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Algebraic Geometry 2

1a) As $\mathbb{C}$ is a field, $\mathbb{C}[X]$ is a PID, all ideals are principal.

$$\operatorname{maxSpec}(\mathbb{C}[X]) = \{(f) \mid f \in \mathbb{C}[X] \setminus \mathbb{C}, \text{ irreducible}\}$$

The Fundamental Theorem of Algebra tells us that the only irreducible polynomials will be degree one polynomials, of the form $X - c$, $c \in \mathbb{C}$.

$$\therefore \operatorname{maxSpec}(\mathbb{C}[X]) = \{(X-c) \mid c \in \mathbb{C}\}$$

For $\operatorname{maxSpec}(\mathbb{C}[X, 1/X])$, we note that

$\mathbb{C}[X, 1/X] \cong \dfrac{\mathbb{C}[X,Y]}{(XY-1)}$. To find $\operatorname{maxSpec}\left(\dfrac{\mathbb{C}[X,Y]}{(XY-1)}\right)$, we use $V = \mathbb{V}(XY-1) \subseteq \mathbb{A}^2$.

$$\therefore \operatorname{maxSpec}\left(\frac{\mathbb{C}[X,Y]}{(XY-1)}\right) = \cancel{\{X-a, Y-1/a \mid a \in \mathbb{C}\}}$$
$$= \{X-a, Y-1/a \mid a \in \mathbb{C} \setminus \{0\}\}$$

As $Y = 1/X$, $\quad X^{-1} - \frac{1}{a} = -\frac{1}{aX}(X-a) \in (X-a)$

$$\therefore \operatorname{maxSpec}(\mathbb{C}[X, X^{-1}]) = \{(X-a) \mid a \in \mathbb{C}^\times\}$$

We can use these 2 previous results to say that:

$$\operatorname{maxSpec}(\mathbb{C}[X, X^{-1}, Y]) = \{(X-\alpha, Y-\beta), \alpha \in \mathbb{C}^\times, \beta \in \mathbb{C}\}$$

b) For $a \in \mathbb{A}^1 \setminus \{0\}$, $\varphi^*(f)(a) = (f \circ \varphi)(a)$
$= f(\varphi(a))$

As we take $f := X^{-1}$, $f_*(\varphi(a))$
$= f_*(a^{-1})$
$= a$

$\therefore$ ~~φ^{-1}~~ $\varphi^*(X^{-1}) = Y$

Let $g = 2x^2 + \dfrac{2x^3 + 4X}{X^5}$

$$\varphi^*(g(a)) = (g \circ \varphi)(a) = g(\varphi(a))$$
$$= \frac{2}{a^2} + \frac{\frac{2}{a^3} + \frac{4}{a}}{\frac{1}{a^5}}$$
$$= \frac{2}{a^2} + 2a^2 + 4a^4$$

$\therefore \varphi^*(g) = 2Y^{-2} + 2Y^2 + 4Y^4$

Let $h = 2 - X$,

$$\varphi^*(h)(a) = (h \circ \varphi)(a) = h(\varphi(a))$$
$$= h\left(\tfrac{1}{a}\right) = 2 - \tfrac{1}{a}$$

$\varphi^*(h) = 2 - Y^{-1}$

2a) Let $\pi : \mathbb{A}^3 \to \mathbb{A}^2$. We want to show that this restricts to a morphism, $\varphi : V \to \mathbb{A}^2$. $(V = \mathbb{V}(Y - uX))$

Let $\varphi = \pi|_V : V \to \mathbb{A}^2$.

As this is a restriction of a polynomial map, it is by definition a morphism.

Let $\psi : \mathbb{A}^2 \to V$

$(x, u) \to (x, xu, u)$

This is well defined as $(x)(u) - (xu) = 0$, so is in $V = \mathbb{V}(Y - uX)$

Let $(x, u) \in \mathbb{A}^2$, then $(\varphi \circ \psi)(x, u)$

$= \varphi(x, xu, u)$

$= (x, u)$

So, $\varphi \circ \psi = Id_{\mathbb{A}^2}$

Let $(x, y, u) \in V$, $(\psi \circ \varphi)(x, y, u)$

$= \psi(x, xu, u) = (x, y, u)$

So, $\psi \circ \varphi = Id_V$

So, φ and ψ are ~~was~~ inverse morphisms of each other, so are isomorphisms.

$\therefore$ π restricts to an isomorphism $\varphi : V \to \mathbb{A}^2$ as required

b) We have $\pi' : \mathbb{A}^3 \to \mathbb{A}^2$. By the same logic in part (a), this restricts to a morphism $\varphi' : V \to \mathbb{A}^2$.

For contradiction, we assume $\psi' : \mathbb{A}^2 \to V$ is an inverse morphism

For $(x, y, u) \in V$,

$$(\psi' \circ \varphi')(x, y, u) = \psi'(x, y)$$
$$= \left(x, y, \frac{y}{x}\right)$$

So, $u = \frac{y}{x}$, $\forall x, y \in \mathbb{C}^*$, and so u cannot be a polynomial

So, ~~ψ~~ ψ' is not a restriction ~~of~~ a polynomial map.
Hence, we have our contradiction as ψ' is not a morphism.
so φ' is not an ~~an~~ isomorphism.

3a)

Using Theorem 3.28, which says, $\overline{V} = \mathbb{V}(\tilde{I})$, and as we have:
$\overline{\mathbb{V}(g)} = \mathbb{V}((\tilde{g})) \subseteq \mathbb{P}^2$, Then By Nullstellensatz, we need to show $\widetilde{(g)} = (\tilde{g})$

$(\tilde{g}) \subseteq \widetilde{(g)}$ comes as $g \in (g)$, So $\tilde{g} \in \widetilde{(g)}$.

For $h \in \widetilde{(g)}$, we can write h as:

$$h = \alpha_1 \tilde{x}_1 + \dots + \alpha_n \tilde{x}_n, \quad \alpha_i \in \mathbb{C}[X,Y,Z], \quad x_i \in (g).$$

We can rewrite each x_i as:

$$x_i = \beta_i g, \quad \text{for } \beta_i \in \mathbb{C}[x,y]$$

So now,

$$h(X,Y,1) = \alpha_1(X,Y,1) \tilde{x}_1(X,Y,1) + \dots + \alpha_n(X,Y,1) \tilde{x}_n(X,Y,1)$$

$$= \alpha_1(X,Y,1)\beta_1 g + \dots + \alpha_n(X,Y,1)\beta_n g$$
$$= (\alpha_1(X,Y,1)\beta_1 + \dots + \alpha_n(X,Y,1)\beta_n) g$$

So, h is just the homogenisation of an element of g, so we have $\widetilde{(g)} \subseteq (\tilde{g})$, and By double inclusion,
$$(\tilde{g}) = \widetilde{(g)}.$$

3b) part (a) gives $\overline{\mathbb{V}(f_i)} = \mathbb{V}(\tilde{f_i})$.

Homogenising each f_i into $\mathbb{C}[X, Y, W]$,

$$\tilde{f}_1 = X + Y + W$$
$$\tilde{f}_2 = X^2 + 6Y^2 + W^2$$
$$\tilde{f}_3 = X^2 + 3YW + W^2$$

$$\tilde{f}_4 = X^3 + 3XY^2 + 4W^3$$

Want to see if substituting $[1:0:0]$, $[0:1:0]$ or $[0:0:1]$ equates any $\tilde{f}_i$ to 0.

Only $\tilde{f}_3$ and $\tilde{f}_4$ vanish during the substitution of $[0:1:0]$.

3c) If we let $g = g_1(X) + \cancel{g_2(Y)}\, g_2(Y)$, a non zero polynomial of $\mathbb{C}[X, Y]$

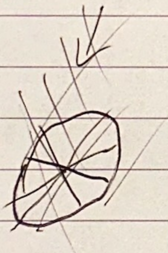
A necessary condition is that $\deg(g_1) = \deg(g_2)$, with a non zero constant term to avoid $[0:0:1]$ passing through.
Also, the leading coefficients of x^d and y^d are nonzero so $[1:0:0]$ and $[0:1:0]$ doesn't pass through.

4a) Take the unit sphere, $S^n = \{ x \in A^{n+1} : ||x|| = 1 \}$
in ~~blah~~ A^{n+1}. Heine-Borel tells us S^n is compact, as its
closed and Bounded.

We can rewrite every nonzero vector in ~~A^{n+1}~~ $A^{n+1}\setminus\{0\}$, as $x = \lambda p$, $\lambda \neq 0$, $p \in S^n$.
Each line will intersect S^n at p and $-p$.

Let f be the map of $S^n \to \mathbb{P}^n$
$p \to [p]$, the equivalence class of p in $\mathbb{P}^n$.
So, $f(p) = f(-p)$, $\forall p \in S^n$.

So, we get a continuous and surjective map between
$\frac{S^n}{\sim} \to \mathbb{P}^n$. As $S^n/\sim$ is compact, then $\mathbb{P}^n$ is also
compact in the quotient Euclidean Topology.



4b) As $\sin x$ is Transcendental, then for the set $\mathbb{V}(y - \sin x)$, it is not algebraic and the only ~~all~~ Polynomial vanishing at every point is the zero polynomial in Zariski Topology.
So, the Zariski closure of $\mathbb{V}(y - \sin x)$ in $\mathbb{A}^2$ is $\mathbb{A}^2$.

Going from $\mathbb{A}^2$ to $\mathbb{P}^2$, taking the Zariski closure of the image of $\mathbb{V}(y - \sin x)$ will leave us with all of $\mathbb{P}^2$, as no nonzero polynomial will vanish on $\{[x : \sin x : 1]\}$.

As $\overline{\mathbb{V}(y - \sin x)}$ is compact, but not algebraic then Chow's Lemma tells us that it is not an analytic subvariety of $\mathbb{P}^2$.

5a) we can equate $\mathbb{P}^2 = \mathbb{C}^2 \sqcup \mathbb{P}^1$

We Let $L_1 = \mathbb{V}(aX+bY+cZ)$,
$L_2 = \mathbb{V}(\alpha X+\beta Y+\gamma Z)$, $(a,b,c) \neq (\alpha, \beta, \gamma)$ as our lines are distinct.

$|L_1 \cap L_2 \cap \mathbb{C}^2| \leq 1$, for $[x:y:1] \in L_1 \cap L_2 \cap \mathbb{C}^2$

For ~~$\gamma \neq 0$~~, $\beta y = -\alpha x - \gamma$, $by = -ax - c$.

So, if $\frac{\alpha}{\beta} \neq \frac{a}{b}$, then the gradients of our two lines are different, and will, by definition, intersect once in $\mathbb{C}^2$.

However, if $\frac{\alpha}{\beta} = \frac{a}{b}$, we have two lines with the same gradient,

So, $\exists$ a point, $p \in L_1 \cap L_2 \cap \mathbb{P}^1$.

(We let $[x:y:0] \in L_1 \cap \mathbb{P}^1$,
so $ax+by=0$, $y = -\frac{a}{b}x$

and $y = -\frac{\alpha}{\beta}x$

So $\alpha x + \beta y = 0$, $[x:y:0] \in L_1 \cap L_2 \cap \mathbb{P}^1$)

We want to find $\overbrace{\mathbb{V}(aX+bY+cZ)}^{L_{1,a,b,c}} \cap \mathbb{P}^1$,
Take $[x:y:0] \in L_{1,a,b,c} \cap \mathbb{P}^1$,
then $aX+bY=0$, so $y = \frac{-a}{b}x$

So, $[X:y:0] = [1:\frac{a}{b}:0]$

Which is a solution, so $L_{1, a,b,c} = \{[1: \frac{a}{b}: 0]\}$

So, L_1 and L_2 intersect in $\mathbb{C}^2$ iff $\frac{a}{b} \neq \frac{\alpha}{\beta}$ by Euclidean Geometry, and only intersect once.

If $\frac{a}{b} = \frac{\alpha}{\beta}$, then we showed that L_1 and L_2 intersect in $\mathbb{P}^2$ at most once.

5bi) Each C_i is of the form $\mathbb{V}(f_i)$, $f_i \in \mathbb{C}[x,y]$

(3a) gives $\overline{C_1} = \overline{\mathbb{V}(f_1)} = \mathbb{V}(\tilde{f_1}) \subseteq \mathbb{P}^2$.

$$\overline{C_2} = \overline{\mathbb{V}(f_2)} = \mathbb{V}(\tilde{f_2}) \subseteq \mathbb{P}^2$$

$$C_1 \cap C_2 = \mathbb{V}(f_1) \cap \mathbb{V}(f_2) = \mathbb{V}(f_1, f_2)$$

$$\Rightarrow \overline{C_1 \cap C_2} = \mathbb{V}(\widetilde{(f_1, f_2)})$$

$$\mathbb{V}(\widetilde{(f_1, f_2)}) \subseteq \mathbb{V}((\tilde{f_1}) + (\tilde{f_2}))$$

Nullstellensatz tells us the inclusion holds if and only if

$\widetilde{(f_1, f_2)} \supseteq (\tilde{f_1}, \tilde{f_2})$

which comes by the homogenisation of an ideal trivially.

5bii) We want to show $\widetilde{(f_1,f_2)} \supset (\tilde{f_1},\tilde{f_2})$.

we need ~~on~~ to show that there is an element of $\widetilde{(f_1,f_2)}$, that isn't an element of $(\tilde{f_1},\tilde{f_2})$.

Let $f_1 = x^3+1$, $\tilde{f_1} = x^3+z^3$

$f_2 = x^3+y+1$, $\tilde{f_2} = x^3+yz^2+z^3$

$f_2 - f_1 = y$, so $y \in (f_1,f_2) \Rightarrow y \in \widetilde{(f_1,f_2)}$.

$(\tilde{f_1},\tilde{f_2})$ However, doesn't have y as an element,

$y \notin (\tilde{f_1},\tilde{f_2})$, so we have the strict inclusion, $(\tilde{f_1},\tilde{f_2}) \subset \widetilde{(f_1,f_2)}$

Nullstellensatz: $\mathbb{V}\overline{(f_1,f_2)} \subset \mathbb{V}\overline{(f_1)} \cap \mathbb{V}\overline{(f_2)}$

$\Rightarrow C_1 \cap C_2 \subset \overline{C_1} \cap \overline{C_2}$

6(a) To show that $\mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra, we need to show it is a $\mathbb{C}$-vector space and a ring.

If we let $f = \lambda g + h$.
we need that for all points, $p \in O$, we have a U_p is an open neighbourhood of p.

So, $h|_{U_p} = \frac{A}{B}\Big|_{U_p}$ and $g|_{V_p} = \frac{X}{Y}\Big|_{V_p}$,

For V_p an open neighbourhood of p.

We take the intersection, $U_p \cap V_p = W_p$, another open neighbourhood of p.

So, $f|_{W_p} = (\lambda g + h)|_{W_p} = \left(\lambda \frac{X}{Y} + \frac{A}{B}\right)\Big|_{W_p}$

$$= \left(\frac{\lambda XB + AY}{BY}\right)\Big|_{W_p} \in \mathcal{O}_Y(O),$$

So $\lambda g + h \in \mathcal{O}_Y(O)$, $\mathcal{O}_Y(O)$ is a $\mathbb{C}$-vector space.

We need to show it is closed under multiplication to be a ring.

$$(gh)|_{W_p} = (g|_{W_p})(h|_{W_p}) = \left(\frac{X}{Y}\frac{A}{B}\right)\Big|_{W_p} = \left(\frac{XA}{YB}\right)\Big|_{W_p},$$

So closed under multiplication, $gh \in \mathcal{O}_Y(O)$, so $\mathcal{O}_Y(O)$ is a ring and so a $\mathbb{C}$-algebra.

6bi) $f \in \mathcal{O}_X(V) \Rightarrow f|_U \in \mathcal{O}_X(U)$, $U \subseteq V$

For $f \in \mathcal{O}_X(V)$, then taking a point, $x \in U$, then we have an open neighbourhood U_x, and we write f as a quotient of polynomials, (non with a nonzero denominator), then $f|_U$ will also be regular, so $f|_U \in \mathcal{O}_X(U)$

6bii) (6a) gives us that $O_X(U)$ is a $\mathbb{C}$-algebra, which means it is a ring, and satisfies (i)

(6bi) satisfies part (ii) as it goes into detail about the restriction map.

Take an open cover of U, call it $\{U_i\}_i$.
We need f_i as an element of $O_X(U_i)$ to agree on an overlap.
We let $f(p) = f_i(p)$ if $p \in U_i$, $f: U \to \mathbb{C}$, as this comes from f_i agreeing on intersections as taking a point, q.
$q \in U$ means we have some U_i, with $q \in U_i$, containing the open neighbourhood of q, with $f|_{U_i} = f_i$ by some rational function.
This satisfies (iii) as it tells us that $f \in O_X(U)$

Take distinct $f, f' \in O_X(U)$. Then for some point $p \in U$, where $f(p) \neq f'(p)$. Take an open cover $\{U_i\}_i$ of U, with p as an element of U_i, so $f|_{U_i} \neq f'|_{U_i}$, satisfying part (iv)