CONTINUITY OF THE SUPERPOTENTIALS AND SLICES OF TROPICAL CURRENTS

FARHAD BABAEE AND TIEN CUONG DINH

ABSTRACT.

CONTENTS

1. INTRODUCTION

Let X be a complex manifold of dimension n, and p, q non-negative integers with $n = p + q$. We denote by $\mathcal{C}^q(X) = \mathcal{C}_p(X)$ the cone of positive closed bidegree (q, q), or bidimension $(p, p)-$ currents on X. We also consider $\mathcal{D}^q(X) = \mathcal{D}_p(X)$, the $\mathbb{R}$-vector space spanned by $\mathcal{C}^q(X)$. It is well-known that the intersection of two positive closed currents is not always defined. The main intial progress were due to the works of Federer [Fed69] and Bedford and Taylor in [BT82]. Federer define the a generic slicing theory of currents, that is for a dominant holomorphic map $f : X \longrightarrow Y$, and a positive closed currents $\mathcal{T} \in \mathcal{C}_p(X)$, or more generally, a *flat current*, a *slice*

$$\mathcal{T} \wedge [f^{-1}(y)]$$

is well-defined for a generic $y \in Y$. Following Bedford and Taylor, Demailly [Dem12], and Fornaess–Sibony [FS95] if $\mathcal{S} = dd^c u$ is a bidegree $(1, 1)$-current, then

$$\mathcal{S} \wedge \mathcal{T} := dd^c(u\mathcal{T}),$$

can be defined or it is *admissible*, when

- The potential, u, is bounded
- u is unbounded but its unbounded locus has a small intersection with $\mathrm{supp}(\mathcal{T})$.

For instance, when $\mathcal{S} = dd^c \log|f|$ and $\mathcal{T}$ are two integration currents, such that their supports intersect in the expected dimension, then

$$\mathcal{S} \wedge \mathcal{T} = \sum c_i[C_i],$$

an irreducible

where each C_i is a component of the intersection, and c_i is the corresponding vanishing number. This intersection coincides with the slicing of integration currents.

Demailly in [Dem92] asked the question of generalising the intersection theory to the case where $\mathcal{T}$ is of a higher bidegree. In several works, the second-named author and Sibony introduced *superpotential theory* and *density* of currents to answer this question. In this article we adopt the approach ~~of Dinh and Sibony~~ for our intersection theory. See also the works of Anderson, Eriksson, Kalm, Wulcan and Yger [AESK+21], and [ASKW22] a non-proper intersection theory.

fact

In [DS09] completely discussed the situation where X is a homogeneous space, and in [DS10] investigated the intersection theory for currents with continuous superpotentials, which is a generalisation of the case of bounded potentials in bidegree $(1,1)$. Once the intersections are defined one can ask the following *continuity problem:*

Problem 1.1. Let $\mathcal{T}_k$ be a sequence of positive closed currents on X converging to $\mathcal{T}$. Let $\mathcal{S}$ be also a positive close current on X. Find sufficient conditions such that

$$\lim_{k\to\infty} (\mathcal{S} \wedge \mathcal{T}_k) = \mathcal{S} \wedge (\lim_{k\to\infty} \mathcal{T}_k).$$

Roughly speaking, it ~~we say that~~ $\mathcal{S}$ is a current on a compact Kähler manifold with a *continuous superpotential*, ~~when~~ for a current $\mathcal{T}$, the wedge product then

$$\mathcal{S} \wedge \mathcal{T} := \lim_{n\to\infty} (\mathcal{S} \wedge \mathcal{T}_n),$$

is independent of the choice of smooth approximation $\mathcal{T}_n \longrightarrow \mathcal{T}$. Consequently, by the Regularisation Theorem for any bidegree we can partially answer Problem 1.1.

Proposition 1.2. Let X be a compact Kähler manifold, $\mathcal{T}_k \longrightarrow \mathcal{T}$ be a convergent sequence in $\mathcal{D}^p(X)$. If a current $\mathcal{S}$ has a continuous superpotential, then

$$\mathcal{S} \wedge \mathcal{T}_n \longrightarrow \mathcal{S} \wedge \mathcal{T}.$$

Proof. The main result ~~of Dinh and Sibony's result~~ in [DS04] implies any current $\mathcal{D}^q(X)$ can be weakly approximated by a difference of smooth closed positive of bidegree (p,p)-forms of bounded mass. The result then follows from the definition of continuity of super-potentials. □

Problem 1.1 becomes more difficult when one considers continuity for slices, and the current $\mathcal{S}$ is an integration current. Borrowing ideas in tropical geometry, we discuss this problem for the very specific case where $\lim_{k\to\infty} \mathcal{T}_k$ is a *complex tropical current* [Bab14, BH17]. (Complex) tropical currents are closed currents on complex tori $(\mathbb{C}^*)^n$ or on a toric variety associated to a *tropical cycle*. Recall that a tropical cycle is a weighted polyhedral complex satisfying the *balancing condition* (see Definition 3.1). For a tropical cycle $\mathcal{C} \subseteq \mathbb{R}^n$, of dimension p, the associated tropical current $\mathcal{T}_\mathcal{C} \in \mathcal{D}_p((\mathbb{C}^*)^n)$, is a closed current with support $\mathrm{Log}^{-1}(\mathcal{C})$, where

$$\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \quad (z_1, \dots, z_n) \longmapsto (-\log|z_1|, \dots, -\log|z_n|).$$

The tropical current $\mathcal{T}_\mathcal{C}$ can be naturally presented as a locally fibration of $\mathrm{Log}^{-1}(\mathcal{C})$, and we say $\mathcal{C}$ is compatible with the fan Σ, if the fibres of $\overline{\mathcal{T}}_\mathcal{C}$ intersect the toric invariant divisors of the toric variety X_Σ transversely. Here $\overline{\mathcal{T}}_\mathcal{C}$ denotes the extension by zero of $\mathcal{T}_\mathcal{C}$ to the toric variety X_Σ.

Theorem 1.3. Let X_Σ be a smooth projective toric variety, and let $\mathcal{C}$ be a tropical cycle compatible with Σ. Then $\overline{\mathcal{T}}_{\mathcal{C}}$ has a continuous super-potential.

The preceding theorem allows for defining the intersection product of a tropical current with any current on a compatible toric variety, we can then restrict the intersection product to the complex torus $T_N \subseteq X_\Sigma$ and use the isomorphism $T_N \simeq (\mathbb{C}^*)^n$, to define the intersection product of two tropical currents in $(\mathbb{C}^*)^n$. On the tropical geometry side, there exists a *stable intersection theory* of tropical cycles. The word stable here precisely corresponds to the continuity of the definition with respect to generic translations of tropical cycles. With the stable intersection and natural addition of tropical cycles, we have the ring of tropical cycles.

Theorem 1.4. The assignment $\mathcal{C} \longmapsto \mathcal{T}_{\mathcal{C}}$ induces a $\mathbb{Z}$-algebra homomorphism between

(a) The $\mathbb{Z}$-algebra of tropical cycles in $\mathbb{R}^n$ with the natural addition (Definition ??) and stable intersection (Definition 3.3) as the multiplication. /

(b) The $\mathbb{Z}$-algebra of tropical currents on $(\mathbb{C}^*)^n$ with the usual addition of currents and the wedge product of currents.

We also address Problem 1.1 in a very particular case of slicing of currents converging to a tropical current. The theorem is inspired by works in [BJS+07], [OP13] and [Jon16]. Add more theorems.

Theorem 1.5. Let $D, W \subseteq (\mathbb{C}^*)^n$ be an algebraic subtorus, and an algebraic subvariety respectively. Assume that $\operatorname{Log}(D)$ intersects $\operatorname{trop}(W)$ properly. Then,

Recall the bi-dimension of W?

$$\lim_{m\to\infty}\Big(\frac{1}{m^{n-p}}\Phi_m^*[W]\wedge[D]\Big) = \Big(\lim_{m\to\infty}\frac{1}{m^{n-p}}\Phi_m^*[W]\Big)\wedge[D],$$

where $\Phi_m : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$ is the m-th power map $(z_1,\dots,z_n) \longmapsto (z_1^m,\dots,z_n^m)$.

The proof relies on a theorem of Berteloot and the second-named author [BD20] that the limit of slices satisfies a certain continuity of harmonic functions, and we can use Fourier analysis to prove the theorems about tropical currents.

2. Tools from Superpotential Theory

Let (X,ω) be a compact Kähler manifold of dimension n. Assume that $\mathcal{S}$ is either a positive or a negative current of bidegree (q,q) on X. The quantity

$$|\langle \mathcal{S}, \omega^{n-q}\rangle|$$

is referred to as the *total mass* of $\mathcal{S}$. For $0 \le r \le n$, we consider the de Rham cohomology groups $H^r(X,\mathbb{C}) = H^r(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ with coefficients in $\mathbb{C}$. Recall that Hodge theory provides the following decomposition of the de Rham cohomology group into Dolbeault cohomology groups:

$$H^r(X,\mathbb{C}) \simeq \bigoplus_{p+q=r} H^{p,q}(X,\mathbb{C}).$$

We denote by $\mathcal{C}^q(X)$ the cone of positive closed bidegree (q,q)-currents or bidimension $(n-q,n-q)$ in X. We denote by $\mathcal{D}^q(X) = \mathcal{D}_{n-q}(X)$ the $\mathbb{R}$-vector space spanned by $\mathcal{C}^q(X)$, which is the space of closed real currents of bidegree (q,q). Every current $\mathcal{T} \in \mathcal{D}^q(X)$ has a cohomology class:

$$\{\mathcal{T}\} \in H^{q,q}(X,\mathbb{R}) = H^{q,q}(X,\mathbb{C}) \cap H^{2q}(X,\mathbb{R}).$$

We define $\mathcal{D}^{q,0}(X) = \mathcal{D}^0_{n-q}(X)$ to be the subspace of $\mathcal{D}^q(X)$, consisting of currents with vanishing cohomology. The $*$-topology on $\mathcal{D}^q(X)$ is defined by the norm:

$$\|\mathcal{S}\|_* := \min(\|\mathcal{S}^+\| + \|\mathcal{S}^-\|),$$

where the minimum is taken over positive currents $\mathcal{S}^+$ and $\mathcal{S}^-$ in $\mathcal{C}^q(X)$ that satisfy $\mathcal{S} = \mathcal{S}^+ - \mathcal{S}^-$. We say that $\mathcal{S}_n$ converges to $\mathcal{S}$ in $\mathcal{D}^q(X)$ if $\mathcal{S}_n$ converges weakly to $\mathcal{S}$ and moreover, $\|\mathcal{S}_n\|_*$ is bounded by a constant independent of n.

Let $h := \dim H^{q,q}(X,\mathbb{R})$, and fix a set of smooth forms $\alpha = (\alpha_1, \ldots, \alpha_{\beta_h})$ such that their cohomology classes $\{\alpha\} = (\{\alpha_1\}, \ldots, \{\alpha_h\})$ form a basis for $H^{q,q}(X,\mathbb{R})$. By Poincaré duality, there exists a set of smooth forms $\alpha^\vee = (\alpha_1^\vee, \ldots, \alpha_h^\vee)$ such that their cohomology classes $\{\alpha^\vee\}$ form the dual basis of $\{\alpha\}$, with respect to the cup-product. By adding $U_\mathcal{S}$ to a suitable combination of $\alpha_i^\vee$, we can assume that $\langle U_\mathcal{S}, \alpha_i \rangle = 0$, for all $i = 1, \ldots, h$. In this case, we say that $U_\mathcal{S}$ is *α-normalised.*

Definition 2.1. Let $\mathcal{T} \in \mathcal{D}^q(X)$ and $\mathcal{S}$ be a smooth form in $\mathcal{D}^{n-q+1,0}(X)$.

(i) The *α-normalised super-potential* $\mathcal{U}_\mathcal{T}$ of $\mathcal{T}$ is given by the function

$$\mathcal{U}_\mathcal{T} : \{\mathcal{S} \in \mathcal{D}^{n-q+1,0}(X) : \text{smooth}\} \longrightarrow \mathbb{R}$$
$$\mathcal{S} \longmapsto \langle \mathcal{T}, U_\mathcal{S} \rangle,$$

where $U_\mathcal{S}$ is the α-normalised potential of $\mathcal{S}$.

(ii) We say $\mathcal{T}$ has a *continuous super-potential,* if $\mathcal{U}_\mathcal{T}$ can be extended to a function on $\mathcal{D}^{n-q+1,0}(X)$ which is continuous with respect to the $*$-topology.

In general, consider $\mathcal{T} \in \mathcal{D}^q(X)$ and $\mathcal{S} \in \mathcal{D}^r(X)$. Assume that $q + r \leq n$ and $\mathcal{T}$ has a continuous super-potential. Let $\mathcal{U}_\mathcal{T}$ be the α-normalised super-potential of $\mathcal{T}$. Let $\beta \in \mathrm{Span}_\mathbb{R}\{\alpha\}$ such that $\{\beta\} = \{\mathcal{T}\}$. For any compactly supported smooth form φ of bidegree $(n-q-r, n-q-r)$, we define

$$(1) \qquad \langle \mathcal{T} \wedge \mathcal{S}, \varphi \rangle := \mathcal{U}_\mathcal{T}(\mathcal{S} \wedge dd^c\varphi) + \langle \beta \wedge \mathcal{S}, \varphi \rangle.$$

Now assume that if $f : X \longrightarrow Y$, is a biholomorphism between smooth compact Kähler manifolds, then we have

$$f_*\mathcal{U}_{\mathcal{R}_1} = \mathcal{U}_{f_*\mathcal{R}_1}, \quad f^*\mathcal{U}_{\mathcal{R}_2} = \mathcal{U}_{f^*\mathcal{R}_2},$$

for $\mathcal{R}_1 \in \mathcal{D}^q(X)$ and $\mathcal{R}_2 \in \mathcal{D}^q(Y)$. You had some comments here, but not clear whether I need to add something.

Definition 2.2. Let $(\mathcal{T}_n)$ be a sequence of currents in $\mathcal{D}^q(X)$ weakly converging to $\mathcal{T}$. Let $\mathcal{U}_\mathcal{T}$ and $\mathcal{U}_{\mathcal{T}_n}$ be their α-normalised super-potentials. If $\mathcal{U}_{\mathcal{T}_n}$ converges to $\mathcal{U}_\mathcal{T}$ uniformly on any $*$-bounded sets of smooth forms in $\mathcal{D}^{n-q+1,0}(X)$, then the convergence is called *SP-uniform.*

It is shown in [DS10, Proposition 3.2.8] that any current with continuous super-potentials can be SP-uniformly approximated by smooth forms. Moreover, currents with continuous super-potentials have other nice properties:

Theorem 2.3 ([DNV18, Theorem 1.1]). *Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two positive currents in $\mathcal{D}_q(X)$, such that $\mathcal{T} \leq \mathcal{T}'$, i.e., $\mathcal{T}' - \mathcal{T}$ is a positive current. Then, if $\mathcal{T}'$ has a continuous super-potential, then so does $\mathcal{T}$.*

$f^*: \mathcal{D}^{n-q-1,0} \to \mathcal{D}^{n-q-1,0}$, $\mathcal{U}_{R_1}$ is a function on $\mathcal{D}^{n-q-1,0}$. So the notation $f_*\mathcal{U}_{R_1}$ is understandable but it is the composition of $\mathcal{U}_{R_1}$ with f^*, nc
This is just about the notation $f_*\mathcal{U}_{R_1}$

Theorem 2.4. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two positive closed currents, and $\mathcal{T}_1$ has a continuous superpotentials, then $\mathcal{T}_1 \wedge \mathcal{T}_2$ is well-defined. Moreover, if $\mathcal{T}_2$ has also a continuous superpotential, then

(a) [DS10, Proposition 3.3.3] $\mathcal{T}_1 \wedge \mathcal{T}_2$ has a continuous superpotential;
(b) [DS10, Proposition 3.3.3] This wedge product is continuous with respect to the SP-uniform convergence.
(c) [DS09, Theorem 4.2.4] $\operatorname{supp}(\mathcal{T}_1 \wedge \mathcal{T}_2) \subseteq \operatorname{supp}(\mathcal{T}_1) \cap \operatorname{supp}(\mathcal{T}_2)$.

Theorem 2.5 ([DS10, Proposition 3.3.4]). Assume that $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are closed positive currents, and $\mathcal{T}_1$ and $\mathcal{T}_2$ have continuous superpotentials. Then,

$$\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_2 \wedge \mathcal{T}_1 \quad \text{and} \quad (\mathcal{T}_1 \wedge \mathcal{T}_2) \wedge \mathcal{T}_3 = \mathcal{T}_1 \wedge (\mathcal{T}_2 \wedge \mathcal{T}_3).$$

Proposition 2.6. Let X be a compact Kähler manifold, $S_n \longrightarrow S$ be a convergent sequence in $\mathcal{D}^q(X)$. If a current $\mathcal{T}$ has a continuous superpotential, then

$$\mathcal{T} \wedge \mathcal{S}_n \longrightarrow \mathcal{T} \wedge \mathcal{S}.$$

Proof. The main result ~~of Dinh and Sibony~~ in [DS04] implies any current $\mathcal{T} \in \mathcal{D}^p(X)$ can be weakly approximated by a difference of smooth closed positive of bidegree (p,p)-forms. The result then follows from the definition of continuity of super-potentials. □ and (1)

Lemma 2.7. Let $\mathcal{T}, \mathcal{T}'$ be positive closed currents such that $\mathcal{T}_{|\Omega} = \mathcal{T}'_{|\Omega}$ in an open subset $\Omega \subseteq X$, and both $\mathcal{T}$ and $\mathcal{T}'$ have continuous super-potentials. Then, for any $\mathcal{S} \in \mathcal{D}^r(X)$,

$$(\mathcal{T} \wedge \mathcal{S})_{|\Omega} = (\mathcal{T}' \wedge \mathcal{S})_{|\Omega}.$$

Proof. In [DS10], for any current $\mathcal{S}$ with continuous super-potential, a family $\{\mathcal{T}_\theta\}_{\theta \in \mathbb{C}^*}$ is constructed that $\mathcal{T}_\theta$ converges SP-uniformly to $\mathcal{S}$ as $|\theta| \to 0$. Let $\epsilon := |\theta| > 0$, be a small positive number, and $V \subseteq U$ be any open set such that V_ϵ, the ϵ-neighbourhood of V, is contianed entirely in Ω. Therefore, by the hypothesis of the lemma, we can construct families of smooth forms $\mathcal{T}_n$ and $\mathcal{T}'_n$ converging SP-uniformly to $\mathcal{T}$ and $\mathcal{T}'$ respectively. Moreover,

$$\mathcal{T}_{n|V_\epsilon} = \mathcal{T}'_{n|V_\epsilon}.$$

Now, for any $(n-q-r, n-q-r)$ smooth form φ with compact support on Ω, we can cover the support of φ with an open set of the form V_ϵ and deduce

$$(\mathcal{T}_n \wedge \mathcal{S}) \wedge \varphi = (\mathcal{T}'_n \wedge \mathcal{S}) \wedge \varphi.$$

This, together with Theorem 2.4(b) implies the assertion.

□

We also have a very useful local version of Theorem 2.3.

Lemma 2.8. If $\mathcal{T}$ is a positive closed current on a compact Kähler manifold X, which is locally bounded by a product of positive closed bidegree $(1,1)$-currents of continuous potentials, resp. Hölder continuous potentials, then $\mathcal{T}$ has continuous superpotentials, respectively Hölder continuous potentials in X.

Proof. Fix a point a in X. In an open neighbourhood of a that we identify with the ball $B(0,2)$ in $\mathbb{C}^n$, we have

$$\mathcal{T} \leq dd^c u_1 \wedge \cdots \wedge dd^c u_p$$

with u_i continuous or Hölder continuous. Without loss of generality, we can assume that these functions are strictly negative. On $B(0,1)$, define

$$u_i' := \max(u_i, A\log\|z\|)$$

with A sufficiently large so that $u_i' = u_i$ on $B(0,1/2)$. Observe that $u_i' = A\log\|z\|$ near $\partial B(0,1)$. Hence we can extend it to a function which is smooth in a neighbourhood of $X \setminus B(0,1)$. Thus, this function is quasi-plurisubharmonic. We have

$$\mathcal{T} \leq (B\omega + dd^c u_1') \wedge \cdots \wedge (B\omega + dd^c u_p')$$

in a neighbourhood W_a of a if B is large enough.

Since we can cover X using a finite number of open sets W_{a_k}, we can add up all obtained quasi-plurisubharmonic functions together and obtain a quasi-plurisubharmonic function u. It is clear that

$$\mathcal{T} \leq (C\omega + u)^p$$

if C is large enough. The function u is continuous or Hölder continuous, and we deduce by Theorem 2.3. □

I have changed this into a proposition. It can be a corollary of the previous theorem, if we prove it for the hypersurfaces. I have written a new proof now. Please check.

Proposition 2.9. Let $q : \widehat{X} \longrightarrow X$, be the blowing up of the compact Kähler manifold X along a submanifold. Assume that $\mathcal{T} \in \mathcal{D}_p(\widehat{X})$ is such that the support of $\mathcal{T}$ does not intersect the exceptional divisors of $\widehat{X}$. If the current $\mathcal{T}$ has a continuous superpotential then $q_*\mathcal{T}$ has the same property.

Proof. Please check this proof. By the hypothesis, q restricts to a biholomorphism in an open neighbourhood $\cancel{U}$ Ω containing supp($\mathcal{T}$). Assume that $\mathcal{S} \in \mathcal{D}^0_{n-q+1}$, and $\mathcal{S}_\epsilon \longrightarrow \mathcal{S}$ is a smooth approximation. Let χ be a test function with support in Ω, but equal to ~~identity~~ 1 in supp($\mathcal{T}$). $f(\Omega)$ and $f_*(T)$?

$$\langle f_*\mathcal{T}, U_{\mathcal{S}_\epsilon}\rangle = \langle \chi\, f_*\mathcal{T}, U_{\mathcal{S}_\epsilon}\rangle = \langle f_*\mathcal{T}, \chi\, U_{\mathcal{S}_\epsilon}\rangle = \langle f_*\mathcal{T} - f_*\beta, \chi U_{\mathcal{S}_\epsilon}\rangle + \langle f_*\beta, \chi U_{\mathcal{S}_\epsilon}\rangle.$$

β not introduce

The term $\langle f_*\mathcal{T} - f_*\beta, \chi U_{\mathcal{S}_\epsilon}\rangle$ equals

$$\langle dd^c U_{f_*\mathcal{T}}, \chi U_{\mathcal{S}_\epsilon}\rangle = \langle U_{f_*\mathcal{T}}, dd^c\chi \wedge \mathcal{S}_\epsilon\rangle = \langle U_{\mathcal{T}}, f^*(dd^c\chi \wedge \mathcal{S}_\epsilon)\rangle.$$

On $f(\cancel{U})$ Ω, f^* acts continuously on currents, and by continuity of super-potential of $\mathcal{T}$ on $X_{\widehat{\Sigma}}$ implies that

$$\langle U_{\mathcal{T}}, f^*(dd^c\chi \wedge \mathcal{S}_\epsilon)\rangle \longrightarrow \langle U_{\mathcal{T}}, f^*(dd^c\chi \wedge \mathcal{S})\rangle.$$

Reassembling yields the result.

$$\mathcal{U}_{f_*\mathcal{T}}(\mathcal{S}_\epsilon) = \langle U_{\mathcal{S}_\epsilon}, \chi f_*\mathcal{T}\rangle = (\mathcal{S}_\epsilon \wedge \chi) = \mathcal{U}_{\mathcal{T}}(f^*(\mathcal{S}_\epsilon \wedge dd^c\chi) \longrightarrow \mathcal{U}_{\mathcal{T}}(f^*\mathcal{S}) = \mathcal{U}_{f_*\mathcal{T}}(\mathcal{S}).$$

$$\langle \mathcal{T} \wedge \mathcal{S}, \varphi\rangle := \mathcal{U}_{\mathcal{T}}(\mathcal{S} \wedge dd^c\varphi) + \langle \mathcal{S} \wedge \varphi, \beta\rangle. \tag{2}$$

□

Problem

Theorem 2.10. For two complex manifolds X and Y, consider two convergent sequences of currents $\mathcal{T}_n \longrightarrow \mathcal{T}$ in $\mathcal{D}^q(X)$ and $\mathcal{S}_n \longrightarrow \mathcal{S}$ in $\mathcal{D}^r(Y)$. We have that

$$\mathcal{T}_n \otimes \mathcal{S}_n \longrightarrow \mathcal{T} \otimes \mathcal{S},$$

weakly in $\mathcal{D}^{q+r}(X \times Y)$.

Sketch of the proof. Let us denote by (x, y) the coordinates on $X \times Y$. Using local coordinates and a partition of unity and Weierstrass theorem we can approximate any smooth forms on $X \times Y$ with forms with polynomial coefficients in (x, y). The approximation is in C^∞. As a result, the convergence, we only need test forms with monomial coefficients. Thus, the variables x, y are separated and the convergence of the tensor products becomes the convergence of each factor. $\square$

2.1. **Semi-continuity of slices.** Let $f : X \longrightarrow Y$ be a dominant holomorphic map between complex manifolds, not necessarily compact, of dimension n and m respectively. Let $\mathcal{T}$ be a positive closed current on X of bi-dimension (p, p) with $p \geq m$. Then a slice

$$\mathcal{T}_y = \langle \mathcal{T} | f | y \rangle$$

obtained by restricting $\mathcal{T}$ to $f^{-1}(y)$ exists for almost every $y \in Y$; see [Dem, Page 171]. This is a positive closed current of bi-dimension $(p-m, p-m)$ on X supported by $f^{-1}(y)$. If Ω is a smooth form of maximal bi-degree on Y and α a smooth $(q-m, q-m)$-form with compact support in X, then we have

$$\langle \mathcal{T}, \alpha \wedge f^*(\Omega) \rangle = \int_{y \in Y} \langle \mathcal{T}_y, \alpha \rangle \Omega(y).$$

In general, if $\mathcal{T}$ and $\mathcal{T}'$ are such that $\mathcal{T}_y = \mathcal{T}'_y$ for almost every y, we do not necessarily have $\mathcal{T} = \mathcal{T}'$. However, the following is true: Let $f_1, \ldots, f_k$ be dominant holomorphic maps from X to $Y_1, \ldots, Y_k$. Consider the vector space spanned by all the differential forms of type $\alpha \wedge f_i^*(\Omega_i)$ for some α as above and for some smooth form Ω_i on Y_i of maximal degree. Assume this space is equal to space of all (q, q)-forms of compact support in X. Then if $\langle \mathcal{T} | f_i | y_i \rangle = \langle \mathcal{T}' | f_i | y_i \rangle$ for every i and almost every $y_i \in Y_i$, we have $\mathcal{T} = \mathcal{T}'$. The proof is a consequence of the above discussion. let

Let $U \subseteq \mathbb{C}^m$ and $V \subseteq \mathbb{C}^n$ be two bounded open sets. Assume that $\pi_1 : U \times V \longrightarrow U$ and $\pi_2 : U \times V \longrightarrow V$ are the canonical projections. Consider two closed positive currents $\mathcal{T}$ and $\mathcal{S}$ on $U \times V$ of bi-dimension (m, m) and (n, n) respectively. We say that $\mathcal{T}$ horizontal-like if $\pi_2(\mathrm{supp}(\mathcal{T}))$ is relatively compact in V. Similarly, if $\pi_1(\mathrm{supp}(\mathcal{S}))$ is relatively compact in U, $\mathcal{S}$ is called vertical-like.

Theorem 2.11 ([BD20, Lemma 3.7]). Let $(\mathcal{T}_n) \longrightarrow \mathcal{T}$ be a convergent sequence of horizontal-like positive closed currents to a horizontal-like current $\mathcal{T}$ in $U \times V$. Let $a \in U$ and assume that the sequence of measures $(\langle \mathcal{T}_n, \pi_1, a \rangle)_n$ is also convergent. Then,

$\lim_{n \to \infty} \langle T_n | \pi_1 | a \rangle (\phi)$

$$\langle \lim_{n \to \infty} \mathcal{T}_n | \pi_1 | a \rangle (\phi) \leq \langle \mathcal{T} | \pi_1 | a \rangle (\phi)$$

for every plurisubharmonic function ϕ on $\mathbb{C}^n$.

There is an simple version of the above theorem for supports which will be useful later.

3.2. The $\mathbb{Z}$-algebra of Tropical Cycles. Recall that, generally speaking, the star of a cone in a complex is the extension of the local p-dimensional fan surrounding it. More precisely:

You're right that when τ and σ are cones we don't need λ. There was a type in the previous version too that σ was assumed to be a cone.

Definition 3.2. Given a polyhedral complex $\Sigma \subseteq \mathbb{R}^n$ and a cell $\tau \in \Sigma$, define the star of τ in Σ, denoted by $\mathrm{star}_\Sigma(\tau)$, is a fan in $\mathbb{R}^n$. The cones of $\mathrm{star}_\Sigma(\tau)$ are the *extensions* of cells σ that include τ as a face. Here, by extension, we mean

$$\bar{\sigma} = \{\lambda(x-y) : \lambda \geq 0, x \in \sigma,\, y \in \tau\}.$$

Definition 3.3 (Stable Intersection). (a) Let $C_1, C_2 \subseteq \mathbb{R}^n$ be two tropical cycles of dimension p and q, intersecting transversely. That is, the top dimensional cells $\sigma_1 \in C_1$ and $\sigma_2 \in C_2$ intersect in dimension $p+q-n$. Then the stable intersection of $C_1 \cdot C_2$ is the tropical cycles supported on finitely many cells $C_1 \cap C_2$. In this case, the weight of a cell $\sigma_1 \cap \sigma_2$ if defined by

$$w_{C_1 \cdot C_2}(\sigma_1 \cap \sigma_2) = w_{\sigma_1} w_{\sigma_2} [N : N_{\sigma_1} + N_{\sigma_2}],$$

where $N = \mathbb{Z}^n$ here.

(b) When C_1 and C_2 do not intersect transversely, then $C_1 \cdot C_2$ as a set is the Hausdorff limit of

$$C_1 \cap (\epsilon b + C_2), \quad \text{as } \epsilon \to 0,$$

for a fixed generic $b \in \mathbb{R}^n$, and the weights are the sum of all the tropical multiplicities of the cells in the transversal intersection $C_1 \cap (\epsilon b + C_2)$ which converge to the same $p+q-n$-dimensional cell in the Hausdorff metric. Equivalently, for top dimensional cones $\sigma_1 \in C_1$ and $\sigma_2 \in C_2$

$$w_{C_1 \cdot C_2}(\sigma_1 \cap \sigma_2) = \sum_{\tau_1, \tau_2} w_{\tau_1} w_{\tau_2} [N : N_{\tau_1} + N_{\tau_2}],$$

where the sum is taken over all $\tau_1 \in \mathrm{star}_{C_1}(\sigma_1 \cap \sigma_2), \tau_2 \in \mathrm{star}_{C_2}(\sigma_1 \cap \sigma_2)$ with $\tau_1 \cap (v + \tau_2) \neq \varnothing$, for some fixed generic vector $b \in \mathbb{R}^n$. $\rightarrow v = b?$

(c) When $p + q < n$, then the stable intersection of C_1 and C_2 is the empty set.

The following result is proved in tropical geometry; see [MS15, Lemmas 3.6.4 and 3.6.9]. We will revisit its proof later through the lens of superpotential theory.

Theorem 3.4. When $p+q \geq n$, the stable intersection, defined above, yields a balanced polyhedral complex of dimension $p + q - n$.

We also need the following for turning the set of tropical cycles into a $\mathbb{Z}$-algebra.

Definition 3.5 (Addition of Tropical Cycles). For two p-dimensional tropical cycles C_1, C_2 in $\mathbb{R}^n$, the addition $C_1 + C_2$ is the tropical cycle obtained by the common refinement of the support $|C_1| \cup |C_2|$ where the weights of a cone σ in the refinement are determined by $w_{C_1+C_2}(\sigma) = w_{C_1}(\sigma) + w_{C_2}(\sigma)$.

Let us end this section with an example of the stable intersection.

We define

$$\Pi_H : (\mathbb{C}^*)^n \simeq \mathbb{C}^* \otimes \left((H \cap \mathbb{Z}^n) \oplus \mathbb{Z}^n/(\mathbb{Z}^n \cap H)\right) \longrightarrow T_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)} .$$

$H \cap \mathbb{Z}^n$

One has

$$\ker(\Pi_H) = \ker(\pi_H) = T_{H\cap\mathbb{Z}^n} \subseteq (\mathbb{C}^*)^n.$$

As a result, when H is of dimension p, the set $\mathrm{Log}^{-1}(H)$ is naturally foliated by the $\pi_H^{-1}(x) = T_{H\cap\mathbb{Z}^n} \cdot x \simeq (\mathbb{C}^*)^p$ for $x \in S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}$. For a lattice basis $u_1, \ldots, u_p$, of $H\cap\mathbb{Z}^n$, the tori $T_{H\cap\mathbb{Z}^n}.x$ can be parametrised by the monomial map

$$(\mathbb{C}^*)^p \longrightarrow (\mathbb{C}^*)^n, \quad z \longmapsto x.z^{[u_1,\ldots,u_p]^t}$$

where $U = [u_1, \ldots, u_p]$ is the matrix with column vectors $u_1, \ldots, u_p$, and z^{U^t} denotes that $z \in (\mathbb{C}^*)^p$ is taken to have the exponents with rows of the matrix U. Accordingly, one can easily check that

$$T_{H\cap\mathbb{Z}^n} \cdot x = \{z \in (\mathbb{C}^*)^n : z^{m_i} = x^{m_i},\ i = 1, \ldots, n-p\}.$$

for any choice of a $\mathbb{Z}$-basis $\{m_1, \ldots, m_{n-p}\}$ of $\mathbb{Z}^n/(H \cap \mathbb{Z}^n)$. checked

Definition 4.1. Let H be a rational subspace of dimension p, and μ be the Haar measure of mass 1 on $S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}$. We define a (p,p)-dimensional closed current $\mathcal{T}_H$ on $(\mathbb{C}^*)^n$ by

$$\mathcal{T}_H := \int_{x\in S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}} \left[\pi_H^{-1}(x)\right] d\mu(x).$$

When A is a rational affine subspace of $\mathbb{R}^n$ parallel to the linear subspace $H = A - a$ for $a \in A$, we define $\mathcal{T}_A$ by translation of $\mathcal{T}_H$. Namely, we define the submersion π_A as the composition

$$\pi_A : \mathrm{Log}^{-1}(A) \xrightarrow{e^a} \mathrm{Log}^{-1}(H) \xrightarrow{\pi_H} S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}.$$

We will call $T^A := \pi_A^{-1}(1) = \ker \pi_A = e^{-a}T_{H\cap\mathbb{Z}^n}$, the *distinguished fibre* of $\mathcal{T}_A$.

remove the comma

Definition 4.2. Let $\mathcal{C}$, be a weighted polyhedral complex of dimension p. The tropical current $\mathcal{T}_{\mathcal{C}}$ associated to $\mathcal{C}$ is given by

$$\mathcal{T}_{\mathcal{C}} = \sum_\sigma w_\sigma\ \mathbb{1}_{\mathrm{Log}^{-1}(\sigma)}\mathcal{T}_{\mathrm{aff}(\sigma)},$$

where the sum runs over all p-dimensional cells σ of $\mathcal{C}$.

Theorem 4.3 ([Bab14]). A weighted complex $\mathcal{C}$ is balanced, if and only if, $\mathcal{T}_{\mathcal{C}}$ is closed.

Theorem 4.4 ([Bab14]). Any tropical current $\mathcal{T}_{\mathcal{C}} \in \mathcal{D}'_{n-1,n-1}((\mathbb{C}^*)^n)$ is of the form $dd^c[\mathfrak{q} \circ \mathrm{Log}]$, where $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$, is a tropical Laurent polynomial, that is $\mathfrak{q}(x) = \max_{\alpha\in A}\{c_\alpha + \langle\alpha, x\rangle\}$, for $A \subseteq \mathbb{Z}^n$ a finite subset and $c_\alpha \in \mathbb{R}$.

Remark 4.5. Note that the support of $dd^c[\mathfrak{q}\circ\mathrm{Log}]$, is given by $\mathrm{Log}^{-1}(\mathrm{Trop}(\mathfrak{q}))$, where $\mathrm{Trop}(\mathfrak{q})$ is the set of points $x \in \mathbb{R}^n$ where $\mathfrak{q}$ is not smooth at x. This set can be balanced with natural weights which coincides with the weights of the closed current $dd^c[\mathfrak{q}\circ\mathrm{Log}]$ and it is called the tropical variety associated to $\mathfrak{q}$.

Proposition 4.6 ([Bab23, Proposition 4.6]). Assume that $\mathcal{T} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$ is a closed positive $(S^1)^n$-invariant current whose support is given by $\mathrm{Log}^{-1}(|\mathcal{C}|)$, for a polyhedral complex $\mathcal{C} \subseteq \mathbb{R}^n$ of pure dimension p. Then $\mathcal{T}$ is a tropical current.

5. Continuity of Superpotentials

Let $q : \mathbb{R}^n \longrightarrow \mathbb{R}$, be a tropical polynomial function, and $\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$, as before. The current $dd^c[q \circ \mathrm{Log}] \in \mathcal{D}'_{n-1,n-1}((\mathbb{C}^*)^n)$ has a bounded potential on any relatively compact open set, and by Bedford–Taylor theory, for any positive closed current $\mathcal{T} \in \mathcal{D}'_{p,p}(\mathbb{C}^*)^n$, the product $((\mathbb{C}^*)^n)$

$$dd^c[q \circ \mathrm{Log}] \wedge \mathcal{T} = dd^c([q \circ \mathrm{Log}]\, \mathcal{T}),$$

is well-defined. See [Dem, Section III.3]. In higher codimensions though, to prove that any two tropical currents have a well-defined wedge product, we utilise the superpotential theory [DS09] on a compact Kähler manifold, and as a result, we extend the tropical currents to smooth compact toric varieties.

5.1. Tropical Currents on Toric Varieties. In a toric variety X_Σ, for a cone $\sigma \in \Sigma$, we denote by $\mathcal{O}_\sigma$, the toric orbit associated with σ. We have

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{O}_\sigma.$$

and the of Σ

We also set D_σ to be the closure of $\mathcal{O}_\sigma$ in the X_Σ. $\Sigma(p)$ p-dimensional skeleton, that is, the union of p-dimensional cells of Σ. Fibres of tropical currents are algebraic varieties with finite degrees and can be extended by zero to any toric variety, in consequence, any tropical current can be extended by zero to toric varieties. Moreover, with the following compatibility condition, we can ask for the extension of the fibres to intersect the toric invariant divisors transversally.

Definition 5.1. (i) For a polyhedron σ, its *recession cone* is the convex polyhedral cone

$$\mathrm{rec}(\sigma) = \{b \in \mathbb{R}^n : \sigma + b \subseteq \sigma\} \subseteq H_\sigma.$$

(ii) Let $\mathcal{C}$ be a p-dimensional balanced weighted complex in $\mathbb{R}^n$, and Σ a p-dimensional fan. We say that $\mathcal{C}$ is *compatible* with Σ, if $\mathrm{rec}(\sigma) \in \Sigma$ for all $\sigma \in \mathcal{C}$.

(iii) We say the tropical current $\mathcal{T}_\mathcal{C}$ is *compatible* with X_Σ, if all the closures of the fibers $\pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ in X_Σ of $\mathcal{T}_\mathcal{C}$ intersect the torus invariant divisors of X_Σ transversely.

Theorem 5.2 ([BH17, Lemma 4.10]). *Let $\mathcal{C}$ be a p-dimensional tropical cycle and Σ be a fan. Assume that $\sigma \in \mathcal{C}$ is a p-dimensional polyhedron and $\rho \in \Sigma$ is a one-dimensional cone. Then*

(a) *The intersection $D_\rho \cap \overline{\pi^{-1}_{\mathrm{aff}(\sigma)}(x)}$ is non-empty and transverse, if and only if, $\rho \in \mathrm{rec}(\sigma)$. Here $\overline{\pi^{-1}_{\mathrm{aff}(\sigma)}(x)}$ corresponds the closure of a fiber of $\mathcal{T}_{\mathrm{aff}(\sigma)}$ in the toric variety X_Σ.*

(b) *In particular, $\mathcal{C}$ is compatible with Σ, if and only if, $\mathcal{T}_\mathcal{C}$ is compatible with X_Σ.*

For a tropical current $\mathcal{T}_\mathcal{C} \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$, and given a toric variety X_Σ we denote its extension by zero $\overline{\mathcal{T}}_\mathcal{C} \in \mathcal{D}'_{p,p}(X_\Sigma)$.

Proposition 5.3. *For every tropical variety $\mathcal{C}$, there exists a smooth projective toric fan Σ compatible with a subdivision of $\mathcal{C}$.*

This, together with Equation (3) implies that the extension of $z^{-\beta}$ and $z^{-\gamma}$ (because $\mathrm{Log} = (-\log, \dots, -\log)$, we get $-\beta$ and $-\gamma$) as rational functions to X_Σ have the same vanishing order along D_ρ, and we write $z^{-\beta} = f^\kappa \frac{g_1}{h_1}$ and $z^{-\gamma} = f^\kappa \frac{g_2}{h_2}$. Now note that in $\Omega \setminus D_\rho$,

$$\mathrm{q} \circ \mathrm{Log} = \max \log\{|e^{c_\beta} z^{-\beta}|, |e^{c_\gamma} z^{-\gamma}|\} = \kappa \log |f| + \max\{|e^{c_\beta} \frac{g_1}{h_1}|, |e^{c_\gamma} \frac{g_2}{h_2}|\},$$

we must have $\kappa < 0$, otherwise $q \circ \mathrm{Log} = -\infty$ in $\Omega \setminus D_\rho$. Consequently, $\mathrm{q} \circ \mathrm{Log} : \Omega \setminus D_\rho \longrightarrow \mathbb{R}$, can be extended to

$$u := \kappa \log |f| + \max\{|e^{c_\beta} \frac{g_1}{h_1}|, |e^{c_\gamma} \frac{g_2}{h_2}|\}$$

on Ω. Setting

$$g = \max\{|e^{-c_\beta} \frac{g_1}{h_1}|, |e^{-c_\gamma} \frac{g_2}{h_2}|\},$$

implies (a).

We have

$$dd^c[\mathrm{q} \circ \mathrm{Log}]_{|\Omega \setminus D_\rho} = \left(dd^c \log |f|^\kappa \; dd^c \log |g|\right)_{|\Omega \setminus D_\rho} = dd^c \log |g|_{|\Omega \setminus D_\rho},$$

since $dd^c \log |f|^\kappa$ is holomorphic in $\Omega \setminus D_\rho$. As a result of compatibility with X_Σ, $\overline{dd^c[\mathrm{q} \circ \mathrm{Log}]}$ does not charge any mass in D_ρ, and we obtain

$$\overline{dd^c[\mathrm{q} \circ \mathrm{Log}]} = dd^c \log |g|.$$

This together with Theorem 4.4 implies (c) and (b). □

Lemma 5.6. Assume that σ is p-dimensional and $\mathrm{aff}(\sigma) = H_1 \cap \cdots \cap H_{n-p}$, is given as the transveral intersection hyperplanes $H_i \subseteq \mathbb{R}^n$. If Σ is a smooth projective fan compatible with $\bigcup_i H_i$, then

$$\overline{\mathcal{T}}_{\mathrm{aff}(\sigma)} \leq \overline{\mathcal{T}_{H_1} \wedge \cdots \wedge \mathcal{T}_{H_{n-p}}} = \overline{\mathcal{T}}_{H_1} \wedge \cdots \wedge \overline{\mathcal{T}}_{H_{n-p}}.$$

Proof. By the definition of tropical currents, we have the inequality

$$\mathcal{T}_{\mathrm{aff}(\sigma)} \leq \mathcal{T}_{H_1} \wedge \cdots \wedge \mathcal{T}_{H_{n-p}},$$

as currents in $(\mathbb{C}^*)^n$, since the right-hand side might have multiplicities but the currents have the same support. Now, the wedge products in X_Σ are well-defined by Lemma 5.5 and Theorem 2.4. As both currents on both sides of the equation coincide on $(\mathbb{C}^*)^n$, the support of the current on the right-hand side contains the closure of the support of $\mathcal{T}_{\mathrm{aff}(\sigma)}$ in X_Σ. For the equality, note that compatibility with Σ, implies $\overline{\mathcal{T}}_{H_i}$. □

Theorem 5.7. Let C be a positively weighted tropical cycle of dimension p compatible with a smooth, projective fan Σ, then $\overline{\mathcal{T}}_C$ has a continuous superpotential in X_Σ.

We need the following definition.

Definition 5.8. We define the affine extension p-dimensional a tropical cycle C, by as the addition of tropical cycles

$$\widehat{C} := \sum_{\sigma \in C} w_\sigma \mathrm{aff}(\sigma).$$

It is clear that if C is a positively weighted tropical cycle, then $\mathcal{T}_{\widehat{C}} - \mathcal{T}_C \geq 0$.

Proof of 5.7 . Let $\widehat{\mathcal{C}}$ be the affine extension of $\mathcal{C}$, and $\widehat{\Sigma}$ be a smooth projective fan which is a refinement of Σ and compatible with $\widehat{\mathcal{C}}$. By the preceding lemma and repeated application of Theorem 2.4 for any $\sigma \in \mathcal{C}$, $\overline{\mathcal{T}}_{\mathrm{aff}(\sigma)}$ has a bounded superpotential, which implies this property for $\overline{\mathcal{T}}_{\widehat{\mathcal{C}}}$. Now, since $\mathcal{T}_{\widehat{\mathcal{C}}} - \mathcal{T}_{\mathcal{C}}$ is a positive closed tropical current in $(\mathbb{C}^*)^n$,

$$\overline{\mathcal{T}_{\widehat{\mathcal{C}}} - \mathcal{T}_{\mathcal{C}}} = \overline{\mathcal{T}}_{\widehat{\mathcal{C}}} - \overline{\mathcal{T}}_{\mathcal{C}} \geq 0$$

in $X_{\widehat{\Sigma}}$. Continuity of the superpotential of $\overline{\mathcal{T}}_{\mathcal{C}}$ in $X_{\widehat{\Sigma}}$ follows from Theorem 2.3.

We now show that $\overline{\mathcal{T}}_{\mathcal{C}}$ has also a continuous super-potential on X_{Σ} as well. We consider the proper map $f : X_{\widehat{\Sigma}} \longrightarrow X_{\Sigma}$, which can be understood as a composition of multiple blow-ups along toric points with exceptional divisors D_ρ for any ray $\rho \in \widehat{\Sigma} \setminus \Sigma$. These divisors satisfy $D_\rho \cap \mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}}) = \varnothing$. We deduce by Proposition 2.9. □

added this remark:

OK

Remark 5.9. When $\mathcal{C}$ is a tropical cycle which is not positively weighted, then there exist positively weighted tropical cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ such that

$$\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2.$$

Therefore, if Σ is compatible with both $\mathcal{C}_1$ and $\mathcal{C}_2$, then by preceding lemma, $\mathcal{T}_{\mathcal{C}}$ has a continuous superpotential in X_{Σ}.

Proposition 5.10. In a toric variety X_{Σ} compatible with the tropical cylce $\mathcal{C}_1 + \mathcal{C}_2$,

$$\overline{\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}} = \overline{\mathcal{T}}_{\mathcal{C}_1} \wedge \overline{\mathcal{T}}_{\mathcal{C}_2}.$$

Proof. The proof is clear since both $\overline{\mathcal{T}}_{\mathcal{C}_1}$ and $\overline{\mathcal{T}}_{\mathcal{C}_2}$ have continuous superpotentials with no mass on the boundary divisors $X_{\Sigma} \setminus T_N$. □

do you assume C_1, C_2 positive? Otherwise $C_1 + C_2$ may be 0

Proposition 5.11. For any two tropical currents $\mathcal{C}_1$ and $\mathcal{C}_2$, the intersection product

$$\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2} := \overline{\mathcal{T}}_{\mathcal{C}_1} \wedge \overline{\mathcal{T}}_{\mathcal{C}_2}|_{(\mathbb{C}^*)^n},$$

does not depend on the choice of a smooth projective toric variety of the fan Σ compatible with $\mathcal{C}_1 + \mathcal{C}_2$, where $(\mathbb{C}^*)^n$ is identified with $T_N \subseteq X_{\Sigma}$. Moreover, this product coincides with the definition of wedge products with bi-degree $(1,1)$ tropical currents in Bedford–Taylor Theory in $(\mathbb{C}^*)^n$.

Proof. This is a consequence of Lemma 2.7, and the fact that intersection product with a bidegree $(1,1)$ current in super-potential theory, in an open set of compact Kähler manifold, coincides with the Bedford–Taylor theory. □

Proposition 5.12. Stable intersection of tropical cycles is associative and commutative

Proof. This is the application of Theorem 5.7 and Theorem 2.5. □

5.2. Proof of $\mathcal{T}_{C_1} \wedge \mathcal{T}_{C_2} = \mathcal{T}_{C_1 \cdot C_2}$. We prove of our main intersection theorems here, and it is not hard to visit the proof of Theorem 3.4 using tools from superpotential theory.

Theorem 5.13. For two tropical varieties C and C' of dimension p and q, respectively, we have

$$\mathcal{T}_{C_1} \wedge \mathcal{T}_{C_2} = \mathcal{T}_{C_1 \cdot C_2}$$

where the $C \cdot C'$ is the stable intersection C_1 and C_2, defined in Definition 3.3. Moreover, the $C_1 \cdot C_2$ is a balanced polyhedral complex of dimension $p+q-n$.

Proposition 5.14 ([Kat09, Propositions 6.1]). Let $H_1, H_2 \subseteq \mathbb{R}^n$ be two rational planes of dimension p and q with $p+q=n$ that intersect transversely. Then, the complex tori $T_{H_1 \cap \mathbb{Z}^n}$ and $T_{H_2 \cap \mathbb{Z}^n}$ intersect at $[N : N_{H_1} + N_{H_2}]$ distinct points.

Proof of Theorem 5.13. Note that $\mathcal{T}_{C_1} \wedge \mathcal{T}_{C_2}$ is well-defined by Proposition 5.11. Assume that C_1 and C_2 are two tropical cycles of dimension p and q, respectively. Note that when $p+q<n$, both sides of the equality are zero. Therefore, we assume that $p+q \geq n$. We proceed with the following steps:

(a) In the transversal case $\operatorname{supp}(\mathcal{T}_{C_1} \wedge \mathcal{T}_{C_2}) = \operatorname{Log}^{-1}(C_1 \cdot C_2) = \operatorname{Log}^{-1}(C_1 \cap C_2)$.
(b) When $p+q=n$, $\mathcal{T}_{C_1} \wedge \mathcal{T}_{C_2} = \mathcal{T}_{C_1 \cdot C_2}$, in the transversal case.
(c) When $p+q>n$, $\mathcal{T}_{C_1} \wedge \mathcal{T}_{C_2} = \mathcal{T}_{C_1 \cdot C_2}$, in the transversal case.
(d) $\operatorname{Log}(\operatorname{supp}(\mathcal{T}_{C_1} \wedge \mathcal{T}_{C_2})) = C_1.C_2$ when $p+q=n$, also in the non-trasverse case.
(e) $\operatorname{Log}(\operatorname{supp}(\mathcal{T}_{C_1} \wedge \mathcal{T}_{C_2})) = C_1.C_2$ when $p+q>n$, also in the non-trasverse case.

To see (a), note the by Theorem 2.4(c), $\operatorname{supp}(\mathcal{T}_{C_1} \wedge \mathcal{T}_{C_2}) \subseteq \operatorname{Log}^{-1}(C_1 \cdot C_2) = \operatorname{Log}^{-1}(C_1 \cap C_2)$. Moreover, on $C_1 \cap C_2$, the fibres of $\mathcal{T}_{C_1}$ and $\mathcal{T}_{C_2}$ have a non-zero intersection.

To prove (b), let $a \in C_1 \cap C_2$ be an isolated point of intersection. We can choose a small ball $B_\epsilon(a) \subseteq \mathbb{R}^n$ such that a is an isolated point of intersection $\sigma_1 \cap \sigma_2 \cap B$, where σ_1 and σ_2 are cells of dimension p and q in C_1 and C_2 respectively. For any rational polyhedran σ, let

$$N_\sigma := N \cap \operatorname{aff}(\sigma),$$
$$S^1(\sigma) := S^1 \otimes_{\mathbb{Z}} (\mathbb{Z}^n/(\mathbb{Z}^n \cap \operatorname{aff}(\sigma))),$$
$$\pi_\sigma := \pi_{\operatorname{aff}(\sigma)},$$

where $\pi_{\operatorname{aff}(\sigma)}$ was defined after Definition 4.1. By Lemma 2.7

$$\mathcal{T}_{C_1} \wedge \mathcal{T}_{C_2}|_{\operatorname{Log}^{-1}(B)} = w_{\sigma_1} w_{\sigma_2} \mathbf{1}_{\operatorname{Log}^{-1}(B)} \int_{(x_1,x_2)\in S^1(\sigma_1)\times S^1(\sigma_2)} [\pi_{\sigma_1}^{-1}(x_1)] \wedge [\pi_{\sigma_2}^{-1}(x_2)]\, d\mu_{\sigma_1}(x_1) \otimes d\mu_{\sigma_2}(x_2).$$

Transversality of the fibres implies

$$[\pi_{\sigma_1}^{-1}(x_1)] \wedge [\pi_{\sigma_2}^{-1}(x_2)] = [\pi_{\sigma_1}^{-1}(x_1) \cap \pi_{\sigma_2}^{-1}(x_2)].$$

By Proposition 5.14, we have $\kappa = [\mathbb{Z}^n : \mathbb{Z}^n_{\operatorname{aff}(\sigma)} + \mathbb{Z}^n_{\operatorname{aff}(\sigma')}]$ distinct intersection points covering $\operatorname{Log}^{-1}(a) \simeq S^1(\sigma_1) \times S^1(\sigma_2)$. When $(x_1, x_2) \in \operatorname{Log}^{-1}(a) \simeq$ vary with respect to the normalised Haar measure, these κ points cover $S^1(\sigma_1) \times S^1(\sigma_2) \simeq (S^1)^n$ with speed κ. Here I don't mean a κ-covering of $(S^1)^n$, as κ is the Jacobian rather than the degree of the map. With (x, x') we cover $(S^1)^n \simeq S^1(\sigma_1) \times S^1(\sigma_2)$ once, but with speed κ. As a result,

$$\int_{(x_1,x_2)\in S^1(\sigma_1)\times S^1(\sigma_2)} [\pi_{\sigma_1}^{-1}(x)] \wedge [\pi_{\sigma_2}^{-1}(x')]\, d\mu_{\sigma_1}(x) \otimes d\mu_{\sigma_2}(x')$$
$$= \int_{y\in(S^1)^n} \kappa[\pi_{\sigma_1\cap\sigma_2}^{-1}(y)]d\mu_{\sigma_1\cap\sigma_2}(y).$$

This proves (b).

To understand (c), let $\sigma = \sigma_1 \cap \sigma_2$ be a $(p+q-n)$-dimensional cell in the intersection. Assume that $0 \in \sigma$, by a translation, and $L := \operatorname{aff}(\sigma)^{\perp}$. By Proposition 5.12,

$$(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) \wedge \mathcal{T}_L = \mathcal{T}_{\mathcal{C}_1} \wedge (\mathcal{T}_{\mathcal{C}_2} \wedge \mathcal{T}_L).$$

Note that $[N : N_\sigma + N_L] = 1$. Assume that $w_{\sigma_1} = w_{\sigma_2} = 1$. As a result, if the multiplicity of $\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}$ at σ equals κ, we have that the multiplicity of $(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) \wedge \mathcal{T}_L$ at the origin is also κ. We have that $[N : N_{\sigma_1} + N_{\sigma_2\cap L}] = [N/N_\sigma : (N_{\sigma_1} + N_{\sigma_2\cap L})/N_\sigma]$, which equals $[N/N_\sigma : N_{\sigma_1\cap L} + N_{\sigma_2\cap L}] = [N : N_{\sigma_1} + N_{\sigma_2}]$. As a consequence, the intersection multiplicity induced on σ by $\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}$ equals the intersection multiplicity in Definition 3.3.

To prove (d) note that if $\mathcal{C}_1 + \epsilon b$ is the translation of the the tropical variety, where $b \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}_{\geq 0}$, then $(e^{\epsilon b})^*\mathcal{T}_{\mathcal{C}_1} = \mathcal{T}_{\mathcal{C}_1+\epsilon b}$. Moreover, we have the SP-convergence of currents with continuous superpotentials.

$$(e^{\epsilon b})^*\mathcal{T}_{\mathcal{C}_1} \longrightarrow \mathcal{T}_{\mathcal{C}_1}, \qquad \text{as } \epsilon \to 0.$$

Therefore, by Theorem 2.4,

$$(e^{\epsilon b})^*\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2} = \mathcal{T}_{\mathcal{C}_1+\epsilon b} \wedge \mathcal{T}_{\mathcal{C}_2} \longrightarrow \mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}, \qquad \text{as } \epsilon \to 0. \tag{4}$$

Considering the support, we obtain the Hausdorff limit

$$\lim \operatorname{supp}((e^{\epsilon b})^*\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) \supseteq \operatorname{supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}).$$

We now note that for all ϵ, the number of intersection points in $(\mathcal{C}_1+\epsilon b)\cap\mathcal{C}_2$ is uniformly bounded by the number of p-dimensional cells in $\mathcal{C}_1$ and q-dimensional cells in $\mathcal{C}_2$, the Hausdorff limit of $(\mathcal{C}_1 + \epsilon b) \cap \mathcal{C}_2$ is also zero dimensional. Now, by definition of $\mathcal{C}_1 \cdot \mathcal{C}_2$ it suffices to show that

$$\lim \operatorname{supp}((e^{\epsilon b})^*\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) = \operatorname{supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}),$$

for any fixed generic b. This is also easy. Let $a_\epsilon \in (\mathcal{C}_1+\epsilon b)\cap\mathcal{C}_2$. Since the translation by ϵb does not change the slopes of the cells, as $a_\epsilon \to a$, the multiplicity for all a_ϵ remains constant for $\epsilon > 0$, therefore $\lim(e^{\epsilon b})^*\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}$ has a non-zero mass at $\operatorname{Log}^{-1}(a)$.

For Part (e), first observe that any $\lim(\mathcal{C}_1 + \epsilon b) \cap \mathcal{C}_2$ is obtained by a translation ϵb, as $\epsilon \to 0$, of finitely many $(p + q - n)$-dimensional cells. Therefore, $\mathcal{C}_1 \cdot \mathcal{C}_2$ is also of dimension $p+q-n$, and the SP-convergence readily implies that the limit is independent of generic b. Now, it only remains to show that

$$\lim \operatorname{supp}((e^{\epsilon b})^*\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}) = \operatorname{supp}(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}).$$

Let σ be a $p+q-n$ dimensional cell in $\mathcal{C}_1 \cdot \mathcal{C}_2$, and by translation, assume that $0 \in \sigma$. Let $L = \operatorname{aff}(\sigma)^{\perp}$. By Proposition 2.6,

$$\big((e^{\epsilon b})^{*}\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}\big) \wedge \mathcal{T}_L \longrightarrow \big(\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}\big) \wedge \mathcal{T}_L, \quad \text{as } \epsilon \to 0.$$

By Part (d), the left-hand-side has mass at the origin. This show (e).

To deduce Part (f), recall that in Equation (4) since we can choose b generically and the previous discussion. The balancing condition is also deduced by the fact that $\mathcal{T}_{\mathcal{C}_1} \wedge \mathcal{T}_{\mathcal{C}_2}$ is closed and Theorem 4.3. □

Proposition 5.12 and Theorem 5.7 imply the following:

Theorem 5.15. The assignment $\mathcal{C} \mapsto \mathcal{T}_{\mathcal{C}}$ induces an isomorphism of $\mathbb{Z}$-algebras between effective tropical cycles and positive tropical currents on $(\mathbb{C}^*)^n$.

5.2.1. *Calculating Intersection Multiplicities Using Monge-Ampère Measures.* Using the equality of the supports in the previous section, we only need to prove the intersection multiplicities in the transversal case locally.

5.2.2. *Real Monge–Ampere Measures.* Let $\Omega \subseteq \mathbb{R}^n$ be an open subset and $u : \Omega \longrightarrow \mathbb{R}$ be a convex (hence continuous) function. The *generalised gradient* of u at $x_0 \in \Omega$ is defined by

$$\nabla u(x_0) = \big\{\xi \in (\mathbb{R}^n)^* : u(x) - u(x_0) \geq \langle \xi, x - x_0\rangle, \text{ for all } x \in \Omega\big\}.$$

In the above, $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{R}^n$, and $(\mathbb{R}^n)^*$ is the dual. The real Monge-Ampère measure associated to a convex function u on a Borel set $E \subseteq \Omega$, is given by

$$\mathrm{MA}[u](E) = \mu\Big(\bigcup_{y \in E} \nabla u(y)\Big),$$

where μ is the Lebesgue measure on $(\mathbb{R}^n)^*$.

It is interesting that for the tropical polynomials, one can compute the associate real Monge–Ampère measures explicitly. Recall that, for any tropical polynomial, there is a natural subdivision of its Newton polytope which is dual to the tropical variety of it. See Figure for an example and [BS14, MS15] for details.

Lemma 5.16 ([Yge13, Page 59],[BGPS14, Proposition 2.7.4])**.** Let $\mathfrak{q} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tropical polynomial associated tropical variety $\mathcal{C} = V_{\mathrm{trop}}(\mathfrak{q})$, one has

$$\mathrm{MA}[\mathfrak{q}] = \sum_{a \in \mathcal{C}(0)} \mathrm{Vol}(\{a\}^*)\delta_a,$$

where $\mathcal{C}(0)$ is the 0-dimensional skeleton of $\mathcal{C}$, and $\{a\}^*$ is the dual of the vertex $a \in \mathcal{C}(0)$.

A detailed discussion of the preceding theorem can be also found in [Bab14].

5.3. **Polarisation.** For n convex functions $u_1, \ldots, u_n : \mathbb{R}^n \longrightarrow \mathbb{R}$, their *mixed Monge-Ampère measure* is defined by

$$\widetilde{\mathrm{MA}}[u_1, \ldots, u_n] = \frac{1}{n!}\sum_{k=1}^{n} \sum_{1 \leq j_1 < \cdots < j_k \leq n} (-1)^{n-k}\, \mathrm{MA}[u_{j_1} + \cdots + u_{j_k}].$$

do you assume that the cycle is effective ?

where $\mathcal{R}_C$ is $(p-1,p-1)$-dimensional current. We claim that $\mathcal{R}_C$ is *normal, i.e.* $\mathcal{R}_C$ and $d\mathcal{R}_C$ have measure coefficients; $\mathcal{R}_C$ is a difference of two normal currents, where the first current $[S]\wedge\mathcal{T}_C$ is a positive closed current, and the second current is an addition of normal pieces. Moreover, the support of $\mathcal{R}_C$ is a subset of S as it is a difference of two currents that both vanish outside S. As a result, the current $\mathcal{R}_C$ is supported on $S\cap\bigcup_\sigma \partial\mathrm{Log}(\sigma)$. This set is a real manifold of Cauchy–Riemann dimension less than $p-1$, therefore by Demailly's first theorem of support the normal current $\mathcal{R}_C$ vanishes; see also the discussion following [BH17, Proposition 4.11]. □

Corollary 6.2. Let $H\subseteq\mathbb{R}^n$ be a rational plane of dimension $r\geq n-p$ and $A:=a+H$, a translation of H for $a\in\mathbb{R}^n$. Assume also that $C\subseteq\mathbb{R}^n$ is a tropical variety of dimension p that intersects A transversely. Then

$$[(e^{-a})T_{H\cap\mathbb{Z}^n}]\wedge\mathcal{T}_C$$

can be viewed as a tropical current of dimension $p-(n-r)$ in the complex subtori $T^A:=(e^{-a})\,T_{H\cap\mathbb{Z}^n}\subseteq(\mathbb{C}^*)^n$.

Proof. Note that the hypothesis implies that the intersection $T^A\cap\pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ is transversal for any $x\in S_{N(\sigma)}$. By translation, it is sufficient to prove the statement for $a=0$. By preceding theorem,

$$[T^A]\wedge\mathcal{T}_C=\sum_{\sigma\in C}w_\sigma\int_{x\in S_{N(\sigma)}}\mathbf{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}[T^A\cap\pi^{-1}_{\mathrm{aff}(\sigma)}(x)]\;d\mu(x).$$

The sets $T^A\cap\pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ can be understood as a translation toric sets in T^A and $d\mu(x)$ are Haar measures, which imply the assertion. □

is ?

Theorem 6.3. Let $M\subseteq(\mathbb{C}^*)^{n-p}$ and $N\subseteq(\mathbb{C}^*)^p$ be two bounded open subsets such that N contains the real torus $(S^1)^p$. Let $\pi: M\times N\longrightarrow M$ be the canonical projection. Let $\mathcal{T}_n$ be a sequence of positive closed (p,p)-bidimensional currents on $M\times N$ such that $\mathrm{supp}(\mathcal{T}_n)\cap(M\times\partial\overline{N})=\varnothing$. Assume that $\mathcal{T}_n\longrightarrow\mathcal{T}$ and $\mathrm{supp}(\mathcal{T})\subseteq M\times(S^1)^p$. Then we have the following convergence of slices

$$\langle\mathcal{T}_n|\pi|x\rangle\longrightarrow\langle\mathcal{T}|\pi|x\rangle\quad\text{for every } x\in M.$$

Note that all the above slices are well-defined for all $x\in M$.

Proof. Since all the currents $\mathcal{T}_n$ and $\mathcal{T}$ are horizontal-like, the slices are well-defined, and we prove that the slices have the same cluster value. Let $\mathcal{S}$ be any cluster value of $\langle\mathcal{T}_n|\pi|x\rangle$. Note that such $\mathcal{S}$ always exists by Banach–Alaoglu theorem. As both measures $\mathcal{S}$ and $\langle\mathcal{T}|\pi|x\rangle$, are supported $\{x\}\times(S^1)^p$ to prove their equality, it suffices to prove that they have the same Fourier coefficients. By Theorem 2.11, we have

$$\langle\mathcal{S},\phi\rangle\leq\langle\mathcal{T}|\pi|x\rangle(\phi),$$

for every plurisubharmonic function ϕ on $\mathbb{C}^n$, and the mass of $\mathcal{S}$ coincides with the mass of $\langle\mathcal{T}|\pi|x\rangle$. Now, note that if ϕ is pluriharmonic, then $-\phi$ and ϕ are plurisubharmonic. As a result,

$$\langle\mathcal{S},\phi\rangle=\langle\mathcal{T}|\pi|x\rangle(\phi),$$

for every pluriharmonic function. Recall that if f is a holomorphic function, then $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are pluriharmonic. We now consider the elements of the Fourier basis $f(\theta) = \exp 2\pi i\langle \nu, \theta\rangle$ for $\nu \in \mathbb{Z}^n$. Then we have the equality

$$\langle \mathcal{S}, f\rangle = \langle \mathcal{T}|\pi|x\rangle(f).$$

This implies that the Fourier measure coefficients of both $\mathcal{S}$ and $\langle \mathcal{T}|\pi|x\rangle$ coincide. □

Lemma 6.4. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a tropical variety of dimension p, and L be a rational $(n-p)$-dimensional plane such that L is transveral to all the affine extensions $\mathrm{aff}(\sigma)$ for $\sigma \in \mathcal{C}$. Assume that $\mathcal{T}$ is a positive closed current of bidimension (p,p) on a smooth projective toric variety X_Σ compatible with $\mathcal{C} + L$ such that $\mathrm{supp}(\mathcal{T}) \subseteq \mathrm{supp}(\mathcal{T}_\mathcal{C})$. Further, for all $a \in \mathbb{R}^n$,

$$\overline{\mathcal{T}}_{L+a} \wedge \mathcal{T} = \overline{\mathcal{T}}_{L+a} \wedge \overline{\mathcal{T}}_\mathcal{C}.$$

Then $\mathcal{T} = \mathcal{T}_\mathcal{C}$ in T_N.

Proof. Let us first remark that $\mathrm{rec}(L+a) = \mathrm{rec}(L)$ for all $a \in \mathbb{R}^n$ and therefore, all $\mathcal{T}_{a+L}$ are compatible with X_Σ and have a continuous super-potenrial in X_Σ and as a result, all the above wedge products are well-defined. By Demailly's second theorem of support [Dem, III.2.13], there are measures $\mu_\sigma^\mathcal{T}$ such that

$$\mathcal{T} = \sum_\sigma \int_{x \in S(Z^n \cap H_\sigma)} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} [\pi_\sigma^{-1}(x)]\, d\mu_\sigma^\mathcal{T}(x).$$

By repeated application of Proposition 6.1,

$$\mathcal{T}_L \wedge \mathcal{T} = \sum_\sigma \int_{(x,y) \in S(\mathbb{Z}^n \cap H_L) \times S(\mathbb{Z}^n \cap H_\sigma)} [\pi_H^{-1}(x) \cap \pi_\sigma^{-1}(y)]\, d\mu_L(x) \otimes \mu_\sigma^\mathcal{T}(y).$$

Applying both sides of the equality $\mathcal{T}_L \wedge \mathcal{T} = \mathcal{T}_L \wedge \mathcal{T}_\mathcal{C}$ on test-functions of the form

$$\omega_\nu = \exp(-i\langle \nu, \theta\rangle)\rho(r)$$

where $\rho : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support of $r \in \mathbb{R}^n$ and $\theta \in [0, 2\pi)^n$, and $\nu \in \mathbb{Z}^n$, completely determines the Fourier coefficients of $\mu_\sigma^\mathcal{T}$ which have to coincide with the normalised Haar measures multiplied by the weight of σ, *i.e.*, $\mu_\sigma^\mathcal{T} = w_\sigma \mu_\sigma$. □

Note that any subtorus of $(\mathbb{C}^*)^n$, can be understood as a fibre of a tropical current. We have the following slicing theorem.

Theorem 6.5. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a tropical variety and $A \subseteq \mathbb{R}^n$ be a rational hyperplane intersecting $\mathcal{C}$ transversely. Let Σ be a fan compatible with $\mathcal{C} + A$. Assume that $\overline{\mathcal{S}}_n$ is a sequence of positive closed currents on X_Σ, and denote by $\mathcal{S}_n$ the restriction to T_N. Further,

- $\overline{\mathcal{S}}_n \longrightarrow \overline{\mathcal{T}}_\mathcal{C}$;
- $\mathrm{supp}(\overline{\mathcal{S}}_n) \longrightarrow \mathrm{supp}(\overline{\mathcal{T}}_\mathcal{C})$.

Then

$$\lim_{n\to\infty} (\mathcal{S}_n \wedge [T^A]) = \mathcal{T}_\mathcal{C} \wedge [T^A],$$

as currents on $T_N \subseteq X_\Sigma$.

Proof. Assume that $L \subseteq \mathbb{R}^n$ is an $(n-p-1)$-dimensional affine plane intersecting all $\operatorname{aff}(\sigma)$ for all $\sigma \in C \cap A$ transversely. Then, on a projective smooth toric variety $X_{\Sigma'}$ compatible with $C + L + A$ the tropical currents $\overline{\mathcal{T}}_{a+L}$, $a \in \mathbb{R}^n$ have continuous super-potentials. Therefore, by Proposition 2.6, we have

$$\lim_{n\to\infty} \left(\overline{\mathcal{S}}_n \wedge \overline{\mathcal{T}}_{a+L}\right) = \overline{\mathcal{T}}_C \wedge \overline{\mathcal{T}}_{a+L}.$$

Now, for any $x \in C \cap L \cap A$, let $B \subseteq \mathbb{R}^n$ containing x be a bounded open set containing only x as an isolated point of the intersection. By a translation we can assume that $x = 0$. Consider the isomorphism

$$\xi : (\mathbb{C}^*)^n \xrightarrow{\ \sim\ } T_{\mathbb{Z}^n/(\mathbb{Z}^n\cap A)} \times T_{\mathbb{Z}^n\cap A},$$

and let π_1 and π_2 be the respective projections. Note that $\pi_1^{-1}(1) = T^A$. We now set

$$\begin{aligned}
U &:= \pi_1 \circ \xi\big(\operatorname{Log}^{-1}(U) \cap \operatorname{supp}(\mathcal{T}_C \wedge \mathcal{T}_{a+L})\big),\\
V &:= \pi_2 \circ \xi\big(\operatorname{Log}^{-1}(U) \cap T^A\big),\\
\mathcal{T}_n &:= \xi_*\big(\mathcal{S}_n \wedge \mathcal{T}_{a+L}\big), \text{ in } T_N,\\
\mathcal{T} &:= \xi_*\big(\mathcal{T}_C \wedge \mathcal{T}_{a+L}\big).
\end{aligned}$$

Therefore, for large n, $\mathcal{T}_n$ and $\mathcal{T}_C$ are horizontal-like as in the setting of Theorem 6.5. By Theorem 6.5, we obtain

$$\lim_{n\to\infty} \left(\mathcal{S}_n \wedge [T^A]\right) \wedge \mathcal{T}_{a+L} = \mathcal{T}_C \wedge [T^A] \wedge \mathcal{T}_{a+L},$$

for every a. We now deduce the convergence on T_N by Lemma 6.4. □

Theorem 6.6. In the situation of Theorem 6.5,

$$\lim_{n\to\infty} \left(\overline{\mathcal{S}}_n \wedge [\overline{T}^A]\right) = \overline{\mathcal{T}}_C \wedge [\overline{T}^A],$$

where the extension is considered in a smooth projective toric variety X_Σ compatible with $C + A$.

Lemma 6.7. Let $U \subseteq \mathbb{C}^n$ be an open subset and D be an analytic subset of $\mathbb{C}^n$. Assume $\mathbb{C}^n$
that we have the convergence of closed positive currents $\mathcal{V}_n \longrightarrow \mathcal{V}$ in $U \setminus D$, and $\mathcal{V}_n$'s and $\mathcal{V}$ have a uniformly bounded local masses near D. Further, assume that for any cluster value $\mathcal{W}$ of the sequence $\{\overline{\mathcal{V}}_n\}_n$, we have

(a) $\operatorname{supp}(\mathcal{W}) \subseteq \operatorname{supp}(\overline{\mathcal{V}})$,
(b) $\operatorname{supp}(\overline{\mathcal{V}}) \cap D$ has the expected Cauchy–Riemann dimension.

Then

$$\overline{\mathcal{V}}_n \longrightarrow \overline{\mathcal{V}}.$$

Proof. $\mathcal{W} - \overline{\mathcal{V}}$ is a positive closed current with the Cauchy–Riemann dimension less than or equal to p, therefore, it must be zero by Demailly's first theorem of support [Dem, Theorem III.2.10]. □

Proof of Theorem 6.6. Applying Theorem 5.2 (or [OP13, Proposition 3.3.2] to each fibre of $\overline{\mathcal{T}}_C$ separately), we obtain $\operatorname{supp}(\overline{\mathcal{T}}_C) \cap \overline{T}_A \cap [D_\rho]$ has the expected Cauchy–Riemann dimension $p - 2$. By Demailly's first theorem of support [Dem, Theorem III.2.10],

$$\overline{\mathcal{S}}_C \wedge [\overline{T}_A] = \overline{\mathcal{T}_C \wedge [T_A]}.$$

By assumption $\overline{\mathcal{S}}_n \longrightarrow \overline{\mathcal{T}}_C$ and $\mathrm{supp}(\mathcal{T}_n) \longrightarrow \mathrm{supp}(\overline{\mathcal{T}}_C)$. The observation in Lemma 2.12,

$$\lim_{n\to\infty} \mathrm{supp}(\overline{\mathcal{S}}_n \wedge [\overline{T}^A]) \subseteq \mathrm{supp}(\overline{\mathcal{T}}_C \wedge [\overline{T}^A]).$$

Therefore, any cluster value of $\overline{\mathcal{S}_n \wedge [T^A]} \subseteq \overline{\mathcal{S}}_n \wedge [\overline{T}^A]$ has a support in $\mathrm{supp}(\overline{\mathcal{T}}_C \wedge [\overline{T}^A])$. Now by setting

(a) $\mathcal{V}_n := \mathcal{S}_n \wedge [\overline{T}^A]$,
(b) $\mathcal{V} := \mathcal{T}_C \wedge [\overline{T}^A]$,
(c) $\mathcal{W}$ a cluster value of $\overline{\mathcal{T}_n \wedge [T^A]}$,

we are in the situation of Lemma 6.7, and conclude. □

Lemma 6.8. Let X_Σ be a smooth projective toric variety, and $\bar{\Delta} \subseteq X_\Sigma \times X_\sigma$ be the diagonal. Let $\mathcal{S}$ and $\mathcal{T}$ be two positive currents on X. Then, for any ray $\rho \in \Sigma$,

Σ

$$\mathrm{supp}(\mathcal{S}) \cap \mathrm{supp}(\mathcal{T}) \cap D_\rho \subseteq X_\Sigma$$

has a Cauchy–Riemann dimension ℓ, if and only if,

$$\mathrm{supp}(\mathcal{S} \otimes \mathcal{T}) \cap \bar{\Delta} \cap D_{(0,\rho)} \subseteq X_\Sigma \times X_\Sigma,$$

has a Cauchy–Riemann dimension ℓ, where $D_{(0,\rho)}$ is the toric invariant divisor corresponding to the ray $(0,\rho)$ in $\Sigma \times \Sigma$.

Proof. The fan of $X_\Sigma \times X_\Sigma$ is $\Sigma \times \Sigma$, we have that $D_{(0,\rho)} \simeq X_\Sigma \times D_\rho$ and the assertion follows. □

Theorem 6.9. Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{R}^n$ be two tropical cycles intersecting properly. Assume that X_Σ is a smooth toric projective variety compatible with $\mathcal{C}_1 + \mathcal{C}_2$. If moreover, for two sequence of positive closed currents $\overline{\mathcal{V}}_n$ and $\overline{\mathcal{W}}_n$ we have

Are W_n, V_n analytic? do you mean Cauchy-R. dimension

(a) $\overline{\mathcal{W}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}_1}$ and $\overline{\mathcal{V}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}_2}$,
(b) $\mathrm{supp}(\overline{\mathcal{W}}_n) \longrightarrow \mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}_1})$ and $\mathrm{supp}(\overline{\mathcal{V}}_n) \longrightarrow \mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}_2})$,
(c) for any n, $\mathrm{supp}(\overline{\mathcal{W}}_n) \cap \mathrm{supp}(\overline{\mathcal{V}}_n)$ has the expected dimension.
(d) for any n, and any ray $\rho \in \Sigma$, $\mathrm{supp}(\overline{\mathcal{W}}_n) \cap \mathrm{supp}(\overline{\mathcal{V}}_n) \cap D_\rho$ has the expected dimension.

necessary? usually you ask such condition for limits only

Then

Say something about the definition of $\overline{W_n} \wedge \overline{V_n}$

$$\overline{\mathcal{W}}_n \wedge \overline{\mathcal{V}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}_1} \wedge \overline{\mathcal{T}}_{\mathcal{C}_2}.$$

Proof. For two closed currents $\mathcal{S}$ and $\mathcal{T}$ on X_Σ we naturally identify $\mathcal{S} \wedge \mathcal{T} = \pi_*(\mathcal{S} \otimes \mathcal{T} \wedge [\bar{\Delta}])$, where $\pi : X_\Sigma \times X_\Sigma \longrightarrow X_\Sigma$ is the projection. Let $\mathcal{W}_n$ and $\mathcal{V}_n$ be the restriction of $\overline{\mathcal{W}}_n$ and $\overline{\mathcal{V}}_n$ to $T_N \subseteq X_\Sigma$. In $T_N \times T_N$ we set $\mathcal{T}_n := \mathcal{W}_n \otimes \mathcal{V}_n$ and $\mathcal{T}_C := \mathcal{T}_{\mathcal{C}_1} \otimes \mathcal{T}_{\mathcal{C}_2}$. Now note that the diagonal in the open torus is the complete intersection of the tori $x_i = y_i$, $i = 1, \ldots, n$. This, together with assumption (c) allows for a repeated application of Theorem 6.5 to obtain

on the first factor?

$$\mathcal{W}_n \otimes \mathcal{V}_n \wedge [\Delta] \longrightarrow \mathcal{T}_{\mathcal{C}_1} \otimes \mathcal{T}_{\mathcal{C}_2} \wedge [\Delta].$$

By assumption (c), and Lemma 6.8, for large n and rays $\rho \in \Sigma$,

$$\mathrm{supp}(\overline{\mathcal{W}}_n \otimes \overline{\mathcal{V}}_n) \cap [\overline{\Delta}] \cap D_\rho$$

have the expected dimension. Lemma 6.8, and the compatibility assumption imply that $\mathrm{supp}(\mathcal{W}_n \otimes \mathcal{V}_n) \cap \bar{\Delta} \cap D_{(0,\rho)}$ and $\mathrm{supp}(\mathcal{T}_C \otimes \mathcal{T}_{C'}) \cap \bar{\Delta} \cap D_{(0,\rho)}$ have the expected Cauchy–Riemann dimension. Therefore, Lemma 2.12 brings us to the situation of Lemma 6.7, and we conclude. □

7. Dynamical Tropicalisation with non-Trivial Valuations

7.1. Dynamical tropicalisation with a non-trivial valuation. Recall that for a field $\mathbb{K}$, $\nu : \mathbb{K} \longrightarrow \mathbb{R} \cup \{\infty\}$, is called a valuation if it satisfies the following properties for every $a, b \in \mathbb{K}$:

(a) $\nu(a) = \infty$ if and only if $a = 0$;
(b) $\nu(ab) = \nu(a) + \nu(b)$;
(c) $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$.

A valuation is called *trivial*, if the valuation of any non-zero element is 0. For an element $a \in \mathbb{K}$, we denote by $\bar{a}$ its image in the residue field. We are interested in the case where $\mathbb{K} = \mathbb{C}((t))$, is the field of *formal Laurent series* with the variable t, with the usual valuation. That is, for $g(t) = \sum_{j \geq k} a_j t^j$, with $a_k \neq 0$, the valuation equals the minimal exponent $\nu(g) = k \in \mathbb{Z}$.

Definition 7.1. (a) Let $f = \sum_{\alpha \in \mathrm{N}} c_\alpha z^\alpha \in \mathbb{K}[z^{\pm 1}]$, be a Laurent polynomial in n variables. The tropicalisation of f with respect to ν, $\alpha \in \mathbb{N}^n$?

$$\mathrm{trop}_\nu(f) : \mathbb{R}^n \longrightarrow \mathbb{R},$$ or $\mathbb{Z}^n$

$$x \mapsto \max\{-\nu(c_\alpha) + \langle x, \alpha \rangle\}.$$

(b) Let $I \subseteq \mathbb{K}[z^{\pm 1}]$ be an ideal. The tropical variety associated to I, as a set, is defined as

$$\mathrm{Trop}_\nu(I) := \bigcap_{f \in I} \mathrm{Trop}(\mathrm{trop}_\nu(f)),$$

where $\mathrm{Trop}(\mathrm{trop}_\nu(f))$ is the set of points where $\mathrm{trop}_\nu(f)$ is not differentiable; see Remark 4.5.

(c) For an algebraic subvariety of the torus $Z \subseteq (\mathbb{K}^*)^n$, with the associated ideal $\mathbb{I}(Z)$, the tropicalisation of Z, as a set, is $\mathrm{Trop}_\nu(Z) := \mathrm{Trop}_\nu(\mathbb{I}(Z))$.

(d) In all the situations above, trop_0 denotes the tropicalisation with respect to the trivial valuation.

We need to relate a non-trivial valuation to the trivial valuation. Compare to [BJS+07, Lemma 1.1]. $z^{\pm 1}$?

Lemma 7.2. Consider the ideal $I \subseteq \mathbb{C}[t^{\pm 1}, z^{\pm 1}] \xhookrightarrow{\iota} \mathbb{C}((t))[z]$. Assume that (u, x) are the coordinates in $\mathbb{R} \times \mathbb{R}^n$. Then, we have the following equality of sets

$$\mathrm{Trop}_0(I) \cap \{u = -1\} = \mathrm{Trop}_\nu(\iota(I)),$$

where ν is the the usual valuation in $\mathbb{C}((t))$. In other words, the tropicalisation of I as an ideal in $\mathbb{C}[t, x]$ with respect to the trivial valuation intersected with $\{u = -1\}$ coincides with the tropicalisation of $I = \iota(I)$ with respect to the usual valuation in $\mathbb{C}((t))$.

The proof of the lemma becomes clear with the following example.

Example 7.3. Let

$$f(x,t) = 4(t^3 + t^{-1})z_1z_2 + (1 + t + t^2)z_1.$$

$t^{\pm 1}$?

Then, the tropicalisation of $f \in \mathbb{C}[t,z]$, with respect to the trivial valuation equals:

$$\mathrm{trop}_0(f) = \max\big\{\max\{3u + x_1 + x_2, -u + x_1 + x_2\}, \max\{x_1, u + x_1 + \underline{2u + x_1}\}\big\}.$$

?

Letting $u := -1$, $\mathrm{trop}_0(f)(-1,x) = \max\{1 + x_1 + x_2, x_1\}$. The latter equals $\mathrm{trop}_\nu(f)$ as an element of $\mathbb{C}((t))[z]$.

Proof of Lemma 7.2. If f is a monomial in $\mathbb{C}[t][z]$, then it is clear that

$$\mathrm{trop}_0(f)(-1,x) = \mathrm{trop}_\nu(\iota(f)).$$

Therefore, we have the equality for any polynomial in $f \in \mathbb{C}[t,z]$. To prove the main statement, note that

$$\begin{aligned}\mathrm{Trop}_\nu(\iota(I)) &= \bigcap_{f\in\iota(I)} \mathrm{Trop}(\mathrm{trop}_\nu(f))\\ &= \bigcap_{f\in I} (\mathrm{Trop}(\mathrm{trop}_0(f)) \cap \{u = -1\})\\ &= \mathrm{Trop}_0(I) \cap \{u = -1\}.\end{aligned}$$

□

Remark 7.4. Bergman in [Ber71], shows that for an algebraic subvariety $Z \subseteq (\mathbb{C}^*)^n$, one has

$$\lim_{t\to\infty} \mathrm{Log}_t(Z) \subseteq \mathrm{Trop}_0(\mathbb{I}(Z)),$$

and he conjectured the equality. This conjecture was later proved by Bieri and Groves in [BG84]. More precisely, Bieri and Grove prove that $\lim \mathrm{Log}_t(Z) \cap (S^1)^n$ is a polyhedral sphere of real dimension equal to (the complex dimension) $\dim(Z) - 1$. Therefore, the fan $\lim \mathrm{Log}_t(Z)$ is a cone over their spherical complex. See also [MS15, Theorem 1.4.2].

Remark 7.5. The above lemma is related to the results of Markwig and Ren in [MR20]. They considered the tropicalisation of an ideal $J \subseteq R[[t]][x]$, where R is the ring of integers of a discrete valuation ring $\mathbb{K}$, which is non-trivially valued. To obtain finiteness properties, however, the authors consider the associated tropical variety in the half-space $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$. Note that such a variety is almost never balanced. The authors also prove that for an ideal $I \subseteq \mathbb{K}[x]$, the tropicalisation of the natural inverse image $\pi^{-1}I \subseteq R[[t]][x]$ with respect to trivial valuation, intersected with $\{u = -1\}$ equals $\mathrm{trop}_\nu(I)$; [MR20, Theorem 4].

Let us also recall the main result of [Bab23].

? a

Theorem 7.6. Let $Z \subseteq (\mathbb{C}^*)^n$ be an irreducible subvariety of dimension p, and $\overline{Z}$ be the closure of Z in ~~the~~ compatible smooth projective toric variety X_Σ. Define $\Phi_m : X_\Sigma \longrightarrow X_\Sigma$ to be the unique continuous extension of

$$\begin{aligned}(\mathbb{C}^*)^n &\longrightarrow (\mathbb{C}^*)^n,\\ z &\longmapsto z^m.\end{aligned}$$

Then,

$$\frac{1}{m^{n-p}}\Phi^*_m[\overline{Z}] \longrightarrow \overline{\mathcal{T}}_{\mathrm{Trop}_0(Z)}, \quad \text{as } m \to \infty,$$

where $\overline{\mathcal{T}}_{\mathrm{Trop}_0(Z)}$ is the extension by zero of $\mathcal{T}_{\mathrm{Trop}_0(Z)}$ to X_Σ. Moreover, the supports also converge in Hausdorff metric.

Note that since the limit of a sequence of closed currents is closed, the above theorem implies that $\mathrm{trop}_0(Z)$ can be equipped with weights to become balanced. Note that the compatibility is in the following sense of Tevelev and Sturmfels:

Theorem 7.7. (a) The closure $\overline{Z}$ of Z in X_Σ is complete, if and only if, $\mathrm{Trop}_0(Z) \subseteq |\Sigma|$; see [Tev07].
(b) We have $|\Sigma| = \mathrm{Trop}_0(Z)$, if and only if, for every $\sigma \in \Sigma$ the intersection $\mathcal{O}_\sigma \cap \overline{Z}$ is non-empty and of pure dimension $p - \dim(\sigma)$; see [ST08].

Theorem 7.8. Let $I \subseteq \mathbb{C}[t^{\pm1}, x^{\pm1}]$ be an ideal with the associated $(p+1)$-dimensional algebraic variety $W = \mathbb{V}(I) \subseteq (\mathbb{C}^*)^{n+1}$. Assume that the projection onto the first coordinate $\pi_1 : W \longrightarrow \mathbb{C}^*$ is surjective and Zariski closed. We denote the fibers as $W_t := \pi_1^{-1}(t)$. We have that $x \to z$?

(a)

$$\frac{1}{m^{n-p}}\Phi^*_m[W_{e^m}] \longrightarrow \mathcal{T}_{\mathrm{Trop}_\nu(I)}, \quad \text{as } m \to \infty\ ,$$

in the sense of currents in $\mathcal{D}_p((\mathbb{C}^*)^n)$, and we have identified $\iota(I) = I$.

(b) $\mathrm{Trop}_\nu(I)$ can be equipped with weights to become balanced.
(c) $\lim \mathrm{supp}(\frac{1}{m^{n-p}}\Phi^*_m[W_{e^m}]) = \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_\nu(I)})$.
(d) On a toric variety X_Σ compatible with $\mathrm{trop}_0(W) + \{u = -1\}$,

$$\frac{1}{m^{n-p}}\Phi^*_m[W_{e^m}] \longrightarrow \overline{\mathcal{T}}_{\mathrm{Trop}_\nu(I)}, \quad \text{as } m \to \infty$$

We need the following:

Lemma 7.9. Let $W \subseteq (\mathbb{C}^*)^{n+1}$ be a $(p+1)$-dimensional smooth subvariety, such that the projection onto the first factor, $\pi_1 : (\mathbb{C}^*)^{n+1} \longrightarrow \mathbb{C}^*$ is surjective and a Zariski closed morphism. Then for a sufficiently large $|t_0| >> 0$

$$[W_{t_0}] = [\pi_1^{-1}(t_0)] = [\{t = t_0\}] \wedge [W].$$

Proof. We first prove that the set of singular points of W, together with the set of points where $[\{t = t_0\}] \wedge [W]$ has a multiplicity greater than 1, is contained in a Zariski closed set in W. We define the *critical set*, notation not introduced

$$C = \{w \in W_{\mathrm{reg}} : \dim (T_w W \cap \ker\ \nabla_w t) = p + 1\},$$

which is the set of points where the tangent space of $T_w W_{reg}$ is included in the tangent space of $T_w\{t = t_0\}$, and this set contains the set of points $w \in W_{\mathrm{reg}}$ points the intersection multiplicity of $\{t = t_0\}$ and W exceeds 1. We fix an ideal associated to $I = \mathbb{I}(W) = \langle f_1, \ldots, f_k\rangle \subseteq \mathbb{C}[t, x]$. At any regular point $w \in W_{\mathrm{reg}}$, $T_w W$ is of dimension $p+1$, and the rank of the Jacobian matrix $J(f)(w) = \big(\frac{\partial f_i}{\partial z_j}(w)\big)_{k\times(n+1)}$ equals codimension of W, $(n+1) - (p+1) = n - p$. We have that $\nabla_w t = e_1$, where e_1 is the first element of the standard basis for the $\mathbb{C}$-vector space $\mathbb{C}^{n+1}$. We have $w \in C$, if and only if,

$$\ker \begin{pmatrix} e_1 \\ Jf(w) \end{pmatrix} = \ker \left(Jf(w) \right).$$

As a result, C is an algebraic variety given as the intersection of $W \setminus W_{\text{sing}}$ with the intersection of zero loci of $(q+1)\times(q+1)$-minors of $\begin{pmatrix} e_1 \\ Jf(w) \end{pmatrix}$. Therefore, the closure of C in W, $\overline{C}$ union W_{sing} is a Zariski-closed subset of W. Since W is not contained in $\{t = t_0\}$, as π_1 is surjective, then $\pi_1(\overline{C} \cup W_{\text{sing}})$ is a Zariski closed proper subset in $\mathbb{C}^* \subseteq \mathbb{C}$, and hence finite. $\square$ is it true?

Proof of Theorem 7.8. By the preceding lemma, and the fact that Φ_m preserves transversal intersection, we have

$$\frac{1}{m^{n-p}}\Phi_m^*[W_{e^m}] = \frac{1}{m^{n-(p+1)}}\Phi_m^*[W] \wedge \frac{1}{m}\Phi_m^*[\{t = e^m\}],$$

for a large m. Since $\mathrm{Trop}_0(W)$ is a fan and it is transversal to the plane $\{u = -1\} \subset \mathbb{R}^{n+1}$ are transversal, we can use Theorem 6.5 to write

?

$$\lim \frac{1}{m^{n-p}}\Phi_m^*[W_{e^m}] = \left(\lim \frac{1}{m^{n-(p+1)}}\Phi_m^*[W]\right) \wedge \left(\lim \frac{1}{m}\Phi_m^*[\{t = e^m\}]\right)$$

By Theorem 7.6, restricted to $(\mathbb{C}^*)^{n+1}$, and the fact that we used $\mathrm{Log} = (-\log|\cdot|, \dots, -\log|\cdot|)$ in the definition of tropical currents, the above limits yield

$$\lim \frac{1}{m^{n-p}}\Phi_m^*[W_{e^m}] = \mathcal{T}_{\mathrm{Trop}_0(W)} \wedge \mathcal{T}_{\{u=-1\}}.$$

Applying Theorems 5.13 and Lemma 7.2, we obtain the equality.

For the assertion (b), note that the limit $\mathcal{T}_{\mathrm{Trop}_\nu(I)}$ is a closed current and Theorem 4.3 implies that $\mathrm{Trop}_\nu(I)$ is naturally balanced. To observe (c), note that (a) implies

$$\lim \mathrm{supp}\left(\frac{1}{m^{n-p}}\Phi_m^*[W_{e^m}]\right) \supseteq \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_\nu(I)}).$$

However, because of transversality, $\mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_\nu(I)}) = \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_0(W)}) \cap \mathrm{supp}(\mathcal{T}_{\{u=-1\}})$. At the same time,

$$\lim \mathrm{supp}(\Phi_m^*[W_{e^m}]) = \lim \mathrm{supp}(\Phi_m^*[W]) \cap \mathrm{supp}(\Phi_m^*[\{t = e^m\}]).$$

Moreover, for the Hausdorff limit of sets $\lim(A_i \cap B_i) \subseteq (\lim A_i) \cap (\lim B_i)$. This implies

$$\lim \mathrm{supp}(\Phi_m^*[W_{e^m}]) \subseteq \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_0(W)}) \cap \mathrm{supp}(\mathcal{T}_{\{u=-1\}}),$$

which implies (c). Now, (d) is implied by Theorem 6.6. $\square$

New: In the setting of the previous theorem, a generalisation of Bergman's theorem (see Remark 7.4) asserts that

$$\mathrm{Log}_t(W_t) \longrightarrow \mathrm{Trop}(I), \quad \text{as } t \longrightarrow \infty.$$

where Log_t is the logarithm with base t. This theorem can be understood as a counterpart of Lemma 7.2 for tropicalisation with Log. This generalisation was finally proved by Jonsson in [Jon16], and we can now deduce a sequential analogue:

Corollary 7.10. In the setting of the previous theorem

$$\frac{1}{m}\mathrm{Log}(W_{e^m}) \longrightarrow \mathrm{Trop}_\nu(I), \quad \text{as } m \to \infty,$$

in the Hausdorff metric in compact subsets of $\mathbb{R}^n$.

Proof. Note that for any variety $Z \subseteq (\mathbb{C}^*)^{n+1}$, as a set $\mathrm{Log}(\Phi_m^{-1}(Z)) = \frac{1}{m}\mathrm{Log}(Z)$. Therefore,

$$\mathrm{Log}\ \mathrm{supp}\Big(\frac{1}{m^{n-p}}\Phi_m^*[W_{e^m}]\Big) = \frac{1}{m}\mathrm{Log}(W_{e^m}).$$

The assertion now follows from Theorem 7.8(c) and continuity of Log with respect to the Hausdorff metric on compact sets. □

Let us first prove the analogous to [BJS+07, Lemma 3.2] and [OP13, Theorem 1.2].

Theorem 7.11. Assume that W and V are two algebraic subvarieties of $(\mathbb{C}^*)^n$ with respective dimensions p and q, with $p+q \geq n$. Assume that $\mathrm{Trop}_0(V)$ and $\mathrm{Trop}_0(W)$ intersect properly, then

$$\frac{1}{m^{2n-(p+q)}}\Phi_m^*([W]\wedge[V]) \longrightarrow \mathscr{T}_{\mathrm{Trop}_0(W)\cdot\mathrm{Trop}_0(V)}, \quad \text{as } m \to \infty.$$

Moreover, $\mathscr{T}_{\mathrm{Trop}_0(W\cap V)} \leq \mathscr{T}_{\mathrm{Trop}_0(W)\cdot\mathrm{Trop}_0(V)}$. In particular, we have the inequality of corresponding induced multiplicities for $\tau \in \mathrm{Trop}_0(W\cap V)$, the multiplicity induced on τ is less than or equal to the multiplicity of τ induced by $\mathrm{Trop}_0(W)\cdot\mathrm{Trop}_0(V)$.

Example 7.12. To see that the inequality in the previous theorem can be strict, let us consider the subvarieties of $(\mathbb{C}^*)^2$, $W = \mathbb{V}(z_2-1)$ and $V = \mathbb{V}(z_2 - z_1^2 - 1)$. We have that

$$\mathrm{Trop}_0(W) = \mathrm{Trop}(\max\{x_2, 2x_1, 0\}),$$
$$\mathrm{Trop}_0(V) = \mathrm{Trop}(\max\{x_2, 0\}).$$

In Example 3.6, we discussed that the stable intersection of these two cycles is the origin $(0,0) \in \mathbb{R}^2$ with multiplicity 2. Note that the (set-theoretic) intersection $W\cap V = \{(0,1)\}$, thus $\mathrm{Trop}_0(W\cap V) = (0,0)$ with multiplicity 1, whereas $\mathrm{Trop}_0(W)\cdot\mathrm{Trop}_0(V) = (0,0)$ with multiplicity 2. We obtain

$$2\,\mathscr{T}_{\mathrm{Trop}_0(W\cap V)} = \mathscr{T}_{\mathrm{Trop}_0(W)\cdot\mathrm{Trop}_0(V)}.$$

Note that this inequality, gladly, does not contradict [OP13, 3.3.1] (a consequence of [Ful98, 8.2]), which the scheme-theoretic intersection $W \cap V$ is considered.

Proof of Theorem 7.11. Assume that X_Σ is a toric variety compatible with $\mathrm{Trop}_0(W) + \mathrm{Trop}_0(V)$. We need to show that the hypothesis (a), …, (d) of Theorem 6.9 for $\overline{W}_m = m^{p-n}\Phi_m^*[\overline{W}]$ and $\overline{V}_m = m^{q-n}\Phi_m^*[\overline{V}]$ is satisfied. Note that hypotheses (a) and (b) are implied by Theorem 7.6. The hypotheses (c) and (d) for $\overline{W}_1$ and $\overline{V}_1$ are a result of [OP13, Proposition 3.3.2]. The hypotheses (c) and (d) for any m are implied by the fact that $\Phi_m^*([\overline{W}]\wedge[\overline{V}]) = \Phi_m^*[\overline{W}]\wedge\Phi_m^*[\overline{V}]$. Any quick explanation for this?. Therefore,

as

are

$$\overline{W}_m \wedge \overline{V}_m \longrightarrow \overline{\mathscr{T}}_{\mathrm{Trop}_0(W)} \wedge \overline{\mathscr{T}}_{\mathrm{Trop}_0(V)}, \quad \text{as } m \to \infty,$$

otherwise dim(V∩W) is ~~big~~ bigger than expected
Contradiction with the hypothesis
on Trop₀(W), Trop₀(V)?

and the first assertion is obtained by restricting to $T_N \subseteq X_\Sigma$ and Theorem 5.13. To see the second assertion, note that by Theorem 7.6,

$$\frac{1}{m^{2n-(p+q)}}\Phi_m^*[W\cap V] \longrightarrow \mathcal{T}_{\mathrm{Trop}_0(W\cap V)}, \quad \text{as } m\to\infty.$$

However, as currents $[W\cap V] \leq [W]\wedge[V]$ since the right-hand side might produce multiplicities on the intersection. Applying Φ_m^* to both sides preserves this inequality. Therefore, for every m,

$$\Phi_m^*[W\cap V] \leq \Phi_m^*[W]\wedge\Phi_m^*[V].$$

We can conclude by taking the limit $m \longrightarrow \infty$. □

The tropical version of the following was observed in various places [OP13, MS15], which does not need to assume the proper intersection of tropicalisations.

Lemma 7.13. Let W and V be two algebraic subvarieties of $(\mathbb{C}^*)^n$, and for $0<\epsilon<1$, let $U_\epsilon((S^1)^n)$ be an ϵ-neighbourhood of $(S^1)^n$. Then,

$$\frac{1}{m^{2n-(p+q)}}\int_{(t_1,t_2)\in U_\epsilon((S^1)^{2n})}\Phi_m^*[t_1V\cap t_2W]\,d\nu(t_1)\otimes d\nu(t_2) \longrightarrow \mathcal{T}_{\mathrm{Trop}_0(V)}\wedge\mathcal{T}_{\mathrm{Trop}_0(W)},$$

as $m\to\infty$. Here, the $d\nu$ ~~are~~ the normalised Lebesgue measure~~s~~ on $U_\epsilon((S^1)^n)$. 's

Proof. Note that $[t_1V\cap t_2W]$ is transversal for generic t_1 and t_2, and since Φ_m^* preserves transversality, we can separate the above integrand into $\Phi_m^*[t_1V]\wedge\Phi_m^*[t_2W]$. Using polar coordinates (r_i,θ_i) for t_i, we have

$$\int_{t_1\in U_\epsilon((S^1)^n)}\Phi_m^*[t_1V]\,d\nu(t_1) = \int_{t_1\in U_\epsilon((S^1)^n)} t_1^{1/m}\,\Phi_m^*[V]\,d\nu(t_1).$$

Here, with abuse of notation, we choose the m-th root $t_1^{1/m}$ to be the first root of $t^{1/m}$. We can also obtain a similar equation for t_2W. Further,

$$(\Phi_m^*[t_1V]\otimes\Phi_m^*[t_2W])\wedge[\Delta] = (\Phi_m^*[V]\otimes\Phi_m^*[W])\wedge(t_1)^{-1/m}(t_2)^{-1/m}[\Delta].$$

$$\int_{U_\epsilon(S^1)^{2n}}(t_1)^{-1/m}(t_2)^{-1/m}[\Delta] = \int_{[(1-\epsilon),1+\epsilon]}\mathcal{T}_\Delta^{R_m}\,dR,$$

where $R_m = (|t_1|^{-1/m}, |t_2|^{-1/m})$, and $\mathcal{T}_\Delta^{R_m}$ is the tropical current associated with the diagonal $\Delta\subseteq\mathbb{R}^n\times\mathbb{R}^n$, where the compact torus is rescaled to $R_m(S^1)^{2n}$. In a toric variety compatible with $\Delta\subseteq\mathbb{R}^n\times\mathbb{R}^n$, for any R, $\overline{\mathcal{T}_\Delta^{R_m}}$ has a continuous superpotential, and $\overline{\mathcal{T}_\Delta^{R_m}}$ is SP-convergent to $\overline{\mathcal{T}}_\Delta$. Using Proposition 2.6, Theorem 2.10, and restricting yields:

$$\frac{1}{m^{2n-(p+q)}}(\Phi_m^*[t_1V]\otimes\Phi_m^*[t_2W])\wedge\int_{[(1-\epsilon),1+\epsilon]}\mathcal{T}_\Delta^{R_m}\,dR \longrightarrow (\mathcal{T}_{\mathrm{Trop}_0(V)}\otimes\mathcal{T}_{\mathrm{Trop}_0(W)})\wedge\mathcal{T}_\Delta,$$

as $m\to\infty$. The latter equals

$$\mathcal{T}_{\mathrm{Trop}_0(V)\times\mathrm{Trop}_0(W).\Delta},$$

which can be identified with $\mathcal{T}_{\mathrm{Trop}_0(V)}\wedge\mathcal{T}_{\mathrm{Trop}_0(W)}$. □

notation OK?

We can also prove $\Phi_m^*[tV \cap W] \longrightarrow \mathcal{T}_{\mathrm{Trop}_0(V)} \wedge \mathcal{T}_{\mathrm{Trop}_0(W)}$, for a generic t, if we have an inequality $\lim \Phi_m^*[tV \cap W] \leq$ or $\geq \mathcal{T}_{\mathrm{Trop}_0(V)} \wedge \mathcal{T}_{\mathrm{Trop}_0(W)}$. Do we have such formula from considering cohomology classes?

Not clear to me

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