## 4.2.2 Obtaining $\mathbb{P}^2$

Let  $[x_0:x_1:x_2]$  denote the homogeneous coordinates of the space  $\mathbb{P}^2$ . It is covered by three coordinate charts:

- $U_0$  corresponding to  $x_0 \neq 0$ , with affine coordinates  $(\frac{x_1}{x_0}, \frac{x_2}{x_0}) = (a_1, a_2)$ .
- $U_1$  corresponding to  $x_1 \neq 0$ , with affine coordinates  $(\frac{x_0}{x_1}, \frac{x_2}{x_1}) = (a_1^{-1}, a_1^{-1}a_2)$ .
- $U_2$  corresponding to  $x_2 \neq 0$ , with affine coordinates  $(\frac{x_0}{x_2}, \frac{x_1}{x_2}) = (a_2^{-1}, a_1 a_2^{-1})$ .

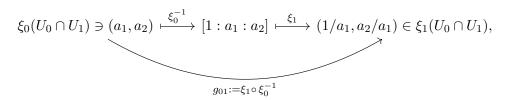
As before, let  $X_i = \xi_i(U_i)$ , and  $X_{ij} = \xi_i(U_i \cap U_j)$ . We have  $\mathbb{C}[X_0] = \mathcal{O}_{X_0}(X_0) = \mathbb{C}[a_1, a_2]$ , and  $\mathbb{C}[X_{01}] = \mathcal{O}_{X_0}(X_{01}) = \mathbb{C}[a_1^{-1}, a_1, a_2]$ . Since on  $X_1, a_1 \neq 0$ , we can write

$$\mathbb{C}[X_1] = \mathcal{O}_{X_1}(X_1) = \mathbb{C}[a_1^{-1}, a_1^{-1}a_2].$$

As a result,

$$\mathbb{C}[X_{10}] = \mathcal{O}_{X_{10}}(X_{10}) = \mathbb{C}[a_1, a_1^{-1}, a_1^{-1}a_2].$$

The isomorphism



provides the information for gluing of  $X_{01} \simeq \mathbb{C}^* \times \mathbb{C}$  and  $X_{10} \simeq X_{01} \simeq \mathbb{C}^* \times \mathbb{C}$  and their corresponding coordinate rings. We can similarly understand the isomorphisms between other charts.

## Torus Actions.

- On  $U_0 \ni a = (a_1, a_2)$  the action of  $t = (t_1, t_2) \in (\mathbb{C}^*)^2$  is obtained by  $t \cdot a = (t_1 a_1, t_2 a_2)$ .
- On  $U_1 \ni b = (b_1, b_2)$  the action of  $t = (t_1, t_2) \in (\mathbb{C}^*)^2$  is obtained by  $t \cdot b = (t_1^{-1}b_1, t_1^{-1}t_2b_2)$ .
- On  $U_2 \ni c = (c_1, c_2)$  the action of  $t = (t_1, t_2) \in (\mathbb{C}^*)^2$  is obtained by  $t\dot{c} = (t_2^{-1}c_1, t_1t_2^{-1}c_2)$ .

This is compatible with the gluing.

**Exercise 4.19.** (a) Find the rest of the isomorphisms  $g_{ij} := \xi_j \circ \xi_i^{-1}$  for gluing in the above example.

(b) Can you use the set of vectors  $\{(1,0),(0,1)\},\{(-1,0),(-1,1)\},\{(1,-1),(0,-1)\}$  to simplify your description? Hint: Use  $(a,b) \in \mathbb{Z}^2$  for  $x^a y^b$ .