Algebraic Geometry

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Plan for today

- Finish the proof about homogenisation
- Some important theorems in Algebraic Geometry
- Regular functions

Theorem

Let $V \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ be a closed affine algebraic variety, and $I := \mathbb{I}(V) \subseteq \mathbb{C}[x_1, \dots, x_n]$. Define the homogenised ideal

$$\tilde{I} = {\{\tilde{f} \in \mathbb{C}[x_0, \dots, x_n] : f \in I\}}.$$

Then,

$$\overline{V} = \mathbb{V}(\tilde{I}) \subseteq \mathbb{P}^n$$
.

Proof.

- If $f \in \mathbb{I}(V)$ then $\overline{V} \subseteq \mathbb{V}(\tilde{f})$.
- If $G \in \tilde{I}$, then $g := G(1, x_1, \dots, x_n) \in I$ (Why?). Do we have $\tilde{g} = G$?

Example

The twisted cubic is given by $C = \mathbb{V}(y - x^2, z - xy)$. $C \subseteq \mathbb{A}^3$ can be parametrised by $\mathbb{A}^1 \ni t \longrightarrow (t, t^2, t^3) \in \mathbb{A}^3$. Homogenisation of the generators of this ideal are $wy - x^2$ and wz - xy.

Check that

$$\mathbb{V}(wy - x^2) \cap \mathbb{V}(wz - xy) \supseteq \{ [x : y : z : w] \in \mathbb{P}^3 : w = x = 0 \}.$$

This shows that

Morphisms of Projective Varieties

Definition

Let $V\subseteq \mathbb{P}^n$ and $W\subseteq \mathbb{P}^n$ be projective algebraic varieties. We say that the map $\varphi:V\longrightarrow W$ is a *morphism of projective varieties* if for each $p\in V$, there exist

- (a) an open subset $U \subseteq V$ with $p \in U$;
- (b) homogeneous polynomials $\varphi_0, \ldots, \varphi_m : U \longrightarrow W$ of the same degree,

such that
$$\varphi_{|_{II}} = [\varphi_0 : \cdots : \varphi_m].$$

• (Exercise 3.28) Prove that $\mathbb{V}(y) \subseteq \mathbb{A}^2$ and $\mathbb{V}(y-x^3) \subseteq \mathbb{A}^2$ are isomorphic, but their projective closures are not.

Why do we care about Projective Varieties?

Theorem (Chow Lemma)

Assume that $X \subseteq \mathbb{P}^n$ is an analytic subvariety of \mathbb{P}^n , that is, X is locally given by an analytic equation. Then $X \subseteq \mathbb{P}^n$ is algebraic.

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Theorem (Bézout Theorem)

Let $f_1, f_2 \in \mathbb{C}[x_0, x_1, x_2]$ two homogeneous polynomials of degree d_1 and d_2 , respectively. Let $Z_1 = \mathbb{V}(f_1) \subseteq \mathbb{P}^2$ and $Z_2 = \mathbb{V}(f_2) \subseteq \mathbb{P}^2$, be the projective curves associated to f_1 and f_2 . Then, the number of intersection points of Z_1 and Z_2 counted with multiplicity is given by d_1d_2 .

Definition

- (a) The dimension of an irreducible projective variety is the dimension of any affine open subsets.
- (b) The *degree* of an irreducible projective variety $Y \subseteq \mathbb{P}^n$ is the number of intersection points (counted with multiplicity) of V with any linear subvariety $L \subseteq \mathbb{P}^n$ such that $\dim(L) + \dim(Y) = n$.

Quasi-Affine and quasi-projective varieties

Definition

- (a) Any open subset of an affine algebraic variety is called a *quasi-affine variety*.
- (b) Any open subset of a projective variety is called a *quasi-projective variety*.

A Basis for Zariski Topology of Affine Varieties

Recall that a basis for a topology is a collection \mathcal{B} of open subsets of a topological space X such that every open set U in X can be written as a union of elements from \mathcal{B} . Note that for any polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, the set

$$D(f) := \mathbb{A}^n \setminus \mathbb{V}(f),$$

is an open subset in \mathbb{A}^n .

- Claim 1. The collection of open sets D(f) for $f \in \mathbb{C}[x_1, \dots, x_n]$ forms a basis for Zariski on \mathbb{A}^n .
- Claim 2. If $V\subseteq \mathbb{A}^n$, is a c.a.a.v. then the open sets of the form $D(g)=V\setminus \mathbb{V}(g)$, where $g\in \mathbb{C}[V]$ form a basis for the Zariski topology on V.

Proof of claim 2

Proof.

- Any open set is of the form $V \setminus \mathbb{V}(J)$ for some $J \subseteq \mathbb{C}[V]$.
- We can find $g_1, \ldots, g_\ell \in \mathbb{C}[V]$ such that $J = (g_1, \ldots, g_\ell)$.

Observe. $(V \setminus V(g_1)) \cup (V \setminus V(g_2)) = \dots$

• Use induction.

Regular functions

Definition

Let $V\subseteq \mathbb{A}^n$, a (closed) affine algebraic variety, and $U\subseteq V$ open. A function $f:U\longrightarrow \mathbb{C}$, is called *regular at a point* $p\in V$, if there is an open neighbourhood $U'\subseteq U$, and polynomials $g,h\in \mathbb{C}[x_1,\ldots,x_n]$, such that $h(p)\neq 0$, for any $p\in U'$, and $f_{|_{U'}}(p)=\frac{g(p)}{h(p)}$. We say that f is *regular* on U if it is regular at every point of U. The set of regular functions on $U\subseteq V$ is denoted by $\mathcal{O}_V(U)$.

Examples of regular functions

(a) The function

$$f_1: \mathbb{A}^1 \setminus \{0,1\} \longrightarrow \mathbb{C}$$
 $z \longmapsto \frac{(z-2)(z-3)}{(z-1)}$

is a in $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1\setminus\{0,1\})$.

(b) Let $f_2: \mathbb{A}^1 \longrightarrow \mathbb{C}$,

$$f_2(z) = \begin{cases} \frac{(z-1)(z-3)}{(z-1)} & z \in \mathbb{A}^1 \setminus \{1\} \\ \frac{(z-2)(z-3)}{(z-2)} & z \in \mathbb{A}^1 \setminus \{2\} \end{cases}.$$

Then $f_2 \in$ We can see that the values of $f_2(z)$ coincides with $\in \mathbb{C}[\mathbb{A}^1]$.

(c) Let $g = xy - 1 \in \mathbb{C}[x, y]$. Give two examples of a regular function on $\mathcal{O}_{\mathbb{V}(g)}(\mathbb{V})(g)$ and a non-example.

Lemma

A regular function $f \in \mathcal{O}_V(U)$ is continuous when \mathbb{C} is identified with \mathbb{A}^1 .

Proof.

- It suffices to show that $f^{-1}(a)$ is closed, for $a \in \mathbb{A}^1$, because
- For every point $p \in U$ there exists U_p such that
- A set V is closed, if and only if, $V \cap U_i$ is closed in U_i , where $\bigcup U_i$ is an open cover for V.

Two remarks

Example

The twisted cubic is given by $C = \mathbb{V}(y - x^2, z - xy)$. $C \subseteq \mathbb{A}^3$ can be parametrised by $\mathbb{A}^1 \ni t \longrightarrow (t, t^2, t^3) \in \mathbb{A}^3$. Homogenisation of the generators of this ideal are $wy - x^2$ and wz - xy.

• Check that $\mathbb{V}(wy - x^2) \cap \mathbb{V}(wz - xy) = \mathbb{V}(xz - y^2) \cap \mathbb{V}(z(yw - z^2) - w(xw - yz))$ $\cup \{[x:y:z:w] \in \mathbb{P}^3: w = x = 0\}.$ This shows that $\mathbb{V}(wy - x^2) \cap \mathbb{V}(wz - xy) \neq \overline{C}.$

Remark

Homogenisation of an ideal is the ideal generated by the homogenisation of its elements.

Regular functions on a closed affine algebraic variety

Theorem

Let V be an irreducible Zariski closed subset of \mathbb{A}^n . Then

$$\mathcal{O}_V(V) = \mathbb{C}[V].$$

Proof.

- $\mathcal{O}_V(V) \supseteq \mathbb{C}[V]$.
- $\mathcal{O}_V(V) \subseteq \mathbb{C}[V]$.
 - Let $g \in \mathcal{O}_V(V)$. By definition every point $p \in V$ has a neighbourhood U_p such that on $g_{|_{U_p}} = \frac{h}{k}$ where $h, k \in \mathbb{C}[V]$ and k does not vanish on U_p .
 - By making U_p possibly smaller, we can assume that U_p is of the form D(f).
 - We can do this for every p ∈ V, and cover it with open sets, but V is compact with respect to the Zariski topology. We deduce that

- On these finitely many open sets, we can write f as.....
- The $\bigcap \mathbb{V}(k_i) = \emptyset$, therefore by Nullstellensatz....
- On $D(f_i) \cap D(f_j)$, we have $g = \dots$, therefore on on entire V.
- On $D(f_1)$, we have $g = g \cdot 1$.

• If g, G are two regular functions and g = G in $D(f_1)$, then...

Definition

Let $Y \subseteq \mathbb{P}^n$, a projective algebraic variety, and $U \subseteq Y$ open. A function $f: U \longrightarrow \mathbb{C}$, is called *regular at a point* $p \in Y$, if there is an open neighbourhood $U' \subseteq U$, and homogeneous polynomials $g, h \in \mathbb{C}[x_1, \dots, x_n]$, of the same degree, such that $h(p) \neq 0$, for any $p \in U'$ and $f_{n-1}(p) = g(p)$. We say that $f_{n-1}(p) = g(p)$ we say that $f_{n-1}(p) = g(p)$.

 $g,h\in\mathbb{C}[x_1,\ldots,x_n]$, of the same degree, such that $h(p)\neq 0$, for any $p\in U'$, and $f_{|_{U'}}(p)=\frac{g(p)}{h(p)}$. We say that f is *regular* on U if it is regular at every point of U. The set of regular functions on $U\subseteq Y$ is denoted by $\mathcal{O}_Y(U)$.

Definition

Let X, Y be two algebraic varieties (*i.e.*, affine, quasi-affine, projective or quasi-projective). A morphism $\varphi: X \longrightarrow Y$, a map such that

- (a) φ is continuous;
- (b) For any for every open set $U \subseteq Y$, and for every regular function $f \in \mathcal{O}_Y(U)$, $\varphi^*(f) = f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(U))$.

Theorem

Let X be an algebraic variety, $Y \subseteq \mathbb{A}^n$ a closed affine algebraic variety, and $\varphi: X \longrightarrow Y$ a map of sets. Then, $\varphi = (\varphi_1, \dots, \varphi_n)$ is a morphism, if and only if, for all $i, \varphi_i \in \mathcal{O}_X(X)$.

Question

How do you compare this with the isormorphisms between closed affine algebraic varieties?

Global regular functions on projective varieties

Theorem

Let Y be an irreducible Zariski closed subset of \mathbb{P}^n . Then

$$\mathcal{O}_Y(Y)=\mathbb{C}.$$

Example

Let $V = \mathbb{V}(xy - 1) \subseteq \mathbb{A}^2$, and $D(x) = \mathbb{A}^1 \setminus \{0\}$. By definition the map

$$\psi: V \longrightarrow D(x)$$
$$(x, y) \longmapsto x,$$

- ullet ψ is an isomorphism.
- $\mathcal{O}_V(D(x)) =$, since

- Any open subset $D(f) \subseteq \mathbb{A}^n$ is isomorphic to a closed subset of \mathbb{A}^{n+1} .
- Any open subset $D(f)\subseteq V=\mathbb{V}(g_1,\ldots,g_\ell)$ is isomorphic to $\subset \mathbb{A}^{n+1}$.

Obtaining \mathbb{P}^1 with gluing

We can construct \mathbb{P}^1 by gluing two copies of \mathbb{A}^1 along $\mathbb{A}^1 \setminus \{0\}$, by the map $x \longmapsto x^{-1}$. We have,

- $\xi_0: U_0 \longrightarrow X_0 := \xi_0(U_0), \ \xi_1: U_1 \longrightarrow X_1 := \xi_1(U_1), \ \text{are isomorphism. (why?)}$
- $X_{01} := \xi_0(U_0 \cap U_1) \subseteq X_0$.
- $X_{10} := \xi_1(U_1 \cap U_0) \subseteq X_1$.
- $g_{01} := \xi_1 \circ \xi_0^{-1} : X_{01} \longrightarrow X_{10}, \quad x \longmapsto y = x^{-1}.$

Note that all these sets are open subsets of \mathbb{P}^1 and isomorphic to closed affine algebraic varieties. We have

- $\mathbb{C}[X_0] = \mathcal{O}_{X_0}(X_0) = \mathbb{C}[x],$
- $\mathbb{C}[X_1] = \dots$
- $\mathbb{C}[X_{01}] = \mathcal{O}_{X_0}(X_{01}) = \frac{\mathbb{C}[x,x']}{(xx'-1)} \simeq \dots \supseteq \mathbb{C}[x].$
- $\mathbb{C}[X_{10}] = \mathcal{O}_{X_1}(X_{10}) = \dots \simeq \mathbb{C}[y, y^{-1}] \supseteq \mathbb{C}[y].$

We have now the isomorphism of $\mathbb{C}\text{-algebras}$ induced by φ :

$$g_{01}^*: \mathbb{C}[X_{10}] \longrightarrow \mathbb{C}[X_{01}]$$
 $f \longmapsto f \circ g_{01} = f(y^{-1})$

Therefore, we can also think of \mathbb{P}^1 as $X_0\simeq \mathbb{A}^1$ and $X_1\simeq \mathbb{A}^1$, where X_{01} and X_{10} are glued by the isomorphism g_{01} .

 $y \longmapsto x = y^{-1}$.

Let $[x_0: x_1: x_2]$ denote the homogeneous coordinates of the space

- \mathbb{P}^2 . It is covered by three coordinate charts:
 - U_0 corresponding to $x_0 \neq 0$, with affine coordinates
- $\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) = (a_1, a_2).$

• U_1 corresponding to $x_1 \neq 0$, with affine coordinates

• U_2 corresponding to $x_2 \neq 0$, with affine coordinates

 $(\frac{x_0}{x_1}, \frac{x_2}{x_1}) = (a_1^{-1}, \dots).$

 $(\frac{x_0}{y_0}, \frac{x_1}{y_0}) = (\dots, \dots).$

As before, let $X_i = \xi_i(U_i)$, and $X_{ij} = \xi_i(U_i \cap U_j)$. We have

- $\mathbb{C}[X_0] = \mathcal{O}_{X_0}(X_0) = \mathbb{C}[a_1, a_2],$
- $\mathbb{C}[X_{01}] = \mathcal{O}_{X_0}(X_{01}) = \mathbb{C}[....,...].$

and Since on X_1 , $a_1 \neq 0$, we can write

$$\mathbb{C}[X_1] = \mathcal{O}_{X_1}(X_1) = \mathbb{C}[a_1^{-1}, a_1^{-1}a_2].$$

As a result,

$$\mathbb{C}[X_{10}] = \mathcal{O}_{X_{10}}(X_{10}) = \mathbb{C}[\dots, a_1^{-1}, a_1^{-1}a_2].$$

The isomorphism from $X_{01} \longrightarrow X_{10}$ by

$$(a_1, a_2) \longmapsto [1 : a_1 : a_2] \longmapsto (1/a_1, a_2/a_1),$$

provides the information for gluing of $X_{01}\simeq \mathbb{C}^*\times \mathbb{C}$ and $X_{10}\simeq X_{01}\simeq \mathbb{C}^*\times \mathbb{C}$ and their corresponding coordinate rings. We can similarly understand the isomorphisms between other charts.