Q1. (a) (**15 marks**) Find all the elements of $\mathrm{maxSpec}(\mathbb{C}[x])$, $\mathrm{maxSpec}(\mathbb{C}[x,1/x])$, and $\mathrm{maxSpec}(\mathbb{C}[x,1/x,y])$ explicitly.

(b) (**5 marks**) Consider the isomorphism $\varphi : \mathbb{A}^1\backslash\{0\} \longrightarrow \mathbb{A}^1\backslash\{0\}$, $a \longmapsto b = 1/a$, and the pullback map on the coordinate rings $\varphi^* : \mathbb{C}[x,1/x] \longmapsto \mathbb{C}[y,1/y]$. Compute $\varphi^*(1/x)$, $\varphi^*(2x^2 + \frac{2x^3+4x}{x^5})$, $\varphi^*(2-x)$.

1a) • $\mathrm{maxSpec}(\mathbb{C}[x]) = \{ m \subseteq \mathbb{C}[x] : m \text{ is a maximal ideal} \}$

m is a maximal if the only ideal strictly containing it is the unit ideal $\mathbb{C}[x]$

$m = (x+a)$ $\quad a \in \mathbb{C}$ $\quad$ is maximal $\quad$ $(m = \{(x+a)f : f \in \mathbb{C}[x]\})$

let n be an ideal such that $m \subsetneq n \subsetneq \mathbb{C}[x]$

$\mathbb{C}$ is a field so $\mathbb{C}[x]$ is a PID $\Rightarrow n = (g) = \{ g \cdot f : f \in \mathbb{C}[x]\}$ for a fixed $g \in \mathbb{C}[x]$

$m = (x-a) \subseteq (g) = n \Rightarrow (x-a) = gf$

g is not a unit so is non-constant $\Rightarrow g = x-a$ $\quad \Rightarrow n = (g) = (x-a) = m$

so m is maximal

$\Rightarrow \mathrm{maxSpec}(\mathbb{C}[x]) = \{ (x-a) \subseteq \mathbb{C}[x] : a \in \mathbb{C} \}$

• $\mathrm{maxSpec}(\mathbb{C}[x,1/x]) = \{ m \subseteq \mathbb{C}[x,1/x] : m \text{ is a maximal ideal} \}$

by the evaluation map $E_\alpha : \mathbb{C}[x,1/x] \to \mathbb{C}$

$f \mapsto f(\alpha_1, \alpha_2 = 1/\alpha_1)$

$\mathbb{C} \cong \mathbb{C}[x,1/x] / \mathrm{Ker}(E_\alpha)$ $\quad$ by the first isomorphism theorem (E_α is surjective)

$\mathbb{C}$ is a field $\Rightarrow \mathrm{Ker}(E_\alpha)$ is maximal $\quad \Rightarrow$ maximal ideals are $m_\alpha = (x-\alpha_1, 1/x - \alpha_2)$

(R/m is a field $\Leftrightarrow$ m is a maximal ideal in R)

$\alpha_1, \alpha_2 \in \mathbb{C}$

$$\begin{aligned} m_\alpha &= (x-\alpha_1, 1/x - \alpha_2) && \alpha_1,\alpha_2 \in \mathbb{C} \\ &= (x-\alpha_1, 1/x - 1/\alpha_1) && \alpha_1 \in \mathbb{C}^* \quad (\alpha_2 = 1/\alpha_1) \\ &= (x-\alpha_1, x-\alpha_1) && \alpha_1 \in \mathbb{C}^* \quad (1/x = 1/\alpha_1) \\ m_\alpha &= (x-\alpha_1) && \alpha_1 \in \mathbb{C}^* \end{aligned}$$

$\mathrm{maxSpec}(\mathbb{C}[x,1/x]) = \{ (x-a) \subseteq \mathbb{C}[x,1/x] : a \in \mathbb{C}^* \}$

• $\mathrm{maxSpec}(\mathbb{C}[x,1/x,y]) = \{ m \subseteq \mathbb{C}[x,1/x,y] : m \text{ is a maximal ideal} \}$

evaluation map $E_\alpha : \mathbb{C}[x,1/x,y] \to \mathbb{C}$

$f \mapsto f(\alpha_1, \alpha_2 = 1/\alpha_1, \alpha_3)$

$\Rightarrow \mathbb{C} \cong \mathbb{C}[x,1/x] / \mathrm{Ker}(E_\alpha)$ $\quad$ by the first isomorphism theorem

$\mathbb{C}$ is a field $\Rightarrow \mathrm{Ker}(E_\alpha)$ maximal ideal $\Rightarrow$ maximal ideals are $m_\alpha = (x-\alpha_1, 1/x - \alpha_2, y-\alpha_3)$

$\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$

$$\begin{aligned} m_\alpha &= (x-\alpha_1, 1/x - \alpha_2, y-\alpha_3) && \alpha_1,\alpha_2,\alpha_3 \in \mathbb{C} \\ &= (x-\alpha_1, 1/x - 1/\alpha_1, y-\alpha_3) && \alpha_1 \in \mathbb{C}^*, \alpha_3 \in \mathbb{C} \quad (\alpha_2 = 1/\alpha_1) \\ &= (x-\alpha_1, x-\alpha_1, y-\alpha_3) && \alpha_1 \in \mathbb{C}^*, \alpha_3 \in \mathbb{C} \quad (1/x = 1/\alpha_1) \\ m_\alpha &= (x-\alpha_1, y-\alpha_3) && \alpha_1 \in \mathbb{C}^*, \alpha_3 \in \mathbb{C} \end{aligned}$$

$\mathrm{maxSpec}(\mathbb{C}[x,1/x,y]) = \{ (x-a, y-b) \subseteq \mathbb{C}[x,1/x,y] : a \in \mathbb{C}^*, b \in \mathbb{C} \}$

Q1. (a) (**15 marks**) Find all the elements of $\operatorname{maxSpec}(\mathbb{C}[x])$, $\operatorname{maxSpec}(\mathbb{C}[x,1/x])$, and $\operatorname{maxSpec}(\mathbb{C}[x,1/x,y])$ explicitly.

(b) (**5 marks**) Consider the isomorphism $\varphi : \mathbb{A}^1\backslash\{0\} \longrightarrow \mathbb{A}^1\backslash\{0\}$, $a \longmapsto b = 1/a$, and the pullback map on the coordinate rings $\varphi^* : \mathbb{C}[x,1/x] \longmapsto \mathbb{C}[y,1/y]$. Compute $\varphi^*(1/x)$, $\varphi^*(2x^2 + \frac{2x^3+4x}{x^5})$, $\varphi^*(2-x)$.

1b) $\varphi : \mathbb{A}^1\backslash\{0\} \longrightarrow \mathbb{A}^1\backslash\{0\}, \ a \longmapsto b = 1/a$

$\varphi^* : \mathbb{C}[x, 1/x] \longmapsto \mathbb{C}[y, 1/y]$

$$\varphi^*(1/x)(a) = (1/x \circ \varphi)(a) = (1/x)(\varphi(a)) = (1/y)(1/a) = a$$

$$\varphi^*(1/x) = y$$

$$\varphi^*\left(2x^2 + \frac{2x^3+4x}{x^5}\right) = \left(2x^2 + \frac{2x^3+4x}{x^5}\right) \circ \varphi = 2(1/y)^2 + \frac{2(1/y)^3 + 4(1/y)}{(1/y)^5}$$

$$= \frac{2}{y^2} + \left(\frac{2}{y^3} + \frac{4}{y}\right)y^5$$

$$= \frac{2}{y^2} + 2y^2 + 4y^4$$

$$= \frac{2 + 2y^4 + 4y^6}{y^2}$$

$$\varphi^*(2-x) = (2-x)\circ\varphi = 2 - 1/y = \frac{2y-1}{y}$$

Q2. (**20 marks**) Consider the affine algebraic hypersurface $V := \mathbb{V}(y - ux) \subseteq \mathbb{A}^3$.

(a) Prove that the projection $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$, $(x, y, u) \longmapsto (x, u)$ restricts to an isomorphism from V to $\mathbb{A}^2$.

(b) Prove that the projection $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$, $(x, y, u) \longmapsto (x, y)$ does not restrict to isomorphism from V to $\mathbb{A}^2$.

$\mathbb{V}(y-ux) \subseteq \mathbb{A}^3$

2a) A morphism of algebraic varieties $\varphi: V \to W$ is an isomorphism if it has an inverse, if there exists a morphism of algebraic varieties $\varphi^{-1}: W \to V$ such that $\varphi^{-1} \circ \varphi = id_V$ and $\varphi \circ \varphi^{-1} = id_W$

$$p_1 : \mathbb{A}^3 \longrightarrow \mathbb{A}^2$$
$$(x,y,u) \longmapsto (x,u)$$

$$\varphi_1 = p_1|_V : V \longrightarrow \mathbb{A}^2$$
$$(x,y,u) \longmapsto (x,u)$$

let $\varphi_1^{-1} : \mathbb{A}^2 \to V$ a morphism of algebraic varieties

$(x,u) \longmapsto (x,ux,u)$

in V, $y = ux$

domain is V, $ux = y$

then $\varphi_1 \circ \varphi_1^{-1}(x,u) = \varphi_1(\varphi_1^{-1}(x,u)) = \varphi_1(x,ux,u) = \underline{\varphi_1(x,y,u)} = (x,u)$

and $\varphi_1^{-1} \circ \varphi_1(x,y,u) = \varphi_1^{-1}(\varphi_1(x,y,u)) = \varphi_1^{-1}(x,u) = (x,ux,u) = (x,y,u)$

Codomain is V, $ux = y$

$\Rightarrow \varphi_1 \circ \varphi_1^{-1} = id_{\mathbb{A}^2}$ and $\varphi_1^{-1} \circ \varphi_1 = id_V$

$\Rightarrow \varphi_1$ is an isomorphism

the projection restricts to an isomorphism

2b)

$$p_2 : \mathbb{A}^3 \to \mathbb{A}^2$$
$$(x,y,u) \longmapsto (x,y)$$

$$\varphi_2 = p_2|_V : V \longrightarrow \mathbb{A}^2$$
$$(x,y,u) \longmapsto (x,y)$$

$$Ker(\varphi_2) = \varphi_2^{-1}(0) = \{(x,y,u) \mid x = y = 0,\ y = ux = 0\}$$
$$= \{(0,0,u) \mid u \cdot 0 = 0\}$$
$$Ker(\varphi_2) = \{(0,0,u) \mid u \in \mathbb{C}\} \neq \{(0,0,0)\}$$

$\Rightarrow \varphi_2$ is not injective

$\Rightarrow \varphi_2$ cannot have an inverse

$\Rightarrow \varphi_2$ cannot be an isomorphism

the projection does not restrict to an isomorphism

Q3. (**25 marks**)

(a) Prove that if $g \in \mathbb{C}[x,y]$ then the projective closure of its variety $\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g}) \subseteq \mathbb{P}^2$ where $\tilde{g} \in \mathbb{C}[x,y,z]$ is the homogenisation of g.

(b) Consider the polynomials $f_1(x,y) = x+y+1$, $f_2(x,y) = x^2+6y^2+1$, $f_3(x,y) = x^2+3y+1$, $f_4(x,y) = x^3+3xy^2+4$. Determine whether or not each of the projective closures includes the points

(i) $[1:0:0]$;
(ii) $[0:1:0]$;
(iii) $[0:0:1]$.

(c) Can you find a general necessary and sufficient condition on $g \in \mathbb{C}[x,y]$ such that its homogenisation $\tilde{g} \in \mathbb{C}[x,y,z]$ does not pass through any of the three points in item (b)?

3a) $g \in \mathbb{C}[x,y]$, $\tilde{g} \in \mathbb{C}[x,y,z]$ the homogenisation of g

$\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g})$ proof:

- $\overline{\mathbb{V}(g)} \subseteq \mathbb{V}(\tilde{g})$: let $G = (g)$ and $\tilde{G} = (\tilde{g})$

$\tilde{I} = (A)$, $A = \{\tilde{f} \in \mathbb{C}[x,y,z] : f \in G\}$ all polynomials in (g) homogenised

let $\tilde{f} \in A$, then $\tilde{f}$ is a homogenised element of (g) so $\tilde{f}(x,y,1) = f(x,y)$

$U_z = \{[x,y,z] \in \mathbb{P}^2 : z \neq 0\}$

then $\tilde{f}|_{U_z} = f = gh \quad h \in \mathbb{C}[x,y]$

$\tilde{f}|_{U_z}(\mathbb{V}(g)) = f(\mathbb{V}(g)) = \overset{=0}{\cancel{g(\mathbb{V}(g))}}h(\mathbb{V}(g)) = 0$

$\overline{\mathbb{V}(g)} \cap U_z = \mathbb{V}(g) \Rightarrow \mathbb{V}(\tilde{f}) \supseteq \mathbb{V}(g) \Rightarrow \mathbb{V}(\tilde{f}) \supseteq \overline{\mathbb{V}(g)}$ because $\mathbb{V}(\tilde{f})$ is closed, for arbitrary $\tilde{f} \in A$

$\tilde{f}$ is a homogenised element of (g), $g \in (g)$ so let $\tilde{f} = \tilde{g}$ then $\mathbb{V}(\tilde{g}) \supseteq \overline{\mathbb{V}(g)}$

- $\mathbb{V}(\tilde{g}) \subseteq \overline{\mathbb{V}(g)}$:

Hilbert's correspondence $\mathbb{V}(\tilde{g}) \subseteq \overline{\mathbb{V}(g)} \Leftrightarrow \mathbb{I}(\overline{\mathbb{V}(g)}) \subseteq \mathbb{I}(\mathbb{V}(\tilde{g})) = \sqrt{(\tilde{g})} = (\tilde{g})$

$\overline{\mathbb{V}(g)}$ is a closed set in $\mathbb{P}^2 \Rightarrow (g) = \sqrt{(g)} = \mathbb{I}(\overline{\mathbb{V}(g)})$ is a radical homogeneous ideal
$\Rightarrow \mathbb{I}(\overline{\mathbb{V}(g)})$ has a finite set of homogenous generators

$\mathbb{I}(\overline{\mathbb{V}(g)}) = (g)$ a principal ideal, so is generated by a single generator that is unique up to multiplication by units
$\Rightarrow g$ is homogenous so $g \in (\tilde{g})$

$\Rightarrow \mathbb{I}(\overline{\mathbb{V}(g)}) = (g) \subseteq (\tilde{g})$

$\Rightarrow \mathbb{V}(\tilde{g}) \subseteq \overline{\mathbb{V}(g)}$

$\overline{\mathbb{V}(g)} \subseteq \mathbb{V}(\tilde{g})$ and $\mathbb{V}(\tilde{g}) \subseteq \overline{\mathbb{V}(g)} \Rightarrow \overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g})$

Q3. **(25 marks)**

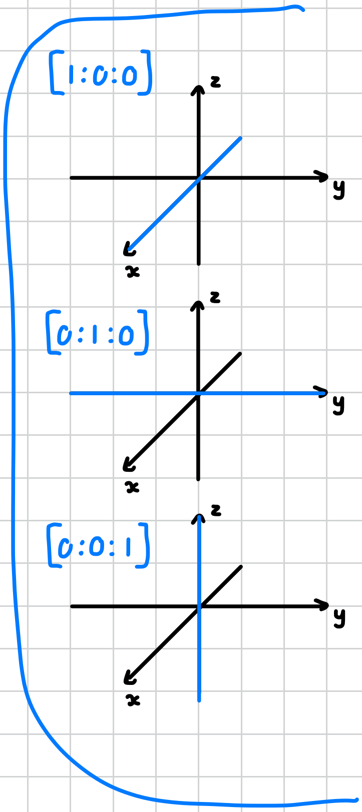
(a) Prove that if $g \in \mathbb{C}[x,y]$ then the projective closure of its variety $\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g}) \subseteq \mathbb{P}^2$ where $\tilde{g} \in \mathbb{C}[x,y,z]$ is the homogenisation of g.

(b) Consider the polynomials $f_1(x,y) = x+y+1$, $f_2(x,y) = x^2+6y^2+1$, $f_3(x,y) = x^2+3y+1$, $f_4(x,y) = x^3+3xy^2+4$. Determine whether or not each of the projective closures includes the points

(i) $[1:0:0]$;
(ii) $[0:1:0]$;
(iii) $[0:0:1]$.

(c) Can you find a general necessary and sufficient condition on $g \in \mathbb{C}[x,y]$ such that its homogenisation $\tilde{g} \in \mathbb{C}[x,y,z]$ does not pass through any of the three points in item (b)?

3b) from Q3a if $g \in \mathbb{C}[x,y]$ then the projective closure of its variety $\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g})$ where $\tilde{g} \in \mathbb{C}[x,y,z]$ is the homogenisation of g



- $f_1(x,y) = x+y+1 \in \mathbb{C}[x,y]$, $\tilde{f}_1(x,y,z) = x+y+z \in \mathbb{C}[x,y,z]$

  $\overline{\mathbb{V}(f_1)} = \mathbb{V}(\tilde{f}_1) = \mathbb{V}(x+y+z)$ (by Q3a)

  (i) $[1:0:0]$ not contained in projective closure
  (ii) $[0:1:0]$ not contained in projective closure
  (iii) $[0:0:1]$ not contained in projective closure

- $f_2(x,y) = x^2+6y^2+1 \in \mathbb{C}[x,y]$, $\tilde{f}_2(x,y,z) = x^2+6y^2+z^2 \in \mathbb{C}[x,y,z]$

  $\overline{\mathbb{V}(f_2)} = \mathbb{V}(\tilde{f}_2) = \mathbb{V}(x^2+6y^2+z^2)$

  (i) $[1:0:0]$ not contained in projective closure
  (ii) $[0:1:0]$ not contained in projective closure
  (iii) $[0:0:1]$ not contained in projective closure

- $f_3(x,y) = x^2+3y+1 \in \mathbb{C}[x,y]$, $\tilde{f}_3(x,y,z) = x^2+3yz+z^2 \in \mathbb{C}[x,y,z]$

  $\overline{\mathbb{V}(f_3)} = \mathbb{V}(\tilde{f}_3) = \mathbb{V}(x^2+3yz+z^2)$

  (i) $[1:0:0]$ not contained in projective closure
  (ii) $[0:1:0]$ is contained in projective closure
  (iii) $[0:0:1]$ not contained in projective closure

- $f_4(x,y) = x^3+3xy^2+4 \in \mathbb{C}[x,y]$, $\tilde{f}_4(x,y,z) = x^3+3xy^2+4z^3 \in \mathbb{C}[x,y,z]$

  $\overline{\mathbb{V}(f_4)} = \mathbb{V}(\tilde{f}_4) = \mathbb{V}(x^3+3xy^2+4z^3)$

  (i) $[1:0:0]$ not contained in projective closure
  (ii) $[0:1:0]$ is contained in projective closure
  (iii) $[0:0:1]$ not contained in projective closure

3c) general necessary and sufficient condition on $g \in \mathbb{C}[x,y]$ such that its homogenisation $\tilde{g} \in \mathbb{C}[x,y,z]$ does not pass through any of the points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$:

$\tilde{g}$ doesn't pass through the points if all $\tilde{g}(1,0,0) \neq 0$, $\tilde{g}(0,1,0) \neq 0$, $\tilde{g}(0,1,0) \neq 0$ so iff $\tilde{g}(x,y,z) = ax^n+by^n+cz^n \in \mathbb{C}[x,y,z]$

so the condition is that g must be of the form $ax^n+by^n+c \in \mathbb{C}[x,y]$ $a,b,c \in \mathbb{C}^*$

Q4. **(15 marks)**

(a) Prove that $\mathbb{P}^n$ is compact with respect to the quotient Euclidean topology from $\mathbb{A}^{n+1} \setminus \{0\}$.

(b) What is the projective Zariski-closure of the $\mathbb{V}(y - \sin(x))$ in $\mathbb{P}^2$? How do you compare this to the Chow's Lemma? **Hint.** In Example 3.44 we have seen that this curve is not algebraic.

4a) $\mathbb{P}^n = (\underbrace{\mathbb{C}^{n+1} \setminus \{0\}}_{=\mathbb{A}^{n+1}\setminus\{0\}})/\sim$ where $a, b \in \underbrace{\mathbb{C}^{n+1} \setminus \{0\}}_{=\mathbb{A}^{n+1}\setminus\{0\}}$, $a \sim b$ iff there is $\lambda \in \mathbb{C}^*$ such that $a = \lambda b$

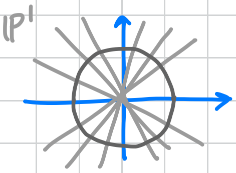
$q : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow (\mathbb{A}^{n+1} \setminus \{0\})/\sim$

$\mathbb{P}^0$

$\mathbb{C}^1 \cong \mathbb{R}^2$ $[x_0]$

unit sphere in $\mathbb{R}^2$ S^1

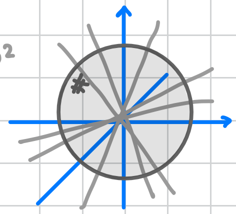
$\mathbb{P}^1$



$\mathbb{C}^2 \cong \mathbb{R}^4$ $[x_0 : x_1]$

unit sphere in $\mathbb{R}^4$ S^3

$\mathbb{P}^2$



$\mathbb{C}^3 \cong \mathbb{R}^6$ $[x_0 : x_1 : x_2]$

unit sphere in $\mathbb{R}^6$ S^5

$\mathbb{P}^n$ $\mathbb{C}^n \cong \mathbb{R}^{2n}$ unit sphere in $\mathbb{R}^{2n}$ S^{2n-1} $[x_0 : x_1 : \cdots : x_{n-1}]$

$\mathbb{P}^n = S^{2n-1}/\sim$ = unit hemisphere in $\mathbb{R}^{2n}$

map $p : S^{2n-1} \longrightarrow S^{2n-1}/\sim \; = \mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\sim$

S^{2n-1} is homeomorphic to $\mathbb{A}^{n+1} \setminus \{0\}$

is continuous and surjective so is a quotient map

S^{2n-1} is compact in the Euclidean topology $\Rightarrow$ $S^{2n-1}/\sim \; = \mathbb{P}^n$ is compact in the quotient Euclidean topology

4b)

Chow's Lemma: $X \subseteq \mathbb{P}^n$ an analytic subvariety of $\mathbb{P}^n$ (X is locally given by an analytic equation). Then if X is compact in the Euclidean topology then $X \subseteq \mathbb{P}^n$ is algebraic

Zariski-closure of $\mathbb{V}(y - \sin(x))$ in $\mathbb{P}^2$: $\overline{\mathbb{V}(y - \sin(x))} = \mathbb{P}^2$
because the smallest closed set containing $\mathbb{V}(y - \sin(x))$ is $\mathbb{P}^2$

$\overline{\mathbb{V}(y - \sin(x))}$ is analytic because $\mathbb{V}(y - \sin(x))$ is analytic

$\overline{\mathbb{V}(y - \sin(x))}$ is compact in the Euclidean topology because it is a closed subset of $\mathbb{P}^2$ and $\mathbb{P}^2$ is compact in the Euclidean topology from 4a

$\overline{\mathbb{V}(y - \sin(x))}$ is not algebraic

$\overline{\mathbb{V}(y - \sin(x))}$ is analytic and compact but not algebraic so Chow's Lemma doesn't also work for the Zariski topology

Q5. (**20 marks**)

(a) The variety of a polynomial of the form $ax + by + cz \in \mathbb{C}[x, y, z]$ for $a, b, c \in \mathbb{C}$ is called a *line* in $\mathbb{P}^2$. Prove that any two distinct lines in $\mathbb{P}^2$ intersect exactly at one point.

(b) Assume that $C_1, C_2 \subseteq \mathbb{A}^2$ are two closed affine algebraic curves.

(i) Prove that we have the inclusion $\overline{C_1 \cap C_2} \subseteq \overline{C}_1 \cap \overline{C}_2$ of projective closures.

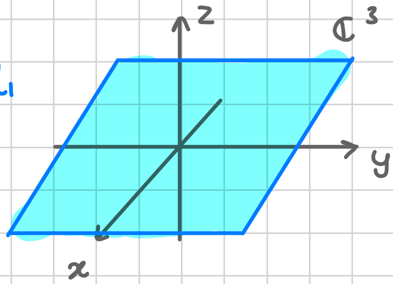
(ii) Find two curves such that the above inclusion is strict.

5a) let $l_1 = \mathbb{V}(a_1x + b_1y + c_1z)$ and $l_2 = \mathbb{V}(a_2x + b_2y + c_2z)$, $a_i, b_i, c_i \in \mathbb{C}$ be two distinct lines in $\mathbb{P}^2$

$\mathbb{P}^2 = (\mathbb{C}^3 \setminus \{0\})/\sim$ (where $a, b \in \mathbb{C}^3 \setminus \{0\}$, $a \sim b$ iff there exists $\lambda \in \mathbb{C}^*$ such that $a = \lambda \cdot b$)

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \end{array}\right]$$

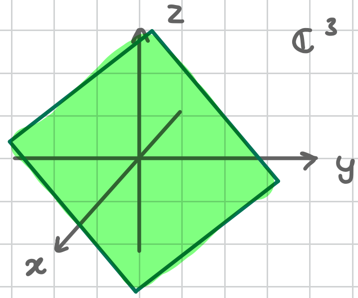
l_1



$$\left[\begin{array}{ccc|c} 1 & 0 & d_1 & e_1 \\ 0 & 1 & d_2 & e_2 \end{array}\right]$$

$$x + d_1z = e_1$$
$$y + d_2z = e_2$$

l_2

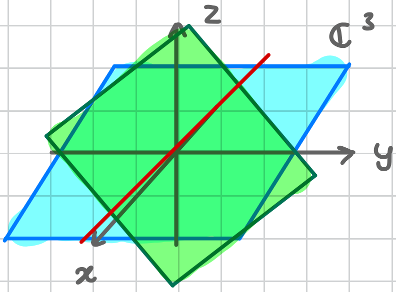


$$\left.\begin{array}{l} x + d_1z = e_1 \\ y + d_2z = e_2 \end{array}\right\} \quad x + \frac{e_2 - y}{d_2} d_1 - e_1 = 0$$

this line passes through $(0,0,0)$

$$\Rightarrow 0 + \frac{e_2 - 0}{d_2} d_1 - e_1 = 0$$

$l_1 = l_2$



$$\Rightarrow e_2d_1 - e_1d_2 = 0$$

$$\Rightarrow e_1 = \frac{e_2d_1}{d_2}$$

$$x + \frac{e_2d_1}{d_2} - \frac{d_1}{d_2}y - \frac{e_2d_1}{d_2} = 0$$

$$x + \frac{d_1}{d_2}y = 0 \quad \text{a point in } \mathbb{P}^2$$

$$\left[1 : \frac{d_1}{d_2} : 0\right]$$

Q5. (**20 marks**)

(a) The variety of a polynomial of the form $ax + by + cz \in \mathbb{C}[x, y, z]$ for $a, b, c \in \mathbb{C}$ is called a *line* in $\mathbb{P}^2$. Prove that any two distinct lines in $\mathbb{P}^2$ intersect exactly at one point.

(b) Assume that $C_1, C_2 \subseteq \mathbb{A}^2$ are two closed affine algebraic curves.

(i) Prove that we have the inclusion $\overline{C_1 \cap C_2} \subseteq \overline{C}_1 \cap \overline{C}_2$ of projective closures.

(ii) Find two curves such that the above inclusion is strict.

5b) $C_1, C_2 \subseteq \mathbb{A}^2$ are closed affine algebraic curves

5bi) $C_1 = \mathbb{V}(a_1x + b_1y + c_1)$ and $C_2 = \mathbb{V}(a_2x + b_2y + c_2)$

$\overline{C}_1 = \mathbb{V}(a_1x + b_1y + c_1z)$ and $\overline{C}_2 = \mathbb{V}(a_2x + b_2y + c_2z)$ $\subseteq \mathbb{P}^2$ by Q3a

$C_1 \cap C_2 = \mathbb{V}(a_1x + b_1y + c_1) \cap \mathbb{V}(a_2x + b_2y + c_2)$

$= \mathbb{V}(a_1x + b_1y + c_1, a_2x + b_2y + c_2)$

$\Rightarrow \overline{C_1 \cap C_2} = \overline{\mathbb{V}(a_1x + b_1y + c_1, a_2x + b_2y + c_2)}$

$= \mathbb{V}(\tilde{I}) \quad \tilde{I} = \{ \tilde{f} \in \mathbb{C}[x,y,z] : f \in (a_1x + b_1y + c_1, a_2x + b_2y + c_2) \}$

$\overline{C}_1 \cap \overline{C}_2 = \mathbb{V}(a_1x + b_1y + c_1z) \cap \mathbb{V}(a_2x + b_2y + c_2z)$

$\overline{C_1 \cap C_2}$ is the smallest closed set containing $C_1 \cap C_2$, $\overline{C}_1 \cap \overline{C}_2$ contains $C_1 \cap C_2$ when $z=1$ and is closed

$\Rightarrow \overline{C_1 \cap C_2} \subseteq \overline{C}_1 \cap \overline{C}_2$

5bii) $C_1 = \mathbb{V}(x^2 + y + 1)$ $\quad \overline{C}_1 = \mathbb{V}(x^2 + yz + z)$

$C_2 = \mathbb{V}(x + y^2 + 1)$ $\quad \overline{C}_2 = \mathbb{V}(xz + y^2 + z)$

$\overline{C}_1 \cap \overline{C}_2 = \mathbb{V}(x^2 + yz + z) \cap \mathbb{V}(xz + y^2 + z)$

$= \mathbb{V}(I) \quad I = (x^2 + yz + z, xz + y^2 + z)$

$\overline{C_1 \cap C_2} = \mathbb{V}(\tilde{I}) \quad \tilde{I} = \{ \tilde{f} \in \mathbb{C}[x,y,z] : f \in (x^2 + y + 1, x + y^2 + 1) \}$

C_1 and C_1 such that $\exists [x:y:z] \in \overline{C}_1 \cap \overline{C}_2$, $[x:y:z] \notin \overline{C_1 \cap C_2}$

$\overline{C}_1 \cap \overline{C}_2 = \mathbb{V}(a_1x + b_2y + c_1z) \cap \mathbb{V}(a_2x + b_2y + c_2z)$

$\overline{C_1 \cap C_2}$

$f = (a_1x + b_2y + c_1)h_1 + (a_2x + b_2y + c_2)h_2 \quad h_1, h_2 \in \mathbb{C}[x,y]$

Q6. (**Bonus 10 marks**)

(a) Let Y be a closed affine algebraic variety and $O \subseteq Y$ an open subset. Prove that $\mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra.

(b) A *sheaf* $\mathcal{F}$ of rings associated to a topological space X consists of the following data:

(i) To each open set $U \subseteq X$, it associates a ring $\mathcal{F}(U)$.

(ii) To each inclusion of open sets $U \hookrightarrow V$, there exists a map $\mathrm{res}_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ called the restriction map from $\mathcal{F}(V)$ to $\mathcal{F}(U)$. These maps satisfy the property that $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ and $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$, where $U \subseteq V \subseteq W$ are open sets.

These data satisfy the following properties:

(iii) Suppose that $f_i \in \mathcal{F}(U_i)$ are a collection of sections that agree on overlaps (formally, $\mathrm{res}_{U_i,U_i\cap U_j} f_i = \mathrm{res}_{U_j,U_i\cap U_j} f_j$ whenever the intersection exists). Then they lift to a section $f \in \mathcal{F}(U)$ which has the property that $\mathrm{res}_{U,U_i} f = f_i$ for all $i \in I$.

(iv) Suppose that $f, f' \in \mathcal{F}(U)$ and that $\mathrm{res}_{U,U_i} f = \mathrm{res}_{U,U_i} f'$ for all $i \in I$. Then $f = f'$.

Let X be an irreducible quasi-projective variety.

(i) Assume that U and V are open subsets of X with $U \subseteq V$. Briefly explain why $f \in \mathcal{O}_X(V)$ implies that $f_{|_U} \in \mathcal{O}_X(U)$.

(ii) Briefly explain why the collection of sets of functions $\mathcal{O}_X(U)$, where U ranges over all open subsets of X, forms a sheaf on X.

Q6a) Let Y be a closed affine algebraic variety and $O \subseteq Y$ an open subset

$\mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra proof:

$\mathcal{O}_V(U)$ = the set of regular functions on $U \subseteq V$

$\mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra if $(\mathcal{O}_Y(O),+,\cdot)$ is a ring and $(\mathcal{O}_Y(O),+)$ is also a $\mathbb{C}$-vector space

$\mathcal{O}_Y(O)$ is a ring: Subring of ring of all complex rational functions $f(x) = \frac{f_1(x)}{f_2(x)}$ $f_1(x), f_2(x)$ complex polynomials $\in \mathbb{C}[x_1,\dots,x_n]$

for $x \in O$ $id_O(x) = \frac{id_O(x)}{1} \in \mathcal{O}_Y(O) \Rightarrow \mathcal{O}_Y(O) \neq \emptyset$

$\forall f, g \in \mathcal{O}_Y(O)$, $x \in O$: $f(x) = \frac{f_1(x)}{f_2(x)}$ and $g = \frac{g_1(x)}{g_2(x)}$ then $f(x) - g(x) = \frac{f_1(x)}{f_2(x)} - \frac{g_1(x)}{g_2(x)} = \frac{f_1(x)g_2(x) - g_1(x)f_2(x)}{f_2(x)g_2(x)} \in \mathcal{O}_Y(O)$

($f_2(x)g_2(x) \neq 0$ because $f_2(x) \neq 0$ and $g_2(x) \neq 0$ $\forall x \in O$)

$\forall f, g \in \mathcal{O}_Y(O)$ $f(x)g(x) = \frac{f_1(x)}{f_2(x)} \cdot \frac{g_1(x)}{g_2(x)} = \frac{f_1(x)g_1(x)}{f_2(x)g_2(x)} \in \mathcal{O}_Y(O)$

$\Rightarrow \mathcal{O}_Y(O)$ is a ring

$\mathcal{O}_Y(O)$ is a $\mathbb{C}$-vector space:

- $\mathcal{O}_Y(O)$ is a ring so is an abelian group with respect to addition
- $1, a, b \in \mathbb{C}$ and $f(x) = \frac{f_1(x)}{f_2(x)} \in \mathcal{O}_Y(O)$ then $1 \cdot f(x) = 1 \cdot \frac{f_1(x)}{f_2(x)} = \frac{f_1(x)}{f_2(x)} = f(x)$

  and $(ab)f(x) = a \cdot (b \cdot f(x))$
- $a, b \in \mathbb{C}$ and $f(x) \in \mathcal{O}_Y(O)$ then $(a+b) \cdot f(x) = (a+b) \cdot \frac{f_1(x)}{f_2(x)} = a \frac{f_1(x)}{f_2(x)} + b \frac{f_1(x)}{f_2(x)} = a \cdot f(x) + b \cdot f(x)$
- $a \in \mathbb{C}$ and $f(x) = \frac{f_1(x)}{f_2(x)}$, $g(x) = \frac{g_1(x)}{g_2(x)} \in \mathcal{O}_Y(O)$ then $a \cdot (f(x) + g(x)) = a \cdot f(x) + a \cdot g(x)$

$\Rightarrow \mathcal{O}_Y(O)$ is also a $\mathbb{C}$-vector space

$\Rightarrow \mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra

Q6. (**Bonus 10 marks**)

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6bi) U, V open $\quad U \subseteq V \subseteq X$

$f \in \mathcal{O}_X(V)$ then there is an open neighbourhood $V' \subseteq V$ for every point in V and polynomials $g, h \in \mathbb{C}[x_1, \ldots, x_n]$ such that $h(p) \neq 0$ for any $p \in V'$ and $f|_{V'}(p) = \frac{g(p)}{h(p)}$

then for every point in $p \in V$ that is contained in a neighbourhood V' then $U \cap V'$ is an open neighbourhood in U containing p.
so for every point in $U \cap V' \subseteq U$ there are polynomials $g, h \in \mathbb{C}[x_1, \ldots, x_n]$ such that $h(p) \neq 0$ and $f|_{U \cap V'}(p) = \frac{g(p)}{h(p)}$ for any $p \in V' \Rightarrow$ for any $p \in U \cap V' \subseteq U$

so $f \in \mathcal{O}_X(V) \Rightarrow f|_U \in \mathcal{O}_X(U)$

6bii) the collection of sets of functions $\mathcal{O}_X(U)$ where U ranges over all open subsets of X forms a sheaf on X because
for $f \in \mathcal{O}_X(U)$ and $f' \in \mathcal{O}_X(U')$ then $f(p) = f'(p)$ for any $p \in U \cap U'$

and if $f(p) = f'(p)$ for any $p \in U_i \cap U_j$, $f(p) \in \mathcal{O}_X(U_i)$ $f'(p) \in \mathcal{O}_X(U_j)$

then for $f'' \in \mathcal{O}_X(U_i \cap U_j)$, $f''|_{U_i} = f_i$ and $f''|_{U_j} = f_j$