4.2.2 Obtaining $\mathbb{P}^2$

Let $[x_0 : x_1 : x_2]$ denote the homogeneous coordinates of the space $\mathbb{P}^2$. It is covered by three coordinate charts:

- U_0 corresponding to $x_0 \neq 0$, with affine coordinates $(\frac{x_1}{x_0}, \frac{x_2}{x_0}) = (a_1, a_2)$.
- U_1 corresponding to $x_1 \neq 0$, with affine coordinates $(\frac{x_0}{x_1}, \frac{x_2}{x_1}) = (a_1^{-1}, a_1^{-1}a_2)$.
- U_2 corresponding to $x_2 \neq 0$, with affine coordinates $(\frac{x_0}{x_2}, \frac{x_1}{x_2}) = (a_2^{-1}, a_1 a_2^{-1})$.

As before, let $X_i = \xi_i(U_i)$, and $X_{ij} = \xi_i(U_i \cap U_j)$. We have $\mathbb{C}[X_0] = \mathcal{O}_{X_0}(X_0) = \mathbb{C}[a_1, a_2]$, and $\mathbb{C}[X_{01}] = \mathcal{O}_{X_0}(X_{01}) = \mathbb{C}[a_1^{-1}, a_1, a_2]$. Since on X_1, $a_1 \neq 0$, we can write

$$\mathbb{C}[X_1] = \mathcal{O}_{X_1}(X_1) = \mathbb{C}[a_1^{-1}, a_1^{-1}a_2].$$

As a result,

$$\mathbb{C}[X_{10}] = \mathcal{O}_{X_{10}}(X_{10}) = \mathbb{C}[a_1, a_1^{-1}, a_1^{-1}a_2].$$

The isomorphism

$$\begin{tikzcd} \xi_0(U_0 \cap U_1) \ni (a_1, a_2) \arrow[r, mapsto, "\xi_0^{-1}"] \arrow[rr, bend right, "g_{01} := \xi_1 \circ \xi_0^{-1}"'] & {[1 : a_1 : a_2]} \arrow[r, mapsto, "\xi_1"] & (1/a_1, a_2/a_1) \in \xi_1(U_0 \cap U_1), \end{tikzcd}$$

provides the information for gluing of $X_{01} \simeq \mathbb{C}^* \times \mathbb{C}$ and $X_{10} \simeq X_{01} \simeq \mathbb{C}^* \times \mathbb{C}$ and their corresponding coordinate rings. We can similarly understand the isomorphisms between other charts.

Torus Actions.

- On $U_0 \ni a = (a_1, a_2)$ the action of $t = (t_1, t_2) \in (\mathbb{C}^*)^2$ is obtained by $t \cdot a = (t_1 a_1, t_2 a_2)$.
- On $U_1 \ni b = (b_1, b_2)$ the action of $t = (t_1, t_2) \in (\mathbb{C}^*)^2$ is obtained by $t \cdot b = (t_1^{-1} b_1, t_1^{-1} t_2 b_2)$.
- On $U_2 \ni c = (c_1, c_2)$ the action of $t = (t_1, t_2) \in (\mathbb{C}^*)^2$ is obtained by $t\dot{c} = (t_2^{-1} c_1, t_1 t_2^{-1} c_2)$.

This is compatible with the gluing.

Exercise 4.19. (a) Find the rest of the isomorphisms $g_{ij} := \xi_j \circ \xi_i^{-1}$ for gluing in the above example.

(b) Can you use the set of vectors $\{(1,0), (0,1)\}, \{(-1,0), (-1,1)\}, \{(1,-1), (0,-1)\}$ to simplify your description?
Hint: Use $(a, b) \in \mathbb{Z}^2$ for $x^a y^b$.