Algebraic Geometry: Assessed Homework 2

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Question 1.

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(a) Find all the elements of $\operatorname{maxSpec}(\mathbb{C}[X])$, $\operatorname{maxSpec}(\mathbb{C}[X, X^{-1}])$, and $\operatorname{maxSpec}(\mathbb{C}[X, X^{-1}, Y])$ explicitly.

(b) Let $U = \mathbb{A}^1 \backslash \{0\}$. Consider the isomorphism $\varphi : U \to U$; $a \mapsto 1/a$, and the pullback $\varphi^* : \mathbb{C}[X, X^{-1}] \to \mathbb{C}[Y, Y^{-1}]$. Compute:

$$\varphi^* \underbrace{(X^{-1})}_{f_1}, \qquad \varphi^* \underbrace{\left(2X^2 + \frac{2X^3 + 4X}{X^5}\right)}_{f_2}, \qquad \varphi^* \underbrace{(2 - X)}_{f_3}.$$

Solution. (a) We note that since $\mathbb{C}$ is a field, we know that $\mathbb{C}[X]$ is a PID. Thus all ideals in $\mathbb{C}[X]$ are of the form (f), for $f \in \mathbb{C}[X]$. Further, (f) is maximal if and only if $f \in \mathbb{C}[X] \setminus \mathbb{C}$ is irreducible. By the Fundamental Theorem of Algebra, the only irreducibles in $\mathbb{C}[X] \setminus \mathbb{C}$ are those polynomials of the form $X - \alpha$ for some $\alpha \in \mathbb{C}$. Thus

$$\operatorname{maxSpec}(\mathbb{C}[X]) = \{(f) \mid f \in \mathbb{C}[X] \setminus \mathbb{C} \text{ is irreducible}\} = \{(X - \alpha) \mid \alpha \in \mathbb{C}\}.$$

Now consider $\mathbb{C}[X, X^{-1}] \cong \mathbb{C}[X, Z]/(XZ - 1)$. We can see that $\operatorname{maxSpec}(\mathbb{C}[X, Z]/(XZ - 1))$ can be identified with the affine variety $V := \mathbb{V}(XZ - 1) \subseteq \mathbb{A}^2$, and thus the points of $\operatorname{maxSpec}(\mathbb{C}[X, Z]/(XZ - 1))$ are exactly the maximal ideals corresponding to the points $(a, 1/a) \in V$. That is,

$$\operatorname{maxSpec}(\mathbb{C}[X, Z]/(XZ - 1)) = \{(X - a, Z - 1/a) \mid a \in \mathbb{C}^\times\}.$$

Going back through the identification, in $\mathbb{C}[X, X^{-1}]$ we have that

$$X^{-1} - 1/a = -\frac{1}{aX}(X - a) \in (X - a),$$

and thus

$$\operatorname{maxSpec}(\mathbb{C}[X, X^{-1}]) = \{(X - a) \mid a \in \mathbb{C}^\times\}.$$

Then $\operatorname{maxSpec}(\mathbb{C}[X, X^{-1}, Y])$ can be identified with $V' = \mathbb{V}(XZ - 1) \subsetneq \mathbb{A}^3$, so $(x, z, y) \in V'$ if and only if $(x, z) \in V$. So $(X - a, Y - b) \in \operatorname{maxSpec}(\mathbb{C}[X, X^{-1}, Y])$ if and only if $(X - a) \in \operatorname{maxSpec}(\mathbb{C}[X, X^{-1}])$. Hence

$$\operatorname{maxSpec}(\mathbb{C}[X, X^{-1}, Y]) = \{(X - a, Y - b) \mid a \in \mathbb{C}^\times, b \in \mathbb{C}\}.$$

(b) By definition, for $f \in \mathbb{C}(X)$ we have that $\varphi^*(f) = f \circ \varphi$, and thus we have that for all $a \in U$,

$$\varphi^*(f_1)(a) = (f_1 \circ \varphi)(a) = f_1(a^{-1}) = a, \implies \varphi^*(X^{-1}) = Y.$$

We note also that the pullback of a constant function is $\varphi^*(\beta) = \beta$, and thus since φ^* is a $\mathbb{C}$-algebra homomorphism, we have that

$$\varphi^*(f_2) = 2\varphi^*(X)^2 + \frac{2\varphi^*(X)^3 + 4\varphi^*(X)}{\varphi^*(X)^5} = 2Y^{-2} + 2Y^2 + 4Y^4, \qquad \varphi^*(f_3) = 2 - \varphi^*(X) = 2 - Y^{-1}. \qquad \square$$

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Question 2. Consider the affine algebraic hypersurface $V := \mathbb{V}(Y - UX) \subseteq \mathbb{A}^3$.

(a) Prove that the projection $\pi : \mathbb{A}^3 \to \mathbb{A}^2;\ (x, y, u) \mapsto (x, u)$ restricts to an isomorphism $V \to \mathbb{A}^2$;

(b) Prove that the projection $\pi' : \mathbb{A}^3 \to \mathbb{A}^2;\ (x, y, u) \mapsto (x, y)$ does not restrict to an isomorphism $V \to \mathbb{A}^2$.

Solution. (a) By definition of a morphism of CAAVs, $\varphi := \pi|_V : V \to \mathbb{A}^2$ is a morphism of varieties. We then consider the map $\iota : \mathbb{A}^2 \to \mathbb{A}^3$ given by $(x, u) \mapsto (x, ux, u)$. By construction, $\iota(\mathbb{A}^2) \subseteq V$, and thus ι corestricts to a morphism $\psi : \mathbb{A}^2 \to V$. Now let $(x, u) \in \mathbb{A}^2$, then

$$(\varphi \circ \psi)(x, u) = \varphi(x, ux, u) = (x, u) \implies \varphi \circ \psi = \mathrm{id}_{\mathbb{A}^2}.$$

Similarly if $(x, y, u) \in V$ then by definition of V we have $y = ux$, and thus

$$(\psi \circ \varphi)(x, y, u) = \psi(x, u) = (x, ux, u) = (x, y, u) \implies \psi \circ \varphi = \mathrm{id}_V.$$

Thus φ, ψ are a pair of mutually inverse morphisms, thus they are both isomorphisms. So π restricts to an isomorphism $\varphi : V \xrightarrow{\sim} \mathbb{A}^2$.

(b) We note once again that π' restricts to a morphism $\varphi' : V \to \mathbb{A}^2$. Suppose now that $\psi' = (\psi'_1, \psi'_2, \psi'_3) : \mathbb{A}^2 \to V$ is an inverse morphism. Then for all $(x, y, u) \in V$ we must have that

$$(x, y, u) = (\psi' \circ \varphi')(x, y, u) = \psi'(x, y),$$

and thus $\psi'_3(x, y) = u = y/x$ for all $x, y \in \mathbb{C}^\times$. So $\psi'_3 = Y/X$ on all of $\mathbb{C}^\times \times \mathbb{C}^\times$, which trivially no polynomial satisfies. Thus any such map cannot be a morphism of CAAVs. Thus φ' is not an isomorphism. □

Question 3.

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$\frac{6}{8}$ (a) Prove that if $g \in \mathbb{C}[X, Y]$, then the projective closure of the variety

$$\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g}) \subseteq \mathbb{P}^2,$$

where $\tilde{g} \in \mathbb{C}[X, Y, Z]$ is the homogenisation of g.

$\frac{8}{8}$ (b) Consider the following polynomials in $\mathbb{C}[X, Y]$:

$$f_1 = X + Y + 1, \quad f_2 = X^2 + 6Y^2 + 1, \quad f_3 = X^2 + 3Y + 1, \quad f_4 = X^3 + 3XY^2 + 4.$$

Determine whether or not each of the projective closures of their respective varieties in $\mathbb{P}^2$ includes the points:

(i) $[1 : 0 : 0]$,

(ii) $[0 : 1 : 0]$,

(iii) $[0 : 0 : 1]$.

$\frac{2}{7}$ (c) Can you find a general necessary and sufficient condition on $g \in \mathbb{C}[X, Y]$ such that its homogenisation does not pass through any of these three points?

Solution. (a) We appeal to Theorem 3.28 from the lecture notes: we have that $\overline{\mathbb{V}(g)} = \mathbb{V}\left(\widetilde{(g)}\right) \subseteq \mathbb{P}^2$, where $\widetilde{(g)}$ is the homogenisation of the ideal (g). Thus, by the Nullstellensatz, it suffices to show that

$$\widetilde{(g)} = (\tilde{g}), \quad ✓$$

Short proof.

as ideals in $\mathbb{C}[X, Y, Z]$. Since $g \in (g)$, we have that $\widetilde{g} \in \widetilde{(g)}$, and thus by properties of ideals we get $(\widetilde{g}) \subseteq \widetilde{(g)}$. Now suppose that $f \in \widetilde{(g)}$, that is suppose that

$$f = \lambda_1 \widetilde{h_1} + \cdots + \lambda_n \widetilde{h_n} \qquad \widetilde{(g)}$$

↳ $f = \widetilde{\alpha g} = \tilde{\alpha}\tilde{g}$ ☒

↳ we can take $n = 1$

$f|_{U_z} = \alpha g \Rightarrow f \underset{(*)}{=} z^m \tilde{\alpha} \tilde{g} \Rightarrow f \in (\tilde{g})$

for some $\lambda_j \in \mathbb{C}[X, Y, Z]$ and some $h_j \in (g)$. Since each $h_j \in (g)$, there exists $\alpha_j \in \mathbb{C}[X, Y]$ so that $h_j = \alpha_j g$. Then, for $\mu_j = \lambda_j(X, Y, 1) \in \mathbb{C}[X, Y]$ we have that

Similar To proof of Thm 3.28

$$\begin{aligned} f(X, Y, 1) &= \lambda_1(X, Y, 1)\widetilde{h_1}(X, Y, 1) + \cdots + \lambda_n(X, Y, 1)\widetilde{h_n}(X, Y, 1) \\ &= \mu_1 h_1 + \cdots + \mu_n h_n \\ &= \mu_1 \alpha_1 g + \cdots + \mu_n \alpha_n g \\ &= \underbrace{(\mu_1 \alpha_1 + \cdots + \mu_n \alpha_n)}_{\in \mathbb{C}[X,Y]} g \end{aligned}$$

✓

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thus f is the homogenisation of some element of (g), thus $(\widetilde{g}) = \widetilde{(g)}$, as required.

(b) We use part (a) to conclude that the projective closure of $\mathbb{V}(f_j)$ passes through one of these points iff $\widetilde{f_j}$ vanishes on that point. We thus homogenise:

$$\widetilde{f_1} = X + Y + Z, \quad \widetilde{f_2} = X^2 + 6Y^2 + Z^2, \quad \widetilde{f_3} = X^2 + 3YZ + Z^2, \quad \widetilde{f_4} = X^3 + 3XY^2 + 4Z^2.$$

We can thus directly compute that $\widetilde{f_1}$ and $\widetilde{f_2}$ do not vanish on any of these three points, whereas $\widetilde{f_3}$ and $\widetilde{f_4}$ vanish only on (ii).

(c) The reason for some of these projective curves passing through these points is that their homogenisations have 'cross-terms', that is their monomial summands are not all monomials solely in either X, Y, or Z. Thus, a nonzero polynomial $g \in \mathbb{C}[X, Y]$ has projective closure not passing through points (i-iii) if and only if g has a decomposition of the form

$$g = aX^d + bY^d + c,$$

where $a, b, c \in \mathbb{C}^\times$ and $d > 0$ is a positive integer. This constant c has to be nonzero as otherwise no Z term will appear on homogenisation, and thus the corresponding curve will paass through $[0 : 0 : 1]$. □

Try $x^3 + y^3 + c + \alpha x^2 y + \beta x y + \gamma y + \delta x + \cdots$

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Question 4.

(a) Prove that $\mathbb{P}^n$ is compact with respect to the quotient Euclidean topology from $\mathbb{A}^{n+1} \setminus \{0\}$.

(b) What is the projective Zariski-closure of $\mathbb{V}(Y - \sin(X))$ in $\mathbb{P}^2$? How do you compare this to Chow's Lemma? **Hint.** In Example 3.44 we have seen that this curve is not algebraic.

Solution. (a) Let $G = (0, \infty)$ the multiplicative group of positive real numbers. Then G acts on $\mathbb{C}^{n+1} \setminus \{0\}$ naturally by scaling, and we can associate the $(n + 1)$-sphere $S^{n+1} \subseteq \mathbb{C}^{n+1} \setminus \{0\}$ with the quotient by this action. That is,

$$S^{n+1} = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{G}.$$

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Then, naturally, since $\mathbb{C}^\times = G \oplus \mu$ (with μ the multiplicative group of complex numbers of modulus 1) we must have that

$$\mathbb{P}^n = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C}^\times} = \frac{S^{n+1}}{\mu},$$

so $\mathbb{P}^n$ (under the Euclidean topology) is a quotient of S^{n+1}. We know also that each S^{n+1} is compact. Thus $\mathbb{P}^n$ must too be compact.

why? proof?

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(b) Let $V = \mathbb{V}(Y - \sin(X)) \subseteq \mathbb{A}^2$ and let $Z \subseteq \mathbb{P}^2$ denote its projective closure. We have seen that V is not algebraic, and it intersects infinitely many distinct irreducible algebraic curves. Thus its Zariski closure must be all of $\mathbb{A}^2$. Hence Z contains $\mathbb{P}^2$, so is equal to $\mathbb{P}^2$. By part (a) we have that $Z = \mathbb{P}^2$ is compact in the Euclidean topology, and thus since $V \subseteq Z \subseteq \mathbb{P}^2$ is closed it too much be compact. Thus $V \subseteq \mathbb{P}^2$ is compact and non-algebraic, so by the Chow Lemma $V = \mathbb{V}(Y - \sin(X)) \subseteq \mathbb{A}^2 \subseteq \mathbb{P}^2$ cannot be an analytic subvariety of $\mathbb{P}^2$.

why?

(*) (*Out of interest:* I've seen something about 'holomorphicity at infinity', and I can see that sin is not holomorphic at infinity – is this related?) □

it might be included in some algebraic curves

$V(y - \sin(x))$ analytic (obv. given by analytic eqns) but not compact.

(*) If it had an annalytic closure in $\mathbb{P}^2$ then closed in $\mathbb{P}^2$ ~~+ analytic~~ $\Rightarrow$ compact $\Big\}$ $\xRightarrow{\text{Chow}}$ algebraic.

Question 5.

(a) The variety of a polynomial of the form $aX + bY + cZ \in \mathbb{C}[X, Y, Z]$ (for $a, b, c \in \mathbb{C}$) is called a *line* in $\mathbb{P}^2$. Prove that any two distinct lines in $\mathbb{P}^2$ intersect exactly at one point.

(b) Assume that $C_1, C_2 \subseteq \mathbb{A}^2$ are two CAA curves.

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(i) Prove that

$$\overline{C_1 \cap C_2} \subseteq \overline{C_1} \cap \overline{C_2}.$$

closed and contains $C_1 \cap C_2$ ✓

(ii) Find two curves such that the above inclusion is strict.

Solution. (a) We first recall that we can decompose $\mathbb{P}^2$ as a disjoint union

$$\mathbb{P}^2 = U_Z \sqcup U'_Z,$$

where $U_Z = \{[x : y : 1] \in \mathbb{P}^2\}$ and $U'_Z = \{[x : y : 0] \in \mathbb{P}^2\}$. Then $U_Z \cong \mathbb{A}^2$ and $U'_Z \cong \mathbb{P}^1$. Let (a, b, c) be a 3-tuple of complex numbers, and $V_{a,b,c}$ the projective variety $V_{a,b,c} = \mathbb{V}(aX + bY + cZ) \subsetneq \mathbb{P}^2$. We understand the behaviour of lines in $\mathbb{A}^2$, so consider $V_{a,b,c} \cap U'_Z$. If $b = 0$ then this is exactly the singleton $\{[0 : 1 : 0]\}$, so suppose that $b \neq 0$. Suppose that $[x : y : 0] \in V_{a,b,c} \cap U'_Z$, then by definition

$$ax + by = 0 \implies y = -\frac{a}{b}x,$$

and thus $[x : y : 0] = [1 : -\frac{a}{b} : 0]$. We can check that this is always a solution, and thus (with some abuse of notation)

$$V_{a,b,c} \cap U'_Z = \begin{cases} [0 : 1 : 0], & \text{if } b = 0 \\ [1 : -\frac{a}{b} : 0], & \text{if } b \neq 0 \end{cases}$$

So any two distinct lines in $\mathbb{P}^2$ intersect in U'_Z at most once, and they intersect if and only if the gradient of their de-homogenisations is the same. But for two distinct lines in $\mathbb{P}^2$, they intersect in $U_Z \cong \mathbb{A}^2$ if and only if the gradients of their de-homogenisations are *different*, and we know that two lines in $\mathbb{A}^2$ can intersect at most once. Thus two distinct projective lines intersect in U_Z if and only if they do not intersect in U'_Z, and in both sets they can intersect at most once. Thus, they must intersect exactly once.

(b) (i) Closed affine algebraic curves are exactly of the form $C_i = \mathbb{V}(f_i)$ for $f_i \in \mathbb{C}[X, Y]$. We then appeal to Q3(a) and Theorem 3.28. We have that $\overline{C_i} = \mathbb{V}(\widetilde{f_i})$, and that

$$\overline{C_1 \cap C_2} = \overline{\mathbb{V}(f_1, f_2)} = \mathbb{V}\left(\widetilde{(f_1, f_2)}\right),$$

and thus by the Nullstellensatz we have that this inclusion holds if and only if

$$\widetilde{(f_1, f_2)} \supseteq (\widetilde{f_1}, \widetilde{f_2}),$$

but by definition of the homogenisation of an ideal, this holds trivially.

(ii) Consider $f_1 = X^2$, $f_2 = X^2 + Y$. Then $\widetilde{f_1} = X^2$ and $\widetilde{f_2} = X^2 + YZ$. We note that $Y = f_2 - f_1$ is homogeneous and an element of (f_1, f_2), thus is an element of $\widetilde{(f_1, f_2)}$. However, $(\widetilde{f_1}, \widetilde{f_2}) \not\ni Y$. Thus

$$\widetilde{(f_1, f_2)} \supset (\widetilde{f_1}, \widetilde{f_2}),$$

and so we must have that

$$\overline{\mathbb{V}(f_1, f_2)} \subset \overline{\mathbb{V}(f_1)} \cap \overline{\mathbb{V}(f_2)}.$$ □

or take two lines that don't intersect in $\mathbb{A}^2$.

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Question 6. (*Bonus Question*)

(a) Let Y be a closed affine algebraic variety and $U \subseteq Y$ an open subset. Prove that $\mathcal{O}_Y(U)$ is a $\mathbb{C}$-algebra.

(b) A *sheaf* $\mathcal{F}$ of rings on a topological space X consists of the following data:

I. To each open set $U \subseteq X$, it associates a ring $\mathcal{F}(U)$, whose elements are called *sections*;

II. To each inclusion $U \hookrightarrow V$ of open sets, it associates a map

$$\operatorname{res}_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$$

called the *restriction map*, satisfying $\operatorname{res}_U^U = \operatorname{id}_{\mathcal{F}(U)}$ and $\operatorname{res}_U^V \circ \operatorname{res}_V^W = \operatorname{res}_U^W$ for all open sets $U \subseteq V \subseteq W$;

III. For each collection $f_i \in \mathcal{F}(U_i)$ agreeing on the intersections, there exists a mutual lift $f \in \mathcal{F}(U)$ of all of the f_i, where $U = \bigcup_i U_i$;

IV. If $f, f' \in \mathcal{F}(U)$ are such that $\operatorname{res}_{U_i}^U f = \operatorname{res}_{U_i}^U f'$ for all i, where $\{U_i\}_i$ is an open cover of U, then $f = f'$.

Let X be an irreducible quasi-projective variety.

(i) Assume that $U \subseteq V$ are open subsets of X. Briefly explain why $f \in \mathcal{O}_X(V)$ implies that $f|_U \in \mathcal{O}_X(U)$;

(ii) Briefly explain why $\mathcal{O}_X$ forms a sheaf of rings on X.

Solution. (a) Since the set of functions $U \to \mathbb{C}$ is a $\mathbb{C}$-algebra (with 0 the zero function and 1 the constant function with value 1), we need only show that $\mathcal{O}_Y(U)$ is a $\mathbb{C}$-subalgebra, i.e. that it is both a linear subspace and a subring. Clearly $0, 1 \in \mathcal{O}_Y(U)$, and so to show this, it will suffice to show that whenever $f, g \in \mathcal{O}_Y(U)$ and $\lambda \in \mathbb{C}$, we have that $f + \lambda g, fg \in \mathcal{O}_Y(U)$. Let $p \in U$, then there exists some open neighbourhoods $U_p, V_p \subseteq U$ of p and some polynomials $A_p, B_p, C_p, D_p \in \mathbb{C}[X_1, \dots, X_n]$ such that

$$f|_{U_p} = \left.\frac{A_p}{B_p}\right|_{U_p} \qquad g|_{V_p} = \left.\frac{C_p}{D_p}\right|_{V_p}$$

Note that $W_p := U_p \cap V_p$ is also an open neighbourhood of p, since the intersection of finitely many open sets is open. Then

$$(f + \lambda g)|_{W_p} = f|_{W_p} + \lambda(g|_{W_p}) = \left.\frac{A_p}{B_p}\right|_{W_p} + \lambda\left(\left.\frac{C_p}{D_p}\right|_{W_p}\right) = \left.\left(\frac{A_p}{B_p} + \frac{\lambda C_p}{D_p}\right)\right|_{W_p} = \left.\left(\frac{A_p D_p + \lambda B_p C_p}{B_p D_p}\right)\right|_{W_p}$$

and thus $f + \lambda g \in \mathcal{O}_Y(U)$. Similarly,

$$(fg)|_{W_p} = (f|_{W_p})(g|_{W_p}) = \left(\left.\frac{A_p}{B_p}\right|_{W_p}\right)\left(\left.\frac{C_p}{D_p}\right|_{W_p}\right) = \left.\left(\frac{A_p}{B_p}\frac{C_p}{D_p}\right)\right|_{W_p} = \left.\frac{A_p C_p}{B_p D_p}\right|_{W_p}$$

and so $fg \in \mathcal{O}_Y(U)$. Thus $\mathcal{O}_Y(U)$ is a $\mathbb{C}$-subalgebra of $\{f \mid f : U \to \mathbb{C}\}$, and so is a $\mathbb{C}$-algebra.

(b) (i) Let $f \in \mathcal{O}_X(V)$. If $p \in U \subseteq V$ and f is given locally (say in neighbourhood U_p) by the rational function Q_p, then f is given by Q_p in the open neighbourhood $U_p \cap U$, and thus $f|_U$ is locally given by rational functions. That is, $f|_U \in \mathcal{O}_X(U)$.

(ii) By almost identical reasoning to Q6(a), each $\mathcal{O}_X(U)$ is a $\mathbb{C}$-algebra and thus is a ring. By Q6(b.i), $\mathcal{O}_X$ (with $\operatorname{res}_U^V = \cdot|_U$) satisfies point II (thus $\mathcal{O}_X$ is a *presheaf* of rings on X).

✓

Now suppose that $\{U_i\}_i$ is an open cover of U. If $f_i \in \mathcal{O}_X(U_i)$ agree on overlaps (when intersections exist), then we can define a function $f : U \to \mathbb{C}$ by $f(x) = f_i(x)$ whenever $x \in U_i$. This is well-defined by our assumption that the f_i behave nicely on intersections, for any $p \in U$, there is some $U_i \ni p$, which must contain an open neighbourhood of p on which $f|_{U_i} = f_i$ is given by a rational function. Thusw $f \in \mathcal{O}_X(U)$.

 ✓

So $\mathcal{O}_X$ satisfies property III. Supposing that $f, f' \in \mathcal{O}_X(U)$ are distinct, then by definition of equality of functions, there is some $x \in U$ such that $f(x) \neq f'(x)$. But since $\{U_i\}_i$ is an open cover, there is some $U_i \ni x$, and thus in particular $f|_{U_i} \neq f'|_{U_i}$, so $\mathcal{O}_X$ satisfies property IV.

Since $\mathcal{O}_X$ is a presheaf of rings on X satisfying properties III-IV, it must be a *sheaf* of rings on X, as required. $\square$
