Binary Search Trees

Contents

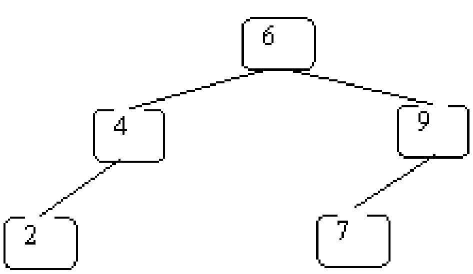
Introduction

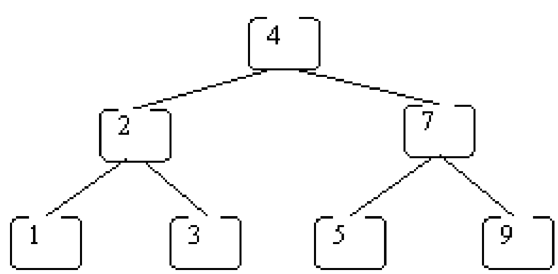
An important special kind of binary tree is the **binary search tree (BST)**. In a BST, each node stores some information including a unique **key value**, and perhaps some associated data. A binary tree is a BST iff, for every node n in the tree:

- All keys in n's left subtree are less than the key in n, and
- all keys in n's right subtree are greater than the key in n.

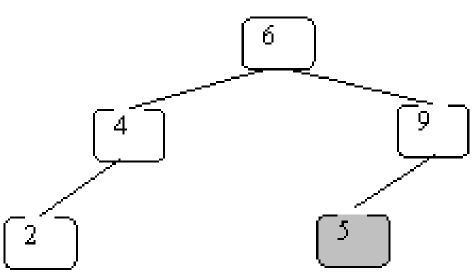
Note: if duplicate keys are allowed, then nodes with values that are equal to the key in node n can be either in n's left subtree or in its right subtree (but not both). In these notes, we will assume that duplicates are not allowed.

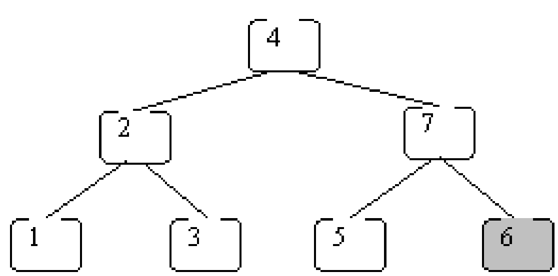
Here are some BSTs in which each node just stores an integer key:





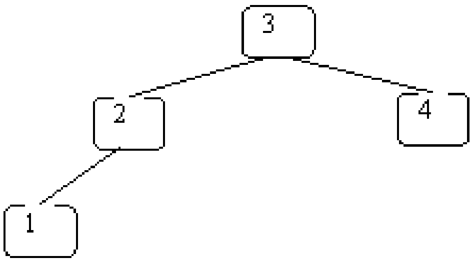
These are not BSTs:

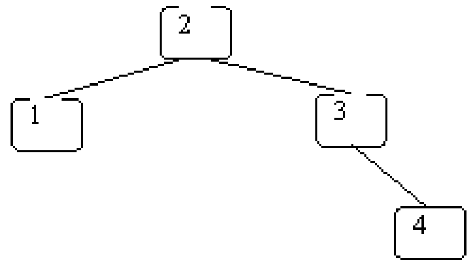




In the left one 5 is not greater than 6. In the right one 6 is not greater than 7.

Note that more than one BST can be used to store the same set of key values. For example, both of the following are BSTs that store the same set of integer keys:





The reason binary-search trees are important is that the following operations can be implemented efficiently using a BST:

- insert a key value
- determine whether a key value is in the tree
- remove a key value from the tree
- print all of the key values in sorted order

<u>TEST YOURSELF #1</u>

Question 1: Which of the following binary trees are BSTs? If a tree is **not** a BST, say why.

```
    A                 10               cat              15
   / \               /                /   \            /  \
  B   C             5              bat    rat         5    22
                   /              /                    \     \
                 -3             ant                    20    30
```

Question 2: Using which kind of traversal (preorder, postorder, inorder, or level-order) visits the nodes of a BST in sorted order?

[solution]

Implementing BSTs

To implement a binary search tree, we will use two classes: one for the individual tree nodes, and one for the BST itself. The following class definitions assume that the BST will store only key values, no associated data. Because most of the BST operations require comparing key values, the type used for the key is **Comparable** (not Object).

```
class BinaryTreenode {
   // *** fields ***
   private Comparable key;
   private BinaryTreenode left, right;

   // *** methods ***

   // constructor
```

```
    public BinaryTreenode(Comparable k, BinaryTreenode l, BinaryTreenode r) {
        key = k;
        left = l;
        right = r;
    }

    // access to fields
    public Comparable getKey() {return key;}
    public BinaryTreenode getLeft() {return left;}
    public BinaryTreenode getRight() {return right;}

    // change fields
    public void setKey(Comparable k) {key = k;}
    public void setLeft(BinaryTreenode l) {left = l;}
    public void setRight(BinaryTreenode r) {right = r;}
}

class BST {
  // *** fields ***
  private BinaryTreenode root; // ptr to the root of the BST

  // *** methods ***
  public BST() { root = null; } // constructor
  public void insert(Comparable key) throws DuplicateException {...}
      // add key to this BST; error if it is already there
  public void delete(Comparable key) {...}
      // remove the node containing key from this BST if it is there;
      // otherwise, do nothing
  public boolean lookup(Comparable key) {...}
     // if key is in this BST, return true; otherwise, return false
  public void print(PrintWriter p) {...}
     // print the values in this BST in sorted order (to p)
}
```

To implement a BST that stores some data with each key, we would use the following class definitions (changes are in red):

```
class BinaryTreenode {
   // *** fields ***
   private Comparable key;
   private Object data;
   private BinaryTreenode left, right;

   // *** methods ***

   // constructor
   public BinaryTreenode(Comparable k, Object d,
                         BinaryTreenode l, BinaryTreenode r) {
       key = k;
       data = d;
       left = l;
       right = r;
   }

   ...
   public Object getData() {return data;}
   public void setData(Object ob) { data = ob; }
   ...
}
```

```
class BST {
  // *** fields ***
  private BinaryTreenode root; // ptr to the root of the BST

  // *** methods ***
  public BST() { root = null; } // constructor
  public void insert(Comparable key, Object data) throws DuplicateException {...}
      // add key and associated data to this BST;
      // error if key is already there
  public void delete(Comparable key) {...}
      // remove the node containing key from this BST if it is there;
      // otherwise, do nothing
  public Object lookup(Comparable key) {...}
     // if key is in this BST, return its associated data; otherwise, return null
  public void print(PrintWriter p) {...}
     // print the values in this BST in sorted order (to p)
}
```

From now on, we will assume that BSTs only store key values, **not** associated data. We will also assume that null is not a valid key value (i.e., if someone tries to insert or lookup a null value, that should cause an exception).

The lookup method

In general, to determine whether a given value is in the BST, we will start at the root of the tree and determine whether the value we are looking for:

- is in the root
- might be in the root's left subtree
- might be in the root's right subtree

There are actually **two** base cases:

1. The tree is empty; return false.
2. The value is in the root node; return true.

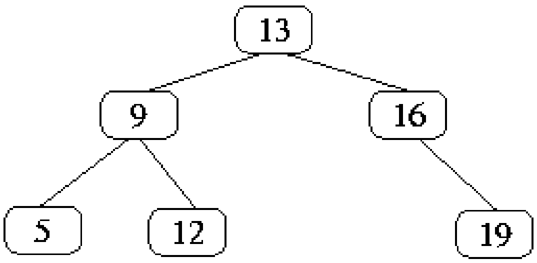
If neither base case holds, a recursive lookup is done on the appropriate subtree. Since all values less than the root's value are in the left subtree, and all values greater than the root's value are in the right subtree, there is no point in looking in **both** subtrees: if the value we're looking for is less than the value in the root, it can only be in the left subtree (and if it is greater than the value in the root, it can only be in the right subtree).

The code for the lookup method uses an auxiliary, recursive method with the same name (i.e., the lookup method is overloaded):

```
public boolean lookup(Comparable k) {
    return lookup(root, k);
}

private static boolean lookup(BinaryTreenode T, Comparable k) {
    if (T == null) return false;
    if (T.getKey().equals(k)) return true;
    if (k.compareTo(T.getKey()) < 0) {
        // k < this node's key; look in left subtree
        return lookup(T.getLeft(), k);
```

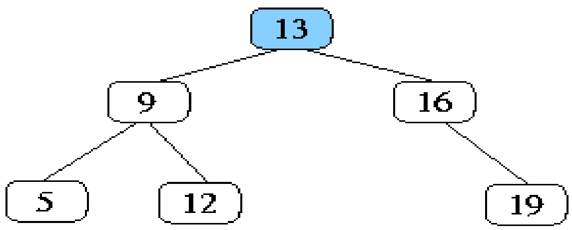
```
    }
    else {
        // k > this node's key; look in right subtree
        return lookup(T.getRight(), k);
    }
}
```

Let's illustrate what happens using the following BST:

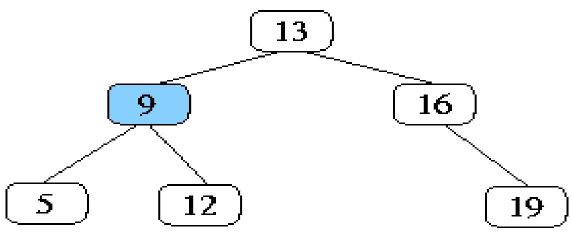


and searching for 12:

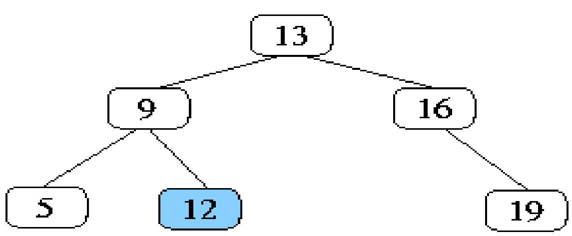
12 < 13 so go to left subtree



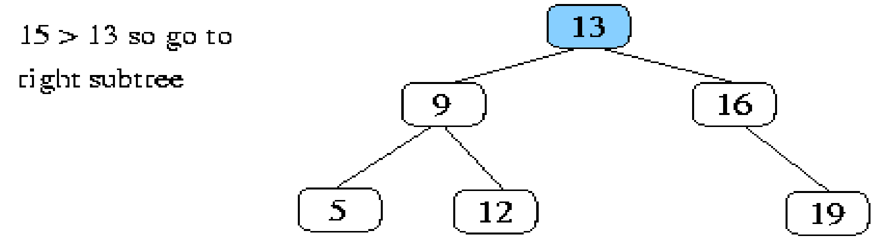
12 > 9 so go to right subtree.

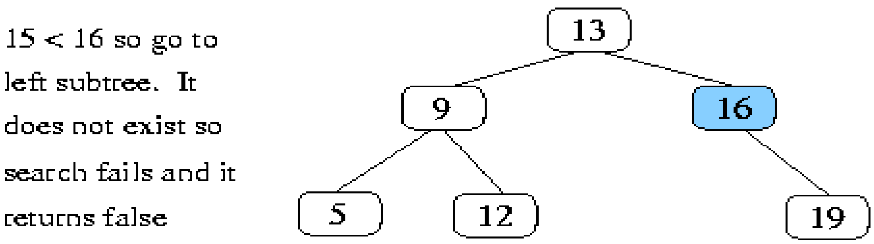


found!



What if we search for 15:





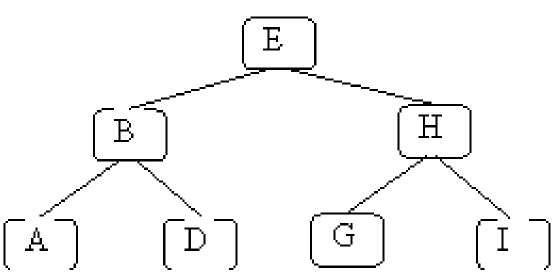
How much time does it take to search for a value in a BST? Note that lookup always follows a path from the root down towards a leaf. In the worst case, it goes all the way to a leaf. Therefore, the worst-case time is proportional to the length of the longest path from the root to a leaf (the height of the tree).

In general, we'd like to know how much time is required for lookup as a function of the number of values stored in the tree. In other words, what is the relationship between the number of nodes in a BST and the height of the tree? This depends on the "shape" of the tree. In the worst case, all nodes have just one child, and the tree is essentially a linked list. For example:

```
     50
    /
   10
     \
      15
        \
         30
        /
      20
```

This tree has 5 nodes, and also has height = 5. Searching for values in the range 16-19, and 21-19 will require following the path from the root down to the leaf (the node containing the value 20); i.e., will require time proportional to the number of nodes in the tree.

In the best case, all nodes have 2 children, and all leaves are at the same depth; for example:



This tree has 7 nodes, and height = 3. In general, a tree like this (a "full" tree) will have height approximately $\log_2(N)$, where N is the number of nodes in the tree. The value $\log_2(N)$ is (roughly) the number of times you can divide N by two, before you get to zero. For example:

```
7/2 = 3      // divide by 2 once
3/2 = 1      // divide by 2 a second time
1/2 = 0      // divide by 2 a third time, the result is zero so quit
```

So $\log_2(7)$ is approximately equal to 3.

The reason we use $\log_2$. (rather than say $\log_3$) is because every non-leaf node in a full BST has **two** children. The number of nodes in each of the root's subtrees is (approximately) 1/2 of the nodes in the whole tree, so the length of a path from the root to a leaf will be the same as the number of times we can divide N (the total number of nodes) by 2.

However, when we use big-O notation, we just say that the height of a full tree with N nodes is O(log N) -- we drop the "2" subscript, because $\log_2(N)$ is proportional to $\log_k(N)$ for any constant k; i.e., for any constants B and k, and any value N:

$$\log_B(N) = \log_k(N) / \log_k(B)$$

and with big-O notation we always ignore constant factors.

To summarize: The worst-case time required to do a lookup in a BST is O(height of tree). In the worst case (a "linear" tree), this is O(N), where N is the number of nodes in the tree. In the best case (a "full" tree), this is O(log N).

The insert method

Where should a new item go in a BST? The answer is easy: it needs to go where you would have found it using lookup! If you don't put it there then you won't find it later.

The code for insert is given below. Note that:

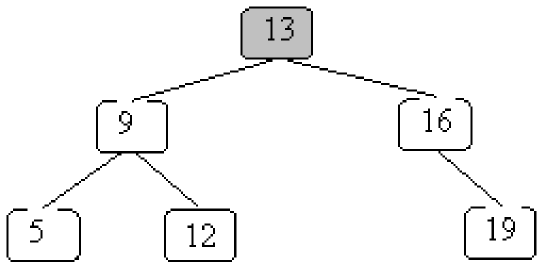
- We assume that duplicates are not allowed (an attempt to insert a duplicate value causes an exception).
- The BST class's insert method only inserts the key if the tree is empty, otherwise, it uses an auxiliary recursive method to do the insertion.
- The node containing the new value is always inserted as a **leaf** in the BST.

```
public void insert(Comparable k) throws DuplicateException {
    if (root == null) {
        root = new BinaryTreenode(k, null, null);
    }
    else insert(root, k);
}

private static void insert(BinaryTreenode T, Comparable k) throws DuplicateException {
// precondition: T != null
    if (T.getKey().equals(k)) throw new DuplicateException();
    if (k.compareTo(T.getKey()) < 0) {
        // add k as left child of T if it doesn't already have one
        // else insert into T's left subtree
        if (T.getLeft() == null) T.setLeft( new BinaryTreenode(k, null, null) );
        else insert(T.getLeft(), k);
    }
```

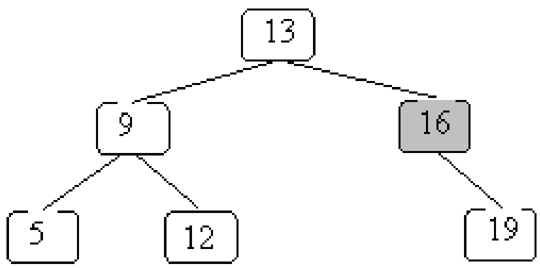
```
    else {
        // here when k > T's key
        // insert k as right child of T if it doesn't already have one
        // else insert into T's right subtree
        if (T.getRight() == null) T.setRight( new BinaryTreenode(k, null, null) );
        else insert(T.getRight(), k);
    }
}
```

Here are pictures illustrating what happens when we insert the value 15 into the example tree used above.

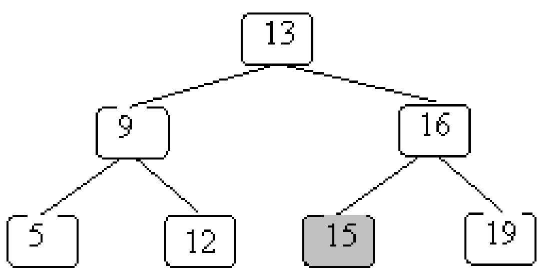
15 > 13 so go to right subtree



15 < 16 and no left subtree



So insert 15 as left child



It is easy to see that the complexity for insert is the same as for lookup: in the worst case, a path is followed all the way to a leaf.

TEST YOURSELF #2

As mentioned above, the order in which values are inserted determines what BST is built (inserting the same values in different orders can result in different final BSTs). Draw the BST that results from inserting the values 1 to 7 in each of the following orders (reading from left to right):

1. 5 3 7 6 2 1 4
2. 1 2 3 4 5 6 7
3. 4 3 5 2 6 1 7

solution

The delete method

As you would expect, deleting an item involves a search to locate the node that contains the value to be deleted. Here is an outline of the code for the delete method.

```
public void delete(Comparable k) {
    root = delete(root, k);
}

private static BinaryTreenode delete(BinaryTreenode T, Comparable k) {
    if (T == null) return null;
    if (k.equals(T.getKey())) {
       // T is the node to be removed
       // code must be added here
    }
    else if (k.compareTo(T.getKey()) < 0) {
        T.setLeft( delete(T.getLeft(), k) );
        return T;
    }
    else {
        T.setRight( delete(T.getRight(), k) );
        return T;
    }
}
```

There are several things to note about this code:

- As for the lookup and insert methods, the BST delete method uses an auxiliary, overloaded delete method to do the actual work.
- If k in not in the tree, then eventually the auxiliary method will be called with T==null. That is not considered an error; the tree is simply unchanged in that case.
- The auxiliary delete method returns a value (a pointer to the updated tree). The reason for this is explained below.

If the search for the node containing the value to be deleted succeeds, there are three cases to deal with:

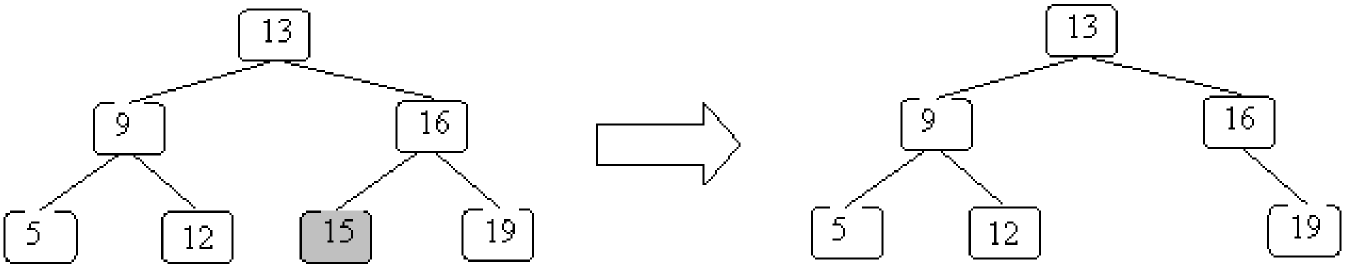
1. The node to delete is a leaf (has no children).
2. The node to delete has one child.
3. The node to delete has two children

When the node to delete is a leaf, we want to remove it from the BST by setting the appropriate child pointer of its parent to null (or by setting root to null if the node to be deleted is the root, and it has no children). Note that the call to delete was one of the following:

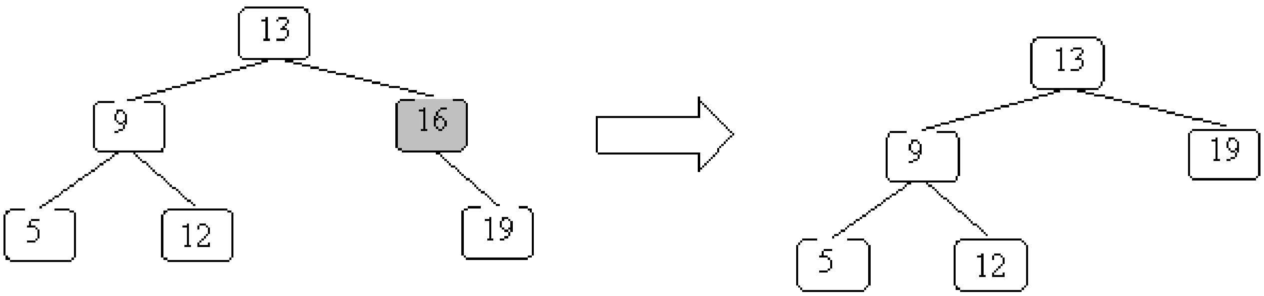
- root = delete(root, k);
- T.setLeft(delete(T.getLeft(), k));
- T.setRight(delete(T.getRight(), k));

So in all three cases, the right thing happens if the delete method just returns null.

Here's what happens when the node containing the value 15 is removed from the example BST:



When the node to delete has one child, we can simply replace that node with its child by returning a pointer to that child. As an example, let's delete 16 from the BST just formed:



Here's the code for delete, handling the two cases we've discussed so far (the new code is shown in red):

```
private static BinaryTreenode delete(BinaryTreenode T, Comparable k) {
    if (T == null) return null;
    if (k.equals(T.getKey())) {
       // T is the node to be removed
       if (T.getLeft() == null && T.getRight() == null) return null;
       if (T.getLeft() == null) return T.getRight();
       if (T.getRight() == null) return T.getLeft();

       // here if T has 2 children
       // code still needs to be added here...
    }
    else if (k.compareTo(T.getKey()) < 0) {
        T.setLeft( delete(T.getLeft(), k) );
        return T;
    }
    else {
        T.setRight( delete(T.getRight(), k) );
        return T;
    }
}
```

The hard case is when the node to delete has two children. We'll call the node to delete n. We can't replace node n with one of its children, because what would we do with the other child? Instead, we will replace node n with another node, x, lower down in the tree, then (recursively) delete node x.

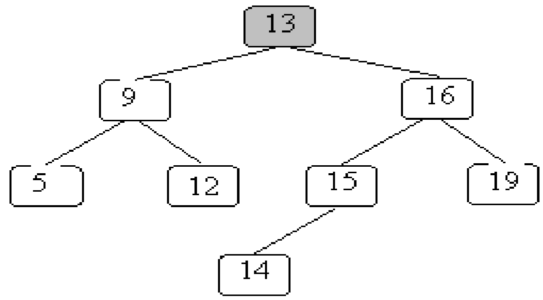
The question is what node can we use to replace node n? We have to choose that node so that the tree is still a BST; i.e., so that all of the values in n's left subtree are less than the value in n, and all of the values in n's right subtree are greater than the value in n. There are two possibilities that work: the node in the left subtree with the **largest** value, or the node in the right subtree with the **smallest** value. We'll arbitrarily decide to use the node in the right subtree (with the smallest value).

To find that node, we just follow a path in the right subtree, always going to the **left** child, since smaller values are in left subtrees. Once the node is found, we copy its key into node n, then we recursively delete the copied node. Here's the final version of the delete method:

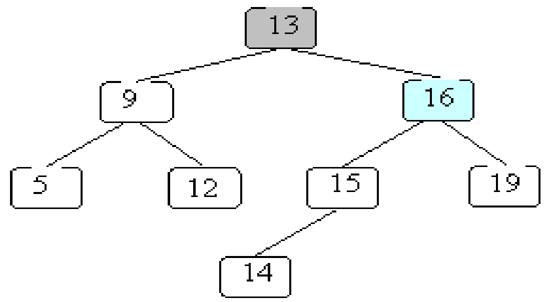
```
private static BinaryTreenode delete(BinaryTreenode T, Comparable k) {
    if (T == null) return null;
    if (k.equals(T.getKey())) {
       // T is the node to be removed
       if (T.getLeft() == null && T.getRight() == null) return null;
       if (T.getLeft() == null) return T.getRight();
       if (T.getRight() == null) return T.getLeft();

       // here if T has 2 children
       BinaryTreenode tmp = smallestNode(T.getRight());
       // copy key field from tmp to T
       T.setKey( tmp.getKey() );

       // now delete tmp from T's right subtree and return
       T.setRight( delete(T.getRight(), tmp.getKey()) );
       return T;

    }
    else if (k.compareTo(T.getKey()) < 0) {
        T.setLeft( delete(T.getLeft(), k) );
        return T;
    }
    else {
        T.setRight( delete(T.getRight(), k) );
        return T;
    }
}
```

Below is a slightly different example BST; let's see what happens when we delete 13 from that tree.

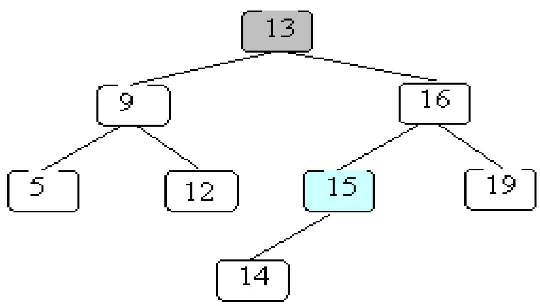
Original BST with 13 located



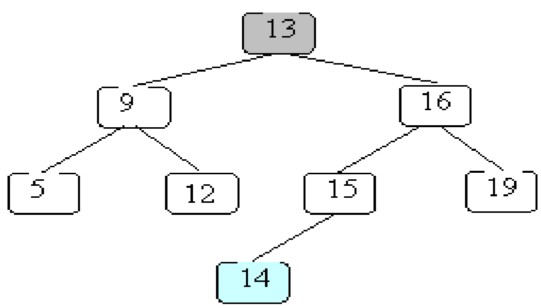
Step into right subtree.



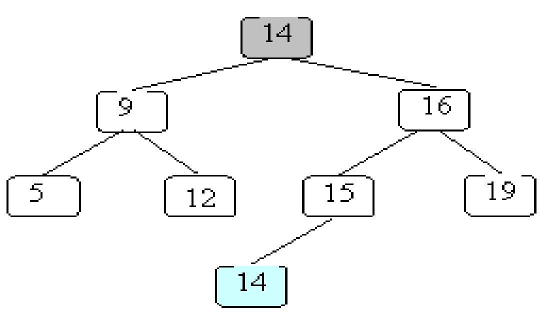
Go to left child.



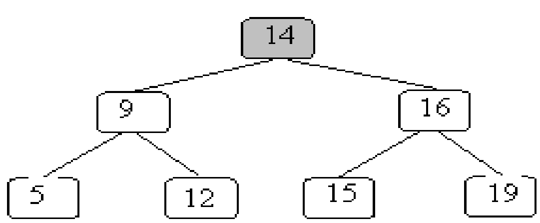
Continue to left child. This is last one.



Replace node to delete with far left child of right subtree.



Remove far left child of right subtree.



TEST YOURSELF #3

Write the auxiliary method **smallestNode** used by the delete method given above. The header for smallestNode is:

```
private static BinaryTreenode smallestNode(BinaryTreenode T)
// precondition: T is not null
// postcondition: return the node in the subtree rooted at T that
//                has the smallest value
```

solution

What is the complexity of the BST delete method?

If the node to be deleted has zero or one child, then the delete method will "follow a path" from the root to that node. So the worst-case time is proportional to the height of the tree (just like for lookup and insert).

If the node to be deleted has two children, the following steps are performed:

1. Find the node to be deleted (follow the path from the root to that node).
2. Find the node x in the right subtree with the smallest value (continue down the path toward a leaf).
3. Recursively delete node x (follow the same path followed in step 2).

So in the worst case, a path from the root to a leaf is followed twice. Since we don't care about constant factors, the time is still proportional to the height of the tree.

Summary

A binary-search tree can be used to store any objects that implement the Comparable interface (i.e., that define the compareTo method). A BST can also be used to store Comparable objects plus some associated data. The advantage of using a binary-search tree (instead of, say, a linked list) is that, if the tree is reasonably balanced (shaped more like a "full" tree than like a "linear" tree) the insert, lookup, and delete operations can all be implemented to run in O(log N) time, where N is the number of stored items. For a linked list, although insert can be implemented to run in O(1) time, lookup and delete take O(N) time.

Logarithmic time is generally **much** faster than linear time. For example, for N = 1,000,000: $\log_2 N = 20$.

Of course, it is important to remember that for a **"linear"** tree (one in which every node has one child), the worst-case times for insert, lookup, and delete will be O(N).