Tropical and Complex Geometry

Farhad Babaee

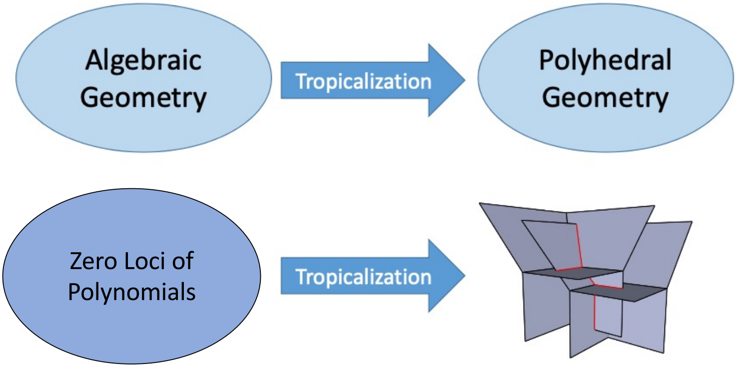
University of Bristol

QMUL
March 15, 2024

Plan of the talk

- Basics of tropical geometry
- Currents
- Some equidistribution statements
- Approximation of currents
- Intersection theory of currents

Tropical Geometry



Tropicalisation by taking logarithm

$$\begin{array}{rccc} \mathrm{Log}_t : & (\mathbb{C}^*)^2 & \rightarrow & \mathbb{R}^2 \\ & (z_1, z_2) & \mapsto & (\log_t |z_1|, \log_t |z_2|) \end{array}$$

What happens to $\mathrm{Log}_t\{\text{Algebraic Variety}\}$ as $t \rightarrow \infty$?

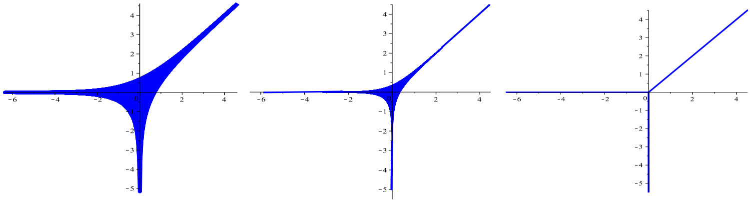
Tropicalisation by taking logarithm

$$\begin{array}{rccc} \operatorname{Log}_t : & (\mathbb{C}^*)^2 & \to & \mathbb{R}^2 \\ & (z_1, z_2) & \mapsto & (\log_t |z_1|, \log_t |z_2|) \end{array}$$

What happens to $\operatorname{Log}_t\{\text{Algebraic Variety}\}$ as $t \to \infty$?

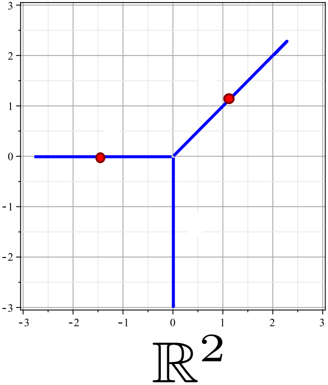
$$\ell = \{(z_1, z_2) \in (\mathbb{C}^*)^2 : z_1 + z_2 + 1 = 0\}$$

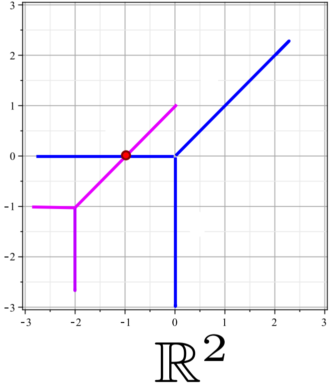
$$\operatorname{Log}_t(\ell) \xrightarrow[t\to\infty]{\text{Hausdorff Metric}} \text{“Tropical Line” in } \mathbb{R}^2$$



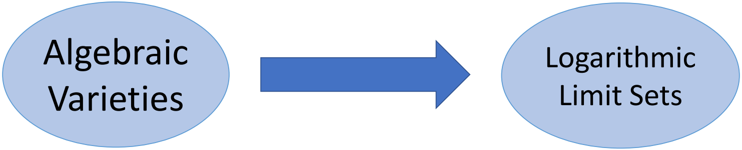
$t = 3$ $t = 10$ $t \to \infty$

Tropical lines behave like lines!





Tropicalisation captures a lot of information



- Dimension
- Degree
- Genus
- Intersection theory
- Hodge index theorem
- Chow class in toric varieties…

Some applications of tropical geometry

- Enumerative Geometry: Gromov-Witten Invariants
- Mirror Symmetry
- Read, Rota–Heron–Welsh Conjecture, Mason Conjecture, Top-Heavy Conjecture

Some applications of tropical geometry

- Enumerative Geometry: Gromov-Witten Invariants

  [Mikhalkin 2005, Following Kontsevich]

- Mirror Symmetry

  [Kontsevich, Gross–Siebert]

- Read, Rota–Heron–Welsh Conjecture, Mason Conjecture, Top-Heavy Conjecture

  [Huh et al. 2012–2023]

Tropical algebra

$$(\mathbb{R} \cup \{-\infty\}, \oplus, \odot) = (\mathbb{R} \cup \{-\infty\}, \max, +)$$

Example

$$2 \oplus 3 = \max\{2, 3\}, \quad 2 \odot 3 = 2 + 3.$$

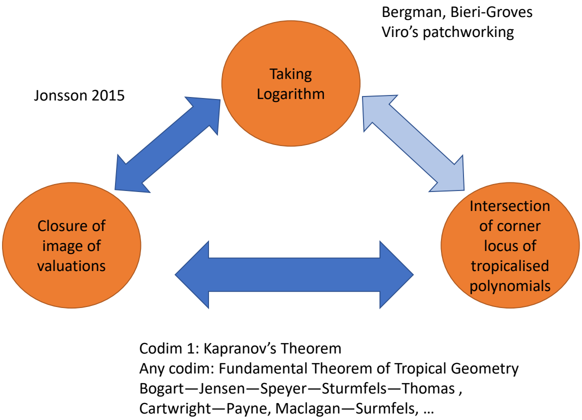
Tropicalise the polynomial

$$f : \mathbb{C}^2 \longrightarrow \mathbb{C}, \quad (z_1, z_2) \longmapsto z_1 + z_2 + 1,$$

and look at the *corner locus* of

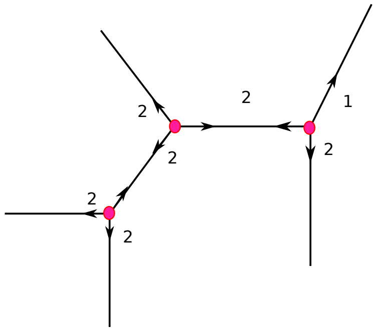
$$\mathrm{trop}(f) := \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad (x_1, x_2) \longmapsto \max\{x_1, x_2, 0\}.$$

Different ways of tropicalisation, non-trivial valuation

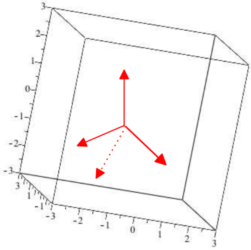


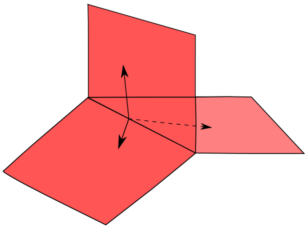
Objects: tropical varieties

After tropicalisation we get polyderal complexes with nice properties: Rational Slopes, Balanced, etc.



Higher Dimensions

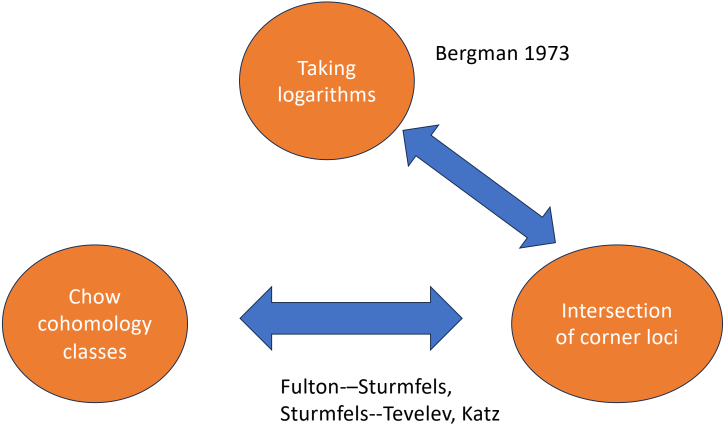




An Application

Mikhalkin 2005: There is a correspondence between the complex and tropical plane curves of degree d and genus g passing through $3d + g - 1$ points in a general position. Therefore, the Gromov–Witten Invariants can be counted tropically.

Different ways of tropicalisation, trivial valuation



A Realisability Question

- We can define **Tropical Varieties** to be balanced rational polyhedral complexes.
- **Question:** Can we obtain all the tropical varieties by tropicalising the algebraic varieties?

A Realisability Question

- We can define **Tropical Varieties** to be balanced rational polyhedral complexes.
- **Question:** Can we obtain all the tropical varieties by tropicalising the algebraic varieties?

  Yes for hypersurfaces, but no in general.

A Realisability Question

- We can define **Tropical Varieties** to be balanced rational polyhedral complexes.
- **Question:** Can we obtain all the tropical varieties by tropicalising the algebraic varieties?

  Yes for hypersurfaces, but no in general.

- Tropical varieties obtained by tropicalisation of an algebraic variety are called *realisable.*

How can we proceed with the non-realisable cases?

(a) Prove analogues of algebraic geometry theorems for matroids/ (smooth) tropical varieties

(b) Lift to analytic objects

How can we proceed with the non-realisable cases?

(a) Prove analogues of algebraic geometry theorems for matroids/ (smooth) tropical varieties

e.g. Huh et al.

(b) Lift to analytic objects

How can we proceed with the non-realisable cases?

(a) Prove analogues of algebraic geometry theorems for matroids/ (smooth) tropical varieties

e.g. Huh et al.

(b) Lift to analytic objects

- Lagerberg's Supercurrents, works of Lagerberg, Gubler, Chambert-Loir, Ducros, Künnemann...
- Complex tropical currents: interactions with complex geometry problems

What Are Complex Currents?

X complex smooth manifold of complex dimension n

- $\mathcal{D}^{p,q}(X)$: Smooth (p,q)-forms with compact support

Example

$$dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2$$

is a $(1,2)$-form.

What Are Complex Currents?

X complex smooth manifold of complex dimension n

- $\mathcal{D}^{p,q}(X)$: Smooth (p,q)-forms with compact support

Example

$$dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2$$

is a $(1,2)$-form.

- Currents $\mathcal{D}'_{p,q}(X) :=$ Topological Dual to $\mathcal{D}^{p,q}(X)$
- A current $\mathscr{T}$ acts on a form $\varphi \in \mathcal{D}^{p,q}(X)$,

$$\langle \mathscr{T}, \varphi \rangle \in \mathbb{C},$$

  and the action is linear and continuous.

Example (Integration Currents)

Let X be a complex smooth manifold, and $Z \subset X$ be a smooth submanifold of complex dimension p, define the (p,p)-current

$$\langle [Z], \varphi \rangle := \int_Z \varphi \in \mathbb{C}$$

This definition extends to analytic subsets Z.

Operations on currents are defined by duality

- Convergence:

$$\mathscr{T}_j \to \mathscr{T}, \quad \text{if } \langle \mathscr{T}_j, \varphi \rangle \to \langle \mathscr{T}, \varphi \rangle \text{ in } \mathbb{C}.$$

- Closedness:
  A current $\mathscr{T}$ is called **closed**, if

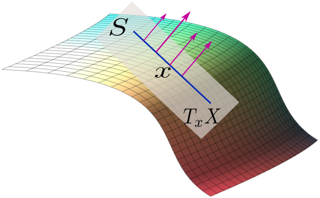
$$\langle d\mathscr{T}, \varphi \rangle = \pm \langle \mathscr{T}, d\varphi \rangle = 0, \quad \forall \varphi.$$

Positivity

Recall that every complex manifold is canonically oriented.

Definition

A smooth differential (p,p)-form φ is *positive* if any restriction $\varphi(x)|_S$ is a nonnegative volume form for all complex p-planes $S \subseteq T_xX$ and $x \in X$.



Definition

A current $\mathscr{T} \in \mathcal{D}'_{p,p}(X)$ is called *positive* if

$$\langle \mathscr{T}, \varphi \rangle \geq 0, \quad \forall \varphi \text{ positive.}$$

Complex tropical currents

When $\mathscr{C} \subseteq \mathbb{R}^n$ is a p-dimensional tropical variety, $\mathscr{T}_{\mathscr{C}}$ is a (p, p) on $(\mathbb{C}^*)^n$ with support $\mathrm{Log}^{-1}(\mathscr{C})$.

Definition (B)

Let $\mathscr{C}$ be a weighted rational polyhedral complex of dimension p. The tropical current $\mathscr{T}_{\mathscr{C}}$ associated to $\mathscr{C}$ is given by

$$\mathscr{T}_{\mathscr{C}} = \sum_{\sigma} w_{\sigma} \ \mathbb{1}_{\mathrm{Log}^{-1}(\sigma^{\circ})} \int_{(S^1)^{n-p}} [\mathrm{fibers}] \, d\mu_{\sigma}(x),$$

where the sum runs over all p dimensional cells σ of $\mathscr{C}$.

If $\mathscr{C}$ is positively weighted, then the associated current $\mathscr{T}_{\mathscr{C}}$ is positive.

Theorem (B)

A weighted complex $\mathscr{C}$ is balanced if and only if $\mathscr{T}_{\mathscr{C}}$ is closed.

Dynamical Tropicalisation in the Trivial Valuation Case

$$\begin{aligned} \Phi_m : (\mathbb{C}^*)^n &\longrightarrow (\mathbb{C}^*)^n \\ (z_1, \ldots, z_n) &\longmapsto (z_1^m, \ldots, z_n^m), \end{aligned}$$

Theorem (B)

Let $Z \subseteq (\mathbb{C}^*)^n$ be an irreducible subvariety of dimension p, then

$$\frac{1}{m^{n-p}} \Phi_m^*[Z] \longrightarrow \mathscr{T}_{\mathrm{trop}(Z)}, \quad \textit{as } m \to \infty,$$

where $\mathscr{T}_{\mathrm{trop}(Z)}$ is the complex tropical current associated to $\mathrm{trop}(Z)$.

Theorem (B)

Let $Z \subseteq (\mathbb{C}^*)^n$ be a an irreducible subvariety of dimension p, and $\overline{Z}$ the tropical compactification of Z in the compatible smooth toric variety X. Then,

$$\frac{1}{m^{n-p}}\Phi_m^*[\overline{Z}] \longrightarrow \overline{\mathscr{T}}_{\mathrm{trop}(Z)}, \quad \textit{as } m \to \infty,$$

where $\Phi_m : X \longrightarrow X$ is the continuous extension of $\Phi_m : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$, and $\overline{\mathscr{T}}_{\mathrm{trop}(Z)}$ is the extension by zero of $\mathscr{T}_{\mathrm{trop}(Z)}$ to X.

Theorem (B)

Let $Z \subseteq (\mathbb{C}^*)^n$ be a an irreducible subvariety of dimension p, and $\overline{Z}$ the tropical compactification of Z in the compatible smooth toric variety X. Then,

$$\frac{1}{m^{n-p}} \Phi_m^*[\overline{Z}] \longrightarrow \overline{\mathscr{T}}_{\mathrm{trop}(Z)}, \quad \textit{as } m \to \infty,$$

where $\Phi_m : X \longrightarrow X$ is the continuous extension of $\Phi_m : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$, and $\overline{\mathscr{T}}_{\mathrm{trop}(Z)}$ is the extension by zero of $\mathscr{T}_{\mathrm{trop}(Z)}$ to X.

- Compare to Kajiwara–Payne tropicalisation.

Reviewing some theorems in tropical geometry

- Dynamical Kapranov Theorem

  Applying $\frac{1}{m}\Phi_m^*$ to Poincaré–Lelong Equation.

$$dd^c \log|z_1 + z_2 + 1| = [\ell].$$

Reviewing some theorems in tropical geometry

- Dynamical Kapranov Theorem

  Applying $\frac{1}{m}\Phi_m^*$ to Poincaré–Lelong Equation.

$$dd^c \log|z_1 + z_2 + 1| = [\ell].$$

  We obtain $dd^c \log\max\{|z_1|, |z_2|, 1\} = \mathscr{T}_{\mathrm{trop}(\ell)}$

Reviewing some theorems in tropical geometry

- Dynamical Kapranov Theorem

  Applying $\frac{1}{m}\Phi_m^*$ to Poincaré–Lelong Equation.

  $$dd^c \log|z_1 + z_2 + 1| = [\ell].$$

  We obtain $dd^c \log \max\{|z_1|, |z_2|, 1\} = \mathscr{T}_{\mathrm{trop}(\ell)}$

- $\frac{1}{m^{n-p}}\Phi_m^*$ preserves the cohomology class $\implies$ cohomology class of the closure algebraic subvarieties of $(\mathbb{C}^*)^n$ on a compatible toric variety is given by tropicalisation.

Reviewing some theorems in tropical geometry

- Dynamical Kapranov Theorem

  Applying $\frac{1}{m}\Phi_m^*$ to Poincaré–Lelong Equation.

  $$dd^c \log|z_1 + z_2 + 1| = [\ell].$$

  We obtain $dd^c \log \max\{|z_1|, |z_2|, 1\} = \mathscr{T}_{\mathrm{trop}(\ell)}$
- $\frac{1}{m^{n-p}}\Phi_m^*$ preserves the cohomology class $\implies$ cohomology class of the closure algebraic subvarieties of $(\mathbb{C}^*)^n$ on a compatible toric variety is given by tropicalisation.
- Tropicalisation gives a balanced complex.

Proof.

- $\{m^{p-n}\Phi_m^*[Z]\}$ have the same mass, so there is a convergent subsequence.
- Any cluster value has a support $\mathrm{Log}^{-1}(\mathrm{trop}(Z))$.
- Demailly's Theorem of Support: any cluster value $\mathscr{S}$ has the form

$$\mathscr{S} = \sum_{\sigma \in \Sigma} \int_{x \in S_{N(\sigma)}} \left[\mathbb{1}_{\mathrm{Log}^{-1}(\sigma^\circ)} \pi^{-1}_{\mathrm{aff}(\sigma)}(x)\right] \, d\eta_\sigma(x),$$

  for some measures $d\eta_\sigma$.
- $d\eta_\sigma$ have to be Haar measures.
- $w_\sigma(\overline{Z}) = \left\{[D_\sigma] \wedge \overline{m^{p-n}\Phi_m^*[Z]}\right\} \longrightarrow \left\{[D_\sigma] \wedge \overline{\mathscr{S}}\right\} = w_\sigma(\overline{\mathscr{S}})$

$\square$

General equidistribution theorem/conjecture

Let $\mathscr{H}_d(\mathbb{P}^n)$ denote the set of holomorphic endomorphisms of degree d on $\mathbb{P}^n$, and assume that $d \geq 2$.

General equidistribution theorem/conjecture

Let $\mathscr{H}_d(\mathbb{P}^n)$ denote the set of holomorphic endomorphisms of degree d on $\mathbb{P}^n$, and assume that $d \geq 2$.

Conjecture (Dinh–Sibony 2010)

For any $f \in \mathscr{H}_d(\mathbb{P}^n)$, any integer p with $1 \leq p \leq n-1$, and generic subvariety $Z \subseteq \mathbb{P}^n$ of dimension p, we have

$$\frac{1}{\deg Z}\frac{1}{d^{(n-p)k}}(f^k)^*[Z] \longrightarrow \mathscr{T}_f^{n-p}, \quad \text{as } k \to \infty,$$

where

$$\mathscr{T}_f := \lim_{k\to\infty} \frac{1}{d^k}(f^k)^*(\omega),$$

where ω is the Fubini–Study form cohomologous to a hyperplane in $\mathbb{P}^n$.

Dinh–Sibony's Theorem: True for "generic" f.

Conjecture (Dinh–Sibony 2010)

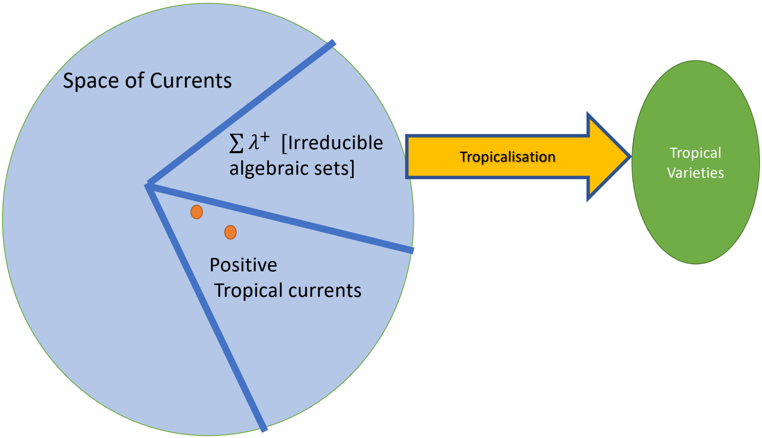
For any $f \in \mathscr{H}_d(\mathbb{P}^n)$*, any integer* p *with* $1 \leq p \leq n-1$*, and generic subvariety* $Z \subseteq \mathbb{P}^n$ *of dimension* p*, we have*

$$\frac{1}{\deg Z}\frac{1}{d^{(n-p)k}}(f^k)^*[Z] \longrightarrow \mathscr{T}_f^{n-p}, \quad \text{as } k \to \infty,$$

where $\mathscr{T}_f$ *is the Green current of* f*.*

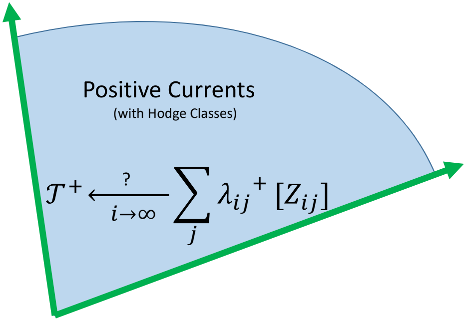
Theorem (Sturmfels–Tevelev 2007)

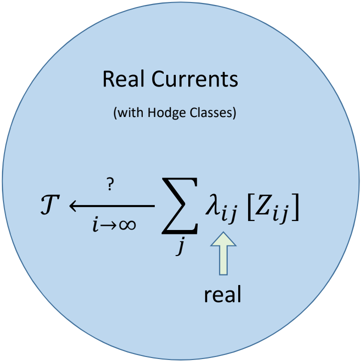
Let Σ *be a complete (rational) fan in* $\mathbb{R}^n$ *and* Z *be a* p*-dimensional subvariety of* $(\mathbb{C}^*)^n$*. Assume that the closure* $\bar{Z} \subseteq X_\Sigma$*, does not intersect any of the toric orbits of* X_Σ *of codimension greater than* p*. Then,* $\mathrm{trop}(Z)$ *equals the union of all* p*-dimensional cones* $\sigma \in \Sigma$ *such that* $\mathcal{O}_\sigma$ *intersects* $\bar{Z}$*.*



Space of Currents
$\sum \lambda^{+}$ [Irreducible algebraic sets]
Positive Tropical currents
Tropicalisation
Tropical Varieties

"Realisability" Question in Complex Geometry?





(HC^+) The Generalized Hodge Conjecture for Positive Currents (Demailly 1982)

An equivalent version of the Hodge Conjecture (Demailly 2012)

Tropical Approach to HC^+

- B 2014 : Introduction of the Tropical Currents
- B – Huh 2017:
  First Counter-example: A $(2,2)$-tropical current on a 4-dimensional toric variety
- Adiprasito – B 2018:
  A family of counterexamples in any dimension and codimension higher than 1

  Demailly 1982: True for $p = 0, n-1, n$.

Extremality

Example

- For (invariant) measures: Ergodicity

Extremality

Example

- For (invariant) measures: Ergodicity
- Integration currents along analytic sets: Sullivan, Lelong

Extremality

Example

- For (invariant) measures: Ergodicity
- Integration currents along analytic sets: Sullivan, Lelong
- Dynamical systems, codim 1: Demailly, Bedford–Lyubich–Smillie, Cantat, Diller–Favre, Fornaess–Sibony, ...
- Dynamical systems, any codim: Dinh–Sibony, Geudj
- Tropical approach in any dimension and codimension: B , B-Huh:

Extremality

Example

- For (invariant) measures: Ergodicity
- Integration currents along analytic sets: Sullivan, Lelong
- Dynamical systems, codim 1: Demailly, Bedford–Lyubich–Smillie, Cantat, Diller–Favre, Fornaess–Sibony, ...
- Dynamical systems, any codim: Dinh–Sibony, Geudj
- Tropical approach in any dimension and codimension: B , B-Huh:

$$\mathscr{C} \text{ strongly extremal } \iff \mathscr{T}_{\mathscr{C}} \text{ strongly extremal.}$$

Extremality

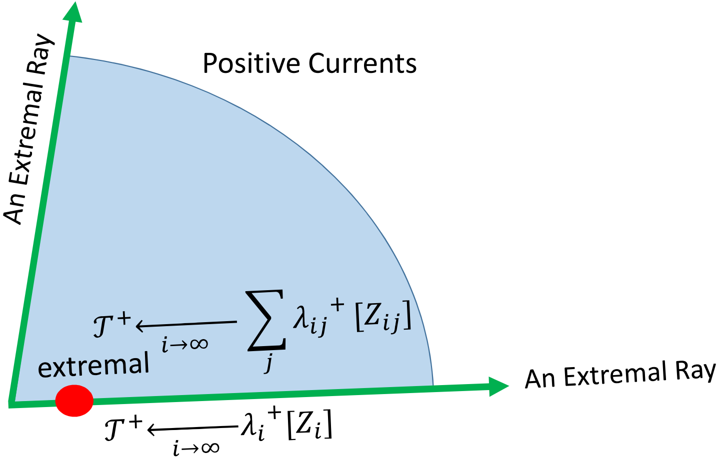
Example

- For (invariant) measures: Ergodicity
- Integration currents along analytic sets: Sullivan, Lelong
- Dynamical systems, codim 1: Demailly, Bedford–Lyubich–Smillie, Cantat, Diller–Favre, Fornaess–Sibony, ...
- Dynamical systems, any codim: Dinh–Sibony, Geudj
- Tropical approach in any dimension and codimension: B , B-Huh:

  $$\mathscr{C} \text{ strongly extremal } \iff \mathscr{T}_{\mathscr{C}} \text{ strongly extremal.}$$

- Extremal decomposition + a relation to rigidity theory with Sean Dewar and James Maxwell

What was the question?



Z_i: irreducible analytic

In summary, we find

A current on a 4-dimensional smooth projective variety which is

- Closed
- Positive
- with Hodge cohomology class
- Extremal
- With some cohomological obstructions

Tropical Geom	**Complex Geometry**
Varieties	Tropical Currents
Balancing Condition	Closedness
Trop	Dyn. Trop
Kapranov	Poincaré–Lelong
Intersection Theory	Superpotential Thoery
Tropicalisation of a family	Dyn. Trop.
Com. trop and and inters.	Commut lim and inters.

Three theorems

- Stable intersection of tropical currents:

$$(\mathscr{C}_1 + \epsilon v) \cap \mathscr{C}_2 \longrightarrow \mathscr{C}_1 \cdot \mathscr{C}_2,$$

  as $\epsilon \to 0$ and $v \in \mathbb{R}^n$ generic.

- Commuting intersection and tropicalisation (Osserman–Payne 2013)

$$\mathrm{trop}(Z_1 \cap Z_2) = \mathrm{trop}(Z_1) \cdot \mathrm{trop}(Z_2). \quad \text{(when proper)}$$

- Convergence of families (Jonsson's 2016): $V \subseteq (\mathbb{C}^*)^{n+1}$, $\pi : V \longrightarrow \mathbb{C}^*$ is subjective and flat:

$$\mathrm{trop}(V \cap \{z_{n+1} = t\}) = \mathrm{trop}(V) \cap \mathrm{trop}(\{z_{n+1} = t\}).$$

If $\mathscr{R}_n \longrightarrow \mathscr{R}$, and $\mathscr{T}_n \longrightarrow \mathscr{T}$ do we have

$$\mathscr{R}_n \wedge \mathscr{T}_n \longrightarrow \mathscr{R} \wedge \mathscr{T} \ ?$$

Equivalently,

$$\mathscr{R}_n \otimes \mathscr{T}_n \wedge [\Delta] \longrightarrow \mathscr{R} \otimes \mathscr{T} \wedge [\Delta] \ ?$$

where $\Delta \subseteq X \times X$ is the diagonal. More generally, when

$$\mathscr{S}_n \longrightarrow \mathscr{S} \implies \mathscr{S}_n \wedge [D] \longrightarrow \mathscr{S} \wedge [D] \ ?$$

where D is a prime divisor.

Intersection Theory of Currents

- Bedford–Taylor (1982) and Demailly: Codimension 1
- Dinh–Sibony's Superpotential Theory: Any dimension and codimension on $\mathbb{P}^n$ (2008) some case of Kähler manifolds.
- Dinh–Sibony's Densities of Currents (2010)
- Andersson–Samuelsson–Wulcan–Yger (2012)

Bedford–Taylor Theory

$\mathscr{T}$ closed, positive current of bidimension (p, p), $dd^c u$ a positive current.

$$dd^c\, u \wedge \mathscr{T} := dd^c\, (u\mathscr{T}).$$

The wedge product is well-defined, if

- u has is bounded
- u unbounded, but the unbounded locus of u intersects $\mathrm{supp}(\mathscr{T})$ with Cauchy–Riemann dimension less that p.

Dinh–Sibony's Superpotential Theory

$\mathscr{R}$ closed, positive current of bidimension (q, q) $q + p \geq n$. Choose ω is a differential form with $\{\omega\} = \{\mathscr{R}\}$. Hodge Theory implies that a current $U_{\mathscr{R}}$ exists such that

$$\mathscr{R} - \omega = dd^c U_{\mathscr{R}}$$

$$\mathscr{R} \wedge \mathscr{T} := dd^c(U_{\mathscr{R}} \wedge \mathscr{T}) + \omega \wedge \mathscr{T}.$$

The wedge product is well-defined if

- $\mathscr{R}$ has a continuous superpotential (the above product extends continuously from $\mathscr{T}$ smooth to all positive currents.)
- Good intersections of supports (the theory is complete for $\mathbb{P}^n$)

Theorem (B–Dinh)

Let $\mathscr{C}$ be a tropical cycle of dimension p compatible with a smooth, projective fan Σ, then $\overline{\mathscr{T}_{\mathscr{C}}}$ has a continuous superpotential in X_Σ.

Theorem (B-Dinh)

Assume that $\mathscr{C}$ and $\mathscr{C}'$ are two tropical varieties, then

$$\mathscr{T}_{\mathscr{C}} \wedge \mathscr{T}_{\mathscr{C}'} = \mathscr{T}_{\mathscr{C}\cdot\mathscr{C}'}$$

Theorem (B-Dinh)

Assume that $\mathscr{C}$ and $\mathscr{C}'$ are two tropical varieties, then

$$\mathscr{T}_{\mathscr{C}} \wedge \mathscr{T}_{\mathscr{C}'} = \mathscr{T}_{\mathscr{C}\cdot\mathscr{C}'}$$

Proof.

- The product makes sense
- The supports coincide
- The multiplicities coincide

$\square$

Theorem (B-Dinh)

Assume that $\mathscr{C}$ and $\mathscr{C}'$ are two tropical varieties, then

$$\mathscr{T}_{\mathscr{C}} \wedge \mathscr{T}_{\mathscr{C}'} = \mathscr{T}_{\mathscr{C}\cdot\mathscr{C}'}$$

Proof.

- The product makes sense
- The supports coincide
- The multiplicities coincide

$\square$

Analogous results to Osserman–Payne and Jonsson to appear soon.

Thanks for your attention!