ and hence φ is in the span and we are done. Indeed, note that $\psi(e_i) = \lambda_i = \varphi(e_i)$ for all $i = 1, \ldots, n$ and since ψ and φ are linear maps $V \to \mathbb{R}$ that agree on the basis elements of V they must be the same map (a linear map is defined by its action on the basis elements). □

Exercise 5.6 Prove Lemma 5.5 in the case of $k = 2$ and $n = 3$.

As an immediate consequence we have:

Corollary 5.7 *Let V be a vector space of dimension n. Then the dimension of $\mathrm{Alt}^k(V)$ is $\binom{n}{k}$. In particular, $\mathrm{Alt}^n(V)$ is 1-dimensional and hence for any non-zero $\varphi, \psi \in \mathrm{Alt}^n(V)$ we have $\varphi = c\psi$ for some $c \in \mathbb{R}$.* □

We also have, as already observed in Exercise 5.1 above, that $\mathrm{Alt}^m(V)$ is 0-dimensional for any $m > n$.

Next we define the *pullback* of an alternating k-form:

Definition 5.8 (Pullback of an alternating form) Given a linear map $L : V \to W$ between vector spaces, we obtain a linear map

$$L^* : \mathrm{Alt}^k(W) \to \mathrm{Alt}^k(V)$$

(for any k) defined as follows: for each $\varphi \in \mathrm{Alt}^k(W)$, $L^*(\varphi) \in \mathrm{Alt}^k(V)$ is the map defined by

$$L^*(\varphi)(v_1, \ldots, v_k) = \varphi(L(v_1), \ldots, L(v_k)).$$

We say that $L^*(\varphi)$ is the *pullback of φ (by L)*.

Now consider the case when $L : V \to V$, set $k = n = \dim(V)$ and consider $L^* : \mathrm{Alt}^n(V) \to \mathrm{Alt}^n(V)$. Fix a non-zero $\psi \in \mathrm{Alt}^n(V)$. Since $\mathrm{Alt}^n(V)$ is 1-dimensional by Corollary 5.7 and ψ is non-zero, $\{\psi\}$ is a basis for $\mathrm{Alt}^n(V)$. Hence, since $L^*(\psi) \in \mathrm{Alt}^n(V)$ there is a $c \in \mathbb{R}$ such that $L^*(\psi) = c\psi$. Now let $\varphi \in \mathrm{Alt}^n(V)$ be arbitrary. Then $\varphi = a\psi$ for some $a \in \mathbb{R}$ and since L^* is linear we have

$$L^*(\varphi) = L^*(a\psi) = aL^*(\psi) = a \cdot c\psi = c \cdot a\psi = c \cdot \varphi.$$

Hence the constant $c \in \mathbb{R}$ satisfies $L^*(\varphi) = c\varphi$ for all $\varphi \in \mathrm{Alt}^n(V)$. That is, c is independent of the φ and only depends on L. We call this constant the *determinant of* L and denote it by $\det(L)$.

To summarize:

Definition 5.9 (Determinant) Given a linear map $L : V \to V$ from an n-dimensional vector space to itself, then the determinant of L is the constant $det(L)$ such that the map $L^* : \mathrm{Alt}^n(V) \to \mathrm{Alt}^n(V)$ is given by

$$L^*(\varphi) = \det(L)\varphi. \tag{5.1}$$

Next we compute $\det(L)$ and get an expression which should look familiar to you from Linear Algebra:

Lemma 5.10 *Let $A = (a_{ij})$ be a (any) matrix representing the linear map $L : V \to V$. Then*

$$\det(L) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}.$$

That is, the determinant of L is the determinant of a (any) matrix A representing it.

Proof. Choose a basis $\{e_1, \ldots, e_n\}$ for V and let $A = (a_{ij})$ be the matrix representation of L with respect to this basis. That is, the j^{th} column of A is given by the coefficients a_{ij} in $L(e_j) = a_{1,j}e_1 + a_{2,j}e_2 + \ldots + a_{n,j}e_n$. Finally, let $\{e_1^*, \ldots, e_n^*\}$ be the dual basis to $\{e_1, \ldots, e_n\}$.

Now, let $\det(L)$ denote the determinant of L as in Definition 5.9, i.e. for any $\varphi \in \mathrm{Alt}^n(V)$ we have $L^*(\varphi) = (\det L)\varphi$. Let $\varphi = v_1^* \wedge \cdots \wedge e_n^*$. We have

$$L^*(e_1^* \wedge \cdots \wedge e_n^*)(e_1, \ldots, e_n) = (\det(L))(e_1^* \wedge \cdots \wedge e_n^*)(e_1, \ldots, e_n) = \det(L)$$

since

$$(e_1^* \wedge \cdots \wedge e_n^*)(e_1, \ldots, e_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) e_1^*(e_{\sigma(1)}) \cdots e_n^*(e_{\sigma(n)}) = 1$$

because each summand is 0 when $\sigma \neq \mathrm{id}$ and it is 1 when $\sigma = \mathrm{id}$.

On the other hand, using the definition of the pullback we get

$$\begin{aligned} L^*(e_1^* \wedge \cdots \wedge e_n^*)(e_1, \ldots, e_n) &= e_1^* \wedge \cdots \wedge e_n^*(L(e_1), \ldots, L(e_n)) \\ &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) e_1^*(L(e_{\sigma(1)})) \cdots e_n^*(L(e_{\sigma(n)})) \\ &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}. \end{aligned}$$

Putting the two expressions for $L^*(e_1^* \wedge \cdots \wedge e_n^*)(e_1, \ldots, e_n)$ together, we get that $\det(L) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$ as desired.

□

We end this subsection with an exercise:

Exercise 5.11 Let V be a k-dimensional vector space.

(a) Suppose $v_1 \dots, v_k \in V$ are linearly dependent. Prove that $\varphi(v_1, \dots, v_k) = 0$ for all $\varphi \in \mathrm{Alt}^k(V)$.

(b) Suppose $v_1^*, \dots, v_k^*$ are linearly dependent. Prove that $v_1^* \wedge \cdots \wedge v_k^* = 0$.

(c) Suppose $v_1^* \wedge \cdots \wedge v_k^* = 0$. Pr0ve that $v_1^*, \dots, v_k^*$ are linearly dependent.

5.2 Differential forms, wedge product, and pull-back

Let TM the tangent bundle of M. Fix k and for each $p \in M$ and consider the vector space $\mathrm{Alt}^k(T_pM)$ of alternating k-forms on the tangent space T_pM. We define $\mathrm{Alt}^k(TM)$ as the disjoint union:

$$\mathrm{Alt}^k(TM) := \bigsqcup_{p \in M} \mathrm{Alt}^k(T_pM)$$

The space $\mathrm{Alt}^k(TM)$ is in fact a smooth manifold, this can be proved in a similar way to how we proved the tangent bundle is a smooth manifold in Proposition 4.2. The smooth atlas will be (V_α, ψ_α) defined be $V_\alpha = \pi^{-1}(U_\alpha)$, $\psi_\alpha(p, \eta) = (\varphi_\alpha(p), a_1, \dots, a_{\binom{n}{k}})$ where $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas for M and $(a_1, \dots, a_{\binom{n}{k}})$ are the coordinates for $\eta \in \mathrm{Alt}^k(T_pM)$ when expressed in the basis for $\mathrm{Alt}^k(T_pM)$. According to the manifold construction lemma, there is a unique topology on $\mathrm{Alt}^k(TM)$ that makes this into a smooth manifold (with this atlas) and from now on we will always assume $\mathrm{Alt}^k(TM)$ is equipped with this topology. In particular, we will view it as a smooth manifold.

Exercise 5.12 Work out the details of the above to show that $\mathrm{Alt}^k(TM)$ is a smooth manifold.

In fact, it has more structure than this: Recall that we said that the tangent bundle is an example of the more general concept of a vector bundle—the same is true for $\mathrm{Alt}^k(TM)$, it is also a vector bundle. It comes with a *projection* $\pi : \mathrm{Alt}^k(TM) \to M$ given by $\varphi_p \mapsto p$ for all $\varphi_p \in \mathrm{Alt}^k(T_pM)$.

Remark 5.13 Let $E = \bigsqcup V_p$ be a vector bundle over M (as in Definition A.1 in the appendix), where each V_p is a vector space of dimension n. Then one can use the the Vector Bundle Construction Lemma (also in the appendix) to show that

$$\mathrm{Alt}^k(E) := \bigsqcup_{p \in M} \mathrm{Alt}^k(V_p)$$

is also a vector bundle over M. Hence, since TM is a vector bundle, so is $\mathrm{Alt}^k(TM)$. The proof of this fact is very similar to why the dual bundle is a vector bundle as in Example A.9), but we encourage you to prove it for yourself in Exercise **??**.

Remark 5.14 The notation $\Lambda^k(TM^*)$ is often used to denote what we call $\mathrm{Alt}^k(TM)$.

Definition 5.15 (Differential form) Let M be a smooth n-manifold. A *differential form of degree k*, or a *differential k-form*, is a smooth section ω of the bundle $\mathrm{Alt}^k(TM)$. That is, a smooth map

$$\omega : M \to \mathrm{Alt}^k(TM)$$

such that

$$\pi(\omega(x)) = x.$$

We denote the set of all differential k-forms on M by $\Omega^k(M)$ (which is an infinite dimensional vector space).

Remark 5.16 One can also define a differential form to be a *continuous* section and most of what we do with them here will work also for this case—in particular it is enough for integration. It will be pointed out if continuous is not enough.

Note that for each $x \in M$,

$$\omega_x := \omega(x) \in \mathrm{Alt}^k(T_xM),$$

i.e. ω_x is an alternating k-form on $T_xM \times \cdots \times T_xM$.

We define a 0-form to be a smooth function $f : M \to \mathbb{R}$, that is $\Omega^0(M) = C^\infty(M)$. Also, recall by Exercise 5.1 or Colorally 5.7 that the dimension of $\mathrm{Alt}^k(T_xM)$ is 0 if $k > n = \dim(M)$ and accordingly a differential k-form on an n-dimensional manifold is 0 whenever $k > n$ as we ask you to verify below. For this reason, differential n-forms on an n-manifold are often referred to as *forms of top degree*.

Exercise 5.17 Suppose M is a smooth manifold of dimension n and $k > n$. Then any differential k-form on M is identically 0.

Since a differential form ω is defined in terms of alternating forms (at each x, ω_x is an alternatingl form), we can extend notions from the world of alternating forms in the previous section to the world of differential forms; in particular wedge products and pullbacks. We start with the wedge product:

Definition 5.18 (Wedge product of differential forms) The *wedge product of two differential forms* $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$ is the differential $(k+l)$-form $\omega \wedge \eta \in \Omega^{k+l}(M)$ defined by $(\omega \wedge \eta)_x = \omega_x \wedge \eta_x$.

Next we define the pullback of a differential form by a smooth map. Let M and N be smooth manifolds and $f : M \to N$ a smooth map. Then, for each $x \in M$, we have the differential

$$df_x : T_xM \to T_{f(x)}N$$

which is a linear map, and hence induces the map

$$(df_x)^* : \mathrm{Alt}^k(T_{f(x)}N) \to \mathrm{Alt}^k(T_xM)$$

as defined in Definition 5.8. This in turn induces a map

$$f^* : \Omega^k(N) \to \Omega^k(M)$$

where $f^*\omega$ is defined by

$$(f^*\omega)_x = (df_x)^*\omega_{f(x)},$$

i.e. at each x it is the pullback of $\omega_{f(x)}$ by df_x. We define this to be the pullback of ω:

Definition 5.19 (Pullback of a differential form) Let $f : M \to N$ be a smooth map and $\omega \in \Omega^k(N)$ a differential k-form on N. The *pullback of* ω (by f) is the differential k-form $f^*\omega \in \Omega^k(M)$ on M defined by

$$(f^*\omega)_x = (df_x)^*\omega_{f(x)}$$

We note the pullback and wedge product commute in the following way:

$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta \tag{5.2}$$

which we ask you to verify:

Exercise 5.20 Verify equation 5.2 above.

Finally we will get an expression for any $\omega \in \Omega^k(U)$ for $U \subset \mathbb{R}^n$ open in terms of the bases of $\mathrm{Alt}^k(T_pU)$. For each $i = 1, \ldots, n$, recall that x^i is the the coordinate function (which is smooth)

$$x^i : U \to \mathbb{R} \quad \text{such that } (p_1, \ldots, p_n) \mapsto p_i.$$

For each $p \in U$ we get the differential

$$dx^i := (dx^i)_p : T_pU \to T_{p_i}\mathbb{R}.$$

Note that we can drop the subscript p since $(dx^i)_p = (dx^i)_q$ for all p, q. Hence (by identifying $T_pU = \mathbb{R}^n$ and $T_{p_i}\mathbb{R} = \mathbb{R}$ as usual) we can view $dx^i \in (\mathbb{R}^n)^* = \mathrm{Alt}^1(TU)$. It follows that we can (and will) also view each dx^i as a differential 1-form, that is $dx^i \in \Omega^1(U)$: it is a section

$$dx^i : U \to \mathrm{Alt}^1(U)$$

defined by

$$(dx^i)_p = dx^i$$

for all p where the right hand side should be interpreted as the linear map $dx^i : \mathbb{R}^n \to \mathbb{R}$.

Now, viewing $dx^i \in (\mathbb{R}^n)^*$ for each i and letting $(e_1, \ldots, e_n)$ denote the standard basis for $\mathbb{R}^n$, note that we have $dx^i(e_j) = 0$ for all $i \neq j$ and $dx^i(e_i) = 0$. Hence $\{dx^1, \ldots, dx^n\}$ is the standard dual basis for $(\mathbb{R}^n)^*$. We have shown:

Lemma 5.21 *The set* $\{dx^1, \ldots, dx^n\}$ *is the standard basis for* $(\mathbb{R}^n)^*$.

□

It follows by Lemma 5.5 that

$$\{dx^{i_1} \wedge \cdots \wedge dx^{i_k} \mid i_1 < i_2 < \cdots < i_k\}$$

is a basis for $\mathrm{Alt}^k(T_pU)$ for each p. Hence, for any $\omega \in \Omega^k(U)$, we have that

$$w_x = \sum_{i_1 < \cdots < i_k} c_{i_1, \ldots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for some $c_{i_1,\dots,i_k} \in \mathbb{R}$ and so

$$\omega = \sum_{i_1<\cdots<i_k} f_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for smooth functions $f_{i_1,\dots,i_k} : U \to \mathbb{R}$.

This is a very useful expression and we summarize it below for easier reference:

Proposition 5.22 *Let $\omega \in \Omega^k(U)$ be a differential k-form on an open $U \subset \mathbb{R}^n$. Then*

$$\omega = \sum_{i_1<\cdots<i_k} f_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for some smooth functions $f_{i_1,\dots,i_k} : U \to \mathbb{R}$. That is

$$\omega_x = \sum_{i_1<\cdots<i_k} f_{i_1,\dots,i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

In particular, any n-form on $\mathbb{R}^n$ is of the form

$$\omega = f dx^1 \wedge \cdots \wedge dx^n$$

for some smooth function $f : U \to \mathbb{R}$. That is,

$$\omega_x = f(x) dx^1 \wedge \cdots \wedge dx^n.$$

□

Remark 5.23 If we had defined differential forms as continuous sections (instead of smooth) we would have the same result as above with the only difference being that the functions $f_{i_1,\dots,i_k}$ would only be required to be continuous.

5.3 Integrating forms of top degree

*Throughout this and next section we will assume that M be a smooth **oriented** manifold of dimension n, that is, we assume the determinant of the Jacobian of all transition maps is positive—it will be required when we define integration on a manifold.*

Our goal is to define what it means to integrate a differential n-form ω over an n-dimensional manifold M, i.e. we want to make sense of the expression

$$\int_M \omega.$$

We note that we will only define this expression when the dimension of M and the degree of ω agree, and when M is oriented.

As usual we start with the Euclidean case, i.e. when $M = U$ is an open subset of $\mathbb{R}^n$. In fact, with Proposition 5.22 in mind, we will simply define the integral of an n-form $\omega \in \Omega^n(U)$ as follows:

Definition 5.24 (Integrating n-forms in $\mathbb{R}^n$) Let $U \subset \mathbb{R}^n$. Let $\omega \in \Omega^n(U)$ be a differential n-form and $X \subset U$. The integral of ω over X is given by

$$\int_X \omega = \int_X f dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = \int_X f dx^1 dx^2 \cdots dx^n$$

where $\omega = f dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$, $f : U \to \mathbb{R}$, and the right hand side is to be interpreted as a multiple integral.

We note that the order of the dx^i in the wedge product in the definition above is important, since if we change the order we might change the sign of the wedge (while not of the multiple integral). We only define the integral when the wedge product is in the order as above, hence to use the above to compute an integral, we first have to make sure the n-form is written in this exact way.

Remark 5.25 Note that the above definition would also make sense if differential forms were defined to be continuous (instead of smooth) sections, i.e. if f above was just continuous instead of smooth (in fact, it is enough to assume that f is measurable).

Example 5.26 Consider the n form $\omega = dx^1 \wedge \cdots \wedge dx^n$ on $\mathbb{R}^n$. This is called the *Euclidean volume form* since, for a region $X \subset \mathbb{R}^n$

$$\int_X \omega = \int_X dx^1 \wedge \cdots \wedge dx^n = \int_X dx^1 \cdots dx^n = Vol(X).$$

□

Now we consider the case when M is a general smooth n-manifold. Let $\omega \in \Omega^n(M)$. As usual, we will define what ω looks like locally (i.e. in charts, which brings us to the familiar setting of $\mathbb{R}^n$) and first define the integral there and then show that this definition is actually independent of the choice of chart.

Let (U, φ) be a chart of M. Then $\varphi : U \to \tilde{U}$ is a diffeomorphism, where $\tilde{U} = \varphi(U)$ is an open set in $\mathbb{R}^n$. In particular $\varphi^{-1} : \tilde{U} \to U$ is a smooth map and hence gives rise to the pullback map $\left(\varphi^{-1}\right)^* : \Omega^n(U) \to \Omega^n(\tilde{U})$. Let η be the pullback of ω by φ^{-1}, that is

$$\eta = (\varphi^{-1})^*(\omega)$$

which is an n-form on $\tilde{U}$. We define the integral of ω on M to be the integral of η on $\mathbb{R}^n$:

Now we are ready to define the integral of an n-form.

Definition 5.27 (Integrating n-forms on M) Let M be an oriented smooth n-manifold and $\omega \in \Omega^n(M)$. Let (U, φ) be any smooth chart and $X \subset U$. Then

$$\int_X \omega = \int_{\varphi(X)} (\varphi^{-1})^* \omega.$$

More generally, the integral over M of a compactly supported n-form ω (i.e when the support of ω is a compact set) is defined as

$$\int_M \omega = \sum_i \int_M \Psi_i \omega$$

where $\{\Psi_i\}$ is a partition of unity subordinate to a finite open cover $\{U_i\}$.

We will prove below that this definition is indeed well-defined, i.e. is independent of the choice of chart (and this is where the assumption on orientation will be important).

First we make an observation: Letting $\{\Psi_i\}$ be as in the definition and setting $\omega_i = \Psi_i \omega$ note that $\omega = \sum_i \omega_i$ and that $\int_M \omega_i = \int_{\text{supp}(\Psi_i)} \omega_i$ where $\text{supp}(\Psi_i) \subset U_i$. Hence, given a partition of unity, we can integrate inside each open set and then sum the corresponding terms. However, finding a partition of unity is often hard, and the above definition is much easier to apply if we can cover the domain we want to integrate over by a single chart. This can of course not always be done, but in several cases it can "almost" be done as in the following example:

Example 5.28 Consider the 1-manifold $\mathbb{S}^1 = \{(x, y) \,|\, x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Suppose we want to integrate a 1-form ω over $\mathbb{S}^1$. We cannot cover S^1 by a single chart but $\mathbb{S}^1 \setminus \{(0, 1)\}$ can be covered by $(U = \mathbb{S}^1 \setminus \{(0, 1)\}, \varphi)$, where $\varphi : U \to (0, 2\pi) \subset \mathbb{R}$ is defined by $\varphi((\cos t, \sin t)) = t$. Now, it's a fact from calculus that when integrating over a set, we can ignore a single point, so particular

$$\int_{\mathbb{S}^1} \omega = \int_{\mathbb{S}^1 \setminus \{(0,1)\}} \omega = \int_{\varphi(\mathbb{S}^1 \setminus \{(0,1)\})} (\varphi^{-1})^*(\omega) = \int_0^{2\pi} (\varphi^{-1})^*(\omega)$$

More generally, deleting a set of lower dimension does not effect integration. (Even more generally, for those who have studied measure theory: we can always delete a set of measure 0 from the domain we are integrating over without effecting the result.) We will return to this example in Example 5.36 below.

Now, we discuss why the definition is independent of the choice of chart. So let (U, ϕ) and ω be as in the definition and let (V, ψ) be another chart of M. Let η be the pullback of ω by ϕ^{-1} and η' be the pullback of ω by ψ^{-1}, i.e.

$$\eta = (\varphi^{-1})^*(\omega), \quad \eta' = (\psi^{-1})^*(\omega).$$

First we will compare η and η'.

Since η, η' are n-forms on (subsets of) $\mathbb{R}^n$, we have

$$\eta = f dx^1 \wedge \cdots \wedge dx^n$$

and

$$\eta' = g dx^1 \wedge \cdots \wedge dx^n$$

for some smooth $f : \tilde{U} \to \mathbb{R}$ and $g : \tilde{V} \to \mathbb{R}$. Now, the transition map $\psi \circ \varphi^{-1} : \tilde{U} \cap \tilde{V} \to \tilde{U} \cap \tilde{V}$, which is smooth (in fact a diffeomorphism), induces the map

$$(\psi \circ \varphi^{-1})^* : \Omega^n(\tilde{U} \cap \tilde{V}) \to \Omega^n(\tilde{U} \cap \tilde{V})$$

which maps η' to η (interpreted as their restrictions to $U \cap V \subset M$). That is η is the pullback of η' by $\psi \circ \varphi^{-1}$:

$$(\psi \circ \varphi^{-1})^* \eta' = \eta.$$

We have for each $x \in \tilde{U}$,

$$\begin{aligned}
\eta_x &= ((\psi \circ \varphi^{-1})^* \eta')_x = \left((\psi \circ \varphi^{-1})^* (g dx^1 \wedge \cdots \wedge dx^n)\right)_x \\
&= \left(d(\psi \circ \varphi^{-1})_x\right)^* (g dx^1 \wedge \cdots \wedge dx^n)_{\psi \circ \varphi^{-1}(x)} \\
&= \left(d(\psi \circ \varphi^{-1})_x\right)^* (g(\psi \circ \varphi^{-1}(x)) dx^1 \wedge \cdots \wedge dx^n) \\
&= g(\psi \circ \varphi^{-1}(x)) \cdot \left(d(\psi \circ \varphi^{-1})_x\right)^* (dx^1 \wedge \cdots \wedge dx^n) \\
&= g(\psi \circ^{-1} (x)) \cdot \det \left(d(\psi \circ \varphi^{-1})_x\right) (dx^1 \wedge \cdots \wedge dx^n)
\end{aligned}$$

where the first four equalities follow by definition, the fifth by linearity, and the last by (5.1). We summarize the above discussion:

Lemma 5.29 *Let $\tilde{U}, \tilde{V} \subset \mathbb{R}^n$ and $\phi : \tilde{U} \to \tilde{V}$ a smooth map. If $\eta = f dx^1 \wedge \cdots \wedge dx^n$ and $\eta' = g dx^1 \wedge \cdots \wedge dx^n$ are two n-forms on $\tilde{U}$ and $\tilde{V}$, respectively, such that η is the pullback of η' under ϕ, then*

$$f(x) = g(\phi(x)) \cdot \det(d\phi_x).$$

□

Lemma 5.30 *Definition 5.27 above is well-defined, i.e. is independent of the choice of chart.*

Proof. Let $X \subset M$ and let (U, φ) and (V, ψ) be two charts of M both containing X. Say $(\varphi^{-1})^* \omega = f\, dx^1 \wedge \cdots \wedge dx^n$ where $f : \varphi(U) \to \mathbb{R}$ and $(\psi^{-1})^* \omega = g\, dx^1 \wedge \cdots \wedge dx^n$ where $g : \psi(V) \to \mathbb{R}$. Then

$$\begin{aligned}
\int_{\varphi(X)} (\varphi^{-1})^* \omega &= \int_{\varphi(X)} f dx^1 \wedge \cdots \wedge dx^n \quad \text{by definition} \\
&= \int_{\varphi(X)} f dx^1 \cdots dx^n \quad \text{by definition} \\
&= \int_{\varphi(X)} g(\psi \circ \varphi^{-1}) \cdot \det(d(\psi \circ \varphi^{-1})_x) dx^1 \cdots dx^n \quad \text{by Lemma 5.29} \\
&= \int_{(\psi \circ \varphi^{-1}) \varphi(X)} g dx^1 \dots dx^n \quad \text{by change of variables/substitution for multiple integrals} \\
&= \int_{\psi(X)} g dx^1 \dots dx^n \quad \text{by definition} \\
&= \int_{\psi(X)} g dx^1 \wedge \dots \wedge dx^n \quad \text{by definition} \\
&= \int_{\psi(X)} (\psi^{-1})^* \omega \quad \text{by definition}
\end{aligned}$$

where we used the assumption that M is oriented when applying the change of variable formula, that is, we used that $|\det(d(\psi \circ \varphi^{-1})_x)| = \det(d(\psi \circ \varphi^{-1})_x)$.

To conclude, to compute

$$\int_X \omega$$

we get the same result whether we use the chart (U, φ) or the chart (V, ψ) in Definition 5.27. Hence the definition is well-defined.

□

5.4 Exterior derivative

In this section we will define a map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying certain properties, mapping a differential k-form ω to a differential $(k+1)$-form $d\omega$ which we call the *exterior derivative* of ω.

As usual we start with the base case when $M = U \subset \mathbb{R}^n$ an open subset of $\mathbb{R}^n$. First consider the case when $k = 0$. Let $f \in \Omega^0(U) = C^\infty(U)$, i.e. $f : U \to \mathbb{R}$ is a smooth function. We define

$$d : \Omega^0(U) \to \Omega^1(U),$$

by setting df to be the 1-form $df \in \Omega^1(U)$ defined by

$$(df)_x = df_x.$$

That is, at each x, $(df)_x$ is the differential of f at x. Recall that this is the linear map $df_x : T_xU \to T_{f(x)}\mathbb{R}$. To see that df defines this way does indeed define a differential 1-form, note that $T_{f(x)}\mathbb{R}$ is isomorphic to $\mathbb{R}$, so for each x we can view df_x as a linear map $T_xU \to \mathbb{R}$, that is, for each x we have $df_x \in (T_xU)^* = \mathrm{Alt}^1(T_xU)$. Hence

$$df : U \to \mathrm{Alt}^1(TU) = (TU)^*$$

where

$$df_x : T_xU \to T_{f(x)}\mathbb{R} = \mathrm{Alt}^1(TU),$$

does indeed define a section of $\mathrm{Alt}^1(TU)$ so is a differential 1-form. We call df the *differential of* f:

Definition 5.31 (Differential) Let $U \subset \mathbb{R}^n$ be any open subset. Given a smooth map $f : U \to \mathbb{R}$, the *differential of* f is the differential 1-form $df : U \to \mathrm{Alt}^1(TU)$ defined by $(df)_x = df_x$.

Note that this agrees with the notion of dx^i in the previous section: if $x^i : \mathbb{R}^n \to \mathbb{R}$ is the i^{th} coordinate function then dx^i is the differential of x^i, interpreted as the 1-form with $(dx^i)_p = dx^i \in (\mathbb{R}^n)^*$ for all p.

Remark 5.32 Contrary to the case for integration, when defining the exterior derivative it is not enough to use differential form that are just continuous sections, we do need some smoothness here. However, it is enough that they are C^1, i.e. once differentiable. In particular, the differential df of a function f makes sense as long as $f \in C^1(U)$.

Recall that at each x the differential of f at x is the Jacobian of f at x, which in this case is the row vector

$$df_x = \left(\left.\frac{\partial f}{\partial x^1}\right|_x, \dots, \left.\frac{\partial f}{\partial x^n}\right|_x \right).$$

Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and $\{dx^1, \dots, dx^n\}$ the dual basis as in the previous section (see Lemma 5.21).

Lemma 5.33 *Let $U \subset \mathbb{R}^n$ be any open subset and $f : U \to \mathbb{R}$ a smooth map. Then*

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i. \tag{5.3}$$

That is, for all $p \in U$ we have

$$df_p = \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x^i}\right|_p dx^i.$$

Proof. To see this note that, for all p and for $j \in \{1, \dots, n\}$ we have

$$(df)_p(e_j) = df_p(e_j) = \left(\left.\frac{\partial f}{\partial x^1}\right|_p, \dots, \left.\frac{\partial f}{\partial x^n}\right|_p \right)(e_j) = \left.\frac{\partial f}{\partial x^j}\right|_p$$

and

$$\sum_{i=1}^{n} \left.\frac{\partial f}{\partial x^i}\right|_p dx^i(e_j) = \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x^i}\right|_p \left(dx^i(e_j)\right) = \left.\frac{\partial f}{\partial x^j}\right|_p$$

Since the two linear maps agree on the basis elements they are the same map, showing (5.3). □

Note that with this notation, we get a useful formula for how to compute the pullback of a differential form (recall Definition 5.19):

Proposition 5.34 (Pullback of differential forms) *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. Let $F : U \to V$ be a smooth map and ω a k-form on V. We use the coordinates $(y^1, \dots, y^m)$ for points in V and write $F = (F_1, \dots, F_m)$. By Proposition 5.22 we can write*

$$w = \sum_{i_1 < \dots < i_k} f_{i_1,\dots,i_k} dy^{i_1} \wedge \dots \wedge dy^{i_k}$$

for some $f_{i_1,\dots,i_k} : V \to \mathbb{R}$, where $\{dy^1, \dots, dy^m\}$ is the dual basis for $\mathbb{R}^m$. Then the pullback of ω (by F) is the k-form on U given by

$$F^*\omega = \sum (f_{i_1,\dots,i_k} \circ F)\, dF_{i_1} \wedge \dots \wedge dF_{i_k}.$$

That is

$$(F^*\omega)_x = \sum f_{i_1,\dots,i_k}(F(x)) d(F_{i_1})_x \wedge \dots \wedge d(F_{i_k})_x$$

for all $x \in U$.

Proof. Let $\omega \in \Omega^k(V)$,

$$w = \sum_{i_1 < \dots < i_k} f_{i_1,\dots,i_k} dy^{i_1} \wedge \dots \wedge dy^{i_k}$$

be as in the statement. Recall that by Definition 5.19 $F^*\omega$ is the k-form on U given by

$$F^*\omega : U \to \mathrm{Alt}^k(TU) \text{ such that } x \mapsto (F^*\omega)_x$$

where $(F^*\omega)_x = (dF_x)^*\omega_{F(x)}$. Now,

$$\omega_{F(x)} = \sum_{i_1<\cdots<i_k} f_{i_1,\ldots,i_k}(F(x))dy^{i_1} \wedge \cdots \wedge dy^{i_k},$$

so

$$\begin{aligned}(dF_x)^*\omega_{F(x)} &= (dF_x)^*\left(\sum_{i_1<\cdots<i_k} f_{i_1,\ldots,i_k}(F(x))dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right)\\ &= \sum_{i_1<\cdots<i_k} f_{i_1,\ldots,i_k}(F(x))(dF_x)^*\left(dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right)\end{aligned}$$

where the last equality follows by linearity of $(dF_x)^*$. Now, since the pullback and wedge products commute by (5.2) we have

$$(dF_x)^*\left(dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) = (dF_x)^*(dy^{i_1}) \wedge \cdots \wedge (dF_x)^*(dy^{i_k}).$$

Moreover, $(dF_x)^*(dy^j)$ is given by

$$(dF_x)^*(dy^j)(v) = dy^j(dF_x(v)) = d(y^j \circ F)_x(v)$$

for each $v \in T_xV$. Here the first equality follows by the definition of pullback (Definition 5.8) and the second by the chain rule for the differential at a point (Lemma 2.8). Hence we have that

$$(dF_x)^*(dy^j) = d(y^j \circ F)_x.$$

Recall that $y^j : \mathbb{R}^m \to \mathbb{R}$ is the coordinate map that gives the j^{th} coordinate. Hence $y^j \circ F = F_j$. Putting all this together we get

$$(F^*\omega)_x = \sum_{i_1<\cdots<i_k} f_{i_1,\ldots,i_k}(F(x))\, d(F_{i_1})_x \wedge \cdots \wedge d(F_{i_k})_x$$

as desired.

□

Note that in particular, if $U, V \subset \mathbb{R}^n$ is open and $F : U \to V$ and ω is an n-form then

$$\omega = f dy^1 \wedge \cdots \wedge dy^n$$

and the pullback is

$$F^*\omega = (f \circ F)\, dF_1 \wedge \cdots \wedge dF_n.$$

Here we have used coordinates $(y^1, \ldots y^n)$ for V. Using coordinates $(x^1, \ldots, x^n)$ for U, a computation shows that this can be written as

$$F^*\omega = (f \circ F)(\det(DF))\, dx^1 \wedge \cdots \wedge dx^n$$

where DF denotes the Jacobian of F, which we leave as an exercise to verify:

Exercise 5.35 Let $F : U \to V$ be a smooth map and let ω be the n-form

$$\omega = f dy^1 \wedge \cdots dy^n.$$

With notation as above, prove that

$$F^*\omega = (f \circ F)(\det(DF))\, dx^1 \wedge \cdots \wedge dx^n.$$

Knowing how to find the pullback we can now integrate n-forms:

Example 5.36 Let $\mathbb{S}^1 = \{(x, y) \,|\, x^2 + y^2 = 1\} \subset \mathbb{R}^2$ and consider the 1-form $\omega = x\, dy - y\, dx$. Let us compute

$$\int_{\mathbb{S}^1} \omega.$$

By Example 5.28 we know we need to find

$$\int_0^{2\pi} (\varphi^{-1})^*(\omega)$$

where $\varphi : U \to (0, 2\pi) \subset \mathbb{R}$ such that $\varphi(\cos t, \sin t) = t$, i.e. $\varphi^{-1}(t) = (\cos t, \sin t)$. We compute the pullback:

$$\begin{aligned}(\varphi^{-1})^*(\omega) =& (\varphi^{-1})^*(x\, dy - y\, dx) = \cos t\, d(\sin t) - \sin t\, d(\cos t)\\ =& \cos t\, \cos t\, dt - \sin t\, (-\sin t)\, dt = \cos^2 t\, dt + \sin^2 t\, dt\\ =& dt.\end{aligned}$$

Hence

$$\int_0^{2\pi} (\varphi^{-1})^*(\omega) = \int_0^{2\pi} dt = 2\pi.$$

That is

$$\int_{\mathbb{S}^1} \omega = 2\pi.$$

In fact, $\omega = x\, dy - y\, dx$ is the volume form on $\mathbb{S}^1$.

Next we define $d\omega$ for $\omega \in \Omega^k(U)$ for open $U \subset \mathbb{R}^n$:

Definition 5.37 (Exterior derivative in $\mathbb{R}^n$) Let $U \subset \mathbb{R}^n$ be open and $\omega \in \Omega^k(U)$ be the differential k-form

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1, \cdots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

The *exterior derivative* of ω is $dw \in \Omega^{k+1}(U)$ defined by

$$d\omega = \sum_{i_1 < \cdots < i_k} df_{i_1, \cdots, i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{5.4}$$

Note that $d\omega$ is indeed a $k + 1$ form since each $df_{i_1, \cdots, i_k}$ is a 1-form. The exterior derivative satisfies the following properties:

Proposition 5.38 (Properties of Exterior Derivative on $\mathbb{R}^n$) *The operation d satisfies:*

(a) d is linear over $\mathbb{R}$. That is, $d(a\omega + b\eta) = ad\omega + bd\eta$ for $a, b \in \mathbb{R}$.

(b) If ω is a k-form and η is an l-form, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) $d \circ d = 0$

(d) d commutes with pullbacks, i.e. if $F : U \to V$ is smooth, where $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ are open sets, and ω a k-form, then

$$F^*(d\omega) = d(F^*\omega).$$

Proof. We note that linearity is clear from the definition (5.4).

We prove part (b): Suppose ω is a k-form and η an l form. We can write each one as a linear combination of basis elements, but by linearity, we can assume each sum only have one term. Say $\omega = f\,dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $\eta = g\,dx^{j_1} \wedge \cdots \wedge dx^{j_l}$, where the indices are strictly increasing in each. Suppose $dx^S := d_{s_1} \wedge \cdots \wedge d_{s_p}$ is any wedge product where the index set S is not necessarily increasing. If it has repeating indices then $dx^S = 0$. Suppose it has no repeating indices. Let σ be a permutation such that $\sigma(S)$ is strictly increasing. Then, for any smooth $h : M \to \mathbb{R}$ we have

$$d(hdx^S) = (\mathrm{sgn}(\sigma))d(hdx^{\sigma(S)}) = (\mathrm{sgn}(\sigma))dg \wedge dx^{\sigma(S)} = dg \wedge dx^S.$$

Hence $d(hdx^S) = dh \wedge dx^S$ for any index set S (not only for strictly increasing, where it is true by definition). We get:

$$\begin{aligned}
d(\omega \wedge \eta) &= d((f\,dx^{i_1} \wedge \cdots \wedge dx^{i_k})(g\,dx^{j_1} \wedge \cdots \wedge dx^{j_l})) \\
&= d(fg\,dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}) \\
&= (f\,dg + g\,df) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}) \\
&= du \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge v\,dx^{j_1} \wedge \cdots \wedge dx^{j_l} + dv \wedge u\,dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\
&= \big(du \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\big) \wedge v\,dx^{j_1} \wedge \cdots \wedge dx^{j_l} + (-1)^k (u\,dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \wedge \big(dv \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}\big) \\
&= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta
\end{aligned}$$

We leave part (c) as an exercise to verify (to do so, start with proving that it is true for 0-forms, using the definition of its differential above, then use that result together with part (b) to prove the general case).

For part (d), write ω as a linear combination of the basis elements, and as above, by linearity, it is enough to prove the claim when there is only one term. Accordingly, assume $\omega = f\,dx^{i_1} \wedge \ldots \wedge dx^{i_k}$.

First note that for any smooth function $h : \mathbb{R}^n \to \mathbb{R}$ we have that F^*dh is defined by $(F^*dh)_x(v) = (dF_x)^* dh_{F(x)}(v) = dh_{F(x)}(dF_x(v)) = d(h \circ F)_x(v)$, i.e $F^*dh = d(h \circ F)$. Now,

$$d\omega = d(f\,dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

so

$$\begin{aligned}
F^*(d\omega) &= F^*(df) \wedge F^*(dx^{i_1}) \wedge \cdots \wedge F^*(dx^{i_k}) \\
&= d(f \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F) \\
&= d(f \circ F) \wedge dF_{i_1} \wedge \cdots \wedge dF_{i_k}.
\end{aligned}$$

Lastly,

$$F^*\omega = (f \circ F)\, dF_{i_1} \wedge \cdots \wedge dF_{i_k}$$

so

$$d(F^*\omega) = d(f \circ F) \wedge dF_{i_1} \wedge \cdots \wedge dF_{i_k}.$$

Hence $F^*(d\omega) = d(F^*\omega)$ as desired, proving part (d).

□

Next we will define the exterior derivative a differential k-form on a general smooth manifold M, which satisfies the same properties as above.

Definition 5.39 (Exterior derivative) Let M be a smooth n manifold and $\omega \in \Omega^k(M)$ be a differential k-form on M. Then the *exterior derivative* of ω is the $(k+1)$-form $d\omega \in \Omega^{k+1}(M)$ defined by

$$(d\omega)_x = (\varphi^* d((\varphi^{-1})^*\omega))_x$$

for any smooth chart (U, φ) of M at p.

As always when we define something in terms of charts, we need to make sure it is well-defined. So suppose ω and (U, φ) is as in the statement and (V, ψ) is another chart at x. We need to show that $\varphi^* d((\varphi^{-1})^*\omega) = \psi^* d((\varphi^{-1})^*\omega)$ on $U \cap V$.

Since $\varphi \circ \psi^{-1}$ is a diffeomorphism of open sets in Euclidean spaces and since the pull-back and exterior derivatives commute in that setting by part (d) in Proposition 5.38, we have that

$$(\varphi \circ \psi^{-1})^* d((\varphi^{-1})^*\omega) = d\big((\varphi \circ \psi^{-1})^*(\varphi^{-1})^*\omega\big).$$

Note that $(\varphi \circ \psi^{-1})^* = (\psi^{-1})^* \circ \varphi^*$. Hence we have

$$((\psi^{-1})^* \circ \varphi^*) d((\varphi^{-1})^*\omega) = d((\psi^{-1})^* \circ \varphi^* \circ (\varphi^{-1})^*\omega)$$

that is,

$$(\psi^{-1})^*\big((\varphi^*)(d((\varphi^{-1})^*\omega))\big) = d((\psi^{-1})^*\omega)$$

and so

$$\varphi^* d((\varphi^{-1})^*\omega) = \psi^* d((\psi^{-1})^*\omega)$$

as we needed to show.

Since the exterior derivative of forms on M is defined locally in terms of forms on Euclidean space, it is not hard to check that it satisfies the same properties:

Theorem 5.40 (Properties of Exterior Derivative) *Let M be a smooth manifold. For each k the exterior differentiation operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying the following:*

(a) d is linear over $\mathbb{R}$.

(b) If ω is a k-form and η is an l-form, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) $d \circ d = 0$

(d) For any smooth $F : M \to N$ *we have* $F^* d\omega = d(F^*(\omega))$

(e) For $f \in \Omega^0(M) = C^\infty(M)$ df *is the* 1*-form satisfying* $(df)_x = df_x$*.*

Proof. All properties follow by the definition of ω together with Proposition 5.38. We only prove (d), that pullback and d commute, and leave the rest as an exercise.

Suppose M is an n-dimensional manifold and N is k-dimensional. Let (U, φ) be a smooth chart of M and (V, ψ) a smooth chart of N. Consider $\psi \circ F \circ \varphi^{-1} : \tilde{U} \to \tilde{V}$, where $\tilde{U} = \varphi(U) \in \mathbb{R}^n$ and $\tilde{V} = \psi(V) \in \mathbb{R}^k$. By Proposition 5.38 part (d) we know that

$$(\psi \circ F \circ \varphi^{-1})^*(d\eta) = d((\psi \circ F \circ \varphi^{-1})^* \eta) \tag{5.5}$$

for all $\eta \in \Omega^k(\tilde{V})$. We will apply this below to $\eta = (\psi^{-1})^* \omega$. Also, by definition of d we have that

$$d\omega = \psi^* d((\psi^{-1})^* \omega) \tag{5.6}$$

and that

$$d(F^*\omega) = \varphi^* d((\varphi^{-1})^* F^* \omega). \tag{5.7}$$

Putting all this together we get that

$$\begin{aligned}
F^*(d\omega) =& F^*(\psi^* d((\psi^{-1})^*\omega)) \quad \text{by (5.6)} \\
=& \varphi^* \circ (\varphi^{-1})^* \circ F^* \circ \psi^* d((\psi^{-1})^* \omega) \\
=& \varphi^* \circ (\psi \circ F \circ \varphi^{-1})^* d((\psi^{-1})^* \omega) \\
=& \varphi^* d((\psi \circ F \circ \varphi^{-1})^*((\psi^{-1})^* \omega) \quad \text{by (5.5)} \\
=& \varphi^* d((\varphi^{-1})^* \circ F^* \circ \psi^* \circ (\psi^{-1})^* \omega) \\
=& \varphi^* d((\varphi^{-1})^* \circ F^* \omega) \\
=& d(F^* \omega) \quad \text{by (5.7)}
\end{aligned}$$

as desired.

□

We end by defining some commonly used terms:

Definition 5.41 (Closed and exact forms) Let ω be a differential form. We say that ω is *closed* if $d\omega = 0$. We say that ω is *exact* if there exists another form η such that $\omega = d\eta$.

We note that every differential n-form on an n-manifold is closed (since any $(n+1)$ form must be 0), and that every exact form is closed (since $d \circ d = 0$).

5.5 Stokes' theorem

In this section we will put together what we have learned in the previous sections of this chapter. Recall that we know how to integrate an n-form on an oriented smooth n-manifold, and that if ω is an $(n-1)$-form, then $d\omega$ is an n-form. Moreover, we will now allow our manifold M to have boundary, which we denote by ∂M. Recall that a manifold with boundary is a topological space (which is Hausdorff, second countable, and paracompact) which is locally homeomorphic to open sets in an n-dimensional closed half-space of $\mathbb{R}^n$. If M is n-dimensional then ∂M is $(n-1)$-dimensional, and is in fact a (possibly disconnected) manifold in its own right equipped with the subspace topology, which inherits the Hausdorff and paracompact properties from M, and is locally homeomorphic to the boundary of the half-space, which is a copy of $\mathbb{R}^{n-1}$.

There is a subtlety we need to take care of: recall that we have only defined integration on *oriented* manifolds and the goal of this section, Stokes' Theorem, involves integration over the boundary of a manifold. Hence we need to define an orientation on ∂M induced by the orientation on M. We do this as follows: Given an orientation on the n-dimensional manifold M which we call the positive orientation, then the induced orientation on the $(n-1)$-manifold ∂M is the orientation such that combined with the *inward pointing* normal vector gives the positive orientation on M. For example, if M is a cylinder, the two boundary curves are oriented in opposite direction of each other.

We can now state Stokes' Theorem:

Theorem 5.42 (Stokes' Theorem) *Let M be an oriented smooth n-manifold. Let $\omega \in \Omega^{n-1}(M)$ be compactly supported. Then*

$$\int_{\partial M} \omega = \int_M d\omega$$

We will prove this theorem below, but first we give some consequences.

Note that

$$\int_{\emptyset} \omega = 0$$

for any ω and hence we have:

Lemma 5.43 *Suppose M is a smooth n-manifold* without boundary *and ω any $(n-1)$-form on M. Then*

$$\int_M d\omega = 0.$$

Another direct consequence is the following:

Lemma 5.44 *Suppose η is an* exact *n-form on a smooth n-manifold M* without boundary*. Then $\int_M \omega = 0$.*

Proof. Since $\omega \in \Omega^n(M)$ is exact, there exists $\eta \in \Omega^{n-1}(M)$ such that $\omega = d\eta$. Hence

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0.$$

□

Also, note that in the particular case that $M = \mathbb{R}$ then Stoke's Theorem is nothing other than the Fundamental Theorem of Calculus: Let $f \in \Omega^0(\mathbb{R}) = C^\infty(\mathbb{R})$ be compactly supported, say on $[a,b]$, then

$$\int_a^b \frac{\partial f}{\partial x} dx = \int_{[a,b]} \frac{\partial f}{\partial x} dx = \int_{[a,b]} df = \int_{\partial[a,b]} f = \int_{\{a,b\}} f = f(b) - f(a)$$

where the minus side on the right hand side comes from the different orientations on the two endpoints and the second equality follows from (5.3).

In fact Stokes' Theorem recovers several other theorems as special cases, including Green's Theorem:

Theorem 5.45 (Green's Theorem) *Let D be a compact domain in $\mathbb{R}^2$ bounded by a piecewise smooth simple closed curve. Let P, Q be smooth real-valued functions on D. Then*

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_{\partial D} P\, dx + Q\, dy.$$

Exercise 5.46 Derive Green's Theorem from Stokes' Theorem.

Now we prove Stokes' Theorem:

Proof. (Proof of Stokes' Theorem) Since ω is compactly supported, for ease of notation we will suppose M itself is compact (else, replace M with the compact subset which is the support of ω). Let $\mathcal{A}$ be a smooth atlas and take a finite subcover, whose corresponding charts are $(U_1, \varphi_1), \ldots, (U_k, \varphi_k)$. Let $\{\Psi_i\}$ be a partition of unity subordinate to this cover. Then $\omega = \sum \Psi_i \omega := \sum \omega_i$ where we have defined $\omega_i = \Psi_i \omega$. Since both d and integration are linear operations, it is hence enough to prove the theorem for each ω_i, or equivalently, in the case when ω is supported in a single chart.

So let (U, φ) be a smooth chart and suppose $\mathrm{supp}(\omega) \subset U$. Let $\eta = (\varphi^{-1})^* \omega$ which is an $(n-1)$-form on $\varphi(U)$. We have $\omega = \varphi^* \eta$. M is locally homeomorphic to a half space in $\mathbb{R}^n$, we will assume that this half-space is $(-\infty, 0] \times \mathbb{R}^{n-1}$. That is $\varphi(U) \subset (-\infty, 0] \times \mathbb{R}^{n-1}$ and we can view η as a form on $(-\infty, 0] \times \mathbb{R}^{n-1}$ which is supported on $\varphi(U)$.

We have

$$\int_M d\omega = \int_M d(\varphi^* \eta) = \int_M \varphi^* d\eta = \int_U \varphi^* d\eta = \int_{\varphi(U)} d\eta = \int_{(-\infty,0] \times \mathbb{R}^{n-1}} d\eta$$

where the second equality follows because the pullback and exterior derivative commute, the third and last because of where the form is supported, and the fourth from the definition of integrating forms.

Also,

$$\int_{\partial M} \omega = \int_{\partial M} \varphi^* \eta = \int_{\partial M \cap U} \varphi^* \eta = \int_{\varphi(\partial M \cap U)} \eta = \int_{\{0\} \times \mathbb{R}^{n-1}} \eta.$$

Hence, we need to show that

$$\int_{(-\infty,0] \times \mathbb{R}^{n-1}} d\eta = \int_{\{0\} \times \mathbb{R}^{n-1}} \eta.$$

Since η is an $(n-1)$-form on $(-\infty, 0] \times \mathbb{R}^{n-1}$ we have that

$$\eta = \sum f_i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

where $\widehat{x^i}$ denotes that x^i is omitted, and each $f_i : (-\infty, 0] \times \mathbb{R}^{n-1} \to \mathbb{R}$ is a smooth function (supported on $\varphi(U)$). Denote

$$\eta_i = f_i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

so that $\eta = \sum \eta_i$. Then $d\eta = \sum d\eta_i$ and $\int d\eta = \sum \int d\eta_i$.

Also, since η is compactly supported, so is each η_i and hence each f_i. In particular, there exists an $R > 0$ such that the support of all f_i are contained in $[-R, 0] \times [-R, R]^{n-1}$. In particular $f_1(-R, x^2, \ldots, x^n) = 0$ and $f_i(x^1, \ldots, x^{i-1}, \pm R, x^{i+1}, \ldots, x^n) = 0$ for all $i = 2, \ldots n$. Hence, using the fundamental theorem of calculus, we have that

$$\int_{-\infty}^{0} \frac{\partial f_1}{\partial x^1} dx^1 = \int_{-R}^{0} \frac{\partial f_1}{\partial x^1} dx^1 = f_1(0, x^2, x^3, \ldots, x^n)$$

and

$$\int_{\mathbb{R}} \frac{\partial f_i}{\partial x^i} dx^i = \int_{-R}^{R} \frac{\partial f_i}{\partial x^i} dx^i = 0$$

for all $i = 2, \ldots, n$.

First consider η_1. We have $\eta_1 = f_1\, dx^2 \wedge \cdots \wedge dx^n$ and so

$$d\eta_1 = df_1 \wedge dx^2 \wedge \cdots \wedge dx^n = \left(\sum \frac{\partial f_1}{\partial x^j} dx^j \right) \wedge dx^2 \wedge \cdots \wedge dx^n = \frac{\partial f_1}{\partial x^1} dx^1 \wedge \cdots \wedge dx^n$$

where we have used the fact that $dx^j \wedge dx^j = 0$. Hence we have that

$$\begin{aligned} \int_{(-\infty,0] \times \mathbb{R}^{n-1}} d\eta_1 &= \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{0} \frac{\partial f_1}{\partial x^1} dx^1 \right) dx^2 \cdots dx^n \\ &= \int_{\mathbb{R}^{n-1}} f_1(0, x^2, \ldots, x^n) dx^2 \cdots dx^n \\ &= \int_{\{0\} \times \mathbb{R}^{n-1}} f_1 dx^1 \wedge \cdots \wedge dx^n \\ &= \int_{\{0\} \times \mathbb{R}^{n-1}} \eta_1. \end{aligned}$$

Now consider η_i for $i \neq 1$. Then $\eta_i = f_i dx^1 \wedge \cdots \widehat{dx^i} \wedge \cdots \wedge dx^n$ and hence note that

$$\int_{\{0\} \times \mathbb{R}^{n-1}} \eta_i = \int_{\{0\} \times \mathbb{R}^{n-1}} f_i\, dx^1 \ldots \widehat{dx^i} \cdots dx^n = 0$$

since $x^1 \equiv 0$ on $\{0\} \times \mathbb{R}^{n-1}$. Also

$$\begin{aligned} d\eta_i &= \left(\sum \frac{\partial f_i}{\partial x^j} dx^j \right) \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \frac{\partial f_i}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge \cdots \wedge dx^n \\ &= \pm \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Hence we have

$$\begin{aligned} \int_{(-\infty,0] \times \mathbb{R}^{n-1}} d\eta_i &= \int_{(-\infty,0] \times \mathbb{R}^{n-1}} \pm \frac{\partial f_i}{\partial x^i} dx^1 \cdots dx^n \\ &= \int_{(-\infty,0] \times \mathbb{R}^{n-2}} \left(\int_{-R}^{R} \pm \frac{\partial f_i}{\partial x^i} dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= 0 \end{aligned}$$

where we have used Fubini's Theorem to change the order of integration.

Hence

$$\int_{(-\infty,0]\times\mathbb{R}^{n-1}} d\eta_i = \int_{\{0\}\times\mathbb{R}^{n-1}} \eta_i$$

for all i and we are done.

$\square$

We end with some applications.

Example 5.47 Let $\mathbb{S}^1 = \{(s,t)\,|\,s^2+t^2=1\}$ be the unit circle and $\omega = x\,dy - y\,dx$. Let us compute

$$\int_{\mathbb{S}^1} \omega$$

using Stokes' Theorem. (Note that we already did this, using a different method, in Example 5.36). Since $\mathbb{S}^1 = \partial B^2$ where B^2 is the 2-dimensional unit ball (i.e. the unit disk) we have by Stokes' Theorem that

$$\int_{\mathbb{S}^1} \omega = \int_{B^2} d\omega.$$

Now, $d\omega = dx\wedge dy - dy\wedge dx = dx\wedge dy + dx\wedge dy = 2dx\wedge dy$ (and note that this is twice the volume form on $\mathbb{R}^2$) and so

$$\int_{\mathbb{S}^1} \omega = \int_{B^2} 2dx\wedge dy = \int_{B^2} 2dxdy = 2\int_{B^2} dxdy = 2Vol(B^2) = 2\pi.$$

Exercise 5.48 Let $\mathbb{S}^{n-1}\subset\mathbb{R}^n$. Follow the above example to compute

$$\int_{\mathbb{S}^{n-1}} \omega$$

where ω is the following $(n-1)$-form on $\mathbb{S}^{n-1}$

$$\omega = \sum_{i=1}^n (-1)^{i+1}x^i\,dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots dx^n.$$

Theorem 5.49 (Brouwer's Fixed Point Theorem) *Let B^n be a closed n-dimensional ball in $\mathbb{R}^n$. If $f: B^n\to B^n$ is continuous, then f has a fixed point.*

Proof. We prove the theorem under the stronger assumption that f is smooth. In order to get a contradiction, suppose that $f(x)\neq x$ for all $x\in B^n$. We define $F: B^n\to\partial B^n$ as follows: for each $x\in B^n$ consider the straight line segment starting at $f(x)$ and going through x; let $F(x)$ be the point where this line intersects the boundary of the ball. This is a smooth function. Note that $F(x)=x$ for all $x\in\partial B^n$, i.e. F is the identity function on ∂B^n. Let ω be the volume form on $\mathbb{S}^{n-1}=\partial B^n$ (in fact $\omega = \sum_{i=1}^n(-1)^{i-1}x^idx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n$). Then $\int_{\partial B^n}\omega = Vol(\mathbb{S}^{n-1})>0$. But on the other hand, using Stokes' Theorem,

$$\int_{\partial B^n}\omega = \int_{\partial B^n} F^*(\omega) = \int_{B^n} dF^*(\omega) = \int_{B^n} F^*(d\omega) = \int_{B^n} F^*(0) = 0$$

where the second to last equality follows by $d\omega=0$ since it is an n form on the $(n-1)$-manifold $\mathbb{S}^{n-1}$.

In the general case, when f is only continuous, the same argument applies where F can be taken to be smooth because a continuous function with no fixed points can be perturbed an arbitrarily small amount to make it smooth without changing its integral.

$\square$

Chapter 6

Riemannian Manifolds

In the first half of the unit, we studied the smooth structure of a manifold. The diffeomorphism keeps the differential structure of the manifold unchanged. However, it does not preserve the geometry. By geometry, we mean properties like length, shortest path, curvature etc. It easy to check that two spheres of different radius are diffeomorphic. But their Gauss curvatures are not the same. In the unit of Introduction to Geometry, you have learned about the geometry of smooth surfaces embedded in $\mathbb{R}^3$, e.g. the induced metric on them, different notions of curvatures, Gauss-Bonnet theorem etc. We introduce these notions in a more general sense and study the theory of Riemannian manifolds. Roughly speaking, a Riemannian manifold is smooth manifolds equipped with an inner product on the tangent space at each point which varies smoothly. We begin with a few familiar examples.
Recall that an *inner product* on an n-dimensional vector space[1] V is a map $g : V \times V \to \mathbb{R}$ satisfying the following property. For every $u, v, w \in V$ and $\lambda \in \mathbb{R}$

- (Symmetry) $g(u, v) = g(v, u)$;
- (Bilinearity) $g(u + v, w) = g(u, w) + g(v, w)$ and $g(\lambda u, v) = \lambda g(u, v)$;
- (Positivity) $g(u, u) \geq 0$ with equality if and only if $u = 0$.

Example 6.1

1. The Euclidean space $\mathbb{R}^n$, is a simple example of a smooth manifold that has a natural inner product on the tangent space at each point p is. Recall from Proposition 2.4 that the tangent space $T_p\mathbb{R}^n$, $p \in \mathbb{R}^n$, is isomorphic to $\mathbb{R}^n_p$ which is the set of all vectors based at p. For any two vectors $v|_p = \sum_{i=1}^n v^i e_i|_p$ and $w|_p = \sum_{i=1}^n w^i e_i|_p$ in $\mathbb{R}^n_p$, we have the natural dot product $v|_p.w|_p = \sum_{i=1}^n v^i w^i$. Hence, we can define a natural inner product on $T_p\mathbb{R}^n$. For every two vectors $D_v|_p = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}|_p$ and $D_w|_p = \sum_{i=1}^n w^i \frac{\partial}{\partial x^i}|_p$ we define the Euclidean inner product (or dot product) as follows.
$$g_{\text{Euc}}(D_v|_p, D_w|_p) := \sum_{i=1}^n v^i w^i.$$
2. Another example that you have seen in Introduction to Geometry unit is an embedded surface S in $\mathbb{R}^3$. The tangent space T_pS at each point p is a subspace of $T_p\mathbb{R}^3$ (verify this). One can define a natural inner product using the restriction of the Euclidean inner product on $T_p\mathbb{R}^3$ to T_pS.

[1] We only consider real vector spaces in this unit.

We did not yet explain what it means that an inner product varies smoothly. This will be defined below.

6.1 Riemannian metric

Definition 6.2 A *Riemannian metric* g on a smooth manifold M is a correspondence which associates to each point $p \in M$ an inner product g_p on T_pM (i.e. for every $X_p, Y_p, Z_p \in T_pM$ and $\lambda \in \mathbb{R}$

i) $g_p(X_p, Y_p) = g_p(Y_p, X_p)$,

ii) $g_p(X_p + \lambda Y_p, Z_p) = g_p(X_p, Z_p) + \lambda g_p(Y_p, Z_p)$,

iii) $g_p(X_p, X_p) \geq 0$ for every $X_p \in T_pM$ with equality if and only $X_p = 0$.)

and it varies smoothly in p in the following sense. Let (U, ϕ) be a coordinate chart around p, and $\{\frac{\partial}{\partial x^i}\}$ be the corresponding local frame. Then

$$g_{ij}(q) := g_q\left(\frac{\partial}{\partial x^i}\bigg|_q, \frac{\partial}{\partial x^j}\bigg|_q\right)$$

is a smooth function on U.
A smooth manifold M with a Riemannian metric g on it is called a *Riemannian manifold* and is usually denoted by the pair (M, g).

Note the smoothness property does not depend on the choice of a coordinate chart. Let $(\tilde{U}, \tilde{\phi})$ be another coordinate chart around p. We show that if $g_{ij}(q)$ is smooth for every $q \in U \cap \tilde{U}$, then $\tilde{g}_{ij}(q) := g_q\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_q, \frac{\partial}{\partial \tilde{x}^j}\bigg|_q\right)$ is smooth for every $\tilde{q} \in U \cap \tilde{U}$.
Consider the indentity map $I : U \cap \tilde{U} \to U \cap \tilde{U}$. We write the differential of the identity map in local coordinates. On one side we use $(U \cap \tilde{U}, \tilde{\phi})$ as the coordinate chart around q and on the other side we use $(U \cap \tilde{U}, \phi)$. Now by Proposition 2.14 we get

$$\frac{\partial}{\partial \tilde{x}^i}\bigg|_q = dI_p(\frac{\partial}{\partial \tilde{x}^i})\bigg|_q = \sum_{j=1}^{n} \frac{\partial F^j}{\partial \tilde{x}^i}(\tilde{\phi}(q))\frac{\partial}{\partial x^j}\bigg|_q,$$

where $F := \phi \circ \tilde{\phi}^{-1} : \tilde{\phi}(U \cap \tilde{U}) \to \phi(U \cap \tilde{U})$ is a smooth map. Hence,

$$\begin{aligned}
\tilde{g}_{ij}(q) &= g_q\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_q, \frac{\partial}{\partial \tilde{x}^j}\bigg|_q\right) \\
&= g_q\left(\sum_{k=1}^{n} \frac{\partial F^k}{\partial \tilde{x}^i}(\tilde{\phi}(q))\frac{\partial}{\partial x^k}\bigg|_q, \sum_{m=1}^{n} \frac{\partial F^m}{\partial \tilde{x}^j}(\tilde{\phi}(q))\frac{\partial}{\partial x^m}\bigg|_q\right) \\
&= \sum_{k=1}^{n}\sum_{m=1}^{n} \frac{\partial F^k}{\partial \tilde{x}^i}(\tilde{\phi}(q))\frac{\partial F^m}{\partial \tilde{x}^j}(\tilde{\phi}(q))g_q\left(\frac{\partial}{\partial x^k}\bigg|_q, \frac{\partial}{\partial x^m}\bigg|_q\right)
\end{aligned}$$

which shows that $\tilde{g}_{ij}(q)$ is smooth.

Notation 6.3 1. For simplicity of notations, we denote the local frame $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ on U by $\{\partial_i\}_{i=1}^n$, and $\partial_i|_p = \frac{\partial}{\partial x^i}\Big|_p$. When the local frame comes from a coordinate chart we call it a *coordinate frame.*

2. *Einstein Summation convention.* We follow the Einstein summation convention for the indices and sum over indices:

   - The vectors always have lower indices while the components of a vector have upper indices.
   - Covectors (i.e. the dual of a vector) always have upper indices while the components of a covector have lower indices.
   - When the same index appears twice once in upper and once in lower index, it means that it is summed over 1 to the dimension of the space, i.e. if the dimension of the space is n then $a_i b^i := \sum_{i=1}^n a_i b^i$.

Example 6.4 We always denote the components of a vector $v \in T_pM$ in coordinates as $(v^1, \cdots, v^n)$. So, in the notation introduced above, we can write vector v in coordinates basis

$$v = v^i \partial_i = \sum_{i=1}^n v^i \partial_i.$$

Similarly, the differentials $dx^i|_p$ have upper indices which is consistent with the convention. And a one form ω in this basis is

$$w = w_i dx^i = \sum_{i=1}^n w_i dx^i.$$

Exercise 6.5 Show that the smoothness property of the Riemannian metric g is equivalent to say that for any pair of smooth vector fields X and Y on any neighbourhood $U \subset M$ (which is in the atlas of M) $g(X, Y) : U \to \mathbb{R}$, $g(X, Y)(q) = g_q(X_q, Y_q)$, is smooth.

Let denote the space of all smooth vector fields on M by $\Gamma(TM)$. We can view a Riemannian metric g on M as a bilinear map $g : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ over $C^\infty(M)$ such that $g_q : T_qM \times T_qM \to \mathbb{R}$ is symmetric and positive definitive. Bilinearity over $C^\infty(M)$ means that for every $f, g \in C^\infty(M)$ and any smooth vector fields $X, Y \in \Gamma(TM)$ we have $F(\cdots, fX + gY, \cdots) = f\,F(\cdots, X, \cdots) + g\,F(\cdots, Y, \cdots)$. We leave the proof of this statement as an exercise.

Exercise 6.6 Show that g is a Riemannian metric on M as it is defined in Definition 6.2 if and only if $g : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ is a bilinear map over $C^\infty(M)$ such that $g_q : T_qM \times T_qM \to \mathbb{R}$ is symmetric and positive definitive.

In the literature, sometime the language of tensor fields is used to define the Riemannian metric. A bilinear map $F : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ over $C^\infty(M)$ is called a $(2, 0)$-tensor field. So a Riemannian metric is a $(2, 0)$-tensor field satisfying the extra properties mentioned above. A detailed discussion of the tensor fields and tensor bundles are given in Appendix **??**.

Definition 6.7 Let M and N be two smooth manifolds and $F : M \to N$ be an immersion. Assume that N has a Riemannian metric h. We define the *pullback* F^*h of h by F as follows.

$$(F^*h)_p(X_p, Y_p) = h_{F(p)}(dF_pX_p, dF_pY_p), \qquad p \in M,\ X_p, Y_p \in T_pM.$$

We leave it as an exercise to show that the pullback F^*h of h gives a Riemannian metric on M.

Definition 6.8 Let (M, g) and (N, h) be two Riemannian manifolds. An immersion (respectively embedding) $F : M \to N$ is called an *isometric immersion (respectively isometric embedding)* if $F^*h = g$.

We define the isometry between Riemannian manifolds. We shall see that the geometric properties such at length, volume, curvature will be preserved under an isometry.

Definition 6.9 Let (M, g) and (N, h) be Riemannian manifolds. A diffeomorphism $F : M \to N$ is called an *isometry* if

$$g_p(X_p, Y_p) = h_{F(p)}(dF_p(X_p), dF_p(Y_p)), \qquad \forall p \in M,\ X_p, Y_p \in T_pM.$$

In other words, $g = F^*h$.
We say that (M, g) is *locally isometric* to (N, h) if there exists a smooth map $F : M \to N$ so that for every $p \in M$ there exist an open neighbourhood U_p of p so that $F : U_p \to F(U_p)$ is an isometry.

Exercise 6.10 i) Show that the isometry is an equivalence relation.

ii) Show that the local isometry is not an equivalence relation.

Exercise 6.11 Let $A : \mathbb{S}^n \to \mathbb{S}^n$ be the antipodal map given by

$$A : \mathbb{S}^n \to \mathbb{S}^n$$

$$x \mapsto -x.$$

Show that A is an isometry of $\mathbb{S}^n$ with the induced metric $\mathring{g} = i^*g_{\text{Euc}}$.

Solution. We first observe that A can be extended to the whole R^{n+1}. By abuse of notation we denote it by A.

$$A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

$$x \mapsto -x.$$

Then $A : \mathbb{S}^n \to \mathbb{S}^n$ can be viewed the composition of : $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and the inclusion map $i : \mathbb{S}^n \to \mathbb{R}^{n+1}$. They are both smooth so is A; see the definition of smooth maps into $\mathbb{R}^n$ (Definition **??**). Note that the inverse of A is itself. So it is a diffeomorphism.

We now show that

$$A^*\mathring{g} = \mathring{g}.$$

Consider A to the whole R^{n+1}Let $\{\partial_i := \frac{\partial}{\partial x^i}\}_{i=1}^{n+1}$ be the standard coordinate frame of $\mathbb{R}^{n+1}$. Note that for every $p \in \mathbb{R}^{n+1}$

$$dA_p : T_p\mathbb{R}^{n+1} \to T_{A(p)}\mathbb{R}^{n+1}$$

$$\partial_i|_p \mapsto -\partial_i|_{-p}.$$

Hence,

$$dA_p(v) = dA_p(v^i\partial_i|_p) = -v^i\partial_i|_{-p} \qquad \forall v \in T_p\mathbb{R}^{n+1}.$$

Therefore,

$$g_{\text{Euc}}|_p(v^i\partial_i|_p, w^j\partial_j|_p) = \sum_{i=1}^{n+1} v^i w^i = g_{\text{Euc}}|_{-p}(v^i\partial_i|_{-p}, w^j\partial_j|_{-p}) = A^*g_{\text{Euc}}|_p(v^i\partial_i|_p, w^j\partial_j|_p) \quad (6.1)$$

for any $v, w \in T_p\mathbb{R}^{n+1}$. I.e. A is an isometry of $\mathbb{R}^{n+1}$.
For any $p \in \mathbb{S}^n$, let us calculate $A^*\mathring{g}_p(X_p, Y_p)$, where $X_p, Y_p \in T_p\mathbb{S}^n \subset T_p\mathbb{R}^{n+1}$.

$$\begin{aligned}
A^*\mathring{g}_p(X_p, Y_p) &= \mathring{g}_{A(p)}(dA_p(X_p), dA_p(Y_p)) \\
&= i^*g_{\text{Euc}}|_{-p}(dA_p(X_p), dA_p(Y_p)) \\
&= g_{\text{Euc}}|_{-p}(dA_p(X_p), dA_p(Y_p)) \\
&= A^*g_{\text{Euc}}|_p(X_p, Y_p) \\
&= g_{\text{Euc}}|_p(X_p, Y_p) \\
&= \mathring{g}_p(X_p, Y_p),
\end{aligned}$$

where we use (7.9) in the second to the last identity.

Let (U, ϕ) be a coordinate chart. Then it is an immediate consequence of the above definition that

$$g_{ij}(q) = g_q(\partial_i|_q, \partial_j|_q) = ((\phi^{-1})^*g)_{\phi(q)}\left(\frac{\partial}{\partial x^i}|_{\phi(q)}, \frac{\partial}{\partial x^j}|_{\phi(q)}\right), \qquad \forall q \in U$$

Remark 6.12 Let $\tilde{U} \subset \mathbb{R}^n$ and $\psi : \tilde{U} \to M$ be a diffeomorphism. Then we say $(\tilde{U}, \psi)$ is a smooth parametrisation of $U := \psi(\tilde{U}) \subset M$ (or a smooth parametrisation of M for short). More precisely, $(\tilde{U}, \psi)$ gives a smooth parametrisation if and only if (U, ψ^{-1}) is a smooth coordinate chart.
Let $(\tilde{U}, \psi)$ be a smooth parametrisation. Then wer can rewrite $g_{ij}(q)$, with $q = \psi(x)$ as follows.

$$g_{ij}(\psi(x)) = (\psi^*g)_x\left(\frac{\partial}{\partial x^i}|_x, \frac{\partial}{\partial x^j}|_x\right) = g_{\psi(x)}(\partial_i|_{\psi(x)}, \partial_j|_{\psi(x)}), \qquad x \in \tilde{U}.$$

Let (U, ϕ) be a coordinate chart and $\{\partial_i\}_{i=1}^n$ the coordinate frame. The corresponding local coframe is $\{dx^1, \cdots, dx^n\}$, where t $\{dx^1_p, \cdots, dx^n_p\}$ gives a basis for T^*_pM for any $p \in U$. Let

$$dx^i \otimes dx^j : \Gamma(TU) \times \Gamma(TU) \to C^\infty(U)$$

$$dx^i \otimes dx^j(X, Y) := dx^i(X)dx^j(Y).$$

It defines a $(2,0)$ tensor field. Then the coordinate representation of metric g can be given by (here, we follow the Einstein summation formula)

$$g = g_{ij} dx^i \otimes dx^j (= \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j),$$

where g_{ij} are a collection of n^2 smooth functions given by

$$g_{ij}(p) = g_p(\partial_i|_p, \partial_j|_p).$$

Exercise 6.13 We denote $\frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$ by $dx^i dx^j$. Show that

$$g = g_{ij} dx^i \otimes dx^j = g_{ij} dx^i dx^j \tag{6.2}$$

We now give some examples of Riemannian manifolds.

Example 6.14

Euclidean space. The most trivial example is $M = \mathbb{R}^n$. The Euclidean metric g_{Euc} gives the usual dot product of $\mathbb{R}^n$ on $T_p\mathbb{R}^n$ for any $p \in \mathbb{R}^n$. In a coordinate frame, it is given by

$$g_{\text{Euc}} = \sum_i dx^i dx^i = (dx^1)^2 + \cdots + (dx^n)^2.$$

Exercise 6.15 Consider $\mathbb{R}^2$ with the Euclidean metric g_{Euc}. Take the following parametrisation of $\mathbb{R}^2$

$$\psi : \mathbb{R}_{>0} \times (0, 2\pi) \to \mathbb{R}^2$$
$$(r, \theta) \mapsto (r\cos\theta, r\sin\theta).$$

a) Check that $\psi : \mathbb{R}_{>0} \times (0, 2\pi) \to U := \mathbb{R}^2 \setminus \{(0, x) : x \geq 0\}$ is a diffeomorphism.

b) Give the coordinate expression of the Euclidean metric g_{Euc} in local coordinate (U, ϕ), where $\phi = \psi^{-1}$.

Example 6.16

1. *Submanifolds of* $\mathbb{R}^n$. Let $S \subset \mathbb{R}^n$ be a smooth submanifold in $\mathbb{R}^n$ and $i : S \to \mathbb{R}^n$ be the inclusion map. The pullback of the Euclidean metric $i^* g_{\text{Euc}}$ gives a Riemannian metric on S. We called it the *induced Riemannian metric.*

2. Let $F : M \to N$ be an smooth map between two manifolds. Let $q \in N$ be a regular value of F. Then $F^{-1}(q)$ is a submanifold of M. Hence, the pullback of the Riemannian metric on M under the inclusion map $i : F^{-1}(q) \to M$ gives the induced Riemannian metric on $F^{-1}(q)$.

3. *Product metrics.* Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds. Consider the product $M_1 \times M_2$. One can define a natural metric $g_1 \oplus g_2$ on the product $M_1 \times M_2$ called the *product metric*: let $X_i + Y_i \in T_{p_1}M_1 \oplus T_{p_2}M_2 = T_{(p_1,p_2)}(M_1 \times M_2)$, for $i = 1, 2$.

   $$g_1 \oplus g_2(X_1 + Y_1, X_2 + Y_2) := g_1(X_1, X_2) + g_2(Y_1, Y_2).$$

   For example we can consider the product metric on $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ by choosing the induced metric from $\mathbb{R}^2$ on $\mathbb{S}^1 \subset \mathbb{R}^2$. The tours with this product metric is called a *flat* torus.

4. *Left-invariant metrics on a Lie group.* Let G be a lie group. A Riemannian metric g on G is called *left-invariant* if L_q is an isometry for any $q \in G$. Let e be its identity element. Take an arbitrary inner product g_e on T_eG. We can define a left-invariant metrig g on G as follow.
$$g_p(X,Y) := g_e(dL_{p^{-1}}(X), dL_{p^{-1}}(Y)), \qquad p \in G, X, Y \in T_pG.$$
Check that it defines a left-invariant Riemannian metric.

Theorem 6.17 *Every smooth manifolds admits a Riemannian metric.*

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ (for some index set A) be an atlas for M. Hence, on each U_α, the pullback of the Euclidean metric, $\varphi_\alpha^* g_{\text{Euc}}$, defines a Riemannian metric on U_α. Let $\{\psi_\alpha\}_{\alpha \in A}$ be a smooth partition of unity subordinate to $\mathcal{A}$ which is a family of smooth functions $\psi_\alpha : M \to \mathbb{R}$ satisfying the properties listed in Definition 1.27. We now define

$$g_p(X_p, Y_p) := \sum_{\alpha \in A} \psi_\alpha(p)(\varphi_\alpha^* g_{\text{Euc}})_p(X_p, Y_p), \qquad \forall p \in M, \quad \forall X_p, Y_p \in T_pM.$$

This defines a Riemannian metric on M. □

Another way to prove the existence of a Riemannian metric on a manifold is by using Theorem 1.33 and Remark 1.34. Recall that it states that every n-dimensional smooth manifold can be embedded in $\mathbb{R}^k$ for some $k > n$. Then we use Example 6.16.
Nash proved that all Riemannian manifolds can be seen in this way.

Theorem 6.18 (Nash embedding theorem) *Let (M, g) be a Riemannian manifold. Then there exists an isometric embedding $F : M \to \mathbb{R}^k$, for some $k > n$.*

Nash gave a precise value for k in terms of n. His exponent has been then improved to $k = \frac{(n+2)(n+3)}{2}$. The proof of Nash's theorem is beyond the scope of this unit.

6.2 Riemannian volume form

A Riemannian metric provides a tool to measure the volume and length. We conclude this chapter by defining the volume form of a Riemannian manifold. The next chapter will be devoted to the discussion of the length of a curve and geodesics in a Riemannian manifold.
Let (M, g) be an oriented Riemannian manifold of dimension n and $U \subset M$ be an open subset of M. Let $(E_1, \cdots, E_n)$ be a local frame on U. Recall from Definition 4.8 that this means for all $p \in U$, $\{E_1|_p, \ldots, E_n|_p\}$ is a basis for T_pM. If in addition we assume $g_p(E_i|_p, E_j|_p) = \delta_{ij}$, then $(E_1, \cdots, E_n)$ is called a *local orthonormal frame.*

Theorem 6.19 *Let (M, g) be an oriented Riemannian manifold of dimension n. There exists a unique smooth differential n-form $\omega \in \Omega^n(M)$ such that*

$$\omega(E_1, \cdots, E_n) = 1,$$

for any oriented local orthonormal frame $(E_1, \cdots, E_n)$ on M.

Proof. Let $(E_1, \cdots, E_n)$ be a local orthonormal frame on $U \subset M$ and $(\mathcal{E}^1, \cdots, \mathcal{E}^n)$ be the dual coframe, i.e. $\mathcal{E}^i(E_j) = \delta_{ij}$. We define $\omega := \mathcal{E}^1 \wedge \cdots \wedge \mathcal{E}^n$ on U. Let be $(\tilde{E}_1, \cdots, \tilde{E}_n)$ another local orthonormal frame on M preserving the orientation, and $(\tilde{\mathcal{E}}^1, \cdots, \tilde{\mathcal{E}}^n)$ be its dual coframe. Then there exists $A = (A_i^j) \in O(n, \mathbb{R})$ so that $\tilde{E}_i = A_i^j E_j$ and $\det(A) = 1$. Therefore,

$$\omega(\tilde{E}_1, \cdots, \tilde{E}_n) = \det(A) = 1 = \omega(E_1, \cdots, E_n),$$

This defines a global form satisfying the desired condition. To prove the uniqueness, let $\tilde{\omega}$ be another Riemannian volume form. We can locally write as $\tilde{\omega} = f\mathcal{E}_1 \wedge \cdots \wedge \mathcal{E}_n$. Since by definition $\tilde{\omega}(E_1, \cdots, E_n) = 1$. We conclude that $f \equiv 1$. $\square$

Definition 6.20 (Riemannian volume form) The unique differential n-form ω defined in Theorem 6.19 is called the *Riemannian volume form*. Let $(E_1, \cdots, E_n)$ be a local orthonormal frame on $U \subset M$ and $(\mathcal{E}^1, \cdots, \mathcal{E}^n)$ be the dual coframe. Then the Riemannian volume has the following local expression

$$\omega = \mathcal{E}^1 \wedge \cdots \wedge \mathcal{E}^n.$$

To emphasis the role of the metric in the definition of the Riemannian volume form ω, it is usually denoted by dv_g.

When M is a two dimensional Riemannian manifold, we use the the *Riemannian area form* instead of the Riemannian volume form.

Using Definition 6.20, we show that the coordinate expression of the Riemannian volume form dv_g on (M, g) in any local coordinates is given by

$$dv_g = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n.$$

Let (U, ϕ) be an oriented smooth chart, $\phi = (x^1, \cdots, x^n)$ around $p \in M$. We know from Chapter 5 that every differential n-form, where n is the dimension of the manifold, can be written as $fdx^1 \wedge \cdots \wedge dx^n$. Hence, in the coordinate we fix, there exists a positive function $f \in C^\infty(M)$ such that

$$dv_g = fdx^1 \wedge \cdots \wedge dx^n.$$

To compute f, it is enough to compute $dv_g(\partial_1, \cdots, \partial_n)$, where $\partial_i = \frac{\partial}{\partial x^i}$. Indeed

$$dv_g(\partial_1|p, \cdots, \partial_n|_p) = f(p)dx^1 \wedge \cdots \wedge dx^n(\partial_1|_p, \cdots, \partial_n|_p) = f(p).$$

we use the expression of dv_g in an oriented local orthonormal frame (E_i) on U. Let $(\mathcal{E}^i)$ be the dual coframe. By the definition of the volume form we have $dv_g = \mathcal{E}^1 \wedge \cdots \wedge \mathcal{E}^n$. Therefore,

$$f = dv_g(\partial_1, \cdots, \partial_n) = \mathcal{E}^1 \wedge \cdots \wedge \mathcal{E}^n(\partial_1, \cdots, \partial_n).$$

We write $\partial_i|_p$ in the basis of $\{E_i|_p\}$.

$$\partial_i|_p = A_i^j(p)E_i|_p.$$

Hence,

$$f = \mathcal{E}^1 \wedge \cdots \wedge \mathcal{E}^n(\partial_1, \cdots, \partial_n) = \mathcal{E}^1 \wedge \cdots \wedge \mathcal{E}^n(A_1^{i_1}E_{i_1}, \cdots, A_n^{i_n}E_{i_n}) = \det(A_j^i).$$

See Chapter 5 of the lecture note for the last identity. We now want to relate $\det(A^i_j)$ to $\det(g_{ij})$. We observe that

$$g_{ij} = g(\partial_i, \partial_j) = g(A^k_i E_k, A^l_j E^l_j) = A^k_i A^l_j g(E_i, E_k) = A^k_i A^l_j \delta_{kl} = \sum_{k=1}^{n} A^k_i A^k_j.$$

Note that the (i, j)-entry of the matrix $A^t A$ is equal to $\sum_{k=1}^n A^k_i A^k_j$. Thus

$$\det(g_{ij}) = \det(A^t A) = \det A^t \det A = (\det A)^2.$$

We conclude that $f = \det A = \pm\sqrt{\det(g_{ij})}$. Since both frames (∂_i) and (E_i) are oriented, the sign must be positive.

Exercise 6.21 Consider the local coordinate (U, ϕ) given in Exercise 6.15. Calculate the Riemannian volume form of $(\mathbb{R}^2, g_{\text{Euc}})$ in this local coordinate .

Exercise 6.22 Let $\gamma : \mathbb{R} \to \mathbb{R}_{>0}$ be a C^∞ function. Consider the surface of revolution

$$S := (x, \gamma(x)\cos\theta, \gamma(x) sin\theta) : x, \theta \in \mathbb{R}.$$

a) Show that S is an embedded 2-dimensional submanifold of $\mathbb{R}^3$ and for any interval $I = (a, a + 2\pi)$ the inverse of the map

$$\psi : \mathbb{R} \times I \to S$$

$$(x, \theta) \mapsto (x, \gamma(x)\cos\theta, \gamma(x)\sin\theta)$$

gives a coordinate chart.

b) Equip S with the Riemannian metric induced from the Euclidean metric on $\mathbb{R}^3$. Calculate the Riemannian metric and the Riemannian volume form on S in the coordinate chart given in part a).

Chapter 7

Connections and Geodesics

In this chapter and the next, we study one of the fundamental concepts in Riemannian Geometry, i.e. geodesics. Geodesics in a Riemannian manifold are the generalisation of straight lines in $\mathbb{R}^n$. In the Introduction to Geometry unit, you learned the concept of geodesics on a surface S in $\mathbb{R}^3$. A geodesic in S has been defined to be a smooth unit-speed curve $\gamma(s)$ whose geodesic curvature κ_g is identically zero. It is equivalent to saying that $\ddot{\gamma}(s)$ is orthogonal to the surface. We call $\ddot{\gamma}(s)$ the *acceleration* of curve γ. So, geodesics are curves that have no acceleration in any direction tangent to the surface. It can be shown that geodesics are locally length minimising curves in S. You may review Introduction to Geometry for more details on the concept of geodesics in embedded surfaces in $\mathbb{R}^3$. We shall extend this definition on a Riemannian manifold. This chapter is self-contained, and you do not need to recall the notations and definitions from Introduction to Geometry. However, comparing them with the definitions and statements in this chapter would be useful.
To define the acceleration of a curve on a Riemannian manifold, we need to define the directional derivative of $\dot{\gamma}$ along γ. We first define vector fields along a curve. Then we introduce a new object called connections. A connection gives us a tool to define the acceleration of a curve on a manifold M.

7.1 Vector fields along curves

A *curve* γ in a manifold M is a smooth map $\gamma : I \to M$ where I is some open interval in $\mathbb{R}$. A *segment* is the restriction of a curve γ to a closed interval $[a, b]$. A *vector field along a curve* γ is a smooth map $V : I \to TM$ so that $V(t) \in T_{\gamma(t)}M$ for every $t \in I$. We denote the space of all vector fields along γ by $\mathcal{T}(\gamma)$.

Example 7.1 The *velocity* $\dot{\gamma} : I \to TM$ of a curve γ is defined as $\dot{\gamma}(t) = d\gamma_t(\frac{d}{dt}|_t)$.

$$I \xrightarrow{\frac{d}{dt}} TI \xrightarrow{d\gamma} TM.$$

Note that $\dot{\gamma}(t) \in T_{\gamma(t)}M$ for every $t \in I$. Thus, it is a vector field along γ. Let $f : M \to \mathbb{R}$ be a smooth function. $\dot{\gamma}$ acts on functions by

$$\dot{\gamma}(t)f = \frac{d}{dt}(f \circ \gamma). \tag{7.1}$$

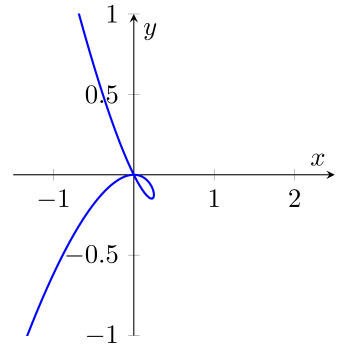

Figure 7.1: $\gamma(t) = (t(1-t), (t-1)t^2)$ for $t \in (-1, 2)$

Let $(\gamma^1(t), \cdots, \gamma^n(t))$ be the representation of $\gamma(t)$ in coordinates. Then

$$\dot\gamma(t) = d\gamma_t(\frac{d}{dt}|_t) = \dot\gamma^i(t)\partial_i|_{\gamma(t)}, \qquad \text{(we use Einstein summation convention)}$$

where $\dot\gamma^i(t)$ is the ordinary derivative of $\gamma^i : I \to \mathbb{R}$. Check the last identity using Proposition 2.14.

Definition 7.2 Let $V \in \Gamma(TM)$. We define the map $V : I \to TM$ by $V(t) := V(\gamma(t)) = V_{\gamma(t)}$. It is smooth since it is the composition of two smooth maps. Therefore, it gives a vector field along γ. $V(t)$ is called the *restriction of* V *along* γ.
A vector field along a curve γ is *extendible* if it is the restriction of a vector field on M along γ.

Example 7.3 Consider the smooth curve $\gamma : \mathbb{R} \to \mathbb{R}^2$

$$\gamma(t) = (t(1-t), (t-1)t^2).$$

We shall see that the vector field $\dot\gamma$ along curve γ is not extendable, i.e. there is no smooth vector field $V \in \Gamma(TM)$ so that $V(\gamma(t)) = \dot\gamma(t)$. Note that $\gamma(0) = \gamma(1) = 0$. But

$$\dot\gamma(0) = (1, 0) \neq (-1, 1) = \dot\gamma(1).$$

So $\dot\gamma$ cannot be a restriction of any smooth section $V : \mathbb{R}^2 \to TR^2$ along γ. Any example of a smooth curve $\gamma : I \to M$ for which there exists $t_0, t_1 \in I$ so that $\gamma(t_0) = \gamma(t_1)$ but $\dot\gamma(t_0) \neq \dot\gamma(t_1)$ works.

7.2 Linear connections

For a curve γ in $\mathbb{R}^n$, we can define the acceleration $\ddot\gamma$ by differentiating the components of the velocity $\dot\gamma$ in standard coordinate frame $\{\partial_i := \frac{\partial}{\partial x^i}\}$ in $\mathbb{R}^n$. Then we define a geodesic to be a curve γ whose acceleration $\ddot\gamma(t) = \ddot\gamma^i(t)\partial_i \equiv 0$. Indeed, solving the system of ODEs $\ddot\gamma^i(t) = 0$, $i = 1, \cdots, n$, shows that γ should be a straight line. However, this definition of the acceleration and hence the definition of geodesics depends on the choice of the coordinate chart. Indeed, let consider the polar coordinate chart $(\mathbb{R}^2 \setminus \{0\}, \phi)$, where $\phi(p) = (r(p), \theta(p))$, $p \in \mathbb{R}^2$ in $\mathbb{R}^2$. Let consider $\gamma : (0, 2\pi) \to \mathbb{R}$, $\gamma(t) = (\cos(t), \sin(t))$ in standard coordinate. Thus

$$\dot{\gamma}(t) = (-\sin t, \cos t), \qquad \ddot{\gamma}(t) = (-\cos t, -\sin t).$$

We now write γ in the polar coordinates and get $\gamma(t) = (1, t)$. Hence,

$$\dot{\gamma}(t) = (0, 1), \qquad \ddot{\gamma}(t) = (0, 0)!$$

We see that $\ddot{\gamma}$ in the polar coordinates is zero, but not in standard coordinate. So, if we define the acceleration to be the derivative of components of $\dot{\gamma}$ in a coordinate chart, then curves with zero "acceleration" in polar coordinates are different than the ones in the standard coordinates. Although the definition of $\dot{\gamma}$ does not require fixing any coordinate chart (see (7.1)), it is not clear how to define $\ddot{\gamma}$. We want to define the acceleration of a curve so that it does not depend on coordinate charts. More generally, we shall study how to differentiate vector fields on a manifold.
Linear connections capture the essential properties of differentiation of vector fields in $\mathbb{R}^n$ and extend it to a general manifold.

Definition 7.4 (Linear Connection[a]) A *linear connection* (in TM) is a map

$$\begin{aligned} \nabla \quad : \quad & \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM) \\ & (X, Y) \mapsto \nabla_X Y \end{aligned}$$

satisfying the following properties.

a. $\nabla_X Y$ is linear in X over $C^\infty(M)$

$$\nabla_{fX_1 + gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y, \qquad \forall f, g \in C^\infty(M);$$

b. $\nabla_X Y$ is linear in Y over $\mathbb{R}$

$$\nabla_X (aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2, \qquad \forall a, b \in \mathbb{R}$$

c. ∇ satisfies the following product rule

$$\nabla_X (fY) = f\nabla_X Y + (Xf)Y, \quad \forall f \in C^\infty(M).$$

See Section 4.3 to recall the definition of Xf. The symbol ∇ is read "del" and $\nabla_X Y$ is called the *covariant derivative* of Y in the direction of X.

[a] *Affine connections* is another name which is used for linear connections.

The linear connection ∇ is defined on the vector fields which are global sections of TM. But we can show that ∇ is actually a local operator in the following sense.

Lemma 7.5 (Locality of the covariant derivative) *Let ∇ be a linear connection on M. Then for every $p \in M$ and $X, Y \in \Gamma(TM)$, the covariant derivative $\nabla_X Y|_p$ depends only on the value of X and Y in an arbitrary small neighbourhood of p, i.e. if $X = \tilde{X}$ and $Y = \tilde{Y}$ on a neighbourhood U of p, where $\tilde{X}, \tilde{Y} \in \Gamma(TM)$ then*

$$\nabla_X Y|_p = \nabla_{\tilde{X}} \tilde{Y}|_p.$$

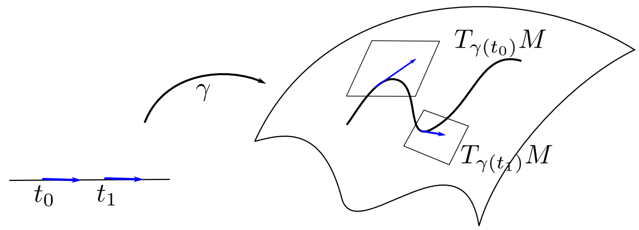

Figure 7.2: A curve γ in M

Proof. We prove that $\nabla_X Y|_p = \nabla_X \tilde{Y}|_p$ and $\nabla_X Y|_p = \nabla_{\tilde{X}} Y|_p$. Then the statement follows. We only prove the first identity and leave the second one as an exercise. Let $\bar{Y} = Y - \tilde{Y}$. Then $\bar{Y}$ is zero on U. We show that $\nabla_X \bar{Y}|_p = 0$. Let ϕ be a smooth bump function with support in U and $\phi(p) = 1$. Then $\phi\bar{Y} = 0$.

$$\nabla_X \phi\bar{Y}|_p = 0 = \phi(p)\nabla_X \bar{Y}|p + (X\phi)(p)\bar{Y}|_p = \nabla_X \bar{Y}|p.$$

This complete the proof.

□

We now give the coordinate expression of $\nabla_X Y$. Let (U, ϕ) be a coordinate chart and $\{\partial_i := \frac{\partial}{\partial x^i}\}$ the corresponding local coordinate frame for TM. We can write $X, Y \in \Gamma(TM)$ in this coordinate chart

$$X = X^i \partial_i \qquad Y = Y^i \partial_i.$$

We now calculate $\nabla_X Y$ in coordinates using the properties of ∇ listed above.

$$\begin{aligned} \nabla_X Y &= \nabla_{X^i \partial_i}(Y^j \partial_j) = X^i \nabla_{\partial_i}(Y^j \partial_j) \\ &= X^i \partial_i Y^j \partial_j + X^i Y^j \nabla_{\partial_i} \partial_j. \end{aligned}$$

Let Γ_{ij}^k be smooth functions such that $\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$. Thus

$$\nabla_X Y = X^i \partial_i Y^j \partial_j + X^i Y^j \Gamma_{ij}^k \partial_k. \tag{7.2}$$

This shows that $\nabla_X Y|_p$ depends only on the value of $X(p)$.
The smooth functions Γ_{ij}^k are called the *Christoffel symbols* of ∇. Note that Γ_{ij}^k depends on the coordinate chart you choose.

Lemma 7.6 *Let M be a manifold covered by a single coordinate chart. There is a one-to-one correspondence between linear connections on M and the choices of n^3 smooth functions $\{\Gamma_{ij}^k\}$ by the rule* (7.2).

Proof. Since M is covered by a single coordinate chart, given n^3 smooth functions $\{\Gamma_{ij}^k\}$, we define ∇ by (7.2). As an exercise you can check that $\nabla_X Y \in \Gamma(TM)$ and satisfies the properties of a linear connection in Definition 7.4.
For any given connection ∇, the components of $\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$ give n^3 smooth functions. □

Example 7.7 (Euclidean connection) We define the Euclidean connection $\overline{\nabla}$ by choosing its Christoffel symbols Γ_{ij}^k to be identically zero in the standard Euclidean coordinate chart. Thus for any two smooth vector fields X, Y in $\mathbb{R}^n$ we have

$$\overline{\nabla}_X Y^i \partial_i = X^j (\partial_j Y^i) \partial_i.$$

Notice the components of $\overline{\nabla}_X Y$ are directional derivative of the components Y^i of Y in the direction X.

The Christoffel symbols of a linear connection depends on the coordinate chart we choose. The following example shows that the Christoffel symbols of the Euclidean connection may not zero if we choose a different coordinate system.

Example 7.8 Consider $\mathbb{R}^2$ with the Euclidean metric g_{Euc}. Take the following parametrisation of $\mathbb{R}^2$

$$\psi : \mathbb{R}_{>0} \times (0, 2\pi) \to \mathbb{R}^2$$
$$(r, \theta) \mapsto (r\cos\theta, r\sin\theta).$$

We calculate the Christoffel symbols of the Euclidean connection $\overline{\nabla}$ in the local coordinate (U, ϕ).

Let denote $x^1 := r$ and $x^2 := \theta$. Thus

$$\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\} = \{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\} = \{\partial_1, \partial_2\}$$

denotes the coordinate frame. To avoid confusion with the standard coordinate frame of $\mathbb{R}^2$, we denote the standard coordinate frame of $\mathbb{R}^2$ by $\{D_{e_1}, D_{e_2}\}$. By the definition of the Euclidean connection, its Christoffel symbols $\bar{\Gamma}^k_{ij} \equiv 0$, i.e. $\overline{\nabla}_{D_{e_i}} D_{e_j} \equiv 0$ We now calculate $\overline{\nabla}_{\partial_i}\partial_j$. Recall from Problem 8, Sheet 6 that

$$\frac{\partial}{\partial r} = \partial_1 = \cos\theta D_{e_1} + \sin\theta D_{e_2}, \quad \frac{\partial}{\partial\theta} = -r\sin\theta D_{e_1} + r\cos\theta D_{e_2}.$$

Hence,

$$\begin{aligned}
\overline{\nabla}_{\partial_1}\partial_1 &= \overline{\nabla}_{\cos\theta D_{e_1} + \sin\theta D_{e_2}}(\cos\theta D_{e_1} + \sin\theta D_{e_2}) \\
&= \cos\theta\overline{\nabla}_{D_{e_1}}(\cos\theta D_{e_1} + \sin\theta D_{e_2}) + \sin\theta\overline{\nabla}_{D_{e_2}}(\cos\theta D_{e_1} + \sin\theta D_{e_2}) \\
&= \cos\theta(D_{e_1}\cos\theta)D_{e_1} + \cos\theta(D_{e_1}\sin\theta)D_{e_2} \\
&\quad + \sin\theta(D_{e_2}\cos\theta)D_{e_1} + \sin\theta(D_{e_2}\sin\theta)D_{e_2} \\
&= (\cos\theta(D_{e_1}\cos\theta) + \sin\theta(D_{e_2}\cos\theta))\, D_{e_1} + (\cos\theta(D_{e_1}\sin\theta) + \sin\theta(D_{e_2}\sin\theta))\, D_{e_2}
\end{aligned}$$

Note that $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$. Therefore,

$$\cos\theta D_{e_1}\cos\theta = \frac{x}{\sqrt{x^2+y^2}} D_{e_1}\frac{x}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x}(\frac{x}{\sqrt{x^2+y^2}}) = \frac{xy^2}{(x^2+y^2)^2};$$

$$\sin\theta D_{e_2}\cos\theta = \frac{y}{\sqrt{x^2+y^2}} D_{e_2}\frac{x}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}(\frac{x}{\sqrt{x^2+y^2}}) = -\frac{xy^2}{(x^2+y^2)^2}$$

Hence, the coefficient of D_{e_1} is zero. Similarly, we obtain that $\cos\theta(D_{e_1}\sin\theta) + \sin\theta(D_{e_2}\sin\theta) = 0$. Therefore, $\Gamma^1_{11} = \Gamma^2_{11} = 0$.
Note that $r\cos\theta = x$ and $r\sin\theta = y$. Thus we get

$$\begin{aligned}
\overline{\nabla}_{\partial_2}\partial_2 &= \overline{\nabla}_{-r\sin\theta D_{e_1} + r\cos\theta D_{e_2}}(-r\sin\theta D_{e_1} + r\cos\theta D_{e_2}) \\
&= y(D_{e_1}y)D_{e_1} - y(D_{e_1}x)D_{e_2} - x(D_{e_2}y)D_{e_1} + x(D_{e_2}x)D_{e_2} \\
&= -yD_{e_2} - xD_{e_1}.
\end{aligned}$$

In order to recover the Christoffel symbol we need to write $\overline{\nabla}_{\partial_2}\partial_2$ as $\overline{\nabla}_{\partial_2}\partial_2 = \Gamma^1_{22}\partial_1 + \Gamma^2_{22}\partial_2$. We need the expression of D_{e_1} and D_{e_2} in terms of ∂_1 and ∂_2. In the notation of Problem 8, Sheet6, we need to calculate the inverse of $d\psi$, since $D_{e_i} = d\psi^{-1}(\partial_i)$ gives the expression D_{e_i} in terms of $\{\partial_i\}$.

$$d\psi^{-1} = d\phi = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Hence,

$$D_{e_1} = \cos\theta\partial_1 - \frac{1}{r}\sin\theta\partial_2, \qquad D_{e_2} = \sin\theta\partial_1 - \frac{1}{r}\cos\theta\partial_2.$$

In conclusion, we get

$$\begin{aligned} \overline{\nabla}_{\partial_2}\partial_2 &= -yD_{e_2} - xD_{e_1} \\ &= -r\sin\theta(\sin\theta\partial_1 - \frac{1}{r}\cos\theta\partial_2) - r\cos\theta(\cos\theta\partial_1 - \frac{1}{r}\sin\theta\partial_2) \\ &= -r\partial_1 \end{aligned}$$

Therefore, $\Gamma^1_{22} = -r$ and $\Gamma^2_{22} = 0$. Doing a similar calculation for $\overline{\nabla}_{\partial_2}\partial_1$ and $\overline{\nabla}_{\partial_1}\partial_2$, we get

$$\Gamma^1_{12} = \Gamma^1_{12} = 0, \qquad \Gamma^2_{12} = \Gamma^2_{12} = \frac{1}{r}.$$

See Example 7.28 for a shorter and much simpler solution.

Theorem 7.9 *Every manifold admits a linear connection.*

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ (for some index set A) be an atlas for M. By Lemma 7.6 on each U_α, we can define a connection ∇^α. For example we can choose Γ^k_{ij} to be 0 on U_α. Let $\{\psi_\alpha\}_{\alpha\in A}$ be a smooth partition of unity subordinate to $\{U_\alpha\}_{\alpha\in A}$ (see Definition 1.27). We now define

$$\nabla_X Y := \sum_{\alpha\in A} \psi_\alpha \nabla^\alpha_X Y, \quad \forall X, Y \in \Gamma(TM).$$

It is easy to check that $\nabla_X Y$ defined above is in $\Gamma(TM)$ and satisfies conditions a-b in Definition 7.4. It remains to check the product rule. Note that a linear combination of connections does not necessarily give a connections. See Exercise 7.10. So, we verify it by a direct calculation.

$$\begin{aligned} \nabla_X(fY) &= \sum_{\alpha\in A} \psi_\alpha \nabla^\alpha_X(fY) \\ &= \sum_{\alpha\in A} \psi_\alpha \left(f\nabla^\alpha_X Y + (Xf)Y\right) \\ &= f\sum_{\alpha\in A} \psi_\alpha \nabla^\alpha_X Y + \left(\sum_{\alpha\in A}\psi_\alpha\right)(Xf)Y \\ &= f\nabla_X Y + (Xf)Y \end{aligned}$$

$\square$

Exercise 7.10 Let ∇^1 and ∇^2 be two linear connections on M. Show that $\frac{1}{2}\nabla^1$ and $\nabla^1 + \nabla^2$ do not satisfy property c in Definition 7.4. Therefore they are not linear connections.

Example 7.11 (Submanifolds of $\mathbb{R}^n$) Let (M, g) be a submanifold of $\mathbb{R}^n$ with $g = i^* g_{Euc}$, where $i : M \to \mathbb{R}^n$ is the inclusion map. There is a natural way to define a linear connection on M. Let $\pi^\top : T_p\mathbb{R}^n \to T_pM$ be the orthogonal projection on T_pM. We define the *tangential connection* $\nabla^\top$ by

$$\nabla^\top_X Y = \pi^\top(\overline{\nabla}_{\tilde{X}}\tilde{Y}),$$

where $\tilde{X}$ and $\tilde{Y}$ are smooth extension of X and Y to a neighborhood of M.

Exercise 7.12 Show that $\nabla^\top$ defines a linear connection on M.

We can now define the directional derivative of vector fields along a curve.

Theorem 7.13 (Covariant Derivatives along a curve) *Let ∇ be a linear connection on M and $\gamma : I \to M$ be a curve in M. There exists a map $D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$ which is uniquely determined by ∇ and satisfies the following properties.*

a) $D_t(aX + bY) = aD_tX + bD_tY, \qquad a, b \in \mathbb{R}.$

b) $D_t(fX) = \dot{f}X + fD_tX, \qquad f \in C^\infty(I).$

c) If $X \in \mathcal{T}(\gamma)$ is extendable to $\tilde{X} \in \Gamma(TM)$. Then $D_tX = \nabla_{\dot{\gamma}}\tilde{X}|_\gamma$.

Proof. We show that if such map exists then it is unique.

$$\begin{aligned} D_tV &= D_tV^i\partial_i = \dot{V}^i\partial_i + V^iD_t\partial_i \\ &= \dot{V}^i\partial_i + V^i\overline{\nabla}_{\dot{\gamma}^j\partial_j}\partial_i \\ &= \dot{V}^i\partial_i + V^i\dot{\gamma}^j\Gamma^k_{ij}\partial_k \\ &= \left(\dot{V}^k + V^i\dot{\gamma}^j\Gamma^k_{ij}\right)\partial_k. \end{aligned} \tag{7.3}$$

This shows that if such operator exists, then it is unique.
For the existence, first assume that the image of γ is subset of a smooth chart (U, ϕ). Then use (7.3) to define D_t. As an exercise show that it satisfies all properties a)-c). If the image of γ is not in a single coordinate chart, we cover its image by with coordinate charts and define D_tV by (7.3). Then we use the uniqueness to conclude that the the definitions are agree when two or more charts intersect. □

Definition 7.14 Let $\gamma : I \to M$ be a curve in M. For any $V \in \mathcal{T}(\gamma)$, D_tV is called the *covariant derivative of V along γ.*

From the proof of Theorem 7.13, we have the expression of the covariant derivative of V along γ in a coordinate chart is given by (7.3).

Exercise 7.15 Let Y and $\tilde{Y}$ be two smooth vector fields on M. Assume that the restriction of Y and $\tilde{Y}$ along γ are equal, i.e. $Y(t) = \tilde{Y}(t)$. Show that

$$\nabla_{\dot{\gamma}(t)}Y = \nabla_{\dot{\gamma}(t)}\tilde{Y}.$$

Example 7.16 Consider $\mathbb{R}^n$ with the Euclidean connection $\overline{\nabla}$. Let $\gamma : I \to \mathbb{R}^3$ be a smooth cure in $\mathbb{R}^n$ and $V \in \mathcal{T}(\gamma)$. We denote the corresponding coveriant derivative along γ by $\overline{D}_t$, and calculate $\overline{D}_t V$ in the standard coordinate chart of $\mathbb{R}^n$:

$$\overline{D}_t V = \overline{D}_t V^i \partial_i = \dot{V}^i(t)\partial_i + V^i \overline{D}_t \partial_i = \dot{V}^i(t)\partial_i + V^i \overline{\nabla}_{\dot{\gamma}^j(t)\partial_j}\partial_i = \dot{V}^i(t)\partial_i.$$

In particular, when $V(t) = \dot{\gamma}(t)$, then $\overline{D}_t \dot{\gamma} = \ddot{\gamma}^i(t)\partial_i$. This is basically the acceleration of γ in $\mathbb{R}^n$.

Exercise 7.17 Let $\gamma : I \to \mathbb{R}^2$. Calculate $\overline{D}_t V$ in the polar coordinate chart.

We now ready to define a geodesic.

Definition 7.18 The covariant derivative $D_t\dot{\gamma}$ of velocity $\dot{\gamma}$ along γ is called the *acceleration* of γ. A curve $\gamma : I \to M$ is a *geodesic with respect to* ∇ if its acceleration is zero:

$$D_t \dot{\gamma} \equiv 0.$$

If we calculate the coordinate expression of $D_t\dot{\gamma}$ by replacing in (7.3) we get

$$D_t\dot{\gamma} = \left(\ddot{\gamma}^k(t) + \dot{\gamma}^i(t)\dot{\gamma}^j(t)\Gamma_{ij}^k\right)\partial_k.$$

The second-order system of ODEs

$$\ddot{\gamma}^k(t) + \dot{\gamma}^i(t)\dot{\gamma}^j(t)\Gamma_{ij}^k = 0, \qquad k = 1, \cdots, n$$

is called the geodesic equation.

Exercise 7.19 Show that geodesics with respect to the Euclidean connection $\overline{\nabla}$ in $\mathbb{R}^n$ are straight lines.

Theorem 7.20 (Existence and uniqueness of geodesics) *Let M be a smooth manifold with a linear connection ∇. For any $p \in M$, any $v \in T_pM$ and any $t_0 \in I \subset \mathbb{R}$, there exist a geodesic $\gamma : I \to M$*

$$\gamma(t_0) = p, \qquad \dot{\gamma}(t_0) = v. \tag{7.4}$$

If $\gamma_1 : J \to M$ is another geodesic such that $t_0 \in J$ and satisfies (7.4)*, then $\gamma|_{I\cap J} = \gamma_1|_{I\cap J}$.*

The proof of this theorem uses the existence and uniqueness of solutions to a second-order system of ODEs. We omit this proof.

Definition 7.21 A geodesic $\gamma : I \to M$ is said to be *maximal* if if there is no geodesic $\tilde{\gamma} : J \to \mathbb{R}$ so that $I \subset J$ and $\tilde{\gamma}|_I = \gamma$.

Corollary 7.22 *Let M be a smooth manifold with a linear connection in TM. For any $p \in M$ and $v \in T_pM$, there exists a unique maximal geodesic $\gamma : I \to M$ so that $0 \in I$ and*

$$\gamma(0) = p, \qquad \dot{\gamma}(0) = v. \tag{7.5}$$

Proof. By Theorem 7.20, we know that there exist a geodesic, satisfying (7.5). Let I be the union of all open interval containing 0 on which there is a geodesic, satisfying (7.5). Again by Theorem 7.20 all such geodesics agree on their common domain. It defines a geodesic $\gamma : I \to M$ which is the unique maximal geodesic. □

7.3 Parallel Translation

The covariant derivative of a vector field along γ enables us to define a parallel translation of a vector along γ which generalises the concept of parallel translation in the Euclidean space to any Riemannian manifold.

Definition 7.23 Let M be a manifold with a linear connection ∇. A vector field V along γ is said to be *parallel along* γ if $D_t V = 0$.

The following theorem is a fundamental fact about parallel vector fields.

Theorem 7.24 (Parallel translation) *Let $\gamma : I \to M$ be a curve in M. Given $t_0 \in I$ and $V_0 \in T_{\gamma(t_0)}M$, there exists a unique vector field $V(t)$ along γ such that $V(t_0) = V_0$ and $D_t V \equiv 0$. The vector field V is called the* parallel translate *of vector V_0 along γ.*

The proof of Theorem 7.24 uses a theorem on the existence and uniqueness of linear ODEs. You can find the proof of this theorem in Chapter 4 of Introduction to Riemannian Manifolds by J. Lee.

Exercise 7.25 Let $\gamma : I \to \mathbb{R}^3$ be a curve in $\mathbb{R}^3$. Show that the parallel translate of a vector $V_0 \in T_{\gamma(t_0)}M$, $t_0 \in I$, is the constant vector field V_0.

7.4 Levi-Civita connection and Riemannian geodesics

We define a linear connection on M independent of its Riemannian structure, i.e. the Riemannian metric. A Riemannian metric gives us a tool to measure the length $\ell(\gamma)$ of a curve $\gamma : I \to M$:

$$\ell(\gamma) = \int_I g(\dot\gamma(t), \dot\gamma(t))^{1/2} dt.$$

If we want that the geodesics in a Riemannian manifold reflect the Riemannian structure and represent locally length minimising curves, we somehow need to single out a unique connection that reflects the Riemannian structure. Geodesics with respect to the Euclidean connection are straight lines. So, it reflects the Euclidean geometry. In addition to the properties of a linear connection, the Euclidean connection satisfies some extra conditions which make it uniquely determined by the Euclidean metric. Namely, the Euclidean inner product of a two vector remains constant under the parallel translation. Note that we defined the parallel translation on any manifold using the covariant derivative associated with a linear connection. The theorem below gives additional properties which define a unique connection on a Riemannian manifold called the Levi-Civita connection.

Theorem 7.26 (Levi-Civita) *Let (M, g) be a Riemannian manifold. There exists a unique linear connection ∇ satisfying the following conditions.*

(i) ∇ *is* symmetric, *i.e.*

$$\nabla_X Y - \nabla_Y X = [X, Y], \qquad X, Y \in \Gamma(TM);$$

(ii) ∇ *is* compatible *with the Riemannian metric g, i.e.*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

The connection ∇ which satisfies conditions (i) and (ii) is called the *Levi-Civita connection.*

Remark 7.27 Let M be an n-dimensional smooth manifold with a symmetric linear connections ∇. Let (U, ϕ) be a coordinate chart around $p \in M$. The symmetry of the connection ∇ implies that for all $1 \leq i, j \leq n$

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0.$$

For the last identity see Example 4.23. This show that the Christoffel symbols of the Levi-Civita connection have the following symmetry.

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \qquad 1 \leq i, j \leq n.$$

Proof. [Proof of Theorem 7.26] Assume that such connection ∇ exists. We prove that it is therefore unique. Let $X, Y, Z \in \Gamma(TM)$. We have the following identities by part (i).

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$
$$Yg(Z, X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X),$$
$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

We now calculate $Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$. Using the symmetry of ∇ and g and the bilinearity of g, we get

$$\begin{aligned} &Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\qquad = \quad g([X, Z], Y) + g([Y, Z], X) - g([X, Y], Z) + 2g(\nabla_X Y, Z). \end{aligned}$$

Therefore,

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}\Big(&Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g([X, Z], Y) - g([Y, Z], X) + g([X, Y], Z)\Big) \end{aligned} \tag{7.6}$$

The above expression shows that $\nabla_Y X$ is uniquely determined by the metric. Indeed, if $\tilde{\nabla}$ is another linear connection satisfying $i) - ii)$, then for every $X, Y, Z \in \Gamma(TM)$

$$\begin{aligned} g(\tilde{\nabla}_X Y, Z) = \frac{1}{2}\Big(&Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g([X, Z], Y) - g([Y, Z], X) + g([X, Y], Z)\Big) \end{aligned} \tag{7.7}$$

Hence, $g(\nabla_X Y - \tilde{\nabla}_X Y, Z) \equiv 0$ for every $Z \in \Gamma(TM)$. Take $Z = \nabla_X Y - \tilde{\nabla}_X Y$ to conclude $\nabla_X Y = \tilde{\nabla}_X Y$ for every $X, Y \in \Gamma(TM)$.
To prove the existence, we define $\nabla_X Y$, $X, Y \in \Gamma(TM)$ using the formula (7.7) and then check that it satisfies properties (i) and (ii). □

We now calculate the Christoffel symbols of the Levi-Civita connection in coordinates. From (7.7), it follows that

$$g(\nabla_{\partial_i}\partial_j, \partial_m) = \Gamma^l_{ij} g_{lm} = \frac{1}{2}\Big(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\Big),$$

where $g_{ij} = g(\partial_i, \partial_j)$. Notice that we use $[\partial_i, \partial_j] = 0$. We denote the inverse of matrix (g_{ij}) by (g^{ij}). Since $g_{ik}g^{kj} = \delta_{ij}$, we get

$$\Gamma^k_{ij} = \Gamma^l_{ij} g_{lm} g^{mk} = \frac{1}{2} g^{mk}\Big(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\Big). \tag{7.8}$$

Example 7.28 We calculate the Christoffel symbol of $(\mathbb{R}^2, g_{\text{Euc}})$ in polar coordinates given in Example 7.8 using the formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}\left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell}\right), \qquad i, j, k \in \{1, 2\} \tag{7.9}$$

where $x^1 := r$ and $x^2 := \theta$. Note that $\overline{g}^{11} = 1, \overline{g}^{22} = r^{-2}$ and $\overline{g}^{12} = 0$. We also have $\frac{\partial \overline{g}_{22}}{\partial r} = 2r$ and the others partial derivatives are zero. Therefore,

$$\Gamma^1_{11} = \sum_{\ell=1}^{2} \frac{1}{2}\overline{g}^{1\ell}\left(\frac{\partial \overline{g}_{1\ell}}{\partial r} + \frac{\partial \overline{g}_{1\ell}}{\partial r} - \frac{\partial \overline{g}_{11}}{\partial x^\ell}\right) = 0;$$

$$\Gamma^1_{22} = \sum_{\ell=1}^{2} \frac{1}{2}\overline{g}^{1\ell}\left(\frac{\partial \overline{g}_{2\ell}}{\partial \theta} + \frac{\partial \overline{g}_{2\ell}}{\partial \theta} - \frac{\partial \overline{g}_{22}}{\partial x^\ell}\right) = -r;$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \sum_{\ell=1}^{2} \frac{1}{2}\overline{g}^{2\ell}\left(\frac{\partial \overline{g}_{2\ell}}{\partial r} + \frac{\partial \overline{g}_{1\ell}}{\partial \theta} - \frac{\partial \overline{g}_{12}}{\partial x^\ell}\right) = 1/r;$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \sum_{\ell=1}^{2} \frac{1}{2}\overline{g}^{1\ell}\left(\frac{\partial \overline{g}_{2\ell}}{\partial r} + \frac{\partial \overline{g}_{1\ell}}{\partial \theta} - \frac{\partial \overline{g}_{12}}{\partial x^\ell}\right) = 0;$$

$$\Gamma^2_{11} = \sum_{\ell=1}^{2} \frac{1}{2} g^{2\ell}\left(\frac{\partial g_{1\ell}}{\partial x} + \frac{\partial g_{1\ell}}{\partial x} - \frac{\partial g_{22}}{\partial x^\ell}\right) = 0,$$

$$\Gamma^2_{22} = \sum_{\ell=1}^{2} \frac{1}{2} g^{2\ell}\left(\frac{\partial g_{2\ell}}{\partial \theta} + \frac{\partial g_{1\ell}}{\partial \theta} - \frac{\partial g_{22}}{\partial x^\ell}\right) = 0.$$

Definition 7.29 A curve γ is a *Riemannian geodesic* if it is a geodesic with respect to the Levi-Civita connection, i.e. its coveriant derivative $D_t\dot{\gamma}$ with respect to the Levi-Civita connection is zero.

Example 7.30 Let $\gamma : I \to M$ be a Riemannian geodesic. We show that the length of $\dot\gamma(t)$ is constant, i.e. $g(\dot\gamma(t), \dot\gamma(t)) \equiv c$, where $c > 0$ is a positive real number.

We show that $\frac{d}{dt} g(\dot\gamma(t), \dot\gamma(t)) = 0$. For any $t \in I$, let (U, ϕ) be a coordinate chart around $\gamma(t)$. Let $\{\partial_i\}$ be the coordinate frame. Then we can write $\dot\gamma = \dot\gamma^i(t)\partial_i$. Hence,

$$\begin{aligned}
\frac{d}{dt} g(\dot\gamma(t), \dot\gamma(t)) &= \frac{d}{dt} g(\dot\gamma^i(t)\partial_i, \dot\gamma^j(t)\partial_j) \\
&= \frac{d}{dt}\left(\dot\gamma^i(t)\dot\gamma^j(t) g(\partial_i, \partial_j)\right) \\
&= \ddot\gamma^i(t)\dot\gamma^j(t) g(\partial_i, \partial_j) + \dot\gamma^i(t)\ddot\gamma^j(t) g(\partial_i, \partial_j) + \dot\gamma^i(t)\dot\gamma^j(t)\frac{d}{dt} g(\partial_i, \partial_j) \\
\text{we use } \tfrac{d}{dt} g(\partial_i, \partial_j) = \dot\gamma(t) g(\partial_i, \partial_j) \quad &= \ddot\gamma^i(t)\dot\gamma^j(t) g(\partial_i, \partial_j) + \dot\gamma^i(t)\ddot\gamma^j(t) g(\partial_i, \partial_j) + \dot\gamma^i(t)\dot\gamma^j(t)(\dot\gamma(t) g(\partial_i, \partial_j)) \\
&= \ddot\gamma^i(t)\dot\gamma^j(t) g(\partial_i, \partial_j) + \dot\gamma^i(t)\ddot\gamma^j(t) g(\partial_i, \partial_j) \\
&\quad + \dot\gamma^i(t)\dot\gamma^j(t)\left(g(\nabla_{\dot\gamma^k\partial_k}\partial_i, \partial_j) + g(\nabla_{\dot\gamma^k\partial_k}\partial_j, \partial_i)\right) \\
&= g(\ddot\gamma^i(t)\partial_i, \dot\gamma^j(t)\partial_j) + g(\dot\gamma^i(t)\partial_i, \ddot\gamma^j(t)\partial_j) \\
&\quad + \dot\gamma^i(t)\dot\gamma^j(t)\dot\gamma^k(t)\left(g(\Gamma^l_{ki}\partial_l, \partial_j) + g(\Gamma^m_{kj}\partial_m, \partial_i)\right)
\end{aligned}$$

We now use the fact that γ is a geodesic and therefore satisfied the geodesic equation.

$$\ddot\gamma^k + \dot\gamma^i\dot\gamma^j\Gamma^k_{ij} = 0, \qquad 1 \le k \le n.$$

After rearrangement we have

$$\begin{aligned}
\frac{d}{dt} g(\dot\gamma(t), \dot\gamma(t)) &= g\left((\ddot\gamma^i(t)\partial_i + \dot\gamma^i(t)\dot\gamma^k(t)\Gamma^l_{ki}\partial_l, \dot\gamma^j(t)\partial_j)\right) \\
&\quad + g\left(\dot\gamma^i(t)\partial_i, \ddot\gamma^j(t)\partial_j + \dot\gamma^j(t)\dot\gamma^k(t)\Gamma^m_{kj}\partial_m\right) \\
&= 0.
\end{aligned}$$

This completes the proof.

Exercise 7.31 Let X and Y be two vector field parallel along γ. Show that $g(X(t), Y(t)) \equiv c$.

Exercise 7.32 Consider the upper half-plane model of the hyperbolic spaces $(\mathbb{H}^2, \breve{g})$ where

$$\mathbb{H}^2 = \{(x, y) : y > 0\}, \quad \text{and} \quad \breve{g} = \frac{(dx)^2 + (dy)^2}{y^2}.$$

1. Calculate the Christoffel symbols for the Levi-Civita connection ∇ on $(\mathbb{H}^2, \breve{g})$.
2. Let $\gamma(t) = (x(t), y(t))$ be a geodesic in $\mathbb{H}^2$. Write the geodesic equations for γ in coordianates.

Exercise 7.33 Consider $\mathbb{S}^2 \subset \mathbb{R}^3$ with the standard metric $\mathring{g}_R$.

a) Show that $\mathring{g}_R$ in the polar coordinate chart has the following expression.

$$\mathring{g}_R = R(d\varphi)^2 + R^2\sin^2\varphi(d\theta)^2.$$

b) Compute the Christoffel symbols of $\mathring{g}_R$ in spherical coordinate chart.

c) Using the geodesic equation in spherical coordinate chart, verify that all meridians curves

$$\gamma(t) = (\theta(t), \varphi(t)) = (\theta_0, ct)$$

passing through north pole $(0, R)$ are geodesics.

Exercise 7.34 a. Let M be a submanifold of $\mathbb{R}^n$ with induced metric. Show that $\nabla^\top$ defined in Example 7.11 is the Levi-Civita connection on M.

b. Let γ be a smooth curve in M. Show that the corresponding covariant derivative $D_t^\top$ along γ is given by

$$D_t^\top V(t) = \pi^\top(\overline{D}_t V(t)),$$

where $\overline{D}_t$ is the covariant derivative along γ associate with $\overline{\nabla}$.

We end this chapter by giving the equation in a coordinate chart for a Riemannian geodesic. Let γ be a Riemannian geodesic and $(\gamma^1(t), \cdots, \gamma^n(t))$ its coordinates representation in a coordinate chart (U, ϕ) around $\gamma(t_0)$, $t_0 \in I$. Using (7.3) and (7.8), we obtain that γ satisfies the following system of second-order ODEs.

$$\begin{aligned} 0 &= D_t\dot{\gamma} = \big(\ddot{\gamma}^k(t) + \dot{\gamma}^i\dot{\gamma}^j(t)\Gamma_{ij}^k\big)\partial_k \\ &= \big(\ddot{\gamma}^k(t) + \frac{1}{2}g^{mk}\Big(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\Big)\dot{\gamma}^i\dot{\gamma}^j(t)\big)\partial_k. \end{aligned}$$

Therefore,

$$\ddot{\gamma}^k(t) + \frac{1}{2}g^{mk}\Big(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\Big)\dot{\gamma}^i\dot{\gamma}^j(t) = 0, \qquad \forall k = 1, \dots, n \tag{7.10}$$

is the equation for the Riemannian geodesics.

Chapter 8

The exponential map and normal coordinates

8.1 The exponential map

In previous chapter, we have seen that for every $p \in M$ and $v \in T_pM$, there is a unique maximal geodesic γ_v with initial conditions $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$. It implicitly defines a map from TM to the set of geodesics in M. We can use this map to define a map between TM and M. This map is called the *exponential map* and encodes significant information about the collective behaviour of geodesics and more importantly about the (Riemannian) manifold M itself. The study of the exponential map further our understanding of geodesics and the Riemannian structure. In this chapter, we only consider Riemannian geodesics with respect to the Levi-Civita connection, even though some of the results may also hold on a manifold with an arbitrary linear connection.
We first define the domain of the exponential map. Let

$$\mathcal{U} = \{v \in TM : \gamma_v \text{ is defined on } [0,1]\}.$$

The *exponential map* $\exp : \mathcal{U} \to M$ is given by

$$\exp(v) = \gamma_v(1),$$

where γ_v is the maximal geodesic with initial conditions $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v \in T_pM$. For each $p \in M$, the *restricted exponential map* $\exp_p$ is the restriction of exp to $\mathcal{U}_p := \mathcal{U} \cap T_pM$.

$$\exp_p : \mathcal{U}_p \to M.$$

The following theorem summarises the main properties of the exponential map.

Theorem 8.1 *(a) The domain of* exp, $\mathcal{U}$, *contains* $0 \in T_pM$ *for every* $p \in M$, *and for every* $p \in M$, $\mathcal{U}_p$ *is star-shaped*[a] *with respect to* $0 \in T_pM$.

(b) For every $v \in \mathcal{U}_p$, *the geodesic* γ_v *is given by* $\gamma_v(t) = \exp(tv) = \exp_p(tv)$ *for all* t *such that either side is defined.*

(c) For each point $p \in M$, *the differential* $d(\exp_p)_0 : T_0(T_pM) \cong T_pM \to T_pM$ *is the identity map under the natural identification of* T_pM *and* $T_0(T_pM)$.

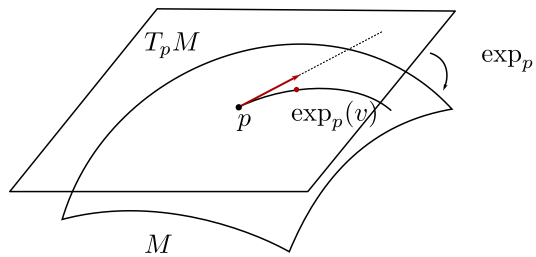

Figure 8.1: Exponential map

(d) $\mathcal{U}$ is an open subset of TM and $\exp : \mathcal{U} \to M$ is smooth.

[a] A subset U of a vector space V is called star-shaped with respect to a point v_0 if for every $t \in (0,1)$, $v_0 + t(v - v_0) \in U$

We only prove properties (a)-(c) for which we first prove the following lemma.

Lemma 8.2 (Rescaling Lemma) *Let γ_v be the maximal geodesic with initial conditions $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. Then For every $c, t \in \mathbb{R}$*

$$\gamma_v(ct) = \gamma_{cv}(t)$$

required at least one of the two sides is defined.

Proof. We use Corollary 7.22 which states that for any $p \in M$ and $v \in T_pM$, there exists a unique maximal geodesic $\gamma_v : I \to M$ so that $0 \in I$ and

$$\gamma_v(0) = p, \qquad \dot\gamma_v(0) = v.$$

If $c = 0$, then $\gamma_0(t) = p = \gamma_v(0)$ for all $t \in \mathbb{R}$. Assume that $c \neq 0$ and $\gamma_v(ct)$ is defined. Let define a new curve $\tilde\gamma$ by $\tilde\gamma(t) := \gamma_v(ct)$, defined on $J = \{t : ct \in I\}$. We show that $\tilde\gamma$ is a geodesic with initial point p and initial velocity cv. Then we use the uniqueness of a (maximal) geodesic to conclude that it has to be equal to γ_{cv}. It is immediate from the definition that $\tilde\gamma(0) = \gamma_v(0) = p$. The chain rule gives

$$\dot{\tilde\gamma}(t) = c\dot\gamma_v(ct).$$

Thus, $\dot{\tilde\gamma}(0) = c\dot\gamma_v(0) = cv$. Let D_t and $\tilde D_t$ denote the covariant differentiation along γ and $\tilde\gamma$ respectively. We use equation (7.3). Writing $\tilde\gamma(t) = (\tilde\gamma^1(t), \cdots, \tilde\gamma^n(t))$ in a local coordinates, we get

$$\begin{aligned}
\tilde D_t \dot{\tilde\gamma}(t) &= \big(\ddot{\tilde\gamma}^k(t) + \dot{\tilde\gamma}^i(t)\dot{\tilde\gamma}^j(t)\Gamma^k_{ij}(\tilde\gamma(t))\big)\partial_k|_{\tilde\gamma(t)} \\
&= \big(c^2\ddot\gamma^k(ct) + c^2\dot\gamma^i(ct)\dot\gamma^j(ct)\Gamma^k_{ij}(\gamma_v(ct))\big)\partial_k|\gamma_v(ct) \\
&= c^2 D_s\dot\gamma(s) = 0,
\end{aligned}$$

where $s = ct$. Thus, $\tilde\gamma$ is a geodesic and it should be equal to $\gamma_{cv}(t)$ by the uniqueness of the maximal geodesic.
Now assume that $\gamma_{cv}(t)$ exists. Then define $u := cv$ and $s := ct$. So it is equivalent to say $\gamma_u(c^{-1}s)$ exists. By the above argument, $\gamma_u(c^{-1}s) = \gamma_{c^{-1}u}(s) = \gamma_v(ct)$. $\square$

Proof. [Proof of Theorem 8.1]

(a) It is clear that $\mathcal{U}$ contains $0 \in T_pM$ for every $p \in M$. If $v \in \mathcal{U}_p$, then by the definition, $\gamma_v(t)$ is defined for any $t \in [0,1]$. Recall that we say $\mathcal{U}_p$ is star-shaped with respect to $0 \in T_pM$ if $tv \in \mathcal{U}_p$ for all $t \in [0,1]$. By Lemma 8.2, we have $\gamma_{tv}(s) = \gamma_v(ts)$ for any $s \in [0,1]$. Therefore, $tv \in \mathcal{U}_p$.

(b) We apply Lemma 8.2 for $t = 1$. It states that $\gamma_v(t) = \gamma_{tv}(1) = \exp(tv)$ provided either side is defined.

(c) We will use the following simple fact. Let $F : M \to N$ be a smooth map between smooth manifolds M and N, and $\gamma : I \to M$ be a curve in M. Then

$$\frac{d}{dt}(F \circ \gamma)|_{t=t_0} = dF_{\gamma(t)}(\dot{\gamma}(t))|_{t=t_0}. \tag{8.1}$$

For any given $v \in T_pM$, consider $\gamma : \mathbb{R} \to T_pM$, $\gamma(t) = tv$. Notice $\gamma(0) = 0$ and $\dot{\gamma}(0) = v$. We now calculate $d(\exp_p)_0$ using (8.1) and part (b).

$$d(\exp_p)_0(v) = \frac{d}{dt}(\exp_p(tv))|_{t=0} = \frac{d}{dt}\gamma_v(t)|_{t=0} = \dot{\gamma}_v(0) = v$$

d) The proof of this part is related to the proof of uniqueness and existence of geodesics. You can see Chapter 5 of the book of Introduction to Riemannian Geometry, by J. Lee.

□

Definition 8.3 A Riemannian manifold (M, g) is said to be *geodesically complete* if for every $p \in M$, the restricted exponential map $\exp_p$ is defined for all $v \in T_pM$.

The restricted exponential map $\exp_p$ defines a special coordinate chart centred at $p \in M$. It is called the *normal coordinate chart centred at* p. To define it we first introduce a *normal neighbourhood* of $p \in M$.

Theorem 8.4 *For any $p \in M$, there exists a neighbourhood V of $0 \in T_pM$ and a neighbourhood U of $p \in M$ such that $\exp_p : V \to U$ is a diffeomorphism.*

Definition 8.5 a) Any open neighbourhood of p which is diffeomorphic to an open neighbourhood of $0 \in T_pM$ under the $\exp_p$ is called a *normal neighbourhood*.

b) Let $\epsilon > 0$ be such that $\exp_p$ is a diffeomorphism on the ball $B_\epsilon(0) \subset \mathcal{U}_p \subset T_pM$ of radius ϵ centred at $0 \in T_pM$. Here, the radius of the ball is measured by the norm induced by g_p. Then the image of $B_\epsilon(0)$ under the restricted exponential map is called a *geodesic ball*. If moreover $\overline{B_\epsilon}(0) \subset \mathcal{U}_p$, then $\exp_p(\partial \overline{B_\epsilon}(0))$ is called a *geodesic sphere*.

Proof. [Proof of Theorem 8.4] Theorem 8.1 part (c) shows that $d\exp_p$ is invertible at 0. Then the inverse functions theorem, see Theorem 3.5, implies that there is a a neighbourhood V of $0 \in T_pM$ such that $\exp_p : V \to T(V)$ is a diffeomorphism. So, we take $U = T(V)$ and the statement follows. □

We now ready to define a normal coordinate chart.

Definition 8.6 Let $\{E_i\}$ be an orthonormal basis for T_pM. We define an isomorphism $F : \mathbb{R}^n \to T_pM$ by $F(e_i) = E_i$. A coordinate chart (U, ϕ) with U a normal neighbourhood of p and $\phi = F^{-1} \circ \exp_p^{-1} : U \to \mathbb{R}^n$ is called *(Riemannian) normal coordinate chart centred at p.*

What is so special about a normal coordinate chart? The following theorem summarise some of the main properties of a normal coordinate chart.

Proposition 8.7 *Let (U, ϕ), $\phi = (x^1, \cdots, x^n)$, be a normal coordinate chart centred at $p \in M$.*

i) The coordinate of p is $(0, \cdots, 0)$, and any geodesic γ_v with initial conditions $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$ in normal coordinates is represented by

$$\gamma_v(t) = (tv^1, \cdots, tv^n), \qquad v = v^i\partial_i.$$

ii) At point p, $g_{ij}(p) = \delta_{ij}$. (Notice It does not hold at other points $q \neq p$ in general.)

Proof. The proof is an immediate consequence of Theorem 8.1 parts (b) and (c). We leave it as an exercise. □

We call the geodesics passing through p and lying in a normal coordinate chart centred at p *radial geodesics*. By Proposition 8.7, they have a simple expression $\gamma_v(t) = (tv^1, \cdots, tv^n)$. Note that other geodesic lying in a normal neighbourhood of p and not passing through the centre p do not have this simple expression in general.

Definition 8.8 In a normal coordinate chart (U, ϕ) centred at p, we define the *radial distance function r*

$$r(q) := (\sum_{i=1}^{n}(x^i(q))^2)^{1/2}, \qquad q \in U$$

and the radial vector field

$$\partial_r := \frac{x^i}{r}\partial_i.$$

In $\mathbb{R}^n$ with standard coordinate chart, the radial distance function $r(x)$ is the distance to the origin, and $\frac{\partial}{\partial r}$ is the unit vector field tangent to straight lines passing through the origin. We shall see below that they have a similar geometric meaning on a Riemannian manifold. We first need to show that it is independent of the choice of a normal coordinate chart.

Proposition 8.9 *In any normal neighbourhood U of $p \in M$, the definition of the radial distance functions $r(x)$ and the radial vector field ∂_r is independent of the choice of a normal coordinate chart.*

Exercise 8.10 Prove the above proposition.
Hint: show that any two normal coordinates (x^i) and $(\tilde{x}^i)$ are related by $x^i(q) = A_i^j\tilde{x}^j(q)$, where $(A_i^j) \in O(n, \mathbb{R})$ is a fixed matrix independent of $q \in U$.

The following proposition and exercise summarise some properties of ∂_r. Later, we shall see that the normal coordinates and the radial vector field will be used to show that geodesics are locally minimising curves.

Proposition 8.11 *The radial vector field ∂_r is a unit vector field.*

Proof. We fix a normal coordinate chart centred at p on U and $q \in U \setminus \{p\}$. Denote

$$q^i := x^i(q), \qquad r_q := r(q) = \sqrt{(q^1)^2 + \cdots + (q^n)^2}.$$

Thus,

$$\partial_r|_q = \frac{q^i}{r_q}\partial_i|_q.$$

Let $v = \frac{q^i}{r_q}\partial_i|_p \in T_pM$. Then by Proposition 8.1 part i), the radial geodesic $\gamma_v(t)$ with $\gamma_v(0) = p$ and $\dot{\gamma}_v(0)$ has the following expression

$$\gamma_v(t) = (t\frac{q^1}{r_q}, \cdots, t\frac{q^n}{r_q}).$$

Hence, $\gamma_v(r_q) = q$ and $\dot{\gamma}_v(r_q) = \partial_r|_q$. Note that

$$|v|_g := \sqrt{g_p(\dot{\gamma}_v(0), \dot{\gamma}_v(0))} = \sqrt{(v^1)^2 + \cdots + (v^n)^2} = \frac{1}{r_q}\sqrt{(q^1)^2 + \cdots + (q^n)^2} = 1.$$

Recall from Exercise 7.30, that the norm of $\dot{\gamma}_v(t)$ is constant. Therefore, $\left|\dot{\gamma}_v(r_q)\right|_g = \left|\partial_r|_q\right| = 1$. Since point $q \in U$ is taken arbitrarily, we conclude that ∂_r is a unit vector field. □

Exercise 8.12 a) Show that $dr = \sum_{i=1}^n \frac{x^i}{r}dx^i$, where $r : M \to \mathbb{R}_{>0}$ is the radial distance function. Conclude that $dr_q(\partial_r|_q) = 1$ for any $q \in U \setminus \{p\}$. Here, U is a normal neighbourhood centred at p.
Hint: Use the expression given in (5.3).

Solution. Let (U, ϕ) be a normal coordinate chart centred at p. Consider

$$f := r \circ \phi^{-1} : B_\delta(0) \to \mathbb{R},$$

$$(x^1, \cdots, x^n) \mapsto \sqrt{(x^1)^2 + \cdots + (x^n)^2},$$

By equation (5.3) we have

$$df = d(r \circ \phi^{-1}) = d((\phi^{-1})^* r) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}dx^i.$$

Hence, $df = \sum \frac{x^i}{\sqrt{(x^1)^2+\cdots+(x^n)^2}}dx^i = \sum \frac{x^i}{r\circ\phi^{-1}}dx^i$. Now by Proposition 5.38, we have

$$dr = d(\phi^*(\phi^{-1})^* r) = \phi^* d((\phi^{-1})^* r) = \phi^* df = \sum \frac{x^i}{r \circ \phi^{-1} \circ \phi}dx^i = \sum \frac{x^i}{r}dx^i,$$

where x^i denotes the ith component of $\phi(q)$, $q \in M$, and dx^i denotes $\phi^* dx^i$ which is the dual of $\partial_i|_q$. Indeed,

$$\begin{aligned}\phi^* dx^i|_q(\partial_j|_q) = dx^i|_{\phi(p)}(d\phi_p(\partial_i|_p)) &= dx^i|_{\phi(p)}(d\phi_p(d(\phi^{-1})|_{\phi_p}(\frac{\partial}{\partial x^j})|_\phi(p)) \\ &= dx^i|_{\phi(p)}(\frac{\partial}{\partial x^j})|_\phi(p)) = \delta_{ij}.\end{aligned}$$

We conclude

$$dr|_q(\partial_r|_q) = \sum \frac{x^i(q)}{r(q)} dx^i|_q(\sum \frac{x^i(q)}{r(q)} \partial_i|_q) = \frac{r(q)^2}{r(q)^2} = 1.$$

b) Let $\delta > 0$ be so that $\exp_p(\overline{B_\delta}(0)) \subset U$. Then use the radial distance function $r : M \to \mathbb{R}$ to show that the geodesic sphere $\exp_p(\partial \overline{B_\delta}(0)) \subset U$ is an embedded submanifold of M of dimension $n-1$. Here, n is the dimension of M.
Hint: Use the regular value theorem.

Solution. We first show that $r^{-1}(\delta) = \exp_p(\partial \overline{B_\delta}(0))$. Let $F : \mathbb{R}^n \to T_pM$ be an isometry so that $\phi = F^{-1} \circ \exp_p^{-1}$. Thus, $q \in \exp_p(\partial \overline{B_\delta}(0))$ if and only if $\phi(q) = (x^1(q), \cdots, x^n(q)) \in F^{-1}(\partial \overline{B_\delta}(0))$ or $\exp_p^{-1}(q) = F(x^1(q), \cdots, x^n(q)) \in \partial \overline{B_\delta}(0)$ But since F is an isometry, we have $(x^1)^2 + \cdots + (x^n)^2 = \delta^2$.
We now show that $\delta > 0$ is a regular value of r. For any $q \in r^{-1}(\delta)$, we have $dr|_q(\partial_r|q) = 1$. Hence, for any $q \in r^{-1}(\delta)$, $dr_q : T_qM \to T_\delta\mathbb{R} \cong \mathbb{R}$ has rank one and is surjective. Therefore, δ is a regular value. By the regular value theorem, we conclude that $r^{-1}(\delta) = \exp_p(\partial \overline{B_\delta}(0))$ is an embedded submanifold of dimension $n-1$.

c) Recall the definition of $\operatorname{grad} r$ from the assessed homework. Show that $\operatorname{grad} r$ is orthogonal to to the geodesic spheres in $U \setminus \{p\}$.
Hint: Use Proposition 3.16.

Solution. Let $\exp_p(\partial \overline{B_\delta}(0))$ be a geodesic sphere in $U \setminus \{p\}$. By part (b) and Proposition 3.16, we know that $T_q(\exp_p(\partial \overline{B_\delta}(0))) = \ker(dr|_q)$ for any $q \in r^{-1}(\delta)$. Thus for any $X \in T_q(\exp_p(\partial \overline{B_\delta}(0)))$

$$dr_q(X) = g_q(\operatorname{grad} r, X) = 0.$$

8.2 Minimising property of geodesics

In $\mathbb{R}^n$, geodesics are (globally) length minimising curves. We have defined a geodesic on a Riemannian manifold to be a smooth curve γ satisfying the geodesic equations $D_t\dot{\gamma} \equiv 0$. This is a local property. In this section, we study some minimising properties of geodesics e.g. we shall see any geodesic is locally a length minimising curve. On a Riemannian manifold (M, g), we can measure the length $\ell_g(\gamma)$ of a smooth curve $\gamma : I \to M$:

$$\ell_g(\gamma) := \int_I |\dot{\gamma}(t)|_g dt,$$

where $|\dot{\gamma}(t)|_g^2 := g(\dot{\gamma}(t), \dot{\gamma}(t))$. Notice that the length of a curve is independent of parametrisation. See Introduction to Geometry unit. The definition of the length of curve can be extended to a piecewise smooth curve. A curve $\gamma : (a, b) \to M$ is piecewise smooth for there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that $\gamma|_{[t_i,t_{i+1}]}$ is smooth. We can now introduce the Riemannian distance function.

Definition 8.13 (Riemannian distance function) Let (M, g) be a Riemannian manifold. The distance function $d : M \times M \to \mathbb{R}_{>0}$ is defined by

$$d_g(p, q) = \inf\{\ell_g(\gamma) : \gamma : [0, 1] \to M \text{ piecewise smooth curve}, \gamma(0) = p, \gamma(1) = q\}.$$

A geodesic segment $\gamma : [a, b] \to M$ is called minimising if $\ell_g(\gamma) \leq \ell_g(c)$ for any piecewise smooth curve c joining $\gamma(a)$ to $\gamma(b)$.

Theorem 8.14 *Let $p \in M$ and U is a normal neighbourhood of p, and $B \subset U$ a geodesic ball centred at p and it closure also in U. Let $\gamma : [0, 1] \to B$ be geodesic segment with $\gamma(0) = p$. If $c : [a, b] \to M$ is any piecewise smooth curve joining $\gamma(0)$ to $\gamma(1)$ then $\ell_g(\gamma) \leq \ell_g(c)$, and if equality holds then the image of c coincides with the image of γ, i.e. $\gamma([0, 1]) = c([a, b])$.*

Let U be a normal neighbourdhood of $p \in M$ and $\delta > 0$ be so that $\exp_p(\overline{B}_\delta(0)) \subset U$. Since the restricted exponential map $\exp_p : B_\delta(0) \to V := \exp_p(B_\delta(0))$ is a diffeomorphism, any coordinate chart on $B_\delta(0)$ gives rise to a coordinate chart on V. We only need to compose it with $\exp_p^{-1}$. A very useful coordinate chart on a ball is polar coordinate chart, where the coordinate map is given by $(r, \theta^1, \cdots, \theta^{n-1})$, with $0 < r < \delta$ and $\theta^i \in (a, \pi + a)$ for some $a \in \mathbb{R}$. For any $q \in U \setminus \{p\}$, consider $\Sigma_{r_q} = \exp_p(\partial \overline{B_{r_q}}(0))$ the geodesic sphere containing q. Σ_{r_q} in the normal coordinate is given by

$$\Sigma_{r_q} = \{x \in U : \sqrt{(x^1)^2 + \cdots + (x^n)^2} = r_q\}.$$

Theorem 8.15 (Gauss Lemma) *Let (M, g) be a Riemannian manifold and B a geodesic ball centred at $p \in M$. Let r be the radial distance function and ∂_r the radial vector field. Then ∂_r is orthogonal to the geodesic spheres in $B \setminus \{p\}$. Moreover, $\partial_r = \operatorname{grad} r$.*

Proof. Let $q \in B \setminus \{p\}$ and γ be the radial geodesic connecting p to q. Note that $\gamma(t) = \exp_p(tv)$ where tv is the curve connecting 0 to $\exp_p^{-1}(q)$ and $g_p(v, v) = 1$. Fix $i \in \{1, \ldots, n-1\}$ and consider $\frac{\partial}{\partial\theta^i}$ and ∂_r along γ. We first show that $g(\frac{\partial}{\partial\theta^i}, \partial_r)$ is constant along γ. In other word we show that $\frac{d}{dt} g(\frac{\partial}{\partial\theta^i}, \partial_r) = 0$. From the proof of Proposition 8.11, we have

$$\dot\gamma(t) = \partial_r|_{\gamma(t)}.$$

We calculate the derivative using the above identity.

$$\begin{aligned}
\frac{d}{dt} g(\frac{\partial}{\partial\theta^i}, \partial_r) &= \dot\gamma g(\frac{\partial}{\partial\theta^i}, \dot\gamma) \\
\big(\text{compatibility of } \nabla \text{ with } g\big) &= g(\nabla_{\dot\gamma} \frac{\partial}{\partial\theta^i}, \dot\gamma) + g(\frac{\partial}{\partial\theta^i}, \nabla_{\dot\gamma}\dot\gamma) \\
\big(\text{symmetric property of } \nabla\big) &= g(\nabla_{\frac{\partial}{\partial\theta^i}} \dot\gamma, \dot\gamma) + 0 \\
\big(\text{compatibility of } \nabla \text{ with } g\big) &= \frac{1}{2} \frac{\partial}{\partial\theta^i} g(\dot\gamma, \dot\gamma) \\
\big(\text{since } g(\dot\gamma, \dot\gamma) = g(\partial_r, \partial_r) = 1\big) &= 0.
\end{aligned}$$

Therefore, $g(\frac{\partial}{\partial\theta^i},\partial_r)$ is constant along γ. We now take the limit as $t\to 0$:

$$\lim_{t\to 0} g(\frac{\partial}{\partial\theta^i},\partial_r)=0, \quad \text{(why?)}$$

Since θ^i and q were arbitrary, this show that $g(\frac{\partial}{\partial\theta^i},\partial_r)=0$.
Let prove that $\partial_r=\operatorname{grad} r$. We write $\operatorname{grad} r$ in polar coordinate frame. $\operatorname{grad} r = a^r\partial_r + \sum_{i=1}^{n-1} a^i \frac{\partial}{\partial\theta^i}$. We know from the exercise above that $g(\operatorname{grad} r, \frac{\partial}{\partial\theta^i}) = dr(\frac{\partial}{\partial\theta^i})=0$. Hence,

$$\sum_{j=1}^{n-1} a^j g(\frac{\partial}{\partial\theta^j},\frac{\partial}{\partial\theta^i})=0.$$

Since $(g_{ij})=(g(\frac{\partial}{\partial\theta^j},\frac{\partial}{\partial\theta^i}))$ is invertible, we conclude that $a^i=0$, $i=1,\cdots,n-1$. Again from the exercise we know

$$g(\operatorname{grad} r,\partial_r)=a^r=dr(\partial_r)=1.$$

Therefore, $\operatorname{grad} r=\partial_r$. □

Proof. [Proof of Theorem 8.14] The length of c is invariant under re-parametrisation. Thus, we can assume that $c:[0,1]\to M$ after reparametrisation. We first assume that $c([0,1])\subset B$. Since $\exp_p$ is diffeomorphism on U and $\overline{B}\subset U$, we can consider the geodesic polar coordinate chart on $B\setminus\{p\}$. So, c in this coordinate chart can be written as

$$c(t)=(r(t),\theta^1(t),\cdots,\theta^{n-1}(t)).$$

Hence, except for finite number of point at which c is not differentiable we have

$$\dot c(t)=(\dot r(t),\dot\theta^1(t),\cdots,\dot\theta^{n-1}(t))).$$

Note that in polar coordinate chart $\gamma(t)$ is a radial geodesic. So all θ^i components are constants and $\dot r(t)$ is also constant. Moreover $r(1)=\ell_g(\gamma)$ (why?).
By the Gauss Lemma we get

$$\begin{aligned}\ell_g(c) &= \int_\epsilon^1 |\dot c(t)|_g dt = \int_\epsilon^1 \sqrt{\dot r(t)^2+g_{ij}(c(t))\dot\theta^i(t)\dot\theta^j(t)}dt\\ &\geq \int_\epsilon^1 |\dot r(t)|dt \geq \int_\epsilon^1 \dot r(t)dt = r(1)-r(\epsilon) \xrightarrow[\epsilon\to 0]{} r(1)=\ell_g(\gamma).\end{aligned}$$

We have the equality if and only if all θ^i component of c are constant and $|\dot r(t)|=\dot r(t)$. Hence, c is a monotonic reparametrisation of γ and we conclude $\gamma([0,1])=c([0,1])$. If $c([0,1])$ is not a subset of $\overline{B}$, then let $t_0\in(0,1)$ be the first time so that $c(t_0)\in\partial B$. Therefore,

$$\ell_g(c)\geq \ell_g(c|_{[0,t_0]})\geq r\geq \ell_g(\gamma).$$

□

Theorem 8.16 (Hopf-Rinow theorem) *Let (M,g) be a connected Riemannian manifold and $p\in M$. The following statements are equivalent.*

a) $\exp_p$ *is defined on all* T_pM.

b) The closed and bounded sets of M *are compact.*

c) M *is geodesically complete. See Definition 8.3.*

d) (M, d_g) is complete as a metric space.

Moreover, any of the statements above implies that

f) For any $q \in M$, there exists geodesic γ joining p to q with $\ell_g(\gamma) = d(p, q)$.

From the Unit of Metric spaces, we know that any compact metric space is complete. A compact manifold without boundary is usually called a *closed* manifold. Therefore,

Corollary 8.17 *a) Any compact connected Riemannian manifold is complete.*

b) A connected closed submanifold of a complete Riemannian manifold is complete with the induced metric. In particular, any closed connected submanifold of Euclidean space is complete.

Proof. The first part of the statement is an immediate corollary of Theorem 8.16 and the discussion above. For the second part of the statement, by Nash's embedding theorem 6.18, any Riemannian manifold can be embedded isometrically in $\mathbb{R}^m$ for some $m \in \mathbb{N}$. Therefore, since it is closed and connected then it is complete as a metric space. $\square$

8.3 Geodesics of the model spaces

We study geodesics and the exponential map on model Riemannian manifolds. These are

- the Euclidean space $(\mathbb{R}^n, g_{Euc})$,
- spheres $(\mathbb{S}^n_R, \mathring{g}_R)$, where $\mathbb{S}^n_R = \{x \in \mathbb{R}^{n+1}, (x^1)^2 + \cdots + (x^{n+1})^2 = R^2\}$ and $\mathring{g}_R$ is the induced metric from $\mathbb{R}^{n+1}$,
- hyperbolic spaces $(\mathbb{H}^n, \breve{g}_R)$ where $\mathbb{H}^n_R = \{(x^1, \cdots, x^{n-1}, x^n) : x^n > 0\}$ and

$$\breve{g}_R = R^2 \frac{(dx^1)^2 + \cdots + (dx^1)^2}{(x^n)^2}.$$

  It is called the Poincaré upper half-space model of the hyperbolic space.

Two dimensional models spaces have been discussed in the unit of Introduction to Geometry.

Euclidean space $(\mathbb{R}^n, g_{Euc})$ We have already discussed that a curve γ in $\mathbb{R}^n$ is a geodesic if and only if it is part of a straight line. See Exercise 7.19. And maximal geodesics are straight lines. It is easy to check that the restricted exponential map, $\exp_p$, $p \in \mathbb{R}^n$ is the identity and is defined on the whole $T_p\mathbb{R}^n \cong \mathbb{R}^n_p$. Therefore, $\mathbb{R}^n$ is a geodesically complete Riemannian manifold.

Spheres $(\mathbb{S}^n_R, \mathring{g}_R)$**.** We first analyse the 2-dimensional sphere. We consider spherical coordinate chart $p \mapsto (\theta(p), \varphi(p))$, $p \in \mathbb{S}^2_R \subset \mathbb{R}^3$. Then we have the following parametrisation.

$$(\theta, \varphi) \mapsto (R \sin\varphi \cos\theta, R \sin\varphi \sin\theta, R \cos\varphi).$$

Exercise 8.18 Consider $\mathbb{S}^2 \subset \mathbb{R}^3$ with the standard metric $\mathring{g}_R$.

a) Show that $\mathring{g}_R$ in the polar coordinate chart has the following expression.

$$\mathring{g}_R = R(d\varphi)^2 + R^2 \sin^2 \varphi (d\theta)^2.$$

b) Compute the Christoffel symbols of $\mathring{g}_R$ in spherical coordinate chart.

c) Using the geodesic equation in spherical coordinate chart, verify that all meridians curves

$$\gamma(t) = (\theta(t), \varphi(t)) = (\theta_0, ct)$$

passing through north pole $(0, R)$ are geodesics.

We now state a very useful lemma to conclude that all great circles are geodesics.

Lemma 8.19 *Let (M, g) and (N, h) be Riemannian manifolds and $F : M \to N$ an isometry. Then if $\gamma_v : I \to M$ is a maximal geodesic in M with initial conditions $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$. Then $\tilde{\gamma}(t) = F(\gamma(t))$ is a maximal geodesic in N with initial conditions $\tilde{\gamma}(0) = F(p)$ and $\dot{\tilde{\gamma}}(0) = dF_p(v)$.*

We use this lemma to conclude that any geodesic on $\mathbb{S}^2_R$ is periodic constant-speed curves whose image is a great circle. Let $\gamma : I \to \mathbb{S}^2_R$ be a maximal geodesic with initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Notice that any element of $O(2, \mathbb{R})$ is an isometry of $\mathbb{S}^2_R$. And there exists $A \in O(2, \mathbb{R})$ which maps p to $(0, R)$. By Lemma 8.19, $A \circ \gamma$ is a geodesic. Therefore, by the uniqueness of geodesics it should be of the form $\gamma(t) = (\theta(t), \varphi(t)) = (\theta_0, ct)$. Any great circle is mapped to another great circle by an element of $O(2, \mathbb{R})$. Therefore, a curve $\gamma : \mathbb{R} \to \mathbb{S}^2_R$ is a maximal Reimannian geodesic if and only if it is a periodic constant-speed curve whose image is a great circle.
We show that this is the case in any dimension.

Theorem 8.20 *A curve $\gamma : \mathbb{R} \to \mathbb{S}^n_R$ is a maximal Reimannian geodesic if and only if it is a periodic constant-speed curve whose image is a great circle. Thus, $\mathbb{S}^n_R$ is geodesically complete.*

Proof. Take an arbitrary point $p \in \mathbb{S}^n_R \subset \mathbb{R}^{n+1}$. Recall that in one of the homework you showed that $v \in T_p\mathbb{S}^n_R \subset \mathbb{R}^{n+1}$ if and only if $g_{Euc}(p, v) = 0$ (note that we here consider p and v as two vectors in $\mathbb{R}^{n+1}$). Let $v \in T_p\mathbb{S}^n_R$ and Π be the plane generated by p, v. We define a smooth curve $\gamma : \mathbb{R} \to \mathbb{S}^n_R \cap \Pi$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$ and we claim that it is also a geodesic.

$$\gamma(t) = \cos(\tfrac{|v|}{R}t)p + \sin(\tfrac{|v|}{R}t)\frac{Rv}{|v|}.$$

It is clear that γ satisfied the initial conditions. We need to verify that $\gamma(t) \in \mathbb{S}^n_R$.

$$|\gamma(t)|^2 = |p|^2 \cos^2(\tfrac{|v|}{R}t) + R^2 \sin^2(\tfrac{|v|}{R}t) = R^2.$$

Let D_t be a the covariant derivative along curve γ. To show that $\gamma(t)$ is a geodesic, we use the statement Exercise 7.34. Hence,

$$D_t\dot{\gamma}(t) = \pi^\top(\overline{D}_t\dot{\gamma}(t)) = \pi^\top(\ddot{\gamma}^1(t), \cdots, \ddot{\gamma}^{n+1}(t)),$$

where $\overline{D}_t$ is the Euclidean covariant derivative along γ. Let us calculate the second derivative of γ.

$$\dot\gamma(t) = -\frac{|v|}{R}\sin(\tfrac{|v|}{R}t)p + \cos(\tfrac{|v|}{R}t)v.$$

$$\ddot\gamma(t) = -\frac{|v|^2}{R^2}\cos(\tfrac{|v|}{R}t)p + \frac{|v|^2}{R^2}\sin(\tfrac{|v|}{R}t)\frac{Rv}{|v|} = \frac{|v|^2}{R^2}\gamma(t).$$

Since $T_{\gamma(t)}\mathbb{S}^n_R$ consists of all vectors orthogonal to $\gamma(t)$, $\ddot\gamma(t) = 0$ is orthogonal to $T_{\gamma(t)}\mathbb{S}^n_R$ for every $t \in \mathbb{R}$. We conclude

$$D_t\dot\gamma(t) = \pi^\top(\frac{|v|^2}{R^2}\gamma(t)) = 0.$$

The fact that $|\dot\gamma(t)|_{\mathring{g}_R}$ is constant immediately follows. It is also clear that γ is periodic.
Conversely, let C be a great circle. It lies in the intersection of $\mathbb{S}^n_R$ with a plane Π. Let v, w be an orthonormal basis for Π. Then C is the image of $\gamma_v(t)$, as described above, with initial values $\gamma_v(0) = Rw$ and $\dot\gamma(0) = v$. □

In particular, Theorem 8.20 shows that $\mathbb{S}^n_R$ is geodesically complete.

Theorem 8.21 *A nonconstant curve in $\mathbb{H}^n_R$ is a maximal geodesic if and only if it is a constant speed curve which gives an embedding of $\mathbb{R}$ and its image is a line parallel to x^n-axis or a Euclidean circle with centre on $\{x^n = 0\}$.*

Proof. We leave it as an exercise to show the statement of the theorem for $\mathbb{H}^2 = \{(x^1, x^n) : x^n > 0\}$ with $\breve{g} = \frac{(dx)^2+(dy)^2}{y^2}$. You can easily show the same result for $\mathbb{H}^2_R$.
Claim. Every geodesic in $\mathbb{H}^2_R$ is a geodesic in $\mathbb{H}^n_R$.
Proof of the Claim. Let $\gamma : \mathbb{R} \to \mathbb{H}^2_R \subset \mathbb{H}^n_R$ be a geodesic on $\mathbb{H}^2_R$. We show that it satisfied the geodesic equation in $\mathbb{H}^n_R$. Since the image of γ is in $\mathbb{H}^2_R$, therefore

$$\gamma(t) = (\gamma^1(t), 0, \cdots, 0, \gamma^n(t)).$$

Recall the expression of $D_t\dot\gamma$ in coordinates. Here, our coordinate chart is the standard coordinate chart $(x^1, \cdots, x^n)$.

$$D_t\dot\gamma(t) = \big(\ddot\gamma^1(t) + \dot\gamma^i\dot\gamma^j(t)\Gamma^1_{ij}, 0, \cdots, 0, \ddot\gamma^n(t) + \dot\gamma^i\dot\gamma^j(t)\Gamma^n_{ij}\big).$$

Notice that $\dot\gamma^i(0) = 0$ for $i \neq 1, n$. Hence, we recover the geodesic equation for γ in $\mathbb{H}^2_R$ and conclude that $D_t\dot\gamma(t) = 0$ on $\mathbb{H}^n_R$ which proves the claim.
Special case. Let $\gamma : \mathbb{R} \to \mathbb{H}^n_R$ be a maximal geodesic such that $\gamma(0)$ lies on the x^n-axis and $\dot\gamma(0)$ is in the span of $\{\partial_1, \partial_n\}$. There is a maximal geodesic $\tilde\gamma : \mathbb{R} \to \mathbb{H}^2_R$ in $\mathbb{H}^2_R$ with this initial conditions. By Theorem 7.20, we know that maximal geodesics are unique. Therefore, $\tilde\gamma = \gamma$ and satisfied the desired properties.
For general case, we use the isometries of $\mathbb{H}^n_R$ to reduces it to the special case. It is an easy exercise to show that translations and orthogonal transformation in $x' := (x^1, \cdots, x^{n-1})$ variables are isometries of $\mathbb{H}^n_R$:

$$T_a : (x', x^n) \mapsto (x' + a, x^n), \qquad a \in \mathbb{R}^{n-1};$$

$$C_A : (x', x^n) \mapsto (Ax', x^n), \qquad A \in O(n-1, \mathbb{R}).$$

Moreover, these isometries map an Euclidean circle with centre on $\{x^n = 0\}$ to another Euclidean circle with centre on $\{x^n = 0\}$. They also map a line parallel to x^n-axis to another line parallel to x^n axis. Let $\gamma : \mathbb{R} \to \mathbb{H}^n_R$ be a maximal geodesic with initial condition $\gamma(0) = p \in \mathbb{H}^n_R$ and $\dot\gamma(0) = v \in T_p\mathbb{H}^n_R$. There exists $a \in \mathbb{R}^n$ and $A \in O(n-1, \mathbb{R})$ so that $q := C_A \circ T_a(p)$ lies on the x^n-axis and $w := C_A \circ T_a(v)$ lies in the span of $\{\partial_1, \partial_n\}$. By Lemma 8.19, $\tilde{\gamma} := C_A \circ T_a \circ \gamma$ is a geodesic. We are in the special case for which the statement holds. Since isometries does not change any of the properties stated in the theorem, γ has the same property as $\tilde{\gamma}$. $\square$

We end this chapter with the following useful definition.

Definition 8.22 Consider a submanifold N of a Riemannian manifold (M, g) with the induced metric $h = i^*g$. The submanifold (N, h) is called totally geodesic if any geodesic on (N, h) is a geodesic on (M, g).

Example 8.23 Any k dimensional linear subspace of $\mathbb{R}^n$ with the induced metric is totally geodesic. The 2-dimensional hyperbolic plane $\mathbb{H}^2_R$ as discussed above is a totally geodesic submanifold of $\mathbb{H}^n_R$. Any k-dimensional sphere $\mathbb{S}^k_R$ as a submanifold of $\mathbb{S}^n_R$, $n \geq k$, is totally geodesic (why?).

Chapter 9

Curvature

In the unit of Introduction to geometry, the Gaussian curvature of a surface has been defined. Intuitively, it measures how the surface is far from being “flat”. A Riemannian manifold is called *flat* if it is locally isometric to an open subset of $\mathbb{R}^n$. A notion of curvature on a Riemannian manifold was first introduced by Riemann using the concept of the Gaussian curvature of a surface. He took any two dimensional subspace Π of T_pM, and considered the image of the intersection of Π with $\mathcal{U}_p$ (domain of $\exp_p$) under the restricted exponential map $\exp_p$. The image is an embedded 2-dimensional submanifold S of M (why?). Since Gauss had previously proved that the curvature of a surface can be expressed in term of the metric, Riemann could speak of the Gaussian curvature of S in terms of the induced metric from M. The notion of the curvature that Riemann studied is called the Sectional curvature of Π, see below. However, a more general and workable formula for the curvature of Riemannian manifold was developed after Riemann. This is called the (Riemann) curvature tensor.

9.1 Curvature tensor

Let (M,g) be an n-dimensional Riemannian manifold and ∇ be the Levi-Civita connection.

Definition 9.1 The curvature R of a Riemannian manifold (M,g) is a map defined by

$$R:\Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\to\Gamma(TM)$$

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z.$$

When we want to emphasis the role of the metric we sometimes denote R by R_g.

Proposition 9.2 *The curvature map is multilinear map $R:\Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\to\Gamma(TM)$ over $C^\infty(M)$.*

Proof. Let $f\in C^\infty(M)$. We show

$$R(fX,Y)Z=fR(X,Y)Z$$

$$R(X,fY)Z=fR(X,Y)Z$$

$$R(X,Y)fZ = fR(X,Y)Z.$$

We show the first identity. The second identity follows from the first one because

$$R(X,Y)Z = -R(Y,X)Z.$$

We leave the last identity as an exercise. Let us prove the first identity.

$$\begin{aligned}
R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z \\
&= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]+(Yf)X}Z \\
&= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]}Z - (Yf)\nabla_X Z + (Yf)\nabla_X Z \\
&= fR(X,Y)Z.
\end{aligned}$$

□

A multilinear map $F : \Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\to\Gamma(TM)$ over $C^\infty(M)$ is called a $(3,1)-$tensor field. So, the curvature R is a $(3,1)$-field. See Appendix for more details on tensor fields. **??**

Exercise 9.3 Show that the components of the curvature tensor in any smooth coordinate chart are given by

$$R^l_{ijk} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm},$$

where $R(\partial_i,\partial_j)\partial_k = R^l_{ijk}\partial_l$.

Definition 9.4 We define the *Riemann curvature tensor field* of a Riemannian manifold (M,g) to be

$$\mathrm{Rm}(X,Y,Z,W) := g(R(X,Y)Z,W).$$

To show its dependency to the metric we sometimes use Rm_g instead of Rm.

The Riemann curvature Rm a mulitilinear map $\mathrm{Rm}:\Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\to C^\infty(M)$. In the other word, we say that it is a $(4,0)$-tensor field. See Appendix **??** for more details on tensor fields.

Note that R and Rm are both local invariants. Let (U,ϕ) be a coordinate chart with $\phi = (x^1,\cdots,x^n)$ and $\{\partial_i\}$ coordinate frame. In a coordinate chart the curvature map and the Riemann curvature are identified by R^l_{ijk} where it is defined by

$$R(\partial_i,\partial_j)\partial_k = R^l_{ijk}\partial_l;$$

and

$$R_{ijkl} = \mathrm{Rm}(\partial_i,\partial_j,\partial_k,\partial_l) = g(R(\partial_i,\partial_j)\partial_k,\partial_l).$$

Exercise 9.5 Show that $R_{ijkl} = g_{lm}R^m_{ijk}$ and therefore

$$R_{ijkl} = g_{lm}R^m_{ijk} = g_{lm}\left(\partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}\right).$$

Example 9.6 Let consider $(\mathbb{R}^n, g_{_{Euc}})$. Let us calculate the coefficients $\overline{R}_{ijk}^{l}$ and $\overline{R}_{ijkl}$ of the curvature tensor field $\overline{R}$ and Riemann tensor of $\mathbb{R}^n$ in standard coordinate chart $(x^1, \cdots, x^n)$.

$$\overline{R}(\partial_i, \partial_j)\partial_k = \overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_j}\partial_k - \overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_i}\partial_k - \overline{\nabla}_{[\partial_i,\partial_j]}\partial_k = 0.$$

Therefore, $\overline{R} \equiv 0$ and $\overline{\mathrm{Rm}} \equiv 0$.

Definition 9.7 We say that a Riemannian manifold (M, g) is *flat* if every point $p \in M$ has a neighbourhood that is isometric to an open set in $(\mathbb{R}^n, g_{\mathrm{Euc}})$, i.e. it is locally isometric to a Euclidean space.

How this notion of curvature measures the flatness of a Riemannian manifold? The following theorem answers this question.

Theorem 9.8 *A Riemannian manifold is flat if and only if its curvature R vanishes identitcaly.*

The following proposition summarise some properties of R and Rm.

Proposition 9.9 *Let X, Y, Z, T be smooth vector fields on M.*

a) (Bianchi Identity)

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$

$$Rm(X,Y,Z,T) + Rm(Y,Z,X,T) + Rm(Z,X,Y,T) = 0.$$

b) $Rm(X,Y,Z,T) = -Rm(Y,X,Z,T)$.

c) $Rm(X,Y,Z,T) = -Rm(X,Y,T,Z)$.

d) $Rm(X,Y,Z,T) = Rm(Z,T,X,Y)$

Proof.

a) To prove the Bianchi identity, we use the symmetry of the Levi-Civita connection and the Jacobi identity in Lemma 4.24.

$$\begin{aligned} R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \\ &\quad +\nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X \\ &\quad +\nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y \\ &= \nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y] \\ &\quad -\nabla_{[X,Y]}Z - \nabla_{[Y,Z]}X - \nabla_{[Z,X]}Y \\ &= [X,[Y,Z]] + [X,[Y,Z]] + [X,[Y,Z]] = 0. \end{aligned}$$

Note that the last identity is the consequence of the Jacobi identity.

b) This is just a direct consequence of the definition.

c) We prove that

$$\mathrm{Rm}(X,Y,W,W) = 0 \tag{9.1}$$

for any $W \in \Gamma(TM)$. Then the statement follows from $\mathrm{Rm}(X, Y, Z + T, Z + T) = 0$. We will make use of the following identities.

$$g(\nabla_X \nabla_Y Z, Z) = Y g(\nabla_X Z, Z) - g(\nabla_X Z, \nabla_Y Z),$$

and

$$g(\nabla_{[X,Y]} Z, Z) = \frac{1}{2}[X, Y] g(Z, Z).$$

We leave the proof of the above identities and the remaining details of the proof as an exercise.

d) We use the Bianchi identity for Rm to prove part d).

$$\mathrm{Rm}(X, Y, Z, T) + \mathrm{Rm}(Y, Z, X, T) + \mathrm{Rm}(Z, X, Y, T) = 0;$$

$$\mathrm{Rm}(Y, Z, T, X) + \mathrm{Rm}(Z, T, Y, X) + \mathrm{Rm}(T, Y, Z, X) = 0;$$

$$\mathrm{Rm}(Z, T, X, Y) + \mathrm{Rm}(T, X, Z, Y) + \mathrm{Rm}(X, Z, T, Y) = 0;$$

$$\mathrm{Rm}(T, X, Y, Z) + \mathrm{Rm}(X, Y, T, Z) + \mathrm{Rm}(Y, T, X, Z) = 0.$$

Adding the identities above, we obtain

$$2\mathrm{Rm}(Z, X, Y, T) + 2\mathrm{Rm}(T, Y, Z, X) = 0.$$

This completes the proof.

□

Exercise 9.10 Complete the proof of Proposition 9.9 b) and c).

The identities above can be written in coordinates as follows.

$$R_{ijkl} + R_{jkli} + R_{klij} = 0$$

$$R_{ijkl} = -R_{jikl}, \quad R_{ijkl} = -R_{ijlk}, \quad R_{ijkl} = R_{klij}.$$

9.2 Sectional curvature

Let (M, g) be a Riemannian manifold and $p \in M$. Take any 2-dimensional subspace Π of T_pM. Let X, Y be a basis for Π. We define the *sectional curvature* of Π as follows

$$K(\Pi) = \frac{\mathrm{Rm}(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2}.$$

If different metrics are considered on the same manifold we use $K_g(\Pi)$ instead of $K(\Pi)$ to emphasis the role of the metric.
If X and Y are an orthonormal basis for Π, then $K(\Pi) = \mathrm{Rm}(X, Y, Y, X)$. When M is a 2-dimensional manifold we call the sectional curvature, the *Gaussian curvature*. Let we consider the normal neighbourhood around p. We denote $\exp_p(\Pi \cap \mathcal{U}_p)$ by S_Π. Then $K(\Pi)$ gives the Gaussian curvature of S_Π at p.
We need to check that the definition is independent of the choice of a basis for Π.

Lemma 9.11 *Let $\Pi \subset T_pM$ be a 2-dimensional subspace of T_pM. The sectional curvature $K(\Pi)$ is independent of the choice of a basis for Π.*

Proof. Let $\{X, Y\}$ and $\{X', Y'\}$ be two basis for Π. It is enough to show that $\frac{\mathrm{Rm}(X,Y,Y,Y)}{g(X,X)g(Y,Y)-g(X,Y)^2}$ is invariant under the following transformations

$$\{X, Y\} \to \{Y, X\}, \quad \{X, Y\} \to \{\lambda X, Y\}, \quad \{X, Y\} \to \{X + \lambda Y, Y\}.$$

Because any changes of basis can be written as a composition of the above transformation. It is easy to check that $\frac{\mathrm{Rm}(X,Y,Y,X)}{g(X,X)g(Y,Y)-g(X,Y)^2}$ is invariant under these transformations and we leave it as an exercise. $\square$

The sectional curvature is also invariant under the isometries, i.e. if M and N are isometric and $F : M \to N$ is an isometry between them, then for any $p \in M$ and $\Pi \subset T_pM$, $K(\Pi) = K(dF_p\Pi)$. For the proof of this fact see the last exercise sheet.

Exercise 9.12 Show that the Gaussian curvature of $\mathbb{S}^2_R$ at every point is $1/R^2$ and the Gaussian curvature of $\mathbb{H}^2_R$ at every point is $-1/R^2$.

We show that the sectional curvatures actually determine the curvature tensor R.

Lemma 9.13 *Let T and $\tilde{T}$ be two $(4,0)$-tensors on M (i.e. $T, \tilde{T} : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ multilinear maps over $C^\infty(M)$) such that they satisfies the properties of Proposition 9.9. If for all 2-dimensional plane $\Pi \subset T_pM$ generated by X, Y we have*

$$K(\Pi) := \frac{T(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} = \frac{\tilde{T}(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} =: \tilde{K}(\Pi).$$

Then $T = \tilde{T}$.

Proof. We show that for every $X, Y, Z, W \in T_pM$ we have $T(X, Y, Z, W) = \tilde{T}(X, Y, Z, W)$. By the assumption we have

$$T(X + Z, Y, Y, X + Z) = \tilde{T}(X + Z, Y, Y, X + Z).$$

Thus, after simplification we get

$$T(X, Y, Y, Z) = \tilde{T}(X, Y, Y, Z).$$

We replace Y by $Y + W$ in the above identity and we get.

$$T(X, Y + W, Y + W, Z) = \tilde{T}(X, Y + W, Y + W, Z);$$

$$T(X, Y, Z, W) + T(X, W, Z, Y) = \tilde{T}(X, Y, Z, W) + \tilde{T}(X, W, Z, Y);$$

$$\begin{aligned} T(X, Y, Z, W) - \tilde{T}(X, Y, Z, W) &= \tilde{T}(X, W, Z, Y) - T(X, W, Z, Y) \\ &= T(Y, Z, X, W) - \tilde{T}(Y, Z, X, W) \end{aligned}$$

The last identity show that $T(X, Y, Z, W) - \tilde{T}(X, Y, Z, W)$ is invariant under cyclic permutation of X, Y, Z. Moreover, T and $\tilde{T}$ satisfy the Bianchi identity. After adding up the cyclic permutations and applying the Bianchi identity, we conclude

$$3(T(X, Y, Z, W) - \tilde{T}(X, Y, Z, W)) = 0.$$

□

A Riemannian manifold (M^n, g) is called a Riemann manifold with constant sectional curvature $\kappa \in \mathbb{R}$ if for all $p \in M$ and $\Pi \subset T_pM$, $K(\Pi) = \kappa$, for some $\kappa \in \mathbb{R}$. Riemannian manifolds with constant sectional curvatures play an important role in Riemannian geometry. We show how we can characterise these manifolds by components $R_{ijkl}(p)$ of the Riemann curvature tensor of an orthonormal basis of T_pM.

Theorem 9.14 *Let (M, g) be a Riemannian manifold and define a multilinear map*

$$T : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$$

over $C^\infty(M)$ as follows:

$$T(X, Y, Z, W) := g(X, W)g(Y, Z) - g(Y, W)g(X, Z), \qquad X, Y, W, Z \in \Gamma(TM).$$

Then M has constant sectional curvature $\kappa \in \mathbb{R}$ if and only if $Rm = \kappa T$, where Rm is the Riemann curvature tensor.

Proof. Let assume that M has constant sectional curvature κ. It is easy to check that T satisfies all properties in Proposition 9.9. Moreover, for every $X, Y \in T_pM$ we have

$$R(X, Y, Y, X) = \kappa(g(X, X)g(Y, Y) - g(X, Y)^2) = \kappa T(X, Y, Y, X).$$

Therefore, Lemma 9.13 implies that $Rm = \kappa T$. The converse is obvious. □

Let $\{\partial_i\}$ be the normal coordinate chart centred at p. Then for any $\Pi \subset T_pM$, $K(\Pi) = \kappa \in \mathbb{R}$ if and only if

$$R_{ijkl} = \kappa(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{9.2}$$

This is the same as saying that $K(\Pi)$ has constant sectional curvature κ if and only if $R_{ijij} = -R_{ijji} = \kappa$ for all $i \neq j$ and $R_{ijkl} = 0$ otherwise.
In the following theorem we show that our models spaces have constant sectional curvature.

Theorem 9.15 (Curvature of model spaces) *a) The Euclidean space $(\mathbb{R}^n, g_{Euc})$ has constant sectional curvature 0.*

b) The sphere $(\mathbb{S}^n_R, \mathring{g}_R)$ has constant sectional curvature $1/R^2$.

c) $(\mathbb{H}^n_R, \breve{g}_R)$ has constant sectional curvature $-1/R^2$.

Proof.

a) This is a trivial consequence of Example 9.6.

b) Because of the isometries of $\mathbb{S}^n_R$, it is enough to compute the sectional curvature for the plane Π spanned by $\{\partial_1, \partial_2\}$ at the north pole. The geodesics with initial velocity in Π are great circles in the (x^1, x^2, x^{n+1})-subspace. It is easy to show that S_Π is isometric to $\mathbb{S}^2_R$. Therefore, the Gaussian curvature of S_Π is the same as the Gaussian curvature of $\mathbb{S}^2_R$ which is $1/R^2$.

c) We make the calculation for a more general form of the metric. Two metrics g and $\tilde{g}$ on a manifold are *conformal* if there exists a positive smooth function $f \in C^\infty(M)$ such that $g_p = f(p)\tilde{g}_p$ where . For example, $\breve{g}_R$ is conformal to the Euclidean metric

$$\breve{g}_R = \frac{R^2}{(x^n)^2} g_{Euc}.$$

Let consider a metric g so that $g = \frac{g_{Euc}}{F^2}$. Equivalently $g_{ij} = \frac{\delta_{ij}}{F^2}$ in the standard coordinate chart. We have $g^{ij} = F^2\delta_{ij}$, where g^{ij} are elements of the inverse of g. For the sake of simplicity in notations we define $f := \log F$. We now calculate the Christoffel symbols of metric g.

$$\begin{aligned}\Gamma_{ij}^k &= \frac{1}{2}g^{mk}\Big(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\Big)\\ &= \frac{1}{2}F^2\Big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\Big)\\ &= -\delta_{jk}\partial_i f - \delta_{ki}\partial_j f + \delta_{ij}\partial_k f.\end{aligned}$$

Note that we use $\partial_j g_{ik} = -\delta_{ik}\frac{2}{F^3}\partial_j F = -2\frac{\delta_{ik}}{F^2}\partial_j f$. Let us denote $\partial_i f$ by f_i and $\partial_i f_j = f_{ij}$. Hence, if all three indices are distinct then $\Gamma_{ij}^k = 0$. Otherwise we have

$$\Gamma_{ij}^i = -f_j, \quad \Gamma_{ii}^j = f_j, \quad \Gamma_{ij}^j = -f_i, \quad \Gamma_{ii}^i = -f_i.$$

We now calculate R_{ijkl}.

Exercise 9.16 Using the definition of the Riemann curvature tensor, show that the coordinate expression of Rm is given by

$$R_{ijkl} = g_{lm}R_{ijk}^m = \frac{1}{F^2}R_{ijk}^l = \frac{1}{F^2}\left(\partial_i\Gamma_{jk}^l - \partial_j\Gamma_{ik}^l + \Gamma_{jk}^m\Gamma_{im}^l - \Gamma_{ik}^m\Gamma_{jm}^l\right).$$

Using the above formula, it is clear that when all four indices are distinct then $R_{ijkl} = 0$. When three indices are distinct then

$$R_{ijj}^k = f_i f_k + f_{ki}, \quad R_{iji}^k = -f_j f_k - f_{kj}, \quad R_{ijk}^k = 0.$$

And

$$R_{ijji} = \frac{1}{F^2}(-\sum_l f_l^2 + f_i^2 + f_j^2 + f_{ii} + f_{jj}).$$

Let denote by K_{ij} the sectional curvature of the plane generated by ∂_i, ∂_j. Then

$$K_{ij} = \frac{R_{ijji}}{g_{ii}g_{ij}} = F^4 R_{ijji} = F^2(-\sum_l f_l^2 + f_i^2 + f_j^2 + f_{ii} + f_{jj}).$$

We now take $F = x^n$. Thus $f = \log x^n$. We replace in the above equations and get that $K_{ij} = -1/R^2$. Moreover, R_{ijkl} satisfies the conditions in (9.2). We conclude that $\mathbb{H}_R^n$ has constant sectional curvature $-1/R^2$.

□

The following theorem shows the importance of the Model spaces.

Theorem 9.17 *Let (M, g) be a complete Riemannian manifold of dimension n with constant sectional curvature K. Then M is locally isometric to $\mathbb{R}^n$, $\mathbb{S}_R^n$ and $\mathbb{H}_R^n$ when $K = 0, 1/R^2, -1/R^2$ respectively.*

Appendix A

Vector Bundles

In Section 4.1 we defined the tangent bundle, and mentioned that it is an example of a more general notion of *vector bundles* which we define here. Intuitively, a *vector bundle of rank k* over a manifold M is, *as a set*, a disjoint union E of k-dimensional vector spaces

$$E = \bigsqcup_{p \in M} V_P,$$

parametrized by points in M. We will put more structure on how these vector spaces fit together (in particular, we will want E to be a smooth manifold itself), but before we give the precise definition we look at an example.

Let $U \subset \mathbb{R}^n$ be an open set, and hence a smooth manifold (as it is an open submanifold of $\mathbb{R}^n$). Let $E = U \times \mathbb{R}^k$. Then we can write

$$E = \bigsqcup_{p \in U} \{p\} \times \mathbb{R}^k.$$

Setting

$$V_p = \{p\} \times \mathbb{R}^k,$$

we see that E as a disjoint union of k-dimensional vector space V_p, one for each $p \in U$. Moreover, we can define a map (the *projection*)

$$\pi : U \times \mathbb{R}^k \to U, \quad (p, x) \mapsto p$$

i.e. $\pi(V_p) = \{p\}$. We say that E is *fibered* over M. Each fiber of the map π, i.e. the preimage of a point $p \in M$, is a vector space of dimension k, namely V_p. That is, $\pi^{-1}(p) = V_p$ for each $p \in U$.

Now, a vector bundle is a pair (E, π) that *locally* behaves like the above. More precisely:

Definition A.1 (Vector bundle) Let M be a smooth manifold. A *smooth vector bundle of rank k over M* is a smooth manifold E (called the *total space*) together with a surjective map $\pi : E \to M$ (called the *projection*) such that

(1) For all $p \in M$, the fiber $V_p = \pi^{-1}(p)$ is a k-dimensional vector space (over $\mathbb{R}$).

(2) For all $p \in M$ there exist neighborhood U of p and a diffeomorphism

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$

(called a *local trivialization of E over U*) such that

- $\pi \circ \Phi^{-1}(x, t) = x$ for all $x \in U$ (i.e. "the diagram commutes")
- for all $q \in U$, the restriction of Φ to V_q, $\Phi|_{V_q} : V_q \to \{q\} \times \mathbb{R}^k$ is an isomorphism.

Remark A.2 Here we have only defined *smooth* vector bundles since this is all we will encounter in this course. However, one can define a (topological) vector bundle similarly. For example, in the above definition letting M be a manifold, E any topological space, and the local trivializations being homeomorphisms, then (E, π) is a vector bundle over M. In this case it actually follows that E is a (topological) manifold so this is not needed in the assumption. But vector bundles can be defined in even more general settings, for example, M could be a graph.

Technically, a vector bundle is a pair (E, π) as above (or even a triple (E, M, π)) but we usually refer to E as the vector bundle. If there exist a local trivialization of E over all of M (i.e. a *global* trivialization) then E is said to be the *trivial bundle*. In this case E is homeomorphic to $M \times \mathbb{R}^k$. An example is the example $E = U \times \mathbb{R}^k$ given above. More generally we have the following trivial bundle for any manifold:

Example A.3 (Trivial Bundle) Let M be a smooth manifold and let $E = M \times \mathbb{R}^k$ and $\pi : M \times \mathbb{R}^k \to M$ defined by $\pi(m, x) = m$. Then setting $\Phi = id$ (the identity map) gives a global trivialization, and so E is the trivial bundle.

In general we will need more than one local trivialization. In this case, the transition map at each point is linear:

Lemma A.4 *Let E be a vector bundle of rank k over M. Suppose $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ and $\Phi_\beta : \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^k$ are two smooth local trivialization and $U_\alpha \cap U_\beta \neq \emptyset$. Then there is a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that the map $\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ has the form*

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v).$$

Proof. Since $\pi \circ \Phi_\alpha^{-1}(p, v) = p$ and $\pi \circ \Phi_\beta^{-1}(p, v) = p$ for all (p, v) we have that

$$\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

is of the form

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \sigma(p, v))$$

for some map

$$\sigma : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to \mathbb{R}^k.$$

Note that $\Phi_\alpha \circ \Phi_\beta^{-1}$ is smooth since Φ_α and Φ_β are diffeomorphisms and hence σ is a smooth map.

Moreover, recall that for each $p \in U_\alpha \cap U_\beta$ both

$$\Phi_\alpha|_{V_p} : V_p \to \{p\} \times \mathbb{R}^k \text{ and } \Phi_\beta|_{V_p} : V_p \to \{p\} \times \mathbb{R}^k$$

are isomorphisms. It follows that the map

$$\Phi_\alpha \circ \Phi_\beta^{-1}|_{\{p\}\times\mathbb{R}^k} : \{p\} \times \mathbb{R}^k \to \{p\} \times \mathbb{R}^k, \quad (p, v) \mapsto (p, \sigma(p, v))$$

is an isomorphism. Equivalently, for each p we have

$$\sigma(p) := \sigma_{\{p\}\times\mathbb{R}^k} : \{p\} \times \mathbb{R}^k \to \mathbb{R}^k$$

is an isomorphism (an invertible linear map) and hence given by a matrix $\tau_{\alpha\beta}(p) \in GL(k, \mathbb{R})$. Hence $\sigma(p, v) = \tau_{\alpha\beta}(p)v$ as we needed to show.

□

We recall the definition of the tangent bundle:

Definition A.5 (Tangent Bundle) Let M be a smooth manifold. The *tangent bundle* of M is the vector bundle whose total space is

$$TM = \bigsqcup_{p\in M} T_pM$$

and projection $\pi : TM \to M$ given by $\pi(v_p) = p$ for all $v_p \in T_pM$.

Before moving on, recall also that we define a vector space as a *smooth section* of the tangent bundle. Smooth sections can be defined for any vector bundle:

Definition A.6 (Section) Let (E, π) be a vector bundle over M. A *section* of the bundle is a continuous map $\sigma : M \to E$ such that $\pi(\sigma(x)) = x$ for all $x \in M$. We say σ is a *smooth section* if it is a smooth map.

Next we will show that the tangent bundle is in fact a vector bundle. This can be done using the definition directly, or using the following lemma which provides a slight shortcut: it is enough to construct the local trivializations and ensure they overlap with smooth transition functions.

Proposition A.7 (Vector Bundle Construction Lemma) *Let M be a smooth manifold and for each $p \in M$ let V_p be a vector space of dimension k. Let*

$$E = \bigsqcup_{p\in M} V_p$$

and

$$\pi : E \to M, \quad v_p \to p$$

for all $v_p \in V_p$. Suppose we have the following:

(1) an open cover $\{U_\alpha\}_{\alpha \in A}$ of M

(2) for each $\alpha \in A$, a bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each V_p is a vector space isomorphism from V_p to $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$

(3) for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that the map $\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ has the form

$$\Phi_\alpha \circ {\Phi_\beta}^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v).$$

Then E is a smooth vector bundle over M of rank k with π as a projection and $\{(U_\alpha, \Phi_\alpha)\}$ as local trivializations.

Let M be a smooth n-manifold. We use the above to show that TM is indeed a smooth vector bundle (of rank n) over M. By definition $TM = \bigsqcup_{p \in M} T_pM$ and we know each T_pM is a vector space of dimension n. Let $\pi : TM \to M$ be the map above which maps every point in T_pM to p.

First, let $\mathcal{A} = \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ be a smooth atlas for M. Then $\{U_\alpha\}$ is an open cover of M so (1) is satisfied.

Now, fix $\alpha \in A$. As usual, write $\psi_\alpha = (x^1, \ldots, x^n)$, giving us the coordinate basis

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

for T_pM. Hence for any $v \in T_pM$ there is a unique vector $(v^1, \ldots, v^n) \in \mathbb{R}^n$ such that

$$v = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p.$$

Now, define

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n, \quad (p, v) \mapsto (p, v^1, \ldots v^n).$$

Note that $\pi^{-1}(U_\alpha) = \bigsqcup_{p \in U_\alpha} T_pM$ and for each p, the restriction of Φ_α to T_pM is

$$\Phi_\alpha|_{T_pM} : T_pM \to \{p\} \times \mathbb{R}^n, \quad (p, v) \mapsto (p, v^1, \ldots v^n)$$

is clearly linear and bijective, showing (2).

It remains to check the transition map condition in (3). Suppose $U_\alpha \cap U_\beta \neq \emptyset$. As above $\psi_\alpha = (x^1, \ldots, x^n)$ and say $\psi_\beta = (\tilde{x}^1, \ldots, \tilde{x}^n)$. Let

$$J_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}), \quad p \mapsto \begin{pmatrix} \frac{\partial \tilde{x}^1}{\partial x^1}(p') & \cdots & \frac{\partial \tilde{x}^1}{\partial x^n}(p') \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{x}^n}{\partial x^1}(p') & \cdots & \frac{\partial \tilde{x}^n}{\partial x^n}(p') \end{pmatrix}$$

be the map that sends p to the Jacobian at $p' = \psi_\alpha(p)$ of the map $\psi_\beta : U_\beta \to \mathbb{R}^n$, in the basis given by the ψ_α-coordinates. Let $p \in U_\alpha \cap U_\beta$ and $v \in T_pM$. Writing v as a linear combination of the basis elements, with respect to each basis, we have

$$v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \text{ and } v = \sum_{i=1}^n \tilde{v}^i \frac{\partial}{\partial \tilde{x}^i}$$

where, by the change of coordinate formula,

$$\tilde{v}^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}(p')v^j.$$

Then

$$\Phi_\beta \circ \Phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is given by

$$\begin{aligned} \Phi_\beta \circ \Phi_\alpha^{-1}((p, v^1, \ldots, v^n)) &= \Phi_\beta(p, v) = (p, \tilde{v}^1, \ldots, \tilde{v}^n) && \text{(A.1)} \\ &= \left(p, \sum \frac{\partial \tilde{x}^1}{\partial x^j}(p')v^j, \ldots, \sum \frac{\partial \tilde{x}^1}{\partial x^n}(p')v^j\right) && \text{(A.2)} \\ &= (p, J_{\alpha\beta}(p)v) && \text{(A.3)} \end{aligned}$$

and we are done.

Remark A.8 Let $M = \mathbb{R}^n$. Then the tangent bundle $TM = T\mathbb{R}^n$ is the trivial bundle. Identifying $T_p\mathbb{R}^n = \mathbb{R}^n$ we can identify $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ and so as in Example A.3 we have that TM is the trivial bundle. In fact, the tangent bundle of a Lie group is always trivial, see Lemma 4.10.

The Vector Bundle Construction Lemma can be used to construct new bundles from other vector bundles. All natural operations on vector spaces, such as taking the dual vector space, or direct sums of vector spaces, carry over to operations on vector bundles.

Example A.9 (Vector Bundles) We give some examples:

(1) Let $E = \bigsqcup_{p \in M} V_p$ be a vector bundle over M with projection $\pi : E \to M$. Then the *dual vector bundle* of E is defined as

$$E^* = \bigsqcup V_p^*$$

where V_p^* is the dual of V_p, with projection map

$$\pi^* : E^* \to M$$

whose fibers are $(\pi^*)^{-1}(p) = V_p^*$.

Given that (E, π) is a vector bundle we prove that (E^*, π^*) is indeed also a vector bundle, using the Vector Bundle Construction Lemma:

First, recall that the dual of a vector space V is the vector space

$$V^* = \{f : V \to \mathbb{R} \mid f \text{ is linear}\}.$$

Also, if V, W are vector spaces, and $L : V \to W$ is a linear map, L induces a map (its dual)

$$L^* : W^* \to V^*$$

given by $w^* \to L^*(w^*)$ defined by $L^*(w^*)(v) = w^*(L(v))$. Moreover, if L is given by the matrix A, then L^* is given by the transpose matrix A^t. In particular, if L is an isomorphism, so is L^*.

Now, since $E = \bigsqcup V_p$ is a vector bundle over M there is an open cover $\{U_\alpha\}$ of M and, for each α, a homeomorphism

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$

whose restriction to V_p is an isomorphism. In particular

$$\left(\Phi_\alpha|_{V_p}\right) : V_p \to \{p\} \times \mathbb{R}^k$$

is a linear map and we let

$$\Phi_\alpha^* : \{p\} \times \mathbb{R}^k \to V_p^*$$

denote its dual (for ease of notation we omit to explicitly write the restriction, and assume Φ_α is restricted to the appropriate domain when taking its dual; we also identify the dual of $\mathbb{R}^k$ with itself). Define, for each α the map

$$\Psi_\alpha^* : (\pi^*)^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$

defined by

$$v^* \mapsto (p, (\Phi_\alpha^*)^{-1}(v^*)) \text{ for all } v^* \in V_p^*$$

Now, Ψ_α^* is bijective since Φ_α is and its restriction to V_p^* is an isomorphism (since the restriction of Φ_α to V_p is). Hence condition (2) is satisfied.

Suppose $U_\alpha \cap U_\beta \neq \emptyset$. Since E is a vector bundle it satisfies Lemma A.4; let

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$$

be the map given by that lemma, meaning that the map

$$\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

is given by

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v).$$

That is, for each p, the linear map

$$\left.\left(\Phi_\alpha \circ \Phi_\beta^{-1}\right)\right|_{\{p\}\times\mathbb{R}^k}$$

is nothing but the matrix $\tau_{\alpha\beta}(p)$ and hence its dual is given by the transpose of this matrix. Equivalently,

$$(\Phi_\alpha \circ \Phi_\beta^{-1}|_{\{p\}\times\mathbb{R}^k})^*(p, v^*) = (p, (\tau_{\alpha\beta}(p))^t v^*).$$

for each p.

Now, define

$$\tau_{\alpha\beta}^* : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}), \quad \tau_{\alpha\beta}^*(p) = \left((\tau_{\alpha\beta}(p))^{-1}\right)^t.$$

It follows that

$$\Psi_\alpha^* \circ (\Psi_\beta^*)^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

has the form

$$\begin{aligned}\Psi^*_\alpha \circ (\Psi^*_\beta)^{-1}(p, v^*) &= \Psi^*_\alpha(p, \Phi^*_\beta(v^*)) = (p, (\Phi^*_\alpha)^{-1}(\Phi^*_\beta(v^*))) \\ &= (p, (\Phi^*_\alpha)^{-1} \circ \Phi^*_\beta(v^*)) = (p, (\Phi_\beta \circ \Phi_\alpha^{-1})^*(v^*)) \\ &= (p, \left((\Phi_\alpha \circ \Phi_\beta^{-1})^{-1}\right)^* (v^*)) \\ &= (p, ((\tau_{\alpha\beta}(p))^{-1})^t v^*) \\ &= (p, \tau^*_{\alpha\beta}(p)v^*).\end{aligned}$$

(where, for ease of notation, we have omitted the to indicate that each map is restricted to V_p and hence linear and so dual defined). Hence (3) is also satisfied.

(2) $Bil(E) = \bigsqcup Bil(V_p)$ is a vector bundle over M, where $Bil(V_p)$ is the vector space consisting of all bi-linear maps $f : V_p \times V_p \to \mathbb{R}$. (The proof of this is very similar to the above). More generally, one can consider *multi-linear* maps and do the same thing.

In particular, define $\mathrm{Alt}^k(V)$ to be the vector space consisting of *alternating k-forms* which consists of multi-linear maps $\varphi : V \times \cdots \times V \to \mathbb{R}$ (where we have k copies of V) such that

$$\varphi(v_1, \ldots, v_i, \ldots, v_j, \ldots v_k) = -\varphi(v_1, \ldots, v_j, \ldots, v_i, \ldots v_k).$$

Then $\mathrm{Alt}^k(E) := \bigsqcup \mathrm{Alt}^k(V_p)$ is a vector bundle. This vector bundle will be important to us in the next chapter.

(3) The *direct sum* $E \oplus F$ is a vector bundle over M given by $E \oplus F = \bigsqcup V_p \oplus W_p$.

Recall that the direct sum of two vector spaces V, W is the vector space $V \oplus W = \{(v, w) \,|\, v \in V, w \in W\}$ with the operations $(v, w) + (v', w') = (v + v', w + w')$ and $\lambda(v, w) = (\lambda v, \lambda w)$.

We leave it as an exercise to verify that the above examples are indeed vector bundles.

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