Let X be a compact Kähler manifold and T a positive closed (p,p)-current on X. We have the following.

Proposition 0.1. *If T is bounded above by a positive closed current of continuous or Hölder continuous super-potential, then T satisfies the same property.*

As I remember, this should be written in my paper with Nessim or by Duc-Viet Vu.

Lemma 0.2. *If T is locally bounded by a product of positive closed $(1,1)$-currents of continuous potentials or Hölder continuous potentials, then this holds globally.*

Proof. Fix a point a in X. In an open neighbourhood of a that we identify with the ball $\mathbb{B}(0,2)$ in $\mathbb{C}^n$, we have

$$T \leq dd^c u_1 \wedge \ldots \wedge dd^c u_p$$

with u_i continuous or Hölder continuous. Without loss of generality, we can assume that these functions are strictly negative. Define on $\mathbb{B}(0,1)$

$$u_i' := \max(u_i, A \log \|z\|)$$

with A big enough so that $u_i' = u_i$ on $\mathbb{B}(0,1/2)$. Observe that $u_i' = A \log \|z\|$ near $b\mathbb{B}(0,1)$. Hence we can extend it to a function which is smooth in a neighbourhood of $X \setminus \mathbb{B}(0,1)$. Thus, this function is quasi-psh. We have

$$T \leq (B\omega + dd^c u_1') \wedge \ldots \wedge (B\omega + dd^c u_p')$$

in a neighbourhood W_a of a if B is large enough.

Since we can cover X using a finite number of open sets W_{a_k}, we can add up all obtained quasi-psh functions together and obtain a quasi-psh function u. It is clear that $T \leq (C\omega + u)^p$ if C is large enough. The function u is continuous or Hölder continuous. □

Remark 0.3. The lemma still holds if T is locally bounded by a finite sum of such products of $(1,1)$-currents. Indeed, it is enough to take the sum of all these psh functions together in order to reduce the problem to the lemma setting.

Corollary 0.4. *If T is locally bounded by finite sum of products of positive closed $(1,1)$-currents of continuous potentials or Hölder continuous potentials, then it has a continuous or Hölder continuous super-potential.*

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Example 0.5. Let $Z \subset Y$ be a real analytic manifold of dimension m in a complex manifold Y of complex dimension m. Assume that the tangent space of Z at each point is totally real, i.e. it doesn't contain any complex line (we say that Z is totally real). Let $K \Subset Z$ be a Borel set. Consider the Lebesgue measure of Z defined by a smooth volume form of Z. Then the restriction of this Lebesgue measure to K is locally a Monge-Ampère measure with Hölder potentials, i.e. it is equal to $(dd^c u)^n$ with u Hölder psh on some open subset of Y. Indeed, the problem is local, so we can use local coordinates so that Y is identified to the unit ball in $\mathbb{C}^m$ and Z is the intersection of this ball with $\mathbb{R}^m$. The Lebesgue measure on Z is equal up to a constant

$$dd^c \max(y_1, 0) \wedge \ldots \wedge dd^c \max(y_m, 0)$$

where we write the coordinates $z_k = x_k + iy_k$. The measure on K is bounded by a constant times the last measure. Hence it is a Monge-Ampère of a Hölder psh function. Recall that (in the local setting) if a measure is bounded by a Monge-Ampère of Hölder potential then it is a Monge-Ampère of Hölder potential. This is a consequence of Dinh-Nguyen J. Funct Analysis 266 (2014), no. 1, 67-84, but it is probably known before.

The following lemma is obvious as we just need to pull-back the potentials. The condition on f guarantees that the pull-back operator is well-defined on positive closed currents and is continuous.

Lemma 0.6. *Let $f : X \to Y$ be a holomorphic surjective map such that all fibers are of dimension $\dim X - \dim Y$, e.g. a submersion. Let T be a positive closed (p,p)-current on Y which is equal (or bounded by) a product of $(1,1)$-currents of continuous or Hölder continuous potentials. Then $f^*(T)$ satisfies the same property.*

Example 0.7. If T is locally (up to a change of coordinates) the product of a measure as in Example 0.5 with an open subset of $\mathbb{C}^{n-m}$, then it is of Hölder super-potential in X, thanks to the last lemma. The same property holds if T is locally bounded by a finite sum of such products.

It seems that this example is very close to the case of tropical currents.