3) Master Theorem

Works for recurrences of the form

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$a \geq 1$, $b > 1$, $f(n)$ is asymptotically +ve function.

$f(n)$ polynomially smaller than $n^{\log_b a}$ by factor n^ϵ

1. If $f(n) = O(n^{(\log_b a) - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$

$f(n)$ polynomially greater than $n^{\log_b a}$ by the factor n^ϵ

3. If $f(n) = \Omega(n^{(\log_b a) + \epsilon})$ for some constant $\epsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

We are comparing $f(n)$ with func. $n^{\log_b a}$. The larger of the two, determines solution to the recursion.

Case 3 $\quad f(n) > n^{\log_b a}$ so soln is $f(n)$

Case 2 $\quad f(n) = n^{\log_b a}$ so soln is $\Theta(n^{\log_b a} \lg n)$ or $\Theta(f(n) \lg n)$

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$\Rightarrow T(n) = 3T\left(\frac{n}{4}\right) + n\log n$

$a = 3$, $b = 4$, $f(n) = n\log n$

$n^{\log_b a} = n^{\log_4 3} = O(n^{0.793})$

$f(n) > n^{\log_b a}$ So, <u>case 3</u>

<u>To find ϵ:</u>

By def of case 3 :

$f(n) = \Omega(n^{\log_b a + \epsilon}) \rightarrow$ (A)

By def of Ω, the RHS in eq. (A) should be lower bound of f(n). f(n) = nlogn, so n can be a lower bound.

$n\log n \geq n^{0.793 + \epsilon}$

for $\epsilon \approx 0.2$.

$n\log n \geq n^{0.793 + 0.2}$

$\boxed{n\log n \geq n}$

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To find c: $(c < 1)$

$$a f\left(\frac{n}{b}\right) \leq c f(n)$$

$$3\frac{n}{4} \log \frac{n}{4} \leq cn\log n$$

$$\frac{3n}{4}\log n - \frac{3n}{4}\log 4 \leq cn\log n. \rightarrow (B)$$

If we take $c = \frac{3}{4}$, the R.H.S in eq (B) will always be greater than L.H.S.

So,

$$T(n) = \Theta(n\log n)$$

4) $T(n) = 2T(\frac{n}{2}) + n\log n$

$a = 2, \quad b = 2, \quad f(n) = n\log n.$

$n^{\log_b a} = n^{\log_2 2} = n$

Pitfall $f(n) > n^{\log_b a}$ asymptotically.

But it is not polynomially larger

The ratio $\frac{f(n)}{n^{\log_b a}} = \frac{n\log n}{n} = \log n$

is asymptotically less than n^{ϵ} for any +ve constant ϵ.

5) Merge Sort, Maximum Subarray

$T(n) = 2T(\frac{n}{2}) + \Theta(n)$

$a = 2 \quad b = 2 \quad f(n) = \Theta(n)$

$n^{\log_b a} = n^{\log_2 2} = n$

$f(n) = n^{\log_b a}$

Case 2

$T(n) = \Theta(n\log n).$

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Extended Master Theorem:

$T(n) = a\,T(\frac{n}{b}) + \theta(n^k \log^p n)$ $a \geq 1,\ b > 1,\ k > 0$
p is a real no.

① if $a > b^k$ then $T(n) = \theta(n^{\log_b a})$

② if $a = b^k$ then

ⓐ $p > -1 \Rightarrow T(n) = \theta(n^{\log_b a} \log^{p+1} n)$

ⓑ $p = -1 \Rightarrow T(n) = \theta(n^{\log_b a} \log \log n)$

ⓒ $p < -1 \Rightarrow T(n) = \theta(n^{\log_b a})$

③ if $a < b^k$

ⓐ $p \geq 0 \Rightarrow T(n) = \theta(n^k \log^p n)$

ⓑ $p < 0 \Rightarrow T(n) = \theta(n^k)$

Example:

- $T(n) = 2T(\frac{n}{2}) + n$

$a = 2,\ b = 2,\ k = 1\ \ p = 0$

$a = b^k$

$2 = 2^1$

Case 2

$p > -1$

$T(n) = \theta(n^{\log_2 2} \log^{0+1} n)$

$T(n) = \theta(n \log n)$