Chapter 3

Tropical Varieties

We now introduce the main player of this book: the tropical variety. The two main results of this chapter are the Fundamental Theorem 3.2.3 and the Structure Theorem 3.3.5. The Fundamental Theorem gives several equivalent definitions of a tropical variety. We discuss this first for hypersurfaces and then for general varieties in Sections 3.1 and 3.2, respectively. The Structure Theorem strengthens the connection between tropical and polyhedral geometry. The main ideas are introduced in Section 3.3, with the proofs following in Sections 3.4 and 3.5. In Section 3.6 we develop the theory of stable intersections, which was previewed for tropical curves in Section 1.3.

We restrict our usage of the name *tropical variety* to mean the tropicalization of a classical variety over a field with a valuation. A more inclusive notion of tropical varieties allows for balanced polyhedral complexes that do not necessarily lift to a classical variety. In Chapter 4, we will see this distinction in the context of linear spaces. For now, we always start with Laurent polynomial ideals or, equivalently, with subvarieties of an algebraic torus.

3.1. Hypersurfaces

Let K be an arbitrary field with a possibly trivial valuation. We work in the ring $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ of Laurent polynomials over K. Given a Laurent polynomial $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}}$, we define its *tropicalization* $\operatorname{trop}(f)$ as in (2.4.1). Namely, $\operatorname{trop}(f)$ is the real-valued function on $\mathbb{R}^n$ that is obtained by replacing each coefficient $c_{\mathbf{u}}$ by its valuation and by performing all additions and multiplications in the tropical semiring $(\mathbb{R}, \oplus, \odot)$. Explicitly,

$$\mathrm{trop}(f)(\mathbf{w}) \;=\; \min_{\mathbf{u}\in\mathbb{Z}^n}\Big(\mathrm{val}(c_{\mathbf{u}}) + \sum_{i=1}^{n} u_i w_i\Big) \;=\; \min_{\mathbf{u}\in\mathbb{Z}^n}\big(\mathrm{val}(c_{\mathbf{u}}) + \mathbf{u}\cdot\mathbf{w}\big).$$

The tropical polynomial $\mathrm{trop}(f)$ is a piecewise linear concave function $\mathbb{R}^n \to \mathbb{R}$. For an illustration of the graph of $\mathrm{trop}(f)$ when $n = 2$, see Figure 1.3.2.

The classical variety of the Laurent polynomial $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a hypersurface in the algebraic torus T^n over the algebraic closure of K:

$$V(f) \;=\; \big\{\, \mathbf{y} \in T^n : f(\mathbf{y}) = 0 \,\big\}.$$

We now define the tropical hypersurface associated with the same f.

Definition 3.1.1. The *tropical hypersurface* $\mathrm{trop}(V(f))$ is the set

$$\{\mathbf{w} \in \mathbb{R}^n : \text{ the minimum in } \mathrm{trop}(f)(\mathbf{w}) \text{ is achieved at least twice}\}.$$

This is the locus in $\mathbb{R}^n$ where the piecewise linear function $\mathrm{trop}(f)$ fails to be linear. When the valuation on K has a splitting $w \mapsto t^w$, this can be rephrased in terms of the initial forms we introduced in (2.6.1):

$$\mathrm{in}_{\mathbf{w}}(f) \;=\; \sum_{\substack{\mathbf{u}:\mathrm{val}(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u} \\ = \mathrm{trop}(f)(\mathbf{w})}} \overline{t^{-\mathrm{val}(c_{\mathbf{u}})}\, c_{\mathbf{u}}}\, x^{\mathbf{u}}. \tag{3.1.1}$$

The tropical hypersurface $\mathrm{trop}(V(f))$ is the set of weight vectors $\mathbf{w} \in \mathbb{R}^n$ for which the initial form $\mathrm{in}_{\mathbf{w}}(f)$ is not a unit in $\Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The equivalence of these two definitions is the easy direction in Theorem 3.1.3 below.

When F is a tropical polynomial, we write $V(F)$ for the set

$$\{\mathbf{w} \in \mathbb{R}^n : \text{ the minimum in } F(\mathbf{w}) \text{ is achieved at least twice}\}.$$

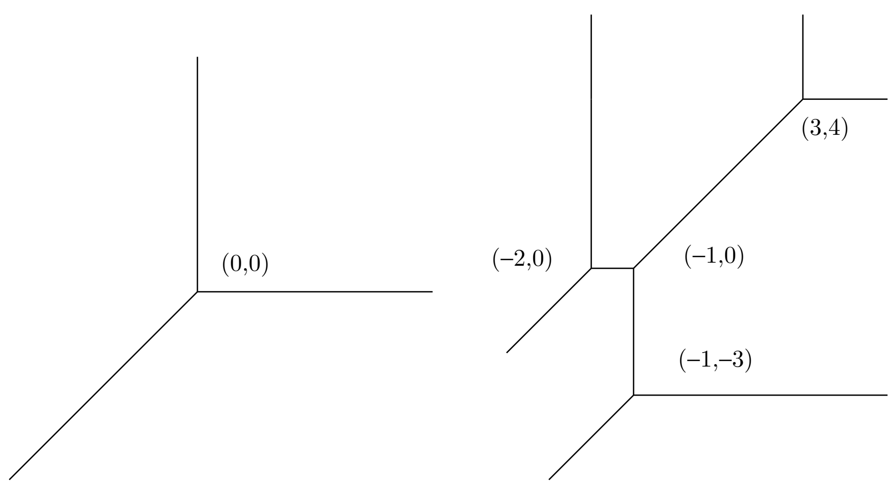

Figure 3.1.1. A tropical line and a tropical quadric.

With this notation, we have

$$\operatorname{trop}(V(f)) \;=\; V(\operatorname{trop}(f)).$$

Example 3.1.2. Let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series with complex coefficients. We examine bivariate Laurent polynomials $f \in K[x^{\pm 1}, y^{\pm 1}]$.

(1) Let $f = x + y + 1$. Then $\operatorname{trop}(f)(\mathbf{w}) = \min(w_1, w_2, 0)$, so

$$\operatorname{trop}(V(f))(\mathbf{w}) \;=\; \{w_1 = w_2 \leq 0\} \cup \{w_1 = 0 \leq w_2\} \cup \{w_2 = 0 \leq w_1\}.$$

This is the tropical line shown on the left in Figure 3.1.1.

(2) Let $f = t^2x^2 + xy + (t^2 + t^3)y^2 + (1 + t^3)x + t^{-1}y + t^3$. Then $\operatorname{trop}(f)(\mathbf{w}) = \min(2 + 2w_1, w_1 + w_2, 2 + 2w_2, w_1, -1 + w_2, 3)$, so $\operatorname{trop}(V(f))$ consists of the three line segments joining the pairs $\{(-1, 0), (-2, 0)\}$, $\{(-1, 0), (-1, -3)\}$, $\{(-1, 0), (3, 4)\}$, and the six rays $\{(-2, 0) + \lambda(0, 1)\}$, $\{(-2, 0) - \lambda(1, 1)\}$, $\{(-1, -3) - \lambda(1, 1)\}$, $\{(-1, -3) + \lambda(1, 0)\}$, $\{(3, 4) + \lambda(0, 1)\}$, $\{(3, 4) + \lambda(1, 0)\}$. In these sets, λ runs over $\mathbb{R}_{\geq 0}$. This is shown on the right in Figure 3.1.1. $\Diamond$

The following theorem was stated in the early 1990s in an unpublished manuscript by Mikhail Kapranov. A proof appeared in [**EKL06**]. It establishes the link between classical hypersurfaces over a field K and tropical hypersurfaces in $\mathbb{R}^n$. In the next section, we present the more general Fundamental Theorem which works for varieties of arbitrary codimension. Kapranov's Theorem for hypersurfaces will serve as the base case for its proof.

We place the extra conditions here on the field K that it is algebraically closed and has a nontrivial valuation with a splitting. If K is an arbitrary field with a valuation, then we may pass to its algebraic closure $\overline{K}$ with an extension of the valuation. If the valuation on K is trivial, we further pass to the field of generalized power series $\overline{K}((\mathbb{R}))$. This yields a field satisfying these conditions. Note that passing to an extension field does not change the function $\operatorname{trop}(f)$, so does not alter the tropical hypersurface $\operatorname{trop}(V(f))$.

Theorem 3.1.3 (Kapranov's Theorem). *Let K be an algebraically closed field with a nontrivial valuation. Fix a Laurent polynomial $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}}$ in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The following three sets coincide:*

(1) *the tropical hypersurface* $\operatorname{trop}(V(f))$ *in* $\mathbb{R}^n$;

(2) *the set* $\{\mathbf{w} \in \mathbb{R}^n : \operatorname{in}_{\mathbf{w}}(f)$ *is not a monomial* $\}$;

(3) *the closure in* $\mathbb{R}^n$ *of* $\big\{(\operatorname{val}(y_1), \ldots, \operatorname{val}(y_n)) : (y_1, \ldots, y_n) \in V(f)\big\}$.

Furthermore, if f is irreducible and $\mathbf{w}$ is any point in $\Gamma_{\mathrm{val}}^n \cap \operatorname{trop}(V(f))$, then the set $\{\mathbf{y} \in V(f) : \operatorname{val}(\mathbf{y}) = \mathbf{w}\}$ is Zariski dense in the hypersurface $V(f)$.

Example 3.1.4. Let $f = x - y + 1 \in K[x^{\pm 1}, y^{\pm 1}]$, where K is as above. Then $V(f) = \{(z, z + 1) : z \in K, z \neq 0, -1\}$, and $\operatorname{trop}(V(f))$ is the tropical line in Figure 3.1.1. Note that $\text{in}_{\mathbf{w}}(f)$ is a monomial unless either $\mathbf{w}$ is $(0,0)$, or $\mathbf{w}$ is a positive multiple of $(1,0),(0,1)$ or $(-1,-1)$. In the former case, $\text{in}_{\mathbf{w}}(f)$ is $x-y+1$. In the latter cases, it is $-y+1, x+1$ or $x-y$. We have

$$\big(\text{val}(z), \text{val}(z+1)\big) \;=\; \begin{cases} (\text{val}(z), 0) & \text{if } \text{val}(z) > 0, \\ (\text{val}(z), \text{val}(z)) & \text{if } \text{val}(z) < 0, \\ (0, \text{val}(z+1)) & \text{if } \text{val}(z) = 0,\ \text{val}(z+1) > 0, \\ (0,0) & \text{otherwise.} \end{cases}$$

As z runs over $K\backslash\{0,-1\}$, the above case distinction describes all points of Γ_{val}^2 that lie in the tropical line $\text{trop}(V(f))$. Since K is algebraically closed, the value group Γ_{val} is dense in $\mathbb{R}$, so the closure of these points in Γ_{val}^2 is the entire tropical line in $\mathbb{R}^2$. This confirms Theorem 3.1.3 for this f. $\diamondsuit$

Proof of Theorem 3.1.3. Let $\mathbf{w} = (w_1, \ldots, w_n) \in \text{trop}(V(f))$. By definition, the minimum in $W = \min_{\mathbf{u}: c_{\mathbf{u}} \neq 0}(\text{val}(c_{\mathbf{u}}) + \mathbf{u} \cdot \mathbf{w}) = \text{trop}(f)(\mathbf{w})$ is achieved at least twice. Therefore $\text{in}_{\mathbf{w}}(f)$, as seen in (3.1.1), is not a monomial. Thus, set (1) is contained in set (2). Conversely, if $\text{in}_{\mathbf{w}}(f)$ is not a monomial, then the minimum in W is achieved at least twice, so $\mathbf{w} \in \text{trop}(V(f))$. This shows the other containment, and so the equality of sets (1) and (2).

We now prove that set (1) contains set (3). Since set (1) is closed, it is enough to consider points in set (3) of the form $\text{val}(\mathbf{y}) := (\text{val}(y_1), \ldots, \text{val}(y_n))$ where $\mathbf{y} = (y_1, \ldots, y_n) \in (K^*)^n$ satisfies $f(\mathbf{y}) = \sum_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} \mathbf{y}^{\mathbf{u}} = 0$. This means $\text{val}(\sum_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} \mathbf{y}^{\mathbf{u}}) = \text{val}(0) = \infty > \text{val}(c_{\mathbf{u}'} \mathbf{y}^{\mathbf{u}'})$ for all $\mathbf{u}'$ with $c_{\mathbf{u}'} \neq 0$. Lemma 2.1.1 implies that the minimum of $\text{val}(c_{\mathbf{u}'} \mathbf{y}^{\mathbf{u}'}) = \text{val}(c_{\mathbf{u}'}) + \mathbf{u}' \cdot \text{val}(\mathbf{y})$ for $\mathbf{u}'$ with $c_{\mathbf{u}'} \neq 0$ is achieved at least twice. Thus $\text{val}(\mathbf{y}) \in \text{trop}(V(f))$.

It remains to be seen that set (3) contains set (1). This is the hard part of Kapranov's Theorem. It will be the content of Proposition 3.1.5. That proposition also shows that $\{\mathbf{y} \in V(f) : \text{val}(\mathbf{y}) = \mathbf{w}\}$ is Zariski dense when f is irreducible, so it completes our proof. $\square$

The next result, which finishes the proof of Theorem 3.1.3, states that every zero of an initial form lifts to a zero of the given polynomial.

Proposition 3.1.5. *Fix* $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$*, and let* $\mathbf{w} \in \Gamma_{\text{val}}^n$*. Suppose* $\text{in}_{\mathbf{w}}(f)$ *is not a monomial and* $\alpha \in (\Bbbk^*)^n$ *satisfies* $\text{in}_{\mathbf{w}}(f)(\alpha) = 0$*. There exists* $\mathbf{y} \in (K^*)^n$ *satisfying* $f(\mathbf{y}) = 0$*,* $\text{val}(\mathbf{y}) = \mathbf{w}$*, and* $\overline{t^{-w_i} y_i} = \alpha_i$ *for* $1 \leq i \leq n$*. If* f *is irreducible, then the set of such* $\mathbf{y}$ *is Zariski dense in the hypersurface* $V(f)$*.*

Proof. We use induction on n. The base case is $n = 1$. After multiplying by a unit, we may assume that $f = \sum_{i=0}^{s} c_i x^i = \prod_{j=1}^{s}(a_j x - b_j)$, where $c_0, c_s \neq 0$. Then $\text{in}_w(f) = \prod_{j=1}^{s} \text{in}_w(a_j x - b_j)$ by Lemma 2.6.2. Since

$\alpha \in \Bbbk^*$ and $\mathrm{in}_w(f)(\alpha) = 0$, the initial form $\mathrm{in}_w(f)$ is not a monomial, and $\mathrm{in}_w(a_j x - b_j)(\alpha) = 0$ for some j. This implies that $\mathrm{in}_w(a_j x - b_j)$ is not a monomial. Hence $\mathrm{val}(a_j) + w = \mathrm{val}(b_j)$, and $\alpha = \overline{t^{-w} b_j / a_j}$. Set $y = b_j / a_j \in K^*$. Then $f(y) = 0$, $\mathrm{val}(y) = w$, and $\overline{t^{-\mathrm{val}(y)} y} = \alpha$ as required.

We now assume $n > 1$ and that the proposition holds for smaller dimensions. We first reduce to the case where no two monomials appearing in f are divisible by the same power of x_n. This has the consequence that, when f is regarded as a polynomial in x_n with coefficients in $K[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$, the coefficients are all monomials of the form $cx^{\mathbf{u}}$ for $c \in K$ and $\mathbf{u} \in \mathbb{Z}^{n-1}$.

Consider the automorphism $\phi_l^* \colon K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ given by $\phi_l^*(x_j) = x_j x_n^{l^j}$ for $1 \leq j \leq n-1$, and $\phi_l^*(x_n) = x_n$, where $l \in \mathbb{N}$. For $\mathbf{u} \in \mathbb{Z}^{n-1}$, we have $\phi_l^*(x^{\mathbf{u}} x_n^i) = x^{\mathbf{u}} x_n^{i + \sum_{j=1}^{n-1} u_j l^j}$. For $l \gg 0$ each monomial in $\phi_l^*(f)$ is divisible by a different power of x_n as required. Suppose that $\mathbf{y} = (y_1, \ldots, y_n) \in T^n$ satisfies $\phi_l^*(f)(\mathbf{y}) = 0$, $\mathrm{val}(y_i) = w_i - l^i w_n$ and $\overline{t^{-w_i + l^i w_n} y_i} = \alpha_i \alpha_n^{-l^i}$ for $1 \leq i \leq n-1$, as well as $\mathrm{val}(y_n) = w_n$ and $\overline{t^{-w_n} y_n} = \alpha_n$. Define $\mathbf{y}' \in T^n$ by $y_i' = y_i y_n^{l^i}$ for $1 \leq i \leq n$ and $y_n' = y_n$. We then have $f(\mathbf{y}') = 0$, $\mathrm{val}(\mathbf{y}') = \mathbf{w}$, and $\overline{t^{-w_i} y_i'} = \alpha_i$. Hence it suffices to prove Proposition 3.1.5 for $\phi_l^*(f)$.

We now assume that f has the special form described above. Consider the set of all $(y_1, \ldots, y_{n-1})$ in T^{n-1} with $\mathrm{val}(y_i) = w_i$ and $\overline{t^{-w_i} y_i} = \alpha_i$ for $1 \leq i \leq n-1$. By Lemma 2.2.12, this set is Zariski dense in T^{n-1}. Moreover, for all such choices, $g(x_n) = f(y_1, \ldots, y_{n-1}, x_n)$ is not the zero polynomial.

Write $\mathbf{u}'$ for the projection of $\mathbf{u} \in \mathbb{Z}^n$ onto the first $n-1$ coordinates. Writing $g = \sum d_i x_n^i$, we have $d_i = \mathbf{c_u} \mathbf{y}^{\mathbf{u}'}$ for a unique $\mathbf{u} \in \mathbb{Z}^n$ that has $u_n = i$. Note that $\mathrm{val}(d_i) + w_n i = \mathrm{val}(c_{\mathbf{u}}) + \mathrm{val}(\mathbf{y}^{\mathbf{u}'}) + w_n i = \mathrm{val}(c_{\mathbf{u}}) + \mathbf{w}' \cdot \mathbf{u}' + w_n u_n = \mathrm{val}(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u}$. Therefore $\mathrm{trop}(g)(w_n) = \mathrm{trop}(f)(\mathbf{w})$, and

$$
\begin{aligned}
\mathrm{in}_{w_n}(g) &= \sum_{i : \mathrm{val}(d_i) + w_n i = \mathrm{trop}(g)(w_n)} \overline{t^{-\mathrm{val}(d_i)} d_i} x_n^i \\
&= \sum_{\mathbf{u} : \mathrm{val}(c_{\mathbf{u}} y^{\mathbf{u}'}) + w_n u_n = \mathrm{trop}(g)(w_n)} \overline{t^{-\mathrm{val}(c_{\mathbf{u}})} c_{\mathbf{u}} t^{-\mathbf{u}' \cdot \mathbf{w}'} y^{\mathbf{u}'}} x_n^{u_n} \\
&= \sum_{\mathbf{u} : \mathrm{val}(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} = \mathrm{trop}(f)(\mathbf{w})} \overline{t^{-\mathrm{val}(c_{\mathbf{u}})} c_{\mathbf{u}}} \cdot \alpha^{\mathbf{u}'} x_n^{u_n} \\
&= \mathrm{in}_{\mathbf{w}}(f)(\alpha_1, \ldots, \alpha_{n-1}, x_n).
\end{aligned}
$$

Thus $\mathrm{in}_{w_n}(g)(\alpha_n) = 0$. By the $n = 1$ case there is $y_n \in K^*$ with $\mathrm{val}(y_n) = w_n$ and $\overline{t^{-w_n} y_n} = \alpha_n$ for which $g(y_n) = 0$, and thus $f(y_1, \ldots, y_{n-1}, y_n) = 0$. We conclude $\mathbf{y} = (y_1, \ldots, y_n)$ is the required point in the hypersurface $V(f)$.

We now show that if f is an irreducible polynomial, then the set $\mathcal{Y}$ of $\mathbf{y}$ with $\mathrm{val}(\mathbf{y}) = \mathbf{w}$ and $\overline{t^{-w_i} y_i} = \alpha_i$ for $1 \leq i \leq n$ is Zariski dense in $V(f)$. For any $(y_1, \ldots, y_{n-1}) \in T^n$, with $\mathrm{val}(y_i) = w_i$ and $\overline{t^{-w_i} y_i} = \alpha_i$ for all i, we constructed a point $\mathbf{y} = (y_1, \ldots, y_{n-1}, y_n) \in \mathcal{Y}$. The set of such $(y_1, \ldots, y_{n-1})$ is Zariski dense in T^{n-1} by Lemma 2.2.12. Hence the projection of $\mathcal{Y}$ onto the first $n-1$ coordinates is not contained in any hypersurface in T^{n-1}. Consider any $g \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with $g(\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathcal{Y}$. Then $\langle f, g\rangle \cap K[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}] = \{0\}$. Since f is irreducible, this implies that g is a multiple of f. We conclude that the set $\mathcal{Y}$ is Zariski dense in $V(f)$. $\square$

In the rest of Section 3.1 we study the combinatorics of tropical hypersurfaces. This uses the notion of *regular subdivisions* from Section 2.3. By the *k-skeleton* of a polyhedral complex Σ, we mean the polyhedral complex consisting of all cells $\sigma \in \Sigma$ with $\dim(\sigma) \leq k$. The field K is again arbitrary.

Proposition 3.1.6. *Let $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial. The tropical hypersurface* $\mathrm{trop}(V(f))$ *is the support of a pure Γ_{val}-rational polyhedral complex of dimension $n-1$ in $\mathbb{R}^n$. It is the $(n-1)$-skeleton of the polyhedral complex dual to the regular subdivision of the Newton polytope of $f = \sum c_{\mathbf{u}} x^{\mathbf{u}}$ given by the weights* $\mathrm{val}(c_{\mathbf{u}})$ *on the lattice points in* $\mathrm{Newt}(f)$.

Proof. By definition, $\mathrm{trop}(V(f))$ is the set of $\mathbf{w} \in \mathbb{R}^n$ for which the minimum in $\mathrm{trop}(f)(\mathbf{w}) = \min_{\mathbf{u}}(\mathrm{val}(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u})$ is achieved at least twice. Let $P = \mathrm{Newt}(f) = \mathrm{conv}\{\mathbf{u} : c_{\mathbf{u}} \neq 0\} \subset \mathbb{R}^n$ be the Newton polytope of f, and define $P_{\mathrm{val}} = \mathrm{conv}\{(\mathbf{u}, \mathrm{val}(c_{\mathbf{u}})) : c_{\mathbf{u}} \neq 0\} \subset \mathbb{R}^{n+1}$. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection onto the first n coordinates. The regular subdivision of P induced by the weights $\mathrm{val}(c_{\mathbf{u}})$, $c_{\mathbf{u}} \neq 0$, consists of the polytopes $\pi(F)$ as F varies over all lower faces of P_{val}. Being a *lower face* of P_{val} means that

$$F \;=\; \mathrm{face}_{\mathbf{v}}(P_{\mathrm{val}}) \;=\; \left\{ \mathbf{x} \in P_{\mathrm{val}} : \mathbf{v} \cdot \mathbf{x} \leq \mathbf{v} \cdot \mathbf{y} \text{ for all } \mathbf{y} \in P_{\mathrm{val}} \right\}$$

for some $\mathbf{v} \in \mathbb{R}^{n+1}$ with last coordinate v_{n+1} positive. For such an F, let $\mathcal{N}(F) = \{\mathbf{v} : \mathrm{face}_{\mathbf{v}}(P_{\mathrm{val}}) = F\}$ be the normal cone. We denote by $\tilde{\pi}(\mathcal{N}(F))$ the restricted projection $\{\mathbf{w} \in \mathbb{R}^n : (\mathbf{w}, 1) \in \mathcal{N}(F)\}$. The collection of $\tilde{\pi}(\mathcal{N}(F))$ as F varies over all lower faces of P_{val} forms a polyhedral complex in $\mathbb{R}^n$ that is dual to the regular subdivision of P induced by the $\mathrm{val}(c_{\mathbf{u}})$.

If $\mathbf{v} = (v_1, \ldots, v_n, 1) \in \mathcal{N}(F)$, then $\mathrm{in}_{\pi(\mathbf{v})}(f)$ is a sum of monomials with exponents in $\pi(F)$, and all vertices of the polytope $\pi(F)$ appear with nonzero coefficient. This means that $\mathbf{w} = (w_1, \ldots, w_n) \in \mathrm{trop}(V(f))$ if and only if $\mathbf{w} \in \tilde{\pi}(\mathcal{N}(F))$ for some face F of P_{val} that has more than one vertex. So $\mathbf{w} \in \mathrm{trop}(V(f))$ if and only if $F = \mathrm{face}_{(\mathbf{w},1)}(P_{\mathrm{val}})$ is not a vertex. This happens if and only if the face $\tilde{\pi}(\mathcal{N}(F))$ of the dual complex that contains $\mathbf{w}$ is not full dimensional. We conclude that $\mathrm{trop}(V(f))$ is the $(n-1)$-skeleton of the dual complex. This is a pure Γ_{val}-rational polyhedral complex. $\square$

Remark 3.1.7. The proof shows that the tropical hypersurface $\mathrm{trop}(V(f))$ is precisely the $(n-1)$-skeleton of the complex $\Sigma_{\mathrm{trop}(f)}$ in Definition 2.5.5.

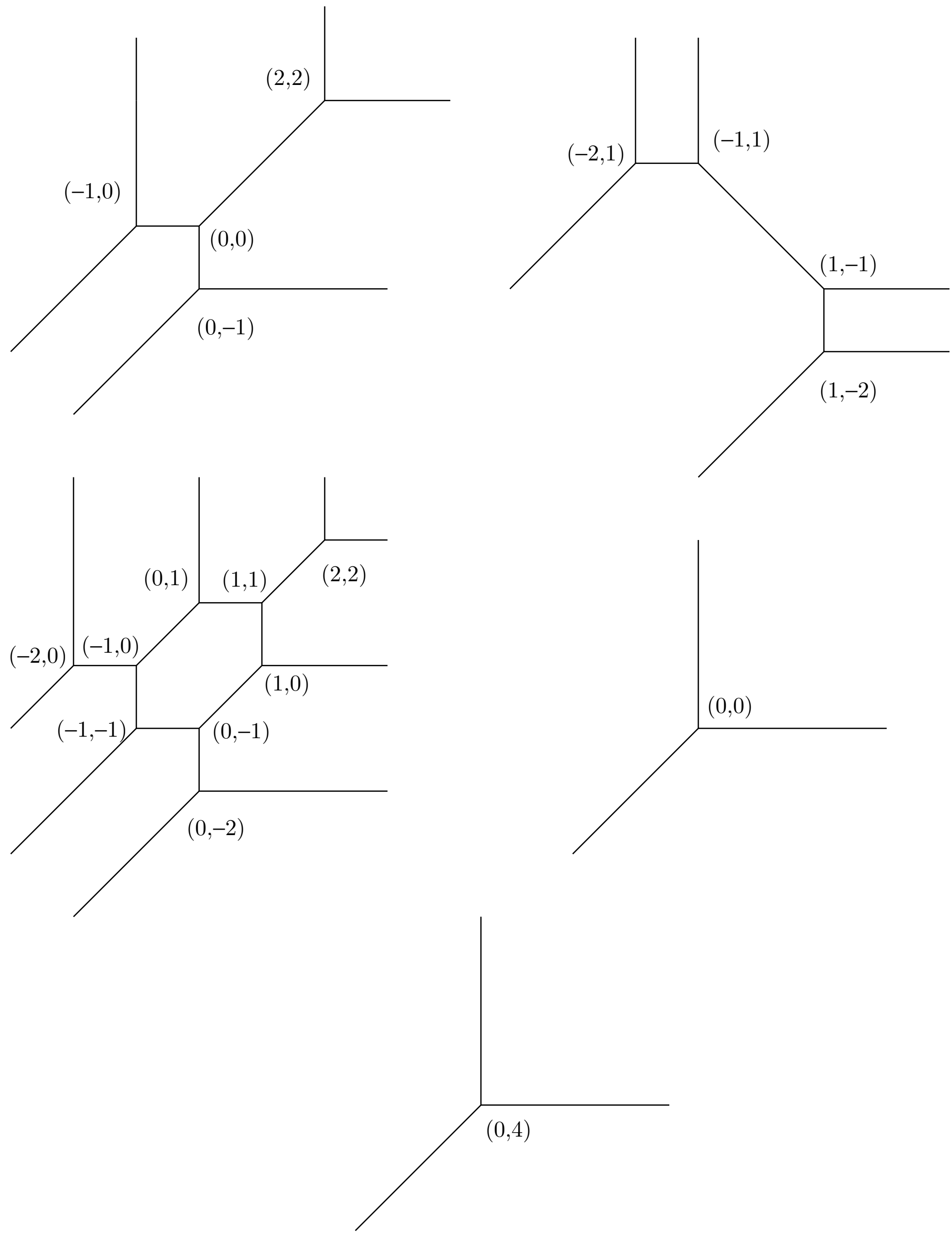

Figure 3.1.2. Five tropical curves from Example 3.1.8.

Example 3.1.8. Let $K = \mathbb{C}\{\{t\}\}$ and $n = 2$. Tropical hypersurfaces in $\mathbb{R}^2$ are tropical curves. The following five examples are depicted in Figure 3.1.2.

(1) If $f_1 = 3tx^2+5xy-7ty^2+8x-y+t^2$, then $\mathrm{trop}(V(f_1))$ is dual to the regular subdivision of $\mathcal{A}_2 = \{(2,0),(1,1),(0,2),(1,0),(0,1),(0,0)\}$ induced by $\mathbf{w} = (1,0,1,0,0,2)$. This subdivision is shown on the left in Figure 2.3.8. The curve $\mathrm{trop}(V(f_1))$ is shown in the upper left in Figure 3.1.2.

(2) Let $f_2 = 3t^3x^2 + 5xy - 7t^3y^2 + 8tx - ty + 1$. The tropical curve $\mathrm{trop}(V(f_2))$ is dual to the regular subdivision of $\mathcal{A}_2$ induced by $\mathbf{w} = (3,0,3,1,1,0)$. This subdivision is shown second in Figure 2.3.8, and $\mathrm{trop}(V(f_2))$ is second in Figure 3.1.2.

(3) Let $f_3 = 5t^3x^3+7tx^2y-8txy^2+9t^3y^3+8tx^2+5xy-ty^2+4tx+8ty+t^3$. The tropical curve $\mathrm{trop}(V(f_3))$ is dual to the regular subdivision of $\mathcal{A}_3=\{(3,0),(2,1),(1,2),(0,3),(2,0),(1,1),(0,2),(1,0),(0,1),(0,0)\}$ induced by $\mathbf{w} = (3,1,1,3,1,0,1,1,1,3)$. It consists of nine triangles. Note that $V(f_3)$ is an elliptic curve with nine points removed, and $\mathrm{trop}(V(f_3))$ has a cycle. See the second row of Figure 3.1.2.

(4) Let $f_4 = 5x^3+7x^2y+8xy^2+9y^3+8x^2+5xy-y^2+4x+8y+1$. The tropical cubic $\mathrm{trop}(V(f_4))$ is dual to the regular subdivision of $\mathcal{A}_3$ induced by $\mathbf{w} = (0,0,0,0,0,0,0,0,0,0)$. The subdivision consists of just the single triangle $\mathrm{conv}(\mathcal{A}_3)$. The picture of $\mathrm{trop}(V(f_4))$, on the right of the second row of Figure 3.1.2, looks like a tropical line. In Section 3.4 we will attach weights to tropical varieties. Those weights will distinguish our tropical cubic from a tropical line.

(5) Let $f_5 = (3t^3+5t^2)xy^{-1} + 8t^2y^{-1} + 4t^{-2}$. The curve $\mathrm{trop}(V(f_5))$ is dual to the regular triangulation of $\{(1,-1),(0,-1),(0,0)\}$ induced by $\mathbf{w} = (2,2,-2)$. This consists of a single triangle. The curve $\mathrm{trop}(V(f_5))$ is a tropical line, shifted so that the vertex is at $(0,4)$. This is shown at the bottom of Figure 3.1.2. $\diamondsuit$

It is instructive to also examine some tropical surfaces in $\mathbb{R}^3$.

Example 3.1.9. Let $K = \mathbb{Q}$, and fix the 2-adic valuation. The following polynomial defines a smooth surface in the three-dimensional torus T_K^3:

$$f \;=\; 12x^2 + 20y^2 + 8z^2 + 7xy + 22xz + 3yz + 5x + 9y + 6z + 4.$$

Its Newton polytope $P = \mathrm{Newt}(f)$ is the tetrahedron $\mathrm{conv}((2,0,0),(0,0,2),$ $(0,0,2),(0,0,0))$. The 2-adic valuations of the coefficients of f define a regular triangulation of P into eight tetrahedra of volume $1/6$. That triangulation has 24 triangles, 25 edges, and ten vertices. It is a good exercise to verify these numbers. Eight triangles and one edge lie in the interior of P. The dual complex $\Sigma_{\mathrm{trop}(f)}$ is a subdivision of $\mathbb{R}^3$ into ten unbounded full-dimensional regions. Its 2-skeleton is the tropical quadric surface $\mathrm{trop}(V(f))$. That tropical surface consists of 25 two-dimensional polyhedra (24 unbounded

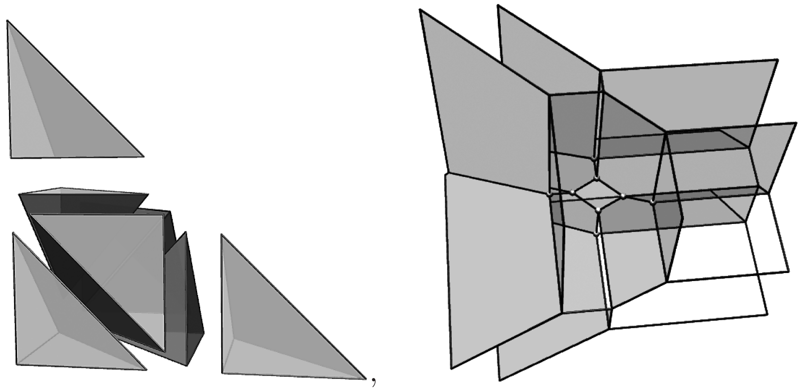

Figure 3.1.3. The regular triangulation and tropical surface of Example 3.1.9.

and one bounded square). It has eight vertices and 24 edges (16 unbounded and eight bounded). The regular triangulation of P is shown on the left of Figure 3.1.3, and the tropical surface is shown on the right. See Proposition 4.5.4 for the classification of all tropical surfaces of degree 2 in $\mathbb{R}^3$. $\diamondsuit$

An important special case of Proposition 3.1.6 arises when the valuations of the coefficients of f are all zero. In that case, the tropical hypersurface is a fan in $\mathbb{R}^n$. We saw an $n = 2$ instance of this in part (4) of Example 3.1.8.

Proposition 3.1.10. *Let $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial whose coefficients have valuation zero. The tropical hypersurface* $\mathrm{trop}(V(f))$ *is the support of an $(n-1)$-dimensional polyhedral fan in $\mathbb{R}^n$. That fan is the $(n-1)$-skeleton of the normal fan to the Newton polytope of f.*

Proof. Let $f = \sum c_{\mathbf{u}} x^{\mathbf{u}}$. If $\mathrm{val}(c_{\mathbf{u}}) = 0$ whenever $c_{\mathbf{u}} \neq 0$, then the regular subdivision of $\mathrm{Newt}(f)$ induced by the vector with coordinates $\mathrm{val}(c_{\mathbf{u}})$ is just the polytope $\mathrm{Newt}(f)$. The complex $\Sigma_{\mathrm{trop}(f)}$ of Definition 2.5.5 is the normal fan of $\mathrm{Newt}(f)$, so the claim follows from Proposition 3.1.6. $\square$

Example 3.1.11. Let f denote the determinant of an $n \times n$-matrix (x_{ij}) whose entries are variables. We regard f as a polynomial of degree n with $n!$ terms in $K[x_{11}, x_{12}, \ldots, x_{nn}]$. Each coefficient is -1 or 1, so has valuation zero. The Newton polytope $P = \mathrm{Newt}(f)$ is the $(n-1)^2$-dimensional *Birkhoff polytope* of bistochastic matrices. The piecewise-linear function $\mathrm{trop}(f)$ is the tropical determinant from (1.2.6). The dual complex $\Sigma_{\mathrm{trop}(f)}$ is the normal fan of the Birkhoff polytope P. The polytope P has one vertex for each permutation of n. The normal fan thus divides $\mathbb{R}^{n^2}$ into $n!$ cones. The cones indexed by two permutations π and π' intersect in a common facet if and only if $\pi^{-1} \circ \pi'$ is a cycle. Checking this is a good exercise.

The tropical hypersurface $\mathrm{trop}(V(f))$ is an (n^2-1)-dimensional fan with lineality space of dimension $2n-1$. The dimension of the lineality space comes from the calculation $n^2-(n-1)^2 = 2n-1$ for the codimension of the affine span of P. That fan has n^2 rays, one for each matrix entry, and its maximal cones are indexed by pairs (π, π') such that $\pi^{-1} \circ \pi'$ is a cycle.

If $n = 3$, then the Birkhoff polytope P is the cyclic 4-polytope with six vertices, whose f-vector is $(6, 15, 18, 9)$. The f-vector records the number of faces of P of each dimension; here P has 15 edges, 18 two-dimensional faces, and nine facets (three-dimensional faces). The tropical determinantal hypersurface $\mathrm{trop}(V(f))$ is an eight-dimensional fan in $\mathbb{R}^9$. Modulo its five-dimensional lineality space, this fan has nine rays, 18 two-dimensional cones, and 15 maximal cones. It is the fan over a two-dimensional polyhedral complex with nine squares and six triangles, namely the 2-skeleton of the product of two triangles. ♢

3.2. The Fundamental Theorem

The goal of this section is to prove the Fundamental Theorem of Tropical Algebraic Geometry, which establishes a tight connection between classical varieties and tropical varieties. We must begin by defining the latter objects.

Definition 3.2.1. Let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let $X = V(I)$ be its variety in the algebraic torus T^n. The *tropicalization* $\mathrm{trop}(X)$ of the variety X is the intersection of all tropical hypersurfaces defined by elements of I:

$$\mathrm{trop}(X) \;=\; \bigcap_{f \in I} \mathrm{trop}(V(f)) \;\subseteq\; \mathbb{R}^n. \tag{3.2.1}$$

We shall see that the set $\mathrm{trop}(X)$ depends only on the radical $\sqrt{I}$ of the ideal I. By a *tropical variety* in $\mathbb{R}^n$ we mean any subset of the form $\mathrm{trop}(X)$ where X is a subvariety of the torus T^n over a field K with valuation.

In Definition 3.2.1, it does not suffice to take the intersection over the tropical hypersurfaces $\mathrm{trop}(V(f))$ where f runs over a generating set of I. We usually have to pass to a larger set of Laurent polynomials in the ideal I. In other words, tropicalization of varieties does not commute with intersections. This fact is a salient feature of tropical geometry. A finite intersection of tropical hypersurfaces is known as a *tropical prevariety*.

Example 3.2.2. Let $n = 2$, $K = \mathbb{C}\{\{t\}\}$, and $I = \langle\, x + y + 1,\, x + 2y \,\rangle$. Then $X = V(I) = \{(-2, 1)\}$ and hence $\mathrm{trop}(X) = \{(0, 0)\}$. However, the intersection of the two tropical lines given by the ideal generators equals

$$\mathrm{trop}(V(x + y + 1)) \cap \mathrm{trop}(V(x + 2y)) \;=\; \bigl\{(w_1, w_2) \in \mathbb{R}^2 : w_1 = w_2 \leq 0\bigr\}.$$

This half-ray is not a tropical variety. It is just a tropical prevariety. ♢

This example shows that a tropical variety $\mathrm{trop}(X)$ is generally not the intersection of the tropical hypersurfaces corresponding to a given generating set of the ideal I of X. This brings us back to the notion of a tropical basis, as in Section 2.6. Definition 2.6.3 can be restated as follows: a finite generating set $\mathcal{T}$ for an ideal I in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a *tropical basis* for I if

$$\mathrm{trop}(V(I)) \; = \; \bigcap_{f \in \mathcal{T}} \mathrm{trop}(V(f)).$$

Theorem 2.6.6 states that every Laurent ideal has a finite tropical basis, and this implies that every tropical variety is a tropical prevariety.

In Example 3.2.2, the two given generators are not yet a tropical basis of the ideal I. However, we get a tropical basis if we add one more polynomial:

$$\mathcal{T} \; = \; \big\{ x + y + 1,\, x + 2y,\, y - 1 \big\}.$$

We now come to the main result of this section, which is the direct generalization of Theorem 3.1.3 from hypersurfaces to arbitrary varieties.

Theorem 3.2.3 (Fundamental Theorem of Tropical Algebraic Geometry)**.** *Let K be an algebraically closed field with a nontrivial valuation, let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let $X = V(I)$ be its variety in the algebraic torus $T^n \cong (K^*)^n$. Then the following three subsets of $\mathbb{R}^n$ coincide:*

(1) *the tropical variety* $\mathrm{trop}(X)$ *as defined in equation* (3.2.1);
(2) *the set of all vectors* $\mathbf{w} \in \mathbb{R}^n$ *with* $\mathrm{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle$;
(3) *the closure of the set of coordinatewise valuations of points in X,*

$$\mathrm{val}(X) = \{(\mathrm{val}(y_1), \ldots, \mathrm{val}(y_n)) : \, (y_1, \ldots, y_n) \in X\}.$$

Furthermore, if X is irreducible and $\mathbf{w}$ is any point in $\Gamma_{\mathrm{val}}^n \cap \mathrm{trop}(X)$, then the set $\{\mathbf{y} \in X : \mathrm{val}(\mathbf{y}) = \mathbf{w}\}$ is Zariski dense in the classical variety X.

The rest of this section is devoted to proving Theorem 3.2.3. We first explain why the assumptions that K is algebraically closed and that the valuation is nontrivial are not serious restrictions for tropical geometry.

Fix a field extension L/K. If $Y \subset T_K^n$ is a variety defined by an ideal $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, then the *extension of Y to T_L^n* is the subvariety Y_L of T_L^n defined by the ideal $I_L = IL[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Recall that L/K is a valued field extension if the valuations satisfy $\mathrm{val}_L|_K = \mathrm{val}_K$.

Theorem 3.2.4. *Let K be a field with a possibly trivial valuation, and let L/K be a valued field extension. Let $X \subset T_K^n$ be a subvariety of the torus T_K^n, and let X_L be its extension to T_L^n. Then*

$$\mathrm{trop}(X_L) \; = \; \mathrm{trop}(X) \; \subset \; \mathbb{R}^n.$$

Proof. Let $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ideal of X so $I_L = IL[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the ideal of X_L. By definition we have $\mathrm{trop}(X_L) = \bigcap_{f \in I_L} \mathrm{trop}(V(f))$. Since every polynomial $f \in I$ is also a polynomial in I_L, we have $\mathrm{trop}(X_L) \subseteq \mathrm{trop}(X)$, so it suffices to show the other inclusion. By Theorem 2.6.6 the ideal I_L has a finite tropical basis, and by Lemma 2.6.5 there is a tropical basis $\mathcal{T}$ for I_L with all coefficients in K. Thus if $\mathbf{w} \notin \mathrm{trop}(X_L)$, there is $f \in \mathcal{T}$ with the minimum in $\mathrm{trop}(f)(\mathbf{w})$ achieved only once, so $\mathbf{w} \notin \mathrm{trop}(V(f))$. Since $f \in I$, we have $\mathbf{w} \notin \mathrm{trop}(X)$, which shows the other inclusion. $\square$

Remark 3.2.5. Theorem 3.2.4 allows us to work over an extension field when this is necessary. In particular, if K has the trivial valuation, then we can take the field $L = K((\mathbb{R}))$ of generalized power series with coefficients in K (or the simpler field of Puiseux series if K has characteristic zero). We can thus assume that the given field has a nontrivial valuation. It also does not change the tropical variety to pass to the algebraic closure of the field K. This lets us assume that the value group Γ_{val} is dense in $\mathbb{R}$. We may also pass to an extension field L/K for which the valuation map $L^* \to \Gamma_{\mathrm{val}}$ splits; by Lemma 2.1.15 we can take $L = \overline{K}$, but smaller fields may also suffice. This allows us to use the Gröbner theory developed in Sections 2.4 and 2.5 when studying tropical varieties over an arbitrary field.

For the rest of this section we assume that K is an algebraically closed field with a nontrivial valuation that splits. We begin with a sequence of lemmas whose purpose is to prepare for the proof of Theorem 3.2.3.

Recall from commutative algebra that a *minimal associated prime* of an ideal I in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a prime ideal $P \supset I$ for which there is no prime ideal Q with $P \supsetneq Q \supset I$. The variety $V(I)$ decomposes as $\bigcup_{P \text{ minimal}} V(P)$. See [**Eis95**, Chapter 3] or [**CLO07**, §4.7] for more details.

Lemma 3.2.6. *Let $X \subset T^n$ be an irreducible variety with prime ideal $I \subset K[x_1^{\pm}, \ldots, x_n^{\pm 1}]$, and fix $\mathbf{w}$ with $\mathrm{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle$. Then all minimal associated primes of the initial ideal $\mathrm{in}_{\mathbf{w}}(I)$ in $\Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ have the same dimension as X.*

Proof. Let $d = \dim(X)$. The ideal $I_{\mathrm{proj}} \subseteq K[x_0, x_1, \ldots, x_n]$, as in Definition 2.2.4, is prime of dimension $d + 1$. Since we are assuming that K is algebraically closed with a nontrivial valuation, Γ_{val} is dense in $\mathbb{R}$. By Theorem 2.5.3, the Gröbner complex $\Sigma(I_{\mathrm{proj}})$ is Γ_{val}-rational, so the cell of $\Sigma(I_{\mathrm{proj}})$ containing $(0, \mathbf{w})$ contains a point $(0, \mathbf{w}') \in \Gamma_{\mathrm{val}}^{n+1}$. We may thus assume that $\mathbf{w} \in \Gamma_{\mathrm{val}}^n$. Hence, by Lemma 2.4.12, all minimal primes of $\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})$ have dimension $d + 1$. By the Principal Ideal Theorem, all minimal primes of $\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}}) + \langle x_0 - 1 \rangle$ have dimension at least d. Since $\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})$ is homogeneous by Lemma 2.4.2, all minimal primes are homogeneous and contained in $\langle x_0, \ldots, x_n \rangle$. None of them contains $x_0 - 1$.

Thus, the minimal primes of $\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}}) + \langle x_0 - 1\rangle$ have dimension exactly d. By Proposition 2.6.1 we have $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})|_{x_0=1}$, viewed as an ideal in $\Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The minimal primes of $\mathrm{in}_{\mathbf{w}}(I)$ are the images in $\Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ of those primes minimal over $\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}}) + \langle x_0 - 1\rangle$ that do not contain any monomial in $x_1, \ldots, x_n$. These all have dimension d. □

The proof of Theorem 3.2.3 will proceed by projecting to the hypersurface case. The next result ensures that a sufficiently nice projection exists.

Proposition 3.2.7. *Fix a subvariety X in T^n and $m \geq \dim(X)$. There exists a morphism $\psi\colon T^n \to T^m$ whose image $\psi(X)$ is Zariski closed in T^m and satisfies $\dim(\psi(X)) = \dim(X)$. This map can be chosen so that the following hold.*

(1) *The kernel of the linear map $\mathrm{trop}(\psi)\colon \mathbb{R}^n \to \mathbb{R}^m$ intersects trivially with a fixed finite arrangement of m-dimensional subspaces in $\mathbb{R}^n$.*

(2) *When $n > m$, if we change coordinates so that ψ is the projection onto the first m coordinates, then the ideal I of X is generated by polynomials in $x_{m+1}, \ldots, x_n$ whose coefficients are monomials in $x_1, \ldots, x_m$.*

Proof. To prove this we derive a version of *Noether normalization* for the Laurent polynomial ring $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. We proceed by induction on $n - m$, the case $n = m$ being trivial. For $n > m$, the ideal I of X is not the zero ideal. For $l \in \mathbb{N}$, we define a monomial change of variables in T^n by

$$\phi_l^*(x_1) = x_1 x_n^l,\ \phi_l^*(x_2) = x_2 x_n^{l^2}, \ldots,\ \phi_l^*(x_{n-1}) = x_{n-1} x_n^{l^{n-1}},\ \phi_l^*(x_n) = x_n.$$

For any f, choosing l sufficiently large, the transformed Laurent polynomial

$$g \;=\; \phi_l^*(f) \;=\; f\big(x_1 x_n^l, x_2 x_n^{l^2}, \ldots, x_{n-1} x_n^{l^{n-1}}, x_n\big)$$

has the property that its monomials have distinct degrees in the variable x_n. Since ϕ^* is invertible, we may replace I by $\phi^*(I)$, and assume that I is generated by a set of polynomials with this property.

This suffices to show that the image of X under the monomial map

$$\pi \,:\, T^n \to T^{n-1},\ (x_1, x_2, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$$

is closed. By [**CLO07**, Theorem 3.2.2], the closure of $\pi(X)$ is the variety in T^{n-1} defined by $I \cap K[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$. The difference $\overline{\pi(X)} \backslash \pi(X)$ is contained in the variety of the leading coefficients of the polynomials in a generating set of I when viewed as polynomials in x_n. As the leading coefficient of each generator is a monomial in $x_1, \ldots, x_{n-1}$, the variety in T^{n-1} defined by these polynomials is empty. We conclude that $\overline{\pi(X)} = \pi(X)$.

To see that $\dim(X) = \dim(\pi(X))$, we note that the ideal I contains a polynomial that is monic when regarded as a polynomial in x_n. Hence $K[X]$

is generated by x_n as a $K[\pi(X)]$-algebra, and the field of fractions $K(X)$ is a finite extension of $K(\pi(X))$. This shows that their transcendence degrees agree, and thus $\dim(X) = \dim(\pi(X))$ (see [**CLO07**, Theorem 9.5.6]).

By induction on $n-m$, there is a morphism $\psi : T^{n-1} \to T^m$ with the desired properties. The claims on dimension, on the image being closed, and the second requirement on the form of the generators all follow.

We can choose our change of coordinates so that the kernel of $\mathrm{trop}(\pi)$ avoids some subspaces, because in the original coordinates the kernel of $\mathrm{trop}(\pi) : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the line spanned by $(1, l, l^2, \ldots, l^{n-1})$ in $\mathbb{R}^n$. For $l \gg 0$, this line intersects any fixed finite number of hyperplanes only in the origin. We obtain the general case using again induction on $n-m$. This shows that we can guarantee the map ψ to also satisfy property (1). $\square$

A key point of tropical geometry is that $\mathrm{trop}(X)$ is the support of a Γ_{val}-rational polyhedral complex. One particular Γ_{val}-rational polyhedral structure on $\mathrm{trop}(X)$ is derived from the Gröbner characterization in the Fundamental Theorem (part (2) of Theorem 3.2.3). In the following statement we identify $\mathbb{R}^n$ with $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ via the map $\mathbf{w} \mapsto (0, \mathbf{w})$.

Proposition 3.2.8. *Let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let $X = V(I)$ be its variety. Then $\{\mathbf{w} : \mathrm{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle\}$ is the support of a subcomplex of the Gröbner complex $\Sigma(I_{\mathrm{proj}})$ and is thus the support of a Γ_{val}-rational polyhedral complex.*

Proof. The Gröbner complex $\Sigma(I_{\mathrm{proj}})$ is a Γ_{val}-rational polyhedral complex in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ by Theorem 2.5.3. Let I_{proj} be as in Proposition 2.6.1. By Proposition 2.6.1 we have $\mathrm{in}_{\mathbf{w}}(I) = \langle 1 \rangle$ if and only if $1 \in \mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})|_{x_0=1}$. This occurs if and only if there is an element in $\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})$ that is a polynomial in x_0 times a monomial in $x_1, \ldots, x_n$, and thus if and only if there is a monomial in $\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})$, since $\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})$ is homogeneous by Lemma 2.4.2. So $\{\mathbf{w} \in \mathbb{R}^n : \mathrm{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle\}$ equals the set $\{\mathbf{w} \in \mathbb{R}^n : \mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})$ does not contain a monomial$\}$. This is a union of cells in the Gröbner complex $\Sigma(I_{\mathrm{proj}})$. The set of $\mathbf{w} \in \mathbb{R}^n$ for which $\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})$ contains a monomial is open by Corollary 2.4.10, so the complement is closed. Hence $\mathrm{trop}(X)$ is the support of a subcomplex of the Gröbner complex. $\square$

The polyhedral complex structure defined by Proposition 3.2.8 depends on the choice of coordinates in T^n. The following example illustrates this.

Example 3.2.9. Let $K = \mathbb{C}$, let $n = 5$, and consider the ideal

$$I = \langle\, x_1 + x_2 + x_3 + x_4 + x_5,\ 3x_2 + 5x_3 + 7x_4 + 11x_5 \,\rangle \ \subset\ K[x_1^{\pm}, \ldots, x_5^{\pm}].$$

The generators are linear forms, so we can identify I with its homogenization I_{proj}. The tropical variety $\mathrm{trop}(V(I_{\mathrm{proj}}))$ is a three-dimensional fan with one-dimensional lineality space. It is a fan over the complete graph K_5 (cf. Example 4.2.13). That fan has ten maximal cones and five ridges. We consider the isomorphism $\phi : T^5 \to T^5$ defined by

$$\phi^* : x_1 \mapsto x_1,\ x_2 \mapsto x_2x_3,\ x_3 \mapsto x_3x_4,\ x_4 \mapsto x_4x_5,\ x_5 \mapsto x_5.$$

The transformed ideal $J = (\phi^*)^{-1}(I)$ has a finer Gröbner fan structure on its tropical variety $\mathrm{trop}(V(J_{\mathrm{proj}}))$. The support is still a fan over the complete graph K_5, but now two edges are subdivided, so the fan has 12 maximal cones. This can be verified using the software `Gfan` [**Jen**]. $\Diamond$

We now embark on the proof of the Fundamental Theorem 3.2.3. At this point we must treat the three sets described in Theorem 3.2.3 as distinct objects. We begin with proving a bound on the dimension of the polyhedral set $\{\mathbf{w} \in \mathbb{R}^n : \mathrm{in}_{\mathbf{w}}(I) \neq \langle 1\rangle\}$, not yet knowing that it equals $\mathrm{trop}(X)$. That bound will be further improved to an equality in Theorem 3.3.8, whose proof in the next section will rely on Theorem 3.2.3.

Lemma 3.2.10. *Let X be a d-dimensional subvariety of T^n, with ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Every cell in the Gröbner complex $\Sigma = \Sigma(I_{\mathrm{proj}})$ whose support lies in the set $\{\mathbf{w} \in \mathbb{R}^n : \mathrm{in}_{\mathbf{w}}(I) \neq \langle 1\rangle\}$ has dimension at most d.*

Proof. Let $\mathbf{w} \in \Gamma_{\mathrm{val}}^n$ lie in the relative interior of a maximal cell $P \in \Sigma$. The affine span of P is $\mathbf{w}+L$, where L is a subspace of $\mathbb{R}^n$. By Lemma 2.2.7 and Corollary 2.6.12 we may assume that L is the span of $\mathbf{e}_1, \dots, \mathbf{e}_k$ for some k. We need to show that $k = \dim(L) \leq d$. Since $\mathbf{w}$ lies in the relative interior of P, $\mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I) \neq \langle 1\rangle$ for all $\mathbf{v} \in \mathbb{Z}^n \cap L$ and ϵ sufficiently small. Lemma 2.4.6 and Proposition 2.6.1 imply $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\mathbf{w}}(I)$ for all $\mathbf{v} \in L \cap \mathbb{Z}^n$. Choose a set $\mathcal{G}$ of generators for $\mathrm{in}_{\mathbf{w}}(I)$ so that no generator is the sum of two other polynomials in $\mathrm{in}_{\mathbf{w}}(I)$ having fewer monomials. Then $f \in \mathcal{G}$ implies that $\mathrm{in}_{\mathbf{v}}(f) = f$ for all $\mathbf{v} \in L$, as $\mathrm{in}_{\mathbf{v}}(f)$ is otherwise a polynomial in $\mathrm{in}_{\mathbf{w}}(I)$ having fewer monomials. In particular, we have $\mathrm{in}_{\mathbf{e}_i}(f) = f$ for $1 \leq i \leq k$, so $f = m\tilde{f}$, where m is a monomial, and $x_1, \dots, x_k$ do not appear in $\tilde{f}$. Since monomials are units in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, this means that $\mathrm{in}_{\mathbf{w}}(I)$ is generated by elements not containing $x_1, \dots, x_k$. Hence $k \leq \dim(\mathrm{in}_{\mathbf{w}}(I)) \leq \dim(X) = d$ as required. $\square$

We now use Theorem 3.1.3 to prove Theorem 3.2.3.

Proof of Theorem 3.2.3. The points in set (3) are $(\mathrm{val}(y_1), \dots, \mathrm{val}(y_n))$ for $\mathbf{y} = (y_1, \dots, y_n) \in X$. For any $f \in I$, these satisfy $f(\mathbf{y}) = 0$. By Theorem 3.1.3, $(\mathrm{val}(y_1), \dots, \mathrm{val}(y_n))$ is in $\mathrm{trop}(V(f))$. Hence $(\mathrm{val}(y_1), \dots, \mathrm{val}(y_n))$ lies in set (1). Since set (1) is closed by construction, set (1) contains set (3).

Next, let $\mathbf{w}$ lie in set (1). Then, for any $f = \sum c_{\mathbf{u}} x^{\mathbf{u}} \in I$, the minimum of $\{\mathrm{val}(c_{\mathbf{u}}) + \mathbf{u}\cdot\mathbf{w} : c_{\mathbf{u}} \neq 0\}$ is achieved twice. Thus $\mathrm{in}_{\mathbf{w}}(f)$ is not a monomial. By Lemma 2.6.2, we see that $\mathrm{in}_{\mathbf{w}}(I)$ is not equal to $\langle 1\rangle$, so $\mathbf{w}$ lies in set (2).

It remains to prove that set (2) is contained in set (3). We first reduce to the case where I is prime. Since $\mathrm{in}_{\mathbf{w}}(f^r) = \mathrm{in}_{\mathbf{w}}(f)^r$ for all f, r by part (3) of Lemma 2.6.2, we have $\mathrm{in}_{\mathbf{w}}(I) = \langle 1\rangle$ if and only if $\mathrm{in}_{\mathbf{w}}(\sqrt{I}) = \langle 1\rangle$, so we may assume that I is radical. Thus we can write $I = \bigcap_{i=1}^{s} P_i$, where P_i is prime, and $V(P_1), \dots, V(P_s)$ are the irreducible components of X. Note that if $\mathbf{w} \in \mathbb{R}^n$ has $\mathrm{in}_{\mathbf{w}}(I) \neq \langle 1\rangle$, then there is $j \in \{1, 2, \dots, s\}$ with $\mathrm{in}_{\mathbf{w}}(P_j) \neq \langle 1\rangle$. Indeed, if not, by Lemma 2.6.2 there are $f_1, \dots, f_s$ with $f_i \in P_i$ and $\mathrm{in}_{\mathbf{w}}(f_i) = 1$. Set $f = \prod_{i=1}^{s} f_i$. Then $\mathrm{in}_{\mathbf{w}}(f) = 1$ and $f \in I$, so $\mathrm{in}_{\mathbf{w}}(I) = \langle 1\rangle$, contradicting our assumption.

We have shown that if $\mathbf{w}$ lies in set (2) for X, then $\mathbf{w}$ lies in set (2) for some irreducible component $V(P_j)$ of X. Thus to show that $\mathbf{w} = \mathrm{val}(\mathbf{y})$ for some $\mathbf{y} \in X$ it suffices to show that $\mathbf{w} = \mathrm{val}(\mathbf{y})$ for some $\mathbf{y} \in V(P_j)$. This remaining case is the content of Proposition 3.2.11 below. ☐

Proposition 3.2.11. *Let X be an irreducible subvariety of T^n, with prime ideal $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Fix $\mathbf{w} \in \Gamma_{\mathrm{val}}^n$ with $\mathrm{in}_{\mathbf{w}}(I) \neq \langle 1\rangle$, and let $\alpha \in V(\mathrm{in}_{\mathbf{w}}(I)) \subset (\Bbbk^*)^n$. Then there exists a point $\mathbf{y} \in X$ with $\mathrm{val}(\mathbf{y}) = \mathbf{w}$ and $\overline{t^{-\mathbf{w}}\mathbf{y}} = \alpha$. The set of such $\mathbf{y}$ is dense in the Zariski topology on X.*

Proof. Let $d = \dim(X)$. The cases $n = 1$ and $n - d = 1$ follow from Proposition 3.1.5. So, we can assume $0 \le d \le n-2$. We shall use induction on n. By Lemma 3.2.10, the set $\{\mathbf{v} \in \mathbb{R}^n : \mathrm{in}_{\mathbf{v}}(I) \neq \langle 1\rangle\}$ is the support of a polyhedral complex Σ, and every cell P in Σ has dimension at most $d = \dim(X)$. Let L_P denote the linear span of $P - \mathbf{w}$ in $\mathbb{R}^n$. Then $\dim(L_P) \le d + 1 < n$, and $\mathbf{w} + L_P$ is the affine subspace spanned by P and $\mathbf{w}$.

Choose a monomial projection $\phi\colon T^n \to T^{n-1}$ so that the linear map $\mathrm{trop}(\phi) : \mathbb{R}^n \to \mathbb{R}^{n-1}$ satisfies $\ker(\mathrm{trop}(\phi)) \cap L_P = \{\mathbf{0}\}$ for all $P \in \Sigma$. This is possible by Proposition 3.2.7. We may also assume, after a change of coordinates, that ϕ maps onto the first $n-1$ coordinates, and the image $\phi(X)$ is closed in T^{n-1}. These assumptions ensure that we can uniquely recover $\mathbf{w}$ from its image under $\mathrm{trop}(\phi)$. Indeed, suppose some other vector $\mathbf{w}' \in \Gamma_{\mathrm{val}}^n$ satisfies $\mathrm{in}_{\mathbf{w}'}(I) \neq \langle 1\rangle$ and $\mathrm{trop}(\phi)(\mathbf{w}') = \mathrm{trop}(\phi)(\mathbf{w})$. The first condition says that $\mathbf{w}' \in P$ for some cell $P \in \Sigma$. This implies $\mathbf{w}' \in \mathbf{w} + L_P$ and hence $\mathbf{w}' - \mathbf{w} \in L_P$. The kernel condition then gives $\mathbf{w} = \mathbf{w}'$.

Let $I' = \phi^{*-1}(I) = I \cap K[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$ and $X' = V(I')$. Since $\phi(X)$ is closed, we have $X' = \phi(X)$. By Lemma 2.6.10, $\mathrm{in}_{\mathrm{trop}(\phi)(\mathbf{w})}(I') \neq \langle 1\rangle$. By induction, there is $\mathbf{y}' = (y_1, \dots, y_{n-1}) \in X' \subset T^{n-1}$ with $\mathrm{val}(y_i) = w_i$, and $\overline{t^{-w_i} y_i} = \alpha_i$ for $1 \le i \le n-1$. Let $J = \langle f(y_1, \dots, y_{n-1}, x_n) : f \in I\rangle \subseteq K[x_n^{\pm 1}]$. Since $K[x_n^{\pm 1}]$ is a principal ideal domain (PID), there exists a single polynomial $f \in I$ whose specialization generates the principal ideal J. By Proposition 3.2.7 we may assume that $f = x_n^l + f'$, where x_n^l does not divide any monomial in f'. So, the degree l is positive, hence $J \neq \langle 1 \rangle$, and we can find a point in $V(J)$.

By Proposition 3.2.7 we may also assume that all coefficients of f, regarded as a polynomial in x_n, are monomials, so $f = \sum_i c_i x^{\mathbf{u}_i} x_n^i$ for $\mathbf{u}_i \in \mathbb{Z}^{n-1}$. Let $g = f(y_1, \ldots, y_{n-1}, x_n) = \sum_i c_i y^{\mathbf{u}_i} x_n^i$. As in the proof of Proposition 3.2.11, $\operatorname{trop}(f)(\mathbf{w}) = \operatorname{trop}(g)(w_n)$, and thus $\operatorname{in}_{\mathbf{w}}(f)(\alpha_1, \ldots, \alpha_{n-1}, x_n) = \operatorname{in}_{w_n}(g)(x_n)$. This implies $\operatorname{in}_{w_n}(g)(\alpha_n) = 0$. As $\alpha_n \neq 0$, the polynomial $\operatorname{in}_{w_n}(g)$ is not a monomial. By the $n = 1$ case in Proposition 3.1.5, there is $y_n \in K^*$ with $g(y_n) = 0$, $\operatorname{val}(y_n) = w_n$ and $\overline{t^{-w_n} y_n} = \alpha_n$. We then have $\mathbf{y} = (y_1, \ldots, y_n) \in X$ with $\operatorname{val}(\mathbf{y}) = \mathbf{w}$, and $\overline{t^{w_i} y_i} = \alpha_i$ for all i, as required. This last step uses the "unique recovery" property from two paragraphs ago.

To finish the proof, we now argue that the set $\mathcal{Y}$ of all such $\mathbf{y}$ is Zariski dense in X. Suppose that there was a subvariety $X' \subsetneq X$ containing $\mathcal{Y}$. The projection $\phi(X')$ contains $\phi(\mathcal{Y})$, and so all $\mathbf{y}' \in \phi(X)$ with $\operatorname{val}(y_i') = w_i$ and $\overline{t^{-w_i} y_i} = \alpha_i$ for $1 \leq i \leq n-1$. By our choice of the map ϕ, we have $\dim(\phi(X)) = \dim(X) > 0$. This set is Zariski dense in $\phi(X)$, by induction, and thus $\phi(X') = \phi(X)$. This contradicts the fact that $\dim(\phi(X')) \leq \dim(X') < \dim(X)$. The last inequality uses that X is irreducible. We conclude that there is no such X', and so $\mathcal{Y}$ is Zariski dense. $\square$

The observation that the set of preimages of $\mathbf{w}$ under the valuation map is Zariski dense is due to Payne; see [**Pay09b**], [**Pay12**].

Remark 3.2.12. It is a consequence of Theorems 3.2.3 and 3.2.4 that if L_1/K and L_2/K are two algebraically closed valued field extensions and $X \subset T_K^n$ is a variety, then the closures of $\operatorname{val}(X(L_1))$ and $\operatorname{val}(X(L_2))$ agree. Here $X(L_i)$ is the set of L_i-valued points of X; if I is the ideal of X in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $I_{L_i} = IL_i[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, then $X(L_i) = V(I_{L_i}) \subset T_{L_i}^n$. To see the equality, note that $\operatorname{trop}(V(I_{L_1})) = \operatorname{trop}(V(I_{L_2}))$ by Theorem 3.2.4. By Theorem 3.2.3, we know that $\operatorname{trop}(V(I_{L_i}))$ is the closure of the valuations of points in $X(L_i)$ for each i, so these sets also agree.

At the end of Section 2.6, we introduced the tropicalization $\operatorname{trop}(\phi)$ of a monomial map ϕ. In this section we studied three equivalent characterizations of the tropicalization $\operatorname{trop}(X)$ of an algebraic variety X in a torus. The next corollary states that these two notions of tropicalization are compatible.

Corollary 3.2.13. *Let $\phi : T^n \to T^m$ be a monomial map. Consider any subvariety X of T^n and the Zariski closure $\overline{\phi(X)}$ of its image in T^m. Then*

$$\operatorname{trop}\big(\, \overline{\phi(X)} \,\big) \;=\; \operatorname{trop}(\phi)\big(\operatorname{trop}(X)\,\big). \tag{3.2.2}$$

Proof. If I is the ideal of X, then $I' = (\phi^*)^{-1}(I)$ is the ideal of $\overline{\phi(X)}$. By Lemma 2.6.10 if $\mathrm{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle$, then $\mathrm{in}_{\mathrm{trop}(\phi)(\mathbf{w})}(I') \neq \langle 1 \rangle$, which shows that $\mathrm{trop}(\phi)(\mathrm{trop}(X)) \subseteq \mathrm{trop}(\overline{\phi(X)})$. For the converse we use the Fundamental Theorem. By part (3) of Theorem 3.2.3, $\mathrm{trop}(\overline{\phi(X)})$ is the closure of the set $\{\mathrm{val}(\mathbf{z})\colon \mathbf{z} \in \overline{\phi(X)}\}$. Since $\mathrm{trop}(\phi)(\mathrm{trop}(X))$ is a closed subset of $\mathbb{R}^m$, it thus suffices to show that every $\mathbf{w} \in \Gamma_{\mathrm{val}}^m \cap \mathrm{trop}(\overline{\phi(X)})$ lies in $\mathrm{trop}(\phi)(\mathrm{trop}(X))$. By Theorem 3.2.3, the set of $\mathbf{z} \in \overline{\phi(X)}$ for which $\mathrm{val}(z) = \mathbf{w}$ is Zariski dense in $\overline{\phi(X)}$, so there is $\mathbf{z} = \phi(\mathbf{y})$ for some $\mathbf{y} \in X$ with $\mathrm{val}(\mathbf{z}) = \mathbf{w}$. Since $\mathrm{val}(\phi(\mathbf{y})) = \mathrm{trop}(\phi)(\mathrm{val}(\mathbf{y}))$, this shows that $\mathbf{w} \in \mathrm{trop}(\phi)(\mathrm{trop}(X))$. □

Remark 3.2.14. Corollary 3.2.13 says that tropicalization commutes with morphisms of tori. This statement is *not* true if the morphism ϕ is replaced by a rational map of tori, with $\mathrm{trop}(\phi)$ defined in each coordinate by the corresponding tropical polynomial as in (2.4.1).

For a simple example consider the map $t \mapsto (t, -1-t)$ from the affine line to the affine plane. This defines a rational map of tori $\phi\colon X = T^1 \dashrightarrow T^2$, which is undefined at $t = -1$. Its image is $\overline{\phi(X)} = V(x + y + 1) \subset T^2$, and hence $\mathrm{trop}(\overline{\phi(X)})$ is the standard tropical line seen in Figure 3.1.1. On the other hand, the tropicalization of ϕ is the piecewise-linear map

$$\mathrm{trop}(\phi) : \mathrm{trop}(X) = \mathbb{R} \to \mathbb{R}^2,\ w \mapsto (w, \min(w, 0)).$$

The image of this is the union of two of the three rays of the tropical line,

$$\mathrm{trop}(\phi)(\,\mathrm{trop}(X)\,) \;=\; \{(a, a) : a \leq 0\} \,\cup\, \{(a, 0) : a \geq 0\}.$$

In this little example, the following inclusion holds and is strict:

$$\mathrm{trop}\big(\,\overline{\phi(X)}\,\big) \;\supset\; \mathrm{trop}(\phi)(\,\mathrm{trop}(X)\,). \tag{3.2.3}$$

The inclusion (3.2.3) holds for every rational map ϕ of tori, but the inclusion is generally strict unless ϕ is a monomial map. How to fill the gap is of central importance in *tropical implicitization*, which aims to compute the tropicalization of a rational variety directly from a parametric representation ϕ. In other words, one first computes $\mathrm{trop}(V(I))$, and then one uses that balanced polyhedral complex in deriving generators for I. For more information see Theorems 5.5.1 and 6.5.16 as well as [**STY07, SY08**].

3.3. The Structure Theorem

We next explore the question of which polyhedral complexes are tropical varieties. The main result in this section is the Structure Theorem 3.3.5 which says that if X is an irreducible subvariety of T^n of dimension d, then $\mathrm{trop}(X)$ is the support of a pure d-dimensional weighted balanced Γ_{val}-rational polyhedral complex that is connected through codimension 1.

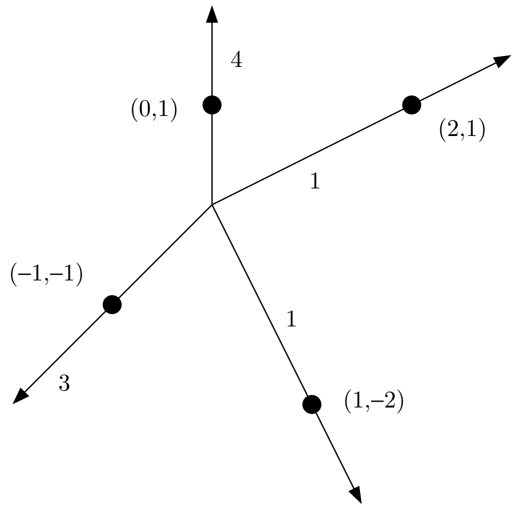

Figure 3.3.1. A balanced rational fan in $\mathbb{R}^2$.

We begin by defining these concepts, starting with the notion of a weighted balanced polyhedral complex. Let $\Sigma \subset \mathbb{R}^n$ be a one-dimensional rational fan with s rays. Let $\mathbf{v}_i$ be the first lattice point on the ith ray of Σ. We give Σ the structure of a *weighted fan* by assigning a positive integer weight $m_i \in \mathbb{N}$ to the ith ray of Σ. We say that the fan Σ is *balanced* if

$$\sum m_i \mathbf{v}_i = 0.$$

This is sometimes called the *zero-tension condition*: a tug-of-war game with ropes in the directions $\mathbf{v}_i$ and participants of strength m_i would have no winner. See Figure 3.3.1 for an example, where the weights are $1, 1, 3$, and 4. We now extend this concept to arbitrary weighted polyhedral complexes.

Definition 3.3.1. Let Σ be a rational fan in $\mathbb{R}^n$, pure of dimension d. Fix weights $m(\sigma) \in \mathbb{N}$ for all cones σ of dimension d. Given a cone $\tau \in \Sigma$ of dimension $d-1$, let L be the linear space parallel to τ. Thus L is a $(d-1)$-dimensional subspace of $\mathbb{R}^n$. Since τ is a rational cone, the abelian group $L_{\mathbb{Z}} = L \cap \mathbb{Z}^n$ is free of rank $d-1$, with $N(\tau) = \mathbb{Z}^n / L_{\mathbb{Z}} \cong \mathbb{Z}^{n-d+1}$. For each $\sigma \in \Sigma$ with $\tau \subsetneq \sigma$, the set $(\sigma + L)/L$ is a one-dimensional cone in $N(\tau) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\mathbf{v}_\sigma$ be the first lattice point on this ray. The fan Σ is *balanced at* τ if

$$\sum m(\sigma)\mathbf{v}_\sigma \;=\; 0. \tag{3.3.1}$$

The fan Σ is *balanced* if it is balanced at all $\tau \in \Sigma$ with $\dim(\tau) = d-1$. If Σ is a pure Γ_{val}-rational polyhedral complex of dimension d with weights $m(\sigma) \in \mathbb{N}$ on each d-dimensional cell in Σ, then for each $\tau \in \Sigma$ the fan $\text{star}_\Sigma(\tau)$ inherits a weighting function m. The complex Σ is *balanced* if the fan $\mathrm{star}_\Sigma(\tau)$ is balanced for all $\tau \in \Sigma$ with $\dim(\tau) = d - 1$. This condition is vacuous for zero-dimensional polyhedral complexes, so all are balanced.

We next explain the combinatorial meaning of the balancing condition for fans and complexes of codimension 1. Let P be a lattice polytope in $\mathbb{R}^n$ with normal fan $\mathcal{N}_P$, and let Σ denote the $(n-1)$-skeleton of $\mathcal{N}_P$. According to Proposition 3.1.10, the fan Σ is the tropical hypersurface $\mathrm{trop}(V(f))$ of any constant-coefficient polynomial f with Newton polytope P. Equivalently, $\Sigma = V(F)$, where F is a tropical polynomial for which the coefficients are all 0 and the exponents of the monomials have convex hull P.

We can turn Σ into a weighted fan as follows. Each maximal cone $\sigma \in \Sigma$ is the inner normal cone of an edge $e(\sigma)$ of the polytope P. We define $m(\sigma)$ to be the *lattice length* of the edge $e(\sigma)$. This is one less than the number of lattice points in $e(\sigma)$. Proposition 3.3.2 below says that Σ is balanced.

For general tropical hypersurfaces we generalize from a lattice polytope P to regular subdivisions Δ of P. We can also define multiplicities from edge lengths in this case. Following Definition 2.3.8, Δ is given by a weight vector $\mathbf{c}$ with one entry $c_{\mathbf{u}}$ for each lattice point $\mathbf{u}$ in P. We construct from this a tropical polynomial $F = \min_{\mathbf{u}\in P}(c_{\mathbf{u}} + \mathbf{x}\cdot\mathbf{u})$. The subdivision Δ is dual to the polyhedral complex Σ_F. The tropical hypersurface $V(F)$ is the $(n-1)$-skeleton of Σ_F by Remark 3.1.7. Every facet σ of Σ corresponds to an edge $e(\sigma)$ of Δ, and we define $m(\sigma)$ to be the lattice length of $e(\sigma)$.

Proposition 3.3.2. *The $(n-1)$-dimensional polyhedral complex $V(F)$ given by a tropical polynomial F in n unknowns is balanced for the weights $m(\sigma)$ defined above.*

Proof. This statement is trivial for $n = 1$. If $n = \dim(P) = 2$, then $d = 1$ in Definition 3.3.1. We claim that $\mathrm{star}_{V(F)}(\sigma)$ is balanced for all zero-dimensional cells σ. Such a cell is dual to a two-dimensional convex polygon Q in the regular subdivision Δ. The vectors $\mathbf{u}_\sigma$ in (3.3.1) are the primitive lattice vectors perpendicular to the edges of Q, and the vectors $m(\sigma)\mathbf{u}_\sigma$ are precisely the edges of Q rotated by 90 degrees. The equation (3.3.1) holds because the edge vectors of any convex polygon Q sum to zero. For $d \geq 3$ we reduce to the case $d = 2$ by working modulo L as in Definition 3.3.1. Here, L is the linear space parallel to σ. This is the lineality space of $\mathrm{star}_{V(F)}(\sigma)$. Hence L is perpendicular to the polygon Q dual to σ in the regular subdivision Δ induced by F. Again, the edges of Q sum to zero. $\square$

Example 3.3.3. Let P be the Newton polytope of the *discriminant* of a univariate quartic $ax^4 + bx^3 + cx^2 + dx + e$. That discriminant equals

$$\begin{aligned} \underline{256a^3e^3} &- 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e + 144ab^2ce^2 \\ &- 80abc^2de - 6ab^2d^2e - \underline{27a^2d^4} + 18abcd^3 + \underline{16ac^4e} \\ &- \underline{4ac^3d^2} - \underline{27b^4e^2} + 18b^3cde - \underline{4b^3d^3} - \underline{4b^2c^3e} + \underline{b^2c^2d^2}. \end{aligned}$$

Its Newton polytope P is a three-dimensional cube that lives in $\mathbb{R}^5$. The eight vertices of P correspond to the underlined monomials. Here Σ is a fan with 12 cones σ of dimension 4, six cones τ of dimension 3, and one cone of dimension 2 (the lineality space). Eleven of the edges of P have lattice length 1, so $m(\sigma) = 1$ for these σ. However, the edge corresponding to $256a^3e^3 + 16ac^4e = 16ae(4ae + ic^2)(4ae - ic^2)$ has lattice length 2, so $m(\sigma) = 2$ for that maximal cone σ of Σ. To check that the fan Σ is balanced, we must examine the cones τ normal to the six square facets of P. The fan $\mathrm{star}_\Sigma(\tau)$ is the normal fan of such a square, and (3.3.1) holds because the four edges of the square form a closed loop. The Newton polygon of the discriminant specialized with $b = 0$ is one of the two facets of P that contain the special edge above. For more information on such discriminants see Section 5.5. $\diamondsuit$

Every tropical polynomial F with coefficients in Γ_{val} has the form $F = \mathrm{trop}(f)$ for some classical polynomial $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Proposition 3.1.6 says that the tropical hypersurface $\mathrm{trop}(V(f))$ is the balanced polyhedral complex Σ_F. We shall see in Lemma 3.4.6 that the combinatorial definition of multiplicity, where $m(\sigma)$ is the length of the edge in the subdivision Δ, is consistent with the general definition of multiplicities for tropical varieties. In Proposition 3.3.10 we prove a combinatorial converse to Proposition 3.3.2.

We next define what it means to be connected through codimension 1.

Definition 3.3.4. Let Σ be a pure d-dimensional polyhedral complex in $\mathbb{R}^n$. Then Σ is *connected through codimension* 1 if for any two d-dimensional cells $P, P' \in \Sigma$ there is a chain $P = P_1, P_2, \ldots, P_s = P'$ for which P_i and P_{i+1} share a common facet F_i for $1 \leq i \leq s-1$. Since the P_i are facets of Σ and the F_i are ridges, we call this a *facet-ridge path* connecting P and P'.

Every zero-dimensional polyhedral complex is connected through codimension 1. A pure one-dimensional polyhedral complex is connected through codimension 1 if and only if it is connected. An example of a connected two-dimensional polyhedral complex that is not connected through codimension 1 is shown in Figure 3.3.2.

This lets us state the second main theorem of this chapter. Its proof will straddle three sections.

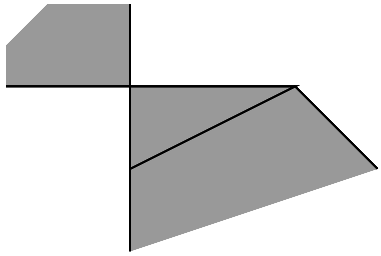

Figure 3.3.2. This complex is not connected through codimension 1.

Theorem 3.3.5 (Structure Theorem for Tropical Varieties)**.** *Let X be an irreducible d-dimensional subvariety of T^n. Then* $\operatorname{trop}(X)$ *is the support of a balanced weighted Γ_{val}-rational polyhedral complex pure of dimension d. Moreover, that polyhedral complex is connected through codimension* 1.

Proof. That $\operatorname{trop}(X)$ is a pure Γ_{val}-rational d-dimensional polyhedral complex is Theorem 3.3.8. That it is balanced is Theorem 3.4.14. Theorem 3.5.1 shows that $\operatorname{trop}(X)$ is connected through codimension 1. □

In the remainder of this section we prove the dimension part of the Structure Theorem. This will be stated separately in Theorem 3.3.8. Its proof will use the following lemma, which says that the star of any cell in a polyhedral complex with support $\operatorname{trop}(X)$ is itself a tropical variety.

Lemma 3.3.6. *Let $X = V(I) \subset T^n_K$ where $I \subseteq K[x_1^{\pm}, \ldots, x_n^{\pm 1}]$, and let Σ be a polyhedral complex with support* $\operatorname{trop}(X) = \{\mathbf{w} \in \mathbb{R}^n : \operatorname{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle\} \subset \mathbb{R}^n$. *Fix* $\mathbf{w} \in \Sigma$. *If $\sigma \in \Sigma$ has* $\mathbf{w}$ *in its relative interior, then*

$$\operatorname{star}_{\Sigma}(\sigma) \;=\; \big\{\mathbf{v} \in \mathbb{R}^n : \operatorname{in}_{\mathbf{v}}(\operatorname{in}_{\mathbf{w}}(I)) \neq \langle 1 \rangle\big\}.$$

Thus $\operatorname{trop}(V(\operatorname{in}_{\mathbf{w}}(I))) = \operatorname{star}_{\operatorname{trop}(X)}(\sigma)$.

Proof. We have

$$\begin{aligned}
&\{\mathbf{v} \in \mathbb{R}^n : \operatorname{in}_{\mathbf{v}}(\operatorname{in}_{\mathbf{w}}(I)) \neq \langle 1 \rangle\} \\
&\qquad = \; \{\mathbf{v} \in \mathbb{R}^n : \operatorname{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I) \neq \langle 1 \rangle \text{ for sufficiently small } \epsilon > 0\} \\
&\qquad = \; \{\mathbf{v} \in \mathbb{R}^n : \mathbf{w} + \epsilon\mathbf{v} \in \; \Sigma \text{ for sufficiently small } \epsilon > 0\} \\
&\qquad = \; \operatorname{star}_{\Sigma}(\sigma),
\end{aligned}$$

where the first equality follows from Lemma 2.4.6 and Proposition 2.6.1, and the third equality follows from Exercise 2.7(13). □

Example 3.3.7. Let $I = \langle tx^2 + x + y + xy + t\rangle$ in $\mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$ and $X = V(I)$. The tropical curve $\operatorname{trop}(X)$ is shown in Figure 3.3.3. The tropical curve of the initial ideal $\operatorname{in}_{(1,1)}(I) = \langle x+y+1\rangle$ is the tropical line, with rays $(1, 0), (0, 1)$, and $(-1, -1)$. This is the star of the vertex $(1, 1)$. It is also the

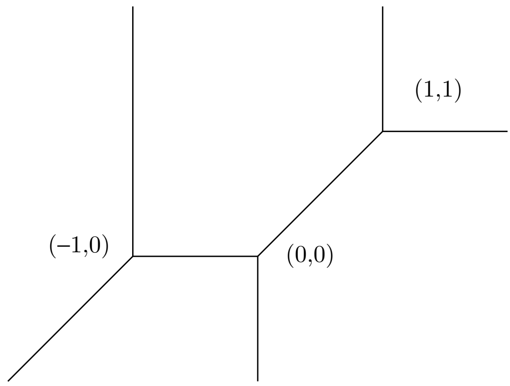

Figure 3.3.3. The tropical curve discussed in Example 3.3.7.

star of the vertex $(-1,0)$, since $\mathrm{in}_{(-1,0)}(I) = \langle x^2+x+xy\rangle = \langle x+1+y\rangle$. At the vertex $(0,0)$, the star has rays $(1,1)$, $(-1,0)$, and $(0,-1)$. This is the tropicalization of $V(\mathrm{in}_{(0,0)}(I)) = V(\langle x+y+xy\rangle)$. $\diamondsuit$

Theorem 3.3.8. *Let X be an irreducible subvariety of dimension d in the algebraic torus T^n over the field K. The tropical variety* $\mathrm{trop}(X)$ *is the support of a pure d-dimensional Γ_{val}-rational polyhedral complex in $\mathbb{R}^n$.*

Proof. Let I be the ideal of X. By part (2) of Theorem 3.2.3, the tropical variety $\mathrm{trop}(X)$ is the support of a Γ_{val}-rational polyhedral complex Σ. Lemma 3.2.10 shows that the dimension of each cell in Σ is at most d. It thus remains to show that each maximal cell in Σ has dimension at least d.

Let σ be a maximal cell in Σ, and fix $\mathbf{w} \in \mathrm{relint}(\sigma)$. Suppose that $\dim(\sigma) = k$. By Lemma 3.3.6, we have $\mathrm{trop}(V(\mathrm{in}_{\mathbf{w}}(I))) = |\,\mathrm{star}_\Sigma(\sigma)|$. This is the linear space parallel to σ, and thus a subspace L of $\mathbb{R}^n$ of dimension k. After a change of coordinates, we may assume that L is spanned by $\mathbf{e}_1,\dots,\mathbf{e}_k$. Since $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I) = \mathrm{in}_{\mathbf{w}}(I)$ for all $\mathbf{v} \in L$ and small $\epsilon > 0$, the ideal $\mathrm{in}_{\mathbf{w}}(I)$ is homogeneous with respect to the grading given by $\deg(x_i) = \mathbf{e}_i$ for $1 \le i \le k$ and $\deg(x_i) = 0$ for $i > k$. Hence, $\mathrm{in}_{\mathbf{w}}(I)$ is generated by Laurent polynomials which use only the variables $x_{k+1},\dots,x_n$.

Let $J = \mathrm{in}_{\mathbf{w}}(I) \cap \Bbbk[x_{k+1}^{\pm1},\dots,x_n^{\pm1}]$. We claim that $\mathrm{trop}(V(J)) = \{\mathbf{0}\}$. Indeed, let $\mathbf{v}' \in \mathrm{trop}(V(J))$ and $\mathbf{v} = (0,\mathbf{v}') \in \mathbb{R}^n$ with first k coordinates zero. If $\mathbf{v}' \neq \mathbf{0}$, then $\mathbf{v} \notin L$ and $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) = \langle 1\rangle$, since σ is a maximal cell in Σ. Hence there is $f \in \mathrm{in}_{\mathbf{w}}(I)$ with $\mathrm{in}_{\mathbf{v}}(f) = 1$. We may choose f in J, as we can take it to be homogeneous in the $\mathbb{Z}^k$-grading discussed above. This shows that $\mathrm{in}_{\mathbf{v}'}(J) = \langle 1\rangle$, so $\mathrm{trop}(V(J)) \subseteq \{\mathbf{0}\}$. Since $1 \notin \mathrm{in}_{\mathbf{w}}(I)$, we have $\mathrm{in}_{\mathbf{0}}(J) = J \neq \langle 1\rangle$, and thus $\mathrm{trop}(V(J)) = \{\mathbf{0}\}$. Lemma 3.3.9 below then implies that $V(J)$ is finite, and so $\dim(\mathrm{in}_{\mathbf{w}}(I)) \le k$. From Lemma 3.2.6 we know that $\dim(\mathrm{in}_{\mathbf{w}}(I)) = d$, and hence $k = \dim(\sigma) \ge d$ as required. $\square$

To complete the proof of Theorem 3.3.8, it now remains to show

Lemma 3.3.9. *Let X be a subvariety of T^n. If the tropical variety* $\mathrm{trop}(X)$ *is a finite set of points in $\mathbb{R}^n$, then X is a finite set of points in T^n.*

Proof. The proof is by induction on n. For $n = 1$, all nontrivial subvarieties are finite, and $\mathrm{trop}(T^1) = \mathbb{R}^1$. Suppose $n > 1$ and the lemma is true for all smaller n. If X is a hypersurface, then Proposition 3.1.6 implies that $\mathrm{trop}(X)$ is not finite. We thus assume $\dim(X) < n - 1$. Choose a map $\pi\colon T^n \to T^{n-1}$ with $Y := \overline{\pi(X)} = \pi(X)$ as guaranteed by Proposition 3.2.7. By changing coordinates, we may assume that π is the projection onto the first $n-1$ coordinates. By Corollary 3.2.13 we know that $\mathrm{trop}(Y)$ is a finite set of points in $\mathbb{R}^{n-1}$. By the induction hypothesis, the variety Y is finite: $Y = \{\mathbf{y}_1, \ldots, \mathbf{y}_r\} \subset T^{n-1}$. Since $\lambda \mathbf{e}_n \notin \mathrm{trop}(X)$ for $\lambda \gg 0$, the ideal I of X contains a polynomial of the form $1 + \sum_{i=1}^{s} f_i x_n^i$ with $f_i \in K[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$. Each $\mathbf{y}_i \in Y$ has at most s preimages $\mathbf{z} \in X$ with $\pi(\mathbf{z}) = \mathbf{y}_i$, so X is a finite set of points in T^n. $\square$

According to the Structure Theorem 3.3.5, every tropical variety $\mathrm{trop}(X)$ is the support of a weighted balanced polyhedral complex Σ. One may wonder whether the converse is true: given such a complex Σ, can we always find a matching variety X with $\mathrm{trop}(X) = |\Sigma|$? We shall see in Chapter 4 that the answer is *no*, even in the context of linear spaces. That is why we distinguish between tropicalized linear spaces and tropical linear spaces. See Example 4.2.15 for a balanced fan Σ that is not $\mathrm{trop}(X)$ for any $X \subset T^n$. We close this section by showing that the answer is *yes* for hypersurfaces.

Proposition 3.3.10. *Let Σ be a weighted balanced Γ_{val}-rational polyhedral complex in $\mathbb{R}^n$ that is pure of dimension $n-1$. Then there exists a tropical polynomial F with coefficients in Γ_{val} such that $\Sigma = V(F)$. This ensures that $|\Sigma| = \mathrm{trop}(V(f))$ for some Laurent polynomial $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.*

Proof. We construct a tropical polynomial F in $u_1, \ldots, u_n$, with coefficients in Γ_{val}, such that $V(F) = \{\, \mathbf{w} \in \mathbb{R}^n : \text{the minimum in } F \text{ is achieved twice}\}$ equals Σ, and the weights in Σ are the edge lengths in the corresponding regular subdivision of the Newton polytope of F. Any Laurent polynomial f with $\mathrm{trop}(f) = F$ will satisfy the conclusion in the last sentence.

Fix an arbitrary generic basepoint $\mathbf{u}_0$ in $\mathbb{R}^n \backslash \Sigma$. For any facet σ of Σ let ℓ_σ be the unique primitive linear polynomial that vanishes on σ and satisfies $\ell_\sigma(\mathbf{u}_0) > 0$. Here *primitive* means that the coefficients of ℓ_σ are relatively prime integers. We also write $m(\sigma)$ for the multiplicity of σ in Σ. The linear forms ℓ_σ determine hyperplanes H_σ, which need not all be distinct. Let $\mathcal{A}$ be the hyperplane arrangement consisting of all these hyperplanes H_σ. By construction we have $\Sigma \subseteq \mathcal{A}$ and $\mathbf{u}_0 \notin \mathcal{A}$. We may refine the polyhedral complex structure on $\mathcal{A}$ so that Σ is a subcomplex of $\mathcal{A}$. For cells σ of $\mathcal{A}$ that are not contained in Σ, we set $m(\sigma) = 0$.

The complement $\mathbb{R}^n \backslash \mathcal{A}$ is the disjoint union of open convex polyhedra P. For each such polyhedron P we choose a path from $\mathbf{u}_0$ to P that crosses each hyperplane in $\mathcal{A}$ at most once and does so transversally. We define $\ell_P = \sum_{i=1}^n a_{P,i} x_i + b_P$ to be the sum of linear forms $m(\sigma)\ell_\sigma$, where σ is crossed by the path from $\mathbf{u}_0$ to P. The desired tropical polynomial is then

$$F(u) \ := \ \bigoplus_P b_P \odot u_1^{a_{P,1}} u_2^{a_{P,2}} \cdots u_n^{a_{P,n}},$$

where P ranges over all connected components of $\mathbb{R}^n \backslash \mathcal{A}$.

Since Σ is balanced, the definition of the linear form ℓ_P is independent of the choice of path from $\mathbf{u}_0$ to P. Indeed, any two such paths are connected by moves that cross codimension-2 faces τ of Σ. The balancing condition implies that $\sum_{\sigma \supset \tau} m(\sigma) \cdot \ell_\sigma = 0$ which ensures invariance of ℓ_P as τ is crossed. This means that the tropical polynomial $F(u)$ depends only on the choice of the basepoint $\mathbf{u}_0$. If $\mathbf{u}_0$ moves to a different component of $\mathbb{R}^n \backslash \Sigma$, then $F(u)$ is changed by tropical multiplication by a monomial, so the tropical hypersurface $V(F)$ remains unchanged.

By construction, the support of Σ is contained in $V(F)$ because F bends along each facet σ of Σ. This can be seen by choosing $\mathbf{u}_0$ just off σ. We need to show the reverse inclusion. Consider any region on which F is linear. That region corresponds to a vertex in the regular subdivision Δ that is dual to $V(F)$ in Proposition 3.1.6. By the remark above, we may assume that this vertex is the zero vector, and hence F is nonnegative. The region where F is zero lies in some connected component of $\mathbb{R}^n \backslash \Sigma$. By construction, every nonzero linear function ℓ_σ used in F is strictly positive on that connected component. Hence they are equal. Moreover, the linear function ℓ_σ is the corresponding edge direction, away from the vertex zero, in the regular subdivision Δ. The lattice length of that edge in Δ equals $m(\sigma)$. Hence Σ and $V(F)$ agree as weighted polyhedral complexes in $\mathbb{R}^n$. $\square$

Remark 3.3.11. In this proof we described an algorithm for reconstructing a tropical polynomial F from the tropical hypersurface Σ it defines. This is interesting even in the constant coefficient case, when the input is a weighted balanced fan Σ of codimension 1, and the output is the corresponding Newton polytope P. In fact, that algorithm for computing P from Σ plays a central role in applications of tropical geometry, notably in implicitization [**STY07, SY08**]. Note that P is unique only up to translation.

3.4. Multiplicities and Balancing

In this section we define multiplicities that give a tropical variety the structure of a weighted balanced polyhedral complex. Another important result here is the Transverse Intersection Theorem (Theorem 3.4.12), which gives some control on the tropicalization of an intersection.

Given a subvariety $X \subset T^n$ with ideal $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, Proposition 3.2.8 implies that the tropical variety $\mathrm{trop}(X)$ is the support of a polyhedral complex Σ. While $|\Sigma| = \mathrm{trop}(X)$ is determined by I, the choice of Σ is not, as seen in Example 3.2.9. By Proposition 3.2.8 the polyhedral complex Σ can be chosen so that, for every $\sigma \in \Sigma$, we have $\mathrm{in}_{\mathbf{w}}(I)$ constant for all $\mathbf{w} \in \mathrm{relint}(\sigma)$. In what follows, we fix such a choice of Σ. Our aim is to define multiplicities on Σ that make it a weighted polyhedral complex.

We first recall some concepts from commutative algebra.

Definition 3.4.1. Let $S = \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. An ideal $Q \subset S$ is *primary* if $fg \in Q$ implies $f \in Q$ or $g^m \in Q$ for some $m > 0$. If Q is primary, then the radical of Q is a prime ideal P. Given an ideal $I \subset S$, we can write $I = \bigcap_{i=1}^{s} Q_i$ where each Q_i is primary with radical P_i, no P_i is repeated, and no Q_i can be removed from the intersection. This *primary decomposition* is not unique in general, but the set $\mathrm{Ass}(I)$ of primes P_i that appear is determined by I. These are the *associated primes*. We are most interested in the *minimal (associated) primes* of I, i.e., those P_i that do not contain any other P_j. We denote this set by $\mathrm{Ass}^{\min}(I) = \{P_1, \ldots, P_t\}$. The primary ideal Q_i corresponding to a minimal prime P_i does not depend on the choice of a primary decomposition for I. For more information see [**Eis95**, Chapter 3], [**CLO07**, §4.7], or [**Stu02**, Chapter 5].

The *multiplicity* of a minimal prime $P_i \in \mathrm{Ass}^{\min}(I)$ is the positive integer

$$\mathrm{mult}(P_i, I) := \ell((S/Q_i)_{P_i}) = \ell(((I : P_i^{\infty})/I)_{P_i}). \tag{3.4.1}$$

Here $\ell(M)$ denotes the *length* of an S_{P_i}-module M. This is the length s of the longest chain of submodules $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_s$.

Example 3.4.2. Let $f = \alpha \prod_{i=1}^{r} (x - \lambda_i)^{m_i}$ with $\alpha, \lambda_i \in \mathbb{k}$ be a univariate polynomial in factored form. The set of associated primes of the ideal $\langle f \rangle$ is $\{\langle x - \lambda_i \rangle : 1 \leq i \leq r\}$, and the multiplicities are $\mathrm{mult}(\langle x - \lambda_i \rangle, \langle f \rangle) = m_i$. $\Diamond$

Definition 3.4.3. Let I be a (not necessarily radical) ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let Σ be a polyhedral complex with support $|\Sigma| = \mathrm{trop}(V(I))$ such that $\mathrm{in}_{\mathbf{w}}(I)$ is constant for $\mathbf{w} \in \mathrm{relint}(\sigma)$ for all $\sigma \in \Sigma$. For a top-dimensional cell $\sigma \in \Sigma$ the *multiplicity* $\mathrm{mult}(\sigma)$ is defined by

$$\mathrm{mult}(\sigma) = \sum_{P \in \mathrm{Ass}^{\min}(\mathrm{in}_{\mathbf{w}}(I))} \mathrm{mult}(P, \mathrm{in}_{\mathbf{w}}(I)) \qquad \text{for any } \mathbf{w} \in \mathrm{relint}(\sigma).$$

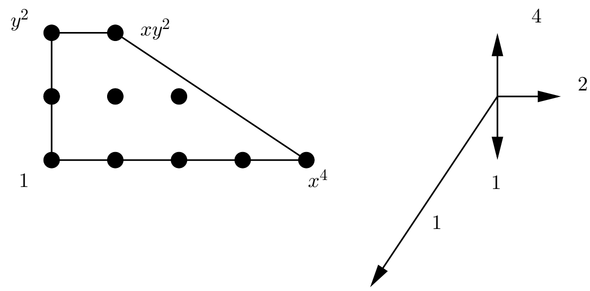

Figure 3.4.1. Multiplicities on the tropical curve in Example 3.4.5.

Remark 3.4.4. If $V(I)$ is irreducible of dimension d and σ is a maximal cell in Σ, then $\mathrm{in}_{\mathbf{w}}(I)$ is homogeneous with respect to a $\mathbb{Z}^d$-grading, so $V(\mathrm{in}_{\mathbf{w}}(I))$ has a d-dimensional torus action on it and is thus a union of d-dimensional torus orbits. The multiplicity $\mathrm{mult}(\sigma)$ is the number of such orbits, counted with multiplicity. See Lemma 3.4.7 below for an algebraic formulation.

Example 3.4.5. Let $f = xy^2 + 4y^2 + 3x^2y - xy + 8y + x^4 - 5x^2 + 4 \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. Then $\mathrm{trop}(V(f))$ consists of four rays perpendicular to the edges of the Newton polygon of f. The rays are generated by $\mathbf{u}_1 = (1, 0)$, $\mathbf{u}_2 = (0, 1)$, $\mathbf{u}_3 = (-2, -3)$, and $\mathbf{u}_4 = (0, -1)$; see Figure 3.4.1. The multiplicities on the rays of $\mathrm{trop}(V(f))$ are shown in Table 3.4.1.

Table 3.4.1

ray	$\mathrm{in}_{\mathbf{u}_i}(\langle f\rangle)$	$\mathrm{mult}(\mathrm{pos}(\mathbf{u}_i))$
$\mathbf{u}_1$	$\langle 4y^2 + 8y + 4\rangle = \langle (y+1)^2\rangle$	2
$\mathbf{u}_2$	$\langle x^4 - 5x^2 + 4\rangle = \langle x-2\rangle \cap \langle x-1\rangle \cap \langle x+1\rangle \cap \langle x+2\rangle$	4
$\mathbf{u}_3$	$\langle xy^2 + x^4\rangle = \langle y^2 + x^3\rangle$	1
$\mathbf{u}_4$	$\langle xy^2 + 4y^2\rangle = \langle x + 4\rangle$	1

The variety of the initial ideal for $\mathbf{u}_1$ is one torus orbit with multiplicity two, the variety for $\mathbf{u}_2$ consists of four torus orbits, and the varieties for $\mathbf{u}_3$ and $\mathbf{u}_4$ are each a single torus orbit. The tropical curve $\mathrm{trop}(V(f))$ is balanced with these multiplicities because $2\mathbf{u}_1 + 4\mathbf{u}_2 + 1\mathbf{u}_3 + 1\mathbf{u}_4 = (0, 0)$. $\Diamond$

Example 3.4.5 is a special case of the following fact which holds for all tropical hypersurfaces and which was already addressed in Proposition 3.3.2.

Lemma 3.4.6. *Let $f = \sum c_{\mathbf{u}} x^{\mathbf{u}} \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, let Δ be the regular subdivision of* $\mathrm{Newt}(f)$ *induced by* $(\mathrm{val}(c_{\mathbf{u}}))$*, and let Σ be the polyhedral complex supported on* $\mathrm{trop}(V(f))$ *that is dual to Δ. The multiplicity of a maximal cell σ of Σ is the lattice length of the edge $e(\sigma)$ of Δ dual to σ.*

Proof. Pick $\mathbf{w}$ in the relative interior of σ. The initial ideal $\mathrm{in}_{\mathbf{w}}(\langle f \rangle)$ is generated by $\mathrm{in}_{\mathbf{w}}(f) = \sum_{\mathbf{u} \in e(\sigma)} \overline{t^{-\mathrm{val}(c_{\mathbf{u}})} c_{\mathbf{u}}} x^{\mathbf{u}}$. The sum here is over those $\mathbf{u} \in e(\sigma)$ with $\mathrm{val}(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} = \mathrm{trop}(f)(\mathbf{w})$. Since $e(\sigma)$ is one dimensional, the vector $\mathbf{u} - \mathbf{u}'$ for $\mathbf{u}, \mathbf{u}' \in e(\sigma)$ is unique up to scaling, and there is a choice $\mathbf{v} = \mathbf{u} - \mathbf{u}'$ for which this has minimal length. The polynomial $\mathrm{in}_{\mathbf{w}}(f)$ is then a monomial in $x_1, \ldots, x_n$ times a Laurent polynomial g in the variable $y = x^{\mathbf{v}}$. After multiplying f by a monomial, we may assume that $\mathrm{in}_{\mathbf{w}}(f)$ is a (non-Laurent) polynomial in y with nonzero constant term. The degree of g in y is then the lattice length of the edge $e(\sigma)$. It follows from Example 3.4.2 that the multiplicity of σ is the lattice length of $e(\sigma)$, as required. $\square$

Note that Lemma 3.4.6 and Proposition 3.3.2 together imply that tropical hypersurfaces are balanced with the multiplicities of Definition 3.4.3.

We now translate the geometric content of Remark 3.4.4 into a precise algebraic form. After a multiplicative change of variables, we may transport any cell in Σ to one with affine span parallel to the span of $\mathbf{e}_1, \ldots, \mathbf{e}_d$. The following lemma gives one method for computing the multiplicity of σ. This should be compared with the formula for multiplicity in Exercise 3.7(34).

Lemma 3.4.7. *Let $X \subset T^n$ be irreducible of dimension d with ideal $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let Σ be a polyhedral complex on* $\mathrm{trop}(X)$ *as above. Let σ be a maximal cell in Σ with affine span parallel to $\mathbf{e}_1, \ldots, \mathbf{e}_d$, and let $\mathbf{w} \in \mathrm{relint}(\sigma) \cap \Gamma_{\mathrm{val}}^n$. If $S' = \Bbbk[x_{d+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$, then* $\mathrm{mult}(\sigma) = \dim_{\Bbbk}(S'/(\mathrm{in}_{\mathbf{w}}(I) \cap S'))$.

Proof. Since $\mathbf{w} \in \mathrm{relint}(\sigma)$, by Corollary 2.4.10 and Proposition 2.6.1 we have $\mathrm{in}_{\mathbf{w}+\epsilon \mathbf{e}_i}(I) = \mathrm{in}_{\mathbf{w}}(I)$ for all sufficiently small $\epsilon > 0$ and $1 \le i \le d$. Thus, by part (2) of Lemma 2.6.2, the initial ideal $\mathrm{in}_{\mathbf{w}}(I)$ is homogeneous with respect to the grading $\deg(x_i) = \mathbf{e}_i$ for $i \le d$ and $\deg(x_i) = 0$ for $i > d$. Hence $\mathrm{in}_{\mathbf{w}}(I)$ has a generating set $\{f_1, \ldots, f_r\}$ not containing the variables $x_1, \ldots, x_d$. Let $\bigcap_{i=1}^s Q_i$ be a primary decomposition of $\mathrm{in}_{\mathbf{w}}(I)$. Each Q_i is also generated by polynomials in $x_{d+1}, \ldots, x_n$, as they are also homogeneous with respect to the $\mathbb{Z}^d$-grading, so $\mathrm{in}_{\mathbf{w}}(I) \cap S' = \bigcap_{i=1}^s (Q_i \cap S')$ is a primary decomposition of the zero-dimensional ideal $\mathrm{in}_{\mathbf{w}}(I) \cap S'$, and $\mathrm{mult}(P_i, Q_i) = \mathrm{mult}(P_i \cap S', Q_i \cap S')$. This implies that each P_i is a minimal prime of $\mathrm{in}_{\mathbf{w}}(I)$. Since $Q_i \cap S'$ is a zero-dimensional ideal in S', its multiplicity is its colength. Therefore,

$$\begin{aligned} \mathrm{mult}(\sigma) &= \sum_{i=1}^{s} \mathrm{mult}(P_i, Q_i) \\ &= \sum_{i=1}^{s} \dim_{\Bbbk} S'/(Q_i \cap S') = \dim_{\Bbbk} S'/(\mathrm{in}_{\mathbf{w}}(I) \cap S'). \end{aligned}$$ $\square$

The goal of the rest of this section is to show that the multiplicities in Definition 3.4.3 force the polyhedral complex $\mathrm{trop}(X)$ to be balanced. We will reduce the proof of this to the case of constant coefficient curves. For this, we need following result about zero-dimensional ideals I. As before we write S_K and $S_{\Bbbk}$ for the Laurent polynomial rings in variables $x_1, \ldots, x_n$ with coefficients in K and $\Bbbk$, respectively. We denote by $\widetilde{S}_K$ and $\widetilde{S}_{\Bbbk}$ the corresponding polynomial rings with $n+1$ variables $x_0, x_1, \ldots, x_n$.

Proposition 3.4.8. *Let $I = \bigcap_{\mathbf{y}} Q_{\mathbf{y}} \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, where each $Q_{\mathbf{y}}$ is primary to a maximal ideal $P_{\mathbf{y}} = \langle x_1 - y_1, \ldots, x_n - y_n \rangle$.*

(1) *Assume further that all $\mathbf{y} \in V(I) \subset T^n$ have the same tropicalization $\mathrm{val}(\mathbf{y}) = \mathbf{w}$, for some fixed $\mathbf{w} \in \Gamma_{\mathrm{val}}^n$. Then $\dim_{\Bbbk} S_{\Bbbk}/\mathrm{in}_{\mathbf{w}}(I) = \sum_{\mathbf{y}} \mathrm{mult}(P_{\mathbf{y}}, Q_{\mathbf{y}}) = \dim_K S_K/I$.*

(2) *Without that assumption, for $\mathbf{w} \in \mathrm{trop}(V(I))$, let*

$$I_{\mathbf{w}} = \bigcap_{\mathbf{y}:\mathrm{val}(\mathbf{y})=\mathbf{w}} Q_{\mathbf{y}}.$$

The multiplicity of the point $\mathbf{w}$ equals $\dim_K S_K/I_{\mathbf{w}}$.

Proof. The equation

$$\dim_K S_K/I \;=\; \sum_{\mathbf{y} \in V(I)} \mathrm{mult}(P_{\mathbf{y}}, Q_{\mathbf{y}})$$

holds for any zero-dimensional ideal $I = \bigcap_{\mathbf{y}} Q_{\mathbf{y}}$ where $P_{\mathbf{y}} = \mathrm{rad}(Q_{\mathbf{y}})$. The homogenization I_{proj} of such an ideal satisfies $\dim_K S_K/I = \dim_K(\widetilde{S}_K/I_{\mathrm{proj}})_d$ for any $d \gg 0$. These two facts also hold for ideals in $S_{\Bbbk}$. By Corollary 2.4.9 we have $\dim_K(\widetilde{S}_K/I_{\mathrm{proj}})_d = \dim_{\Bbbk}(\widetilde{S}_K/\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}}))_d$, so to show that $\dim_K S_K/I = \dim_{\Bbbk} S_{\Bbbk}/\mathrm{in}_{\mathbf{w}}(I)$ it suffices to show

$$(\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}}))_d = (\mathrm{in}_{\mathbf{w}}(I)_{\mathrm{proj}})_d \text{ for } d \gg 0.$$

The inclusion $\subseteq$ follows from Proposition 2.6.1, since $J \subseteq (J|_{x_0=1})_{\mathrm{proj}}$ for any homogeneous ideal $J \subset \Bbbk[x_0, \ldots, x_n]$. For the reverse inclusion, we first note that Proposition 2.6.1 also implies that $\mathrm{in}_{\mathbf{w}}(I)_{\mathrm{proj}} = (\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}}) : \prod_{i=0}^n x_i^{\infty})$. Saturating by the irrelevant ideal $\langle x_0, \ldots, x_n \rangle$ does not change $(\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}}))_d$ for $d \gg 0$, and this saturation has only one-dimensional associated primes. These associated primes have the form $P_{\mathbf{y}'} = \langle y'_j x_i - y'_i x_j : 0 \leq i < j \leq n \rangle$ for some $\mathbf{y}' = (y'_0 : \cdots : y'_n) \in \mathbb{P}^n$. Write $(\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}}) : \langle x_0, \ldots, x_n \rangle^{\infty}) = \bigcap_{\mathbf{y}} Q_{\mathbf{y}'}$, where $Q_{\mathbf{y}'}$ is primary to $P_{\mathbf{y}'}$ Now

$$\left(\bigcap Q_{\mathbf{y}'} : \prod x_i^{\infty}\right) = \bigcap_{\mathbf{y}'} (Q_{\mathbf{y}'} : \prod x_i^{\infty}) = \bigcap_{\mathbf{y}' : x_i \notin P_{\mathbf{y}'}} Q_{\mathbf{y}'},$$

so it suffices to show that $x_i \notin P_{\mathbf{y}'}$ for all i and all $Q_{\mathbf{y}'}$.

Since each primary component $Q_{\mathbf{y}}$ of I is $P_{\mathbf{y}}$ primary, it contains $(x_i - y_i)^d$ for some $d \gg 0$. The product $\prod_{\mathbf{y}}(x_i - y_i x_0)^d$ is thus in I_{proj} for $d \gg 0$, and so since $\text{val}(\mathbf{y}) = \mathbf{w}$ for all $\mathbf{y}$, we have $\prod_{\mathbf{y}}(x_i - \tilde{y}_i x_0)^d \in \text{in}_{(0,\mathbf{w})}(I_{\text{proj}})$, where $\tilde{y}_i = \overline{t^{-w_i} y_i}$. This shows that $x_i \notin P_{\mathbf{y}'}$ for all $\mathbf{y}'$ and $0 \leq i \leq n$. Indeed, for each i the product $\prod_{\mathbf{y}}(x_i - \tilde{y}_i x_0)^d \in \text{in}_{(0,\mathbf{w})}(I_{\text{proj}})$, so for each $\mathbf{y}'$ there is there is $\tilde{y}_i$ with $x_i - \tilde{y}_i x_0 \in P_{\mathbf{y}'}$. If $x_i \in P_{\mathbf{y}'}$ for some i, then $x_j \in P_{\mathbf{y}'}$ for $0 \leq j \leq n$, since each $\tilde{y}_i$ is nonzero as $y_i \neq 0$. This contradicts the fact that $\mathbf{y}' \in \mathbb{P}^n$, so we conclude that the first claim holds.

For Proposition 3.4.8(2), we claim that $\text{in}_{\mathbf{w}}(I) = \text{in}_{\mathbf{w}}(I_{\mathbf{w}})$. The inclusion $\subseteq$ is immediate from $I \subseteq I_{\mathbf{w}}$. For the inclusion $\supseteq$, note that for any $\mathbf{y}$ with $\text{val}(\mathbf{y}) \neq \mathbf{w}$ we have $\mathbf{w} \notin \text{trop}(Q_{\mathbf{y}})$, so there is $f_{\mathbf{y}} \in Q_{\mathbf{y}}$ with $1 = \text{in}_{\mathbf{w}}(f)$. Given $f \in I_{\mathbf{w}}$, we then have $g = f \prod_{\text{val}(\mathbf{y}) \neq \mathbf{w}} f_{\mathbf{y}} \in I$ with $\text{in}_{\mathbf{w}}(g) = \text{in}_{\mathbf{w}}(f)$. This gives the other inclusion. The result now follows from the first part using the interpretation of the multiplicity of Lemma 3.4.7. $\square$

The concept of transverse intersection is fundamental in algebraic, differential, and symplectic geometry. The same holds in tropical geometry.

Definition 3.4.9. Let Σ_1 and Σ_2 be two polyhedral complexes in $\mathbb{R}^n$, and let $\mathbf{w} \in \Sigma_1 \cap \Sigma_2$. The point $\mathbf{w}$ lies in the relative interior of a unique cell σ_i in Σ_i for $i = 1, 2$. The complexes Σ_1, Σ_2 *meet transversely* at $\mathbf{w} \in \Sigma_1 \cap \Sigma_2$ if the affine span of σ_i is $\mathbf{w} + L_i$ for $i = 1, 2$, and $L_1 + L_2 = \mathbb{R}^n$. Two tropical varieties $\text{trop}(X)$ and $\text{trop}(Y)$ *intersect transversely* at some $\mathbf{w} \in \text{trop}(X) \cap \text{trop}(Y)$ if there is *some* choice of polyhedral complexes Σ_1, Σ_2, with $\text{trop}(X) = |\Sigma_1|$ and $\text{trop}(Y) = |\Sigma_2|$, and these meet transversely at $\mathbf{w}$.

We next show, in Theorem 3.4.12, that if the tropicalizations of two varieties meet transversely at $\mathbf{w} \in \mathbb{R}^n$, then $\mathbf{w}$ lies in the tropicalization of the intersection of the varieties. This requires the following lemma.

Lemma 3.4.10. *Let I, J be homogeneous ideals in $K[x_0, \ldots, x_n, y_0, \ldots, y_m]$, and fix $\mathbf{w} \in \mathbb{R}^{n+m+2}$. If $\text{in}_{\mathbf{w}}(I)$ has a generating set only involving $x_0, \ldots, x_n$ and $\text{in}_{\mathbf{w}}(J)$ has a generating set only involving $y_0, \ldots, y_m$, then*

$$\text{in}_{\mathbf{w}}(I + J) \;=\; \text{in}_{\mathbf{w}}(I) + \text{in}_{\mathbf{w}}(J).$$

Proof. Suppose that this is not the case. Then there is some homogeneous polynomial $f + g$ in $I + J$ of degree d with $f \in I_d$, $g \in J_d$ and $\text{in}_{\mathbf{w}}(f + g) \notin \text{in}_{\mathbf{w}}(I) + \text{in}_{\mathbf{w}}(J)$. Fix a monomial term order $\prec$ on $\Bbbk[x_0, \ldots, y_m]$. We may further assume that $\text{in}_{\prec}(\text{in}_{\mathbf{w}}(f + g)) \notin \text{in}_{\prec}(\text{in}_{\mathbf{w}}(I) + \text{in}_{\mathbf{w}}(J))$. This implies

$$\text{in}_{\prec}(\text{in}_{\mathbf{w}}(f + g)) \;\notin\; \text{in}_{\prec}(\text{in}_{\mathbf{w}}(I)) + \text{in}_{\prec}(\text{in}_{\mathbf{w}}(J)). \tag{3.4.2}$$

Let $x^{\mathbf{u}_1} y^{\mathbf{v}_1}$ and $x^{\mathbf{u}_2} y^{\mathbf{v}_2}$ be the monomials in $\text{in}_{\prec}(\text{in}_{\mathbf{w}}(f))$ and $\text{in}_{\prec}(\text{in}_{\mathbf{w}}(g))$, respectively, and let $\alpha_1, \alpha_2 \in K$ be their coefficients in f and g. From (3.4.2) we conclude that $x^{\mathbf{u}_1} y^{\mathbf{v}_1} = x^{\mathbf{u}_2} y^{\mathbf{v}_2}$, and $\text{val}(\alpha_1 + \alpha_2) > \text{val}(\alpha_1) = \text{val}(\alpha_2)$.

We may assume that this counterexample is maximal in the following sense: if $f' \in I_d, g' \in J_d$ is any other pair with $f + g = f' + g'$, then either $\operatorname{trop}(f')(\mathbf{w}) < \operatorname{trop}(f)(\mathbf{w})$ or $\operatorname{trop}(f')(\mathbf{w}) = \operatorname{trop}(f)(\mathbf{w})$ and $\operatorname{in}_\prec(\operatorname{in}_{\mathbf{w}}(f')) \succ \operatorname{in}_\prec(\operatorname{in}_{\mathbf{w}}(f))$. To see that such a maximal pair exists, note that if there were no such pair, we could find a sequence $f_i = f + h_i \in I, g_i = g - h_i \in J$ with $f_i + g_i = f + g$ for all i, and $\operatorname{trop}(f_i)(\mathbf{w})$ strictly increasing. The strict increase comes from the fact that there are only finitely many monomials of degree d, so we cannot have $\operatorname{trop}(f_{i+1})(\mathbf{w}) = \operatorname{trop}(f_i)(\mathbf{w})$ and $\operatorname{in}_\prec(\operatorname{in}_{\mathbf{w}}(f_{i+1})) \succ \operatorname{in}_\prec(\operatorname{in}_{\mathbf{w}}(f_i))$ an infinite number of times. By passing to a subsequence we may assume that the support of each f_i is the same.

Since $\operatorname{supp}(f + h_i) = \operatorname{supp}(f + h_{i+1})$, there are $\alpha, \beta \in K^*$ for which $\alpha(f + h_i) + \beta(f + h_{i+1}) = (\alpha + \beta)f + (\alpha h_i + \beta h_{i+1})$ has strictly smaller support. Since $f + h_i \neq f + h_{i+1}$, we may assume that one of the monomials removed from $\operatorname{supp}(f + h_i)$ in this manner has different coefficients in h_i and h_{i+1}, and thus $\alpha + \beta \neq 0$. Note that for any two polynomials p_1, p_2 we have $\operatorname{trop}(p_1 + p_2)(\mathbf{w}) \geq \min(\operatorname{trop}(p_1)(\mathbf{w}), \operatorname{trop}(p_2)(\mathbf{w}))$. Since $f_i - f = g - g_i \in I \cap J$ for all i, the resulting polynomial $h'_i = 1/(\alpha + \beta)(\alpha h_{i+1} + \beta h_i)$ is also in $I \cap J$, so $f'_i = f + h'$ lies in I and has $\operatorname{trop}(f'_i)(\mathbf{w}) \geq \operatorname{trop}(f_i)(\mathbf{w})$ and $\operatorname{supp}(f'_i) \subsetneq \operatorname{supp}(f_i)$. By passing to another subsequence, we may assume that the sequence $\operatorname{trop}(f + h'_i)(\mathbf{w})$ is again increasing. Continuing to iterate this procedure would eventually yield the support of the new f_i being empty, which is impossible since $\operatorname{in}_{\mathbf{w}}(f_i + g_i) \notin \operatorname{in}_{\mathbf{w}}(J)$. This shows that the infinite increasing sequence does not exist, so we may assume that the pair f, g is maximal in the required sense.

Now $f \in I$ implies that $x^{\mathbf{u}_1} y^{\mathbf{v}_1} \in \operatorname{in}_\prec(\operatorname{in}_{\mathbf{w}}(I))$, so there is $f_1 \in I$ with $\operatorname{in}_{\mathbf{w}}(f_1) \in \Bbbk[x_0, \ldots, x_n]$, and $\operatorname{in}_\prec(\operatorname{in}_{\mathbf{w}}(f_1)) = x^{\mathbf{u}_3}$ dividing $x^{\mathbf{u}_1}$. We may assume that the coefficient of $x^{\mathbf{u}_3}$ in f_1 is one. We can thus write $f = \alpha_1 x^{\mathbf{u}_1 - \mathbf{u}_3} y^{\mathbf{v}_1} f_1 + f_2$ where $\operatorname{trop}(f_2)(\mathbf{w}) \geq \operatorname{trop}(f)(\mathbf{w})$, and if equality holds, then $\operatorname{in}_\prec(\operatorname{in}_{\mathbf{w}}(f_2)) \prec \operatorname{in}_\prec(\operatorname{in}_{\mathbf{w}}(f))$. Similarly, $g = \alpha_2 x^{\mathbf{u}_1} y^{\mathbf{v}_1 - \mathbf{v}_3} g_1 + g_2$ where $\operatorname{trop}(g_2)(\mathbf{w}) \geq \operatorname{trop}(g)(\mathbf{w})$, and if equality holds, then $\operatorname{in}_\prec(\operatorname{in}_{\mathbf{w}}(g_2)) \prec \operatorname{in}_\prec(\operatorname{in}_{\mathbf{w}}(g))$. Since $\operatorname{val}(\alpha_1 + \alpha_2) > \operatorname{val}(\alpha_1) = \operatorname{val}(\alpha_2)$, we can write $\alpha_2 = \alpha_1(-1 + \beta)$, where $\operatorname{val}(\beta) > 0$. Then

$$\begin{aligned} f + g &= \alpha_1 x^{\mathbf{u}_1 - \mathbf{u}_3} y^{\mathbf{v}_1} f_1 + f_2 + \alpha_2 x^{\mathbf{u}_1} y^{\mathbf{v}_1 - \mathbf{v}_3} g_1 + g_2 \\ &= \alpha_1 x^{\mathbf{u}_1 - \mathbf{u}_3} y^{\mathbf{v}_1 - \mathbf{v}_3} (y^{\mathbf{v}_3} f_1 - x^{\mathbf{u}_3} g_1 + \beta x^{\mathbf{u}_3} g_1) + f_2 + g_2 \\ &= \alpha_1 x^{\mathbf{u}_1 - \mathbf{u}_3} y^{\mathbf{v}_1 - \mathbf{v}_3} (-(g_1 - y^{\mathbf{v}_3}) f_1 + (f_1 - x^{\mathbf{u}_3}) g_1 + \beta x^{\mathbf{u}_3} g_1) + f_2 + g_2. \end{aligned}$$

Set

$$f' = \alpha_1 x^{\mathbf{u}_1 - \mathbf{u}_3} y^{\mathbf{v}_1 - \mathbf{v}_3} (-(g_1 - y^{\mathbf{v}_3}) f_1) + f_2,$$

and

$$g' = \alpha_1 x^{\mathbf{u}_1 - \mathbf{u}_3} y^{\mathbf{v}_1 - \mathbf{v}_3} ((f_1 - x^{\mathbf{u}_3}) g_1 + \beta x^{\mathbf{u}_3} g_1) + g_2.$$

Then, by construction $f' \in I$, $g' \in J$, and $f'+g' = f+g$. In addition, either $\mathrm{trop}(f')(\mathbf{w}) > \mathrm{trop}(f)(\mathbf{w})$ or $\mathrm{in}_{\prec}(\mathrm{in}_{\mathbf{w}}(f')) \prec \mathrm{in}_{\prec}(\mathrm{in}_{\mathbf{w}}(f))$. This contradicts our choice of a maximal counterexample, so we conclude that none exists, and hence $\mathrm{in}_{\mathbf{w}}(I+J) = \mathrm{in}_{\mathbf{w}}(I) + \mathrm{in}_{\mathbf{w}}(J)$. □

Remark 3.4.11. Lemma 3.4.10 is a variant of Buchberger's second criterion for S-pairs. See, for example, [**CLO07**, §2.9] for details of the standard case. See [**CM13**] for more on this criterion for the Gröbner bases studied here.

We now use Lemma 3.4.10 to prove the Transverse Intersection Theorem. This states that when two tropical varieties meet transversely, their intersection equals the tropicalization of the intersections. This is a very useful tool for nontrivial computations. For some generalizations, see [**OP13**].

Theorem 3.4.12. *Let X and Y be subvarieties of T_K^n. If* $\mathrm{trop}(X)$ *and* $\mathrm{trop}(Y)$ *meet transversely at $\mathbf{w} \in \Gamma_{\mathrm{val}}^n$, then $\mathbf{w} \in \mathrm{trop}(X \cap Y)$. Therefore*

$$\mathrm{trop}(X \cap Y) \;=\; \mathrm{trop}(X) \cap \mathrm{trop}(Y)$$

if the polyhedral intersection on the right-hand side is transverse everywhere.

Proof. Let Σ_1 and Σ_2 be polyhedral complexes in $\mathbb{R}^n$ with $\mathrm{trop}(X) = |\Sigma_1|$ and $\mathrm{trop}(Y) = |\Sigma_2|$. Let $I, J \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ideals of X and Y. Let $\sigma_i \in \Sigma_i$ be the cell containing $\mathbf{w}$ in its relative interior for $i = 1, 2$, with the affine span of σ_i equal to $\mathbf{w} + L_i$. Our hypothesis says that $L_1 + L_2 = \mathbb{R}^n$.

We now reduce to the case that L_1 contains $\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{e}_{r+1}, \ldots, \mathbf{e}_s$, and L_2 contains $\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{e}_{s+1}, \ldots, \mathbf{e}_n$. By the assumption $L_1 + L_2 = \mathbb{R}^n$, there exists a basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$ for $\mathbb{R}^n$ where $\mathbf{a}_1, \ldots, \mathbf{a}_r \in L_1 \cap L_2$, $\mathbf{a}_{r+1}, \ldots, \mathbf{a}_s \in L_1$, $\mathbf{a}_{s+1}, \ldots, \mathbf{a}_n \in L_2$ and all $\mathbf{a}_i \in \mathbb{Z}^n$. Write these as the rows of an $n \times n$-matrix A, with A_i the ith column of A. Let $\phi\colon T^n \to T^n$ be the morphism given by $\phi^*(x_i) = x^{A_i}$, so $\mathrm{trop}(\phi)$ is given by the matrix A^T. The morphism ϕ is finite but is not an isomorphism if $\det(A) \neq \pm 1$. Since A has full rank by construction, however, the linear map $\mathrm{trop}(\phi)\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. Let $I' = \phi^*(I)$, $X' = V(I')$, $J' = \phi^*(J)$, and $Y' = V(J')$. Then $\phi(X') = X$ and $\phi(Y') = Y$. By Corollary 3.2.13 we have

$$\mathrm{trop}(X) = \mathrm{trop}(\phi)(\mathrm{trop}(X')),$$
$$\mathrm{trop}(Y) = \mathrm{trop}(\phi)(\mathrm{trop}(Y')),$$
$$\mathrm{trop}(\phi)(\mathrm{trop}(X' \cap Y')) = \mathrm{trop}(X \cap Y).$$

By construction

$$\mathrm{trop}(\phi)(\mathrm{span}(\mathbf{e}_1, \ldots, \mathbf{e}_r)) \subseteq L_1 \cap L_2,$$
$$\mathrm{trop}(\phi)(\mathrm{span}(\mathbf{e}_{r+1}, \ldots, \mathbf{e}_s)) \subseteq L_1,$$
$$\mathrm{trop}(\phi)(\mathrm{span}(\mathbf{e}_{s+1}, \ldots, \mathbf{e}_n)) \subseteq L_2,$$

and $\operatorname{trop}(X')$ and $\operatorname{trop}(Y')$ intersect transversely at $\operatorname{trop}(\phi)^{-1}(\mathbf{w})$. It suffices to show that $\operatorname{trop}(\phi)^{-1}(\mathbf{w}) \in \operatorname{trop}(X' \cap Y')$. By replacing X and Y with X' and Y', we may thus assume that L_1 contains $\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{e}_{r+1}, \ldots, \mathbf{e}_s$ and L_2 contains $\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{e}_{s+1}, \ldots, \mathbf{e}_n$.

As in the proof of Theorem 3.3.8, $\operatorname{in}_{\mathbf{w}}(I)$ is homogeneous with respect to a $\mathbb{Z}^{\dim(L_1)}$-grading, and we can find polynomials $f_1, \ldots, f_l$ in $x_{s+1}, \ldots, x_n$ that generate $\operatorname{in}_{\mathbf{w}}(I)$. Similarly, there is a generating set $g_1, \ldots, g_m$ for $\operatorname{in}_{\mathbf{w}}(J)$ only using $x_{r+1}, \ldots, x_s$. Let $I_{\text{proj}} \subseteq K[x_0, \ldots, x_{n+1}]$ be the ideal obtained by homogenizing $I \cap K[x_1, \ldots, x_n]$ using the variable x_{n+1}, and let J_{proj} be the ideal obtained by homogenizing $J \cap K[x_1, \ldots, x_n]$ using the variable x_0.

For $\overline{\mathbf{w}} = (0, \mathbf{w}, 0) \in \mathbb{R}^{n+2}$, the initial ideal $\operatorname{in}_{\overline{\mathbf{w}}}(I_{\text{proj}})$ has a generating set only using $x_{s+1}, \ldots, x_{n+1}$, and $\operatorname{in}_{\overline{\mathbf{w}}}(J_{\text{proj}})$ has a generating set only using $x_0, x_{r+1}, \ldots, x_s$. Thus by Lemma 3.4.10 we have $\operatorname{in}_{\overline{\mathbf{w}}}(I_{\text{proj}} + J_{\text{proj}}) = \operatorname{in}_{\overline{\mathbf{w}}}(I_{\text{proj}}) + \operatorname{in}_{\overline{\mathbf{w}}}(J_{\text{proj}})$. Furthermore, after setting $x_0 = x_{n+1} = 1$ as in Proposition 2.6.1, we obtain

$$\operatorname{in}_{\mathbf{w}}(I + J) \; = \; \operatorname{in}_{\mathbf{w}}(I) + \operatorname{in}_{\mathbf{w}}(J). \tag{3.4.3}$$

Since $\operatorname{in}_{\mathbf{w}}(I)$ and $\operatorname{in}_{\mathbf{w}}(J)$ are proper ideals, by the Nullstellensatz, there exist $\mathbf{y} = (y_{r+1}, \ldots, y_s) \in (\Bbbk^*)^{s-r}$ and $\mathbf{z} = (z_{s+1}, \ldots, z_n) \in (\Bbbk^*)^{n-s}$ with $f_i(\mathbf{y}) = g_j(\mathbf{z}) = 0$ for all i, j. Now, for any $(t_1, \ldots, t_r) \in (\Bbbk^*)^r$, the vector $(t_1, \ldots, t_r, y_{r+1}, \ldots, y_s, z_{s+1}, \ldots, z_n)$ lies in the variety $V(\operatorname{in}_{\mathbf{w}}(I)) \cap V(\operatorname{in}_{\mathbf{w}}(J)) = V(\operatorname{in}_{\mathbf{w}}(I) + \operatorname{in}_{\mathbf{w}}(J)) = V(\operatorname{in}_{\mathbf{w}}(I + J))$. We conclude that $\operatorname{in}_{\mathbf{w}}(I + J) \neq \langle 1 \rangle$, and hence $\mathbf{w} \in \operatorname{trop}(V(I + J)) = \operatorname{trop}(X \cap Y)$. $\square$

If the two tropical varieties $\operatorname{trop}(X)$ and $\operatorname{trop}(Y)$ do not meet transversely at the point $\mathbf{w}$, then $\mathbf{w}$ may fail to lie in $\operatorname{trop}(X \cap Y)$. For instance, suppose X is a line and Y is a conic, both in the plane, and their tropicalizations intersect as in Figure 1.3.6. Then $\operatorname{trop}(X) \cap \operatorname{trop}(Y)$ contains the line segment $[A, B]$, while $\operatorname{trop}(X \cap Y) = \{A, B\}$ consists only of the two endpoints.

We next prove that the tropicalizations of constant coefficient curves are balanced. This is a key step in establishing the balancing property in Theorem 3.3.5. Our proof of Proposition 3.4.13 rests on basic commutative algebra and is self-contained. A shorter argument can be given if one uses intersection theory on toric varieties. This will be explained in Remark 6.7.8.

Proposition 3.4.13. *If C is a curve in $T^n_{\Bbbk} \cong (\Bbbk^*)^n$, then the one-dimensional fan* $\operatorname{trop}(C)$ *is balanced using the multiplicities of Definition* 3.4.3.

Proof. Let $\mathbf{u}_1, \ldots, \mathbf{u}_s$ be the first lattice points on the rays of $\operatorname{trop}(C)$, let $m_i = \operatorname{mult}(\operatorname{pos}(\mathbf{u}_i))$, and set $\mathbf{u} = \sum_{i=1}^{s} m_i \mathbf{u}_i$. We will show that if $\mathbf{v} \in \mathbb{Z}^n$ is primitive, in the sense that $\gcd(v_1, \ldots, v_n) = 1$, then $\mathbf{v} \cdot \mathbf{u} = 0$. This implies $\mathbf{v} \cdot \mathbf{u} = 0$ for all $\mathbf{v} \in \mathbb{Z}^n$, and so $\mathbf{u} = \mathbf{0}$. By Lemma 2.2.7 and Corollary 2.6.12, after a change of coordinates, it suffices to consider the case $\mathbf{v} = \mathbf{e}_1$.

Let $I \subset \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ideal of C. Let K' be the algebraic closure of $\Bbbk(t)$. Since $\Bbbk$ is algebraically closed, K' has residue field $\Bbbk$. We denote by I' the extension of I to $K'[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and write $C_{K'} \subset T^n_{K'}$ for the variety of I'. We have $\operatorname{trop}(C) = \operatorname{trop}(C_{K'})$ by Theorem 3.2.4.

Consider the ideal $J'_\alpha = I' + \langle x_1 - \alpha \rangle$ for $\alpha \in K'^*$. There exists $L \in \mathbb{N}$ and a finite subset $\mathcal{D} \subset K'^*$ such that $\dim_K(S_{K'}/J'_\alpha) = L$ for all $\alpha \in K'^* \backslash \mathcal{D}$. To see this, we apply classical Gröbner bases to the homogenization $(J'_\alpha)_{\text{proj}}$ of J'_α. The initial ideal is constant for all α outside a finite set $\mathcal{D}$. The number L equals the Hilbert polynomial of this initial ideal, which is constant because $x_1 - \alpha$ cannot be a zerodivisor on $S_{K'}/I'$ for infinitely many α.

Choose $\alpha_1, \alpha_2 \in K'^* \backslash \mathcal{D}$ with $\operatorname{val}(\alpha_1) = 1$ and $\operatorname{val}(\alpha_2) = -1$. Let $X^+ = V(I' + \langle x_1 - \alpha_1 \rangle) \subset T^n_{K'}$, and $X^- = V(I' + \langle x_1 - \alpha_2 \rangle) \subset T^n_{K'}$. The desired identity $u_1 = 0$ will be obtained by computing L tropically.

Set $\beta_1 = \overline{t^{-1}\alpha_1} \in \Bbbk^*$ and $\beta_2 = \overline{t^1 \alpha_2} \in \Bbbk^*$. From (3.4.3) in the proof of Theorem 3.4.12, we conclude

$$\begin{array}{rcll} \operatorname{in}_{\mathbf{w}}(I' + \langle x_1 - \alpha_1 \rangle) & = & \operatorname{in}_{\mathbf{w}}(I') + \langle x_1 - \beta_1 \rangle \neq \langle 1 \rangle & \text{for } \mathbf{w} \in \operatorname{trop}(X^+), \\ \operatorname{in}_{\mathbf{w}}(I' + \langle x_1 - \alpha_2 \rangle) & = & \operatorname{in}_{\mathbf{w}}(I') + \langle x_1 - \beta_2 \rangle \neq \langle 1 \rangle & \text{for } \mathbf{w} \in \operatorname{trop}(X^-). \end{array}$$

We now focus on α_1. Let $H = \operatorname{trop}(V(x_1 - \alpha_1)) = \{\mathbf{w} \in \mathbb{R}^n : w_1 = 1\}$. We claim that $\operatorname{trop}(X^+) = \operatorname{trop}(C) \cap H$. Indeed, for any $\mathbf{w} \in \operatorname{trop}(C) \cap H$ the cone of $\operatorname{trop}(C)$ containing $\mathbf{w}$ in its relative interior is $\operatorname{pos}(\mathbf{w})$, so $\operatorname{trop}(C)$ intersects H transversely at $\mathbf{w}$. Since $\mathbf{w}$ was an arbitrary intersection point, the claim follows from Theorem 3.4.12. We now decompose $I' + \langle x_1 - \alpha_1 \rangle = \bigcap_{\mathbf{y}} Q_{\mathbf{y}}$, where $Q_{\mathbf{y}}$ is $P_{\mathbf{y}}$ primary for $\mathbf{y} \in T^n_{K'}$. The $\mathbf{y}$ appearing here are precisely the points of X^+. Let $X^+_{\mathbf{w}} = \{\mathbf{y} \in X^+ : \operatorname{val}(\mathbf{y}) = \mathbf{w}\}$. Note that for $\mathbf{w} \in \operatorname{trop}(X^+)$, we have $\operatorname{in}_{\mathbf{w}}(\bigcap_{\mathbf{y} \in X^+} Q_{\mathbf{y}}) = \operatorname{in}_{\mathbf{w}}(\bigcap_{\mathbf{y} \in X^+_{\mathbf{w}}} Q_{\mathbf{y}})$. The inclusion $\subseteq$ is automatic. For the other inclusion, note that for all $\mathbf{y} \in X^+ \setminus X^+_{\mathbf{w}}$, there is $f_{\mathbf{y}} \in Q_{\mathbf{y}}$ with $\operatorname{in}_{\mathbf{w}}(f_{\mathbf{y}}) = 1$. For any $g \in \bigcap_{\mathbf{y} \in X^+_{\mathbf{w}}} Q_{\mathbf{y}}$, we set $g' = g \prod_{\mathbf{y} \in X^+ \setminus X^+_{\mathbf{w}}} f_{\mathbf{y}}$ to get $\operatorname{in}_{\mathbf{w}}(g) = \operatorname{in}_{\mathbf{w}}(g')$. Combined with the first of the above equations, this gives $\operatorname{in}_{\mathbf{w}}(\bigcap_{\mathbf{y} \in X^+_{\mathbf{w}}} Q_{\mathbf{y}}) = \operatorname{in}_{\mathbf{w}}(I') + \langle x_1 - \beta_1 \rangle$.

At this point, Proposition 3.4.8 implies that

$$\begin{aligned} \dim_{K'} S_{K'} / \left(\bigcap_{\mathbf{y} \in X^+_{\mathbf{w}}} Q_{\mathbf{y}} \right) &= \sum_{\mathbf{y} \in X^+_{\mathbf{w}}} \operatorname{mult}(Q_{\mathbf{y}}, P_{\mathbf{y}}) \\ &= \dim_{\Bbbk}(S_{\Bbbk} / (\operatorname{in}_{\mathbf{w}}(I') + \langle x_1 - \beta_1 \rangle)). \end{aligned}$$

By summing these identities over all $\mathbf{w} \in \operatorname{trop}(X^+)$, we find

$$L \;=\; \sum_{\mathbf{y} \in X^+} \operatorname{mult}(Q_{\mathbf{y}}, P_{\mathbf{y}}) \;=\; \sum_{\mathbf{w} \in \operatorname{trop}(X^+)} \dim_{\Bbbk}(S_{\Bbbk} / (\operatorname{in}_{\mathbf{w}}(I') + \langle x_1 - \beta_1 \rangle)).$$

The same identities hold for X^- and β_2.

Let $\mathbf{u_w}$ be the first lattice point on the ray $\mathrm{pos}(\mathbf{w})$ of $\mathrm{trop}(C)$. Then $\lambda = (\mathbf{u_w})_1$ satisfies $\mathbf{u_w} = \lambda \mathbf{w}$ because $w_1 = 1$. We now claim that

$$(3.4.4) \qquad \lambda \cdot \mathrm{mult}(\mathrm{pos}(\mathbf{w})) \; = \; \dim_{\Bbbk}(S_{\Bbbk}/(\mathrm{in}_{\mathbf{w}}(I') + \langle x_1 - \beta_1 \rangle)).$$

That claim implies

$$L \;\; = \sum_{\mathbf{w} \in \mathrm{trop}(X^+)} \mathrm{mult}(\mathrm{pos}(\mathbf{w})) \cdot (\mathbf{u_w})_1 \;\; = \sum_{i:(\mathbf{u}_i)_1 > 0} m_i \cdot (\mathbf{u}_i)_1,$$

and similarly $L = \sum_{i:(\mathbf{u}_i)_1<0} -m_i(\mathbf{u}_i)_1$. This completes the proof as follows:

$$u_1 \;=\; \sum_{i:(\mathbf{u}_i)_1>0} m_i(\mathbf{u}_i)_1 \;-\; \sum_{i:(\mathbf{u}_i)_1<0} m_i|(\mathbf{u}_i)_1| \;=\; L - L \;=\; 0.$$

Thus it remains to prove (3.4.4). To this end, we perform a change of coordinates that takes x_1 to $x^{\mathbf{u_w}}$, and thus $\mathbf{w}$ to $\lambda^{-1}\mathbf{e}_1$. Now, our claim (3.4.4) states $\lambda \cdot \mathrm{mult}(\mathrm{pos}(\mathbf{w})) = \dim_{\Bbbk}(S_{\Bbbk}/(\mathrm{in}_{\lambda^{-1}\mathbf{e}_1}(I') + \langle x^{\mathbf{u_w}} - \beta_1 \rangle))$. The initial ideal $\mathrm{in}_{\lambda^{-1}\mathbf{e}_1}(I')$ has a generating set that does not contain x_1. Since $V(I')$ is a curve, by Corollary 2.4.9, the initial ideal is one dimensional, so for each $2 \leq i \leq n$ it contains a polynomial in $\Bbbk[x_i]$ with constant term one. After dividing by x_i, we obtain $x_i^{-1} - p_i' \in \mathrm{in}_{\lambda^{-1}\mathbf{e}_1}(I')$ for some $p_i' \in \Bbbk[x_i]$. Now $\langle x^{\mathbf{u_w}} - \beta_1 \rangle = \langle x_1^{\lambda} - \beta_1 x^{\mathbf{u}'} \rangle$, where $u_1' = 0$ and $u_i' = -(\mathbf{u_w})_i$ for $2 \leq i \leq n$. This implies $\mathrm{in}_{\lambda^{-1}\mathbf{e_1}}(I') + \langle x^{\mathbf{u_w}} - \beta_1 \rangle = \mathrm{in}_{\lambda^{-1}\mathbf{e}_1}(I') + \langle x_1^{\lambda} - f \rangle$ for some $f \in \Bbbk[x_2, \ldots, x_n]$. We next use the fact that $\dim_{\Bbbk} \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]/J = \dim_{\Bbbk} \Bbbk[x_1, \ldots, x_n]/J_{\mathrm{aff}}$ for any zero-dimensional Laurent ideal J. Fix the lexicographic term order $x_1 \succ x_2 \succ \cdots \succ x_n$ on $\Bbbk[x_1, \ldots, x_n]$. By Buchberger's criterion, the initial ideal of $(\mathrm{in}_{\lambda^{-1}\mathbf{e}_1}(I') + \langle x_1^{\lambda} - f \rangle)_{\mathrm{aff}}$ is generated by x_1^{λ} and the monomial generators of $\mathrm{in}_{\mathrm{lex}}((\mathrm{in}_{\lambda^{-1}\mathbf{e}_1}(I'))_{\mathrm{aff}})$. The right-hand side of (3.4.4) is λ times the $\Bbbk$-dimension of $\Bbbk[x_2^{\pm 1}, \ldots, x_n^{\pm 1}]/\mathrm{in}_{\lambda^{-1}\mathbf{e}_1}(I')$. But, that last $\Bbbk$-dimension equals the multiplicity of $\mathrm{pos}(\mathbf{w})$ by Lemma 3.4.7. □

Last but not least, here is the main theorem in this section.

Theorem 3.4.14. *Let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that all irreducible components of $V(I)$ have the same dimension d. Fix a polyhedral complex Σ with support* $\mathrm{trop}(V(I))$ *such that* $\mathrm{in}_{\mathbf{w}}(I)$ *is constant for* $\mathbf{w}$ *in the relative interior of each cell in Σ. Then Σ is a weighted balanced polyhedral complex with the weight function* mult *of Definition* 3.4.3.

Proof. Write $\sqrt{I} = \bigcap P_i$ where each P_i is a d-dimensional prime ideal. By Theorem 3.2.3, the tropical variety $\mathrm{trop}(V(I))$ is the union $\bigcup \mathrm{trop}(V(P_i))$. By Theorem 3.3.8, $\mathrm{trop}(V(I))$ a pure d-dimensional polyhedral complex.

Fix a $(d-1)$-dimensional cell $\tau \in \Sigma$. Lemma 2.2.7 and Corollary 2.6.12 guarantee that, after a multiplicative change of coordinates, the affine span of τ is a translate of the span of $\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}$. Fix $\mathbf{w} \in \mathrm{relint}(\tau)$. Part (2) of

Lemma 2.6.2 implies that $\mathrm{in}_{\mathbf{w}}(I)$ is homogeneous with respect to the $\mathbb{Z}^{d-1}$-grading given by $\deg(x_i) = \mathbf{e}_i$ for $1 \leq i \leq d-1$, and $\deg(x_i) = \mathbf{0}$ for $i \geq d$. This means that $\mathrm{in}_{\mathbf{w}}(I)$ has a generating set in which $x_1, \ldots, x_{d-1}$ do not appear.

Let $J = \mathrm{in}_{\mathbf{w}}(I) \cap \Bbbk[x_d^{\pm 1}, \ldots, x_n^{\pm 1}]$. By Lemma 3.3.6 the tropical variety of $V(\mathrm{in}_{\mathbf{w}}(I)) \subset T^n_{\Bbbk}$ is the star of τ in Σ, which has lineality space spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}$. Since $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) \cap \Bbbk[x_d^{\pm 1}, \ldots, x_n^{\pm 1}] = \mathrm{in}_{\overline{\mathbf{v}}}(J)$, where $\overline{\mathbf{v}}$ is the projection of $\mathbf{v}$ onto the last $n-d+1$ coordinates, the fact that $\mathrm{trop}(V(I))$ is pure of dimension d implies that $\mathrm{trop}(V(J))$ is one dimensional.

Let $P_1, \ldots, P_r$ be the minimal associated primes of J. Then $V(J) = \bigcup_{i=1}^r V(P_i)$, so

$$\begin{aligned}\mathrm{trop}(V(J)) &= \mathrm{cl}(\mathrm{val}(\mathbf{y}) : \mathbf{y} \in V(J)) \\ &= \bigcup_{i=1}^r \mathrm{cl}(\mathrm{val}(\mathbf{y}) : \mathbf{y} \in V(P_i)) = \bigcup_{i=1}^r \mathrm{trop}(V(P_i)).\end{aligned}$$

By Theorem 3.3.8 we thus have $\dim(P_i) \leq 1$ and at least one index i satisfies $\dim(P_i) = 1$. Thus $\dim(V(J)) = 1$.

Suppose $\mathbf{v} \in \mathbb{Q}^n$ satisfies $\mathbf{w} + \epsilon\mathbf{v} \in \sigma$ for all sufficiently small $\epsilon > 0$, where σ is a maximal cell of Σ that has τ as a facet. The equality $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\overline{\mathbf{v}}}(J)$ and Lemma 3.4.7 imply that the multiplicity of the cone $\mathrm{pos}(\overline{\mathbf{v}})$ in $\mathrm{trop}(V(J))$ equals the multiplicity of σ in $\mathrm{trop}(X)$. Thus, showing that Σ is balanced at τ is exactly the same as showing that $\mathrm{trop}(V(J))$ is balanced at $\mathbf{0}$. Thus proving the theorem for J suffices, so we may assume that X is a curve in $(\Bbbk^*)^n$. This is Proposition 3.4.13, so the result follows. □

Remark 3.4.15. In the statement of Theorem 3.4.14 we do not assume that I is radical. We have $\mathrm{trop}(V(I)) = \mathrm{trop}(V(\sqrt{I}))$, but the multiplicities might differ. If I is not radical, then the tropical variety together with its multiplicities records information about the affine scheme $X = \mathrm{Spec}(S/I)$.

3.5. Connectivity and Fans

The polyhedral complex underlying a tropical variety has a strong connectedness property, introduced in Definition 3.3.4.

Theorem 3.5.1. *Let X be an irreducible subvariety of T^n of dimension d. Then* $\mathrm{trop}(X)$ *is the support of a pure d-dimensional polyhedral complex that is connected through codimension* 1.

This result is important for the algorithmic computation of tropical varieties. Given a variety $X \subset T^n$, we can define a graph whose nodes are the d-dimensional cells of $\mathrm{trop}(X)$ and where two nodes are connected by an edge if the corresponding cells share a facet. Theorem 3.5.1 states that this graph is connected. To compute $\mathrm{trop}(X)$, one can start with one node and identify all neighbors using the initial ideal techniques of Section 2.5. This method is described in [**BJS$^+$07**] and implemented in `Gfan` [**Jen**].

The proof of Theorem 3.5.1 is by induction on dimension d of X. The base case $d = 1$ is surprisingly nontrivial. It is proved in Proposition 6.6.22.

Proposition 3.5.2. *Let X be a one-dimensional irreducible subvariety of the torus T^n. Then* $\operatorname{trop}(X)$ *is connected.*

Let Δ be the standard tropical hyperplane $\operatorname{trop}(V(x_1+\cdots+x_n+1))\subset\mathbb{R}^n$. The tropicalization of any hyperplane $H_{\mathbf{a}}=\{\mathbf{x} : a_1x_1 + \cdots + a_nx_n + a_0 = 0\} \subset T^n$ with all $a_i \neq 0$ equals $-\mathbf{v} + \Delta$, where $v_i = \operatorname{val}(a_i) - \operatorname{val}(a_0)$.

Proof of Theorem 3.5.1. Theorem 3.3.8 states that $\operatorname{trop}(X)$ is the support of a pure d-dimensional polyhedral complex Σ. We need to show that Σ is connected through codimension 1. The proof is by induction on $d = \dim(X)$. The base case $d = 1$ is Proposition 3.5.2. Indeed, a one-dimensional polyhedral complex Σ is a graph, and a graph is connected if and only if it is connected through codimension 1. Next, suppose that $d = \dim(X)$ satisfies $2 \leq d < n$, and that the theorem is true for all smaller dimensions. After a multiplicative change of coordinates in T^n, we may also assume that no facet σ of Σ lies in a tropical hyperplane $-\mathbf{v} + \Delta$.

Fix facets $\sigma, \sigma' \in \Sigma$. Pick $\mathbf{w} \in \operatorname{relint}(\sigma) \cap \Gamma_{\text{val}}^n$ and $\mathbf{w}' \in \operatorname{relint} \sigma' \cap \Gamma_{\text{val}}^n$. Choose $\mathbf{v} \in \Gamma_{\text{val}}^n$ for which $-\mathbf{v} + \Delta$ contains $\mathbf{w}, \mathbf{w}'$. To see that this is possible, note that if $\mathbf{y}, \mathbf{y}' \in T^n$ with $\operatorname{val}(\mathbf{y}) = \mathbf{w}$, $\operatorname{val}(\mathbf{y}') = \mathbf{w}'$ and $H_{\mathbf{a}} = V(a_1x_1 + \cdots + a_nx_n + a_0)$ is any hyperplane passing through both $\mathbf{y}$ and $\mathbf{y}'$, then $\operatorname{trop}(H_{\mathbf{a}}) = -\mathbf{v} + \Delta$ is a tropical hyperplane passing through $\mathbf{w}$ and $\mathbf{w}'$. Since $\mathbf{w}, \mathbf{w}'$ lie in the relative interior of d-dimensional cells in Σ and these cells are not contained in $-\mathbf{v} + \Delta$, by replacing $\mathbf{w}, \mathbf{w}'$ with other points in the relative interior of their respective cells if necessary, we may assume that $\mathbf{w}$ and $\mathbf{w}'$ lie in top-dimensional cells of $-\mathbf{v} + \Delta$.

By Part 4 of Theorem 6.3 of [**Jou83**], the set U of $\mathbf{a} \in \mathbb{P}^n$ for which the intersection $X \cap H_{\mathbf{a}}$ is irreducible is Zariski open in $\mathbb{P}^n$. We write $U = \mathbb{P}^n \backslash V(f_1, \ldots, f_r)$ as the complement of a subvariety. By Lemma 2.2.12, there is $\mathbf{a} = (1 : a_1 : \cdots : a_n) \in U$ with $\operatorname{val}(a_i) = v_i$ for $i = 1, \ldots, n$.

The intersection $\operatorname{trop}(X)\cap\operatorname{trop}(H_{\mathbf{a}})$ inherits a polyhedral complex structure $\overline{\Sigma}$ from Σ and $-\mathbf{v}+\Delta$. Fix $\tilde{\mathbf{w}} \in \operatorname{trop}(X)\cap\operatorname{trop}(H_{\mathbf{a}})$ for which $\tilde{\mathbf{w}}$ lies in the relative interior of a top-dimensional cell σ of Σ and σ' of $-\mathbf{v} + \Delta$. By our assumption on $\operatorname{trop}(X)$, the cell σ does not lie in $-\mathbf{v} + \Delta$, so $\operatorname{trop}(X)$ and $\operatorname{trop}(H_{\mathbf{a}})$ intersect transversely at $\tilde{\mathbf{w}}$. Theorem 3.4.12 implies that $\operatorname{trop}(X) \cap \operatorname{trop}(H_{\mathbf{a}}) = \operatorname{trop}(X \cap H_{\mathbf{a}})$. By construction, $Y = X \cap H_{\mathbf{a}}$ is irreducible, so $\operatorname{trop}(Y)$ is connected through codimension 1 by induction.

If $\overline{\sigma}$ is a $(d-1)$-dimensional cell in $\overline{\Sigma}$, then $\overline{\sigma}$ is the intersection of $-\mathbf{v}+\Delta$ with a d-dimensional cell σ in Σ, by our assumption on $\mathrm{trop}(X)$. If $\overline{\sigma}$ and $\overline{\sigma}'$ are adjacent top-dimensional cells in $\overline{\Sigma}$, then either $\sigma = \sigma'$ or σ and σ' are adjacent in Σ. By construction, $\mathbf{w}$ and $\mathbf{w}'$ lie in the relative interiors of top-dimensional cells $\overline{\sigma}$ and $\overline{\sigma}'$ in $\overline{\Sigma}$, so there is a path $\overline{\sigma} = \overline{\sigma}_1, \overline{\sigma}_2, \ldots, \overline{\sigma}_r = \overline{\sigma}'$ in $\overline{\Sigma}$ connecting $\mathbf{w}$ to $\mathbf{w}'$. Lifting these and removing adjacent duplicates, we find that $\sigma_1, \ldots, \sigma_r$ is a path of adjacent top-dimensional cells in Σ connecting $\mathbf{w}$ to $\mathbf{w}'$. We conclude that $\mathrm{trop}(X)$ is connected through codimension 1. □

Remark 3.5.3. Theorem 3.5.1 is stronger than it may seem at first glance, as the property of being connected through codimension 1 in fact only depends on the underlying set. *Every* polyhedral complex Σ with support $|\Sigma| = \mathrm{trop}(X)$ is connected through codimension 1. To see this, it suffices to note that a polyhedral complex is connected through codimension 1 if and only if a refinement of it is connected through codimension 1. For the "if" direction, note that a path of adjacent top-dimensional cells in the refinement lifts to a path of adjacent or identical top-dimensional cells in the original complex. For the "only if" direction, it suffices to note that any subdivision of a cell is connected through codimension 1. Now let Σ' be an arbitrary polyhedral complex with support $\mathrm{trop}(X)$, and let Σ be a connected-through-codimension-1 polyhedral complex with support $\mathrm{trop}(X)$ whose existence is guaranteed by Theorem 3.5.1. Then the common refinement of Σ and Σ' is connected through codimension 1 since Σ is, and so Σ' is also connected through codimension 1.

When first entering the field of tropical geometry, a student might get the impression that every tropical variety $\mathrm{trop}(X)$ is the support of a *unique coarsest* polyhedral complex Σ. This would mean that Σ' refines Σ for any balanced complex Σ' with $|\Sigma'| = \mathrm{trop}(X)$. For instance, such a coarsest Σ exists when X is a hypersurface and also when $\dim(X) \leq 2$. However, it does not exist in general. The following is an explicit counterexample.

Example 3.5.4. Fix $K = \mathbb{C}$ with the trivial valuation. We present a three-dimensional variety $X \subset T^5$ for which there is no coarsest fan Σ in $\mathbb{R}^5$ with $|\Sigma| = \mathrm{trop}(X)$. Consider the torus T^3 with coordinates (x, y, z) and define X to be the closure of the image of the rational map

$$T^3 \dashrightarrow T^5,\ (x,y,z) \mapsto \big(x(1-x), x(1-y), x(1-z), y(1-z), z(1-z)\big).$$

The tropicalization of X is a three-dimensional fan in $\mathbb{R}^5$. It is constructed geometrically as follows. Start with three copies of the standard tropical line in $\mathbb{R}^2$. Consider their direct product. This is a three-dimensional fan in $\mathbb{R}^6 = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. It has nine rays and 27 maximal cones. Then $\mathrm{trop}(X)$

is the image of this fan under the classical linear map given by the matrix

$$A \;=\; (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_6) \;=\; \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The following two three-dimensional simplicial cones lie in $\mathrm{trop}(X)$:

$$\mathrm{pos}\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\} \quad \text{and} \quad \mathrm{pos}\{\mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_6\}.$$

These two cones intersect in one ray. That ray is spanned by $\mathbf{1}{=}(1,1,1,1,1)^T$, and it lies in the relative interior of each of the two cones.

To see that there is no coarsest fan structure on $\mathrm{trop}(X)$, we note that $\mathrm{trop}(X)$ is the support of the cone over a two-dimensional polyhedral complex Π. That complex contains two triangles which meet in one point in their relative interiors. Any coarsest polyhedral subdivision of $|\Pi|$ would use that point as a 0-cell. Each triangle must be divided into three polygons that are either triangles or quadrilaterals. These coarsest subdivisions of a triangle are not unique. Hence no unique coarsest fan structure exists on $\mathrm{trop}(X)$. ♢

Our second topic in this section is the role of fans in tropical geometry. In Proposition 3.1.10 we saw that the tropicalization of a constant-coefficient hypersurface is a pure fan of codimension 1. We begin by generalizing this result to constant-coefficient varieties of arbitrary codimension.

Corollary 3.5.5. *Let $X \subset T^n$ be an irreducible d-dimensional variety where K is a field with the trivial valuation. Then the tropical variety* $\mathrm{trop}(X)$ *is the support of a balanced polyhedral fan of dimension d.*

Proof. We can choose the Gröbner fan structure given by Corollary 2.5.12. The statements about dimension and balancing follow from the Structure Theorem 3.3.5. Alternatively, we can choose a finite tropical basis $\mathcal{T}$ consisting of Laurent polynomials f whose coefficients have valuation zero. For each $f \in \mathcal{T}$, the tropical hypersurface $\mathrm{trop}(V(f))$ is the support of a fan, by Proposition 3.1.10. By taking the common refinement of these fans, we obtain a fan structure on the intersection $\mathrm{trop}(X) = \bigcap_{f \in \mathcal{T}} \mathrm{trop}(V(f))$. □

On the other hand, suppose $X \subset T^n$ is a d-dimensional variety over a field whose valuation is nontrivial. Then $\mathrm{trop}(X)$ is a polyhedral complex in $\mathbb{R}^n$ but usually not a fan. However, there are three different ways of associating fans to this complex. We summarize these below.

(1) By Lemma 3.3.6, $\mathrm{star}_{\mathrm{trop}(X)}(\sigma) = \mathrm{trop}(\mathrm{in}_{\mathbf{w}}(I))$ supports a fan for any cell σ of $\mathrm{trop}(X)$. Its dimension modulo the lineality space is $d - |\sigma|$. Every vertex of $\mathrm{trop}(X)$ determines a fan of dimension d.

(2) If $K = \mathbb{k}(t)$ and $X \subset T_K^n$, then we can construct the lift $X_t \subset T_{\mathbb{k}}^{n+1}$ by regarding t as a variable. The tropical variety $\mathrm{trop}(X_t)$ is a fan of dimension $d+1$ in $\mathbb{R}^{n+1}$ whose intersection with the affine hyperplane $w_t = 1$ is the tropical variety $\mathrm{trop}(X)$.

(3) By Theorem 3.5.6 below, the *recession fan* of $\mathrm{trop}(X)$ is the tropicalization of the same variety X, but with trivial valuation on K.

We need to explain what is meant by the recession fan. Fix a polyhedron

$$P \;=\; \big\{\mathbf{x} \in \mathbb{R}^n \,:\, A\mathbf{x} \leq \mathbf{b}\big\}.$$

The *recession cone* of P is

$$\mathrm{rec}(P) \;=\; \big\{\mathbf{x} \in \mathbb{R}^n \,:\, A\mathbf{x} \leq \mathbf{0}\big\}. \tag{3.5.1}$$

This is the unique cone satisfying $P = \mathrm{rec}(P) + Q$ for some polytope Q. In this decomposition, the polytope Q is not unique, but the recession cone is. Furthermore, $\mathrm{rec}(P)$ is the cone dual to the support of the normal fan $\mathcal{N}_P$.

If Σ is a polyhedral complex in $\mathbb{R}^n$, then its *recession fan* $\mathrm{rec}(\Sigma)$ is the union of all cones $\mathrm{rec}(P)$ where P runs over Σ. The set $\mathrm{rec}(\Sigma)$ is the support of a polyhedral fan. Burgos, Gil, and Sombra [**BGS11**] identify situations when the fan structure on Σ is not canonical. In particular, there is generally no unique coarsest fan structure on Σ. Our usage of the term "recession fan" simply means that such a fan structure exists, but it does not refer to any specific fan. With this understanding, the recession fan depends only on the support Σ, and we can write $\mathrm{rec}(|\Sigma|) = \mathrm{rec}(\Sigma)$.

Every field can be given a trivial valuation, for which $\mathrm{val}(a) = 0$ if $a \neq 0$. For $X \subset T^n$, we write $\mathrm{trop}(X_{\mathrm{triv}})$ for the tropicalization of X with respect to the trivial valuation. This may be different from the tropicalization $\mathrm{trop}(X)$ of X with respect to the original valuation on K, but there is a connection between the two.

Theorem 3.5.6. *Let X be a subvariety of T^n. Then the tropical variety* $\mathrm{trop}(X_{\mathrm{triv}})$ *is the recession fan of* $\mathrm{trop}(X)$:

$$\mathrm{trop}(X_{\mathrm{triv}}) \;=\; \mathrm{rec}(\mathrm{trop}(X)). \tag{3.5.2}$$

Proof. First suppose that $X = V(f)$ is a hypersurface with defining polynomial $f \in K[x_1^{\pm}, \ldots, x_n^{\pm 1}]$. Then $\mathrm{trop}(X)$ is the $(n-1)$-skeleton of the complex $\Sigma_{\mathrm{trop}(f)}$. Each unbounded cell in $\mathrm{trop}(X)$ corresponds to a face F of dimension ≥ 1 of the Newton polytope $\mathrm{Newt}(f)$, and its recession cone is the normal cone $\mathcal{N}(F)$. Moreover, every positive-dimensional face F occurs. Hence the right-hand side of (3.5.2) is the $(n-1)$-skeleton of the normal fan of $\mathrm{Newt}(f)$. By Proposition 3.1.10, this is also the left-hand side of (3.5.2).

For the general case, we use the identity

$$\mathrm{rec}(P \cap P') \;=\; \mathrm{rec}(P) \,\cap\, \mathrm{rec}(P'),$$

which holds for the recession cones of any two polyhedra P and P' in $\mathbb{R}^n$. As this extends to finite intersections of polyhedra in $\mathbb{R}^n$, we derive

$$\begin{aligned}\mathrm{trop}(X_{\mathrm{triv}}) = \bigcap_{f \in \mathcal{T}} \mathrm{trop}(V(f)_{\mathrm{triv}}) \;&=\; \bigcap_{f \in \mathcal{T}} \mathrm{rec}(\mathrm{trop}(V(f))) \\ = \mathrm{rec}\Big(\bigcap_{f \in \mathcal{T}} \mathrm{trop}(V(f))\Big) \;&=\; \mathrm{rec}\big(\mathrm{trop}(X)\big).\end{aligned}$$

Here, as before, the set $\mathcal{T}$ is a finite tropical basis for the variety X. □

3.6. Stable Intersection

In this section we introduce the notion of *stable intersection*. This was discussed for plane curves in Section 1.3. In general, it gives a combinatorial way to intersect any pair of weighted balanced polyhedral complexes so that the result is again a weighted balanced polyhedral complex. If the complexes are the tropicalizations of classical varieties, then the stable intersection represents their intersection after a generic multiplicative perturbation:

Theorem 3.6.1. *Let X_1, X_2 be subvarieties of the torus T^n, and let Σ_1, Σ_2 be weighted balanced Γ_{val}-rational polyhedral complexes whose supports are* $\mathrm{trop}(X_1)$ *and* $\mathrm{trop}(X_2)$*, respectively. There exists a Zariski dense subset $U \subset T^n$, consisting of elements $\mathbf{t} = (t_1, \ldots, t_n)$ with $\mathrm{val}(\mathbf{t}) = \mathbf{0}$, such that*

$$\mathrm{trop}(X_1 \cap \mathbf{t}X_2) \;=\; \Sigma_1 \cap_{st} \Sigma_2 \qquad \text{for all } \; \mathbf{t} \in U. \tag{3.6.1}$$

Here $\Sigma_1 \cap_{st} \Sigma_2$ is the stable intersection of balanced polyhedral complexes. This purely combinatorial notion will be introduced in Definition 3.6.5. The set $\mathbf{t}X_2$ on the left-hand side is the translated variety $\{\mathbf{tx} : \mathbf{x} \in X_2\}$. A proof of Theorem 3.6.1 will be presented at the end of the section.

We begin by developing the formal theory of stable intersections. This requires some preliminary material. Let N denote the lattice $\mathbb{Z}^n$ of $\mathbb{R}^n$, and let N_σ be the sublattice of N generated by the lattice points in the linear space parallel to a Γ_{val}-rational polyhedron σ. The *index* $[N : N']$ of a sublattice $N' \subset N$ of the same rank is the order of the quotient group N/N'. A refinement Σ' of a weighted polyhedral complex Σ inherits a weighting from the complex Σ: if σ' is a maximal dimensional cell in Σ' with $\sigma' \subseteq \sigma$ for $\sigma \in \Sigma$, then we assign to σ' the weight of σ. We first note that if the complex Σ is balanced, then so is the refinement Σ'.

Lemma 3.6.2. *Let Σ be a pure weighted balanced Γ_{val}-rational polyhedral complex in $\mathbb{R}^n$, and let Σ' be a Γ_{val}-rational refinement of Σ. Then Σ' is balanced.*

Proof. For a codimension-1 cell τ' in Σ', let τ be the smallest cell in Σ containing τ'. If τ has codimension 1 in Σ, then balancing at τ' follows immediately from the balancing condition on Σ, since $\mathrm{star}_{\Sigma'}(\tau') = \mathrm{star}_{\Sigma}(\tau)$. If τ is top dimensional, then $\mathrm{star}_{\Sigma'}(\tau')$ has two cones that meet along the affine span of τ'. The generators $\mathbf{v}_1$ and $\mathbf{v}_2$ for the lattices of these two cones, modulo the lattice of τ', satisfy the balancing condition $\mathbf{v}_1 = -\mathbf{v}_2$. Since both cones come from the same cone τ of Σ, they have the same multiplicity m. Therefore, the weighted sum $m\mathbf{v}_1 + m\mathbf{v}_2$ equals zero, as required. □

We now consider how polyhedral complexes behave under projections. Let Σ be a pure weighted Γ_{val}-rational polyhedral complex in $\mathbb{R}^n$, and let $\phi : N \to N' \cong \mathbb{Z}^m$ be a homomorphism of lattices. We suppose that ϕ is given by an $m \times n$ integer matrix A. After refining Σ, we may assume that the projected polyhedra $\{\phi(\sigma) : \sigma \in \Sigma\}$ again form a polyhedral complex.

This image complex need not be pure. For example, consider a fan whose support is the union of the two planes $x_1 = x_2 = 0$ and $x_3 = x_4 = 0$ in $\mathbb{R}^4$. Let ϕ be the projection onto the first three coordinates. The image of the fan is the union of the plane $x_3 = 0$ and the line $x_1 = x_2 = 0$ in $\mathbb{R}^3$.

Let Σ' be the subcomplex of the image containing the projected polyhedra of maximum dimension and all their faces. Then Σ' inherits a weighting from Σ. Namely, we assign to a maximal cell σ' of Σ' the multiplicity

$$\mathrm{mult}(\sigma') \quad = \sum_{\sigma\in\Sigma, \phi(\sigma)=\sigma'} \mathrm{mult}(\sigma)\cdot[N'_{\sigma'} : \phi(N_\sigma)]. \tag{3.6.2}$$

If V is a matrix whose columns form a basis for N_σ, then the columns of the matrix AV form a basis for $\phi(N_\sigma)$. The lattice index $[N'_{\sigma'} : \phi(N_\sigma)]$ is the greatest common divisor of the maximal minors of the matrix AV. The sum of two sublattices of N is the smallest sublattice containing both.

Lemma 3.6.3. *If Σ is balanced, then the projection Σ' of Σ is balanced.*

Proof. Let τ' be a codimension-1 cell in Σ', and let $\tau_1, \dots, \tau_r$ be the codimension-1 cells in Σ with $\phi(\tau_i) = \tau'$. There may be cells in Σ of both larger and smaller dimension that map to τ', but we consider only those of codimension 1. For each τ_i let $\sigma_{i1}, \dots, \sigma_{il}$ be the maximal cells of Σ containing τ_i. For each i, j the quotient $N_{\sigma_{ij}}/N_{\tau_i}$ is isomorphic to $\mathbb{Z}$. Let $\mathbf{v}_{ij} \in N$ restrict to the generator for $N_{\sigma_{ij}}/N_{\tau_i}$ pointing in the direction of σ_{ij}. The balancing condition for Σ ensures that $\sum_j \mathrm{mult}(\sigma_{ij})\mathbf{v}_{ij}$ lies in N_{τ_i}.

Let $\sigma'_1, \ldots, \sigma'_s$ be the top-dimensional cells of Σ' containing τ'. Fix a vector $\mathbf{v}^k \in N'_{\sigma'_k} \subseteq N'$ whose image generates $N'_{\sigma'_k}/N'_{\tau'}$ and points in the direction of σ'_k. For each σ_{ij}, we have either $\phi(\sigma_{ij}) = \tau'$ or $\phi(\sigma_{ij}) = \sigma'_k$ for some $k = k(ij)$. In the former case we set $\mathbf{v}^{k(ij)} = \mathbf{0}$. In the latter case the projection of $\mathbf{v}_{ij}$ in $N'_{\sigma'_k}/N'_{\tau'}$ is a multiple of the corresponding $\mathbf{v}^{k(ij)}$ by the factor $[N'_{\sigma'_k} : N'_{\tau'} + \mathrm{span}_{\mathbb{Z}}(\phi(\mathbf{v}_{ij}))]$. Since

$$\begin{aligned}[N'_{\sigma'_k} : \phi(N_{\sigma_{ij}})] &= [N'_{\sigma'_k} : \phi(N_\tau + \mathrm{span}_{\mathbb{Z}}(\mathbf{v}_{ij}))] \\ &= [N'_{\sigma'_k} : \phi(N_\tau) + \mathrm{span}_{\mathbb{Z}}(\phi(\mathbf{v}_{ij}))] \\ &= [N'_{\sigma'_k} : N'_{\tau'} + \mathrm{span}_{\mathbb{Z}}(\phi(\mathbf{v}_{ij}))][N'_{\tau'} : \phi(N_{\tau_i})],\end{aligned}$$

this factor is $[N'_{\sigma'_{k(ij)}} : \phi(N_{\sigma_{ij}})]/[N'_{\tau'} : \phi(N_{\tau_i})]$.

Thus for each fixed index $i \in \{1, \ldots, r\}$ we have

$$\begin{aligned}&\sum_j \mathrm{mult}(\sigma_{ij}) \cdot [N'_{\sigma'_{k(ij)}} : \phi(N_{\sigma_{ij}})] \cdot \mathbf{v}^{k(ij)} \\ &\quad = [N'_{\tau'} : \phi(N_{\tau_i})] \cdot \big(\sum_j \mathrm{mult}(\sigma_{ij})[N'_{\sigma'} : N'_{\tau'} + \mathrm{span}_{\mathbb{Z}}(\phi(\mathbf{v}_{ij}))]\mathbf{v}^{k(ij)}\big) \\ &\quad = [N'_{\tau'} : \phi(N_{\tau_i})] \cdot \phi\big(\sum_j \mathrm{mult}(\sigma_{ij})\mathbf{v}_{ij}\big) \\ &\quad = \mathbf{0} \ \in \ N'/N'_{\tau'}.\end{aligned}$$

Summing this expression over all choices of τ'_i, we find

$$\sum_{ij} \mathrm{mult}(\sigma_{ij}) \cdot [N'_{\sigma'_{k(ij)}} : \phi(N_{\sigma_{ij}})] \cdot \mathbf{v}^{k(ij)} \ = \ \mathbf{0} \ \in \ N'/N'_{\tau'}.$$

The coefficient of $\mathbf{v}^k$ is $\sum_{i,j:k(ij)=k} \mathrm{mult}(\sigma_{ij})[N'_{\sigma'_k} : \phi(N'_{\sigma_{ij}})]$, which is the multiplicity we assigned to σ'_k in (3.6.2). This shows that Σ' is balanced. $\square$

A key consequence of the balancing condition is the following technical lemma. This will be used to show that our definition of stable intersection is well defined. For two polyhedra σ_1, σ_2, recall from (2.3.1) that the Minkowski sum $\sigma_1 + \sigma_2$ is the polyhedron $\{\mathbf{a} + \mathbf{b} : \mathbf{a} \in \sigma_1, \mathbf{b} \in \sigma_2\}$.

Lemma 3.6.4. *Let Σ_1 and Σ_2 be pure weighted balanced Γ_{val}-rational polyhedral complexes in $\mathbb{R}^n$. Let $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$ be top-dimensional cells with $\dim(\sigma_1 + \sigma_2) = n$ and $\dim(\sigma_1 \cap \sigma_2) = \dim(\sigma_1) + \dim(\sigma_2) - n$. Choose refinements of Σ_1 and Σ_2 so that $\sigma_1 \cap \sigma_2$ is a cell in both complexes. For $\mathbf{v} \in \mathbb{R}^n$, consider the following sum over all maximal cones $\tau_1 \in \mathrm{star}_{\Sigma_1}(\sigma_1 \cap \sigma_2)$ and $\tau_2 \in \mathrm{star}_{\Sigma_2}(\sigma_1 \cap \sigma_2)$ with $\dim(\tau_1 + \tau_2) = n$ and $\tau_1 \cap (\mathbf{v} + \tau_2) \neq \emptyset$:*

$$\sum_{\tau_1, \tau_2} \mathrm{mult}(\tau_1)\,\mathrm{mult}(\tau_2)[N : N_{\tau_1} + N_{\tau_2}]. \tag{3.6.3}$$

This sum is constant for all vectors $\mathbf{v}$ in a dense open subset of $\mathbb{R}^n$.

Proof. Let $\tilde{\Sigma}_i = \operatorname{star}_{\Sigma_i}(\sigma_1 \cap \sigma_2)$ for $i = 1, 2$. Consider the product fan $\tilde{\Sigma}_1 \times \tilde{\Sigma}_2 \subseteq \mathbb{R}^{2n}$. This has cones $\tau_1 \times \tau_2$ for $\tau_1 \in \operatorname{star}_{\Sigma_1}(\sigma_1 \cap \sigma_2)$ and $\tau_2 \in \operatorname{star}_{\Sigma_2}(\sigma_1 \cap \sigma_2)$. This fan is balanced with the weight on $\tau_1 \times \tau_2$ given by $\operatorname{mult}(\tau_1)\operatorname{mult}(\tau_2)$. Balancing holds because each codimension-1 cone in this fan is the product of a maximal cone of one of the factors with a codimension-1 cone of the other. The balancing equation for this cone comes from the second factor.

Consider the projection $\pi : \mathbb{R}^{2n} \to \mathbb{R}^n$ given by $\pi(x, y) = x - y$. After refining $\widetilde{\Sigma}_1 \times \widetilde{\Sigma}_2$, this induces a map of fans: for each pair (τ_1, τ_2) with $\tau_i \in \Sigma_i$ for $i = 1, 2$, the Minkowski sum $\tau_1 + (-\tau_2)$ is a union of cones in the image. The condition $\dim(\sigma_1 + \sigma_2) = n$ means that the cones $\bar{\sigma}_1, \bar{\sigma}_2$ of the two stars corresponding to σ_1 and σ_2 satisfy $\dim(\bar{\sigma}_1 + (-\bar{\sigma}_2)) = n$. Let τ be a cone of the image fan. Each cone $\tau_1 \times \tau_2$ of the product fan that projects to τ contributes $\operatorname{mult}(\tau_1)\operatorname{mult}(\tau_2)[N : N_{\tau_1} + N_{\tau_2}]$ to the multiplicity of τ. The final multiplicity is obtained by adding up all these contributions. This image fan is balanced by Lemma 3.6.3.

Let V be the interior of a top-dimensional cone of the image fan. Now, $\mathbf{v}$ lies in the projection of a top-dimensional cone $\tau_1 \times \tau_2$ of $\widetilde{\Sigma}_1 \times \widetilde{\Sigma}_2$ if and only if $\mathbf{v} \in \tau_1 - \tau_2$, which occurs if and only if $\tau_1 \cap (\mathbf{v} + \tau_2) \neq \emptyset$. Thus, for $\mathbf{v} \in V$, the sum (3.6.3) is the multiplicity of the top-dimensional cone of the image fan that contains $\mathbf{v}$. Since that image is an n-dimensional balanced fan in $\mathbb{R}^n$, the multiplicity does not depend on the choice of cone (see Exercise 3.7(24)). The sum thus does not depend on the choice of $\mathbf{v}$, as long as $\mathbf{v}$ lies in the interior of a top-dimensional cone of the image fan. $\square$

Definition 3.6.5. Let Σ_1 and Σ_2 be pure weighted balanced polyhedral complexes in $\mathbb{R}^n$. The *stable intersection* $\Sigma_1 \cap_{st} \Sigma_2$ is the polyhedral complex

$$\Sigma_1 \cap_{st} \Sigma_2 \quad = \bigcup_{\substack{\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2 \\ \dim(\sigma_1 + \sigma_2) = n}} \sigma_1 \cap \sigma_2. \tag{3.6.4}$$

The multiplicity of a top-dimensional cell $\sigma_1 \cap \sigma_2$ in $\Sigma_1 \cap_{st} \Sigma_2$ is

$$\operatorname{mult}_{\Sigma_1 \cap_{st} \Sigma_2}(\sigma_1 \cap \sigma_2) \; = \; \sum_{\tau_1, \tau_2} \operatorname{mult}_{\Sigma_1}(\tau_1)\operatorname{mult}_{\Sigma_2}(\tau_2)[N : N_{\tau_1} + N_{\tau_2}], \tag{3.6.5}$$

where the sum is over all $\tau_1 \in \operatorname{star}_{\Sigma_1}(\sigma_1 \cap \sigma_2), \tau_2 \in \operatorname{star}_{\Sigma_2}(\sigma_1 \cap \sigma_2)$ with $\tau_1 \cap (\mathbf{v} + \tau_2) \neq \emptyset$, for some fixed generic $\mathbf{v}$. This is independent of the choice of $\mathbf{v}$ by Lemma 3.6.4. In equation (3.6.4), the sum $\sigma_1 + \sigma_2$ is the Minkowski sum.

Note that the stable intersection of two polyhedral complexes is contained in their set-theoretic intersection. This containment can be strict.

We illustrate the concept of stable intersection with some examples.

Example 3.6.6. The standard tropical plane is the fan Σ with rays spanned by the vectors $\mathbf{e}_0 = (-1,-1,-1)$, $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$. The two-dimensional cones C_{ij} of Σ are spanned by the pairs $\mathbf{e}_i, \mathbf{e}_j$. The multiplicity on each of the six cones C_{ij} is one. Two cones σ_1, σ_2 of Σ have $\dim(\sigma_1 + \sigma_2) = 3$ if and only if one is two dimensional and the other has dimension at least one and is not a ray of the first. For example, when $\sigma_1 = C_{12}$, then σ_2 can be any cone that contains $\mathbf{e}_0$ or $\mathbf{e}_3$. The intersection $\sigma_1 \cap \sigma_2$ in that case is either $\{\mathbf{0}\}$ or one of the rays of σ_1. The latter case occurs when $\sigma_2 = C_{ij}$ with $i \in \{1,2\}$ and $j \in \{0,3\}$. *The stable intersection $\Sigma \cap_{st} \Sigma$ is thus the one-skeleton of the fan Σ. The multiplicity of each ray is one.* To show this, it suffices to consider the case $C_{12} \cap C_{13}$. For $\mathbf{v} = (1,1,-1)$ the cones C_{12} and $\mathbf{v} + C_{13}$ intersect in the ray $(1,1,0) + \mathrm{pos}((1,0,0))$. The lattice $N_{C_{12}}$ is the span of $\{\mathbf{e}_1, \mathbf{e}_2\}$, while the lattice $N_{C_{13}}$ is the span of $\{\mathbf{e}_1, \mathbf{e}_3\}$, so $N_{C_{12}} + N_{C_{13}} = \mathbb{Z}^3 = N$. The lattice index $[N : N_{C_{12}} + N_{C_{13}}]$ equals one.

Next we consider the tropical curves shown in Figure 3.6.1, where we denote the solid curve by Σ_1 and the dotted curve by Σ_2. The stable intersection $\Sigma_1 \cap_{st} \Sigma_2$ consists of three points: $(-1,2)$ with multiplicity one, $(1,1)$ with multiplicity two, and $(3,-1)$ with multiplicity one. We verify the multiplicity of $\sigma = \{(1,1)\}$ using the formula (3.6.5). After refining Σ_1 and Σ_2 appropriately, $\mathrm{star}_{\Sigma_1}(\sigma)$ consists of three rays $\mathrm{pos}\{(1,0)\}$, $\mathrm{pos}\{(0,1)\}$, and $\mathrm{pos}\{(-1,-1)\}$. Likewise, $\mathrm{star}_{\Sigma_2}(\sigma)$ consists of two rays $\mathrm{pos}\{(1,-1)\}$ and $\mathrm{pos}\{(-1,1)\}$. For $\mathbf{v} = (1,1)$, the fan $\mathrm{star}_{\Sigma_1}(\sigma)$ intersects $\mathbf{v} + \mathrm{star}_{\Sigma_2}(\sigma)$ in two points, $(1,0)$ and $(0,1)$. The first of these comes from the rays $\tau_1 = \mathrm{pos}\{(1,0)\}$ and $\tau_2 = \mathrm{pos}\{(1,-1)\}$, while the second comes from the rays $\tau_1 = \mathrm{pos}\{(0,1)\}$ and $\tau_2 = \mathrm{pos}\{(-1,1)\}$. Since the weights of all cells in Σ_1 and Σ_2 are one, the multiplicity (3.6.5) is two:

$$(1)(1)[\mathbb{Z}^2 : \mathrm{span}_{\mathbb{Z}}((1,0),(1,-1))] + (1)(1)[\mathbb{Z}^2 : \mathrm{span}_{\mathbb{Z}}((0,1),(-1,1))] \;=\; 1+1.$$

For $\mathbf{v} = (-1,0)$, the only cones in the respective stars that intersect are $\mathrm{pos}\{(-1,-1)\}$ and $\mathrm{pos}\{(1,-1)\}$. The multiplicity (3.6.5) is now computed as $(1)(1)[\mathbb{Z}^2 : \mathrm{span}_{\mathbb{Z}}\{(-1,-1),(1,-1)\}] = 2$. Note that we get the same answer for the two different $\mathbf{v}$. $\Diamond$

The stable intersection of two pure weighted balanced polyhedral complexes is again pure and balanced. This requires the following three lemmas.

Lemma 3.6.7. *Let Σ_1 and Σ_2 be pure weighted balanced polyhedral complexes, and let σ be a cell of $\Sigma_1 \cap_{st} \Sigma_2$. We have the equality of weighted fans*

$$\mathrm{star}_{\Sigma_1 \cap_{st} \Sigma_2}(\sigma) \;=\; \mathrm{star}_{\Sigma_1}(\sigma) \cap_{st} \mathrm{star}_{\Sigma_2}(\sigma). \tag{3.6.6}$$

Proof. We first show the equality of sets. A vector $\mathbf{v}$ is in $\mathrm{star}_{\Sigma_1 \cap_{st} \Sigma_2}(\sigma)$ if and only if there is $\mathbf{w} \in \sigma$, a top-dimensional cell $\tau \in \Sigma_1 \cap_{st} \Sigma_2$, and an

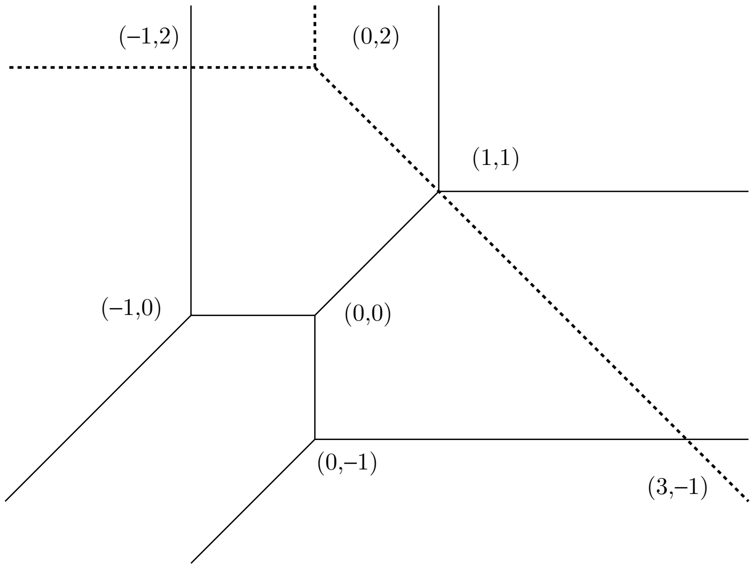

Figure 3.6.1. The stable intersection of two curves in Example 3.6.6.

$\epsilon > 0$ with $\mathbf{w} + \epsilon\mathbf{v} \in \tau$. By the definition of stable intersection, we can write $\tau = \tau_1 \cap \tau_2$ for $\tau_1 \in \Sigma_1$, $\tau_2 \in \Sigma_2$ with $\dim(\tau_1 + \tau_2) = n$. We have $\mathbf{w} + \epsilon\mathbf{v} \in \tau_i$ for $i = 1, 2$, so $\mathbf{v} \in \mathrm{star}_{\Sigma_1}(\sigma) \cap \mathrm{star}_{\Sigma_2}(\sigma)$. Then $\mathbf{v}$ is in the cone $\bar{\tau}_i$ of $\mathrm{star}_{\Sigma_i}(\sigma)$, which contains a translate of τ_i, for $i = 1, 2$, so $\dim(\bar{\tau}_1 + \bar{\tau}_2) = n$. Thus $\bar{\tau}_1 \cap \bar{\tau}_2 \in \mathrm{star}_{\Sigma_1}(\sigma) \cap_{st} \mathrm{star}_{\Sigma_2}(\sigma)$, and this shows "$\subseteq$" in (3.6.6).

For the reverse inclusion, it suffices to show that, for $\tau_1 \in \Sigma_1$, $\tau_2 \in \Sigma_2$, we have $\dim(\bar{\tau}_1 + \bar{\tau}_2) = n$ if and only if $\dim(\tau_1 + \tau_2) = n$. Since the linear space parallel to the sum of two polyhedra is the sum of the linear spaces parallel to the summands, it suffices to observe that the linear spaces parallel to $\bar{\tau}_i$ and to τ_i are equal. The linear space parallel to $\bar{\tau}_i$ is the span of $\mathbf{x} - \mathbf{y}$ with $\mathbf{x} \in \tau_i$ and $\mathbf{y} \in \sigma$, which is contained in the linear space parallel to τ_i. The opposite inclusion comes from the fact that $\bar{\tau}_i$ contains a translate of τ.

We now show that the multiplicities on the two fans in (3.6.6) agree. Let $\bar{\tau}$ be a top-dimensional cone in $\mathrm{star}_{\Sigma_1 \cap_{st} \Sigma_2}(\sigma)$. Its multiplicity is that of the corresponding cell τ in $\Sigma_1 \cap_{st} \Sigma_2$. This is the sum, over choices $\tau_1 \in \mathrm{star}_{\Sigma_1}(\sigma)$ and $\tau_2 \in \mathrm{star}_{\Sigma_2}(\sigma)$ with $\tau_1 \cap (\mathbf{v} + \tau_2) \neq \emptyset$ for fixed generic $\mathbf{v}$, of the quantities

$$\mathrm{mult}_{\Sigma_1}(\tau_1) \cdot \mathrm{mult}_{\Sigma_2}(\tau_2) \cdot [N : N_{\tau_1} + N_{\tau_2}].$$

The multiplicity on $\bar{\tau}$ in $\mathrm{star}_{\Sigma_1}(\sigma) \cap_{st} \mathrm{star}_{\Sigma_2}(\sigma)$ is the sum over all choices $\bar{\tau}_i \in \mathrm{star}_{\mathrm{star}_{\Sigma_i}(\sigma)}(\bar{\tau}) = \mathrm{star}_{\Sigma_i}(\tau)$ for $i = 1, 2$ with $\bar{\tau}_1 \cap (\mathbf{v} + \bar{\tau}_2) \neq \emptyset$ of

$$\mathrm{mult}_{\mathrm{star}_{\Sigma_1}(\sigma)}(\bar{\tau}_1) \cdot \mathrm{mult}_{\mathrm{star}_{\Sigma_2}}(\sigma)(\bar{\tau}_2) \cdot [N : N_{\bar{\tau}_1} + N_{\bar{\tau}_2}].$$

They are equal because $\mathrm{mult}_{\mathrm{star}_{\Sigma_i}(\sigma)}(\bar{\tau}_i) = \mathrm{mult}_{\Sigma_i}(\tau_i)$, and $N_{\tau_1} = N_{\bar{\tau}_2}$. $\square$

In this section we relax the notion of the lineality space of a polyhedral complex Σ to mean the largest subspace L for which if $\mathbf{x} \in |\Sigma|$ and $\mathbf{v} \in L$, then $\mathbf{x} + \mathbf{v} \in |\Sigma|$. This notion only depends on the support $|\Sigma|$ of the complex. For example, the fan in $\mathbb{R}$ consisting of the three cones pos(1), pos(−1), and $\{0\}$ has lineality space $\mathbb{R}$ for this definition.

Lemma 3.6.8. *Let Σ be a pure weighted balanced Γ_{val}-rational polyhedral complex in $\mathbb{R}^n$ of codimension d, and let H be the (classical) hyperplane $\{\mathbf{x} : x_1 = 0\}$. The stable intersection $\Sigma \cap_{st} H$ is either empty or a pure weighted balanced polyhedral complex of codimension $d + 1$.*

Let l be the dimension of the intersection of the lineality space of Σ with H. If $d + l > n - 1$, then the stable intersection is empty.

Proof. If $d+l > n-1$, then there is no pair $\tau \in \Sigma$ with $\dim(\tau+H) = n$, so the stable intersection is empty. Indeed, $\dim(\tau + H) = \dim(\tau) + \dim(H) - \dim(\operatorname{aff}(\tau) \cap H) \leq (n - d) + (n - 1) - l = 2n - (d + 1 + l)$ is less than n.

We now assume $d + l \leq n - 1$. Let τ be a top-dimensional cell of $\Sigma \cap_{st} H$, so $\tau = \sigma \cap H$, where σ is cell of Σ with $\dim(\sigma) = n - d$ and $\dim(\sigma + H) = n$. The image of H modulo the linear space parallel to σ has dimension d, so we can choose a d-dimensional subspace H' of H with $\dim(H' + \sigma) = n$. The fan $\operatorname{star}_\Sigma(\tau)$ is balanced with the weights inherited from Σ. By Lemma 3.6.3, its image modulo H' is a balanced weighted $(n - d)$-dimensional polyhedral fan in $\mathbb{R}^n/H' \simeq \mathbb{R}^{n-d}$. Its support is all of $\mathbb{R}^{n-d}$, which means that $(\operatorname{star}_\Sigma(\tau)+H')\cap H = (\operatorname{star}_\Sigma(\tau)\cap H)+H'$ is all of H, and thus $\operatorname{star}_\Sigma(\tau)\cap H$ has dimension at least $n-d-1$. Since $\dim(\sigma+H) = n$, we do not have $\sigma \subset H$, and so τ has dimension $n - d - 1$. This shows that the stable intersection $\Sigma \cap_{st} H$ is pure of the expected codimension.

Let σ be a codimension-1 cell in $\Sigma \cap_{st} H$. To prove balancing at σ, we must show that the fan $\operatorname{star}_{\Sigma\cap_{st}H}(\sigma)$ is balanced. By Lemma 3.6.7, it equals $\operatorname{star}_\Sigma(\sigma) \cap_{st} \operatorname{star}_H(\sigma)$. Since stable intersection commutes with projections, we can quotient by the linear space parallel to σ. This reduces balancing to the case where Σ is a two-dimensional fan, and hence $\Sigma \cap_{st} H$ is a one-dimensional fan. For generic small $\mathbf{v} \in \mathbb{R}^n$, the intersection $\Sigma \cap (\mathbf{v} + H)$ is transverse, so its relatively open one-dimensional cells lie in the relative interiors of two-dimensional cones of Σ. Therefore, the stable intersection $\Sigma \cap_{st} (\mathbf{v} + H)$ equals the actual intersection, and the multiplicity of a cone $\tau \cap (\mathbf{v} + H)$ is the lattice index $[N : N_\tau + N_H]$ times the multiplicity of τ. Each unbounded ray of $\Sigma \cap (\mathbf{v} + H)$ corresponds to a ray of $\Sigma \cap_{st} H$ plus the choice of a two-dimensional cone $\tau \in \Sigma$ with $\dim(\tau + H) = n$ and $\tau \cap (\mathbf{v} + H) \neq \emptyset$. The sum of the multiplicities of rays in $\Sigma \cap (\mathbf{v} + H)$ corresponding to a fixed ray σ of $\Sigma \cap_{st} H$ thus equals the multiplicity of σ.

We claim that it suffices to show that $\Sigma \cap (\mathbf{v} + H)$ is balanced. Indeed, when summing the left-hand side of the balancing equation (3.3.1) over all vertices of the intersection, each bounded edge contributes to two summands, with direction vectors $\pm\mathbf{u}_\sigma$ and the same multiplicity m_σ. These contributions cancel. Balancing implies that the sub-sum coming from each vertex adds to $\mathbf{0}$, so the entire sum is $\mathbf{0}$, and this equals the contribution coming from the unbounded rays, which is the equation (3.3.1) for $\Sigma \cap_{st} H$.

Let $\mathbf{u}$ be a vertex of $\Sigma \cap (\mathbf{v} + H)$. Let σ be the ray of Σ containing $\mathbf{u}$, and let $\tau_1, \dots, \tau_s$ be the two-dimensional cones of Σ containing σ. Write $\mathbf{u}_i$ for the element of N_{τ_i} that projects to a generator of N_{τ_i}/N_σ, and m_i for the multiplicity of τ_i in Σ. Since Σ is balanced, we have $\sum_i m_i \mathbf{u}_i \in N_\sigma$.

We write $\mathbf{u}^i$ for the first lattice point of the ray of $\operatorname{star}_{\Sigma\cap(\mathbf{v}+H)}(\mathbf{u})$ corresponding to τ_i. The multiplicity of this cone in $\Sigma \cap_{st} (\mathbf{v} + H)$ is $m_i[N : N_{\tau_i} + N_H]$. We have

$$\mathbf{u}^i = [N_{\tau_i} : \mathbb{Z}\mathbf{u}^i + N_\sigma]\mathbf{u}_i + \mathbf{u}_{\sigma,i},$$

where $\mathbf{u}_{\sigma,i} \in N_\sigma$. By the second and third isomorphism theorems,

$$\begin{aligned} N/(N_H + N_{\tau_i}) \;&\cong\; \big(N/(N_H + N_\sigma)\big)/\big((N_H + N_{\tau_i})/(N_H + N_\sigma)\big) \\ &\cong\; \big(N/(N_H + N_\sigma)\big)/\big(N_{\tau_i}/((N_H \cap N_{\tau_i}) + N_\sigma)\big). \end{aligned}$$

Hence $[N : N_H + N_\sigma] = [N : N_H + N_{\tau_i}][N_{\tau_i} : \mathbb{Z}\mathbf{u}^i + N_\sigma]$. Thus,

$$\begin{aligned} &\sum_i m_i[N : N_{\tau_i} + N_H]\mathbf{u}^i \\ &\qquad = \sum_i m_i[N : N_{\tau_i} + N_H]([N_{\tau_i} : \mathbb{Z}\mathbf{u}^i + N_\sigma]\mathbf{u}_i + \mathbf{u}_{\sigma,i}) \\ &\qquad = \sum_i [N : N_H + N_\sigma]m_i\mathbf{u}_i + \sum_i m_i[N : N_{\tau_i} + N_H]\mathbf{u}_{\sigma,i} \\ &\qquad \in\; N_\sigma \cap N_H. \end{aligned}$$

Since $N_\sigma \cap N_H = \mathbf{0}$, we have that $\Sigma \cap_{st} (\mathbf{v} + H)$ is balanced as required. □

Lemma 3.6.9. *Let Σ be a pure weighted balanced Γ_{val}-rational polyhedral complex in $\mathbb{R}^n$. Let $H_1, \dots, H_d$ be hyperplanes in $\mathbb{R}^n$ whose normal vectors are linearly independent. Write L for the linear space $\bigcap_{i=1}^d H_i$. We have*

$$\Sigma \cap_{st} L \;=\; ((\Sigma \cap_{st} H_1) \cap_{st} H_2) \cdots \cap_{st} H_d.$$

Proof. Without loss of generality we may assume that $H_i = \{\mathbf{x} \in \mathbb{R}^n : x_i = 0\}$. The proof is by induction on d. The base case $d = 1$ is a tautology. Let $L' = \{\mathbf{x} \in \mathbb{R}^n : x_1 = \cdots = x_{d-1} = 0\}$. By induction, $\Sigma \cap_{st} L' = ((\Sigma \cap_{st} H_1) \cap_{st} H_2) \cdots \cap_{st} H_{d-1}$, so we need to show $\Sigma \cap_{st} L = (\Sigma \cap_{st} L') \cap_{st} H_d$. Let σ be a maximal cell in $\Sigma \cap_{st} L$. Then there is a maximal cell $\tau \in \Sigma$ with $\sigma = \tau \cap L$ and $\dim(\tau + L) = n$, so the projection of $\operatorname{star}_\Sigma(\sigma) + L$ to $\mathbb{R}^n/L \cong \mathbb{R}^d$

is d-dimensional. As the projection is balanced by Lemma 3.6.3, it must be all of $\mathbb{R}^{n-d}$ (see Exercise 3.7(24)). This means that there is $\mathbf{x} \in \mathrm{star}_\Sigma(\sigma)+L$ with $x_1 = \cdots = x_{d-1} = 0$ and $x_d \neq 0$. We may assume that $\mathbf{x} \in \mathrm{star}_\Sigma(\sigma)$, as adding an element of L does not change the first d coordinates. Recall from Definition 2.3.6 that the cones of $\mathrm{star}_\Sigma(\sigma)$ have the form $\bar{\tau}$ for $\tau \supset \sigma$. We must have $\mathbf{x} \in \bar{\tau}'$ for some cell τ' of Σ containing σ. The cell τ' has the property that $\bar{\tau}'+L$ has dimension n, and thus $\dim(\tau'+L) = n$. Since $\mathbf{x} \in \bar{\tau}'$, there is $\mathbf{x}' \in \tau'$ with $x'_1 = \cdots = x'_{d-1} = 0$ and $x'_d \neq 0$. Let $\sigma' = \tau \cap L'$. Since $L \subset L'$, we have $\dim(\tau + L') = n$, and so $\sigma' \in \Sigma \cap_{st} L'$. By construction, $\sigma = \sigma' \cap H_d$. Since $x'_d \neq 0$, we have $\sigma' \neq \sigma$, so $\dim(\sigma' + H_d) = n$. Thus $\sigma \in (\Sigma \cap_{st} L') \cap_{st} H_d$.

For the reverse inclusion, note that if σ is a maximal cell in $(\Sigma \cap_{st} L') \cap_{st} H_d$, then there is a maximal cell $\sigma' \in \Sigma \cap_{st} L'$ with $\sigma = \sigma' \cap H_d$ and $\dim(\sigma' + H_d) = n$. Furthermore, there is a maximal cell $\tau \in \Sigma$ with $\sigma' = \tau \cap L'$ and $\dim(\tau + L') = n$. We thus have $\tau \cap L = \tau \cap (L' \cap H_d) = \sigma$, and as before $\dim(\tau + L) = n$. This shows the equality as sets.

To see the equality of multiplicities, note that the multiplicities of L, L', and H_d are all one, and $N_{L'} + N_{H_d} = N_L$. This means that the multiplicity in both descriptions of a cell σ is the sum over all τ mentioned above with $\tau \cap (\mathbf{v}+L) \neq \emptyset$ for fixed generic $\mathbf{v}$ of the quantity $\mathrm{mult}_\Sigma(\tau)[N : N_\tau + L]$. ☐

Theorem 3.6.10. *Let Σ_1 and Σ_2 be pure weighted balanced Γ_{val}-rational polyhedral complexes in $\mathbb{R}^n$ of codimensions d and e, respectively, and let l be the dimension of the intersection of their lineality spaces. Then the stable intersection $\Sigma_1 \cap_{st} \Sigma_2$ is either empty or it is a pure weighted balanced Γ_{val}-rational polyhedral complex of codimension $d+e$. If $d+e+l > n$, then the stable intersection is empty.*

Proof. Let Δ be the diagonal linear subspace $\{(\mathbf{w},\mathbf{w}) : \mathbf{w} \in \mathbb{R}^n\} \subset \mathbb{R}^{2n}$. There is a natural identification of Δ with $N_\mathbb{R}$. We claim that

$$\Sigma_1 \cap_{st} \Sigma_2 \;=\; (\Sigma_1 \times \Sigma_2) \cap_{st} \Delta \;\subseteq\; \Delta \;\cong\; N_\mathbb{R}.$$

To see this, consider a cell $\tau_1 \times \tau_2$ of $\Sigma_1 \times \Sigma_2$. Let A_1 and A_2 be matrices whose columns form a basis for N_{τ_1} and N_{τ_2}, respectively. Then the matrix $A^{12} = \left(\begin{smallmatrix} A_1 & 0 & I \\ 0 & A_2 & I \end{smallmatrix}\right)$ has columns forming a basis for $N_{(\tau_1\times\tau_2)+\Delta}$, so the Minkowski sum $(\tau_1 \times \tau_2) + \Delta$ has dimension $2n$ if and only if this matrix has rank $2n$. This is the case if and only if the matrix $A_{12} = \begin{pmatrix} A_1 & -A_2 \end{pmatrix}$ has rank n, which occurs if and only if $\dim(\tau_1 + \tau_2) = n$. This means that $(\tau_1 \times \tau_2) \cap \Delta \in (\Sigma_1 \times \Sigma_2) \cap_{st} \Delta$ if and only if $\tau_1 \cap \tau_2 \in \Sigma_1 \cap_{st} \Sigma_2$. Also, $(\tau_1 \times \tau_2) \cap ((\mathbf{0}, -\mathbf{v}) + \Delta) \neq \emptyset$ if and only if $\tau_1 \cap (\mathbf{v} + \tau_2) \neq \emptyset$. So, to compute the multiplicity of $\tau_1 \cap \tau_2$ or $(\tau_1 \times \tau_2) \cap \Delta$ we sum over the same pairs (τ_1, τ_2). To see that the multiplicity is the same in both cases, it suffices to observe that the index of $N_{(\tau_1\times\tau_2)+\Delta}$ in $N\oplus N$ is the greatest common divisor of the maximal minors of the matrix A^{12}, while the index $[N : N_{\tau_1} + N_{\tau_2}]$ is the greatest common divisor of the maximal minors of the matrix A_{12}. Since these coincide, the multiplicities coincide, and so the claim follows.

Change coordinates so that $\Delta = \{\mathbf{x} \in \mathbb{R}^{2n} : x_1 = \cdots = x_n = 0\}$. Write H_i for the hyperplane $\{\mathbf{x} \in \mathbb{R}^n : x_i = 0\}$, and $\Sigma = \Sigma_1 \times \Sigma_2$. By Lemma 3.6.9 we have $\Sigma \cap_{st} \Delta = ((\Sigma \cap_{st} H_1) \cap_{st} H_2) \cdots \cap_{st} H_n$. The result then follows from Lemma 3.6.8 since $\operatorname{codim}(\Sigma_1 \times \Sigma_2) = d + e$ and $\operatorname{codim}(\Delta) = n$ in $\mathbb{R}^{2n}$. Thus, if the stable intersection is nonempty, it has codimension $d + e + n$ in $\mathbb{R}^{2n}$, and so has codimension $d + e$ in $N_{\mathbb{R}} \cong \mathbb{R}^n$. □

Definition 3.6.11. Let Σ_1, Σ_2 be pure weighted balanced Γ_{val}-rational polyhedral complexes in $\mathbb{R}^n$ that meet transversely at a point $\mathbf{w}$ that lies in the relative interior of maximal cells $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$. Here we use the notion of *meeting transversely* from Definition 3.4.9. The *tropical multiplicity* of the intersection at $\mathbf{w}$ is the product $\operatorname{mult}_{\Sigma_1}(\sigma_1)\operatorname{mult}_{\Sigma_2}(\sigma_2)[N : N_{\sigma_1}+N_{\sigma_2}]$.

If Σ_1 and Σ_2 intersect transversely at every point $\mathbf{w}$ of their intersection, then the stable intersection $\Sigma_1 \cap_{st} \Sigma_2$ equals the intersection $\Sigma_1 \cap \Sigma_2$, and the multiplicity of the stable intersection at $\mathbf{w}$ is the tropical multiplicity.

We now make the link to the construction for curves in Section 1.3. The stable intersection can be obtained by translating each Σ_i by a small amount so that the intersection is transverse, computing the intersection together with its tropical multiplicity, and then taking the limit as the translation becomes smaller and smaller. This definition is made precise as follows.

Recall that the *Hausdorff metric* on subsets of $\mathbb{R}^n$ is given by $d(A, B) = \max(\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|)$. This lets us speak about the limit of a sequence of subsets of $\mathbb{R}^n$. If the subsets are weighted polyhedral complexes Σ_i that converge to a polyhedral complex Σ, then the limit inherits a weighting in the following way. A top-dimensional cell σ of the limit complex Σ is the limit of top-dimensional cells σ_i of Σ_i if $\lim_{i\to\infty}\sigma_i = \sigma$. We consider the set of all such sequences of σ_i limiting to σ, where we identify cofinal sequences. If $\lim_{i\to\infty}\operatorname{mult}_{\Sigma_i}(\sigma_i)$ exists for all such sequences, then we define the multiplicity of σ to be the sum of all these limits.

We often apply these concepts to finite collections of weighted points. In this case the multiplicity of a limit point $\mathbf{u}$ is the sum of the multiplicities of all points that tend to $\mathbf{u}$. The following result, however, works in general.

Proposition 3.6.12. *Let Σ_1 and Σ_2 be weighted balanced polyhedral complexes that are pure of codimension d and e. For general $\mathbf{v} \in \mathbb{R}^n$, the limit*

$$\lim_{\epsilon\to 0}\Sigma_1 \cap (\epsilon\mathbf{v} + \Sigma_2)$$

exists and it equals $\Sigma_1 \cap_{st} \Sigma_2$ as a weighted polyhedral complex. In particular, this intersection is independent of the choice of translate $\mathbf{v}$.

Proof. We first give the condition for $\mathbf{v}$ to be generic. Consider any pair of cells $\tau_i \in \Sigma_1$ and $\tau_j \in \Sigma_2$ with nontrivial intersection. If $\dim(\tau_i + \tau_j) < n$, then there is a vector $\mathbf{u}_{ij}$ perpendicular to the affine spans of both τ_i and τ_j. For any $\mathbf{v}$ with $\mathbf{u}_{ij} \cdot \mathbf{v} \neq 0$, we have $\tau_i \cap (\mathbf{v} + \tau_j) = \emptyset$. Choose one such vector $\mathbf{u}_{ij}$ for each pair $\tau_i \in \Sigma_1$, $\tau_j \in \Sigma_2$ with $\tau_i \cap \tau_j \neq \emptyset$ and $\dim(\tau_i + \tau_j) < n$. Let V be the open set in $\mathbb{R}^n$ consisting of vectors $\mathbf{v}$ with $\mathbf{v} \cdot \mathbf{u}_{ij} \neq 0$ for all i, j.

Fix $\mathbf{v} \in V$. Suppose $\mathbf{w}$ lies in $\lim_{\epsilon \to 0} \Sigma_1 \cap (\epsilon \mathbf{v} + \Sigma_2)$. For all $\epsilon > 0$ there is $\mathbf{w}_\epsilon \in \Sigma_1 \cap (\epsilon' \mathbf{v} + \Sigma_2)$ with $\|\mathbf{w}_\epsilon - \mathbf{w}\| < \epsilon$. Here $\epsilon' < \epsilon$ depends on ϵ. Since Σ_1 is closed, we have $\mathbf{w} \in \Sigma_1$. Similarly, since $\mathbf{w}_\epsilon - \epsilon' \mathbf{v} \in \Sigma_2$, and Σ_2 is closed, we have $\mathbf{w} \in \Sigma_2$, so $\mathbf{w} \in \Sigma_1 \cap \Sigma_2$. Let σ be the smallest cell of Σ_1 containing $\mathbf{w}$. After refining if necessary, σ is also a cell of Σ_2. For sufficiently small ϵ, the point $\mathbf{w}_\epsilon$ lies in a cell τ_1 of Σ_1 containing σ. Similarly, for small ϵ, the point $\mathbf{w}_\epsilon - \epsilon' \mathbf{v}$ lies in a cell τ_2 of Σ_2 which must also have σ as a face. Since $\mathbf{v} \in V$, we have $\epsilon' \mathbf{v} \in V$. Since $\mathbf{w}_\epsilon \in \tau_1 \cap (\epsilon' \mathbf{v} + \tau_2)$, we must have $\dim(\tau_1 + \tau_2) = n$, which means that $\mathbf{w} \in \Sigma_1 \cap_{st} \Sigma_2$.

For the converse, let σ be a top-dimensional cell of $\Sigma_1 \cap_{st} \Sigma_2$, and let $\mathbf{w} \in \sigma$. There are $\tau_1 \in \Sigma_1$, $\tau_2 \in \Sigma_2$ with $\tau_1 \cap \tau_2 = \sigma$, $\dim(\tau_1 + \tau_2) = n$, and $\tau_1 \cap (\mathbf{v} + \tau_2) \neq \emptyset$. Choose $\mathbf{w}' \in \tau_1 \cap (\mathbf{v} + \tau_2)$. For any $0 < \epsilon' < 1$, we have $\mathbf{w}_{\epsilon'} = (1 - \epsilon')\mathbf{w} + \epsilon' \mathbf{w}' \in \tau_1 \cap (\epsilon' \mathbf{v} + \tau_2)$, since τ_1 and τ_2 are convex, and $\mathbf{w}' - \mathbf{v} \in \tau_2$. Given $\epsilon > 0$ we can choose $\epsilon' < \epsilon / \|\mathbf{w}' - \mathbf{w}\|$. Then $\|\mathbf{w}_{\epsilon'} - \mathbf{w}\| < \epsilon$. We conclude that $\mathbf{w} \in \lim_{\epsilon \to 0} \Sigma_1 \cap (\epsilon \mathbf{v} + \Sigma_2)$.

For the multiplicities, note that the intersection $\Sigma_1 \cap (\epsilon \mathbf{v} + \Sigma_2)$ is transverse for generic $\mathbf{v}$. A top-dimensional cell is the intersection of unique maximal cells $\tau_1 \in \Sigma_1$ and $\epsilon \mathbf{v} + \tau_2$ for $\tau_2 \in \Sigma_2$ with $\dim(\tau_1 + \tau_2) = n$. The multiplicity of such an intersection is $\mathrm{mult}_{\Sigma}(\tau_1)\, \mathrm{mult}_{\Sigma_2}(\tau_2)[N : N_{\tau_1} + N_{\tau_2}]$. Since the multiplicity of a top-dimensional cell σ in $\Sigma_1 \cap_{st} \Sigma_2$ is the sum of this quantity over all pairs τ_1, τ_2 with $\tau_1 \cap (\mathbf{v} + \tau_2) \neq \emptyset$, and such pairs are exactly those for which $\lim_{\epsilon \to 0} \tau_1 \cap (\epsilon \mathbf{v} + \tau_2) = \sigma$, this shows the equality. $\square$

Example 3.6.13. Fix $K = \mathbb{Q}$ with the 2-adic valuation. Consider first $\Sigma_1 = \mathrm{trop}(V(4x^2 + xy + 12y^2 + y + 3))$ and $\Sigma_2 = \mathrm{trop}(V(4x + y + 4))$ in $\mathbb{R}^2$. This is shown in the first picture in Figure 3.6.2, with Σ_2 drawn with dotted lines. The vertical ray of Σ_1 has multiplicity 2. The second picture shows the intersections $\Sigma_1 \cap ((1,0) + \Sigma_2)$ and $\Sigma_1 \cap ((0,1) + \Sigma_2)$. Both intersection points have multiplicity 2. These are these cases $\epsilon = 1$ of the translations $\epsilon(1,0) + \Sigma_2$ and $\epsilon(0,1) + \Sigma_2$. As ϵ goes to zero, the intersection point in both cases approaches the point $(-1,1)$. The multiplicity also does not change, so the stable intersection $\Sigma_1 \cap_{st} \Sigma_2$ is the point $(-1,1)$ with multiplicity 2.

Consider next the tropical line $\Sigma_3 = \mathrm{trop}(V(x+8y+1))$. This is shown in the third picture in Figure 3.6.2. The intersection $\Sigma_1 \cap \Sigma_3$ is not transverse. The translations $(1, 1/2) + \Sigma_3$ and $(-1/2, 0) + \Sigma_3$ are drawn in the last picture in Figure 3.6.2, and these give transverse intersections in two points. In both cases the tropical multiplicity is one at each point. As ϵ goes to zero, the limits of $\Sigma_1 \cap (\epsilon(1, 1/2) + \Sigma_3)$ and $\Sigma_1 \cap (\epsilon(-1/2, 0) + \Sigma_3)$ are both the two points $(0, 0)$ and $(0, -2)$. The limiting multiplicity is one in both cases, so the stable intersection $\Sigma_1 \cap_{st} \Sigma_2$ is these two points with multiplicity one.

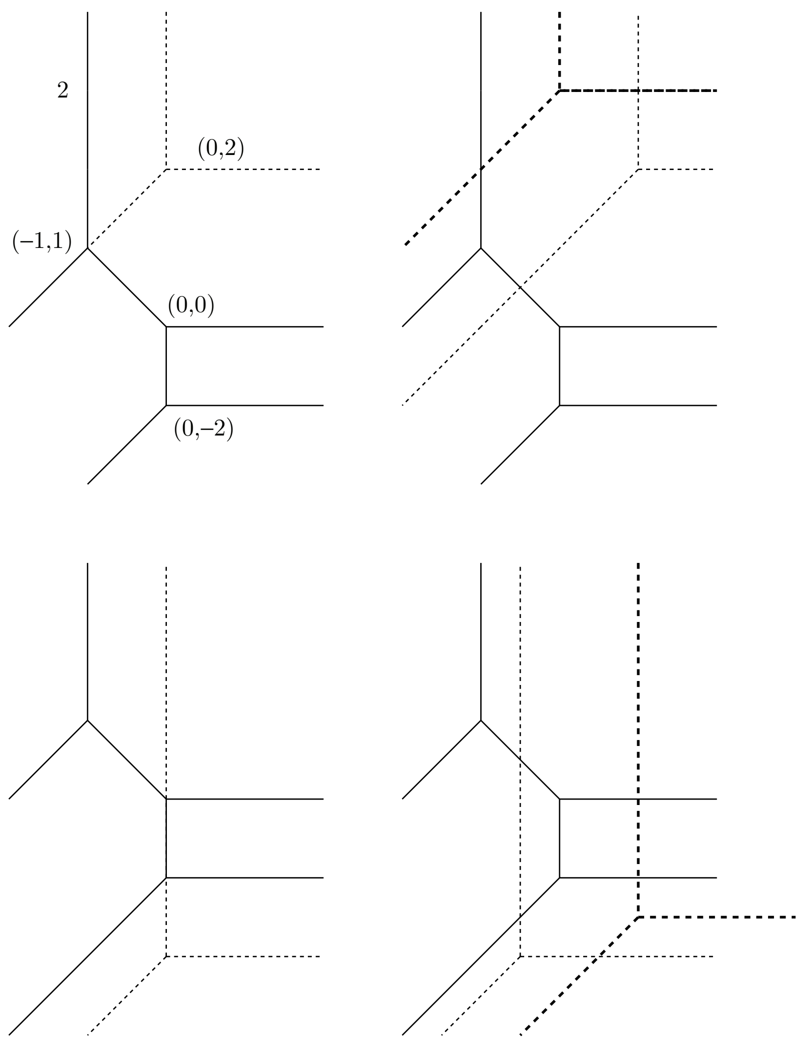

Figure 3.6.2. Stable intersections of lines and quadrics in the plane.

Both $\Sigma_1 \cap_{st} \Sigma_2$ and $\Sigma_1 \cap_{st} \Sigma_3$ are stable intersections of a quadric with a line. The intersections consist of two points, counted with multiplicity. This is a preview of the tropical complete intersections studied in Section 4.6. $\Diamond$

Remark 3.6.14. It can be shown that stable intersection is associative: if $\Sigma_1, \Sigma_2, \Sigma_3$ are weighted balanced Γ_{val}-rational polyhedral complexes, then

$$(\Sigma_1 \cap_{st} \Sigma_2) \cap_{st} \Sigma_3 \;=\; \Sigma_1 \cap_{st} (\Sigma_2 \cap_{st} \Sigma_3). \tag{3.6.7}$$

Thus stable intersection defines a multiplication on the set of weighted balanced Γ_{val}-rational polyhedral complexes, where complexes with the same support and weight function are identified. We can define an addition on this set by taking the unions of complexes (appropriately subdivided if necessary). If we also allow arbitrary real weights on maximal cells, then this makes this set into an $\mathbb{R}$-algebra. The subalgebra where all polyhedral complexes are fans appeared earlier in the work of McMullen as the *polytope algebra*. See [**JY13**] for details on the connection. Versions of this algebra have also arisen in the work of Allermann and Rau on tropical intersection theory [**AR10**] and Fulton and Sturmfels on toric intersection theory [**FS97**].

We now come to the derivation of Theorem 3.6.1 from the beginning of this section. We start with the following important special case.

Proposition 3.6.15. *Fix $X \subset T^n$. There is a finite set $B \subset \Bbbk$ for which for all $\alpha \in K$ with $\mathrm{val}(\alpha) = 0$ and $\overline{\alpha} \notin B$ the hyperplane $H_\alpha = V(x_1 - \alpha)$ satisfies*

$$\mathrm{trop}(X \cap H_\alpha) \;=\; \mathrm{trop}(X) \cap_{st} \mathrm{trop}(H_\alpha). \tag{3.6.8}$$

Proof. This proof is in two parts. We first show (3.6.8) set-theoretically and then check that the multiplicities coincide. Let I be the ideal of X in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Fix $\mathbf{w} \in \mathbb{R}^n$ with $w_1 = 0$. We first claim that there is a finite set $B \subset \Bbbk$ for which if $\alpha \in K$ with $\mathrm{val}(\alpha) = 0$ and $\overline{\alpha} \notin B$, then

$$\mathrm{in}_{\mathbf{w}}(I + \langle x_1 - \alpha \rangle) \;=\; \mathrm{in}_{\mathbf{w}}(I) + \langle x_1 - \overline{\alpha} \rangle. \tag{3.6.9}$$

To do this, we first consider the Gauss valuation on the field $K(s)$. This is given by setting $\mathrm{val}(p) = \min(\mathrm{val}(a_i))$ for a polynomial $p = \sum a_i s^i \in K[s]$, and then setting $\mathrm{val}(p/q) = \mathrm{val}(p) - \mathrm{val}(q)$ for $p, q \in K[s]$. The valuation ring is $R(s)$, and the residue field is $\Bbbk(s)$. Let $I_s = IK(s)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let $J_s = (I_s)_{\mathrm{proj}} + \langle x_1 - sx_0 \rangle \subset K(s)[x_0, \ldots, x_n]$. Note that $\mathrm{in}_{(0,\mathbf{w})}(J_s) = \mathrm{in}_{(0,\mathbf{w})}((I_s)_{\mathrm{proj}}) + \langle x_1 - sx_0 \rangle$. The containment $\supseteq$ is immediate. By Corollary 2.4.9 the Hilbert function of $(I_s)_{\mathrm{proj}}$ and $\mathrm{in}_{(0,\mathbf{w})}((I_s)_{\mathrm{proj}})$ agree. Since $x_1 - sx_0$ is a nonzerodivisor on both $(I_s)_{\mathrm{proj}}$ and $\mathrm{in}_{(0,\mathbf{w})}((I_s)_{\mathrm{proj}})$, the Hilbert functions of J_s and $\mathrm{in}_{(0,\mathbf{w})}((I_s)_{\mathrm{proj}})) + \langle x_1 - sx_0 \rangle$ also agree, which implies the equality.

Let $\mathcal{G}$ be a Gröbner basis for J_s with respect to $(0, \mathbf{w})$. We may assume by the previous paragraph that $\mathcal{G}$ is the union of a Gröbner basis $\mathcal{G}'$ for $(I_s)_{\text{proj}}$ and the polynomial $x_1 - sx_0$. We may also assume, after multiplying by a common denominator, that all $g \in \mathcal{G}$ lie in $K[s][x_0, \ldots, x_n]$. Each $g \in \mathcal{G}'$ has the form $\sum h_i f_i$ for $f_i \in (I_s)_{\text{proj}} \cap K[x_0, \ldots, x_n]$ and $h_i \in K(s)[x_0, \ldots, x_n]$. Let $\mathcal{S} \subset K[s]$ be the set of all polynomials occurring in a numerator or denominator of a coefficient of some h_i. The set $\mathcal{S}$ only depends on the initial ideal $\text{in}_{(0,\mathbf{w})}((I_s)_{\text{proj}})$, and not on the particular choice of $\mathbf{w}$.

Fix $p = \sum a_i s^i \in K[s]$, so $p(\alpha) = \sum a_i \alpha^i \in K$ for $\alpha \in K$. There is a finite set $B_p \subset \Bbbk$ for which $\text{val}_K(p(\alpha)) = \text{val}_{K(s)}(p) = \min(\text{val}(a_i))$ for any $\alpha \in K$ with $\text{val}(\alpha) = 0$ and $\overline{\alpha} \notin B_p$. Let $B' = \bigcup_{p \in \mathcal{S}} B_p$. Then for any α with $\overline{\alpha} \notin B'$ we have $\text{in}_{(0,\mathbf{w})}(g)|_{s=\overline{\alpha}} = \text{in}_{(0,\mathbf{w})}(g|_{s=\alpha})$ for all $g \in \mathcal{G}$. The equality (3.6.9) then follows after setting $x_0 = 1$ and applying Proposition 2.6.1.

The classical hyperplane $H = \{\mathbf{w} : w_1 = 0\}$ is $\text{trop}(H_\alpha)$ for all $\alpha \in K$ with $\text{val}(\alpha) = 0$. Choose a polyhedral complex Σ with support $|\Sigma| = \text{trop}(X)$. We next show the following equivalence holds for α with $\text{val}(\alpha) = 0$ and $\overline{\alpha}$ outside a finite set B'':

$$\mathbf{w} \in \Sigma \cap_{st} H \text{ if and only if } \text{in}_{\mathbf{w}}(I) + \langle x_1 - \overline{\alpha} \rangle \neq \langle 1 \rangle.$$

For $\mathbf{w} \in \Sigma \cap H$ we have $\mathbf{w} \in \Sigma \cap_{st} H$ if and only if $\dim(\sigma + H) = n$ for some cone σ of Σ containing $\mathbf{w}$. This occurs if and only if $\text{star}_\Sigma(\sigma') \not\subseteq H$, where σ' is the cone of Σ containing $\mathbf{w}$ in its relative interior. Recall from Lemma 3.3.6 that $\text{star}_\Sigma(\sigma')$ has support $\text{trop}(V(\text{in}_{\mathbf{w}}(I)))$. Write $\pi : T^n \to K^*$ for the projection onto the first coordinate, and $Y = V(\text{in}_{\mathbf{w}}(I)) \subseteq T^n_{\Bbbk}$. By Corollary 3.2.13, the projection π satisfies $\text{trop}(\pi(Y)) = \pi(\text{trop}(Y))$, so $\text{trop}(Y) \subseteq H$ if and only if $\text{trop}(\pi(Y)) \subseteq \{0\}$, which by elimination theory [**CLO07**, Theorem 2, §3.2] is equivalent to the existence of a polynomial $f \in \text{in}_{\mathbf{w}}(I) \cap \Bbbk[x_1^{\pm 1}]$. Thus $\mathbf{w} \in \Sigma \cap_{st} H$ if and only if $\text{in}_{\mathbf{w}}(I) \cap \Bbbk[x_1^{\pm 1}] = \{0\}$.

If $f \in \text{in}_{\mathbf{w}}(I) \cap \Bbbk[x_1^{\pm 1}]$, then $f(\overline{\alpha}) \in \text{in}_{\mathbf{w}}(I) + \langle x_1 - \overline{\alpha} \rangle$. For a given nonzero polynomial $f \in \Bbbk[x_1^{\pm 1}]$, we have $f(\overline{\alpha}) \neq 0$ for all but finitely many $\overline{\alpha}$. Thus, if $\text{in}_{\mathbf{w}}(I) \cap \Bbbk[x_1^{\pm 1}] \neq \{0\}$, then $\text{in}_{\mathbf{w}}(I) + \langle x_1 - \overline{\alpha} \rangle = \langle 1 \rangle$ for all but finitely many $\overline{\alpha}$. Conversely, if $\text{in}_{\mathbf{w}}(I) \cap \Bbbk[x_1^{\pm 1}] = \{0\}$, the closure of $\pi(Y)$ is $\Bbbk^*$, so for all but finitely many $\overline{\alpha}$ there is $y_{\overline{\alpha}} \in (\Bbbk^*)^{n-1}$ with $(\overline{\alpha}, y_{\overline{\alpha}}) \in Y$. Since $(\overline{\alpha}, y_{\overline{\alpha}})$ also lies in $V(\text{in}_{\mathbf{w}}(I) + \langle x_1 - \overline{\alpha} \rangle)$, we conclude that $\text{in}_{\mathbf{w}}(I) + \langle x_1 - \overline{\alpha} \rangle \neq \langle 1 \rangle$. This gives a finite set $B'' \subset \Bbbk$ for which if $\overline{\alpha} \notin B''$ we have $\mathbf{w} \in \Sigma \cap_{st} H$ if and only if $\text{in}_{\mathbf{w}}(I) + \langle x_1 - \overline{\alpha} \rangle \neq \langle 1 \rangle$. Note that B'' only depends on $\text{in}_{\mathbf{w}}(I)$ and not on the particular choice of $\mathbf{w}$. Set $B_{\mathbf{w}} = B' \cup B''$. Since the Gröbner complex of $(I_s)_{\text{proj}}$ is finite, there are only a finite number of different choices for $\text{in}_{(0,\mathbf{w})}((I_s)_{\text{proj}})$, so there are only a finite number of $B_{\mathbf{w}}$ as $\mathbf{w}$ varies over $\Sigma \cap H$. Let B be the union of these finite sets. Thus if $\overline{\alpha} \notin B$, and $\mathbf{w} \in \Sigma \cap H$, we have $\mathbf{w} \in \Sigma \cap_{st} H$ if and only if $\mathrm{in}_{\mathbf{w}}(I + \langle x_1 - \alpha \rangle) \neq \langle 1 \rangle$, so if and only if $\mathbf{w} \in \mathrm{trop}(X \cap H_\alpha)$. This completes the first half of the proof.

For the second half of the proof we check that the multiplicities agree on the two sides of (3.6.8). Fix $\mathbf{w}$ in the relative interior of a maximal cell σ of $\mathrm{trop}(X) \cap_{st} \mathrm{trop}(H_\alpha)$. Since $\sigma \subset \{\mathbf{w} : w_1 = 0\}$, we may change coordinates while fixing w_1 so that the affine span of σ is $\mathrm{span}(\mathbf{e}_{n-d+2}, \ldots, \mathbf{e}_n)$, where $d = \dim(X)$. Part (2) of Lemma 2.6.2 implies that $\mathrm{in}_{\mathbf{w}}(I)$ is generated by polynomials in the variables $x_1, \ldots, x_{n-d+1}$. Let $J = \mathrm{in}_{\mathbf{w}}(I) \cap S_{n-d+1}$, where $S_{n-d+1} = \Bbbk[x_1^{\pm 1}, \ldots, x_{n-d+1}^{\pm 1}]$. The multiplicity of σ in $\mathrm{trop}(X \cap H_\alpha)$ equals

$$(3.6.10) \qquad \mathrm{mult}_{\mathrm{trop}(X \cap H_\alpha)}(\sigma) \;=\; \dim_{\Bbbk}(S_{n-d+1}/(J + \langle x_1 - \overline{\alpha} \rangle)),$$

by (3.6.9) and Lemma 3.4.7. We shall finish by showing that this is also the multiplicity of σ in $\Sigma \cap_{st} H$. We do this by computing the multiplicity of the stable intersection using the limit formulation of Proposition 3.6.12. By Lemma 3.6.7 we may pass to the star of σ and quotient by the linear space parallel to σ. This means that the stable intersection we need to consider is that of $V(\mathrm{in}_{\mathbf{w}}(J))$ and $\{\mathbf{w} \colon w_1 = 0\}$ in $\mathbb{R}^{n-d}$.

The dimension (3.6.10) equals $\dim_{K'} S_{K',n-d+1}/(J' + \langle x_1 - \overline{\alpha} \rangle)$ where $K' = \Bbbk((\mathbb{R}))$ is the field in Example 2.1.7, $S_{K',n-d+1} = K'[x_1^{\pm 1}, \ldots, x_{n-d+1}^{\pm 1}]$, and J' is the ideal in $S_{K',n-d+1}$ with the same generators as J. Indeed, this dimension can be computed using Buchberger's algorithm, which depends only on the field of definition of its input. Similar arguments show that $\dim_{K'} S_{K',n-d+1}/(J' + \langle x_1 - \beta \rangle)$ is a constant D for all but finitely many $\beta \in K'$. By Theorem 3.2.4 we have $\mathrm{trop}(V(J')) = \mathrm{trop}(V(J))$.

Choose $\alpha_\epsilon \in K'$ with $\mathrm{val}(\alpha_\epsilon) = \epsilon > 0$ that is generic in the sense above. Proposition 3.4.8 implies that $\dim_{K'}(S_{K',n-d+1}/J' + \langle x_1 - \alpha_\epsilon \rangle)$ is the sum of the multiplicities of the points in the finite set $\mathrm{trop}(V(J' + \langle x_1 - \alpha_\epsilon \rangle))$. Since the intersection of $\mathrm{trop}(V(J'))$ and $\mathrm{trop}(V(x_1 - \alpha_\epsilon)) = \{\mathbf{w} : w_1 = \epsilon\}$ is transverse at all points of their intersection, by Theorem 3.4.12 we have $\mathrm{trop}(V(J' + \langle x_1 - \alpha_\epsilon \rangle)) = \mathrm{trop}(V(J')) \cap \mathrm{trop}(V(x_1 - \alpha_\epsilon))$. Now $\mathrm{trop}(V(x_1 - \alpha_\epsilon)) = \epsilon \mathbf{v} + H$ for any generic $\mathbf{v}$ with $v_1 = 1$. By Proposition 3.6.12, the multiplicity of the origin in $\mathrm{trop}(V(J')) \cap_{st} H$ equals the limit as $\epsilon \to 0$ of the sum of the multiplicities of $\mathrm{trop}(V(J' + \langle x_1 - \alpha_\epsilon \rangle))$. For all but finitely many $\alpha \in K$, this is the dimension D of $S_{n-d+1}/(J + \langle x_1 - \overline{\alpha} \rangle)$ as required. □

Proof of Theorem 3.6.1. Write $x_1, \ldots, x_n, y_1, \ldots, y_n$ for coordinates on $\mathbb{R}^{2n}$. By Lemma 3.6.9, for any balanced weighted complex $\Sigma \in \mathbb{R}^{2n}$, $(\Sigma \cap_{st} \{\mathbf{w} : w_1{=}0\}) \cap_{st} \cdots \cap_{st} \{\mathbf{w} : w_n{=}0\} \;=\; \Sigma \cap_{st} \{\mathbf{w} : w_1{=}\cdots{=}w_n{=}0\}$. Using Proposition 3.6.15 and the change of coordinates $x_i \mapsto x_i/y_i$, $y_i \mapsto y_i$, this identity implies the following fact. For any variety $Z \subset T^{2n}$ there exists a dense set $U \subset T^n$ such that $\operatorname{val}(\alpha) = \mathbf{0}$ and

$$(\operatorname{trop}(Z) \cap_{st} \operatorname{trop}(V(x_1 - \alpha_1 y_1))) \cap_{st} \cdots \cap_{st} \operatorname{trop}(V(x_n - \alpha_n y_n)) \\ = \operatorname{trop}(Z \cap V(x_1 - \alpha_1 y_1, \ldots, x_n - \alpha_n y_n))$$

for all $\alpha \in U$. The fact that U is dense follows from Lemma 2.2.12.

Let I and J be the ideals for X_1 and X_2, respectively, in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Given $\mathbf{t} = (t_1, \ldots, t_n) \in T^n$, write $J' = \mathbf{t}^{-1} J = \langle f(t_i^{-1} y_i) : f \in J \rangle$ for the ideal of $\mathbf{t}Y$. By the proof of Theorem 3.6.10, we have $\operatorname{trop}(X) \cap_{st} \operatorname{trop}(\mathbf{t}Y) \cong (\operatorname{trop}(X) \times \operatorname{trop}(\mathbf{t}Y)) \cap_{st} \Delta$, where $\Delta = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n} : x_i = y_i \text{ for } 1 \leq i \leq n\}$ is the diagonal in $\mathbb{R}^{2n}$. So, the stable intersection we are interested in equals

$$(3.6.11) \qquad (\operatorname{trop}(X) \times \operatorname{trop}(\mathbf{t}Y)) \cap_{st} \operatorname{trop}(V(x_i - y_i : 1 \leq i \leq n)).$$

The transformation $y_i' = t_i^{-1} y_i$ changes none of these tropical varieties when $\operatorname{val}(t_i) = 0$ for all i, so (3.6.11) equals

$$(\operatorname{trop}(X) \times \operatorname{trop}(Y)) \cap_{st} \operatorname{trop}((V(x_i - t_i y_i') : 1 \leq i \leq n)).$$

By the first paragraph, there is a dense set $U \subset T^n$ such that (3.6.11) equals $\operatorname{trop}((X \times \mathbf{t}Y) \cap V(x_i - y_i : 1 \leq i \leq n)) \simeq \operatorname{trop}(X \cap \mathbf{t}Y)$ for all $\mathbf{t} \in U$. $\square$

We close this section with an application of Theorem 3.6.1. Further applications will be seen in Section 4.6. Recall that the *degree* of a projective variety $\overline{X} \subset \mathbb{P}^n$ of dimension d is the number of intersection points, counted with multiplicity, of $\overline{X}$ with a generic subspace of dimension $n-d$. Let L_{n-d} be the standard tropical linear space of dimension $n - d$ in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$; this consists of all cones $\operatorname{pos}(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{n-d}})$ where $0 \leq i_1 < \cdots < i_{n-d} \leq n$.

Corollary 3.6.16. *Let $\overline{X} \subseteq \mathbb{P}^n_K$ be an irreducible projective variety of dimension d, and let $X = \overline{X} \cap T^n$. Let K have the trivial valuation. The degree of $\overline{X}$ is the multiplicity of the origin in the stable intersection of* $\operatorname{trop}(X)$ *with the tropical linear space L_{n-d}:*

$$\deg(\overline{X}) \;=\; \operatorname{mult}_{\mathbf{0}}(\operatorname{trop}(X) \cap_{st} L_{n-d}).$$

Proof. We first show that there is an open set U_1 in the Grassmannian $G(n-d, n+1)$ parameterizing codimension-d subspaces of K^{n+1} for which if $L \in U_1$, with $L^\circ = (L \cap (K^*)^{n+1})/K^* \subset (K^*)^{n+1}/K^* \cong T^n$, then $\operatorname{trop}(L^\circ) = L_{n-d}$. Indeed, let U_1 be the open set consisting of those $L \in G(n-d, n+1)$ for which all Plücker coordinates are nonzero. Such an L has the property that for any subset $J = \{j_1, \ldots, j_d\} \subset \{0, 1, \ldots, n\}$ the ideal I_L of L has a generating set $\ell_1, \ldots, \ell_d$ with $\operatorname{supp}(\ell_i) \cap J = \{x_{j_i}\}$. Any $\mathbf{w} \notin L_{n-d}$ has $\min(w_0, \ldots, w_n)$ achieved at at most d indices, so choosing J to contain these indices, we see that $\operatorname{in}_{\mathbf{w}}(I_L)$ contains a monomial, and so $\mathbf{w} \notin \operatorname{trop}(L^\circ)$. This shows that $\operatorname{trop}(L^\circ) \subseteq L_{n-d}$. As L_{n-d} has the same dimension as L°, and no subfan can be balanced, we conclude that $\operatorname{trop}(L^\circ) = L_{n-d}$. (This is generalized in Example 4.2.13.) Since $\dim(\overline{X}\backslash X) < \dim(X)$, there is an open set $U_2 \subset G(n-d, n+1)$ for which if $L \in U_2$, then $L \cap \overline{X} = L \cap X$. There is also an open set $U_3 \subset G(n-d, n+1)$ for which if $L \in U_3$, then $\deg(\overline{X}) = |\overline{X} \cap L|$, where the latter is counted with multiplicity.

Fix $L \in U_1 \cap U_2 \cap U_3$. Let $U_4 \subset T^n$ be the open subset consisting of those $\mathbf{t} \in T^n$ for which $\mathbf{t}L \in U_1 \cap U_2 \cap U_3$. By Theorem 3.6.1 there is $\mathbf{t} \in U_4$ for which $\operatorname{trop}(X \cap \mathbf{t}L^\circ) = \operatorname{trop}(X) \cap_{st} \operatorname{trop}(L^\circ) = \operatorname{trop}(X) \cap_{st} L_{n-d}$. By Proposition 3.4.8 $\operatorname{trop}(X \cap \mathbf{t}L^\circ)$ is the origin with multiplicity equal to the number of points in $X \cap \mathbf{t}L^\circ$, counted with multiplicity. Since $\mathbf{t}L \in U_2 \cap U_3$, the multiplicity of the origin is thus the degree of $\overline{X}$, as required. $\square$

3.7. Exercises

(1) Draw $\operatorname{trop}(V(f))$ for the following $f \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$:
   (a) $f = t^3x + (t + 3t^2 + 5t^4)y + t^{-2}$;
   (b) $f = (t^{-1} + 1)x + (t^2 - 3t^3)y + 5t^4$;
   (c) $f = t^3x^2 + xy + ty^2 + tx + y + 1$;
   (d) $f = 4t^4x^2 + (3t + t^3)xy + (5 + t)y^2 + 7x + (-1 + t^3)y + 4t$;
   (e) $f = tx^2 + 4xy - 7y^2 + 8$;
   (f) $f = t^6x^3 + x^2y + xy^2 + t^6y^3 + t^3x^2 + t^{-1}xy + t^3y^2 + tx + ty + 1$.

(2) By Example 3.1.11, the tropical hypersurface of the 3×3-determinant has 15 maximal cones. These come in two symmetry classes. Pick two representatives σ_1 and σ_2, and find matrices $\mathbf{w}_1$ and $\mathbf{w}_2$ in $\mathbb{Q}^{3\times 3}$ that satisfy $\mathbf{w}_i \in \operatorname{relint}(\sigma_i)$ for $i = 1, 2$. Next construct rank 2 matrices M_1 and M_2 in $\mathbb{C}\{\{t\}\}^{3\times 3}$ with $\operatorname{val}(M_i) = \mathbf{w}_i$ for $i = 1, 2$.

(3) The Pfaffian of a skew-symmetric 6×6-matrix is a polynomial of degree 3 in 15 variables. Compute its tropical hypersurface.

(4) Verify (as much as possible) the Fundamental Theorem 3.2.3 and the Structure Theorem 3.3.5 for the six curves in Exercise 3.7(1).

(5) Draw the recession fan for the six plane curves in Exercise 3.7(1).

(6) Let $K = \mathbb{Q}$ with the 3-adic valuation. Construct two explicit distinct quadratic polynomials $f, g \in K[x_1, x_2, x_3, x_4]$ which form a tropical basis for the Laurent polynomial ideal they generate.

(7) Using your f and g in Exercise 3.7(6), compute the elimination ideal $\langle f, g\rangle \cap K\left[x_1x_2^{-1}, x_2x_3^{-1}, x_3x_4^{-1}\right]$. Interpret your result geometrically.

(8) Let $Y = V(x_1 + x_2 + x_3 + x_4 + 1, x_2 - x_3 + x_4) \subseteq (\mathbb{C}^*)^4$. Compute $\mathrm{trop}(Y)$ and a polyhedral fan Σ with support $\mathrm{trop}(Y)$. Show that Σ is balanced if we put the weight one on each maximal cone.

(9) Give an example to show that the tropicalization of a hypersurface might be a fan even if some of the coefficients have nonzero valuation. What sort of converse can you give to Proposition 3.1.10?

(10) What is the largest multiplicity of any edge in the tropicalization of any plane curve of degree d? How about surfaces in 3-space?

(11) For f in Example 3.1.2(2) and the vertex $\mathbf{w} = (-1, 0)$ on the right in Figure 3.1.1, describe all points $\mathbf{y} \in V(f)$ with $\mathrm{val}(\mathbf{y}) = \mathbf{w}$. For this example, verify that the set of such $\mathbf{y}$ is Zariski dense in $V(f)$.

(12) Let I be the ideal in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_4^{\pm 1}]$ generated by the five elements

$$\begin{aligned}
&(x_1 + x_3)^2(x_3 + x_4),\\
&(x_1 + x_2)(x_1 + x_4)^2,\\
&(x_1 + x_3)^2(x_1 + x_4),\\
&(x_1 + x_2)(x_1 + x_3)(x_1 + x_4),\\
&(x_1 + x_2)(x_1 + x_3)(x_3 + x_4)^2.
\end{aligned}$$

Find all associated primes of I and an explicit primary decomposition. Compute the tropical variety $\mathrm{trop}(V(I))$ with multiplicities.

(13) Let X and Y be subvarieties of T^n, and let $\Sigma = \mathrm{trop}(X) + \mathrm{trop}(Y)$ be the Minkowski sum of their tropicalizations in $\mathbb{R}^n$. Show that Σ is a tropical variety. Explain how to construct a subvariety $Z \subset T^n$ such that $\Sigma = \mathrm{trop}(Z)$.

(14) Compute generators for the ideal J_{proj} in Example 3.2.9. List the 12 maximal cones in the Gröbner fan structure on $\mathrm{trop}(V(J_{\mathrm{proj}}))$.

(15) True or false: The transverse intersection of two balanced polyhedral complexes in $\mathbb{R}^n$ is again a balanced polyhedral complex?

(16) Show that the k-skeleton of any n-dimensional polytope is connected through codimension 1. Get started with k=1 and n=3.

(17) Let $f(x, y)$ be the polynomial in Example 1.5.1. Compute the multiplicities of all rays in the one-dimensional fan $\mathrm{trop}(V(f))$.

(18) Describe a method for computing the multiplicity $\mathrm{mult}(P_i, I)$ defined in (3.4.1). Try it on some examples, e.g., using `Macaulay2`.

(19) Let P be the prime ideal generated by the 2×2-minors of a 3×3-matrix of unknowns. Compute $\mathrm{mult}(P, P^n)$ for $n = 1, 2, 3, \ldots$.

(20) Fix $\mathbf{w} = (1, 1)$ and the polynomials

$$f = xy - tx - ty + t^2,$$
$$g = x^2 - (t^2 + 2t)x + t^3 + t^2,$$
$$h_1 = y^2 - (t^2 + t)y + t^3,$$
$$h_2 = y^2 - (t^2 + 2t)y + t^3 + t^2.$$

Which of the two ideals $I_1 = \langle f, g, h_1\rangle$ and $I_2 = \langle f, g, h_2\rangle$ in $\mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$ satisfies the conclusion of Proposition 3.4.8?

(21) Consider the two tropical planes in $\mathbb{R}^3$ defined by

$$a_1 \odot x \,\oplus\, a_2 \odot y \,\oplus\, a_3 \odot z \,\oplus\, a_4$$
$$\text{and} \qquad b_1 \odot x \,\oplus\, b_2 \odot y \,\oplus\, b_3 \odot z \,\oplus\, b_4.$$

Find necessary and sufficient conditions, in terms of $a_1, a_2, \ldots, b_4$, for these to meet transversally at every point in their intersection.

(22) Let $X \subset T^{12}$ be the variety of 3×4-matrices of rank at most 2. Determine a fan structure on $\operatorname{trop}(X)$. Verify that it is connected through codimension 1. Draw the graph on the maximal cones.

(23) Given two polyhedral complexes Σ and Σ' in $\mathbb{R}^n$, show that

$$\operatorname{rec}(\Sigma \cap \Sigma') \;=\; \operatorname{rec}(\Sigma) \,\cap\, \operatorname{rec}(\Sigma'),$$

and explain how to construct a fan structure on this set.

(24) Show that if Σ is an n-dimensional weighted balanced Γ_{val}-rational polyhedral complex in $\mathbb{R}^n$, then the support $|\Sigma|$ is all of $\mathbb{R}^n$ and the weight on each n-dimensional polyhedron is the same.

(25) Show that if L is a (classical) linear space contained in the lineality space of two weighted balanced Γ_{val}-rational polyhedral complexes $\Sigma_1, \Sigma_2 \subseteq \mathbb{R}^n$, then L is contained in the lineality space of the stable intersection $\Sigma_1 \cap_{st} \Sigma_2$, and $(\Sigma_1/L) \cap_{st} (\Sigma_2/L) = (\Sigma_1 \cap_{st} \Sigma_2)/L$.

(26) Let L be a sublattice of rank n in $\mathbb{Z}^n$ that is generated by the columns of an $n \times r$-matrix A. Show that the index $[\mathbb{Z}^n : L]$ is the greatest common divisor of the maximal nonzero minors of A.

(27) Let L_1 and L_2 be tropical linear spaces in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. Show that their stable intersection $L_1 \cap_{st} L_2$ is a tropical linear space. Express the Plücker coordinates of $L_1 \cap_{st} L_2$ in terms of those of L_1 and L_2.

(28) According to Example 3.1.11, the tropical 3×3-determinant $X = \operatorname{trop}(V(f))$ is an eight-dimensional fan in $\mathbb{R}^9$. Compute the fans $X \cap_{st} X$ and $X \cap_{st} X \cap_{st} X$. Realize these two fans as the tropicalizations of two explicit varieties in the torus of 3×3-matrices.

(29) The Grassmannian $\overline{X} = G(2, 5)$ is a variety of dimension 6 in $\mathbb{P}^9$. See Proposition 2.2.10. Use Corollary 3.6.16 to compute $\deg(\overline{X})$.

(30) Find two tropical surfaces in $\mathbb{R}^3$ whose stable intersection is empty. Show that your surfaces arise from projective surfaces in $\mathbb{P}^3$. Find an example where your two projective surfaces have degree 1000.

(31) Compute $\operatorname{trop}(X)$ for $X = V(\pi x^2 + ey^2 + \sqrt{2}, \zeta(3)xyz + 1) \subseteq T^3_{\mathbb{C}}$.

(32) Fix $K = \mathbb{Q}$ with the p-adic valuation where $p = 2$ or $p = 3$. The discriminant of Example 3.3.3 is a polynomial in $K[a, b, c, d, e]$ whose tropicalization F now has some nonzero coefficients. For both primes, compute the polyhedral complex Σ_F and the tropical variety $V(F)$. Find the weights, and explain why $V(F)$ is balanced.

(33) Let $X \subset T^5$ be the variety given by the parameterization in Example 3.5.4. Find the ideal of X and compute the tropicalization $\operatorname{trop}(X)$.

(34) Let I be a homogeneous ideal in $K[x_1, \dots, x_n]$, and let $\mathbf{w}$ be in the relative interior of a maximal cell σ of $\operatorname{trop}(V(I))$. Let P be the toric ideal associated with the lattice $\left\{\mathbf{u} \in \mathbb{Z}^n : \operatorname{in}_{\mathbf{u}}(\operatorname{in}_{\mathbf{w}}(I)) = \operatorname{in}_{\mathbf{w}}(I)\right\}$, as in [**MS05, Stu96**]. Show that the multiplicity of σ can be computed in `Macaulay2` by the formula

$$\operatorname{mult}(\sigma) = \operatorname{degree}(\operatorname{in}_{\mathbf{w}}(I))/\operatorname{degree}(P).$$