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Algebraic Geometry
Coursework 2

1. (a) Consider the objects $\operatorname{maxSpec}(\mathbb{C}[x])$, $\operatorname{maxSpec}(\mathbb{C}[x, 1/x])$ and $\operatorname{maxSpec}(\mathbb{C}[x, 1/x, y])$. They can be described as follows.

Proof. Noting that the maximal spectrum is given by the maximal ideals of the ring, we can see that

$$\begin{aligned}
\operatorname{maxSpec}(\mathbb{C}[x]) &= \{(x-a) \mid a \in \mathbb{C}\}, \\
\operatorname{maxSpec}(\mathbb{C}[x, 1/x]) &= \{(x-a, 1/x-b) \mid a, b \in \mathbb{C}\}, \\
\operatorname{maxSpec}(\mathbb{C}[x, 1/x, y]) &= \{(x-a, 1/x-b, y-c) \mid a, b, c \in \mathbb{C}\}.
\end{aligned}$$

Knowing our correspondence between the maximal spectrum and points in a variety, we can now instead describe the spectra as

$$\begin{aligned}
\operatorname{maxSpec}(\mathbb{C}[x]) &= \mathbb{C}, \\
\operatorname{maxSpec}(\mathbb{C}[x, 1/x]) &= \mathbb{C} \setminus \{0\} = \mathbb{C}^*, \text{ and} \\
\operatorname{maxSpec}(\mathbb{C}[x, 1/x, y]) &= \mathbb{C}^* \times \mathbb{C}.
\end{aligned}$$

□

(b) Consider the isomorphism $\varphi : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$ given by $a \mapsto 1/a$ and the pullback map between coordinate rings $\varphi^* : \mathbb{C}[x, 1/x] \to \mathbb{C}[y, 1/y]$. We have that $\varphi^*(1/x) = y$, $\varphi^*\left(2x^2 + \frac{2x^3+4x}{x^5}\right) = \frac{1}{y^2} + 2y^2 + 4y^4$ and $\varphi^*(2-x) = 2 - 1/y$.

Proof. We can explicitly compute each of these.

$$\varphi^*(1/x) = 1/\varphi(y) = y.$$

$$\begin{aligned}
\varphi^*\left(2x^2 + \frac{2x^3+4x}{x^5}\right) &= 2\varphi(y)^2 + \frac{2\varphi(y)^3 + 4\varphi(y)}{\varphi(y)^5} \\
&= \frac{1}{y^2} + \frac{\frac{2}{y^3} + \frac{4}{y}}{\frac{1}{y^5}} \\
&= \frac{1}{y^2} + 2y^2 + 4y^4.
\end{aligned}$$

$$\varphi^*(2-x) = 2 - \varphi(y) = 2 - 1/y.$$

□

2. Consider the affine algebraic hypersurface $V = \mathbb{V}(y - ux) \subset \mathbb{A}^3$.

   (a) The projection $\phi : \mathbb{A}^3 \to \mathbb{A}^2$ given by $\phi : (x, y, u) \mapsto (x, u)$ restricts to an isomorphism from V to $\mathbb{A}^2$.

   Proof. Clearly, ϕ is a morphism as each component is trivially a polynomial. Let $\psi(x, u) = (x, ux, u)$. We claim that this is an inverse to ϕ, and so is an isomorphism. Observe that $\phi(\psi(x, u)) = \phi(x, ux, u) = (x, u)$ and $\psi(\phi(x, y, u)) = \psi(x, u) = (x, ux, u)$. Noting that $y - ux = 0$ and thus $y = ux$, we see that these are indeed mutually inverse and so isomorphisms. □

   (b) The projection $\phi : \mathbb{A}^3 \to \mathbb{A}^2$ given by $\phi : (x, y, u) \mapsto (x, y)$ does not restrict to an isomorphism from V to $\mathbb{A}^2$.

   Proof. First, we note that, clearly, ϕ is a morphism. However, we immediately see that it cannot restrict to an isomorphism as it does not have an inverse; we have that $\phi(x, y, u) = (x, ux)$ and so an inverse φ could not injectively map to V. □

3. (a) Let $g \in \mathbb{C}[x, y]$ with homogenization $\tilde{g} \in \mathbb{C}[x, y, z]$. Then $\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g})$.

   Proof. Noting that $\mathbb{V}(g)$ is a closed affine algebraic variety, and that the homogenization of an ideal generated by a single element is the same as the ideal generated by the homogenization of that element, the result follows directly from Theorem 3.28 in the lecture notes. □

   (b) Consider the following 4 polynomials:

$$\begin{aligned}
f_1(x, y) &= x + y + 1 \\
f_2(x, y) &= x^2 + 6y^2 + 1 \\
f_3(x, y) &= x^2 + 3y + 1 \\
f_4(x, y) &= x^3 + 3xy^2 + 4.
\end{aligned}$$

   For the points $[1 : 0 : 0], [0 : 1 : 0]$ and $[0 : 0 : 1]$, we have that the only cases where any of these points is contained in the projective closures of the given polynomials is that $[0 : 1 : 0] \in \overline{\mathbb{V}(f_3)}$ and $[0 : 1 : 0] \in \overline{\mathbb{V}(f_4)}$.

   Proof. From the previous part, we know that $\overline{\mathbb{V}(f_i)} = \mathbb{V}(\tilde{f}_i)$ for each polynomial f_i. Thus, noting that

$$\begin{aligned}
\tilde{f}_1(x, y, z) &= x + y + z \\
\tilde{f}_2(x, y, z) &= x^2 + 6y^2 + z^2 \\
\tilde{f}_3(x, y, z) &= x^2 + 3yz + z^2 \\
\tilde{f}_4(x, y, z) &= x^3 + 3xy^2 + 4z^3,
\end{aligned}$$

   we can examine the varieties of each homogenization. To determine if each variety includes the provided points, we can, in a sense, "plug in" our values. Converting from homogenous coordinates to lines in $\mathbb{C}^3$, we take our points now as the lines $(t, 0, 0)$, $(0, t, 0)$ and $(0, 0, t)$ where $t \in \mathbb{C} \cup \{\infty\}$. Plugging in our values, we get the following results:

	$(t,0,0)$	$(0,t,0)$	$(0,0,t)$
$\mathbb{V}(\tilde{f}_1)$	$t=0$	$t=0$	$t=0$
$\mathbb{V}(\tilde{f}_2)$	$t^2=0$	$6t^2=0$	$t^2=0$
$\mathbb{V}(\tilde{f}_3)$	$t^2=0$	$0t=0$	$t^2=0$
$\mathbb{V}(\tilde{f}_4)$	$t^3=0$	$0t^2=0$	$4t^3=0$

We note that we haven't simplified the two highlighted terms. This is because, unlike every other variety, these equations are satisfied for all $t \in \mathbb{C} \cup \{\infty\}$, not just at $t=0$. Thus, we can see that the entire line, and so the projective point $[0:1:0]$ is included in these varieties, whereas none of the other points are included in any other variety. That is, $[0:1:0] \in \mathbb{V}(\tilde{f}_3)$ and $[0:1:0] \in \mathbb{V}(\tilde{f}_3)$ only. □

(c) We claim that a necessary and sufficient condition for $g \in \mathbb{C}[x,y]$ such that $\mathbb{V}(\tilde{g})$ does not pass through any of $[1:0:0],[0:1:0]$ and $[0:0:1]$ is that every term in g consists of either a constant term or of precisely one variable of degree equal to $\deg \tilde{g}$.

4. (a) The space $\mathbb{P}^n$ is compact with respect to the quotient Euclidean topology from $\mathbb{A}^{n+1}\setminus\{0\}$.

Proof. We first note that, as already shown in previous coursework, any closed affine algebraic variety is compact, and thus $\mathbb{A}^{n+1}\setminus\{0\}$ is compact. It's a standard topological result that the quotient of a compact space is compact, and so, as $\mathbb{P}^n$ is defined as the quotient space of the compact space $\mathbb{A}^{n+1}\setminus\{0\}$, it is compact. □

(b) There is no projective closure of $\mathbb{V}(y-\sin(x))$.

Proof. This follows from Chow's Lemma. Assume $\overline{V}$ is the projective closure of the variety $V=\mathbb{V}(y-\sin(x))$. We know, as $y=\sin x$ is analytic, $\overline{V}$ is an analytic subvariety of $\mathbb{P}^1$. By Chow's Lemma, we then conclude that $\overline{V}$ is algebraic. However, as shown in Example 3.43, V, and thus its closure, cannot be algebraic, as this would contradict Bézout's Theorem. Thus, no such $\overline{V}$ can exist. □

5. (a) A line in $\mathbb{P}^2$ is a variety given by $ax+by+cz \in \mathbb{C}[x,y,z]$ for $a,b,c\in\mathbb{C}$. Two distinct lines intersect at exactly one point.

Proof. Let $\ell_1,\ell_2 \in \mathbb{P}^2$ be two lines given as

$$\begin{aligned}\ell_1 &= \mathbb{V}(a_1x+b_1y+c_1z)\\ \ell_2 &= \mathbb{V}(a_2x+b_2y+c_2z)\end{aligned}$$

for $a_1,a_2,b_1,b_2,c_1,c_2 \in \mathbb{C}$. At their point(s) of intersection, we know that $\ell_1=\ell_2$, and so

$$\begin{aligned}&\ell_1=\ell_2\\ \implies\ &a_1x+b_1y+c_1z = a_2x+b_2y+c_2z\\ \implies\ &(a_1-a_2)x+(b_1-b_2)y+(c_1-c_2)z=0.\end{aligned}$$

From here, let $z=1$ to find a point $[x:y:1]\in\mathbb{P}^2$ on the intersection between the two lines, as

$$\left[x : -\frac{(a_2-a_1)x+(c_2-c_1)}{b_2-b_1} : 1\right].$$

Finally, by similarly letting $y = 1$ and performing the same substitution, we see that ℓ_1 and ℓ_2 intersect at

$$\left[-\frac{c_2 - c_1 + b_2 - b_1}{a_2 - a_1} : 1 : 1\right],$$

precisely one point in $\mathbb{P}^2$. $\square$

(b) Let $C_1, C_2 \subset \mathbb{A}^2$ be two closed affine algebraic curves.

i. We have the inclusion $\overline{C_1 \cap C_2} \subset \overline{C_1} \cap \overline{C_2}$.

Proof. Let $C_1 = \mathbb{V}(\{f_i\}), C_2 = \mathbb{V}(\{g_j\})$ for polynomials f_i, g_j and, for notational ease, let $I = (f_1, ..., f_n), J = (g_1, ..., g_m)$ be the ideals generated by these polynomials. By Theorem 3.28, we know that

$$\overline{C_1} = \mathbb{V}(\tilde{I}), \text{ and } \overline{C_2} = \mathbb{V}(\tilde{J}).$$

Now, observe that

$$\begin{aligned}\overline{C_1 \cap C_2} &= \overline{\mathbb{V}(I) \cap \mathbb{V}(J)} \\ &= \overline{\mathbb{V}(I \cap J)} \\ &= \mathbb{V}(\widetilde{I \cap J})\end{aligned}$$

Letting $a \in \overline{C_1 \cap C_2}$, we can see therefore there exists some $f \in I \cap J$ such that $\tilde{f}(a) = 0$, and thus $a \in \mathbb{V}(\tilde{I})$ and $a \in \mathbb{V}(\tilde{J})$. That is,

$$a \in \mathbb{V}(\tilde{I}) \cap \mathbb{V}(\tilde{J}) = \overline{C_1} \cap \overline{C_2}$$

and so $\overline{C_1 \cap C_2} \subseteq \overline{C_1} \cap \overline{C_2}$. $\square$

ii. The curves C_1, C_2 given by

$$C_1 = \mathbb{V}(y - x^2), \quad C_2 = \mathbb{V}(z - xy)$$

satisfy the strict inclusion

$$\overline{C_1 \cap C_2} \subset \overline{C_1} \cap \overline{C_2}.$$

Proof. We first note that $C = C_1 \cap C_2 = \mathbb{V}(y - x^2, z - xy)$ is the twisted cubic given in Examples 1.8.4, 2.41(a) and 3.34. Through homogenization, we can see that $\overline{C_1} = \mathbb{V}(wy - x^2)$ and $\overline{C_2} = \mathbb{V}(wz - xy)$. Finally, as demonstrated in Example 3.34, we then have that

$$\overline{C_1} \cap \overline{C_2} = \mathbb{V}(wy - x^2) \cap \mathbb{V}(wz - xy) = \overline{C} \cup \{[x : y : z : w] \in \mathbb{P}^3 \mid w = x = 0\} \supset \overline{C}.$$

As $\overline{C} = \overline{C_1 \cap C_2}$, our result holds. $\square$

6. (a) Let Y be a closed affine algebraic variety and $O \subseteq Y$ open. Then $\mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra.

Proof. We use the test given in Example 2.4 of the notes, where any ring containing $\mathbb{C}$ as a subring is a $\mathbb{C}$-algebra. Thus, we proceed by verifying that $\mathbb{C} \subset \mathcal{O}_Y(V)$ and that $\mathcal{O}_Y(V)$ is indeed a ring.

First, let $f_1, f_2 \in \mathcal{O}_Y(V)$ be regular, and so there exists $g_1, g_2, h_1, h_2 \in \mathbb{C}[x_1, ..., x_n]$ such that

$$f_1(p) = \frac{g_1(p)}{h_1(p)}, \text{ and } f_2(p) = \frac{g_2(p)}{h_2(p)}$$

for all $p \in O$, with $h_1(p) \neq 0$ and $h_2(p) \neq 0$. Now, consider,

$$\begin{aligned} f_1(p) + f_2(p) &= \frac{g_1(p)}{h_1(p)} + \frac{g_2(p)}{h_2(p)} \\ &= \frac{g_1(p)h_2(p) + g_2(p)h_1(p)}{h_1(p)h_2(p)} \end{aligned} \qquad \begin{aligned} f_1(p)f_2(p) &= \frac{g_1(p)}{h_1(p)} \frac{g_2(p)}{h_2(p)} \\ &= \frac{g_1(p)g_2(p)}{h_1(p)h_2(p)} \end{aligned}$$

and, knowing that $h_1(p) \neq 0$ and $h_2(p) \neq 0$, we have that $h_1(p)h_2(p) \neq 0$. Thus, $f_1 + f_2$ and $f_1 f_2$ are both regular, and so $\mathcal{O}_Y(O)$ is a ring.
Finally, each $f : O \to C \in \mathcal{O}_Y(O)$ maps into $\mathbb{C}$, and the constant functions

$$(a, 0, 0, ..., 0) \mapsto a \in \mathbb{C}$$

are regular. Thus, identifying each of these constant functions with the complex constant it maps to, we can see that $\mathbb{C} \subset \mathcal{O}_Y(O)$ as a subring, and so our two conditions are met, allowing us to conclude that $\mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra. $\square$

(b) –