(a) $\mathrm{MA}[\mathfrak{q}](\mathbb{R}^n) = \mathrm{Vol}_n(\Delta_{\mathfrak{q}})$, where $\Delta_{\mathfrak{q}}$ is the Newton polytope of $\mathfrak{q}$.
(b) (Tropical Bernstein Theorem) $\widetilde{\mathrm{MA}}[\mathfrak{q}_1, \dots, \mathfrak{q}_n](\mathbb{R}^n) = \widetilde{\mathrm{Vol}}(\Delta_{\mathfrak{q}_1}, \dots, \Delta_{\mathfrak{q}_n})$, where $\widetilde{\mathrm{Vol}}$ is the mixed volume.

Corollary 5.16. Assume that $\alpha_i, \beta_i \in \mathbb{Z}^n$ for $i = 1, \dots, n$. Let $\mathfrak{q}_i = \max\{\langle \alpha_i, x\rangle, \langle \beta_i, x\rangle\}$ be n tropical polynomials. Then,

$$n!\,\widetilde{\mathrm{MA}}[\mathfrak{q}_1, \dots, \mathfrak{q}_n] = \kappa\,\delta_0,$$

where κ is given by the volume *zonotope* of the Minkowski sum of the vectors $\sum_{i=1}^n [\alpha_i - \beta_i]$.

Proof. Note that $\Delta_{\mathfrak{q}_i}$ is the line segment between α_i and β_i. Moreover, in the definition of $\widetilde{\mathrm{MA}}[\mathfrak{q}_1, \dots, \mathfrak{q}_n]$ only $\mathrm{Vol}(\sum_{i=1}^n [\alpha_i - \beta_i])$ possibly has a non-zero n-dimensional volume. Finally, the origin is the only 0-dimensional cell of the tropical variety of polynomial $\mathfrak{q}_1 + \cdots + \mathfrak{q}_n$, if and only if, $\{\alpha_1 - \beta_1, \cdots, \alpha_n - \beta_n\}$ forms a linearly independent set. Therefore, $n!\,\mathrm{MA}[\mathfrak{q}_1 + \cdots + \mathfrak{q}_n] = \kappa\,\delta_0$. □

6. Slicing tropical currents

Proposition 6.1. Let $\mathcal{C}$ be a p-dimensional tropical cycle in $\mathbb{R}^n$, and $S \subseteq (\mathbb{C}^*)^n$ be an algebraic hypersurface with transversal intersection with $\mathcal{T}_{\mathcal{C}}$. Then, $[S] \wedge \mathcal{T}_{\mathcal{C}}$ is admissible and it is a closed positive current of bidimension $(p-1, p-1)$ given by

$p \geq 1$

$$[S] \wedge \mathcal{T}_{\mathcal{C}} = \sum_{\sigma \in \mathcal{C}} w_\sigma \int_{x \in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[S \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\big]\, d\mu(x).$$

Proof. The idea of the proof is similar to that of [BH17, Proposition 4.11]. Let f be a defining function ~~the equation~~ of S in $(\mathbb{C}^*)^n$. Assume that $\mathrm{Log}^{-1}(\sigma^\circ) \cap S \neq \varnothing$, for a p-dimensional cone $\sigma \in \mathcal{C}$, then for each fiber $\pi_\sigma^{-1}(x)$, the transversality assumption allows for application of the Lelong–Poincaré formula to deduce

is σ a cone?

π_σ ?

$$dd^c\big(\log|f|\,\mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\big[\pi_\sigma^{-1}(x)\big]\big)$$
$$= \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)}\big[S \cap \pi_\sigma^{-1}(x)\big] + \mathcal{R}_\sigma(x)$$

comma

where $\mathcal{R}_\sigma(x)$ is a $(p-1, p-1)$-bidimensional current. The support of $\mathcal{R}_\sigma(x)$ lies in the boundary of $\mathrm{Log}^{-1}(\sigma)$, as $\mathcal{R}_\sigma(x)$ is the difference of two currents that coincide in any set of form $\mathrm{Log}^{-1}(B)$, where $B \subseteq \mathbb{R}^n$ is a small ball with

should it be
$B \cap \sigma' = \emptyset$
$\forall \sigma' \in \mathcal{C}, \sigma' \neq \sigma$

$$B \cap \sigma^\circ \neq \varnothing, \quad B \cap \partial\sigma = \varnothing,$$

and both vanish outside $\mathrm{Log}^{-1}(\sigma)$. Integrating along the fibers, and adding for all p-dimensional cones $\sigma \in \mathcal{C}$, we obtain

$$[S] \wedge \mathcal{T}_{\mathcal{C}} = \sum_{\sigma \in \mathcal{C}} w_\sigma \int_{x \in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \big[S \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\big]\, d\mu(x) + \mathcal{R}_{\mathcal{C}},$$

where $\mathcal{R}_{\mathcal{C}}$ is $(p-1, p-1)$-dimensional current. We claim that $\mathcal{R}_{\mathcal{C}}$ is *normal*, i.e. $\mathcal{R}_{\mathcal{C}}$ and $d\mathcal{R}_{\mathcal{C}}$ have measure coefficients; $\mathcal{R}_{\mathcal{C}}$ is a difference of two normal currents, where the first current $[S] \wedge \mathcal{T}_{\mathcal{C}}$ is a positive closed current, and the second current is an addition of normal pieces. Moreover, the support of $\mathcal{R}_{\mathcal{C}}$ is a subset of S as it is a difference of two currents that both vanish outside S. As a result, the current $\mathcal{R}_{\mathcal{C}}$ is supported on

$S \cap \bigcup_\sigma \partial \mathrm{Log}(\sigma)$. This set is a real manifold of Cauchy–Riemann dimension less than $p-1$, therefore by Demailly's first theorem of support the normal current $\mathcal{R}_{\mathcal{C}}$ vanishes; see also the discussion following [BH17, Proposition 4.11]. ☐

Corollary 6.2. Let $H \subseteq \mathbb{R}^n$ be a rational plane of dimension r and $A := a + H$, a translation of H for $a \in \mathbb{R}^n$. Assume also that $\mathcal{C} \subseteq \mathbb{R}^n$ is a tropical variety of dimension p that intersects A transversely. Then

$r \geq n-p$

$$[(e^{-a})T_{H\cap \mathbf{Z}^n}] \wedge \mathcal{T}_{\mathcal{C}}$$

can be viewed as a tropical current of dimension $p-(n-r)$ in the complex subtori $T^A := (e^{-a})\, T_{H \cap Z^n} \subseteq (\mathbf{C}^*)^n$.

Proof. Note that the hypothesis implies that the intersection $T^A \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ is transversal for any $x \in S_{N(\sigma)}$. By translation, it is sufficient to prove the statement for $a = 0$. By preceding theorem,

$$[T^A] \wedge \mathcal{T}_{\mathcal{C}} = \sum_{\sigma \in \mathcal{C}} w_\sigma \int_{x \in S_{N(\sigma)}} \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \, [T^A \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)] \; d\mu(x).$$

The sets $T^A \cap \pi^{-1}_{\mathrm{aff}(\sigma)}(x)$ can be understood as a translation toric sets in T^A and $d\mu_\sigma(x)$ are Haar measures, which imply the assertion. ☐

maybe keep the notation $\mu(x)$

Theorem 6.3. Let $M \subseteq (\mathbf{C}^*)^{n-p}$ and $N \subseteq (\mathbf{C}^*)^p$ be two bounded open subsets such that N contains the real torus $(S^1)^p$. Let $\pi : M \times N \longrightarrow M$ be the canonical projection. Let $\mathcal{T}_n$ be a sequence of positive closed (p,p)-bidimensional currents on $M \times N$ such that $\mathrm{supp}(\mathcal{T}_n) \cap (M \times \partial \overline{N}) = \varnothing$. Assume that $\mathcal{T}_n \longrightarrow \mathcal{T}$ and $\mathrm{supp}(\mathcal{T}) \subseteq M \times (S^1)^p$. Then we have the following convergence of slices

$$\langle \mathcal{T}_n | \pi | x \rangle \longrightarrow \langle \mathcal{T} | \pi | x \rangle \quad \text{for every } x \in M.$$

Note that all the above slices are well-defined for all $x \in M$.

Proof. Since all the currents $\mathcal{T}_n$ and $\mathcal{T}$ are horizontal-like, the slices are well-defined, and we prove that the slices have the same cluster value. Let $\mathcal{S}$ be any cluster value of $\langle \mathcal{T}_n | \pi | x \rangle$. Note that such $\mathcal{S}$ always exists by Banach–Alaoglu theorem. As both measures $\mathcal{S}$ and $\langle \mathcal{T} | \pi | x \rangle$, are supported $\{x\} \times (S^1)^p$, to prove their equality, it suffices to prove that they have the same Fourier coefficients. By Theorem 2.11, we have

$$\langle \mathcal{S}, \phi \rangle \leq \langle \mathcal{T} | \pi | x \rangle (\phi),$$

for every plurisubharmonic function ϕ on $\mathbf{C}^n$, and the mass of $\mathcal{S}$ coincides with the mass of $\langle \mathcal{T} | \pi | x \rangle$. Now, note that if ϕ is pluriharmonic, then $-\phi$ and ϕ are plurisubharmonic. As a result,

$$\langle \mathcal{S}, \phi \rangle = \langle \mathcal{T} | \pi | x \rangle (\phi),$$

for every pluriharmonic function. Recall that if f is a holomorphic function, then $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are pluriharmonic. We now consider the elements of the Fourier basis $f(\theta) = \exp 2\pi i \langle \nu, \theta \rangle$ for $\nu \in \mathbb{Z}^n$. Then we have the equality

$$\langle \mathcal{S}, f \rangle = \langle \mathcal{T} | \pi | x \rangle (f).$$

This implies that the Fourier measure coefficients of both $\mathcal{S}$ and $\langle \mathcal{T} | \pi | x \rangle$ coincide. ☐

Lemma 6.4. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a tropical variety of dimension p, and L be a rational $(n-p)$-dimensional plane such that L is transveral to all the affine extensions $\text{aff}(\sigma)$ for $\sigma \in \mathcal{C}$. Assume that $\mathcal{T}$ be a positive closed current of bidimension (p,p) on a smooth projective toric variety of X_Σ compatible with $\mathcal{C} + L$ such that $\text{supp}(\mathcal{T}) \subseteq \text{supp}(\mathcal{T}_{\mathcal{C}})$. Further, for all $a \in \mathbb{R}^n$,

$$\overline{\mathcal{T}}_{L+a} \wedge \mathcal{T} = \overline{\mathcal{T}}_{L+a} \wedge \overline{\mathcal{T}}_{\mathcal{C}}.$$

Then $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$ in $(\mathbb{C}^*)^n$.

Proof. Let us first remark that $\text{rec}(L+a) = \text{rec}(L)$ for all $a \in \mathbb{R}^n$ and therefore, all $\mathcal{T}_{a+L}$ are compatible with X_Σ and have a continuous super-potenrial in X_Σ and as a result, all the above wedge products are well-defined.

By Demailly's second theorem of support [Dem, III.2.13], there are measures μ_σ such that

$$\mathcal{T} = \sum_{\sigma} \int_{x \in S(Z^n \cap H_\sigma)} \mathbb{1}_{\text{Log}^{-1}(\sigma^\circ)} [\pi_\sigma^{-1}(x)]\, d\mu_\sigma^{\mathcal{T}}(x).$$

not introduced.

By repeated application of Proposition 6.1,

$$\mathcal{T}_L \wedge \mathcal{T} = \sum_{\sigma} \int_{(x,y) \in S(\mathbb{Z}^n \cap H_L) \times S(\mathbb{Z}^n \cap H_\sigma)} [\pi_H^{-1}(x) \cap \pi_\sigma^{-1}(y)]\, d\mu_L(x) \otimes \mu_\sigma^{\mathcal{T}}(y).$$

Applying both sides of the equality $\mathcal{T}_L \wedge \mathcal{T} = \mathcal{T}_L \wedge \mathcal{T}_{\mathcal{C}}$ on test-functions of the form

$$\omega_\nu = \exp(-i\langle \nu, \theta \rangle)\rho(r)$$

introduce the coordinates r.

where $\rho : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support and $\theta \in [0, 2\pi)^n$, and $\nu \in \mathbb{Z}^n$, completely determines the Fourier coefficients of $\mu_\sigma^{\mathcal{T}}$ which have to coincide with the normalised Haar measures multiplied by the weight of σ, *i.e.*, $\mu_\sigma^{\mathcal{T}} = w_\sigma \mu_\sigma$. □

Note that any subtorus of $(\mathbb{C}^*)^n$, can be understood as a fibre of a tropical current. We have the following slicing theorem.

Theorem 6.5. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a tropical variety and $A \subseteq \mathbb{R}^n$ be a rational hyperplane intersecting $\mathcal{C}$ transversely. Let Σ be a fan compatible with $\mathcal{C} + A$. Assume that $\overline{\mathcal{S}}_n$ is a sequence of positive closed currents on X_Σ, and denote by $\mathcal{S}_n$ the restriction to T_N. Further,

- $\overline{\mathcal{S}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}}$;
- $\text{supp}(\overline{\mathcal{S}}_n) \longrightarrow \text{supp}(\overline{\mathcal{T}}_{\mathcal{C}})$.

Then We have that

$$\lim_{n \to \infty} \left(\mathcal{S}_n \wedge [T^A]\right) = \mathcal{T}_{\mathcal{C}} \wedge [T^A],$$

as currents on $T_N \subseteq X_\Sigma$.

Proof. Assume that $L \subseteq \mathbb{R}^n$ is an $(n-p-1)$-dimensional affine plane intersecting all $\text{aff}(\sigma)$ for all $\sigma \in \mathcal{C} \cap A$ transversely. Then, on a projective smooth toric variety $X_{\Sigma'}$ compatible with $\mathcal{C} + L + A$ the tropical currents $\overline{\mathcal{T}}_{a+L}$, $a \in \mathbb{R}^n$ have continuous super-potentials. Therefore, by Proposition 2.6, we have

$$\lim_{m \to \infty} \left(\overline{\mathcal{S}}_n \wedge \overline{\mathcal{T}}_{a+L}\right) = \overline{\mathcal{T}}_{\mathcal{C}} \wedge \overline{\mathcal{T}}_{a+L}.$$

n

So you assume $A = H$?

$a + L = L$

Now, for any $x \in \mathcal{C} \cap L \cap A$, let $B \subseteq \mathbb{R}^n$ containing x be a bounded open set containing only x as an isolated point of the intersection. By a translation we can assume that $x = 0$. Let H be the linear space parallel to A, and

$$\xi : (\mathbb{C}^*)^n \xrightarrow{\sim} T_{\mathbf{Z}^n/(\mathbf{Z}^n \cap H)} \times T_{\mathbf{Z}^n \cap H}$$

$\mathbb{Z}$

be the isomorphism, and π_1 and π_2 be the respective projections. Note that for $x \in S^1_{\mathbf{Z}^n/(\mathbf{Z}^n\cap H)}$, we have $\pi_1^{-1}(1) = T^A$. We now set

~~natural~~ is it unique ?

we don't need x ?

$$\begin{aligned}
U &:= \pi_1 \circ \xi\big(\mathrm{Log}^{-1}(U) \cap \mathrm{supp}(\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{a+L})\big)\\
V &:= \pi_2 \circ \xi\big(\mathrm{Log}^{-1}(U) \cap T^A\big),\\
\mathcal{T}_n &:= \xi_*(\mathcal{S}_n \wedge \mathcal{T}_{a+L}), \text{ in } T_N,\\
\mathcal{T} &:= \xi_*(\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{a+L}).
\end{aligned}$$

maybe : as in the setting of Theorem 6.3

Therefore, for large n, $\mathcal{T}_n$ and $\mathcal{T}_{\mathcal{C}}$ are horizontal-like. By Theorem 6.5, we obtain

$$\lim_{n\to\infty} (\mathcal{S}_n \wedge [T^A]) \wedge \mathcal{T}_{a+L} = \mathcal{T}_{\mathcal{C}} \wedge [T^A] \wedge \mathcal{T}_{a+L},$$

for every a. We now deduce the convergence on $X_{\Sigma'}$ by Lemma 6.4. ~~Finally the convergence on $(\mathbb{C}^*)^n \simeq T_N$ follows from restriction.~~ T_N ? □

Theorem 6.6. In the situation of Theorem 6.5,

$$\lim_{n\to\infty} (\mathcal{S}_n \wedge [\overline{T}^A]) = \overline{\mathcal{T}}_{\mathcal{C}} \wedge [\overline{T}^A],$$

what is W ?

where the extension is considered in a smooth projective toric variety X_Σ compatible with $\mathrm{trop}(\mathcal{W}) + A$.

Lemma 6.7. Let $U \subseteq \mathbb{C}^n$ be an open subset and D be an analytic subset of $\mathbb{C}^n$. Assume that we have the convergence of closed positive currents $\mathcal{V}_n \longrightarrow \mathcal{V}$ in $U \setminus D$, and $\mathcal{V}_n$'s and $\mathcal{V}$ have a finite local mass near D. Further, assume that for any cluster value $\mathcal{W}$ of the sequence $\{\overline{\mathcal{V}}_n\}_n$, $\mathcal{W}$ we have

uniformly bounded local masses

(a) $\mathrm{supp}(\mathcal{W}) \subseteq \mathrm{supp}(\overline{\mathcal{V}})$,
(b) $\mathrm{supp}(\overline{\mathcal{V}}) \cap D$ has the expected Cauchy–Riemann dimension.

Then

$$\overline{\mathcal{V}}_n \longrightarrow \cancel{\mathcal{W}} = \overline{\mathcal{V}}.$$

$\mathcal{W} - \overline{\mathcal{V}}$ is positive closed

Proof. $\overline{\mathcal{V}} - \mathcal{W}$ has the Cauchy–Riemann dimension less than or equal to p, therefore, it must be zero. (Demailly again) □

Proof of Theorem 6.6. Applying Theorem 5.2 (or [OP13, Proposition 3.3.2] to each fibre of $\overline{\mathcal{T}}_{\mathcal{C}}$ separately), we obtain $\mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}}) \cap \overline{T}_A \cap [D_\rho]$ has the expected Cauchy–Riemann dimension $p-2$. By Demailly's first theorem of support [Dem, Theorem III.2.10] $\overline{\mathcal{S}}_{\mathcal{C}} \wedge [\overline{T}_A] = \overline{\mathcal{T}_{\mathcal{C}} \wedge [T_A]}$. By assumption $\overline{\mathcal{S}}_n \longrightarrow \overline{\mathcal{T}}_{\mathrm{trop}(\mathcal{W})}$ and $\mathrm{supp}(\mathcal{T}_n) \longrightarrow \mathrm{supp}(\overline{\mathcal{T}}_{\mathrm{trop}(\mathcal{W})})$. ? The observation in Lemma 2.12,

$$\lim_{n\to\infty} \mathrm{supp}(\overline{\mathcal{S}}_n \wedge [\overline{T}^A]) \subseteq \mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}} \wedge [\overline{T}^A]).$$

Therefore, any cluster value of $\overline{\mathcal{S}_n \wedge [T^A]} \subseteq \overline{\mathcal{S}}_n \wedge [\overline{T}^A]$ has a support in $\mathrm{supp}(\overline{\mathcal{T}}_{\mathcal{C}} \wedge [\overline{T}^A])$. Now by setting

(a) $\mathcal{V}_n := \mathcal{S}_n \wedge [\overline{T^A}]$. →)
(b) $\mathcal{V} := \mathcal{T}_{\mathcal{C}} \wedge [\overline{T^A}]$, →)
(c) $\mathcal{W}$ a cluster value of $\overline{\mathcal{T}_n \wedge [T^A]}$.

we are in the situation of Lemma 6.7, and conclude. □

Lemma 6.8. Let X_Σ be a smooth projective toric variety, and $\bar{\Delta} \subseteq X_\Sigma$ be the diagonal. Let $\mathcal{S}$ and $\mathcal{T}$ be two positive currents on X. Then, for any ray $\rho \in \Sigma$, $X_\Sigma \times X_\Sigma$

$$\operatorname{supp}(\mathcal{S}) \cap \operatorname{supp}(\mathcal{T}) \cap D_\rho \subseteq X_\Sigma$$

has a Cauchy–Riemann dimension ℓ, if and only if,

$$\operatorname{supp}(\mathcal{S} \otimes \mathcal{T}) \cap \bar{\Delta} \cap D_{(0,\rho)} \subseteq X_\Sigma \times X_\Sigma,$$

has a Cauchy–Riemann dimension ℓ, where $D_{(0,\rho)}$ is the toric invariant divisor corresponding to the ray $(0,\rho)$ in $\Sigma \times \Sigma$.

Proof. The fan of $X_\Sigma \times X_\Sigma$ is $\Sigma \times \Sigma$, we have that $D_{(0,\rho)} \simeq X_\Sigma \times D_\rho$ and the assertion follows. □

Theorem 6.9. Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{R}^n$ be two tropical cycles intersecting properly. Assume that X_Σ is a smooth toric projective variety compatible with $\mathcal{C}_1 + \mathcal{C}_2$. If moreover, for two sequences of positive closed currents $\overline{\mathcal{V}}_n$ and $\overline{\mathcal{W}}_n$ we have

(a) $\overline{\mathcal{W}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}_1}$ and $\overline{\mathcal{V}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}_2}$,
(b) $\operatorname{supp}(\overline{\mathcal{W}}_n) \longrightarrow \operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}_1})$ and $\operatorname{supp}(\overline{\mathcal{V}}_n) \longrightarrow \operatorname{supp}(\overline{\mathcal{T}}_{\mathcal{C}_2})$,
(c) For any n, $\operatorname{supp}(\overline{\mathcal{W}}_n) \cap \operatorname{supp}(\overline{\mathcal{V}}_n)$ has the expected dimension. →)
(d) For any n, and any ray $\rho \in \Sigma$, $\operatorname{supp}(\overline{\mathcal{W}}_n) \cap \operatorname{supp}(\overline{\mathcal{V}}_n) \cap D_\rho$ has the expected dimension.

Then

$$\overline{\mathcal{W}}_n \wedge \overline{\mathcal{V}}_n \longrightarrow \overline{\mathcal{T}}_{\mathcal{C}} \wedge \overline{\mathcal{T}}_{\mathcal{C}'}.$$

Proof. For two closed currents $\mathcal{S}$ and $\mathcal{T}$ on X_Σ we naturally identify $\mathcal{S} \wedge \mathcal{T} = \pi_*(\mathcal{S} \otimes \mathcal{T} \wedge [\bar{\Delta}])$, where $\pi : X_\Sigma \times X_\Sigma \longrightarrow X_\Sigma$ is the projection. In $T_N \times T_N \subseteq X_\Sigma \times X_\Sigma$ we $\mathcal{T}_n := \mathcal{W}_n \otimes \mathcal{V}_n$ and $\mathcal{T}_{\mathcal{C}} := \mathcal{T}_{\mathcal{C}_1} \otimes \mathcal{T}_{\mathcal{C}_2}$. Now note that the diagonal in the open torus is the complete intersection of the tori $x_i = y_i$, $i = 1, \dots, n$. This together with assumption (c) allows for a repeated application of Theorem 6.5 to obtain

$$\mathcal{W}_n \otimes \mathcal{V}_n \wedge [\Delta] \longrightarrow \mathcal{T}_{\mathcal{C}_1} \otimes \mathcal{T}_{\mathcal{C}_2} \wedge [\Delta].$$

By assumption (c), and Lemma 6.8, for large n and rays $\rho \in \Sigma$,

$$\operatorname{supp}(\overline{\mathcal{W}}_n \otimes \overline{\mathcal{V}}_n) \cap [\bar{\Delta}] \cap D_\rho$$

have the expected dimension. Lemma 6.8, and the compatibility assumption imply that $\operatorname{supp}(\mathcal{W}_n \otimes \mathcal{V}_n) \cap \bar{\Delta} \cap D_{(0,\rho)}$ and $\operatorname{supp}(\mathcal{T}_{\mathcal{C}} \otimes \mathcal{T}_{\mathcal{C}'}) \cap \bar{\Delta} \cap D_{(0,\rho)}$ have the expected Cauchy–Riemann dimension. Therefore, Lemma 2.12 brings us to the situation of Lemma 6.7 and we conclude.

Can we drop assumption (b)? □

How? Maybe (c)? or (d)

? Maybe (d) should be a similar condition for the limits ?

7. Dynamical Tropicalisation with non-Trivial Valuations

7.1. Dynamical tropicalisation with a non-trivial valuation. Recall that for a field $\mathbb{K}$, $\nu : \mathbb{K} \longrightarrow \mathbb{R} \cup \{\infty\}$, is called a valuation if it satisfies the following properties for every $a, b \in \mathbb{K}$:

(a) $\nu(a) = \infty$ if and only if $a = 0$;
(b) $\nu(ab) = \nu(a) + \nu(b)$;
(c) $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$.

A valuation is called *trivial*, if the valuation of any non-zero element is 0. For an element $a \in \mathbb{K}$, we denote by $\bar{a}$ its image in the residue field. We are interested in the case where $\mathbb{K} = \mathbb{C}((t))$, is the field of *formal Laurent series* with the parameter t. with the usual valuation. That is, for $g(t) = \sum_{j \geq k} a_j t^j$, with $a_k \neq 0$, the valuation equals ~~the~~ the minimal exponent $\nu(g) = k \in \mathbb{Z}$.

Definition 7.1. (a) Let $f = \sum_{\alpha \in \mathbb{N}} c_\alpha z^\alpha \in \mathbb{K}[z^{\pm 1}]$, be a Laurent polynomial in n variables. The tropicalisation of f with respect to ν,

$$\begin{aligned}\operatorname{trop}_\nu(f) : \mathbb{R}^n &\longrightarrow \mathbb{R},\\ x &\mapsto \max\{-\nu(c_\alpha) + \langle x, \alpha\rangle\}.\end{aligned}$$

(b) Let $I \subseteq \mathbb{K}[z^{\pm 1}]$ be an ideal. The tropical variety associated to I, as a set, is defined as

$$\operatorname{Trop}_\nu(I) := \bigcap_{f \in I} \operatorname{Trop}(\operatorname{trop}_\nu(f)),$$

where $\operatorname{Trop}(\operatorname{trop}_\nu(f))$ is the set of points where $\operatorname{trop}_\nu(f)$ is not differentiable; see Remark 4.5.

(c) For an algebraic subvariety of the torus $Z \subseteq (\mathbb{K}^*)^n$, with the associated ideal $\mathbb{I}(Z)$, the tropicalisation of Z, as a set, is $\operatorname{Trop}_\nu(Z) := \operatorname{Trop}_\nu(\mathbb{I}(Z))$.

(d) In all the situations above, trop_0 denotes the tropicalisation with respect to the trivial valuation.

We need to relate a non-trivial valuation to the trivial valuation.

Lemma 7.2. Consider the ideal $I \subseteq \mathbb{C}[t^{\pm 1}, z^{\pm 1}] \xhookrightarrow{\iota} \mathbb{C}((t))[z]$. Assume that (u, x) are the coordinates in $\mathbb{R} \times \mathbb{R}^n$. Then, we have the following equality of sets

$$\operatorname{Trop}_0(I) \cap \{u = -1\} = \operatorname{Trop}_\nu(\iota(I)).$$

where ν is ...

That is, the tropicalisation of I as an ideal in $\mathbb{C}[t, x]$ with respect to the trivial valuation intersected with $\{u = -1\}$ coincides with the tropicalisation of $I = \iota(I)$ with respect to the usual valuation in $\mathbb{C}((t))$.

The proof of the lemma becomes clear with the following example.

Example 7.3. Let

$$f(x, t) = 4(t^3 + t^{-1}) z_1 z_2 + (1 + t + t^2) z_1.$$

Then, the tropicalisation of $f \in \mathbb{C}[t, z]$, with respect to the trivial valuation equals:

$$\operatorname{trop}_0(f) = \max\left\{\max\{3u + x_1 + x_2, -u + x_1 + x_2\}, \max\{x_1, u + x_1 + 2u + x_1\}\right\}$$

Letting $u := -1$, $\operatorname{trop}_0(f)(-1, x) = \max\{1 + x_1 + x_2, x_1\}$. The latter equals $\operatorname{trop}_\nu(f)$ as an element of $\mathbb{C}((t))[z]$.

Proof of Lemma 7.2. If f is a monomial in $\mathbb{C}[t][z]$, then it is clear that

$$\mathrm{trop}_0(f)(-1,x) = \mathrm{trop}_\nu(\iota(f)).$$

Therefore, we have the equality for any polynomial in $f \in \mathbb{C}[t,z]$. To prove the main statement, note that

$$\begin{aligned}\mathrm{Trop}_\nu(\iota(I)) &= \bigcap_{f\in\iota(I)} \mathrm{Trop}(\mathrm{trop}_\nu(f))\\ &= \bigcap_{f\in I} \big(\mathrm{Trop}(\mathrm{trop}_0(f)) \cap \{u=-1\}\big)\\ &= \mathrm{Trop}_0(I) \cap \{u=-1\}.\end{aligned}$$

□

Remark 7.4. Bergman in [Ber71], shows that for an algebraic subvariety $Z \subseteq (\mathbb{C}^*)^n$, one has

t

$$\lim \mathrm{Log}_t(Z) \subseteq \mathrm{Trop}_0(\mathbb{I}(Z)),$$

and he conjectured the equality. This conjecture was later proved by Bieri and Groves in [BG84]. More precisely, Bieri and Grove prove that $\lim \mathrm{Log}_t(Z) \cap (S^1)^n$ is a polyhedral sphere of real dimension equal to (the complex dimension) $\dim(Z) - 1$. Therefore, the fan $\lim \mathrm{Log}_t(Z)$ is a cone over their spherical complex. See also [MS15, Theorem 1.4.2].

Remark 7.5. The above lemma is related to the results of Markwig and Ren in [MR20]. They considered the tropicalisation of an ideal $J \subseteq R[[t]][x]$, where R is the ring of integers of a discrete valuation ring $\mathbb{K}$, which is non-trivially valued. To obtain finiteness properties, however, the authors consider the associated tropical variety in the half-space $\mathbb{R}_{\leq 0} \times \mathbb{R}^n$. Note that such a variety is almost never balanced. The authors also prove that for an ideal $I \subseteq \mathbb{K}[x]$, the tropicalisation of the natural inverse image $\pi^{-1}I \subseteq R[[t]][x]$ with respect to trivial valuation, intersected with $\{u=-1\}$ equals $\mathrm{trop}_\nu(I)$; [MR20, Theorem 4].

Let us also recall the main result of [Bab23].

be

Theorem 7.6. Let $Z \subseteq (\mathbb{C}^*)^n$ be an irreducible subvariety of dimension p, and $\overline{Z}$ the closure of Z in the compatible smooth projective toric variety X. Then,

$$\frac{1}{m^{n-p}} \Phi_m^*[\overline{Z}] \longrightarrow \overline{\mathcal{T}}_C, \quad \text{as } m \to \infty,$$

Φ_m not defined.

where $\Phi_m : X \longrightarrow X$ is the continuous extension of $\Phi_m : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$, and $\overline{\mathcal{T}}_{\mathrm{trop}_0(Z)}$ is the extension by zero of $\mathcal{T}_{\mathrm{trop}_0(Z)}$ to X. Moreover, the supports also converge in Hausdorff metric.

Note that since the limit of a sequence of closed currents is closed, the above theorem implies that $\mathrm{trop}_0(Z)$ can be equipped with weights to become balanced. Note that the compatibility is in the following sense of Tevelev and Sturmfels:

Theorem 7.7. (a) The closure $\bar{Z}$ of Z in X_Σ is complete, if and only if, $\mathrm{trop}(Z) \subseteq |\Sigma|$; see [Tev07].

(b) We have $|\Sigma| = \mathrm{trop}(Z)$, if and only if, for every $\sigma \in \Sigma$ the intersection $\mathcal{O}_\sigma \cap \bar{Z}$ is non-empty and of pure dimension $p - \dim(\sigma)$; see [ST08].

Theorem 7.8. Let $I \subseteq \mathbb{C}[t^{\pm 1}, x^{\pm 1}]$ be an ideal with the associated $(p+1)$-dimensional algebraic variety $W = \mathbb{V}(I) \subseteq (\mathbb{C}^*)^{n+1}$. Assume that the projection onto the first coordinate $\pi_1 : W \longrightarrow \mathbb{C}^*$ is surjective and Zariski closed. We denote the fibers as $W_t := \pi_1^{-1}(t)$. We have that

(a)
$$\frac{1}{m^{n-p}} \Phi_m^*[W_{e^m}] \longrightarrow \mathcal{T}_{\mathrm{Trop}_\nu(I)}, \quad \text{as } m \to \infty ,$$
in the sense of currents in $\mathcal{D}_p((\mathbb{C}^*)^n)$.

(b) $\mathrm{Trop}_\nu(I)$ can be equipped with weights to become balanced.

(c) $\lim \mathrm{supp}(\frac{1}{m^{n-p}} \Phi_m^*[W_{e^m}]) = \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_\nu(I)})$.

(d) On a toric variety X_Σ compatible with $\mathrm{trop}_0(W) + \{u = -1\}$,
$$\frac{1}{m^{n-p}} \Phi_m^* \overline{[W_{e^m}]} \longrightarrow \overline{\mathcal{T}}_{\mathrm{Trop}_\nu(I)}, \quad \text{as } m \to \infty$$

We need the following:

Lemma 7.9. Let $W \subseteq (\mathbb{C}^*)^{n+1}$ be a $(p+1)$-dimensional smooth subvariety, such that the projection onto the first factor, $\pi_1 : (\mathbb{C}^*)^{n+1} \longrightarrow \mathbb{C}^*$ is surjective and a Zariski closed morphism. Assume that W Then for a sufficiently large $|t_0| >> 0$
$$[W_{t_0}] = [\pi_1^{-1}(t_0)] = [\{t = t_0\}] \wedge [W].$$

Proof. We first prove that the set of singular points of W, together with the set of points where $[\{t = t_0\}] \wedge [W]$ has a multiplicity greater than 1, is contained in a Zariski closed set in W. We define the *critical set*,
$$C = \{ w \in W_{\mathrm{reg}} : \dim \left(T_w W \cap \ker \ \nabla_w t \right) = p + 1 \},$$
which is the set of points where the tangent space of $T_w W_{reg}$ is included in the tangent space of $T_w\{t = t_0\}$, and this set contains the set of points $w \in W_{\mathrm{reg}}$ points the intersection multiplicity of $\{t = t_0\}$ and W exceeds 1. We fix an ideal associated to $I = \mathbb{I}(W) = \langle f_1, \dots, f_k \rangle \subseteq \mathbb{C}[t, x]$. At any regular point $w \in W_{\mathrm{reg}}$, $T_w W$ is of dimension $p+1$, and the rank of the Jacobian matrix $J(f)(w) = \left(\frac{\partial f_i}{\partial z_j}(w) \right)_{k \times (n+1)}$ equals codimension of W, $(n+1) - (p+1) = n - p$. We have that $\nabla_w t = e_1$, where e_1 is the first element of the standard basis for the $\mathbb{C}$-vector space $\mathbb{C}^{n+1}$. We have $w \in C$, if and only if,
$$\ker \begin{pmatrix} e_1 \\ Jf(w) \end{pmatrix} = \ker \left(Jf(w) \right).$$
As a result, C is an algebraic variety given as the intersection of $W \setminus W_{\mathrm{sing}}$ with the intersection of zero loci of $(q+1) \times (q+1)$-minors of $\begin{pmatrix} e_1 \\ Jf(w) \end{pmatrix}$. Therefore, the closure of C in W, $\overline{C}$ union W_{sing} is a Zariski-closed subset of W. Since W is not contained in $\{t = t_0\}$, as π_1 is surjective, then $\pi_1(\overline{C} \cup W_{\mathrm{sing}})$ is a Zariski closed proper subset in $\mathbb{C}^* \subseteq \mathbb{C}$, and hence finite.

□

Proof of Theorem 7.8. By the preceding lemma, and the fact that Φ_m preserves transversal intersection, we have
$$\frac{1}{m^{n-p}} \Phi_m^*[W_{e^m}] = \frac{1}{m^{n-(p+1)}} \Phi_m^*[W] \wedge \frac{1}{m} \Phi_m^*[\{t = e^m\}],$$

for a large m. Since $\mathrm{trop}_0(W)$ is a fan and it is transversal to the plane $\{u=-1\} \subset \mathbb{R}^{n+1}$ are transversal, we can use Theorem 6.5 to write

$$\lim \frac{1}{m^{n-p}} \Phi_m^*[W_{e^m}] = \left(\lim \frac{1}{m^{n-(p+1)}} \Phi_m^*[W]\right) \wedge \left(\lim \frac{1}{m} \Phi_m^*[\{t=e^m\}]\right)$$

By Theorem 7.6, restricted to $(\mathbb{C}^*)^{n+1}$, and the fact that we used $\mathrm{Log} = (-\log|\cdot|, \dots, -\log|\cdot|)$ in the definition of tropical currents, the above limits yield

$$\lim \frac{1}{m^{n-p}} \Phi_m^*[W_{e^m}] = \mathcal{T}_{\mathrm{Trop}_0(W)} \wedge \mathcal{T}_{\{u=-1\}}.$$

Applying Theorems 5.11 and Lemma 7.2, we obtain the equality. For the assertion (b), note that the limit $\mathcal{T}_{\mathrm{Trop}_\nu(I)}$ is a closed current and Theorem 4.3 implies that $\mathrm{Trop}_\nu(I)$ is naturally balanced. To observe (c), note that (a) implies

$$\lim \mathrm{supp}\left(\frac{1}{m^{n-p}} \Phi_m^*[W_{e^m}]\right) \supseteq \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_\nu(I)}).$$

However, because of transversality, $\mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_\nu(I)}) = \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_0(W)}) \cap \mathrm{supp}(\mathcal{T}_{\{u=-1\}})$. At the same time,

$$\lim \mathrm{supp}(\Phi_m^*[W_{e^m}]) = \lim \mathrm{supp}(\Phi_m^*[W]) \cap \mathrm{supp}(\Phi_m^*[\{t=e^m\}]).$$

Moreover, for the Hausdorff limit of sets $\lim(A_i \cap B_i) \subseteq (\lim A_i) \cap (\lim B_i)$. This implies

$$\lim \mathrm{supp}(\Phi_m^*[W_{e^m}]) \subseteq \mathrm{supp}(\mathcal{T}_{\mathrm{Trop}_0(W)}) \cap \mathrm{supp}(\mathcal{T}_{\{u=-1\}}),$$

which implies (c). Now, (d) is implied by Theorem 6.6. □

Let us first prove the analogous result to the main result of Bogart Jensen, Speyer, Sturmfels, and Thomas in [BJS+07]. See also [OP13] for generalisation.

Theorem 7.10. Assume that W and Z ? ?
respectively. Further,

(a) the supports converge in the Hausdorff metric ,
(b) $\mathcal{C}$ and $\mathcal{C}'$ intersect properly.

Then, $\mathcal{W}_n \wedge \mathcal{V}_n$ converges to $\mathcal{T}_{\mathcal{C}} \wedge \mathcal{T}_{\mathcal{C}'}$.

Proof to be completed. When $\mathcal{C}$ and $\mathcal{C}'$ intersect properly, it implies that the fibres of $\mathcal{T}_{\mathcal{C}}$ and $\mathcal{T}_{\mathcal{C}'}$ intersect transversely. In this situation,

Let $I \subseteq \mathbb{C}[t^{\pm 1}, x^{\pm 1}]$ be an ideal with the associated $(p+1)$-dimensional algebraic variety $W = \mathbf{V}(I) \subseteq (\mathbb{C}^*)^{n+1}$. Assume that the projection onto the first coordinate $\pi_1 : W \longrightarrow \mathbb{C}^*$ is surjective and Zariski closed. We denote the fibres as $W_t := \pi_1^{-1}(t)$. We have that

$$\frac{1}{m^{n-p}} \Phi_m^*[W_{e^m}] \longrightarrow \mathcal{T}_{\mathrm{trop}_\nu(I)}, \quad \text{as } m \to \infty\,,$$

in the sense of currents in $\mathcal{D}_p((\mathbb{C}^*)^n)$. In particular, $\mathrm{trop}_\nu(I)$ can be equipped with weights to become balanced. Moreover, if Σ is a toric variety compatible with $\mathrm{trop}_0(W)$ and $\{u=-1\}$, then on X_Σ,

$$\frac{1}{m^{n-p}} \Phi_m^*[\overline{W}_{e^m}] \longrightarrow \overline{\mathcal{T}}_{\mathrm{trop}_\nu(I)}, \quad \text{as } m \to \infty\,.$$

□