Algebraic Geometry, Lecture 1

Farhad Babaee

University of Bristol

Admin stuff

- Lectures and Problem Classes
  - Thursdays 13:00 to 14:00 (FRY G.06) Lectures
  - Thursdays 15:00 to 16:00 (CHEM LT4) Lectures
  - Fridays 14:00 to 15:00 (PHYS 3.21 Berry) Alternating Lectures and Problem Classes

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- Assessment
  - Assessed Homework 1 (7.5%), Week 17, Available from 13:00, February 20th to 13:00, February 27th, Individual upload
  - Assessed Homework 2 (7.5%), Week 22, Available from 13:00, April 17th to 13:00, April 24th, individual upload
  - Problem Classes Presentations (5%) along the course, presenting solutions, individual/group presentation Final exam (80%), May/June, Written exam

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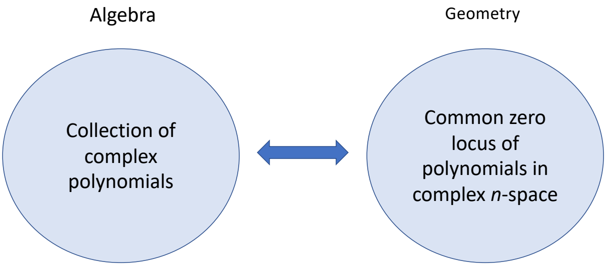
  - Master's students: Final written exam (80%), May/June. PhD/research students: Oral exam or presentation.

- Office Hours (My office Fry 2.13)
  - Fridays 16:00 to 17:00 (might change)

What is the (Complex) Algebraic Geometry?



The goal of our course

- Describe basic objects in algebraic geometry
- Describe dimension, degree, smoothness, etc. in both algebraic and geometric settings
- In toric varieties read off a lot of info from combinatorial data

Some examples

Example

The zero locus of the polynomial $x^2 + y^2 + 1$ is empty in $\mathbb{R}^2$ but non-empty in $\mathbb{C}^2$.

Example (Fermat's last theorem)

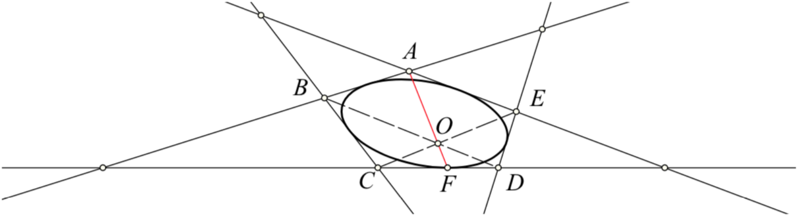
The zero locus of $x^n + y^n + z^n$ is empty in $\mathbb{Q}^3$, for integer $n \geq 3$.

Remark

Algebraic geometry can be done in any field, but for simplicity and intuition, we mostly deal with complex numbers in this course.

Some history

Babylonians (2000-1500BC) seemed to know how to solve $ax^2 + bx = c$.
in $\mathbb{R}$. They also knew Pythagoras Theorem!

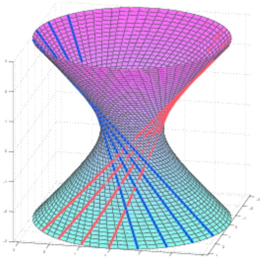
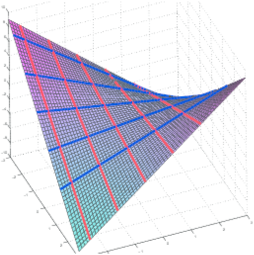


Appolonius (262-190 BC) seemed to know that a non-degenerate plane conic is determined by 5 tangent lines.

Mid-nineteenth century

Bernhard Riemann (1828-1866) showed that compact Riemann surfaces can be described as zero sets of a polynomial function. We later state a generalisation of this statement which called the Chow Theorem.

Mid-nineteenth century

Any quadratic surface (zero set of a degree 2 polynomial in 3 variables) can be covered by lines.

Any cubic surface contains exactly 27 lines!

Turn of the century

- Italian School of Algebraic Geometry asserted many statements, but sometimes they lacked rigor. For example, let us look at the Bézout's Theorem

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Assume that C_1 and C_2 are two curves of degree d_1 and d_2 in the complex projective space $\mathbb{P}^2$. Then, the number of intersection points, counting the multiplicities, is $d_1 d_2$.

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An idea for the proof.

- Proof for $d_1 = 1$, $d_2 = 2$.
- By moving the curves we can assume the intersections take place in $\mathbb{C}^2 \subseteq \mathbb{P}^2$.
- The theorem is true if the defining functions of C_1 and C_2 are of the form $f(x, y) = a_1 y - b_1 x$, and $g(x, y) = (a'_1 y - b'_1 x)(a'_2 x - b'_2 y)$.
- Intuitively, the number of intersection points, taking into account the multiplicities, do not change if we perturb the curves.

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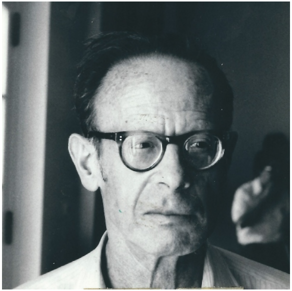
- Not complete!

$\square$

Beginning of 20th century


David Hilbert (1862-1943) and Emmy Noether (1882-1935) set the algebraic foundations for solid algebraic geometry.

20th century

Oscar Zariski 1899-1986, and André Weil 1906-1998 with many others revived the topic and developed it.

Alexander Grothendieck (1928-2014 - Fields Medal 1966) aided by Artin, Mumford (Fields 1974) and many others, introduced Scheme Theory and lifted Algebraic Geometry to a “dizzying heights of abstraction”. This abstraction made algebraic geometry more natural, more general, and often simplified.

What we will study in this course?

- Some flavours of Scheme Theory.
- Basic foundations and many nice constructions/theorems:
  - Affine, Projective, and Quasi-Projective Algebraic Varieties
  - Degree
  - Gluing
  - Smoothness and resolution of singularities
  - Some toric geometry

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  Question: Have you taken any courses on the Geometry of Manifolds? Algebraic Topology