Ex. Why does every alg. variety of $\mathbb{A}^n$ have a finite dimension?

Solution. Assume that is not the case and we have an infinite chain of irreducible ideals.

$$\cdots \subsetneq V_2 \subsetneq V_1 \subseteq V$$

Applying $\mathbb{I}$

$$\cdots \supsetneq \mathbb{I}(V_2) \supsetneq \mathbb{I}(V_1) \supseteq \mathbb{I}(V)$$

gives an ascending chain of ideals in $\mathbb{C}[x_1, \dots, x_n]$, which has to stop by Noetherian property of $\mathbb{C}[x_1, \dots, x_n]$.

Ex. 2.20 Assume that $V = V_1 \cup \cdots \cup V_t$ is a decomposition of $V$ into irreducible alg. varieties, with the property that $V_i \subseteq V_j \Longrightarrow i = j$. (*)

Then $V_i$'s, up to re-ordering, are uniquely determined.

Solution. Assume that

$$V = V_1 \cup \cdots \cup V_t = V_1' \cup \cdots \cup V_s'$$

are two decompositions, and $V_1', \ldots, V_s'$ are also irreducible with the property that

$$V_i' \subseteq V_j' \Rightarrow i = j.$$

Now suppose, on the contrary, that there exists $V_i$ that $V_i \neq V_k'$ for all $k = 1, \ldots, s$.

we have

$$V_i = V_i \cap V = (V_i \cap V_1') \cup \cdots \cup (V_i \cap V_s')$$

By irreducibility of $V_i$, there exists

$l \in \{1, \dots, s\}$ such that

$$V_i \subseteq V_i \cap V_l' \subseteq V_l'.$$

Similarly,

$V_l' \subseteq V_m$, for some $m \in \{1, \dots, t\}$.

Therefore,

$$V_i \subseteq V_l' \subseteq V_m \overset{\text{property } (*)}{\Longrightarrow}$$

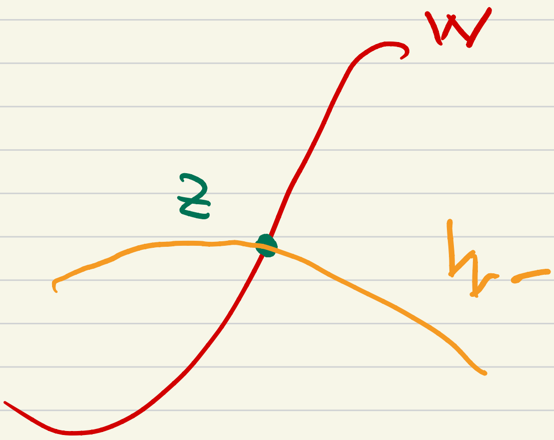
$$V_i = V_l' = V_m .$$

$\blacksquare$

Exercise 2.31.

a) $\varphi : V \longrightarrow W$ is continuous.

<u>Sol.</u> For $Z \subseteq W$ closed, we need to show that $\varphi^{-1}(Z)$ is also closed. Recall that $W \subseteq \mathbb{A}^m$ with induced topology from $\mathbb{A}^m$ is a topological space where its closed sets are of the form $Z = W \cap W'$, where $W' \subseteq \mathbb{A}^m$ is closed.



That implies that

$$Z = W \cap W' = \mathbb{V}(\{f_i\}_i) \cup \mathbb{V}(\{g_j\}_j)$$

Now $x \in \varphi^{-1}(Z)$,

$$\Leftrightarrow \varphi(x) \in Z \Leftrightarrow f_i(\varphi(x)) = 0$$

and $g_j(\varphi(x)) = 0$

for all $i, j$.

Therefore,

$$\varphi^{-1}(Z) = \mathbb{V}(\{f_i \circ \varphi\}_i \cup \{g_j \circ \varphi\}_j)$$

Since $\varphi$ is a restriction of a polynomial morphism $\mathbb{A}^n \to \mathbb{A}^m$,

$f_i \circ \varphi$ and $g_j \circ \varphi$ are all polynomials. i.e. $\varphi^{-1}(Z) \subseteq V$ is also an alg. variety.

b) Consider the projection

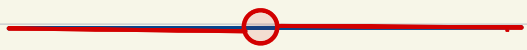
$$\pi : \mathbb{A}^2 \longrightarrow \mathbb{A}^1$$
$$(x,y) \longmapsto x .$$

The set $V = \mathbb{V}(xy-1) \subseteq \mathbb{A}^2$

is closed, but

$$\pi(V) = \mathbb{A}^1 \setminus \{0\} \subseteq \mathbb{A}^1$$

is not closed.

warning. the picture is in $\mathbb{R}^2$ and not $\mathbb{C}^2$.

Ex. 1.10. Zariski topology on $\mathbb{A}^2$ does not coincide with the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$.

<u>Sol.</u>

$\mathbb{V}(x-y) \subseteq \mathbb{A}^2$ is closed, but not closed in $\mathbb{A}^1 \times \mathbb{A}^1$. To see this, note that

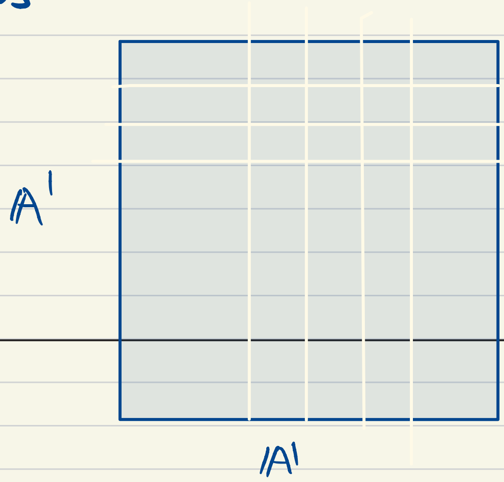
open sets in $\mathbb{A}^1$ are of the form $\mathbb{A}^1 \setminus \{\text{finitely many points}\}$.

Therefore, the open sets of $\mathbb{A}^1 \times \mathbb{A}^1$ are of the form

$$\mathbb{A}^1 \times \mathbb{A}^1 \setminus \left\{ \begin{array}{c} \text{finitely many} \\ \text{horizontal or vertical} \\ \text{lines} \end{array} \right\}$$

As a result,

$\mathbb{A}^1 \times \mathbb{A}^1 \setminus \{x=y\}$




cannot be given as a union of any of such sets.