Algebraic Geometry: Coursework 1.

① $A \subseteq \mathbb{A}^n$.

a). The closure of A in $\mathbb{A}^n$, denoted $\bar{A}$ is:

$$\bar{A} = A \cup \{\text{all limit points of } A\}.$$

Alternatively, the closure of A is the intersection of all closed sets containing A, or the smallest closed set containing A.

b). We will show that the smallest closed set containing A is $\mathbb{V}(\mathbb{I}(A))$. First, note that in the Zariski topology closed sets are exactly closed affine algebraic varieties. $\mathbb{V}(\mathbb{I}(A))$ is clearly a closed affine algebraic variety, and so is closed under the Zariski topology. We have:

$$\mathbb{V}(\mathbb{I}(A)) = \mathbb{V}(\{f \in \mathbb{C}[x_1, \dots, x_n] : f(x) = 0 \ \forall x \in A\})$$
$$= \{z \in \mathbb{C}^n : f(z) = 0 \ \forall f \in \mathbb{C}[x_1, \dots, x_n] \text{ such that } f(x) = 0 \ \forall x \in A\}.$$

So $\mathbb{V}(\mathbb{I}(A))$ is the set of all zeroes of all polynomials which vanish on A, so we must have $A \subseteq \mathbb{V}(\mathbb{I}(A))$. It remains to show that $\mathbb{V}(\mathbb{I}(A))$ is the smallest such Zariski-closed set. Suppose not: Let $B \subseteq \mathbb{A}^n$ be a Zariski-closed set such that: $A \subseteq B \subsetneq \mathbb{V}(\mathbb{I}(A))$. Then we must have that $B = \mathbb{V}(\{g_i\}_{i \in I})$, for some $g_i \in \mathbb{C}[x_1, \dots, x_n]$. By assumption, $A \subseteq B$, i.e. $A \subseteq \mathbb{V}(\{g_i\}_{i \in I})$. This means, by definition, that every g_i vanishes on all of A, i.e. $g_i \in \mathbb{I}(A) \ \forall i \in I$. Hence, we have $\cancel{\mathbb{I}(A) \subseteq \{g_i\}_{i \in I}}$ $\{g_i\}_{i \in I} \subseteq \mathbb{I}(A)$. Then, as Hilbert's correspondence is inclusion-reversing, we have $\mathbb{V}(\mathbb{I}(A)) \subseteq \mathbb{V}(\{g_i\}_{i \in I}) = B$. However, this contradicts our assumption, so we must have that $\mathbb{V}(\mathbb{I}(A))$ is the smallest Zariski-closed set containing A, i.e. is the Zariski closure of A.

c). Let $B = \{z \in \mathbb{C} : |z| \leq 1\}$. This is closed in the Euclidean topology (as it contains its boundary), so in this topology $B = \bar{B}$.
B is not a closed affine algebraic variety as it is a set of infinite cardinality which is not the entire complex plane. Thus, B is not closed in the Zariski topology, so $B \neq \bar{B}$.
Ergo, the closure of B in the Euclidean topology does not coincide with the closure of B in the Zariski topology.

(2)

a) We first define an open cover. Let X be a topological space and let $A \subset X$. A collection of sets $\{B_i\}_{i\in I}$ is an open cover for A if B_i is, for all $i \in I$, an open set in X, and $A \subseteq \bigcup_{i\in I} B_i$.
We can now define compactness:
Let $C \subset X$. C is compact if every open cover of C has a finite subcover, i.e. if $\{B_i\}_{i\in I}$ is an open cover of C, then C can be covered by a finite subcollection of these open sets, i.e. there exist $B_1, \ldots, B_n \in \{B_i\}_{i\in I}$ (without loss of generality) such that $C = \bigcup_{j=1}^{n} B_j$.

b) First, note that $\mathbb{C}^2$ with the Euclidean topology is a metric space. In a metric space, a set is compact if and only if it is closed and bounded. Thus, $\mathbb{V}(x^2-y^3)$ is not compact in the Euclidean topology as it is not a bounded set.
In the Zariski topology, we will prove the more general result that any subset $V \subseteq \mathbb{A}^n$ is compact.
Let $\mathcal{B} = \{B_i\}_{i\in I}$ be an open cover of V. Each $B_i \in \mathcal{B}$ is by definition the complement of some closed affine algebraic variety, call it V_i. Pick some $B_i \in \mathcal{B}$, and then if $V_i \cap V = \emptyset$, $V \subseteq B_i$, so B_i is a finite subcover and we are done. Otherwise, we can iterate this process to get (without loss of generality) $V_1 \cap \cdots \cap V_n$ and $B_1 \cup \cdots \cup B_n$. As before, if $V \cap V_1 \cap \cdots \cap V_n = \emptyset$, then $V \subseteq B_1 \cup \cdots \cup B_n$ and thus $\{B_1, \ldots, B_n\}$ is a finite subcover. If this isn't the case, we get the inclusion chain:

$$\cdots \subseteq V_1 \cap \cdots \cap V_2 \subseteq \cdots \subseteq V_2 \cap V_1 \subseteq V_1$$

and so using Hilbert's correspondence,

$$\mathbb{I}(V_1) \subseteq \cdots \subseteq \mathbb{I}(V_1 \cap \cdots \cap V_n) \subseteq \cdots$$

By Hilbert's Basis theorem, $\mathbb{C}[x_1, \ldots, x_m]$ is Noetherian, so this chain of ideals must stabilise, assume at $\mathbb{I}(V_1 \cap \cdots \cap V_m)$.
As, for every B_j we pick, $V_j \cap V \cap V_1 \cap \cdots \cap V_m \subseteq V \cap V_1 \cap \cdots \cap V_m$, we have two cases to consider:

- $V_j \cap V \cap V_1 \cap \cdots \cap V_m = V \cap V_1 \cap \cdots \cap V_m$. In this case, the intersection never shrinks to the empty set, so $V \not\subseteq B_1 \cup B_2 \cup \cdots$, contradicting that $\mathcal{B}$ is an open cover for V;
- $V_j \cap V \cap V_1 \cap \cdots \cap V_m \subsetneq V \cap V_1 \cap \cdots \cap V_m$. In this case, as Hilbert's correspondence is inclusion reversing, we get $\mathbb{I}(V \cap V_1 \cap \cdots \cap V_m) \subsetneq \mathbb{I}(V_j \cap V \cap V_1 \cap \cdots \cap V_m)$, which contradicts our assumption that the chain of ideals had stabilised.

Hence, there must be some n such that $V \cap V_1 \cap \cdots \cap V_n = \emptyset$, meaning $\{B_1, \ldots, B_n\}$ is a finite subcover. As

$\mathbb{V}(x^2-y^3)$ is a subset of $\mathbb{A}^1$, this general result immediately implies that $\mathbb{V}(x^2-y^3)$ is compact in the Zariski topology.

③

a) First, we recall that a curve is the zero-set of a single non-constant polynomial. Let $W = \mathbb{V}(x^2-y^2) \subseteq \mathbb{A}^2$. Then by theorem 2.24, $\dim W = 2-1 = 1$. Hence, W is irreducible, as we have by the definition of dimension, that $W = W_1 \supsetneq W_0 = \{pt\}$ is the largest chain of irreducible subvarieties, i.e. W has no irreducible subvarieties, i.e. W is itself irreducible.
Define the morphism $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$, $(x,y) \mapsto (x^2, y^2)$.
Then $\varphi^{-1}(W) = \varphi^{-1}(\mathbb{V}(x^2-y^2)) = \mathbb{V}(x-y, x+y) = \mathbb{V}(x+y) \cup \mathbb{V}(x-y)$, so the pre-image of W under φ is not irreducible, as $\varphi^{-1}(W)$ can be decomposed.

b) Let X be irreducible, but suppose the closure of X, $\bar{X}$, is not. Then:

$$\bar{X} = Y_1 \cup Y_2 \quad \text{for some closed sets } Y_1, Y_2.$$

Then we have:

$$X = X \cap \bar{X} = X \cap (Y_1 \cup Y_2) = (X \cap Y_1) \cup (X \cap Y_2),$$

i.e.

$$X = (Y_1 \cap X) \cup (Y_2 \cap X),$$

~~which would imply that X is reducible, a contradiction.~~
and as X is irreducible without loss of generality we must then have

$X \subseteq Y_1$, but as Y_1 is a closed set, this would imply $\bar{X} \subseteq Y_1$ (equivalently $\bar{X} \subseteq Y_2$), which precisely gives that $\bar{X}$ is itself irreducible.

c) Let V, W be two closed affine algebraic varieties, and assume V is irreducible. Also, suppose $\varphi : V \to W$ is an isomorphism. Then, by theorem 2.39, there exists an ~~isomorphism~~ morphism $\Phi = \varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$.
As V is irreducible, $\mathbb{I}(V)$ is prime (by theorem 2.16).
Hence, $\mathbb{C}[V] = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{\mathbb{I}(V)}$
is an integral domain.
Using a result in the notes, as φ is an isomorphism, Φ is also an isomorphism.
Now, let $a, b \in \mathbb{C}[W]$ such that $ab = 0 \in \mathbb{C}[W]$.
Then, using the properties of isomorphisms, we have

$\Phi(ab) = \Phi(a)\,\Phi(b) = \Phi(0) = 0 \in \mathbb{C}[V]$.
As $\mathbb{C}[V]$ is an integral domain, we must have either $\Phi(a) = 0$ or $\Phi(b) = 0$. Then, as Φ is an isomorphism, either $a = 0$ or $b = 0$ (in $\mathbb{C}[W]$), and Hence,

$$\mathbb{C}[W] = \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbb{I}(W)}$$

is an integral domain. Hence, $\mathbb{I}(W)$ is a prime ideal, so by theorem 2.16, W is irreducible, i.e. isomorphisms preserve irreducibility.
Now, assume $\dim(V) = d$. Then, there exists a chain of subvarieties:

$$V = V_d \supsetneq V_{d-1} \supsetneq \cdots \supsetneq V_0 = \{pt\}.$$

As φ is an isomorphism, it is inclusion preserving:

$$\varphi(V) = W = \varphi(V_d) \supsetneq \varphi(V_{d-1}) \supsetneq \cdots \supsetneq \varphi(V_0) = \varphi(\{pt\}).$$

By the above work, we know φ preserves the irreducibility of (sub) varieties, so then $\dim(W) \geq d$.
Assume there exists $e > d$ such that $\dim(W) = e$. Then there exists a chain of irreducible subvarieties:

$$W = W_e \supsetneq W_{e-1} \supsetneq \cdots \supsetneq W_0 = \{pt\}.$$

Note that φ^{-1} exists and is an isomorphism as φ is an isomorphism. Thus:

$$\varphi^{-1}(W) = V = \varphi^{-1}(W_e) \supsetneq \varphi(W_{e-1}) \supsetneq \cdots \supsetneq \varphi(W_0) = \varphi^{-1}(\{pt\}).$$

But this implies that $\dim V = e > d$, which is a contradiction.
Hence, $\dim W = d$, i.e. isomorphisms preserve dimension.

d) We note that we ~~gan~~ can use the relations $y = zx$, $y^2 - x^2(x+1) = 0$ to substitute in and show $x^2(z^2 - x - 1) = 0$ and hence that:

$$\mathbb{V}(zx - y,\ y^2 - x^2(x+1)) = \mathbb{V}(zx - y,\ x^2(z^2 - x - 1))$$
$$= \mathbb{V}(zx - y,\ x^2) \cup \mathbb{V}(zx - y,\ z^2 - x - 1) = \mathbb{V}(zx - y,\ x) \cup \mathbb{V}(zx - y,\ z^2 - x - 1)$$
$$= \mathbb{V}(x, y) \cup \mathbb{V}(zx - y,\ z^2 - x - 1). \qquad (*)$$

As $\mathbb{C}[x,y,z]/(x,y) \cong \mathbb{C}[z]$ is an integral domain, (x,y) must be prime and so $\mathbb{V}(x,y)$ is irreducible.
Similarly, we consider the map $\varphi : \mathbb{C}[x,y,z] \to \mathbb{C}[t]$, $x \mapsto t^2 - 1$, $y \mapsto t^3 - t$, $z \mapsto t$. It is easy to see that φ is surjective and $\ker \varphi = (zx - y,\ z^2 - x - 1)$. By the isomorphism theorems, we thus have:

$$\frac{\mathbb{C}[x,y,z]}{(zx - y,\ z^2 - x - 1)} \cong \mathbb{C}[t].$$

As $\mathbb{C}[t]$ is an integral domain, $(zx - y,\ z^2 - x - 1)$ is prime and thus $\mathbb{V}(z^2 - x - 1)$ is irreducible. Hence, the irreducible decomposition is:

$$\mathbb{V}(zx - y,\ y^2 - x^2(x+1)) = \mathbb{V}(x,y) \cup \mathbb{V}(zx - y,\ z^2 - x - 1). \qquad (*).$$

(4)

a) First, we note that as V is Zariski-closed and $\mathbb{A}^n \setminus V \neq 0$, we ~~then~~ have that V is a finite subset, i.e. $V = \{v_1, \dots, v_k\} \subseteq \mathbb{A}^n$. We will index the components of each v_i thusly:

$$v_i = (v_{i_1}, \dots, v_{i_n}).$$

Similarly, we let $a = (a_1, \dots, a_n)$. It is then easy to see that the polynomial

$$g(x_1, \dots, x_n) = \prod_{i=1}^{k} (x_1 - v_{i_1}) \cdots \prod_{i=1}^{k} (x_n - v_{i_n})$$

vanishes on all of V, i.e. $g \in \mathbb{I}(V)$. This is because for every $v \in V$, at least one bracket in the above expansion will evaluate to 0. However, we have

$$g(a_1, \dots, a_n) = \prod_{i=1}^{k} (a_1 - v_{i_1}) \cdots \prod_{i=1}^{k} (a_n - v_{i_n})$$
$$= \prod_{j=1}^{n} \left[\prod_{i=1}^{k} (a_j - v_{j_i}) \right],$$

which generally will not ~~be~~ ~~on~~ evaluate to 1.
However, we can guarantee this evaluation by multiplying by a constant term:

$$\alpha = \left[\prod_{j=1}^{n} \left[\prod_{i=1}^{k} (a_j - v_{j_i}) \right] \right]^{-1},$$

which is possible as $a_\ell, v_{m,k} \in \mathbb{C}$, and $\mathbb{C}$ is a field.
Now, let $f(\underline{x}) = \alpha \cdot g(\underline{x}) \in \mathbb{C}[x_1, \dots, x_n]$. Then, $\forall v \in V$ $f(v) = \alpha \cdot g(v) = \alpha \cdot 0 = 0$, so $f \in \mathbb{I}(V)$. Further,

$$f(a) = \alpha \cdot g(a) = \frac{\prod_{j=1}^{n} \left[\prod_{i=1}^{k} (a_j - v_{j_i}) \right]}{\prod_{j=1}^{n} \left[\prod_{i=1}^{k} (a_j - v_{j_i}) \right]} = 1,$$

so this polynomial satisfies the required conditions. (I feel like there should be a less constructive solution, this is the only intuition I had).

b)(i) Let $I = (f_1, \dots, f_k)$ and assume $\mathbb{V}(g) \supseteq \mathbb{V}(I)$. Then $\mathbb{V}(g) \supseteq \mathbb{V}(f_1, \dots, f_k) = \mathbb{V}(f_1) \cap \dots \cap \mathbb{V}(f_k)$ in $\mathbb{A}^n$. We note that the same inclusion applies in $\mathbb{A}^{n+1}$, i.e. $\forall a \in \mathbb{A}^{n+1}$, if $f_i(a) = 0$ for all $1 \leq i \leq k$, then $g(a) = 0$. Now, let $a = (a_1, \dots, a_{n+1}) \in \mathbb{V}(f_1, \dots, f_k) \subseteq \mathbb{A}^{n+1}$. Then $a_{n+1} g(a) - 1 = -1 \neq 0$, and hence $\mathbb{V}(x_{n+1} g - 1) \cap \mathbb{V}(f_1, \dots, f_k) = \emptyset$. Let $J = (f_1, \dots, f_k, x_{n+1} g - 1)$. Then we have $\mathbb{V}(J) = \mathbb{V}(f_1, \dots, f_k, x_{n+1} g - 1) = \mathbb{V}(f_1, \dots, f_k) \cap \mathbb{V}(x_{n+1} g - 1) = \emptyset$. Using the Nullstellensatz, we then have $\mathbb{I}(\mathbb{V}(J)) = \mathbb{I}(\emptyset) = \mathbb{C}[x_1, \dots, x_{n+1}] = \sqrt{J}$.
From here, we see that $1 \in \sqrt{J}$, and hence that $1 \in J$. So we now have that J is an ideal of $\mathbb{C}[x_1, \dots, x_{n+1}]$ which contains the multiplicative identity, hence

$$J = (f_1, \dots, f_k, x_{n+1} g - 1) = \mathbb{C}[x_1, \dots, x_{n+1}].$$

(ii)

ver

(5)

a) We first note that $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is injective if and only if $\ker \varphi^*$ is trivial. Further, we note that $\varphi(V)$ being dense in W is precisely the same as the closure of $\varphi(V)$, $\overline{\varphi(V)}$, being W. We recall the form of φ^*:

$$\varphi^* : \frac{\mathbb{C}[x_1,\dots,x_m]}{\mathbb{I}(W)} \to \frac{\mathbb{C}[x_1,\dots,x_n]}{\mathbb{I}(V)}, \quad g \mapsto g \circ \varphi.$$

We see that $\ker \varphi^*$ being trivial is precisely the same as the map $\psi : \mathbb{C}[x_1,\dots,x_m] \to \mathbb{C}[V]$, $f \mapsto f \circ \varphi$ having $\ker \psi = \mathbb{I}(W)$. We define a further map:

$$\theta : \mathbb{C}[x_1,\dots,x_m] \to \mathbb{C}[x_1,\dots,x_n], \quad f \mapsto f \circ \varphi.$$

We now note that $\ker(\psi) = \theta^{-1}(\mathbb{I}(V))$, so $\ker \varphi^*$ is trivial (i.e. φ^* is injective) if and only if $\theta^{-1}(\mathbb{I}(V)) = \mathbb{I}(W)$:

$$\theta^{-1}(\mathbb{I}(V)) = \{f \in \mathbb{C}[x_1,\dots,x_m] : f \circ \varphi(v) = 0 \ \forall v \in V\}$$
$$= \{f \in \mathbb{C}[x_1,\dots,x_m] : f(w) = 0 \ \forall w \in \varphi(V)\} = \mathbb{I}(\varphi(V)).$$

Now, we can see φ^* is injective if and only if $\mathbb{I}(W) = \mathbb{I}(\varphi(V))$. Also, we have the implication chain:

$$\mathbb{I}(W) = \mathbb{I}(\varphi(V)) \iff \mathbb{V}(\mathbb{I}(W)) = \mathbb{V}(\mathbb{I}(\varphi(V))) \iff \mathbb{V}(\mathbb{I}(\varphi(V))) = W.$$

By question (1b), $\mathbb{V}(\mathbb{I}(\varphi(V))) = \overline{\varphi(V)}$. So, we can conclude that φ^* is injective if and only if $\varphi(V)$ is dense in W.

b) We will prove one implication. Assume that $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective. Then, by the homomorphism theorem, $\frac{\mathbb{C}[W]}{\ker \varphi^*} \cong \operatorname{im} \varphi^*$.

Now, let $\pi : \mathbb{C}[x_1, \dots, x_m] \to \mathbb{C}[W]$ be the quotient map. Then using the correspondence theorem, we know the preimage of $\ker \varphi^*$ under π is an ideal, I, of $\mathbb{C}[x_1, \dots, x_m]$ containing $\mathbb{I}(W)$, and

$$\frac{\mathbb{C}[x_1, \dots, x_m]}{I} \cong \frac{\mathbb{C}[W]}{\ker \varphi^*} \cong \operatorname{im} \varphi^* = \mathbb{C}[V]$$

By definition, $\mathbb{I}(W) \subset I$, so, by Hilbert's correspondence, $W = \mathbb{V}(\mathbb{I}(W)) \supseteq \mathbb{V}(I) =: U$.

As I is the preimage of $\mathbb{I}(V)$ under the map $\Phi : \mathbb{C}[x_1, \dots, x_m] \to \mathbb{C}[x_1, \dots, x_n]$, $f \mapsto f \circ \varphi$, (note that this is distinct from φ^* as it is defined over the whole ~~coordinate~~ polynomial rings) I is radical. Thus, by the Nullstellensatz, $\mathbb{I}(\mathbb{V}(I)) = I$. By our definition of the coordinate rings, we hence have

$$\mathbb{C}[U] = \frac{\mathbb{C}[x_1, \dots, x_m]}{\mathbb{I}(U)} = \frac{\mathbb{C}[x_1, \dots, x_m]}{\mathbb{I}(\mathbb{V}(I))} = \frac{\mathbb{C}[x_1, \dots, x_m]}{I}.$$

Thus,

$$\mathbb{C}[U] = \frac{\mathbb{C}[x_1, \dots, x_m]}{I} \cong \frac{\mathbb{C}[W]}{\ker \varphi^*} \cong \operatorname{im} \varphi^* = \mathbb{C}[V],$$

i.e. $\mathbb{C}[U] \cong \mathbb{C}[V]$. But by a result in the notes, $\mathbb{C}[U] \cong \mathbb{C}[V]$ if and only if $U \cong V$. Thus, if φ^* is surjective, then φ defines an isomorphism between V and some algebraic subvariety of W.