Algebraic Geometry Coursework 2

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1)

Q1. (a) (**15 marks**) Find all the elements of maxSpec($\mathbb{C}[x]$), maxSpec($\mathbb{C}[x, 1/x]$), and maxSpec($\mathbb{C}[x, 1/x, y]$) explicitly.

(b) (**5 marks**) Consider the isomorphism $\varphi : \mathbb{A}^1\backslash\{0\} \longrightarrow \mathbb{A}^1\backslash\{0\}$, $a \longmapsto b = 1/a$, and the pullback map on the coordinate rings $\varphi^* : \mathbb{C}[x, 1/x] \longmapsto \mathbb{C}[y, 1/y]$. Compute $\varphi^*(1/x)$, $\varphi^*(2x^2 + \frac{2x^3+4x}{x^5})$, $\varphi^*(2-x)$.

a) We first show that any maximal ideal in $\mathbb{C}[x_1,\dots,x_n]$ is of the form $m_a = (x_1 - a_1, \dots, x_n - a_n)$ for some $a = (a_1,\dots,a_n) \in \mathbb{C}^n$.

To show such an ideal is maximal, define $\phi : \mathbb{C}[x_1,\dots,x_n] \to \mathbb{C}$ by $\phi(p) = p(a)$, then:

$$\begin{aligned}\ker\phi &= \{p \in \mathbb{C}[x_1,\dots,x_n] \mid p(a_1,\dots,a_n) = 0\} \\ &= \{(x_1-a_1)p_1 + \dots + (x_n - a_n)p_n \mid p_1,\dots,p_n \in \mathbb{C}[x_1,\dots,x_n]\} \\ &= (x_1 - a_1, \dots, x_n - a_n) \\ &= m_a,\end{aligned}$$

and for any $c \in \mathbb{C}$, $\phi(c) = c$, so $\operatorname{im}\phi = \mathbb{C}$.
Hence by the first isomorphism theorem:

$\mathbb{C}[x_1,\dots,x_n]/m_a \cong \mathbb{C}$ is a field, so m_a is maximal.

Now let I be a maximal ideal. Thus there are no non-trivial ideals that are a proper subset of I.
By the order-reversing property of $\mathbb{V}$, $\mathbb{V}(I)$ must be minimal with respect to inclusion. Since $\mathbb{V}(I) \neq \emptyset$ (otherwise $I = \mathbb{C}[x_1,\dots,x_n]$), $\mathbb{V}(I)$ is a singleton, $\{(a_1,\dots,a_n)\}$. So:

$$\begin{aligned}I &= \mathbb{I}(\mathbb{V}(I)) \qquad (I \text{ maximal} \Rightarrow I \text{ radical}) \\ &= \mathbb{I}(\{(a_1,\dots,a_n)\}) \\ &= (x_1 - a_1, \dots, x_n - a_n) \\ &= m_a.\end{aligned}$$

- $\operatorname{MaxSpec}(\mathbb{C}[x]) = \{(x-a) \mid a \in \mathbb{C}\}$.
- $\operatorname{MaxSpec}(\mathbb{C}[x, \frac{1}{x}]) = \{(x-a, \frac{1}{x} - b) \mid a, b \in \mathbb{C}\}$

Note that we cannot have $a = 0$ or $b = 0$. If $a = 0$, then $x \in (x, \frac{1}{x} - b) \Rightarrow \frac{1}{x}\cdot x = 1 \in (x, \frac{1}{x} - b)$
$\Rightarrow (x, \frac{1}{x} - b) = \mathbb{C}[x, \frac{1}{x}]$

and if $b = 0$, then $\frac{1}{x} \in (x - a, \frac{1}{x}) \Rightarrow x \cdot \frac{1}{x} = 1 \in (x-a, \frac{1}{x})$
$\Rightarrow (x-a, \frac{1}{x}) = \mathbb{C}[x, \frac{1}{x}]$

Thus $\operatorname{MaxSpec}(\mathbb{C}[x, \frac{1}{x}]) = \{(x-a, \frac{1}{x} - b) \mid a, b \in \mathbb{C}^\times\}$
$= \{(x-a, x - \frac{1}{b}) \mid a, b \in \mathbb{C}^\times\}$

We have $(x-a) - (x - \frac{1}{b}) = \frac{1}{b} - a \in (x-a, x-\frac{1}{b})$ which must be 0 (otherwise $(x-a, x-\frac{1}{b}) = \mathbb{C}[x, \frac{1}{x}]$) and so $a = \frac{1}{b}$. Therefore $(x-a, x-\frac{1}{b}) = (x-a)$ giving:

$$\operatorname{MaxSpec}(\mathbb{C}[x, \tfrac{1}{x}]) = \{(x-a) \mid a \in \mathbb{C}^\times\}.$$

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- $\text{MaxSpec}(\mathbb{C}[x, \frac{1}{x}, y])$
$= \{(x-a, \frac{1}{x} - b, y-c) \mid a, b, c \in \mathbb{C}\}$

Again we require $a = 1/b$, however we may take any $c \in \mathbb{C}$ since y does not depend on x or $1/x$, thus:

$\text{MaxSpec}(\mathbb{C}[x, \frac{1}{x}, y]) = \{(x-a, y-c) \mid a, c \in \mathbb{C},\ a \neq 0\}$

b) $\varphi : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$, $a \mapsto b = \frac{1}{a}$
$\varphi^* : \mathbb{C}[x, 1/x] \to \mathbb{C}[y, 1/y]$

By theorem 2.38, φ^* is a $\mathbb{C}$-algebra homomorphism, and if $f(x) = x$, then $\varphi^*(f) = \varphi^*(x) = \varphi(x) = 1/y$.

$$\varphi^*(1/x) = 1/\varphi^*(x)$$
$$= 1/(1/y) = y$$

$$\varphi^*\left(2x^2 + \frac{2x^3 + 4x}{x^5}\right) = 2\varphi^*(x)^2 + \frac{2\varphi^*(x)^3 + 4\varphi^*(x)}{\varphi^*(x)^5}$$
$$= 2(1/y)^2 + \frac{2(1/y)^3 + 4(1/y)}{(1/y)^5}$$
$$= 2(1/y)^2 + 2y^2 + 4y^4$$

$$\varphi^*(2-x) = 2 - \varphi^*(x)$$
$$= 2 - 1/y.$$

2)

Q2. (**20 marks**) Consider the affine algebraic hypersurface $V := \mathbb{V}(y - ux) \subseteq \mathbb{A}^3$.

(a) Prove that the projection $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$, $(x, y, u) \longmapsto (x, u)$ restricts to an isomorphism from V to $\mathbb{A}^2$.

(b) Prove that the projection $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$, $(x, y, u) \longmapsto (x, y)$ does not restrict to isomorphism from V to $\mathbb{A}^2$.

a) Let $\varphi_1 : V \to \mathbb{A}^2$ be the restriction of the projection $(x, y, u) \mapsto (x, u)$.
This is certainly a morphism as it is the restriction of polynomial maps.

Then $\psi_1 : \mathbb{A}^2 \to V$, $(a, b) \mapsto (a, ab, b)$ is a well-defined morphism (as $ab - (a \times b) = 0$) and:

$$\psi_1 \circ \varphi_1(x, y, u) = \psi_1(x, u)$$
$$= (x, xu, u)$$
$$= (x, y, u) \quad (\text{since } y = xu \text{ on } V)$$

$$\varphi_1 \circ \psi_1(a, b) = \varphi_1(a, ab, b)$$
$$= (a, b)$$

So φ_1 is an isomorphism.

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b) Let $\varphi_2 : V \to \mathbb{A}^2$ be the restriction of the projection $(x, y, u) \longmapsto (x, y) = (x, ux)$.

This is indeed a morphism, but it is not an isomorphism since $(0,0,1), (0,0,-1) \in V$, however:

$\varphi_2((0,0,1)) = (0,0)$
$\varphi_2((0,0,-1)) = (0,0)$.

So φ_2 is not injective, and therefore not an isomorphism.

3)

Q3. (**25 marks**)

(a) Prove that if $g \in \mathbb{C}[x, y]$ then the projective closure of its variety $\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{g}) \subseteq \mathbb{P}^2$ where $\tilde{g} \in \mathbb{C}[x, y, z]$ is the homogenisation of g.

(b) Consider the polynomials $f_1(x, y) = x + y + 1, f_2(x, y) = x^2 + 6y^2 + 1, f_3(x, y) = x^2 + 3y + 1, f_4(x, y) = x^3 + 3xy^2 + 4$. Determine whether or not each of the projective closures includes the points

(i) $[1 : 0 : 0]$;
(ii) $[0 : 1 : 0]$;
(iii) $[0 : 0 : 1]$.

(c) Can you find a general necessary and sufficient condition on $g \in \mathbb{C}[x, y]$ such that its homogenisation $\tilde{g} \in \mathbb{C}[x, y, z]$ does not pass through any of the three points in item (b)?

a) By theorem 3.28, $\overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{I})$, where $I = \mathbb{I}(\mathbb{V}(g)) = \sqrt{(g)}$.
So $\tilde{I} = (\{\tilde{f} \mid f \in \sqrt{(g)}\})$.

$(\subseteq)$: Let $(a_1, \dots, a_n) \in \overline{\mathbb{V}(g)} = \mathbb{V}(\tilde{I})$
Then $\tilde{f}(a_1, \dots, a_n) = 0$ for all $f \in \sqrt{(g)}$.
In particular, since $g \in \sqrt{(g)}$, $\tilde{g}(a_1, \dots, a_n) = 0$.
So $(a_1, \dots, a_n) \in \mathbb{V}(\tilde{g})$, and hence $\overline{\mathbb{V}(g)} \subseteq \mathbb{V}(\tilde{g})$.

$(\supseteq)$: Let $(a_1, \dots, a_n) \in \mathbb{V}(\tilde{g})$ then $\tilde{g}(a_1, \dots, a_n) = 0$.
Let $f \in \sqrt{(g)}$, then $f^m \in (g)$ for some $m \in \mathbb{N}$.

$\Rightarrow f^m = h \cdot g$ for some $h \in \mathbb{C}[x_1, \dots, x_n]$
$\Rightarrow \widetilde{f^m} = \widetilde{h \cdot g}$
$\Rightarrow \tilde{f}^m = \tilde{h}\tilde{g}$ (Since homogenisation is a ring homomorphism)

$\Rightarrow \tilde{f}(a_1, \dots, a_n)^m = \tilde{h}(a_1, \dots, a_n) \cdot \tilde{g}(a_1, \dots, a_n) = 0$

$\Rightarrow \tilde{f}(a_1, \dots, a_n) = 0$ (Since $\mathbb{C}[x,y,z]$ an integral domain)
$\Rightarrow (a_1, \dots, a_n) \in \mathbb{V}(\tilde{f})$.

Since this is true for all $f \in \sqrt{(g)} = I$, $(a_1, \dots, a_n) \in \mathbb{V}(\tilde{I}) = \overline{\mathbb{V}(g)}$. So $\mathbb{V}(\tilde{g}) = \overline{\mathbb{V}(g)}$.

b) $f_1(x, y) = x + y + 1 \Rightarrow \tilde{f}_1(x, y, z) = x + y + z$
$f_2(x, y) = x^2 + 6y^2 + 1 \Rightarrow \tilde{f}_2(x, y, z) = x^2 + 6y^2 + z^2$
$f_3(x, y) = x^2 + 3y + 1 \Rightarrow \tilde{f}_3(x, y, z) = x^2 + 3yz + z^2$
$f_4(x, y) = x^3 + 3xy^2 + 4 \Rightarrow \tilde{f}_4(x, y, z) = x^3 + 3xy^2 + 4z^3$

We have that $[x : y : z] \in \overline{\mathbb{V}(f)}$ iff $\tilde{f}(x, y, z) = 0$ by part (a).

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i) $\tilde{f}_1(1,0,0) = 1 \neq 0$
$\tilde{f}_2(1,0,0) = 1 \neq 0$
$\tilde{f}_3(1,0,0) = 1 \neq 0$
$\tilde{f}_4(1,0,0) = 1 \neq 0$

So $[1:0:0] \notin \overline{\mathbb{V}(f_i)}$ for all $i = 1,2,3,4$.

ii) $\tilde{f}_1(0,1,0) = 1 \neq 0$
$\tilde{f}_2(0,1,0) = 6 \neq 0$
$\tilde{f}_3(0,1,0) = 0$
$\tilde{f}_4(0,1,0) = 0$

So $[0:1:0] \in \overline{\mathbb{V}(f_3)}, \overline{\mathbb{V}(f_4)}$, but $[0:1:0] \notin \overline{\mathbb{V}(f_1)}, \overline{\mathbb{V}(f_2)}$.

iii) $\tilde{f}_1(0,0,1) = 1 \neq 0$
$\tilde{f}_2(0,0,1) = 1 \neq 0$
$\tilde{f}_3(0,0,1) = 1 \neq 0$
$\tilde{f}_4(0,0,1) = 4 \neq 0$

So $[0:0:1] \notin \overline{\mathbb{V}(f_i)}$ for $i = 1,2,3,4$.

c) If $\tilde{g}$ is a homogeneous polynomial of degree n, then $\tilde{g}(1,0,0)$, $\tilde{g}(0,1,0)$, $\tilde{g}(0,0,1) \neq 0$ if and only if the coefficients of x^n, y^n, z^n in $\tilde{g}$ are all non-zero.

Equivalently, if and only if the constant coefficient, and the coefficients of x^n and y^n are non-zero in g.

Indeed, if we write $\tilde{g}(x,y,z) = ax^n + by^n + cz^n + \tilde{h}(x,y,z)$, where $\tilde{h}$ consists of all cross-terms (i.e., each summand in $\tilde{h}$ contains at least two of x, y, and z), then:

$\tilde{h}(1,0,0) = \tilde{h}(0,1,0) = \tilde{h}(0,0,1) = 0$.

So $\tilde{g}(1,0,0) = a$, $\tilde{g}(0,1,0) = b$, $\tilde{g}(0,0,1) = c$, and thus $\tilde{g}(1,0,0), \tilde{g}(0,1,0), \tilde{g}(0,0,1) \neq 0 \iff a,b,c \neq 0$.

Hence we require $g(x,y) = ax^n + by^n + c +$ (cross-terms) with $a,b,c \neq 0$.

4)

Q4. **(15 marks)**

(a) Prove that $\mathbb{P}^n$ is compact with respect to the quotient Euclidean topology from $\mathbb{A}^{n+1} \setminus \{0\}$.

(b) What is the projective Zariski-closure of the $\mathbb{V}(y - \sin(x))$ in $\mathbb{P}^2$? How do you compare this to the Chow's Lemma? **Hint.** In Example 3.44 we have seen that this curve is not algebraic.

a) We first show that $\mathbb{P}^n \cong S^{2n+1}/\sim$, where $\underline{x} \sim \underline{y}$ iff $\underline{x} = \lambda \underline{y}$ for some $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$.

Indeed, if $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$, then we define $\varphi : \mathbb{P}^n \to S^{2n+1}/\sim$ by:

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$$\varphi([z_1:\dots:z_{n+1}]) = [\frac{1}{\sqrt{|z_1|^2+\dots+|z_{n+1}|^2}}(x_1, y_1, \dots, x_{n+1}, y_{n+1})],$$

where the RHS is the equivalence class of the $(2n+2)$-tuple. This is a bijection with inverse:

$$\varphi^{-1}([(a_1, a_2, \dots, a_{2n+2})]) = [a_1+ia_2 : \dots : a_{2n+1}+ia_{2n+2}].$$

This is well-defined because if $[(a_1,\dots,a_{2n+2})] = [(b_1,\dots,b_{2n+2})]$ then $b_i = \lambda a_i$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$, for all $i = 1,\dots,2n+2$. Thus $[b_1+ib_2 : \dots : b_{2n+1}+ib_{2n+2}] = [\lambda(a_1+ia_2):\dots:\lambda(a_{2n+1}+ia_{2n+2})]$
$= [a_1+ia_2 : \dots : a_{2n+1}+ia_{2n+2}]$.

We can see this is a bijection since:

$$\begin{aligned}\varphi^{-1}\circ\varphi([z_1:\dots:z_{n+1}]) &= \varphi^{-1}([\frac{1}{\sqrt{|z_1|^2+\dots+|z_{n+1}|^2}}(x_1,y_1,\dots,x_{n+1},y_{n+1})]) \\ &= \frac{1}{\sqrt{|z_1|^2+\dots+|z_{n+1}|^2}}[x_1+iy_1:\dots:x_{n+1}+iy_{n+1}] \\ &= [x_1+iy_1:\dots:x_{n+1}+iy_{n+1}] \\ &= [z_1:\dots:z_{n+1}]\end{aligned}$$

$$\begin{aligned}\varphi\circ\varphi^{-1}((a_1,\dots,a_{2n})) &= \varphi([a_1+ia_2:\dots:a_{2n+1}+ia_{2n+2}]) \\ &= \frac{1}{\sqrt{a_1^2+a_2^2+\dots+a_{2n+2}^2}}(a_1,\dots,a_{2n+2}) \\ &= (a_1,\dots,a_{2n+2}) \quad (\text{since } (a_1,\dots,a_{2n+2}) \in S^{2n+1})\end{aligned}$$

and it is continuous with continuous inverse, so we do have a homeomorphism.

Now $S^{2n+1} \subseteq \mathbb{R}^{2n+2}$ is compact because it is closed:

- $\mathbb{R}^{2n+2}\setminus S^{2n+1} = \{\underline{x} = (x_1,\dots,x_n) \mid |\underline{x}| \neq 1\}$, so if $\underline{x} \in \mathbb{R}^{2n+2}\setminus S^{n+1}$, $|\underline{x}| = 1+\varepsilon$ for some $\varepsilon \neq 0$, then $B_{|\varepsilon/2|}(\underline{x})$ is an open ball such that for any $\underline{y} \in B_{|\varepsilon/2|}(\underline{x})$, $|\underline{y}| \in [1-\varepsilon/2, 1+\varepsilon/2]$, so $\underline{y} \notin \mathbb{R}^{2n+2}\setminus S^{n+1}$. Thus $\mathbb{R}^{2n+2}\setminus S^{2n+1}$ is open, and so its complement S^{n+1} is closed.

and bounded:

- For any $\underline{x} \in S^{2n+1}$, $|\underline{x}| = 1$.

So by the Heine-Borel theorem it is compact. Therefore $S^{n+1}/\sim$ is compact since if $\{U_i \mid i \in I\}$ is an open cover of $S^{n+1}/\sim$, then as the quotient map $q: S^{n+1} \to S^{n+1}/\sim$ is continuous, $\{q^{-1}(U_i) \mid i \in I\}$ is an open cover of S^{n+1} (as all of the $q^{-1}(U_i)$ are open and for any $\underline{x} \in S^{n+1}$, $[\underline{x}] \in U_j$ for some $j \in I$, so $\underline{x} \in q^{-1}(U_j)$, and thus $\underline{x} \in \bigcup_{i\in I} q^{-1}(U_i)$).

Hence $\mathbb{P}^n$ is compact as it is homeomorphic to a compact space.

b) The projective closure of $\mathbb{V}(y-\sin x)$ is the smallest Zariski closed set that contains $\mathbb{V}(y-\sin x)$.

Since $\mathbb{V}(f_1,\dots,f_n) = \bigcap_{i=1}^{n} \mathbb{V}(f_i) \subseteq \mathbb{V}(f_j)$, $j=1,\dots,n$ (informally, considering the variety of more polynomials makes the variety 'smaller'), if we cannot find a homogeneous non-zero polynomial f such that $\overline{\mathbb{V}(y-\sin x)} = \mathbb{V}(f)$, then we must have $\overline{\mathbb{V}(y-\sin x)} = \mathbb{P}^n$.

Indeed suppose such a polynomial f exists, then $\mathbb{V}(y-\sin x) \subseteq \mathbb{V}(f)$.

Let $g(x) = f(x,0)$. This is a polynomial in x. However, since $\sin(n\pi)=0$ for all $n\in\mathbb{Z}$, $g(n\pi)=0$ for all $n\in\mathbb{Z}$, and hence g has infinitely many zeroes. This is a contradiction unless g is the zero function.

Therefore $\overline{\mathbb{V}(y-\sin x)} = \mathbb{V}(0) = \mathbb{P}^2$.

We had by Example 3.44 that $\mathbb{V}(y-\sin x)$ is not algebraic, and therefore by the Chow lemma, $\mathbb{V}(y-\sin x)$ is not compact in the Euclidean topology, as it is analytic.

We therefore expect that $\overline{\mathbb{V}(y-\sin x)}$ should not be compact, which would contradict the previous result.

However, since the projective closure was taken in the Zariski topology, we do not have such a contradiction.

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5)

Q5. (**20 marks**)

(a) The variety of a polynomial of the form $ax+by+cz \in \mathbb{C}[x,y,z]$ for $a,b,c \in \mathbb{C}$ is called a *line* in $\mathbb{P}^2$. Prove that any two distinct lines in $\mathbb{P}^2$ intersect exactly at one point.

Let $ax+by+cz=0$ and $dx+ey+fz=0$ be distinct lines in $\mathbb{P}^2$. This means that $(d,e,f) \neq \lambda(a,b,c)$ for some $\lambda \in \mathbb{C}$.
Points on the intersection correspond to solutions of the matrix equation:

$$\underbrace{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}}_{M}\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\underline{x}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $\operatorname{rank}(M) \leq 2$, we must have that $\dim(\ker M) = \operatorname{nullity}(M) = 3 - \operatorname{rank}(M) \geq 1$. So there exists at least one intersection in $\mathbb{P}^2$.

Moreover, since (a,b,c) and (d,e,f) are linearly independent, we have that $\dim(\ker M) \leq 1$. So $\dim(\ker M) = 1$.

Thus if (x,y,z) and (x',y',z') are solutions of the intersection, then $(x',y',z') = \lambda(x,y,z)$, and so $[x:y:z] = [x':y':z']$.

b)

(b) Assume that $C_1, C_2 \subseteq \mathbb{A}^2$ are two closed affine algebraic curves.

(i) Prove that we have the inclusion $\overline{C_1 \cap C_2} \subseteq \overline{C_1} \cap \overline{C_2}$ of projective closures.

(ii) Find two curves such that the above inclusion is strict.

i) Let $C_1, C_2 \subseteq \mathbb{A}^2$ be closed affine algebraic curves. Then $C_1 = \mathbb{V}(f_1)$, $C_2 = \mathbb{V}(f_2)$ for some polynomials $f_1, f_2 \in \mathbb{C}[x,y]$, f_1, f_2 non-constant.

So $C_1 \cap C_2 = \mathbb{V}(f_1, f_2)$.
Let $(a,b,c) \in \overline{C_1 \cap C_2} = \mathbb{V}(\tilde{I})$, where $\tilde{I} = (\{\tilde{f} \in \mathbb{C}[x,y,z] \mid f \in \mathbb{I}(C_1 \cap C_2)\})$

Thus $F(a,b,c) = 0$ for all $F \in \tilde{I}$. In particular, since $f_1, f_2 \in \mathbb{I}(C_1 \cap C_2)$, $\tilde{f}_1, \tilde{f}_2 \in \tilde{I}$.

So $(a,b,c) \in \mathbb{V}(\tilde{f}_1) = \overline{C_1}$ and $(a,b,c) \in \mathbb{V}(\tilde{f}_2) = \overline{C_2}$.
Hence $\overline{C_1 \cap C_2} \subseteq \overline{C_1} \cap \overline{C_2}$.

ii) Let $C_1 = \mathbb{V}(x)$ and $C_2 = \mathbb{V}(y)$ be curves in $\mathbb{A}^2$.
Then $\overline{C_1} = \mathbb{V}(x)$, $\overline{C_2} = \mathbb{V}(y)$, where $x, y \in \mathbb{C}[x,y,z]$.
So $\overline{C_1} \cap \overline{C_2} = \mathbb{V}(x,y) = \{(0,0,z) \mid z \in \mathbb{C}\}$.

$C_1 \cap C_2 = \mathbb{V}(x,y) = \{(0,0)\}$

$I = \mathbb{I}(\{(0,0)\}) = (x,y)$
$A = \{\tilde{f} \in \mathbb{C}[x,y,z] \mid f \in I\} \subseteq \mathbb{C}[x,y,z]$
$\tilde{I} = (A)$
$= (\tilde{f}_1, \dots, \tilde{f}_n)$ (By Hilbert's basis theorem there exists a finite set of generators).

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Since $f_i \in I$, f_i has no constant term, therefore every summand in $\tilde{f_i}$ has a factor of x or of y. Thus $\tilde{f_i}(0,0,z) = 0$. Hence $(0,0,z) \in \mathbb{V}(\tilde{f_i})$. Since this is true for $i = 1, \dots, n$, $(0,0,z) \in \mathbb{V}(\tilde{I}) = \overline{C_1 \cap C_2}$.

Hence $\overline{C_1} \cap \overline{C_2} \subseteq \overline{C_1 \cap C_2}$ and by Q5(b)(i), $\overline{C_1} \cap \overline{C_2} = \overline{C_1 \cap C_2}$.

6)

Q6. **(Bonus 10 marks)**

(a) Let Y be a closed affine algebraic variety and $O \subseteq Y$ an open subset. Prove that $\mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra.

a) Since $\{f : O \to \mathbb{C}\}$ forms a $\mathbb{C}$-algebra and $\mathcal{O}_Y(O)$ is a subset of this set, it suffices to show that $\mathcal{O}_Y(O)$ is non-empty and closed under addition, multiplication, and scalar multiplication:

- The identity map is a polynomial, and therefore in $\mathcal{O}_Y(O)$. Hence $\mathcal{O}_Y(O)$ is non-empty.

Let $f_1, f_2 \in \mathcal{O}_Y(O)$, then there exist $g_1, g_2, h_1, h_2 \in \mathbb{C}[x_1, \dots, x_n]$ such that for all $q \in O$, there is are open neighbourhoods O_1, O_2 of q with $h_1(p) \neq 0$ for all $p \in O_1$, $h_2(p) \neq 0$ for all $p \in O_2$, and $f_1|_{O_1}(p) = g_1(p)/h_1(p)$, $f_2|_{O_2}(p) = g_2(p)/h_2(p)$.
Since O_1, O_2 are open, so is $O_1 \cap O_2$. Thus for any $q \in O$:

- $f_1 + f_2|_{O_1 \cap O_2}(p) = \frac{g_1(p)}{h_1(p)} + \frac{g_2(p)}{h_2(p)} = \frac{(g_1h_2 + g_2h_1)(p)}{h_1h_2(p)}$ for all $p \in O_1 \cap O_2$, with $g_1h_2 + g_2h_1, h_1h_2 \in \mathbb{C}[x_1, \dots, x_n]$, and $h_1h_2(p) \neq 0$ (since $h_1(p) \neq 0$ for $p \in O_1$, $h_2(p) \neq 0$ for $p \in O_2$).
Thus $f_1 + f_2 \in \mathcal{O}_Y(O)$.

- $\lambda f_1|_{O_1}(p) = \frac{\lambda g_1(p)}{h_1(p)}$ for all $p \in O_1$, with $\lambda g_1, h_1 \in \mathbb{C}[x_1, \dots, x_n]$ and $h_1(p) \neq 0$ for all $p \in O_1$. So $\lambda f_1 \in \mathcal{O}_Y(O)$ for all $\lambda \in \mathbb{C}$.

- $f_1 f_2|_{O_1 \cap O_2}(p) = \frac{g_1(p)}{h_1(p)} \cdot \frac{g_2(p)}{h_2(p)} = \frac{g_1g_2(p)}{h_1h_2(p)}$ with $g_1g_2, h_1h_2 \in \mathbb{C}[x_1, \dots, x_n]$ and $h_1h_2(p) \neq 0$ for all $p \in O_1 \cap O_2$.

Hence $\mathcal{O}_Y(O)$ is a $\mathbb{C}$-algebra.

b) Let X be an irreducible quasi-projective variety.

(i) Assume that U and V are open subsets of X with $U \subseteq V$. Briefly explain why $f \in \mathcal{O}_X(V)$ implies that $f_{|_U} \in \mathcal{O}_X(U)$.

(ii) Briefly explain why the collection of sets of functions $\mathcal{O}_X(U)$, where U ranges over all open subsets of X, forms a sheaf on X.

i) Let $U, V \subseteq X$ be open subsets with $U \subseteq V$. Let $f \in \mathcal{O}_X(V)$. Then for all $p \in V$, there is an open neighbourhood $V' \subseteq V$ and homogeneous polynomials $g, h \in \mathbb{C}[x_1, \dots, x_n]$ of the same degree such that $h(p) \neq 0$ for all $p \in V'$ and $f|_{V'}(p) = g(p)/h(p)$.

Since $U \subseteq V$, this is true for all $p \in U$, and since U, V' open, so is $U \cap V'$. Thus there is an open neighbourhood $U' = U \cap V' \subseteq U$ for which $f|_U : U \to \mathbb{C}$ satisfies the

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above results, with homogeneous polynomials $g|_u$, $h|_u$ of the same degree (g, h as above) such that $h|_u(p) \neq 0$ for all $p \in U' \subseteq V'$, and $(f|_u)|_{u'}(p) = g|_u(p)/h|_u(p)$.
Thus $f|_u \in \mathcal{O}_X(U)$.

ii) We can define the map $\mathcal{F}$ on X which associates to each open set $U \subseteq X$ the ring $\mathcal{O}_X(U)$ (it is indeed a ring since it is a $\mathbb{C}$-algebra by Q6(a)).
For each inclusion $U \hookrightarrow V$, we can also define the restriction:

$$\begin{aligned} \mathrm{res}_{V,U} : \mathcal{O}_X(V) &\to \mathcal{O}_X(U) \\ f &\mapsto f|_U . \end{aligned}$$

This is well-defined since $f|_U \in \mathcal{O}_X(U)$ by Q6(b)(i). We now show this satisfies (ii), (iii), and (iv):

(ii):
- Let $f: U \to \mathbb{C}$, then $\mathrm{res}_{U,U}(f) = f|_U = f$, so $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{O}_X(U)}$.
- Let $f: W \to \mathbb{C}$, then
$$\begin{aligned} \mathrm{res}_{V,U} \circ \mathrm{res}_{W,V}(f) &= \mathrm{res}_{V,U}(f|_V) \\ &= (f|_V)|_U \\ &= f|_U \quad (\text{as } U \subseteq V) \\ &= \mathrm{res}_{W,U}(f). \end{aligned}$$

(iii): Let $\{U_i \mid i \in I\}$ be an open cover for U.
Suppose $f_i \in \mathcal{O}_X(U_i)$ $\forall i \in I$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$ with $U_i \cap U_j \neq \emptyset$.

Define $f: U \to \mathbb{C}$ by $f(p) = f_j(p)$ if $p \in U_j$.
This is well-defined since if $p \in U_i$ and $p \in U_j$, then $p \in U_i \cap U_j$ and so $f_i(p) = f_j(p)$.
This is a regular function as if $p \in U_i$ for some i, then $f(p) = f_j(p)$ is a rational function in U_i, and $\mathrm{res}_{U,U_i}(f) = f|_{U_i} = f_i$.

(iv): Let $f, f' \in \mathcal{O}_X(U)$ be such that $f|_{U_i} = f'|_{U_i}$ for all $i \in I$. Then since $\{U_i \mid i \in I\}$ is an open cover for U, any $p \in U$ is in U_j for some $j \in I$.
Thus $f(p) = f|_{U_j}(p) = f'|_{U_j}(p) = f'(p)$.
So $f = f'$.

Hence $\mathcal{O}_X$ forms a sheaf on X.