The Smoothed Satisfaction of Voting Axioms

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Abstract

We initiate the work towards a comprehensive picture of the smoothed satisfaction of voting axioms, to provide a finer and more realistic foundation for comparing voting rules. We adopt the *smoothed social choice framework* introduced in a 2020 NeurIPS paper [51], where an adversary chooses arbitrarily correlated "ground truth" preferences for the agents, on top of which random noises are added. We focus on characterizing the smoothed satisfaction of two well-studied voting axioms: *Condorcet criterion* and *participation*. We prove that for any fixed number of alternatives, when the number of voters n is sufficiently large, the smoothed satisfaction of the Condorcet criterion under a wide range of voting rules is 1, $1-\exp(-\Theta(n))$, $\Theta(n^{-0.5})$, $\exp(-\Theta(n))$, or being $\Theta(1)$ and $1-\Theta(1)$ at the same time; and the smoothed satisfaction of participation is $1-\Theta(n^{-0.5})$. Our results address open questions by Berg and Lepelley [3] in 1994, and also confirm the following high-level message: the Condorcet criterion is a bigger concern than participation under realistic models.

1 Introduction

The *"widespread presence of impossibility results"* [45] is one of the most fundamental and significant challenges in social choice theory. These impossibility results often assert that no "perfect" voting rule exists for three or more alternatives [1, 25, 42]. Nevertheless, an (imperfect) voting rule must be designed and used in practice for agents to make a collective decision. In the social choice literature, the dominant paradigm of doing so has been the *axiomatic approach*, i.e., voting rules are designed, evaluated, and compared to each other w.r.t. their satisfaction of desirable normative properties, known as *(voting) axioms*.

Most definitions of dissatisfaction of voting axioms are based on worst-case analysis. For example, a voting rule r does not satisfy CONDORCET CRITERION (CC for short), if there exists a collection of votes, called a profile, where the *Condorcet winner* exists but is not chosen by r as a winner. The Condorcet winner is the alternative who beats all other alternatives in their head-to-head competitions. As another example, a voting rule r does not satisfy PARTICIPATION (PAR for short), if there exist a profile and a voter who has incentive to abstain from voting. An instance of dissatisfaction of PAR is also known as the *no-show paradox* [16]. Unfortunately, when the number of alternatives m is at least four, no irresolute voting rule satisfies CC and PAR simultaneously [36].

While the classical worst-case analysis of (dis)satisfaction of axioms can be desirable in high-stakes applications such as political elections, it is often too coarse to serve as practical criteria for comparing different voting rules in more frequent, low-stakes applications of social choice, such as business decision-making [5], crowdsourcing [31], informational retrieval [30], meta-search engines [12], recommender systems [48], etc. A decision maker who desires both axioms would find it hard to choose between a voting rule that satisfies CC but not PAR, such as Copeland, and a voting rule that satisfies PAR but not CC, such as plurality. A finer and more quantitative measure of satisfaction of axioms is therefore called for.

One natural and classical approach is to measure the likelihood of satisfaction of axioms under a probabilistic model of agents' preferences, in particular the independent and identically distributed (i.i.d.) uniform distribution over all rankings, known as *Impartial Culture (IC)* in social choice. This line of research was initiated and established by Gehrlein and Fishburn in a series of work in the 1970's [21, 23, 22], and has become a "*new sub-domain of the theory of social choice*" [11]. Some classical results were summarized in the 2011 book by Gehrlein and Lepelley [24], and recent progresses can be found in the 2021 book edited by Diss and Merlin [11].

While this line of work is highly significant and interesting from a theoretical point of view, its practical implications may not be as strong, because most previous work focused on a few specific distributions, especially IC, which has been widely criticized to be unrealistic (see, e.g., [39, p. 30], [20, p. 104], and [27]). Indeed, conclusions drawn under any specific distribution may not hold in practice, as "*all models are wrong*" [7]. Technically, characterizing the likelihood of satisfaction of CC and of PAR are already highly challenging w.r.t. IC, and despite that Berg and Lepelley [3] explicitly posed them as open questions in 1994, not much is known beyond a few voting rules. Therefore, the following question largely remains open.

How likely are voting axioms satisfied under realistic models?

The importance of successfully answering this question is two-fold. First, it tells us whether the worst-case violation of an axiom is a significant concern in practice. Second, it provides a finer and more quantitative foundation for comparing voting rules.

We believe that the *smoothed analysis* proposed by Spielman and Teng [46] provides a promising framework for addressing the question. In this paper, we adopt the *smoothed social choice* framework by Xia in a NeurIPS-20 paper [51], which models the satisfaction of a *per-profile* voting axiom X by a function $X(r, P) \in \{0, 1\}$, where r is a voting rule and P is a profile, such that r satisfies X if $\min_P X(r, P) = 1$. Let Π denote a set of distributions over all rankings over the m alternatives (denoted by $\mathcal{L}(\mathcal{A})$), which represents the "ground truth" preferences for a single agent that the adversary can choose from. Let n denote the number of agents. Because a higher value of $X(r, P)$ is more desirable to the decision maker, the adversary aims at minimizing expected $X(r, P)$ by choosing $\vec{\pi} \in \Pi^n$—the profile P is generated from $\vec{\pi}$. The smoothed satisfaction of X under r with n agents, denoted by $\widetilde{X}^{\min}_{\Pi}(r, n)$, is defined as follows [51]:

$$\widetilde{X}^{\min}_{\Pi}(r, n) = \inf_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}} X(r, P) \tag{1}$$

Notice that agents' ground truth preferences can be arbitrarily correlated, while the noises are independent, which is a standard assumption in the literature and in practice [51].

Example 1 (Smoothed CC under plurality). *Let $X = \text{CC}$ and $r = \overline{Plu}$ denote the irresolute plurality rule, which chooses all alternatives that are ranked at the top most often as the (co-)winners. Suppose there are three alternatives, denoted by $\mathcal{A} = \{1, 2, 3\}$, and suppose $\Pi = \{\pi^1, \pi^2\}$, where π^1 and π^2 are distributions shown in Table 1.*

Then, we have $\widetilde{\text{CC}}^{\min}_{\Pi}(\overline{Plu}, n) = \inf_{\vec{\pi} \in \{\pi^1, \pi^2\}^n} \Pr_{P \sim \vec{\pi}} \text{CC}(\overline{Plu}, P)$. When $n = 2$, the adversary has four choices of $\vec{\pi}$, i.e., $\{(\pi^1, \pi^1), (\pi^1, \pi^2), (\pi^2, \pi^1), (\pi^2, \pi^2)\}$.

	123	132	231	321	213	312
π^1	1/4	1/4	1/8	1/8	1/8	1/8
π^2	1/8	1/8	3/8	1/8	1/8	1/8

Table 1: Π in Example 1.

Each $\vec{\pi}$ leads to a distribution over the set of all profiles of two agents, i.e., $\mathcal{L}(\mathcal{A})^2$. We have $\widetilde{\text{CC}}^{\min}_{\Pi}(\overline{Plu}, 2) = 1$, because CC is satisfied at all profiles of two agents. As we will see later in Example 3, for all sufficiently large n, $\widetilde{\text{CC}}^{\min}_{\Pi}(\overline{Plu}, n) = \exp(-\Theta(n))$.

1.1 Our Contributions

We initiate the work towards a comprehensive picture of smoothed satisfaction of voting axioms under commonly-studied voting rules, by focusing on CC and PAR in this paper due to their importance, popularity, and incompatibility [36]. Recall that m is the number of alternatives and n is the number of agents. Our technical contributions are two-fold.

First, smoothed satisfaction of CC **(Theorem 1 and 2).** We prove that, under mild assumptions, for any fixed $m \geq 3$ and any sufficiently large n, the smoothed satisfaction of CC under a wide range of voting rules is 1, $1 - \exp(-\Theta(n))$, $\Theta(n^{-0.5})$, $\exp(-\Theta(n))$, or being $\Theta(1)$ and $1 - \Theta(1)$ at the same time (denoted by $\Theta(1) \wedge (1 - \Theta(1))$). The $1 - \exp(-\Theta(n))$ case is positive news, because it states that CC is satisfied almost surely when n is large, regardless of the adversary's choice. The remaining three cases are negative news, because they state that the adversary can make CC to be violated with non-negligible probability, no matter how large n is.

Second, smoothed satisfaction of PAR **(Theorems 3, 4, 5, 6).** We prove that, under mild assumptions, for any fixed $m \geq 3$ and any sufficiently large n, the smoothed satisfaction of PAR under a wide range of voting rules is $1 - \Theta(n^{-0.5})$. These are positive news, because they state that PAR is satisfied almost surely for large n, regardless of the adversary's choice. While this message may not be surprising at a high level, as the probability for a single agent to change the winner vanishes as $n \to \infty$, the theorems are useful and non-trivial, as they provide asymptotically tight rates.

In particular, straightforward corollaries of our theorems to IC address open questions posed by Berg and Lepelley [3] in 1994, and also provides a mathematical justification of two common beliefs related to PAR: first, IC exaggerates the likelihood for paradoxes to happen, and second, the dissatisfaction of PAR is not a significant concern in practice [28], especially when it is compared to our results on smoothed CC. Table 2 summarizes corollaries of our results under some commonly-studied voting rules w.r.t. IC as well as the satisfaction of CC and PAR on Preflib data [32].

Table 2: Satisfaction of CC and PAR w.r.t. IC and w.r.t. 315 Preflib profiles of linear orders under elections category. Experimental results are presented in Appendix G.

	Axiom	**Plu.**	**Borda**	**Veto**	**STV**	**Black**	**MM**	**Sch.**	**RP**	**Copeland$_{0.5}$**
Theory	CC	$\Theta(1) \wedge (1 - \Theta(1))$				always satisfied				
	PAR	always satisfied			$1 - \Theta\left(n^{-0.5}\right)$					
Preflib	CC	96.8%	92.4%	74.2%	99.7%	100%	100%	100%	100%	100%
	PAR	100%	100%	100%	99.7%	99.4%	100%	100%	100%	99.7%

Table 2 provides a more quantitative way of comparing voting rules. Suppose the decision maker puts 50% weight (or any fixed non-zero ratio) on both CC and PAR, and assume that the preferences are generated from IC. Then, when n is sufficiently large, the last five voting rules in the table (that satisfy CC) outperform the first five voting rules in the table (the first four satisfies PAR).

Beyond CC **and** PAR**.** Theorems 1–6 are proved by (non-trivial) applications of a *categorization lemma* (Lemma 1), which characterizes smoothed satisfaction of a large class of axioms that can be represented by unions of finitely many polyhedra, including CC and PAR. We believe that Lemma 1 is a promising tool for analyzing other axioms in future work.

1.2 Related Work and Discussions

The Condorcet criterion (CC) was proposed by Condorcet in 1785 [9], has been one of the most classical and well-studied axioms, and has *"nearly universal acceptance"* [41, p. 46]. CC is satisfied by many commonly-studied voting rules, except positional scoring rules [15] and multi-round-score-based elimination rules, such as STV. Most previous work focused on characterizing the *Condorcet efficiency*, which is the probability for the Condorcet winner to win conditioned on its existence [14, 13, 40, 22, 37]. Beyond positional scoring rules, the study was mostly based on computer simulations, see, e.g., [17, 18, 34, 38].

The participation axiom (PAR) was motivated by the *no-show paradox* [16] and was proved to be incompatible with CC for every $m \geq 4$ [36]. The likelihood of PAR under commonly studied voting rules w.r.t. IC was posed as an open question by Berg and Lepelley [3] in 1994, and has been investigated in a series of works including [29, 28, 49], see [24, Chapter 4.2.2]. In particular, Lepelley and Merlin [28] analyzed the likelihood of various no-show paradoxes for three alternatives under *scoring runoff rules*, which includes STV, w.r.t. IC and other distributions, and "*strongly believe that the no-show paradox is not an important flaw of the scoring run-off voting systems*".

Our work vs. previous work on CC **and** PAR**.** Our results address open questions by Berg and Lepelley [3] about the likelihood of satisfaction of CC and PAR in two dimensions: first, we conduct smoothed analysis, which extends i.i.d. models and is believed to be significantly more general and realistic. Second, our results cover a wide range of voting rules whose likelihood of satisfaction under CC or PAR even w.r.t. IC were not mathematically characterized before, including CC under STV, and PAR under maximin, Copeland, ranked pairs, Schulze, and Black's rule. While all results in this paper assume that the number of alternatives m is fixed, they are already more general than many previous work that focused on $m = 3$.

Smoothed analysis. There is a large body of literature on the applications of smoothed analysis to computational problems [47]. Its main idea, i.e., the worst average-case analysis, has been proposed and investigated in other disciplines as well. For example, it is the central idea in frequentist statistics (as in the *frequentist expected loss* and *minimax decision rules* [4]) and is also closely related to the *min of means* criteria in decision theory [26].

Recently, Baumeister et al. [2] and Xia [51] independently proposed to conduct smoothed analysis in social choice. We adopt the framework in the latter work, though our motivation and goal are quite different. We aim at providing a comprehensive picture of smoothed satisfaction of voting axioms, while [51] focused on analyzing smoothed likelihood of Condorcet's voting paradox and the ANR impossibility on *anonymity* and *neutrality*. On the technical level, while Lemma 1 is a straightforward corollary of [52, Theorem 2], applications of results like Lemma 1 can be highly non-trivial and problem dependent as commented in [52], which is the case of this paper. We believe that Lemma 1's main merit is conceptual, as it provides a general categorization of smoothed satisfaction of a large class of per-profile axioms beyond CC and PAR for future work.

2 Preliminaries

For any $q \in \mathbb{N}$, we let $[q] = \{1, \ldots, q\}$. Let $\mathcal{A} = [m]$ denote the set of $m \geq 3$ *alternatives*. Let $\mathcal{L}(\mathcal{A})$ denote the set of all linear orders over $\mathcal{A}$. Let $n \in \mathbb{N}$ denote the number of agents (voters). Each agent uses a linear order $R \in \mathcal{L}(\mathcal{A})$ to represent his or her preferences, called a *vote*, where $a \succ_R b$ means that the agent prefers alternative a to alternative b. The vector of n agents' votes, denoted by P, is called a *(preference) profile*, sometimes called an n-profile. The set of n-profiles for all $n \in \mathbb{N}$ is denoted by $\mathcal{L}(\mathcal{A})^* = \bigcup_{n=1}^{\infty} \mathcal{L}(\mathcal{A})^n$. A *fractional* profile is a profile P coupled with a possibly non-integer and/or negative weight vector $\vec{\omega}_P = (\omega_R : R \in P) \in \mathbb{R}^n$ for the votes in P. It follows that a non-fractional profile is a fractional profile with uniform weight, namely $\vec{\omega}_P = \vec{1}$. Sometimes the weight vector is omitted when it is clear from the context or when $\vec{\omega}_P = \vec{1}$.

For any (fractional) profile P, let $\text{Hist}(P) \in \mathbb{R}_{\geq 0}^{m!}$ denote the anonymized profile of P, also called the *histogram* of P, which contains the total weight of every linear order in $\mathcal{L}(\mathcal{A})$ according to P. An *irresolute voting rule* $\overline{r} : \mathcal{L}(\mathcal{A})^* \to (2^{\mathcal{A}} \setminus \{\emptyset\})$ maps a profile to a non-empty set of winners in $\mathcal{A}$. A *resolute* voting rule r is a special irresolute voting rule that always chooses a single alternative as the (unique) winner. We say that a voting rule r is a *refinement* of another voting rule $\overline{r}$, if for every profile P, $r(P) \subseteq \overline{r}(P)$.

(Un)weighted majority graphs and (weak) Condorcet winners. For any (fractional) profile P and any pair of alternatives a, b, let $P[a \succ b]$ denote the total weight of votes in P where a is preferred to b. Let $\text{WMG}(P)$ denote the *weighted majority graph* of P, whose vertices are $\mathcal{A}$ and whose weight on edge $a \to b$ is $w_P(a, b) = P[a \succ b] - P[b \succ a]$. Let $\text{UMG}(P)$ denote the *unweighted majority graph*, which is the unweighted directed graph that is obtained from $\text{WMG}(P)$ by keeping the edges with strictly positive weights. Sometimes a distribution π over $\mathcal{L}(\mathcal{A})$ is viewed as a fractional profile, where for each $R \in \mathcal{L}(\mathcal{A})$ the weight on R is $\pi(R)$. In such cases, we let $\text{WMG}(\pi)$ denote the weighted majority graph of the fractional profile represented by π.

The *Condorcet winner* of a profile P is the alternative that only has outgoing edges in $\text{UMG}(P)$. A *weak Condorcet winner* is an alternative that does not have incoming edges in $\text{UMG}(P)$. Let $\text{CW}(P)$ and $\text{WCW}(P)$ denote the set of Condorcet winners and weak Condorcet winners in P, respectively. Notice that $\text{CW}(P) \subseteq \text{WCW}(P)$ and $|\text{CW}(P)| \leq 1$. The domain of $\text{CW}(\cdot)$ and $\text{WCW}(\cdot)$ can be naturally extended to all weighted or unweighted directed graphs.

For example, a distribution $\hat{\pi}$, $\text{WMG}(\hat{\pi})$, and $\text{UMG}(\hat{\pi})$ for $m = 3$ are illustrated in Figure 1. We have $\text{CW}(\hat{\pi}) = \emptyset$ and $\text{WCW}(\hat{\pi}) = \{1, 2\}$. As another example, let π_{uni} denote the uniform distribution over $\mathcal{L}(\mathcal{A})$. Then, the weight on every edge in $\text{WMG}(\pi_{\text{uni}})$ is 0 and $\text{UMG}(\pi_{\text{uni}})$ does not contain any edge.

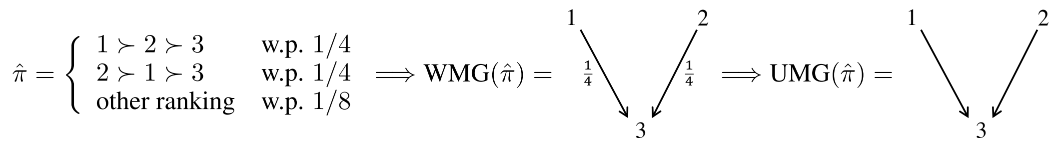


Figure 1: $\hat{\pi}$, WMG($\hat{\pi}$) (only positive edges are shown), and UMG($\hat{\pi}$).

Due to the space constraint, we focus on presenting smoothed CC on positional scoring rules and MRSE rules in the main text, whose irresolute versions are defined below. Their resolute versions can be obtained by applying a tie-breaking mechanism on the co-winners. See Section A for definitions of other rules studied in Section 3 for PAR.

Integer positional scoring rules. An *(integer) positional scoring rule* $\overline{r}_{\vec{s}}$ is characterized by an integer scoring vector $\vec{s} = (s_1, \dots, s_m) \in \mathbb{Z}^m$ with $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m$. For any alternative a and any linear order $R \in \mathcal{L}(\mathcal{A})$, we let $\vec{s}(R, a) = s_i$, where i is the rank of a in R. Given a profile P with weights $\vec{\omega}_P$, the positional scoring rule $\overline{r}_{\vec{s}}$ chooses all alternatives a with maximum $\sum_{R \in P} \omega_R \cdot \vec{s}(R, a)$. For example, *plurality* uses the scoring vector $(1, 0, \dots, 0)$, *Borda* uses the scoring vector $(m-1, m-2, \dots, 0)$, and *veto* uses the scoring vector $(1, \dots, 1, 0)$.

Multi-round score-based elimination (MRSE) rules. An irresolute MRSE rule $\overline{r}$ for m alternatives is defined by a vector of $m-1$ rules $(\overline{r}_2, \dots, \overline{r}_m)$, where for every $2 \leq i \leq m$, $\overline{r}_i$ is a positional scoring rule over i alternatives that outputs a *total preorder* over them in the decreasing order of their scores. Given a profile P, $\overline{r}(P)$ is selected in $m-1$ rounds. For each $1 \leq i \leq m-1$, in round i, a loser (an alternative with the lowest score) under $\overline{r}_{m+1-i}$ is eliminated. We use the *parallel-universes tie-breaking (PUT)* [10] to select winners—an alternative a is a winner if there is a way to break ties among the losers in each round, so that a is the remaining alternative after $m-1$ rounds. If an MRSE rule $\overline{r}$ only uses integer position scoring rules, then it is called an *int-MRSE rule*. Commonly studied int-MRSE rules include *STV*, which uses plurality in each round, *Coombs*, which uses veto in each round, and *Baldwin's rule*, which uses Borda in each round.

Example 2 (Irresolute STV). *Figure 2 illustrates the execution of irresolute STV, denoted by $\overline{STV}$, under π_{uni} (the uniform distribution) and $\hat{\pi}$ (the distribution in Figure 1), where each node represents the (tied) losers of the corresponding round, and each edge represents the loser to be eliminated. We have $\overline{STV}(\pi_{uni}) = \{1, 2, 3\}$ and $\overline{STV}(\hat{\pi}) = \{1, 2\}$.*

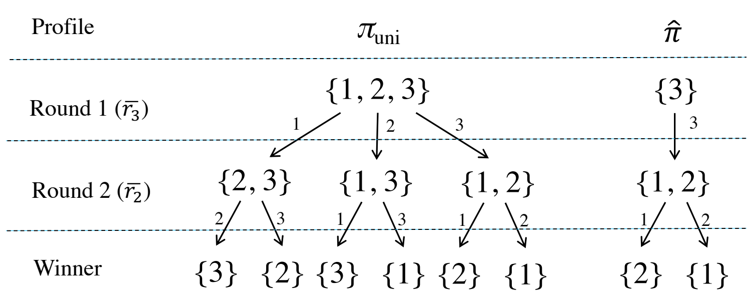


Figure 2: $\overline{\text{STV}}$ under π_{uni} and $\hat{\pi}$ (defined in Figure 1).

Axioms of voting. We focus on *per-profile axioms* [51] in this paper. A per-profile axiom is defined as a function X that maps a voting rule $\overline{r}$ and a profile P to $\{0, 1\}$, where 0 (respectively 1) means that $\overline{r}$ dissatisfies/violates (respectively, satisfies) the axiom at P. Then, the classical (worst-case) satisfaction of the axiom under $\overline{r}$ is defined to be $\min_{P \in \mathcal{L}(\mathcal{A})^*} X(\overline{r}, P)$.

For example, a (resolute or irresolute) rule $\overline{r}$ satisfies CC, if $\min_{P \in \mathcal{L}(\mathcal{A})^*} \text{CC}(\overline{r}, P) = 1$, where $\text{CC}(\overline{r}, P) = 1$ if and only if either (1) there is no Condorcet winner under P, or (2) the Condorcet winner is a co-winner of P under $\overline{r}$. A resolute rule r satisfies PAR, if $\min_{P \in \mathcal{L}(\mathcal{A})^*} \text{PAR}(r, P) = 1$, where $\text{PAR}(r, P) = 1$ if and only if no voter has incentive to abstain from voting. Formally, let $P = (R_1, \dots, R_n)$, then $[\text{PAR}(r, P) = 1] \iff \left[\forall j \leq n, r(P) \succeq_{R_j} r(P - R_j)\right]$, where $P - R_j$ is the $(n-1)$-profile that is obtained from P by removing the j-th vote. For any pair of alternatives a and b, we write $\{a\} \succeq_{R_j} \{b\}$ if and only if agent j, whose preferences are R_j, prefers a to b. See Appendix B for a list of 13 well-studied per-profile axioms and one non-per-profile axiom.

Smoothed satisfaction of axioms. Given a per-profile axiom X, a set Π of distributions over $\mathcal{L}(\mathcal{A})$, a voting rule $\overline{r}$, and $n \in \mathbb{N}$, the *smoothed satisfaction of* X under $\overline{r}$ with n agents, denoted by $\widetilde{X}^{\min}_{\Pi}(\overline{r}, n)$, is defined in Equation (1) in the Introduction. We note that the "min" in the superscript means that the adversary aims at minimizing the satisfaction of X. Formally, Π is part of the single-agent preference model defined as follows.

Definition 1 (Single-Agent Preference Model [51]). *A* single-agent preference model *is denoted by $\mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi)$, where Θ is the parameter space, $\mathcal{L}(\mathcal{A})$ is the sample space, and Π consists of distributions indexed by Θ. $\mathcal{M}$ is* strictly positive *if there exists $\epsilon > 0$ such that the probability of any linear order under any distribution in Π is at least ϵ. $\mathcal{M}$ is* closed *if Π (which is a subset of the probability simplex in $\mathbb{R}^{m!}$) is a closed set in $\mathbb{R}^{m!}$.*

Example 1 illustrates a strictly positive and closed single-agent preference model for $m = 3$, where $\Pi = \{\pi^1, \pi^2\}$ and $\epsilon = 1/8$. Other examples can be found in [51, Example 2 in the appendix].

3 The Smoothed Satisfaction of CC and PAR

Smoothed CC under Integer Positional Scoring Rules. To present the results, we first define *almost Condorcet winners (ACW)* of a profile P, which are the two alternatives (whenever they exist) that are tied in the UMG and beat all other alternatives in head-to-head competitions.

Definition 2 (Almost Condorcet Winners). *For any unweighted directed graph G over $\mathcal{A}$, a pair of alternatives a, b are* almost Condorcet winners (ACWs)*, denoted by ACW(G), if (1) a and b are tied in G, and (2) for any other alternative $c \notin \{a, b\}$, G has $a \to c$ and $b \to c$. For any profile P, let ACW(P) = ACW(UMG(P)).*

For example, 1 and 2 are ACWs of $\hat{\pi}$ (as a fractional profile) in Figure 1. By definition, for any profile P, $|\text{ACW}(P)|$ is either 0 or 2, and when it is 2, $\text{WCW}(P) = \text{ACW}(P)$.

We now present a full characterization of smoothed CC under integer positional scoring rules.

Theorem 1 (Smoothed CC: Integer Positional Scoring Rules). *For any fixed $m \geq 3$, let $\mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi)$ be a strictly positive and closed single-agent preference model, let $\overline{r}_{\vec{s}}$ be an irresolute integer positional scoring rule, and let $r_{\vec{s}}$ be a refinement of $\overline{r}_{\vec{s}}$. For any $n \geq 8m + 49$ with $2 \mid n$,*

$$\widetilde{\text{CC}}^{\min}_{\Pi}(r_{\vec{s}}, n) = \begin{cases} 1 - \exp(-\Theta(n)) & \textit{if } \forall \pi \in \textit{CH}(\Pi), |\textit{WCW}(\pi)| \times |\overline{r}_{\vec{s}}(\pi) \cup \textit{WCW}(\pi)| \leq 1 \\ \Theta(n^{-0.5}) & \textit{if} \begin{cases} \textit{(1) } \forall \pi \in \textit{CH}(\Pi), \textit{CW}(\pi) \cap (\mathcal{A} \setminus \overline{r}_{\vec{s}}(\pi)) = \emptyset \textit{ and} \\ \textit{(2) } \exists \pi \in \textit{CH}(\Pi) \textit{ s.t. } |\textit{ACW}(\pi) \cap (\mathcal{A} \setminus \overline{r}_{\vec{s}}(\pi))| = 2 \end{cases} \\ \exp(-\Theta(n)) & \textit{if } \exists \pi \in \textit{CH}(\Pi) \textit{ s.t. } \textit{CW}(\pi) \cap (\mathcal{A} \setminus \overline{r}_{\vec{s}}(\pi)) \neq \emptyset \\ \Theta(1) \wedge (1 - \Theta(1)) & \textit{otherwise} \end{cases}$$

For any $n \geq 8m + 49$ with $2 \nmid n$,

$$\widetilde{\text{CC}}^{\min}_{\Pi}(r_{\vec{s}}, n) = \begin{cases} 1 - \exp(-\Theta(n)) & \textit{same as the } 2 \mid n \textit{ case} \\ \exp(-\Theta(n)) & \textit{if } \exists \pi \in \textit{CH}(\Pi) \textit{ s.t. } \begin{cases} \textit{(1) } \textit{CW}(\pi) \cap (\mathcal{A} \setminus \overline{r}_{\vec{s}}(\pi)) \neq \emptyset \textit{ or} \\ \textit{(2) } |\textit{ACW}(\pi) \cap (\mathcal{A} \setminus \overline{r}_{\vec{s}}(\pi))| = 2 \end{cases} \\ \Theta(1) \wedge (1 - \Theta(1)) & \textit{otherwise} \end{cases}$$

Generality. We believe that Theorem 1 is quite general, as it can be applied to *any* refinement of *any* irresolute integer positional scoring rule (i.e., using *any* tie-breaking mechanism) w.r.t. *any* Π that satisfies mild conditions. The power of Theorem 1 is that it converts complicated probabilistic arguments about smoothed CC to deterministic arguments about properties of (fractional) profiles in $\text{CH}(\Pi)$, i.e., $\overline{r}_{\vec{s}}(\pi)$, $\text{CW}(\pi)$, $\text{ACW}(\pi)$, and $\text{WCW}(\pi)$, which are much easier to check. In particular, Theorem 1 can be easily applied to i.i.d. distributions (including IC) as shown in Example 3 below.

Intuitive explanations of the conditions. While the conditions for the cases in Theorem 1 may appear technical, they have intuitive explanations. Take the $2 \mid n$ case for example. **The $\mathbf{1 - \exp(-\Theta(n))}$ case** happens if every $\pi \in \text{CH}(\Pi)$ is a "robust" instance of CC satisfaction, in the sense that after any small perturbation is introduced to π, it is still an instance of CC satisfaction. For **the $\mathbf{\Theta(n^{-0.5})}$ case,** condition (1) states that every $\pi \in \text{CH}(\Pi)$ is an instance of CC satisfaction, and condition (2) requires that some $\pi \in \text{CH}(\Pi)$ corresponds to a "non-robust" instance of CC satisfaction, in the sense that after a small perturbation $\vec{\eta}$ is added to π, CC is violated at $\pi + \vec{\eta}$.

The $\exp(-\Theta(n))$ case happens if there exists a "robust" instance of CC dissatisfaction $\pi \in \text{CH}(\Pi)$, in the sense that after any small perturbation is introduced to π, it is still an instance of CC dissatisfaction. **The $\Theta(1) \wedge (1-\Theta(1))$** case holds if none of the other cases hold.

Odd vs. even n. The $2 \nmid n$ case has similar explanations. The main difference is that when $2 \nmid n$, the UMG of any n-profile must be a complete graph. Therefore, when $\text{ACW}(\pi) \neq \emptyset$, with high probability an alternative in $\text{ACW}(\pi)$ is the Condorcet winner in the randomly-generated n-profile. Then, the $\Theta(n^{-0.5})$ case in $2 \mid n$ becomes part of the $\exp(-\Theta(n))$ case in $2 \nmid n$.

Example 3 (**Applications of Theorem 1 to plurality).** *In the setting of Example 1, we apply Theorem 1 to any sufficiently large n with $2 \mid n$ and any refinement of irresolute plurality, denoted by Plu, for the following sets of distributions.*

• $\Pi = \{\pi^1, \pi^2\}$. *We have* $\widetilde{\text{CC}}_{\Pi}^{\min}(Plu, n) = \exp(-\Theta(n))$*, because let* $\pi' = \frac{3\pi^1+\pi^2}{4}$*, we have* $CW(\pi') = WCW(\pi') = \{2\}$, $ACW(\pi') = \emptyset$*, and* $\overline{Plu}(\pi') = \{1\}$.

• $\Pi_{IC} = \{\pi_{uni}\}$*, i.e., smoothed* CC *becomes likelihood of* CC *w.r.t. IC. We have* $\widetilde{\text{CC}}_{\Pi_{IC}}^{\min}(Plu, n) = \Theta(1) \wedge (1-\Theta(1))$*, because* $CW(\pi_{uni}) = \emptyset$, $WCW(\pi_{uni}) = \{1,2,3\}$*, and* $ACW(\pi_{uni}) = \emptyset$.

Smoothed CC **under int-MRSE Rules.** Smoothed CC under an MRSE rule $\overline{r}$ depends on whether the positional scoring rules it uses satisfy the CONDORCET LOSER (CL) criterion, which requires that the Condorcet loser, whenever it exists, never wins. The Condorcet loser is the alternative that loses to all head-to-head competitions. For any voting rule $\overline{r}$, we write $\text{CL}(\overline{r}) = 1$ if and only if $\overline{r}$ satisfies CONDORCET LOSER.

To present the result, we first define *parallel universes* under an MRSE rule $\overline{r}$ at $\vec{x} \in \mathbb{R}^{m!}$, denoted by $\text{PU}_{\overline{r}}(\vec{x})$, to be the set of all elimination orders in the execution of $\overline{r}$ at $\vec{x}$. Then, for any alternative a, let the *possible losing rounds*, denoted by $\text{LR}_{\overline{r}}(\vec{x}, a) \subseteq [m-1]$, be the set of all rounds in the parallel universes where a drops out. The formal definitions can be found in Definition 26 in Appendix E.3.

Example 4. *In the setting of Example 2, we let* $\overline{r} = \overline{STV}$*.* $PU_{\overline{STV}}(\pi_{uni})$ *consists of linear orders that correspond to all paths from the root to leaves in Figure 2. Therefore,* $PU_{\overline{STV}}(\pi_{uni}) = \mathcal{L}(\mathcal{A})$*. For every* $a \in \mathcal{A}$, $LR_{\overline{STV}}(\pi_{uni}, a)$ *corresponds to the rounds where a is in a node of that round in Figure 2. Therefore, for every* $a \in \mathcal{A}$*, we have* $LR_{\overline{STV}}(\pi_{uni}, a) = \{1,2\}$.

For $\hat{\pi}$ in Figure 1, we have: $PU_{\overline{STV}}(\hat{\pi}) = \{[3 \rhd 1 \rhd 2], [3 \rhd 2 \rhd 1]\}$[1]$, LR_{\overline{STV}}(\hat{\pi}, 1) = LR_{\overline{STV}}(\hat{\pi}, 2) = \{2\}$, *and* $LR_{\overline{STV}}(\hat{\pi}, 3) = \{1\}$.

We are now ready to present the $2 \mid n$ case of our characterization of smoothed CC under MRSE rules. The full version can be found in Appendix E.3.

Theorem 2 (**Smoothed** CC**: int-MRSE rules, $2 \mid n$).** *For any fixed $m \geq 3$, let $\mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi)$ be a strictly positive and closed single-agent preference model, let $\overline{r} = (\overline{r}_2, \ldots, \overline{r}_m)$ be an int-MRSE rule and let r be a refinement of $\overline{r}$. For any $n \in \mathbb{N}$ with $2 \mid n$, we have*

$$\widetilde{\text{CC}}_{\Pi}^{\min}(r, n) = \begin{cases} 1 & \text{if } \forall 2 \leq i \leq m, CL(\overline{r}_i) = 1 \\ 1-\exp(-\Theta(n)) & \text{if} \begin{cases} (1)\ \exists 2 \leq i \leq m \text{ s.t. } CL(\overline{r}_i) = 0 \text{ and} \\ (2)\ \forall \pi \in CH(\Pi), \forall a \in WCW(\pi), \forall i^* \in LR_{\overline{r}}(\pi, a), \\ \quad\quad \text{we have } CL(\overline{r}_{m+1-i^*}) = 1 \end{cases} \\ \Theta(n^{-0.5}) & \text{if} \begin{cases} (1)\ \forall \pi \in CH(\Pi), CW(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi)) = \emptyset \text{ and} \\ (2)\ \exists \pi \in CH(\Pi) \text{ s.t. } |ACW(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi))| = 2 \end{cases} \\ \exp(-\Theta(n)) & \text{if } \exists \pi \in CH(\Pi) \text{ s.t. } CW(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi)) \neq \emptyset \\ \Theta(1) \wedge (1-\Theta(1)) & \text{otherwise} \end{cases}$$

The most interesting cases are the 1 case and the $1-\exp(-\Theta(n))$ case. The 1 case happens when all positional scoring rules used in $\overline{r}$ satisfy CONDORCET LOSER. In this case, if the Condorcet winner exists, then it cannot be a loser in any round, which means that it is the unique winner under $\overline{r}$. The $1-\exp(-\Theta(n))$ case happens when (1) the 1 case does not happen, and (2) for every distribution $\pi \in \text{CH}(\Pi)$, every weak Condorcet winner a, and every possible losing round i^* for a, the positional scoring rule used in round i^*, i.e. $\overline{r}_{m+1-i^*}$, must satisfy CONDORCET LOSER. (2) guarantees that when a small permutation is added to π, if a weak Condorcet winner a becomes the Condorcet winner, then it will be the unique winner under $\overline{r}$.

[1] We use $\rhd$ to indicate the elimination order to avoid confusion with $\succ$.

Example 5 (Applications of Theorem 2 to STV). *In the setting of Example 4, let STV denote an arbitrary refinement of* $\overline{STV} = (\overline{r}_2, \overline{r}_3)$. *The 1 case does not hold for sufficiently large* n, *because* $\overline{r}_3$ *(plurality) does not satisfy* CONDORCET LOSER.

When $\Pi_{IC} = \{\pi_{uni}\}$, *Theorem 2 implies that for any sufficiently large* n *with* $2 \mid n$, *the* $\Theta(1) \wedge (1-\Theta(1))$ *case holds. The* $1-\exp(-\Theta(n))$ *case does not hold, because its condition (2) fails:* $1 \in WCW(\pi_{uni})$ *and round* 1 *is a possible losing round for alternative* 1 *(i.e.,* $1 \in LR_{\overline{STV}}(\pi_{uni}, 1)$*), yet* $\overline{r}_3$ *does not satisfy* CONDORCET LOSER. *The* $\Theta(n^{-0.5})$ *case does not hold, because its condition (2) fails:* $ACW(\pi_{uni}) = \emptyset$. *The* $\exp(-\Theta(n))$ *case does not hold because* $CW(\pi_{uni}) = \emptyset$.

Like Theorem 1, Theorem 2 can also be easily applied to i.i.d. distributions. Like Example 5, we have the following corollary w.r.t. IC, which corresponds to $\Pi_{\text{IC}} = \{\pi_{\text{uni}}\}$.

Corrollary 1 (Likelihood of CC under int-MRSE rules w.r.t. IC). *For any fixed* $m \geq 3$, *any refinement* r *of any int-MRSE rule* $\overline{r}$, *and any* $n \in \mathbb{N}$,

$$\Pr_{P\sim(\pi_{uni})^n}(\text{CC}(r, P) = 1) = \begin{cases} 1 & \text{if } \forall 2 \leq i \leq m, CL(\overline{r}_i) = 1 \\ \Theta(1) \wedge (1-\Theta(1)) & \text{otherwise} \end{cases}$$

Proof sketches for Theorem 1 and 2. In light of various multivariate central limit theorems (CLTs), when n is large, the profile is approximately $n \cdot \pi^*$ for $\pi^* = (\sum_{j=1}^{n} \pi_j)/n \in \text{CH}(\Pi)$ with high probability. Despite this high-level intuition, the conditions of the cases are quite differently from smoothed CC by definition. To see this, note that (i) the adversary may not be able to set any agent's ground truth preferences to be $\pi^* \in \text{CH}(\Pi)$, because π^* may not be in Π as shown in Example 3, and (ii) in the definition of smoothed CC, agent j's vote is a random variable distributed as π_j, instead of the fractional vote π_j. Standard CLTs can probably be applied to prove the $1-\exp(-\Theta(n))$ case and the $\Theta(1) \wedge (1-\Theta(1))$ case, but they are too coarse for other cases.

To address this challenge, we model the satisfaction of CC by the union of multiple polyhedra $\mathcal{C}$ as exemplified in Section 4. This converts the smoothed CC problem to a *PMV-in-*$\mathcal{C}$ problem [52] (Definition 3). Then, we refine [52, Theorem 2] to prove a categorization lemma (Lemma 1), and apply it to obtain Lemma 2 that characterizes smoothed CC for a large class of voting rules called *generalized irresolute scoring rules (GISRs)* [19, 50] (Definition 7 in Appendix D.1). Finally, we apply Lemma 2 to integer positional scoring rules and int-MRSE rules to obtain Theorem 1 and Theorem 2. The full proof can be found in Appendix E.2 and E.3, respectively. □

The smoothed satisfaction of PAR. Due to the space constraint, we briefly introduce our characterizations of smoothed PAR under commonly-studied voting rules defined in Appendix A, which belong to a large class of voting rules called *generalized scoring rules (GSRs)* [53] (Definition 7 in Appendix D.1). Formal statements and proofs of the theorems can be found in Appendix F.2–F.5.

Theorems 3, 4, 5, 6 (Smoothed PAR **under commonly-studied rules).** *For any fixed* $m \geq 4$, *any GSR* r *that is a refinement of maximin, STV, Schulze, ranked pairs, Copeland, any int-MRSE, or any Condocetified positional scoring rule, and any strictly positive and closed* Π *over* $\mathcal{L}(\mathcal{A})$ *with* $\pi_{uni} \in CH(\Pi)$, *there exists* $N \in \mathbb{N}$ *such that for every* $n \geq N$, $\widetilde{\text{PAR}}_{\Pi}^{\min}(r, n) = 1 - \Theta(\frac{1}{\sqrt{n}})$.

In fact, if $\pi_{\text{uni}} \notin \text{CH}(\Pi)$, then smoothed PAR converges to 1 at a faster rate, which is more positive news, as shown in Lemma 3 (Appendix F.1).

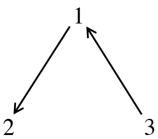
4 Beyond CC and PAR: The Categorization Lemma

In this section, we present a general lemma that characterizes smoothed satisfaction of per-profile axioms that can be represented by unions of polyhedra, including CC and PAR. To develop intuition, we start with an example of modeling CC under irresolute plurality as the union of the following two types of polyhedra in $\mathbb{R}^{m!}$.

• $\mathcal{C}_{\text{NCW}}$ represents that there is no Condorcet winner, which is the union of polyhedra $\mathcal{H}^G$, where G is an unweighted graph over $\mathcal{A}$ that does not have a Condorcet winner, as exemplified in Example 6.

• $\mathcal{C}_{\text{CWW}}$ represents that the Condorcet winner exists and also wins the plurality election, which is the union of polyhedra $\mathcal{H}^a$ for every $a \in \mathcal{A}$, that represents a being the Condorcet winner as well as a $\overline{\text{Plu}}$ co-winner, as exemplified in Example 7.

Example 6 ($\mathcal{H}^G$)**.** *Let $m = 3$ and let x_{abc} denote the number of $[a \succ b \succ c]$ votes in a profile. The following figure shows G (left) and $\mathcal{H}^G$ (right).*

$G =$ 

$\iff$

$$(x_{213} + x_{231} + x_{321}) - (x_{123} + x_{132} + x_{312}) \leq -1 \tag{2}$$
$$(x_{123} + x_{132} + x_{213}) - (x_{231} + x_{321} + x_{312}) \leq -1 \tag{3}$$
$$(x_{132} + x_{312} + x_{321}) - (x_{123} + x_{213} + x_{231}) \leq 0 \tag{4}$$
$$(x_{123} + x_{213} + x_{231}) - (x_{132} + x_{312} + x_{321}) \leq 0 \tag{5}$$

Among the four inequalities, (2) represents the $1 \to 2$ edge in G, (3) represents the $3 \to 1$ edge in G, and (4) and (5) represent the tie between 2 and 3 in G.

Example 7 ($\mathcal{H}^a$)**.** *Let $m = 3$. $\mathcal{H}^1$ is the polyhedron represented by the following four inequalities:*

$$\left.\begin{array}{r}(x_{213} + x_{231} + x_{321}) - (x_{123} + x_{132} + x_{312}) \leq -1\\(x_{231} + x_{321} + x_{312}) - (x_{123} + x_{132} + x_{213}) \leq -1\end{array}\right\} \text{1 is the Condorcet winner}$$
$$\left.\begin{array}{r}(x_{213} + x_{231}) - (x_{123} + x_{132}) \leq 0\\(x_{321} + x_{312}) - (x_{123} + x_{132}) \leq 0\end{array}\right\} \text{1 is a } \overline{\text{Plu}} \text{ co-winner}$$

It is not hard to see that $\overline{\text{Plu}}$ satisfies CC at a profile P if and only if Hist(P) is in $\mathcal{C} = \mathcal{C}_{\text{NCW}} \cup \mathcal{C}_{\text{CWW}}$, where $\mathcal{C}_{\text{NCW}} = \bigcup_{G:\text{CW}(G)=\emptyset} \mathcal{H}^G$ and $\mathcal{C}_{\text{CWW}} = \bigcup_{a \in \mathcal{A}} \mathcal{H}^a$. An example of Par under Copeland can be found in Appendix C.1. In general, the satisfaction of a wide range of axioms can be represented by unions of finitely many polyhedra. Then, the smoothed satisfaction problem reduces to the lower bound of the following PMV-in-$\mathcal{C}$ problem.

Definition 3 (The PMV-in-$\mathcal{C}$ problem [52]). *Given $q, I \in \mathbb{N}$, $\mathcal{C} = \bigcup_{i \leq I} \mathcal{H}_i$, where $\forall i \leq I$, $\mathcal{H}_i \subseteq \mathbb{R}^q$ is a polyhedron, and a set Π of distributions over $[q]$, we are interested in*

the upper bound $\sup_{\vec{\pi} \in \Pi^n} \Pr(\vec{X}_{\vec{\pi}} \in \mathcal{C})$, *and* **the lower bound** $\inf_{\vec{\pi} \in \Pi^n} \Pr(\vec{X}_{\vec{\pi}} \in \mathcal{C})$,

*where $\vec{X}_{\vec{\pi}}$ is the (n, q)-*Poisson multinomial variable (PMV) *that corresponds to the histogram of n independent random variables distributed as $\vec{\pi}$.*

See Example 9 in Appendix C.2 for an example of PMV. The following lemma provides an asymptotic characterization on the lower bound of the PMV-in-$\mathcal{C}$ problem.

Lemma 1 (Categorization lemma, simplified). *For any PMV-in-$\mathcal{C}$ problem and any $n \in \mathbb{N}$, $\inf_{\vec{\pi} \in \Pi^n} \Pr(\vec{X}_{\vec{\pi}} \in \mathcal{C})$ is 0, $\exp(-\Theta(n))$, $\text{poly}^{-1}(n)$, $\Theta(1) \wedge (1 - \Theta(1))$, $1 - \text{poly}^{-1}(n)$, $1 - \exp(-\Theta(n))$, or 1.*

The full version of Lemma 1 (Appendix C.2) also characterizes the condition for each case, the degree of polynomial, and $\sup_{\vec{\pi} \in \Pi^n} \Pr(\vec{X}_{\vec{\pi}} \in \mathcal{C})$. Lemma 1's main merit is conceptual, as it categorizes the smoothed likelihood into seven cases for quantitative comparisons, summarized in the increasing order in the table below, which are *0*, *very unlikely (VU)*, *unlikely (U)*, *medium (M)*, *likely (L)*, *very likely (VL)*, and *1*. The first three cases (0, VU, U) are negative news, where the adversary can set the ground truth so that the axiom is almost surely violated in large elections ($n \to \infty$). The last three cases (L, VL, and 1) are positive news, because the axiom is satisfied almost surely in large elections, regardless of the adversary's choice. The M case can be interpreted positively or negatively, depending on the context.

Name	**0**	**VU**	**U**	**M**	**L**	**VL**	**1**
Lem. 1	0	$\exp(-\Theta(n))$	$\text{poly}^{-1}(n)$	$\Theta(1) \wedge (1 - \Theta(1))$	$1 - \text{poly}^{-1}(n)$	$1 - \exp(-\Theta(n))$	1

5 Future work

There are many open questions for future work. What are the smoothed CC and smoothed Par for voting rules not sutdied in this paper, such as Bucklin? What is the smoothed satisfaction of Par when a group of agents can simultaneously abstain from voting [28]? More generally, we believe that drawing a comprehensive picture of smoothed satisfactions of other voting axioms and/or paradoxes, such as those described in Appendix B, is an important, promising, and challenging mission, and the categorization lemma (Lemma 1) can be a useful conceptual and technical tool to start with.

References

[1] Kenneth Arrow. *Social choice and individual values*. New Haven: Cowles Foundation, 2nd edition, 1963. 1st edition 1951.

[2] Dorothea Baumeister, Tobias Hogrebe, and Jörg Rothe. Towards Reality: Smoothed Analysis in Computational Social Choice. In *Proceedings of AAMAS*, pages 1691–1695, 2020.

[3] S. Berg and D. Lepelley. On probability models in voting theory. *Statistica Neerlandica*, 48 (2):133–146, 1994.

[4] James O. Berger. *Statistical Decision Theory and Bayesian Analysis*. Springer, 2nd edition, 1985.

[5] Debarun Bhattacharjya and Jeffrey Kephart. Bayesian interactive decision support for multi-attribute problems with even swaps. In *Proceedings of the Thirtieth Conference Annual Conference on Uncertainty in Artificial Intelligence*, 2014.

[6] Duncan Black. *The Theory of Committees and Elections*. Cambridge University Press, 1958.

[7] George E. P. Box. Robustness in the strategy of scientific model building. In R. L. Launer and G. N. Wilkinson, editors, *Robustness in Statistics*, pages 201–236. Academic Press, 1979.

[8] Felix Brandt, Johannes Hofbauer, and Martin Strobel. Exploring the No-Show Paradox for Condorcet Extensions. In Mostapha Diss and Vincent Merlin, editors, *Evaluating Voting Systems with Probability Models*. Springer, 2021.

[9] Marquis de Condorcet. *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*. Paris: L'Imprimerie Royale, 1785.

[10] Vincent Conitzer, Matthew Rognlie, and Lirong Xia. Preference functions that score rankings and maximum likelihood estimation. In *Proceedings of the Twenty-First International Joint Conference on Artificial Intelligence (IJCAI)*, pages 109–115, Pasadena, CA, USA, 2009.

[11] Mostapha Diss and Vincent Merlin, editors. *Evaluating Voting Systems with Probability Models*. Studies in Choice and Welfare. Springer International Publishing, 2021.

[12] Cynthia Dwork, Ravi Kumar, Moni Naor, and D. Sivakumar. Rank aggregation methods for the web. In *Proceedings of the 10th World Wide Web Conference*, pages 613–622, 2001.

[13] Peter C. Fishburn. Aspects of One-Stage Voting Rules. *Management Science*, 21(4):422–427, 1974.

[14] Peter C. Fishburn. Simple voting systems and majority rule. *Behavioral Science*, 19(3):166–176, 1974.

[15] Peter C. Fishburn. Paradoxes of voting. *The American Political Science Review*, 68(2):537–546, 1974.

[16] Peter C. Fishburn and Steven J. Brams. Paradoxes of Preferential Voting. *Mathematics Magazine*, 56(4):207–214, 1983.

[17] Peter C. Fishburn and William V. Gehrlein. An analysis of simple two-stage voting systems. *Behavioral Science*, 21(1):1–12, 1976.

[18] Peter C. Fishburn and William V. Gehrlein. An analysis of voting procedures with nonranked voting. *Behavioral Science*, 22(3):178–185, 1977.

[19] Rupert Freeman, Markus Brill, and Vincent Conitzer. General Tiebreaking Schemes for Computational Social Choice. In *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems*, pages 1401–1409, 2015.

[20] William V. Gehrlein. *Condorcet's Paradox*. Springer, 2006.

[21] William V. Gehrlein and Peter C. Fishburn. The probability of the paradox of voting: A computable solution. *Journal of Economic Theory*, 13(1):14–25, 1976.

[22] William V. Gehrlein and Peter C. Fishburn. Coincidence probabilities for simple majority and positional voting rules. *Social Science Research*, 7(3):272–283, 1978.

[23] William V. Gehrlein and Peter C. Fishburn. Probabilities of election outcomes for large electorates. *Journal of Economic Theory*, 19(1):38–49, 1978.

[24] William V. Gehrlein and Dominique Lepelley. *Voting Paradoxes and Group Coherence: The Condorcet Efficiency of Voting Rules*. Springer, 2011.

[25] Allan Gibbard. Manipulation of voting schemes: A general result. *Econometrica*, 41:587–601, 1973.

[26] Itzhak Gilboa and David Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, 1989.

[27] Aki Lehtinen and Jaakko Kuorikoski. Unrealistic Assumptions in Rational Choice Theory. *Philosophy of the Social Sciences*, 37(2):115–138, 2007.

[28] Dominique Lepelley and Vincent Merlin. Scoring run-off paradoxes for variable electorates. *Economic Theory*, 17:53–80, 2001.

[29] Dominique Lepelley, Frederic Chantreuil, and Sven Berg. The likelihood of monotonicity paradoxes in run-off elections. *Mathematica Slocial Science*, 31:133–146, 1996.

[30] Tie-Yan Liu. *Learning to Rank for Information Retrieval*. Springer, 2011.

[31] Andrew Mao, Ariel D. Procaccia, and Yiling Chen. Better human computation through principled voting. In *Proceedings of the National Conference on Artificial Intelligence (AAAI)*, Bellevue, WA, USA, 2013.

[32] Nicholas Mattei and Toby Walsh. PrefLib: A Library of Preference Data. In *Proceedings of Third International Conference on Algorithmic Decision Theory*, Lecture Notes in Artificial Intelligence, 2013.

[33] David C. McGarvey. A theorem on the construction of voting paradoxes. *Econometrica*, 21 (4):608–610, 1953.

[34] Samuel Merrill. A statistical model for Condorcet efficiency based on simulation under spatial model assumptions. *Public Choice*, 47(2):389–403, 1985.

[35] Elchanan Mossel, Ariel D. Procaccia, and Miklos Z. Racz. A Smooth Transition From Powerlessness to Absolute Power. *Journal of Artificial Intelligence Research*, 48(1):923–951, 2013.

[36] Hervé Moulin. Condorcet's principle implies the no show paradox. *Journal of Economic Theory*, 45(1):53–64, 1988.

[37] Jill Van Newenhizen. The Borda Method Is Most Likely to Respect the Condorcet Principle. *Economic Theory*, 2(1):69–2–83, 1992.

[38] Hannu Nurmi. An Assessment of Voting System Simulations. *Public Choice*, 73(4):459–487, 1992.

[39] Hannu Nurmi. *Voting Paradoxes and How to Deal with Them*. Springer-Verlag Berlin Heidelberg, 1999.

[40] David C. Paris. Plurality distortion and majority rule. *Behavioral Science*, 20(2):125–133, 1975.

[41] Donald G. Saari. *Basic Geometry of Voting*. Springer, 1995.

[42] Mark Satterthwaite. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10:187–217, 1975.

[43] Alexander Schrijver. *Theory of Linear and Integer Programming*. Wiley, 1998.

[44] Markus Schulze. A new monotonic, clone-independent, reversal symmetric, and condorcet-consistent single-winner election method. *Social Choice and Welfare*, 36(2):267—303, 2011.

[45] Amartya Sen. The Possibility of Social Choice. *American Economic Review*, 89(3):349–378, 1999.

[46] Daniel A. Spielman and Shang-Hua Teng. Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time. *Journal of the ACM*, 51(3), 2004.

[47] Daniel A. Spielman and Shang-Hua Teng. Smoothed Analysis: An Attempt to Explain the Behavior of Algorithms in Practice. *Communications of the ACM*, 52(10):76–84, 2009.

[48] Shuaiqiang Wang, Shanshan Huang, Tie-Yan Liu, Jun Ma, Zhumin Chen, and Jari Veijalainen. Ranking-Oriented Collaborative Filtering: A Listwise Approach. *ACM Transactions on Information Systems*, 35(2):Article No. 10, 2016.

[49] Mark C. Wilson and Geoffrey Pritchard. Probability calculations under the IAC hypothesis. *Mathematical Social Sciences*, pages 244–256, 2007.

[50] Lirong Xia. Generalized Decision Scoring Rules: Statistical, Computational, and Axiomatic Properties. In *Proceedings of the Sixteenth ACM Conference on Economics and Computation*, pages 661–678, Portland, Oregon, USA, 2015.

[51] Lirong Xia. The Smoothed Possibility of Social Choice. In *Proceedings of NeurIPS*, 2020.

[52] Lirong Xia. How Likely Are Large Elections Tied? In *Proceedings of ACM EC*, 2021.

[53] Lirong Xia and Vincent Conitzer. Generalized scoring rules and the frequency of coalitional manipulability. In *Proceedings of the ACM Conference on Electronic Commerce*, pages 109–118, 2008.

[54] Lirong Xia and Vincent Conitzer. Finite local consistency characterizes generalized scoring rules. In *Proceedings of the Twenty-First International Joint Conference on Artificial Intelligence*, pages 336–341, 2009.

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(c) Did you discuss any potential negative societal impacts of your work? [No] We are not aware of negative social impact of this work.

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(a) Did you state the full set of assumptions of all theoretical results? [Yes]

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(a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes]

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We didn't use crowdsourcing or conducted research with human subjects

Contents

A Definitions of More Voting Rules

WMG-based rules. A voting rule is said to be *weighted-majority-graph-based (WMG-based)* if its winners only depend on the WMG of the input profile. In this paper we consider the following commonly-studied WMG-based irresolute rules.

- **Copeland.** The Copeland rule is parameterized by a number $0 \leq \alpha \leq 1$, and is therefore denoted by $\overline{\text{Cd}}_\alpha$. For any profile P, an alternative a gets 1 point for each other alternative it beats in head-to-head competitions, and gets α points for each tie. $\overline{\text{Cd}}_\alpha$ chooses all alternatives with the highest total score as winners.
- **Maximin.** For each alternative a, its *min-score* is defined to be $\text{MS}_P(a) = \min_{b\in\mathcal{A}} w_P(a,b)$. Maximin, denoted by $\overline{\text{MM}}$, chooses all alternatives with the max min-score as winners.
- **Ranked pairs.** Given a profile P, an alternative a is a winner under ranked pairs (denoted by $\overline{\text{RP}}$) if there exists a way to fix edges in $\text{WMG}(P)$ one by one in a non-increasing order w.r.t. their weights (and sometimes break ties), unless it creates a cycle with previously fixed edges, so that after all edges are considered, a has no incoming edge. This is known as the *parallel-universes tie-breaking (PUT)* [10].
- **Schulze.** The *strength* of any directed path in the WMG is defined to be the minimum weight on single edges along the path. For any pair of alternatives a, b, let $s[a,b]$ denote the highest weight among all paths from a to b. Then, we write $a \succeq b$ if and only if $s[a,b] \geq s[b,a]$, and Schulze [44] proved that the strict version of this binary relation, denoted by $\succ$, is transitive. The Schulze rule, denoted by $\overline{\text{Sch}}$, chooses all alternatives a such that for all other alternatives b, we have $a \succeq b$.

Condorcetified (integer) positional scoring rules. The rule is defined by an integer scoring vector $\vec{s} \in \mathbb{Z}^m$ and is denoted by $\overline{\text{Cond}}_{\vec{s}}$, which selects the Condorcet winner when it exits, and otherwise uses $\overline{r}_{\vec{s}}$ to select the (co)-winners. For example, *Black's rule* [6] is the Condorcetified Borda rule.

B Per-Profile and Non-Per-Profile Axioms

In this section, we provide an (incomplete) list of 14 commonly-studied per-profile axioms and one commonly-studied non-per-profile axiom that we do not see a clear per-profile representation.

Per-Profile Axioms. We present the definitions of the per-profile axioms in the alphabetical order. Their equivalent X definition is often straightforward unless explicitly discussed below.

1. ANONYMITY states that the winner is insensitive to the identities of the voters. It is a per-profile axiom as shown in [51].
2. CONDORCET CRITERION is a per-profile axiom as discussed in the Introduction.
3. CONDORCET LOSER requires that a *Condorcet loser*, which is the alternative who *loses* to every head-to-head competition with other alternatives, should not be selected as the winner. It is a per-profile axiom in the same sense as CC.
4. CONSISTENCY requires that for any profile P and any sub-profile P' of P, if $r(P') = r(P \setminus P')$, then $r(P) = r(P')$. Therefore, for any profile P, we can define
$$[Consistency(r,P) = 1] \iff [\forall P' \subset P, [r(P') = r(P \setminus P')] \Rightarrow [r(P) = r(P')]]$$
5. GROUP-NON-MANIPULABLE is defined similarly to NON-MANIPULABLE below, except that multiple voters are allowed to simultaneously change their votes, and after doing so, at least one of them strictly prefers the old winner.
6. INDEPENDENT OF CLONES requires that the winner does not change when *clones* of an alternative is introduced. The clones and the original alternative must be ranked consecutively in each vote. Let IoC denote INDEPENDENT OF CLONES. For any profile P, we let $IoC(r,P) = 1$ if and only if for every alternative a and every profile P' obtain from P by introducing clones of a, we have $r(P) = r(P')$.

7. MAJORITY CRITERION requires that any alternative that is ranked at the top place in more than 50% of the votes must be selected as the winner. *Majority criterion* is stronger than CONDORCET CRITERION.

8. MAJORITY LOSER requires that any alternative who is ranked at the bottom place in more than 50% of the votes should not be selected as the winner. MAJORITY LOSER is weaker than CONDORCET LOSER.

9. MONOTONICITY requires raising up the position of the current winner in any vote will not cause it to lose. Let MONO denote MONOTONICITY. One way to define $Mono$ is the following.Let $Mono^1(r, P) = 1$ if and only if for every profile P' that is obtained from P by raising the position of $r(P)$ in one vote, we have $r(P') = r(P)$. Another definition is: $Mono^2(r, P) = 1$ if and only if for every profile P' that is obtained from P by raising the position of $r(P)$ in arbitrarily many votes, we have $r(P') = r(P)$. Notice that the classical (worst-case) MONOTONICITY is satisfied if and only if $\min_P Mono^1(r, P) = 1$ or equivalently, $\min_P Mono^2(r, P) = 1$. The smoothed satisfaction of $\min_P Mono^1$ might be different from $\min_P Mono^2$, which is beyond the scope of this paper.

10. NEUTRALITY states that the winner is insensitive to the identities of the alternatives. It is a per-profile axiom as shown in [51].

11. NON-MANIPULABLE requires that no agent has incentive to unilaterally change his/her vote to improve the winner w.r.t. his/her true preferences. More precisely, for any profile $P = (R_1, \dots, R_n)$, we have

$$[Non-Manipulable(r, P) = 1] \Leftrightarrow \left[\forall j \leq n, \forall R'_j \in \mathcal{L}(\mathcal{A}), r(P) \succeq_{R_j} r(P \cup \{R'_j\} \setminus \{R_j\})\right]$$

12. PARTICIPATION is a per-profile axiom as discussed in the Introduction.

13. REVERSAL SYMMETRY requires that the winner of any profile should not be the winner when all voters' rankings are inverted.

Non-Per-Profile Axiom(s). We were not able to model NON-DICTATORSHIP (ND) as a per-profile axiom studied in this paper. A voting rule is not a dictator if for each $j \leq n$, there exists a profile P whose winner is not ranked at the top of agent j's preferences.

C Materials for Section 4: The Categorization Lemma

While the categorization lemma (Lemma 1) was presented after Theorems 1 through 6 in the main text, the proofs of the theorems depend on the lemma. Therefore, we present materials for the categorization letter before the proofs for the theorems in the appendix.

C.1 Modeling Satisfaction of PAR as A Union of Polyhedra

PAR **under Copeland$_\alpha$.** We now show how to approximately model the satisfaction of PAR under Copeland$_\alpha$. For every pair of unweighted directed graphs G_1, G_2 over $\mathcal{A}$ and every $R \in \mathcal{L}(\mathcal{A})$, we define a polyhedron $\mathcal{H}^{G_1,R,G_2}$ to represent the histograms of profile P that contains an R-vote, $G_1 = \text{UMG}(P)$, and $G_2 = \text{UMG}(P \setminus \{R\})$. The linear inequalities used to specify the UMGs of P and $(P \setminus \{R\})$ are similar to $\mathcal{H}^G$ defined above, as illustrated in the following example.

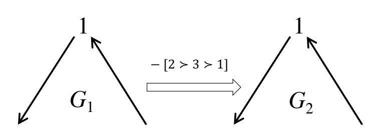


Figure 3: G_1, G_2, and R.

Example 8. *Let $m = 3$, $R = [2 \succ 3 \succ 1]$, and let G_1, G_2 denote the graphs in Figure 3. $\mathcal{H}^{G_1,R,G_2}$ is represented by the following inequalities.*

$$-x_{231} \leq -1 \tag{6}$$

$$\left.\begin{aligned}(x_{213}+x_{231}+x_{321})-(x_{123}+x_{132}+x_{312}) &\leq -1\\(x_{123}+x_{132}+x_{213})-(x_{231}+x_{321}+x_{312}) &\leq -1\\(x_{132}+x_{312}+x_{321})-(x_{123}+x_{213}+x_{231}) &\leq -1\end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned}(x_{213}+x_{231}-1+x_{321})-(x_{123}+x_{132}+x_{312}) &\leq -1\\(x_{123}+x_{132}+x_{213})-(x_{231}-1+x_{321}+x_{312}) &\leq -1\\(x_{132}+x_{312}+x_{321})-(x_{123}+x_{213}+x_{231}-1) &\leq 0\\(x_{123}+x_{213}+x_{231}-1)-(x_{132}+x_{312}+x_{321}) &\leq 0\end{aligned}\right\} \tag{8}$$

(6) guarantees that P contains an R-vote. The three inequalities in (7) represent UMG$(P) = G_1$, and the four inequalities in (8) represent UMG$(P) = G_2$.

We do not require x_R's to be non-negative, which does not affect the results of the paper, because the histograms of randomly-generated profiles are always non-negative.

By enumerating G_1, R, and G_2 that correspond to a violation of PAR, the polyhedra that represent satisfaction of PAR under Copeland$_\alpha$ are:

$$\mathcal{C} = \bigcup\nolimits_{G_1,R,G_2:\text{Copeland}_\alpha(G_1)\succeq_R\text{Copeland}_\alpha(G_2)} \mathcal{H}^{G_1,R,G_2}$$

C.2 Formal Statement of the Categorization Lemma and Proof

We first introduce notation for polyhedra. Given $q \in \mathbb{N}, L \in \mathbb{N}$, an $L \times q$ integer matrix $\mathbf{A}$, a q-dimensional row vector $\vec{\mathbf{b}}$, we define

$$\mathcal{H} \triangleq \left\{\vec{x} \in \mathbb{R}^q : \mathbf{A} \cdot (\vec{x})^\top \leq \left(\vec{b}\right)^\top\right\}, \quad \mathcal{H}_{\leq 0} \triangleq \left\{\vec{x} \in \mathbb{R}^q : \mathbf{A} \cdot (\vec{x})^\top \leq \left(\vec{0}\right)^\top\right\}$$

That is, $\mathcal{H}$ is the polyhedron represented by $\mathbf{A}$ and $\vec{\mathbf{b}}$ and $\mathcal{H}_{\leq 0}$ is the *characteristic cone* of $\mathcal{H}$.

Example 9 (Poisson multinomial variable (PMV) $\vec{X}_{\vec{\pi}}$). *In the setting of Example 1, we have $q = m! = 6$. Let $n = 2$ and $\vec{\pi} = (\pi^2, \pi^1)$. $\vec{X}_{\vec{\pi}}$ is the histogram of two random variables Y_1, Y_2 over $[q]$, where Y_1 (respectively, Y_2) is distributed as π^2 (respectively, π^1).*

For example, let $\vec{x} \in \{0,1,2\}^{\mathcal{L}(\mathcal{A})}$ denote the vector whose 123 and 231 components are 1 and all other components are 0. We have $\Pr(\vec{X}_{\vec{\pi}} = \vec{x}) = \frac{1}{4} \times \frac{3}{8} + \frac{1}{8} \times \frac{1}{8} = \frac{7}{64}$.

Definition 4 (Almost complement). *Let $\mathcal{C}$ denote a union of finitely many polyhedra. We say that a union of finitely many polyhedra $\mathcal{C}^*$ is an* almost complement *of $\mathcal{C}$, if (1) $\mathcal{C} \cap \mathcal{C}^* = \emptyset$ and (2) $\mathbb{Z}^q \subseteq \mathcal{C} \cup \mathcal{C}^*$.*

$\mathcal{C}^*$ is called an "almost complement" (instead of "complement") of $\mathcal{C}$ because $\mathcal{C}^* \cup \mathcal{C} \neq \mathbb{R}^q$. Effectively, $\mathcal{C}^*_{\leq 0}$ can be viewed as the complement of $\mathcal{C}$ when only integer vectors are concerned. It it not hard to see that $\mathcal{C}$ is an almost complement of $\mathcal{C}^*$. The following result states that the characteristic cones of $\mathcal{C}$ and $\mathcal{C}^*$, which may overlap, cover $\mathbb{R}^q$.

Proposition 1. *For any union of finitely many polyhedra $\mathcal{C}$ and any almost complement $\mathcal{C}^*$ of $\mathcal{C}$, we have $\mathcal{C}_{\leq 0} \cup \mathcal{C}^*_{\leq 0} = \mathbb{R}^q$.*

Proof. Suppose for the sake of contradiction that $\mathcal{C}_{\leq 0} \cup \mathcal{C}^*_{\leq 0} \neq \mathbb{R}^q$. Let $\vec{x} \in \mathbb{R}^q \setminus (\mathcal{C}_{\leq 0} \cup \mathcal{C}^*_{\leq 0})$ with $|\vec{x}|_1 = 1$. Because $\mathcal{C}_{\leq 0}$ and $\mathcal{C}^*_{\leq 0}$ are unions of polyhedra, there exists an $\delta > 0$ neighborhood $B_\delta = \{\vec{x}' \in \mathbb{R}^q : |\vec{x}' - \vec{x}|_\infty \leq \delta\}$ of $\vec{x}$ in $\mathbb{R}^q$ that is $\eta > 0$ away from $\mathcal{C}_{\leq 0} \cup \mathcal{C}^*_{\leq 0}$. Therefore, there exists $n \in \mathbb{N}$ with $n > \frac{1}{\delta}$ such that $nB_\delta = \{n\vec{x}' : \vec{x}' \in B_\delta\}$ do not overlap $\mathcal{C} \cup \mathcal{C}^*$. Because the radius of nB_δ is larger than 1, there exists an integer vector in nB_δ, which contradicts the assumption that $\mathbb{Z}^q \subseteq \mathcal{C} \cup \mathcal{C}^*$. $\square$

W.l.o.g., in this paper we assume that all polyhedra are represented by integer matrices $\mathbf{A}$ where the entries of each row are coprimes, which means that the greatest common divisor of all entries in the row is 1. For any $\mathcal{C} = \bigcup_{i \leq I} \mathcal{H}_i$ where $\mathcal{H}_i$ is the polyhedron characterized by integer matrices $\mathbf{A}_i$ with coprime entries and $\vec{\mathbf{b}}_i$, its almost complement always exists and is not unique. Let us define an specific almost complement of $\mathcal{C}$ that will be commonly used in this paper.

Definition 5 (Standard almost complement). *Let $\mathcal{C} = \cup_{i \leq I} \mathcal{H}_i$ denote a union of I rational polyhedra characterized by $\mathbf{A}_i$ and $\vec{\mathbf{b}}_i$, we define its* standard almost complement*, denoted by $\hat{\mathcal{C}}$, as follows.*

$$\hat{\mathcal{C}} = \bigcup_{\vec{a}_i \in \mathbf{A}_i : \forall i \leq I} \bigcap_{i \leq I} \{\vec{x} \in \mathbb{R}^q : -\vec{a}_i \cdot \vec{x} \leq -b'_i - 1\},$$

where $\vec{a}_i$ is a row in $\mathbf{A}_i$ and b'_i is the corresponding component in $\vec{\mathbf{b}}_i$. We write $\hat{\mathcal{C}} = \bigcup_{i^* \leq \hat{I}} \hat{\mathcal{H}}_{i^*}$, where $\hat{I} \in \mathbb{N}$ and each $\hat{\mathcal{H}}_{i^*}$ is a rational polyhedron.

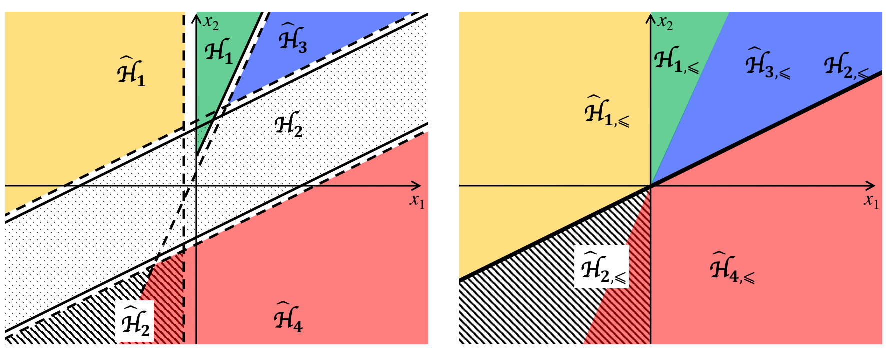
It is not hard to verify that $\hat{\mathcal{C}}$ is indeed an almost complement of $\mathcal{C}$. Let us take a look at a simple example for $q = 2$.

Example 10. *Let $\mathcal{C} = \mathcal{H}_1 \cup \mathcal{H}_2$, where $\mathcal{H}_1 = \left\{\vec{x} \in \mathbb{R}^2 : \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \cdot (\vec{x})^\top \leq \begin{bmatrix} 0 \\ -2 \end{bmatrix}\right\}$ and $\mathcal{H}_2 = \left\{\vec{x} \in \mathbb{R}^2 : \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \cdot (\vec{x})^\top \leq \begin{bmatrix} 8 \\ 8 \end{bmatrix}\right\}$. It follows that $\hat{\mathcal{C}} = \hat{\mathcal{H}}_1 \cup \hat{\mathcal{H}}_2 \cup \hat{\mathcal{H}}_3 \cup \hat{\mathcal{H}}_4$, where*

$$\hat{\mathcal{H}}_1 = \left\{\vec{x} \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \cdot (\vec{x})^\top \leq \begin{bmatrix} -1 \\ -9 \end{bmatrix}\right\}, \hat{\mathcal{H}}_2 = \left\{\vec{x} \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \cdot (\vec{x})^\top \leq \begin{bmatrix} -1 \\ -9 \end{bmatrix}\right\}$$

$$\hat{\mathcal{H}}_3 = \left\{\vec{x} \in \mathbb{R}^2 : \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \cdot (\vec{x})^\top \leq \begin{bmatrix} 1 \\ -9 \end{bmatrix}\right\}, \hat{\mathcal{H}}_4 = \left\{\vec{x} \in \mathbb{R}^2 : \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix} \cdot (\vec{x})^\top \leq \begin{bmatrix} 1 \\ -9 \end{bmatrix}\right\}$$

Figure 4 (a) shows $\mathcal{C}$ and $\hat{\mathcal{C}}$. Figure 4 (b) shows $\mathcal{C}_{\leq 0}$ and $\hat{\mathcal{C}}_{\leq 0}$, where $\hat{\mathcal{H}}_2$ is a one-dimensional polyhedron, i.e., a straight line. Note that $\mathcal{C} \cup \hat{\mathcal{C}} \neq \mathbb{R}^q$ and $\mathcal{C}_{\leq 0} \cup \hat{\mathcal{C}}_{\leq 0} = \mathbb{R}^q$.



(a) $\mathcal{C}$ and $\hat{\mathcal{C}}$. (b) $\mathcal{C}_{\leq 0}$ and $\hat{\mathcal{C}}_{\leq 0}$.

Figure 4: In (a), $\mathcal{C} = \mathcal{H}_1 \cup \mathcal{H}_2$, where $\mathcal{H}_1$ is the green area and $\mathcal{H}_2$ is a shaded area, and $\hat{\mathcal{C}} = \hat{\mathcal{H}}_1 \cup \hat{\mathcal{H}}_2 \cup \hat{\mathcal{H}}_3 \cup \hat{\mathcal{H}}_4$, where $\hat{\mathcal{H}}_2$ is a shaded area, and $\hat{\mathcal{H}}_1$, $\hat{\mathcal{H}}_3$, and $\hat{\mathcal{H}}_4$ are the yellow, red, and blue areas, respectively. In (b), $\mathcal{C}_{\leq 0} \cup \hat{\mathcal{C}}_{\leq 0} = \mathbb{R}^q$, where $\hat{\mathcal{H}}_2$ is a straight line.

To present the categorization lemma, we recall the definitions of α_n, β_n, and Theorem 2 in [52]. We first define the *activation graph*.

Definition 6 (Activation graph [52]). *For each Π, $\mathcal{H}_i$, and $n \in \mathbb{N}$, the* activation graph*, denoted by $\mathcal{G}_{\Pi,\mathcal{C},n}$, is defined to be the complete bipartite graph with two sets of vertices CH(Π) and $\{\mathcal{H}_i : i \leq I\}$, and the weight on the edge $(\pi, \mathcal{H}_i)$ is defined as follows.*

$$w_n(\pi, \mathcal{H}_i) \triangleq \begin{cases} -\infty & \text{if } \mathcal{H}_{i,n}^{\mathbb{Z}} = \emptyset \\ -\frac{n}{\log n} & \text{otherwise, if } \pi \notin \mathcal{H}_{i,\leq 0} \\ \dim(\mathcal{H}_{i,\leq 0}) & \text{otherwise} \end{cases},$$

where $\mathcal{H}_{i,n}^{\mathbb{Z}}$ is the set of non-negative integer vectors in $\mathcal{H}_i$ whose L_1 norm is n.

Definition 6 slightly abuses notation, because its vertices $\{\mathcal{H}_i : i \leq I\}$ are not explicitly indicated in the subscript of $\mathcal{G}_{\Pi,\mathcal{C},n}$. This does not cause confusion when they are clear from the context.

When $\mathcal{H}_{i,n}^{\mathbb{Z}} = \emptyset$ we say that $\mathcal{H}_i$ is *inactive (at n)*, and when $\mathcal{H}_{i,n}^{\mathbb{Z}} \neq \emptyset$ we say that $\mathcal{H}_i$ is *active (at n)*. In addition, if the weight on any edge $(\pi, \mathcal{H}_i)$ is positive, then we say that π is *active* and is *activated* by $\mathcal{H}_i$ (which must be active at n).

Roughly speaking, for any sufficiently large n and $\vec{\pi} = (\pi_1, \ldots, \pi_n) \in \Pi^n$, let $\pi = \frac{1}{n}\sum_{j=1}^n \pi_j$, then [52, Theorem 1] implies

$$\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{H}_i) \approx n^{w_n(\pi,\mathcal{H}_i)-q}$$

It follows that $\Pr(\vec{X}_{\vec{\pi}} \in \mathcal{C})$ is mostly determined by the heaviest weight on edges connected to π, denoted by $\dim_{\mathcal{C},n}^{\max}(\pi)$, which is formally defined as follows:

$$\dim_{\mathcal{C},n}^{\max}(\pi) \triangleq \max_{i \leq I} w_n(\pi, \mathcal{H}_i)$$

Then, a max-(respectively, min-) adversary aims to choose $\vec{\pi} = (\pi_1, \ldots, \pi_n) \in \Pi^n$ to maximize (respectively, minimize) $\dim_{\mathcal{C},n}^{\max}(\frac{1}{n}\sum_{j=1}^n \pi_j)$, which are characterized by α_n (respectively, β_n) defined as follows.

$$\alpha_n \triangleq \max_{\pi \in \text{CH}(\Pi)} \dim_{\mathcal{C},n}^{\max}(\pi)$$
$$\beta_n \triangleq \min_{\pi \in \text{CH}(\Pi)} \dim_{\mathcal{C},n}^{\max}(\pi)$$

We further define the following notation that will be frequently used in the proofs of this paper. Let $\mathcal{C}_n^{\mathbb{Z}}$ denote the set of all non-negative integer vectors in $\mathcal{C}$ whose L_1 norm is n. That is,

$$\mathcal{C}_n^{\mathbb{Z}} = \bigcup_{i \leq I} \mathcal{H}_{i,n}^{\mathbb{Z}}$$

By definition, $\mathcal{C}_n^{\mathbb{Z}} = \emptyset$ if and only if all $\mathcal{H}_i$'s are inactive at n. Therefore, we have

$$(\alpha_n = -\infty) \iff (\beta_n = -\infty) \iff (\mathcal{C}_n^{\mathbb{Z}} = \emptyset)$$

For completeness, we recall [52, Theorem 2] below.

Theorem 2 in [52] (Smoothed likelihood of PMV-in-$\mathcal{C}$). *Given any $q, I \in \mathbb{N}$, any closed and strictly positive Π over $[q]$, and any set $\mathcal{C} = \bigcup_{i \leq I} \mathcal{H}_i$ that is the union of finitely many polyhedra with integer matrices, for any $n \in \mathbb{N}$,*

$$\sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{C}\right) = \begin{cases} 0 & \text{if } \alpha_n = -\infty \\ \exp(-\Theta(n)) & \text{if } -\infty < \alpha_n < 0 \\ \Theta\left(n^{\frac{\alpha_n - q}{2}}\right) & \text{otherwise (i.e. } \alpha_n \geq 0) \end{cases},$$

$$\inf_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{C}\right) = \begin{cases} 0 & \text{if } \beta_n = -\infty \\ \exp(-\Theta(n)) & \text{if } -\infty < \beta_n < 0 \\ \Theta\left(n^{\frac{\beta_n - q}{2}}\right) & \text{otherwise (i.e. } \beta_n \geq 0) \end{cases}.$$

For any almost complement $\mathcal{C}^*$ of $\mathcal{C}$, let α_n^* and β_n^* denote the counterparts of α_n and β_n for $\mathcal{C}^*$, respectively. We note that α_n^* and β_n^* depend on the polyhedra used to representation $\mathcal{C}^*$. We are now ready to present the full version of the categorization lemma as follows.

Lemma 1. (Categorization Lemma, Full Version). *Given any $q, I \in \mathbb{N}$, any closed and strictly positive Π over $[q]$, any $\mathcal{C} = \bigcup_{i \leq I} \mathcal{H}_i$ and its almost complement $\mathcal{C}^* = \bigcup_{i^* \leq I^*} \mathcal{H}_{i^*}^*$, for any $n \in \mathbb{N}$,*

$$\inf_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{C}\right) = \begin{cases} 0 & \text{if } \beta_n = -\infty \\ \exp(-\Theta(n)) & \text{if } -\infty < \beta_n < 0 \\ \Theta\left(n^{\frac{\beta_n - q}{2}}\right) & \text{if } 0 \leq \beta_n < q \\ \Theta(1) \wedge (1 - \Theta(1)) & \text{if } \alpha_n^* = \beta_n = q \\ 1 - \Theta\left(n^{\frac{\alpha_n^* - q}{2}}\right) & \text{if } 0 \leq \alpha_n^* < q \\ 1 - \exp(-\Theta(n)) & \text{if } -\infty < \alpha_n^* < 0 \\ 1 & \text{if } \alpha_n^* = \infty \end{cases}$$

$$\sup_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{C}\right) = \begin{cases} 0 & \textit{if } \alpha_n = -\infty \\ \exp(-\Theta(n)) & \textit{if } -\infty < \alpha_n < 0 \\ \Theta\left(n^{\frac{\alpha_n - q}{2}}\right) & \textit{if } 0 \le \alpha_n < q \\ \Theta(1) \wedge (1 - \Theta(1)) & \textit{if } \alpha_n = \beta_n^* = q \\ 1 - \Theta\left(n^{\frac{\beta_n^* - q}{2}}\right) & \textit{if } 0 \le \beta_n^* < q \\ 1 - \exp(-\Theta(n)) & \textit{if } -\infty < \beta_n^* < 0 \\ 1 & \textit{if } \beta_n^* = -\infty \end{cases}$$

Proof. We present the proof for the $\inf$ part of Lemma 1 and the proof for the $\sup$ part is similar. Notice that $\mathbb{Z}^q \subseteq \mathcal{C} \cup \mathcal{C}^*$, we have:

$$\inf_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{C}\right) = 1 - \sup_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{C}^*\right)$$

The proof is done by combining the $\inf$ part of [52, Theorem 2] (applied to $\mathcal{C}$) and one minus the $\sup$ part of [52, Theorem 2] (applied to $\mathcal{C}^*$).

- **The 0, $\exp(-\Theta(n))$ and $\Theta\left(n^{\frac{\beta_n - q}{2}}\right)$ cases** follow after the corresponding $\inf$ part of [52, Theorem 2] applied to $\mathcal{C}$.
- **The $\Theta(1) \wedge (1 - \Theta(1))$ case.** The condition of this case implies that the polynomial bounds in the $\inf$ part of [52, Theorem 2] (applied to $\mathcal{C}$) hold, which means that $\inf_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{C}\right) = \Theta(1)$, and the polynomial bounds in the $\sup$ part of [52, Theorem 2] (applied to $\mathcal{C}^*$) hold, which means that
$$\inf_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{C}\right) = 1 - \sup_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{C}^*\right) = 1 - \Theta(1)$$
- **The $1 - \Theta\left(n^{\frac{\alpha_n^* - q}{2}}\right)$, $1 - \exp(-\Theta(n))$, and 1 cases** follow after one minus the $\sup$ part of [52, Theorem 2] (applied to $\mathcal{C}^*$).

$\square$

Remarks. The conditions for all, except 0 and 1, cases are different between $\sup$ and $\inf$ parts of the lemma. Moreover, the degrees of polynomial in the L and U cases may be different between $\sup$ and $\inf$ parts. Let us use the setting in Example 10 and Figure 5 to illustrate the conditions for the $\inf$ case. For the purpose of illustration, we assume that all polyhedra in $\mathcal{C}$ and $\mathcal{C}^*$ are active at n.

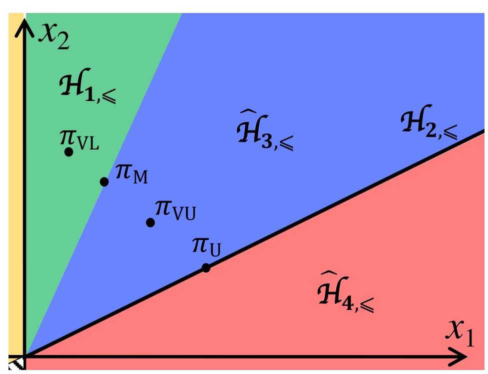


Figure 5: An Illustration of π_{VU}, π_{U}, π_{M}, and π_{VL} for the $\inf$ part of Lemma 1.

• **The 0 (respectively, 1) case** holds when no non-negative integer with L_1 norm n is in $\mathcal{C}$ (respectively, in $\mathcal{C}^*$).

• **The VU case.** Given that the 0 and 1 cases do not hold, the VU case holds when $\mathrm{CH}(\Pi)$ contains a distribution π_{VU} that is not in $\mathcal{C}_{\le 0}$. Notice that $\mathcal{C}_{\le 0}$ is a closed set and $\mathcal{C}_{\le 0} \cup \mathcal{C}^*_{\le 0} = \mathbb{R}^q$. This means that π_{VU} is an interior point of $\mathcal{C}^*_{\le 0}$. For example, in Figure 5, π_{VU} is not in $\mathcal{C}_{\le 0}$ and is an interior point of $\hat{\mathcal{H}}_{3,\le 0}$.

• **The U case** holds when $\mathrm{CH}(\Pi) \subseteq \mathcal{C}_{\le 0}$, and $\mathrm{CH}(\Pi)$ contains a distribution π_{U} that lies on a (low-dimensional) boundary of $\mathcal{C}_{\le 0}$. For example, in Figure 5, π_{U} lies in a 1-dimensional polyhedron $\mathcal{H}_{2,\le 0} \subseteq \mathcal{C}_{\le 0}$, and is not in any 2-dimensional polyhedron in $\mathcal{C}_{\le 0}$.

• **The M case** holds when the U case does not hold, and $\text{CH}(\Pi)$ contains a distribution π_{M} that lies in the intersection of a q-dimensional subspace of $\mathcal{C}_{\leq 0}$ and a q-dimensional subspace of $\mathcal{C}^*_{\leq 0}$. For example, in Figure 5, π_{U} lies in $\mathcal{H}_{1,\leq 0}$ and $\hat{\mathcal{H}}_{3,\leq 0}$, both of which are 2-dimensional.

• **The L case holds** when every distribution in $\text{CH}(\Pi)$ is in a q-dimensional subspace of $\mathcal{C}_{\leq 0}$, and there exists $\pi_{\text{L}} \in \text{CH}(\Pi)$ that lies in a (low-dimensional) boundary of $\mathcal{C}^*_{\leq 0}$. No such π_{L} exists in Figure 5's example, but if we apply Lemma 1 to $\mathcal{C}^*$, then π_{U} in Figure 5 is an example of π_{L} for $\mathcal{C}^*$.

• **The VL case holds** when every distribution in $\text{CH}(\Pi)$ is an inner point of $\mathcal{C}_{\leq 0}$. For example, in Figure 5, π_{VL} is an inner point of $\mathcal{H}_{1,\leq 0} \subseteq \mathcal{C}$.

D GISRs and Their Algebraic Properties

D.1 Definition of GISRs

All irresolute voting rules studied in this paper are generalized irresolute scoring rules (GISR) [19, 50], whose resolute versions are known as *generalized scoring rules (GSRs)* [53]. We recall the definition of GISRs based on separating hyperplanes [54, 35].

For any real number x, let $\text{Sign}(x) \in \{+, -, 0\}$ denote the sign of x. Given a set of K hyperplanes in the q-dimensional Euclidean space, denoted by $\vec{H} = (\vec{h}_1, \ldots, \vec{h}_K)$, for any $\vec{x} \in \mathbb{R}^q$, we let $\text{Sign}_{\vec{H}}(\vec{x}) = (\text{Sign}(\vec{x} \cdot \vec{h}_1), \ldots, \text{Sign}(\vec{x} \cdot \vec{h}_K))$. In other words, for any $k \leq K$, the k-th component of $\text{Sign}_{\vec{H}}(\vec{x})$ equals to 0, if $\vec{p}$ lies in hyperplane $\vec{h}_k$; and it equals to $+$ (respectively, $-$) if $\vec{p}$ lies in the positive (respectively, negative) side of $\vec{h}_k$. Each element in $\{+, -, 0\}^K$ is called a *signature*.

Definition 7 (Generalized irresolute scoring rule (GISR)). *A* generalized irresolute scoring rule (GISR) $\overline{r}$ *is defined by (1) a set of* $K \geq 1$ *hyperplanes* $\vec{H} = (\vec{h}_1, \ldots, \vec{h}_K) \in (\mathbb{R}^{m!})^K$ *and (2) a function* $g : \{+, -, 0\}^K \to (2^{\mathcal{A}} \setminus \emptyset)$. *For any profile* P, *we let* $\overline{r}(P) = g(\text{Sign}_{\vec{H}}(\text{Hist}(P)))$. $\overline{r}$ *is called an* integer GISR (int-GISR) *if* $\vec{H} \in (\mathbb{Z}^{m!})^K$. *If for all profiles* P, *we have* $|\overline{r}(P)| = 1$, *then* $\overline{r}$ *is called a generalized scoring rule (GSR). Int-GSRs are defined similarly to int-GISRs.*

Definition 8 (Feasible and atomic signatures). *Given integer* $\vec{H}$ *with* $K = |\vec{H}|$, *let* $\mathcal{S}_K = \{+, -, 0\}^K$. *A signature* $\vec{t} \in \mathcal{S}_K$ *is* feasible, *if there exists* $\vec{x} \in \mathbb{R}^d$ *such that* $\text{Sign}_{\vec{H}}(\vec{x}) = \vec{t}$. *Let* $\mathcal{S}_{\vec{H}} \subseteq \mathcal{S}_K$ *denote the set of all feasible signatures.*

A signature $\vec{t}$ *is called an* atomic signature *if and only if* $\vec{t} \in \{+, -\}^K$. *Let* $\mathcal{S}^{\circ}_{\vec{H}}$ *denote the set of all feasible atomic signatures.*

The domain of any GISR $\overline{r}$ can be naturally extended to $\mathbb{R}^{m!}$ and to $\mathcal{S}_{\vec{H}}$. Specifically, for any $\vec{t} \in \mathcal{S}_{\vec{H}}$ we let $\overline{r}(\vec{t}) = g(\vec{t})$. It suffices to define g on the feasible signatures, i.e., $\mathcal{S}_{\vec{H}}$.

Notice that the same voting rule can be represented by different combinations of $(\vec{H}, g)$. In the following section we recall int-GISR representations of the voting rules studied in this paper.

D.2 Commonly-Studied Voting Rules as GISRs

As discussed in [52], the irresolute versions of Maximin, Copeland_{α}, Ranked Pairs, and Schulze belong to the class of *edge-order-based (*EO-based) rules, which are defined over the weak order on edges in $\text{WMG}(P)$. We recall its formal definition below.

Definition 9 (Edge-order-based rules). *A (resolute or irresolute) voting rule* $\overline{r}$ *is* edge-order-based (EO-based), *if for any pair of profiles* P_1 *and* P_2 *such that for every combination of four different alternatives* $\{a, b, c, d\} \subset \mathcal{A}$, $[w_{P_1}(a, b) \geq w_{P_1}(c, d)] \Leftrightarrow [w_{P_2}(a, b) \geq w_{P_2}(c, d)]$, *we have* $\overline{r}(P_1) = \overline{r}(P_2)$.

All EO-based rules can be represented by a GISR using a set of hyperplanes that represents the orders over WMG edges. We first recall pairwise difference vectors as follows.

Definition 10 (Pairwise difference vectors [51]). *For any pair of different alternatives* a, b, *let* $\text{Pair}_{a,b}$ *denote the* $m!$*-dimensional vector indexed by rankings in* $\mathcal{L}(\mathcal{A})$: *for any* $R \in \mathcal{L}(\mathcal{A})$, *the* R*-component of* $\text{Pair}_{a,b}$ *is* 1 *if* $a \succ_R b$; *otherwise it is* -1.

We now define the hyperplanes for edge-order-based rules.

Definition 11 (**$\vec{H}_{\text{EO}}$**). *$\vec{H}_{EO}$ consists of $\binom{m(m-1)}{2}$ hyperplanes indexed by $\vec{h}_{e_1,e_2}$, where $e_1 = (a_1, a_2)$ and $e_2 = (a_2, b_2)$ are two different pairs of alternatives, such that*

$$\vec{h}_{e_1,e_2} = \textit{Pair}_{a_1,b_1} - \textit{Pair}_{a_2,b_2}$$

That is, for any (fractional) profile P, $\vec{h}_{e_1,e_2} \cdot \text{Hist}(P) \leq 0$ if and only if the weight on e_1 in WMG(P) is no more than the weight on e_2 in WMG(P). Therefore, given $\text{Sign}_{\vec{H}_{\text{EO}}}(P)$, we can compare the weights on pairs of edges, which leads to the weak order on edges in WMG(P) w.r.t. their weights. Consequently, for any profile P, $\text{Sign}_{\vec{H}}(P)$ contains enough information to determine the (co-)winners under any edge-order-based rules. Formally, the GISR representations of these rules used in this paper are defined by $\vec{H}_{\text{EO}}$ and the following g functions that mimic the procedures of choosing the winner(s).

Definition 12. *Let $\overline{MM}$, $\overline{Cd_\alpha}$, $\overline{RP}$, $\overline{Sch}$ denote the int-GISRs defined by $\vec{H}_{EO}$ and the following g functions. Given a feasible signature $\vec{t} \in \mathcal{S}_{\vec{H}_{EO}}$,*

- ***g_{MM}** first picks a representative edge e_a whose weight is no more than all other outgoing edges of a, then compare the weights of e_a's for all alternatives and choose alternatives a whose e_a has the highest weight as the winners.*
- ***g_{Cd_α}** compares weights on pairs of edges $a \to b$ and $b \to a$, and then calculate the Copeland$_\alpha$ scores accordingly. The winners are the alternatives with the highest Copeland$_\alpha$ score.*
- ***g_{RP}** mimics the execution of PUT-Ranked Pairs, which only requires information about the weak order over edges w.r.t. their weights in WMG.*
- ***g_{Sch}** first computes an edge e_p with the minimum weight on any given directed path p, then for each pair of alternatives a and b, computes an edge $e_{(a,b)}$ that represents the strongest edge among all paths from a to b. g_{Sch} then mimics Schulze to select the winner(s).*

While Copeland can be represented by $\vec{H}_{\text{EO}}$ and g_{Cd_α} as in the definition above, in this paper we use another set of hyperplanes, denoted by $\vec{H}_{\text{Cd}_\alpha}$, that represents the UMG of the profile. The reason is that in this way any refinement of Cd$_\alpha$ would break ties according to the UMG of the profile, which is needed in the proof of Theorem 4.

Definition 13 (**$\overline{\text{Cd}_\alpha}$ as a GISR**). *$\overline{Cd_\alpha}$ is represented by $\vec{H}_{Cd_\alpha}$ and g_{Cd_α} defined as follows. For every pair of different alternatives (a, b), $\vec{H}_{Cd_\alpha}$ contains a hyperplane $\vec{h}_{(a,b)} = \textit{Pair}_{a,b} - \textit{Pair}_{b,a}$. For any profile P, g_{Cd_α} first computes the outcome of each head-to-head elections between alternatives a and b by checking $\vec{h}_{(a,b)} \cdot \textit{Hist}(P)$, then calculate the Copeland$_\alpha$ score, and finally choose all alternatives with the maximum score as the winners.*

The GISR representation of MRSE rules is based on the fact that the winner(s) can be computed from comparing the scores between any pair of alternatives (a, b) after a set of alternatives B is removed. This idea is formalized in the following definition. For any $R \in \mathcal{L}(\mathcal{A})$ and any $B \subset \mathcal{A}$, let $R|_{\mathcal{A}\setminus B}$ denote the linear order over $(\mathcal{A} \setminus B)$ that is obtained from R by removing alternatives in B.

Definition 14 (**MRSE rules as GISRs**). *Any MRSE $\overline{r} = (\overline{r}_2, \ldots, \overline{r}_m)$ is represented by $\vec{H}$ and $g_{\overline{r}}$ defined as follows. Given an int-MRSE rule $\overline{r} = (\overline{r}_2, \ldots, \overline{r}_m)$, for any pair of alternatives a, b and any subset of alternatives $B \subseteq (\mathcal{A} \setminus \{a, b\})$, we let $\textit{Score}^{\Delta}_{B,a,b}$ denote the vector, where for every $R \in \mathcal{L}(\mathcal{A})$, the R-th component of $\textit{Pair}_{B,a,b}$ is $s_i^{m-|B|} - s_j^{m-|B|}$, where i and j are the ranks of a and b in $R|_{\mathcal{A}\setminus B}$, respectively.*

For any pair of different alternatives $\{a, b\} \subseteq (\mathcal{A} \setminus B)$, $\vec{H}$ contains a hyperplane $\textit{Score}^{\Delta}_{B,a,b}$. For any profile P, $g_{\overline{r}}$ mimics $\overline{r}$ to compute the PUT winners based on whether $\vec{h}_{(B,a,b)} \cdot \textit{Hist}(P)$ is < 0, $= 0$, or > 0.

In fact, the GISR representation of $\overline{r}$ in Definiton 14 corresponds to the *PUT structure* [52], which we do not discuss in this paper for simplicity of presentation. Any GSR refinement of $\overline{r}$, denoted by r, uses the same $\vec{H}$ in Definiton 14 and a different g function that always chooses a single loser to be eliminated in each round. The constraint is, for any profile P, the break-tie mechanisms used in g only depends on $\text{Sign}_{\vec{H}}(P)$ (but not any other information contained in P). For example, lexicographic tie-breaking w.r.t. a fixed order over alternatives is allowed but using the first agent's vote to break ties is not allowed.

D.3 Minimally Continuous GISRs

Next, we define (minimally) continuous GISR in a similar way as Freeman et al. [19], except that in this paper the domain of GISR is $\mathbb{R}^{m!}$ (in contrast to $\mathbb{R}^{m!}_{\geq 0}$ in [19]).

Definition 15 **((Minimally) continuous GISR).** *A GISR $\overline{r}$ is* continuous*, if for any $\vec{x} \in \mathbb{R}^{m!}$, any alternative a, and any sequence of vectors $(\vec{x}_1, \vec{x}_2 \ldots)$ that converges to $\vec{x}$,*

$$[\forall j \in \mathbb{N}, a \in \overline{r}(\vec{x}_j)] \implies [a \in \overline{r}(\vec{x})]$$

A GISR $\overline{r}$ is called minimally continuous*, if it is continuous and there does not exist a continuous GISR $\overline{r}^*$ such that (1) for all $\vec{x} \in \mathbb{R}^{m!}$, $\overline{r}^*(\vec{x}) \subseteq \overline{r}(\vec{x})$, and (2) the inclusion is strict for some $\vec{x}$.*

Equivalently, a continuous GISR $\overline{r}$ is minimally continuous if and only if the (fractional) profiles with unique winners is a dense subset of $\mathbb{R}^{m!}$. That is, for any vector in $\mathbb{R}^{m!}$, there exists a sequence of profiles with unique winners that converge to it. As commented by Freeman et al. [19], many commonly-studied irresolute voting rules are continuous GISRs. It is not hard to verify that positional scoring rules and MRSE rules are minimally continuous GISRs, which is formally proved in the following proposition.

Proposition 2. *Positional scoring rules and MRSE rules are minimally continuous.*

Proof. Let $\vec{s} = (s_1, \ldots, s_m)$ denote the scoring vector. We first prove that $\overline{r}_{\vec{s}}$ is continuous. For any $\vec{x} \in \mathbb{R}^{m!}$, any $a \in \mathcal{A}$, and any sequence $(\vec{x}_1, \vec{x}_2, \ldots)$ that converges to $\vec{x}$ such that for all $j \geq 1$, $a \in \overline{r}(\vec{x}_j)$, we have that for every $b \in \mathcal{A}$, $\vec{s}(\vec{x}_j, a) \geq \vec{s}(\vec{x}_j, b)$. Notice that $\vec{s}(\vec{x}_j, a)$ (respectively, $\vec{s}(\vec{x}_j, b)$) converges to $\vec{s}(\vec{x}, a)$ (respectively, $\vec{s}(\vec{x}, b)$). Therefore, $\vec{s}(\vec{x}, a) \geq \vec{s}(\vec{x}, b)$, which means that $a \in \overline{r}_{\vec{s}}(\vec{x})$, i.e., $\overline{r}_{\vec{s}}$ is continuous.

To prove that $\overline{r}_{\vec{s}}$ is minimally continuous, it suffices to prove that for any $\vec{x} \in \mathbb{R}^{m!}$ and any $a \in \overline{r}_{\vec{s}}(\vec{x})$, there exists a sequence $(\vec{x}_1, \vec{x}_2, \ldots)$ that converges to $\vec{x}$ such that for all $j \geq 1$, $\overline{r}(\vec{x}_j) = \{a\}$. Let σ denote an arbitrary cyclic permutation among $\mathcal{A} \setminus \{a\}$ and P denote the following $(m-1)$-profile.

$$P = \left\{\sigma^i(a \succ \text{others}) : 1 \leq i \leq m-1\right\}$$

Then, for every $j \in \mathbb{N}$, we let $\vec{x}_j = \vec{x} + \frac{1}{j}\text{Hist}(P)$. It is easy to check that $\overline{r}(\vec{x}_j) = \{a\}$, which proves the minimal continuity of $\overline{r}_{\vec{s}}$.

Let $\overline{r} = (\overline{r}_2, \ldots, \overline{r}_m)$ denote the MRSE rule. We will use notation in Section E.3 to prove the proposition for $\overline{r}$. We first prove that $\overline{r}$ is continuous. Let $\vec{x} \in \mathbb{R}^{m!}$, $a \in \mathcal{A}$, and $(\vec{x}_1, \vec{x}_2, \ldots)$ be a sequence that converges to $\vec{x}$ such that for all $j \geq 1$, $a \in \overline{r}(\vec{x}_j)$. Because the number of different parallel universes is finite (more precisely, $m!$), there exists a subsequence of $(\vec{x}_1, \vec{x}_2, \ldots)$, denoted by $(\vec{x}'_1, \vec{x}'_2, \ldots)$, and a parallel universe $O \in \mathcal{L}(\mathcal{A})$ where a is ranked in the last position (i.e., a is the winner), such that for all $j \in \mathbb{N}$, O is a parallel universe when executing $\overline{r}$ on $\vec{x}'_j$. Therefore, for all $1 \leq i \leq m-1$, in round i, $O[i]$ has the lowest $\overline{r}_{m+1-i}$ score in $\vec{x}'_j|_{O[i,m]}$ among alternatives in $O[i, m]$. It follows that $O[i]$ has the lowest $\overline{r}_{m+1-i}$ score in $\vec{x}|_{O[i,m]}$ among alternatives in $O[i, m]$, which means that O is also a parallel universe when executing $\overline{r}$ on $\vec{x}$. This proves that $\overline{r}$ is continuous.

The proof of minimal continuity of $\overline{r}$ is similar to the proof for positional scoring rules presented above. For any $\vec{x} \in \mathbb{R}^{m!}$ and any $a \in \overline{r}_{\vec{s}}(\vec{x})$, let O denote a parallel universe where a is ranked in the last position. Let P denote the following profile of $(m-1)! + (m-2)! + \cdots + 2!$ votes, where O is the unique parallel universe.

$$P = \bigcup\nolimits_{i=1}^{m-1} \{O[1] \succ \cdots \succ O[i] \succ R_i : \forall R_i \in \mathcal{L}(O[i+1, m])\}$$

For any $j \in \mathbb{N}$, let $\vec{x}_j = \vec{x} - \frac{1}{j}\text{Hist}(P)$. It is not hard to verify that $(\vec{x}_1, \vec{x}_2, \ldots)$ converges to $\vec{x}$, and for every $1 \leq i \leq m-1$ and every $j \in \mathbb{N}$, alternative $O[i]$ is the unique loser in round i, where

$-\frac{1}{j}\text{Hist}(P)$ is used as the tie-breaker. This means that for all $j \in \mathbb{N}$, $\overline{r}(\vec{x}_j) = \{a\}$, which proves the minimal continuity of $\overline{r}$. $\square$

D.4 Algebraic Properties of GISRs

We first define the refinement relationship among (feasible or infeasible) signatures.

Definition 16 (**Refinement relationship $\trianglelefteq$**). *For any pair of signatures $\vec{t}_1, \vec{t}_2 \in \mathcal{S}_K$, we say that $\vec{t}_1$* refines *$\vec{t}_2$, denoted by $\vec{t}_1 \trianglelefteq \vec{t}_2$, if for every $k \leq K$, if $[\vec{t}_2]_k \neq 0$ then $[\vec{t}_1]_k = [\vec{t}_2]_k$. If $\vec{t}_1 \trianglelefteq \vec{t}_2$ and $\vec{t}_1 \neq \vec{t}_2$, then we say that $\vec{t}_1$* strictly refines *$\vec{t}_2$, denoted by $\vec{t}_1 \triangleleft \vec{t}_2$.*

In words, $\vec{t}_1$ refines $\vec{t}_2$ if $\vec{t}_1$ differs from $\vec{t}_2$ only on the 0 components in $\vec{t}_2$. By definition, $\vec{t}_1$ refines itself. Next, given $\vec{H}$ and a feasible signature $\vec{t}$, we define a polyhedron $\mathcal{H}^{\vec{H},\vec{t}}$ to represent profiles whose signatures are $\vec{t}$.

Definition 17 (**$\mathcal{H}^{\vec{H},\vec{t}}$ ($\mathcal{H}^{\vec{t}}$ in short)**). *For any $\vec{H} = (\vec{h}_1, \ldots, \vec{h}_K) \in (\mathbb{R}^d)^K$ and any $\vec{t} \in \mathcal{S}_{\vec{H}}$, we let $\mathbf{A}^{\vec{t}} = \begin{bmatrix} \mathbf{A}^{\vec{t}}_+ \\ \mathbf{A}^{\vec{t}}_- \\ \mathbf{A}^{\vec{t}}_0 \end{bmatrix}$, where*

- *$\mathbf{A}^{\vec{t}}_+$ consists of a row $-\vec{h}_i$ for each $i \leq K$ with $t_i = +$.*
- *$\mathbf{A}^{\vec{t}}_-$ consists of a row $\vec{h}_i$ for each $i \leq K$ with $t_i = -$.*
- *$\mathbf{A}^{\vec{t}}_0$ consists of two rows $-\vec{h}_i$ and $\vec{h}_i$ for each $i \leq K$ with $t_i = 0$.*

Let $\vec{\mathbf{b}}^{\vec{t}} = [\underbrace{-\vec{1}}_{\text{for } \mathbf{A}^{\vec{t}}_+}, \underbrace{-\vec{1}}_{\text{for } \mathbf{A}^{\vec{t}}_-}, \underbrace{\vec{0}}_{\text{for } \mathbf{A}^{\vec{t}}_0}]$. The corresponding polyhedron is denoted by $\mathcal{H}^{\vec{H},\vec{t}}$, or $\mathcal{H}^{\vec{t}}$ in short when $\vec{H}$ is clear from the context.

The following proposition follows immediately after the definition.

Proposition 3. *Given $\vec{H}$, for any pair of feasible signatures $\vec{t}_1, \vec{t}_2 \in \mathcal{S}_{\vec{H}}$, $\vec{t}_1 \trianglelefteq \vec{t}_2$ if and only if $\mathcal{H}^{\vec{t}_1}_{\leq 0} \supseteq \mathcal{H}^{\vec{t}_2}_{\leq 0}$.*

Proposition 4 (**Algebraic characterization of (minimal) continuity**). *A GISR $\overline{r}$ is continuous, if and only if*

$$\forall \vec{t} \in \mathcal{S}_{\vec{H}}, \text{ we have } \overline{r}(\vec{t}) \supseteq \bigcup\nolimits_{\vec{t}' \in \mathcal{S}_{\vec{H}} : \vec{t}' \trianglelefteq \vec{t}} \overline{r}(\vec{t}')$$

$\overline{r}$ is minimally continuous, if and only if

$$\forall \vec{t} \in \mathcal{S}_{\vec{H}}, \text{ we have } \overline{r}(\vec{t}) = \bigcup\nolimits_{\vec{t}' \in \mathcal{S}^{\circ}_{\vec{H}} : \vec{t}' \trianglelefteq \vec{t}} \overline{r}(\vec{t}'), \text{ and (2) } \forall \vec{t} \in \mathcal{S}^{\circ}_{\vec{H}}, \text{ we have } |\overline{r}(\vec{t})| = 1$$

The "continuity" part of Proposition 4 states that for any feasible signature $\vec{t}$ and its refinement $\vec{t}'$, we must have $\overline{r}(\vec{t}') \subseteq \overline{r}(\vec{t})$. The "minimal continuity" part states that any minimally continuous GISR is uniquely determined by its winners under atomic signatures (where a single winner is chosen for any atomic signature).

Proof. **The "if" part for continuity.** Suppose for the sake of contradiction that there exists $\vec{t} \in \mathcal{S}_{\vec{H}}$ such that $\overline{r}(\vec{t}) \supseteq \bigcup_{\vec{t}' \in \mathcal{S}^{\circ}_{\vec{H}} : \vec{t}' \trianglelefteq \vec{t}} \overline{r}(\vec{t}')$ but $\overline{r}$ is not continuous. This means that there exists $\vec{x} \in \mathbb{R}^{m!}$ with $\text{Sign}_{\vec{H}}(\vec{x}) = \vec{t}$, an infinite sequence $(\vec{x}_1, \vec{x}_2, \ldots)$ that converge to $\vec{x}$, and an alternative $a \notin \overline{r}(\vec{x})$, such that for every $j \in \mathbb{N}$, $a \in \overline{r}(\vec{x}_j)$. Because the total number of (feasible) signatures is finite, there exists an infinite subsequence of $(\vec{x}_1, \vec{x}_2, \ldots)$, denoted by $(\vec{x}'_1, \vec{x}'_2, \ldots)$, and $\vec{t}' \in \mathcal{S}_{\vec{H}}$ such that for all $j \in \mathbb{N}$ we have $\text{Sign}_{\vec{H}}(\vec{x}'_j) = \vec{t}'$. Note that $(\vec{x}'_1, \vec{x}'_2, \ldots)$ also converges to $\vec{x}$. Therefore, the following holds for every $k \leq K$.

- If $t'_k = 0$, then for every $j \in \mathbb{N}$ we have $\vec{h}_k \cdot \vec{x}_j = 0$, which means that $\vec{h}_k \cdot \vec{x} = 0$, i.e. $t_k = 0$.
- If $t'_k = +$, then for every $j \in \mathbb{N}$ we have $\vec{h}_k \cdot \vec{x}_j > 0$, which means that $\vec{h}_k \cdot \vec{x} \geq 0$, i.e. $t_k \in \{0, +\}$.
- Similarly, if $t'_k = -$, then for every $j \in \mathbb{N}$ we have $\vec{h}_k \cdot \vec{x}_j < 0$, which means that $\vec{h}_k \cdot \vec{x} \leq 0$, i.e. $t_k \in \{0, -\}$.

This means that $\vec{t'} \trianglelefteq \vec{t}$. Recall that we have assumed $\overline{r}(\vec{t}) \supseteq \bigcup_{\vec{t'} \in \mathcal{S}_{\vec{H}} : \vec{t'} \trianglelefteq \vec{t}} \overline{r}(\vec{t'})$, which means that $a \in \overline{r}(\vec{t'}) \subseteq \overline{r}(\vec{t}) = \overline{r}(\vec{x})$. This contradicts the assumption that $a \notin \overline{r}(\vec{x})$.

The "only if" part for continuity. Suppose for the sake of contradiction that $\overline{r}$ is continuous but there exists $\vec{t} \in \mathcal{S}_{\vec{H}}$ such that $\bigcup_{\vec{t'} \in \mathcal{S}_{\vec{H}} : \vec{t'} \trianglelefteq \vec{t}} \overline{r}(\vec{t'}) \not\subseteq \overline{r}(\vec{t})$. This means that there exist $\vec{t'} \triangleleft \vec{t}$ and an alternative a such that $a \in \overline{r}(\vec{t'})$ but $a \notin \overline{r}(\vec{t})$. Because both $\vec{t}$ and $\vec{t'}$ are feasible, there exists $\vec{x}, \vec{x}' \in \mathbb{R}^{m!}$ such that $\text{Sign}_{\vec{H}}(\vec{x}) = \vec{t}$ and $\text{Sign}_{\vec{H}}(\vec{x}') = \vec{t'}$. It is not hard to verify that the infinite sequence $(\vec{x} + \vec{x}', \vec{x} + \frac{1}{2}\vec{x}', \vec{x} + \frac{1}{3}\vec{x}', \ldots)$ converge to $\vec{x}$, and for every $j \in \mathbb{N}$, $\text{Sign}_{\vec{H}}(\vec{x} + \frac{1}{j}\vec{x}') = \vec{t'}$, which means that $a \in \overline{r}(\vec{x} + \frac{1}{j}\vec{x}')$. By continuity of $\overline{r}$ we have $a \in \overline{r}(\vec{x}) = \overline{r}(\vec{t})$, which contradicts the assumption that $a \notin \overline{r}(\vec{t})$.

The "if" part for minimal continuity. To simplify the presentation, we formally define refinements of GISRs as follows.

Definition 18 (Refinements of GISRs). *Let $\overline{r}^*$ and $\overline{r}$ be a pair of GISR such that for every $\vec{x} \in \mathbb{R}^{m!}$, $\overline{r}^*(\vec{x}) \subseteq \overline{r}(\vec{x})$. $\overline{r}^*$ is called a* refinement *of $\overline{r}$. If additionally there exists $\vec{x} \in \mathbb{R}^{m!}$ such that $\overline{r}^*(\vec{x}) \subset \overline{r}(\vec{x})$, then $\overline{r}^*$ is called a* strict refinement *of $\overline{r}$.*

Suppose for every $\vec{t} \in \mathcal{S}_{\vec{H}}$ we have $\overline{r}(\vec{t}) = \bigcup_{\vec{t'} \in \mathcal{S}^{\circ}_{\vec{H}} : \vec{t'} \trianglelefteq \vec{t}} \overline{r}(\vec{t'})$, and for every $\vec{t} \in \mathcal{S}^{\circ}_{\vec{H}}$ we have $|\overline{r}(\vec{t})| = 1$. By the "continuity" part proved above, $\overline{r}$ is continuous. To prove that $\overline{r}$ is minimally continuous, suppose for the sake of contradiction that $\overline{r}$ has a strict refinement, denoted by $\overline{r}^*$. Clearly for every atomic feasible signature $\vec{t} \in \mathcal{S}^{\circ}_{\vec{H}}$ we have $\overline{r}^*(\vec{t}) = \overline{r}(\vec{t})$. Therefore, by the "continuity" part proved above, for every feasible signature $\vec{t} \in \mathcal{S}_{\vec{H}}$, we have

$$\overline{r}^*(\vec{t}) \supseteq \bigcup_{\vec{t'} \in \mathcal{S}_{\vec{H}} : \vec{t'} \trianglelefteq \vec{t}} \overline{r}^*(\vec{t'}) \supseteq \bigcup_{\vec{t'} \in \mathcal{S}^{\circ}_{\vec{H}} : \vec{t'} \trianglelefteq \vec{t}} \overline{r}^*(\vec{t'}) = \bigcup_{\vec{t'} \in \mathcal{S}^{\circ}_{\vec{H}} : \vec{t'} \trianglelefteq \vec{t}} \overline{r}(\vec{t'}) = \overline{r}(\vec{t}),$$

which contradicts the assumption that $\overline{r}^*$ is a strict refinement of $\overline{r}$.

The "only if" part for minimal continuity. Suppose $\overline{r}$ is a minimally continuous GISR. We define another GISR $\overline{r}^*$ as follows.

- For every $\vec{t} \in \mathcal{S}^{\circ}_{\vec{H}}$ we let $\overline{r}^*(\vec{t}) \subseteq \overline{r}(\vec{t})$ and $|\overline{r}^*(\vec{t})| = 1$.
- For every $\vec{t} \in \mathcal{S}_{\vec{H}}$, we let $\overline{r}^*(\vec{t}) = \bigcup_{\vec{t'} \in \mathcal{S}^{\circ}_{\vec{H}} : \vec{t'} \trianglelefteq \vec{t}} \overline{r}^*(\vec{t'})$.

By the continuity part proved above, $\overline{r}^*$ is continuous. It is not hard to verify that $\overline{r}^*$ refines $\overline{r}$. Therefore, if either condition for minimal continuity does not hold, then $\overline{r}^*$ is a strict refinement of $\overline{r}$, which contradicts the minimality of $\overline{r}$.

This proves Proposition 4. □

Next, we prove some properties about $\mathcal{H}^{\vec{t}}$ that will be frequently used in the proofs of this paper. The proposition has three parts. Part (i) characterizes profiles P whose histogram is in $\mathcal{H}^{\vec{t}}$; part (ii) characterizes vectors in $\mathcal{H}^{\vec{t}}_{\leq 0}$; and part (iii) states that for every atomic signature $\vec{t}$, $\mathcal{H}^{\vec{t}}_{\leq 0}$ is a full dimensional cone in $\mathbb{R}^{m!}$.

Claim 1 (Properties of $\mathcal{H}^{\vec{t}}$). *Given integer $\vec{H}$, any $\vec{t} \in \mathcal{S}_{\vec{H}}$,* *(i) for any integral profile P, $Hist(P) \in \mathcal{H}^{\vec{t}}$ if and only if $Sign_{\vec{H}}(Hist(P)) = \vec{t}$;*

(ii) for any $\vec{x} \in \mathbb{R}^{m!}$, $Hist(\vec{x}) \in \mathcal{H}^{\vec{t}}_{\leq 0}$ if and only if $\vec{t} \trianglelefteq Sign_{\vec{H}}(\vec{x})$;

(iii) if $\vec{t} \in \mathcal{S}^{\circ}_{\vec{H}}$ then $\dim(\mathcal{H}^{\vec{t}}_{\leq 0}) = m!$.

Proof. Part (i) follows after the definition. More precisely, $\text{Sign}_{\vec{H}}(\text{Hist}(P)) = \vec{t}$ if and only if for every $k \leq K$, (1) $t_k = +$ if and only if $\vec{h}_k \cdot \text{Hist}(P) > 0$, which is equivalent to $-\vec{h}_k \cdot \text{Hist}(P) \leq -1$ because $\vec{h}_k \in \mathbb{Z}^{m!}$; (2) likewise, $t_k = -$ if and only if $\vec{h}_k \cdot \text{Hist}(P) \leq -1$, and (3) if $t_k = 0$ if and only if $\vec{h}_k \cdot \text{Hist}(P) \leq 0$ and $-\vec{h}_k \cdot \text{Hist}(P) \leq 0$. This proves Part (i).

Part (ii) also follows after the definition. More precisely, $\vec{x} \in \mathcal{H}^{\vec{t}}_{\leq 0}$ if and only if for every $k \leq K$, (1) $t_k = +$ if and only if $-\vec{h}_k \cdot \vec{x} \leq 0$, which is equivalent to $[\text{Sign}_{\vec{H}}(\vec{x})]_k \in \{0, +\}$; (2) likewise, $t_k = -$ if and only if $\vec{h}_k \cdot \vec{x} \leq 0$, which is equivalent to $[\text{Sign}_{\vec{H}}(\vec{x})]_k \in \{0, -\}$, and (3) if $t_k = 0$ if and only if $\vec{h}_k \cdot \vec{x} \leq 0$ and $-\vec{h}_k \cdot \vec{x} \leq 0$, which is equivalent to $[\text{Sign}_{\vec{H}}(\vec{x})]_k = 0$. This is equivalent to $\vec{t} \trianglelefteq \text{Sign}_{\vec{H}}(\vec{x})$.

We now prove Part (iii). Suppose $\vec{t} \in \mathcal{S}^{\circ}_{\vec{H}}$. Let $\vec{x} \in \mathcal{H}^{\vec{t}} \cap \mathbb{R}^{m!}_{\geq 0}$ denote an arbitrary non-negative vector whose existence is guaranteed by the assumption that $\vec{t} \in \mathcal{S}^{\circ}_{\vec{H}}$. Therefore, for every $k \leq K$, either $\vec{h}_k \cdot \vec{x} \leq -1$ or $-\vec{h}_k \cdot \vec{x} \leq -1$, which means that there exists $\delta > 0$ such that any $\vec{x}'$ with $|\vec{x}' - \vec{x}|_\infty < \delta$, we have $\vec{h}_k \cdot \vec{x} < 0$ or $-\vec{h}_k \cdot \vec{x} < 0$. This means that $\vec{x}$ is an interior point of $\mathcal{H}^{\vec{t}}_{\leq 0}$ in $\mathbb{R}^{m!}$, which implies that $\dim(\mathcal{H}^{\vec{t}}_{\leq 0}) = m!$. $\square$

E Materials for Section 3: Smoothed CONDORCET CRITERION

E.1 Lemma 2 and Its Proof

For any GISR $\overline{r}$, we first define $\mathcal{R}^{\overline{r}}_{\text{CWW}}$ (respectively, $\mathcal{R}^{\overline{r}}_{\text{CWL}}$) that corresponds to fractional profiles where a Condorcet winner exists and is a co-winner (respectively, not a co-winner) under $\overline{r}$. CWW (respectively, CWL) stands for "Condorcet winner wins" (respectively, "Condorcet winner loses").

$$\mathcal{R}^{\overline{r}}_{\text{CWW}} = \{\vec{x} \in \mathbb{R}^{m!} : \text{CW}(\vec{x}) \cap \overline{r}(\vec{x}) \neq \emptyset\}$$
$$\mathcal{R}^{\overline{r}}_{\text{CWL}} = \{\vec{x} \in \mathbb{R}^{m!} : \text{CW}(\vec{x}) \cap (\mathcal{A} \setminus \overline{r}(\vec{x})) \neq \emptyset\}$$

For any set $\mathcal{R} \subseteq \mathbb{R}^{m!}$, let Closure$(\mathcal{R})$ denote the *closure* of $\mathcal{R}$ in $\mathbb{R}^{m!}$, that is, all points in $\mathcal{R}$ and their limiting points. Next, we introduce four conditions to present Lemma 2 below.

Definition 19. *Given a GISR $\overline{r}$ and $n \in \mathbb{N}$, we define the following conditions, where $\vec{x} \in \mathbb{R}^{m!}$.*

- **Always satisfaction: $\mathbf{C_{AS}(\overline{r}, n)}$** *holds if and only if for all $P \in \mathcal{L}(\mathcal{A})^n$,* $\text{CC}(\overline{r}, P) = 1$.
- **Robust satisfaction: $\mathbf{C_{RS}(\overline{r}, \vec{x})}$** *holds if and only if $\vec{x} \notin Closure(\mathcal{R}^{\overline{r}}_{CWL})$.*
- **Robust dissatisfaction: $\mathbf{C_{RD}(\overline{r}, \vec{x})}$** *holds if and only if $CW(\vec{x}) \cap (\mathcal{A} \setminus \overline{r}(\vec{x})) \neq \emptyset$.*
- **Non-Robust satisfaction: $\mathbf{C_{NRS}(\overline{r}, \vec{x})}$** *holds if and only if $ACW(\vec{x}) \neq \emptyset$ and $\vec{x} \notin Closure(\mathcal{R}^{\overline{r}}_{CWW})$.*

In words, $\text{C}_{\text{AS}}(\overline{r}, n)$ means that $\overline{r}$ always satisfies CC for n agents. Robust satisfaction $\text{C}_{\text{RS}}(\overline{r}, \vec{x})$ states that $\vec{x}$ is away from the dissatisfaction instances (i.e., $\mathcal{R}^{\overline{r}}_{\text{CWL}}$) by a constant margin. Robust dissatisfaction $\text{C}_{\text{RD}}(\overline{r}, \vec{x})$ states that the Condorcet winner exists under $\vec{x}$ and is not a co-winner under $\overline{r}$. Robust satisfaction and robust dissatisfaction are not "symmetric", because there are two sources of satisfaction: (1) no Condorcet winner exists and (2) the Condorcet winner exists and is also a winner, while there is only one source of dissatisfaction: the Condorcet winner exists but is not a winner.

The intuition behind Non-Robust satisfaction $\text{C}_{\text{NRS}}(\overline{r}, \vec{x})$ may not be immediately clear by definition. It is called "satisfaction", because $\text{ACW}(\vec{x}) \neq \emptyset$ implies that $\text{CW}(\vec{x}) = \emptyset$, which means that $\overline{r}$

satisfies CC at $\vec{x}$. The reason behind "non-robust" is that when a small perturbation $\vec{x}'$ is introduced, UMG$(\vec{x}+\vec{x}')$ often contains a Condorcet winner that is not a co-winner under $\vec{x}$, because $\vec{x}$ is constantly far away from $\mathcal{R}^{\overline{r}}_{\text{CWW}}$.

Example 11 (The four conditions in Definition 19). *Let $m = 3$ and $n = 14$. Table 3 illustrates four distributions, their UMG, the irresolute plurality winners, and their (dis)satisfaction of the four conditions introduced defined in Definition 19. π^1, π^2, and π' are the same as in Example 1 and 3. Notice that π' is a linear combination of π^1 and π^2.*

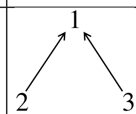
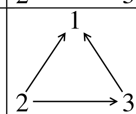
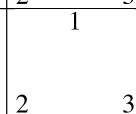
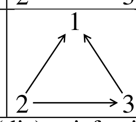
	123	132	231	321	213	312	UMG	$\overline{\text{Plu}}$ winner(s)	C_{AS}	C_{RS}	C_{RD}	C_{NRS}
π^1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$		$\{1\}$	N	N	N	Y
π^2	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$		$\{2\}$	N	Y	N	N
π_{uni}	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$		$\{1,2,3\}$	N	N	N	N
$\frac{3\pi^1+\pi^2}{4}$	$\frac{7}{32}$	$\frac{7}{32}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$		$\{1\}$	N	N	Y	N

Table 3: Distributions and their (dis)satisfaction of conditions in Definition 19.

Let P_{14} denote the 14-profile $\{6 \times [1 \succ 2 \succ 3], 4 \times [2 \succ 3 \succ 1], 4 \times [2 \succ 1 \succ 3]\}$. It is not hard to verify that alternative 2 is the Condorcet winner under P_{14} and $\overline{Plu}(P_{14}) = \{1\}$. Therefore, $C_{AS}(\overline{Plu}, 14) = N$.

- $\boldsymbol{\pi^1}$. *$C_{RS}(\overline{Plu}, \pi^1) = N$. To see this, let $\vec{x}'$ denote the vector that corresponds to the single-vote profile $\{2 \succ 3 \succ 1\}$. For any sufficiently small $\delta > 0$, $\pi^1 + \delta\vec{x}' \in \mathcal{R}^{\overline{Plu}}_{CWL}$, because 2 is the Condorcet winner and 1 is the unique plurality winner. $C_{RD}(\overline{Plu}, \pi^1) = N$ because $CW(\pi^1) = \emptyset$. $C_{NRS}(\overline{Plu}, \pi^1) = Y$ because $ACW(\pi^1) = \{2, 3\}$, and for any $\vec{x}' \in \mathbb{R}^6$ and any $\delta > 0$ that is sufficiently small, in $\pi^1 + \delta\vec{x}'$ we have that 2 or 3 is Condorcet winner and 1 is the unique plurality winner, which means that $\pi^1 + \delta\vec{x}' \notin \mathcal{R}^{\overline{r}}_{CWW}$.*

- $\boldsymbol{\pi^2}$. *$C_{RS}(\overline{Plu}, \pi^2) = Y$ because the plurality score of 2 is strictly higher than the plurality score of any other alternative, which means that for any $\vec{x}' \in \mathbb{R}^{m!}$, for any $\delta > 0$ that is sufficiently small, 2 is the Condorcet winner as well as the unique plurality winner in $\pi^2 + \delta\vec{x}'$. This means that π^2 is not in the closure of vectors where CC is violated. $C_{RD}(\overline{Plu}, \pi^2) = N$ because $CW(\pi^2) \cap (\mathcal{A} \setminus \overline{Plu}(\pi^2)) = \{2\} \cap \{1, 3\} = \emptyset$. $C_{NRS}(\overline{Plu}, \pi^2) = N$ because $ACW(\pi^2) = \emptyset$.*

- $\boldsymbol{\pi_{uni}}$. *$C_{RS}(\overline{Plu}, \pi_{uni}) = N$. To see this, let $\vec{x}'$ denote the vector that corresponds to the 14-profile P_{14} defined earlier in this example to prove $C_{AS}(\overline{Plu}, 14) = N$. For any $\delta > 0$ that is sufficiently small, we have $\pi_{uni} + \delta\vec{x}' \in \mathcal{R}^{\overline{Plu}}_{CWL}$, because 2 is the Condorcet winner and 1 is the unique plurality winner. $C_{RD}(\overline{Plu}, \pi_{uni}) = N$ because $CW(\pi_{uni}) = \emptyset$. $C_{NRS}(\overline{Plu}, \pi_{uni}) = N$ because $ACW(\pi_{uni}) = \emptyset$.*

- $\boldsymbol{\frac{3\pi^1+\pi^2}{4}}$. *Let $\pi' = \frac{3\pi^1+\pi^2}{4}$. $C_{RS}(\overline{Plu}, \pi') = N$ because $\pi' \in \mathcal{R}^{\overline{Plu}}_{CWL}$. $C_{RD}(\overline{Plu}, \pi') = Y$ because $CW(\pi') \cap (\mathcal{A} \setminus \overline{Plu}(\pi')) = \{2\} \cap \{2, 3\} \neq \emptyset$. $C_{NRS}(\overline{Plu}, \pi') = N$ because $ACW(\pi') = \emptyset$.*

For any condition Y, we use $\neg Y$ to indicate that Y does not hold. For example, $\neg\text{C}_{\text{AS}}(\overline{r}, n)$ means that $\text{C}_{\text{AS}}(\overline{r}, n)$ does not hold, i.e., there exists $P \in \mathcal{L}(\mathcal{A})^n$ with $\text{CC}(\overline{r}, P) = 0$. A GISR rule r_1

is a *refinement* of another voting rule r_2, if for all $\vec{x} \in \mathbb{R}^{m!}$, we have $r_1(\vec{x}) \subseteq r_2(\vec{x})$. We note that while the four conditions in Definition 19 are not mutually exclusive by definition, they provide a complete characterization of smoothed CC under any refinement of any minimally continuous int-GISR as shown in the lemma below.

Lemma 2 (Smoothed CC: Minimally Continuous Int-GISRs). *For any fixed $m \geq 3$, let $\mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi)$ be a strictly positive and closed single-agent preference model, let $\overline{r}$ be a minimally continuous int-GISR and let r be a refinement of $\overline{r}$. For any $n \in \mathbb{N}$ with $2 \mid n$, we have*

$$\widetilde{\mathrm{CC}}_{\Pi}^{\min}(r,n) = \begin{cases} 1 & \text{if } C_{AS}(\overline{r},n) \\ 1-\exp(-\Theta(n)) & \text{if } \neg C_{AS}(\overline{r},n) \text{ and } \forall \pi \in CH(\Pi), C_{RS}(\overline{r},\pi) \\ \Theta(n^{-0.5}) & \text{if } \begin{cases} \text{(1) } \forall \pi \in CH(\Pi), \neg C_{RD}(\overline{r},\pi) \text{ and} \\ \text{(2) } \exists \pi \in CH(\Pi) \text{ s.t. } C_{NRS}(\overline{r},\pi) \end{cases} \\ \exp(-\Theta(n)) & \text{if } \exists \pi \in CH(\Pi) \text{ s.t. } C_{RD}(\overline{r},\pi) \\ \Theta(1) \wedge (1-\Theta(1)) & \text{otherwise} \end{cases}$$

For any $n \in \mathbb{N}$ with $2 \nmid n$, we have

$$\widetilde{\mathrm{CC}}_{\Pi}^{\min}(r,n) = \begin{cases} 1 & \text{same as the } 2 \mid n \text{ case} \\ 1-\exp(-\Theta(n)) & \text{same as the } 2 \mid n \text{ case} \\ \exp(-\Theta(n)) & \text{if } \exists \pi \in CH(\Pi) \text{ s.t. } C_{RD}(\overline{r},\pi) \text{ or } C_{NRS}(\overline{r},\pi) \\ \Theta(1) \wedge (1-\Theta(1)) & \text{otherwise} \end{cases}$$

Lemma 2 can be applied to a wide range of resolute voting rules because it works for any refinement r (i.e., using any tie-breaking mechanism) of any minimally continuous GISR (which include all voting rules discussed in this paper). Notice that r is not required to be a GISR, the L case and the 0 case never happen, and the conditions of all cases depend on $\overline{r}$ but not r.

Example 12 (Applications of Lemma 2 to plurality). *Continuing the setting of Example 11, we let Plu denote any refinement of $\overline{Plu}$. We first apply the $2 \mid n$ part of Lemma 2 to the following four cases of Π for sufficiently large n using Table 3. The first three cases correspond to i.i.d. distributions, i.e., $|\Pi| = 1$. In particular, $\Pi = \{\pi_{uni}\}$ corresponds to IC.*

- $\Pi = \{\pi^1, \pi^2\}$. *We have $\widetilde{\mathrm{CC}}_{\Pi}^{\min}(Plu,n) = \exp(-\Theta(n))$, that is, the VU case holds. This is because let $\pi' = \frac{3\pi^1+\pi^2}{4}$, we have $\pi' \in CH(\Pi)$ and $C_{RS}(\overline{Plu},\pi') = N$ according to Table 3.*
- $\Pi_1 = \{\pi^1\}$. *We have $\widetilde{\mathrm{CC}}_{\Pi_1}^{\min}(Plu,n) = \Theta(n^{-0.5})$, that is, the U case holds.*
- $\Pi_2 = \{\pi^2\}$. *We have $\widetilde{\mathrm{CC}}_{\Pi_2}^{\min}(Plu,n) = 1-\exp(-\Theta(n))$, that is, the VL case holds.*
- $\Pi_{IC} = \{\pi_{uni}\}$. *We have $\widetilde{\mathrm{CC}}_{\Pi_{IC}}^{\min}(Plu,n) = \Theta(1) \wedge (1-\Theta(1))$, that is, the M case holds.*

When $2 \nmid n$ and $\Pi_1 = \{\pi^1\}$, we have $\widetilde{\mathrm{CC}}_{\Pi_1}^{\min}(Plu,n) = \exp(-\Theta(n))$, that is, the VU case holds.

Intuitive explanations. The conditions in Lemma 2 can be explained as follows. Take the $2 \mid n$ case for example. In light of various multivariate central limit theorems, the histogram of the randomly-generated profile when the adversary chooses $\vec{\pi} = (\pi_1, \ldots, \pi_n)$ is concentrated in a $\Theta(n^{-0.5})$ neighborhood of $\sum_{j=1}^n \pi_j$, denoted by $B_{\vec{\pi}}$. Let $\text{avg}(\vec{\pi}) = \frac{1}{n}\sum_{j=1}^n \pi_j$, which means that $\text{avg}(\vec{\pi}) \in \text{CH}(\Pi)$. The condition for the 1 case is straightforward. Suppose the 1 case does not happen, then the VL case happens if all distributions in $\text{CH}(\Pi)$, which includes $\text{avg}(\vec{\pi})$, are far from instances of dissatisfaction, so that no instance of dissatisfaction is in $B_{\vec{\pi}}$. Suppose the VL case does not happen. The U case happens if the min-adversary can find a non-robust satisfaction instance ($\text{C}_{\text{NRS}}(\overline{r},\pi)$) but cannot find a robust dissatisfaction instance ($\neg\text{C}_{\text{RD}}(\overline{r},\pi)$). And if the min-adversary can find a robust dissatisfaction instance ($\text{C}_{\text{RD}}(\overline{r},\pi)$), then $B_{\vec{\pi}}$ does not contain any instance of satisfaction, which means that the VU case happens. All remaining cases are M cases.

Odd vs. even n. The $2 \nmid n$ case also admits a similar explanation. The main difference is that when $2 \nmid n$, the UMG of any n-profile must be a complete graph, i.e., no alternatives are tied in the UMG. Therefore, when $\text{C}_{\text{NRS}}(\overline{r},\pi)$ is satisfied, a Condorcet winner (who is one of the two ACWs in π) must exist and constitutes an instance of robust dissatisfaction when $2 \nmid n$. On the other hand, it is possible that the two ACWs in π are tied in an n-profile when $2 \mid n$, which constitutes a case where CC is satisfied because the Condorcet winner does not exist. This happens with probability $\Theta(n^{-0.5})$. This difference leads to the $\Theta(n^{-0.5})$ case when $2 \mid n$, and it becomes part of the $\exp(-\Theta(n))$ case when $2 \nmid n$.

Proof sketch. Before presenting the formal proof in the following subsection, we present a proof sketch here.

We first prove the special case $r = \overline{r}$, which is done by applying Lemma 1 in the following three steps. **Step 1.** Define $\mathcal{C}$ that characterizes the satisfaction of CC under $\overline{r}$, and an almost complement $\mathcal{C}^*$ of $\mathcal{C}$. In fact, we will let $\mathcal{C} = \mathcal{C}_{\text{NCW}} \cup \mathcal{C}_{\text{CWW}}$ as in Section 4 and Section C.1, and prove that one choice of $\mathcal{C}^*$ is the union of polyhedra that represent profiles where the Condorcet winner exists but is not an $\overline{r}$ co-winner. **Step 2.** Characterize α_n^* and β_n, which is technically the most involved part due to the generality of the theorem. **Step 3.** Formally apply Lemma 1.

Then, let r denote an arbitrary refinement of $\overline{r}$. We define a slightly different version of CC, denoted by CC^*, whose satisfaction under $\overline{r}$ will be used as a lower bound on the satisfaction of CC under r. For any GISR $\overline{r}$ and any profile P, we define

$$\text{CC}^*(\overline{r}, P) = \begin{cases} 1 & \text{if } \text{CW}(P) = \emptyset \text{ or } \text{CW}(P) = \overline{r}(P) \\ 0 & \text{otherwise} \end{cases}$$

Compared to CC, CC^* rules out profiles P where a Condorcet winner exists and is not the unique winner under $\overline{r}$. Therefore, for any $\vec{\pi} \in \Pi^n$, we have

$$\Pr_{P \sim \vec{\pi}}(\text{CC}^*(\overline{r}, P) = 1) \leq \Pr_{P \sim \vec{\pi}}(\text{CC}(r, P) = 1) \leq \Pr_{P \sim \vec{\pi}}(\text{CC}(\overline{r}, P) = 1)$$

Then, we prove that smoothed CC^*, i.e., $\widetilde{\text{CC}^*}^{\min}_{\Pi}(\overline{r}, n)$, asymptotically matches $\widetilde{\text{CC}}^{\min}_{\Pi}(\overline{r}, n)$, which concludes the proof of Lemma 2.

E.1.1 Proof of Lemma 2

Proof. The 1 cases of the theorem is trivial. **In the rest of the proof, we assume that the 1 case does not hold.** That is, there exists an n-profile P such that $\text{CW}(P)$ exists but is not in $\overline{r}(P)$. We will prove that the theorem holds for any $n > N_{\overline{r}}$, where $N_{\overline{r}} \in \mathbb{N}$ is a constant that only depends on $\overline{r}$ that will be defined later (in Definition 24). This is without loss of generality, because when n is bounded above by a constant, the 1 case belongs to the U case (i.e., $\Theta(n^{-0.5})$) and the VU case (i.e., $\exp(-\Theta(n))$).

Let $\overline{r}$ be defined by $\vec{H}$ and g. We first prove the theorem for the special case where $r = \overline{r}$, and then show how to modify the proof for general r. For any irresolute voting rule $\overline{r}$, we recall that $\text{CC}(\overline{r}, P) = 1$ if and only if either P does not have a Condorcet winner, or the Condorcet winner is a co-winner under $\overline{r}$.

Proof for the special case $r = \overline{r}$. Recall that in this case $\overline{r}$ is a minimally continuous GISR. In light of Lemma 1, the proof proceeds in the following three steps. **Step 1.** Define $\mathcal{C}$ that characterizes the satisfaction of CONDORCET CRITERION of $\overline{r}$ and an almost complement $\mathcal{C}^*$ of $\mathcal{C}$. **Step 2.** Characterize $\Pi_{\mathcal{C},n}$, $\Pi_{\mathcal{C}^*,n}$, β_n, and α_n^*. **Step 3.** Apply Lemma 1.

Step 1: Define $\mathcal{C}$ and $\mathcal{C}^*$. The definition is similar to the ones presented in Section 4 for plurality. We will define $\mathcal{C} = \mathcal{C}_{\text{NCW}} \cup \mathcal{C}_{\text{CWW}}$, where $\mathcal{C}_{\text{NCW}}$ represents the histograms of profiles that do not have a Condorcet winner, and $\mathcal{C}_{\text{CWW}}$ represents histograms of profiles where a Condorcet winner exists and is a co-winner under $\overline{r}$. $\mathcal{C}_{\text{NCW}}$ is similar to the set defined in [51, Proposition 5 in the Appendix]. For completeness we recall its definition using the notation of this paper.

Recall that $\text{Pair}_{a,b}$ is the pairwise difference vector defined in Definition 10. It follows that for any profile P and any pair of alternatives a, b, $\text{Pair}_{a,b} \cdot \text{Hist}(P) > 0$ if and only if there is an edge $a \to b$ in $\text{UMG}(P)$; $\text{Pair}_{a,b} \cdot \text{Hist}(P) = 0$ if and only if a and b are tied in $\text{UMG}(P)$. Then, we use $\text{Pair}_{a,b}$'s to define polyhedra that characterize histograms of profiles whose UMGs equal to a given graph G.

Definition 20 (**$\mathcal{H}^G$**)**.** *Given an unweighted directed graph G over $\mathcal{A}$, let $\mathbf{A}^G = \begin{bmatrix} \mathbf{A}^G_{edge} \\ \mathbf{A}^G_{tie} \end{bmatrix}$, where $\mathbf{A}^G_{edge}$ consists of rows $Pair_{b,a}$ for all edges $a \to b \in G$, and $\mathbf{A}^G_{edge}$ consists of two rows $Pair_{b,a}$ and $Pair_{a,b}$ for each tie $\{a, b\}$ in G. Let $\vec{\mathbf{b}}^G = [\underbrace{-\vec{1}}_{\text{for } \mathbf{A}^G_{edge}}, \underbrace{\vec{0}}_{\text{for } \mathbf{A}^G_{tie}}]$ and*

$$\mathcal{H}^G = \left\{\vec{x} \in \mathbb{R}^{m!} : \mathbf{A}^G \cdot (\vec{x})^\top \leq \left(\vec{\mathbf{b}}^G\right)^\top\right\}$$

Next, we define polyhedra indexed by an alternative a and a feasible signature $\vec{t} \in \mathcal{S}_{\vec{H}}$ that characterize the histograms of profiles P where a is the Condorcet winner and $\text{Sign}_{\vec{H}}(P) = \vec{t}$.

Definition 21 (**$\mathcal{H}^{a,\vec{t}}$**)**.** *Given $\vec{H} = (\vec{h}_1, \ldots, \vec{h}_K) \in (\mathbb{R}^d)^K$, $a \in \mathcal{A}$, and $\vec{t} \in \mathcal{S}_{\vec{H}}$, we let $\mathbf{A}^{a,\vec{t}} = \begin{bmatrix} \mathbf{A}^{CW=a} \\ \mathbf{A}^{\vec{t}} \end{bmatrix}$, where $\mathbf{A}^{CW=a}$ consists of pairwise difference vectors $Pair_{b,a}$ for each alternative $b \neq a$, and $\mathbf{A}^{\vec{t}}$ is the matrix used to define $\mathcal{H}^{\vec{t}}$ in Definition 17. Let $\vec{\mathbf{b}}^{a,\vec{t}} = [\underbrace{-\vec{1}}_{\text{for } \mathbf{A}^{CW=a}}, \underbrace{\vec{\mathbf{b}}^{\vec{t}}}_{\text{for } \mathbf{A}^{\vec{t}}}]$ and*

$$\mathcal{H}^{a,\vec{t}} = \{\vec{x} \in \mathbb{R}^{m!} : \mathbf{A}^{a,\vec{t}} \cdot (\vec{x})^\top \leq \left(\vec{\mathbf{b}}^{a,\vec{t}}\right)^\top\}$$

Next, we use $\mathcal{H}^G$ and $\mathcal{H}^{a,\vec{t}}$ as building blocks to define $\mathcal{C} = \mathcal{C}_{\text{NCW}} \cup \mathcal{C}_{\text{CWW}}$ and an almost complement of $\mathcal{C}$, denoted by $\mathcal{C}_{\text{CWL}}$. At a high level, $\mathcal{C}_{\text{NCW}}$ corresponds to the profiles where no Condorcet winner exists (NCW represents "no Condorcet winner"), $\mathcal{C}_{\text{CWW}}$ corresponds to profiles where the Condorcet winner exists and is also an $\overline{r}$ co-winner (CWW represents "Condorcet winner wins"), and $\mathcal{C}_{\text{CWL}}$ corresponds to profiles where the Condorcet winner exists and is not an $\overline{r}$ co-winner (CWL represents "Condorcet winner loses").

Definition 22 (**$\mathcal{C}$ and $\mathcal{C}_{\text{CWL}}$**)**.** *Given an int-GISR characterized by $\vec{H}$ and g, we define*

$$\mathcal{C} = \mathcal{C}_{NCW} \cup \mathcal{C}_{CWW}, \quad \text{where } \mathcal{C}_{NCW} = \bigcup\nolimits_{G: CW(G)=\emptyset} \mathcal{H}^G \text{ and } \mathcal{C}_{CWW} = \bigcup\nolimits_{a \in \mathcal{A}, \vec{t} \in \mathcal{S}_{\vec{H}}: a \in \overline{r}(\vec{t})} \mathcal{H}^{a,\vec{t}}$$
$$\mathcal{C}_{CWL} = \bigcup\nolimits_{a \in \mathcal{A}, \vec{t} \in \mathcal{S}_{\vec{H}}: a \notin \overline{r}(\vec{t})} \mathcal{H}^{a,\vec{t}}$$

We note that some $\mathcal{H}^{a,\vec{t}}$ can be empty. To see that $\mathcal{C}_{\text{CWL}}$ is indeed an almost complement of $\mathcal{C} = \mathcal{C}_{\text{NCW}} \cup \mathcal{C}_{\text{CWW}}$, we note that $\mathcal{C} \cap \mathcal{C}_{\text{CWL}} = \emptyset$, and for any integer vector $\vec{x}$,

- if $\vec{x}$ does not have a Condorcet winner then $\vec{x} \in \mathcal{C}_{\text{NCW}} \subseteq \mathcal{C}$;
- if $\vec{x}$ has a Condorcet winner a, which is also an $\overline{r}$ co-winner, then $\vec{x} \in \mathcal{H}^{a,\text{Sign}_{\vec{H}}(\vec{x})} \subseteq \mathcal{C}_{\text{CWW}} \subseteq \mathcal{C}$;
- otherwise $\vec{x}$ has a Condorcet winner a, which is not an $\overline{r}$ co-winner. Then $\vec{x} \in \mathcal{H}^{a,\text{Sign}_{\vec{H}}(\vec{x})} \subseteq \mathcal{C}_{\text{CWL}}$.

Therefore, $\mathbb{Z}^q \subseteq \mathcal{C} \cup \mathcal{C}_{\text{CWL}}$.

Step 2: Characterize $\Pi_{\mathcal{C},n}$, $\Pi_{\mathcal{C}_{\text{CWL}},n}$, β_n, and α_n^*. Recall that β_n and α_n^* are defined by $\dim^{\max}_{\mathcal{C},n}(\pi)$ and $\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi)$ for $\pi \in \text{CH}(\Pi)$ as follows:

$$\beta_n = \min\nolimits_{\pi \in \text{CH}(\Pi)} \dim^{\max}_{\mathcal{C},n}(\pi) = \min\nolimits_{\pi \in \text{CH}(\Pi)} \max\left(\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi), \dim^{\max}_{\mathcal{C}_{\text{CWW}},n}(\pi)\right)$$
$$\alpha_n^* = \max\nolimits_{\pi \in \text{CH}(\Pi)} \dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi)$$

For convenience, we let $\Pi_{\mathcal{C},n}$ denote the distributions in $\text{CH}(\Pi)$, each of which is connected to an edge with positive weight in the activation graph (Definition 6). Formally, we have the following definition.

Definition 23 ($\Pi_{\mathcal{C},n}$)**.** *Given a set of distributions Π over q, $\mathcal{C} = \bigcup_{i \le I} \mathcal{H}_i$, and $n \in \mathbb{N}$, let*

$$\Pi_{\mathcal{C},n} = \{\pi \in \mathit{CH}(\Pi) : \exists i \le I \text{ s.t. } \mathcal{H}^{\mathbb{Z}}_{i,n} \neq \emptyset \text{ and } \pi \in \mathcal{H}_{i,\le 0}\}$$

Table 4 gives an overview of the rest of the proof in Step 2, which characterizes $\dim^{\max}_{\mathcal{C},n}(\pi)$ and $\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi)$ by the membership of $\pi \in \text{CH}(\Pi)$ in $\Pi_{\mathcal{C}_{\text{NCW}},n}, \Pi_{\mathcal{C}_{\text{CWW}},n}$, and $\Pi_{\mathcal{C}_{\text{CWL}},n}$, respectively, where $n \ge N_{\overline{r}}$ for a constant $N_{\overline{r}}$ that will be defined momentarily (in Definition 24).

$\pi \in \Pi_{\mathcal{C}_{\text{NCW}},n}$	$*$	$*$	N	Y	Y	N
$\pi \in \Pi_{\mathcal{C}_{\text{CWW}},n}$	Y	Y	N	N	N	N
$\pi \in \Pi_{\mathcal{C}_{\text{CWL}},n}$	Y	N	Y	Y	N	N
$\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi)$ (Claim 3)	$*$	$*$	$-\frac{n}{\log n}$	$m!$ or $m!-1$	$m!$	N/A
$\dim^{\max}_{\mathcal{C}_{\text{CWW}},n}(\pi)$ (Claim 6)	$m!$	$m!$	$\le -\frac{n}{\log n}$	<0	<0	
$\dim^{\max}_{\mathcal{C},n}(\pi) = \max\left(\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi), \dim^{\max}_{\mathcal{C}_{\text{CWW}},n}(\pi)\right)$	$m!$	$m!$	$-\frac{n}{\log n}$	$\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi)$	$m!$	
$\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi)$ (Claim 6)	$m!$	$-\frac{n}{\log n}$	$m!$	$m!$	$-\frac{n}{\log n}$	

Table 4: $\dim^{\max}_{\mathcal{C},n}(\pi)$ and $\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi)$ for CC for $\pi \in \text{CH}(\Pi)$ and sufficiently large n.

We will first specify $N_{\overline{r}}$ in Step 2.1. Then in Step 2.2, we will characterize $\Pi_{\mathcal{C}_{\text{NCW}},n}$ and $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi)$ in Claim 3, and characterize $\Pi_{\mathcal{C}_{\text{CWW}},n}$, $\dim^{\max}_{\mathcal{C}_{\text{CWW}},n}(\pi)$, $\Pi_{\mathcal{C}_{\text{CWL}},n}$, and $\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi)$ in Claim 6. Finally, in Step 2.3 we will verify $\dim^{\max}_{\mathcal{C},n}(\pi)$ and $\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi)$ in Table 4.

Step 2.1. Specify $N_{\overline{r}}$. We first prove the following claim, which provides a sufficient condition for a polyhedron to be active for sufficiently large N.

Claim 2. *For any polyhedron $\mathcal{H}$ characterized by integer matrix $\mathbf{A}$ and $\vec{\mathbf{b}} \le \vec{0}$, if $\dim(\mathcal{H}_{\le 0}) = m!$ and $\mathcal{H} \cap \mathbb{R}^{m!}_{>0} \neq \emptyset$, then there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $\mathcal{H}$ is active at n.*

Proof. By Minkowski-Weyl theorem (see e.g., [43, p. 100]), $\mathcal{H} = \mathcal{V} + \mathcal{H}_{\le 0}$, where $\mathcal{V}$ is a finitely generated polyhedron. Therefore, any affine space containing $\mathcal{H}$ can be shifted to contain $\mathcal{H}_{\le 0}$, which means that $\dim(\mathcal{H}) \ge \dim(\mathcal{H}_{\le 0}) = m!$. Because $\mathcal{H} \cap \mathbb{R}^{m!}_{>0} \neq \emptyset$, it contains an interior point (inner point with an full dimensional neighborhood), denoted by $\vec{x}$, whose δ neighborhood (for some $0 < \delta < 1$) in L_∞ is contained in $\mathcal{H} \cap \mathbb{R}^{m!}_{>0}$. Let B denote the δ neighborhood of $\vec{x}$. Let $N = \frac{m!|\vec{x}|_1}{\delta}$. Then, because $\vec{\mathbf{b}} \le \vec{0}$ and $\frac{N}{|\vec{x}|_1} \ge 1$, for every $n > N$ and every $\vec{x}' \in B$ we have

$$\mathbf{A} \cdot \left(\frac{n}{|\vec{x}|_1}\vec{x}'\right)^\top < \frac{n}{|\vec{x}|_1}\left(\vec{\mathbf{b}}\right)^\top \le \left(\vec{\mathbf{b}}\right)^\top$$

This means that $\frac{n}{|\vec{x}|_1}B \subseteq \mathcal{H} \cap \mathbb{R}^{m!}_{>0}$. Moreover, it is not hard to verify that $\frac{n}{|\vec{x}|_1}B$ contains the following non-negative integer n vector

$$\left(\left\lfloor \frac{n}{|\vec{x}|_1}x_1 \right\rfloor, \ldots, \left\lfloor \frac{n}{|\vec{x}|_1}x_{m!-1} \right\rfloor, n - \sum_{i=1}^{m!-1}\left\lfloor \frac{n}{|\vec{x}|_1}x_i \right\rfloor\right)$$

This proves Claim 2. □

We now define the constant $N_{\overline{r}}$ used throughout the proof.

Definition 24 ($N_{\overline{r}}$)**.** *Let $N_{\overline{r}}$ denote a number that is larger than m^4 and the maximum N obtain from applying Claim 2 to all polyhedra $\mathcal{H}$ in $\mathcal{C}_{NCW}$, $\mathcal{C}_{CWW}$, or $\mathcal{C}_{CWL}$ where $\dim(\mathcal{H}_{\le 0}) = m!$ and $\mathcal{H} \cap \mathbb{R}^{m!}_{>0} \neq \emptyset$.*

Step 2.2. Characterize $\Pi_{\mathcal{C}_{\text{NCW}},n}$, $\Pi_{\mathcal{C}_{\text{CWW}},n}$, and $\Pi_{\mathcal{C}_{\text{CWL}},n}$.

Claim 3 (**Characterizations of $\Pi_{\mathcal{C}_{\text{NCW}},n}$ and $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi)$**). *For any $n \geq m^4$ such that $\neg C_{AS}(\overline{r}, n)$ and any distribution π over $\mathcal{A}$, we have*

- *if $2 \mid n$, then $\pi \in \Pi_{\mathcal{C}_{NCW},n}$ if and only if $CW(\pi) = \emptyset$, and*

$$\dim^{\max}_{\mathcal{C}_{NCW},n}(\pi) = \begin{cases} -\frac{n}{\log n} & \textit{if } CW(\pi) \neq \emptyset \\ m! - 1 & \textit{if } ACW(\pi) \neq \emptyset \\ m! & \textit{otherwise (i.e. } CW(\pi) \cup ACW(\pi) = \emptyset\textit{)} \end{cases}$$

- *if $2 \nmid n$, then $\pi \in \Pi_{\mathcal{C}_{NCW},n}$ if and only if $CW(\pi) \cup ACW(\pi) = \emptyset$, and*

$$\dim^{\max}_{\mathcal{C}_{NCW},n}(\pi) = \begin{cases} -\frac{n}{\log n} & \textit{if } CW(\pi) \cup ACW(\pi) \neq \emptyset \\ m! & \textit{otherwise (i.e. } CW(\pi) \cup ACW(\pi) = \emptyset\textit{)} \end{cases}$$

Proof. In the proof we assume that $n \geq m^4$. We first recall the following characterization of $\mathcal{H}^G$, where part (i)-(iii) are due to [51, Claim 3 in the Appendix] and part (iv) follows after [51, Claim 6 in the Appendix].

Claim 4 (**Properties of $\mathcal{H}^G$ [51]**). *For any UMG G,*

(i) for any integral profile P, $Hist(P) \in \mathcal{H}^G$ if and only if $G = UMG(P)$;

(ii) for any $\vec{x} \in \mathbb{R}^{m!}$, $\vec{x} \in \mathcal{H}^G_{\leq 0}$ if and only if $UMG(\vec{x})$ is a subgraph of G.

(iii) $\dim(\mathcal{H}^G_{\leq 0}) = m! - Ties(G)$.

(iv) For any $n \geq m^4$, $\mathcal{H}^G$ is active at n if (1) n is even, or (2) n is odd and G is a complete graph.

The $2 \mid n$ case. By Claim 4 (iv), when $n \geq m^4$ and $2 \mid n$, every $\mathcal{H}^G$ is active. This means that $\pi \in \Pi_{\mathcal{C}_{\text{NCW}},n}$ if and only if $\pi \in \mathcal{H}^G_{\leq 0}$ for some graph G that does not have a Condorcet winner. According to Claim 4 (ii), this holds if and only if there exists a supergraph of UMG(π) (which can be UMG(π) itself) that not have a Condorcet winner, which is equivalent to UMG(π) does not have a Condorcet winner, i.e. CW$(\pi) = \emptyset$. It follows that $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi) = -\frac{n}{\log n}$ if and only if CW$(\pi) \neq \emptyset$.

To characterize the $m! - 1$ case and the $m!$ case for $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi)$, we first prove the following claim to characterize graphs whose complete supergraphs all have Condorcet winners.

Claim 5. *For any unweighted directed graph G over $\mathcal{A}$, the following conditions are equivalent. (1) Every complete supergraph of G has a Condorcet winner. (2) $CW(G) \cup ACW(G) \neq \emptyset$.*

Proof. We first prove (1)⇒(2) in the following three cases.

- **Case 1: $|\text{WCW}(G)| = 1$.** In this case we must have CW$(G) =$ WCW(G), otherwise there exists an alternative b that is different from the weak Condorcet winner, denoted by a, such that a and b are tied in G. Notice that b is not a weak Condorcet winner. Therefore, we can complete G by adding $b \to a$ and breaking other ties arbitrarily, and it is not hard to see that the resulting graph does not have a Condorcet winner, which is a contradiction.
- **Case 2: $|\text{WCW}(G)| = 2$.** Let WCW$(G) = \{a, b\}$. We note that a and b are not tied with any other alternative. Otherwise for the sake of contradiction suppose a is tied with $c \neq b$. Then, we can extend G to a complete graph by assigning $c \to a$ and $a \to b$. The resulting complete graph does not have a Condorcet winner, which is a contradiction. This means that a and b are the almost Condorcet winners, and hence (2) holds.
- **Case 3: $|\text{WCW}(G)| \geq 3$.** In this case, we can assign directions of edges between WCW(G) to form a cycle, and then assign arbitrary direction to other missing edges in G to form a complete graph, which does not have a Condorcet winner and is thus a contradiction.

(2)⇒(1) is straightforward. If $\mathrm{CW}(G) \neq \emptyset$, then any supergraph of G has the same Condorcet winner. If $\mathrm{ACW}(G) = \{a, b\} \neq \emptyset$, then any complete supergraph of G either has a as the Condorcet winner or has b as the Condorcet winner. This proves Claim 5. □

The $\dim^{\max}_{\mathcal{C}_{\mathrm{NCW}},n}(\pi) = m! - 1$ case when $2 \mid n$. Suppose $\mathrm{ACW}(\pi) = \{a, b\}$. Let G^* denote a supergraph of $\mathrm{UMG}(\pi)$ where ties in $\mathrm{UMG}(\pi)$ except $\{a, b\}$ are broken arbitrarily. By Claim 4 (ii), $\pi \in \mathcal{H}^{G^*}_{\leq 0}$ and by Claim 4 (iii), $\mathcal{H}^{G^*}_{\leq 0} = m! - 1$. Recall from Claim 4 (iv) that $\mathcal{H}^{G^*}$ is active at n because we assumed that $n > m^4$. Therefore, $\dim^{\max}_{\mathcal{C}_{\mathrm{NCW}},n}(\pi) \geq m! - 1$. To see that $\dim^{\max}_{\mathcal{C}_{\mathrm{NCW}},n}(\pi) \leq m! - 1$, we note that for every graph G that does not have a Condorcet winner such that $\pi \in \mathcal{H}^{G}_{\leq 0}$. By Claim 4 (ii), G is a supergraph of $\mathrm{UMG}(\pi)$. This means that G is not a complete graph, because by Claim 5, any complete supergraph of $\mathrm{UMG}(\pi)$ must have a Condorcet winner. It follows that $\mathrm{Ties}(G) \geq 1$ and by Claim 4 (iii), $\mathcal{H}^{G}_{\leq 0} \leq m! - 1$. Therefore, $\dim^{\max}_{\mathcal{C}_{\mathrm{NCW}},n}(\pi) = m! - 1$.

The $\dim^{\max}_{\mathcal{C}_{\mathrm{NCW}},n}(\pi) = m!$ case when $2 \mid n$. Suppose $\mathrm{CW}(\pi) \cup \mathrm{ACW}(\pi) = \emptyset$. By Claim 5 there exists a complete supergraph G of $\mathrm{UMG}(\pi)$ that does not have a Condorcet winner, which means that $\mathcal{H}^G \subseteq \mathcal{C}_{\mathrm{NCW}} \subseteq \mathcal{C}$. We have $\pi \in \mathcal{H}^{G}_{\leq 0}$ (Claim 4 (ii)), $\dim(\mathcal{H}^{G}_{\leq 0}) = m!$ (Claim 4 (iii)), and $\mathcal{H}^G$ is active at n (Claim 4 (iv)). Therefore, $\dim^{\max}_{\mathcal{C}_{\mathrm{NCW}},n}(\pi) = m!$.

The $2 \nmid n$ case. By Claim 4 (iv), when $n \geq m^4$ and $2 \nmid n$, $\mathcal{H}^G$ is active if and only if G is a complete graph. It follows from Claim 4 (ii) that $\pi \in \Pi_{\mathcal{C}_{\mathrm{NCW}},n}$ if and only if $\pi \in \mathcal{H}^{G}_{\leq 0}$, where G is complete supergraph of $\mathrm{UMG}(\pi)$ that does not have a Condorcet winner. By Claim 4 (iii), $\dim(\mathcal{H}^{G}_{\leq 0}) = m!$. Therefore, by Claim 5, $\pi \in \Pi_{\mathcal{C}_{\mathrm{NCW}},n}$ if and only if $\mathrm{CW}(\pi) \cup \mathrm{ACW}(\pi) = \emptyset$. Moreover, whenever $\pi \in \Pi_{\mathcal{C}_{\mathrm{NCW}},n}$ we have $\dim^{\max}_{\mathcal{C}_{\mathrm{NCW}},n}(\pi) = m!$.

This proves Claim 3. □

Recall that we have assumed the 1 case of the theorem does not hold, that is, $\neg \mathrm{C}_{\mathrm{AS}}(\vec{r}, n)$. The following claim characterizes $\Pi_{\mathcal{C}_{\mathrm{CWW}},n}$, $\dim^{\max}_{\mathcal{C}_{\mathrm{CWW}},n}(\pi)$, $\Pi_{\mathcal{C}_{\mathrm{CWL}},n}$, and $\dim^{\max}_{\mathcal{C}_{\mathrm{CWL}},n}(\pi)$, when $\neg \mathrm{C}_{\mathrm{AS}}(\vec{r}, n)$.

Claim 6 (Characterizations of $\Pi_{\mathcal{C}_{\mathrm{CWW}},n}$, $\dim^{\max}_{\mathcal{C}_{\mathrm{CWW}},n}(\pi)$, $\Pi_{\mathcal{C}_{\mathrm{CWL}},n}$, and $\dim^{\max}_{\mathcal{C}_{\mathrm{CWL}},n}(\pi)$). *Given any strictly positive Π and any minimally continuous int-GISR $\vec{r}$, for any $n \geq N_{\vec{r}}$ (see Definition 24) such that $\neg C_{AS}(\vec{r}, n)$ and any $\pi \in CH(\Pi)$,*

$$[\pi \in \Pi_{\mathcal{C}_{CWW},n}] \Leftrightarrow \left[\pi \in \mathit{Closure}(\mathcal{R}^{\vec{r}}_{CWW})\right] \Leftrightarrow \left[\dim^{\max}_{\mathcal{C}_{CWW},n}(\pi) = m!\right], \text{ and}$$
$$[\pi \in \Pi_{\mathcal{C}_{CWL},n}] \Leftrightarrow \left[\pi \in \mathit{Closure}(\mathcal{R}^{\vec{r}}_{CWL})\right] \Leftrightarrow \left[\dim^{\max}_{\mathcal{C}_{CWL},n}(\pi) = m!\right]$$

Proof. We first prove properties of $\mathcal{H}^{a,\vec{t}}$ in the following claim, which has three parts. Part (i) states that $\mathcal{H}^{a,\vec{t}}$ characterizes histograms of the profiles whose signature is $\vec{t}$ and where alternative a is the Condorcet winner. Part (ii) characterizes the characteristic cone of $\mathcal{H}^{a,\vec{t}}$. Part (iii) characterizes the dimension of the characteristic cone for some cases.

Claim 7 (Properties of $\mathcal{H}^{a,\vec{t}}$). *Given $\vec{H}$, for any $a \in \mathcal{A}$ and any $\vec{t} \in \mathcal{S}_{\vec{H}}$,*

(i) for any integral profile P, $Hist(P) \in \mathcal{H}^{a,\vec{t}}$ if and only if a is the Condorcet winner under P and $Sign_{\vec{H}}(P) = \vec{t}$;

(ii) for any $\vec{x} \in \mathbb{R}^{m!}$, $\vec{x} \in \mathcal{H}^{a,\vec{t}}_{\leq 0}$ if and only if a is a weak Condorcet winner under $\vec{x}$ and $\vec{t} \trianglelefteq Sign_{\vec{H}}(\vec{x})$;

(iii) if $\vec{t} \in \mathcal{S}^{\circ}_{\vec{H}}$ and $\mathcal{H}^{a,\vec{t}} \neq \emptyset$, then $\dim(\mathcal{H}^{a,\vec{t}}_{\leq 0}) = m!$.

Proof. Part (i) follows after the definition. More precisely, $\mathbf{A}^{\mathrm{CW}=a} \cdot (\mathrm{Hist}(P))^\top \leq \left(-\vec{1}\right)^\top$ if and only if a is the Condorcet winner under P, and by Claim 1 (i), $\mathbf{A}^{\vec{t}} \cdot (\mathrm{Hist}(P))^\top \leq \left(\vec{\mathbf{b}}^{\vec{t}}\right)^\top$ if and only if $\mathrm{Sign}_{\vec{H}}(\mathrm{Hist}(P)) = \vec{t}$.

Part (ii) also follows after the definition. $\mathbf{A}^{\text{CW}=a} \cdot (\vec{x})^\top \leq \left(\vec{0}\right)^\top$ if and only if a is a weak Condorcet winner under P, and by Claim 1 (ii), $\mathbf{A}^{\vec{t}} \cdot (\vec{x})^\top \leq \left(\vec{0}\right)^\top$ if and only if $\vec{t} \trianglelefteq \text{Sign}_{\vec{H}}(\vec{x})$.

To prove Part (iii), suppose $\vec{x} \in \mathcal{H}^{a,\vec{t}}$. Because $\vec{t} \in \mathcal{S}^{\circ}_{\vec{H}}$, we have $\vec{\mathbf{b}}^{a,\vec{t}} = -\vec{1}$ (Definition 21). Therefore, there exists $\delta > 0$ such that for all vector $\vec{x}'$ such that $|\vec{x}' - \vec{x}|_1 < \delta$, $\mathbf{A}^{a,\vec{t}} \cdot (\vec{x}')^\top < \left(\vec{0}\right)^\top$, which means that $\vec{x}' \in \mathcal{H}^{a,\vec{t}}_{\leq 0}$. Therefore, $\mathcal{H}^{a,\vec{t}}_{\leq 0}$ contains the δ neighborhood of $\vec{x}$, whose dimension is $m!$. This means that $\dim(\mathcal{H}^{a,\vec{t}}_{\leq 0}) = m!$. □

$\left[\boldsymbol{\pi \in \Pi_{\mathcal{C}_{\text{CWW}},n}}\right] \Leftarrow \left[\boldsymbol{\pi \in}\ \textbf{Closure}\boldsymbol{(\mathcal{R}^{\overline{r}}_{\text{CWW}})}\right]$**.** Suppose $\pi \in \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWW}})$ and let $(\vec{x}_1, \vec{x}_2, \ldots)$ denote an infinite sequence in $\mathcal{R}^{\overline{r}}_{\text{CWW}}$ that converges to π. Because the number of alternatives and the number of feasible signatures are finite, there exists an infinite subsequence $(\vec{x}'_1, \vec{x}'_2, \ldots)$ such that (1) there exists $a \in \mathcal{A}$ such that for all $j \in \mathbb{N}$, $\text{CW}(\vec{x}'_j) = \{a\}$, and (2) there exists $\vec{t} \in \mathcal{S}_{\vec{H}}$ such that $a \in \overline{r}(\vec{t})$ and for all $j \in \mathbb{N}$, $\text{Sign}_{\vec{H}}(\vec{x}'_j) = \vec{t}$. Because $\overline{r}$ is minimally continuous, by Proposition 4, there exists a feasible atomic refinement of $\vec{t}$, denoted by $\vec{t}_a \in \mathcal{S}^{\circ}_{\vec{H}}$, such that $\overline{r}(\vec{t}_a) = \{a\}$. Therefore, to prove that $\pi \in \Pi_{\mathcal{C}_{\text{CWW}},n}$, it suffices to prove that (i) for every $n > N_{\overline{r}}$, $\mathcal{H}^{a,\vec{t}_a}$ is active, and (ii) $\pi \in \mathcal{H}^{a,\vec{t}_a}_{\leq 0}$, which will be done as follows.

(i) $\boldsymbol{\mathcal{H}^{a,\vec{t}_a}}$ is active. By Claim 2, it suffices to prove that $\mathcal{H}^{a,\vec{t}_a} \cap \mathbb{R}^{m!}_{>0} \neq \emptyset$. This is proved by explicitly constructing a vector in $\mathcal{H}^{a,\vec{t}_a} \cap \mathbb{R}^{m!}_{\geq 0}$ as follows. Because $\vec{t}_a$ is feasible, there exists $\vec{x}^a \in \mathbb{R}^{m!}$ such that $\text{Sign}_{\vec{H}}(\vec{x}^a) = \vec{t}_a$. Recall that π is strictly positive and $(\vec{x}'_1, \vec{x}'_2, \ldots)$ converges to π, there exists $j \in \mathbb{N}$ such that $\vec{x}'_j > \vec{0}$. For any $\delta > 0$, let $\vec{x}_\delta = \vec{x}'_j + \delta \vec{x}^a$. We let $\delta > 0$ denote a sufficiently small number such that the following two conditions hold.

- $\vec{x}_\delta > \vec{0}$. The existence of such δ follows after noticing that $\vec{x}'_j > \vec{0}$.
- $\text{CW}(\vec{x}_\delta) = \{a\}$. The existence of such δ is due to the assumption that $\text{CW}(\vec{x}'_j) = \{a\}$, which means that $\mathbf{A}^{\text{CW}=a} \cdot \left(\vec{x}'_j\right)^\top < \left(\vec{0}\right)^\top$, where $\mathbf{A}^{\text{CW}=a}$ is defined in Definition 21. Therefore, for any sufficiently small $\delta > 0$ we have $\mathbf{A}^{\text{CW}=a} \cdot (\vec{x}_\delta)^\top < \left(\vec{0}\right)^\top$, which means that a is the Condorcet winner under $\vec{x}_\delta$.

Because $\vec{t}_a$ is a refinement of $\vec{t}$, we have $\text{Sign}_{\vec{H}}(\vec{x}_\delta) = \vec{t}_a$. Therefore, $\vec{x}_\delta \in \mathcal{H}^{a,\vec{t}_a} \cap \mathbb{R}^{m!}_{>0}$. Following Claim 2 and the definition of $N_{\overline{r}}$ (Definition 24), we have that $\mathcal{H}^{a,\vec{t}_a}$ is active for all $n > N_{\overline{r}}$.

(ii) $\boldsymbol{\pi \in \mathcal{H}^{a,\vec{t}_a}_{\leq 0}}$. Because for all $j \in \mathbb{N}$, $\mathbf{A}^{\text{CW}=a} \cdot \left(\vec{x}'_j\right)^\top < \left(\vec{0}\right)^\top$ and $(\vec{x}'_1, \vec{x}'_2, \ldots)$ converge to π, we have $\mathbf{A}^{\text{CW}=a} \cdot (\pi)^\top \leq \left(\vec{0}\right)^\top$, which means that a is a weak Condorcet winner under π. It is not hard to verify that for every $k \leq K$, if $t_k = +$ (respectively, $-$ and 0), then we have $[\text{Sign}_{\vec{H}}(\pi)]_k \in \{0, +\}$ (respectively, $\{0, -\}$ and $\{0\}$). Therefore, $\vec{t} \trianglelefteq \text{Sign}_{\vec{H}}(\pi)$, which means that $\vec{t}_a \trianglelefteq \text{Sign}_{\vec{H}}(\pi)$ because $\vec{t}_a \trianglelefteq \vec{t}$. By Claim 7 (ii), we have $\pi \in \mathcal{H}^{a,\vec{t}_a}_{\leq 0}$.

$\left[\boldsymbol{\pi \in \Pi_{\mathcal{C}_{\text{CWW}},n}}\right] \Rightarrow \left[\boldsymbol{\pi \in}\ \textbf{Closure}\boldsymbol{(\mathcal{R}^{\overline{r}}_{\text{CWW}})}\right]$**.** Suppose $\pi \in \Pi_{\mathcal{C}_{\text{CWW}},n}$, which means that there exists $a \in \mathcal{A}$ and $\vec{t} \in \mathcal{S}_{\vec{H}}$ such that $\pi \in \mathcal{H}^{a,\vec{t}}_{\leq 0}$, $a \in \overline{r}(\vec{t})$, $\text{CW}(\vec{t}) = \{a\}$, and $\mathcal{H}^{a,\vec{t}}$ contains a non-negative integer n-vector, denoted by $\vec{x}'$. By Proposition 4, because $\overline{r}$ is minimally continuous, there exists $\vec{t}_a \in \mathcal{S}^{\circ}_{\vec{H}}$ such that $\vec{t}_a \trianglelefteq \vec{t}$ and $\overline{r}(\vec{t}_a) = \{a\}$. Let $\vec{x}^* \in \mathcal{H}^{\vec{t}_a}$ denote an arbitrary vector, which is guaranteed to exist because $\vec{t}_a \in \mathcal{S}^{\circ}_{\vec{H}}$. Because $\vec{x}' \in \mathcal{H}^{a,\vec{t}}$, we have $\mathbf{A}^{\text{CW}=a} \cdot (\vec{x}')^\top \leq \left(-\vec{1}\right)^\top$. Therefore, there exists δ_a such that $\mathbf{A}^{\text{CW}=a} \cdot (\vec{x}' + \delta_a \vec{x}^*)^\top < \left(\vec{0}\right)^\top$. Let $\vec{x} = \vec{x}' + \delta_a \vec{x}^*$. Recall that

$\pi \in \mathcal{H}_{\leq 0}^{a,\vec{t}}$, which means that $\mathbf{A}^{\text{CW}=a} \cdot (\pi)^\top \leq \left(\vec{0}\right)^\top$. Therefore, for all $\delta > 0$ we have

$$\mathbf{A}^{\text{CW}=a} \cdot (\pi + \delta\vec{x})^\top = \mathbf{A}^{\text{CW}=a} \cdot (\pi)^\top + \delta\mathbf{A}^{\text{CW}=a} \cdot (\vec{x})^\top < \left(\vec{0}\right)^\top,$$

which means that $\text{CW}(\pi + \delta\vec{x}) = \{a\}$. It is not hard to verify that $\text{Sign}_{\vec{H}}(\pi + \delta\vec{x}) = \vec{t}_a$, which means that $\overline{r}(\pi + \delta\vec{x}) = \{a\}$. Consequently, for every $\delta > 0$ we have $\pi + \delta\vec{x} \in \mathcal{R}_{\text{CWW}}^{\overline{r}}$. Notice that the sequence $(\pi + \vec{x}, \pi + \frac{1}{2}\vec{x}, \ldots)$ converges to π. Therefore, $\pi \in \text{Closure}(\mathcal{R}_{\text{CWW}}^{\overline{r}})$.

$\left[\boldsymbol{\pi \in} \textbf{Closure}\boldsymbol{(\mathcal{R}_{\textbf{CWW}}^{\overline{r}})}\right] \Rightarrow \left[\textbf{dim}_{\boldsymbol{\mathcal{C}_{\textbf{cww}},n}}^{\textbf{max}}\boldsymbol{(\pi) = m!}\right]$**.** Continuing the proof of the $[\pi \in \Pi_{\mathcal{C}_{\text{cww}},n}] \Rightarrow \left[\pi \in \text{Closure}(\mathcal{R}_{\text{CWW}}^{\overline{r}})\right]$ part, because π is strictly positive and $(\pi + \vec{x}, \pi + \frac{1}{2}\vec{x}, \ldots)$ converges to π, there exists $j \in \mathbb{N}$ such that $\pi + \frac{1}{j}\vec{x} > \vec{0}$. Recall that $\text{CW}(\pi + \frac{1}{j}\vec{x}) = \{a\}$, $\text{Sign}_{\vec{H}}(\pi + \frac{1}{j}\vec{x}) = \vec{t}_a$, and $\vec{t}_a$ is atomic, we have

$$\mathbf{A}^{\text{CW}=a} \cdot \left(\pi + \frac{1}{j}\vec{x}\right)^\top < \left(\vec{0}\right)^\top \text{ and } \mathbf{A}^{\vec{t}_a} \cdot \left(\pi + \frac{1}{j}\vec{x}\right)^\top < \left(\vec{0}\right)^\top$$

Therefore, there exists $\ell > 0$ such that

$$\mathbf{A}^{\text{CW}=a} \cdot \left(\ell(\pi + \frac{1}{j}\vec{x})\right)^\top \leq \left(-\vec{1}\right)^\top \text{ and } \mathbf{A}^{\vec{t}_a} \cdot \left(\ell(\pi + \frac{1}{j}\vec{x})\right)^\top \leq \left(-\vec{1}\right)^\top,$$

which means that $\ell(\pi + \frac{1}{j}\vec{x}) \in \mathcal{H}^{a,\vec{t}_a} \cap \mathbb{R}_{>0}^{m!} \neq \emptyset$. by Claim 7 (iii), we have $\dim_{\mathcal{C}_{\text{cww}},n}^{\max}(\pi) = m!$.

$\left[\textbf{dim}_{\boldsymbol{\mathcal{C}_{\textbf{cww}},n}}^{\textbf{max}}\boldsymbol{(\pi) = m!}\right] \Rightarrow \left[\boldsymbol{\pi \in \Pi_{\mathcal{C}_{\textbf{cww}},n}}\right]$ follows after the definition of $\Pi_{\mathcal{C}_{\text{cww}},n}$. More concretely, $\dim_{\mathcal{C}_{\text{cww}},n}^{\max}(\pi) = m!$ means that there exists a polyhedron $\mathcal{H} \subseteq \mathcal{C}_{\text{CWW}}$ such that the weight on the edge $(\pi, \mathcal{H})$ in the activation graph is $m!$, which implies that $\pi \in \Pi_{\mathcal{C}_{\text{cww}},n}$.

The proofs for $\Pi_{\mathcal{C}_{\text{CWL}},n}$ and $\dim_{\mathcal{C}_{\text{CWL}},n}^{\max}(\pi)$ are similar to the proofs for $\Pi_{\mathcal{C}_{\text{cww}},n}$ and $\dim_{\mathcal{C}_{\text{cww}},n}^{\max}(\pi)$. For completeness, we include the full proofs below.

$\left[\boldsymbol{\pi \in \Pi_{\mathcal{C}_{\textbf{CWL}},n}}\right] \Leftarrow \left[\boldsymbol{\pi \in} \textbf{Closure}\boldsymbol{(\mathcal{R}_{\textbf{CWL}}^{\overline{r}})}\right]$**.** Suppose $\pi \in \text{Closure}(\mathcal{R}_{\text{CWL}}^{\overline{r}})$ and let $(\vec{x}_1, \vec{x}_2, \ldots)$ denote an infinite sequence in $\mathcal{R}_{\text{CWL}}^{\overline{r}}$ that converges to π. Because the number of alternatives and the number of feasible signatures are finite, there exists an infinite subsequence $(\vec{x}_1', \vec{x}_2', \ldots)$ such that (1) there exists $a \in \mathcal{A}$ such that for all $j \in \mathbb{N}$, $\text{CW}(\vec{x}_j') = \{a\}$, and (2) there exists $\vec{t} \in \mathcal{S}_{\vec{H}}$ such that $a \notin \overline{r}(\vec{t})$ and for all $j \in \mathbb{N}$, $\text{Sign}_{\vec{H}}(\vec{x}_j') = \vec{t}$. Let $b \in \overline{r}(\vec{t})$ denote an arbitrary winner. Because $\overline{r}$ is minimally continuous, by Proposition 4, there exists a feasible atomic refinement of $\vec{t}$, denoted by $\vec{t}_b$, such that $\overline{r}(\vec{t}_b) = \{b\}$. Therefore, to prove that $\pi \in \Pi_{\mathcal{C}_{\text{CWL}},n}$, it suffices to show that (i) for every $n > N$, $\mathcal{H}^{a,\vec{t}_b}$ is active, and (ii) $\pi \in \mathcal{H}_{\leq 0}^{a,\vec{t}_b}$.

(i) $\boldsymbol{\mathcal{H}^{a,\vec{t}_b}}$ **is active.** We will apply Claim 2 to prove that $\mathcal{H}^{a,\vec{t}_b}$ is active at every $n > N$. In fact, it suffices to prove that $\mathcal{H}^{a,\vec{t}_b} \cap \mathbb{R}_{>0}^{m!} \neq \emptyset$. This will be proved by explicitly constructing a vector in $\mathcal{H}^{a,\vec{t}_b} \cap \mathbb{R}_{>0}^{m!}$ as follows. Because $\vec{t}_b$ is feasible, there exists $\vec{x}^b \in \mathbb{R}^{m!}$ such that $\text{Sign}_{\vec{H}}(\vec{x}^b) = \vec{t}_b$. Recall that π is strictly positive and $(\vec{x}_1', \vec{x}_2', \ldots)$ converges to π, there exists $j \in \mathbb{N}$ such that $\vec{x}_j' > \vec{0}$. For any $\delta > 0$, let $\vec{x}_\delta = \vec{x}_j' + \delta\vec{x}^b$. We let $\delta > 0$ denote a sufficiently small number such that the following two conditions hold.

- $\vec{x}_\delta > \vec{0}$. The existence of such δ follows after noticing that $\vec{x}_j' > \vec{0}$.
- $\text{CW}(\vec{x}_\delta) = \{a\}$. The existence of such δ is due to the assumption that $\text{CW}(\vec{x}_j') = \{a\}$, which means that $\mathbf{A}^{\text{CW}=a} \cdot (\vec{x}_j')^\top < \left(\vec{0}\right)^\top$, where $\mathbf{A}^{\text{CW}=a}$ is defined in Definition 21. Therefore, for any sufficiently small $\delta > 0$ we have $\mathbf{A}^{\text{CW}=a} \cdot (\vec{x}_\delta)^\top < \left(\vec{0}\right)^\top$, which means that a is the Condorcet winner under $\vec{x}_\delta$.

Because $\vec{t}_b$ is a refinement of $\vec{t}$, we have $\text{Sign}_{\vec{H}}(\vec{x}_\delta) = \vec{t}_b$. Therefore, $\vec{x}_\delta \in \mathcal{H}^{a,\vec{t}_b} \cap \mathbb{R}^{m!}_{>0}$. Following Claim 2 and the definition of $N_{\overline{r}}$ (Definition 24), we have that $\mathcal{H}^{a,\vec{t}_a}$ is active for all $n > N_{\overline{r}}$.

(ii) $\boldsymbol{\pi \in \mathcal{H}^{a,\vec{t}_b}_{\leq 0}}$. Because for all $j \in \mathbb{N}$, $\mathbf{A}^{\text{CW}=a} \cdot (\vec{x}'_j)^\top < \left(\vec{0}\right)^\top$ and $(\vec{x}'_1, \vec{x}'_2, \ldots)$ converge to π, we have $\mathbf{A}^{\text{CW}=a} \cdot (\pi)^\top \leq \left(\vec{0}\right)^\top$, which means that π is a weak Condorcet winner. It is not hard to verify that for every $k \leq K$, if $t_k = +$ (respectively, $-$ and 0), then we have $[\text{Sign}_{\vec{H}}(\pi)]_k \in \{0, +\}$ (respectively, $\{0, -\}$ and $\{0\}$). Therefore, $\vec{t} \trianglelefteq \text{Sign}_{\vec{H}}(\pi)$, which means that $\vec{t}_b \trianglelefteq \text{Sign}_{\vec{H}}(\pi)$ because $\vec{t}_b \trianglelefteq \vec{t}$. It follows that $\mathbf{A}^{\vec{t}_b} \cdot (\pi)^\top \leq \left(\vec{0}\right)^\top$. This means that $\pi \in \mathcal{H}^{a,\vec{t}_b}_{\leq 0}$.

$\boldsymbol{[\pi \in \Pi_{\mathcal{C}_{\text{CWL}},n}] \Rightarrow \left[\pi \in \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWL}})\right]}$. Suppose $\pi \in \Pi_{\mathcal{C}_{\text{CWL}},n}$, which means that there exists $a \in \mathcal{A}$ and $\vec{t} \in \mathcal{S}_{\vec{H}}$ such that $\pi \in \mathcal{H}^{a,\vec{t}}_{\leq 0} \subseteq \mathcal{C}_{\text{CWL}}$, $a \notin \overline{r}(\vec{t})$, $\text{CW}(\pi) = \{a\}$, and $\mathcal{H}^{a,\vec{t}}$ contains a non-negative integer n-vector, denoted by $\vec{x}'$. Let $b \in \overline{r}(\vec{t})$ denote an arbitrary co-winner. By Proposition 4, because $\overline{r}$ is minimally continuous, there exists $\vec{t}_b \in \mathcal{S}^\circ_{\vec{H}}$ such that $\vec{t}_b \trianglelefteq \vec{t}$ and $\overline{r}(\vec{t}_b) = \{b\}$. Let $\vec{x}^* \in \mathcal{H}^{\vec{t}_b}$ denote an arbitrary vector whose existence is guaranteed by the assumption that $\vec{t}_b \in \mathcal{S}^\circ_{\vec{H}}$. Because $\vec{x}' \in \mathcal{H}^{a,\vec{t}}$, we have $\mathbf{A}^{\text{CW}=a} \cdot (\vec{x}')^\top \leq \left(-\vec{1}\right)^\top$. Therefore, there exists δ_a such that $\mathbf{A}^{\text{CW}=a} \cdot (\vec{x}' + \delta_a \vec{x}^*)^\top < \left(\vec{0}\right)^\top$. Let $\vec{x} = \vec{x}' + \delta_a \vec{x}^*$. Recall that $\pi \in \mathcal{H}^{a,\vec{t}}_{\leq 0}$, which means that $\mathbf{A}^{\text{CW}=a} \cdot (\pi)^\top \leq \left(\vec{0}\right)^\top$. Therefore, for all $\delta > 0$ we have $\mathbf{A}^{\text{CW}=a} \cdot (\pi + \delta\vec{x})^\top < \left(\vec{0}\right)^\top$, which means that $\text{CW}(\pi + \delta\vec{x}) = \{a\}$. It is not hard to verify that $\text{Sign}_{\vec{H}}(\pi + \delta\vec{x}) = \vec{t}_b$, which means that $\overline{r}(\pi + \delta\vec{x}) = \{b\}$. This means that for every $\delta > 0$ we have $\pi + \delta\vec{x} \in \mathcal{R}^{\overline{r}}_{\text{CWL}}$. Notice that π is the limit of the sequence $(\pi + \vec{x}, \pi + \frac{1}{2}\vec{x}, \ldots)$. Therefore, $\pi \in \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWL}})$.

$\boldsymbol{\left[\pi \in \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWL}})\right] \Rightarrow \left[\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi) = m!\right]}$. Continuing the proof of the $[\pi \in \Pi_{\mathcal{C}_{\text{CWL}},n}] \Rightarrow \left[\pi \in \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWL}})\right]$ part, because π is strictly positive and $(\pi + \vec{x}, \pi + \frac{1}{2}\vec{x}, \ldots)$ converges to π, there exists $j \in \mathbb{N}$ such that $\pi + \frac{1}{j}\vec{x} > \vec{0}$. Recall that $\text{CW}(\pi + \frac{1}{j}\vec{x}) = \{a\}$, $\text{Sign}_{\vec{H}}(\pi + \frac{1}{j}\vec{x}) = \vec{t}_b$, and $\vec{t}_b$ is atomic, which means that $\mathbf{A}^{\text{CW}=a} \cdot \left(\pi + \frac{1}{j}\vec{x}\right)^\top < \left(\vec{0}\right)^\top$ and $\mathbf{A}^{\vec{t}_b} \cdot \left(\pi + \frac{1}{j}\vec{x}\right)^\top < \left(\vec{0}\right)^\top$. Therefore, there exists $\ell > 0$ such that

$$\mathbf{A}^{\text{CW}=a} \cdot \left(\ell(\pi + \frac{1}{j}\vec{x})\right)^\top \leq \left(-\vec{1}\right)^\top \text{ and } \mathbf{A}^{\vec{t}_b} \cdot \left(\ell(\pi + \frac{1}{j}\vec{x})\right)^\top \leq \left(-\vec{1}\right)^\top,$$

which means that $\ell(\pi + \frac{1}{j}\vec{x}) \in \mathcal{H}^{a,\vec{t}_b} \cap \mathbb{R}^{m!}_{>0} \neq \emptyset$. by Claim 7 (iii), we have $\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi) = m!$.

$\boldsymbol{\left[\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi) = m!\right] \Rightarrow [\pi \in \Pi_{\mathcal{C}_{\text{CWL}},n}]}$ follows after the definition.

This proves Claim 6. $\square$

We are now ready to verify Table 4 column by column as follows.

- **$*$YY:** $\dim^{\max}_{\mathcal{C},n}(\pi) = \max(\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi), \dim^{\max}_{\mathcal{C}_{\text{CWW}},n}(\pi))$, and by Claim 6 we have $\dim^{\max}_{\mathcal{C}_{\text{CWW}},n}(\pi) = m!$. The $\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi)$ part also follows after Claim 6.
- **$*$YN:** The $\dim^{\max}_{\mathcal{C},n}(\pi)$ part follows after Claim 6. Recall that we have assumed $\neg \text{C}_{\text{AS}}(\overline{r}, n)$. This means that there exists an n-profile P such that $\text{CW}(P) \neq \emptyset$ and $\text{CW}(P) \not\subseteq \overline{r}(P)$. Let $\{a\} = \text{CW}(P)$ and $\vec{t} = \text{Sign}_{\vec{H}}(P)$. It follows that $\text{Hist}(P) \in \mathcal{H}^{a,\vec{t},\mathbb{Z}}_n \neq \emptyset$ and $\mathcal{H}^{a,\vec{t}} \subseteq \mathcal{C}_{\text{CWL}}$. Because $\pi \notin \Pi_{\mathcal{C}_{\text{CWL}},n}$, according to the definition of the activation graph (Definition 6), the weight on the edge $(\pi, \mathcal{H}^{a,\vec{t}})$ is $-\frac{n}{\log n}$, and the weight on any edge connected to π is not positive. Therefore, $\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi) = -\frac{n}{\log n}$.

- **NNY:** The $\dim^{\max}_{\mathcal{C},n}(\pi)$ part follows after the definition. The $\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi)$ part follows after Claim 6.

- **YNY:** Recall that the "N" means that $\pi \notin \Pi_{\mathcal{C}_{\text{CWW}},n}$, which implies that $\dim^{\max}_{\mathcal{C}_{\text{CWW}},n}(\pi) < 0$. Therefore, $\dim^{\max}_{\mathcal{C},n}(\pi) = \max(\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi), \dim^{\max}_{\mathcal{C}_{\text{CWW}},n}(\pi))$, which means that $\dim^{\max}_{\mathcal{C},n}(\pi) = \dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi)$. The $\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi)$ part follows after Claim 6.

- **YNN:** We first prove the $\dim^{\max}_{\mathcal{C},n}(\pi)$ part. Because in this case $\pi \in \Pi_{\mathcal{C}_{\text{NCW}},n}$ and $\pi \notin \Pi_{\mathcal{C}_{\text{CWW}},n}$, by the definition of $\Pi_{\mathcal{C}_{\text{NCW}},n}$ and $\Pi_{\mathcal{C}_{\text{CWW}},n}$, we have $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi) \geq 0$ and $\dim^{\max}_{\mathcal{C}_{\text{CWW}},n}(\pi) \leq -\frac{n}{\log n}$. Therefore, $\dim^{\max}_{\mathcal{C},n}(\pi) = \dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi)$. It suffices to prove that $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi) = m!$. Recall from Proposition 1 that

$$\mathcal{C}_{\text{NCW},\leq 0} \cup \mathcal{C}_{\text{CWW},\leq 0} \cup \mathcal{C}_{\text{CWL},\leq 0} = \mathbb{R}^{m!}$$

Therefore, there exists a polyhedron $\mathcal{H}$ in $\mathcal{C}_{\text{NCW}}$, $\mathcal{C}_{\text{CWW}}$, or $\mathcal{C}_{\text{CWL}}$ such that $\pi \in \mathcal{H}_{\leq 0}$ and $\dim(\mathcal{H}_{\leq 0}) = m!$. We now prove that $\mathcal{H}$ is indeed active. Because π is strictly positive and $\mathcal{H}_{\leq 0}$ is convex, $\mathcal{H}_{\leq 0}$ contains an interior point in $\mathbb{R}^{m!}_{>0}$, denoted by $\vec{x}$. Formally, let $\vec{x}'$ denote an arbitrary interior point of $\mathcal{H}_{\leq 0}$. It is not hard to verify that for some sufficiently small $\delta > 0$, $\vec{x} = \dfrac{\pi + \delta \vec{x}'}{1+\delta} \in \mathbb{R}^{m!}_{>0}$ is an interior point of $\mathcal{H}_{\leq 0}$.

Suppose $\mathcal{H}$ is characterized by $\mathbf{A}$ and $\vec{\mathbf{b}}$. Then, we have $\mathbf{A} \cdot (\vec{x})^\top < \left(\vec{0}\right)^\top$. Therefore, there exists $\ell > 0$ such that $\mathbf{A} \cdot (\ell \vec{x})^\top \leq \left(\vec{\mathbf{b}}\right)^\top$, which means that $\ell \vec{x} \in \mathcal{H} \cap \mathbb{R}^{m!}_{>0} \neq \emptyset$. By Claim 2 and the definition of $N_{\overline{r}}$ (Definition 24), $\mathcal{H}$ is active at every $n > N_{\overline{r}}$.

Recall that in the YNN case we have $\pi \notin \Pi_{\mathcal{C}_{\text{CWW}},n}$ and $\pi \notin \Pi_{\mathcal{C}_{\text{CWL}},n}$. Therefore, $\mathcal{H} \subseteq \mathcal{C}_{\text{NCW}}$, which means that $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi) = m! = \dim^{\max}_{\mathcal{C},n}(\pi)$. Following a similar reasoning as in the "$*$YN" case, we have $\dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi) = -\frac{\log n}{n}$.

- **NNN:** This case is impossible because as proved in the "YNN" case, for all $n > N_{\overline{r}}$, $\pi \notin \Pi_{\mathcal{C}_{\text{CWW}},n}$ and $\pi \notin \Pi_{\mathcal{C}_{\text{CWL}},n}$ implies that $\pi \in \Pi_{\mathcal{C}_{\text{NCW}},n}$.

Step 3: Apply Lemma 1. In this step, we apply the inf part of Lemma 1 by combining and simplifying conditions in Table 4.

- **The 0 case** never holds when $n \geq m^4$, because any complete graph is the UMG of some n-profile [51, Claim 6 in the Appendix]. In particular, any complete graph where there is no Condorcet winner is the UMG of an n-profile.

- **The 1 case** holds if and only if $\overline{r}$ satisfies CC for all n profile P, i.e. $\text{C}_{\text{AS}}(\overline{r}, n)$.

- **The VU case.** According to the inf part of Lemma 1, the VU case holds if and only if $\beta_n = -\frac{n}{\log n}$. Note that we do not need to assume $\text{C}_{\text{AS}}(\overline{r}, n)$ in the VU case. According to Table 4, $\beta_n = -\frac{n}{\log n}$ if and only if there exists $\pi \in \text{CH}(\Pi)$ such that the "NNY" column holds. Recall that the "NNN" column is impossible for any $n > N_{\overline{r}}$. Therefore, the "NNY" column holds for $\pi \in \text{CH}(\Pi)$ if and only if $\pi \notin \Pi_{\mathcal{C}_{\text{NCW}},n}$ and $\pi \notin \Pi_{\mathcal{C}_{\text{CWW}},n}$, which is equivalent to the following condition by Claim 6

$$\pi \notin \Pi_{\mathcal{C}_{\text{NCW}},n} \text{ and } \pi \notin \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWW}}) \tag{9}$$

Next, we simplify (9) for $2 \mid n$ and $2 \nmid n$, respectively.

  - **$2 \mid n$.** By the $2 \mid n$ part of Claim 3, $\pi \notin \Pi_{\mathcal{C}_{\text{NCW}},n}$ if and only if π has a Condorcet winner. We prove that in this case (9) is equivalent to:

$$\text{CW}(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi)) \neq \emptyset \tag{10}$$

(9)⇒(10). Suppose π has a Condorcet winner, denoted by a, and (9) holds. For the sake of contradiction suppose that (10) does not hold, which means that $a \in \overline{r}(\pi)$. Then, following a similar construction as in the proof of Claim 6, the minimal continuity of $\overline{r}$ implies that there exist $\vec{t}_a \in \mathcal{S}^{\circ}_{\vec{H}}$ with $\vec{t}_a \trianglelefteq \text{Sign}_{\vec{H}}(\pi)$ and $\overline{r}(\vec{t}_a) = \{a\}$, and

$\vec{x} \in \mathcal{H}^{\vec{t}_a}$ such that for every $\delta > 0$ we have $\pi + \delta\vec{x} \in \mathcal{R}^{\overline{r}}_{\text{CWW}}$. Then $(\pi + \vec{x}, \pi + \frac{1}{2}\vec{x}, \ldots)$ converges to π, which contradicts the assumption that $\pi \notin \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWW}})$.

(10)⇒(9). Let $a \in \text{CW}(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi))$, which means that $\{a\} = \text{CW}(\pi)$ and $a \notin \overline{r}(\pi)$. Suppose for the sake of contradiction that (9) does not hold. Due to Claim 3, we have $\pi \notin \Pi_{\mathcal{C}_{\text{NCW}},n}$. Therefore, $\pi \in \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWW}})$. This means that there exists a sequence $(\vec{x}_1, \vec{x}_2, \ldots)$ in $\mathcal{R}^{\overline{r}}_{\text{CWW}}$ that converge to π. It follows that there exists $j^* \in \mathbb{N}$ such that for all $j > j^*$, a is the Condorcet winner under $\vec{x}_j$, which means that $a \in \overline{r}(\vec{x}_j)$ because $\vec{x}_j \in \mathcal{R}^{\overline{r}}_{\text{CWW}}$. Therefore, by the continuity of $\overline{r}$, we have $a \in \overline{r}(\pi)$, which means that $\text{CW}(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi)) = \emptyset$. This is a contradiction to (10).

Therefore, when $2 \mid n$, the VU case holds if and only if there exists $\pi \in \text{CH}(\Pi)$ such that (10) holds, which is as described in the statement of the theorem, i.e.

$$\exists \pi \in \text{CH}(\Pi) \text{ s.t. } \text{C}_{\text{RD}}(\overline{r}, \pi)$$

– **$2 \nmid n$.** By the $2 \nmid n$ part of Claim 3, $\pi \notin \Pi_{\mathcal{C}_{\text{NCW}},n}$ is equivalent to $\text{CW}(\pi) \cup \text{ACW}(\pi) \neq \emptyset$, i.e. either $\text{CW}(\pi) \neq \emptyset$ or $\text{ACW}(\pi) \neq \emptyset$. When $\text{CW}(\pi) \neq \emptyset$, as in the $2 \mid n$ case, (9) becomes (10). When $\text{ACW}(\pi) \neq \emptyset$, (9) becomes $\text{C}_{\text{NRS}}(\overline{r}, \pi) = 1$. Therefore, when $2 \nmid n$ the VU case holds if and only if the condition in the statement of the theorem holds, i.e.

$$\exists \pi \in \text{CH}(\Pi) \text{ s.t. } \text{C}_{\text{RD}}(\overline{r}, \pi) \text{ or } \text{C}_{\text{NRS}}(\overline{r}, \pi)$$

- **The U case.** According to the $\inf$ part of Lemma 1, the U case holds if and only if $0 \leq \beta_n < m!$. According to Table 4, $0 \leq \beta_n < m!$ if and only if

  (i) for every $\pi \in \text{CH}(\Pi)$ the NNY column of Table 4 does not hold, and

  (ii) there exists $\pi \in \text{CH}(\Pi)$ such that the YNY column of Table 4 holds and $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi) < m!$.

  Part (ii) can be simplified as follows. By Claim 3, $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi) < m!$ if and only if $2 \mid n$ and $\text{ACW}(\pi) \neq \emptyset$, and in this case $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi) = m! - 1$. We show that it suffices to additionally require that $\pi \notin \Pi_{\mathcal{C}_{\text{CWW}},n}$ (i.e. the "N"), or in other words, given $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi) = m! - 1$, $\pi \notin \Pi_{\mathcal{C}_{\text{CWW}},n}$ implies $\pi \in \Pi_{\mathcal{C}_{\text{CWL}},n}$ (i.e. the second "Y"). Suppose for the sake of contradiction that $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi) = m! - 1$, $\pi \notin \Pi_{\mathcal{C}_{\text{CWW}},n}$, and $\pi \notin \Pi_{\mathcal{C}_{\text{CWL}},n}$. Notice that this corresponds to the "YNN" column in Table 4, which means that $\dim^{\max}_{\mathcal{C}_{\text{NCW}},n}(\pi) = m!$, which is a contradiction. By Claim 6, $\pi \notin \Pi_{\mathcal{C}_{\text{CWW}},n}$ if and only if $\pi \notin \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWW}})$. Therefore, part (ii) is equivalent to

  $$\exists \pi \in \text{CH}(\Pi) \text{ s.t. } \text{C}_{\text{NRS}}(\overline{r}, \pi)$$

  Summing up, the U case holds if and only if the condition in the statement of the theorem holds, i.e.

  $$2 \mid n, \text{ and (1) } \forall \pi \in \text{CH}(\Pi), \neg\text{C}_{\text{RD}}(\overline{r}, \pi), \text{ and (2) } \exists \pi \in \text{CH}(\Pi) \text{ s.t. } \text{C}_{\text{NRS}}(\overline{r}, \pi)$$

- **The L case** never holds when $n \geq m^4$, because according to Table 4, $\alpha^*_n = \max_{\pi \in \text{CH}(\Pi)} \dim^{\max}_{\mathcal{C}_{\text{CWL}},n}(\pi)$ is either $-\frac{n}{\log n}$ or $m!$, which means that it is never in $[0, m!)$.

- **The VL case.** According to the $\inf$ part of Lemma 1, the VL case holds if and only if the 1 case does not hold and $\alpha^*_n = -\frac{n}{\log n}$. According to Table 4, this happens in the "$*$YN" column or the "YNN" column, which is equivalent to only requiring that the last "N" holds (because "NNN" is impossible), i.e. for all $\pi \in \text{CH}(\Pi)$, $\pi \notin \Pi_{\mathcal{C}_{\text{CWL}},n}$. By Claim 6, the VL case holds if and only if if and only if the condition in the statement of the theorem holds, i.e.

  $$\neg\text{C}_{\text{AS}}(\overline{r}, n) \text{ and } \forall \pi \in \text{CH}(\Pi), \text{C}_{\text{RS}}(\overline{r}, \pi)$$

- **The M case** corresponds to the remaining cases.

Proof for general refinement r of $\overline{r}$. We now turn to the proof of the theorem for an arbitrary refinement of $\overline{r}$, denoted by r. We first define a slightly different version of CC, denoted by CC^*, which will be used as the lower bound on the (smoothed) satisfaction of the regular CC. For any GISR $\overline{r}$ and any profile P, we define

$$\text{CC}^*(\overline{r}, P) = \begin{cases} 1 & \text{if } \text{CW}(P) = \emptyset \text{ or } \text{CW}(P) = \overline{r}(P) \\ 0 & \text{otherwise} \end{cases}$$

In words, $\text{CC}^*(\overline{r}, P) =$ if and only if (1) the Condorcet winner does not exist, or (2) the Condorcet winner exists and is the *unique* winner under P according to $\overline{r}$. Compared to the standard Condorcet criterion CC, CC^* rules out profiles P where a Condorcet winner exists and is not the unique winner. CC^* and CC coincide with each other when $\overline{r}$ is a resolute rule. Because for any profile P we have $r(P) \subseteq \overline{r}(P)$, for any $\vec{\pi} \in \Pi^n$ we have

$$\Pr_{P\sim\vec{\pi}}(\text{CC}^*(\overline{r}, P) = 1) \leq \Pr_{P\sim\vec{\pi}}(\text{CC}(r, P) = 1) \leq \Pr_{P\sim\vec{\pi}}(\text{CC}(\overline{r}, P) = 1)$$

Therefore,

$$\widetilde{\text{CC}^*}^{\min}_{\Pi}(\overline{r}, n) \leq \widetilde{\text{CC}}^{\min}_{\Pi}(r, n) \leq \widetilde{\text{CC}}^{\min}_{\Pi}(\overline{r}, n) \tag{11}$$

n order to prove the theorem, it suffices to prove that the lower bound in (11), i.e., $\widetilde{\text{CC}^*}^{\min}_{\Pi}(\overline{r}, n)$, has the same dichotomous characterization as $\widetilde{\text{CC}}^{\min}_{\Pi}(\overline{r}, n)$. To this end, we first define a union of polyhedra, denoted by $\mathcal{C}'$, and its almost complement $\mathcal{C}'_{\text{CWL}}$ that are similar to Definition 22 as follows.

Definition 25 (**$\mathcal{C}'$ and $\mathcal{C}'_{\text{CWL}}$**)**.** *Given an int-GISR characterized by $\vec{H}$ and g, we define*

$$\mathcal{C}' = \mathcal{C}_{NCW} \cup \mathcal{C}'_{CWW}, \quad \textit{where } \mathcal{C}'_{CWW} = \bigcup\nolimits_{a\in\mathcal{A}, \vec{t}\in\mathcal{S}_{\vec{H}}:\overline{r}(\vec{t})=\{a\}} \mathcal{H}^{a,\vec{t}}$$

$$\mathcal{C}'_{CWL} = \bigcup\nolimits_{a\in\mathcal{A}, \vec{t}\in\mathcal{S}_{\vec{H}}:\overline{r}(\vec{t})\neq\{a\}} \mathcal{H}^{a,\vec{t}}$$

Notice that $\mathcal{C}_{\text{NCW}}$ used in Definition 25 was define in Definition 22. Just like $\mathcal{C}_{\text{CWL}}$ is an almost complement of $\mathcal{C}$, $\mathcal{C}'_{\text{CWL}}$ is an almost complement of $\mathcal{C}'$. Formally, we first note that $\mathcal{C}' \cap \mathcal{C}'_{\text{CWL}} = \emptyset$, and for any integer vector $\vec{x}$,

- if $\vec{x}$ does not have a Condorcet winner then $\vec{x} \in \mathcal{C}_{\text{NCW}} \subseteq \mathcal{C}'$;
- if $\vec{x}$ has a Condorcet winner a, which is the unique $\overline{r}$ winner, then $\vec{x} \in \mathcal{H}^{a,\text{Sign}_{\vec{H}}(\vec{x})} \subseteq \mathcal{C}'_{\text{CWW}} \subseteq \mathcal{C}$;
- otherwise $\vec{x}$ has a Condorcet winner a, which is either not a $\overline{r}$ co-winner or $|\overline{r}(\vec{x})| \geq 2$. In both cases $\vec{x} \in \mathcal{H}^{a,\text{Sign}_{\vec{H}}(\vec{x})} \subseteq \mathcal{C}'_{\text{CWL}}$.

Therefore, $\mathbb{Z}^q \subseteq \mathcal{C}' \cup \mathcal{C}'_{\text{CWL}}$. The proof for $\widetilde{\text{CC}^*}^{\min}_{\Pi}(\overline{r}, n)$ is similar to the proof for $\widetilde{\text{CC}}^{\min}_{\Pi}(\overline{r}, n)$ presented earlier. The main difference is that $\mathcal{C}$, $\mathcal{C}_{\text{CWW}}$, and $\mathcal{C}_{\text{CWL}}$ are replaced by $\mathcal{C}'$, $\mathcal{C}'_{\text{CWW}}$, and $\mathcal{C}'_{\text{CWL}}$, respectively. The key part is to prove a counterpart to Table 4, which follows after proving $\Pi_{\mathcal{C}'_{\text{CWW}},n} = \Pi_{\mathcal{C}_{\text{CWW}},n}$ and $\Pi_{\mathcal{C}'_{\text{CWL}},n} = \Pi_{\mathcal{C}_{\text{CWL}},n}$ for every $n > N_{\overline{r}}$, as formally shown in the following claim.

Claim 8. *For any $n > N_{\overline{r}}$, we have $\Pi_{\mathcal{C}'_{CWW},n} = \Pi_{\mathcal{C}_{CWW},n}$ and $\Pi_{\mathcal{C}'_{CWL},n} = \Pi_{\mathcal{C}_{CWL},n}$.*

Proof. The main difference between $\mathcal{C}'_{\text{CWW}}$ (respectively, $\mathcal{C}'_{\text{CWL}}$) and $\mathcal{C}_{\text{CWW}}$ (respectively, $\mathcal{C}_{\text{CWL}}$) is the memberships of polyhedra $\mathcal{H}^{a,\vec{t}}$, where $a \in \overline{r}(\vec{t})$ and $\overline{r}(\vec{t}) \geq 2$. Therefore, to prove the claim, it suffices to show that the membership of $\mathcal{H}^{a,\vec{t}}$ does not affect $\Pi_{\mathcal{C}'_{\text{CWW}},n}$ (respectively, $\Pi_{\mathcal{C}'_{\text{CWL}},n}$) compared to $\Pi_{\mathcal{C}_{\text{CWW}},n}$ (respectively, $\Pi_{\mathcal{C}_{\text{CWL}},n}$).

It suffices to show that for any polyhedron $\mathcal{H}^{a,\vec{t}}$, where $a \in \overline{r}(\vec{t})$ and $\overline{r}(\vec{t}) \geq 2$, for any $\pi \in \text{CH}(\Pi)$ and any $n > N_{\overline{r}}$, if $\mathcal{H}^{a,\vec{t}}$ is active and $\pi \in \mathcal{H}^{a,\vec{t}}_{\leq 0}$, then there exist $\mathcal{H}^{a,\vec{t}_a}_{\leq 0} \subseteq \mathcal{C}_{\text{CWW}} \cap \mathcal{C}'_{\text{CWW}}$ and $\mathcal{H}^{a,\vec{t}_b}_{\leq 0} \subseteq \mathcal{C}_{\text{CWL}} \cap \mathcal{C}'_{\text{CWL}}$ such that (1) $\mathcal{H}^{a,\vec{t}_a}_{\leq 0}$ and $\mathcal{H}^{a,\vec{t}_b}_{\leq 0}$ are active at n, and (2) $\pi \in \mathcal{H}^{a,\vec{t}_a}_{\leq 0} \cap \mathcal{H}^{a,\vec{t}_b}_{\leq 0}$.

In other words, if a distribution $\pi \in \text{CH}(\Pi)$ is in $\mathcal{C}'_{\text{CWW}}$, $\mathcal{C}'_{\text{CWL}}$, $\mathcal{C}_{\text{CWW}}$, or $\mathcal{C}_{\text{CWL}}$ due to $\mathcal{H}^{a,\vec{t}}$, then it is also in the same set without considering its edge to $\mathcal{H}^{a,\vec{t}}$ in the activation graph. As we will see soon, (1) follows after the assumption that $n > N_{\overline{r}}$ and (2) follows after the minimal continuity of $\overline{r}$. Formally, the proof proceeds in the following three steps.

(i) Define $\vec{t}_a$ and $\vec{t}_b$. Let $b \neq a$ denote a co-winner under π, i.e., $\{a, b\} \subseteq \overline{r}(\pi)$. Because $\overline{r}$ is minimally continuous, by Proposition 4, there exists a feasible atomic signature $\vec{t}_a \in \mathcal{S}^{\circ}_{\vec{H}}$ (respectively, $\vec{t}_b \in \mathcal{S}^{\circ}_{\vec{H}}$) such that $\vec{t}_a \trianglelefteq \vec{t}$ (respectively, $\vec{t}_b \trianglelefteq \vec{t}$) and $\overline{r}(\vec{t}_a) = \{a\}$ (respectively, $\overline{r}(\vec{t}_b) = \{b\}$).

(ii) Prove that $\mathcal{H}^{a,\vec{t}_a}_{\leq 0}$ and $\mathcal{H}^{a,\vec{t}_b}_{\leq 0}$ are active at any $n > N_{\overline{r}}$. Because $\vec{t}_a$ is feasible, there exists $\vec{x} \in \mathbb{R}^{m!}$ such that $\text{Sign}_{\vec{H}}(\vec{x}) = \vec{t}_a$. Therefore, recall that π is strictly positive (by ϵ), for some sufficiently small $\delta > 0$, we have $\pi + \delta\vec{x} \in \mathbb{R}^{m!}_{>0}$, $\text{CW}(\pi + \delta\vec{x}) = \{a\}$, and $\text{Sign}_{\vec{H}}(\pi + \delta\vec{x}) = \vec{t}_a$. This means that $\pi + \delta\vec{x}$ is an interior point of $\mathcal{H}^{a,\vec{t}_a}$ (which also means that $\dim(\mathcal{H}^{a,\vec{t}_a}) = m!$). Recall that the $\vec{\text{b}}$ part of $\mathcal{H}^{a,\vec{t}_a}$ (Definition 17 and 21) is non-positive, we have $\mathcal{H}^{a,\vec{t}_a} \subseteq \mathcal{H}^{a,\vec{t}_a}_{\leq 0}$, which means that $\dim(\mathcal{H}^{a,\vec{t}_a}_{\leq 0}) = m!$ as well. Therefore, according to Claim 2 and the definition of $N_{\overline{r}}$ (Definition 24), $\mathcal{H}^{a,\vec{t}_a}$ is active at any $n > N_{\overline{r}}$. Similarly, we have that $\mathcal{H}^{a,\vec{t}_b}$ is active at any $n > N_{\overline{r}}$.

(iii) Prove that $\pi \in \mathcal{H}^{a,\vec{t}_a}_{\leq 0} \cap \mathcal{H}^{a,\vec{t}_b}_{\leq 0}$. Recall that $\pi \in \mathcal{H}^{a,\vec{t}}_{\leq 0}$. Therefore, according to Claim 7 (ii), we have $\vec{t} \trianglelefteq \text{Sign}_{\vec{H}}(\pi)$, which means that $\vec{t}_a \trianglelefteq \text{Sign}_{\vec{H}}(\pi)$, because $\vec{t}_a \trianglelefteq \vec{t}$. By Claim 7 (ii) again, we have $\pi \in \mathcal{H}^{a,\vec{t}_a}_{\leq 0}$. Similarly, we can prove that $\pi \in \mathcal{H}^{a,\vec{t}_b}_{\leq 0}$.

This completes the proof of Claim 8. $\square$

Therefore, $\widetilde{\text{CC}^*}^{\min}_{\Pi}(\overline{r}, n)$ has the same characterization as $\widetilde{\text{CC}}^{\min}_{\Pi}(\overline{r}, n)$, which concludes the proof of Lemma 2 due to (11). $\square$

E.2 Proof of Theorem 1

Theorem 1. (Smoothed CC: Integer Positional Scoring Rules). *Let $\mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi)$ be a strictly positive and closed single-agent preference model, let $\overline{r}_{\vec{s}}$ be a minimally continuous int-GISR and let $r_{\vec{s}}$ be a refinement of $\overline{r}_{\vec{s}}$. For any $n \geq 8m + 49$ with $2 \mid n$, we have*

$$\widetilde{\text{CC}}^{\min}_{\Pi}(r_{\vec{s}}, n) = \begin{cases} 1 - \exp(-\Theta(n)) & \textit{if } \forall \pi \in \textit{CH}(\Pi), |\textit{WCW}(\pi)| \times |\overline{r}(\pi) \cup \textit{WCW}(\pi)| \leq 1 \\ \Theta(n^{-0.5}) & \textit{if } \begin{cases} \textit{(1) } \forall \pi \in \textit{CH}(\Pi), \textit{CW}(\pi) \cap (\mathcal{A} \setminus \overline{r}_{\vec{s}}(\pi)) = \emptyset \textit{ and} \\ \textit{(2) } \exists \pi \in \textit{CH}(\Pi) \textit{ s.t. } |\textit{ACW}(\pi) \cap (\mathcal{A} \setminus \overline{r}_{\vec{s}}(\pi))| = 2 \end{cases} \\ \exp(-\Theta(n)) & \textit{if } \exists \pi \in \textit{CH}(\Pi) \textit{ s.t. } \textit{CW}(\pi) \cap (\mathcal{A} \setminus \overline{r}_{\vec{s}}(\pi)) \neq \emptyset \\ \Theta(1) \textit{ and } 1 - \Theta(1) & \textit{otherwise} \end{cases}$$

For any $n \geq 8m + 49$ with $2 \nmid n$, we have

$$\widetilde{\text{CC}}^{\min}_{\Pi}(r_{\vec{s}}, n) = \begin{cases} 1 - \exp(-\Theta(n)) & \textit{same as the } 2 \mid n \textit{ case} \\ \exp(-\Theta(n)) & \textit{if } \exists \pi \in \textit{CH}(\Pi) \textit{ s.t. } \begin{cases} \textit{(1) } \textit{CW}(\pi) \cap (\mathcal{A} \setminus \overline{r}_{\vec{s}}(\pi)) \neq \emptyset \textit{ or} \\ \textit{(2) } |\textit{ACW}(\pi) \cap (\mathcal{A} \setminus \overline{r}_{\vec{s}}(\pi))| = 2 \end{cases} \\ \Theta(1) \textit{ and } 1 - \Theta(1) & \textit{otherwise} \end{cases}$$

Proof. We apply Lemma 2 to prove the theorem. For any integer irresolute positional scoring rule $\overline{r}_{\vec{s}}$, we prove the following claim to simplify $\text{Closure}(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\text{CWW}})$ and $\text{Closure}(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\text{CWL}})$.

Claim 9. *For any $\pi \in \textit{CH}(\Pi)$,*

$$\left[\pi \in \textit{Closure}(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\textit{CWW}})\right] \Leftrightarrow [\textit{WCW}(\pi) \cap \overline{r}_{\vec{s}}(\pi) \neq \emptyset]$$

$$\left[\pi \in \textit{Closure}(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\textit{CWL}})\right] \Leftrightarrow [\exists a \neq b \textit{ s.t. } a \in \textit{WCW}(\pi) \textit{ and } b \in \overline{r}_{\vec{s}}(\pi)]$$

Proof. The proof is done in the following steps.

$\left[\boldsymbol{\pi \in} \textbf{Closure}\boldsymbol{(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\textbf{CWW}})}\right] \Rightarrow \textbf{[WCW}\boldsymbol{(\pi) \cap \overline{r}_{\vec{s}}(\pi) \neq \emptyset}\textbf{]}$. Suppose $\pi \in \text{Closure}(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\text{CWW}})$, which means that there exists a sequence $(\vec{x}_1, \vec{x}_2, \ldots)$ in $\mathcal{R}^{\overline{r}_{\vec{s}}}_{\text{CWW}}$ that converges to π. It follows that there exists an alternative $a \in \mathcal{A}$ and a subsequence of $(\vec{x}_1, \vec{x}_2, \ldots)$, denoted by $(\vec{x}'_1, \vec{x}'_2, \ldots)$ such that for every $j \in \mathbb{N}$, $\text{CW}(\vec{x}'_j) = \{a\}$ and $a \in \overline{r}_{\vec{s}}(\vec{x}'_j)$. This means that the following holds.

- a is a weak Condorcet winner under π. Notice that for any $b \neq a$ and any $j \in \mathbb{N}$, we have $\text{Pair}_{b,a} \cdot \vec{x}'_j < 0$, which means that $\text{Pair}_{b,a} \cdot \pi \leq 0$.
- $a \in \overline{r}_{\vec{s}}(\pi)$. Notice that for any $b \neq a$ and any $j \in \mathbb{N}$, the total score of a is higher than or equal to the total score of b in $\vec{x}'_j$. Therefore, the same holds for π, which means that $a \in \overline{r}_{\vec{s}}(\pi)$.

Therefore, a is a weak Condorcet winner as well as a $\overline{r}_{\vec{s}}$ co-winner, which implies $\text{WCW}(\pi) \cap \overline{r}_{\vec{s}}(\pi) \neq \emptyset$.

$\left[\boldsymbol{\pi \in} \textbf{Closure}\boldsymbol{(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\textbf{CWW}})}\right] \Leftarrow \textbf{[WCW}\boldsymbol{(\pi) \cap \overline{r}_{\vec{s}}(\pi) \neq \emptyset}\textbf{]}$. Suppose $\text{WCW}(\pi) \cap \overline{r}_{\vec{s}}(\pi) \neq \emptyset$ and let $a \in \text{WCW}(\pi) \cap \overline{r}_{\vec{s}}(\pi)$. We will explicitly construct a sequence of vectors in $\mathcal{R}^{\overline{r}_{\vec{s}}}_{\text{CWW}}$ that converges to π. Let σ denote a cyclic permutation among $\mathcal{A} \setminus \{a\}$ and let P denote the following $(m-1)$-profile

$$P = \{\sigma^i(a \succ \text{others}) : 1 \leq i \leq m-1\} \tag{12}$$

It is not hard to verify that $\text{CW}(P) = \overline{r}_{\vec{s}}(P) = \{a\}$. Therefore, for any $\delta > 0$ we have

$$\text{CW}(\pi + \delta \cdot \text{Hist}(P)) = \overline{r}_{\vec{s}}(\pi + \delta \cdot \text{Hist}(P)) = \{a\},$$

which means that $\pi + \delta \cdot \text{Hist}(P) \in \text{Closure}(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\text{CWW}})$. It follows that $(\pi + \frac{1}{j}\text{Hist}(P) : j \in \mathbb{N})$ is a sequence in $\text{Closure}(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\text{CWW}})$ that converges to π, which means that $\pi \in \text{Closure}(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\text{CWW}})$.

$\left[\boldsymbol{\pi \in} \textbf{Closure}\boldsymbol{(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\textbf{CWL}})}\right] \Rightarrow \boldsymbol{[\exists a \neq b}$ **s.t.** $\boldsymbol{a \in} \textbf{WCW}\boldsymbol{(\pi)}$ **and** $\boldsymbol{b \in \overline{r}_{\vec{s}}(\pi)]}$. Suppose $\pi \in \text{Closure}(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\text{CWL}})$, which means that there exists a sequence $(\vec{x}_1, \vec{x}_2, \ldots)$ in $\mathcal{R}^{\overline{r}_{\vec{s}}}_{\text{CWL}}$ that converges to π. It follows that there exists a pair of different alternatives $a, b \in \mathcal{A}$ and a subsequence of $(\vec{x}_1, \vec{x}_2, \ldots)$, denoted by $(\vec{x}'_1, \vec{x}'_2, \ldots)$ such that for every $j \in \mathbb{N}$, $\text{CW}(\vec{x}'_j) = \{a\}$ and $b \in \overline{r}_{\vec{s}}(\vec{x}'_j)$. Following a similar proof as in the $\mathcal{R}^{\overline{r}_{\vec{s}}}_{\text{CWL}}$ part, we have that a is a weak Condorcet winner under π and $b \in \overline{r}_{\vec{s}}(\pi)$.

$\left[\boldsymbol{\pi \in} \textbf{Closure}\boldsymbol{(\mathcal{R}^{\overline{r}_{\vec{s}}}_{\textbf{CWL}})}\right] \Leftarrow \boldsymbol{[\exists a \neq b}$ **s.t.** $\boldsymbol{a \in} \textbf{WCW}\boldsymbol{(\pi)}$ **and** $\boldsymbol{b \in \overline{r}_{\vec{s}}(\pi)]}$. Let $a \neq b$ be two alternatives such that $a \in \text{WCW}(\pi)$ and $b \in \overline{r}_{\vec{s}}(\pi)$. We define a profile P where $\text{CW}(P) = \{a\}$ and $\overline{r}_{\vec{s}}(P) = \{b\}$, whose existence is guaranteed by the following claim, which is slightly different from [15, Theorem 6] for scoring vectors $\vec{s} = (s_1, \ldots, s_m)$ with $s_1 > s_2 > \cdots > s_m$.

Claim 10. *For any $m \geq 3$, any positional scoring rule with scoring vector $\vec{s} = (s_1, \ldots, s_m)$ where $s_1 > s_m$, any $n \geq 8m + 49$, and any pair of different alternatives $a \neq b$, there exists an n-profile P such that $\text{CW}(P) = \{a\}$ and $\overline{r}_{\vec{s}}(P) = \{b\}$.*

Proof. We explicitly construct an n-profile P where the Condorcet winner exists and is different from the unique $\overline{r}_{\vec{s}}$ winner. Then, we apply a permutation over $\mathcal{A}$ to P to make a the Condorcet and b the unique $\overline{r}_{\vec{s}}$ winner. The construction is done in two cases: $s_2 = s_m$ and $s_2 > s_m$.

- **Case 1:** $\boldsymbol{s_2 = s_m}$**.** In this case $\overline{r}_{\vec{s}}$ corresponds to the plurality rule. We let

$$\begin{aligned} P = &\left\lfloor \frac{n-1}{2} \right\rfloor \times [2 \succ 1 \succ 3 \succ \text{others}] + \left\lfloor \frac{n-3}{2} \right\rfloor \times [3 \succ 1 \succ 2 \succ \text{others}] \\ &+ \left(n + 1 - 2\left\lfloor \frac{n-1}{2} \right\rfloor\right) \times [1 \succ 2 \succ 3 \succ \text{others}] \end{aligned}$$

It is not hard to verify that the alternative 1 is the Condorcet winner and 2 is the unique plurality winner.

- **Case 2: $s_2 > s_m$.** Let $2 \leq k \leq m-1$ denote the smallest number such that $s_k > s_{k+1}$. Let $A_1 = [4 \succ \cdots \succ k+1]$ and $A_2 = [k+2 \succ \cdots \succ m]$, and let P^* denote the following 7-profile.

$$P^* = \{3 \times [1 \succ 2 \succ A_1 \succ 3 \succ A_2] + 2 \times [2 \succ 3 \succ A_1 \succ 1 \succ A_2] \\ +[3 \succ 1 \succ A_1 \succ 2 \succ A_2] + [2 \succ 1 \succ A_1 \succ 3 \succ A_2]\}$$

It is not hard to verify that 1 is the Condorcet winner under P^*, and the total score of 1 is $3s_1 + 2s_2 + 2s_{k+1} < 3s_1 + 3s_2 + s_{k+1}$, which is the total score of 2. Note that the total score of any alternative in A_1 is $7s_k$, which might be larger than the score of 2. If $3s_1 + 3s_2 + s_{k+1} \geq 7s_k$, then we let $b = 2$; otherwise we let $b = 4$. Let P_b denote the following $(m-1)$-profile that will be used as a tie-breaker. Let σ denote an arbitrary cyclic permutation among $\mathcal{A} \setminus \{b\}$.

$$P_b = \{\sigma^i([b \succ \text{others}]) : 1 \leq i \leq m-1\}$$

Let

$$P = \left\lfloor \frac{n-m+1}{7} \right\rfloor \times P^* + P_b + \left(n-m+1-7\left\lfloor \frac{n-m+1}{7} \right\rfloor\right) \times [b \succ \text{others}]$$

It is not hard to verify that when $n \geq 8m+49$, $\text{CW}(P) = \{1\}$, $\overline{r}_{\vec{s}}(P) = \{b\}$, and $b \neq 1$.

This proves Claim 10. □

Let P denote the profile guaranteed by Claim 10. For any $\delta > 0$ we have

$$\text{CW}(\pi + \delta \cdot \text{Hist}(P)) = \{a\} \text{ and } \overline{r}(\pi + \delta \cdot \text{Hist}(P)) = \{b\},$$

which means that $\pi + \delta \cdot \text{Hist}(P) \in \text{Closure}(\mathcal{R}_{\text{CWL}}^{\overline{r}_{\vec{s}}})$. It follows that $(\pi + \frac{1}{j}\text{Hist}(P) : j \in \mathbb{N})$ is a sequence in $\mathcal{R}_{\text{CWL}}^{\overline{r}_{\vec{s}}}$ that converges to π, which means that $\pi \in \text{Closure}(\mathcal{R}_{\text{CWL}}^{\overline{r}_{\vec{s}}})$. This proves Claim 9. □

Claim 9 implies that for all $n \geq 8m+49$, the 1 case doe not hold, i.e., $\text{C}_{\text{AS}}(\overline{r}_{\vec{s}}, n) = 0$. We now apply Claim 9 to simplify the conditions in Lemma 2.

- $\text{C}_{\text{RS}}(\overline{r}_{\vec{s}}, \pi)$. By definition, this holds if and only if $\pi \notin \text{Closure}(\mathcal{R}_{\text{CWL}}^{\overline{r}_{\vec{s}}})$, which is equivalent to $\nexists a \neq b$ s.t. $a \in \text{WCW}(\pi)$ and $b \in \overline{r}_{\vec{s}}(\pi)$. In other words, either $\text{WCW}(\pi) = \emptyset$ or $(\text{WCW}(\pi) = \overline{r}_{\vec{s}}(\pi)$ and $|\text{WCW}(\pi)| = 1)$. Notice that $\overline{r}_{\vec{s}}(\pi) \neq \emptyset$. Therefore, $\text{C}_{\text{RS}}(\overline{r}_{\vec{s}}, \pi)$ is equivalent to $|\text{WCW}(\pi)| \times |\overline{r}_{\vec{s}}(\pi) \cup \text{WCW}(\pi)| \leq 1$.
- $\text{C}_{\text{NRS}}(\overline{r}_{\vec{s}}, \pi)$. By definition, this holds if and only if $\text{ACW}(\pi) \neq \emptyset$ and $\pi \notin \text{Closure}(\mathcal{R}_{\text{CWW}}^{\overline{r}_{\vec{s}}})$, which is equivalent to $\text{ACW}(\pi) \neq \emptyset$ and $\text{WCW}(\pi) \cap \overline{r}_{\vec{s}}(\pi) = \emptyset$. The latter is equivalent to $\text{WCW}(\pi) \cap (\mathcal{A} \setminus \overline{r}_{\vec{s}}(\pi)) = \text{WCW}(\pi)$. We note that when $\text{ACW}(\pi) \neq \emptyset$, we have $\text{WCW}(\pi) = \text{ACW}(\pi)$. Therefore, $\text{C}_{\text{NRS}}(\overline{r}_{\vec{s}}, \pi)$ is equivalent to $|\text{ACW}(\pi) \cap (\mathcal{A} \setminus \overline{r}_{\vec{s}}(\pi))| = 2$.

Theorem 1 follows after Lemma 2 with the simplified conditions discussed above. □

E.3 Definitions, Full Statement, and Proof for Theorem 2

For any $O \in \mathcal{L}(\mathcal{A})$, any $1 \leq i < i' \leq m$, and any $a \in \mathcal{A}$, let $O[i]$ denote the alternative ranked at the i-th place in O, let $O[i, i']$ denote the set of alternatives ranked from the i-th place to the i'-th place in O, and let $O^{-1}[a]$ denote the rank of a in O. For any $A \subseteq \mathcal{A}$ and any $\vec{x} \in \mathbb{R}^{m!}$ that represents the histogram of a profile, let $\vec{x}|_A \in \mathbb{R}^{|A|!}$ denote the histogram of the profile restricted to alternatives in A.

Example 13. *Let* $O = [3 \rhd 1 \rhd 2]$.[2] *We have* $O[2] = 1$, $O^{-1}(2) = 3$, *and* $O[2,3] = \{1,2\}$. *Let* $\hat{\pi}$ *denote the (fractional) profile in Figure 1. We have* $\hat{\pi}|_{O[2,3]} = (\underbrace{0.5}_{1 \succ 2}, \underbrace{0.5}_{2 \succ 1})$.

Definition 26 (Parallel universes and possible losing rounds under MRSE rules). *For any MRSE rule* $\overline{r} = (\overline{r}_2, \ldots, \overline{r}_m)$ *and any* $\vec{x} \in \mathbb{R}^{m!}$, *the set of* parallel universes under $\overline{r}$ at $\vec{x}$, *denoted by* $PU_{\overline{r}}(\vec{x}) \subseteq \mathcal{L}(\mathcal{A})$, *is the set of all elimination orders under PUT. Formally,*

$$PU_{\overline{r}}(\vec{x}) = \{O \in \mathcal{L}(\mathcal{A}) : \forall 1 \leq i \leq m-1, O[i] \in \arg\min_a Score_{\overline{r}_{m+1-i}}(\vec{x}|_{O[i,m]}, a)\},$$

where $Score_{\overline{r}_{m+1-i}}(\vec{x}|_{O[i,m]}, a)$ *is the total score of* a *under the positional scoring rule* $\overline{r}_{m+1-i}$, *where the profile is* $\vec{x}|_{O[i,m]}$.

For any alternative a, *let the* possible losing rounds, *denoted by* $LR_{\overline{r}}(\vec{x}, a) \subseteq [m-1]$, *be the set of all rounds in the parallel universes where* a *drops out. Formally,*

$$LR_{\overline{r}}(\vec{x}, a) = \{O^{-1}[a] : O \in PU_{\overline{r}}(\vec{x})\}$$

See Example 4 for examples of parallel universes and possible losing rounds under STV.

Theorem 2. (Smoothed CC: int-MRSE rules). *Let* $\mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi)$ *be a strictly positive and closed single-agent preference model, let* $\overline{r} = (\overline{r}_2, \ldots, \overline{r}_m)$ *be an int-MRSE and let* r *be a refinement of* $\overline{r}$. *For any* $n \in \mathbb{N}$ *with* $2 \mid n$, *we have*

$$\widetilde{\text{CC}}^{\min}_{\Pi}(r, n) = \begin{cases} 1 & \text{if } \forall 2 \leq i \leq m, CL(\overline{r}_i) = 1 \\ 1 - \exp(-\Theta(n)) & \text{if } \begin{cases} \text{(1) } \exists 2 \leq i \leq m \text{ s.t. } CL(\overline{r}_i) = 0 \text{ and} \\ \text{(2) } \forall \pi \in CH(\Pi), \forall a \in WCW(\pi) \text{ and } \forall i^* \in LR_{\overline{r}}(\pi, a), \\ \qquad \text{we have } CL(\overline{r}_{m+1-i^*}) = 1 \end{cases} \\ \Theta(n^{-0.5}) & \text{if } \begin{cases} \text{(1) } \forall \pi \in CH(\Pi), CW(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi)) = \emptyset \text{ and} \\ \text{(2) } \exists \pi \in CH(\Pi) \text{ s.t. } |ACW(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi))| = 2 \end{cases} \\ \exp(-\Theta(n)) & \text{if } \exists \pi \in CH(\Pi) \text{ s.t. } CW(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi)) \neq \emptyset \\ \Theta(1) \text{ and } 1 - \Theta(1) & \text{otherwise} \end{cases}$$

For any $n \in \mathbb{N}$ *with* $2 \nmid n$, *we have*

$$\widetilde{\text{CC}}^{\min}_{\Pi}(r, n) = \begin{cases} 1 & \text{same as the } 2 \mid n \text{ case} \\ 1 - \exp(-\Theta(n)) & \text{same as the } 2 \mid n \text{ case} \\ \exp(-\Theta(n)) & \text{if } \exists \pi \in CH(\Pi) \text{ s.t. } \begin{cases} \text{(1) } CW(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi)) \neq \emptyset \text{ or} \\ \text{(2) } |ACW(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi))| = 2 \end{cases} \\ \Theta(1) \text{ and } 1 - \Theta(1) & \text{otherwise} \end{cases}$$

Intuitive explanations. The conditions for U, VU, and M cases are the same as their counterparts in Theorem 1. The most interesting cases are the 1 case and the VL case. The 1 case happens when all positional scoring rule used in $\overline{r}$ satisfy CONDORCET LOSER. This is true because for any positional scoring rule that satisfies CONDORCET LOSER, the Condorcet winner, when it exists, cannot have the lowest score among all alternatives. Therefore, like in Baldwin's rule, the Condorcet winner never loses in any round, which means that it must be the unique winner under $\overline{r}$.

The VL case happens when (1) the 1 case does not happen, and (2) for every distribution $\pi \in \text{CH}(\Pi)$, every weak Condorcet winner a, and every round i^* where a is eliminated in a parallel universe, the positional scoring rule used in round i^*, i.e. $\overline{r}_{m+1-i^*}$ for $m+1-i^*$ alternatives, must satisfy CONDORCET LOSER. (2) makes sense because it guarantees that when a small permutation is added to π, if a weak Condorcet winner a becomes the Condorcet winner, then it will be the unique winner under $\overline{r}$, because in every round i^* where a can possibly be eliminated before the perturbation (i.e. i^* is a possible losing round), the voting rule used in that round, i.e. $\overline{r}_{m+1-i^*}$, will not eliminate a after a has become a Condorcet winner. The following example shows the VL case under $\overline{\text{STV}}$.

Proof. We apply Lemma 2 to prove the theorem. We first prove the following claim, which states that when n is sufficiently large, $\text{C}_{\text{AS}}(\overline{r}, n) = 1$ if and only if all scoring rules used in $\overline{r}$ satisfy the Condorcet loser criterion.

[2] Again, we use $\rhd$ in contrast to $\succ$ to indicate that O is a parallel universe instead of an agent's preferences.

Claim 11. *For int-MRSE $\overline{r}$, there exists $N \in n$ such that for every $n > N$, $C_{AS}(\overline{r}, n)$ holds if and only if for all $2 \le i \le m$, $CL(\overline{r}_i) = 1$.*

Proof. **The $\Leftarrow$ direction.** Suppose for all $2 \le i \le m$, $\mathrm{CL}(\overline{r}_i) = 1$ and for the sake of contradiction, suppose $\mathrm{C}_{\mathrm{AS}}(\overline{r}, n) = 0$, which means that there exists an n-profile P such that $\mathrm{CW}(P) = \{a\}$ and $a \notin \overline{r}(P)$. This means that $\mathrm{LR}_{\overline{r}}(\pi, a) \neq \emptyset$. Let $O \in \mathrm{LR}_{\overline{r}}(\pi, a)$ denote an arbitrary possible losing round of a and let $i^* = O^{-1}[a]$, which means that a has the lowest total score in the restriction of P on the remaining alternatives (i.e. $O[i^*, m]$), when $\overline{r}_{m+1-i^*}$ is used. In other words,

$$a \in \arg\min_b \mathrm{Score}_{\overline{r}_{m+1-i^*}}(P|_{O[i^*,m]}, b)$$

Notice that a is the Condorcet winner under P, which means that a is also the Condorcet winner under $P|_{O[i^*,m]}$. We now obtain a profile P_{i^*} over $O[i^*, m]$ from $P|_{O[i^*,m]}$, which constitutes a violation of CONDORCET LOSER for $\overline{r}_{m+1-i^*}$. Let $n' = |P|$.

$$P_{i^*} = (n' + 1) \times \mathcal{L}(O[i^*, m]) - P$$

That is, P_{i^*} is obtained from $(n' + 1)$ copies of all linear orders over $O[i^*, m]$) by subtracting linear orders in P. It is not hard to verify that a is the Condorcet loser as well as an $\overline{r}_{m+1-i^*}$ co-winner in P_{i^*}, because all alternatives are tied in the WMG of $(n' + 1) \times \mathcal{L}(O[i^*, m])$ and are tied w.r.t. their total $\overline{r}_{m+1-i^*}$ scores under $(n' + 1)\mathcal{L}(O[i^*, m])$. This is a contradiction to the assumption that all $\overline{r}_i$'s satisfies the Condorcet loser criterion.

The $\Rightarrow$ direction. For the sake of contradiction, suppose $\mathrm{CL}(\overline{r}_{i^*}) = 1$ for some $2 \le i^* \le m$, which means that there exist a profile P_1 over $m+1-i^*$ alternatives $\{i^*, \ldots, m\}$, such that alternative i^* is the Condorcet loser and a co-winner of $\overline{r}_{m+1-i^*}$ under P_1. We will construct a profile P over $\mathcal{A}$ to show that $\mathrm{C}_{\mathrm{AS}}(\overline{r}, n) = 0$ for every sufficiently large n. We will show that alternatives in $O[1, i^* - 1]$ are eliminated in the first $i^* - 1$ round of executing $\overline{r}$ on P. Then i^* will be eliminated in the next round.

First, we define a profile P' over $O[i^*, m]$ where i^* is the Condorcet winner as well as the unique $\overline{r}_{m+1-i^*}$ loser. Let σ denote an arbitrary cyclic permutation among $O[i^* + 1, m]$, and let

$$P_2 = \{\sigma^i(a \succ O[i^* + 1, m]) : 1 \le i \le m - i^*\},$$

where alternatives in $O[i^* + 1, m]$ are ranked alphabetically. Let $n_1 = |P_1|$ and

$$P' = m(n_1 + 1) \times \mathcal{L}(O[i^*, m]) - m \times P_1 - P_2$$

It is not hard to verify that P' is indeed a profile, i.e., the weight on each ranking is a non-negative integer. i^* is the Condorcet winner under P' because i^* is the Condorcet loser in P_1, and $|P_2| < m$. i^* is the unique loser under P' because for any other alternative $a \in O[i^*, m]$, we have

$$\begin{aligned}
&\mathrm{Score}_{\overline{r}_{m+1-i^*}}(m(n' + 1) \times \mathcal{L}(O[i^*, m]), i^*) = \mathrm{Score}_{\overline{r}_{m+1-i^*}}(m(n' + 1) \times \mathcal{L}(O[i^*, m]), a),\\
&\mathrm{Score}_{\overline{r}_{m+1-i^*}}(P_1, i^*) \ge \mathrm{Score}_{\overline{r}_{m+1-i^*}}(P_1, a), \text{ and}\\
&\mathrm{Score}_{\overline{r}_{m+1-i^*}}(P_2, i^*) > \mathrm{Score}_{\overline{r}_{m+1-i^*}}(P_2, a).
\end{aligned}$$

Next, we let P^* denote the profile obtained from P' by appending $O[1] \succ O[2] \succ \cdots \succ O[i^* - 1]$ in the bottom. More precisely, we let

$$P^* = \{R \succ O[1] \succ O[2] \succ \cdots \succ O[i^* - 1] : R \in P'\}$$

Finally, we are ready to define P. Let σ_1 denote an arbitrary cyclic permutation among alternatives in $O[1, i^* - 1]$. Let $n' = |P'|$ and $P = P^1 \cup P^2 \cup P^3$, defined as follows.

- P^1 consists of n' copies of $\{\sigma_1^i(P^*) : 1 \le i \le i^* - 1\}$. This part has $(n')^2(i^* - 1)$ rankings and is mainly used to guarantee that $O[1, i^* - 1]$ are removed in the first $i^* - 1$ rounds.
- P^2 consists of $\left\lfloor \frac{n - (n')^2(i^* - 1)}{n'} \right\rfloor$ copies of P^*. This part guarantees that i^* is the Condorcet winner. We require n to be sufficiently large so that $\lfloor \frac{n - (n')^2(i^* - 1)}{n'} \rfloor > n'$.
- P^3 consists of $n - |P_1| - |P_2|$ copies of $[O[m] \succ O[m-1] \succ \cdots \succ O[1]]$, which guarantees that $|P| = n$. Note that the number of rankings in this part is no more than n'.

Let $N=(n')^2$. For any $n>N$, notice that the second part has at least n' copies of P^*, where i^* is the Condorcet winner. Therefore, i^* is the Condorcet winner under P. It is not hard to verify that $O[1,i^*-1]$ are removed in the first i^*-1 rounds under $\overline{r}$, and in the i^*-th round alternative i^* is unique $\overline{r}_{m+1-i^*}$ loser, which means that $i^*\notin\overline{r}(P)$. This concludes the proof of Claim 11. □

We prove the following claim to simplify Closure($\mathcal{R}^{\overline{r}}_{\text{CWW}}$) and Closure($\mathcal{R}^{\overline{r}}_{\text{CWL}}$).

Claim 12. *For any int-MRSE $\overline{r}$ and any $\pi\in CH(\Pi)$,*

$$\left[\pi\in \mathit{Closure}(\mathcal{R}^{\overline{r}}_{\mathit{CWW}})\right]\Leftrightarrow[\mathit{WCW}(\pi)\cap\overline{r}(\pi)\neq\emptyset]$$
$$\left[\pi\in \mathit{Closure}(\mathcal{R}^{\overline{r}}_{\mathit{CWL}})\right]\Leftrightarrow[\exists a\in \mathit{WCW}(\pi)\text{ and }i^*\in \mathit{LR}_{\overline{r}}(\pi,a)\text{ s.t. }\mathit{CL}(\overline{r}_{m+1-i^*})=0]$$

Proof. The proof for the $\mathcal{R}^{\overline{r}}_{\text{CWW}}$ part is similar to the proof of Claim 9. We present the formal proof below for completeness.

$\left[\boldsymbol{\pi\in}\textbf{ Closure}\boldsymbol{(\mathcal{R}^{\overline{r}}_{\text{CWW}})}\right]\ \boldsymbol{\Rightarrow}\ \textbf{[WCW}\boldsymbol{(\pi)\cap\overline{r}(\pi)\neq\emptyset}\textbf{]}$**.** Suppose $\pi\in$ Closure($\mathcal{R}^{\overline{r}}_{\text{CWW}}$), which means that exists a sequence $(\vec{x}_1,x_2,\ldots)$ in $\mathcal{R}^{\overline{r}}_{\text{CWW}}$ that converges to π. It follows that there exists an alternative $a\in\mathcal{A}$ and a subsequence of $(\vec{x}_1,\vec{x}_2,\ldots)$, denoted by $(\vec{x}'_1,x'_2,\ldots)$, and $O\in\mathcal{L}(\mathcal{A})$ where $O[m]=a$, such that for every $j\in\mathbb{N}$, $\text{CW}(\vec{x}'_j)=\{a\}$ and $O\in\text{PU}_{\overline{r}}(\vec{x}'_j)$. This means that the following holds.

- a is a weak Condorcet winner under π.
- $a\in\overline{r}(\pi)$. More precisely, $O\in\text{PU}_{\overline{r}}(\pi)$. To see this, recall that $O\in\text{PU}_{\overline{r}}(\vec{x}'_j)$ is equivalent to
$$\forall 2\le i\le m, O[i]\in\arg\min\nolimits_b \text{Score}_{\overline{r}_i}(\vec{x}'_j|_{O[i,m]},b)$$
Therefore, the same relationship holds for π, namely
$$\forall 2\le i\le m, O[i]\in\arg\min\nolimits_b \text{Score}_{\overline{r}_i}(\pi|_{O[i,m]},b),$$
which means that $O\in\text{PU}_{\overline{r}}(\pi)$.

Therefore, a is a weak Condorcet winner as well as a $\overline{r}$ co-winner, which implies that $\text{WCW}(\pi)\cap\overline{r}(\pi)\neq\emptyset$.

$\left[\boldsymbol{\pi\in}\textbf{ Closure}\boldsymbol{(\mathcal{R}^{\overline{r}}_{\text{CWW}})}\right]\ \boldsymbol{\Leftarrow}\ \textbf{[WCW}\boldsymbol{(\pi)\cap\overline{r}(\pi)\neq\emptyset}\textbf{]}$**.** Suppose $\text{WCW}(\pi)\cap\overline{r}(\pi)\neq\emptyset$ and let $a\in\text{WCW}(\pi)\cap\overline{r}(\pi)$. We will explicitly construct a sequence of vectors in $\mathcal{R}^{\overline{r}}_{\text{CWW}}$ that converges to π. Because $a\in\overline{r}(\pi)$, there exists a parallel universe $O\in\text{PU}_{\overline{r}}(\pi)$ such that $O[m]=a$. Let $\vec{x}=-\text{Hist}(\{O\})$, i.e. we will use "negative" O to break ties, so that for every $1\le i\le m-1$, $O[i]$ is eliminated in round i. For any $\delta>0$, it is not hard to verify that $O\in\text{PU}_{\overline{r}}(\pi+\delta\vec{x})$. In fact, $\text{PU}_{\overline{r}}(\pi+\delta\vec{x})=\{O\}$, i.e.
$$\forall 2\le i\le m, \{O[i]\}=\arg\min\nolimits_b \text{Score}_{\overline{r}_i}((\pi+\delta\vec{x})|_{O[i,m]},b),$$
which means that $\{a\}=\overline{r}(\pi+\delta\vec{x})$. Notice that a is the Condorcet winner under $\pi+\delta\vec{x}$ for any sufficiently small $\delta>0$. Therefore, for any sufficiently small $\delta>0$ we have $\pi+\delta\vec{x}\in\mathcal{R}^{\overline{r}}_{\text{CWW}}$. Because the sequence $(\pi+\vec{x},\pi+\frac{1}{2}\vec{x},\ldots)$ in $\mathcal{R}^{\overline{r}}_{\text{CWW}}$ converges to π, we have $\pi\in\text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWW}})$.

$\left[\boldsymbol{\pi\in}\textbf{ Closure}\boldsymbol{(\mathcal{R}^{\overline{r}}_{\text{CWL}})}\right]\ \boldsymbol{\Rightarrow}\ \boldsymbol{[\exists a\in}\textbf{ WCW}\boldsymbol{(\pi)}\textbf{ and }\boldsymbol{i^*\in}\textbf{ LR}_{\overline{r}}\boldsymbol{(\pi,a)}\textbf{ s.t. CL}\boldsymbol{(\overline{r}_{m+1-i^*})=0]}$**.** Suppose $\pi\in\text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWL}})$, which means that there exists a sequence $(\vec{x}_1,\vec{x}_2,\ldots)$ in $\mathcal{R}^{\overline{r}}_{\text{CWL}}$ that converges to π. It follows that there exists $a\in\mathcal{A}$, $O\in\mathcal{L}(\mathcal{A})$ with $O[m]\neq a$, and a subsequence of $(\vec{x}_1,\vec{x}_2,\ldots)$, denoted by $(\vec{x}'_1,\vec{x}'_2,\ldots)$ such that for every $j\in\mathbb{N}$, $\text{CW}(\vec{x}'_j)=\{a\}$ and $O\in\text{PU}_{\overline{r}}(\vec{x}'_j)$. Let $i^*=O^{-1}[a]$, i.e. i^* is the round where a loses in the parallel universe O, which means that for every $j\in\mathbb{N}$,
$$a\in\arg\min\nolimits_b \text{Score}_{\overline{r}_{m+1-i^*}}(\vec{x}'_j|_{O[i^*,m]},b).$$
Notice that a is the Condorcet winner among $O[i^*,m]$. This means that $\overline{r}_{m+1-i^*}$ does not satisfy the Condorcet loser criterion, because for any sufficiently large $\psi>0$, a is the Condorcet loser as well as a co-winner in $\psi\cdot\text{Hist}(O[i^*,m])-\vec{x}'_j|_{O[i^*,m]}$. Because $(\vec{x}'_1,\vec{x}'_2,\ldots)$ converges to π, it is not hard to verify that $a\in\text{WCW}(\pi)$ and $O\in\text{PU}_{\overline{r}}(\pi)$. Therefore, we have $a\in\text{WCW}(\pi)$, $i^*\in\text{LR}_{\overline{r}}(\pi,a)$, and $\text{CL}(\overline{r}_{m+1-i^*})=0$.

$\left[\boldsymbol{\pi \in \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWL}})}\right] \Leftarrow \left[\boldsymbol{\exists a \in \text{WCW}(\pi) \text{ and } i^* \in \text{LR}_{\overline{r}}(\pi, a) \text{ s.t. } \text{CL}(\overline{r}_{m+1-i^*}) = 0}\right]$**.** Let $a \in \text{WCW}(\pi)$ and $i^* \in \text{LR}_{\overline{r}}(\pi, a)$ such that $\text{CL}(\overline{r}) = 0$. Furthermore, we let $O^* \in \text{PU}_{\overline{r}}(\pi)$ denote the parallel universe such that $O[i^*] = a$. Because $\overline{r}_{m+1-i^*}$ does not satisfy the Condorcet loser criterion, there exists profile P_a over $O[i^*, m]$ where a is the Condorcet loser but $a \in \overline{r}_{m+1-i^*}(P_a)$. In fact, there exists a profile P^*_a where a is the Condorcet loser but $\{a\} = \overline{r}_{m+1-i^*}(P^*)$, i.e. a is the unique winner under P^*_a. To see this, let σ denote an arbitrary cyclic permutation among $O[i^* + 1, m]$, and let

$$P = \{\sigma^i(a \succ O[i^* + 1, m]) : 1 \le i \le m - i^*\}$$

It is not hard to verify that the score of a is strictly larger than the score of any other alternative under P. Therefore, when $\delta > 0$ is sufficiently small, a is the Condorcet loser as well as the unique winner under $P^*_a = P_a + \delta \cdot P$. Now, we define a profile P' over $\mathcal{A}$ by stacking $O[1, i^* - 1]$ on top of each (fractional) ranking in P^*_a. In other words, a ranking $[O[1] \succ \cdots \succ O[i^* - 1] \succ R^*]$ is in P' if and only if $R^* \in P^*_a$, and the two rankings have the same weights (in P' and P^*_a, respectively).

Let $\vec{x} = -\text{Hist}(P')$. It is not hard to verify that for any $\delta > 0$, a is the Condorcet winner under $\pi + \delta\vec{x}$ and in the first i^* rounds of the execution of $\overline{r}$, $O[1], O[2], \ldots, O[i^*]$ are eliminated in order. In particular, $O[i^*] = a$ is eliminated in the i^*-th round, which means that $a \notin \overline{r}(\pi + \delta\vec{x})$. Consequently, $\pi + \delta\vec{x} \in \mathcal{R}^{\overline{r}}_{\text{CWL}}$. Notice that $(\pi + \frac{1}{j}\vec{x} : j \in \mathbb{N})$ is a sequence in $\mathcal{R}^{\overline{r}}_{\text{CWL}}$ that converges to π, which means that $\pi \in \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWL}})$. This proves Claim 12. □

We now apply Claim 12 to simplify the conditions in Lemma 2.

- $\text{C}_{\text{RS}}(\overline{r}, \pi)$. By definition, this holds if and only if $\pi \notin \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWL}})$, which is equivalent to $\nexists a \in \text{WCW}(\pi)$ and $i^* \in \text{LR}_{\overline{r}}(\pi, a)$ s.t. $\text{CL}(\overline{r}_{m+1-i^*}) = 0$. In other words, for all $a \in \text{WCW}(\pi)$ and all $i^* \in \text{LR}_{\overline{r}}(\pi, a)$, $\overline{r}_{m+1-i^*}$ satisfies CONDORCET LOSER, or equivalently, $\forall a \in \text{WCW}(\pi)$ and $\forall i^* \in \text{LR}_{\overline{r}}(\pi, a), \text{CL}(\overline{r}_{m+1-i^*}) = 1$.
- $\text{C}_{\text{NRS}}(\overline{r}, \pi)$. By definition, this holds if and only if $\text{ACW}(\pi) \neq \emptyset$ and $\pi \notin \text{Closure}(\mathcal{R}^{\overline{r}}_{\text{CWW}})$, which is equivalent to $\text{ACW}(\pi) \neq \emptyset$ and $\text{WCW}(\pi) \cap \overline{r}(\pi) = \emptyset$. The latter is equivalent to $\text{WCW}(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi)) = \text{WCW}(\pi)$. We note that when $\text{ACW}(\pi) \neq \emptyset$, we have $\text{WCW}(\pi) = \text{ACW}(\pi)$. Therefore, $\text{C}_{\text{NRS}}(\overline{r}, \pi)$ is equivalent to $|\text{ACW}(\pi) \cap (\mathcal{A} \setminus \overline{r}(\pi))| = 2$.

Theorem 2 follows after Lemma 2 with the simplified conditions discussed above. □

F Materials for Section 3: Smoothed PARTICIPATION

F.1 Lemma 3 and Its Proof

We first introduce some notation to present the theorem.

Definition 27 (**$\oplus$ operator**)**.** *For any pair of signatures $\vec{t}_1, \vec{t}_2 \in \mathcal{S}_K$, we define $\vec{t}_1 \oplus \vec{t}_2$ to be the following signature:*

$$\forall k \le K, [\vec{t}_1 \oplus \vec{t}_2]_k = \begin{cases} [\vec{t}_1]_k & \text{if } [\vec{t}_1]_k = [\vec{t}_2]_k \\ 0 & \text{otherwise} \end{cases}$$

For example, when $K = 3$, $\vec{t}_1 = (+, -, 0)$, and $\vec{t}_2 = (+, 0, 0)$, we have $\vec{t}_1 \oplus \vec{t}_2 = (+, 0, 0)$. By definition, we have $\vec{t}_1 \trianglelefteq \vec{t}_1 \oplus \vec{t}_2$ and $\vec{t}_2 \trianglelefteq \vec{t}_1 \oplus \vec{t}_2$.

Definition 28 (**$\text{Vio}^r_{\text{PAR}}(n)$ and ℓ_n**)**.** *For any GSR r and any $n \in \mathbb{N}$, we define*

$$\text{Vio}^r_{\text{PAR}}(n) = \left\{\text{Sign}_{\vec{H}}(P) \oplus \text{Sign}_{\vec{H}}(P \setminus \{R\}) : P \in \mathcal{L}(\mathcal{A})^n, R \in \mathcal{L}(\mathcal{A}), r(P \setminus \{R\}) \succ_R r(P)\right\}$$

$$\ell_n = m! - \max_{\vec{t} \in \text{Vio}^r_{\text{PAR}}(n) : \exists \pi \in \text{CH}(\Pi), \text{ s.t. } \vec{t} \trianglelefteq \text{Sign}_{\vec{H}}(\pi)} \dim(\mathcal{H}^{\vec{t}}_{\le 0})$$

In words, $\text{Vio}^r_{\text{PAR}}(n)$ consists of all signatures $\vec{t}$ that is obtained by combining two feasible signatures, i.e., $\text{Sign}_{\vec{H}}(P)$ and $\text{Sign}_{\vec{H}}(P \setminus \{R\})$, by the $\oplus$ operator, where P and R constitutes an violation of PAR. Notice that $r(P \setminus \{R\}) \succ_R r(P)$ implicitly assumes that P contains an R vote. Then, ℓ_n is defined to be $m!$ minus the maximum dimension of polyhedron $\mathcal{H}^{\vec{t}}$, among all $\vec{t}$ in $\text{Vio}^r_{\text{PAR}}(n)$ that refines $\text{Sign}_{\vec{H}}(\pi)$ for some $\pi \in \text{CH}(\Pi)$.

Lemma 3 (**Smoothed** PAR**: Int-GSR**)**.** *Let $\mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi)$ be a strictly positive and closed single-agent preference model, let r be an int-GSR. For any $n \in \mathbb{N}$,*

$$\widetilde{\text{PAR}}^{\min}_{\Pi}(r, n) = \begin{cases} 1 & \textit{if } \textit{Vio}^r_{\text{PAR}}(n) = \emptyset \\ 1 - \exp(-\Theta(n)) & \textit{otherwise if } \forall \pi \in \textit{CH}(\Pi) \textit{ and } \vec{t} \in \textit{Vio}^r_{\text{PAR}}(n), \vec{t} \ntrianglelefteq \textit{Sign}_{\vec{H}}(\pi) \\ 1 - \Theta(n^{-\ell_n/2}) & \textit{otherwise, i.e. } \exists \pi \in \textit{CH}(\Pi) \textit{ and } \vec{t} \in \textit{Vio}^r_{\text{PAR}}(n) \textit{ s.t. } \vec{t} \trianglelefteq \textit{Sign}_{\vec{H}}(\pi) \end{cases}$$

Applying Lemma 3 to a voting rule r often involves the following steps. First, we choose an GSR representation of r by specifying the $\vec{H}$ and g, though according to Lemma 3 the asymptotic bound does not depend on such choice. Second, we characterize $\text{Vio}^r_{\text{PAR}}(n)$ and verify whether it is empty. If $\text{Vio}^r_{\text{PAR}}(n)$ is empty then the 1 case holds. Third, if $\text{Vio}^r_{\text{PAR}}(n)$ is non-empty but none of $\vec{t} \in \text{Vio}^r_{\text{PAR}}(n)$ refines $\text{Sign}_{\vec{H}}(\pi)$ for any $\pi \in \text{CH}(\Pi)$, then the VL case holds. Finally, if neither 1 nor VL case holds, then the L case holds, where the degree of polynomial depends on ℓ_n. Characterizing $\text{Vio}^r_{\text{PAR}}(n)$ and ℓ_n can be highly challenging, as it aims at summarizing all violations of PAR for n-profiles (using signatures under $\vec{H}$).

Proof. The high-level idea of the proof is similar to the proof of Lemma 2. In light of Lemma 1, the proof proceeds in the following three steps. **Step 1.** Define $\mathcal{C}$ that characterizes the satisfaction of PARTICIPATION of r, and an almost complement $\mathcal{C}^*$ of $\mathcal{C}$. **Step 2.** Characterize possible values of α^*_n and their conditions, and then notice that α^*_n is at most $m! - 1$, which means that only the 1, VL, or L case in Lemma 1 hold. This means that the value of β_n does not matter. **Step 3.** Apply Lemma 1.

Step 1. Given two feasible signatures $\vec{t}_1, \vec{t}_2 \in \mathcal{S}_{\vec{H}}$ and a ranking $R \in \mathcal{L}(\mathcal{A})$, we first formally define a polyhedron $\mathcal{H}^{\vec{t}_1, R, \vec{t}_2}$ to characterize the profiles whose signature is $\vec{t}_1$ and after removing a voter whose preferences are R, the signature of the new profile becomes $\vec{t}_2$.

Definition 29 ($\boldsymbol{\mathcal{H}^{\vec{t}_1, R, \vec{t}_2}}$)**.** *Given $\vec{H} = (\vec{h}_1, \ldots, \vec{h}_K) \in (\mathbb{Z}^d)^K$, $\vec{t}_1, \vec{t}_2 \in \mathcal{S}_{\vec{H}}$, and $R \in \mathcal{L}(\mathcal{A})$, we let*

$$\mathbf{A}^{\vec{t}_1, R, \vec{t}_2} = \begin{bmatrix} -\textit{Hist}(R) \\ \mathbf{A}^{\vec{t}_1} \\ \mathbf{A}^{\vec{t}_2} \end{bmatrix}, \ \vec{\mathbf{b}}^{\vec{t}_1, R, \vec{t}_2} = [-1, \underbrace{\vec{\mathbf{b}}^{\vec{t}_1}}_{\textit{for } \mathbf{A}^{\vec{t}_1}}, \underbrace{\vec{\mathbf{b}}^{\vec{t}_2} + \textit{Hist}(R) \cdot \mathbf{A}^{\vec{t}_2}}_{\textit{for } \mathbf{A}^{\vec{t}_2}}] \textit{ and}$$

$$\mathcal{H}^{\vec{t}_1, R, \vec{t}_2} = \{\vec{x} \in \mathbb{R}^{m!} : \mathbf{A}^{\vec{t}_1, R, \vec{t}_2} \cdot (\vec{x})^\top \leq \left(\vec{\mathbf{b}}^{\vec{t}_1, R, \vec{t}_2}\right)^\top\}$$

Notice that $\text{Hist}(R) \in \{0, 1\}^{m!}$ is the vector whose R-component is 1 and all other components are 0's. The $\mathbf{A}^{\vec{t}_2}$ part in Definition 29 is equivalent to $\mathbf{A}^{\vec{t}_2} \cdot (\vec{x} - \text{Hist}(R))^\top \leq \left(\vec{\mathbf{b}}^{\vec{t}_2}\right)^\top$. We prove properties of $\mathcal{H}^{\vec{t}_1, R, \vec{t}_2}$ in the following claim.

Claim 13 (**Properties of** $\boldsymbol{\mathcal{H}^{\vec{t}_1, R, \vec{t}_2}}$)**.** *Given integer $\vec{H}$. For any $\vec{t}_1, \vec{t}_2 \in \mathcal{S}_{\vec{H}}$, any $R \in \mathcal{L}(\mathcal{A})$,*

(i) for any integral profile P, $\textit{Hist}(P) \in \mathcal{H}^{\vec{t}_1, R, \vec{t}_2}$ if and only if $\textit{Sign}_{\vec{H}}(P) = \vec{t}_1$ and $\textit{Sign}_{\vec{H}}(P \setminus \{R\}) = \vec{t}_2$;

(ii) for any $\vec{x} \in \mathbb{R}^{m!}_{\geq 0}$, $\vec{x} \in \mathcal{H}^{\vec{t}_1, R, \vec{t}_2}_{\leq 0}$ if and only if $\vec{t}_1 \oplus \vec{t}_2 \trianglelefteq \textit{Sign}_{\vec{H}}(\vec{x})$;

(iii) If there exists $\vec{x} \in \mathcal{H}^{\vec{t}_1, R, \vec{t}_2}_{\leq 0}$ such that $[\vec{x}]_R > 0$, then $\dim(\mathcal{H}^{\vec{t}_1, R, \vec{t}_2}_{\leq 0}) = \dim(\mathcal{H}^{\vec{t}_1 \oplus \vec{t}_2}_{\leq 0})$. Moreover, if $\vec{t}_1 \neq \vec{t}_2$ and $\mathcal{H}^{\vec{t}_1, R, \vec{t}_2}_{\leq 0} \neq \emptyset$, then $\dim(\mathcal{H}^{\vec{t}_1, R, \vec{t}_2}_{\leq 0}) \leq m! - 1$.

Proof. Part (i) follows after the definition. Part (ii) also follows after the definition. Recall that $\vec{x} \in \mathcal{H}^{\vec{t}_1, R, \vec{t}_2}_{\leq 0}$ if and only if $\mathbf{A}^{\vec{t}_1} \cdot (\vec{x})^\top \leq \left(\vec{0}\right)^\top$, $\mathbf{A}^{\vec{t}_2} \cdot (\vec{x})^\top \leq \left(\vec{0}\right)^\top$, and the R component of $\vec{x}$ is non-negative, which is automatically satisfied for every $\vec{x} \in \mathbb{R}^{m!}_{\geq 0}$. The first sets of inequalities holds if and only if $\mathbf{A}^{\vec{t}_1 \oplus \vec{t}_2} \cdot (\vec{x})^\top \leq \left(\vec{0}\right)^\top$.

To prove the first part of Part (iii), let $\mathbf{A}_1^+$ and $\mathbf{A}_2^+$ denote the essential equalities of $\mathbf{A}^{\vec{t}_1,R,\vec{t}_2}$ and $\mathbf{A}^{\vec{t}_1\oplus\vec{t}_2}$, respectively. We show that $\mathbf{A}_1^+$ and $\mathbf{A}_2^+$ contains the same set of row vectors (while some rows may appear different number of times in $\mathbf{A}_1^+$ and $\mathbf{A}_2^+$). As noted in the proof of Part (ii), the set of row vectors in $\mathbf{A}^{\vec{t}_1,R,\vec{t}_2}$ is the same as the set of row vectors in $\mathbf{A}^{\vec{t}_1\oplus\vec{t}_2}$, except that the former contains $-\text{Hist}(R)$. Recall that we have assumed that there exists $\vec{x} \in \mathcal{H}_{\leq 0}^{\vec{t}_1,R,\vec{t}_2}$ such that $[\vec{x}]_R > 0$, which means that $-\text{Hist}(R) \cdot (\vec{x})^\top$ does not hold for every vector in $\mathcal{H}_{\leq 0}^{\vec{t}_1,R,\vec{t}_2}$. Therefore, $-\text{Hist}(R)$ is not a row in $\mathbf{A}_1^+$, which means that $\mathbf{A}_1^+$ and $\mathbf{A}_2^+$ contains the same set of row vectors. Then, we have

$$\dim(\mathcal{H}_{\leq 0}^{\vec{t}_1,R,\vec{t}_2}) = m! - \text{Rank}(\mathbf{A}_1^+) = m! - \text{Rank}(\mathbf{A}_2^+) = \dim(\mathcal{H}_{\leq 0}^{\vec{t}_1\oplus\vec{t}_2})$$

The second part of Part (iii) is proved by noticing that when $\vec{t}_1 \neq \vec{t}_2$, $\vec{t}_1 \oplus \vec{t}_2$ contains at least one 0. Suppose $[\vec{t}_1 \oplus \vec{t}_2]_k = 0$. This means that for all $\vec{x} \in \mathcal{H}_{\leq 0}^{\vec{t}_1,R,\vec{t}_2}$, we have $\vec{h}_k \cdot \vec{x} = 0$, which means that $\dim(\mathcal{H}_{\leq 0}^{\vec{t}_1,R,\vec{t}_2}) \leq m! - 1$. $\square$

We now use $\mathcal{H}^{\vec{t}_1,R,\vec{t}_2}$ to define $\mathcal{C}$ and $\mathcal{C}^*$.

Definition 30 (**$\mathcal{C}$ and $\mathcal{C}^*$ for** PARTICIPATION)**.** *Given an int-GSR r characterized by $\vec{H}$ and g, we define*

$$\mathcal{C} = \bigcup\nolimits_{\vec{t}_1,\vec{t}_2\in\mathcal{S}_{\vec{H}},R\in\mathcal{L}(\mathcal{A}):r(\vec{t}_1)\succeq_R r(\vec{t}_2)} \mathcal{H}^{\vec{t}_1,R,\vec{t}_2}$$
$$\mathcal{C}^* = \bigcup\nolimits_{\vec{t}_1,\vec{t}_2\in\mathcal{S}_{\vec{H}},R\in\mathcal{L}(\mathcal{A}):r(\vec{t}_1)\prec_R r(\vec{t}_2)} \mathcal{H}^{\vec{t}_1,R,\vec{t}_2}$$

In words, $\mathcal{C}$ consists of polyhedra $\mathcal{H}^{\vec{t}_1,R,\vec{t}_2}$ that characterize the histograms of profiles such that after any R-vote is removed, the winner under r is not improved w.r.t. R. $\mathcal{C}^*$ consists of polyhedra $\mathcal{H}^{\vec{t}_1,R,\vec{t}_2}$ that characterize the histograms of profiles such that after removing an R-vote, the winner under r is strictly improved w.r.t. R. It is not hard to see that $\mathcal{C}^*$ is an almost complement of $\mathcal{C}$.

It follows from Claim 13 (i) that for any n-profile P, PAR is satisfied (respectively, dissatisfied) at P if and only if $\text{Hist}(P) \in \mathcal{C}$ (respectively, $\text{Hist}(P) \in \mathcal{C}^*$).

Step 2: Characterize α_n^*. In this step we discuss the values and conditions for α_n^* (for $\mathcal{C}^*$) in the following three cases.

$\alpha_n^* = -\infty$. This case holds if and only if PAR holds for all n-profiles, which is equivalent to $\text{Vio}_{\text{PAR}}^r(n) = \emptyset$.

$\alpha_n^* = -\frac{n}{\log n}$. This case holds if and only if (1) PAR is not satisfied at all n-profiles, which is equivalent to $\text{Vio}_{\text{PAR}}^r(n) \neq \emptyset$, and (2) the activation graph $\mathcal{G}_{\Pi,\mathcal{C}^*,n}$ does not contain any non-negative edges, which is equivalent to $\forall \pi \in \text{CH}(\Pi)$ and $\forall \mathcal{H}^{\vec{t}_1,R,\vec{t}_2} \subseteq \mathcal{C}^*$ that is active at n, we have $\pi \notin \mathcal{H}_{\leq 0}^{\vec{t}_1,R,\vec{t}_2}$. We will prove that part (2) is equivalent to the following:

$$(2) \iff \left[\forall \pi \in \text{CH}(\Pi) \text{ and } \vec{t} \in \text{Vio}_{\text{PAR}}^r(n), \vec{t} \not\trianglelefteq \text{Sign}_{\vec{H}}(\pi)\right] \tag{13}$$

We first prove the "$\Rightarrow$" direction of (13). Suppose for the sake of contradiction that this is not true. That is, $\mathcal{G}_{\Pi,\mathcal{C}^*,n}$ does not contain any non-negative edges and there exist $\pi \in \text{CH}(\Pi)$ and $\vec{t} \in \text{Vio}_{\text{PAR}}^r(n)$ such that $\vec{t} \not\trianglelefteq \text{Sign}_{\vec{H}}(\pi)$. Let P denote the n-profile such that $\text{Sign}_{\vec{H}}(P) = \vec{t}_1$, $\text{Sign}_{\vec{H}}(P \setminus \{R\}) = \vec{t}_2$, $r(P \setminus \{R\}) \succ_R r(P)$, and $\vec{t} = \vec{t}_1 \oplus \vec{t}_2$. By Claim 13 (i), $\text{Hist}(P) \in \mathcal{H}^{\vec{t}_1,R,\vec{t}_2}$, which means that $\mathcal{H}^{\vec{t}_1,R,\vec{t}_2}$ is active at n. By Claim 13 (ii), $\text{Hist}(P) \in \mathcal{H}_{\leq 0}^{\vec{t}_1,R,\vec{t}_2}$. These imply that the weight on the edge $(\pi, \mathcal{H}^{\vec{t}_1,R,\vec{t}_2})$ in $\mathcal{G}_{\Pi,\mathcal{C}^*,n}$ is non-negative (whose weight is $\dim(\mathcal{H}_{\leq 0}^{\vec{t}_1,R,\vec{t}_2})$), which contradicts the assumption that (2) holds.

Next, we prove the "$\Leftarrow$" direction of (13). Suppose for the sake of contradiction that (2) does not hold, which means that there exists an edge $(\pi, \mathcal{H}^{\vec{t}_1,R,\vec{t}_2})$ in $\mathcal{G}_{\Pi,\mathcal{C}^*,n}$ whose weight is non-negative.

Equivalently, $\mathcal{H}^{\vec{t}_1,R,\vec{t}_2}$ is active at n and $\pi \in \mathcal{H}^{\vec{t}_1,R,\vec{t}_2}_{\leq 0}$. By Claim 13 (ii), $\vec{t}_1 \oplus \vec{t}_2 \in \text{Vio}^r_{\text{PAR}}(n)$. Recall that π is strictly positive, and then by Claim 13 (ii), we have $t_1 \oplus \vec{t}_2 \trianglelefteq \text{Sign}_{\vec{H}}(\pi)$. However, this contradict the assumption.

These prove (13).

$\alpha^*_n > 0$. For this case, we prove

$$\alpha^*_n = \max_{\vec{t} \in \text{Vio}^r_{\text{PAR}}(n): \exists \pi \in \text{CH}(\Pi), \text{ s.t. } \vec{t} \trianglelefteq \text{Sign}_{\vec{H}}(\pi)} \dim(\mathcal{H}^{\vec{t}}_{\leq 0}), \tag{14}$$

We first prove the "$\leq$" direction in (14). For any edge $(\pi, \mathcal{H}^{\vec{t}_1,R,\vec{t}_2})$ in $\mathcal{G}_{\Pi,\mathcal{C}^*,n}$ whose weight is non-negative, $\mathcal{H}^{\vec{t}_1,R,\vec{t}_2}$ is active at n. Therefore, there exists an n-profile P such that $\text{Hist}(P) \in \mathcal{H}^{\vec{t}_1,R,\vec{t}_2}$. Let $\vec{t} = \vec{t}_1 \oplus \vec{t}_2$. We have $\vec{t} \in \text{Vio}^r_{\text{PAR}}(n)$. By Claim 13 (ii), we have $\vec{t} \trianglelefteq \text{Sign}_{\vec{H}}(\pi)$. By Claim 13 (iii), we have $\dim(\mathcal{H}^{\vec{t}_1,R,\vec{t}_2}_{\leq 0}) = \dim(\mathcal{H}^{\vec{t}}_{\leq 0})$. Therefore, the "$\leq$" direction in (14) holds.

Next, we prove the $\geq$ direction of (14). For any $\vec{t} \in \text{Vio}^r_{\text{PAR}}(n)$ and $\pi \in \text{CH}(\Pi)$ such that $\vec{t} \trianglelefteq \text{Sign}_{\vec{H}}(\pi)$, let P denote an n-profile and let R denote a ranking that justify $\vec{t}$'s membership in $\text{Vio}^r_{\text{PAR}}(n)$, and let $\vec{t}_1 = \text{Sign}_{\vec{H}}(P)$ and $\vec{t}_2 = \text{Sign}_{\vec{H}}(P \setminus \{R\})$, which means that $\vec{t} = \vec{t}_1 \oplus \vec{t}_2$. By Claim 13 (i), $\text{Hist}(P) \in \mathcal{H}^{\vec{t}_1,R,\vec{t}_2} \subseteq \mathcal{C}^*$, which means that $\mathcal{H}^{\vec{t}_1,R,\vec{t}_2}$ is active at n. By Claim 13 (ii), $\pi \in \mathcal{H}^{\vec{t}_1,R,\vec{t}_2}_{\leq 0}$. By Claim 13 (iii), $\dim(\mathcal{H}^{\vec{t}_1,R,\vec{t}_2}_{\leq 0}) = \dim(\mathcal{H}^{\vec{t}}_{\leq 0})$. This means that the weight on the edge $(\pi, \mathcal{H}^{\vec{t}_1,R,\vec{t}_2})$ in $\mathcal{G}_{\Pi,\mathcal{C}^*,n}$ is $\dim(\mathcal{H}^{\vec{t}}_{\leq 0})$, which implies the "$\geq$" direction in (14) holds.

Therefore, (14) holds. Notice that by Claim 13 (iii), $\alpha^*_n \leq m! - 1$.

Step 3: Applying Lemma 1. Lemma 3 follows after a straightforward application of Lemma 1 and Step 2. Notice that $\Pi_{\mathcal{C},n}$ and β_n are irrelevant in this proof because only the 1, $1 - \exp(n)$, and $1 - \mathcal{H}(n)$ cases will happen. This completes the proof of Lemma 3. □

F.2 Proof of Theorem 3

Recall from Definition 9 that an EO-based rule is determined by the total preorder over edges in WMG w.r.t. their weights. Theorem 3 characterizes smoothed PAR for any EO-based int-GSR refinements of maximin, Ranked Pairs, and Schulze.

Theorem 3 (Smoothed PAR: maximin, Ranked Pairs, Schulze)**.** *For any $m \geq 4$, any EO-based int-GSR r that is a refinement of maximin, STV, Schulze, or ranked Pairs, and any strictly positive and closed Π over $\mathcal{L}(\mathcal{A})$ with $\pi_{uni} \in CH(\Pi)$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,*

$$\widetilde{\text{PAR}}^{\min}_{\Pi}(r, n) = 1 - \Theta(\frac{1}{\sqrt{n}})$$

Proof. Because r is EO-based, w.l.o.g., we assume that its int-GSR representation uses $\vec{H}_{\text{EO}}$ (Definition 11).

Overview. The proof is done by applying Lemma 3 to show that for any sufficiently large n, the 1 case and the VL case do not happen, and $\ell_n = 1$ in the L case. This is done by explicitly constructing an n-profile P, under which PAR is violated when an R-vote is removed (which means that $\vec{t} = \text{Sign}_{\vec{H}_{\text{EO}}}(P) \oplus \text{Sign}_{\vec{H}_{\text{EO}}}(P \setminus \{R\}) \in \text{Vio}^r_{\text{PAR}}(n)$ and therefore the 1 case does not hold), then show that $\vec{t} \trianglelefteq \pi_{\text{uni}}$, or more generally, any signature refines $\text{Sign}_{\vec{H}_{\text{EO}}}(\pi_{\text{uni}})$ (which means that the VL case does not hold), and finally prove that $\dim(\mathcal{H}^{\vec{t}}_{\leq 0}) = m! - 1$, which means that $\ell_n = 1$.

Maximin: r refines $\overline{\text{MM}}$. We first prove the proposition for $2 \nmid n$, then show how to modify the proof for $2 \mid n$. As mentioned in the overview, the proof proceeds in the following steps.

Constructing P_{MM} and R_{MM} that violates PAR. Let G_{MM} denote the following weighted directed graph with weights w_{MM}, where the weights on all edges are odd and different, except on $4 \to 1$ and $3 \to 2$.

- $w_{\text{MM}}(4,1) = w_{\text{MM}}(3,2) = 5$, $w_{\text{MM}}(1,2) = 1$, $w_{\text{MM}}(1,3) = 9$, $w_{\text{MM}}(2,4) = 13$, and $w_{\text{MM}}(3,4) = 17$;
- for every $5 \leq i \leq m$, $w_{\text{MM}}(1,i) \geq 21$, $w_{\text{MM}}(2,i) \geq 21$, $w_{\text{MM}}(3,i) \geq 21$, and $w_{\text{MM}}(4,i) \geq 21$;
- the weights on other edges are assigned arbitrarily. Moreover, the difference between any pair of edges is at least 4, except that the weights on $4 \to 1$ and $3 \to 2$ are the same.

See the middle graph in Figure 6 for an example of $m = 5$.

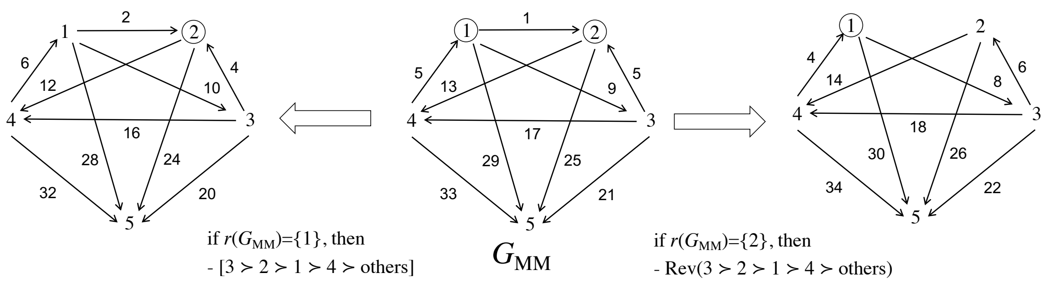


Figure 6: WMGs for minimax. $\overline{\text{MM}}$ (co)-winners are circled.

It follows from McGarvey's theorem [33] that for any $n > m^4$ and $2 \nmid n$, there exists an n-profile P_{MM} whose WMG is G_{MM}. Therefore, for any $n > m^4 + 2$ and $2 \nmid n$, there exists an n-profile P_{MM} whose WMG is G_{MM}, and P_{MM} includes the following two rankings:

$$[3 \succ 2 \succ 1 \succ 4 \succ \text{others}], \text{Rev}\,(3 \succ 2 \succ 1 \succ 4 \succ \text{others}),$$

where for any ranking R, Rev (R) denotes its reverse ranking. We now show that $\text{PAR}(r, P_{\text{MM}}) = 0$, which implies that the 1 case does not happen. Notice that the min-score of alternatives 1 and 2 are the highest, which means that $r(P_{\text{MM}}) \subseteq \{1, 2\}$.

- If $r(P_{\text{MM}}) = \{1\}$, then we let $R_{\text{MM}} = [3 \succ 2 \succ 1 \succ 4 \succ \text{others}]$. It follows that in $P_{\text{MM}} - R_{\text{MM}}$, the min-score of 2 is strictly higher than the min-score of any other alternative, which means that $r(P_{\text{MM}} \setminus \{R_{\text{MM}}\}) = \{2\}$. Notice that $2 \succ_{R_{\text{MM}}} 1$, which means that $\text{PAR}(r, P_{\text{MM}}) = 0$. See the left graph in Figure 6 for an illustration.
- If $r(P_{\text{MM}}) = \{2\}$, then we let $R_{\text{MM}} = \text{Rev}\,(3 \succ 2 \succ 1 \succ 4 \succ \text{others})$. It follows that in $P_{\text{MM}} - R_{\text{MM}}$, the min-score of 1 is strictly higher than any the min-score of other alternatives, which mean that $r(P_{\text{MM}} \setminus \{R_{\text{MM}}\}) = \{1\}$. Notice that $1 \succ_{R_{\text{MM}}} 2$, which again means that $\text{PAR}(r, P_{\text{MM}}) = 0$. See the right graph in Figure 6 for an illustration.

Let $\vec{t}_1 = \text{Sign}_{\vec{H}_{\text{EO}}}(P_{\text{MM}})$, $R = R_{\text{MM}}$ and $\vec{t}_2 = \text{Sign}_{\vec{H}_{\text{EO}}}(P_{\text{MM}} \setminus \{R_{\text{MM}}\})$. We have $\vec{t}_1 \oplus \vec{t}_2 \in \text{Vio}^r_{\text{PAR}}(n) \neq \emptyset$, which means that the 1 case of Lemma 3 does not hold. The VL case of Lemma 3 does not hold because $\vec{t}_1 \oplus \vec{t}_2 \trianglelefteq \text{Sign}_{\vec{H}_{\text{EO}}}(\pi_{\text{uni}})$ and $\pi_{\text{uni}} \in \text{CH}(\Pi)$.

Prove $\dim(\mathcal{H}^{\vec{t}_{\text{MM}}}_{\leq 0}) = m! - 1$. Let $e_1 = (4, 1)$ and $e_2 = (3, 2)$. Notice $[\vec{t}_1]_{(e_1,e_2)} = [\vec{t}_1]_{(e_2,e_1)} = 0$, where $[\vec{t}_1]_{(e_1,e_2)}$ is the (e_1, e_2) component of $\vec{t}_1$, and all other components of $\vec{t}_1$ are non-zero. Also notice that $\vec{t}_2$ is a refinement of $\vec{t}_1$. This means that $\vec{t}_1 \oplus \vec{t}_2 = \vec{t}_1$. Notice that $\text{Hist}(P_{\text{MM}})$ is an inner point of $\mathcal{H}^{\vec{t}_1}_{\leq 0}$, such that all inequalities are strict except the two inequalities about e_1 and e_2. This means that the essential equalities of $\mathbf{A}^{\vec{t}_1 \oplus \vec{t}_2}$ are equivalent to

$$(\text{Pair}_{4,1} - \text{Pair}_{3,2}) \cdot \vec{x} = \vec{0}$$

Therefore, $\dim(\mathcal{H}^{\vec{t}_1 \oplus \vec{t}_2}_{\leq 0}) = m! - 1$.

The maximin part of the proposition when $2 \nmid n$ then follows after Lemma 3. When $2 \mid n$, we only need to modify G_{MM} in Figure 6 by increasing all positive weights by 1.

Ranked Pairs: r refines $\overline{\text{RP}}$. The proof is similar to the proof of the maximin part, except that a different graph G_{RP} (with weight w_{RP}) is used, as shown in the middle graph in Figure 7. Formally, when $2 \nmid n$, let G_{RP} denote the following weighted directed graph, where the weights on all edges are odd and different, except on $4 \to 1$ and $3 \to 4$.

- $w_{\text{RP}}(4,1) = w_{\text{RP}}(3,4) = 9$, $w_{\text{RP}}(1,2) = 5$, $w_{\text{RP}}(1,3) = 13$, $w_{\text{RP}}(2,4) = 17$, and $w_{\text{RP}}(2,3) = 21$;
- for any $5 \leq i \leq m$, $w_{\text{RP}}(1,i) \geq 25$, $w_{\text{RP}}(2,i) \geq 25$, $w_{\text{RP}}(3,i) \geq 25$, and $w_{\text{RP}}(4,i) \geq 25$;
- the weights on other edges are assigned arbitrarily. Moreover, the difference between any pair of edges is at least 4, except that the weights on $4 \to 1$ and $3 \to 4$ are the same.

See the middle graph in Figure 7 for an example of $m = 5$.

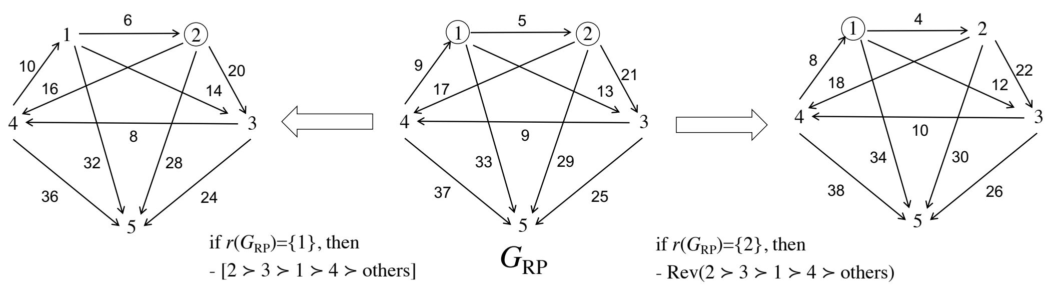


Figure 7: WMGs for ranked pairs. $\overline{\text{RP}}$ (co)-winners are circled.

Again, according to McGarvey's theorem [33] that for any $n > m^4$ and $2 \nmid n$, there exists an n-profile P_{RP} whose WMG is G_{RP}. Therefore, for any $n > m^4 + 2$ and $2 \nmid n$, there exists an n-profile P_{RP} whose WMG is G_{RP}, and P_{RP} includes the following two rankings:

$$[2 \succ 3 \succ 1 \succ 4 \succ \text{others}], \text{Rev}\,(3 \succ 2 \succ 1 \succ 4 \succ \text{others})$$

We now show that $\text{PAR}(r, P_{\text{RP}}) = 0$, which implies that the 1 case does not happen. Notice that depending on how the tie between $3 \to 4$ and $4 \to 1$ are broken, the $\overline{\text{RP}}$ winner can be 1 or 2, which means that $\overline{\text{RP}}(P_{\text{RP}}) = \{1, 2\}$.

- If $r(P_{\text{RP}}) = \{1\}$, then we let $R_{\text{RP}} = [2 \succ 3 \succ 1 \succ 4 \succ \text{others}]$. It follows that in $\text{WMG}(P_{\text{RP}} - R_{\text{RP}})$, $4 \to 1$ has higher weight than $3 \to 4$, which means that $4 \to 1$ is fixed before $3 \to 4$, and therefore $r(P_{\text{RP}} \setminus \{R_{\text{RP}}\}) = \{2\}$. Notice that $2 \succ_{R_{\text{RP}}} 1$, which means that $\text{PAR}(r, P_{\text{RP}}) = 0$. See the left graph in Figure 7 for an illustration.
- If $r(P_{\text{RP}}) = \{2\}$, then we let $R_{\text{RP}} = \text{Rev}\,(2 \succ 3 \succ 1 \succ 4 \succ \text{others})$. It follows that in $\text{WMG}(P_{\text{RP}} \setminus \{R_{\text{RP}}\})$, $3 \to 4$ has higher weight than $4 \to 1$, which means $r(P_{\text{RP}} - R_{\text{RP}}) = \{1\}$. Notice that $1 \succ_{R_{\text{RP}}} 2$, which means that $\text{PAR}(r, P_{\text{RP}}) = 0$. See the right graph in Figure 7 for an illustration.

The proof for $\ell_n = 1$ is similar to the proof for the maximin part. The only difference is that now let $e_1 = (4,1)$, $e_2 = (3,4)$, $\vec{t_1} = \text{Sign}_{\vec{H}_{\text{EO}}}(P_{\text{RP}})$, and $\vec{t_2} = \text{Sign}_{\vec{H}_{\text{EO}}}(P_{\text{RP}} \setminus \{R_{\text{RP}}\})$. When $2 \mid n$, we only need to modify G in Figure 6 (b) such that all positive weights are increased by 1.

Schulze: r refines $\overline{\text{Sch}}$. The proof is similar to the proof of the maximin part, except that a different graph G_{Sch} is used, as shown in the middle graph in Figure 8. Formally, when $2 \nmid n$, let G_{Sch} denote the following weighted directed graph, where the weights on all edges are odd and different, except on $4 \to 1$ and $2 \to 3$.

- $w_{\text{Sch}}(4,1) = w_{\text{Sch}}(2,3) = 9$, $w_{\text{Sch}}(1,2) = 13$, $w_{\text{Sch}}(1,3) = 5$, $w_{\text{Sch}}(2,4) = 1$, and $w_{\text{Sch}}(3,4) = 17$;
- for any $5 \leq i \leq m$, $w_{\text{Sch}}(1,i) \geq 21$, $w_{\text{Sch}}(2,i) \geq 21$, $w_{\text{Sch}}(3,i) \geq 21$, and $w_{\text{Sch}}(4,i) \geq 21$;

- the weights on other edges are assigned arbitrarily. Moreover, the difference between any pair of edges is at least 4, except that the weights on $4 \to 1$ and $3 \to 4$ are the same.

See the middle graph in Figure 8 for an example of $m = 5$.

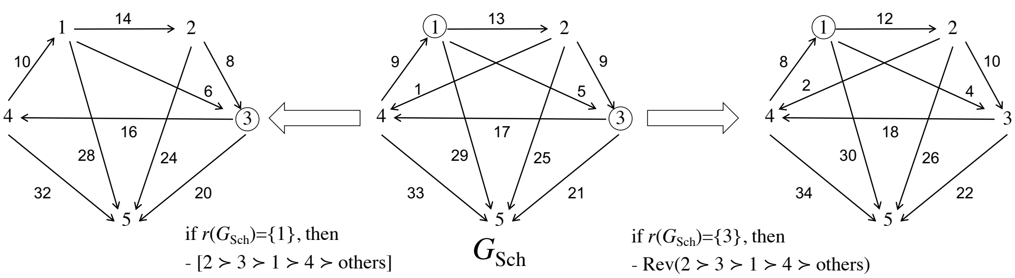


Figure 8: WMGs for Schulze. $\overline{\text{Sch}}$ (co)-winners are circled.

Again, according to McGarvey's theorem [33] that for any $n > m^4$ and $2 \nmid n$, there exists an n-profile P_{Sch} whose WMG is G_{Sch}. Therefore, for any $n > m^4 + 2$ and $2 \nmid n$, there exists an n-profile P_{Sch} whose WMG is G_{Sch} and P_{Sch} includes the following two rankings:

$$[2 \succ 3 \succ 1 \succ 4 \succ \text{others}], \text{Rev}\,(3 \succ 2 \succ 1 \succ 4 \succ \text{others})$$

We now show that $\text{PAR}(r, P_{\text{Sch}}) = 0$, which implies that the 1 case does not happen. Notice that $s[1,3] = s[3,1] = 9$, and for any alternative $a \in \mathcal{A} \setminus \{1,3\}$ we have $s[1,a] > s[a,1]$. Therefore, $\overline{\text{Sch}}(P_{\text{Sch}}) = \{1,3\}$.

- If $r(P_{\text{Sch}}) = \{1\}$, then we let $R_{\text{Sch}} = [2 \succ 3 \succ 1 \succ 4 \succ \text{others}]$. It follows that in $P_{\text{Sch}} - R_{\text{Sch}}$ we have $s[1,3] = 8 < 10 = s[3,1]$, which means that $r(P_{\text{Sch}} \setminus \{R_{\text{Sch}}\}) = \{3\}$. Notice that $3 \succ_{R_{\text{Sch}}} 1$, which means that $\text{PAR}(r, P_{\text{Sch}}) = 0$. See the left graph in Figure 8 for an illustration.
- If $r(P_{\text{Sch}}) = \{3\}$, then we let $R_{\text{Sch}} = \text{Rev}\,(2 \succ 3 \succ 1 \succ 4 \succ \text{others})$. It follows that in $P_{\text{Sch}} \setminus \{R_{\text{Sch}}\}$, we have $s[1,3] = 10 > 9 = s[3,1]$, which means that $r(P_{\text{Sch}} - R_{\text{Sch}}) = \{1\}$. Notice that $1 \succ_{R_{\text{Sch}}} 3$, which means that $\text{PAR}(r, P_{\text{Sch}}) = 0$. See the right graph in Figure 8 for an illustration.

The proof for $\ell_n = 1$ is similar to the proof for the maximin part. The only difference is that now let $e_1 = (4,1)$, $e_2 = (2,3)$, $\vec{t}_1 = \text{Sign}_{\vec{H}_{\text{EO}}}(P_{\text{Sch}})$, and $\vec{t}_2 = \text{Sign}_{\vec{H}_{\text{EO}}}(P_{\text{Sch}} \setminus \{R_{\text{Sch}}\})$. When $2 \mid n$, we only need to modify G_{Sch} in Figure 8 such that all positive weights are increased by 1.

This completes the proof of Theorem 3. □

F.3 Proof of Theorem 4

A voting rule r is said to be *UMG-based*, if the winner only depends on UMG of the profile. Formally, r is UMG-based if for all pairs of profiles P_1 and P_2 such that $\text{UMG}(P_1) = \text{UMG}(P_2)$, we have $r(P_1) = r(P_2)$.

Theorem 4 (**Smoothed PAR: Copeland$_\alpha$**)**.** *For any $m \geq 4$, any UMG-based int-GSR refinement of $\overline{Cd}_\alpha$, denoted by Cd_α, and any strictly positive and closed Π over $\mathcal{L}(\mathcal{A})$ with $\pi_{uni} \in CH(\Pi)$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,*

$$\widetilde{\text{PAR}}_{\Pi}^{\min}(Cd_\alpha, n) = 1 - \Theta(\frac{1}{\sqrt{n}})$$

Proof. Because Cd_α is UMG-based, we can represent Cd_α as a GSR with the $\vec{H}_{\text{Cd}_\alpha}$ defined in Definition 13, which consists of $\binom{m}{2}$ hyperplanes that represents the UMG of the profile. The high-level idea behind the proof is similar to the proof of Theorem 3: we first explicitly construct a violation of PAR under Cd_α, then show that the dimension of the characteristic cone of the corresponding polyhedron is $m! - 1$.

Let G^* denote the complete unweighted directed graph over $\mathcal{A}$ that consists of the following edges.

- $1 \to 2$, $2 \to 3$, $3 \to 1$.
- For any $i \in \{4, \ldots, m\}$, there are three edges $1 \to i$, $2 \to i$, $3 \to i$.
- The edges among alternatives in $i \in \{4, \ldots, m\}$ are assigned arbitrarily.

For example, Figure 9 (a) illustrates G^* for $m = 4$. Let P denote any profile whose UMG is G^*. It

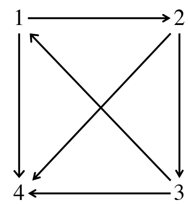


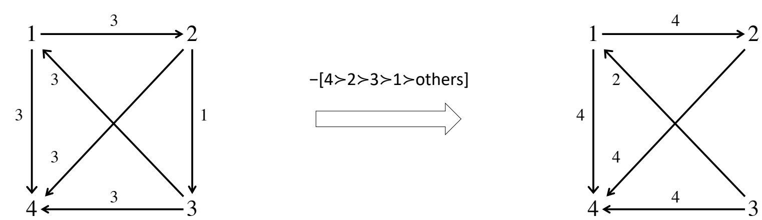
Figure 9: G^* for Copeland with $m = 4$.

is not hard to verify that $\overline{\text{Cd}_\alpha}(P) = \{1, 2, 3\}$. W.l.o.g. let $\text{Cd}_\alpha(P) = \{1\}$.

$2 \nmid n$ case. The proof is done for the following two sub-cases: $\alpha > 0$ and $\alpha = 0$.

$2 \nmid n$ and $\alpha > 0$. Let G_{Cd_α} (with weights w_{Cd_α}) denote the following weighted directed graph over $\mathcal{A}$ whose UMG is G^*, the weight on $2 \to 3$ is 1, and the weights on other edges are 3 or -3.

- $w_{\text{Cd}_\alpha}(2, 3) = 1$ and $w_{\text{Cd}_\alpha}(3, 1) = w_{\text{Cd}_\alpha}(1, 2) = 3$.
- For any $4 \le i \le m$, $w_{\text{Cd}_\alpha}(1, i) = w_{\text{Cd}_\alpha}(2, i) = w_{\text{Cd}_\alpha}(3, i) = 3$.
- The weights on other edges are 3 or -3.



(a) $G_{\text{Cd}_\alpha} = \text{WMG}(P_{\text{Cd}_\alpha})$. (b) $\text{WMG}(P_{\text{Cd}_\alpha} \setminus \{R_{\text{Cd}_\alpha}\})$.

Figure 10: G_{Cd_α} and $\text{WMG}(P_{\text{Cd}_\alpha} \setminus \{P_{\text{Cd}_\alpha}\})$ for $2 \nmid n$ and $\alpha > 0$.

See Figure 10 (a) for an example of G_{Cd_α}. According to McGarvey's theorem [33] that for any $n > m^4$ and $2 \nmid n$, there exists an n-profile P_{Cd_α} whose WMG is G_{Cd_α}. Therefore, for any $n > m^4 + 2$ and $2 \nmid n$, there exists an n-profile P_{Cd_α} whose WMG is G_{Cd_α}, and P_{Cd_α} includes the following two rankings.

$$[4 \succ 2 \succ 3 \succ 1 \succ \text{others}], \text{Rev}\,(4 \succ 2 \succ 3 \succ 1 \succ \text{others})$$

We now show that $\textsc{Par}(r, P_{\text{Cd}_\alpha}) = 0$, which implies that the 1 case Lemma 3 does not hold. Let $R_{\text{Cd}_\alpha} = [4 \succ 2 \succ 3 \succ 1 \succ \text{others}]$. Notice that in the profile $P_{\text{Cd}_\alpha} - R_{\text{Cd}_\alpha}$, the Copeland$_\alpha$ score of alternative 3 is $m - 2 + \alpha$, which is strictly higher than the Copeland$_\alpha$ score of alternative 1, which is $m - 2$. Therefore, $\text{Cd}_\alpha(P_{\text{Cd}_\alpha} \setminus \{R_{\text{Cd}_\alpha}\}) = \{3\}$. See Figure 10 (b) for $\text{WMG}(P_{\text{Cd}_\alpha} \setminus \{R_{\text{Cd}_\alpha}\})$. Notice that $3 \succ_{R_{\text{Cd}_\alpha}} 1$, which means that the $\textsc{Par}(r, P_{\text{Cd}_\alpha}) = 0$.

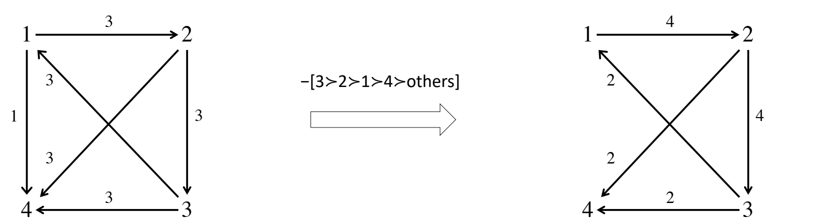
Therefore, the 1 case of Lemma 3 does not hold. Let $\vec{t}_1 = \text{Sign}_{\vec{H}_{\text{Cd}_\alpha}}(P_{\text{Cd}_\alpha})$ and $\vec{t}_2 = \text{Sign}_{\vec{H}_{\text{Cd}_\alpha}}(P_{\text{Cd}_\alpha} \setminus \{R_{\text{Cd}_\alpha}\})$. The VL case of Lemma 3 does not hold because $\vec{t}_1 \oplus \vec{t}_2 \trianglelefteq \text{Sign}_{\vec{H}_{\text{Cd}_\alpha}}(\pi_{\text{uni}})$ and $\pi_{\text{uni}} \in \text{CH}(\Pi)$.

Next, we prove that $\dim(\mathcal{H}_{\leq 0}^{\vec{t}_1 \oplus \vec{t}_2}) = m! - 1$. Notice that $[\vec{t}_1]_{(2,3)} = +$ and $[\vec{t}_2]_{(2,3)} = 0$, and all other components of $\vec{t}_1$ and $\vec{t}_2$ are the same and are non-zero. Therefore, $\vec{t}_1$ is a refinement of $\vec{t}_2$, which means that $\vec{t}_1 \oplus \vec{t}_2 = \vec{t}_2$. Notice that $\text{Hist}(P_{\text{Cd}_\alpha})$ is an inner point of $\mathcal{H}_{\leq 0}^{\vec{t}_2}$, in the sense that all inequalities are strict except the inequalities about $(2,3)$. This means that the essential equalities of $\mathbf{A}^{\vec{t}_1 \oplus \vec{t}_2}$ are equivalent to $\text{Pair}_{2,3} \cdot \vec{x} = \vec{0}$. Therefore, $\dim(\mathcal{H}_{\leq 0}^{\vec{t}_2}) = \dim(\mathcal{H}_{\leq 0}^{\vec{t}_1 \oplus \vec{t}_2}) = m! - 1$. This proves the proposition when $2 \nmid n$, $\alpha > 0$, and $\text{Cd}_\alpha(P) = \{1\}$.

If $\text{Cd}_\alpha(P) = \{2\}$ (respectively, $\text{Cd}_\alpha(P) = \{3\}$), then we simply switch the weights on $2 \to 3$ and $3 \to 1$ (respectively, $2 \to 3$ and $1 \to 2$) in Figure 9 (b), and the rest of the proof is similar to the $\text{Cd}_\alpha(P) = \{1\}$ case. This proves Theorem 4 for $2 \nmid n$ and $\alpha > 0$.

$2 \nmid n$ and $\alpha = 0$. Let G_{Cd_α} (with weights w_{Cd_α}) denote the following weighted directed graph over $\mathcal{A}$ whose UMG is G^* as illustrated in Figure 9 (a).

- $w_{\text{Cd}_\alpha}(2,3) = w_{\text{Cd}_\alpha}(3,1) = w_{\text{Cd}_\alpha}(1,2) = 3$.
- For any $4 \leq i \leq m$, $w_{\text{Cd}_\alpha}(1,i) = w_{\text{Cd}_\alpha}(2,i) = w_{\text{Cd}_\alpha}(3,i) = 3$, except $w_{\text{Cd}_\alpha}(4,1) = 1$.
- The weights on edge between $\{4, \ldots, m\}$ are 3 or -3.



(a) $G_{\text{Cd}_\alpha} = \text{WMG}(P_{\text{Cd}_\alpha})$. (b) $\text{WMG}(P_{\text{Cd}_\alpha} \setminus \{R_{\text{Cd}_\alpha}\})$.

Figure 11: G_{Cd_α} and $\text{WMG}(P_{\text{Cd}_\alpha} \setminus \{P_{\text{Cd}_\alpha}\})$ for $2 \nmid n$ and $\alpha = 0$.

See Figure 11 (a) for an example of G_{Cd_α}. According to McGarvey's theorem [33] that for any $n > m^4$ and $2 \nmid n$, there exists an n-profile P_{Cd_α} whose WMG is G_{Cd_α}. Therefore, for any $n > m^4 + 2$ and $2 \nmid n$, there exists an n-profile P_{Cd_α} whose WMG is G_{Cd_α} and P_{Cd_α} includes the following two rankings.

$$[3 \succ 2 \succ 1 \succ 4 \succ \text{others}], \text{Rev}\,(3 \succ 2 \succ 1 \succ 4 \succ \text{others})$$

We now show that $\textsc{Par}(\text{Cd}_\alpha, P_{\text{Cd}_\alpha}) = 0$, which implies that the 1 case Lemma 3 does not hold. Let $R_{\text{Cd}_\alpha} = [3 \succ 2 \succ 1 \succ 4 \succ \text{others}]$. Notice that in the profile $P_{\text{Cd}_\alpha} \setminus \{R_{\text{Cd}_\alpha}\}$, the Copeland_α score of alternative 1 is $m - 3 + \alpha = m - 3$, which is strictly higher than the Copeland_α score of alternative 2 and 3, which means that $\text{Cd}_\alpha(P_{\text{Cd}_\alpha} - R_{\text{Cd}_\alpha}) \subseteq \{2,3\}$. See Figure 11 (b) for an example of $\text{WMG}(P_{\text{Cd}_\alpha} \setminus \{R_{\text{Cd}_\alpha}\})$. Notice that $2 \succ_{R_{\text{Cd}_\alpha}} 1$ and $3 \succ_{R_{\text{Cd}_\alpha}} 1$, which means that $\textsc{Par}(\text{Cd}_\alpha, P_{\text{Cd}_\alpha}) = 0$.

The proofs for $\ell_n = 1$, the $\text{Cd}_\alpha(P) = \{2\}$ case, and the $\text{Cd}_\alpha(P) = \{3\}$ case are similar to their counterparts for the "$2 \nmid n$ and $\alpha = 0$" case above.

$2 \mid n$. The proof for the $2 \mid n$ case is similar to the proof of the $2 \nmid n$ case with the following modifications. The n-profile P_{Cd_α} where $\textsc{Par}$ is violated is obtained from the profile in the $2 \nmid n$ plus $\text{Rev}\,(R_{\text{Cd}_\alpha})$. Below we present the full proof for the case of $2 \mid n$ and $\alpha > 0$ for example. The other cases can be proved similarly.

$2 \mid n$ and $\alpha > 0$. W.l.o.g. suppose $\text{Cd}_\alpha(G^*) = \{1\}$. Let G_{Cd_α} (with weights w_{Cd_α}) denote the weighted directed graph in Figure 10 (a). According to McGarvey's theorem [33] that for any $n > m^4$ and $2 \mid n$, there exists an $(n-1)$-profile P'_{Cd_α} whose WMG is G_{Cd_α}. Let

$$P_{\text{Cd}_\alpha} = P'_{\text{Cd}_\alpha} + \text{Rev}\,(4 \succ 2 \succ 3 \succ 1 \succ \text{others})$$

It is not hard to verify that in P_{Cd_α}, the Copeland_α score of alternative 3 is $m-2+\alpha$, which is strictly higher than the Copeland_α score of alternative 1, which is $m-2$. Therefore, $\text{Cd}_\alpha(P_{\text{Cd}_\alpha}) = \{3\}$.

Let $R_{\text{Cd}_\alpha} = \text{Rev}\,(4 \succ 2 \succ 3 \succ 1 \succ \text{others})$. Notice that $\text{Cd}_\alpha(P_{\text{Cd}_\alpha} \setminus \{R_{\text{Cd}_\alpha}\}) = \text{Cd}_\alpha(G^*) = \{1\}$ and $1 \succ_{R_{\text{Cd}_\alpha}} 3$, which means that $\text{PAR}(\text{Cd}_\alpha, P_{\text{Cd}_\alpha}) = 0$. Therefore, the 1 case in Lemma 3 does not hold. Let $\vec{t}_1 = \text{Sign}_{\vec{H}_{\text{Cd}_\alpha}}(P_{\text{Cd}_\alpha})$ and $\vec{t}_2 = \text{Sign}_{\vec{H}_{\text{Cd}_\alpha}}(P_{\text{Cd}_\alpha} \setminus \{R_{\text{Cd}_\alpha}\})$. Like in other cases, the VL case of Lemma 3 does not holds because $\vec{t}_1 \oplus \vec{t}_2 \trianglelefteq \text{Sign}_{\vec{H}_{\text{Cd}_\alpha}}(\pi_{\text{uni}})$.

Next, we prove that $\dim(\mathcal{H}_{\leq 0}^{\vec{t}_1 \oplus \vec{t}_2}) = m! - 1$. Notice that $[\vec{t}_1]_{(2,3)} = 0$ and $[\vec{t}_2]_{(2,3)} = +$, and all other components of $\vec{t}_1$ and $\vec{t}_2$ are the same and are non-zero. Therefore, $\vec{t}_1$ is a refinement of $\vec{t}_2$, which means that $\vec{t}_1 \oplus \vec{t}_2 = \vec{t}_1$. Notice that $\text{Hist}(P_{\text{Cd}_\alpha})$ is an inner point of $\mathcal{H}_{\leq 0}^{\vec{t}_1}$, in the sense that all inequalities are strict except the inequalities about $(2,3)$. This means that the essential equalities of $\mathbf{A}^{\vec{t}_1 \oplus \vec{t}_2}$ are equivalent to

$$\text{Pair}_{2,3} \cdot \vec{x} = \vec{0} \text{ and } -\text{Pair}_{2,3} \cdot \vec{x} = \vec{0}$$

Therefore, $\dim(\mathcal{H}_{\leq 0}^{\vec{t}_1 \oplus \vec{t}_2}) = m! - 1$, which means that $\ell_n = -(m! - (m! - 1)) = 1$. The $2 \mid n$ and $\alpha > 0$ case follows after Lemma 3.

The proof for other subcases of $2 \mid n$ are similar to the proof of $2 \mid n$ and $\alpha > 0$ case above. This completes the proof of Theorem 4. $\square$

F.4 Proof of Theorem 5

Theorem 5 (Smoothed PAR: int-MRSE). *Given $m \geq 4$, any int-MRSE $\overline{r}$, any int-GSR r that is a refinement of $\overline{r} = (\overline{r}_2, \ldots, \overline{r}_m)$, and any strictly positive and closed Π over $\mathcal{L}(\mathcal{A})$ with $\pi_{uni} \in CH(\Pi)$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,*

$$\widetilde{\text{PAR}}_{\Pi}^{\min}(r, n) = 1 - \Theta(\frac{1}{\sqrt{n}})$$

Proof. The intuition behind the proof is similar to the proof of Theorem 3. Indeed, Lemma 3 can be applied to r, but it is unclear how to characterize ℓ_n. Therefore, in this proof we do not directly characterize $dim(\mathcal{H}_{\leq 0}^{\vec{t}})$ as in the proof of Theorem 3, but will instead define another polyhedron $\mathcal{H}^r$ to characterize a set of sufficient conditions for PAR to be violated—and the dimension of the new polyhedron is easy to analyze. Let us start with defining sufficient conditions on a profile P for PAR to be violated under any refinement of $\overline{r}$.

Condition 1 (Sufficient conditions: violation of PAR under an MRSE rule). *Given an MRSE $\overline{r}$, a profile P satisfies the following conditions during the execution of $\overline{r}$.*

(1) For every $1 \leq i \leq m-4$, in the i-th round, alternative $i+4$ drops out.

(2) In round $m-3$, 1 has the highest score, 2 has the second highest score, and 3 and 4 are tied for the last place.

(3) If 3 is eliminated in round $m-3$, then 2 and 4 are eliminated in round $m-2$ and $m-1$, respectively, which means that the winner is 1.

(4) If 4 is eliminated in round $m-3$, then 1 and 3 are eliminated in round $m-2$ and $m-1$, respectively, which means that the winner is 2.

(5) P contains at least one vote $[4 \succ 2 \succ 1 \succ 3 \succ \text{others}]$ and at least one vote $[3 \succ 1 \succ 2 \succ 4 \succ \text{others}]$, where "others" represents $5 \succ \cdots \succ m$.

(6) All losers described above, except in (2), are "robust" , in the sense that after removing any vote from P, they are still the unique losers.

Let us verify that for any profile P that satisfies Condition 1, $\text{PAR}(r, P) = 0$. It is not hard to see that $\overline{r}(P) = \{1, 2\}$. If $r(P) = \{1\}$, then let $R_r = [4 \succ 2 \succ 1 \succ 3 \succ \text{others}]$. This means that when any voter whose preferences are R_r abstain from voting, alternative 4 drops out in round $m-3$ of $(P \setminus \{R_r\})$, and consequently 2 becomes the winner. Notice that $2 \succ_{R_r} 1$, which means that $\text{PAR}(r, P) = 0$. Similarly, if $r(P) = \{2\}$, then let $R_r = [3 \succ 1 \succ 2 \succ 4 \succ \text{others}]$, which means that 3 drops out in round $m-3$ of $(P \setminus \{R_r\})$, and 1 becomes the winner. Notice that $1 \succ_{R_r} 2$. Again, we have $\text{PAR}(r, P) = 0$. The procedures of executing $\overline{r}$ under P and $(P \setminus \{R_r\})$ are represented in Figure 12.

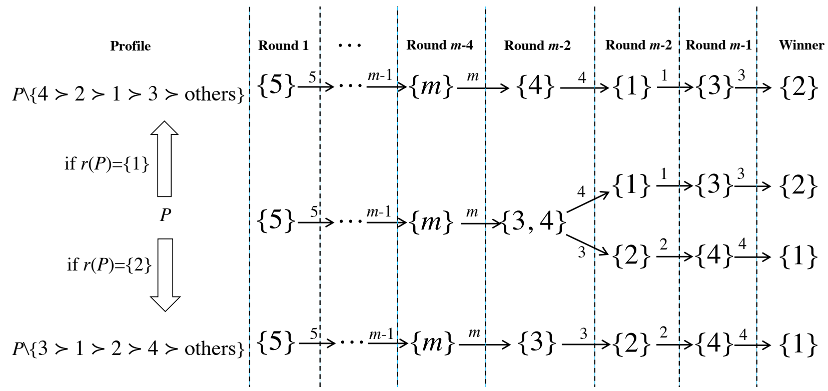


Figure 12: Executing $\overline{r}$ for a profile that satisfies Condition 1.

The rest of the proof proceeds as follows. In Step 1 below, We will prove by construction that for every sufficiently large n, there exists an n-profile P_r that satisfies Condition 1. Then in Step 2, we formally define $\mathcal{H}^{\overline{r}}$ to represent profiles that satisfy Condition 1. Finally, in Step 3, we show that $\dim(\mathcal{H}^{\overline{r}}_{<0}) = m! - 1$ because there is essentially only one equality (in Condition 1 (2)). Theorem 5 then follows after 1 minus the polynomial case of the inf part of [52, Theorem 2].

Step 1: define P_r. Before defining P_r, we first define a profile P^* that consists of a constant and odd number of votes in Steps 1.1–1.3. We then prove that PAR is violated at P^* in Step 1.4 and 1.5, where in Step 1.4 we show that $\overline{r}(P) = \{1, 2\}$ and in Step 1.5 we point out a violation of PAR depending on $r(P^*)$. Then in Step 1.6, we show how to expand P^* to an n-profile P_r for any sufficiently large n.

Let $P^* = P_1 + P_2 + P_3$, where P_1 consists of even number of votes and is designed to guarantee Condition 1 (1), i.e., $5, \ldots, m$ are eliminated in the first $m-4$ rounds, respectively. This means that in the beginning of round $m-3$, the remaining alternatives are $\{1, 2, 3, 4\}$. P_2 consists of an odd number of votes and is designed to guarantee Condition 1 (2), i.e., in round $m-3$, $\overline{r}_4$ outputs the weak order $[1 \succ 2 \succ 3 = 4]$. P_3 consists of an even number of votes and is designed to guarantee Condition 1 (3) and (4), i.e., if 3 (respectively, 4) is eliminated then 1 (respectively, 2) wins.

Step 1.1: define P_1. Let P_1^1 denote the following profile of $(24m(m-4)! + \frac{(m+5)(m-4)}{2}(m-1)!)$ votes.

$$P_1^1 = m \times \{[R_1 \succ R_2 : \forall R_1 \in \mathcal{L}(\{1,2,3,4\}), R_2 \in \mathcal{L}(\{5,\ldots,m\})\} \cup \bigcup_{i=5}^{m} i \times \{[i \succ R_2] : \forall R_2 \in \mathcal{L}(\mathcal{A} \setminus \{i\})\}$$

For every $2 \le i \le m$, let the scoring vector of $\overline{r}_i$ be $(s_1^i, \ldots, s_i^i)$. For example, the scoring vector of $\overline{r}_4$ is $(s_1^4, s_2^4, s_3^4, s_4^4)$. We let $P_1 = (s_1^4 - s_4^4 + 1)|P_2| \times P_1^1$, where $|P_2|$ is the number of votes in P_2, which is a constant and will become clear after Step 1.2.

Step 1.2: define P_2. The main challenge in this step is to use an odd number of votes to define P_2 such that in round $m-3$, the score of 1 is strictly higher than the score of 2, which is strictly higher than the score of 3 and 4. We first define the following 8-profile, denoted by P_2^1.

$$\begin{aligned} P_2^1 = \{&[1 \succ \text{others} \succ 3 \succ 4 \succ 2], [1 \succ \text{others} \succ 4 \succ 3 \succ 2], \\ &3 \times [1 \succ \text{others} \succ 2 \succ 4 \succ 3], 3 \times [2 \succ \text{others} \succ 1 \succ 3 \succ 4]\} \end{aligned}$$

The numbers of times alternatives $\{1, 2, 3, 4\}$ are ranked in each position in $P_2^1|_{\{1,2,3,4\}}$ are indicated in Table 5.

Next, we define a profile P_2^2 that consists of an odd number of votes where the scores of 3 and 4 are equal. Let $d_1 = s_1^4 - s_2^4$ and $d_2 = s_2^4 - s_3^4$. The construction is done in the following three cases.

Alternative	1st	2nd	3rd	4th
1	5	3	0	0
2	3	3	0	2
3	0	1	4	3
4	0	1	4	3

Table 5: Number of times each alternative is ranked in each position in $P_2^1|_{\{1,2,3,4\}}$.

- If $d_1 = 0$, then we let P_2^2 consist of a single vote $[3 \succ 4 \succ 1 \succ 2 \succ \text{others}]$.
- If $d_1 \neq 0$ and $d_2 = 0$, then we let P_2^2 consist of a single vote $[1 \succ 3 \succ 4 \succ 2 \succ \text{others}]$.
- If $d_1 \neq 0$ and $d_2 \neq 0$, then we let $d_1' = d_1 / \gcd(d_1, d_2)$ and $d_2' = d_2 / \gcd(d_1, d_2)$, where $\gcd(d_1, d_2)$ is the greatest common divisor of d_1 and d_2. It follows that at least one of d_1' and d_2' is an odd number.
  - If d_1' is odd, then we let
  $$P_2^2 = (d_1' + d_2') \times [1 \succ 3 \succ 4 \succ 2 \succ \text{others}] + d_2' \times [4 \succ 1 \succ 3 \succ 2 \succ \text{others}]$$
  - Otherwise, we must have d_1' is even and d_2' is odd. Then, we let
  $$P_2^2 = (d_1' + d_2') \times [3 \succ 4 \succ 1 \succ 2 \succ \text{others}] + d_1' \times [4 \succ 1 \succ 3 \succ 2 \succ \text{others}]$$

It is not hard to verify that in either case P_2^2 consists of an odd number of votes, and the score of 3 and 4 are equal under P_2^2. To guarantee that 3 and 4 have the lowest $\overline{r}_4$ scores in $P_2|_{\{1,2,3,4\}}$, we include sufficiently many copies of P_2^1 in P_2. Formally, let

$$P_2 = (|P_2^2| + 1) \times P_2^1 + P_2^2$$

Step 1.3: define P_3. We let $P_3 = ((s_1 - s_3)|P_2| + 1) \times P_3^*$, where $P_3^* = P_3^{*1} + P_3^{*2}$ is the 36-profile defined as follows. P_3^{*1} consists of 12 votes, where each alternative in $\{1, 2, 3, 4\}$ is ranked in the top in three votes, followed by the remaining three alternatives in a cyclic order.

$$\begin{aligned} P_3^{*1} = \{&[1 \succ 2 \succ 3 \succ 4 \succ \text{others}], [1 \succ 3 \succ 4 \succ 2 \succ \text{others}], [1 \succ 4 \succ 2 \succ 3 \succ \text{others}], \\ &[2 \succ 1 \succ 4 \succ 3 \succ \text{others}], [2 \succ 4 \succ 3 \succ 1 \succ \text{others}], [2 \succ 3 \succ 1 \succ 4 \succ \text{others}], \\ &[3 \succ 1 \succ 4 \succ 2 \succ \text{others}], [3 \succ 4 \succ 2 \succ 1 \succ \text{others}], [3 \succ 2 \succ 1 \succ 4 \succ \text{others}], \\ &[4 \succ 1 \succ 2 \succ 3 \succ \text{others}], [4 \succ 2 \succ 3 \succ 1 \succ \text{others}], [4 \succ 3 \succ 1 \succ 2 \succ \text{others}]\} \end{aligned}$$

P_3^{*2} consists of 24 votes that are defined in the following three steps. First, we start with $\mathcal{L}(\{1, 2, 3, 4\})$, which consists of 24 votes. Second, we replace $[3 \succ 2 \succ 4 \succ 1]$ and $[4 \succ 1 \succ 3 \succ 2]$ by $[3 \succ 1 \succ 4 \succ 2]$ and $[4 \succ 2 \succ 3 \succ 1]$, respectively. That is, the locations of 1 and 2 are exchanged in the two votes. This is designed to guarantee that the $\overline{r}_4$ scores of all alternative are the same in $P_3^{*2}|_{\{1,2,3,4\}}$, and after 3 is removed, 1's $\overline{r}_3$ score is higher than 2's $\overline{r}_3$ score; and after 4 is removed, 2's $\overline{r}_3$ score is higher than 1's $\overline{r}_3$score. Third, we append the lexicographic order of $\{5, \ldots, m\}$ to the end of each of the 24 rankings. Formally, we define

$$\begin{aligned} P_3^{*2} = \{&R_4 \succ 5 \succ \cdots \succ m : R_4 \in \mathcal{L}(\{1, 2, 3, 4\})\} - [3 \succ 2 \succ 4 \succ 1 \succ \text{others}] \\ &- [4 \succ 1 \succ 3 \succ 2 \succ \text{others}] + [3 \succ 1 \succ 4 \succ 2 \succ \text{others}] + [4 \succ 2 \succ 3 \succ 1 \succ \text{others}] \end{aligned}$$

Step 1.4: Prove $\overline{r}(P^*) = \{1, 2\}$. Recall that $P^* = P_1 + P_2 + P_3$. Notice that the P_1 part guarantees that $\{5, \ldots, m\}$ are dropped out in the first $m-4$ rounds, and the scores of all alternatives in $\{1, 2, 3, 4\}$ are the same under P_1 no matter what alternatives are dropped out. Therefore, it suffices to calculate the results of the last three rounds based on $P_2 + P_3$, which is done as follows.

In round $m - 3$, it is not hard to check that every alternative in $\{1, 2, 3, 4\}$ gets the same total score under P_3, where each of them is ranked at each position for 9 times. Therefore, due to P_2, alternative 3 and 4 are tied for the last place in round $m - 3$.

If 3 is eliminated in round $m-3$, then $P^*_3|_{\{1,2,4\}} = P^{*1}_3|_{\{1,2,4\}} + P^{*2}_3|_{\{1,2,4\}}$ becomes the following.

$$\begin{aligned}P^{*1}_3|_{\{1,2,4\}} =&\{2\times[1\succ 4\succ 2],[1\succ 2\succ 4],2\times[2\succ 1\succ 4],[2\succ 4\succ 1],\\ &[1\succ 4\succ 2],[4\succ 2\succ 1],[2\succ 1\succ 4],2\times[4\succ 1\succ 2],[4\succ 2\succ 1]\}\\ P^{*2}_3|_{\{1,2,4\}} =&4\times\mathcal{L}(\{1,2,4\})-[2\succ 4\succ 1]-[4\succ 1\succ 2]+[1\succ 4\succ 2]+[4\succ 2\succ 1]\end{aligned}$$

It is not hard to verify that the numbers of times alternatives $\{1,2,4\}$ are ranked in each position in $P^*_3|_{\{1,2,4\}}$ are as indicated in Table 6 (a).

Alternative	1st	2nd	3rd
1	13	12	11
2	11	12	13
4	12	12	12

(a) 3 is removed.

Alternative	1st	2nd	3rd
1	11	12	13
2	13	12	11
3	12	12	12

(b) 4 is removed.

Table 6: Number of times each alternative is ranked in each position in round $m-2$.

This means that the score of alternative 2 is strictly lower than the score of 1 or 3, because $s^3_1 - s^3_3 \geq 1$, where the score vector for $\overline{r}_3$ is (s^3_1, s^3_2, s^3_3). Recall that P_3 consists of sufficiently large number of copies of P^*_3. Therefore, even considering the score difference between alternatives in P_2, the score of 2 is still the strictly lowest among $\{1,2,4\}$ in P^* in round $m-2$. This means that alternative 2 drops in round $m-2$, and it is easy to check that $1\succ 4$ in 20 votes in P^*_3, which is strictly more than half ($=16$). This means that 1 is the r winner if 3 is eliminated in round $m-3$.

If 4 is eliminated in round $m-3$, then $P^*_3|_{\{1,2,3\}} = P^{*1}_3|_{\{1,2,3\}} + P^{*2}_3|_{\{1,2,3\}}$ becomes the following.

$$\begin{aligned}P^{*1}_3|_{\{1,2,3\}} =&\{2\times[1\succ 2\succ 3],[1\succ 3\succ 2],2\times[2\succ 3\succ 1],[2\succ 1\succ 3],\\ &2\times[3\succ 2\succ 1],[3\succ 1\succ 2],[1\succ 2\succ 3],[2\succ 3\succ 1],[3\succ 1\succ 2]\}\\ P^{*2}_3|_{\{1,2,3\}} =&4\times\mathcal{L}(\{1,2,3\})-[3\succ 2\succ 1]-[1\succ 3\succ 2]+[3\succ 1\succ 2]+[2\succ 3\succ 1]\end{aligned}$$

The numbers of times alternatives $\{1,2,3\}$ are ranked in each position in $P^*_3|_{\{1,2,3\}}$ are as indicated in Table 6 (b). Again, it is not hard to verify that alternative 1 drops in round $m-2$, and 2 beats 3 in the last round to become the r winner in this case.

Step 1.5: Prove that PAR is violated at P^*. At a high-level the proof is similar to Step 1.4, and the absent vote is effectively used as a tie breaker between alternatives 3 and 4. Recall that r is a refinement of $\overline{r}$ and it was shown in Step 1.4 that $\overline{r}(P^*) = \{1,2\}$. Therefore, either $r(P^*) = \{1\}$ or $r(P^*) = \{2\}$. The proof is done in the follow two cases.

- If $r(P^*) = \{1\}$, then we let
$$R_r = [4\succ 2\succ 1\succ 3\succ \text{others}],$$
which is a vote in P^2_3. Then in $(P^*\setminus\{R_r\})$, alternative 4 is eliminated in round $m-3$, and following a similar reasoning as in Step 1.4, we have $r(P^*\setminus\{R_r\}) = \{2\}$. Notice that $2\succ_{R_r} 1$, which means that PAR is violated at P^*.
- If $r(P^*) = \{2\}$, then we let
$$R_r = [3\succ 1\succ 2\succ 4\succ \text{others}],$$
which is a vote in P^2_3. Then in $(P^*\setminus\{R_r\})$, alternative 3 is eliminated in round $m-3$, and following a similar reasoning as in Step 1.4, we have $r(P^*\setminus\{R_r\}) = \{1\}$. Notice that $1\succ_{R_r} 2$, which means that PAR is violated at P^*.

Step 1.6: Construct an n-profile P_r. The intuition behind the construction is the following. P_r consists of three parts: P^1_r, P^2_r, and P^3_r. P^1_r consists of multiple copies of P^* defined in Steps 1.1-1.3 above, which is used to guarantee that PAR is violated at P_r and the score difference between any pair of alternatives is sufficiently large so that votes in P^3_r does not affect the execution of r. P^2_r consists of multiple copies of $\mathcal{L}(\mathcal{A})$. P^3_r consists of no more than $m!-1$ votes, and $|P^3_r|$ is an even number.

Define P_r^1. To guarantee that $|P_r^3|$ is even, the definition of P_r^1 depends on the parity of n. Recall that P^* consists of an odd number of votes. When $2 \mid n$, we let

$$P_r^1 = m!\left(s_1^3 - s_3^3\right) \times P^*$$

When $2 \nmid n$, we let

$$P_r^1 = \left(m!\left(s_1^3 - s_3^3\right) + 1\right) \times P^*$$

Define P_r^2. Let $n_1 = |P_r^1|$. P_r^2 consists of as many copies of $\mathcal{L}(\mathcal{A})$ as possible, i.e.

$$P_r^2 = \left\lfloor \frac{n - n_1}{m!} \right\rfloor \times \mathcal{L}(\mathcal{A})$$

Define P_r^3. P_r^3 consists of multiple copies of pairs of rankings defined as follows.

$$P_r^3 = \left(\frac{n - n_1 - |P_r^2|}{2}\right) \times \{[1 \succ 2 \succ 3 \succ 4 \succ \text{others}], [2 \succ 1 \succ 4 \succ 3 \succ \text{others}]\}$$

It is not hard to verify that $P_r = P_r^1 + P_r^2 + P_r^3$ share the same properties as P^*: $\overline{r}(P_r) = \{1, 2\}$; if $[4 \succ 2 \succ 1 \succ 3 \succ \text{others}]$ is removed, then 2 is the unique winner; and if $[3 \succ 1 \succ 2 \succ 4 \succ \text{others}]$ is removed, then 1 is the unique winner. This means that PAR is violated at P_r.

Step 2: define a polyhedron $\mathcal{H}^{\overline{r}}$ to represent profiles that satisfy Condition 1. To define $\mathcal{H}^{\overline{r}}$, we recall from Definition 14 that for any a, b, any $B \subseteq \mathcal{A} \setminus \{a, b\}$, and any profile P, $\text{Score}_{B,a,b}^{\Delta} \cdot \text{Hist}(P)$ is the difference between the $\overline{r}_{m-|B|}$ score of a and the $\overline{r}_{m-|B|}$ score of b in $P|_{\mathcal{A} \setminus B}$. We are now ready to define $\mathcal{H}^{\overline{r}}$ whose $\mathbf{A}$ matrix has five parts that correspond to Condition 1 (1)–(5). Condition 1 (6) will be incorporated in the $\vec{\mathbf{b}}$ vector of $\mathcal{H}^{\overline{r}}$.

Definition 31 ($\mathcal{H}^{\overline{r}}$). *Given $\overline{r} = (\overline{r}_2, \ldots, \overline{r}_m)$, we let* $\mathbf{A}^{\overline{r}} = \begin{bmatrix} \mathbf{A}^{(1)} \\ \mathbf{A}^{(2)} \\ \mathbf{A}^{(3)} \\ \mathbf{A}^{(4)} \\ \mathbf{A}^{(5)} \end{bmatrix}$, *where*

- $\mathbf{A}^{(1)}$: *for every $1 \leq i \leq m-4$ and every $j \in \mathcal{A} \setminus \{i+4\}$, $\mathbf{A}^{(1)}$ has a row $\text{Score}_{\{5,\ldots,i+3\},i+4,j}^{\Delta}$.*
- $\mathbf{A}^{(2)}$, $\mathbf{A}^{(3)}$, *and* $\mathbf{A}^{(4)}$ *are defined as follows.*

$$\mathbf{A}^{(2)} = \begin{bmatrix} \text{Score}_{\{5,\ldots,m\},2,1}^{\Delta} \\ \text{Score}_{\{5,\ldots,m\},3,2}^{\Delta} \\ \text{Score}_{\{5,\ldots,m\},4,3}^{\Delta} \\ \text{Score}_{\{5,\ldots,m\},3,4}^{\Delta} \end{bmatrix}, \mathbf{A}^{(3)} = \begin{bmatrix} \text{Score}_{\{3,5,\ldots,m\},4,1}^{\Delta} \\ \text{Score}_{\{3,5,\ldots,m\},2,4}^{\Delta} \\ \text{Score}_{\{2,3,5,\ldots,m\},4,1}^{\Delta} \end{bmatrix}, \mathbf{A}^{(4)} = \begin{bmatrix} \text{Score}_{\{4,5,\ldots,m\},3,2}^{\Delta} \\ \text{Score}_{\{4,5,\ldots,m\},1,3}^{\Delta} \\ \text{Score}_{\{1,4,5,\ldots,m\},3,2}^{\Delta} \end{bmatrix}$$

- $\mathbf{A}^{(5)}$ *consists of two rows defined as follows.*

$$\mathbf{A}^{(5)} = \begin{bmatrix} -\text{Hist}(4 \succ 2 \succ 1 \succ 3 \succ \text{others}) \\ -\text{Hist}(3 \succ 1 \succ 2 \succ 4 \succ \text{others}) \end{bmatrix}$$

Let

$$\vec{\mathbf{b}}^{\overline{r}} = [\underbrace{\vec{\mathbf{b}}^{(1)}}_{\text{for } \mathbf{A}^{(1)}}, \underbrace{(s_4^4 - s_1^4 - 1, s_4^4 - s_1^4 - 1, 0, 0)}_{\text{for } \mathbf{A}^{(2)}}, \underbrace{(s_3^3 - s_1^3 - 1, s_3^3 - s_1^3 - 1, s_2^2 - s_1^2 - 1)}_{\text{for } \mathbf{A}^{(3)}},$$
$$\underbrace{(s_3^3 - s_1^3 - 1, s_3^3 - s_1^3 - 1, s_2^2 - s_1^2 - 1)}_{\text{for } \mathbf{A}^{(4)}}, \underbrace{(-1, -1)}_{\text{for } \mathbf{A}^{(5)}}],$$

where for every $1 \leq i \leq m-4$ and every $j \in \mathcal{A} \setminus \{i+4\}$, $\vec{\mathbf{b}}^{(1)}$ contains a row $s_{m+1-i}^{m+1-i} - s_1^{m+1-i} - 1$. Let

$$\mathcal{H}^{\overline{r}} = \left\{\vec{x} \in \mathbb{R}^{m!} : \mathbf{A}^{\overline{r}} \cdot (\vec{x})^\top \leq \left(\vec{\mathbf{b}}^{\overline{r}}\right)^\top\right\}.$$

Step 3: Apply Lemma 3 and [52, Theorem 2]. We first prove the following properties of $\mathcal{H}^{\overline{r}}$.

Claim 14 (Properties of $\mathcal{H}^{\overline{r}}$). *Given any integer MRSE rule $\overline{r}$,*

(i) for any integral profile P, if Hist$(P) \in \mathcal{H}^{\overline{r}}$ then $\text{PAR}(r, P) = 0$*;*

(ii) $\pi_{uni} \in \mathcal{H}^{\overline{r}}_{\leq 0}$*;*

(iii) $\dim(\mathcal{H}^{\overline{r}}_{\leq 0}) = m! - 1$.

Proof. Part (i) follows after a similar reasoning as in Step 1 of the proof of Theorem 5. To prove Part (ii), notice that for any $B \subseteq \mathcal{A}$ and $a, b \in (\mathcal{A} \setminus B)$, we have $\text{Score}^{\Delta}_{B,a,b} \cdot \vec{1} = 0$. Also notice that for any $R \in \mathcal{L}(\mathcal{A})$ we have $-\text{Hist}(R) \cdot \vec{1} = -1 < 0$. Therefore, $\mathbf{A}^{\overline{r}} \cdot \left(\vec{1}\right)^{\top} \leq \left(\vec{0}\right)^{\top}$, which means that $\pi_{\text{uni}} \in \mathcal{H}^{\overline{r}}_{\leq 0}$. To prove Part (iii), notice that $\mathbf{A}^{\overline{r}} \cdot (\vec{x})^{\top} \leq \left(\vec{0}\right)^{\top}$ contains one equality in $\mathbf{A}^{(2)}$, i.e.

$$\text{Score}^{\Delta}_{\{5,\ldots,m\},3,4} \cdot (\vec{x})^{\top} = 0 \tag{15}$$

This means that $\dim(\mathcal{H}^{\overline{r}}_{\leq 0}) \leq m! - 1$. Recall that P_r is the n-profile defined in Step 1 that satisfies Condition 1. Notice that $\text{Hist}(P_r)$ is an inner point of $\mathcal{H}^{\overline{r}}_{\leq 0}$ in the sense that all inequalities in $\mathbf{A}^{\overline{r}} \cdot (\vec{x})^{\top} \leq \left(\vec{0}\right)^{\top}$ except Equation (15) are strict, which means that $\dim(\mathcal{H}^{\overline{r}}_{\leq 0}) \geq m! - 1$. This proves Claim 14. □

Because of the existence of P_r defined in Step 1, and Claim 14 (i) and (ii), the 1 case and the VL case of Lemma 3 do not hold for any sufficiently large n. Therefore, it follows from the L case of Lemma 3 that $\widetilde{\text{PAR}}^{\min}_{\Pi}(r, n)$ is at least $1 - O(n^{-0.5})$, because $\ell_n \geq 1$. It remains to show that $\widetilde{\text{PAR}}^{\min}_{\Pi}(r, n)$ is upper-bounded by $1 - \Omega(n^{-0.5})$. We have the following calculations.

$$\begin{aligned}
1 - \widetilde{\text{PAR}}^{\min}_{\Pi}(r, n) &= \sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(\text{PAR}(r, P) = 0) \\
&\geq \sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(\text{Hist}(P) \in \mathcal{H}^{\overline{r}}) && \text{Claim 14 (i)} \\
&= \Theta(n^{-0.5}) && \text{Claim 14 (ii), (iii), and [52, Theorem 2]}
\end{aligned}$$

The last equation follows after applying the sup part of [52, Theorem 2] to $\mathcal{H}^{\overline{r}}$. More concretely, recall that in Step 1 above we have constructed an n-profile P_r for any sufficiently large n and it is not hard to verify that $\text{Hist}(P_r) \in \mathcal{H}^{\overline{r}}$, which means that $\mathcal{H}^{\overline{r}}$ is active at any sufficiently large n. Claim 14 (ii) implies that the polynomial case of [52, Theorem 2] holds, and Claim 14 (iii) implies that $\alpha_n = m! - 1$ for $\mathcal{H}^{\overline{r}}$.

This proves Theorem 5. □

F.5 Proof of Theorem 6

Theorem 6 (Smoothed PAR: Condorcetified Integer Positional Scoring Rules). *Given $m \geq 4$, an integer positional irresolute scoring rule $\overline{r}_{\vec{s}}$, any Condocetified positional scoring rule $Cond_{\vec{s}}$ that is a refinement of $\overline{Cond_{\vec{s}}}$, and any strictly positive and closed Π over $\mathcal{L}(\mathcal{A})$ with $\pi_{uni} \in CH(\Pi)$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,*

$$\widetilde{\text{PAR}}^{\min}_{\Pi}(Cond_{\vec{s}}, n) = 1 - \Theta\left(\frac{1}{\sqrt{n}}\right)$$

Proof. The proof follows the same logic in the proof of Theorem 5. We first prove the theorem for even n then show how to extend the proof to odd n's.

Intuition for $2 \mid n$. Let $\vec{s} = (s_1, \ldots, s_m)$. We first identify a set of sufficient conditions for PAR to be violated.

Condition 2 (Sufficient conditions for the violation of PAR). *Given a Condorcetified irresolute integer positional scoring rule $\overline{\text{Cond}_{\vec{s}}}$, P satisfies the following conditions.*

(1) $\overline{\text{Cond}_{\vec{s}}}(P) = \{2\}$, and the score of 2 is higher than the score of any other alternative by at least $s_1 - s_m + 1$.

(2) Alternative 1 is a weak Condorcet winner, $w_P(1,3) = 0$, and for every $i \in \mathcal{A} \setminus \{1,3\}$, $w_P(1,i) \geq 2$.

(3) P contains at least one vote of $[3 \succ 1 \succ 2 \succ \text{others}]$.

Recall that $\text{Cond}_{\vec{s}}$ is a refinement of $\overline{\text{Cond}_{\vec{s}}}$ and due to Condition 2 (2), P does not contain a Condorcet winner. Therefore, according to Condition 2 (1), we have $\text{Cond}_{\vec{s}} = \{2\}$. Any voter whose preferences are $[3 \succ 1 \succ 2 \succ \text{others}]$ has incentive to abstain from voting, because the voter prefers 1 to 2, and $\{1\}$ is the Condorcet winner in $P - [3 \succ 1 \succ 2 \succ \text{others}]$, which means that

$$\text{Cond}_{\vec{s}}(P - [3 \succ 1 \succ 2 \succ \text{others}]) = \{1\}$$

This means that $\text{PAR}(\text{Cond}_{\vec{s}}, P) = 0$ for any profile P that satisfies Condition 2. The rest of the proof proceeds as follows. In Step 1, for any n that is sufficiently large, we construct an n-profile $P_{\vec{s}}$ that satisfies Condition 2. Then in Step 2, we formally define $\mathcal{H}^{\overline{\text{Cond}_{\vec{s}}}}$ to represent profile that satisfy Condition 2. Finally, in Step 3 we formally prove properties about $\mathcal{H}^{\overline{\text{Cond}_{\vec{s}}}}$ and apply Lemma 3 and [52, Theorem 2] to prove Theorem 5.

Step 1 for $2 \mid n$: define $P_{\vec{s}}$. The construction is similar to the construction in the proof of Claim 10, which is done for the following two cases: $\overline{r}_{\vec{s}}$ is the plurality rule and $\overline{r}_{\vec{s}}$ is not the plurality rule.

- **When $\overline{r}_{\vec{s}}$ is the plurality rule,** i.e. $s_2 = s_m$, we let

$$\begin{aligned} P_{\vec{s}} = &\left(\frac{n}{2} - 6\right) \times [2 \succ 1 \succ 3 \succ \text{others}] + 4 \times [2 \succ 3 \succ 1 \succ \text{others}] \\ &+ \left(\frac{n}{2} - 6\right) \times [3 \succ 1 \succ 2 \succ \text{others}] + 6 \times [1 \succ 2 \succ 3 \succ \text{others}] \end{aligned}$$

  It is not hard to verify that $P_{\vec{s}}$ satisfies Condition 2 for any even number $n \geq 28$.

- **When $\overline{r}_{\vec{s}}$ is not the plurality rule,** i.e., $s_2 > s_m$, like Step 1 in the proof of Theorem 5, we first construct a profile P^* that consists of a constant number of votes and satisfies Condition 2, then extend it to arbitrary odd number n. Let $2 \leq k \leq m-1$ denote the smallest number such that $s_k > s_{k+1}$. Let $A_1 = [4 \succ \cdots \succ k+1]$ and $A_2 = [k+2 \succ \cdots \succ m]$, and let $P^* = P_1^* + P_2^*$, where P_1^* is the following 10-profile that is used to guarantee Condition 2 (2) and (3).

$$\begin{aligned} P_1^* = \{&4 \times [1 \succ 2 \succ A_1 \succ 3 \succ A_2] + 3 \times [2 \succ 3 \succ A_1 \succ 1 \succ A_2] \\ &+2 \times [3 \succ 1 \succ A_1 \succ 2 \succ A_2] + [2 \succ 1 \succ A_1 \succ 3 \succ A_2]\} \end{aligned}$$

  And let P_2^* denote the following $36(m-3)!$-profile, which is used to guarantee that 2 is the unique winner under P^*, i.e., Condition 2 (1).

$$P_2^* = 6 \times \{[R_1 \succ R_2] : \forall R_1 \in \mathcal{L}(\{1,2,3\}), R_2 \in \mathcal{L}(\{4, \ldots, m\}), \}$$

  It is not hard to verify that the following observations hold for P_1^*.

  - 1 is the Condorcet winner, $w_{P_1^*}(1,3) = 0$, and for any $i \in \mathcal{A} \setminus \{1,3\}$, we have $w_{P_1^*}(1,i) \geq 2$.
  - The total score of 1 under P_1^* is $4s_1 + 3s_2 + 3s_{k+1}$, the total score of 2 under P_1^* is $4s_1 + 4s_2 + 2s_{k+1}$, and the total score of 3 under P_1^* is $2s_1 + 3s_2 + 5s_{k+1}$. Recall that we have assumed that $s_2 > s_{k+1}$. Therefore,

$$4s_1 + 4s_2 + 2s_{k+1} > 4s_1 + 3s_2 + 3s_{k+1} > 2s_1 + 3s_2 + 5s_{k+1},$$

    which means that the score of 2 is strictly higher than the scores of 1 and 3 in P_1^*.

Given these observations, it is not hard to verify that $P^* = P_1^* + P_2^*$ satisfies Condition 2. Let $P_{\vec{s}}$ denote as many copies of P^* as possible, plus pairs of rankings $\{[2 \succ 1 \succ 3 \succ \text{others}], [2 \succ 3 \succ 1 \succ \text{others}]\}$. More precisely, let

$$P_{\vec{s}} = \left\lfloor \frac{n}{|P^*|} \right\rfloor \times P^* + \left(\frac{n - |P^*| \cdot \lfloor \frac{n}{|P^*|} \rfloor}{2} \right) \times \{[2 \succ 1 \succ 3 \succ \text{others}], [2 \succ 3 \succ 1 \succ \text{others}]\}$$

It is not hard to verify that $P_{\vec{s}}$ satisfies Condition 2, which concludes Step 1 for the $2 \mid n$ case.

Step 2 for $2 \mid n$: define a polyhedron $\mathcal{H}^{\overline{\text{Cond}_{\vec{s}}}}$ to represent profiles that satisfy Condition 2.

Definition 32 ($\mathcal{H}^{\overline{\text{Cond}_{\vec{s}}}}$)**.** *Given an irresolute integer positional scoring rule $\overline{r}_{\vec{s}} = (s_1, \ldots, s_m)$, we let* $\mathbf{A}^{\vec{s}} = \begin{bmatrix} \mathbf{A}^{(1)} \\ \mathbf{A}^{(2)} \\ \mathbf{A}^{(3)} \end{bmatrix}$*, where*

- $\mathbf{A}^{(1)}$*: for every $i \in \mathcal{A} \setminus \{2\}$, $\mathbf{A}^{(1)}$ contains a row Score$_{i,2}$.*
- $\mathbf{A}^{(2)}$ *contains two rows Pair$_{1,3}$ and Pair$_{3,1}$, and for every $i \in \mathcal{A} \setminus \{1, 3\}$, $\mathbf{A}^{(1)}$ contains a row Pair$_{i,1}$.*
- $\mathbf{A}^{(3)}$ *consists of a single row $-$Hist$(3 \succ 1 \succ 2 \succ \textit{others})$.*

$$\textit{Let} \quad \vec{\mathbf{b}}^{\vec{s}} = \Bigg[\underbrace{(s_m - s_1 - 1) \cdot \vec{1}}_{\textit{for } \mathbf{A}^{(1)}}, \underbrace{(0, 0, -2, \ldots, -2)}_{\textit{for } \mathbf{A}^{(2)}}, \underbrace{-1}_{\textit{for } \mathbf{A}^{(3)}} \Bigg]$$

$$\textit{and} \quad \mathcal{H}^{\vec{s}} = \left\{ \vec{x} \in \mathbb{R}^{m!} : \mathbf{A}^{\vec{s}} \cdot (\vec{x})^\top \leq \left(\vec{\mathbf{b}}^{\vec{s}} \right)^\top \right\}.$$

Step 3 for $2 \mid n$: Apply Lemma 3 and [52, Theorem 2]. We first prove the following properties of $\mathcal{H}^{\overline{\text{Cond}_{\vec{s}}}}$.

Claim 15 (**Properties of $\mathcal{H}^{\overline{\text{Cond}_{\vec{s}}}}$**)**.** *Given any integer positional scoring rule $\vec{s}$,*

(i) for any integral profile P, if Hist$(P) \in \mathcal{H}^{\overline{\textit{Cond}_{\vec{s}}}}$ then $\textsc{Par}(\textit{Cond}_{\vec{s}}, P) = 0$*;*

(ii) $\pi_{\textit{uni}} \in \mathcal{H}_{\leq 0}^{\overline{\textit{Cond}_{\vec{s}}}}$*;*

(iii) $\dim(\mathcal{H}_{\leq 0}^{\overline{\textit{Cond}_{\vec{s}}}}) = m! - 1$.

Proof. The proof for Part (i) and (ii) are similar to the proof of Claim 14. To prove Part (iii), notice that $\mathbf{A}^{\vec{s}} \cdot (\vec{x})^\top \leq \left(\vec{0}\right)^\top$ contains one equality in $\mathbf{A}^{(2)}$, i.e.

$$\text{Pair}_{1,3} \cdot (\vec{x})^\top = (0)^\top \tag{16}$$

This means that $\dim(\mathcal{H}_{\leq 0}^{\overline{\text{Cond}_{\vec{s}}}}) \leq m! - 1$. Notice that Hist$(P_{\vec{s}})$ is an inner point of $\mathcal{H}_{\leq 0}^{\overline{\text{Cond}_{\vec{s}}}}$ in the sense that all other inequalities except Equation (16) are strict, which means that $\dim(\mathcal{H}_{\leq 0}^{\overline{\text{Cond}_{\vec{s}}}}) \geq m! - 1$. This proves Claim 15. ☐

Therefore, we have the following bound.

$$
\begin{aligned}
& 1 - \widetilde{\text{PAR}}_{\Pi}^{\min}(\text{Cond}_{\vec{s}}, n) \\
= & \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(\text{PAR}(\text{Cond}_{\vec{s}}, P) = 0) \\
\geq & \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(\text{Hist}(P) \in \mathcal{H}^{\overline{\text{Cond}_{\vec{s}}}}) && \text{Claim 15 (i)} \\
= & \Theta(n^{-0.5}) && \text{Claim 15 (ii), (iii), and [52, Theorem 2]}
\end{aligned}
$$

Consequently, $\widetilde{\text{PAR}}_{\Pi}^{\min}(\text{Cond}_{\vec{s}}, n) = 1 - \Omega(n^{-0.5})$. Notice that the 1 case and VL case Lemma 3 do not hold because of the existence of $P_{\vec{s}}$ and Claim 15 (ii). Therefore, Theorem 6 for the $2 \mid n$ case follows after the $1 - O(n^{-0.5})$ upper bound proved in Lemma 3.

Proof for the $2 \nmid n$ case. When $2 \nmid n$, we modify the proof as follows.

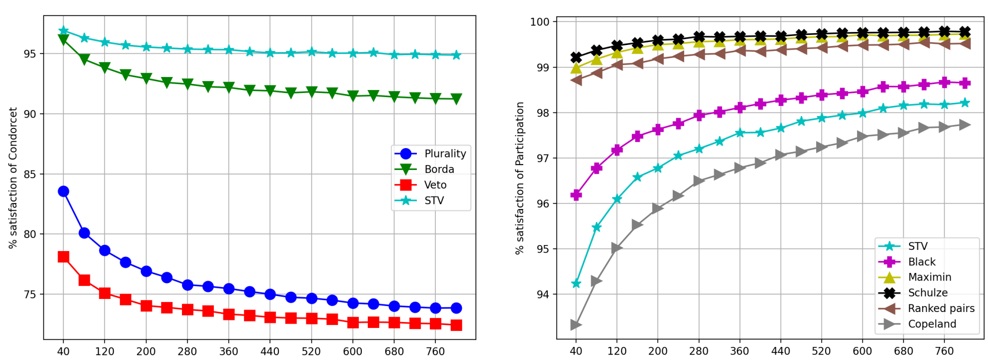
- First, Condition 2 (2) is replaced by the following condition:

  (2′): Alternative 1 is the Condorcet winner under P, $w_P(1, 3) = 1$, and for every $i \in \mathcal{A} \setminus \{1, 3\}$, $w_P(1, i) \geq 3$.
- Second, in Step 1, $P_{\vec{s}}$ has an additional vote $[2 \succ 1 \succ 3 \succ \text{others}]$.
- Third, in Step 2 Definition 32, the $\vec{\mathbf{b}}^{\vec{s}}$ components corresponding to $\mathbf{A}^2$ is $(1, -1, -3, \ldots, -3)$.

A similar claim as Claim 15 can be proved for the $2 \nmid n$ case. This proves Theorem 6. □

G Experimental Results

We report satisfaction of CC and PAR using simulated data and Preflib linear-order data [32] under four classes of commonly-used voting rules studied in this paper, namely positional scoring rules (plurality, Borda, and veto), voting rules that satisfy CONDORCET CRITERION (maximin, ranked pairs, Schulze, and Copeland$_{0.5}$), MRSE (STV), and Condorcetified positional scoring rule (Black's rule). All experiments were implemented in Python 3 and were run on a MacOS laptop with 3.1 GHz Intel Core i7 CPU and 16 GB memory.



(a) CC. (b) PAR.

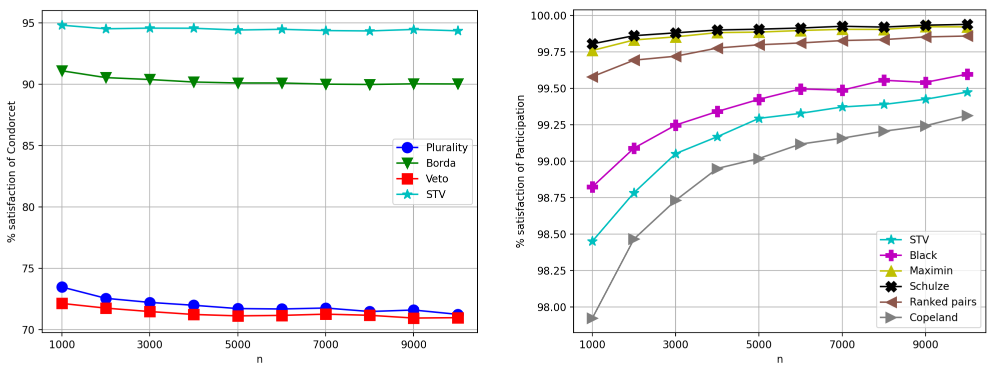
Figure 13: Satisfaction of CC and PAR under IC for $m = 4$, $n = 40$ to 800, 200000 trials.

Synthetic data. We generate profiles of $m = 4$ alternatives under IC.[3] The number of alternatives n ranges from 40 to 800. In each setting we generate 200000 profiles. The satisfaction of CC under plurality, Borda, veto, and STV are presented in Figure 13 (a), and the satisfaction of PAR

[3] See [8] for theoretical results and extensive simulation studies of PAR under the IAC model.

under STV, maximin, ranked pairs, Schulze, Black, and Copeland$_{0.5}$ are presented in Figure 13 (b). Notice that voting rules not in Figure 13 (a) always satisfy CC and voting rules not in Figure 13 (b) always satisfy PAR.

The results provide a sanity check for the theoretical results proved in this paper. In particular, Figure 13 (a) confirms that the satisfaction of CC is $\Theta(1)$ and $1 - \Theta(1)$ under positional scoring rules (Theorem 1) and STV (Corollary 1) w.r.t. IC. Figure 13 (b) confirms that the satisfaction of PAR is $1 - \Theta(n^{-0.5})$ under maximin, ranked pairs, Schulze (Theorem 3), Copeland$_\alpha$ (Theorem 4), STV (Theorem 5), and Black (Theorem 6). Figure 14 in Appendix G summarizes results with large n (1000 to 10000) that further confirm the asymptotic observations described above.



(a) CC. (b) PAR

Figure 14: Satisfaction of CC and PAR under IC for $m = 4$, $n = 1000$ to 10000, 200000 trials.

Preflib data. We also calculate the satisfaction of CC and PAR under all voting rules studied in this paper with lexicographic tie-breaking for all 315 Strict Order-Complete Lists (SOC) under election data category from Preflib [32]. The results are summarized in Table 7, which is the bottom part of Table 2.

Table 7: Satisfaction of CC and PAR in 315 Preflib SOC profiles. Some statistics of the data are shown in Figure 15.

	Plurality	Borda	Veto	STV	Black	Maximin	Schulze	Ranked pairs	Copeland$_{0.5}$
CC	96.8%	92.4%	74.2%	99.7%	100%	100%	100%	100%	100%
PAR	100%	100%	100%	99.7%	99.4%	100%	100%	100%	99.7%

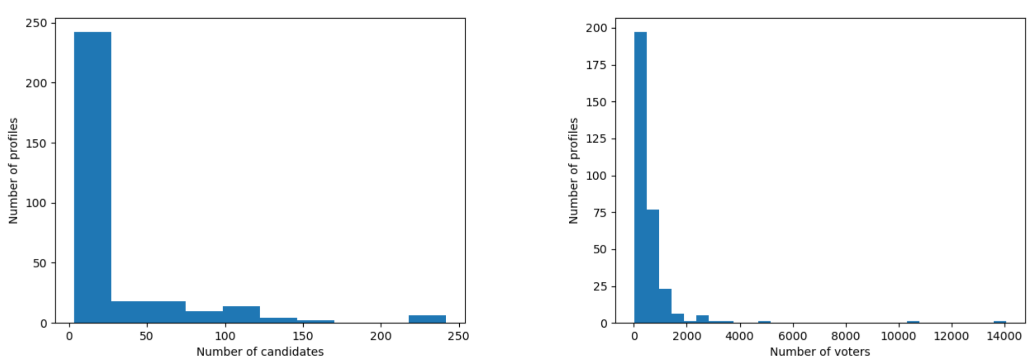


Figure 15: Histograms of number of candidates and number of voters in the 315 Preflib SOC data studied in this paper.

Table 7 delivers the following message, that PAR is less of a concern than CC in Preflib data—all voting rules have close to 100% satisfaction of PAR, while the satisfaction of CC is much lower for plurality, Borda, and Veto. The most interesting observations are: first, maximin, Schulze, and ranked pairs achieve 100% satisfaction of CC and PAR in Preflib data, which is consistent with the belief that Schulze and ranked pairs are superior in satisfying voting axioms, and maximin is doing well in PAR (and indeed, maximin satisfies PAR when $m = 3$). Second, STV does well in CC and PAR, though it does not satisfy either in the worst case. Third, veto has poor satisfaction of CC (74.2%), which is mainly due to the profiles where the number of alternatives is more than the number of voters, so that a Condorcet winner exists and is also a veto co-winner, but loses due to the tie-breaking mechanism.