Algebraic Geometry Notes

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Foreword

The present notes are the lecture notes for the course of Algebraic Geometry in Bristol the second teaching block in 2023/2024. We mainly follow the book of Karen Smith et. al. "An Invitation to Algebraic Geometry ([SKKT00])" but also add other examples, exercises, and topics from other sources. We have several copies of this book in the Maths library in Queens Building. Our other references are

- Joe Harris, Algebraic Geometry, A First Course, [Har95]
- Atiyah and Macdonald, Introduction to commutative algebra, [AM69]
- Miles Ried, Undergraduate algebraic geometry, [Rei88]
- Robin Hartshorne, Algebraic geometry, [Har77]
- Shafarevich, Basic Algebraic Geometry, [Sha74]

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1 Closed Affine Algebraic Varieties

1.1 Definition and Examples

Algebraic geometry is the area of mathematics where the relation between algebraic objects, mostly an *ideal of polynomials* of several variables over a field, and the corresponding geometric object, the zero loci of the polynomials is investigated. These geometric objects are called *algebraic varieties*. One goal is to be able to read off the geometric properties of a given algebraic variety from the algebraic data on its ideal. For instance, this course will show how irreducibility, dimension, and singularities can be defined purely algebraically.

In recent years, interest in computational aspects of algebraic geometry has grown, and also we have seen many applications of algebraic geometry in all the other areas of mathematics, such as combinatorics. This has led to the subject of *combinatorial algebraic geometry* which includes *toric and tropical geometry*. Time permitting, we discuss the basics of these theories and make an effort to describe many algebro-geometric properties of some algebraic varieties from the combinatorial data.

1.2 Closed Affine Algebraic Varieties in $\mathbb{C}^n$

We first define the basic notion of *closed affine algebraic varieties*, For our course, we mainly consider the field of complex numbers as our ground field and look at the common zero loci of n-variable polynomials in $\mathbb{C}^n$, but from time to time we mention whether the theorems hold in other fields or not.

Definition 1.1. An *closed affine algebraic variety* in $\mathbb{C}^n$ or a *Zariski-closed subset* or *closed algebraic subset* of $\mathbb{C}^n$ is the common zero locus of a collection of complex polynomials in $\mathbb{C}^n$. For a collection of complex polynomials $\{f_i\}_{i\in I}$, we write

$$V = \mathbb{V}(\{f_i\}_{i\in I}) = \bigcap_{i\in I} \mathbb{V}(\{f_i\}).$$

We call V the (closed affine) algebraic variety of $\{f_i\}_{i\in I}$.

Example 1.2.

(a) (i) $\mathbb{C}^n = \mathbb{V}(0)$;

(ii) $\varnothing = \mathbb{V}(1)$;

(iii) Every point $(a_1, \ldots, a_n) \in \mathbb{C}^n$ is a closed affine algebraic variety: $\{(a_1, \ldots, a_n)\} = \mathbb{V}(\{x_1 - a_1, \ldots, x_n - a_n\})$.

(b) $\mathbb{C}^1$ *the complex line*, and $\mathbb{C}^2$ the *complex plane* are (closed affine) algebraic varieties. Note that in the courses on Complex Variables, on the contrary, $\mathbb{C}$ is called the *complex plane*. The justification here is that a plane is a 2-dimensional vectors space and $\mathbb{C}^2$ is two dimensional over $\mathbb{C}$.

(c) An *affine plane curve* is the zero set of a non-constant complex polynomial in two variables in the complex plane $\mathbb{C}^2$.

(d) The zero set of one degree-one polynomial in $\mathbb{C}^n$ is called an *affine algebraic hyperplane.*

(e) The zero set of one non-constant polynomial in $\mathbb{C}^n$ is called an *affine hypersurface*

(f) $\mathrm{SL}(n,\mathbb{C}) = \{M \in M_{n,n}(\mathbb{C}) : \det(M) = 1\}$, is a closed algebraic hypersurface in the set $M_{n,n}(\mathbb{C})$ which can be identified with $\mathbb{C}^{n^2}$.

(g) For $k \leq n$, the set of matrices $\boldsymbol{A}_k := \{M \in M_{n,n}(\mathbb{C}) : M \text{ has rank at most } k\ \}$ is a closed affine algebraic variety of $\mathbb{C}^{n^2}$. Since $A \in \boldsymbol{A}_k$, if and only if, the determinant of all of the $(k+1)\times(k+1)$ submatrices of A vanishes. Therefore, $\boldsymbol{A}_k$ is the affine algebraic variety that corresponds $(C^n_{k+1})^2$ polynomial equations. Here, $C^n_k = \frac{n!}{k!(n-k)!}$, equals the number of k-subsets of an n-set.

(h) $y - \sin(x)$ a series and not a polynomial, therefore we do not expect $V = \mathbb{V}(y - sin(x))$ to be an algebraic variety. We can prove later that indeed there is no polynomial whose zero locus is V.

$\triangle$

Remark 1.3. An important tool in differential geometry is the *partition of unity*, where we use functions that are smooth but not analytic. So there is no chance for them to become polynomials and we do not have them in algebraic geometry.

Exercise 1.4. Show that every closed affine algebraic variety in $\mathbb{C}^n$ is closed in the Euclidean topology.

Exercise 1.5. Show that the disc $\{x \in \mathbb{C} : |x| \leq 1\}$ is not an algebraic variety of $\mathbb{C}$.

1.3 The Zariski Topology on $\mathbb{C}^n$

We intend to define a topology on $\mathbb{C}^n$ where the closed sets are the (closed affine) algebraic varieties. We verify immediately after stating the definition that these closed sets do indeed give rise to a topology.

Definition 1.6. The *Zariski topology* on $\mathbb{C}^n$ is a topology whose open sets are given by complements of closed affine algebraic varieties in $\mathbb{C}^n$.[1] The set $\mathbb{C}^n$ endowed with its Zariski topology is denoted by $\mathbb{A}^n$, and it is called the *affine n-space.*

Proposition 1.7. The affine n-space $\mathbb{A}^n$ is a topological space.

Proof. Let $\mathcal{O}$ be the collection of Zariski open sets. We need to check that

- We have that $\varnothing, \mathbb{C}^n \in \mathcal{O}$, as we note that their complements $\mathbb{C}^n = \mathbb{V}(0)$, and $\varnothing = \mathbb{V}(1)$ are indeed closed.
- The union of any collection of open sets in $\mathcal{O}$ is in $\mathcal{O}$. Equivalently, for the intersection of any collection of algebraic varieties $V_i = \mathbb{V}(\{f_{i,j}\}_{j\in J_i})$, $i \in I$, we have
$$\bigcap_{i\in I} V_i = \bigcap_{i\in I} \mathbb{V}(\{f_{i,j}\}_{j\in J_i}) = \mathbb{V}(\bigcup_{i\in I}\{f_{i,j}\}_{j\in J_i}).$$

[1] For the basics of general topology see its Wikipedia page.

- The intersection of any finite number of Zariski open sets is indeed open. To see this, by induction, it suffices to check that the intersection of any two algebraic varieties is an algebraic variety:

$$\mathbb{V}(\{f_i\}_{i\in I}) \cup \mathbb{V}(\{f_j\}_{j\in J}) = \mathbb{V}(\{f_i f_j\}_{(i,j)\in I\times J}).$$

$\square$

Example 1.8. (a) The Euclidean closed set $\left\{\frac{1}{n}\right\}_{n\in\mathbb{Z}_{>0}} \cup \{0\} \subseteq \mathbb{C}^1$ is not a Zariski closed set in $\mathbb{A}^1$. In fact, if $V \subsetneq \mathbb{A}^1$ is closed, it must be a finite set.

(b) All the non-empty Zariski open sets are dense in $\mathbb{A}^n$. In fact, we can even show that all the proper Zariski closed subsets in $\mathbb{A}^n$ are of the Lebesgue measure zero.

(c) Recall $\boldsymbol{A}_k$ from Example 1.2.7 and note that

$$(\boldsymbol{A}_{k-1})^c = \{n \times n - \text{matrices with rank at least } k\}$$

is a Zariski open set in $\mathbb{A}^{n^2}$.

(d) The *twisted cubic* given by $\mathbb{V}(x^2 - y, x^3 - z) = \mathbb{V}(x^2 - y) \cap \mathbb{V}(x^3 - z)$ is a closed affine *algebraic curve*. Note that the twisted cubic can be parametrised by $(t, t^2, t^3) \in \mathbb{C}^3$ for $t \in \mathbb{C}$.

$\triangle$

Exercise 1.9. Prove that polynomials are continuous functions with respect to the Zariski topology.

Exercise 1.10. Show that the union of infinitely many algebraic varieties is not necessarily an algebraic variety. What goes wrong in the last part of the proof of Proposition 1.7 if we take an infinite union?

Exercise 1.11. Show that the Zariski topology in $\mathbb{A}^2$ does not coincide with the product topology in $\mathbb{A}^1 \times \mathbb{A}^1$. Hint. Prove that $V(x - y)$ is Zariski closed in $\mathbb{A}^2$, but not in $\mathbb{A}^1 \times \mathbb{A}^1$ equipped with the *product topology*. Convince yourself that Euclidean product topology in $\mathbb{C}^1 \times \mathbb{C}^1$ coincides with the Euclidean topology on $\mathbb{C}^2$.

Exercise 1.12. Without using the Cayley–Hamilton Theorem, prove that all the matrices satisfying their characteristic polynomials form a Zariski-closed subset of $\mathbb{C}^{n\times n}$.

2 Algebraic Foundations

2.1 A bit of Algebra

We recall some basic definitions from Ring Theory and Commutative Algebra. We trust that the reader knows the definition of ring, field, modules, and vector spaces on their own. Gladly, all the rings in this course are commutative and contain multiplicative identity element 1. Let R be a ring, recall that a nonempty subset $I \subseteq R$ is called an *ideal*, if for all $a, b \in I$ and $r \in R$, we have

$$a + b \in I, \ ra \in I.$$

For any subset $J \subseteq R$, the *ideal generated* by J is given by

$$(J) = \bigcap \big\{\text{all ideals in } R \text{ containing } J\big\},$$

Note that (J) is an ideal, the intersection of any collection of ideals is still an ideal. The reader can verify that (J) is all the linear combinations of elements of J with coefficients in R, *i.e.*,

$$(J) = \big\{r_1 j_1 + \cdots + r_k j_k : \text{for any positive integer } k, \, j_i \in J, \, r_i \in R\big\}.$$

An ideal I is *finitely generated*, if there are finitely many elements of $f_1, \ldots, f_k \in I$ that

$$(f_1, \ldots, f_k) = (\{f_1, \ldots, f_k\}) = J.$$

Given an ideal $I \subseteq R$, we can define the *quotient ring* R/I. The elements of R/I are the cosets of I. The map $\pi : R \longrightarrow R/I$, $r \longmapsto r + I$, is a surjective ring homomorphism. Note under the map π, an ideal $K \subseteq R$, which contains I is mapped to an ideal of R/I. Conversely, if $J \subseteq R/I$, is an ideal, then the preimage $\pi^{-1}(J)$ is an ideal in R containing I. As a result:

Proposition 2.1. The map π induces one-to-one order preserving correspondence between the ideals of R containing I and the ideals of R/I.

Recall also that:

Definition 2.2.

(a) A *zero-divisor* in a ring is an element a for which there exists $b \neq 0$, in R, such that $ab = 0$. An *integral domain* is a ring with no non-zero zero-divisors.

(b) A *field* is a ring where every non-zero element is a *unit*, *i.e.*, it has a multiplicative inverse.

(c) An ideal $\mathfrak{p} \subsetneq R$ is called *prime,* if for $a, b \in R$

$$ab \in \mathfrak{p} \implies a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}.$$

(d) An ideal $\mathfrak{m} \subsetneq R$ is called *maximal,* if the only ideal strictly containing it is the unit ideal R.

(e) The *radical* of an ideal $I \subseteq R$ is defined as

$$\sqrt{I} := \{a \in R : a^n \in I, \text{ for some } n > 0 \}.$$

An ideal $J \subseteq R$ is called *radical* if $\sqrt{J} = J$.

Exercise 2.3. Verify that

(a) $\mathfrak{p}$ is a prime ideal in $R \iff R/\mathfrak{p}$ is an integral domain.

(b) $\mathfrak{m}$ is a maximal ideal in $R \iff R/\mathfrak{m}$ is a field.

(c) A maximal ideal is a prime ideal.

(d) The radical of an ideal is also an ideal.

(e) I is a radical ideal in $R \iff R/I$ is *reduced*, *i.e.*, it has no non-zero nilpotent elements.

Assume that $\alpha : Q \longrightarrow R$, is a ring homomorphism, that is, it preserves sums, products, and maps any multiplicative or additive identity element in Q to a multiplicative or additive identity element in R, respectively. If $\mathfrak{p} \subseteq R$ is prime, then $\alpha^{-1}(\mathfrak{p}) \subseteq Q$ is also prime. To see this, note that

$$\bar{\alpha} : Q \longrightarrow R/\mathfrak{p},$$

is a ring homomorphism, and we have

$$\ker \bar{\alpha} = \alpha^{-1}(\mathfrak{p})$$

and $Q/\ker \bar{\alpha}$ is isomorphic to a subring of $R/\mathfrak{p}$. As a result, $Q/\ker \bar{\alpha}$ is an integral domain, and $\alpha^{-1}(\mathfrak{p})$ is also prime. When $\mathfrak{m} \subseteq R$ is maximal, $\alpha^{-1}(\mathfrak{m})$ is certainly prime, however, not necessarily maximal. (Example: $Q := \mathbb{Z}, R := \mathbb{Q}, \mathfrak{m} = (0)$.)

Let $(R, +, .)$ be a ring and $\mathbb{K}$ be a field. If $(R, +)$ is also a $\mathbb{K}$-vector space, then R is called a $\mathbb{K}$-*algebra*.

Example 2.4.

(a) Any ring R containing $\mathbb{C}$ as a subring is a $\mathbb{C}$-algebra. A $\mathbb{C}$-algebra R is also a $\mathbb{C}$-vector space. In simple words, the difference between understanding R as a $\mathbb{C}$-algebra and a $\mathbb{C}$-vector space is that in a vector space we only add elements of R and multiply them by $\mathbb{C}$, whereas for $\mathbb{C}$-algebra, we multiply the elements of R to each other as well.

(b) The $R = \mathbb{C}[x_1, \dots, x_n]$ of complex polynomials in the n variables $x_1, \dots, x_n$, with the usual addition and multiplication of polynomials, is a ring. R contains $\mathbb{C}$ as a subring, and it is naturally a $\mathbb{C}$-algebra.

$\triangle$

Exercise 2.5. We know that $(\mathbb{R}^3, +)$ is an $\mathbb{R}$-vector space. Prove that $(\mathbb{R}^3, +, \times)$, where $\times$ is the cross product, is not an $\mathbb{R}$-algebra.

Analogous to the case of the rings and ideals one defines:

Definition 2.6. Let R be a $\mathbb{C}$-algebra and $J \subseteq R$,

(a) The $\mathbb{C}$-algebra *generated by* J is defined as the intersection of all $\mathbb{C}$-algebras in R containing J. Note that the $\mathbb{C}$-algebra generated by J is the set of all polynomials in the elements of J and coefficients in $\mathbb{C}$.

(b) An algebra is called finitely generated if it can be generated by some finite set.

(c) Given two $\mathbb{C}$-algebras R and S, a $\mathbb{C}$-*algebra homomorphism* Φ is both

- a ring homomorphism $\Phi : R \longrightarrow S$, and
- a linear homomorphism over $\mathbb{C}$, that is, $\Phi(\lambda r) = \lambda\Phi(r)$, for $\lambda \in \mathbb{C}, r \in R$, and Φ preserves the addition operation.

Example 2.7.

(a) Consider the set $\{x, y\} \subseteq \mathbb{C}[x, y]$. The ideal generated $\{x, y\}$ is the set of all polynomials with the constant term equal to zero: $(\{x, y\}) = xP_1(x, y) + yP_2(x, y)$ for the polynomials $P_1, P_2 \in \mathbb{C}[x, y]$. However, the $\mathbb{C}$-algebra generated by $\{x, y\}$ is exactly $\mathbb{C}[x, y]$.

(b) For every $\mathbb{C}$-algebra R and an ideal $I \subseteq R$, R/I has a natural $\mathbb{C}$-algebra structure.

(c) Similar to the fact that every linear map can be totally understood by its action on a set of basis, a $\mathbb{C}$-algebra homomorphism can be completely determined by its action on a set of generators. For instance, we can define a homomorphism $\Phi : \frac{\mathbb{C}[x,y]}{(x^2+y^3)} \longrightarrow \mathbb{C}[z]$, is determined by $\Phi(\bar{x})$ and $\Phi(\bar{y})$, but we must have

$$\Phi(0) = \Phi(\bar{x})^2 + \Phi(\bar{y})^3 = 0.$$

△

2.2 Hilbert's Basis Theorem

In this section, we show that every algebraic variety can be described as the zero loci of finitely many polynomials.

Definition 2.8. A ring R is *Noetherian* if all its ideals are finitely generated. Equivalently, the ideals of R satisfy the *ascending chain* condition[2], that is, for any sequence of ideals

$$I_1 \subseteq I_2 \subseteq \cdots ,$$

there exists an integer r such that $I_r = I_{r+1} = \cdots$.

We leave it to the reader to verify that the above two definitions of Noetherian property are equivalent. Note that every field is Noetherian since it has only the ideals (0) and (1). Therefore, the following theorem immediately implies that

$$\mathbb{C}[x_1, \ldots, x_{n-1}, x_n] = (\mathbb{C}[x_1, \ldots, x_{n-1}])[x_n]$$

is a Noetherian ring.

[2] I prefer to call this the *stable ascending chain condition* instead, but one doesn't argue with Hartshorne.

Theorem 2.9 (Hilbert's Basis Theorem)**.** Let R be a ring.

$$R \text{ is Noetherian } \implies R[x] \text{ is Noetherian.}$$

Proof. Let $J \subseteq R[x]$ be an ideal. We prove that J is finitely generated. We define the following ideals I_i of R given by the coefficients of the leading terms of polynomials of degree i in J. *I.e.*,

$$I_i := \{a_i \in R : \text{there exists } f = a_i x^i + a_{i-1}x^{i-1} + \cdots \in J\}.$$

We can check that

- I_i are indeed ideals in R;
- The ideals $I_i \subseteq R$ form an ascending chain of ideals

$$I_0 \subseteq I_1 \subseteq \cdots.$$

Since R is Noetherian, the ascending chain of ideals stabilizes, that is, there exists an integer r such that

$$I_0 \subseteq I_1 \subseteq \cdots \subseteq I_r = I_{r+1} = \cdots.$$

For $i = 0, \ldots, r$, we can choose the generators $a_{i1}, \cdots, a_{in_i}$ for each I_i. Now for each $i = 0, \ldots, r$ and $j = 1, \ldots, n_i$, choose $f_{ij} \in J$ with the leading coefficients a_{ij}. We claim that $\{f_{ij}\}_{i,j}$ generates J: for $g \in J$ given by

$$g = bx^m + \text{lower order terms},$$

we can write

$$b = \sum_k c_{\ell k} a_{\ell k}\,, \qquad \text{for some } c_{\ell k} \in R.$$

Here $\ell = m$ when $m \leq r$, otherwise $\ell = r$. Now we conclude the proof by induction. The polynomial

$$g_1 = g - x^{m-\ell} \sum c_{\ell k} f_{\ell k},$$

has a lower degree than g. $g_1 \in J$ since it is a different of two element of the ideal J. In turn, by an induction hypothesis on the degree of the polynomials, g_1 can be written as a linear combination of f_{ij} with coefficients in $R[x]$ and therefore, $g \in (f_{ij})_{i,j}$.

□

2.3 The Ideal of a Variety and Nullstellensatz

For a subset $A \subseteq \mathbb{A}^n$, we define the *ideal corresponding to* A, denoted by $\mathbb{I}(A)$, as the set

$$\mathbb{I}(A) := \{f \in \mathbb{C}[x_1, \ldots, x_n] : f(x) = 0 \text{ for all } x \in A\}.$$

That is, the set of all the polynomial functions vanishing on A. It is clear that $\mathbb{I}(A)$ is an ideal in $\mathbb{C}[x_1, \ldots, x_n]$. For every A, $\mathbb{I}(A)$ is, in fact, radical: If $f^n(x) = 0$, for some integer $n > 0$ and all $x \in V$, then $f(x) = 0$, for all $x \in A$.

By Hilbert's Basis Theorem 2.9,

$$I = \mathbb{I}(A) = (f_1, \dots, f_k),$$

for some positive integer k, and $f_i \in \mathbb{C}[x_1, \dots, x_n]$. Now, if we moreover assume that V is a closed affine algebraic variety, then

$$\mathbb{V}(\mathbb{I}(V)) = V.$$

To see this,

- If $x \in V$, then $f(x) = 0$ for all $f \in \mathbb{I}(V)$ and $x \in \mathbb{V}(\mathbb{I}(V))$.
- If $x \in \mathbb{V}(\mathbb{I}(V))$, then $f(x) = 0$ for all $f \in \mathbb{I}(V)$. But V is an algebraic variety and is given by $V = \mathbb{V}(\{g_i\}_{i \in A})$ for some $g_i \in \mathbb{C}[x_1, \dots, x_n]$. Thus, $g_i \in \mathbb{I}(V)$, and $\mathbb{V}(\mathbb{I}(V)) \subseteq \mathbb{V}(\{g_i\}_{i \in A})$.

Now we can use Hilbert's Basis Theorem 2.9 to show that

> "Any closed affine algebraic variety is an intersection of finitely many closed affine algebraic hypersurfaces."

Exercise 2.10. Prove the preceding statement.

Exercise 2.11. Prove that for two ideals $I, J \subseteq \mathbb{C}[x_1, \dots x_n]$,

(a) $\mathbb{V}(I \cap J) = \mathbb{V}(I) \cup \mathbb{V}(J)$;

(b) $\mathbb{V}(I + J) = \mathbb{V}(I) \cap \mathbb{V}(J)$.

Now a natural question arises: Do we also have the equality $\mathbb{I}(\mathbb{V}(I)) = I$? To experiment, let us take $V = \{1\} \subseteq \mathbb{C} = \mathbb{A}^1$. On the one hand, for the ideal $I := (x-1)^2$, we have $\mathbb{V}(I) = \{1\}$. On the other hand, $f(x) = x - 1 \in \mathbb{I}(V)$, but $f \notin I$. The following theorem answers our question:

Theorem 2.12 (Hilbert's Nullstellensatz)**.** For every ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$,

$$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}.$$

In particular, if I is radical, then

$$\mathbb{I}(\mathbb{V}(I)) = I.$$

In German *Nullstellen* = zero set + *Satz* = theorem. Miles Reid in [Rei88] recommends that one should stick to the German word if they don't want to be considered an "ignorant peasant".

$$\begin{tikzcd}
\boxed{\left\{\text{closed affine algebraic varieties in } \mathbb{A}^n \right\}} \arrow[d, bend left, "\mathbb{I}"] \\
\boxed{\left\{\text{radical ideals in } \mathbb{C}[x_1, \dots, x_n]\right\}} \arrow[u, bend left, "\mathbb{V}"]
\end{tikzcd}$$

Note that $V \subseteq W \subseteq \mathbb{A}^n$, if and only if, $\mathbb{I}(W) \subseteq \mathbb{I}(V)$. Thus, Hilbert's correspondence between ideals and varieties is *inclusion-reversing*.

Remark 2.13. Hilbert's Nullstellensatz, in fact, holds true for every algebraically closed field, and this condition is necessary. (Example: $I = (x^2 + 1) \subseteq \mathbb{R}[x]$, is radical, but $\mathbb{V}(I) = \varnothing$).

Now observe that, on the one hand, the smallest, with respect to inclusion, closed affine algebraic varieties are single points. Therefore, the correspondence implies that the points correspond to the maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$. On the other hand, the ideals of the form $\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$ are maximal: define the surjective ring homomorphism $\delta_a : \mathbb{C}[x_1, \ldots, x_n] \longrightarrow \mathbb{C}$, $\delta_a(f) := f(a)$. Then $\mathfrak{m}_a = \ker(\delta_a)$. As $\mathbb{C}$ is a field, $\mathfrak{m}_a$ has to be maximal. By definition, $\mathfrak{m}_a \subseteq \mathbb{I}(\{a\})$, $\mathbb{I}(\{a\}) \neq \mathbb{C}[x_1, \ldots, x_n]$, therefore, $\mathfrak{m}_a = \mathbb{I}\{a\}$. As a result,

$$a = (a_1, \ldots, a_n) \in \mathbb{A}^n \longleftrightarrow \mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n).$$

Exercise 2.14. Show that any radical ideal I in $\mathbb{C}[x_1, \ldots, x_n]$ is the intersection of all maximal ideals containing I.

2.4 Irreducibility and Dimension

We now notice that the 'indecomposibility' of prime ideals has a geometric meaning.

Definition 2.15. Let $V \subseteq \mathbb{A}^n$ be a Zariski-closed subset. V is called *irreducible* if it cannot be expressed as the union of two proper subsets $V = V_1 \cup V_2$, such that V_1 and V_2 are closed in $\mathbb{A}^n$. Equivalently, if V is irreducible, and $V = V_1 \cup V_2$, for V_1, V_2 closed subsets of $\mathbb{A}^n$, then $V = V_1$ or $V = V_2$.

Theorem 2.16. Let $V \subseteq \mathbb{A}^n$, be a (closed affine) algebraic variety. Then,

$$V \text{ is irreducible} \iff \mathbb{I}(V) \text{ is prime.}$$

Remark 2.17. An equivalent condition for an ideal $\mathfrak{p} \subsetneq R$ to be prime is that for all ideals $J, K \subseteq R$,

$$JK \subseteq \mathfrak{p} \implies J \subseteq \mathfrak{p} \text{ or } K \subseteq \mathfrak{p}.$$

Proof of Theorem 2.16. Assume that $\mathbb{I}(V)$ is prime, and $V \subseteq V_1 \cup V_2$ with $V_1, V_2 \subseteq \mathbb{A}^n$ closed. Then

$$\mathbb{I}(V_1)\,\mathbb{I}(V_2) \subseteq \mathbb{I}(V_1) \cap \mathbb{I}(V_2) = \mathbb{I}(V_1 \cup V_2) \subseteq \mathbb{I}(V).$$

By Remark 2.17, we obtain $\mathbb{I}(V_1) \subseteq \mathbb{I}(V)$ or $\mathbb{I}(V_2) \subseteq \mathbb{I}(V)$. So $V \subseteq V_1$ or $V \subseteq V_2$, that is, V is irreducible. For the converse, assume that V is irreducible, and let $fg \in \mathbb{I}(V)$. Then $V \subseteq \mathbb{V}(fg) = \mathbb{V}(f) \cup \mathbb{V}(g)$. Hence, $V \subseteq V(f)$ or $V \subseteq \mathbb{V}(g)$, by irreducibility of V. As a result, $f \in \mathbb{I}(V)$ or $g \in \mathbb{I}(V)$. That is to say $\mathbb{I}(V)$ is prime. $\square$

Corollary 2.18. $\mathbb{A}^n$ is irreducible.

Proof. The ring $A := \mathbb{C}[x_1, \ldots, x_n]$ is an integral domain. Therefore $(0) \subseteq A$ is a prime ideal and by Theorem 2.16, $\mathbb{V}(0) = \mathbb{A}^n$ is irreducible. $\square$

In summary, for $A := \mathbb{C}[x_1, \ldots, x_n]$

$$\begin{array}{ccccc} \{\text{points in } \mathbb{A}^n\} & \subseteq & \{\text{ irreducible varieties in } \mathbb{A}^n\} & \subseteq & \{\text{varieties in } \mathbb{A}^n\} \\ \updownarrow & & \updownarrow & & \updownarrow \\ \{\text{maximal ideals in } A\} & \subseteq & \{\text{ prime ideals in } A\} & \subseteq & \{\text{ radical ideals in } A\}. \end{array}$$

Now we state the following decomposition theorem.

Proposition 2.19 ([Har77, Proposition 1.5])**.** Let $V \subseteq \mathbb{A}^n$ be a (closed affine) algebraic variety, then there are finitely many irreducible algebraic varieties $V_i \subseteq \mathbb{A}^n$, such that

$$V = V_1 \cup \cdots \cup V_r.$$

Proof. Assume that V is not a finite union of algebraic varieties, then $V = V_1 \cup V_2$ such that at least one of V_1 or V_2 is a union of infinitely many algebraic varieties. Iterating this process, we can find an infinite chain of descending chain of algebraic varieties

$$V \supsetneq V_1' \supsetneq V_2' \supsetneq \cdots,$$

that does not stop. However, this implies that we have a non-stopping chain of ascending ideals

$$\mathbb{I}(V) \subsetneq \mathbb{I}(V_1') \subsetneq \mathbb{I}(V_2') \subsetneq \cdots \subseteq \mathbb{C}[x_1, \ldots, x_n],$$

which is a contradiction to the Noetherian property of $\mathbb{C}[x_1, \ldots, x_n]$. □

Exercise 2.20. Assume that $V = V_1 \cup \cdots \cup V_r$ is a decomposition of V into irreducible algebraic varieties, with the property that $V_i \subseteq V_j \implies i = j$. Then, V_i, up to re-ordering, are uniquely determined.

The above properties allow for defining the *dimension* of closed affine algebraic varieties:

Definition 2.21. If $V \subseteq \mathbb{A}^n$, is irreducible closed affine algebraic variety, then the dimension of V, denoted by $\dim(V)$, is the largest integer d such that there is a chain

$$V = V_d \supsetneq V_{d-1} \supsetneq \cdots \supsetneq V_0 = \{\mathrm{pt}\},$$

where $V_i \subseteq V$ are irreducible algebraic subvarieties of V. The dimension of a closed affine algebraic variety is the maximal dimension of its irreducible varieties which are subsets of V.

Example 2.22. $\dim(\mathbb{A}^1) = 1$. △

We accept the following theorems:

Theorem 2.23 (see [AM69, Chapter 11])**.** $\dim(\mathbb{A}^n) = n$.

Theorem 2.24 ([Har77, Proposition 1.13])**.** A closed affine algebraic variety $V \subseteq \mathbb{A}^n$ has dimension $n-1$, if and only if, it is the zero set $\mathbb{V}(f)$ of a single non-constant irreducible polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$.

Remark 2.25. If an algebraic variety V is smooth, the above definition coincides with the definition of the dimension of V as a smooth (complex) manifold.

2.4.1 An application: Cayley–Hamilton Theorem

As an application, we can prove the Cayley–Hamilton Theorem, borrowed from the beautiful notes of Edixhoven and Taelman [ET09], which I have uploaded on Blackboard.

Theorem 2.26 (Cayley–Hamilton)**.** Let A be an $n \times n$ matrix over $\mathbb{C}$, and I be the identity matrix, and $\chi_A(\lambda) := \det(A - \lambda I) \in \mathbb{C}[\lambda]$. Then $\chi_A(A) = 0 \in M_{n\times n}(\mathbb{C})$.

Lemma 2.27. If a matrix A has distinct eigenvalues, then $\chi_A(A) = 0$.

Proof. We know from linear algebra that eigenvectors corresponding to distinct eigenvalues are linearly independent. Therefore, A is diagonalisable. Moreover, if Q is the invertible matrix whose columns are n linearly independent eigenvectors, then $Q^{-1}AQ$ is a diagonal matrix. Now it's easy to check that we have the matrix equation $\chi_A(A) = \chi_A(Q^{-1}AQ) = 0_{n\times n}$. □

Proof of Theorem 2.26. Let

- V_1 be the set of all matrices in $M_{n\times n}(\mathbb{C}) \simeq \mathbb{A}^{n\times n}$ consisting of all the matrices satisfying $\chi_A(A) = 0$. V_1 can be regarded as a closed affine algebraic variety in $\mathbb{A}^{n\times n}$, given by $n \times n$ polynomial equations in the $n \times n$ variables considered as entries of a matrix A. V_1 contains all the matrices with distinct eigenvalues by Lemma 2.27.

- V_2 be the set of all matrices such that their eigenvalues are not all distinct, *i.e.*, with some eigenvalues of multiplicity more than 1. V_2 is also a closed affine algebraic variety: $A \in V_2$, if and only if, the *discriminant* of the polynomial $\chi_A(\lambda)$ vanishes.

In consequence, have that $\mathbb{A}^{n\times n} = M_{n\times n}(\mathbb{C}) = V_1 \cup V_2$. However, $\mathbb{A}^{n^2}$ is irreducible. Since there exist matrices with distinct eigenvalues, we cannot have $\mathbb{A}^{n^2} \subseteq V_2$,. Therefore

$$\mathbb{A}^{n^2} = V_1.$$

□

Exercise 2.28. Verify the Cayley–Hamilton Theorem for any field k.

2.5 Morphisms of Closed Affine Algebraic Varieties

A polynomial map

$$\begin{aligned} \varphi : \mathbb{A}^n &\longrightarrow \mathbb{A}^m, \\ a &\longmapsto \varphi(a) = (\varphi_1(a), \dots, \varphi_m(a)), \end{aligned}$$

where all φ_i's are polynomials in n variables. More generally, we have the notion of *morphism of algebraic varieties*

Definition 2.29. Let $V \subseteq \mathbb{A}^n$, and $W \subseteq \mathbb{A}^m$ be two closed affine algebraic varieties.

(a) A map $\varphi : V \longrightarrow W$, is called a morphism of algebraic varieties if there exists a polynomial mapping $\Phi : \mathbb{A}^n \longrightarrow \mathbb{A}^m$, such that $\varphi = \Phi_{|_V}$. In other words, morphisms of algebraic varieties are restrictions of polynomial maps between the ambient spaces.

(b) A morphism of algebraic varieties $\varphi : V \longrightarrow W$ is an *isomorphism*, if has an inverse, *i.e.* there exists a morphism of algebraic varieties $\psi : W \longrightarrow V$, such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity maps of V and W, respectively.

(c) Two closed affine algebraic varieties V and W are called *isomorphic*, if there is an isomorphism between them. In this case, we write $V \simeq W$.

Example 2.30. (a) The constant map $\varphi_1 : \mathbb{A}^1 \longmapsto \{a\} \subseteq \mathbb{A}^1$, defined by $x \mapsto a$, is a morphism of algebraic varieties, but it is not an isomorphism since it is not bijective.

(b) $V := \mathbb{V}(y - x^2) \subseteq \mathbb{A}^2$ and $\mathbb{A}^1$ are isomorphic. The inverse to the morphism of algebraic varieties $\varphi_3 : \mathbb{A}^1 \longrightarrow V$, $t \longmapsto (t, t^2)$ is given by $\psi : \mathbb{A}^2 \longrightarrow \mathbb{A}^1$, $(x, y) \longmapsto x$.

(c) A (generalised) complex Hénon map is given by $\varphi_2 : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $(x, y) \longmapsto (y, p(y) - \delta x)$, where $p(y)$ is a polynomial of degree $d \geq 2$ and $\delta \in \mathbb{C}$ is a non-zero constant. One can check that

$$\varphi_2^{-1}(x, y) = \big(\frac{p(x) - y}{\delta}, x\big),$$

is the inverse for f. Hénon maps are an important class of *automorphisms* of the complex plane *i.e.* isomorphism from $\mathbb{C}^2$ to itself, and have been intensively studied in complex dynamical systems. In particular, Hénon maps are isomorphisms of $\mathbb{A}^2 \longrightarrow \mathbb{A}^2$.

△

Exercise 2.31. (a) Show that a morphism $\varphi : V \longrightarrow W$ is a continuous map.

(b) Show, by an example, that a morphism is not necessarily *closed*. Recall that a map is called closed, if and only if, it maps closed sets to closed sets.

Remark 2.32. If $V \subseteq \mathbb{A}^n$, then the Zariski topology of $\mathbb{A}^n$ induces a Zariski topology on V by declaring the open sets in V to be $O \cap \mathbb{A}^n$, where $O \subseteq \mathbb{A}^n$ is an open set. Similarly, the closed sets in V would be of the form $Z \cap V$, for Zariski closed sets $Z \subseteq \mathbb{A}^n$. If $W \subseteq V$ is such a Zariski closed set of V, we call W an *a closed affine algebraic subvariety* of V.

Remark 2.33. We can define a general irreducible X topological space as follows: X is called irreducible, if any decomposition of $X = X_1 \cup X_2$, where X_1 and X_2 are closed in X, implies that $X \subseteq X_1$ or $X \subseteq X_2$.

Exercise 2.34. Check that the above definition generalises the Definition 2.15.

2.6 The Coordinate Ring

Definition 2.35. For a given algebraic variety $V \subseteq \mathbb{A}^n$, the *coordinate ring* of V, denoted by $\mathbb{C}[V]$, is defined by

$$\mathbb{C}[V] = \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbb{I}(V)}.$$

Remark 2.36. In many books such as [Har77, Har95], the coordinate ring of V is denoted by $A[V]$. However, we stick to $\mathbb{C}[V]$ as we mainly follow [SKKT00].

We have that

> "$\mathbb{C}[V]$ is a $\mathbb{C}$-algebra and can be viewed as the set of all the polynomial functions in $\mathbb{C}[x_1, \dots, x_n]$ restricted to V."

To see the above statement, note that $\mathbb{C}[V]$ is a $\mathbb{C}$-algebra and that the restriction

$$\begin{aligned} \mathbb{C}[x_1, \dots, x_n] &\longrightarrow \mathbb{C}[x_1, \dots, x_n]_{|V}, \\ f &\longmapsto f_{|V}. \end{aligned}$$

is a surjective $\mathbb{C}$-algebra homomorphism with kernel $\mathbb{I}(V)$.

Example 2.37. (a) If $W = \mathbb{V}(x) = \{0\}$. We have $\mathbb{C}[W] \simeq \mathbb{C}$. Two polynomial functions $f_{|W} = g_{|W}$, if and only if the constant term of $f(0) = g(0)$. Therefore, on W the polynomials are distinguished only by their constant terms, which again explains why $\mathbb{C}[W] = \mathbb{C}$.

(b) Consider $V = \{(x, y) \in \mathbb{A}^2 : xy - 1 = 0\}$. Since $\mathbb{C}[V] = \frac{\mathbb{C}[x,y]}{\mathbb{I}(V)}$. This coordinate ring is a finitely generated $\mathbb{C}$-algebra, where the function x is identified with $1/y$. For instance, on V we have the equality of the polynomial functions $x^5y^2 = x^3$. Equivalently, $x^5y^2 + \mathbb{I}(V) = x^3 + \mathbb{I}(V)$, since $x^5y^3 - x^2 = x^2(x^3y^3 - 1) \in \mathbb{I}(V)$. Also, on V the function $1/y$ behaves like a polynomial: on V we always have $xy \neq 0$, and $1/y = x$. This makes it reasonable to think of $\mathbb{C}[V]$ as

$$\mathbb{C}[x, 1/x] = \big\{\text{Complex Laurent polynomials with variable } x \in \mathbb{C} \setminus \{0\}\big\}.$$

We will revisit this example later in the course.

△

Recall that a ring is called *reduced* if it has no non-zero nilpotent elements.

Theorem 2.38. Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be two algebraic varieties.

(a) For any algebraic variety V, $\mathbb{C}[V]$ is a reduced, finitely generated $\mathbb{C}$-algebra.

(b) Any morphism of algebraic varieties $\varphi : V \longrightarrow W$, induces a well-defined $\mathbb{C}$-algebra homomorphism, called the *pullback of* φ,

$$\begin{aligned} \varphi^* : \mathbb{C}[W] &\longrightarrow \mathbb{C}[V], \\ g &\longmapsto g \circ \varphi. \end{aligned}$$

Proof. (a) Recall that $\mathbb{I}(V)$ is a radical ideal, and it is easy to see that for a ring R and an ideal $I \subseteq R$, R/I is reduced, if and only if, I is radical[3]. Since $\mathbb{C}[x_1, \dots, x_n]$ is a finitely generated $\mathbb{C}$-algebra, so is $\mathbb{C}[V]$.

(b) It is easy to check that for $f \in \mathbb{C}[W]$, we have $\varphi^*(f)$ is a polynomial function in $\mathbb{C}[V]$

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, "f\circ\varphi"'] & W \arrow[d, "f"] \\ & \mathbb{C} \end{tikzcd}$$

Moreover, φ^* is a $\mathbb{C}$-algebra homomorphism: for $f, g \in \mathbb{C}[W]$ and $\lambda \in \mathbb{C}$, one has

1. $\varphi^*(1) = 1 \circ \varphi = 1$;
2. $\varphi^*(f+g) = \varphi^*(f) + \varphi^*(g)$;
3. $\varphi^*(fg) = \varphi^*(f)\varphi^*(g)$;
4. $\varphi^*(\lambda f) = \lambda\varphi^*(f)$.

Note that Items 1 and 3 imply that if $fg = 1$, then $\varphi^*(f)\varphi^*(g) = 1$.

□

Theorem 2.39. (a) Any reduced, finitely generated $\mathbb{C}$-algebra is isomorphic to the coordinate ring of an algebraic variety.

(b) Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be two algebraic varieties. Any $\mathbb{C}$-algebra morphism $\Phi : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is of the form $\Phi = \varphi^*$ for a uniquely defined map $\varphi : V \longrightarrow W$.

(c) Any morphism $\Theta : S \longrightarrow R$ of reduced, finitely generated $\mathbb{C}$-algebras defines a morphisms of algebraic varieties $\varphi : V \longrightarrow W$, where V and W are unique up to isomorphism.

Proof. (a) Let R be a reduced, $\mathbb{C}$-algebra, with generators $\{u_1, \dots, u_m\}$. We define the surjective $\mathbb{C}$-algebra homomorphism α, such that

$$\begin{aligned} \alpha : \mathbb{C}[y_1, \dots, y_m] &\longrightarrow R, \\ y_i &\longmapsto u_i. \end{aligned}$$

Note that the above information on the generators of a $\mathbb{C}$-algebra completely determines a $\mathbb{C}$-algebra morphism α. For instance, $\alpha(2y_1 + y_2y_3) = 2\alpha(y_1) + \alpha(y_2)\alpha(y_3)$, and more generally, for $g \in \mathbb{C}[y_1, \dots, y_m]$, $\alpha(g) = g(\alpha(y_1), \dots, \alpha(y_m))$.

[3] This is Exercise 2.3(e). Solution: if R/I has a non-zero nilpotent element $f + I$, we have $f + I \neq I \iff f \notin I$, then $(f+I)^n = f^n + I = I$, for some positive integer n. So $f^n \in I$ and I cannot be radical. If I is not radical, then there is an element $f \in R, f \notin I$ and a positive integer n, such that $f^n \in I$. As a result, $(f+I) \neq I$, is a non-zero element of R/I, but it is nilpotent as $(f+I)^n = f^n + I = I$.

Now let $I := \ker(\alpha)$. Since

$$R \simeq \mathbb{C}[y_1, \ldots, y_m]/I$$

is reduced, I is a radical ideal. In turn, $V = \mathbb{V}(I) \subseteq \mathbb{A}^n$ is an (closed affine) algebraic variety, and by Nullstellensatz $\mathbb{I}(V) = \mathbb{I}(\mathbb{V}(I)) = \sqrt{I} = I$. Therefore, $\mathbb{C}[V] = \mathbb{C}[y_1, \ldots, y_m]/I(V) \simeq R$.

(b) Assume that $\mathbb{C}[W] = \mathbb{C}[y_1, \ldots y_m]/\mathbb{I}(W)$ and $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots x_n]/\mathbb{I}(V)$. Let $\bar{y}_i = y_i + \mathbb{I}(W)$, and $\bar{x}_i = x_i + \mathbb{I}(V)$, be the generators of $\mathbb{C}[W]$ and $\mathbb{C}[V]$, respectively. We intend to construct a polynomial mapping $\varphi : \mathbb{A}^n \longrightarrow \mathbb{A}^m$, such that

1. $\varphi^* = \Phi$;
2. $\varphi(V) \subseteq W$;
3. $\varphi_{|_V} : V \longrightarrow W$ is unique.

We have that $\Phi(\bar{y}_i) \in \mathbb{C}[V]$. Let $\varphi_i(x) \in \mathbb{C}[x_1, \ldots, x_n]$ be any function that $\varphi_i(x) + \mathbb{I}(V) = \Phi(\bar{y}_i)$. We claim that

$$\begin{aligned} \varphi : \mathbb{A}^n &\longrightarrow \mathbb{A}^m, \\ a = (a_1, \ldots, a_n) &\longmapsto (\varphi_1(a), \ldots, \varphi_m(a)), \end{aligned}$$

satisfies the required properties:

1. $\varphi^* = \Phi$. Let $f \in \mathbb{C}[W]$, we have $\Phi(f(\bar{y}_1, \ldots, \bar{y}_m)) = f(\Phi(\bar{y}_1), \ldots, \Phi(\bar{y}_m)) = f(\varphi_1(x), \ldots, \varphi_n(x)) + \mathbb{I}(V) = (f \circ \varphi)(x) + \mathbb{I}(V)$. Therefore,

$$\varphi^*(f) = \Phi(f).$$

2. $\varphi(V) \subseteq W$. Let $a \in V$, to see that $\varphi(a) \in W$, we show that $g(\varphi(a)) = 0$, for any $g \in \mathbb{I}(W)$. We have that $f(a) = 0$, for any $f \in \mathbb{I}(V)$. Since $\Phi : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$, maps the 'zero' of $\mathbb{C}[W]$, to the 'zero' of $\mathbb{C}[V]$, *i.e.*, $\mathbb{I}(W) \longmapsto \mathbb{I}(V)$. If $g \in I(W)$, then $\Phi(g) \in \mathbb{I}(V)$. Therefore, $\Phi(g)(a) = g(\varphi(a)) = 0$.
3. Note that we have made a choice for $\varphi : \mathbb{A}^n \longrightarrow \mathbb{A}^m$, and even though this map is not unique, the restriction to V is indeed unique. For, if we made another choice $\varphi' : \mathbb{A}^n \longrightarrow \mathbb{A}^m$, in the above construction with $\varphi'_i(x) + \mathbb{I}(V) = \Phi(\bar{y}_i)$, then $\varphi_i - \varphi'_i \in \mathbb{I}(V)$ for all i. Equivalently, $\varphi(a) - \varphi'(a) = 0$ for any $a \in V$ or $\varphi_{|_V} = \varphi'_{|_V}$.

(c) Using the construction in Part (b) one can show that (see Exercise 2.40) $\mathbb{C}$-algebras $\mathbb{C}[V] \simeq \mathbb{C}[V']$, if and only if the (closed affine) algebraic varieties V and V' are isomorphic. Combining this with Part (a), one finds varieties and morphisms $\varphi : V \longrightarrow W$ and $\varphi' : V' \longrightarrow W'$ such that

$$\alpha : V \xrightarrow{\simeq} V', \quad \alpha' : W \xrightarrow{\simeq} W',$$

and $\varphi' = \alpha' \circ \varphi \circ \alpha^{-1}$. In other words, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\simeq"'] & W \arrow[d, "\simeq"] \\ V' \arrow[r, "\varphi'"'] & W' \end{tikzcd}$$

$\square$

Exercise 2.40. (a) For two morphism of varieties $\varphi : V \longrightarrow W$ and $\psi : W \longrightarrow V$, show that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

(b) $(\mathrm{id}_V)^* = \mathrm{id}_{\mathbb{C}[V]}$.

(c) If $I : \mathbb{C}[V] \longrightarrow \mathbb{C}[V]$ is the identity map, then $I = (\mathrm{id}_V)^*$.

(d) Deduce that $V \simeq V'$ if and only if $\mathbb{C}[V] \simeq \mathbb{C}[V']$.

Example 2.41. (a) The morphism

$$\begin{aligned} \varphi : \mathbb{A}^1 &\longrightarrow \mathbb{V}(y - x^2, z - x^3) \subseteq \mathbb{A}^3, \\ t &\longmapsto (t, t^2, t^3), \end{aligned}$$

is an isomorphism. Moreover, it induces the $\mathbb{C}$-algebra isomorphism

$$\begin{aligned} \Phi : \mathbb{C}[x, y, z]/(y - x^2, z - x^3) &\longrightarrow \mathbb{C}[t] \\ x &\longmapsto t, \\ y &\longmapsto t^2, \\ z &\longmapsto t^3. \end{aligned}$$

Note that if $f \in \mathbb{C}[x, y, z]/(y - x^2, z - x^3)$, then $\varphi^*(f) = f(t, t^2, t^3) \in \mathbb{C}[t]$.

(b) We can verify that

$$\begin{aligned} \mathbb{A}^1 &\longrightarrow \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2, \\ t &\longmapsto (t^2, t^3), \end{aligned}$$

is not an isomorphism, even though it is bijective. Note that the pullback

$$\begin{aligned} \mathbb{C}[x, y]/(y^2 - x^3) &\longrightarrow \mathbb{C}[t], \\ x &\longmapsto t^2, \\ y &\longmapsto t^3, \end{aligned}$$

is not an isomorphism of $\mathbb{C}$-algebras, as t is not in the image.

$\triangle$

Exercise 2.42 ([SKKT00, Exercise 2.5.1-2]).

(a) Show that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is injective if and only if φ is *dominant*. Recall that a map, φ, is called dominant if its image, $\varphi(V)$, is dense in W.

(b) Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if φ defines an isomorphism between V and some algebraic subvariety of W.

Exercise 2.43 ([SKKT00, Exercise 2.5.3]). Show that if $\varphi = (\varphi_1, \ldots, \varphi_n) : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ is an isomorphism, then the *Jacobian determinant*

$$\det \begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_n}{\partial x_1} & \cdots & \frac{\partial \varphi_n}{\partial x_n} \end{bmatrix}$$

is a nonzero constant polynomial. The converse of this statement is known as the *Jacobian Conjecture* and is still open.

2.7 Affine Schemes

In this section, we explain the correspondence $\mathbb{C}[V] \leftrightarrow V$ and will try to understand the idea behind the notion of the *affine schemes*. For a ring R, let us define the *maximal spectrum* of R by

$$\operatorname{maxSpec}(R) = \{\mathfrak{m} \subseteq R : \mathfrak{m} \text{ is a maximal ideal}\}.$$

In the following paragraphs, we discuss that for any closed affine algebraic variety $V \subseteq \mathbb{A}^n$, $\operatorname{maxSpec}(\mathbb{C}[V])$ can be endowed with the Zariski topology, and can be identified with V.

To warm up, note that all the maximal ideals of $A := \mathbb{C}[x_1, \ldots, x_n]$ are of the form $\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$, where $a = (a_1, \ldots, a_n)$ is a point in $\mathbb{A}^n$. As any subset of $\mathbb{A}^n$ is the union of its points, combined Nullstellensatz, and Exercise 2.14[4], we obtain:

$$\mathbb{V}(I) = \bigcup_{a \in \mathbb{V}(I)} \{a\} \quad \longleftrightarrow \quad \mathbb{I}(\mathbb{V}(I)) = \sqrt{I} = \bigcap_{\mathfrak{m}_a \supseteq I} \mathfrak{m}_a.$$

Moreover, for an ideal $I \subseteq A$, let $V = \mathbb{V}(I)$:

$$\operatorname{maxSpec}(\mathbb{C}[V]) \quad \longleftrightarrow \quad \{\mathfrak{m} \in \operatorname{maxSpec}(A) : \mathfrak{m} \supseteq \mathbb{I}(V)\} \quad \longleftrightarrow \quad \{\text{points in } \mathbb{V}(I)\}.$$

Analogously,

(a) We can now define the Zariski closed sets of A. Recall that the Zariski closed sets in $\mathbb{A}^n$ are

$$\begin{aligned} \mathbb{V}(I) &= \{a \in \mathbb{A}^n : f(a) = 0, f \in I\} \\ &= \{a \in \mathbb{A}^n : f \in \mathfrak{m}_a, f \in I\} \\ &= \{a \in \mathbb{A}^n : \mathfrak{m}_a \supseteq I\}, \end{aligned}$$

[4]Solution 1: The inclusion $\sqrt{I} \subseteq \bigcap_{\text{maximal } \mathfrak{m} \supseteq I} \mathfrak{m}$ is clear. For the converse, if $f \in \bigcap_{\mathfrak{m}_a \supseteq I} \mathfrak{m}_a$, then $f(a) = 0$ for every $a \in \mathbb{V}(I)$. By Nullstellensatz $f \in \sqrt{I}$.

Solution 2: Use that $\prod_a \mathfrak{m}_a = \bigcap_a \mathfrak{m}_a$ for distinct maximal ideals $\mathfrak{m}_a$.

for $I \subseteq A$ an ideal. Similarly define the Zariski topology on $\mathrm{maxSpec}(A)$ by declaring the closed sets to be

$$\mathbb{V}(I) = \{\mathfrak{m} \in \mathrm{maxSpec}(A) : \mathfrak{m} \supseteq I\},$$

for any ideal $I \subseteq A$. Similarly, we can define the Zariski topology on $\mathbb{C}[V]$ by declaring the closed sets to be

$$\mathbb{V}(J) = \{\mathfrak{m} \in \mathrm{maxSpec}\big(\mathbb{C}[V]\big) : \mathfrak{m} \supseteq J\},$$

for ideals $J \subseteq \mathbb{C}[V]$.

(b) For a morphism of algebraic varieties $\varphi : V \longrightarrow W$, we can consider the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$. If $a \in V \longmapsto \varphi(a) \in W$, and a and $\varphi(a)$ correspond to the maximal ideals $\mathfrak{m}_a \subseteq \mathbb{C}[V]$ and $\mathfrak{n}_{\varphi(a)} \subseteq \mathbb{C}[W]$, respectively. We have

$$(\varphi^*)^{-1}(\mathfrak{m}_a) = \mathfrak{n}_{\varphi(a)}.$$

To see this, note that

$$\begin{aligned} f \in \mathfrak{n}_{\varphi(a)} \iff f(\varphi(a)) = 0 \iff \varphi^*(f)(a) = 0 \\ \iff \varphi^*(f) \in \mathfrak{m}_a \iff f \in (\varphi^*)^{-1}(\mathfrak{m}_a). \end{aligned}$$

That is, the inverse image of the maximal ideals in this case are maximal. To summarize into a diagram:

$$\begin{tikzcd} \mathbb{C}[V] \supseteq \mathfrak{m}_a \arrow[d, leftrightarrow] & (\varphi^*)^{-1}(\mathfrak{m}_a) \subseteq \mathbb{C}[W] \arrow[l, "\varphi^*"'] \arrow[d, leftrightarrow] \\ V \ni a \arrow[r, "\varphi"'] & \varphi(a) \in W. \end{tikzcd}$$

As a result, any homomorphism $\Phi : R \longrightarrow S$ of finitely generated reduced $\mathbb{C}$-algebras induces a continuous map of associated maximal spectra

$$\Phi^{-1} : \mathrm{maxSpec}(S) \longrightarrow \mathrm{maxSpec}(R).$$

Exercise 2.44. Let $\varphi : V \longrightarrow W$ be a morphism of algebraic varieties. Prove that the $(\varphi^*)^{-1} : \mathrm{maxSpec}(\mathbb{C}[V]) \longrightarrow \mathrm{maxSpec}(\mathbb{C}[W])$ is continuous.

In summary, the Nullstellensatz for $\mathbb{C}[x_1, \dots, x_n]$ and Theorem 2.39, the above Properties (a),(b) allow for turning $\mathrm{maxSpec}(R)$ to a nice topological space when R is finitely generated reduced $\mathbb{C}$-algebra. However, it is impossible to replace $\mathbb{C}[V]$ in the above definition by any commutative ring, since in general the inverse image of the maximal ideals are not maximal. The profound idea in Scheme Theory is that changing the concentration from the maximal ideals to the prime ideals[5] makes this generalisation possible:

[5]From Definition 2.45 to Section 2.8 non-examinable.

Definition 2.45. The *spectrum* of a commutative ring R, denoted by $\mathrm{Spec}(R)$ is the set of all prime ideals in R, that is

$$\mathrm{Spec}(R) = \{\mathfrak{p} : \mathfrak{p} \text{ prime ideal in } R\}.$$

We can naturally equip $\mathrm{Spec}(R)$ with the Zariski topology by defining the closed sets to be

$$\mathbb{V}(I) := \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq I\},$$

for any ideal $I \subseteq R$.

Remark 2.46. (a) The above definition assigns to any commutative ring with unit element 1 a topological space. The points of this topological space are the prime ideals, we have a well-defined topology, and the pre-image of prime ideals under homomorphisms of rings are indeed prime. *I.e.*, we have Properties similar to (a) and (b) in this extended definition too. This exemplifies a remarkable philosophy of Grothendieck about mathematics: by bypassing a few steps and technicalities, the definition of our topological space has been generalised yet simplified.

(b) The spectrum of a ring is called an *affine scheme* by Grothendieck. Affine schemes are the "open" pieces of a *scheme*, and later we get an idea of how to use isomorphisms to "glue" affine schemes.

The spectrum of the ring seems like a natural generalisation of maxSpec, but since our *points* are the prime ideals now, peculiar situations appear as we show in the examples below. Note that maximal ideals still correspond to points, but these points are *closed* since the smallest closed set containing $\mathfrak{m}$, the closure of $\mathfrak{m}$ equals $\mathbb{V}(\mathfrak{m}) = \{\mathfrak{m}\}$.

Example 2.47. It is easy to see that $\mathrm{Spec}(\mathbb{Z}) = \{(0), (2), (3), (5), \dots\}$. The only prime ideal which is not maximal is $(0) \subseteq \mathbb{Z}$. Note that, for instance, $(0) \subset (2)$. More generally, the closure of (0), the smallest closed set containing the point (0), is

$$\mathbb{V}(0) = \{\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}) : \mathfrak{p} \supseteq (0)\} = \mathrm{Spec}(\mathbb{Z}).$$

I.e., everything! In particular, (0) is not closed. Note that for any prime $p \in \mathbb{Z}$, the ideal (p) is maximal and therefore a *closed point*. Moreover, any non-empty open set in $\mathrm{Spec}(\mathbb{Z})$ is of the form

$$\mathrm{Spec}(\mathbb{Z}) \setminus \{\text{finitely many closed points}\}.$$

So, any non-empty open set contains (0). We call (0) a *generic point* of $\mathrm{Spec}(\mathbb{Z})$. △

Example 2.48. Compared to $\mathrm{maxSpec}(\mathbb{C}[x])$, the spectrum $\mathrm{Spec}(\mathbb{C}[x])$ has a spooky new guest (0). The closed points correspond to the maximal ideals which are of the form $(x - a)$, $a \in \mathbb{C}$. Therefore, the closed points do correspond to a specific point $a \in \mathbb{C}$, but (0) cannot be placed anywhere, and at the same time "near" to every other point. See Figure 1. △

Example 2.49. Let us look at the non-reduced $\mathbb{C}$-algebra $R := \mathbb{C}[x]/(x^2)$. The spectrum of R can be understood as the point $\{x = 0\} \subseteq \mathbb{A}$ with multiplicity two, which is an example of a *fat point*. △

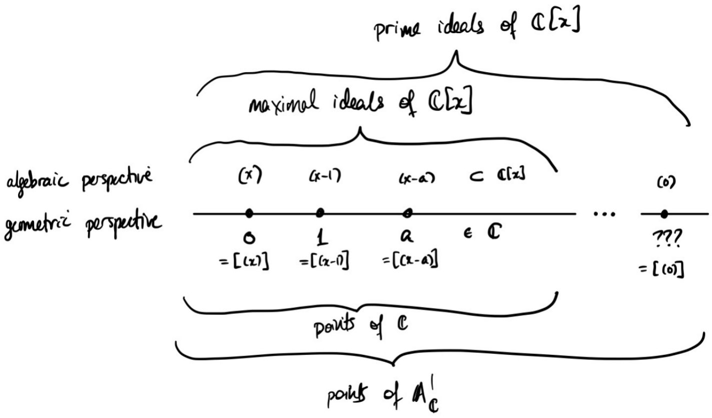


Figure 1: A picture of $\mathrm{Spec}(\mathbb{C}[x])$ from [Vak22, Page 108].

2.8 The Equivalence of Algebra and Geometry

In the language of the categories, the results in Section 2.6 can be rephrased as the *equivalence* of the following two *categories*:

- Var := *the category of (closed affine) algebraic varieties*;
- $\mathsf{Alg}_{\mathbb{C}}$:= *the category of finitely generated, reduced* $\mathbb{C}$*-algebras.*

Roughly speaking, a category has *objects* and *morphisms* between any pairs of objects. Morphisms are also called *maps* or simply *arrows*. For Var the objects are the varieties, and the morphisms are the morphisms of algebraic varieties. For $\mathsf{Alg}_{\mathbb{C}}$ the objects are the reduced, finitely generated $\mathbb{C}$-algebras, and the morphisms are $\mathbb{C}$-algebra homomorphisms. Every object A in a category C, is associated with an *identity morphism* id_A. In each category, there is a natural composition of morphisms that is associative.

A *functor* is a map between two categories. More precisely, A *covariant functor* $\mathcal{F}$ from a category $\mathsf{C} = (\mathrm{Obj}(\mathsf{C}), \mathrm{Mor}(\mathsf{C}))$ to the category $\mathsf{D} = (\mathrm{Obj}(\mathsf{D}), \mathrm{Mor}(\mathsf{D}))$, denoted by $\mathcal{F} : \mathsf{C} \longrightarrow \mathsf{D}$, consists of the information

- A map of objects $\mathrm{Obj}(\mathsf{C}) \longrightarrow \mathrm{Obj}(\mathsf{D})$;
- A map of morphisms $\mathrm{Mor}(\mathsf{C}) \longrightarrow \mathrm{Mor}(\mathsf{D})$,

satisfying

- For every object $A \in \mathsf{C}$, $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$;
- If
$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta\circ\alpha"'] & B \arrow[r, "\beta"] & C, \end{tikzcd}$$
then
$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(\alpha)"] \arrow[rr, bend right, "\mathcal{F}(\beta\circ\alpha)"'] & \mathcal{F}(B) \arrow[r, "\mathcal{F}(\beta)"] & \mathcal{F}(C), \end{tikzcd}$$

is also commutative.

A *contravariant functor* reverses the arrows.

For any category C we can naturally define the identity functor id_{C}, which allows us to define the important notion of the *equivalence of categories.* We say that the two categories C and D are equivalent if there are functors $\mathcal{F} : \mathsf{C} \longrightarrow \mathsf{D}$, and $\mathcal{G} : \mathsf{D} \longrightarrow \mathsf{C}$, such that

$$\mathcal{G} \circ \mathcal{F} \simeq \mathrm{id}_{\mathsf{C}}\,, \quad \mathcal{F} \circ \mathcal{G} \simeq \mathrm{id}_{\mathsf{D}}.$$

That is, for any $A \in \mathrm{Obj}(\mathsf{C})$, the object $\mathcal{G} \circ \mathcal{F}(A) \in \mathrm{Obj}(\mathsf{C})$ is an object isomorphic to A (and not necessarily equal). This holds true similarly for $\mathcal{F} \circ \mathcal{G}$.

Now let us define the functor

$$\begin{aligned} \mathcal{F} : \mathsf{Var} &\longrightarrow \mathsf{Alg}_{\mathbb{C}}, \\ \mathrm{Obj}(\mathsf{Var}) \ni V &\longmapsto \mathcal{F}(V) := \mathbb{C}[V] \in \mathrm{Obj}(\mathsf{Alg}_{\mathbb{C}}), \\ \mathrm{Mor}(\mathsf{Var}) \ni \varphi &\longmapsto \mathcal{F}(\varphi) := \varphi^* \in \mathrm{Mor}(\mathsf{Alg}_{\mathbb{C}}). \end{aligned}$$

Also,

$$\begin{aligned} \mathcal{G} : \mathsf{Alg}_{\mathbb{C}} &\longrightarrow \mathsf{Var}, \\ \mathrm{Obj}(\mathsf{Alg}_{\mathbb{C}}) \ni \mathbb{C}[V] &\longmapsto \mathcal{G}(V) := \mathrm{maxSpec}(\mathbb{C}[V]), \\ \mathrm{Mor}(\mathsf{Alg}_{\mathbb{C}}) \ni \Phi &\longmapsto \mathcal{G}(\Phi) = \varphi, \end{aligned}$$

where φ is defined in Theorem 2.39(b). The results in Section 2.6 verify that

- $\mathcal{F}$ is a functor;
- $\mathcal{G}$ is a functor;
- $\mathcal{G} \circ \mathcal{F} \simeq \mathrm{id}_{\mathsf{Alg}_{\mathbb{C}}}$, and $\mathcal{F} \circ \mathcal{G} \simeq \mathrm{id}_{\mathsf{Var}}$.

In summary, we have the equivalence between Var and $\mathsf{Alg}_{\mathbb{C}}$. This equivalence also highlights a very modern view of mathematics that often to study sets, we study the functions on them. In our case,

> "Studying polynomial functions on algebraic varieties provides enough information to understand algebraic varieties up to isomorphism."

The reader interested in learning more on Category Theory may consult nice and gentle notes of Tom Leinster [Lei14] which are available at https://arxiv.org/pdf/1612.09375.pdf, or very solid notes of Pierre Schapira available on his Webpage (https://webusers.imj-prg.fr/~pierre.schapira/LectNotes/indexLN.html).

Figure 2: Bristol Planitarium

3 Projective Varieties

3.1 Projective Space

In this section, we define the projective spaces. A projective space of dimension n contains a copy of $\mathbb{A}^n$, and it turns out to be a compact set in the Euclidean topology. Let us set $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Definition 3.1. The projective n-space is defined as the set of all classes

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim,$$

where $a, b \in \mathbb{C}^{n+1} \setminus \{0\}$, $a \sim b$, if and only if, there is $\lambda \in \mathbb{C}^*$, such that $a = \lambda \cdot b$.

Let us note the following.

(a) The equivalence relation means $\mathbb{P}^n$ corresponds to the set of all lines in $\mathbb{C}^{n+1}$, passing through the origin. Therefore, any element or point in $\mathbb{P}^n$ corresponds to such a line.

(b) A point $a \in \mathbb{P}^n$ is a class of points in $\mathbb{C}^{n+1} \setminus \{0\}$ under the equivalence relation. In coordinates, such a point is denoted by $a = [x_0 : \cdots : x_n]$, which are called the *homogeneous coordinates* of a.

(c) The equivalence relation $\sim$ can be understood by the orbits of the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$ by $(\lambda, a) \longmapsto \lambda \cdot a$. That is, $a \sim b$ if and only if a and b lie on the same orbit under the action of $\mathbb{C}^*$.

(d) The above projective space can be defined over any field k. Moreover, when $k = \mathbb{R}$, then

$$\mathbb{RP}^n = (\mathbb{S}^n)/\sim,$$

where $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is the n-dimensional sphere, and $a \sim b$ if and only if $a = \pm b$. Note that

$$\mathbb{S}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim_1$$

where $\sim_1$ is the equivalence relation that $a \sim_1 b$ if and only if $a = \lambda b$ for $\lambda > 0$.

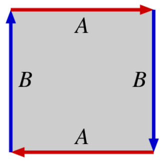


Figure 3: $\mathbb{RP}^2$ obtained by a certain glueing of the sides of a square.

Example 3.2. (a) If $[x_0, \dots, x_n] \in \mathbb{P}^n$, by definition $[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n]$ for any $\lambda \in \mathbb{C}^*$. Therefore, if for instance, $x_0 \neq 0$, then $[x_0 : x_1 : \cdots : x_n] = [1 : \frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0}]$.

(b) $\mathbb{P}^0$ is all the lines in $\mathbb{C}^1$, which is the line $\mathbb{C}^1$ itself. Therefore, $\mathbb{P}^0$ has only one point. In homogeneous coordinates $\mathbb{P}^0 = [x_0] = \{a \in \mathbb{C}^*\}/\sim \; = [1]$.

(c) $\mathbb{P}^1 = \{[x_0 : x_1] : (x_0, x_1) \neq (0,0)\}$.

$$\mathbb{P}^1 = \begin{cases} \{[1 : x_1/x_0]\} = \{[1 : a] : a \in \mathbb{C}\}, & \text{if } x_0 \neq 0; \\ \{[0 : x_1] : x_1 \neq 0\} = \{[0 : 1]\} & \text{if } x_0 = 0. \end{cases}$$

As a result, we may write $\mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$. Note that $\mathbb{C}$ as a vector space over $\mathbb{R}$ is two dimensional. Therefore, $\mathbb{P}^1$ can be compared to a sphere of real dimension two, called the *Riemann Sphere*. See this rather farcical video: https://www.youtube.com/watch?v=l3nlXJHD714.

(d) Similar to above: $\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{P}^{n-1} = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \{\infty\}$.

(e) We can understand $\mathbb{P}^1$ with two open sets $U_0 = \{[x_0 : x_1] : x_0 \neq 0\}, U_1 = \{[x_0 : x_1] : x_1 \neq 0\}$. Each of these open sets are homeomorphic to $\mathbb{C}$. If $[x_0 : x_1] \in U_0 \cap U_1$, then $x_0 \neq 0, x_1 \neq 0$. Therefore $U_0 \cap U_1 \simeq \mathbb{C}^*$. Moreover, if $[x_0 : x_1] \in U_0 \cap U_1, [x_0 : x_1] = [x_0/x_1 : 1] = [1 : x_1/x_0]$. As a consequence, we regard $\mathbb{P}^1$ as two copies of $A_1 := A_2 := \mathbb{C}$ which are glued to each other on $\mathbb{C}^* \subseteq A_1, A_2$ by the map $a \mapsto \frac{1}{a}$.

$\triangle$

Exercise 3.3. Justify yourself that the real projective plane $\mathbb{RP}^2$ is a disc with the sides identified as in Figure 3.

For completeness let us recall the definition of an analytic manifold; see [Voi02].

Definition 3.4. Assume that $U \subseteq \mathbb{C}^n$ is an open subset in the Euclidean topology.

(a) Let $f : U \longrightarrow \mathbb{C}$ be a (complex-valued) differentiable function. We say that f is *analytic* or *holomorphic* at the point $a \in U$ if for all $j \in \{1, \dots, n\}$ the one variable function

$$z_j \longmapsto f(a_1, ..., a_{j-1}, z_j, a_{j+1}, ..., a_n)$$

is analytic at a_j.

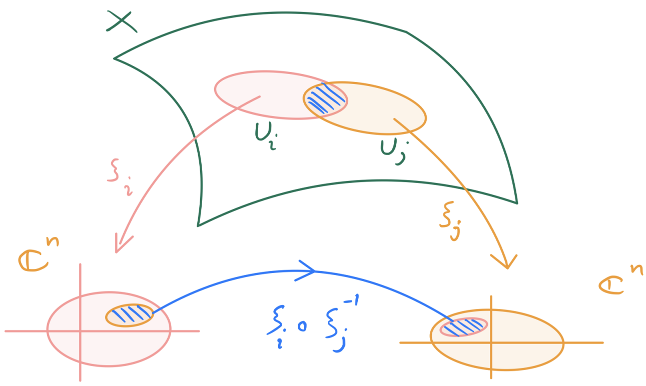


Figure 4: Nicely glued pieces of a manifold X.

(b) A map $\varphi = (\varphi_1, \dots, \varphi_m) : U \longrightarrow \mathbb{C}^m$, is called *analytic*, if all $j = 1, \dots, m$, the functions φ_j are analytic for all $a \in U$.

(c) A complex analytic manifold X of dimension n, is a topological space satisfying the following properties:

- X is a second countable Hausdorff topological space;
- X can be covered with a (countable) collection of open sets $X = \bigcup_i U_i$, such that for each U_i there is a homeomorphism $\xi_i : U_i \longrightarrow V_i \subseteq \mathbb{C}^n$, where V_i is an open subset. (*I.e.*, any complex manifold has an open cover, and it locally looks like open subsets of $\mathbb{C}^n$.) Each pair (U_i, ξ_i) is called a *chart* and the collection of all the charts for the manifold X, $\{(U_i, \xi_i)\}_i$, is called an *atlas*;
- For all i, j the maps of *change of coordinates* $\xi_j \circ \xi_i^{-1} : \xi_i(U_i \cap U_j) \longrightarrow \xi_j(U_i \cap U_j)$ are analytic. (*I.e.*, open pieces are analytically glued.) See Figure 4.

Example 3.5. Any open set of $U \subseteq \mathbb{C}^n$ is a complex n-dimensional manifold. Its atlas can be described by one chart. △

Theorem 3.6. With the natural quotient topology induced from the Euclidean topology of $\mathbb{C}^{n+1}$ on $\mathbb{P}^n$, $\mathbb{P}^n$ is a complex n-dimensional analytic manifold.

Proof. We need to find an atlas $\{(U_i, \xi_i)\}_{i \in I}$ where

(a) Each U_i is an open subset of $\mathbb{P}^n$, $\xi_i : U_i \longrightarrow \mathbb{C}^n$ is a homeomorphism;

(b) $\mathbb{P}^n = \bigcup_{i \in I} U_i$;

(c) For each i and j, the map of change of coordinates $\xi_j \circ \xi_i^{-1} : \xi_i(U_i \cap U_j) \longrightarrow \xi_j(U_i \cap U_j)$ is analytic.

To show these, note that:

(a) For $i = 0, \dots, n$, define $U_i = \{[x_0, \dots x_{i-1}, x_i, x_{i+1}, \dots, x_n] : x_i \neq 0\}$. It is clear that each U_i is open in $\mathbb{P}^n$ in the Euclidean topology induced from $\mathbb{C}^{n+1}$. Let

$$\xi_i : U_i \longrightarrow \mathbb{C}^n,$$
$$[x_0, \dots x_{i-1}, x_i, x_{i+1}, \dots, x_n] \longmapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right).$$

ξ_i is a composition of division by a non-zero number and a linear projection, therefore it is continuous. Clearly, ξ_i^{-1} is also a continuous function.

(b) By definition $\mathbb{P}^n = \bigcup_{i \in I} U_i$.

(c) By renumbering, without loss of generality, we prove Item (c) for $i = 0, j = 1$. In $U_0 \cap U_1$ both x_0, x_1 are non-zero. Therefore, if $a = (a_1, \dots, a_n) \in \xi_0(U_0 \cap U_1) \subsetneq \mathbb{C}^n$, then $a_1 \neq 0$. We have

$$\begin{tikzcd}
(a_1, \dots, a_n) \arrow[r, mapsto, "\xi_0^{-1}"] \arrow[rr, bend right, "\xi_1 \circ \xi_0^{-1}"'] & {[1 : a_1 : \cdots : a_n]} \arrow[r, mapsto, "\xi_1"] & (1/a_1, a_2/a_1, \dots, a_n/a_1).
\end{tikzcd}$$

Since $a_1 \neq 0$, the above map is well-defined and analytic.

□

Exercise 3.7. Rewrite the proof of Theorem 3.6 for yourself when $n = 2$. That is, prove that $\mathbb{P}^2$ is an analytic manifold. Write down all the charts U_0, U_1, U_2 and all the change of coordinates on the intersections explicitly (Pretty please. This is very useful for us later!).

Exercise 3.8. Prove that with the induced Euclidean topology $\mathbb{P}^n$ is compact. Deduce that any analytic function $f : \mathbb{P}^n \longrightarrow \mathbb{C}$ has to be constant. In particular, any polynomial $f : \mathbb{P}^n \longrightarrow \mathbb{C}$ is constant. Hint: check out Theorem 3.13.

3.1.1 A Quick Review: Complex Analysis in One Variable

A function $f : U \longrightarrow \mathbb{C}$, where $U \subseteq \mathbb{C}$ is an open set, is said to be *differentiable* at a point $z_0 \in U$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Theorem 3.9. (a) A function $f(x + iy) = u(x, y) + iv(x, y)$ is complex differentiable at a point $z_0 = x_0 + iy_0$.

(b) partial derivatives u_x, u_y, v_x, v_y exist and satisfy the *Cauchy-Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The first big surprise of the theory of complex functions, which has no direct analogue for real functions $g : \mathbb{R}^2 \to \mathbb{R}^2$, is the following:

Theorem 3.10. Let $f = u + iv : U \longrightarrow \mathbb{C}$, be a complex function.

(a) f is complex differentiable at every point $z_0 \in U$ and its partial derivatives u_x, u_y, v_x, v_y are continuous.

(b) u_x, u_y, v_x, v_y are continuous and satisfy the Cauchy–Riemann equations at every $z_0 \in U$.

(c) u_x, u_y, v_x, v_y are continuous and f is *conformal*. That is to say the Jacobian/derivative of f,

$$D(f) = J_f = \begin{pmatrix} \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y} \end{pmatrix}$$

preserves angles.

(d) f is *analytic* (holomorphic) in U, that is, its Taylor series z_0,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

converges uniformly to $f(z)$ for all $z \in U$, sufficiently close to z_0.

Example 3.11. (a) We can easily check that the function $f : \mathbb{C} \to \mathbb{C}$, given by $f(z) = \bar{z}$, does not satisfy the Cauchy–Riemann equations. Writing $z = x + iy$, we express f as:

$$f(x + iy) = x - iy.$$

Defining $u(x, y) = x$ and $v(x, y) = -y$, the Cauchy–Riemann equations state:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Computing the derivatives $\frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial y} = -1$, which are not equal. Thus, f is not analytic.

(b) Using the geometric series formula for $|r| < 1$,

$$\frac{1}{1 - r} = \sum_{n=0}^{\infty} r^n,$$

we rewrite $\frac{1}{z}$ as:

$$\frac{1}{z} = \frac{1}{1 - [-(z - 1)]}.$$

This series expansion is valid for $|z - 1| < 1$, leading to:

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z - 1)^n.$$

Therefore $\frac{1}{z}$ is analytic outside $\{z = 0\}$.

(d) Any function of the form $g(z)/h(z)$ for two polynomials $h, g : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic outside $\mathbb{V}(h)$.

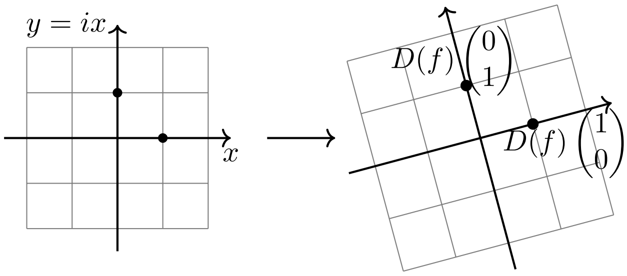
$\triangle$

Exercise 3.12. (a) Derive the Cauchy–Riemann equations from the picture below and Theorem 3.10.

(b) Write $\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ and $\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$$

Thus $\frac{\partial f}{\partial \bar{z}}$ measures how far f is from being analytic.



Theorem 3.13 (Liouville's Theorem)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function (i.e., holomorphic everywhere in $\mathbb{C}$) and suppose that f is bounded, meaning there exists some $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then f must be constant.

The following implies that by having the values of holomorphic f around the point z_0, you can determine $f(z_0)$.

Theorem 3.14 (Cauchy's Theorem and Integral Formula)**.** Let f be holomorphic on a connected open set $U \subseteq \mathbb{C}$. Then, for every $z_0 \in U$,

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0),$$

where γ is any closed simple positively-oriented contour around z_0 that is contained in U.

Sketch of the proof. Assume $z_0 = 0$. Write the Taylor expansion for $f(z)$ and use the polar change of variables $z = re^{i\theta}$. $\square$

3.2 Projective Varieties

Let $f(x_0, x_1) = x_0 + x_1 - 1$. First observe that the variety $\mathbb{V}(f)$ defines a line in $\mathbb{C}^2$. However, in $\mathbb{P}^1$ such a zero set is not well-defined, since $[x_0, x_1] \in \mathbb{P}^1$ has to be exactly the same point as $[\lambda x_0, \lambda x_1]$ for any $\lambda \in \mathbb{C}^*$. But $x_0 + x_1 = 1$ does not imply that $\lambda x_0 + \lambda x_1 = 1$ for all $\lambda \in \mathbb{C}^*$. On the other hand, observe that $\mathbb{V}(x_0^3 + x_0 x_1^2 + x_1^3)$ is, in fact, well-defined in $\mathbb{P}^1$. These observations prompt us to concentrate on the polynomials that their zero sets are *invariant* under $\mathbb{C}^*$-action, and as we will see in Proposition 3.19, are exactly the *homogeneous polynomials*, *i.e.*, a polynomial that all of its monomial summands have the same degree. Now it is easy to see that if $h \in \mathbb{C}[x_0, \ldots, x_n]$, is a homogeneous polynomial of degree d, then

$$h(\lambda x_0, \ldots, \lambda x_n) = \lambda^d h(x_0, \ldots, x_n),$$

has a well-defined zero locus on the projective space.

Definition 3.15. A projective algebraic variety in $\mathbb{P}^n$ is the common zero locus of an arbitrary collection of homogeneous polynomials in $n+1$ variables. That is, $V = \mathbb{V}(\{f_i\}_{i\in I}) \subseteq \mathbb{P}^n$, where $f_i \in \mathbb{C}[x_0, \ldots, x_n]$ are homogeneous.

Example 3.16 ([SKKT00, Page 38])**.** The *conic curve* is the projective variety given by $V = \mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2$. We can cover $\mathbb{P}^2$ as in the proof of Theorem 3.6, by the charts U_x, U_y and U_z, where on each chart we have $x \neq 0$, $y \neq 0$, and $z \neq 0$, respectively. Therefore, V can be covered by the open sets

$$V = (V \cap U_x) \cup (V \cap U_y) \cap (V \cap U_z).$$

If $[x : y : z] \in V \cap U_z$, then $[x : y : z] = [x/z : y/z : 1]$. Therefore, in $V \cap U_z$, $0 = x^2 + y^2 - z^2 = (x/z)^2 + (y/z)^2 - 1^2$. We have

$$V \cap U_z = \{[x/z : y/z : 1] = [a : b, 1] : a^2 + b^2 - 1 = 0 \text{ for all } a, b \in \mathbb{C}\}.$$

This is the *complex circle*. We check that the equations for $(V \cap U_y)$ and $(V \cap U_z)$ look like hyperbola. $\triangle$

Now we have two ways to understand any projective variety using the affine varieties:

- By using the affine charts. For instance, we have $\mathbb{P}^n = \bigcup_{i=0}^n U_i$, where U_i were defined in Theorem 3.6. Therefore,

  $$V = \bigcup_{i=0}^n (V \cap U_i) \subseteq \mathbb{P}^n.$$

  Note that U_i's are only one choice of affine charts for $\mathbb{P}^n$.

- By the *affine cone* over the variety. The affine cone is obtained by looking at the zero set of the homogenous polynomial equations defining $V = \mathbb{V}(\{f_i\}_{i\in I})$ in $\mathbb{C}^{n+1}$. Intuitively, we can consider $\mathbb{P}^n \subseteq \mathbb{A}^{n+1}$, (as a subset and not an algebraic subvariety), and for every point in $V \subseteq \mathbb{P}^n$, assign the line from the origin passing through that point.

3.3 The Homogeneous Nullstellensatz

We intend to use the quotient map $q : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$, to define a topology on $\mathbb{P}^n$. That is easy: the Zariski topology on $\mathbb{A}^{n+1}$ induces a topology on $\mathbb{A}^{n+1} \setminus \{0\}$, which in turn, induces a *quotient topology* on $\mathbb{P}^n$, *i.e.*, the unique topology on $\mathbb{P}^n$ that makes q a continuous map. In other words,

$$\begin{aligned} Y \subseteq \mathbb{P}^n \text{ is closed } &\iff q^{-1}(Y) \subseteq \mathbb{A}^{n+1} \setminus \{0\} \text{ is closed} \\ &\iff \exists\, Z \subseteq \mathbb{A}^{n+1} \text{ closed, such that } q^{-1}(Y) = Z \cap (\mathbb{A}^{n+1} \setminus \{0\}). \end{aligned}$$

Therefore, we have the bijection

$$\begin{aligned} \{\text{closed subsets of } \mathbb{P}^n\} &\xrightarrow{\simeq} \{\text{closed } \mathbb{C}^*\text{-invariant subsets of } \mathbb{A}^{n+1} \text{ containing } 0\}, \\ Y &\mapsto q^{-1}(Y) \cup \{0\}. \end{aligned}$$

By Nullstellensatz, the closed subsets of $\mathbb{A}^{n+1}$ correspond to the radical ideals in $\mathbb{C}[x_0, \ldots, x_n]$. So we ask ourselves, what are the ideals that correspond to the $\mathbb{C}^*$-invariant subsets? Let us discuss the situation with an illuminating example.

Example 3.17. Let $J = (x^3, xy)$. $\mathbb{V}(J)$ defines a closed set in $\mathbb{P}^1$, since

$$(a, b) \in \mathbb{V}(J) \subseteq \mathbb{A}^2 \iff (\lambda a, \lambda b) \in \mathbb{V}(J), \text{ for all } \lambda \in \mathbb{C}^*.$$

To see this, just note that the generators of J are homogeneous polynomials. Note that, since J is an ideal, it does contain non-homogeneous polynomials like $f(x, y) := x^3 + xy$. However, this does not pose a difficulty, since the summands of f, x^3 and xy are already in J. Note that for a point $(x_0, y_0) \in \mathbb{A}^2$, and any $\lambda \in \mathbb{C}^*$ we have $f(\lambda x_0, \lambda y_0) = \lambda^3 x_0^3 + \lambda^2 x_0 y_0$. Moreover,

$$(x_0, y_0) \in \mathbb{V}(J) \iff x_0^3 = 0, x_0 y_0 = 0 \iff \lambda^3 x_0^3 + \lambda^2 x_0 y_0 = 0, \text{for all } \lambda \in \mathbb{C}^*.$$

This observation will be discussed in full generality in Proposition 3.19. △

Motivated by the above example, we define the following.

Definition 3.18. An ideal $J \subseteq \mathbb{C}[x_0, \ldots, x_n]$ is called *homogeneous*, if for all $f = \sum_d f_d \in J$, where f_d is the sum of degree d terms of f, we have that $f_d \in J$.

Proposition 3.19. Let $J \subseteq \mathbb{C}[x_0, \ldots, x_n]$ be an ideal. Then the following are equivalent:

(a) J has a finite set of homogeneous generators;

(b) J is $\mathbb{C}^*$-invariant, that is, $f \in J \iff (\lambda \cdot^* f)(x) = f(\lambda \cdot x) \in J$ for all $\lambda \in \mathbb{C}^*$;

(c) J is homogeneous.

Proof. (a) $\implies$ (b) : Assume that $J = (g_1, \ldots g_k)$, where $g_1, \ldots, g_k$ are homogeneous polynomials of degree $d_1, \ldots, d_k$, respectively. If $f = h_1 g_1 + \ldots h_k g_k$, then

$$(\lambda\cdot)^* f = ((\lambda\cdot)^* h_1)\lambda^{d_1}(g_1) + \cdots + ((\lambda\cdot)^* h_k)\lambda^{d_k}(g_k) \in J.$$

(b) $\implies$ (c) Assume that J is $\mathbb{C}^*$-invariant. Let $f \in J$, and write $f = f_0 + \cdots + f_N$, where f_i is the homogeneous part of degree i in f. We have that $(\lambda\cdot)^* f \in J$, for any $\lambda \in \mathbb{C}^*$. We intend to show that $f_i \in J$, for all $i = 1, \ldots N$. Note that, for any $\lambda \in \mathbb{C}^*$,

$$(\lambda\cdot)^* f = f_0 + \lambda f_1 + \lambda^2 f_2 \cdots + \lambda^N f_N.$$

Choose distinct numbers $\lambda_0, \ldots, \lambda_N \in \mathbb{C}^*$. Gladly this is possible since $\mathbb{C}^*$ has infinitely many elements. Now

$$\begin{pmatrix} (\lambda_0\cdot)^* f \\ (\lambda_1\cdot)^* f \\ \vdots \\ (\lambda_N\cdot)^* f \end{pmatrix} = \begin{pmatrix} 1 & \lambda_0 & \lambda_0^2 & \ldots & \lambda_0^N \\ 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \ldots & \lambda_N^N \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{pmatrix}.$$

Since the above Vandermonde matrix is invertible for distinct λ_i, we can write $f_0, \ldots, f_N$ as a linear combination of $(\lambda_0\cdot)^* f, (\lambda_1\cdot)^* f, \ldots, (\lambda_N\cdot)^* f$ which are in J, by assumption.

(c) $\implies$ (a) If $J \subseteq \mathbb{C}[x_0, \ldots, x_n]$ is an ideal, then is is finitely generated by Hilbert Basis Theorem. If $J = (h_1, \ldots, h_k)$, we can write h_i as the sum of its homogeneous summands $h_i = h_{i,0} + \ldots h_{i,N_i}$.[6] Now Since J is homogeneous, all these summands are in J and they clearly generate J.

□

Let us denote by $\mathfrak{m}_0 = (x_0 - 0, \ldots, x_n - 0)$ is the maximal ideal corresponding to $0 = (0, \ldots, 0) \in \mathbb{A}^{n+1}$, which does not belong to $\mathbb{P}^n$.

Proposition 3.20. All the closed sets of $\mathbb{P}^n$ are of the form $\mathbb{V}(J)$, where J is a radical homogeneous ideal in $\mathbb{C}[x_0, \ldots, x_n]$, $J \neq \mathfrak{m}_0$.

Proof. We have mentioned the bijection between the closed subsets $Y \subseteq \mathbb{P}^n$ in the quotient topology, and closed $\mathbb{C}^*$-invariant subsets of $\mathbb{A}^{n+1}$ containing 0, $q^{-1}(Y) \cup \{0\}$. By Nullstellensatz, there is a radical $J \subseteq \mathbb{C}[x_0, \ldots, x_n]$ such that $\mathbb{V}(J) = q^{-1}(Y) \cup \{0\}$. Since $\mathbb{V}(J)$ is invariant under $\mathbb{C}^*$, then J also has to be invariant under $\mathbb{C}^*$:

- $f \in J \iff (\lambda.)^* f = f(\lambda.\ ) \in J$;
- $f(\lambda \cdot a) = 0 \iff \lambda \cdot a \in \mathbb{V}(f)$.

The statement now follows by Proposition 3.19. □

The preceding proposition justifies the following definition, and proves that it coincides with the quotient topology from $\mathbb{A}^{n+1}$.

Definition 3.21. The *Zariski topology* on $\mathbb{P}^n$ is obtained by declaring the closed sets to be of the form $\mathbb{V}(J)$, for any homogeneous ideal $J \subseteq \mathbb{C}[x_0, \ldots, x_n]$, $J \neq \mathfrak{m}_0$.

We have also the following correspondence:

[6] For instance if $h = (3x^2 + y^2) + (x^4)$, I mean $3x^2 + y^2$ and x^4 are homogeneous summands of h.

$$\begin{tikzcd} \{\text{Projective Algebraic Varieties in } \mathbb{P}^n\} \arrow[d, bend left, "\mathbb{I}"] \\ \{\text{Homogeneous Radical Ideals in } \mathbb{C}[x_0,\dots,x_n] \text{ other than } \mathfrak{m}_0\} \arrow[u, bend left, "\mathbb{V}"] \end{tikzcd}$$

More precisely,

Theorem 3.22 (The Homogeneous Nullstellensatz)**.**

(a) For any projective variety $Y \subseteq \mathbb{P}^n$, we have $\mathbb{V}(\mathbb{I}(Y)) = Y$.

(b) For any homogeneous ideal $J \neq \mathfrak{m}_0$, $\mathbb{I}(\mathbb{V}(J)) = \sqrt{J}$.

Proof. We have already discussed different pieces of the proof, but for clarity and completeness, we include the proof here.

(a) Let $q : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$, be the quotient map. We have $\mathbb{I}(Y) = \mathbb{I}(q^{-1}(Y) \cup \{0\}) \subseteq \mathbb{C}[x_0, \dots, x_n]$, which is a homogeneous (and a radical) ideal. Moreover,

$$\mathbb{V}(\mathbb{I}(q^{-1}(Y) \cup \{0\})) = q^{-1}(Y) \cup \{0\} \subseteq \mathbb{A}^{n+1}.$$

Therefore, in $\mathbb{P}^n$, $\mathbb{V}(\mathbb{I}(Y)) = Y$.

(b) When $J \neq \mathfrak{m}_0$, the variety $\mathbb{V}(J)$ can be considered in both $\mathbb{P}^n$ and $\mathbb{A}^{n+1}$. By Nullstellensatz in the affine case, $\mathbb{I}(\mathbb{V}(J)) = \sqrt{J}$. One can also check that $\sqrt{J}$ is also a homogeneous ideal.

$\square$

Exercise 3.23. Prove that if J is homogeneous, then so is $\sqrt{J}$.

Exercise 3.24. Prove that any two distinct lines in $\mathbb{P}^2$ meet exactly at one point.

Exercise 3.25. Are the homogeneous polynomials honest functions from $\mathbb{P}^n \longrightarrow \mathbb{C}$?

Exercise 3.26. Would you rather be an architect or a structural engineer?

3.4 The Projective Closure of an Affine Variety

Recall from Example 3.2, that we viewed $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$. We have

$$\mathbb{P}^1 = \begin{cases} U_0 := \{[1 : x_1/x_0]\} = \{[1 : a] : a \in \mathbb{C}\} & \text{if } x_0 \neq 0; \\ \{[0 : x_1] : x_1 \neq 0\} = \{[0 : 1]\} & \text{if } x_0 = 0. \end{cases}$$

Obviously, x_0 is a homogeneous polynomial and $\mathbb{V}(x_0)$ is closed in $\mathbb{P}^1$. Therefore, its complement U_0 is open in $\mathbb{P}^1$. We can now ask what is the closure of U_0 in $\mathbb{P}^1$? The answer is $\mathbb{P}^1$, since the smallest closed set in $\mathbb{P}^1$ containing U_0 is $\mathbb{V}(0)$. To see this, assume that $\mathbb{V}(I) \subseteq \mathbb{P}^1$ is the closure of U_0, then if $g(x_0, x_1) \in I$, then $g_{|U_0}$ is a polynomial in $\mathbb{C}[a]$ vanishing on $U_0 \simeq \mathbb{A}^1$. Therefore, it has to be zero!

Now we can ask a more general question. Assume $V \subset \mathbb{A}^n$, and we have identified $\mathbb{A}^n$ as above with U_0. What is the closure of $V \subseteq U_0$ in $\mathbb{P}^n$? This set is called the *projective closure of* V.

Before stating the main theorem, let us go through another example: let $\ell := \{(a, b) : a + b + 1 = 0\} \subseteq \mathbb{A}^2$. To find the closure $\overline{\ell} \subseteq \mathbb{P}^2$, note that if $\overline{\ell} = \mathbb{V}(g)$, for $g \in \mathbb{C}[x, y, z]$, then g must be a homogeneous polynomial such that $g_{|U_0}$ vanishes on ℓ. So a good candidate for g is $x + y + z$, because on U_z, $z \neq 0$ and $[x : y : z] = [x/z : y/z : 1]$. By replacing $a = x/z$ and $b = y/z$, we must $a + b + 1 = 0$ on ℓ. Note that $(x + y + z)(x)$ or $(x + y + z)^2$ also vanish on ℓ when we restrict them to U_z but they are not 'minimal'. We can alternatively identify $U_z = \{z \neq 0\}$ with the points where $z = 1$, by choosing one representative of U_z and think that

$$x + y + z_{|U_z} = x + y + 1.$$

Note also that $x + y + z$ even though it has a well-defined zero set in $\mathbb{P}^2$, it is not a function $\mathbb{P}^2 \longrightarrow \mathbb{C}$, but when we fix a representative on U_z, then it becomes an honest function! We now need a general procedure to go from $x + y + 1$ to $x + y + z$. This process is called *homogenisation*, and it is very simple:

Definition 3.27. Given a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$, of degree d, the *homogenisation* of f gives a homogeneous polynomial $\tilde{f} \in \mathbb{C}[x_0, \dots, x_n]$, of degree d satisfying

$$\tilde{f}(x_0, \dots, x_n) = x_0^d f(x_1/x_0, \dots, x_n/x_0).$$

For instance, homogenising $x + y + z^3 + 4xy$ gives $xu^2 + yu^2 + z^3 + 4xyu$, which is obtained by compensating for the lower-degree terms with powers of a new variable.

Note that if a variety $V = \mathbb{V}(I) \subseteq \mathbb{A}^n$ is defined by an ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$, then by Hilbert's Basis Theorem, there exist finitely many generators $I = (g_1, \dots, g_k)$, and we have $V = \mathbb{V}(g_1, \dots, g_k)$. However, in general, as Example 3.35 below shows, we do not necessarily have

$$\overline{V} = \mathbb{V}(\tilde{g}_1, \dots, \tilde{g}_k).$$

Thus, while $\mathbb{I}(\overline{V}) \subseteq \mathbb{C}[x_0, \dots, x_n]$ is finitely generated, the polynomials $\tilde{g}_1, \dots, \tilde{g}_k$ may not form a generating set. The following theorem, however, shows that one way to find the closure is to homogenise *all* the elements of the ideal I and consider their common zero set.

Theorem 3.28 ([SKKT00, Page 43])**.** Let $V \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ be a closed affine algebraic variety, and $I := \mathbb{I}(V) \subseteq \mathbb{C}[x_1, \ldots, x_n]$. Let

$$A := \{\tilde{f} \in \mathbb{C}[x_0, \ldots, x_n] : f \in I\}$$

We define the *homogenisation* of I obtained as the ideal generated by homogenisation of the elements of I, *i.e.,*

$$\tilde{I} = (A) = (\{\tilde{f} \in \mathbb{C}[x_0, \ldots, x_n] : f \in I\}).$$

Then,

$$\overline{V} = \mathbb{V}(\tilde{I}) \subseteq \mathbb{P}^n.$$

Proof. Note that the set A is almost never an ideal, but the variety of A equals the variety of the ideal generated by A, i.e., $\mathbb{V}(A) = \mathbb{V}(\tilde{I})$.

- $\overline{V} \subseteq \mathbb{V}(A) = \mathbb{V}(\tilde{I})$: Assume that $G \in A$, we show that $\mathbb{V}(G) \supseteq \overline{V}$. By definition, G is the homogenization of a polynomial g in I. Therefore, we can obtain g from G by setting $x_0 = 1$. Define $U_0 = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 \neq 0\}$. Then, on U_0, we have $G_{|U_0} = g$. Since $G_{|U_0}(V) = g(V) = 0$, the closed set $\mathbb{V}(G)$ contains V, and consequently, its closure satisfies $\overline{V} \subseteq \mathbb{V}(G)$.

- $\overline{V} \supseteq \mathbb{V}(\tilde{I})$: We show the equivalent statement $\mathbb{I}(\overline{V}) \subseteq \mathbb{I}(\mathbb{V}(\tilde{I})) = \sqrt{\tilde{I}} = \tilde{I}$[7]. By Proposition 3.20, the ideal $\mathbb{I}(\overline{V})$ is homogeneous, and by Proposition 3.19, it can be generated by homogeneous polynomials. Therefore, if $G \in \mathbb{I}(\overline{V})$ is such a generator, then it is homogeneous and vanishes on $\overline{V}$. Consequently, $G_{|U}$ must vanish on $\overline{V} \cap U = V$. Moreover, $G_{|U}$ is given by a polynomial of the form $G(1, x_1, \ldots, x_n) = g \in \mathbb{C}[x_1, \ldots, x_n]$. Hence, $g \in I = \mathbb{I}(V)$. By definition, the homogenization of g, $\tilde{g} \in A \subseteq \tilde{I}$. Note that

  $$\tilde{g} = x_0^{\deg(g)} G(1, x_1/x_0, \ldots, x_n/x_0) = x_0^{\deg(g)-\deg(G)} G(x_0, x_1, \ldots, x_n).$$

  Since $\deg(G) \geq \deg(g)$, it follows that

  $$G = x_0^{\deg(G)-\deg(g)} \tilde{g} \in \tilde{I}.$$

□

Remark 3.29. The $x+y+z \in \mathbb{C}[x, y, z]$ is not a well-defined function from $\mathbb{P}^2 \longrightarrow \mathbb{C}$. For instance, $[1 : 1 : 1] = [2 : 2 : 2]$ but $1 + 1 + 1 \neq 2 + 2 + 2$. We can, however, consider $x + y + z$ as a function $\mathbb{P}^2 \longrightarrow \mathbb{P}^1$. Such functions can be also treated as *sections of vector bundles* which we do not study in this course.

Exercise 3.30. Prove that if $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is radical, then so is its homogenisation $\tilde{I} \subseteq \mathbb{C}[x_0, x_1, \ldots, x_n]$.

Exercise 3.31. Prove that if V is an irreducible affine variety, then so is its projective closure $\overline{V}$.

[7] See Exercise 3.30.

Exercise 3.32. Prove that projective closure in the Zariski topology coincides with the projective closure in the Euclidean topology.

Exercise 3.33. Consider the varieties of the polynomials $x + y + 1, x^2 + 6y^2 + 1$, $x^2 + 3y + 1$, $x^3 + 3xy^2 + 4$.

(a) Calculate the projective closures in $\mathbb{P}^2$.

(b) Determine whether or not each of the projective closures includes the points

(i) $[1 : 0 : 0]$;

(ii) $[0 : 1 : 0]$;

(iii) $[0 : 0 : 1]$.

(c) Can you find a general necessary and sufficient condition on g such that its homogenisation $\tilde{g}$ does not pass through any of the three points in item (b)?

Exercise 3.34. Prove that $\mathbb{P}^n$ is compact with the induced Euclidean topology from $\mathbb{A}^{n+1}$. Explain why projective closure is sometimes called projective compactification.

Example 3.35 ([SKKT00, Page 43])**.** Let us revisit the example of twisted cubic in Examples 1.8.4 and 2.41.(a). The twisted cubic is given by $C = \mathbb{V}(y - x^2, z - xy)$. $C \subseteq \mathbb{A}^3$ can be paramertrised by $\mathbb{A}^1 \ni t \longrightarrow (t, t^2, t^3) \in \mathbb{A}^3$. Homogenisation of the generators of this ideal are $wy - x^2$ and $wz - xy$. We have that $\mathbb{V}(wy - x^2) \cap \mathbb{V}(wz - xy) = \overline{V} \cup \{[x : y : z : w] \in \mathbb{P}^3 : w = x = 0\}$. As we remarked earlier, this example highlights that homogenising a set of generators of I does not always give rise to the set of generators of $\tilde{I}$. $\triangle$

Exercise 3.36 (Understanding the projective closure of a hypersurface)**.** Consider $V = \mathbb{V}(x^2y + x^3 + 1 + y^2) \subset \mathbb{A}^2$. Homogenise the equation with variable z. What is the defining equation for $\overline{V} \subseteq \mathbb{P}^2$? What is the defining equation of $\overline{V} \cap U_z$? What happens to the equation when $z \longrightarrow 0$? What is the equation of all the extra points in $\overline{V} \setminus V$?

3.5 Morphisms of Projective Varieties

Definition 3.37. Let $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^n$ be projective algebraic varieties. We say that the map $\varphi : V \longrightarrow W$ is a *morphism of projective varieties* if for each $p \in V$, there exist

(a) an open subset $U \subseteq V$ with $p \in U$;

(b) homogeneous polynomials $\varphi_0, \dots, \varphi_m : U \longrightarrow W$ of the same degree,

such that $\varphi_{|_U} = [\varphi_0 : \cdots : \varphi_m]$.

Example 3.38. Let us consider the affine algebraic variety $V = \mathbb{V}(y - x^2) \subseteq \mathbb{A}^2$. It is easy to check that $V \simeq \mathbb{A}^1$. If we take the projective closure of V, $\overline{V} \subset \mathbb{P}^2$, then $\overline{V} = \mathbb{V}(yz - x^2)$. We can therefore understand $\mathbb{V}(yz - x^2)$ as the union of the following sets (which are not open charts)

$$\begin{cases} \{[x : y : z] \in \mathbb{P}^2 : yz - x^2 = 0, z \neq 0\} \\ \{[x : y : z] \in \mathbb{P}^2 : yz - x^2 = 0, z = 0\} = \{[0 : 1 : 0] \in \mathbb{P}^2\}. \end{cases}$$

Recalling that $\mathbb{P}^2 = \mathbb{C}^2 \sqcup \mathbb{P}^1$, $\overline{V}$ is the union of $V = \overline{V} \cap U_z$ and the extra point at infinity. Similarly, the closure of $\mathbb{A}^1 \subseteq \mathbb{P}^1$, has one extra point at infinity and $\overline{\mathbb{A}^1} = \mathbb{P}^1$. We may seek a continuous map

$$\begin{aligned} \mathbb{V}(y - x^2) \sqcup \{[0 : 1 : 0]\} &\longrightarrow \mathbb{A}^1 \sqcup \{\infty\}, \\ (t, t^2) &\longmapsto t, \\ [0 : 1 : 0] &\longmapsto \infty. \end{aligned}$$

In fact, we cannot find a globally defined polynomial map $\varphi : [\varphi_1 : \varphi_2] : \overline{V} \longrightarrow \mathbb{P}^1$, that is continuous. For instance, when $z \neq 0$, $\varphi(x, y, z) = [\varphi_1(x/z, y/z, 1) : \varphi_2(x/z, y/z, 1)]$ however, we want this polynomial map to be well-defined and continuous as $z \longrightarrow 0$. Therefore, in the chart where $z = 0$, we need to 'hide' z so that the polynomials φ_1, φ_2 are well-defined. It is now easy to see that the map

$$\begin{aligned} \varphi : \overline{V} &\longrightarrow \mathbb{P}^1 \\ [x : y : z] &\longmapsto \begin{cases} [y : z] & \text{if } z \neq 0, \\ [x : y] & \text{if } x \neq 0. \end{cases} \end{aligned}$$

defines an isomorphism.

$\triangle$

Example 3.39. When $V \subseteq \mathbb{A}^n$ is an algebraic variety, its coordinate ring $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]$ can be interpreted at all the polynomial functions $\mathbb{C}[x_1, \ldots, x_n]_{|V}$. However, when f is a polynomial and Y is an irreducible projective variety $f_{|Y} : Y \longrightarrow \mathbb{C}$ is constant. We might still want to define the coordinate ring of Y by $\frac{\mathbb{C}[x_0, \ldots, x_n]}{\mathbb{I}(Y)}$, but the difficulty here is that when $Y, Z \in \mathbb{P}^n$ are two projective algebraic varieties then $\mathbb{C}[Y] \simeq \mathbb{C}[Z]$ does not imply $Y \simeq Z$. For instance, in the previous example we saw that $\mathbb{V}(zx - y^2)$ and $\mathbb{P}^1 = \mathbb{I}(0)$, are isomorphic. However, $\frac{\mathbb{C}[x,y,z]}{\mathbb{I}(zx - y^2)}$, and $\mathbb{C}[x, y]$ are not isomorphic, since the corresponding affine cones, *i.e.*, the affine algebraic varieties in $\mathbb{A}^3$ and $\mathbb{A}^2$ given by $\mathbb{V}(xz - y^2)$ and $\mathbb{V}(0)$ are not isomorphic, since $\mathbb{V}(xz - y^2)$ looks like a cone, with an apex at the origin (a singularity), but $\mathbb{A}^2$ has no singular points.[8] $\triangle$

In Section 4 we explain how to define a good notion of "coordinate rings" for projective varieties by glueing 'local' ones.

Exercise 3.40. Prove that $\mathbb{V}(y) \subseteq \mathbb{A}^2$ and $\mathbb{V}(y - x^3) \subseteq \mathbb{A}^2$ are isomorphic, but their projective closures are not.

[8]In any case, some books defined the coordinate ring of a projective variety Y as $\mathbb{C}[Y] = \mathbb{C}[x_0, \ldots, x_n]/\mathbb{I}(Y)$, but as we have discussed it does not have nice functorial properties of the affine case in Section 2.8.

3.6 Why do we care about the Projective Varieties?

It is easy to see that projective spaces are compact with respect to the induced Euclidean topology from $\mathbb{A}^{n+1}$. Therefore, their Zariski closed subsets, the projective varieties, are also compact with respect to the Euclidean topology. Compactness properties simplify many theorems. For instance, two distinct lines in $\mathbb{P}^2$ always meet at exactly one point, which is not true in the affine case. There are also many topological properties of projective varieties, which are not true in the affine case. Several important conjectures, such as Grothendieck Standard Conjectures and The celebrated Hodge Conjecture are proposed in the projective case.

Let us mention the following two important theorems that only hold in the projective setting.

Theorem 3.41 (Chow Lemma)**.** Assume that $X \subseteq \mathbb{P}^n$ is an analytic subvariety of $\mathbb{P}^n$, that is, X is locally given by an analytic equation. Then, if X is compact in the Euclidean topology, then $X \subseteq \mathbb{P}^n$ is algebraic.

Chow's Lemma certainly does not hold in the affine case. For instance, $\mathbb{V}(y - e^x)$ is an affine analytic variety, and it cannot be described as a zero set of any polynomial equation. Chow Lemma has been generalised enormously by Jean-Pierre Serre in a famous article known as GAGA.

Theorem 3.42 (Bézout Theorem)**.** Let $f_1, f_2 \in \mathbb{C}[x_0, x_1, x_2]$ two homogeneous polynomials of degree d_1 and d_2, respectively. Let $Z_1 = \mathbb{V}(f_1) \subseteq \mathbb{P}^2$ and $Z_2 = \mathbb{V}(f_2) \subseteq \mathbb{P}^2$, be the projective curves associated to f_1 and f_2. Assume that Z_1 and Z_2 have no common irreducible components. Then, the number of intersection points of Z_1 and Z_2 *counted with multiplicity* is given by $d_1 d_2$.

Before ending this chapter, let us define the notion of dimension and degree for projective varieties.

Definition 3.43. (a) The dimension of an irreducible projective variety is the dimension of any affine open subsets.

(b) The *degree* of an irreducible projective variety $Y \subseteq \mathbb{P}^n$ is the number of intersection points (counted with multiplicity) of V with any linear subvariety $L \subseteq \mathbb{P}^n$ such that $\dim(L) + \dim(Y) = n$.

Example 3.44. The analytic variety $V = \mathbb{V}(y - \sin(x))$ is not an algebraic variety, since the line $y = 1/2$, intersects V at infinitely many points, but Bézout tells us that algebraic varieties have a finite degree. △

Exercise 3.45. Is it possible to find the projective closure of the analytic variety $\mathbb{V}(y - \sin(x))$ in $\mathbb{P}^2$? Would this contradict the Chow Lemma?

Exercise 3.46. Prove that a morphism of projective varieties defined in 3.37 is a continuous map.
Hint. Prove that if $X = \bigcup U_i$, for U_i open and for every i, $Z \cap U_i$ is closed in U_i, then Z is closed in X.

4 Quasi-Affine and Quasi-Projective Varieties

Definition 4.1.

(a) Any Zariski open subset of an affine algebraic variety is called a *quasi-affine variety.*

(b) Any Zariski open subset of a projective variety is called a *quasi-projective variety.*

In other words, a quasi affine (respectively quasi-projective) variety is a *locally closed* subset of $\mathbb{A}^n$ (respectively $\mathbb{P}^n$). Recall that in a topological space X, a set V is called locally closed, if there exists an open subset $U \subseteq X$ and a closed subset $Z \subseteq X$, such that $V = U \cap Z$. From now on, the word *variety* means any affine, quasi-affine, projective, or quasi-projective variety.

Exercise 4.2. Prove that every quasi-affine variety is a quasi-projective variety.

Exercise 4.3. Prove that any open set in an irreducible projective variety is dense.

4.1 Regular Functions

For a few sections, we mainly follow [Har77, Section I.3].

4.1.1 Regular Functions on Quasi-Affine Algebraic Varieties

Definition 4.4. Let $V \subseteq \mathbb{A}^n$, be a closed affine algebraic variety, and $U \subseteq V$ open. A function $f : U \longrightarrow \mathbb{C}$, is called *regular at a point* $p \in U$, if there is an open neighbourhood $U' \subseteq U$, and polynomials $g, h \in \mathbb{C}[x_1, \dots, x_n]$, such that $h(p) \neq 0$, for any $p \in U'$, and $f_{|_{U'}}(p) = \frac{g(p)}{h(p)}$. We say that f is *regular* on U if it is regular at every point of U. The set of regular functions on $U \subseteq V$ is denoted by $\mathcal{O}_V(U)$.

Example 4.5. (a) The function

$$\begin{aligned} f_1 : \mathbb{A}^1 \setminus \{0, 1\} &\longrightarrow \mathbb{C} \\ z &\longmapsto \frac{(z-2)(z-3)}{(z-1)} \end{aligned}$$

is in $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1 \setminus \{0, 1\})$.

(b) Let $f_2 : \mathbb{A}^1 \longrightarrow \mathbb{C}$,

$$f_2(z) = \begin{cases} \frac{(z-1)(z-3)}{(z-1)} & z \in \mathbb{A}^1 \setminus \{1\} \\ \frac{(z-2)(z-3)}{(z-2)} & z \in \mathbb{A}^1 \setminus \{2\} \end{cases}.$$

Then $f_2 \in \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1)$. We can see that the values of $f_2(z)$ coincides with $z - 3 \in \mathbb{C}[\mathbb{A}^1]$.

(c) For a closed affine algebraic variety V, if $f \in \mathbb{C}[V]$ then $\frac{1}{f} \in \mathcal{O}_V(V \setminus \mathbb{V}(f))$.

(d) f_1 and f_2 are continuous.

△

Lemma 4.6. A regular function is continuous when $\mathbb{C}$ is identified with $\mathbb{A}^1$.

Proof. Let $f : Y \longrightarrow \mathbb{C}$ be a regular function. Since closed sets in $\mathbb{A}^1$ are either the empty set, $\mathbb{A}^1$ or a union of finitely many points, it suffices to show that the inverse image of only one point on $\mathbb{A}^1$ is a closed set in Y. Let $a \in \mathbb{A}^1$, $f^{-1}(a) = \{p \in Y : f(p) = a\}$. Similar to Exercise 3.46, we can check the closed-ness locally. Let $q \in f^{-1}(a)$. Since f is regular at q, there is a neighbourhood U of q such that for all $x \in U$, $f_{|U}(x) = \frac{g(x)}{h(x)} = a$, or $ah(x) - g(x) = 0$. As a result, $f^{-1}(a) \cap U = \mathbb{V}(ah - g) \cap U$, which is closed in U. $\square$

Now, if you have taken the course of Algebraic Geometry to only care about polynomial functions, you might not be happy to see the regular functions. However,

- The local nature of regular functions is essential in algebraic geometry for glueing the pieces affine algebraic varieties;
- (Global) regular functions on affine algebraic varieties (and not only proper open subsets) and projective algebraic varieties, do behave like polynomial functions as we will see in Theorems 4.7, 4.14.

4.1.2 A Basis for Zariski Topology of Affine Varieties

Recall that a basis for a topology is a collection $\mathcal{B}$ of open subsets of a topological space X such that every open set U in X can be written as a union of a collection of elements in $\mathcal{B}$. Note that for any polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, the set

$$D(f) := \mathbb{A}^n \setminus \mathbb{V}(f),$$

is an open subset in $\mathbb{A}^n$. Note that $D(f_1) \cup D(f_2) = \big(\mathbb{A}^n \setminus \mathbb{V}(f_1)\big) \cup \big(\mathbb{A}^n \setminus \mathbb{V}(f_2)\big) = \mathbb{A}^n \setminus (\mathbb{V}(f_1) \cap \mathbb{V}(f_2))$. As a result, if $I = (f_1, \ldots, f_k) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is an ideal, then

$$\mathbb{A}^n \setminus \mathbb{V}(I) = \bigcup_{i=1}^{k} D(f_i).$$

In consequence, the sets of the form $D(f)$ form a basis for the Zariski topology of $\mathbb{A}^n$. Replacing A^n with $V = \mathbb{V}(g_1, \ldots, g_\ell) \subseteq \mathbb{A}^n$, we can simply check that the sets of the form $V \cap D(f) = V \setminus \mathbb{V}(f)$ for $f \in \mathbb{C}[x_1, \ldots, x_n]$ or equivalently $D(g)$ for $g \in \mathbb{C}[V]$,[9] also form a basis for any closed affine algebraic variety V.

Theorem 4.7. Let V be an irreducible Zariski closed subset of $\mathbb{A}^n$. Then

$$\mathcal{O}_V(V) = \mathbb{C}[V].$$

Proof.
- $\mathcal{O}_V(V) \supseteq \mathbb{C}[V]$: this is easy. Recall from Section 2.6 that any function $f \in \mathbb{C}[V]$ can be understood as a polynomial function restricted to V. Therefore, $f = \frac{f}{1} \in \mathcal{O}_V(V)$.

[9] since we can think of $\mathbb{C}[V]$ as polynomial functions restricted to V.

- $\mathcal{O}_V(V) \subseteq \mathbb{C}[V]$. This part has several beautiful ideas. Let $g \in \mathcal{O}_V(V)$. By definition, every point $p \in V$ has a neighbourhood U_p such that on $g_{|U_p} = \frac{h}{k}$ where $h, k \in \mathbb{C}[V]$ and k does not vanish on U_p. Now, since the sets of the form $D(f) = \mathbb{A}^n \setminus \mathbb{V}(f)$ form a basis for the the Zariski topology on V, by shrinking the neighbourhood if necessary, we can assume that each U_p is of the form $D(f)$. Let us now cover V, with the open sets of the form $\{D(f_j)\}_{j\in J}$. We know from the First Assessed Coursework that V is compact with respect to the Zariski topology, and there is a finite subcover of $V = \bigcup_{i=1}^{\ell} D(f_i)$. In sum, for each $i = 1, \dots, \ell$, there the regular function g agrees with some $\frac{h_i}{k_i}$ on $D(f_i)$, and k_i is always non-zero on $D(f_i)$. Since $\bigcup_{i=1}^{\ell} D(f_i)$ is a cover for V, we must have $\bigcap_{i=1}^{\ell} \mathbb{V}(k_i) \cap V = \varnothing$. By Hilbert's Nullstellensatz, the ideal generated by k_i's must be a unit ideal in $\mathbb{C}[V]$. Therefore, there exist $r_i \in \mathbb{C}[V]$, such that

$$1 = r_1k_1 + \cdots + r_\ell k_\ell.$$

Now, note that for all pairs i, j we have

$$g = \frac{h_i}{k_i} = \frac{h_j}{k_j} \quad \text{on } D(f_i) \cap D(f_j).$$

Since $D(f_i) \cap D(f_j) \subseteq V$ is dense, we have the equality $k_jh_i - k_ih_j = 0$, on all V. On each $D(f_i)$, k_i is non-vanishing and $g = \left(\frac{h_i}{k_i}\right)$. We obtain

$$\begin{aligned} g = 1 \cdot g = \big(r_1k_1 + \cdots + r_\ell k_\ell\big) \cdot \big(\frac{h_i}{k_i}\big) &= \big(r_1k_1h_i + \cdots + r_\ell k_\ell h_i\big)\big(\frac{1}{k_i}\big) \\ &= \big(r_1k_ih_1 + \cdots + r_\ell k_ih_\ell\big)\big(\frac{1}{k_i}\big) =: G. \end{aligned}$$

Thus $g = G$ as regular functions on $D(f_i)$, and by Lemma 4.6, $G = g$ on V. Moreover, clearly, $G = r_1h_1 + \cdots + r_\ell h_\ell \in \mathbb{C}[V]$.

□

Exercise 4.8 (glueing property of regular functions)**.** Prove that if V is an affine algebraic variety, $U_1, U_2 \subseteq V$ open subsets, and $f_1 \in \mathcal{O}_V(U_1), f_2 \in \mathcal{O}_V(U_2)$, with $f_{1|_{U_1\cap U_2}} = f_2|_{U_1\cap U_2}$, then there exists a regular function $f \in \mathcal{O}_V(U_1 \cup U_2)$ such that

$$f_{|_{U_1}} = f_1, \quad f_{|_{U_2}} = f_2.$$

4.1.3 Regular Functions on Quasi-Projective Algebraic Varieties

Definition 4.9. Let $Y \subseteq \mathbb{P}^n$, a (closed) projective algebraic variety, and $U \subseteq Y$ open. A function $f : U \longrightarrow \mathbb{C}$, is called *regular at a point* $p \in Y$, if there is an open neighbourhood $U' \subseteq U$, and homogeneous polynomials $g, h \in \mathbb{C}[x_1, \dots, x_n]$, of the same degree, such that $h(p) \neq 0$, for any $p \in U'$, and $f_{|_{U'}}(p) = \frac{g(p)}{h(p)}$. We say that f is *regular* on U if it is regular at every point of U. The set of regular functions on $U \subseteq Y$ is denoted by $\mathcal{O}_Y(U)$.

Exercise 4.10. Let $Y \subseteq \mathbb{P}^n$ be a projective variety. Prove that the open sets of the form $D(f) = Y \setminus \mathbb{V}(f)$, for $f \in \mathbb{C}[x_0, \dots, x_n]$ homogeneous, also form a basis for the Zariski topology in Y.

Exercise 4.11. Let f be a regular function on a quasi-projective variety X. Prove that it is continuous. Deduce that if f and g are regular on an irreducible variety X, and $f_{|_U} = g_{|_U}$ on an open subset $U \subseteq X$, then $f = g$ on X.

The following definition is, in fact, equivalent to Definition 3.37, for projective varieties.

Definition 4.12. Let X, Y be two varieties (*i.e.*, affine, quasi-affine, projective or quasi-projective). A morphism $\varphi : X \longrightarrow Y$, a map such that

(a) φ is continuous;

(b) For any for every open set $V \subseteq Y$, and for every regular function $f \in \mathcal{O}_Y(V)$, $\varphi^*(f) = f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$.

Exercise 4.13. Prove that $\xi_i : U_i \longrightarrow \mathbb{A}^n$, for all i, defined in the proof of Theorem 3.6 are isomorphisms.

We state the following theorem without proof, which implies that regular functions on projective spaces are very restricted.

Theorem 4.14. Let Y be an irreducible Zariski closed subset of $\mathbb{P}^n$. Then

$$\mathcal{O}_Y(Y) = \mathbb{C}.$$

We leave the proof of the following theorem as an exercise.

Theorem 4.15. Let X be an algebraic variety, $Y \subseteq \mathbb{A}^n$ a closed affine algebraic variety, and $\varphi : X \longrightarrow Y$ a map of sets. Then, $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a morphism, if and only if, for all i, $\varphi_i \in \mathcal{O}_X(X)$.

Example 4.16. Let $V = \mathbb{V}(xy - 1) \subseteq \mathbb{A}^2$, and $D(x) = \mathbb{A}^1 \setminus \{0\}$. By definition the map

$$\begin{aligned} \psi : V &\longrightarrow D(x) \\ (x, y) &\longmapsto x, \end{aligned}$$

is a morphism, since

- ψ is the restriction of the projection onto the first coordinate. Note that restriction of continuous maps remains continuous;
- $U \subseteq D(x)$, and $f \in \mathcal{O}_{D(x)}(U)$ is regular, $\psi^*(f) = f \circ \psi = f$ is also regular on $\psi^{-1}(U) = \{(x, y) : x \in U, y = 1/x\}$.

By Theorem 4.15, the inverse of ψ given by

$$\begin{aligned} \varphi : D(x) &\longrightarrow V \\ x &\longmapsto (x, \frac{1}{x}), \end{aligned}$$

is also a morphism, since x and $1/x$ are indeed regular. Therefore, we have an isomorphism of varieties. By Theorem 4.7,

$$\mathcal{O}_V(V) = \mathbb{C}[V] = \frac{\mathbb{C}[x, y]}{(xy - 1)}.$$

Therefore, $\mathcal{O}_{D(x)}(D(x)) = \varphi^*(\mathcal{O}_V(V)) = \varphi^*(\frac{\mathbb{C}[x,y]}{(xy-1)})$, which equals $\mathbb{C}[x, 1/x]$, since $y \in \mathcal{O}_V(V)$ and $\varphi^*(y) = y \circ \varphi = y \circ (x, 1/x) = 1/x$. $\mathcal{O}_{D(x)}(D(x))$ can be understood as the *coordinate ring* of the *quasi-affine* variety $\mathbb{A}^1 \setminus \{0\} \simeq \mathbb{C}^*$. At first sight, $\frac{1}{x}$ does not look like a polynomial, but it does indeed behave like a polynomial on V, as we have $xy - 1 = 0$ and $\frac{1}{x} = y$. The identification $V = \text{maxSpec}(\mathbb{C}[V])$ in Section 2.7, also implies

$$\mathbb{C}^* = \text{maxSpec}(\mathbb{C}[x, 1/x]).$$

$\triangle$

Exercise 4.17.

(a) We have $\mathbb{C}[x] \subseteq \mathbb{C}[x, 1/x]$ as $\mathbb{C}$-algebras. Determine all the elements in $\text{maxSpec}(\mathbb{C}[x]) \setminus \text{maxSpec}(\mathbb{C}[x, 1/x])$.

(b) Consider the isomorphism $\varphi : \mathbb{A}^1 \setminus \{0\} \longrightarrow \mathbb{A}^1 \setminus \{0\}$, $a \longmapsto b = 1/a$, and the pullback map on the coordinate rings $\varphi^* : \mathbb{C}[x, 1/x] \longmapsto \mathbb{C}[y, 1/y]$. Compute $\varphi^*(1/x)$, $\varphi^*(2x^2 + \frac{2x^3+4x}{x^5})$, $\varphi^*(2 - x)$.

We can easily generalise the preceding example to show that:

"Any open subset $D(f) \subseteq \mathbb{A}^n$ is isomorphic to a closed subset of $\mathbb{A}^{n+1}$."

Namely,

$$\begin{aligned} \varphi : D(f) = \mathbb{A}^n \setminus \mathbb{V}(f) &\longrightarrow \mathbb{A}^{n+1} \\ (x_1, \ldots, x_n) &\longmapsto \Big(x_1, \ldots, x_n, \frac{1}{f(x_1, \ldots, x_n)}\Big), \end{aligned}$$

- $\varphi(D(f))$ is closed in $\mathbb{A}^{n+1}$;
- $\varphi : D(f) \longrightarrow \varphi(D(f))$ is a morphism.

To see these, assume that $\mathbb{A}^{n+1}$ has the coordinates $(x_1, \ldots, x_n, z)$. We have

- $\varphi(D(f)) = \mathbb{V}(zf - 1)$;
- Since $x_1, \ldots, x_n, f(x_1, \ldots, x_n)$ are all regular functions on $D(f)$, φ is a morphism by Theorem 4.15.

It is also easy to check that

$$\begin{aligned} \psi : \mathbb{V}(zf - 1) &\longrightarrow D(f) \\ (x_1, \ldots, x_n, z) &\longmapsto (x_1, \ldots, x_n) \end{aligned}$$

is a morphism and the inverse to φ.

More generally, similar to above, we can show that if $V = \mathbb{V}(g_1, \ldots, g_\ell)$, the open subset

$$D(f) \cap V = \mathbb{V}(g_1, \ldots, g_\ell) \setminus \mathbb{V}(f),$$

is isomorphic to $W := \mathbb{V}(g_1, \ldots, g_\ell, zf - 1)$. By Theorem 4.7, $\mathcal{O}_V(V) = \mathbb{C}[V] = \frac{\mathbb{C}[x_1,\ldots,x_n]}{\mathbb{I}(V)} = \frac{\mathbb{C}[x_1,\ldots,x_n]}{(g_1,\ldots,g_\ell)}$, and we can understand the coordinate ring

$$\mathbb{C}[D(f) \cap V] := \mathbb{C}[V]_f := \frac{\mathbb{C}[x_1, \ldots, x_n, 1/f]}{(g_1, \ldots, g_\ell)} = \varphi^*(\mathbb{C}[W]) \simeq \mathbb{C}[W] = \frac{\mathbb{C}[x_1, \ldots, x_n, z]}{(g_1, \ldots, g_\ell, zf - 1)}.$$

Note that by the above isomorphism, we have

$$\mathcal{O}_V[D(f) \cap V] = \varphi^*(\mathcal{O}_W(W)) = \varphi^*(\mathbb{C}[W]) \simeq \mathbb{C}[V]_f.$$

Remark 4.18. When $f \in \mathbb{C}[V]$, then $D(f)$ is naturally defined as $V \setminus \mathbb{V}(f)$.

Exercise 4.19. Show that $\mathrm{GL}_n(\mathbb{C})$ is isomorphic to an affine algebraic variety.

Summary of this section

- We defined quasi affine/projective varieties: open subsets.
- Found a basis for Zariski topology on c.a.a.v $V \subseteq \mathbb{A}^n$: open subsets of the form $D(f) := V \setminus \mathbb{V}(f)$, for $f \in \mathbb{C}[V]$.
- Defined regular functions: roughly, $f(x)/g(x)$ on open subsets, where g doesn't vanish. We needed f, g to be of the same degree in the projective case.
- Defined isomorphism in general: continuous maps and pulling back regular to regular.
- Found the regular functions on closed affine algebraic varieties and their open basis:

$$\mathcal{O}_V(V) = \mathbb{C}[V], \quad \mathcal{O}_V(D(f)) = \mathbb{C}[V]_f = \mathbb{C}[V, 1/f] = \frac{\mathbb{C}[x_1, \ldots, x_n, z]}{(\mathbb{I}(V), zf - 1)}.$$

4.2 Two Examples of Glueing

4.2.1 Obtaining $\mathbb{P}^1$

We have seen in Example 3.2.(e) that we can construct $\mathbb{P}^1$ by glueing two copies of $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\}$, by the map $x \longmapsto x^{-1}$. Let us write this more formally. We have,

- $\xi_0 : U_0 \longrightarrow X_0 := \xi_0(U_0)$, $\xi_1 : U_1 \longrightarrow X_1 := \xi_1(U_1)$, are isomorphisms.
- $X_{01} := \xi_0(U_0 \cap U_1) \subset X_0$.
- $X_{10} := \xi_1(U_1 \cap U_0) \subset X_1$.
- $g_{01} := \xi_1 \circ \xi_0^{-1} : X_{01} \longrightarrow X_{10}, \quad x \longmapsto y = x^{-1}$.

Note that all these sets are open subsets of $\mathbb{P}^1$ and isomorphic to affine algebraic varieties. We have

- $\mathbb{C}[X_0] = \mathcal{O}_{X_0}(X_0) = \mathbb{C}[x]$, $\mathbb{C}[X_1] = \mathcal{O}_{X_1}(X_1) = \mathbb{C}[y]$.

- $\mathbb{C}[X_{01}] = \mathcal{O}_{X_0}(X_{01}) = \frac{\mathbb{C}[x,x']}{(xx'-1)} \simeq \mathbb{C}[x, x^{-1}] \supseteq \mathbb{C}[x]$.
- $\mathbb{C}[X_{10}] = \mathcal{O}_{X_1}(X_{10}) = \frac{\mathbb{C}[y,y']}{(yy'-1)} \simeq \mathbb{C}[y, y^{-1}] \supseteq \mathbb{C}[y]$.

We have now the isomorphism of $\mathbb{C}$-algebras induced by φ :

$$\begin{aligned} g_{01}^* : \mathbb{C}[X_{10}] &\longrightarrow \mathbb{C}[X_{01}] \\ f &\longmapsto f \circ g_{01} = f(y^{-1}) \\ y &\longmapsto x = y^{-1}. \end{aligned}$$

Therefore, we can also think of $\mathbb{P}^1$ as $X_0 \simeq \mathbb{A}^1$ and $X_1 \simeq \mathbb{A}^1$, where X_{01} and X_{10} are glued by the isomorphism g_{01}.

4.2.2 Obtaining $\mathbb{P}^2$

Let $[x_0 : x_1 : x_2]$ denote the homogeneous coordinates of the space $\mathbb{P}^2$. It is covered by three coordinate charts:

- U_0 corresponding to $x_0 \neq 0$, with affine coordinates $(\frac{x_1}{x_0}, \frac{x_2}{x_0}) = (a_1, a_2)$.
- U_1 corresponding to $x_1 \neq 0$, with affine coordinates $(\frac{x_0}{x_1}, \frac{x_2}{x_1}) = (a_1^{-1}, a_1^{-1}a_2)$.
- U_2 corresponding to $x_2 \neq 0$, with affine coordinates $(\frac{x_0}{x_2}, \frac{x_1}{x_2}) = (a_2^{-1}, a_1 a_2^{-1})$.

As before, let $X_i = \xi_i(U_i)$, and $X_{ij} = \xi_i(U_i \cap U_j)$. We have $\mathbb{C}[X_0] = \mathcal{O}_{X_0}(X_0) = \mathbb{C}[a_1, a_2]$, and $\mathbb{C}[X_{01}] = \mathcal{O}_{X_0}(X_{01}) = \mathbb{C}[a_1^{-1}, a_1, a_2]$. Since on X_1, $a_1 \neq 0$, we can write

$$\mathbb{C}[X_1] = \mathcal{O}_{X_1}(X_1) = \mathbb{C}[a_1^{-1}, a_1^{-1}a_2].$$

As a result,

$$\mathbb{C}[X_{10}] = \mathcal{O}_{X_{10}}(X_{10}) = \mathbb{C}[a_1, a_1^{-1}, a_1^{-1}a_2].$$

The isomorphism

$$\begin{tikzcd} \xi_0(U_0 \cap U_1) \ni (a_1, a_2) \arrow[r, mapsto, "\xi_0^{-1}"] \arrow[rr, bend right, "g_{01}:=\xi_1 \circ \xi_0^{-1}"'] & {[1 : a_1 : a_2]} \arrow[r, mapsto, "\xi_1"] & (1/a_1, a_2/a_1) \in \xi_1(U_0 \cap U_1), \end{tikzcd}$$

provides the information for glueing of $X_{01} \simeq \mathbb{C}^* \times \mathbb{C}$ and $X_{10} \simeq X_{01} \simeq \mathbb{C}^* \times \mathbb{C}$ and their corresponding coordinate rings. We can similarly understand the isomorphisms between other charts.

Torus Actions[10]:

- On $U_0 \ni a = (a_1, a_2)$ the action of $t = (t_1, t_2) \in (\mathbb{C}^*)^2$ is obtained by $t \cdot a = (t_1 a_1, t_2 a_2)$.
- On $U_1 \ni b = (b_1, b_2)$ the action of $t = (t_1, t_2) \in (\mathbb{C}^*)^2$ is obtained by $t \cdot b = (t_1^{-1} b_1, t_1^{-1} t_2 b_2)$.

[10] I have added this in relation to the Toric Varieties.

- On $U_2 \ni c = (c_1, c_2)$ the action of $t = (t_1, t_2) \in (\mathbb{C}^*)^2$ is obtained by $t \cdot c = (t_2^{-1} c_1, t_1 t_2^{-1} c_2)$.

This is compatible with the glueing.

Exercise 4.20. (a) Find the rest of the isomorphisms $g_{ij} := \xi_j \circ \xi_i^{-1}$ for glueing in the above example.

(b) Can you use the set of vectors $\{(1,0),(0,1)\},\{(-1,0),(-1,1)\},\{(1,-1),(0,-1)\}$ to simplify your description?
Hint: Use $(a,b) \in \mathbb{Z}^2$ for $x^a y^b$.

4.2.3 Abstract Varieties: Two Perspectives

General abstract varieties can be defined in two ways.

- Defining the abstract varieties similar to manifolds: an *abstract variety* has a set of functions that behave like regular functions, and each point has an open neighbourhood that is isomorphic to an affine algebraic variety. I'd like to compare this to how a structural engineer analyses a building structure.

- Defining the abstract varieties with glueing data: an *abstract variety* is given as
$$X := \bigsqcup X_i / \sim,$$
where X_i's are affine algebraic varieties, such that in their intersections we have compatible glueing data and isomorphisms between $X_i \cap X_j \subseteq X_i$ and $X_j \cap X_i \subseteq X_j$, to define the equivalence or glueing. For instance, in the example of $\mathbb{P}^2$ above, if the sets X_i's and g_{ij}'s were given as the initial data, we could glue them to construct $\mathbb{P}^2$. I'd like to call this an architect point of view.

We discuss the second point of view in more detail in the chapter on Toric Varieties. These two points of view are, in fact, equivalent. See for instance `https://stacks.math.columbia.edu/tag/00AK`, and the details of these two constructions in the lecture notes of Edixhoven and Taelman [ET09] on Blackboard.

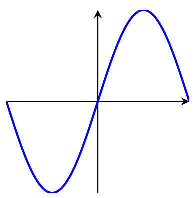
5 Smoothness and Tangent Spaces

Consider the analytic curve $\mathbb{V}(y - sin(x)) \subseteq \mathbb{A}^2$. The Taylor expansion around $x_0 = 0$, is given by

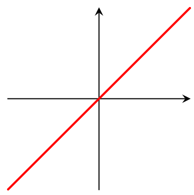
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots.$$

Therefore, the first-order approximation of $sin(x)$ is x and the third-order approximation is $x - (x^3)/6$, which are show in the figures below.

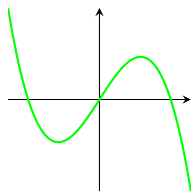
$y = \sin(x)$



$y = x$



$y = x - \frac{x^3}{6}$



The first-order approximation by an affine linear space, having the same dimension as our variety or manifold, defines what is known as the tangent space. Consider, for example, an affine algebraic variety $V = \mathbb{V}(f) \subseteq \mathbb{A}^n$. Intuitively, the tangent space to V at a point $p \in V$ is the affine linear subspace that best approximates V at p.

Formally, a vector $v \in T_pV$ if and only if the point $a + \lambda v$ closely approximates a point in V. In other words, as we slightly move along v, the value $f(a + \lambda v)$ stays near zero. Hence, we have:

$$v \in T_aV \iff (Df(a))v = \langle \nabla f(a), v\rangle = \frac{d}{d\lambda}\bigg|_{\lambda=0} f(a + \lambda v) = 0.$$

The gradient

$$\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a)\right)$$

is perpendicular to the tangent space. If $V = \mathbb{V}(f_1, \dots, f_k) \subseteq \mathbb{A}^n$, then the tangent space at a is the intersection:

$$T_aV = \bigcap_{i=1}^{k} \ker(\nabla f_i(a)).$$

Formally, we state:

Definition 5.1. Let $V = \mathbb{V}(I) = \mathbb{V}(f_1, \dots, f_k) \subseteq \mathbb{A}^n$. For $a \in V$, the tangent space of V at a, denoted T_aV, is given by:

$$\begin{aligned}
T_aV &= \left\{ v \in \mathbb{A}^n : \forall i,\ \frac{\partial f_i}{\partial v}(a) = \frac{d}{d\lambda} f_i(a+\lambda v)\bigg|_{\lambda=0} = 0 \right\} \\
&= \{ v \in \mathbb{A}^n : \forall f \in I,\ \lambda \mapsto f(a+\lambda v) \text{ has order } \geq 2 \} \\
&= \left\{ v \in \mathbb{A}^n : \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1}(a) & \cdots & \frac{\partial f_k}{\partial x_n}(a) \end{pmatrix} v = 0 \right\} \\
&= \left\{ v \in \mathbb{A}^n : \begin{pmatrix} \nabla f_1(a) \\ \vdots \\ \nabla f_k(a) \end{pmatrix} v = 0 \right\}.
\end{aligned}$$

Example 5.2. Consider the affine algebraic variety $V = \mathbb{V}(x^2 + y^2 - z^3) \subseteq \mathbb{A}^3$. At an arbitrary point $a = (a_1, a_2, a_3)$, the gradient is given by:

$$\nabla f(a) = (2a_1,\ 2a_2,\ -3a_3^2).$$

Thus, the tangent space dimension $\dim T_aV$ equals 2 at non-singular points and is smaller otherwise. Particularly, at the singular point $(0,0,0)$, the tangent space dimension is maximal and equal to 3. $\triangle$

Remark 5.3. Note that all of the above definitions make sense over any field even if we do not have a notion of limits as we can symbolically define the partial derivatives for any polynomial over any field.

5.1 Smoothness

Definition 5.4. Assume that $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, is a radical ideal. Choose the generators $I = (f_1, \ldots, f_k)$. Then an affine algebraic subvariety $V = \mathbb{V}(I) \subseteq \mathbb{A}^n$ is *smooth* of dimension d at $x \in V$, if

$$\dim(T_xV) = d.$$

In other words,

$$\operatorname{rank} \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} = n - d.$$

We say that $V = \mathbb{V}(I)$ is a *smooth affine variety*, if it is smooth at all the points $x \in V$.

Definition 5.5 (Smooth Variety)**.** Let X be a variety (affine, quasi-affine, projective, quasi-projective). Then X is said to be *smooth of dimension* d, if for all $a \in X$, there exists an open subset $U \subseteq X$ containing a, which is isomorphic smooth closed affine algebraic variety of dimension d.

Example 5.6. To find the non-smooth points of the curve given az the zero set of $f(x,y) = y^2 - x^3 - x^2$ in $\mathbb{A}^2$, we note that $\mathbb{V}(f)$ is of dimension 1 by Theorem

2.24. We need to find the points where $f(a,b)=0$, and $\dim(\ker(\nabla f(0)))\neq 1$. We compute partial derivatives

$$\frac{\partial f}{\partial x}=-3x^2-2x,\quad \frac{\partial f}{\partial y}=2y.$$

The points where $\ker(\nabla f(0))$ is 2 dimensional is exactly where

$$2y=0\Rightarrow y=0,\quad -3x^2-2x=-x(3x+2)=0\Rightarrow x=0,-\frac{2}{3}.$$

Note that

- $(0,0)$: satisfies the equation.
- $\left(-\frac{2}{3},0\right)$: does not satisfy the equation.

Thus, the only singular point of the curve is $(0,0)$.

△

We state the following without proof:

Theorem 5.7 (Smooth + Connected $\implies$ Irreducible)**.** If X is a connected smooth variety of dimension d, then X is irreducible and $\dim(X)=d$ (as a topological space).

Exercise 5.8. Show that the algebraic subvariety of $\mathbb{A}^2$, given by $xy=0$ is not smooth at the origin.

Exercise 5.9. Show that the conic surface $\mathbb{V}(x^2+y^2-z^2)\subseteq\mathbb{A}^3$, is not smooth at the origin.

5.2 Tangent Spaces: Algebraic Definition

We intend to state an intrinsic and an algebraic definition of the tangent spaces in terms of ideals. To start, as usual, we first define this concept for (closed) affine algebraic varieties. Let $A:=\mathbb{C}[x_1,\ldots,x_n]$, $I=\mathbb{I}(V)\subseteq A$, a radical ideal, and for a point $a=(a_1,\ldots,a_n)\in V\subseteq\mathbb{A}^n$, denote by $\mathfrak{m}:=\mathfrak{m}_a\subseteq A$, the ideal corresponding to $a\in\mathbb{A}^n$, and $\bar{\mathfrak{m}}:=\bar{\mathfrak{m}}_a\subseteq\mathbb{C}[V]=\frac{A}{\mathbb{I}(V)}\simeq\mathcal{O}_V(V)$, the ideal corresponding to $a\in V$. For any $\mathbb{C}$-vector space S, we denote its linear dual by

$$V^*=\operatorname{Hom}(S,\mathbb{C})=\{\text{linear functions from } S \text{ to } \mathbb{C}\}.$$

Example 5.10. In layman's terms, if

$$\mathbb{A}^2=\left\{\begin{pmatrix}a\\b\end{pmatrix}: a,b\in\mathbb{C}\right\},$$

then its dual space is given by

$$(\mathbb{A}^2)^*=\left\{\begin{pmatrix}c & d\end{pmatrix}: c,d\in\mathbb{C}\right\}.$$

Here, an element $\begin{pmatrix}c & d\end{pmatrix}$ from the dual space acts as a linear functional on a vector $\begin{pmatrix}a\\b\end{pmatrix}$, producing:

$$\begin{pmatrix}c & d\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=ca+db.$$

△

We now want to define that the tangent space can be understood completely in the algebraic setting:

- $\mathfrak{m}_a/\mathfrak{m}_a^2$ as a $\mathbb{C}$-vector space, can be identified with $(T_a\mathbb{A}^n)^* \simeq \mathbb{A}^n$;
- $\bar{\mathfrak{m}}_a/\bar{\mathfrak{m}}_a^2$ as a $\mathbb{C}$-vector space, can be identified with $(T_aV)^*$.

In order to show these claims we consider the following binary operation or pairing,

$$\begin{aligned}\langle\cdot,\cdot\rangle : \mathfrak{m} \times T_a\mathbb{A}^n &\longrightarrow \mathbb{C}\,,\\ (f,v) &\longmapsto \frac{\partial f}{\partial v}(a).\end{aligned}$$

Lemma 5.11. $\langle\cdot,\cdot\rangle$ is bilinear and induces a perfect pairing

$$\langle\cdot,\cdot\rangle : \mathfrak{m}_a/\mathfrak{m}_a^2 \times T_a\mathbb{A}^n \longrightarrow \mathbb{C},$$

of $\mathbb{C}$-vector spaces, *i.e.*, each side can be identified with the dual of the other side. In other words, $\mathfrak{m}/\mathfrak{m}^2$ can be identified with the cotangent of $\mathbb{A}^n$ at a.

Example 5.12. Let $\mathbf{0} = (0,0) \in \mathbb{A}^2$ be the origin. Let $\mathfrak{m} := \mathfrak{m}_{\mathbf{0}} = (x,y)$ be the maximal ideal corresponding to the origin. The ideal $\mathfrak{m}_{\mathbf{0}}^2$ has all the polynomials of degree 2 or more. Now consider an example $f = 3x + 5y + x^2 + xyz \in \mathfrak{m}_{\mathbf{0}}$. Then $f - 3x + 5y \in \mathfrak{m}_{\mathbf{0}}/\mathfrak{m}_{\mathbf{0}}^2$, since quotienting by $\mathfrak{m}_{\mathbf{0}}^2$ kills all the degree 2 or higher terms. Now observe that $\nabla f(0,0) = (3,5)$, and we can find a correspondence

$$\nabla f(0,0) = (3,5) \longmapsto 3x + 5y = \overline{f} \in \mathfrak{m}_{\mathbf{0}}/\mathfrak{m}_{\mathbf{0}}^2.$$

Similarly, if $g(0,0) = 0$ then $g \in \mathfrak{m}_{\mathbf{0}}$ if $g(x,y) = ax + by +$ higher order terms, then we have the one-to-one correspondence

$$\nabla g(0,0) = (a,b) \longmapsto ax + by \in \mathfrak{m}_{\mathbf{0}}/\mathfrak{m}_{\mathbf{0}}^2.$$

Since ∇g for $g \in \mathfrak{m}_{\mathbf{0}}$ forms a dual for the tangent space of $\mathbb{A}^2$ at the origin, which is a copy of $\mathbb{A}^2$, we can consider the dual of $\mathfrak{m}_{\mathbf{0}}/\mathfrak{m}_{\mathbf{0}}^2$ as the tangent space, which is the content of the lemma. △

Proof of Lemma 5.11. Without loss of generality, we can assume that $a = 0$.

(a) Note that $T_a\mathbb{A}^n = \mathbb{A}^n$ is a $\mathbb{C}$-vector space.

(b) The dual $(T_a\mathbb{A}^n)^*$ is, by definition, the set of linear functions $f : T_a\mathbb{A}^n \longrightarrow \mathbb{C}$.

(c) $\langle\cdot,\cdot\rangle$ defines a linear map $\Psi : \mathfrak{m}_a \longrightarrow T_a(\mathbb{A}^n)^*$, given by

$$f \longrightarrow \langle f,\cdot\rangle.$$

Obviously,

$$\begin{aligned}\Psi(f) = \langle f,\cdot\rangle : T_a\mathbb{A}^n &\longrightarrow \mathbb{C}\\ v &\longmapsto \langle f,v\rangle = \frac{\partial f}{\partial v}(a).\end{aligned}$$

is linear.

(d) $\ker(\Psi) = \mathfrak{m}_a^2$. Since, we can write any polynomial, as its Taylor expansion at $a = 0 : f(x) = f(a) + \sum b_i x_i +$ (higher degree terms). If $f \in \mathfrak{m}_a$ then $f(a) = 0$, and when $\nabla f(a) = (b_1, \ldots, b_n)$, and $\mathfrak{m}_a^2$ contains all the functions with degree greater than or equal to 2. As a result, the induced map $\overline{\Psi} : \mathfrak{m}_a/\mathfrak{m}_a^2 \longrightarrow (T_a\mathbb{A}^n)^*$ is injective.

(e) As a $\mathbb{C}$-vector space, $\mathfrak{m}_a/\mathfrak{m}_a^2$ is n-dimensional and is spanned by $\{x_1, \ldots, x_n\}$, therefore, $\overline{\Psi} : \mathfrak{m}_a/\mathfrak{m}_a^2 \longrightarrow (T_a\mathbb{A}^n)^*$ is an isomorphism, and we can identify the two vectors spaces.

□

Proposition 5.13. $\langle \cdot, \cdot \rangle$ is bilinear and induces a perfect pairing

$$\langle \cdot, \cdot \rangle : \bar{\mathfrak{m}}_a/\bar{\mathfrak{m}}_a^2 \times T_aV \longrightarrow \mathbb{C},$$

of $\mathbb{C}$-vector spaces

Example 5.14. Assume that $I \subseteq \mathbb{C}[x, y]$ and $0 \in \mathbb{V}(I)$. By definition

$$T_0V = \{v \in \mathbb{A}^2 : \langle \nabla g(0), v \rangle = 0, g \in I\}.$$

Therefore, $\{\nabla g(0), g \in I\}$ is exactly the orthogonal complement of T_0V. However, for $g = ax + by + cx^2 + dxy + ey^2 + ... \in I$,

$$\nabla g(0) = (a, b)$$

Also note that in the quotient $(I + \mathfrak{m}_0^2)/\mathfrak{m}_0^2$ g is exactly given by $ax + by$. The correspondence

$$ax + by \longmapsto (a, b)$$

which is like considering linear functions versus their matrices, now allows us to identify $(I + \mathfrak{m}_0^2)/\mathfrak{m}_0^2$ with $\{\nabla g(0), g \in I\}$ and the orthogonal complement of T_0V. △

Proof. Let us break the proof into three steps for clarity. As before, $\mathfrak{m} := \mathfrak{m}_a, I := \mathbb{I}(V)$.

(a) Recall that

$$T_aV = \big\{v \in \mathbb{A}^n : \forall f \in I \subseteq \mathfrak{m},\ \nabla f(a) \cdot v = 0\big\}.$$

Note that for $f \in \mathfrak{m}^2$, then $\nabla f(a) = 0$, and it does not provide any information for defining T_aV. Therefore, we can take a quotient by $\mathfrak{m}^2$:

$$T_aV = \big\{v \in \mathbb{A}^n : \forall f \in (I + \mathfrak{m}^2)/\mathfrak{m}^2 \subseteq \mathfrak{m}/\mathfrak{m}^2,\ \nabla f(a) \cdot v = 0\big\}.$$

In fact, T_aV can be understood as the orthogonal complement of $(I + \mathfrak{m}^2)/\mathfrak{m}^2$.

(b) Now we use a fact from linear algebra: if $\langle \cdot, \cdot \rangle$ is a perfect pairing between two finite-dimensional $\mathbb{C}$-vectors spaces A and B, $B' \subseteq B$ is a subspace, and $A' \subseteq A$ is the orthogonal complement of B', then we have perfect pairing between A/A' and B' induced by $\langle \cdot, \cdot \rangle$.

(c) We have

$$\big(\mathfrak{m}/\mathfrak{m}^2\big)/\big((I+\mathfrak{m}^2)/\mathfrak{m}^2\big) \simeq \big(\mathfrak{m}/I\big)/\big((I+\mathfrak{m}^2)/I\big) = \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2$$

and the proof is complete by Step (b) by taking $A = \mathfrak{m}/\mathfrak{m}^2$, $A' = \big((I+\mathfrak{m}^2)/\mathfrak{m}^2\big), B = T_a\mathbb{A}^n, B' = T_aV$. We have also used the pairing perfect pairing between A and B, from Lemma 5.11.

$\square$

Definition 5.15. For X a variety, $a \in X$, we define $T_aX = (\mathfrak{m}_a/\mathfrak{m}_a^2)^*$, where $U \subseteq X$ is an open affine variety containing a and $\mathfrak{m}_a \subseteq \mathcal{O}_X(U)$ is the maximal ideal of a (one can show that this is independent of the chosen affine open U).

Definition 5.16. X is smooth of dimension d if and only if $\dim T_aX = d$ for all $a \in X$.

6 Desingularisation and Blowing up

In the field of Algebraic Geometry, two varieties are considered to be *birationally equivalent* if they are isomorphic except for a "small set" of points. Heisuke Hironaka in 1964 proved that any quasi-projective variety over $\mathbb{C}$ that is quasi-projective can be transformed into a smooth quasi-projective variety through a process called the blowing-up (*i.e.* zooming in), in other words, any quasi-projective variety can be *desingularised*. We discuss the blowing up process in the following paragraphs. A video lecture on this topic is available at https://youtu.be/Gkkh_nl7ETw.

Definition 6.1. A morphism $\pi : X \longrightarrow V$, of quasi-projective varieties is called a *birational morphism* if there are open dense subsets $A \subseteq X$ and $B \subseteq V$, such that $\pi_{|_A} : A \longrightarrow B$ is an isomorphism of algebraic varieties.

6.1 Blowing up of $\mathbb{A}^n$ at a Point

6.1.1 Blowing up $\mathbb{A}^2$ at a Point: Intuition

We intend to discuss the following without proof:

"Any curve $C \subseteq \mathbb{A}^2$, can be viewed as the projection of a curve $\tilde{C} \subseteq \mathbb{A}^2 \times \mathbb{A}^1$, such that looking at $\tilde{C}$ along the new direction, we see C, moreover $\tilde{C}$ is less singular."

Let us explain the idea behind the blowing up method by considering $C := \mathbb{V}(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$. This curve looks like the top left curve in Figure 5. Here is what we can do to blow up C :

(i) Note that the tangent space is not well-defined at the origin, but it is well-defined when we approach the origin. So we remove the origin.

(ii) Lift any point $(a, b) \in C \setminus \{(0,0)\}$ to the height equal to the slope of the line passing through (a, b) and the origin, *i.e.*, $(a, b) \longmapsto (a, b, \frac{b}{a}) \in (C \setminus \{(0,0)\}) \times \mathbb{C}$.

(iii) Take the closure of the curve we obtained in $\mathbb{A}^3$.

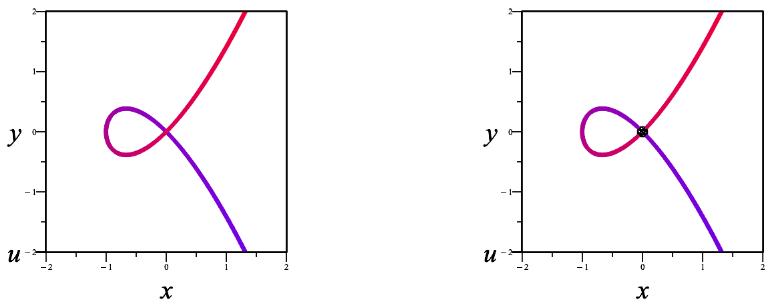


Figure 5: $C := \mathbb{V}(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$.

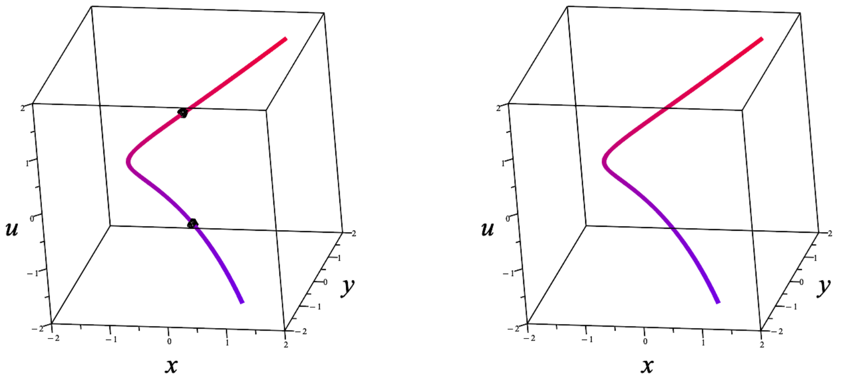


Figure 6: In the last four pictures, the second and third pictures are isomorphic $\Longrightarrow$ the first and the fourth are birational.

If you'd like to look at the blowup curve from different angles, here is the Maple code.

```
intersectplot(u^2 = x + 1, y = x*u, x = -2 .. 2, y = -2 .. 2,
u = -2 .. 2, shading = "z", thickness = 7,
labelfont = ["TimesNewRoman", 40]);
```

Now let us apply the same procedure as above to the disc.

Example 6.2. Questions:

(a) Consider the lines in Figure 7. What is the image of each line under the map

$$\begin{aligned} \varphi : \mathbb{A}^2 \setminus \{a = 0\} &\longrightarrow \mathbb{A}^2 \times (-\infty, \infty) \\ (a, b) &\longmapsto (a, b, \text{ the slope of the line connecting } (0,0) \text{ and } (a, b))? \end{aligned}$$

(b) Do you agree that we have discontinuity when we approach the vertical line from right or left? How can we fix this discontinuity?

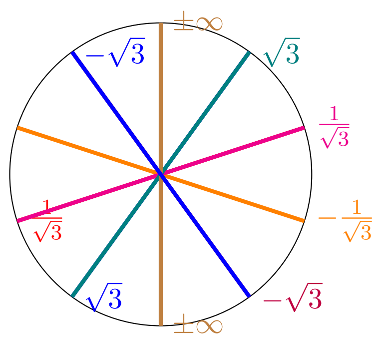


Figure 7: Lines with different slopes passing through the origin; thanks to ChatGPT for writing the code for this figure.

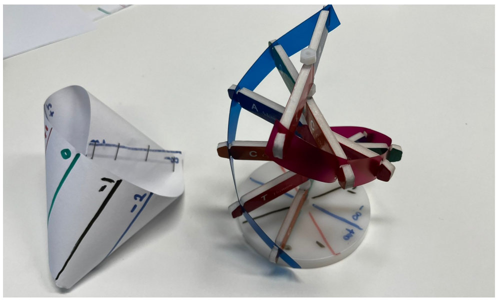


Figure 8: Blowup of $\mathbb{A}^2$ and an affine chart

(c) What shape would you get by considering the blowup of $\mathbb{R}^2$ at the origin, on the slope direction, $\pm\infty$ are identified.

Answers:

(a) The picture is given by a DNA-like helix, we get copies of $\mathbb{C} \setminus \{0\}$ at different heights equal to their slopes.

(b) Yes, and we can identify $\pm\infty$ by considering the new direction as $\mathbb{P}^1$ instead of $\mathbb{A}^1$. In other words, we can think of the slope direction $\mathbb{A}^1$ as an open chart of $\mathbb{A}^1 = U_0 = \{[1 : u] : u = \text{slope} \in \mathbb{C}\} \subseteq \mathbb{P}^1$.

(c) The open Möbios band, as the lines are the entire $\mathbb{R}^1$. See Figure 8.

△

6.1.2 The Algebraic Definition

The idea for blowing up is to think that the objects we obtained above in $\mathbb{A}^2 \times \mathbb{A}^1$ as an open chart of $\mathbb{A}^2 \times \mathbb{P}^1$. For instance consider the points

$$\big((a, b); [1 : \frac{b}{a}]\big) \in \mathbb{A}^2 \times \mathbb{P}^1,$$

and extend the definition to the other chart with the usual identification $[a/b : 1] = [a : b] = [1 : b/a]$, when $a \neq 0, b \neq 0$. However, this is still not defined at $(a, b) = (0, 0)$. The idea is to install an entire copy of $\mathbb{P}^1$ at $(0, 0)$. Another way to formulate this blowup is to think that $\mathbb{P}^1$ is the set of equivalence classes of lines passing through the origin and that the point $[1 : b/a]$ can be understood as the equivalence class of the line which contains the point $(a, b) \in \mathbb{A}^2$. This leads us to define the blowup of $\mathbb{A}^2$ at $(0, 0)$ by:

$$\big\{(p, [\ell]) : p \in \ell \text{ for all points } p \in \mathbb{A}^2, \text{ and lines } \ell \text{ passing through } (0, 0)\big\},$$

where $[\ell] \in \mathbb{P}^1$ denotes the equivalence class of ℓ. Note that above $(0, 0)$ we get a representative of all the lines passing through the origin: $\mathbb{P}^1$. This contains the u -axis or the “slope axis” as an open affine chart, and it is called the *exceptional divisor.*

Remark 6.3. We might be tempted to define the blowing up of $\mathbb{A}^2$ as

$$\{\big((a,b);[a:b]\big):(a,b)\in\mathbb{A}^2\},$$

however, this set is not defined at $(a,b)=(0,0)$. See Exercise 6.7.

Now we can easily generalise this idea to any dimensions: if $p=(x_1,x_2,\ldots,x_n)\in\mathbb{A}^n$ and $\ell=[y_1:y_2:\cdots:y_n]\in\mathbb{P}^{n-1}$, we have that $(x_1,x_2,\ldots,x_n)\in\ell$ if and only if the vectors $(x_1,x_2,\ldots,x_n)$, $(y_1,y_2,\ldots,y_n)\in\mathbb{A}^n$ are along the same line or linearly dependent. In other words,

$$\begin{aligned}&\operatorname{rank}\begin{pmatrix}x_1 & x_2 & \cdots & x_n\\ y_1 & y_2 & \cdots & y_n\end{pmatrix}\leq 1\\ &\iff \text{determinant of all } 2\times 2\text{-matrices vanish}\\ &\iff \{x_iy_j-y_ix_j=0:1\leq i,j\leq n\}.\end{aligned}$$

We now arrive at the following natural equivalent definitions.

Definition 6.4. (a) The blowup of $\mathbb{A}^n$ at the origin $0\in\mathbb{A}^n$ is given by

$$\widetilde{\mathbb{A}^n}:=\mathrm{Bl}_{\{0\}}(\mathbb{A}^n):=\big\{(p,[\ell])\in\mathbb{A}^n\times\mathbb{P}^{n-1}:\ell\text{ is a line passing through }0\text{ and }p\big\}.$$

(b) Equivalently, the blowup of $\mathbb{A}^n$ at the origin is given by

$$\begin{aligned}\widetilde{\mathbb{A}^n}&:=\mathrm{Bl}_{\mathbb{A}^n}(\{0\})\\ &=\big\{\big((x_1,\ldots,x_n);[y_1:\cdots:y_n]\big)\in\mathbb{A}^n\times\mathbb{P}^{n-1},\\ &\qquad x_iy_j-y_ix_j=0,1\leq i,j\leq n\big\}.\end{aligned}$$

(c) Equivalently, $\widetilde{\mathbb{A}^n}$ is the closure of

$$\big\{\big((x_1,\ldots,x_n);[x_1:\cdots:x_n]\big):(x_1,\ldots,x_n)\in\mathbb{A}^n\setminus\{0\}\big\}$$

in $\mathbb{A}^n\times\mathbb{P}^{n-1}$. See Exercise 6.7.

Now let $\pi:\widetilde{\mathbb{A}^n}\longrightarrow\mathbb{A}^n$, $(q,\ell)\longmapsto q$ be the projection map.

(c) Let $V\subseteq\mathbb{A}^n$ be an affine algebraic variety. The preimage $\pi^{-1}(V)\subseteq\widetilde{\mathbb{A}^n}\subseteq\mathbb{A}^n\times\mathbb{P}^{n-1}$ is called the *total transform* of V. Note that if $\Pi:\mathbb{A}^n\times\mathbb{P}^{n-1}\longrightarrow\mathbb{A}^n$, is the projection map, then $\pi=\Pi|_{\widetilde{\mathbb{A}^n}}$, and

$$\pi^{-1}(V)=\Pi^{-1}(V)\cap\widetilde{\mathbb{A}^n}.$$

(d) Let $V\subseteq\mathbb{A}^n$ be an affine algebraic variety passing through the origin. The *blowing up* of V at the origin $\{0\}$ is given by

$$\widetilde{V}:=\overline{\pi^{-1}(V\setminus\{0\})}\subseteq\mathbb{A}^n\times\mathbb{P}^{n-1},$$

i.e., remove the origin, take the pre-image by π, take the closure in $\mathbb{A}^n\times\mathbb{P}^{n-1}$.

(e) $E := \pi^{-1}(0) = (0, \ldots, 0) \times \mathbb{P}^{n-1}$ is called the *exceptional divisor* of $\widetilde{A^n}$.

$$E \cap \widetilde{V} = \pi^{-1}_{|\widetilde{V}}(0)$$

is the exceptional divisor of $\widetilde{V}$.

(f) To blowup $\mathbb{A}^n$ at any point $q \in \mathbb{A}^n$, we make a linear change of coordinates by sending $q \longmapsto 0$.

Example 6.5. Let us compute the blowup of $C := \mathbb{V}(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$ at the origin. To understand the blowup $\tilde{C} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$, we can look at different charts of $\mathbb{P}^1$ and take different snapshots of $\tilde{C}$. Recall that the projection for blowup of $\mathbb{A}^2$ at the origin is given by $\pi : \{\big((x,y) : [t:u]\big) \in \mathbb{A}^2 \times \mathbb{P}^1,\ xu = yt\} \longrightarrow \mathbb{A}^2$, where $\pi\big((x,y);[t:u]\big) = (x,y)$. Let us look at the chart $\{[1:u] : u \in \mathbb{A}^1\} \subseteq \mathbb{P}^1$. We have,

$$\pi^{-1}(C) = \{\big((x,y);[1:u]\big) : (x,y) \in C, y = xu\}.$$

Note that since $0 \in C$, $\pi^{-1}(C)$ contains the exceptional divisor, which we need to remove to obtain the blowup $\pi : \tilde{C} \longrightarrow C$. We have

$$\begin{cases} (x,y) \in C \implies y^2 - x^2(x+1) = 0 \\ xu = y \end{cases}$$

Plugging in the second equation into the first, $(ux)^2 - x^2(x+1) = x^2(u^2 - x - 1) = 0$. The zero set in xu-plane is $x = 0$ or $u^2 - x - 1 = 0$. The line $x = 0$ is the u-axis, which is the slope axis for us, or the exceptional divisor. The parabola $u^2 - x - 1 = 0$ is the blowup of C, in the xu-plane. If we draw the xu-plane with the x-axis and u-axis, the parabola intersects exceptional divisor the line at $x = 1$, $u = 0$, and at $x = -1$ and $u = 0$. If we draw $u^2 - x - 1 = 0, y = ut$, in the xyu-axis, we obtain the Figure 6. For the chart $\{[t:1] : t \in \mathbb{A}^1\} \subseteq \mathbb{P}^1$, we obtain a similar picture. Since in both these charts, the blowup is smooth, we can deduce that $\tilde{C}$ is smooth.

$\triangle$

Example 6.6. Let us blowup the cusp $y^2 = x^3$. In the chart $t = 1$ of $\mathbb{P}^1$, we have

$$\begin{cases} y^2 = x^3 \\ xu = y \end{cases} \implies (xu)^2 = x^3 \implies x^2(u^2 - x) = 0.$$

We obtain $x = 0$, *i.e.* a copy of the exceptional divisor, and the parabola $u^2 - x = 0$ which is the resolved curve. In xyu-coordinates we can draw

$$\begin{cases} u^2 - x = 0 \\ xu = y \end{cases}$$

to look at the blowup curve from different angles. Here is the Maple code

```
h_1 := intersectplot(u^2 - x = 0, y = x*u, x = -2 .. 2,
y = -2 .. 2, u = -2 .. 2, shading = "z",
thickness = 7, labelfont = ["TimesNewRoman", 40]);
h_2 := intersectplot(x = 0, y = 0, x = -2 .. 2,
y = -2 .. 2, u = -2 .. 2, color = "red");
display(h_1, h_2);
```

$\triangle$

Exercise 6.7. Show that the blowup of $\mathbb{A}^2$ a $(0,0)$ is the closure of

$$\{\big((a,b);[a:b]\big) : (a,b) \in \mathbb{A}^2 \setminus (0,0)\}$$

in $\mathbb{A}^2 \times \mathbb{P}^1$.

Lemma 6.8. Let $V \subseteq \mathbb{A}^n$, affine algebraic variety with $0 \in V$.

(a) Total transform = blowup $\bigcup$ exceptional divisor, *i.e.*,

$$\pi^{-1}(V) = \overline{\pi^{-1}(V \setminus \{0\})} \cup \pi^{-1}(\{0\}).$$

(b) For any algebraic variety $V \subseteq \mathbb{A}^n$, we have the birational morphism $\tilde{V} \longrightarrow V$.

Proof. (a) Since $0 \in V$, we have $\pi^{-1}(V) = \pi^{-1}(V \setminus \{0\}) \cup \pi^{-1}(\{0\})$, but $\pi^{-1}(V)$ is closed, given by algebraic equations as above, therefore it contains $\overline{\pi^{-1}(V \setminus \{0\})}$.

(b) It is easy to see that the equations for $\big((x_1,\ldots,x_n);[y_1:\cdots:y_n]\big)$, when $(x_1,\ldots,x_n) \neq (0,,\ldots,0)$, the equations $\{x_iy_j - y_ix_j = 0 : 1 \leq i,j \leq n\}$ imply that $[y_1:\cdots:y_n] = [x_1:\cdots:x_n]$. Now, it is easy to see that

$$\begin{aligned} \pi : \widetilde{\mathbb{A}^n} \setminus \pi^{-1}(\{0\}) &\longrightarrow \mathbb{A}^n \setminus \{0\} \\ \big((x_1,\ldots,x_n);[x_1:\cdots:x_n]\big) &\longmapsto (x_1,\ldots,x_n), \end{aligned}$$

and

$$\begin{aligned} \varphi : \mathbb{A}^n \setminus \{0\} &\longrightarrow \widetilde{\mathbb{A}^n} \setminus \pi^{-1}(0) \\ (x_1,\ldots,x_n) &\longmapsto \big((x_1,\ldots,x_n);[x_1:\cdots:x_n]\big), \end{aligned}$$

are morphisms and inverses to each other. As a result, $\pi|_{\tilde{V}\setminus\pi^{-1}(\{0\})}$ and $\varphi|_{\mathbb{A}^n\setminus\{0\}}$ are also isomorphism.

$\square$

Remark 6.9. Let $\mathrm{Bl}_p\mathbb{A}^n$, be the blowup of $\mathbb{A}^n$ at $p \in \mathbb{A}^n$. Let $V \subseteq \mathbb{A}^n$. If $p \notin V$, then the total transform of V, $\pi^{-1}(V)$, is the same as blowup of V along $p \notin V$.

Exercise 6.10. Describe the blowup at the origin of the conic surface in $\mathbb{A}^3$ given by $x^2 + y^2 = z^2$.

Exercise 6.11. Consider the family of lines $\ell_c = \{(x,y) \in \mathbb{A}^2 : x + y = c\}$, where $c \in \mathbb{C}$ is a parameter. Let $\pi : \mathbb{A}^2 \times \mathbb{P}^1 \longrightarrow \mathbb{A}^2$, be the blowing up map at the origin. Write the equations and sketch the graphs of $\pi^{-1}(\ell_c)$ for $c = 2, 1, 0$, in xu-plane.

6.2 Blowing up of $\mathbb{A}^n$ along a variety

Let us briefly mention the definition of blowup of $\mathbb{A}^n$ along an affine subvariety. Compare to Exercise 6.7. We encourage readers to go through beautifully written Chapter 7 of [SKKT00].

Definition 6.12. (a) Let $I = (f_1, \dots, f_k) \subseteq \mathbb{C}[x_1, \dots, x_n]$, and $(x) = (x_1, \dots, x_n)$. The blowing up of $\mathbb{A}^n$ along the ideal I is the closure of the set

$$\{\big((x); [f_1(x) : \cdots : f_k(x)]\big) : (x) \in \mathbb{A}^n \setminus V\}.$$

in $\mathbb{A}^n \times \mathbb{P}^{k-1}$.

(b) Let $I = \mathbb{I}(V)$ be the radical ideal associated to the affine variety $V \subseteq \mathbb{A}^n$. This definition, up to isomorphism, does not depend on the choice of $f_1, \dots, f_k$.

Exercise 6.13. Find the algebraic equations to define the blowing up of $\mathbb{A}^n$ along a subvariety $V \subseteq \mathbb{A}^n$ similar to Definition 6.4.(b).

7 Toric Geometry

Toric geometry is a subfield of algebraic geometry, where we can construct algebraic varieties from combinatorial and discrete data. The goal is to be able to read algebro-geometric properties of these varieties from the combinatorial data. Miles Reid in 1983, writes "This construction has been of considerable use within algebraic geometry in the last 10 years...and has also been amazingly successful as a tool of algebro-geometric imperialism, infiltrating areas of combinatorics." Fulton writes "toric varieties have provided a remarkably fertile testing ground for general theories."

7.1 Cones and their dual

Let N be a group isomorphic to $\mathbb{Z}^n$, *i.e.,* a finitely generated free abelian group of rank n.

- $N \simeq \mathbb{Z}^n$ as groups.
- The dual lattice $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, *i.e.*, all the group homomorphisms $f : N \longrightarrow \mathbb{Z}$.
- $N_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} M$.
- $M_{\mathbb{R}} := \mathrm{Hom}(M_{\mathbb{R}}, \mathbb{R}))$.
- We have $M \simeq \mathbb{Z}^n$ as groups, $N_{\mathbb{R}} \simeq M_R \simeq \mathbb{R}^n$ as vector spaces. We have the natural inclusion $M \subseteq M_{\mathbb{R}}$.

If we identify $N_{\mathbb{R}} = \mathbb{R}^n$, then $M_{\mathbb{R}}$ would be identified with the dual of $\mathbb{R}^n$, denoted by $(\mathbb{R}^n)^*$. We can denote by $\langle \cdot, \cdot \rangle$ the pairing of $(\mathbb{R}^n)^*$ and $\mathbb{R}^n$. This pairing is simply the dot product on $\mathbb{R}^n$, if we identify $(\mathbb{R}^n)^*$ and $\mathbb{R}^n$.

Recall that a *convex cone* or simply a *cone* $\sigma \subseteq \mathbb{R}^n$ satisfies,

- $v \in \sigma$, $\lambda \in \mathbb{R}_{\geq 0} \implies \lambda v \in \sigma$;
- $v, v' \in \sigma \implies v + v' \in \sigma$.

In these notes, all cones are assumed to be convex.

Definition 7.1.

(a) For $A = \{v_1, \ldots, v_k\} \subseteq \mathbb{R}^n$, we define the cone generated by A as

$$\mathrm{cone(A)} := \{\lambda_1 v_1 + \cdots + \lambda_k v_k : \lambda_i \in \mathbb{R}_{\geq 0}\}.$$

(b) A cone σ is called *polyhedral* if $\sigma = \mathrm{cone(A)}$, for a finite set $A \subseteq \mathbb{R}^n$.

(c) A subset $\sigma \subseteq \mathbb{R}^n$ is called a *rational polyhedral cone* or simply *rational cone* if $\sigma = \mathrm{cone}(\{v_1, \ldots, v_k\})$ for some $v_i \in N$.

(d) A cone σ is called *strongly convex* if it does not contain any line passing through the origin, *i.e.*, $\sigma \cap -\sigma = \{0\}$.

Remark 7.2. All the cones considered in this chapter are convex rational polyhedral cones.

Example 7.3.

(a) $\mathrm{cone}(\{\sqrt{2}\}) \subseteq \mathbb{R}$, is a rational cone, since $\mathrm{cone}(\{\sqrt{2}\}) = \mathbb{R}_{\geq 0} = \mathrm{cone}(\{1\})$.

(b) $\mathrm{cone}(\{(\sqrt{2}, 1)\}) \subseteq \mathbb{R}^2$ is not a rational cone.

(c) $\mathrm{cone}\{(1,0),(3,-2)\} \subseteq \mathbb{R}^2$ is a rational cone.

(d) The rational cone $\mathrm{cone}\{(1,0),(3,-2),(-3,2)\} \subseteq \mathbb{R}^2$ is not strongly convex.

△

Definition 7.4. For any cone $\sigma \subseteq \mathbb{R}^n$, we define its *dual cone* by

$$\sigma^\vee = \{u \in (\mathbb{R}^n)^* : \langle u, v\rangle \geq 0\,,\ \text{for all } v \in \sigma\}.$$

It is easy to see the following:

Proposition 7.5. Assume that σ is a rational cone, then $\sigma^\vee$ is also a rational cone.

Note that any cone can be understood as the intersection of half-spaces containing it. This helps us identify the dual cones easily: For $v \in \mathbb{R}^n$ let us define

$$H_v^+ = \mathrm{cone}(\{v\})^\vee := \{u \in (\mathbb{R}^n)^* : \langle u, v\rangle \geq 0\}.$$

Now, by definition,

$$\sigma^\vee = \bigcap_{v\in\sigma} H_v^+,$$

and since $\sigma^\vee$ is also convex, if $\sigma = \mathrm{cone}\{v_1, \ldots, v_k\}$,

$$\sigma^\vee = \bigcap_{i=1}^{k} H_{v_i}^+.$$

Example 7.6. We know that for $v_1, v_2 \in \mathbb{R}^2$, we have $\langle v_1, v_2\rangle := v_1{\cdot}v_2 = \|v_1\|\|v_2\|\cos\theta$, where θ is the angle between v_1 and v_2. Therefore,

$$\langle v_1, v_2\rangle \geq 0 \iff \theta \leq 90^\circ.$$

△

Example 7.7. Let us draw and identify the dual cone to $\sigma = \mathrm{cone}(\{e_1, 2e_1 - 3e_2\})$. Figure 9. We denote by e_1^*, e_2^* the standard basis for $(\mathbb{R}^2)^*$.

(1) The dual cone to e_1 is given by the half-space $\mathrm{cone}(\{e_1^*, e_2^*, -e_2^*\})$. These are the elements in the shaded red area since their angle with e_1 is less than or equal to 90 degrees. Therefore, their dot product with e_1 is non-negative.

(2) The dual cone to $2e_1 - 3e_2$ is given by the half-space $\mathrm{cone}(\{3e_1^* + 2e_2^*, -3e_1^* - 2e_2^*, 2e_1^* - 3e_2^*\})$.

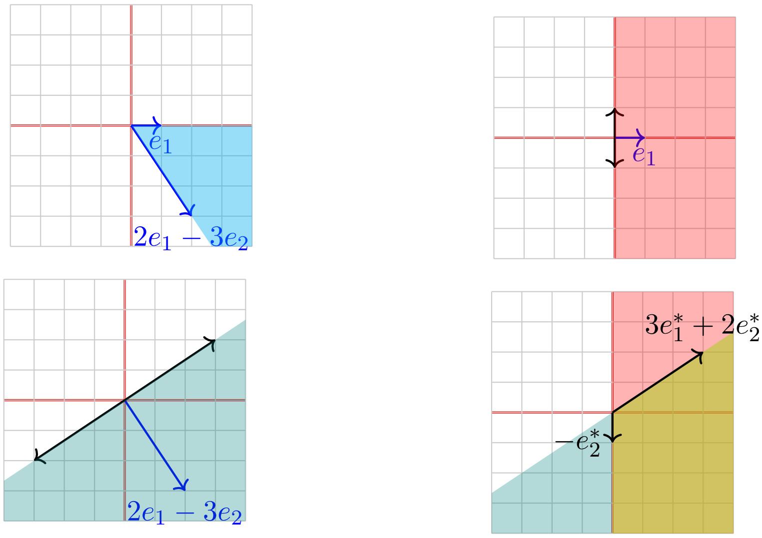


Figure 9: cone($\{2e_1 - 3e_2, e_1\}$) shaded in cyan, and its dual shaded in yellow.

(3) Take the intersection. We see that $\sigma^\vee = \text{cone}(\{-e_2^*, 3e_1^* + 2e_2^*\})$.

(4) With identification $(\mathbb{R}^2)^* = \mathbb{R}^2$, we have $e_i^* = e_i$.

$\triangle$

Example 7.8. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. For $1 \leq r \leq n$, let $\sigma := \text{cone}(\{e_1, \ldots, e_r\}) \subseteq \mathbb{R}^n$ then $\sigma^\vee$ is generated by

$$\{e_1, \ldots, e_r, e_{r+1}, -e_{r+1}, \ldots, e_n, -e_n\} \subseteq \mathbb{R}^n.$$

$\triangle$

Proposition 7.9. Let $\sigma \subseteq \mathbb{R}^n$ be a cone.

(a) $\left(\sigma^\vee\right)^\vee = \sigma$.

(b) If $\sigma = \sigma_1 + \sigma_2$, then
$$\sigma^\vee = \sigma_1^\vee \cap \sigma_2^\vee.$$

(c) If σ is rational then $\sigma^\vee$ is also rational.

We leave the proof as an exercise.

7.2 Monoids

We are interested in additive properties of $\sigma \cap \mathbb{Z}^n$, where σ is a cone makes it a *monoid.*

Definition 7.10. (a) A *semi-group* G is a non-empty set, with an associative binary operation.

(b) A *monoid* S is a semi-group that is commutative, has a unique additive identity element *zero*, and satisfies the *cancellation law*, *i.e.*,

$$s + t = s' + t \implies s = s' \quad \text{for } s, s', t \in S.$$

Example 7.11. (a) The set of $n \times n$ matrices with matrix multiplication is a semi-group.

(b) $\mathbb{Z}_{\geq 0}$ is a monoid.

(c) If $\sigma \subseteq \mathbb{R}^n$ is a cone $\sigma \cap \mathbb{Z}^n$ is a monoid.

△

Note that for a monoid we only have the notion of the addition of its elements, which gives rise to the following notion.

Definition 7.12. A monoid S is *finitely generated* if there are finitely many elements $v_1, \ldots, v_k$ in S, such that they generate S as a monoid, *i.e.*, for any $s \in S$, there are non-negative integers $q_i \in \mathbb{Z}_{\geq 0}$ such that

$$s = q_1 v_1 + \ldots q_k v_k.$$

Theorem 7.13 (Gordan's Lemma)**.** Let σ be a rational (polyhedral) cone in $\mathbb{R}^n$, then $\sigma \cap \mathbb{Z}^n$ is a finitely generated monoid.

Proof. By definition, we can assume that $\sigma = \mathrm{cone}(\{v_1, \ldots, v_k\})$. Therefore, for any $v \in \sigma \cap \mathbb{Z}^n$, there exist $r_i \in \mathbb{R}_{\geq 0}$, such that

$$v = r_1 v_1 + \cdots + r_k v_k \,. \tag{1}$$

Let

$$G = \{t_1 v_1 + \cdots + t_k v_k : \; 0 \leq t_i \leq 1\}.$$

G is a closed and bounded subset of $\mathbb{R}^n$, therefore it is compact. Hence, $G \cap \mathbb{Z}^n$ has finitely many elements. See Figure 10. We claim that the lattice points in $G \cap \mathbb{Z}^n$, generate $\sigma \cap \mathbb{Z}^n$ as a monoid. To see this, we rewrite Equation 1 as

$$v = \big(\lfloor r_1 \rfloor v_1 + \ldots \lfloor r_k \rfloor v_k\big) + \Big((r_1 - \lfloor r_1 \rfloor) v_1 + \ldots (r_k - \lfloor r_k \rfloor) v_k\Big) =: I + J.$$

We have

(a) I is in the set generated by $G \cap \mathbb{Z}^n$ as a monoid;

(b) $J \in G$. In addition, $J = v - I$ and $v, I \in \mathbb{Z}^n$, therefore $J \in G \cap \mathbb{Z}^n$.

□

7.3 Affine Toric Varieties

In this section, we describe how to every rational cone $\sigma \subseteq \mathbb{R}^n$, we can assign an affine variety $X_\sigma \in \mathbb{A}^N$ which is unique up isomorphism (usually $n \neq N$). Here are the steps of this procedure

Step 1. Find $\sigma^\vee$;

Step 2. Find some generators $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n$;

Step 3. Find the *associated* $\mathbb{C}$*-algebra* $\mathbb{C}[S_\sigma]$; (Definition 7.15)

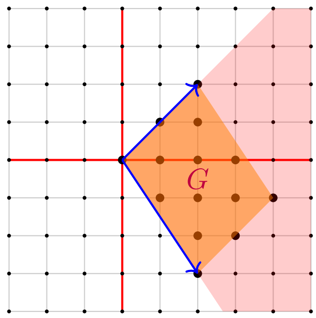


Figure 10: Gordan: the lattice points in the orange area G generate all the lattice points in the shaded areas as a monoid.

Step 4. Use the relation between the generators to find $\text{maxSpec}\big(\mathbb{C}[S_\sigma]\big)$.

So far we have a some ideas on how to deal with Steps 1,2 and Step 4 from Section 2.6. We now explain Step 3, which is fairly easy. For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ let us use the multi-index notation for the monomial $z^m := z_1^{m_1} \ldots z_n^{m_n} \in \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$.

Example 7.14. For $m = (5, -2)$, $z^m = z_1^5 z_2^{-2} \in \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$. △

Definition 7.15. Let $\sigma \subseteq \mathbb{R}^n$ be a rational cone, and $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n$, be the associated monoid. We define

$$\mathbb{C}[S_\sigma] = \mathbb{C}\text{-algebra generated by } \{z^m \colon\ m \in \sigma^\vee \cap \mathbb{Z}^n\}.$$

We can now easily verify that for $m_1, m_2 \in \mathbb{Z}^n$,

$$\begin{aligned} m_1 &\longmapsto z^{m_1} \\ m_2 &\longmapsto z^{m_2} \\ m_1 + m_2 &\longmapsto z^{m_1+m_2} = z^{m_1} z^{m_2}. \end{aligned}$$

This implies that if we have finitely many $\{m_1, \ldots, m_k\}$ generating a monoid S_σ, *i.e.*, with different combinations of their sums, then $z^{m_1}, z^{m_2}, \ldots, z^{m_n}$ generate $\mathbb{C}[S_\sigma]$ as a $\mathbb{C}$-algebra. Let us clarify this with some examples.

Example 7.16.

(a) Let $\sigma_1 = \text{cone}(\{1\}) = \mathbb{R}_{\geq 0}$. We have $\sigma_1^\vee = \mathbb{R}_{\geq 0}$ and $S_{\sigma_1} = \sigma_1^\vee \cap \mathbb{Z} = \mathbb{Z}_{\geq 0}$. The $\mathbb{C}$-algebra generated by $\{1, z_1^1, z_1^2, \ldots\}$ is simply all the polynomials in variables z_1 with coefficients in $\mathbb{C}$. Therefore,

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[z_1].$$

(b) Let $\sigma_2 = \text{cone}(\{e_1, e_2\}) \subseteq \mathbb{R}^2$. We have $\sigma_2^\vee = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and $S_{\sigma_2} = \sigma_2^\vee \cap \mathbb{Z}^2 = \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Therefore,

$$\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[z_1, z_2].$$

(c) Let $\sigma_3 = \text{cone}(\{e_1\}) \subseteq \mathbb{R}^2$. We have $\sigma_3^\vee = \mathbb{R}_{\geq 0} \times \mathbb{R}$ and $S_{\sigma_3} = \sigma_3^\vee \cap \mathbb{Z}^2 = \mathbb{Z}_{\geq 0} \times \mathbb{Z}$. Therefore,

$$\mathbb{C}[S_{\sigma_3}] = \mathbb{C}[z_1, z_2, z_2^{-1}].$$

(d) Bearing in mind the inconsistency in the notation with (a), (b), let us assume $\{e_1, \dots, e_n\}$ is now the standard basis for $\mathbb{R}^n$. For an integer $0 \leq k \leq n$, let $\sigma_4 = \text{cone}(\{e_1, \dots, e_k\}) \subseteq \mathbb{R}^n$. Then

$$\mathbb{C}[S_{\sigma_4}] = \mathbb{C}[z_1, \dots, z_k, z_{k+1}, z_{k+1}^{-1}, \dots, z_n, z_n^{-1}].$$

$\triangle$

Exercise 7.17. Check that for the cone in Figure 9, $S_\sigma \neq \mathbb{C}[z_1^{-1}, z_1^3 z_2^2]$.

In view of the Equivalence of Algebra and Geometry in Section 2.8, the following lemma is crucial.

Lemma 7.18. Let σ be a rational (polyhedral) cone in $\mathbb{R}^n$. Then, the associated $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma]$, is finitely generated and reduced. As a result, the affine toric variety $X_\sigma := \text{maxSpec}\big(\mathbb{C}[S_\sigma]\big)$ is a closed affine algebraic variety.

Proof. By Gordan's Lemma S_σ is finitely generated as a monoid. Hence, $\mathbb{C}[S_\sigma]$ is a finitely generated $\mathbb{C}$-algebra. For reducedness, note that we need to prove that if $f \in \mathbb{C}[S_\sigma]$ is polynomial, then $f^n = 0$, for some positive integer n, then implies $f = 0$. This is rather clear since we cannot make a Laurent polynomial vanish when we take the powers, as the highest positive degree increases and the lowest negative degree decreases. More formally, $\mathbb{C}[S_\sigma]$ is a sub-algebra of $\mathbb{C}[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]$, which is reduced, and therefore $\mathbb{C}[S_\sigma]$ is reduced, as well. The rest of the assertion follows from the discussion in Section 2.7.

$\square$

Example 7.19. Back to the cone in Figure10. We have $\sigma = \text{cone}(\{2e_1 + 2e_2, 2e_1 - 3e_2\}) \subseteq \mathbb{R}^2$. It is easy to see that $\{e_1, 2e_1 - 3e_2\}$ does not generate S_σ as a monoid. However, the proof of Gordan's Lemma implies that all the lattice points in the orange area G do generate S_σ. As a result, the monomials $\{z^m : m \in G\}$ generate the $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma]$. Note that we have some redundancy here and we can indeed find a lower number of generators as well. $\triangle$

Definition 7.20. Let $\sigma \subseteq \mathbb{R}^n$, we define the affine variety X_σ to be the topological space $\text{maxSpec}\big(\mathbb{C}[S_\sigma]\big)$ equipped with Zariski topology defined in Section 2.7.

Let us make the following important remark.

Remark 7.21. Note that by Theorem 2.39(a) and Lemma 7.18, $\mathbb{C}[S_\sigma]$ is isomorphic as a $\mathbb{C}$-algebra to the coordinate ring of an affine algebraic variety V. This affine algebraic variety is not unique. However, if for two varieties $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$, we have $\mathbb{C}[V] \simeq \mathbb{C}[W] \simeq \mathbb{C}[S_\sigma]$, then $V \simeq W$ by Theorem 2.39(c). In other words, we determine the affine toric varieties up to an isomorphism.

Example 7.22. Let us revisit Section 4.2.1.

(a) Let $\sigma_1 = \text{cone}(\{1\}) = \mathbb{R}_{\geq 0} \subseteq \mathbb{R}$. $\mathbb{C}[S_\sigma] = \mathbb{C}[z]$. Thus,

$$\text{maxSpec}(\mathbb{C}[z]) \simeq \mathbb{A}^1.$$

(b) For $\sigma_2 = \text{cone}(\{-1\}) = \mathbb{R}_{\geq 0} \subseteq \mathbb{R}$. $\mathbb{C}[S_\sigma] = \mathbb{C}[z^{-1}]$. Therefore,

$$\text{maxSpec}(\mathbb{C}[z^{-1}]) \simeq \mathbb{A}^1.$$

(c) For $\tau = \text{cone}(\{0\}) \subseteq \mathbb{R}$. $\tau^\vee = \mathbb{R}$, and $S_\tau = \mathbb{Z}$. By Example 4.16, we have

$$\mathbb{C}[S_\tau] = \mathbb{C}[z, z^{-1}] \implies \text{maxSpec}(\mathbb{C}[S_\tau]) \simeq \mathbb{A}^1 \setminus \{0\}.$$

$\triangle$

Example 7.23. (a) Let $\sigma_1 = \text{cone}(1, 2) = \mathbb{R}_{\geq 0} \subseteq \mathbb{R}$. Let us take both generators $\{1, 2\}$ of S_σ. Then, $\mathbb{C}[S_\sigma] = \mathbb{C}[z] = \mathbb{C}[z, z^2] = \frac{\mathbb{C}[x,y]}{(y-x^2)}$. Thus, by Remark 7.21 we have the isomorphism:

$$\text{maxSpec}(\{\mathbb{C}[z]\}) \simeq \mathbb{A}^1 \simeq \mathbb{V}(x^2 - y).$$

This is Example 2.30.(b).

(b) Let $\sigma_2 = \text{cone}(\{(1, 0), (-1, 0)\})$. Then $\sigma_2^\vee = \text{cone}\{(0, 1), (0, -1)\}$. $\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[y, y^{-1}]$. We can also understand this algebra by

$$(0, 1) \longmapsto u$$
$$(0, -1) \longmapsto v$$

and the relation $(0, 1) + (0, -1) = (0, 0)$, which implies $uv = u^0 v^0 = 1$. Thus,

$$\mathbb{C}[S_\tau] \simeq \mathbb{C}[y, y^{-1}] \simeq \frac{\mathbb{C}[u, v]}{(uv - 1)}.$$

By preceding example $\mathbb{C}^* \simeq \mathbb{V}(uv - 1)$. Note that in Chapter 2, we proved $V \simeq W \iff \mathbb{C}[V] \simeq \mathbb{C}[W]$, for affine algebraic subvarieties of $\mathbb{A}^n$ in the sense of Chapter 1. $\mathbb{A}^1 \setminus \{0\}$ does not fall in that category.

$\triangle$

Exercise 7.24. Let $\sigma = \text{cone}(\{1, 2, 3\}) = \mathbb{R}_{\geq 0} \subseteq \mathbb{R}$. Take all $\{1, 2, 3\}$ as generators of S_σ.

(a) Identify $\mathbb{C}[S_\sigma]$.

(b) Identify X_σ.

See Examples 1.8.4, 2.41.(a), 3.35.

7.3.1 Cartesian Product of Affine Algebraic Varieties

Let $I = (f_1, \ldots, f_k) \subseteq \mathbb{C}[z_1, \ldots, z_n]$ and $J = (g_1, \ldots, g_\ell) \subseteq \mathbb{C}[t_1, \ldots, t_m]$, be two radical ideals. Then,

$$\mathbb{V}(I) \times \mathbb{V}(J) = \mathbb{V}(f_1, \ldots, f_k, g_1, \ldots, g_\ell) \subseteq \mathbb{A}^n \times \mathbb{A}^m$$

With the coordinate ring given by $\frac{\mathbb{C}[z_1,\ldots,z_n,t_1,\ldots,t_m]}{(f_1,\ldots,f_k,g_1,\ldots,g_\ell)} = \mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[W]$, with the induced Zariski topology on

$$\text{maxSpec}(\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[W]).$$

Remark 7.25. The Zariski topology on $V \times W$ is larger than the product topology, and for instance, on $\mathbb{A}^1 \times \mathbb{A}^1$ is homeomorphic to $\mathbb{A}^2$ but we have seen in Excercise 1.11 that $\mathbb{V}(x-y)$ is not a closed set in $\mathbb{A}^1 \times \mathbb{A}^1$ with product topology.

Example 7.26. In Example 7.16, $\sigma_4 = \text{cone}(\{e_1, \ldots, e_k\})$, we have

$$X_{\sigma_4} = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}.$$

equipped with Zariski topology. Since $\text{maxSpec}(\mathbb{C}[z_i, z_i^{-1}]) \simeq \mathbb{C}^*$, and $\text{maxSpec}(\mathbb{C}[z_i]) \simeq \mathbb{C}$. △

7.3.2 More Examples

Example 7.27. Let us consider $\sigma = \text{cone}(\{e_1, e_2\}) \subseteq \mathbb{R}^2$, from Figure 9. We can see that $\sigma^\vee = \text{cone}(e_1, e_2)$ is also generated as a monoid by

$$\{(1,0), (1,1), (0,1)\}.$$

The assignment

$$\begin{aligned} X &\longmapsto z^{(1,0)} = z_1 \\ Y &\longmapsto z^{(1,1)} = z_1 z_2 \\ Z &\longmapsto z^{(0,1)} = z_2, \end{aligned}$$

Induces a $\mathbb{C}$-algebra morphism $\mathbb{C}[X,Y,Z] \longrightarrow \mathbb{C}[z_1, z_1 z_2, z_2] = \mathbb{C}[z_1, z_2]$. The relation $(1,0)+(0,1) = (1,1)$, implies that the ideal $(Y - XZ)$, is the kernel of this $\mathbb{C}$-algebra morphism, and we obtain an isomorphism of $\mathbb{C}$-algebras $\frac{\mathbb{C}[X,Y,Z]}{XZ-Y} \simeq \mathbb{C}[z_1, z_2]$. This, in turn, implies the isomorphism of varieties $\mathbb{C}^2 \simeq \mathbb{V}(XZ - Y)$.

△

Question. Given $m_1 = (a,b), m_2 = (c,d) \in \mathbb{Z}^2$, how can we make sure that they generate $\text{cone}(\{m_1, m_2\})$ as a monoid?

Answer. Check whether or not

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1.$$

See Exercise 7.31.

Example 7.28. We can check that $\text{maxSpec}(\mathbb{C}[z_2, z_1 z_2^{-1}]) \simeq \mathbb{C}^2$. In fact, you have already done this in Section 4.2.2. △

Exercise 7.29. For $a, b, c, d \in \mathbb{Z}$, assume that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1.$$

(a) Prove that the assignment

$$
\begin{aligned}
z_1 &\longmapsto z_1^a z_2^b \\
z_2 &\longmapsto z_1^c z_2^d,
\end{aligned}
$$

induces a $\mathbb{C}$-algebra isomorphism between $\mathbb{C}[z_1, z_2]$ and $\mathbb{C}[z_1^a z_2^b, z_1^c z_2^d]$.

(b) If $\operatorname{cone}(\{(a,b),(c,d)\})$ is the dual of a cone σ then V_σ is smooth.

(c) If $\sigma = \operatorname{cone}(\{v_1, v_2\})$ is a rational cone with $\det(v_1|v_2) = \pm 1$, then this property also holds $\sigma^\vee$.

Exercise 7.30. For a finite subset $A \subseteq \mathbb{Z}^n$, let $f(z) = \sum_{\alpha \in A} c_\alpha z^\alpha \in \mathbb{C}[z, z^{-1}] = \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ be a *Laurent polynomial.*

(a) We define the *support* of f, by $\operatorname{supp}(f) = \{\alpha \in \mathbb{Z}^n : c_\alpha \neq 0\}$. Check that $\operatorname{supp}(z_1^2 + z_1 z_2^{-5})$ in $\mathbb{R}^2$ is $\{(1,0),(1,-5)\}$.

(b) Verify that $\mathbb{C}[S_\sigma] = \{f \in \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}] : \operatorname{supp}(f) \subseteq S_\sigma\}$.

(c) We define the *Newton Polytope* of f to be the smallest convex set containing the support of f. Show that

$$\text{Newton polytope of } f^n = n \times \text{Newton polytope of } f.$$

Here is a nice exercise to know when you have a set of generators and in fact works for any dimension.

Exercise 7.31. (a) If $A = \{m_1 = (a,b), m_2 = (c,d)\} \in \mathbb{Z}^2$, then

$$\det(m_1|m_2) := \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1,$$

if and only if,

A generates $\operatorname{cone}(A) \cap \mathbb{Z}^2$ as a monoid.

(b) If $B = \{m_1, \ldots, m_k\} \subseteq \mathbb{Z}^2$ and $\det(w_i|w_{i+1}) = \pm 1$ for $i = 1, \ldots, k$ where $w_1, \ldots, w_k$ are another ordering of $m_1, \ldots, m_k$, then B generates $\operatorname{cone}(B) \cap \mathbb{Z}^2$ as a monoid.

Exercise 7.32. In Figure 11, find a set of generators for $\operatorname{cone}(\{-e_2, 3e_1 + 2e_2\}) \cap \mathbb{Z}^2$ and the relations between these generators.

7.4 Abstract Varieties and Glueing Data

We now intend to create new varieties using existing ones. This process resembles the glueing in topology. To achieve this, we require some data for the glueing process, which includes:

(a) A set I;

(b) For each $i \in I$, an affine algebraic variety X_i;

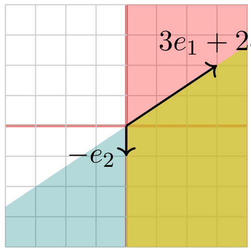


Figure 11: Generators of $\text{cone}(\{-e_2, 3e_1 + 2e_2\})$.

(c) For each pair $i, j \in I$, an open subvariety $X_{ij} \subseteq X_i$;

(d) For each pair $i, j \in I$, an isomorphism of varieties $g_{ij} : X_{ij} \to X_{ji}$.

These data need to be compatible in the following sense:

- For any $i, j, k \in I$, $g_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{ki}$;
- For any $i, j, k \in I$, $g_{jk} \circ g_{ij} = g_{ik}$ on $X_{ij} \cap X_{ik}$;
- For any $i \in I$, $g_{ii} = \mathrm{id}_{X_{ii}}$.

Now using the given glueing data, we define the *abstract algebraic variety* as

$$X := \left(\bigsqcup_{i \in I} X_i \right) / \sim,$$

where $x \sim y$ if and only if there exist $i, j \in I$ such that $x \in X_{ij} \subseteq X_i$ and $y \in X_{ji} \subseteq X_j$ such that $g_{ij}(x) = y$. This gives our space the usual quotient topology.

Definition 7.33. A complex abstract algebraic variety is called

(a) an *abstract affine variety* (respectively, if it is isomorphic to an affine algebraic subvariety of $\mathbb{A}^n$ for some n. An open subset of an abstract affine variety is called *abstract quasi-affine variety.*

(b) an *abstract projective variety* if it is isomorphic to a projective algebraic subvariety of $\mathbb{P}^n$, for some n. An open subset of an abstract projective variety is called *abstract quasi-projective variety.*

(c) *separated* if it is Hausdorff with respect to the Euclidean topology.

(d) *complete* if it is compact with respect to the Euclidean topology.

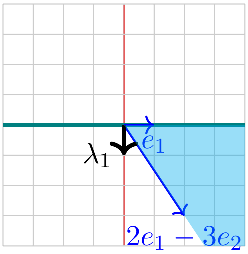
Remark 7.34. There are general definitions of separatedness and completeness that can be verified for any field, but we skip them here for simplicity.

7.5 Faces of a Cone

Definition 7.35. A subset τ of a cone σ is called a *face*, if there exist $\lambda \in \sigma^\vee$, such that

$$\tau = \lambda^\perp \cap \sigma = \{x \in \sigma : \langle \lambda, x \rangle = 0\}.$$

In this case, we write $\tau \preceq \sigma$.



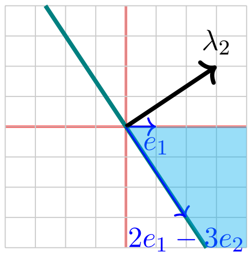


Figure 12: $\lambda_1^\perp \cap \sigma = \mathrm{cone}(\{e_1\})$ and $\lambda_2^\perp \cap \sigma = \mathrm{cone}(\{2e_1 - 3e_2\})$. Note that $\lambda_1^\perp = -\lambda_1^\perp$, but $-\lambda_1 \notin \sigma$, since $\langle \lambda_1, 2e_1 - 3e_2 \rangle < 0$, similarly for λ_2.

Example 7.36. In Figure 12, we can easily verify that $\sigma = \mathrm{cone}(\{e_1, 2e_1 - 3e_2\})$ has 4 faces: $\{0\}, \mathrm{cone}(\{e_1\}), \mathrm{cone}(\{2e_1 - 3e_2\})$, and σ. △

Proposition 7.37. Let σ be a rational cone, then

(a) Every face $\tau = \sigma \cap \lambda^\perp$ is rational cone.

(b) Intersection of every two faces of σ is also a face of σ.

(c) Every face of a face is a face of σ.

(d) If $\tau \preceq \sigma$, then $\sigma^\vee \subseteq \tau^\vee$.

Proof. We leave the proof as an exercise. □

Proposition 7.38. Let σ be a strongly convex cone. If $\tau \preceq \sigma$ is a face, then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-\lambda)$, for some $\lambda \in \sigma^\vee \cap \mathbb{Z}^2$.

Proof. By Proposition 7.9.(a) it suffices to prove the above equality for the dual of each side. By Proposition 7.9.(b), we need to show

$$\tau = \sigma \cap \left(\mathbb{R}_{\geq 0}(-\lambda)\right)^\vee = \sigma \cap \mathrm{cone}(\{-\lambda\})^\vee = \sigma \cap \lambda^\perp.$$

To justify the latter equality, if $x \in \sigma \cap \mathrm{cone}(\{-\lambda\})^\vee$ then $\langle -\lambda, x\rangle \geq 0$. However, by assumption $\lambda \in \sigma^\vee$ and $\langle \lambda, x \rangle \geq 0$. Thus $x \in \lambda^\perp$. □

Example 7.39. Review Figures 9, 12 and 13.

△

Exercise 7.40. Let $\sigma_1 = \mathrm{cone}(\{(e_1, e_1 + e_2)\}), \sigma_2 = \mathrm{cone}(\{(e_2, e_1 + e_2)\}), \tau = \mathrm{cone}(\{(e_1 + e_2)\})$. We have $\tau \preceq \sigma_1$ and $\tau \preceq \sigma_2$. Find $\lambda_i \in \sigma_i^\vee \cap \mathbb{Z}^2$ such that $\tau = \sigma_i \cap \lambda_i^\perp$. Verify that $\mathbb{R}_{\geq 0}(-\lambda_i) + \sigma_i^\vee = \tau^\vee$, for $i = 1, 2$.

Recall that for a cone σ,

$$S_\sigma = \sigma^\vee \cap \mathbb{Z}^n.$$

Proposition 7.41. Let σ be a rational strongly and let τ be a face of σ given by $\tau = \lambda^\perp \cap \sigma$, where $\lambda \in \sigma^\vee \cap \mathbb{Z}^n$. Then, we have $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-\lambda)$.

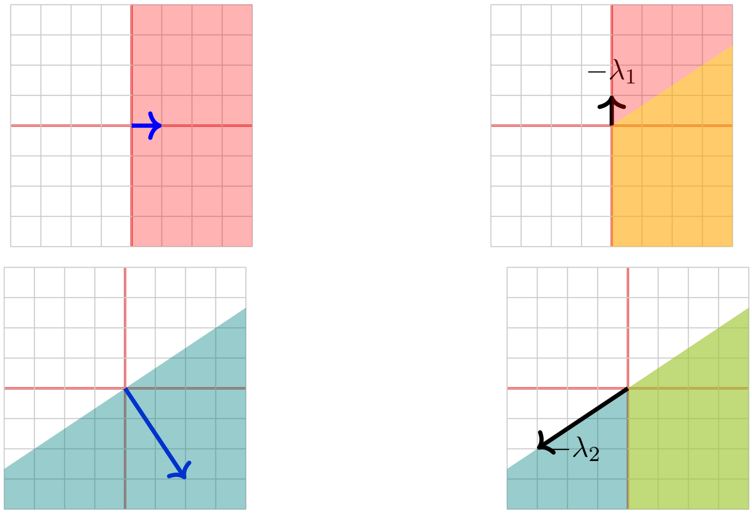


Figure 13: Let $\sigma = \operatorname{cone}(\{e_1, 2e_1 - 3e_2\})$. On the the left we have drawn $\tau_1 = \operatorname{cone}(e_1)^\vee, \tau_2 = \operatorname{cone}(2e_1 - 3e_2)^\vee$. One the right we see the equations $\mathbb{R}_{\geq 0}(-\lambda_i) + \sigma^\vee = \tau_i^\vee$.

Proof. In Proposition 7.38, take the intersection of both sides with $M = \mathbb{Z}^n$. We obtain $S_\sigma = \tau^\vee \cap \mathbb{Z}^n$ on the left-hand side. On the right-hand side, we obtain

$$\left(\sigma^\vee \cap \mathbb{Z}^n\right) \cap \left(\mathbb{R}_{\geq 0}(-\lambda) \cap \mathbb{Z}^n\right).$$

We have $\mathbb{R}_{\geq 0}(-\lambda) \cap \mathbb{Z}^n = \mathbb{R}_{\geq 0}(-\lambda') \cap \mathbb{Z}^n$, where λ' is the smallest lattice vector such that $\operatorname{cone}(\{\lambda\}) = \operatorname{cone}(\{\lambda'\})$, *i.e.*, $\lambda = q\lambda'$, for some $q \in \mathbb{Z}_{\geq 0}$. We claim that, in fact, $-\lambda' \in S_\sigma + \mathbb{Z}_{\geq 0}(-\lambda)$. To see this, note that $\lambda' \in \sigma^\vee \cap \mathbb{Z}^n \implies (q-1)\lambda' \in \sigma^\vee \cap \mathbb{Z}^n$. As a result,

$$-\lambda' = -\lambda + (q-1)\lambda' \in \left(\mathbb{Z}_{\geq 0}(-\lambda)\right) + (\sigma^\vee \cap \mathbb{Z}^n).$$

□

7.6 Fans

Definition 7.42. Let Σ be a finite collection of cones in $\mathbb{R}^2$. Σ is a *fan* if it satisfies,

(a) Each cone $\sigma \in \Sigma$ is a strongly convex rational cone.

(b) If $\sigma \in \Sigma$ then all the faces of σ also belong to Σ.

(c) If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of each.

The *dimension of a cone* in $\mathbb{R}^n$ is the dimensional of minimal subspace of $\mathbb{R}^n$, containing σ.

Example 7.43. In Figure 14, the collection of the cones on the left is not a fan. It consists of 3 two-dimensional cones, 3 one-dimensional cones, and 1 zero-dimensional cone that is not a fan, since Properties (a) for the orange cone, Property (c) also fails since the $y-$axis is the face of the orange cone but not a face of the purple or pink cone. On the right, we have a fan with 3 two-dimensional cones. △

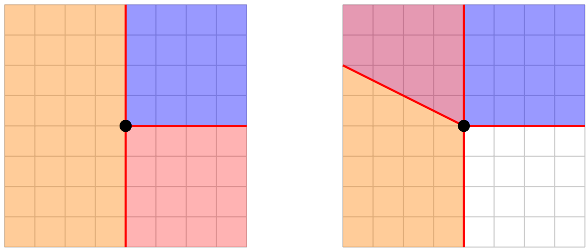


Figure 14: The collection of the cones on the left is not a fan, on the right we have a fan.

For simplicity, in this version of the notes, we only consider the cones in $\mathbb{R}^2$.

Definition 7.44. (a) A fan $\Sigma \subseteq \mathbb{R}^2$ is called *complete* if it covers $\mathbb{R}^2$.

(b) A two dimensional cone $\sigma = \operatorname{cone}(\{v_1, v_2\})$ is called smooth, linear combination of v_1 and v_2 with coefficients in $\mathbb{Z}$ span $\mathbb{Z}^2$. A 2-dimensional fan Σ is called smooth, if for all two dimensional cones $\sigma \in \Sigma$, σ is smooth.

Lemma 7.45. If $\sigma \subseteq \mathbb{R}^2$ is a two dimensional strongly convex smooth cone, then X_σ is a smooth closed affine algebraic variety.

Proof. It is easy to see that a two-dimensional cone σ, is smooth, if there are $v_1, v_2 \in \mathbb{Z}^2$, such that $\sigma = \operatorname{cone}(\{v_1, v_2\})$ and $\left|\det\left(v_1|v_2\right)\right| = 1$. Moreover, if we have $A = \left(v_1|v_2\right) \in \mathrm{GL}_2(\mathbb{Z})$, using the fact that

$$A^{-1} = \frac{1}{\det A}\operatorname{Adj}(A),$$

we find that $A^{-1} \in \mathrm{GL}_2(\mathbb{Z})$. Now it is not hard to see that the dual cone $\sigma^\vee = \operatorname{cone}(\{w_1, w_2\})$ is also smooth, and we obtain a $\mathbb{C}$-algebra homomorphism Φ, such that

$$\begin{aligned} \Phi_A : \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2] &\longrightarrow \mathbb{C}[z_1, z_2] \\ z^{w_1} &\longmapsto z_1 \\ z^{w_2} &\longmapsto z_2 \,. \end{aligned}$$

has the inverse $\Psi := \Phi_{A^{-1}}$. This induces an isomorphism between the corresponding varieties $X_\sigma \simeq \mathbb{C}^2$. Hence, X_σ is smooth. $\square$

Exercise 7.46. Prove that σ is smooth, if and only if, $\sigma^\vee$ is smooth.

7.7 From Fans to Toric Varieties

Now we explain how for a fan $\Sigma \in \mathbb{R}^2$, the associated toric variety X_Σ is defined.

Lemma 7.47. Let σ_1 and σ_2 be 2 two-dimensional strongly convex rational cones. Assume that $\tau = \sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2. Then there exists $\lambda \in \mathbb{Z}^2$, such that

(a) $\tau = \lambda^\perp \cap \sigma_1 = \lambda^\perp \cap \sigma_2$, with $\lambda \in \sigma_1^\vee \cap \mathbb{Z}^2$ and $-\lambda \in \sigma_2^\vee \cap \mathbb{Z}^2$.

(b) $\mathbb{C}[S_\tau] = \mathbb{C}[S_{\sigma_1} + \mathbb{Z}_{\geq 0}(-\lambda)] = \mathbb{C}[S_{\sigma_2} + \mathbb{Z}_{\geq 0}(\lambda)]$.

(c) In addition, we have the inclusion of open subvarieties $X_\tau \subseteq X_{\sigma_1}$ and $X_\tau \subseteq X_{\sigma_2}$.

(d) $\mathcal{O}_{X_{\sigma_i}}(X_{\sigma_i}) \simeq \mathbb{C}[S_{\sigma_i}]$, $\mathcal{O}_{X_{\sigma_1}}(X_\tau) \simeq \mathbb{C}[S_{\sigma_1}]_{z^\lambda}$, $\mathcal{O}_{X_{\sigma_2}}(X_\tau) \simeq \mathbb{C}[S_{\sigma_2}]_{z^{-\lambda}}$.

Proof. (a) There is a unique line $H_\lambda = \lambda^\perp = \{x \in \mathbb{R}^2 : \langle \lambda, x\rangle = 0\}$ with dimension equal to $\dim(\tau) = 1$, containing τ. Since τ is rational, we can assume $\lambda \in \mathbb{Z}^2$. We have that

$$H_\lambda \cap \sigma_1 = H_\lambda \cap \sigma_2 = \tau.$$

Since σ_1, σ_2 are strongly convex, we can assume $\sigma_1 \subseteq H_\lambda^+ = \{x \in \mathbb{R}^2 : \langle \lambda, x\rangle \geq 0\}$, and $\sigma_2 \subseteq H_\lambda^- = \{x \in \mathbb{R}^2 : \langle \lambda, x\rangle \leq 0\}$. Thus, $\langle \lambda, x\rangle \geq 0$, for $x \in \sigma_1$ and $\lambda \in \sigma_1^\vee$. Similarly, $-\lambda \in {\sigma_2}^\vee$.

(b) This is a simple consequence of Part (a) and Proposition 7.41.

(c) Since maxSpec is contravariant, the inclusions $S[\tau] \supseteq S[\sigma_i]$, for $i = 1, 2$, imply that

$$X_\tau \subseteq X_{\sigma_i}.$$

Now assume that $\{v_1, \ldots, v_k\}$ are some generators for S_{σ_1}. Let $f(z) = z^\lambda$, then Part (b) gives

$$\mathbb{C}[S_\tau] = \mathbb{C}[z^{v_1}, \ldots, z^{v_k}, f^{-1}] \simeq \frac{\mathbb{C}[z^{v_1}, \ldots, z^{v_k}, y]}{(yf - 1)},$$

this implies that

$$X_\tau \simeq X_{\sigma_1} \cap \{f \neq 0\} = X_{\sigma_1} \cap D(f),$$

which is an open subset of X_{σ_1}, often denoted as $\left(X_{\sigma_1}\right)_f$ (The complement of $f = 0$ in X_{σ_1} is open.) Compare to Example 4.16 and the following remarks. We have the result similarly for σ_2 :

$$X_\tau \simeq X_{\sigma_2} \cap \{f^{-1} \neq 0\} = X_{\sigma_2} \cap D(f^{-1}),$$

(d) Since $\mathbb{C}[S_{\sigma_i}]$ is the coordinate ring of the affine variety X_{σ_i}, $\mathcal{O}_{X_{\sigma_i}}(X_{\sigma_i}) \simeq \mathbb{C}[S_{\sigma_i}]$ follows from Theorem 4.7. By previous parts, $\mathcal{O}_{X_{\sigma_1}}(X_\tau) \simeq \mathbb{C}[S_{\sigma_1}, f^{-1}]$. Denoting $\mathbb{C}[S_{\sigma_1}, f^{-1}]$ as $\mathbb{C}[S_{\sigma_1}]_f$ is just a notation called *localisation*. We deduce the statement by taking $f = z^\lambda$ and $f = z^{-\lambda}$, respectively and using (b).

□

7.7.1 A Recipe for Constructing Toric Varieties from a Fan

Now we discuss how to obtain an abstract variety from a fan. This abstract variety is separated and unique up to an isomorphism. This process works in any dimension, but for this version of the notes we stick to dimension 2.

Input. A two dimensional smooth fan $\Sigma \subseteq \mathbb{R}^2$.
Output. A two dimensional smooth separated abstract toric surface X_Σ.

Step 1. For each two-dimensional cone σ find $\sigma^\vee$, and $\mathbb{C}[S_\sigma] = \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2]$.

Step 2. If $\tau = \sigma_i \cap \sigma_j$, is a face dimension one, find $\lambda \in \sigma_i^\vee$ such that $-\lambda \in \sigma_j^\vee$ and $\tau = \sigma_i \cap \lambda^\perp = \sigma_j \cap \lambda^\perp$. By Lemma 7.47,

$$\mathbb{C}[S_\tau] = \mathbb{C}[S_{\sigma_i} + \mathbb{Z}_{\geq 0}(-\lambda)] = \mathbb{C}[S_{\sigma_j} + \mathbb{Z}_{\geq 0}(\lambda)]$$

Step 3. Consider the isomorphism of $\mathbb{C}$-algebras Φ_{ji}

$$\begin{array}{ccc} \mathbb{C}[S_\tau] & \xrightarrow{\Phi_{ji}} & \mathbb{C}[S_\tau] \\ \cup\!\shortmid & & \cup\!\shortmid \\ \mathbb{C}[S_{\sigma_j}] & & \mathbb{C}[S_{\sigma_i}] \end{array}$$

that extends the assignment

$$z^{-\lambda} \longmapsto z^{\lambda} \tag{2}$$
$$z^{\mu_j} \longmapsto z^{\mu_i} \tag{3}$$

This induces the isomorphism of open subsets g_{ij}

$$\begin{array}{ccc} X_\tau & \xrightarrow{g_{ij}} & X_\tau \\ \shortmid\!\cap & & \shortmid\!\cap \\ X_{\sigma_i} & & X_{\sigma_j}, \end{array}$$

which we use as glueing data. Note that $g_{12}^* = \Phi_{21}$.

Step 4. Define the glueing abstract toric variety

$$X_\Sigma := \left(\bigsqcup_{\sigma \in \Sigma} X_\sigma \right) / \sim,$$

where $x \sim y$ if and only if there exist $i, j \in I$ such that $x \in X_\tau \subseteq X_{\sigma_i}$ and $y \in X_\tau \subseteq X_{\sigma_j}$ such that $g_{ij}(x) = y$.

This procedure produces a separated toric variety.

Now we can understand the following theorem in combinatorial algebraic geometry:

Theorem 7.48. (a) Σ is smooth, if and only if, X_Σ is smooth.

(b) Σ is complete, if and only if, X_Σ is complete.

Proof. In dimension 2, the 'only if' implication in Part (a) is implied by Lemma 7.45, since the smoothness can be checked locally. For the rest See [CLS11]. $\square$

Example 7.49. Consider the one dimensional fan Σ consisting of $\sigma_1 = \text{cone}(\{1\}), \sigma_2 = \text{cone}(\{1\}), \tau = \text{cone}(\{0\})$. We know from Example 7.22 that $\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[z], \mathbb{C}[S_{\sigma_2}] = \mathbb{C}[z^{-1}], \mathbb{C}[S_\tau] = \mathbb{C}[z, z^{-1}]$. Note that $\mathbb{C}[\tau] = \mathbb{C}[z]_z \supseteq \mathbb{C}[z]$. Similarly, $\mathbb{C}[\tau] = \mathbb{C}[z^{-1}]_{z^{-1}} \supseteq \mathbb{C}[z^{-1}]$. These imply that $X_\tau \subseteq X_{\sigma_i}$, $i = 1, 2$ as an open subset. We have that

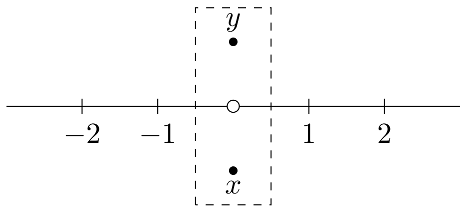


Figure 15: Example of a non-separated topological space (*i.e.*, non-Hausdorff with respect to the Euclidean topology). There is no open set containing x but not containing y.

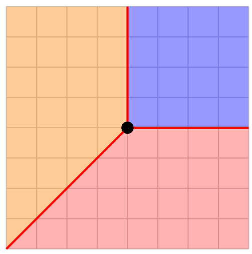


Figure 16: Find the associated toric variety up to an isomorphism.

$X_{\sigma_i} \simeq \mathbb{C}$ and $X_\tau \simeq \mathbb{C}^*$. The above recipe tells us to derive the glueing data based on the homomorphism

$$\begin{array}{ccc} \mathbb{C}[S_\tau] & \xrightarrow{\Phi} & \mathbb{C}[S_\tau] \\ \cup\!| & & \cup\!| \\ \mathbb{C}[S_{\sigma_2}] & & \mathbb{C}[S_{\sigma_1}] \end{array}$$

Such that

$$z \longmapsto z^{-1}.$$

This induces the glueing X_τ

$$\begin{array}{ccc} X_\tau \simeq \mathbb{C}^* & \xrightarrow{z \mapsto z^{-1}} & X_\tau \simeq \mathbb{C}^* \\ | \!\cap & & | \!\cap \\ X_{\sigma_1} \simeq \mathbb{C} & & X_{\sigma_2} \simeq \mathbb{C}, \end{array}$$

which is exactly the description of $\mathbb{P}^1$ in Section 4.2.1. If instead, we use the glueing map $z \longmapsto z$, we get a non-separated variety which is not the correct one (Figure 15). △

Exercise 7.50. Let Σ be the fan given in Figure 16.

(a) What is the toric variety X_Σ?

(b) Is the variety smooth or complete?

Lemma 7.51. In Step 4 of Section 7.7.1 to obtain a separated toric variety Φ_{ji}

$$\begin{array}{ccc} \mathbb{C}[S_\tau] & \xrightarrow{\Phi_{ji}} & \mathbb{C}[S_\tau] \\ \cup\!\mid & & \cup\!\mid \\ \mathbb{C}[S_{\sigma_j}] & & \mathbb{C}[S_{\sigma_i}] \end{array}$$

must extend

$$z^{-\lambda} \longmapsto z^{\lambda} \tag{4}$$

to an isomorphism.

Proof. By a monomial change of coordinates, we can assume that $\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[z^{e_1}, z^{e_2}]$ and $\mathbb{C}[\sigma_2] = \mathbb{C}[z^{-e_1}, z^{\mu}]$. Here $\tau = \text{cone}(\{e_2\})$. We, therefore, obtain $\mathbb{C}[S_\tau] = \mathbb{C}[z^{e_1}, z^{-e_1}, z^{e_2}]$ which contains $\mathbb{C}[S_{\sigma_1}]$ and $\mathbb{C}[S_{\sigma_2}]$. Note that $X_{\sigma_1} \simeq X_{\sigma_2} = \mathbb{C}^2$ and $X_\tau \simeq \mathbb{C}^* \times \mathbb{C}$. ote that the invertible generators must be mapped to each other. As a result, there are only two choices

$$z^{-e_1} \longmapsto z^{e_1} \tag{5}$$

or

$$z^{e_1} \longmapsto z^{e_1} \tag{6}$$

It is easy to check that that the glueing of X_σ and X_{σ_2} along X_τ is separated if and only if the diagonal mapping $X_\tau \longrightarrow X_{\sigma_1} \times X_{\sigma_2}$ given by $z \longmapsto (z, g_{12}(z))$ is a closed embedding. That is, the image of X_τ under the map $(z, g_{12}(z))$ is a closed affine algebraic subvariety of $X_{\sigma_1} \times X_{\sigma_2} \simeq \mathbb{C}^4$. For the choice $\Psi_{21}(z^{e_1}) = z^{e_1}$, and $\Psi_{21}(z^{e_2}) = z^{\mu_2}$, we obtain, $z = (z_1, z_2) \longmapsto \left(z_1, z_2, z_1, z_1^{m_1} z_2^{m_2}\right)$, where $\mu_2 = (m_1, m_2)$. The defining equations are therefore $D = z_1^{m_1} z_2^{m_2}$ and $C = z_1$. However, on V_τ, $z_1 \neq 0$ and the intersection does not yield a closed affine algebraic variety. On the other hand, the image $z = (z_1, z_2) \longmapsto \left(z_1, z_2, z_1^{-1}, z_1^{m_1} z_2^{m_2}\right)$, can be described by $D = z_1^{m_1} z_2^{m_2}$ and $Cz - 1 = 0$, where is a closed affine algebraic subvariety of $X_{\sigma_1} \times X_{\sigma_2}$, and $z \neq 0$.

□

Exercise 7.52. Let Σ be the fan consisting of

- σ_1 cone spanned by $\{(1,0), (1,1)\}$;
- σ_2 cone spanned by $\{(0,1), (1,1)\}$;
- τ cone spanned by $\{(1,1)\}$.

(a) Determine whether or not the toric variety X_Σ has the following properties. Briefly justify your answer.

 (i) smooth;

 (ii) complete.

(b) Describe the coordinate rings of X_{σ_1}, X_{σ_2}, and X_τ.

(c) (i) Explain why we have the inclusions $\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau]$, $\mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau]$;

(ii) Describe the glueing of X_{σ_1} and X_{σ_2} along X_τ.

Solutions. We have that

$$\begin{aligned} \mathbb{C}[S_{\sigma_1}] &= \mathbb{C}[y, \frac{x}{y}] \\ \mathbb{C}[S_{\sigma_2}] &= \mathbb{C}[x, \frac{y}{x}] \\ \mathbb{C}[S_\tau] &= \mathbb{C}[x, y, \frac{x}{y}, \frac{y}{x}] \end{aligned}$$

By Lemma 7.45, we have that $X_{\sigma_1} \simeq X_{\sigma_2} \simeq \mathbb{C}^2$. This is like treating $\frac{y}{x}$ in $\mathbb{C}[S_{\sigma_1}]$ as a new variable t. We easily get that if

$$\begin{aligned} t &\longmapsto \frac{x}{y} \\ y &\longmapsto y \\ x &\longmapsto x, \end{aligned}$$

then $x = ty$. Similarly, $X_{\sigma_2} \simeq \mathbb{V}(ux - y)$. Given the recipe, we can find the unique $\mathbb{C}$-algebra morphism

$$\begin{array}{ccc} \mathbb{C}[S_\tau] & \xrightarrow{\Phi} & \mathbb{C}[S_\tau] \\ \cup\mathrm{I} & & \cup\mathrm{I} \\ \mathbb{C}[S_{\sigma_2}] & & \mathbb{C}[S_{\sigma_1}] \end{array}$$

that assigns

$$\begin{aligned} y &\longmapsto x \\ \frac{x}{y} &\longmapsto \frac{y}{x} \end{aligned}$$

This induces the isomorphism of algebraic varieties

$$\begin{array}{ccc} X_\tau & \xrightarrow{\varphi} & X_\tau \\ \mathrm{I}\cap & & \mathrm{I}\cap \\ X_{\sigma_1} & & X_{\sigma_2}, \end{array}$$

which we can use for glueing. (This part is more than what is asked in the question:) In fact,

$$X_\Sigma \simeq \mathrm{Bl}_0(\mathbb{A}^2).$$

To see this, let us prove that $\mathrm{Bl}_0(\mathbb{A}^2)$ is an analytic manifold and exactly determines the maps of change of coordinates as given by the fan. Recall that

$$\mathrm{Bl}_0(\mathbb{A}^2) = \{\big((x, y); [t : u]\big) \in \mathbb{A}^2 \times \mathbb{P}^1,\ ty - xu = 0\}.$$

We can cover $\mathrm{Bl}_0(\mathbb{A}^2)$ with two charts where $t \neq 0$ or $u \neq 0$.

- When $t \neq 0$, consider the affine chart $\mathbb{A}^1 \simeq \{[1:u] \in \mathbb{P}^1\}$ for $\mathbb{P}^1$. We have the equation $xu = y$ for $(x,y,u) \in \mathbb{A}^2 \times \mathbb{A}^1$. We obtain the isomorphism[11]

$$\begin{aligned} \xi_1 : \{(x, xu, u) : (x,u) \in \mathbb{A}^2\} &\longrightarrow \mathbb{A}^2 \\ (x, xu, u) &\longmapsto (x,u). \end{aligned}$$

- When $u \neq 0$, we have the equation $x = ty$ for $(x,y,t) \in \mathbb{A}^2 \times \mathbb{A}^1$. We have the isomorphism

$$\begin{aligned} \xi_2 : \{(ty, y, t) : (y,t) \in \mathbb{A}^2\} &\longrightarrow \mathbb{A}^2 \\ (ty, y, t) &\longmapsto (y,t). \end{aligned}$$

In the intersection of the open affine charts where $u \neq 0$ and $t \neq 0$, we note that

$$(x, y; [t:u]) = (x, y; [\frac{t}{u} : 1]) = (x, y; [1 : \frac{u}{t}]).$$

Writing $t = x/y$ and $u = y/x$, we obtain

$$\begin{tikzcd} (x, u = y/x) \arrow[r, mapsto, "\xi_0^{-1}"] \arrow[rrr, bend right, "\xi_1 \circ \varphi \circ \xi_0^{-1}"'] & (x, xu = y, u = y/x) \arrow[r, mapsto, "\varphi"] & (x = ty, y, t = y/x) \arrow[r, mapsto, "\xi_1"] & (y, t = x/y). \end{tikzcd}$$

which is exactly the glueing map obtained by the fan above.

Remark 7.53 (The Toric Resolution of Singularities)**.** In the previous example, we see *subdividing* the $\sigma = \text{cone}(\{e_1, e_2\})$ to obtain the fan in the previous example, corresponds to blowing up $X_\sigma \simeq \mathbb{C}^2$ at the origin. This fact is true in general, and by subdividing non-smooth cones, we can obtain smaller cones that are smooth and, as a result, a smooth toric variety. This procedure is the toric version of Hironaka's Theorem and it is called the Toric Resolution of Singularities [CLS11, Theorem 11.1.09]. To prove this for yourself in dimension 2 using Lemma 7.45, convince yourself that any 2-dimensional cone can be subdivided so that the determinant of the generators of smaller cones is ± 1. See [CLS11, Theorem 10.1.10].

Exercise 7.54. Prove that for any rational cone $\sigma \subseteq \mathbb{R}^n$, we have $(\mathbb{C}^*)^n \subseteq X_\sigma$.

Definition 7.55 (General definition of a toric variety)**.** A toric variety is an irreducible variety X such that

(a) $(\mathbb{C}^*)^n$ is a Zariski open subset of X, and

(b) the action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on X.

The action of $(\mathbb{C}^*)^n$ on X partitions X into *orbits*,

[11] Note that projection along u-axis is not an isomorphism but only a birational isomorphism. See Lemma 6.8.

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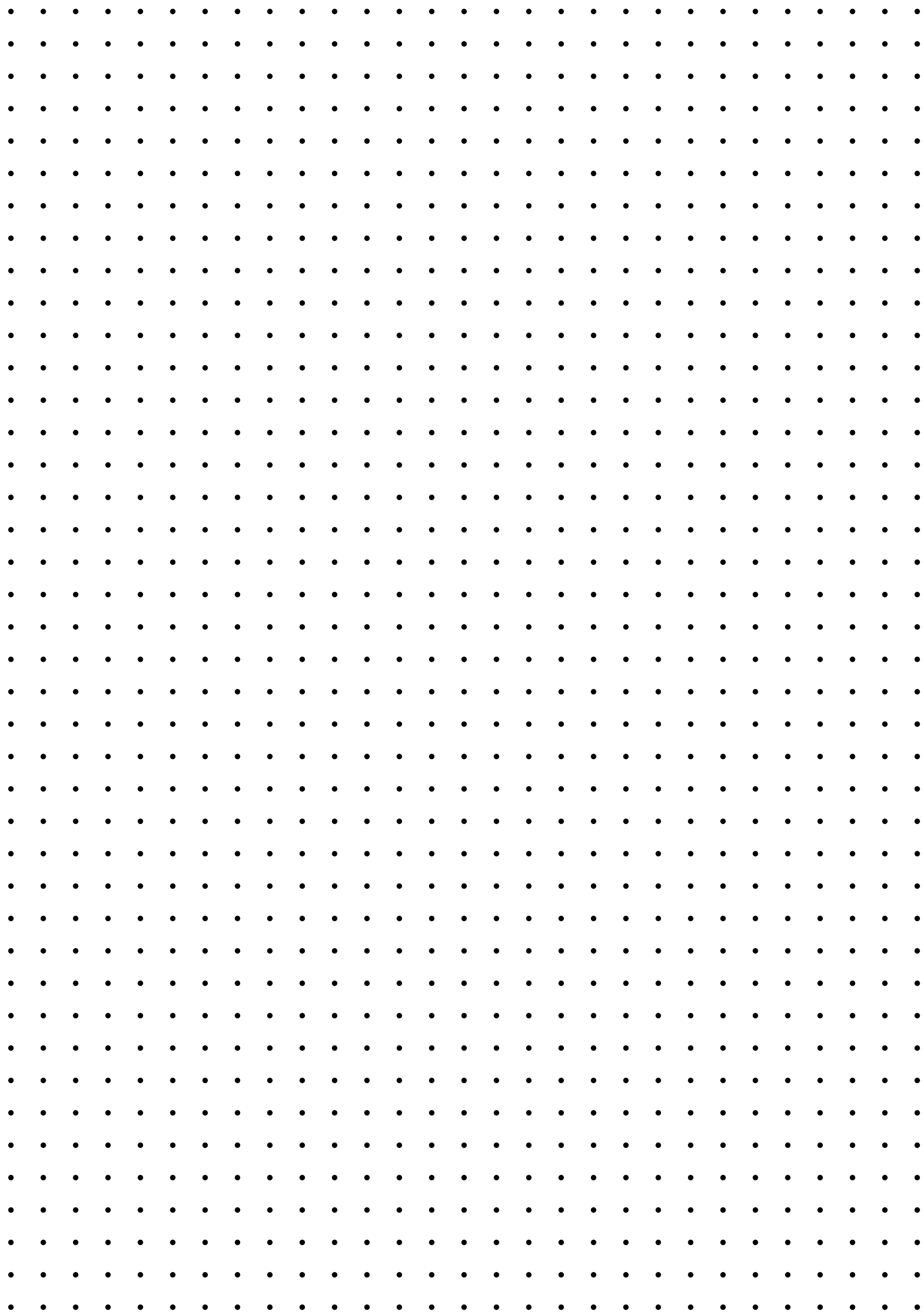

Figure 17: Practice Sheet