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Algebraic Geometry

~~(a)~~

1a) The closure of a subset $A \subset \mathbb{A}^n$, in Zariski Topology, ~~rega~~ Notated by $\bar{A}$, is the smallest set containing A.

$$\bar{A} = \bigcap \{ X \subset \mathbb{A}^n : X \text{ is closed}, A \subset X \}.$$

b) $\mathbb{I}(A)$ is the vanishing Ideal of A, $= \{ f \in \mathbb{C}[x_1, \dots, x_n] : f(a) = 0, \forall a \in A \}$

We want: $\bar{A} = \mathbb{V}(\mathbb{I}(A))$

In Zariski Topology, any set of the form $\mathbb{V}(J)$, for some ideal J, is closed.

As every $f \in \mathbb{I}(A)$ vanishes on A, $A \subset \mathbb{V}(\mathbb{I}(A))$

Let X be any closed subset in $\mathbb{A}^n$, with $A \subset X$, we can say $X = \mathbb{V}(J)$, for some J ideal of $\mathbb{C}[x_1, \dots, x_n]$.

$\forall f \in J$, $a \in A$, as $a \in X$, $f(a) = 0$, so $f \in \mathbb{I}(A)$ and so $J \subset \mathbb{I}(A)$

Taking zero-sets, $\mathbb{V}(\mathbb{I}(A)) \subset \mathbb{V}(J) = X$.

Hence, $\mathbb{V}(\mathbb{I}(A))$ is contained in every closed set containing A, and is also closed and contains A.

$\mathbb{V}(\mathbb{I}(A))$ = intersection of all closed sets containing A, and so is the smallest closed set containing A.

And so, $\bar{A} = \bigcap \{ X \subset \mathbb{A}^n : X \text{ closed}, A \subset X \} = \mathbb{V}(\mathbb{I}(A))$

c) If we let $B = \mathbb{Z} \subset \mathbb{C}$, $\mathbb{C}$ = an affine line $\mathbb{A}^1(\mathbb{C})$

In Zariski Topology on $\mathbb{A}^1(\mathbb{C})$, closed sets are the zero sets of polynomials in one variable, but any non-zero polynomial in $\mathbb{C}$ has finite roots, so the only infinite closed set is $\mathbb{C}$.
Any other infinite subset in $\mathbb{C}$ would be dense in Zariski Topology, so $\bar{B} = \mathbb{C}$

In Euclidean Topology, discrete sets are closed if they have no limits, and taking $B = \mathbb{Z}$, as this is a discrete subset with no limit, B is closed in Euclidean Topology, so $\bar{B} = \mathbb{Z}$.

2a) A subset, K, of a Topological space is compact if every open cover of K has a finite subcover.

Taking $\{U_i\}_{i \in I}$ as a collection of open sets in X, where X is our Topological space, and K is a subset of X, with: $K \subseteq \bigcup_{i \in I} U_i$, then there is a finite set of indices, $i_1, \dots, i_n \in I$, with:

$$K \subseteq U_{i_1} \cup \dots \cup U_{i_n}$$

b) $\mathbb{V}(x^2 - y^3) \subseteq \mathbb{C}$. Let $X = \mathbb{V}(x^2 - y^3)$

$$= \{(x, y) \in \mathbb{C}^2 : x^2 - y^3 = 0\}$$

The Zariski Topology is found with the zero sets of the collections of polynomials.

As the polynomial ring, $\mathbb{C}[x_1, \dots, x_n]$ is Noetherian, and the Zariski Topology on $\mathbb{C}^2$ is Noetherian, then ~~of~~ every subset of $\mathbb{C}^2$ will be compact.

As X is a subset of $\mathbb{C}^2$, it is compact in Zariski Topology.

The Heine-Borel Theorem tells us that ~~#~~ a subset is compact if and only if it is closed and bounded.

$\mathbb{V}(x^2 - y^3)$ is the zero set of a polynomial, and is closed in Euclidean Topology,

But, as $y \to \infty$, and $x = \pm y^{3/2}$, both $|x|$ and $|y|$ become large, so $\|(x, y)\| = \sqrt{|x|^2 + |y|^2}$ grows without bound. Hence, X is unbounded, and not compact in Euclidean Topology.

3a) Let $W = \mathbb{V}(Y+X) \subseteq \mathbb{A}^2$, and $\varphi: \mathbb{A}^2 \to \mathbb{A}^2$
$(x,y) \to (x^2, y^2)$

W is irreducible as it is just the line $\{(a,-a) : a \in \mathbb{C}\}$, defined by the irreducible polynomial $Y+X$.

$\varphi^{-1}(W)$ here is $\mathbb{V}(x^2+y^2)$. x^2+y^2 has factors: $(x+iy)(x-iy)$ over $\mathbb{C}$,
so $\varphi^{-1}(W) = \mathbb{V}(x+iy) \cup \mathbb{V}(x-iy)$, a union of two irreducible curves, hence $\varphi^{-1}(W)$ is reducible.

b) We want to show that if X is irreducible, then $\bar{X}$ is irreducible.

We can do this by showing that if $\bar{X}$ is reducible, then X must be reducible.

Let $\bar{X} = C_1 \cup \cdots \cup C_n$ be reducible, where C_i is a closed set in Y for all i, and $\bar{X} \supseteq X$,
then, $X = \bar{X} \cap X$
$$= \underbrace{(C_1 \cap X) \cup \cdots \cup (C_n \cap X)}_{\text{is reducible}}, \quad C_i \cap X \text{ is closed by Subspace Topology}$$

Hence, If $\bar{X}$ is reducible $\Rightarrow X$ is reducible.

3c) Showing V is irreducible is equivalent to showing that $\mathbb{I}(V)$ is a Prime Ideal, and then taking the coordinate ring:

$$\mathbb{C}[V] := \frac{\mathbb{C}[x_1,..,x_n]}{\mathbb{I}(V)}$$ is ~~also~~ an Integral Domain as $\mathbb{I}(V)$ is Prime.

So we have that $\mathbb{C}[V]$ is an integral domain.

An Isomorphism of $V \xrightarrow{\sim} W$, gives us $\mathbb{C}[W] \xrightarrow{\sim} \mathbb{C}[V]$

Taking $\varphi: R \xrightarrow{\sim} S$, R an ID, then for $a, b \in S$, $ab = 0$,

$\varphi^{-1}(ab) = \varphi^{-1}(0) = 0$

$\Rightarrow \varphi^{-1}(a)\varphi^{-1}(b) = 0$, and as R is an ID, one of $\varphi^{-1}(a)$, or $\varphi^{-1}(b) = 0$.

So the Isomorphism gives: $\varphi(ab) = 0$, $\varphi(a) = 0$ or $\varphi(b) = 0$

$\Rightarrow$ Hence, S is an Integral Domain.

So, As $\mathbb{C}[V]$ is an Integral Domain, so is $\mathbb{C}[W]$

$\Rightarrow W$ is irreducible.

Hence, the Isomorphism preserves irreducibility.

If $\dim(V) = n$, $\Rightarrow V = V_n \supsetneq V_{n-1} \supsetneq \cdots \supsetneq V_0 = \{pt\}$

Where $V_i \subseteq V$ are algebraic, irreducible subvarieties of V.

And so, $\mathbb{I}(V) = P_1 \subset \cdots \subset P_n$ of maximal chain length,

Where $P_i \trianglelefteq \mathbb{C}[x_1,..,x_n]$ are prime ideals.

The correspondence Theorem says that this is equivalent to:

$(0) \subset q_1 \subset \cdots \subset q_n$ of maximal chain length,

with $q_i \trianglelefteq \mathbb{C}[V]$ prime ideals.

Applying $(\varphi^*)^{-1}$ as an Isomorphism,

$(0) \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$ of max chain length, is equivalent to W having dimension n.

Hence, ~~irreducibility~~ ~~preserved~~ Isomorphisms preserve irreducibility and Dimension.

3d) Let $V = \mathbb{V}(zx-y,\ y^2 - x^2(x+1)) \subseteq \mathbb{A}^3$.

If we say $y = zx$, then $y^2 - x^2(x+1)$

$$\Rightarrow (zx)^2 - x^2(x+1)$$
$$= -x^2(x+(1-z^2))$$
$$= -(x)(x)(x+1-z^2)$$

So, $V = \left(\mathbb{V}(x) \cup \mathbb{V}(x+1-z^2)\right) \cap \mathbb{V}(zx-y)$

$$= \underbrace{\mathbb{V}(x, zx-y)} \cup \mathbb{V}(x+1-z^2, zx-y)$$

requires $x=0$, $\Rightarrow zx = 0$, y must also $= 0$.

$$= \mathbb{V}(x,y) \cup \mathbb{V}(x+1-z^2, zx-y)$$

$\mathbb{V}(x,y)$ is just the z axis, and is isomorphic to $\mathbb{A}^1$, which is irreducible in Zariski Topology.

Let $W = \mathbb{V}(x+1-z^2, zx-y)$. To show this is irreducible is equivalent to showing $\mathbb{C}[W]$ is an Integral Domain. ~~by~~

For $\alpha: \mathbb{C}[x,y,z] \to \mathbb{C}[t]$, $\ker(\alpha) = \mathbb{I}(W)$

If $z \to t$, then $x = t^2 - 1$, $y = t(t^2-1)$, so

$$x \to t^2 - 1,$$
$$y \to t^3 - t,$$
$$z \to t$$

which is a surjective map, with kernel,
$\ker(\alpha) = (zx - y,\ x + 1 - z^2)$,

so,

$$\mathbb{C}[t] \cong \frac{\mathbb{C}[x,y,z]}{\ker(\alpha)} = \mathbb{C}[W]$$, with ~~$\mathbb{C}[t]$~~ and $\mathbb{C}[W]$ an integral Domain,

So $(zx - y,\ x + 1 - z^2)$ is a prime Ideal,
and so $\mathbb{V}(zx - y, x + 1 - z^2)$ is irreducible,

So The irreducible components of $\mathbb{V}(zx - y, y^2 - x^2(x+1))$
$= \mathbb{V}(x, y) \cup \mathbb{V}(x + 1 - z^2, zx - y)$

4a) For $a \notin V$, then $a \notin \mathbb{V}(\mathbb{I}(V))$
$$= \mathbb{V}(g_1, \ldots, g_m)$$

Hence, There exists some j, such that
$$g_j(a) \neq 0$$
If we say that $X = g_j(a) \in \mathbb{C}^\times$,
then $f = X^{-1} g_j = 1$ for $f(a)$, as
$$g_j(a) = X,\ X^{-1}X = 1.$$

$f \in \mathbb{I}(V)$ as $g_j \in \mathbb{I}(V)$, and ~~all~~ ~~multiples~~
Ideals are closed under multiplication,

~~and~~ $\cancel{(f(a) = X^{-1} g_i = 1)}$

4bi) We have that $\mathbb{V}(g) \supseteq \mathbb{V}(f_1,\dots,f_k)$
$= \bigcap_{i=1}^{k} \mathbb{V}(f_i)$

And, $\mathbb{V}(f_1,\dots,f_k, x_{n+1}g-1) = \mathbb{V}(f_1,\dots,f_k) \cap \mathbb{V}(x_{n+1}g-1)$

Let's call $f_1,\dots,f_k = I$, $x_{n+1}g-1 = m$,

For $y \in \mathbb{V}(I) \cap \mathbb{V}(m)$, then $y \in \mathbb{V}(I)$ ~~or~~ and $y \in \mathbb{V}(m)$.

Then, $f_j(y) = 0$ for all j, or $y_{n+1}g(y)-1 = 0$

If $y \in \mathbb{V}(f_1,\dots,f_k) \in \mathbb{V}(g)$,

then, $y_{n+1}g(y)-1 = 0-1 = -1 \neq 0$. A contradiction

So, $\mathbb{V}(f_1,\dots,f_k) \cap \mathbb{V}(x_{n+1}g-1) = \emptyset$.

By Nullstellensatz, If $J = (f_1,\dots,f_k, x_{n+1}g-1)$

$\sqrt{J} = \mathbb{I}(\mathbb{V}(J)) = \mathbb{I}(\emptyset) = \mathbb{C}[x_1,\dots,x_{n+1}]$

And, as $1 \in \sqrt{J}$, $1^m \in J$ for some m.

$1^m = 1$ for all m, so

$1 \in J$.

If $\mathbb{C}[x_1,\dots,x_{n+1}] = R$, and so, $R = R\cdot 1 \subseteq RJ \subseteq J$

$\implies R \subseteq J$, $R = J$.

So, $(f_1,\dots,f_k, x_{n+1}g-1) = \mathbb{C}[x_1,\dots,x_{n+1}]$

4b (ii) We know that $(f_1, \dots, f_k, x_{n+1} g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}]$,
and $1 \in (f_1, \dots, f_k, x_{n+1} g - 1)$, so

$$r_1 f_1 + \dots + r_k f_k + r_{k+1}(x_{n+1} g - 1) = 1$$
For any $\underline{x} = (x_1, \dots, x_{n+1}) \in \mathbb{A}^{n+1}$.

We have that $x_{n+1} = \frac{1}{g}$ as a result, so, taking points of

$$y = (x_1, \dots, x_n, x_{n+1}) = \left(x_1, \dots, x_n, \frac{1}{g(x_1, \dots, x_n)}\right),$$

then we have that

$$1 = \sum_{j=1}^{k} r_j(x_1, \dots, x_n) f_j(x_1, \dots, x_n).$$

If we take r_j' as polynomials in $\mathbb{C}[x_1, \dots, x_n]$,
and let m be the maximal degree of x_{n+1} for each
$r_i f_j$, then: $(\underline{x} = x_1, \dots, x_n)$

$$1 = \frac{\sum_{j=1}^{k} r_j'(\underline{x}) f_j(\underline{x})}{g(\underline{x})^m},$$

so, $g(\underline{x})^m = \sum_{j=1}^{k} r_j'(\underline{x}) f_j(\underline{x})$, $\forall \underline{x}$.

And, as $\sum_{j=1}^{k} r_j'(\underline{x}) f_j(\underline{x}) \in (f_1, \dots, f_k) = I$,

$g(\underline{x})^m \in I$, so $g^m \in I$ $\boxtimes$.

5a) For $V \subseteq \mathbb{A}^n$, $\mathbb{C}[x_1, \dots, x_n] = R$
$W \subseteq \mathbb{A}^m$, $\mathbb{C}[x_1, \dots, x_m] = S$
And a morphism $\varphi : V \to W$,

Then $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is injective if and only if $\mathrm{Ker}(\varphi^*)$ is trivial.

$\mathrm{Ker}(\varphi^*)$ comes from the map: $\psi : S \to \mathbb{C}[V]$
$f \to f \circ \varphi$

when it has ~~~~ kernel $\mathbb{I}(W)$

~~Ker~~ $\mathrm{Ker}(\psi)$ comes from the inverse of the image of $S \to R$, call this map ϕ.

So, φ^* is injective if and only if $\phi^{-1}(\mathbb{I}(V)) = \mathbb{I}(W)$

And, $\phi^{-1}(\mathbb{I}(V)) = \{ f \in S \mid f \circ \varphi(x) = 0 \ \forall x \in V \}$
$= \{ f \in S \mid f(y) = 0 \ \forall y \in \varphi(V) \}$
$= \mathbb{I}(\varphi(V))$.

Hence, φ^* is injective ~~iff~~ if and only if
$\mathbb{I}(\varphi(V)) = \mathbb{I}(W)$.

And ~~so~~ we can rewrite this as:
$$\mathbb{V}(\mathbb{I}(\varphi(V))) = \mathbb{V}(\mathbb{I}(W))$$
$$\Rightarrow \mathbb{V}(\mathbb{I}(\varphi(V))) = W$$

And, by 1b, $\mathbb{V}(\mathbb{I}(\varphi(V))) = \overline{\varphi(V)}$, so:

φ^* injective $\Leftrightarrow$ $\mathrm{Ker}(\varphi^*)$ is trivial
$\Leftrightarrow \mathbb{I}(\varphi(V)) = \mathbb{I}(W)$
$\Leftrightarrow \overline{\varphi(V)} = W$

But, if the closure of $\varphi(V) = W$, then this means that $\varphi(V)$ is dense in W, so φ^* is injective $\Leftrightarrow$ φ is dominant.

5b) $\varphi^*: \mathbb{C}[W] \to \mathbb{C}[V]$ surjective $\Leftrightarrow$ φ defines an Iso between V and an algebraic subvariety of W.

By Homomorphism theorem:

$$\frac{\mathbb{C}[W]}{\mathrm{Ker}(\varphi^*)} \cong \mathrm{Im}(\varphi^*)$$

And, the correspondence theorem tells us that for some $J \trianglelefteq S$, containing $\mathbb{I}(W)$, ~~that the preimage of~~ is the preimage of $\mathrm{Ker}(\varphi^*)$ under the map $S \to \mathbb{C}[W]$.

$$\text{So, } \frac{S}{J} \cong \frac{\mathbb{C}[W]}{\mathrm{Ker}(\varphi^*)} \cong \mathrm{im}(\varphi^*)$$

As $\mathbb{I}(W) \subseteq J$, then $W = \mathbb{V}(\mathbb{I}(W)) \supseteq \mathbb{V}(J)$.

If we say that $\mathbb{V}(J) = U$, and is the algebraic subvariety of W, and as J is the preimage of $\mathbb{I}(V)$ under:

$S \to R$, $f \to f \circ \varphi$, then J is radical, so $\sqrt{J} = J$

We use Nullstellensatz: $\sqrt{J} = J = \mathbb{I}(V(J)) = \mathbb{I}(U)$

And so, $\mathbb{C}[U] = \frac{S}{J}$, and so φ^* is surjective.

$$\text{So, } \mathbb{C}[U] \cong \frac{\mathbb{C}[W]}{\mathrm{Ker}(\varphi^*)} \cong \mathrm{im}(\varphi^*) = \mathbb{C}[V]$$

$\therefore \mathbb{C}[U] \cong \mathbb{C}[V]$,

~~and~~ ~~Ara~~ And an Isomorphism between V and U, the algebraic subvariety of W.