UNIVERSITY OF BRISTOL

School of Mathematics

ALGEBRAIC GEOMETRY

MATHM0036

(Paper code MATH–M0036)

May/June 2023   2 hours 30 minutes

The exam contains FOUR questions
All Four answers will be used for assessment

Calculators of an approved type (permissible for A-Level examinations) are permitted.

Candidates may bring four sheets of A4 notes written double-sided into the examination.
Candidates must insert these into their answer booklet(s) for collection at the end of the examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

Q1. A *sheaf* $\mathcal{F}$ of rings associated to a topological space X consists of the following data:

(i) To each open set $U \subseteq X$, it associates a ring $\mathcal{F}(U)$.

(ii) To each inclusion of open sets $U \hookrightarrow V$, there exists a map $\mathrm{res}_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ called the restriction map from $\mathcal{F}(V)$ to $\mathcal{F}(U)$. These maps satisfy the property that $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ and $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$, where $U \subseteq V \subseteq W$ are open sets.

These data satisfy the following properties:

(iii) Suppose that $f_i \in \mathcal{F}(U_i)$ are a collection of sections that agree on overlaps (formally, $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ whenever the intersection exists). Then they lift to a section $f \in \mathcal{F}(U)$ which has the property that $\mathrm{res}_{U,U_i} f = f_i$ for all $i \in I$.

(iv) Suppose that $f, f' \in \mathcal{F}(U)$ and that $\mathrm{res}_{U,U_i} f = \mathrm{res}_{U,U_i} f'$ for all $i \in I$. Then $f = f'$.

Let X be an irreducible quasi-projective variety.

(a) (i) (**5 marks**) Assume that U and V are open subsets of X with $U \subseteq V$. Briefly explain why $f \in \mathcal{O}_X(V)$ implies that $f_{|_U} \in \mathcal{O}_X(U)$.

(ii) (**5 marks**) Briefly explain why the collection of sets of functions $\mathcal{O}_X(U)$, where U ranges over all open subsets of X, forms a sheaf on X.

(b) (**15 marks**) Prove that any regular function on X is continuous.

Q2. Consider the product varieties $X = \mathbb{P}^1 \times \mathbb{P}^1$, $Y = \mathbb{V}(xw - yz) \subseteq \mathbb{P}^3$ and the *Segre* morphism

$$\begin{aligned} \varphi : X &\longrightarrow Y \\ ([s:t] \times [u:v]) &\longmapsto [su : sv : tu : tv]. \end{aligned}$$

(a) (**15 marks**) Find the inverse map $\psi : Y \longrightarrow X$. (Hint. Describe the map φ in some affine charts.)

(b) (**10 marks**) Prove that ψ is a morphism.

Continued...

Q3. (a) (**10 marks**) Consider the family of algebraic varieties, with parameter $t \in \mathbb{C}$, given by

$$V_t := \mathbb{V}(xy - t) \subseteq \mathbb{A}^2.$$

Sketch the variety of V_0, V_1, and V_2 in $\mathbb{R}^2$. Determine whether or not these varieties are smooth. Briefly justify your answers.

(b) (**15 marks**) Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be two affine algebraic varieties, and

$$\varphi : V \longrightarrow W$$

a morphism. Show that the pullback

$$\varphi^* : \mathbb{C}[W] \longrightarrow \mathbb{C}[V]$$

is injective if and only if φ is *dominant.* Recall that a map φ is called dominant if its image, $\varphi(V)$, is dense in W.

Q4. Let Σ be the fan consisting of

- σ_1 cone spanned by $\{(1,0),(0,1)\}$;
- σ_2 cone spanned by $\{(1,0),(1,-1)\}$;
- τ cone spanned by $\{(1,0)\}$.

(a) (**6 marks**) Determine whether or not the toric variety X_Σ has the following properties. Briefly justify your answer.

(i) smooth;

(ii) complete.

(b) (**9 marks**) Describe the coordinate rings of X_{σ_1}, X_{σ_2}, and X_τ.

(c) (i) (**5 marks**) Explain why we have the inclusions $\mathbb{C}[X_{\sigma_1}] \subseteq \mathbb{C}[X_\tau]$, $\mathbb{C}[X_{\sigma_2}] \subseteq \mathbb{C}[X_\tau]$;

(ii) (**5 marks**) Describe the gluing of X_{σ_1} and X_{σ_2} along X_τ.

End of examination.