# Greedy Algorithms

## Huffman Coding

# Encoding

**Example**

Suppose we want to store a given a 100, 000 character data file. The file contains only 6 characters, appearing with the following frequencies:

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| Frequency in '000s | 45 | 13 | 12 | 16 | 9 | 5 |

- A binary code encodes each character as a binary string or codeword over some given alphabet $\Sigma$
  - a code is a set of codewords.
  - e.g., $\{000, 001, 010, 011, 100, 101\}$ and $\{0, 101, 100, 111, 1101, 1100\}$

  are codes over the binary alphabet $\Sigma = \{0, 1\}$.

# Encoding

Given a code $C$ over some alphabet it is easy to encode the message using $C$. Just scan through the message, replacing the characters by the codewords.

**Example**

$\Sigma = \{a, b, c, d\}$

If the code is

$$C_1 = \{a = 00,\ b = 01,\ c = 10,\ d = 11\}$$

then bad is encoded as 01 00 11

If the code is

$$C_2 = \{a = 0,\ b = 110,\ c = 10,\ d = 111\}$$

then bad is encoded as 110 0 111

# Fixed Length vs Variable Length Encoding

- In a fixed-length code each codeword has the same length.
- In a variable-length code codewords may have different lengths.

**Example**

| | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| Freq in '000s | 45 | 13 | 12 | 16 | 9 | 5 |
| fixed-len code | 000 | 001 | 010 | 011 | 100 | 101 |
| variable-len code | 0 | 101 | 100 | 111 | 1101 | 1100 |

*Note: since there are 6 characters, a fixed-length code must use at least 3 bits per codeword).*

- The fixed-length code requires $300,000$ bits to store the file.
- The variable-length code requires only $(45 \cdot 1 + 13 \cdot 3 + 12 \cdot 3 + 16 \cdot 3 + 9 \cdot 4 + 5 \cdot 4) \cdot 1000 = 224,000$bits, saving a lot of space!

- Goal is to save space!

# Decoding

$$C_1 = \{a = 00, b = 01, c = 10, d = 11\}.$$
$$C_2 = \{a = 0, b = 110, c = 10, d = 111\}.$$
$$C_3 = \{a = 1, b = 110, c = 10, d = 111\}$$

Given an encoded message, decoding is the process of turning it back into the original message.
Message is uniquely decodable if it can be decoded in only one way.

**Example**

Relative to $C_1$, 010011 is uniquely decodable to bad.
Relative to $C_2$, 1100111 is uniquely decodable to bad.
Relative to $C_3$, 1101111 is not uniquely decipherable it could have encoded either bad or acad.

In fact, *any* message encoded using $C_1$ or $C_2$ is uniquely decipherable. Unique decipherability property is needed in order for a code to be useful.

# Prefix Codes

Fixed-length codes are always uniquely decipherable. WHY?

We saw before that fixed-length codes do not always give the best compression though, so we prefer to use variable length codes.

**Definition**

A code is called a prefix (free) code if no codeword is a prefix of another one.

**Example**

$\{a = 0,\ b = 110,\ c = 01,\ d = 111\}$ is *not* a prefix code.
$\{a = 0,\ b = 110,\ c = 10,\ d = 111\}$ is a prefix code.

# Prefix Codes

Important Fact: Every message encoded by a prefix free code is uniquely decipherable.

- Because no codeword is a prefix of any other, we can always find the first codeword in a message, peel it off, and continue decoding.

**Example**

code: $\{a = 0, b = 110, c = 10, d = 111\}$.

$$01101100 = 01101100 = abba$$

We are therefore interested in finding *good* (best compression) prefix-free codes.

# The Optimal Source Coding Problem

## Huffman Coding Problem

Given an alphabet $A = \{a_1, \ldots, a_n\}$ with frequency distribution $f(a_i)$, find a binary prefix code $C$ for $A$ that minimizes the number of bits

$$B(C) = \sum_{i=1}^{n} f(a_i) L(c_i)$$

needed to encode a message of $\sum_{i=1}^{n} f(a_i)$ characters, where

- $c_i$ is the codeword for encoding $a_i$, and
- $L(c_i)$ is the length of the codeword $c_i$.

# Example

**Problem**

Suppose we want to store messages made up of 4 characters $a, b, c, d$ with frequencies 60, 5, 30, 5 (percents) respectively. What are the fixed-length codes and prefix-free codes that use the least space?

Solution:

| | | | | |
|---|---|---|---|---|
| characters | $a$ | $b$ | $c$ | $d$ |
| frequency | 60 | 5 | 30 | 5 |
| fixed-length code | 00 | 01 | 10 | 11 |
| prefix code | 0 | 110 | 10 | 111 |

To store these 100 characters,
(1) the fixed-length code requires $100 \times 2 = 200$ bits,
(2) the prefix code uses only

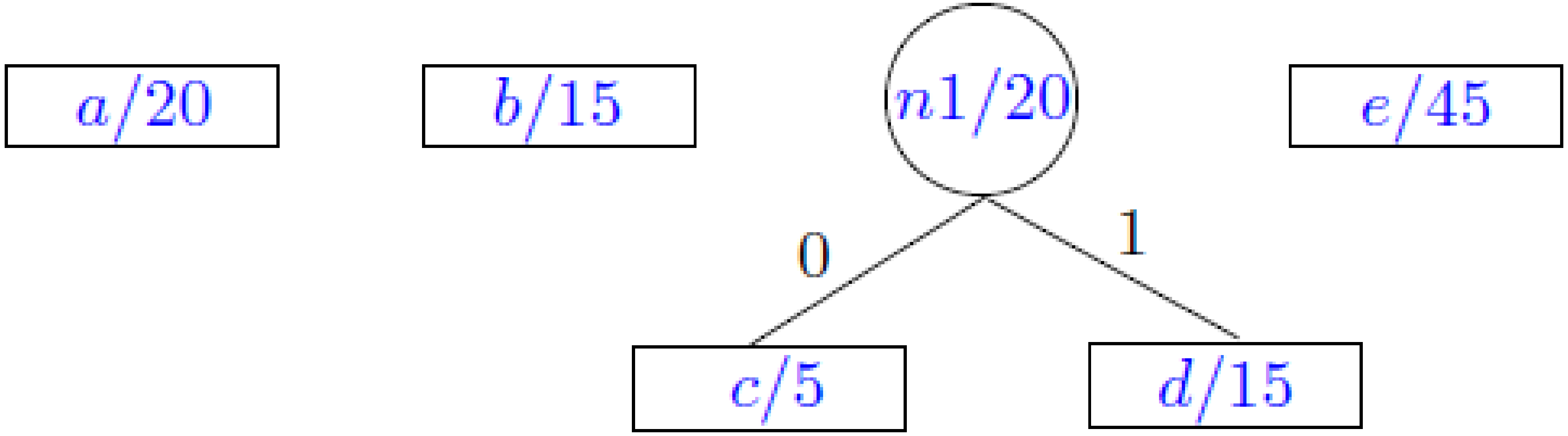
$$60 \times 1 + 5 \times 3 + 30 \times 2 + 5 \times 3 = 150 \text{ bits}$$

a 25% saving.

# Huffman Coding

Let $S = \{a/20, b/15, c/5, d/15, e/45\}$ be the original character set $S$ alphabet and its corresponding frequency distribution.

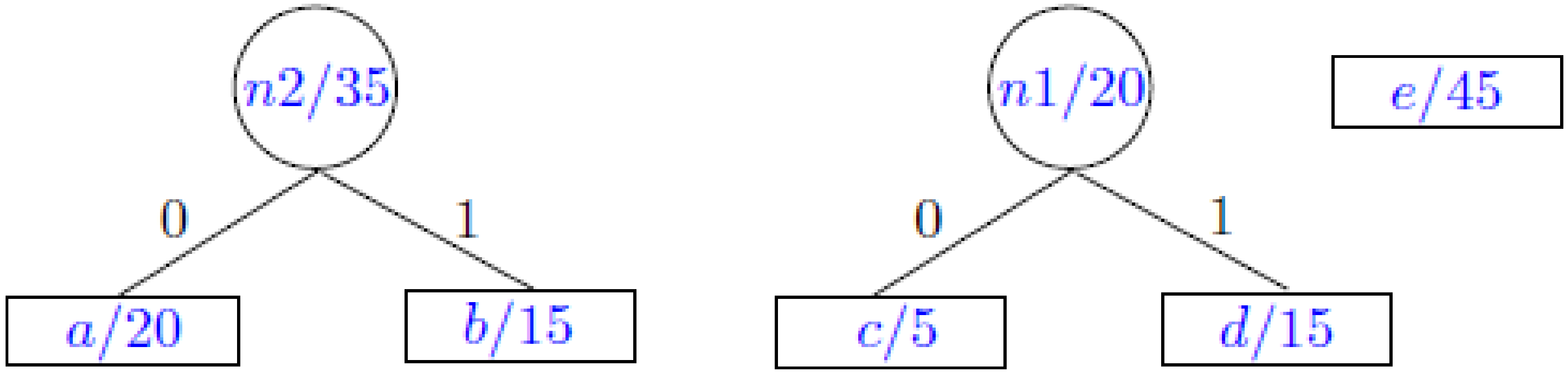
1. The first Huffman coding step merges $c$ **and** $d$. (could also have merged $c$ and $b$).




Now have $S = \{a/20, b/15, \mathbf{n1/20}, e/45\}$.

# Huffman Coding

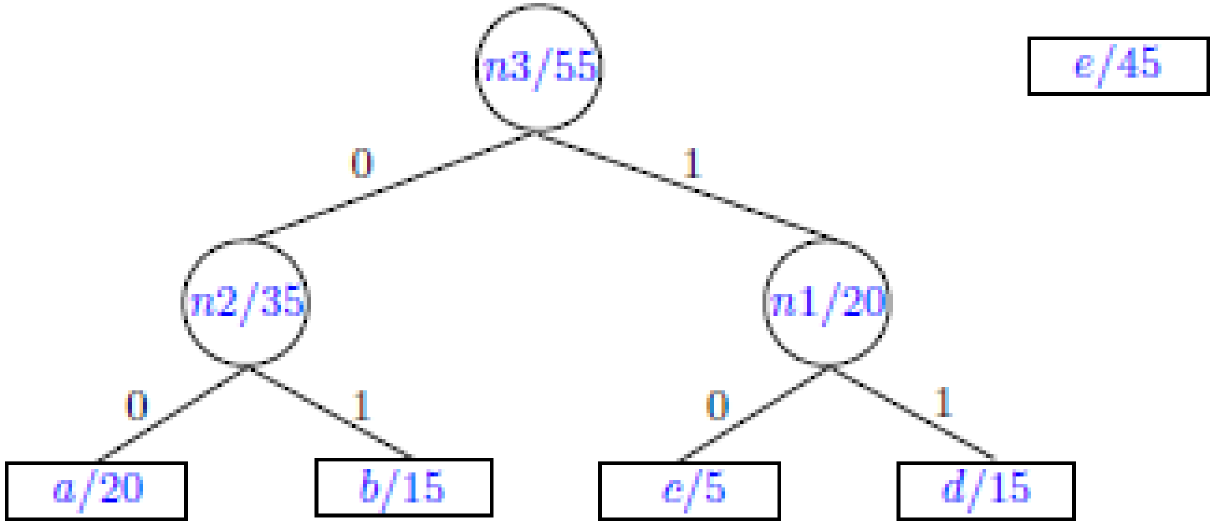
2. Algorithm merges $a$ **and** $b$ (could also have merged $n1$ and $b$)




Now have $S = \{n2/35, n1/20, e/45\}$.

# Huffman Coding

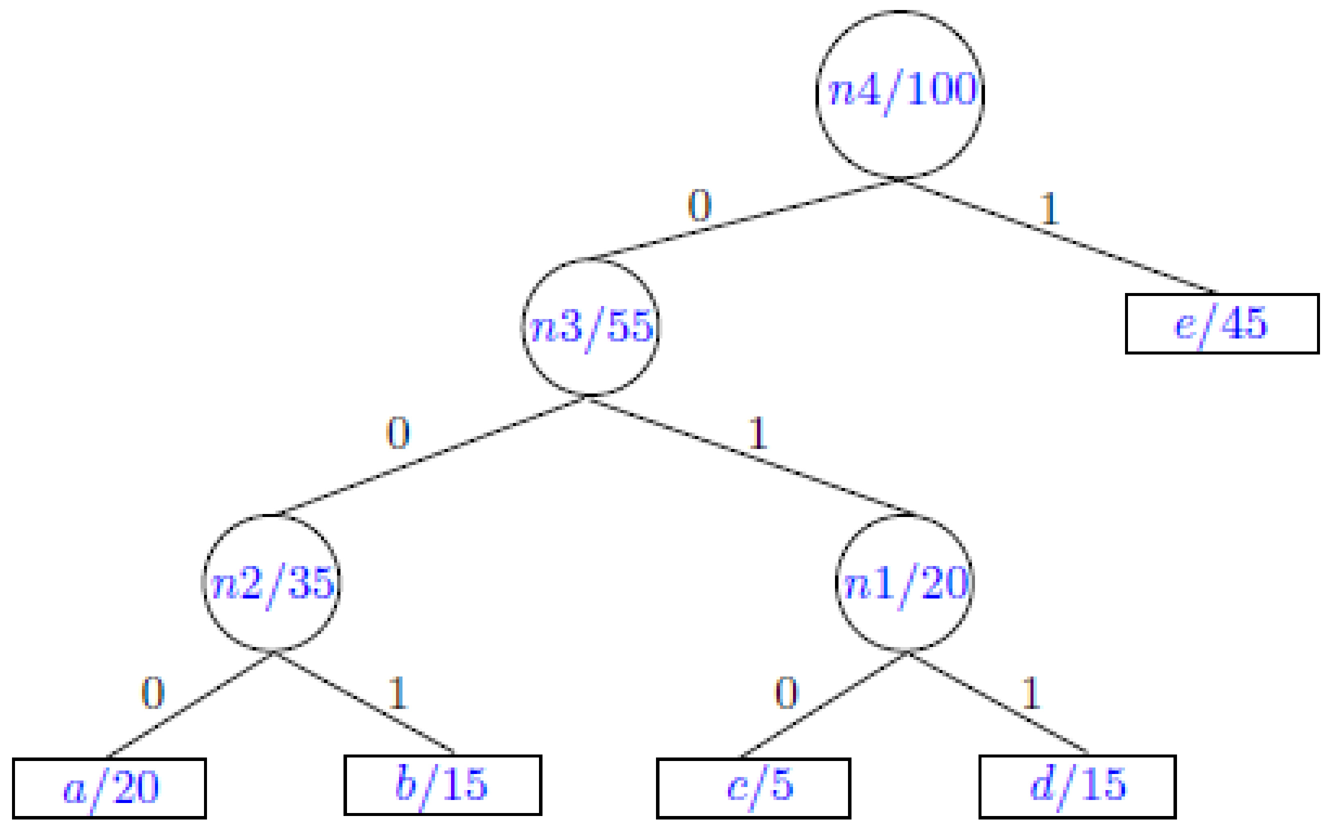
Algorithm merges $n1$ **and** $n2$.




Now have $S_3 = \{\mathbf{n3/55}, e/45\}$.

# Huffman Coding

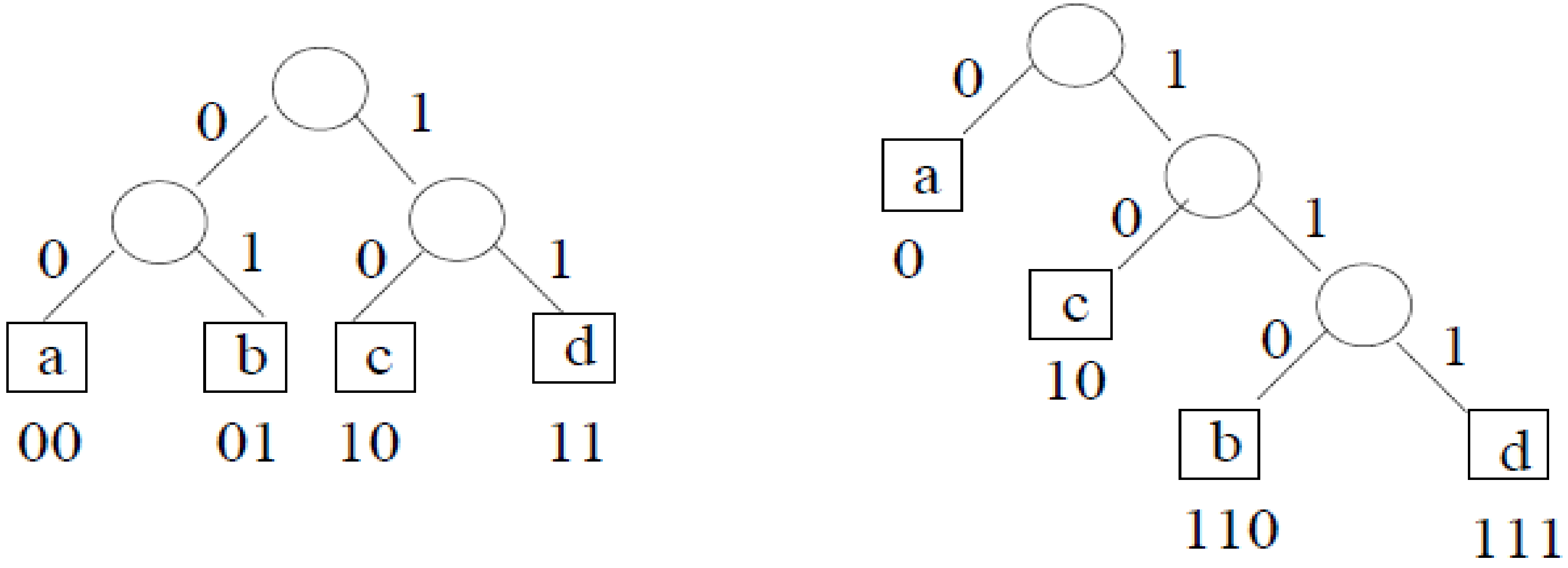
Algorithm next merges $e$ **and** $n3$ and finishes.




The Huffman code is:
$a = 000$, $b = 001$,
$c = 010$, $d = 011$,
$e = 1$.

# Correspondence between Binary Trees and Prefix Codes

1-1 correspondence between leaves and characters.




- Left edge is labeled 0; right edge is labeled 1
- The binary string on a path from the root to a leaf is the codeword associated with the character at the leaf.

# Huffman Coding - Algorithm

Set $S$ be the original set of message characters.

1. (Greedy idea)
   - Pick two smallest frequency characters $x, y$ from $S$.
   - Create a subtree that has these two characters as leaves.
   - Label the root of this subtree as $z$.
2. - Set frequency $f(z) = f(x) + f(y)$.
   - Remove $x, y$ from $S$ and add $z$ to $S$
     - $S = S \cup \{z\} - \{x, y\}$.
     - Note that $|S|$ has just decreased by one.

Repeat this procedure, called *merge*, with the new alphabet $S$, until $S$ has only one character left in it.

The resulting tree is the Huffman code tree.

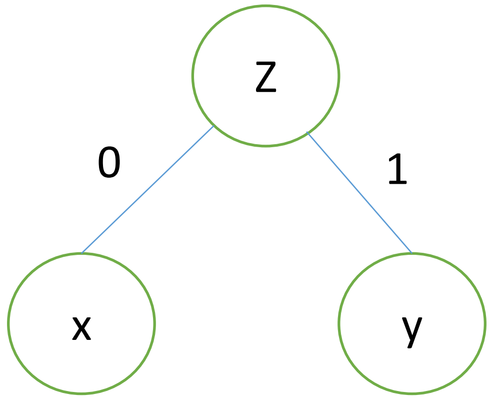
- It encodes the optimum (minimum-cost) prefix code for the given frequency distribution.

# Huffman Coding - Algorithm

Given character set $S$ with frequency distribution $\{f(a) : a \in S\}$:
The binary Huffman tree is constructed using a priority queue, $Q$, of nodes, with frequencies as keys.

Huffman($S$)

$n = |S|$;
$Q = S$; `// the future leaves`
**for** $i = 1$ **to** $n - 1$ **do**
  `// Why` $n - 1$`?`
  $z =$ new node;
  x = $left[z] =$ Extract-Min($Q$);
  y = $right[z] =$ Extract-Min($Q$);
  $f[z] = f[left[z]] + f[right[z]]$;
  Insert($Q, z$);
**end**
**return** Extract-Min($Q$); `// root of the tree`




Running time is $O(n \log n)$, as each priority queue operation takes time $O(\log n)$.

# Slide Credits

- COMP 3711H Design and Analysis of Algorithms
  Fall 2014