# Algebraic Geometry
# Coursework 1, Partial Solutions

Q1. Let $A \subseteq \mathbb{A}^n$ be a subset.

(a) (**5 marks**) What is the definition of the closure of $A$ in $\mathbb{A}^n$?

(b) (**5 marks**) Prove that $\mathbb{V}(\mathbb{I}(A))$ equals the Zariski closure of $A$ in $\mathbb{A}^n$.

(c) (**5 marks**) Give an example of a subset in $B \subseteq \mathbb{C}$ whose closure in the Zariski topology does not coincide with its closure in the Euclidean topology.

**Solution to (a).** Observe first that $\mathbb{I}(A) \subseteq \mathbb{C}[x_1, \dots, x_n]$ is a radical ideal, therefore $\mathbb{V}(\mathbb{I}(A))$ is certainly closed. Suppose $Z$ is any closed set in the Zariski topology containing $A$. As applying $\mathbb{I}(-)$ is order-reversing, $\mathbb{I}(Z) \subseteq \mathbb{I}(A)$. Applying $\mathbb{V}(-)$ to both sides, gives $\mathbb{V}(\mathbb{I}(A)) \subseteq Z$. Since $Z$ was an arbitrary closed set containing $A$, it follows that $\mathbb{V}(\mathbb{I}(A))$ is the smallest closed set containing $A$. $\overline{A} = \mathbb{V}(\mathbb{I}(A))$.

**Solution to (b).** Take any non-empty, non-finite Euclidean-closed set $A \subsetneq \mathbb{C}$, then the closure in $\mathbb{A}^1$ would be $\mathbb{A}^1$. For instance the closure of $\{1/n\} \cup \{0\}$ is $\mathbb{A}^1$ in the Zariski topology.

Q2. (a) (**5 marks**) What is the definition of a compact subset of a topological space?

(b) (**10 marks**) Prove that $\mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}^2$ is compact in the Zariski topology but not in the Euclidean topology.

**Solution to (b).**

- Directly from the definition, and without using Hein-Borel Theorem, we can see that the sequence $\{(n^3, n^2), n \in \mathbb{N}\} \subseteq \mathbb{V}(x^2 - y^3)$ has no convergent subsequence and therefore it is not compact for the Euclidean topology.

- Let $V := \mathbb{V}(x^2 - y^3)$. Assume that $V \subseteq \bigcup U_{\alpha \in I}$ is an open cover for $V$. The index set $I$ is not empty. So you can choose an element $\beta_1 \in I$ arbitrarily, by appealing to the Axiom of Choice if you're fancy. Now either $V \subseteq U_{\beta_1}$ or there exists $\beta_2 \in I$, with

$$(V \cap U_{\beta_2}) \setminus (V \cap U_{\beta_1}) \neq \varnothing.$$

That is, $(V \cap U_{\beta_2})$ contains new points of $V$ not covered by $U_{\beta_1}$. Repeating the same process until step $i$, either $O_i := \bigcup_{j=1}^{i} U_{\beta_j}$ is a cover for $V$ or we can choose $U_{\beta_{i+1}}$ for $\beta_{i+1} \in I$ such that it contains some points of $V$ not included $V \cap O_i$. We therefore obtain

$$O_1 \subsetneq O_2 \subsetneq \cdots \subsetneq O_{i+1} = \bigcup_{j=1}^{i+1} U_{\beta_j},$$

And we obtain a (*countable*) chain of ascending open subsets. Taking complements

$$\mathbb{A}^n \setminus O_1 \supsetneq \mathbb{A}^n \setminus O_2 \supsetneq \cdots \supsetneq \mathbb{A}^n \setminus O_{i+1} \supsetneq \dots.$$

Applying $\mathbb{I}(-)$

$$\mathbb{I}(\mathbb{A}^n \setminus O_1) \supsetneq \mathbb{I}(\mathbb{A}^n \setminus O_2) \supsetneq \cdots \supsetneq \mathbb{I}(\mathbb{A}^n \setminus O_{i+1}) \subsetneq \ldots.$$

This sequence must stabilise by the Noetherian property. As a result, finding such an uncountable collection $\{O_i\}$ with $V \cap O_i \subsetneq V \cap O_{i+1}$ is impossible. Therefore $\{O_i\}$ must be a finite collection and it also must be a cover!

**Remark 1.** We could alternatively take a chain of open sets in $V$

$$V \cap O_1 \subsetneq V \cap O_2 \subsetneq \ldots,$$

and use the Hilbert Basis Theorem in $\mathbb{C}[V]$ to prove that this sequence must stabilise.

**Remark 2.** Consider $\bigcup_{p \in V}\{V \setminus \{p\}\}$. This is an infinite (uncountable) open cover with no repetition. But any two (distinct) open sets of the above is also a cover for $V$.

Q3. (a) (**5 marks**) Find a curve $W \subseteq \mathbb{A}^2$ and a morphism $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$, such that $W$ is irreducible but $\varphi^{-1}(W)$ is not.

(b) (**5 marks**) Let $Y$ be a topological space and consider $X \subseteq Y$ with the subspace topology. Prove that if $X$ is irreducible then so is its closure.

(c) (**5 marks**) Prove that isomorphisms preserve irreducibility and dimension of closed affine algebraic varieties.

**Solution for preserving dimension.** Without loss of generality we can assume that $V$ and hence $W$ are irreducible. Now, if $\varphi : V \longrightarrow W$ is an isomorphism, and

$$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d = V$$

is a maximal ascending sequence of irreducible subsets of $V$ then

$$\varphi(V_0) \subsetneq \varphi(V_1) \subsetneq \cdots \subsetneq \varphi(V_d) = W$$

is an ascending sequence of irreducible subsets of $W$, not necessarily maximal. But this implies that $\dim(W) \geq \dim(V)$. Repeating the same argument for a maximal chain of irreducible subsets of $W$ and using $\psi = \varphi^{-1}$, we obtain $\dim(V) \geq \dim(V)$ and we're done.

(d) (**10 marks**) Find the irreducible components of $\mathbb{V}(zx - y, y^2 - x^2(x+1)) \subseteq \mathbb{A}^3$. You need to justify why each component is irreducible.

**Solution.**

$$\begin{cases} y^2 - x^2(x+1) = 0 \\ xz = y \end{cases}$$

Plugging in the second equation into the first, $(zx)^2 - x^2(x+1) = x^2(z^2 - x - 1) = 0$. Therefore $x = 0$ or $z^2 - x - 1 = 0$. The set of points satisfying $x = 0$ and $xz = y$ is exactly $W_1 := \{(0,0,z) : z \in \mathbb{C}\} = \mathbb{V}(x, y)$.

$$\mathbb{C}[W_1] := \frac{\mathbb{C}[x,y,z]}{(x,y)} \simeq \mathbb{C}[z]$$

which is an integral domain, therefore $\mathbb{I}(W_1)$ is prime and $W_1$ is irreducible. When $z^2 - x - 1 = 0$ and $xz = y$ then $x = z^2 - 1$ and $y = xz = z(z^2 - 1)$. So $W_2 = \{(z^2 - 1, z(z^2 - 1), z) : z \in \mathbb{C}\}$. Now observe that that the maps $W_1 \longrightarrow \mathbb{A}^1$, $(x, y, z) \longmapsto z$ and $\mathbb{A}^1 \longrightarrow W_1$, $z \longrightarrow (z^2 - 1, z(z^2 - 1), z)$ are morphisms and inverses to each other. Therefore $W_2 \simeq \mathbb{A}^1$ and therefore $W_2$ is also irreducible. (Since we have shown in the previous question that an isomorphism preserves irreducibility.)

Q4. (a) (**10 marks**) Let $V \subseteq \mathbb{A}^n$ be a Zariski-closed subset and $a \in \mathbb{A}^n \setminus V$ be a point. Find a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ such that

$$f \in \mathbb{I}(V), \quad f(a) = 1.$$

**Solution.** Since $V \cup \{a\} \supsetneq V$ and both of these sets are closed, we have $\mathbb{I}(V \cup \{a\}) \subsetneq \mathbb{I}(V)$. Choose an element $g \in \mathbb{I}(V) \setminus \mathbb{I}(V \cup \{a\})$. We have $g(a) \neq 0$ and $f(x) := \frac{g(x)}{g(a)}$ satisfies the required property. MB presented a nice solution on the board too.

(b) (**15 marks**) Let $I, (g) \subseteq \mathbb{C}[x_1, \dots, x_n]$ be two ideals. Assume that $\mathbb{V}(g) \supseteq \mathbb{V}(I)$.

(i) Prove that if $I = (f_1, \dots, f_k)$, then

$$(f_1, \dots, f_k, x_{n+1}g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}]. \tag{1}$$

(ii) By only using Equation (1) and not the Nullstellensatz, prove that there exists a positive integer $m$ such that $g^m \in I$.

**Solution to (i).** If $a = (a_1, \dots, a_{n+1}) \in \mathbb{V}(f_1, ..., f_k) = \mathbb{V}(I)$ then $f_1(a) = \dots f_k(a) = g(a) = 0$. Then the polynomial $x_{n+1}g - 1$ evaluated at $a$ equals $a_{n+1} \times 0 - 1 = -1$. If $a \in \mathbb{V}(x_{n+1}g - 1)$ then $g(a)$ cannot be zero, and therefore $a \notin \mathbb{V}(g)$ and $a \notin \mathbb{V}(I)$. As a result $\mathbb{V}(f_1, \dots, f_k, x_{n+1}g - 1) = \varnothing$. Let $A = \mathbb{C}[x_1, \dots, x_{n+1}]$, and $J = \mathbb{V}(f_1, \dots, f_k, x_{n+1}g - 1)$. By nullstellensatz

$$A = \mathbb{V}(\varnothing) = \mathbb{I}(\mathbb{V}(J)) = \sqrt{J}.$$

Hence $1 \in A = \sqrt{J}$ and $1^m = 1 \in J$ for some positive integer $m$, then $J = (1) = A$.

**Solution to (ii).** Since $1 \in A$, Equation (1) implies that there are $h_1, \dots, h_{k+1} \in A$ such that

$$1 = h_1 f_1 + \dots + h_k f_k + h_{k+1}(x_{n+1}g - 1).$$

This is an identity of polynomials in $A$ and holds for any $a \in A$. In particular, it holds for any $a \in \mathbb{V}(x_{n+1}g - 1)$. Now note that if $a \in \mathbb{V}(x_{n+1}g - 1)$ then $g(a) \neq 0$ and we have $a_{n+1} = 1/g(a)$. In the above equation, $h_1, \dots, h_{k+1}$ are polynomials and might contain $x_{n+1}$. Collecting the terms involving $x_{n+1}$ we can rewrite the above expression as

$$1 = k_1 + k_2 x_{n+1} + \dots + k_m x_{n+1}^m.$$

where $k_i \in (f_1, \dots, f_k)$. On $\mathbb{V}(x_{n+1}g - 1)$, we can replace $x_{n+1} = \frac{1}{g}$, and obtain

$$1 = k_1 + \dots + \frac{h_m}{g^m} \implies g^m = g^m k_1 + \dots + k_m.$$

Moreover, this equation is an equality of polynomials and it holds in $\mathbb{A}^n \setminus \mathbb{V}(g)$. Since polynomials are continuous, the above equality also holds on the closure $\overline{\mathbb{A}^n \setminus \mathbb{V}(g)} = \mathbb{A}^n$. Note that the right-hand side of the final equation is in $I$.

**Remark 1.** We could directly do the calculations in $\mathbb{C}[\mathbb{V}(x_{n+1}g - 1)]$ and then obtain the polynomial equation without the density argument above. See Joe Harris' Book Page 59

**Remark 2.** The ideas for Part (b) is classically called the *trick of Rabinowitsch* that we have broken up into this question. Using this trick we can use the Weak Nullstellensatz in $\mathbb{A}^{n+1}$ :

$$\mathbb{V}(I) = \varnothing \iff I = \mathbb{C}[x_1, \dots, x_{n+1}],$$

and prove the Nullstellensatz as follows. Consider $g \in \mathbb{I}(\mathbb{V}(I))$. If $V := \mathbb{I}(V)$, we directly proved in the notes that $\mathbb{V}(g) \supseteq \mathbb{V}(\mathbb{I}(V)) = V$. Since $\mathbb{V}(f_1, \dots, f_k, x_{n+1}g - 1) = \varnothing$, by the Weak Nullstellensatz

$$1 \in (f_1, \dots, f_k, x_{n+1}g - 1).$$

Now Part b(ii) shows that $g^m \in I$, that is $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$.

Q5. Prove at least one implication from each of the following equivalences.

(a) (**10 marks**) Show that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is injective if and only if $\varphi$ is *dominant*. Recall that a map, $\varphi$, is called dominant if its image, $\varphi(V)$, is dense in $W$.

(b) (**10 marks**) Prove that the pullback $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$ is surjective if and only if $\varphi$ defines an isomorphism between $V$ and some algebraic subvariety of $W$.

**Solution to (a)** " $\Longleftarrow$ " Let $f \in \mathbb{C}[W]$. If $\varphi^*(f) = 0$, and $\varphi$ is dominant, then $f \circ \varphi(x) = 0$, for all $x \in V$. Since $\varphi(V)$ is dense in $W$, and $f$ is continuous, $f = 0$ on all $W$, and $f \in \mathbb{I}(W)$.

" $\Longrightarrow$ " Assume that $\varphi$ is not dominant. Then $\overline{\varphi(V)} \subsetneq W$ and by Nullstellensatz $\mathbb{I}(\overline{\varphi(V)}) \supsetneq \mathbb{I}(W)$. Choose $f \in \mathbb{I}(\overline{\varphi(V)}) \setminus \mathbb{I}(W)$. Then, $\varphi^*(f) = 0$, but $f \notin \mathbb{I}(W)$.

**Solution to (b)** " $\Longrightarrow$ ". We claim that $Z := \mathbb{V}(\ker(\varphi^*))$ is a closed affine algebraic subvariety of $W$ isomorphic to $V$. Note that $\ker(\varphi^*) = \{g \in \mathbb{C}[W] : g \circ \varphi \in \mathbb{I}(V)\} = \{g \in \mathbb{C}[W] : g \circ \varphi(x) = 0, \text{ for all } x \in V\}$ which includes $\mathbb{I}(W)$. Since $\varphi^*$ is a homomorphism of $\mathbb{C}$-algebras $\ker(\varphi^*)$ is an ideal, and

$$\mathbb{C}[W]/\ker(\varphi^*) \simeq \mathbb{C}[Z] \simeq \mathbb{C}[V] \implies Z \simeq W.$$

" $\Longleftarrow$ " Assume that $\varphi$ induces an isomorphism $V \simeq \varphi(V)$. Note that isomorphism are closed maps, so $\varphi(V)$ is a closed affine algebraic variety. Therefore, $\varphi^*$ is a $\mathbb{C}$-algebra isomorphism between $\mathbb{C}[\varphi(V)] \subseteq \mathbb{C}[W]$ and $\mathbb{C}[V]$.