Linear Algebra: Sheet 2

Present all your answers in complete sentences

Numbas quiz

Complete the week 3 quiz on Blackboard. This quiz contains questions on Chapters 1, 2 and 3. This contains important practice of the more computational parts of the course and **is worth 2.5% of your grade for this unit**. You can attempt the questions as many times as you like, and the deadline is **1pm on Wednesday 02/10/24**.

Hand-in question

Submit your solutions to this question on Blackboard by **1pm on Wednesday 02/10/24** for feedback from your tutor.

1. Consider the system of linear equations

$$
\begin{aligned}
2x_1 - x_2 + 3x_3 + x_4 &= 2 \\
-4x_1 + 2x_2 - 6x_3 + x_4 &= -7
\end{aligned}
$$

for some unknowns $x_1, x_2, , x_3, x_4, \in \mathbb{R}$.

(i) Write this system in the form $Ax = b$ for some $A \in M_{n,m}(\mathbb{R})$, $x \in \mathbb{R}^k$ and $b \in \mathbb{R}^l$, specifying the values of $m, n, k, l \in \mathbb{N}$.

(ii) Find all solutions to this system of equations.

(iii) Suppose we add a third equation to the system

$$ax_3 + 2x_4 = 3,$$

where $a \in \mathbb{R}$ is a constant. How many solutions does this new system of three equations have? (Your answer may depend on a.)

Additional questions

Try these questions and look at the solutions for feedback. Some of these questions may also be discussed in your tutorial.

2. The Kronecker delta is defined as follows:

$$\delta_{ij} := \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases} ,$$

Notice that this means the identity matrix is of the form $I = (\delta_{ij})$.

(a) Let $b_1, b_2, b_3, \cdots, b_n$ be a set of n real numbers. Show that

$$\sum_{i=1}^{n} b_i \delta_{ij} = b_j \text{ , and } \quad \sum_{j=1}^{n} b_j \delta_{ij} = b_i \ .$$

(b) Show that

$$\sum_{j=1}^{n} \delta_{ij} \delta_{jk} = \delta_{ik} \ ,$$

i.e., $I^2 = I$.

3. (i) Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$. Show that

$$e_i \cdot Ae_j = a_{ij} \ .$$

(ii) Show that if for $A, B \in M_{m,n}(\mathbb{R})$ we have

$$y \cdot Ax = y \cdot Bx \ , \quad \text{for all} \quad y \in \mathbb{R}^m \ , \quad x \in \mathbb{R}^n$$

then $A = B$.

4. Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$. Show that for any $x \in \mathbb{R}^n$ we have

$$\|Ax\| \leq \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} \|x\| \ .$$

(This is an estimate on how much a matrix can change the size of a vector. Hint: Use the Cauchy-Schwarz inequality.)

5. (a) Prove that any square matrix $A \in M_n(\mathbb{R})$ can be decomposed into a sum of a symmetric and an anti-symmetric matrix in a unique way using the decomposition

$$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t).$$

(b) Write $\begin{pmatrix} 12 & 25 \\ -3 & 14 \end{pmatrix}$ as a sum of a symmetric and an anti-symmetric matrix.

6. (a) Let A, B, C be matrices such that AB and ABC are defined. Directly from the definitions of matrix multiplication and transpose, show that $(AB)^t = B^t A^t$. Show also that $(ABC)^t = C^t B^t A^t$.

(b) Use part (a) to show that for $A \in M_n(\mathbb{R})$ and $x, y \in \mathbb{R}^n$

$$x \cdot A^t y = y \cdot Ax \ .$$

Hint: Show that $a \cdot b = (a_1, \cdots, a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ where the right hand side denotes matrix multiplication.

7. Let $x, y \in \mathbb{R}^n$. Consider the matrix $A = xy^t \in M_n(\mathbb{R})$. Show that its elements a_{ij} are $a_{ij} = x_i y_j$. Show that $A^2 = (x \cdot y)A = (x^t y)A$, that is the scalar $(x \cdot y) = x^t y$ multiplies every entry of A to yield $A^2 = AA$. Also calculate the products AA^t and $A^t A$.

8. Solve the following system of equations by Gaussian elimination.

$$\begin{aligned} x_1 + x_2 + 2x_3 - 4x_4 &= -1 \\ -x_1 + x_2 - 2x_3 - 2x_4 &= -2 \\ -x_1 + 2x_2 - 4x_3 + x_4 &= -1 \\ 3x_1 - 3x_4 &= -3 \end{aligned}$$

9. Let $A \in M_n(\mathbb{R})$ and assume that for any $e_i \in \mathbb{R}^n$, $i = 1, \cdots, n$ the equation $Ax = e_i$ has a unique solution which we will denote by x_i (i.e., $Ax_i = e_i$). Show that the matrix B with columns $x_1, \cdots, x_n$ satisfies

$$AB = I \ .$$

Show that, moreover, $BA = I$. (Hint: for the first part consider the columns of the matrix AB.)

10. For 2×2 matrices, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc \neq 0$, verify that $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Now take $A = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$ and calculate A^{-1} in two ways: (i) using the above formula, and (ii) using the Gauss-Jordan procedure.

11. Use the Gauss-Jordan algorithm to compute the inverse of $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 7 & 2 \\ 0 & 1 & -1 \end{pmatrix}$, and check by hand that $A^{-1}A = I$.

Now use A^{-1} to express the components of the solution x of $Ax = b$ explicitly in terms of $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.