Linear Algebra: Sheet 1

Present all your answers in complete sentences

Numbas quiz

Complete the week 2 quiz on Blackboard by 1pm on Wednesday 25/09/24. This quiz contains questions on Chapters 1 and 2. This contains important practice of the more computational parts of the course. You can attempt the questions as many times as you like.

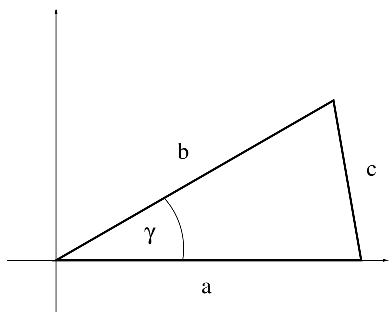
Hand-in question

Submit your solutions to this question on Blackboard by **1pm on Wednesday 25/09/24** for feedback from your tutor.

1. We are given a triangle with side lengths $a, b, c > 0$ and angle γ between legs a and b, see the figure below. We want to prove the law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \ , \tag{1}$$

and derive the triangle inequality.



(i) Consider the vectors $u = \begin{pmatrix} a \\ 0 \end{pmatrix}$ and $v = b\begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix}$. Show that $\|u\| = a$ and $\|v\| = b$.

(ii) Prove that (1) holds. It may help to plot the vectors u and v, show that they span the triangle with sides a, b, c, and consider $c = \|u - v\|$.

(iii) Use the law of cosines to derive the triangle inequality in the form

$$c^2 \le (a+b)^2 \ ,$$

and determine for which $\gamma \in [0, \pi]$ we have equality.

(iv) Use the results above to show that for any $x, y \in \mathbb{R}^2$ we have $\|x + y\| \le \|x\| + \|y\|$.

Additional questions

Try these questions and look at the solutions for feedback. Some of these questions may also be discussed in your tutorial.

2. Sketch the following vectors in $\mathbb{R}^2$ and compute their norm $\|v\|$

(a) $v_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, (b) $v_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, (c) $v_3 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, (d) $v_4 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$, (e) $v_5 = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$ (f) $v_6 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

3. Let $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ y \end{pmatrix}$ with $y \in \mathbb{R}$. Compute $\|u\|$, $\|v\|$ and $\|u + v\|$, and determine for which $y \in \mathbb{R}$ we have

$$\|u + v\| = \|u\| + \|v\| \ .$$

Sketch the vectors u, v and $u + v$ in this case.

4. Let $v_1, v_2, \cdots, v_k \in \mathbb{R}^n$ be k arbitrary vectors in $\mathbb{R}^n$. Use the triangle inequality to show that

$$\|v_1 + v_2 + \cdots + v_k\| \leq \|v_1\| + \|v_2\| + \cdots + \|v_k\|$$

and give an example of k vectors for which there is equality.

5. Use the Cauchy Schwarz inequality to derive the following relation: For any collection of N real numbers $a_1, a_2, \ldots, a_N$ we have

$$\left(\frac{a_1 + a_2 + \cdots + a_N}{N}\right)^2 \leq \frac{a_1^2 + a_2^2 + \cdots + a_N^2}{N} ,$$

i.e., the square of the average is less or equal than the average of the squares. Hint: Consider $v \in \mathbb{R}^N$ with components given by the numbers $a_1, a_2, \cdots, a_N$ and find a suitable $w \in \mathbb{R}^N$ such that $v \cdot w = (a_1 + a_2 + \cdots + a_N)/N$.

6. Let $v, w \in \mathbb{R}^n$. Use the relation between the norm and the dot product, $\|v\|^2 = v \cdot v$, to show
   (i) the parallelogram law:
   $$\|v - w\|^2 + \|v + w\|^2 = 2\|v\|^2 + 2\|w\|^2$$
   (ii) the identity:
   $$v \cdot w = \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2\right)$$

7. Recall that for a complex number $z = x + \mathrm{i}y$ we defined $\bar{z} := x - \mathrm{i}y$ and $|z| = \sqrt{\bar{z}z}$. Show that for any $z, z_1, z_2 \in \mathbb{C}$

(a) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$	(b) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$	(c) $\bar{\bar{z}} = z$
(d) $\overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$	(e) $\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$	(f) $\lvert z_1 + z_2\rvert \leq \lvert z_1\rvert + \lvert z_2\rvert$

8. Prove that for vectors $x, y \in \mathbb{R}^n$ we have $x \cdot y = 0$ if and only if $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

9. Let $v, w \in \mathbb{R}^2$.
   (a) Write these vectors in polar form and draw a sketch that shows how the polar form of the vectors allows you to find the angle between them.
   (b) Write down an explicit expression for $v \cdot w$ in terms of their polar forms and use the identity $\cos\varphi\cos\theta + \sin\varphi\sin\theta = \cos(\varphi - \theta)$ to show that this agrees with the geometric method above.

10. Use De Moivre's formula to derive the following relations:

$$\cos(3\varphi) = 4\cos^3\varphi - 3\cos\varphi \quad \text{and} \quad \sin(3\varphi) = -4\sin^3\varphi + 3\sin\varphi .$$

11. Use Euler's identity $\mathrm{e}^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi$ to show the following representations for trigonometric functions:

$$\sin\varphi = \frac{\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{e}^{-\mathrm{i}\varphi}}{2\mathrm{i}} , \quad \cos\varphi = \frac{\mathrm{e}^{\mathrm{i}\varphi} + \mathrm{e}^{-\mathrm{i}\varphi}}{2} .$$

12. The following extension of the rational numbers is analogous to the construction of the complex numbers from the real numbers.

   Consider numbers of the form $z = x + \sqrt{2}\,y$ where x and y are *rational* numbers. We call the set of all these numbers $\mathbb{Q}(\sqrt{2})$, i.e., $\mathbb{Q}(\sqrt{2}) = \{x + \sqrt{2}\,y\,;\, x, y \in \mathbb{Q}\}$. Show that if $z_1, z_2 \in \mathbb{Q}(\sqrt{2})$ then
   (i) $z_1 + z_2 \in \mathbb{Q}(\sqrt{2})$
   (ii) $z_1 z_2 \in \mathbb{Q}(\sqrt{2})$
   (iii) If $z_1 \neq 0$ then $1/z_1 \in \mathbb{Q}(\sqrt{2})$ (hint: use the fact that $\sqrt{2}$ is irrational.)
   (iv) If $z_1 \neq 0$ then $z_2/z_1 \in \mathbb{Q}(\sqrt{2})$

13. Let n be a positive integer, a complex number z is called an n'th root of unity if

$$z^n = 1 .$$

   (i) Show that if z is an n'th root of unity, then $|z| = 1$.
   (ii) Find all roots of unity for $n = 2$ and $n = 3$ and plot their location in the complex plane.
   (iii) For an arbitrary $n \in \mathbb{N}$, show that there are exactly n different roots of unity and describe their location on the unit circle.