Assessed 2

23 April 2024 16:06

63/100

Q1. (a) (**10 marks**) Find all the elements of maxSpec($\mathbb{C}[x]$) and maxSpec($\mathbb{C}[x, 1/x]$), respectively.

(b) (**10 marks**) Consider the isomorphism $\varphi : \mathbb{A}^1 \setminus \{0\} \longrightarrow \mathbb{A}^1 \setminus \{0\}$, $a \longmapsto b = 1/a$, and the pullback map on the coordinate rings $\varphi^* : \mathbb{C}[x, 1/x] \longmapsto \mathbb{C}[y, 1/y]$. Compute $\varphi^*(1/x)$, $\varphi^*(2x^2 + \frac{2x^3+4x}{x^5})$, $\varphi^*(2 - x)$.

a) $\text{maxSpec}(\mathbb{C}[x]) = \{ m \leq \mathbb{C}[x] \mid m \text{ a maximal ideal} \}$

$\frac{10}{10}$

($\mathbb{C}$ a field $\Rightarrow \mathbb{C}[x]$ a P.I.D) $= \{ (f) \leq \mathbb{C}[x] \mid (f) \text{ maximal} \}$

($\mathbb{C}[x]$ an integral domain) $= \{ (f) \leq \mathbb{C}[x] \mid f \text{ irreducible} \}$

($\mathbb{C}$ algebraically closed) $= \{ (f) \leq \mathbb{C}[x] \mid \deg f = 1 \}$

$= \{ (x-\alpha) \leq \mathbb{C}[x] \mid \alpha \in \mathbb{C} \}$.

Consider the $\mathbb{C}$-algebra homomorphism:

$\varphi : \mathbb{C}[x, y] \longrightarrow \mathbb{C}[x, 1/x]; x \mapsto x, \ y \mapsto 1/x$.

$\Rightarrow \varphi(xy) = x \cdot 1/x = 1, \Rightarrow \varphi(xy - 1) = 0$.

$\Rightarrow \ker \varphi \supseteq (xy-1)$, So by Homomorphism theorem:

$\text{im}\,\varphi = \mathbb{C}[x, 1/x] \cong \frac{\mathbb{C}[x,y]}{\ker \varphi} = \mathbb{C}[x,y]/(xy-1) =: \mathbb{C}[V]$.

~~Since $\mathbb{I}(V) = (xy-1)$, and $\mathbb{V}(\mathbb{I}(V)) = V$, we determine that~~

$V = \mathbb{V}((xy-1)) = \mathbb{V}(xy-1)$.

$\alpha \cdot \beta - 1 = 0 \Rightarrow \alpha\beta = 1 \Rightarrow \beta = \alpha^{-1} \Rightarrow \alpha, \beta \in \mathbb{C}^*$.

So $V = \mathbb{V}(xy-1) = \{ (\alpha, \alpha^{-1}) \mid \alpha \in \mathbb{C}^* \}$

So maximal ideals of $\mathbb{C}[V]$ are of the form $(x-\alpha, y-\alpha^{-1})$ for $\alpha \in \mathbb{C}^*$ (since we consider $\mathbb{C}[x,y]|_V$).

$xy - 1 = 0 \Rightarrow xy = 1$, so $(x-\alpha, y-\alpha^{-1})$

$y(x-\alpha) = 1 - \alpha y \Rightarrow -\alpha^{-1} y(x-\alpha) = \alpha^{-1}(\alpha y - 1) = y - \alpha^{-1}$

$\Rightarrow y - \alpha^{-1} \in (x-\alpha) \Rightarrow (x-\alpha) = (x-\alpha, y-\alpha^{-1})$.

Thus. $\text{maxSpec}(\mathbb{C}[V]) = \text{maxSpec}(\mathbb{C}[x,y]/(xy-1)) = \{ (x-\alpha) \mid \alpha \in \mathbb{C}^* \}$.

Thus, $\operatorname{maxSpec}(\mathbb{C}[V]) = \operatorname{maxSpec}\left(\mathbb{C}[x,y]/(xy-1)\right) = \{(x-\alpha) \mid \alpha \in \mathbb{C}^*\}$,

So $\operatorname{maxSpec}(\mathbb{C}[x, 1/x]) = \{(x-\alpha) \mid \alpha \in \mathbb{C}^*\}$, by the isomorphism $\varphi|_V$. ✓

b) Let $\alpha \in \mathbb{A}^1 \setminus \{0\}$.

10/10

$$\varphi^*(1/x)(\alpha) = 1/x \circ \varphi(\alpha) = 1/x\left(\tfrac{1}{\alpha}\right) = \frac{1}{1/\alpha} = \alpha.$$

So $\varphi^*(1/x) = \mathrm{id}_{\mathbb{A}^1 \setminus \{0\}} \implies \underline{\varphi^*(1/x) = y} \in \mathbb{C}[y, 1/y]$

Given that $\varphi^*(1/x) = y$, and that φ^* is a $\mathbb{C}$-algebra homomorphism:

$$\varphi^*(1/x) = y \implies \varphi^*(x) = \varphi^*\left(\left(1/x\right)^{-1}\right) = \varphi^*(1/x)^{-1} = y^{-1} = 1/y.$$

Thus, additionally given that $\varphi^*(x) = 1/y$:

$$\varphi^*\left(2x^2 + \frac{2x^3+4x}{x^5}\right) = \varphi^*\left(2x^2 + \frac{2x^3}{x^5} + \frac{4x}{x^5}\right) = \varphi^*\left(2x^2 + 2\cdot\frac{1}{x^2} + 4\cdot\frac{1}{x^4}\right)$$

$$= 2\varphi^*(x)^2 + 2\varphi^*(1/x)^2 + 4\varphi^*(1/x)^4$$

$$= 2\left(\tfrac{1}{y}\right)^2 + 2y^2 + 4y^4 = \underline{4y^4 + 2y^2 + 2y^{-2}},$$

and $\varphi^*(2-x) = \varphi^*(2) - \varphi^*(x) = \underline{2 - 1/y}$. ✓□

Q2. Consider the affine algebraic hypersurface $V := \mathbb{V}(y - ux) \subseteq \mathbb{A}^3$.

(a) (**10 marks**) Prove that the projection $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$, $(x, y, u) \longmapsto (x, u)$ restricts to an isomorphism between V and $\mathbb{A}^2$.

(b) (**10 marks**) Prove that the projection $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$, $(x, y, u) \longmapsto (x, y)$ does not restrict to isomorphism between V and $\mathbb{A}^2$.

(c) (**10 marks**) Find $\mathcal{O}_V(D(u))$.

a) Let $\Phi : \mathbb{A}^3 \to \mathbb{A}^2;\ (x,y,u) \longmapsto (x,u)$, and $\Phi|_V =: \varphi : V \to \mathbb{A}^2$.

Since Φ is a polynomial mapping, φ is a morphism of algebraic varieties.

10/10

Define $\psi : \mathbb{A}^2 \to \mathbb{A}^3;\ (\alpha, \beta) \longmapsto (x,y,u) := (\alpha, \alpha\beta, \beta)$

$$(x,y,u) \in V \implies y - ux = 0 \implies y = ux \implies (x,y,u) = (x, ux, u)$$
$$\implies \psi(x,u) = (x,y,u).$$
$$\implies V \subseteq \operatorname{im}\psi$$

For $(x,y,u) = (\alpha, \alpha\beta, \beta) = \psi(\alpha,\beta)$, $y - ux = 0 = \alpha\beta - \beta\alpha = 0$,
$$\implies \psi(\alpha,\beta) \in V$$
$$\implies \operatorname{im}\psi \subseteq V \implies \operatorname{im}\psi = V.$$

$$\Rightarrow \psi(\alpha,\beta) \in V$$
$$\Rightarrow \operatorname{im}\psi \subseteq V \Rightarrow \operatorname{im}\psi = V.$$

So $\psi : \mathbb{A}^2 \to V$.

$\forall (\alpha,\beta) \in \mathbb{A}^2$, $\varphi \circ \psi(\alpha,\beta) = \varphi(\alpha, \alpha\beta, \beta) = (\alpha,\beta)$, so $\varphi\circ\psi = id_{\mathbb{A}^2}$.

$\forall (x,y,u) \in V$, $y - ux = 0 \Rightarrow y = xu$

$$\Rightarrow \psi\circ\varphi(x,y,u) = \psi(x,u) = (x, xu, u) = (x,y,u), \text{ so } \psi\circ\varphi = id_V.$$

So by definition, $\varphi = \Phi|_V$ is an isomorphism between V & $\mathbb{A}^2$ ✓

b) $V = \{(x,y,u) \in \mathbb{A}^3 \mid y = ux\}$. Define $\psi : V \to \mathbb{A}^2 ; (x,y,u) \mapsto (x,y)$. $= (x, ux)$

10/10

$(x,y,u) = (0,0,\alpha)$ satisfies $y = ux$ $\forall \alpha \in \mathbb{A}^1$, as $0 = \alpha \cdot 0$.

Therefore $\psi(0,0,\alpha) = (0,0)$ $\forall \alpha \in \mathbb{C}$ (Specifically, it is defined).

If for $\alpha \in \mathbb{C}$, $\theta : \mathbb{A}^2 \to V$ s.t. $\theta\circ\psi(0,0,\alpha) = \theta(0,0) = (0,0,\alpha)$,
then $\forall \beta \neq \alpha$, $\theta\circ\psi(0,0,\beta) = \theta(0,0) = (0,0,\alpha) \neq (0,0,\beta)$.

Thus, $\nexists\, \theta : \mathbb{A}^2 \to V$ s.t. $\theta\circ\psi = id_V$, so the restriction of the projection $\mathbb{A}^3 \to \mathbb{A}^2 ; (x,y,u) \mapsto (x,y)$ to V, ψ, cannot be an isomorphism. ✓

c) We have that $D(u) = \mathbb{A}^3 \setminus \mathbb{V}(u)$

0/10

$$\mathbb{V}(u) \cancel{=} \{(\alpha,0,0) \mid \alpha \in \mathbb{C}\} \cup \{(0,0,\beta) \mid \beta \in \mathbb{C}\}.$$

$\mathbb{V}(u) = \{(\alpha,\beta,0) \mid \alpha,\beta \in \mathbb{C}\}$

So $D(u) \cancel{=} \{(x,y,u) \in (\mathbb{A}^1 \setminus \{0\})^3 \mid y = ux\}$.

$(0,0,1) \in D(u)$

Q3. (**20 marks**) Prove that if V is an irreducible affine variety, then so is its projective closure $\overline{V}$.

4/20

$V \subseteq \mathbb{A}^n$ irreducible $\iff \mathbb{I}(V)$ prime $\Rightarrow \mathbb{I}(V)$ radical

$$\mathbb{I}(V) = I := (f_1, \dots, f_k), \text{ for some } f_i\text{'s. (Hilbert's Basis theorem)}$$

So $\forall a,b \in \mathbb{C}[x_1,\dots,x_n]$, $ab \in I \Rightarrow a \in I$ or $b \in I$.

We have $\check{I} = (\{\check{f} \in \mathbb{C}[x_0,x_1,\dots,x_n] \mid f \in I\})$

By Ex 3.21, $\check{I}$ is radical

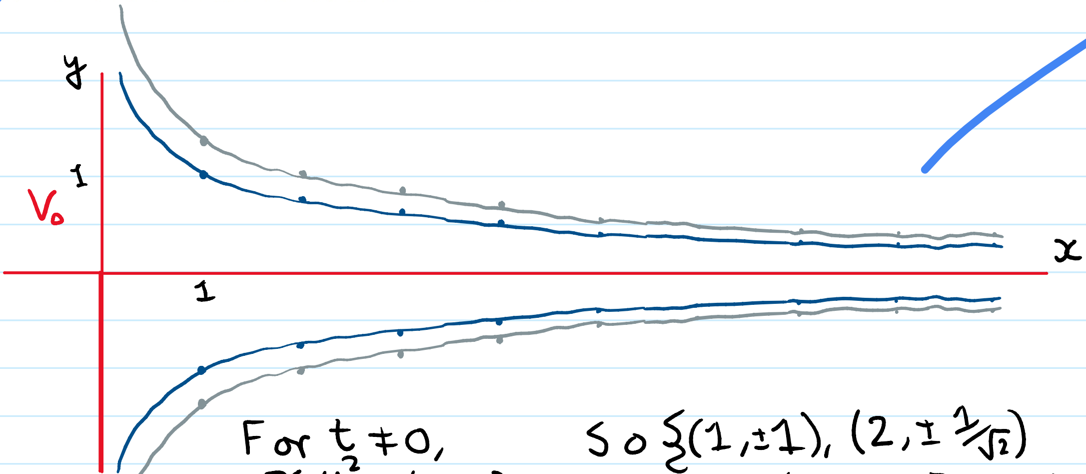
Consider $g, h \in \mathbb{C}[x_0, x_1, \dots, x_n]$, such that $gh \in \hat{I}$.
So $\exists \{f_i\}_{i=1}^{k}$ such that $gh = \cancel{\sum_{i=1}^{k} f_i}$. $\sum_{i=1}^{k} h_i \tilde{f}_i$ for some $h_i \in \mathbb{C}[x_0, \dots, x_n]$.

Q5. (**20 marks**) Consider the family of algebraic varieties, with parameter $t \in \mathbb{C}$, given by

$$V_t := \mathbb{V}(xy^2 - t) \subseteq \mathbb{A}^2.$$

19/20

Sketch the variety of V_0, V_1, and V_2 in $\mathbb{R}^2$. Which one of these varieties are smooth? Which one of these varieties are irreducible?



For $t \neq 0$,
$xy^2 - t = 0$
$\Rightarrow xy^2 = t$
$\Rightarrow y^2 = t/x$
$\Rightarrow y = \pm\sqrt{t/x}$

So $\{(1, \pm 1), (2, \pm 1/\sqrt{2}), (3, \pm 1/\sqrt{3}), \dots\} \subseteq V_1$

$\{(1, \pm\sqrt{2}), (2, \pm \sqrt{2}/\sqrt{2}), (3, \pm \sqrt{2}/\sqrt{3}), \dots\} \subseteq V_2$

For $t = 0$, $xy^2 - 0 = 0 \Leftrightarrow xy^2 = 0 \Leftrightarrow x = 0$ or $y^2 = 0$
$\Leftrightarrow x = 0$ or $y = 0$.

For $f_t = xy^2 - t$, $\nabla f = (y^2, 2xy)$.

For $t \neq 0$ $(a,b) \in V_t \Rightarrow ab^2 - t = 0 \Rightarrow ab^2 = t \neq 0$
$\Rightarrow a, b^2 \neq 0 \Rightarrow a, b \neq 0$
$\Rightarrow (b^2, 2ab) \neq (0,0)$.
So 1 linearly independent vector.

So $\nabla f_t(a,b) : \mathbb{A}^2 \to \mathbb{A}^1 ; \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto b^2 v_1 + 2ab\, v_2$
is of rank 1, so

is of rank 1, so

$$\text{rank} + \text{nullity} = \dim \mathbb{A}^2$$
$$1 + \underline{1} = 2$$

Thus $\forall (a,b) \in V_t$ for $t \neq 0$, $\dim \operatorname{Ker} \nabla f_t(a,b) = 1$

So $\forall x \in V_t, t \neq 0$, V_t is smooth of dimension 1 at x.

Thus, by definition, V_1, V_2 are smooth, of dimension 1.

For $t=0$, $(0,0), (0,1) \in V_0$, so:

$$\nabla f_t(0,0)\begin{pmatrix}v_1\\v_2\end{pmatrix} = (0^2, 2\cdot 0\cdot 0)\begin{pmatrix}v_1\\v_2\end{pmatrix} = (0\ \ 0)\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

So $\text{rank}\, \nabla f_t(0,0) = 0$

$\Rightarrow \dim \ker \nabla f_t(0,0) = 2$

$$\nabla f_t(0,1)\begin{pmatrix}v_1\\v_2\end{pmatrix} = (1^2, 2\cdot 0\cdot 1)\begin{pmatrix}v_1\\v_2\end{pmatrix} = (1\ \ 0)\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

So $\text{rank}\, \nabla f_t(0,0) = 1$

$\Rightarrow \dim \ker \nabla f_t(0,1) = 1$.

Thus, V_0 is not smooth, as it is at different points in V_0, it is smooth of differing dimensions.

Why is f_t irred?

Since $\deg f_t = 3$, f_t irreducible $\Rightarrow f_t$ has a root

Our graph shows that f_1, f_2 have no roots, so.

The "graphs" you sketched consist on the roots!

f_1, f_2 irreducible $\Rightarrow f_1, f_2$ prime

$\Rightarrow (f_1), (f_2)$ prime

Notice that this is not true in general; here we are lucky that C[x,y] is a UFD.

$\Rightarrow (f_1), (f_2)$ radical

$\Rightarrow \mathbb{I}(\mathbb{V}(f_1)) = (f_1)$, $\mathbb{I}(\mathbb{V}(f_2)) = (f_2)$ prime

$\Leftrightarrow V_1 = \mathbb{V}(f_1)$, $V_2 = \mathbb{V}(f_2)$ irreducible.

What about V_0?