

UNIVERSITY OF BRISTOL

School of Mathematics

**SOLUTIONS - Algebraic Geometry**

MATHM0036

(Paper code MATHMATHM0036)

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May/June 2024 2 hour(s) 30 minutes

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**This paper contains two sections: Section A and Section B.**  
**Each section should be answered in a separate booklet.**

All FOUR answers will be used for assessment.

Calculators of an approved type (permissible for A-Level examinations) are permitted.

**Candidates may bring ONE hand-written sheet of A4 notes, written double sided into the examination. Candidates must insert this sheet into their answer booklet(s) for collection at the end of the examination.**

*Do not turn over until instructed.*

Q1. Assume that  $V$  is an affine algebraic variety, and  $U, U_1, U_2 \subseteq V$  are two open subsets.

- (a) **(15 marks) (Standard - Workbook - practicing definition)** Prove that  $\mathcal{O}_V(U)$  is a  $\mathbb{C}$ -algebra.

Solution. The easiest way is to prove that  $\mathcal{O}_V(U)$  includes  $\mathbb{C}$  as a subring. Obviously,  $\mathbb{C} \subseteq \mathcal{O}_V(U)$ . We show that  $\mathcal{O}_V(U)$  has a ring structure. Recall that function  $f : U \rightarrow \mathbb{C}$ , is called *regular at a point*  $p \in V$ , if there is an open neighbourhood  $U' \subseteq U$ , and polynomials  $g, h \in \mathbb{C}[x_1, \dots, x_n]$ , such that  $h(p) \neq 0$ , for any  $p \in U'$ , and  $f|_{U'}(p) = \frac{g(p)}{h(p)}$ . We say that  $f$  is *regular* on  $U$  if it is regular at every point of  $U$ . Therefore, it suffices to show that if  $f, k$  are regular at  $p \in U$  then  $f + k$  and  $fk$  are also regular at  $p \in U$ , this is also clear: assume that  $U' \text{ and } V' \subseteq U$  and  $f|_{U'}(p) = \frac{g(p)}{h(p)}$  and  $k|_{V'}(p) = \frac{g'(p)}{h'(p)}$  then on the open  $U' \cap V'$ ,  $f + k = \frac{gh' + hg'}{hh'}$  and  $fg = \frac{gg'}{hh'}$ .

- (b) **(10 marks) (Workbook - practicing definition)** Assume further that  $f_1 \in \mathcal{O}_V(U_1), f_2 \in \mathcal{O}_V(U_2)$ , with  $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$ . Prove that there exists a regular function  $f \in \mathcal{O}_V(U_1 \cup U_2)$  such that

$$f|_{U_1} = f_1, \quad f|_{U_2} = f_2.$$

Solution. This question means that the regular functions can be glued. Define the well-defined function  $f$  on  $U_1 \cup U_2$  as given. It is clear that for  $i = 1, 2$ , on  $U_i$   $f = f_i$  is regular, since for any point  $p \in U_1 \cup U_2$  so  $f(p) = f_1(p)$  or  $f(p) = f_2(p)$  which are regular by assumption.

Q2. (a) **(15 marks) (Standard - Unseen)** Let  $U = \mathbb{A}^2 \setminus \{0\}$  and  $X = \mathbb{A}^2$ . Compute  $\mathcal{O}_X(U)$ , and show that  $U$  is not an affine algebraic variety.

Solution. We assert that  $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{0\}) = K[x_1, x_2]$ , implying  $\mathcal{O}_X(U) = \mathcal{O}_X(X)$ ; thus, every regular function on  $U$  extends to  $X$ . This is, in fact, rephrasing a result in complex analysis: the Removable Singularity Theorem, which ensures every holomorphic function on  $\mathbb{C}^2 \setminus 0$  extends holomorphically to  $\mathbb{C}^2$ . To demonstrate our assertion, consider  $\phi \in \mathcal{O}_X(U)$ . Notably,  $\phi$  is regular on the two distinguished open subsets  $D(x_1) = (\mathbb{A}^1 \setminus 0) \times \mathbb{A}^1$  and  $D(x_2) = \mathbb{A}^1 \times (\mathbb{A}^1 \setminus 0)$ . By Proposition 3.8,  $\phi = f \cdot x_1^m$  on  $D(x_1)$  and  $\phi = g \cdot x_2^n$  on  $D(x_2)$  for some  $f, g \in K[x_1, x_2]$  and  $m, n \in \mathbb{N}$ , with  $x_1|f$  and  $x_2|g$ . Both representations of  $\phi$  on  $D(x_1) \cap D(x_2)$  yield  $fx_2^n = gx_1^m$ . As the zero locus  $\mathbb{V}(fx_2^n - gx_1^m)$  is closed,  $fx_2^n = gx_1^m$  holds on  $D(x_1) \cap D(x_2) = \mathbb{A}^2$ . In other words,  $fx_2^n = gx_1^m$  in the polynomial ring  $\mathbb{C}[A^2] = K[x_1, x_2]$ . However, the varieties,  $\mathbb{A}^2$  and  $\mathbb{A}^2 \setminus \{0\}$  are not isomorphic.

- (b) **(10 marks) (Standard - Unseen)** Prove that  $\mathbb{V}(y) \subseteq \mathbb{A}^2$  and  $\mathbb{V}(y - x^2) \subseteq \mathbb{A}^2$  are isomorphic, but their corresponding projective closures in  $\mathbb{P}^2$  are not.

Solution. The map  $\mathbb{V}(y - x^2) \subseteq \mathbb{A}^2 \rightarrow \mathbb{V}(y) = \mathbb{A}^1 \times \{0\}, (x, y) \mapsto x$  is an isomorphism with the inverse given by  $t \mapsto (t, t^2)$ . The projective closure of  $\mathbb{V}(y)$ ,  $\overline{\mathbb{V}(y)}$  in  $\mathbb{P}^2$  is given by the  $\{[x : 0 : z] \in \mathbb{P}^2\}$  while the projective closure of  $\mathbb{V}(y - x^2)$  is given by

$$\overline{\mathbb{V}(y - x^2)} = \{[x : y : z] \in \mathbb{P}^2 : yz - x^2\}.$$

On the chart  $U_x$  where  $x = 1$ ,  $\overline{\mathbb{V}(y - x^2)} \cap U_x$  is given by  $yz = 1$ . This set, however, is isomorphic to  $\mathbb{A}^1 \setminus \{0\} \subseteq \mathbb{A}^1$  and cannot be isomorphic to  $\mathbb{A}^1$  itself.

- Q3. (a) **(10 marks) (Workbook - Unseen)** Consider the family of algebraic varieties, with parameter  $t \in \mathbb{C}$ , given by

$$V_t := \mathbb{V}(x^2 + y^2 - t) \subseteq \mathbb{A}^2.$$

Sketch the variety of  $V_0, V_1$ , and  $V_2$  in  $\mathbb{R}^2$ . Determine which one of these three varieties is smooth. Briefly justify your answers.

Solution. Let  $f_t = x^2 + y^2 - t$ .  $\nabla f_0 = \nabla f_1 = \nabla f_2 = (2x, 2y)$ . Note that the kernel of  $\nabla f_i$  is always one dimensional except at  $(0, 0)$ . However,  $(0, 0)$  is in  $V_0$  but not in  $V_1$  nor  $V_2$ . Therefore,  $V_0$  is not smooth, but  $V_1$  and  $V_2$  are.

- (b) **(15 marks) (Standard - Seen)** Let  $V \subseteq \mathbb{A}^n$  and  $W \subseteq \mathbb{A}^m$  be two closed affine algebraic varieties, and

$$\varphi : V \longrightarrow W$$

a morphism. Prove that the pullback  $\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$  is surjective if and only if  $\varphi$  defines an isomorphism between  $V$  and some algebraic subvariety of  $W$ .

Solution.

“  $\implies$  ”. We claim that  $Z := \mathbb{V}(\ker(\varphi^*))$  is a closed affine algebraic subvariety of  $W$  isomorphic to  $V$ . Note that  $\ker(\varphi^*) = \{g \in \mathbb{C}[W] : g \circ \varphi \in \mathbb{I}(V)\} = \{g \in \mathbb{C}[W] : g \circ \varphi(x) = 0, \text{ for all } x \in V\}$  which includes  $\mathbb{I}(W)$ , since  $\varphi^*$  is a homomorphism of  $\mathbb{C}$ -algebras. Moreover,  $\ker(\varphi^*)$  is an ideal, and

$$\mathbb{C}[W]/\ker(\varphi^*) \simeq \mathbb{C}[Z] \simeq \mathbb{C}[V] \implies Z \simeq V.$$

“  $\impliedby$  ” Assume that  $\varphi$  induces an isomorphism  $V \simeq \varphi(V)$ . Note that isomorphism are closed maps, so  $\varphi(V)$  is a closed affine algebraic variety. Therefore,  $\varphi^*$  is a  $\mathbb{C}$ -algebra isomorphism between  $\mathbb{C}[\varphi(V)] \subseteq \mathbb{C}[W]$  and  $\mathbb{C}[V]$ .

*Continued...*

Q4. Let  $\Sigma$  be the fan consisting of

- $\sigma_1$  cone spanned by  $\{(-1, -1), (0, 1)\}$ ;
- $\sigma_2$  cone spanned by  $\{(0, 1), (1, 0)\}$ ;
- $\tau$  cone spanned by  $\{(1, 1)\}$ .

(a) **(6 marks)(Standard - Workbook)** Determine whether or not the toric variety  $X_\Sigma$  has the following properties. Briefly justify your answer.

- (i) smooth;
- (ii) complete.

Solution.

(i) Yes, since the  $\sigma_1 \cap \mathbb{Z}^2$  and  $\sigma_2 \cap \mathbb{Z}^2$  both span  $\mathbb{Z}^2$ .

(ii) No, since  $|\Sigma| \subsetneq \mathbb{R}^2$ .

(b) **(9 marks)(Standard - Workbook)** Describe the coordinate rings of  $X_{\sigma_1}$ ,  $X_{\sigma_2}$ , and  $X_\tau$ .

Solution. We have  $\sigma_1^\vee = \text{cone}(\{(-1, 1), (-1, 0)\})$ ,  $\sigma_2^\vee = \text{cone}(\{(1, 0), (0, 1)\})$ ,  $\tau^\vee = \text{cone}(\{(0, 1), (-1, 0), (1, 0)\})$ . Therefore  $\mathbb{C}[X_{\sigma_2}] = \mathbb{C}[x, y]$ ,  $\mathbb{C}[X_{\sigma_1}] = \mathbb{C}[x^{-1}y, x^{-1}]$ ,  $\mathbb{C}[X_\tau] = \mathbb{C}[y, x, x^{-1}] = \mathbb{C}[yx^{-1}, x, x^{-1}]$ .

(c) (i) **(5 marks)(Standard - Workbook)** Explain why we have the inclusions  $\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau]$ ,  $\mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau]$ ;

(ii) **(5 marks)(Standard - Workbook)** Describe the gluing of  $X_{\sigma_1}$  and  $X_{\sigma_2}$  along  $X_\tau$ .

Solution. Therefore, the equalities  $\mathbb{C}[X_{\sigma_1}]_x = \mathbb{C}[X_\tau] = \mathbb{C}[X_{\sigma_2}]_{x^{-1}}$ . These equalities give rise to the inclusions  $X_\tau \subseteq X_{\sigma_1}$  and  $X_\tau \subseteq X_{\sigma_2}$ . We also have the isomorphisms of  $\mathbb{C}$ -algebras

$$\begin{aligned} \Phi : \mathbb{C}[X_{\sigma_1}] &\supseteq \mathbb{C}[X_\tau] \longrightarrow \mathbb{C}[X_\tau] \subseteq \mathbb{C}[X_{\sigma_2}] \\ x^{-1} &\longmapsto x \\ x^{-1}y &\longmapsto y. \end{aligned}$$

The map  $\Phi$  provides the information for gluing the coordinate rings, as well as the corresponding varieties  $X_\tau \subseteq X_{\sigma_1}$  and  $X_\tau \subseteq X_{\sigma_2}$ .

*End of examination.*