$f(n) = 3n^2 + 8n + 2$

$g(n) = n^2$.

$$3n^2 + 8n + 2 \le 3n^2 + 8n^2 + 2n^2$$

$$3n^2 + 8n + 2 \le \underbrace{13}_{c}\, \underbrace{n^2}_{g(n)}$$

$$\forall\ n \geq 1.$$

Big-Oh

$$f(n) \le c\, g(n)$$

$f(n) = O(n^2)$

Also $f(n) = O(n^3)$
$f(n) = O(2^n)$

Big-Ω

$$3n^2 + 8n + 2 \geq \underbrace{1}_{c} \times \underbrace{n^2}_{g(n)} \quad \forall\ n \geq 1.$$

$f(n) = \Omega(n^2)$

Also $f(n) = \Omega(n)$
$f(n) = \Omega(\sqrt{n})$

Big-Θ (tight bound)

$$1n^2 \le 3n^2 + 8n + 2 \le 13n^2$$

So $c_1 = 1$
$c_2 = 13$
$n_0 = 1$.

$$\forall\ n \geq 1$$

$f(n) = \Theta(n^2)$

$f(n) \neq \Theta(n^3)$ cuz left-hand inequality won't satisfy.

$f(n) \neq \Theta(\lg n)$. cuz right-hand inequality won't satisfy.

Θ notation is also called average notation.

Example 3

$$\frac{1}{2}n^2 + 3n = \Theta(n^2).$$

Big-Oh:

$$\frac{1}{2}n^2 + 3n \leq \frac{1}{2} \times n^2 + 3n^2 \quad \forall\ n \geq 1.$$

$$\frac{1}{2} \times n^2 + 3n \leq \frac{7n^2}{2}$$

Big-Ω:

ignore the +ve term

$$\frac{1}{2} \times n^2 \leq \frac{1}{2}n^2 + 3n. \quad \forall\ n \geq 1.$$

Big-Θ:

$$\underset{\substack{\uparrow \\ c_1}}{\frac{1}{2}}n^2 \leq \frac{1}{2}n^2 + 3n \leq \underset{\substack{\uparrow \\ c_2.}}{\frac{7}{2}}n^2$$

$n_0 = 1.$

Q1 For given functions, indicate whether $f(n)=O(g(n))$, $f(n)=\Omega(g(n))$ or both i.e $f(n)=\Theta(g(n))$

$f(n) = n\log n$ $g(n) = 10n\log(10n)$

Argument: Both $f(n)$ & $g(n)$ are $O(n\log n)$
$f(n)$ & $g(n)$ are functions of same growth rate, so they are just constant multiples of each other and they can be upper & lower bound of each other for different constants, so

$$f(n) = \Theta(g(n))$$

Mathematical proof:

Big-Oh

$f(n) \le c\,g(n)$

$n\log n \le c(10n\log 10n)$

let $c=1$, the above inequality holds for all $n>1$ so, $n_0=1$

Big-Ω

$f(n) \ge c\,g(n)$

$\cancel{n}\log n \ge c\,(10\cancel{n}\log(10n))$

$\log n \ge c10[\log 10 + \log n]$

$\log n \ge c\times 10\,[1+\log n]$

$\frac{1}{10} \times \boxed{\frac{\log n}{(1+\log n)}} \ge c$

this factor will always be less than 1 (At max it will be 1 for $n=1$). So we can choose c to be any value less than $\frac{1}{10}$. let $c=\frac{1}{100}$ & solve for n_0.

So, $f(n) = \Theta(g(n))$.

Q1

$f(n) = 100n + \log n$, $g(n) = n + (\log n)^2$

if $f(n) = O(g(n))$ & $f(n) = \Omega(g(n))$

then $f(n) = \Theta(g(n))$.

Big-Oh

$f(n) \leq c\, g(n)$.

$100n + \log n \leq c\,(n + (\log n)^2)$

let $c = 100$

$100n + \log n \leq 100\,(n + (\log n)^2)$

$\cancel{100n} + \log n \leq \cancel{100n} + 100(\log n)^2$

$\log n \leq 100\,(\log n)^2$

$1 \leq 100\,(\log n)$

$\boxed{n_0 = 10}$

So, $f(n) = O(g(n))$

Big-Ω

$f(n) \geq c\, g(n)$

$100n + \log n \geq c\,(n + (\log n)^2)$

let $c = 1$ & inequality will hold for all $n \geq 10$

So, $n_0 = 10$

So, $f(n) = \Omega(g(n))$

$$\boxed{f(n) = \Theta(g(n))}$$

$$k_2 n^2 \le \frac{n^2}{800} - 400n + 36 \le k_1 n^2$$

Prove that $f(n) = \frac{n^2}{800} - 400n + 36$ is $\Theta(n^2)$

(1) Let's solve for k_1

$$\frac{n^2}{800} - 400n + 36 \le k_1 n^2$$

divide both sides by n^2

$$\frac{1}{800} - \frac{400}{n} + \frac{36}{n^2} \le k_1.$$

$$\frac{400}{n} > \frac{36}{n^2}$$

So, a larger value is being subtracted from $\frac{1}{800}$ and a smaller value is being added. So, overall result on L.H.S would be less than $\frac{1}{800}$. So we can choose $k_1 = \frac{1}{800}$ & inequality will be satisfied. This works for $n_0 = 1$ ($\forall n > n_0$).

(2) Lets solve for k_2

$$k_2 \le \frac{1}{800} - \frac{400}{n} + \frac{36}{n^2}$$

Again by same analysis as above, R.H.S would be smaller than $\frac{1}{800}$ (as a larger value i.e. $\frac{400}{n}$ is being subtracted and a smaller value i.e $\frac{36}{n^2}$ is being added). So k_2 would be less than $\frac{1}{800}$. lets choose $k_1 = \frac{1}{8000}$. Plugging this value of k_1 we can find n_0. ($n_0 = 360{,}000$)

$$\boxed{\text{So, } k_1 = \frac{1}{8000} \quad k_2 = \frac{1}{800} \quad n_0 = 360{,}000}$$

There can be multiple solutions but k_1, k_2, n_0 are all +ve constants.