# Analysis of Algorithms
# CS 477/677

Sorting – Part A

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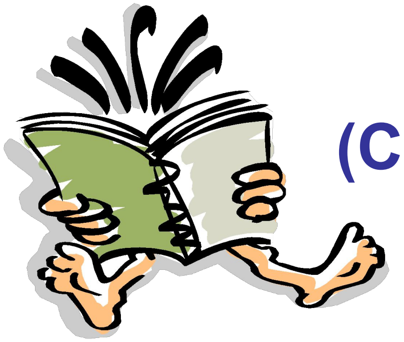


**(Chapter 2)**

# The Sorting Problem

- **Input:**
  - A sequence of $n$ numbers $a_1, a_2, \ldots, a_n$
- **Output:**
  - A permutation (reordering) $a_1', a_2', \ldots, a_n'$ of the input sequence such that $a_1' \leq a_2' \leq \cdots \leq a_n'$

# Structure of data

- Usually, the numbers to be sorted are part of a collection of data called a record

- Each record contains a key, which is the value to be sorted

example of a record

| | |
|---|---|
| **Key** | **other data** |

- Note that when the keys must be rearranged, the data associated with the keys must also be rearranged (time consuming !!)

- Pointers can be used instead (space consuming !!)

# Why Study Sorting Algorithms?

- There are a variety of situations that we can encounter
  - Do we have randomly ordered keys?
  - Are all keys distinct?
  - How large is the set of keys to be ordered?
  - Need guaranteed performance?
- Various algorithms are better suited to some of these situations

# Some Definitions

- Internal Sort
  - The data to be sorted is all stored in the computer’s main memory.
- External Sort
  - Some of the data to be sorted might be stored in some external, slower, device.
- In Place Sort
  - The amount of extra space required to sort the data is constant with the input size.

# Stability

- A STABLE sort preserves relative order of records with equal keys

Sorted on first key:

| | | | | |
|---|---|---|---|---|
| Aaron | 4 | A | 664-480-0023 | 097 Little |
| Andrews | 3 | A | 874-088-1212 | 121 Whitman |
| Battle | 4 | C | 991-878-4944 | 308 Blair |
| Chen | 2 | A | 884-232-5341 | 11 Dickinson |
| Fox | 1 | A | 243-456-9091 | 101 Brown |
| Furia | 3 | A | 766-093-9873 | 22 Brown |
| Gazsi | 4 | B | 665-303-0266 | 113 Walker |
| Kanaga | 3 | B | 898-122-9643 | 343 Forbes |
| Rohde | 3 | A | 232-343-5555 | 115 Holder |
| Quilici | 1 | C | 343-987-5642 | 32 McCosh |

Sort file on second key:

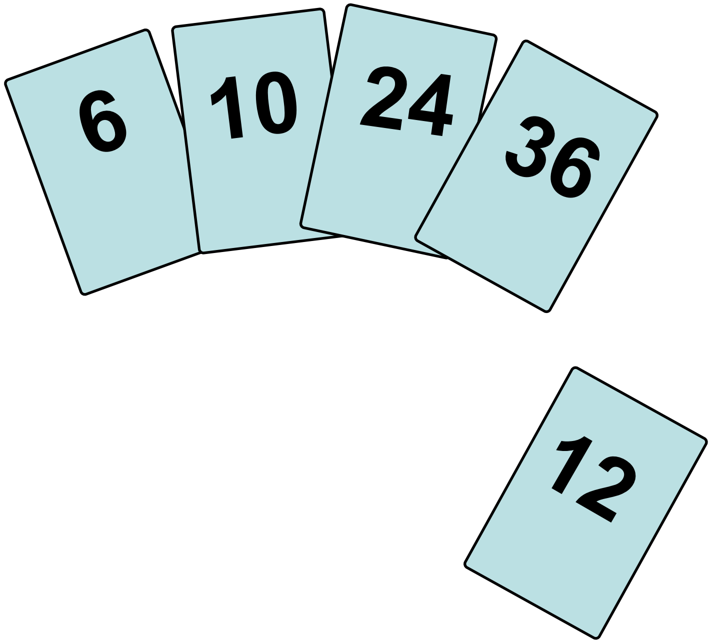
| | | | | |
|---|---|---|---|---|
| Fox | 1 | A | 243-456-9091 | 101 Brown |
| Quilici | 1 | C | 343-987-5642 | 32 McCosh |
| Chen | 2 | A | 884-232-5341 | 11 Dickinson |
| Kanaga | 3 | B | 898-122-9643 | 343 Forbes |
| Andrews | 3 | A | 874-088-1212 | 121 Whitman |
| Furia | 3 | A | 766-093-9873 | 22 Brown |
| Rohde | 3 | A | 232-343-5555 | 115 Holder |
| Battle | 4 | C | 991-878-4944 | 308 Blair |
| Gazsi | 4 | B | 665-303-0266 | 113 Walker |
| Aaron | 4 | A | 664-480-0023 | 097 Little |

Records with key value 3 are not in order on first key!!

# Insertion Sort

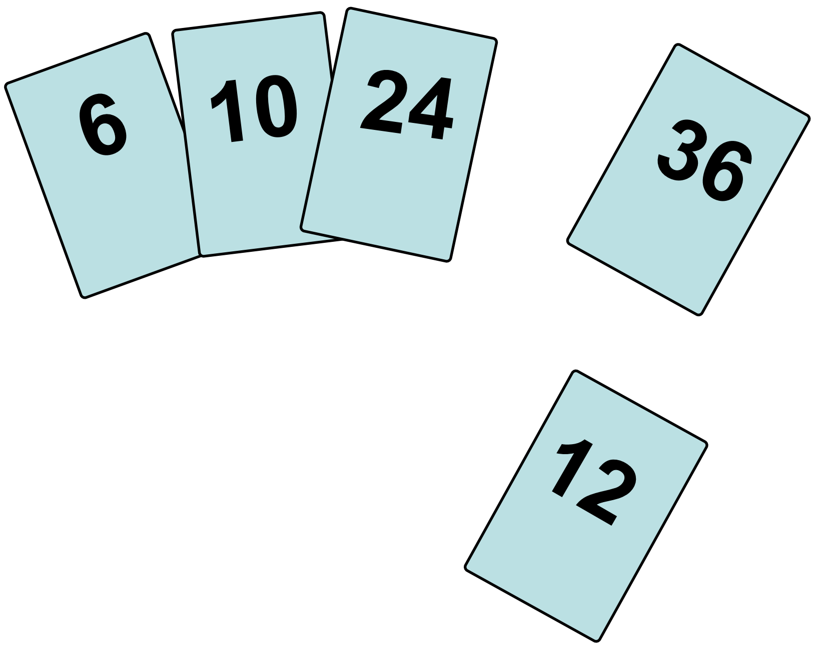
- Idea: like sorting a hand of playing cards
  - Start with an empty left hand and the cards facing down on the table.
  - Remove one card at a time from the table, and insert it into the correct position in the left hand
    - compare it with each of the cards already in the hand, from right to left
  - The cards held in the left hand are sorted
    - these cards were originally the top cards of the pile on the table

# Insertion Sort

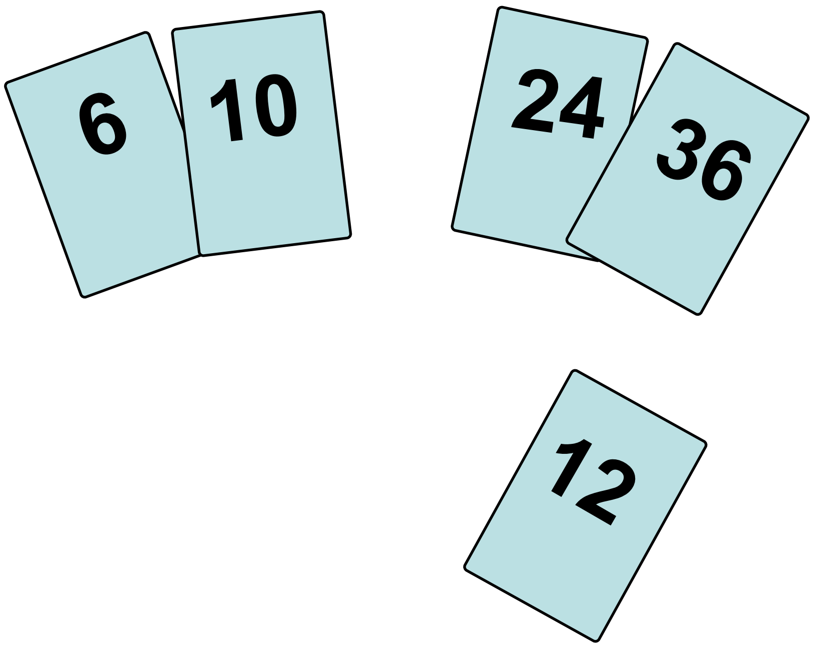



**To insert 12, we need to make room for it by moving first 36 and then 24.**

# Insertion Sort



# Insertion Sort



# Insertion Sort

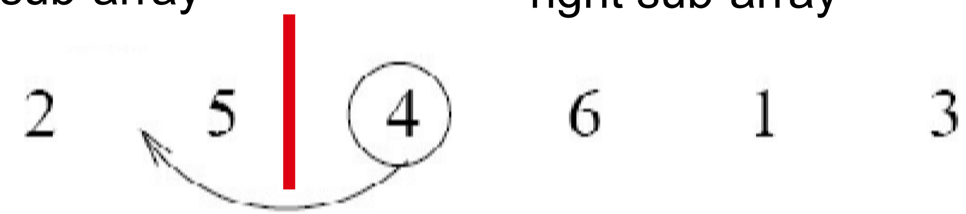
input array

5 2 4 6 1 3

at each iteration, the array is divided in two sub-arrays:

left sub-array

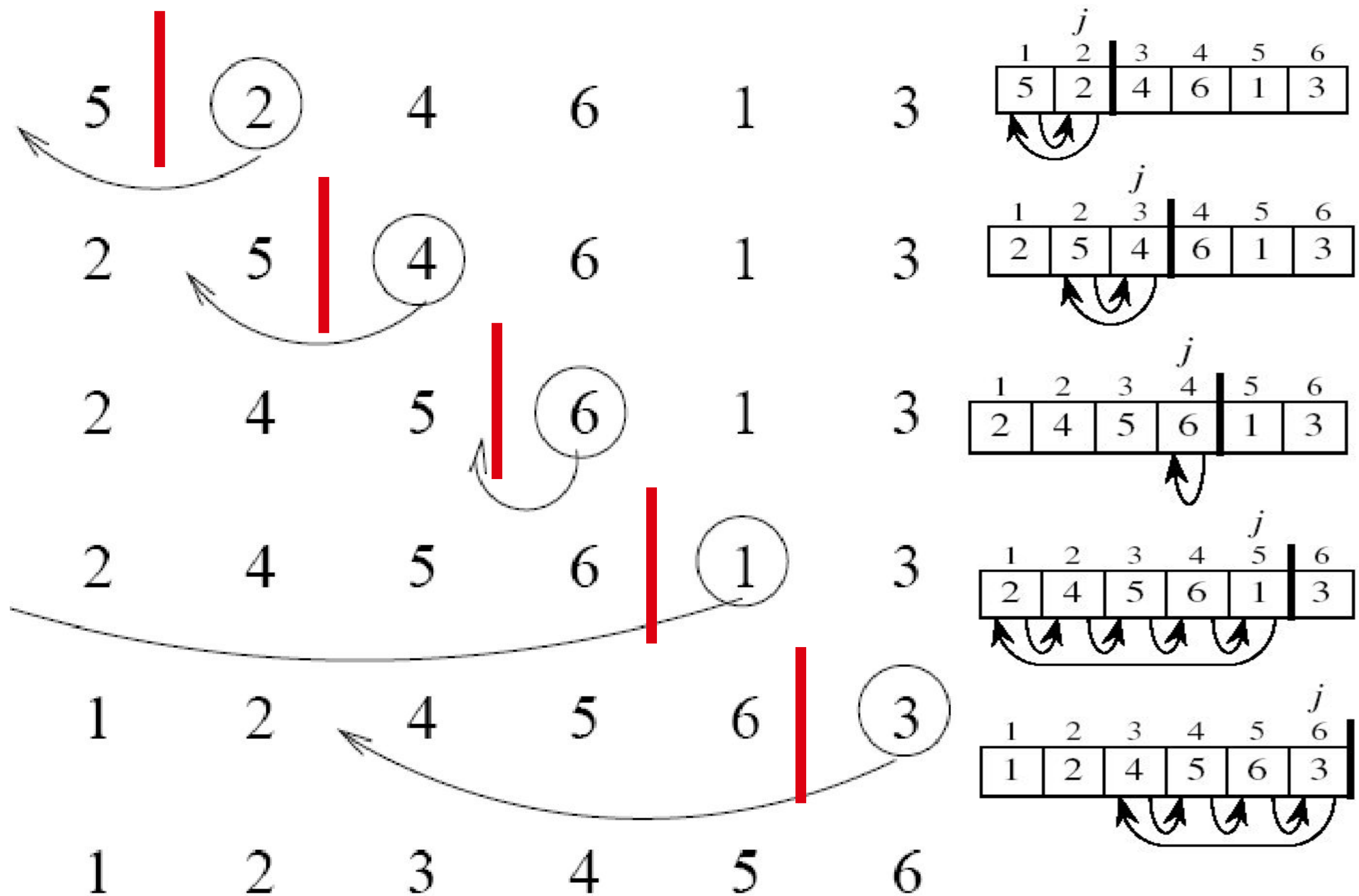
right sub-array




sorted

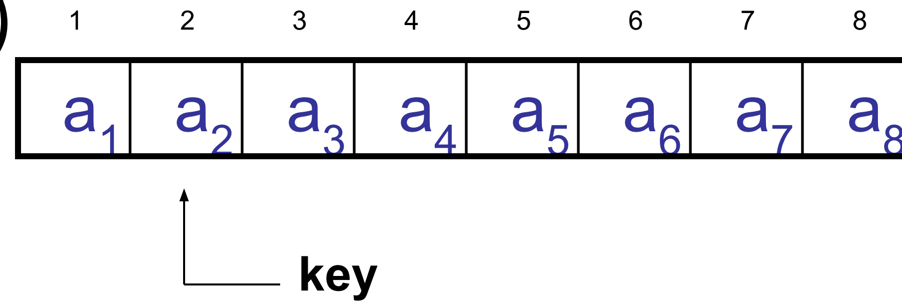
unsorted

# Insertion Sort



# INSERTION-SORT

*Alg.:* INSERTION-SORT(*A*)




```
for j ← 2 to n
  do key ← A[ j ]
       ▷ Insert A[ j ] into the sorted sequence A[1 . . j -1]
       i ← j - 1
       while i > 0 and A[i] > key
       do A[i + 1] ← A[i]
            i ← i - 1
       A[i + 1] ← key
```

- Insertion sort – sorts the elements in place

# Loop Invariant for Insertion Sort

*Alg.:* INSERTION-SORT*(A)*

**for** j ← 2 **to** n

  **do** key ← A[ j ]

   Insert A[ j ] into the sorted sequence A[1 . . j -1]

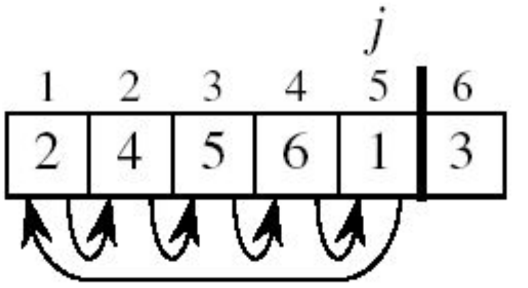
   i ← j - 1

   **while** i > 0 and A[i] > key

    **do** A[i + 1] ← A[i]

     i ← i - 1

   A[i + 1] ← key



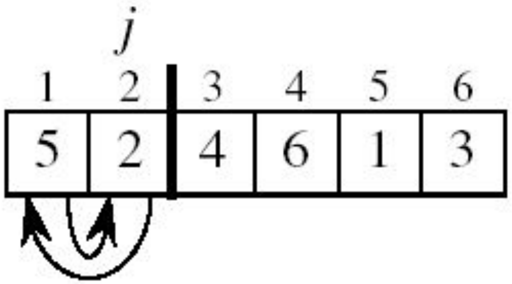

**Invariant**: at the start of the **for** loop the elements in A[1 . . j-1] are in sorted order

# Proving Loop Invariants

- Proving loop invariants works like induction
- **Initialization (base case):**
  - It is true prior to the first iteration of the loop
- **Maintenance (inductive step):**
  - If it is true before an iteration of the loop, it remains true before the next iteration
- **Termination:**
  - When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct
  - Stop the induction when the loop terminates

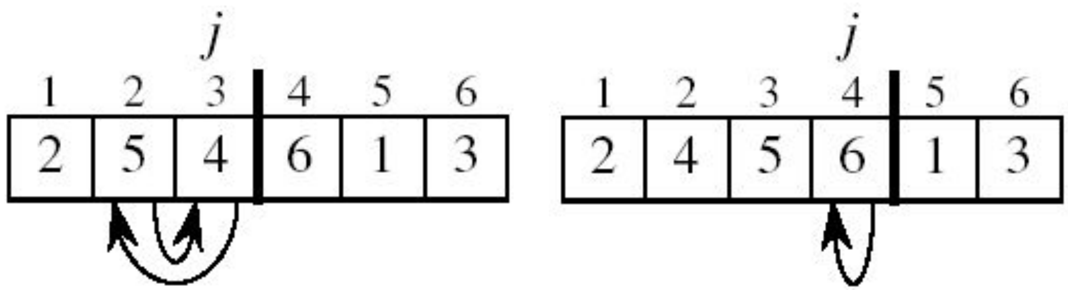
# Loop Invariant for Insertion Sort

- **Initialization:**
  - Just before the first iteration, j = 2: the subarray $A[1 \,..\, j\text{-}1] = A[1]$, (the element originally in $A[1]$) – is sorted



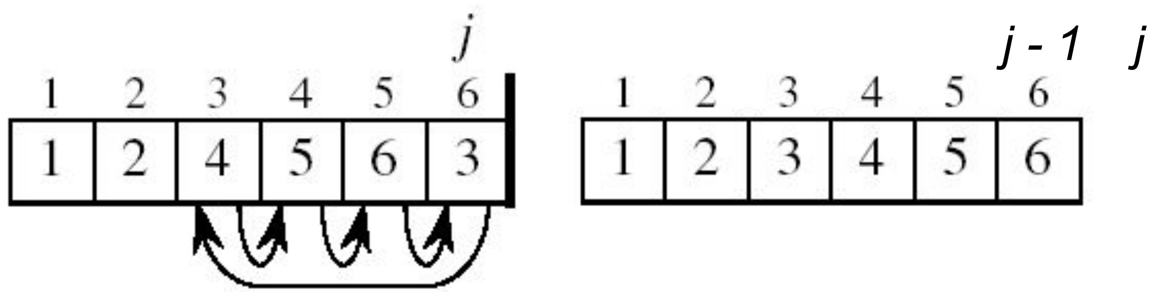
# Loop Invariant for Insertion Sort

- **Maintenance:**
  - the **while** inner loop moves $A[j-1]$, $A[j-2]$, $A[j-3]$, and so on, by one position to the right until the proper position for *key* (which has the value that started out in $A[j]$) is found
  - At that point, the value of *key* is placed into this position.



# Loop Invariant for Insertion Sort

- **Termination:**
  - The outer **for** loop ends when $j = n + 1 \Rightarrow j-1 = n$
  - Replace $n$ with $j-1$ in the loop invariant:
    - the subarray $A[1 \,..\, n]$ consists of the elements originally in $A[1 \,..\, n]$, but in sorted order




- The entire array is sorted!

**Invariant**: at the start of the **for** loop the elements in $A[1 \,..\, j-1]$ are in sorted order

# Analysis of Insertion Sort

| INSERTION-SORT*(A)* | cost | times |
| --- | --- | --- |
| **for** j ← 2 **to** n | $c_1$ | n |
| **do** key ← A[ j ] | $c_2$ | n-1 |
| Insert A[ j ] into the sorted sequence A[1 . . j -1] | 0 | n-1 |
| i ← j - 1 | $c_4$ | n-1 |
| **while** i > 0 and A[i] > key | $c_5$ | $\sum_{j=2}^{n} t_j$ |
| **do** A[i + 1] ← A[i] | $c_6$ | $\sum_{j=2}^{n} (t_j - 1)$ |
| i ← i – 1 | $c_7$ | $\sum_{j=2}^{n} (t_j - 1)$ |
| A[i + 1] ← key | $c_8$ | n-1 |

$t_j$: # of times the while statement is executed at iteration j

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} \left(t_j - 1\right) + c_7 \sum_{j=2}^{n} \left(t_j - 1\right) + c_8(n-1)$$

# Best Case Analysis

- The array is already sorted

**"while** i > 0 and A[i] > key"

  - $A[i] \le$ key upon the first time the **while** loop test is run (when $i = j$ -1)
  - $t_j = 1$

- $T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1)$

$= (c_1 + c_2 + c_4 + c_5 + c_8)n + (c_2 + c_4 + c_5 + c_8)$

$= an + b = \Theta(n)$

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

# Worst Case Analysis

- The array is in reverse sorted order **"while** i > 0 and A[i] > key"
  - Always A[i] > key in **while** loop test
  - Have to compare key with all elements to the left of the j-th position ⇒ compare with j-1 elements ⇒ $t_j = j$

using $\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \Rightarrow \sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1 \Rightarrow \sum_{j=2}^{n} (j-1) = \frac{n(n-1)}{2}$ we have:

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right) + c_6\frac{n(n-1)}{2} + c_7\frac{n(n-1)}{2} + c_8(n-1)$$

$$= an^2 + bn + c$$ a quadratic function of n

- $T(n) = \Theta(n^2)$ order of growth in $n^2$

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\sum_{j=2}^{n} t_j + c_6\sum_{j=2}^{n}(t_j - 1) + c_7\sum_{j=2}^{n}(t_j - 1) + c_8(n-1)$$

# Comparisons and Exchanges in Insertion Sort

| INSERTION-SORT(*A*) | cost | times |
|---|---|---|
| **for** j ← 2 **to** n | $c_1$ | n |
| **do** key ← A[ j ] | $c_2$ | n-1 |
| Insert A[ j ] into the sorted sequence A[1 . . j -1] | 0 | n-1 |
| i ← j - 1 | $c_4$ | n-1 |
| **while** i > 0 and A[i] > key | $c_5$ | $\sum_{j=2}^{n} t_j$ |
| **do** A[i + 1] ← A[i] | $c_6$ | $\sum_{j=2}^{n} (t_j - 1)$ |
| i ← i – 1 | $c_7$ | $\sum_{j=2}^{n} (t_j - 1)$ |
| A[i + 1] ← key | $c_8$ | n-1 |

$\approx n^2/2$ comparisons

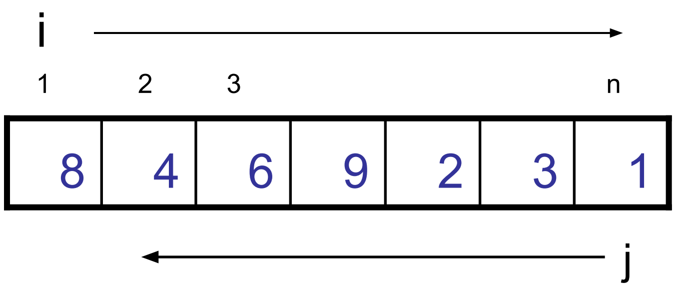
$\approx n^2/2$ exchanges

# Insertion Sort - Summary

- Advantages
  - Good running time for “almost sorted” arrays $\Theta(n)$
- Disadvantages
  - $\Theta(n^2)$ running time in worst and average case
  - $\approx n^2/2$ comparisons and exchanges

# Bubble Sort (Ex. 2-2, page 38)

- Idea:
  - Repeatedly pass through the array
  - Swaps adjacent elements that are out of order




- Easier to implement, but slower than Insertion sort

# Example

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 4 | 6 | 9 | 2 | 3 | 1 |

i = 1 ← j

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 4 | 6 | 9 | 2 | 1 | 3 |

i = 1 ← j

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 4 | 6 | 9 | 1 | 2 | 3 |

i = 1 ← j

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 4 | 6 | 1 | 9 | 2 | 3 |

i = 1 ← j

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 4 | 1 | 6 | 9 | 2 | 3 |

i = 1 ← j

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 1 | 4 | 6 | 9 | 2 | 3 |

i = 1 j

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 8 | 4 | 6 | 9 | 2 | 3 |

i = 1 j

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 8 | 4 | 6 | 9 | 2 | 3 |

i = 2 j

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 8 | 4 | 6 | 9 | 3 |

i = 3 j

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 8 | 4 | 6 | 9 |

i = 4 j

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 8 | 6 | 9 |

i = 5 j

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 8 | 9 |

i = 6 j

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 8 | 9 |

i = 7

j

# Bubble Sort

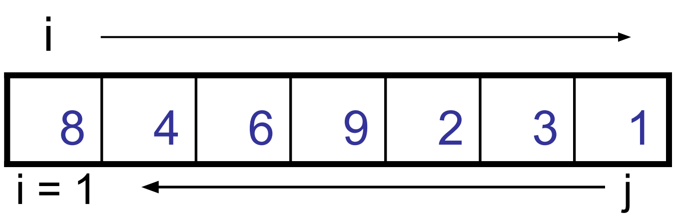
*Alg.:* BUBBLESORT(A)

**for** i ← 1 **to** length[A]

  **do for** j ← length[A] **downto** i + 1

    **do if** A[j] < A[j -1]

      **then** exchange A[j] ↔ A[j-1]



# Bubble-Sort Running Time

*Alg.:* BUBBLESORT(A)

**for** i ← 1 **to** length[$A$] $c_1$

**do for** j ← length[$A$] **downto** i + 1 $c_2$

Comparisons: ≈ $n^2/2$

**do if** $A$[j] < $A$[j -1] $c_3$

**then** exchange $A$[j] ↔ $A$[j-1] $c_4$

Exchanges: ≈ $n^2/2$

$$T(n) = c_1(n+1) + c_2\sum_{i=1}^{n}(n-i+1) + c_3\sum_{i=1}^{n}(n-i) + c_4\sum_{i=1}^{n}(n-i)$$

$$= \Theta(n) + (c_2 + c_2 + c_4)\sum_{i=1}^{n}(n-i)$$

$$where\ \sum_{i=1}^{n}(n-i) = \sum_{i=1}^{n}n - \sum_{i=1}^{n}i = n^2 - \frac{n(n+1)}{2} = \frac{n^2}{2} - \frac{n}{2}$$

Thus, $T(n) = \Theta(n^2)$

# Selection Sort (Ex. 2.2-2, page 27)

- Idea:
  - Find the smallest element in the array
  - Exchange it with the element in the first position
  - Find the second smallest element and exchange it with the element in the second position
  - Continue until the array is sorted
- Disadvantage:
  - Running time depends only slightly on the amount of order in the file

# Example

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 4 | 6 | 9 | 2 | 3 | 1 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 4 | 6 | 9 | 2 | 3 | 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 6 | 9 | 4 | 3 | 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 9 | 4 | 6 | 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 9 | 6 | 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 9 | 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 8 | 9 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 8 | 9 |

# Selection Sort

*Alg.:* SELECTION-SORT*(A)*

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 4 | 6 | 9 | 2 | 3 | 1 |

```
n ← length[A]
for j ← 1 to n - 1
  do smallest ← j
       for i ← j + 1 to n
         do if A[i] < A[smallest]
              then smallest ← i
       exchange A[j] ↔ A[smallest]
```

# Analysis of Selection Sort

| *Alg.:* SELECTION-SORT*(A)* | cost | times |
|---|---|---|
| n ← length[A] | $c_1$ | 1 |
| **for** j ← 1 **to** n - 1 | $c_2$ | n |
| **do** smallest ← j | $c_3$ | n-1 |
| **for** i ← j + 1 **to** n | $c_4$ | $\sum_{j=1}^{n-1}(n-j+1)$ |
| **do if** A[i] < A[smallest] | $c_5$ | $\sum_{j=1}^{n-1}(n-j)$ |
| **then** smallest ← i | $c_6$ | $\sum_{j=1}^{n-1}(n-j)$ |
| exchange A[j] ↔ A[smallest] | $c_7$ | n-1 |

≈$n^2/2$ comparisons

≈n exchanges

$$T(n) = c_1 + c_2 n + c_3(n-1) + c_4\sum_{j=1}^{n-1}(n-j+1) + c_5\sum_{j=1}^{n-1}(n-j) + c_6\sum_{j=2}^{n-1}(n-j) + c_7(n-1) = \Theta(n^2)$$