Hedging performance across styles: Analyzing option portfolio under Constant Elasticity of Variance (CEV) dynamic

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Abstract

We consider a convincingly accepted and empirically supported diffusion model in mathematical finance literature, the Constant Elasticity of Variance (CEV) model for hedging analysis of option portfolio other than the well known Black-Scholes and Merton model (the BSM model). Our concern is to analyze the approximation of option portfolios under four different contract styles namely European, American, Bermudan, and Asian (Arithmetic Average and Geometric Average). This is done in a comparative fashion with respect to the benchmark of Black-Scholes and Merton model. That is, our concern is the investigation of effective hedging performance across styles. For each option style, we study the Greeks using two approaches; Ordinary lattice approach and Auxiliary lattice approach. We illustrate how these Greeks compare with each other then we pick up the best of these Greeks for delta and delta-gamma based approximation. We examine the accuracy of delta and delta-gamma approximations to the valuation of illustrative option portfolios under all four contract types corresponding to various choice of elasticity of the Constant Elasticity of Variance model and compare and contrast them with the benchmark Black-Scholes and Merton model. Our empirical analysis clarifies the effects of low market volatility in delta and delta-gamma based approximation of the option portfolios as well.

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Notations

$\stackrel{\text{def}}{=}$	: Equal by definition
$\Phi(.)$	: cumulative distribution function of standard normal distribution
δ^c	: Call option delta
δ^p	: Put option delta
γ	: Option Gamma
α	: Elasticity factor
$f(S_T, T; S_t, t)$	: Continuous part of the density function of S_T of the asset price under the Q –measure
I_q	: Modified Bessel function of the first kind of order q
A_j	: Arithmetic average of the underlying asset
A_j^g	: Geometric average of the underlying asset
$C(S, N, K, \sigma, T, r, \alpha)$	: Call option price from lattice
Δ	: Delta from ordinary lattice
γ	: Gamma from ordinary lattice
$\delta_{A_{i,j}}$	: Delta at $S_{i,j}$ in auxiliary trinomial tree
P^{δ}	: Delta approximation to portfolio value
$P^{\delta\gamma}$	:Delta-gamma approximation to portfolio value

Introduction

Options are the most important financial derivatives. It is always desirable to find an appropriate model to determine the accurate prices of options. The Black-Scholes and Merton (BSM) model was developed by Fisher Black, Myron Scholes and Robert Merton in 1973 [1], which is the first widely used model for option pricing with the assumption that the underlying stock price follows geometric Brownian motion. But the inability of Black-Scholes and Merton model to accommodate the negative skewness and the high kurtosis has led to the development of a series of alternative models over the years. An alternative model, Constant Elasticity of Variance (CEV) was developed by John Cox (1975) which attempts to incorporate stochastic volatility and the leverage effect [13]. Also CEV model fits market prices of call options significantly better than the Black-Scholes model according to MacBeth and Merville (1980) [15]. So, it is necessary to value options of four different contract styles namely European, American, Bermudan, and Asian (Arithmetic Average and Geometric Average) under CEV dynamic and compare those with the benchmark Black-Scholes and Merton model.

Our main concern is to analyze the Greek-based approximations of option portfolios under four different contact styles. Therefore, calculation of effective Greeks is very important for delta and delta-gamma based approximations to the valuation of option portfolios under all four contract types. This thesis deals with two approaches to calculate Greeks; ordinary lattice approach and auxiliary lattice approach.

We finally estimate the approximation errors across styles for various non-normalities induced by different values of elasticity parameter of CEV model.

Our dissertation is composed as follows:

In chapter 1, we first discussed the widely used Black-Scholes (1973) and Merton (1973) model (the BSM model) for option pricing. Here, we discussed the assumptions of the BSM model, option valuations by risk-neutral method, and studied the derivation of the Greeks of BSM model.

Chapter 2 investigates an alternative to the Black-Scholes (1973) and Merton (1973) (BSM) model as BSM model has serious drawbacks. An empirically supported stochastic process, Constant Elasticity of Variance (CEV) model is generally found to perform better. We described this model and its dynamics. Then we focus on a trinomial model for the CEV process.

In Chapter 3, we implemented option pricing of Asian Arithmetic Average, Asian Geometric Average, European, American and Bermudan options by using recursive trinomial lattice approach under Constant Elasticity of Variance (CEV) dynamics.

In Chapter 4, we turned to the calculation of the sensitivities of the option prices under all four option styles. We focused on the Delta and Gamma as we consider concern delta and delta-gamma based approximation across styles. We here, computed delta and gamma from Ordinary lattice as well as from Auxiliary lattice. We then choose the effective one for hedging option portfolio which is the one based on Auxiliary lattice.

Chapter 5 presents the Greek-based approximations to option portfolios for all four option styles. We then estimate the approximation errors across styles.

In Chapter 6 we carried out propagation of approximation error across styles for both high volatility and low volatility markets. This illustrates the effects of elasticity in defining the hedging performance across styles for various non-normalities induced by different values of elasticity parameter of CEV model.

Finally, Chapter 7 is the summarization of the results. We described what we got from this thesis.

Chapter 1

Black-Scholes and Merton Model

The Black-Scholes and Merton or BSM model is a mathematical model for the dynamics of a financial market containing derivatives investment instrument. The main purpose behind the BSM model is to hedge the option by buying or selling the underlying asset in just the right way and as a consequence, to eliminate risk. In a financial market, we usually assume a stock price process to follow the geometric Brownian motion. In this chapter we discuss the BSM model based on geometric Brownian motion theory of stock price and risk-neutral valuation.

We first discuss the assumptions of Black-Scholes and Merton model, derivation of the BSM partial difference equation (PDE), risk-neutral valuation and the closed form BSM pricing formula for European call option on a non-dividend paying stock. Then we finally study the Greeks.

1.1 The assumptions of Black-Scholes and Merton model

The Black-Scholes and Merton model assumes that the underlying stock price follows Geometric Brownian motion [19]. Now we discuss the Geometric Brownian motion Stochastic Differential Equation.

Let S_t be the price of an asset at time t .The price changes to $S_t + dS_t$ for very short change dt in time. Then the return of the asset after this short time interval is

$$\frac{dS_t}{S_t}$$

The mathematical model of this return consists of two components. One of these is predictable, deterministic called drift component and the other one is diffusion component. So, the Geometric Brownian motion (GBM) is the name for the change in a random process dS_t, in relation to the current value, S_t. This proportional change $\frac{dS_t}{S_t}$, or rate of return, is modeled as an Arithmetic Brownian motion. The Stochastic Differential Equation (SDE) is

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB(t) \tag{1.1}$$

where μ is called the drift. It is a measure of the average rate of growth of the asset price. And $\sigma > 0$ is volatility, standard deviation of the returns. $dB(t)$ is a random variable having normal distribution with mean 0 and variance dt.

Multiplying (1.1) by S_t gives the SDE for the change in S_t itself as

$$dS_t = \mu S_t dt + \sigma S_t dB(t) \tag{1.2}$$

Drift coefficient μS_t and diffusion coefficient σS_t are proportional to the latest known value S_t, and thus change continuously. To find the solution to the GBM, a trial solution is taken, and Itô's formula is applied to derive the corresponding SDE. If that matches the GBM, then the trial solution is the definitive solution. The intuition for the trial solution is that if S_t were deterministic, $\frac{dS_t}{S_t}$ would be the derivative of $\ln[S_t]$ with respect to S_t.

Now,

$$dln[S_t] = \frac{dln[S_t]}{dS_t} dS_t + \frac{1}{2} \frac{d^2 ln[S_t]}{dS_t^2} (dS_t^2)$$

where

$$\frac{dln[S_t]}{dS_t} = \frac{1}{S_t}, \qquad \frac{d^2 ln[S_t]}{dS_t^2} = \frac{-1}{S_t^2}, (dS_t)^2 = \sigma^2 S_t^2 dt$$

Substituting these, together with dS_t, gives

$$\begin{aligned} dln[S_t] &= \frac{1}{S_t}\left(\mu S_t dt + \sigma S_t dB(t)\right) + \frac{1}{2}\left(\frac{-1}{S_t^2}\right)\sigma^2 S_t^2 dt \\ &= \mu dt + \sigma dB - \frac{1}{2}\sigma^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB \end{aligned}$$

In integral form

$$\int_{t=0}^{T} dln[S_t] = \int_{t=0}^{T} \left(\mu - \frac{1}{2}\sigma^2\right) dt + \int_{t=0}^{T} \sigma dB$$

$$ln[S(T)] - ln[S(0)] = \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B(T)$$

$$\ln\left[\frac{S(T)}{S(0)}\right] = \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B(T)$$

Thus $\ln\left[\frac{S(T)}{S(0)}\right] = \left(\mu - \frac{1}{2}\sigma^2\right)$ has a normal distribution with parameters

$$E\left[\ln\left[\frac{S(T)}{S(0)}\right]\right] = E\left[\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B(T)\right] = \left(\mu - \frac{1}{2}\sigma^2\right) T$$

$$Var\left[\ln\left[\frac{S(T)}{S(0)}\right]\right] = Var\left[\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B(T)\right] = \sigma^2 T$$

Taking exponential gives the final expression for $S(T)$

$$\frac{S(T)}{S(0)} = \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B(T)\right]$$

$$S(T) = S(0) \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B(T)\right]$$

It is clearly evident, $S(T)$ cannot become negative and it has the lognormal density

$$\frac{1}{S(T) v \sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\frac{\ln(x) - m}{v}\right]^2\right\}$$

where $m \stackrel{\text{def}}{=} E[\ln[\mathrm{S(T)}]] = \ln[S(0)] + \left(\mu - \frac{1}{2}\sigma^2\right) T$ and $v \stackrel{\text{def}}{=} Stdev[\ln[\mathrm{S(T)}]] = \sigma\sqrt{T}$. So the underlying stock price follows geometric Brownian motion indicates the lognormal property of stock price.

1.2 Option Valuation under BSM model

The most important thing an investor needs to understand is how options are priced. We will discuss two methods for pricing option under BSM model [1]; the Black-Scholes partial differential method and risk-neutral method respectively.

1.2.1 The Black-Scholes Partial Differential Equation Method

We are now able to discuss the Black-Scholes PDE for an option on a non-dividend paying stock with strike K and maturity T [19]. The value of an option on a stock is modeled as a function of two variables- calendar time t, and stock price $S(t)$ −and is denoted by $V(t)$. the stock price is assumed to evolve according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB(t)$$

where μ and σ are constants.

Itô's formula gives the change in the option in the option value resulting from the change in time, dt, and the change in S over dt, as

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2$$

Using $dS = \mu S dt + \sigma S dB$ and $dS^2 = \sigma^2 S^2 dt$ gives

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} [\mu S dt + \sigma S dB\ ] + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 dt \\ &= \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right] dt + \sigma S \frac{\partial V}{\partial S} dB \end{aligned}$$

The partial differential equation method of option valuation is based on the insight that the option and the stock on which it is written have the same source of randomness. Thus, by taking opposite positions in the option and the stock, the randomness of the one asset can offset the randomness of the other. It is therefore possible to form a portfolio of stock and options in such proportion that the overall randomness of this portfolio is zero. Moreover, if the proportion of stock and options in this portfolio is changed as the value of the stock changes, this portfolio can be maintained riskless at all times.

At time t, form a portfolio that is long λ shares and short 1 option. The value P of this portfolio at time t is

$$P(t) = \lambda\, S(t) - V(t)$$

The minus sign comes from the fact that the option is not owned but owed. The value of this portfolio changes according to

$$dP = \lambda dS - dV$$

$$= \lambda\,[\mu S dt + \sigma S dB\,] - \left[\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right]dt - \sigma S\frac{\partial V}{\partial S}dB$$

$$= \left[\lambda\mu S - \frac{\partial V}{\partial t} - \mu S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right]dt + \left[\lambda\sigma S - \sigma S\frac{\partial V}{\partial S}\right]dB$$

The random term can be made to disappear by choosing λ in such a way that the coefficient

$$\left[\lambda\sigma S - \sigma S\frac{\partial V}{\partial S}\right] = 0$$

So, $\lambda = \frac{\partial V}{\partial S}$

Then we have

$$dP = \left[\lambda\mu S - \frac{\partial V}{\partial t} - \mu S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right]dt$$

The portfolio is then riskless so its value must increase in accordance with the risk-free interest rate, otherwise there would be an arbitrage opportunity. The interest accrued on 1 unite of money over a time interval of length dt is $1rdt$. The value of the portfolio thus grows by $Prdt$ over dt. Equating the two expression for the change in the value of P gives

$$\left[\lambda\mu S - \frac{\partial V}{\partial t} - \mu S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right]dt = Prdt$$

$$\lambda\mu S - \frac{\partial V}{\partial t} - \mu S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = Pr$$

Substituting $P = \lambda S - V$ and $\lambda = \frac{\partial V}{\partial S}$ gives

$$\frac{\partial V}{\partial S}\mu S - \frac{\partial V}{\partial t} - \mu S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = \left(\frac{\partial V}{\partial S}S - V\right)r$$

The term $\frac{\partial V}{\partial S}\mu S$ cancels, leaving

$$-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = \frac{\partial V}{\partial S}rS - rV$$

which rearranges to

$$\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S}rS + \frac{\partial V}{\partial t} = rV \tag{1.3}$$

This is the second-order partial differential equation (PDE) in the unknown function V .The fact that this PDE does not contain the growth rate μ of the stock price may be surprising at first sight. The PDE must be accompanied by the specification of the option value at the time of exercise, the so- called option payoff. The PDE method was developed by Black and Scholes using key insights by Merton.

In order to solve (1.3) boundary conditions must also be provided. In case of our call option those conditions are:

$$V(S,T) = \max(S-K,0), V(0,t) = 0 \text{ for all } t$$

$$\text{And } V(S,t) \to S \; as \; S \to \infty.$$

The solution to (1.3) in the case of a call option is

$$C(S,t) = S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2) \tag{1.4}$$

where $d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$

and $d_2 = d_1 - \sigma\sqrt{T-t}$

and $\Phi(.)$ is the CDF of the standard normal distribution. One way to confirm (1.4) is to compute the various partial derivatives using (1.4), then substitute them into (1.3) and check that (1.3) holds. The price of a European put-option can also now be easily computed from put-call parity and (1.4).

$$\begin{aligned} P(S_t,t) &= Ke^{-r(T-t)} - S_t + C(S_t,t) \\ &= \Phi(-d_2)Ke^{-r(T-t)} - \Phi(-d_2)S_t \end{aligned}$$

where $\Phi(.)$ is the cumulative distribution function of the standard normal distribution.

1.2.2 Risk-Neutral Valuation

The most interesting feature of Black- Scholes PDE (1.3) is that μ does not appear anywhere. Note that the PDE (1.3) also holds if we assume $\mu = r$. However, if $\mu = r$ then investors would not demand a premium for holding the stock. Since this would generally only hold if investors were risk-neutral, this method of derivatives pricing came to be known as risk-neutral pricing. The risk-neutral pricing valuation describes that an option can be valued on the assumption that the world is risk-neutral. Therefore, the valuation procedure includes the following procedures:

We consider a market with riskless asset $A = \{A_t : 0 \le t \le T\}$ and a risky asset $S = \{S_t : 0 \le t \le T\}$ respectively. Then the following SDE describes the dynamics of S_t:

$$dS_t = rS_t dt + \sigma S_t dB_t, \forall\ t \in [0,T]$$

$$S_t = S_0 \text{ at } t = 0 \tag{1.5}$$

where $B = \{B_t : 0 \le t \le T\}$ is a Brownian motion under the given probability measure Q, called the probability measure of risk-neutral world and r is the risk-free interest rate.

Note that (1.5) implies

$$S_T = S_t e^{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(B_T - B_t)}$$

so that S_T is log-normally distributed under Q. It is now easily confirmed that the call option price in (1.4) also satisfies

$$C(S_t, t) = E_t\left[e^{-r(T-t)} \max(S_T - K, 0)\right]$$

which is also consistent with martingale pricing.

Graphically, the call and put prices under BSM model is given in Figure 1.1

(a) (b)

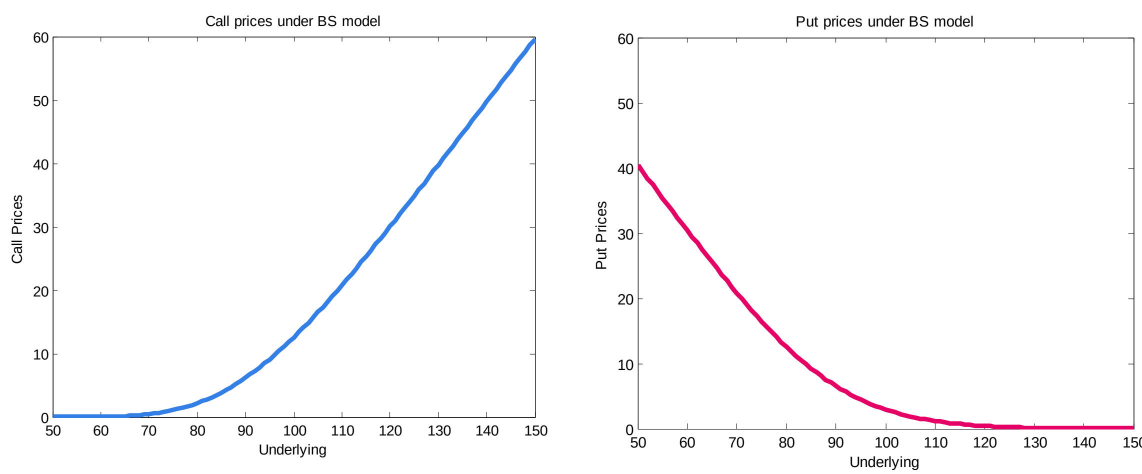


Figure 1.1 (a) Call prices and (b) Put prices for underlying asset price 100 and strike price 95.

1.3 Greeks under BSM model

The Greeks are the quantities representing the sensitivity of the price of options to change in underlying parameters. They are partial derivatives of the price with respect to the parameter values. The Greeks are vital in the mathematical theory of finance. In this section, we will discuss only two important Greeks, delta and gamma of option for Black-Scholes-Merton model [5].

1.3.1 The option Delta

The delta of an option is defined as the partial derivatives of the option price with respect to the underlying asset price, S_t [20].

For puts and calls, we define [5]

$$\delta^c = \frac{\partial C}{\partial S_t}$$

$$\delta^p = \frac{\partial P}{\partial S_t}$$

We recall that (1.4) gives the Black-Scholes-Merton (BSM) formula for a European call option price

$$C(S,t) = S_t \Phi(d_1) - e^{-r(T-t)} K \Phi(d_2)$$

$$\text{where } d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$\text{and } d_2 = d_1 - \sigma\sqrt{T-t}$$

Using basic calculus, we can take the partial derivative of the option price with respect to the underlying asset price, S_t, as follows

$$\frac{\partial C}{\partial S_t} = \delta^c = \Phi(d_1)$$

We refer to this as the delta of the option, and kit has the interpretation that for small changes in S_t the call option price will change by $\Phi(d_1)$. Notice that as $\Phi(.)$ is the cumulative normal density function, which is between zero and one, we have

$$0 < \delta^c < 1$$

so that the call option price in the BSM model will change in the same direction as the underlying asset price, but the change will be less than one-for-one.

The call delta of the European call and put options from the BSM model is shown in figure 1.2 and Figure 1.3 for Strike price $= 95$ and for S_t varying from 40 to 160. Notice that delta changes dramatically when the option is close to at-the-money– that is, when S_t $\approx$ Strike price.

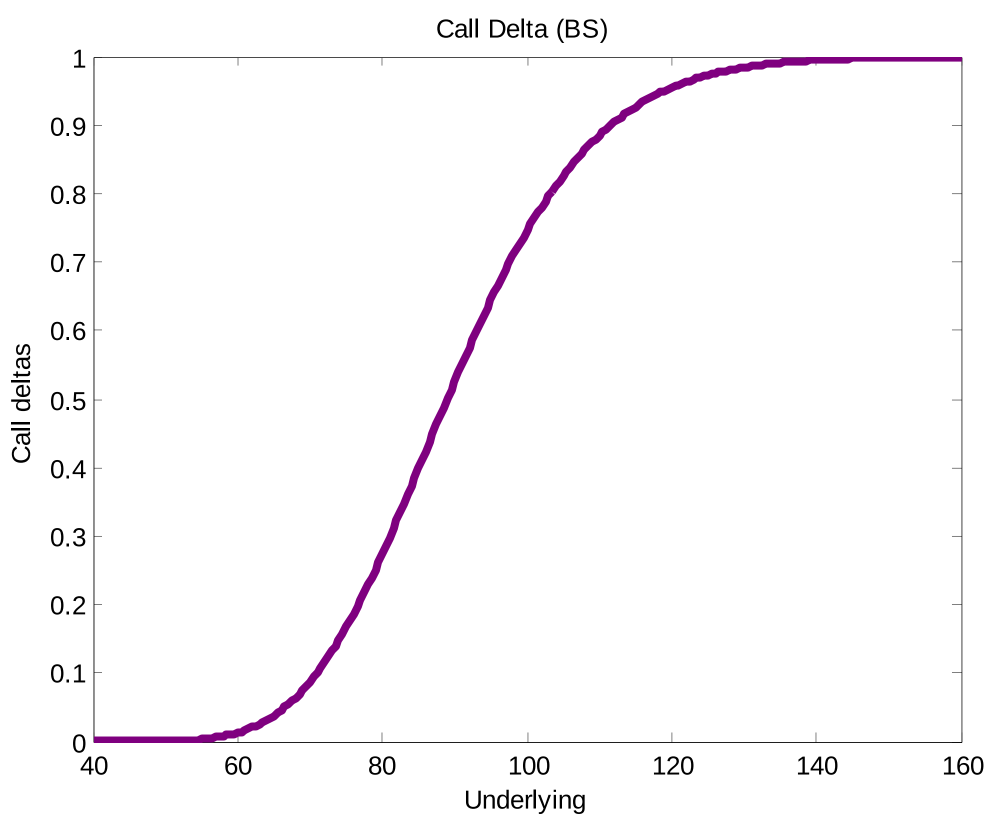


Figure 1.2 Call delta of the European call options from the BSM model

For a European put option, we have the put-call parity stating that

$$P(S_t, t) = Ke^{-r(T-t)} - S_t + C(S_t, t)$$

so that we can easily derive that

$$\frac{\partial P}{\partial S_t} = \delta^P = \Phi(d_1) - 1$$

Here we notice that

$$-1 < \delta^p < 0$$

So, the BSM put option moves in the opposite direction of the underlying asset, and again th option price will change by less (in absolute term) than the underlying asset price.

The put delta of the European call and put options from the BSM model is shown in figure for Strike price $= 95$ and for S_t varying from 40 to 160. Notice that delta changes dramatically when the option is close to at-the-money— that is, when $S_t \approx$ Strike price.

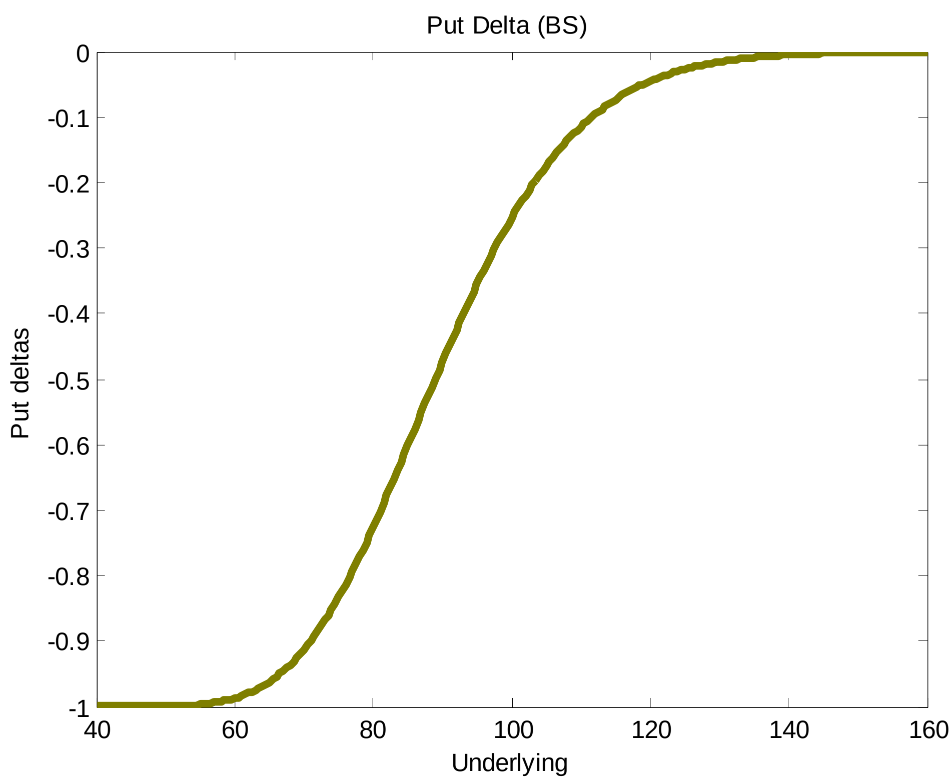


Figure 1.3 Put delta of the European put options from the BSM model

1.3.2 The option Gamma

The option gamma is a measure of the rate of change of its delta. When the underlying asset price makes a large upward move in a short time, the call option price will increase by more than the delta approximation would suggest.

The Greek letter gamma, γ, is used to denote the rate of change of δ with respect to the price of the underlying asset, that is [5]

$$\gamma = \frac{\partial \delta}{\partial S_t} = \frac{\partial^2 C}{\partial S_t^2}$$

For a European call or put option on an underlying asset at the rate r,

$$\gamma^c = \gamma^p = \frac{\Phi(d_1)}{S_t \sigma \sqrt{T-t}}$$

Figure 1.4 shows the gamma for an option using the BSM model with parameters as in Figure 1.2 and Figure 1.3 where we plotted the deltas.

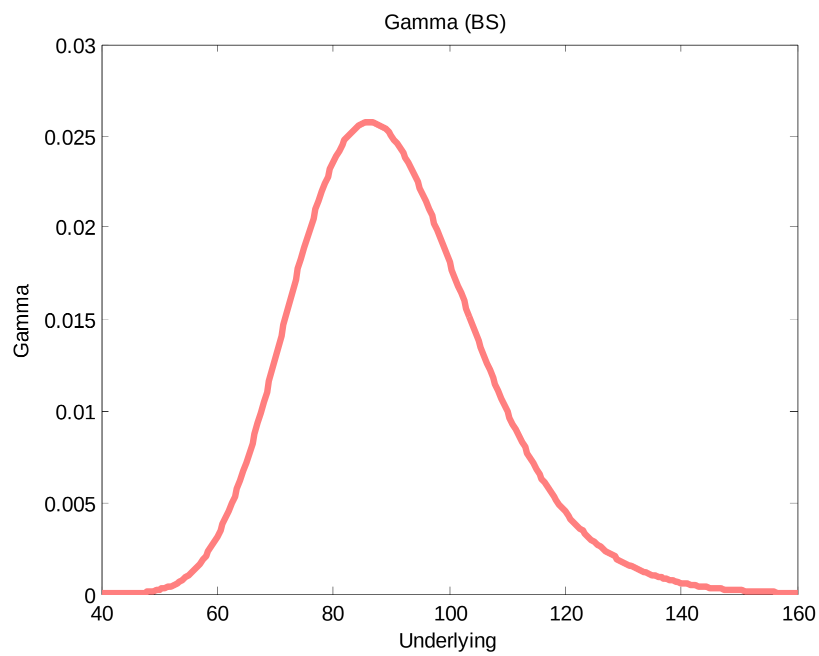


Figure 1.4 Gamma of the European options from the BSM model

Here we notice, when the option is close to at-the-money, the gamma is relatively large and when the option is deep out-the-money the gamma is relatively small. All this implies that ignoring gamma is most crucial for at-the-money options.

Chapter 2

Alternative to Black-Scholes and Merton Model

Option pricing to hedge a portfolio requires a model that is convincingly accepted in the finance literature. The Black and Scholes (1973) and Merton (1973) model (the BSM model) was the first widely used model for option pricing with the assumption that the underlying stock price follows geometric Brownian motion. This actually indicates that an asset's price changes continuously in a way that produces a lognormal distribution for the price at any future time. But according to Jackwerth and Rubinstein (1996) - the log-normal assumption is unable to accommodate the negative skewness and the high kurtosis that are usually implicit in empirical asset return distributions [14]. So the log normality assumption does not hold empirically. When the Black-Scholes model is used to price stock options, certain biases such as the well-known strike price bias (volatility smile) persist.

Because of the drawbacks, there are many alternative processes that can be assumed. One possibility is to retain the property that the asset price changes continuously, but assume a process other than geometric Brownian motion. A model where stock prices change continuously is known as diffusion model.

2.1 The Constant Elasticity of Variances (CEV) Model

An alternative stochastic process, Constant Elasticity of Variance (CEV) was developed by John Cox (1975) which attempts to capture stochastic volatility and the leverage effect and have received empirical support in the literature. For example, we observe the leverage effect by Bekaert and Wu (2000), and the implied volatility skew by Dennis and Mayhew (2002) [14]. The CEV diffusion is a Local Volatility Model, consistent with a complete market setup and therefore it allows hedging of short option positions only through the underlying asset. The model is widely used by practitioners in the financial industry.

According to MacBeth and Merville (1980), the CEV model is generally found to perform better [15]. MacBeth and Merville compare deviations of market from model prices for both the BSM and CEV call pricing models. Based on a daily sample of options on six stocks over one year, they conclude that the CEV model better explains market prices than the BSM model.

The BSM option pricing model assumes that the underlying stock price follows geometric Brownian motion,

$$dS = \mu S dt + \sigma S dz \tag{2.1}$$

where $z(t)$ is a standard Brownian motion. This assumption means that the percentage change in the stock price $\frac{dS}{S}$, over the small interval dt, is normally distributed with mean μdt and instantaneous variance $\sigma^2 dt$.

The CEV model is a diffusion model where the risk-neutral process for a stock price S is

$$dS = \mu S dt + \sigma S^{\frac{\alpha}{2}} dz \tag{2.2}$$

where α is a constant, known as elasticity factor, and $0 \leq \alpha < 2$. For $0 \leq \alpha < 2$, the CEV process has an absorbing barrier when the asset price reaches zero.

In the limiting case when $\alpha = 2$, the equation (2.2) reduces to equation (2.1), implying that geometric Brownian motion is a special case of the CEV process. Following Cox (1975), we restrict the range of α to the interval [0,2]. The process with $\alpha < 0$ does not have reasonable economic properties since the volatility explodes as we approach the origin and the asset price can become negative.

Equation (2.2) is based on nature's probability measure, the P -measure. In order to price securities under the CEV process, we first transform the stochastic process in (2.2) to the Q -measure under which the deflated price processes of all securities are martingales. In this case, the accumulated money market account is used as the numeraire. Under the Q -

measure, the revised process resembles (2.2) except that the risk-free rate r (here, assumed constant) replaces μ,

$$dS = rSdt + \sigma S^{\frac{\alpha}{2}} dz \tag{2.3}$$

Using the martingale property, derivatives can be valued by taking the expectation of the normalized payoff at maturity under the Q -measure.

To implement this procedure for a derivative maturing at time T, we require the density function of the asset price under the Q –measure. The continuous part of the density of S_T, conditional on $S_t (t < T)$, is given by

$$f(S_T, T; S_t, t) = (2-\alpha)\gamma^{\frac{1}{(2-\alpha)}}(ab^{1-2\alpha})^{\frac{1}{4-2\alpha}}\; e^{-a-b}\; I_{\frac{1}{2-\alpha}}\left(2\sqrt{ab}\,\right),$$

Where $\tau = T - t,$

$$\gamma = \frac{2r}{\sigma^2(2-\alpha)(e^{r(2-\alpha)\tau}-1)},$$

$$a = \gamma\, S_t^{2-\alpha}\; e^{r(2-\alpha)\tau},$$

$$b = \gamma\, S_T^{2-\alpha}\;,$$

And I_q is the modified Bessel function of the first kind of order q.

2.2 A Trinomial Model for the CEV Process

One of the first computational models used in the financial mathematics community was the binomial tree model. This model was popular for some time but in the last 15 years has become significantly outdated and is of little practical use. However it is still one of the first models students traditionally were taught. A more advanced model is the trinomial tree model [2].

This improves upon the binomial model by allowing a stock price to move up, down or stay the same with certain probabilities, as shown in the diagram below.

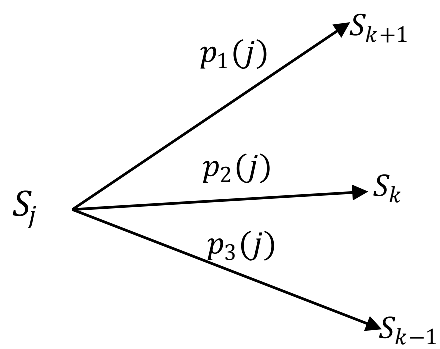


Figure 2.1 Trinomial Tree

In this section, we develop a discrete approximation for the CEV process using the trinomial method [2]. For this we assume the asset price dynamics are expressed in terms of the Q –measure. We transform the variable S so that the transformed process has constant volatility.

Let, $y = y(t, S)$

Now we apply Ito's Lemma and obtain the stochastic differential equation for y as follows

$$dy = q(t, S)dt + \frac{\partial y}{\partial S}\sigma S^{\frac{\alpha}{2}}dz \tag{2.4}$$

where $q(t, S) = \frac{\partial y}{\partial t} + rS\frac{\partial y}{\partial S} + \frac{1}{2}\sigma^2 S^\alpha \frac{\partial^2 y}{\partial S^2}$.

To ensure that the process (2.4) has constant volatility, we must find a transformation such that

$$\frac{\partial y}{\partial S}\sigma S^{\frac{\alpha}{2}} = v,$$

For some positive constant v. This is equivalent to

$$\frac{\partial y}{\partial S} = \frac{v}{\sigma} S^{-\frac{\alpha}{2}}.$$

For $\alpha \neq 2,$ this transformation is given by

$$y = \frac{v}{\sigma\left(1-\frac{\alpha}{2}\right)} S^{\left(1-\frac{\alpha}{2}\right)},$$

And for $\alpha = 2,$ the appropriate transformation is given by

$$y = \frac{v}{\sigma}\log(S).$$

For the latter case, trinomial approximation methods have been examined by Boyle (1986), (1988), Kamrad and Ritchken (1991), and Tian (1993), among others [2]. For the case with $\alpha \neq 2,$ the transformed equation becomes

$$dy = \left[r\left(1-\frac{\alpha}{2}\right)y - \frac{\alpha v^2}{4\left(1-\frac{\alpha}{2}\right)y}\right]dt + vdz \tag{2.5}$$

The transformed process (2.5) has constant volatility, which allows for a straightforward construction of a two-dimensional grid for trinomial trees. However, the transformed process has a more complex drift term, which explodes when $y \to 0$ (with the only exception when $\alpha = 0$). This makes the standard trinomial branching process problematic for the region close to y =0, because the trinomial jumps and probabilities must be chosen to match the drift (and volatility).

To resolve this problem, we modify the standard trinomial approach as suggested by Tian (1994). Under Tian's modification [2], the trinomial branching process simultaneously utilizes both the transformed process (2.5) and the original process (2.3). The procedure is carried out in two steps.

First, the transformed process (2.5) is used to define an evenly spaced, two-dimensional grid in the (t, y) space represented by partitions,

$$t_0, t_0 + \Delta t, t_0 + 2\Delta t, \dots \dots, t_0 + N\Delta t,$$

$$y_0 - M_1\Delta y,\ y_0 - \Delta y,\ y_0, y_0 + \Delta y, \dots \dots, y_0 + M_2\Delta y,$$

where (t_0, y_0) is the starting point of the trinomial process, Δt and Δy are increments in time and state variable y, respectively, and y_0 is related to the initial asset prices S_0 through the transformation by

$$y_0 = \frac{v}{\sigma\left(1-\frac{\alpha}{2}\right)} S_0^{1-\frac{\alpha}{2}},$$

For simplicity, define $t_i = t_0 + i\Delta t$ for $i = 0, 1, 2, \dots \dots N$, and $y_i = y_0 + j\Delta y$ for $j = -M_1, \dots \dots, -1, 0, +1, \dots \dots, M_2$. The following restriction is imposed to ensure stability of the trinomial method

$$\Delta y = \lambda v \sqrt{\Delta t} \tag{2.6}$$

where $\lambda \geq 1$ is an arbitrary constant.

Normally, the trinomial branching process is defined on this two-dimensional grid, which matches the drift and volatility of the transformed process. However, this standard approach will not work for all a values of the CEV process as the drift term of the transformed process is unbounded at zero and there is a positive probability that the zero position $(y = 0\ or\ S = 0)$ is accessible from above. A solution is to project the two-dimensional grid in the $(t, y) -$space onto an equivalent two-dimensional grid in the $(t, S) -$space, using the inverse transformation of (2.6) defined by

$$S = \begin{cases} \left[\dfrac{\sigma\left(1-\frac{\alpha}{2}\right)}{v} y\right], & \text{if } y > 0, \\ 0, \text{otherwise} \end{cases}$$

This new grid is represented by partitions,

$$t_0, t_1, t_2, \dots \dots, t_N,$$

$$S_{M_1}, \dots, S_{-1}, S_0, S_1, \dots \dots, S_{M_2},$$

where

$$S_j = \begin{cases} \left[\dfrac{\sigma\left(1 - \frac{\alpha}{2}\right)}{v} y_j \right], & \text{if } y_j > 0, \\ 0, \text{otherwise} \end{cases}$$

The next step is to specify the trinomial branching process on this new two- dimensional grid in the (t, S) −space. The price of the underlying asset is restricted to move only to points on this grid. This ensures computational efficiency as the trinomial lattice will always recombine properly. Suppose that the current position of the asset price is S_j. The next move will take the asset value to S_{k+1}, S_k, or S_{k-1}, with probability $p_1(j)$, $p_2(j)$, $p_3(j)$, respectively. Such a branching process is illustrated in Figure 2.2 for the case when $k = j$.

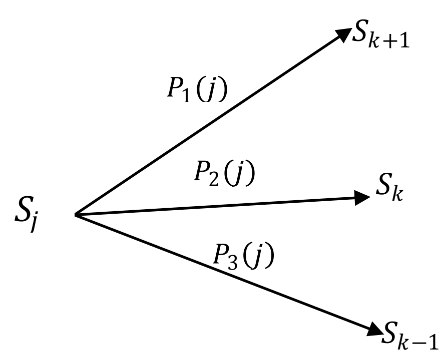


Figure 2.2 A generalized Trinomial Branching Process

In a standard trinomial branching process, we generally have $k = j$. In other words, the underlying asset may move up a level, down a level, or stay the same during the next time period. This standard trinomial branching process works well for the lognormal diffusion process $(\alpha = 2)$. However, for other diffusion processes such as the CEV process, k must be allowed to deviate from j in order to match the drift and the volatility with non-negative probabilities.

For this generalized trinomial branching process, the three probabilities are specified by the following set of equations,

$$p_1(j) + p_2(j) + p_3(j) = 1$$
$$p_1(j)\, S_{k+1} + p_2(j) S_k + p_3(j) S_{k-1} = M_j$$

$$p_1(j)\, S_{k+1}^2 + p_2(j) S_k^2 + p_3(j) S_{k-1}^2 = M_j^2 + V_j^2$$

where

$$M_j = S_j + rS_j \Delta t,$$

$$V_j = \sigma^2 S_j^{\alpha} \Delta t,$$

We solve the above equations from the above equations and get the probabilities

$$\begin{aligned} p_1(j) &= \frac{(S_k - M_j)(S_{k-1} - M_j) + V_j}{(S_k - S_{k+1})(S_{k-1} - S_{k+1})} \\ p_2(j) &= \frac{(S_{k+1} - M_j)(S_{k-1} - M_j) + V_j}{(S_{k+1} - S_k)(S_{k-1} - S_k)} \\ p_1(j) &= \frac{(S_k - M_j)(S_{k-1} - M_j) + V_j}{(S_{k+1} - S_{k-1})(S_k - S_{k-1})} \end{aligned} \qquad (2.7)$$

We will use the symbol S_u, S_m and S_d to denote the jump sizes in place of S_{k-1}, S_k and S_{k+1} respectivley and for probabilities $P_u, P_m,$ and P_d in place of P_1, P_2 and P_3 respectively. The middle move of the branching process (k) is chosen properly to avoid the negative probabilities. To line up the drifts, the central branch of the trinomial tree should be placed as close as possible to the expected asset price at the end of the period based on the continuous process (2.3). In other words, one wishes to select k such that the price of the underlying asset after the central move matches the expected asset price of the underlying stochastic process.

This means that it is optimal to have

$$S_k = S_j + rS_j \Delta t.$$

When $S_j > 0$, the above condition can be simplified to

$$y_k = y_j (1 + r\Delta t)^{1-\frac{\alpha}{2}},$$

or, equivalently,

$$k = \frac{y_j (1 + r\Delta t)^{1-\frac{\alpha}{2}} - y_0}{\lambda v \sqrt{\Delta t}} \qquad (2.8)$$

Of course, the right-hand side of equation (2.8) is usually not an integer. A simple solution is to choose an integer that is as close to it as possible,

$$k = \left[\frac{y_j\,(1 + r\Delta t)^{1-\frac{\alpha}{2}} - y_0}{\lambda v\sqrt{\Delta t}}\right] \tag{2.9}$$

where $[.]$ returns the integer that is closest to a real number.

Once the branching process and the probabilities are properly defined, the standard backward recursive procedure may be used to value call or put of Asian Arithmetic Average, Asian Geometric Average, European, Bermudan and American option prices. Following MacBeth and Merville (1980), the value of σ to be used for models with α values is adjusted to be $\sigma = (\sigma_{BS})S_0^{1-\frac{\alpha}{2}}$.

To implement the valuation procedure, appropriate values must be assigned to the two constants used in the trinomial model, λ and v. Following Boyle (1986), (1988) and Kamrad and Ritchken (1991), we choose $\lambda = \sqrt{1.5}$. This particular value of λ tend to provide the fastest convergence because it leads to the three trinomial probabilities having roughly the same value. The value of v has no impact on the accuracy of the model, and we choose $v = \sigma$ for convenience. We will see in chapter 3 that it is evident that recursive trinomial prices converge to the closed-form solution rapidly for all four option types.

Chapter 3

Option Valuation under CEV model using recursive trinomial lattice approach

Options form a major category of modern financial assets, and are important risk management tools. So, our first requirement is a model to value options. An option derives its value from an underlying asset, but its payoff is not a linear function of the underlying asset price, and so the option price is not a linear function of the underlying asset price either. This nonlinearity adds complications to pricing. This chapter is devoted to the pricing of options. Here we will price Asian arithmetic, Asian geometric, European, American and Bermudan options under CEV model [7] [10].

3.1 Options

An option is a kind of financial derivatives that gives the owner the right to buy or sell another financial asset. There are two basic types of options: the call options and the put options. A call option allows the holder to buy the underlying asset with a predetermined price at or before a certain date. On the other hand, a put option gives the holder the right to sell the underlying asset. The predetermined price mentioned above is called the strike price. The certain date that a option owner is allowed to exercise the option at or before it is known as the maturity date. The financial asset that the option holder can buy or sell with the exercise price is called the underlying asset. Exercising an option denotes that the holder exercises the right to buy or sell the underlying asset. With the rapid growth and the deregulation of financial markets, non standardized options are created by financial institutions to fit their clients' needs. These complex options are usually traded in the rapidly growing over-the-counter markets. Pricing these options accurately and efficiently is an important problem in financial field.

3.2 Asian option

An Asian option is an option whose payoff depends on the arithmetic average price and the geometric average price of the underlying asset. Assume that the average price of the underlying asset between the option initial date and the maturity date is A. Then the option holder has the right to buy (or sell) the underlying asset with price A. This contract is useful for hedging transactions whose cost is related to the average price of the underlying asset. Its price is also less subject to price manipulation; hence variants of Asian options are popular especially in thinly-traded markets. How to price an Asian option accurately and efficiently is important in both financial and academic fields. In this chapter we will price Asian option taking both the arithmetic average price and the geometric average price of the underlying asset.

The Asian option is one of the most representative examples of the options that are hard to be priced in terms of speed and accuracy. Up to now, there is still no simple closed form for pricing Asian options. Numerous approximation methods are suggested in academic literatures. However, most of the existing methods are either inefficient or inaccurate or both. Generally speaking, these approximation methods can be grouped into two different categories: approximation analytical formulae and (quasi-) Monte Carlo simulations. The lattice approach is more general than these since most methods from the first two categories suffer from the inability to price Asian options without bias. Under this consideration, one recursive pricing method will be developed. The difficulty with the lattice method in the case of Asian options lies in its exponential nature; since the price of the underlying asset at each time step influences the option's payoff, it seems that 3^N paths have to be individually evaluated for an $N-$ time-step trinomial lattice. But in other methods we can use 100 to 500 time steps while using recursion allows us to take 13 to 15 time steps maximum and gives us accuracy.

3.2.1 Pricing Asian Arithmetic Average option

An Asian option is an option whose payoff depends on the average price of the underlying asset during a specific period. First we consider pricing of Asian option taking arithmetic average. In a discrete time model, the average price of the underlying asset is defined as $A_j = \frac{\sum_{i=0}^{j} S_i}{j+1}$.

Thus the payoff for a Asian option is

$$\text{Payoff}= \begin{cases} max(A_n - K, 0), & \text{for a call option} \\ max(K - A_n, 0), & \text{for a put option} \end{cases}$$

Now the value of as Asian option can be evaluated by taking expectation of the future discounted payoff. Using the arithmetic average the option value is

$$\text{Option values}= \begin{cases} E[e^{-rT} max(A(T) - K, 0)], & \text{for a call option} \\ E[e^{-rT} max(K - A(T), 0)], & \text{for a put option} \end{cases}$$

Now we will consider pricing Asian Arithmetic average option from trinomial lattice. For that, we construct a 2 −time steps trinomial tree and have a general idea for N −time steps.

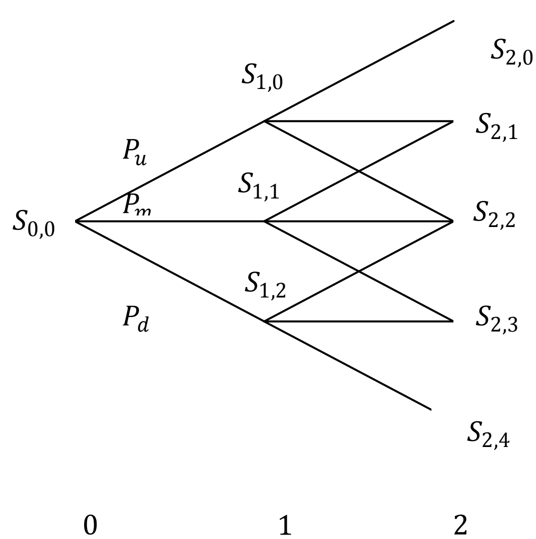


Figure 3.1 A 2 −time steps trinomial tree

In general, we can say that the initial asset price is $S_{0,0}$, $S_{i,j}$ denotes the $(j+1)$th largest asset price at time i. P_u, P_m and P_d denote the branching probabilities. Let S_u, S_m and S_d are jump sizes as we discussed in chapter 2.

Now after building the share price tree for $N-$time steps we calculate the arithmetic average at each node of the tree. Once all the averages at time step N are obtained, the payoffs corresponding to the averages are computed at maturity T that is node N.

$C(A,T) = \max(A(T) - K, 0)$ (Call Option)

$C(A,T) = \max(K - A(T), 0)$ (Put Option)

Then the Asian arithmetic average option price are computed using the usual backward induction scheme that is;

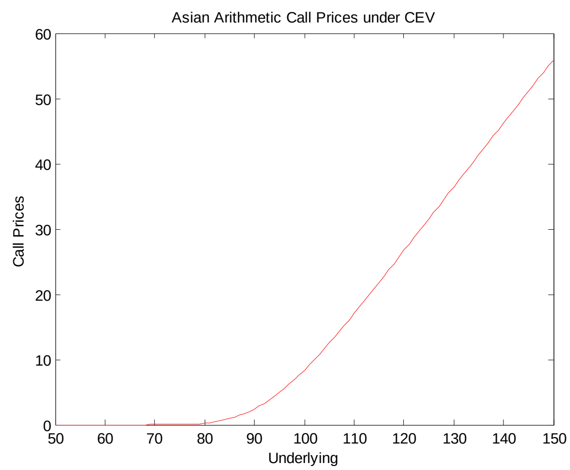
$$C_{i,j} = e^{-r\Delta t}\left[P_u C_{i+1,j+1} + P_m C_{i+1,j} + P_d C_{i+1,j-1}\right] \tag{3.1}$$

where i represents the time position and j the space position.

The backward induction algorithm can be derived from the risk-neutrality and is the same for put and call options. When applied in the context of a trinomial tree, we can calculate the option value at interior nodes of the tree by considering it as a weighting of the option value at the future nodes, discounted by one time step. Thus we can calculate the option price at time i , C_i, as the option price of an up move $P_u C_{i+1}$ plus the option price of the middle move by $P_m C_{i+1}$ plus the option price of a down move by $P_d C_{i+1}$, discounted by one time step, $e^{-r\Delta t}$. So at any node on the tree our backward induction formula, (3.1), is applied to give us the option prices at any node in the tree. The name of the algorithm should now be clear since we only need to value the option at maturity, i.e. the leaf nodes, and then work our way backwards through the tree calculating option values at all the nodes until we reach the root S_0, C_0.

Graphically,

(a)



(b)

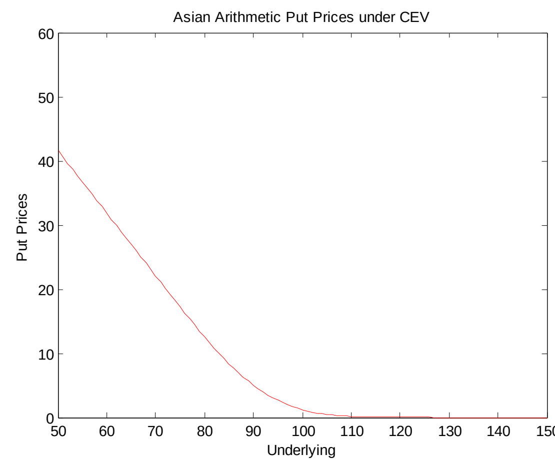


Figure 3.2 Asian Arithmetic Average (a) Call option prices (*left*), (b) Put option prices (*right*)

Note: Here elasticity, $\alpha = 1.9999,$ and volatility $\sigma = 0.25,$ $\lambda = \sqrt{1.5},$ rate, $r = 10\%,$ number of time steps $N = 13,$ strike price, $K = 95,$ time to maturity , $T = \frac{1}{2}$ year.

We can compare the Asian Arithmetic (CEV) with simulated Asian Arithmetic option prices that is the pricing of an Asian option by Monte Carlo simulation (Figure 3.3 and Figure 3.4).

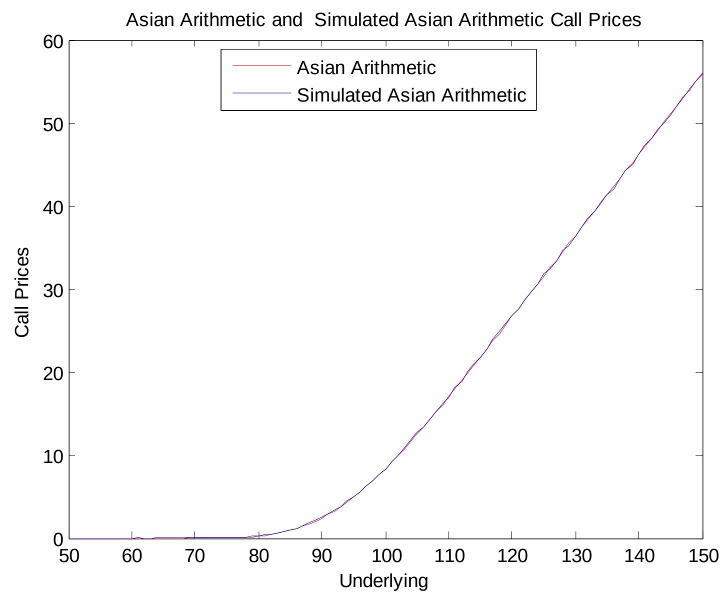


Figure 3.3 Asian Arithmetic and Simulated Asian Arithmetic Call Prices

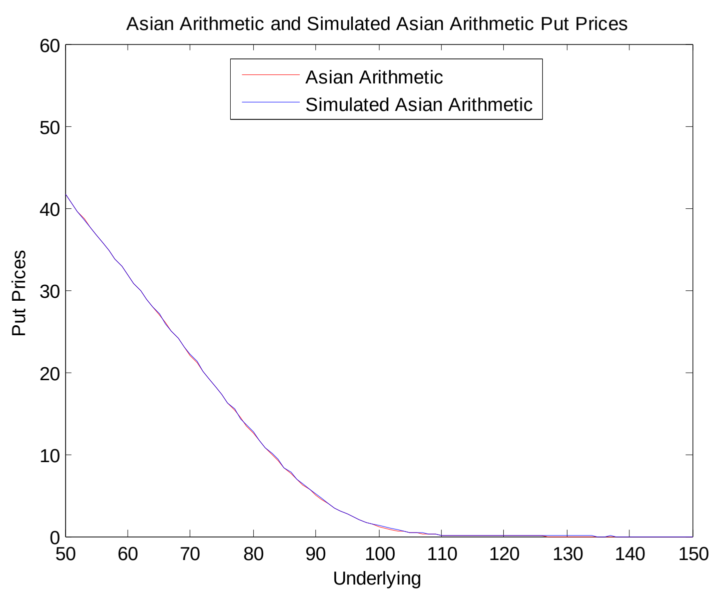


Figure 3.4 Asian Arithmetic and Simulated Asian Arithmetic Put Prices

3.2.2 Pricing Asian Geometric Average option

An Asian geometric average option is an option whose payoff depends on the geometric average price of the underlying asset during a specific period. In a discrete time model, the geometric average price of the underlying asset is defined as

$$A_j^g = \prod_{i=0}^{j} S_i^{\frac{1}{j+1}}$$

Thus the payoff for an Asian geometric average option is

$$\text{Payoff} = \begin{cases} max\left(A_n^g - K, 0\right), & \text{for a call option} \\ max\left(K - A_n^g, 0\right), & \text{for a put option} \end{cases}$$

Now the value of as Asian option can be evaluated by taking expectation of the future discounted payoff.

Using the arithmetic average the option value is

$$\text{Option values} = \begin{cases} E[e^{-rT} max(A^g(T) - K, 0)], & \text{for a call option} \\ E[e^{-rT} max(K - A^g(T), 0)], & \text{for a put option} \end{cases}$$

Now we follow the algorithm exactly what we did in the valuation of Asian arithmetic average option. From the share price tree we calculate the geometric average at each node of the tree. Once all the averages at time step N are obtained, the payoffs corresponding to the averages are computed at maturity T that is node N.

$$C^g(A, T) = \max(A^g(T) - K, 0) \text{ (Call Option)}$$

$$C^g(A, T) = \max(K - A^g(T), 0) \text{ (Put Option)}$$

Then the Asian arithmetic average option price are computed using the usual backward induction scheme that is;

$$C_{i,j}^g = e^{-r\Delta t}\left[P_u C_{i+1,j+1}^g + P_m C_{i+1,j}^g + P_d C_{i+1,j-1}^g\right]$$

where i represents the time position and j the space position.

(a) (b)

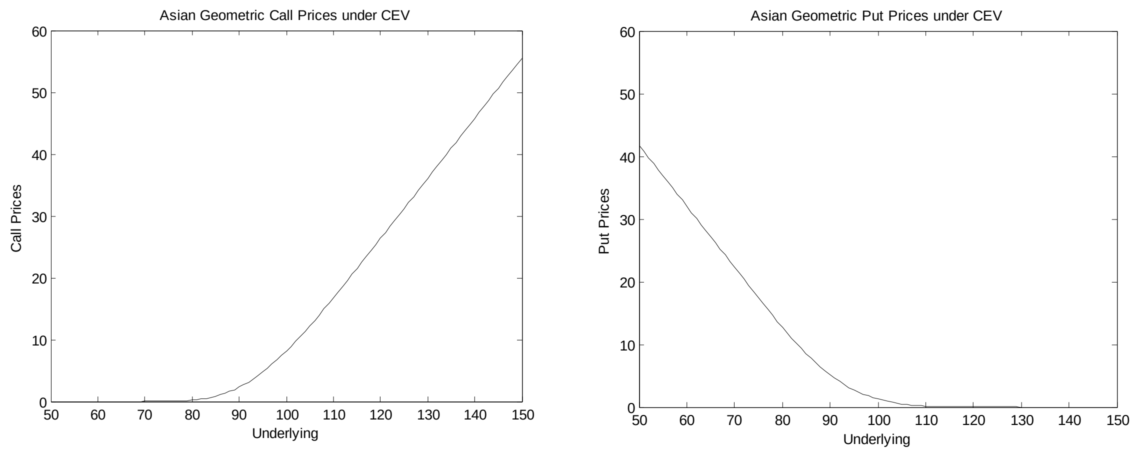


Figure 3.5 Asian Geometric Average (a) Call option prices (*left*), (b) Put option prices (*right*)

Note: Elasticity, $\alpha = 1.9999,$ and volatility $\sigma = 0.25,$ $\lambda = \sqrt{1.5},$ rate, $r = 10\%,$ number of time steps $N = 13,$ strike price, $K = 95,$ time to maturity , $T = \frac{1}{2}$ year.

3.3 European option

By definition, European call option or put option is a contract which gives its owner the right to buy or sell an agreed financial asset (underlying asset) at the agreed price K (Strike price) at the specified time T (Time to maturity).

3.3.1 Pricing European option

We have the initial asset price $S_{0,0}$, the transition probabilities P_u, P_m and P_d and also S_u, S_m and S_d as jump sizes. Now from the trinomial share price tree, calculate the option payoffs at maturity T. Mathematically,

$$\text{Payoff, } C(S,T) = \begin{cases} max(S-K,0), & \text{for a call option} \\ max(K-S,0), & \text{for a put option} \end{cases}$$

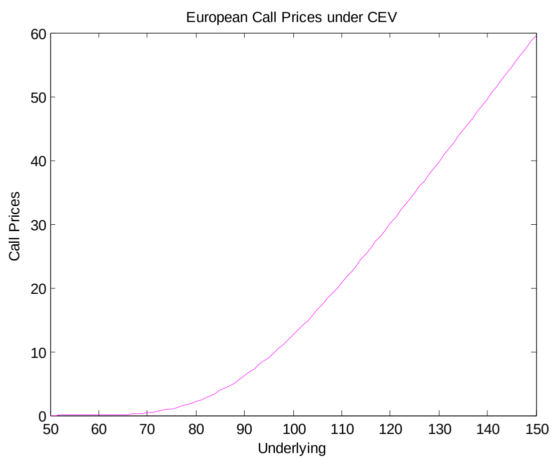
We obtain the payoffs and then use them in the following backward induction where i, j represent the time position and space position respectively:

$$C_{i,j}^{E} = e^{-r\Delta t}\left[P_u C_{i+1,j+1}^{E} + P_m C_{i+1,j}^{E} + P_d C_{i+1,j-1}^{E}\right] \tag{3.2}$$

where the superscript E stands for European option.

The European Call option prices and Put option prices are graphically illustrated as follows:

(a)



(b)

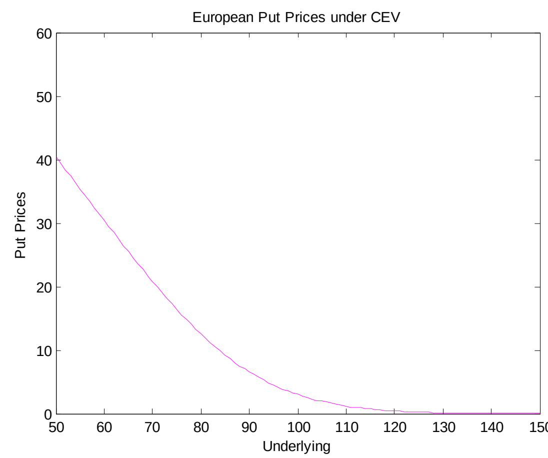


Figure 3.6 European (a) Call option prices (*left*), (b) Put option prices (*right*)

Note: Elasticity, $\alpha = 1.9999,$ and volatility $\sigma = 0.25,$ $\lambda = \sqrt{1.5},$ rate, $r = 10\%,$ number of time steps $N = 13,$ strike price, $K = 95,$ time to maturity , $T = \frac{1}{2}$ year.

We consider same specification with $\alpha = 1.9999$ (very close to 2) to compare the CEV European option prices with Black-Scholes European option prices (both Call and Put). Option prices are illustrated (Figure 3.7 & Figure 3.8) for comparison. We see that CEV European approximately very close to BS or BSM European call and put prices.

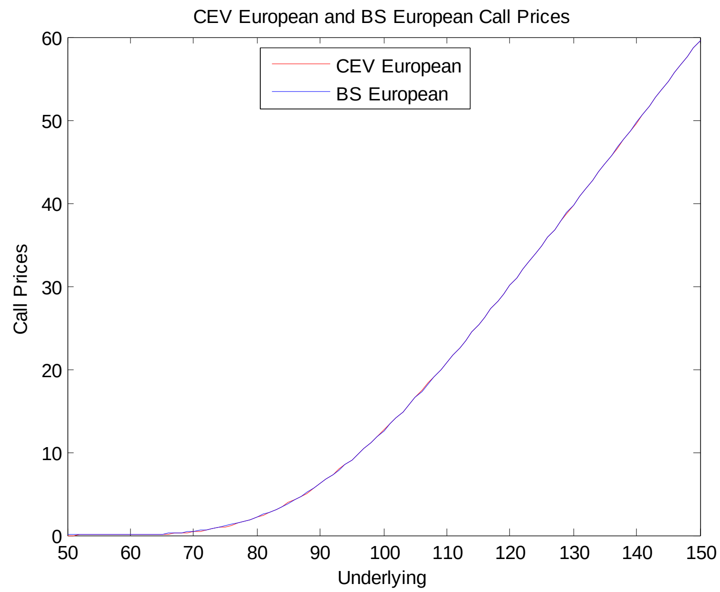


Figure 3.7 CEV European and BS European Call Prices

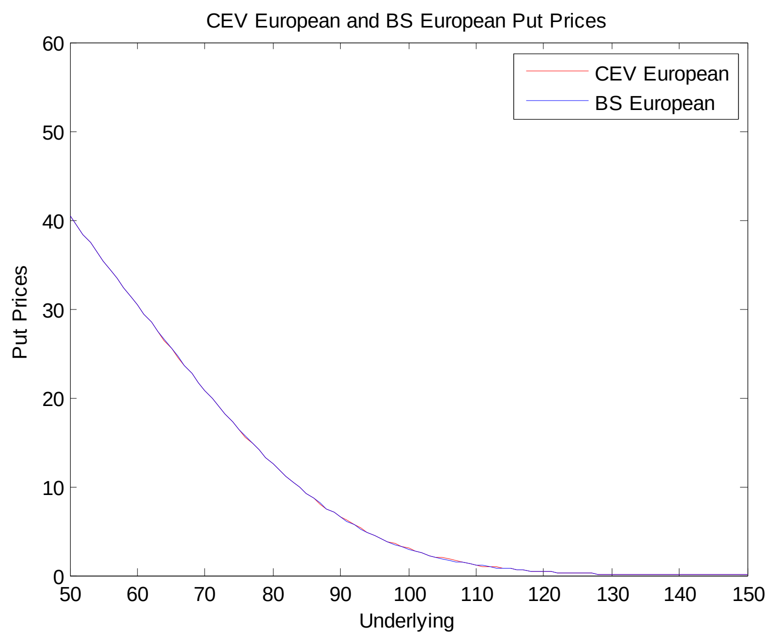


Figure 3.8 CEV European and BS European Put Prices

3.4 American option

An American option is a derivative written on an underlying asset which can be exercised any time up to its maturity. The value of an American option is the maximum value which can be achieved by optimal exercising. This value can be estimated by defining an optimal exercising rule on the option and computing the expected discounted payoff of the option under this rule.

3.4.1 American option pricing

We know, American call or put option is a contract which gives its owner the right to buy or sell an agreed asset (underlying asset) at the agreed price K (strike price) at or before the specified time T (maturity time). Therefore, an American option can be exercised at any time τ, up until maturity T, where $0 < \tau \leq T$ where 0 stands for present time. The pay-off for American put and call options can be now defined at every node of the tree, not just the leafs.

$$\text{Option Payoff, } C(S,T) = \begin{cases} max\left(S_{i,j} - K, 0\right), & \text{for a call option} \\ max\left(K - S_{i,j}, 0\right), & \text{for a put option} \end{cases}$$

The modified backward recursion takes the following form:

$$C_{i,j}^{A} = \max\left(\text{Option Payoff}, e^{-r\Delta t}\left[P_u C_{i+1,j+1}^{A} + P_m C_{i+1,j}^{A} + P_d C_{i+1,j-1}^{A}\right]\right) \qquad (3.3)$$

The backward recursion (3.3) has to be modified accordingly. Fortunately, the modification can be still derived using the principle of price neutrality.

Graphically we represent the American option prices under CEV model for elasticity, $\alpha = 1.9999$, and volatility $\sigma = 0.25$, $\lambda = \sqrt{1.5}$, rate, $r = 10\%$, number of time steps $N = 13$, strike price, $K = 95$, time to maturity , $T = \frac{1}{2}$ year.

(a) (b)

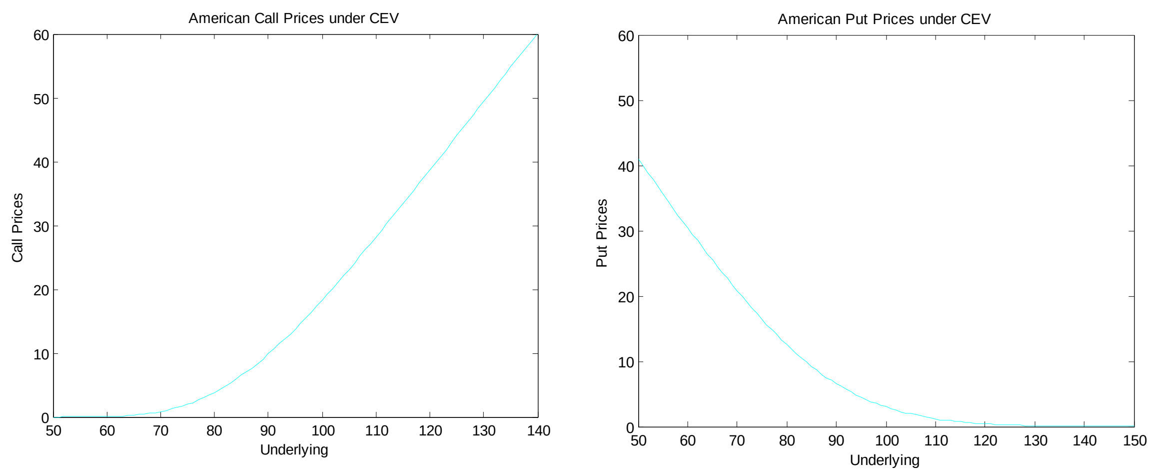


Figure 3.9 American (a) Call option prices (*left*), (b) Put option prices (*right*)

3.5 Bermudan option

The discretized version of American option, which is also called Bermudan option, is an American style derivative that can be exercised only on pre specified dates up to maturity.

3.5.1 Bermudan option pricing

In the case of Bermudan options, trinomial trees behave in similar manner. The value of the options at the nodes where early exercise is allowed is computed as in the case of American options. Otherwise it is calculated as in the case of European options.

The Bermudan options, $C^B_{i,j} = \begin{cases} (3.2), & if\ t_i = exercise\ date \\ (3.3), & otherwise \end{cases}$

Graphically,

(a) (b)

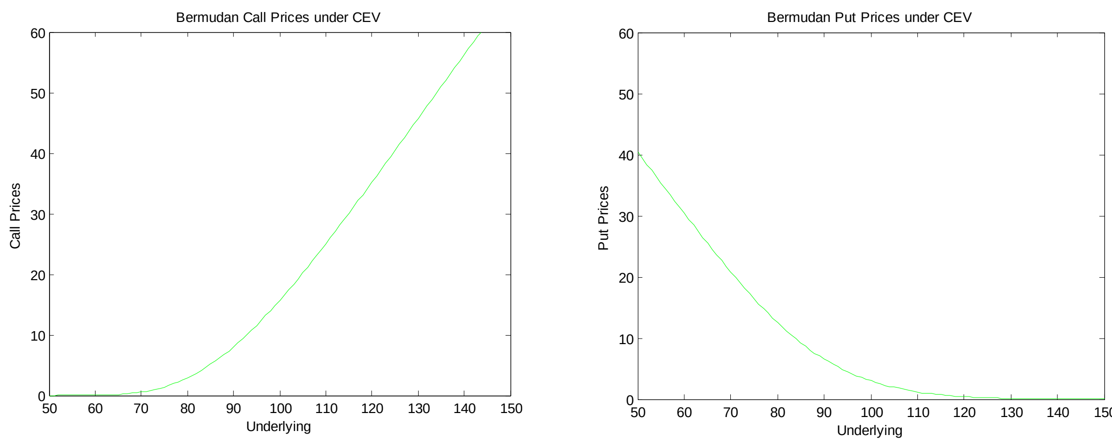


Figure 3.10 Bermudan (a) Call option prices (*left*), (b) Put option prices (*right*)

Note: Here elasticity, $\alpha = 1.9999,$ and volatility $\sigma = 0.25,$ $\lambda = \sqrt{1.5},$ rate, $r = 10\%,$ number of time steps $N = 13,$ strike price, $K = 95,$ time to maturity , $T = \frac{1}{2}$ year.

Finally, we see (Figure 3.7 and Figure 3.8) option prices of all styles and Black-Schole European prices in a same figure window for an overall idea.

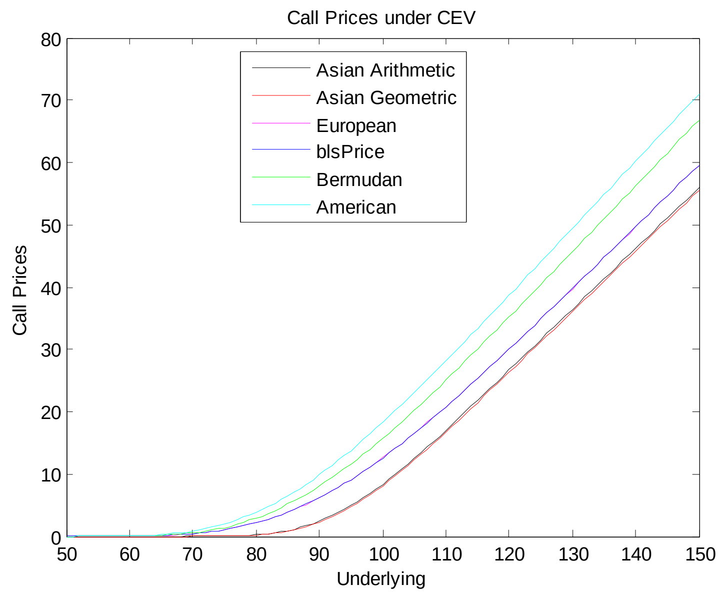


Figure 3.11 Call prices under CEV against stock prices

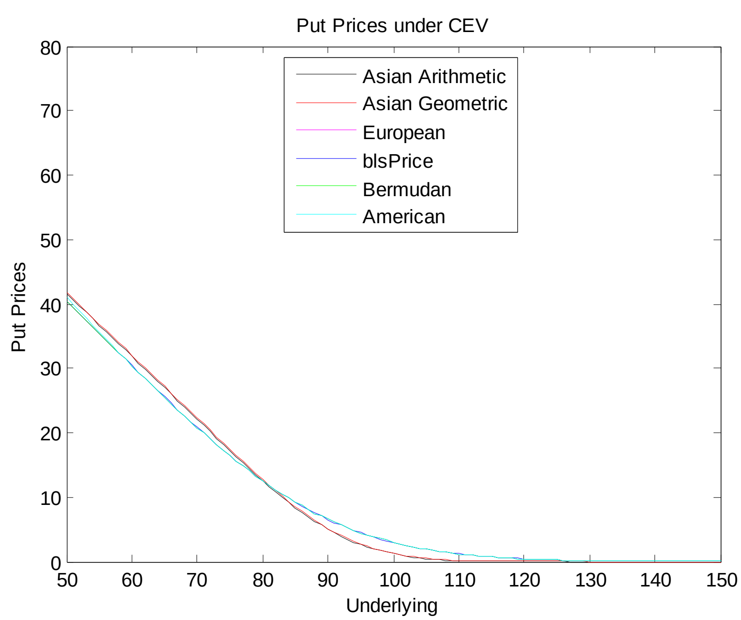


Figure 3.12 Put prices under CEV against stock prices

3.6 Number of Time Step for Accuracy

We investigate the accuracy of the option prices of each style by increasing the number of steps [6] [7] [11] [12] [18]. The Figure 3.9 shows how the option prices converge to the true price.

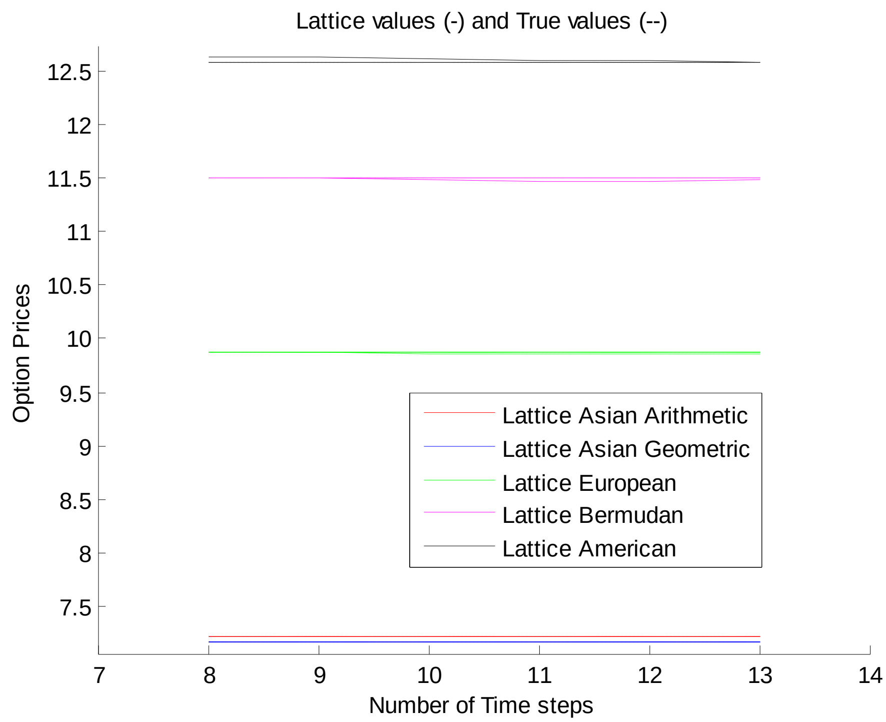


Figure 3.13 Lattice prices for different number of time steps (ranging from 8 to 13) with true values

We clearly see that for number of time step 13, the lattice option prices of each style converge to the actual value. So, in this thesis we will use number of time step 13 to obtain our results.

Chapter 4
Greeks under the CEV model

Information about derivatives of options (commonly known as Greeks) is of practical and theoretical importance. In addition to pricing an option, a dealer of the financial services industry must also be able to hedge it. Thus, a practitioner needs to have knowledge regarding the sensitivity measures of derivative securities for designing hedging strategies to reduce the risk. In this chapter, we will derive the most important Greeks delta and gamma under all four contract styles in two different approach; Ordinary lattice approach and Auxiliary lattice approach.

4.1 The Greeks

The Greeks are the partial derivatives with respect to the underlying parameter to see the sensitivity of small changes in that parameter [13]. Delta measures the rate of change of the option price with respect to the underlying security; $\delta = \frac{\partial V}{\partial S_t}$. Gamma measures the rate of change in the delta with respect to the underlying security, therefore being a second order derivative, $\gamma = \frac{\partial \delta}{\partial S_t} = \frac{\partial^2 V}{\partial S_t^2}$.

4.1.1 Delta and Gamma through the ordinary Trinomial Lattice

We first create a function and denote the trinomial lattice by $C(S, N, K, \sigma, T, r)$, for underlying asset S, number of time step N, Strike K , volatility σ, time to maturity T, risk free interest rate r, and elasticity α.

Then Delta and Gamma [9] are calculated through the lattice Figure 3.1,

$$\delta = \frac{C(S + dS, N, K, \sigma, T, r, \alpha) - C(S, N, K, \sigma, T, r, \alpha)}{dS}$$

$$\gamma = \frac{C(S + dS, N, K, \sigma, T, r, \alpha) - 2C(S, N, K, \sigma, T, r, \alpha) + C(S - dS, N, K, \sigma, T, r, \alpha)}{dS^2}$$

where we adjust dS for CEV model as

$$dS = S\left(\sigma S^{\frac{\alpha}{2}-1}\sqrt{T}\right)$$

dS is chosen such that the amount is proportional to the volatility and the current share price, while taking into consideration time up to maturity.

We plot Delta and Gamma values against share price in Figure 4.1, via ordinary Trinomial Lattice under CEV.

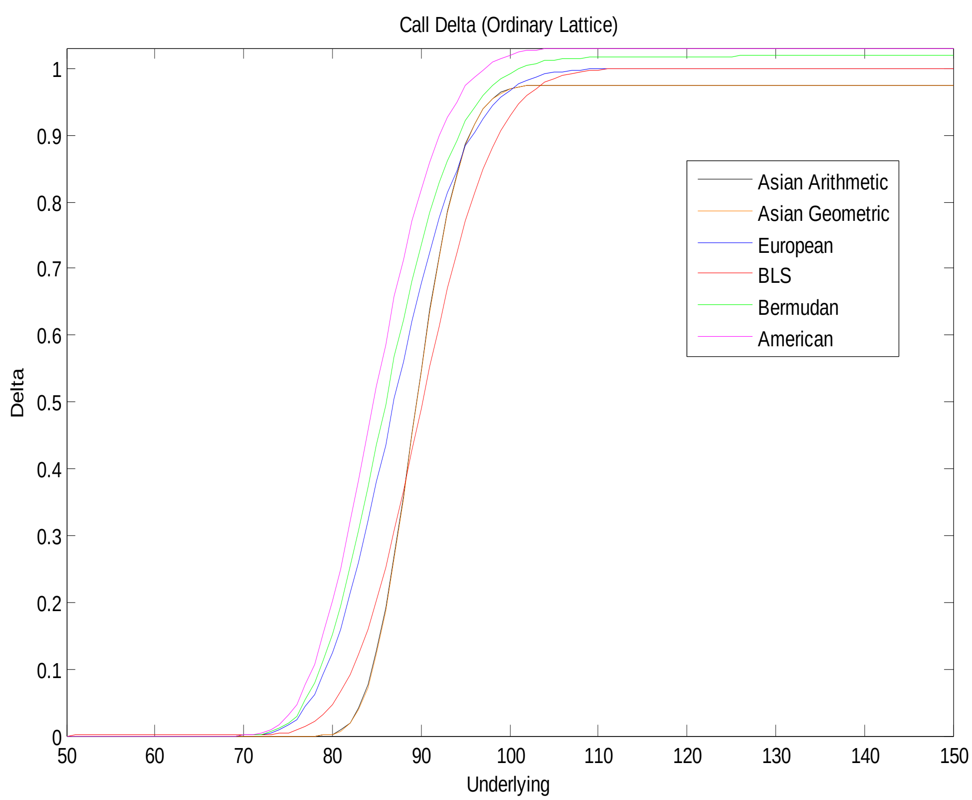


Figure 4.1 Delta via the ordinary Trinomial Lattice

We plot Gamma values against share price in Figure 4.2, via ordinary Trinomial Lattice under CEV.

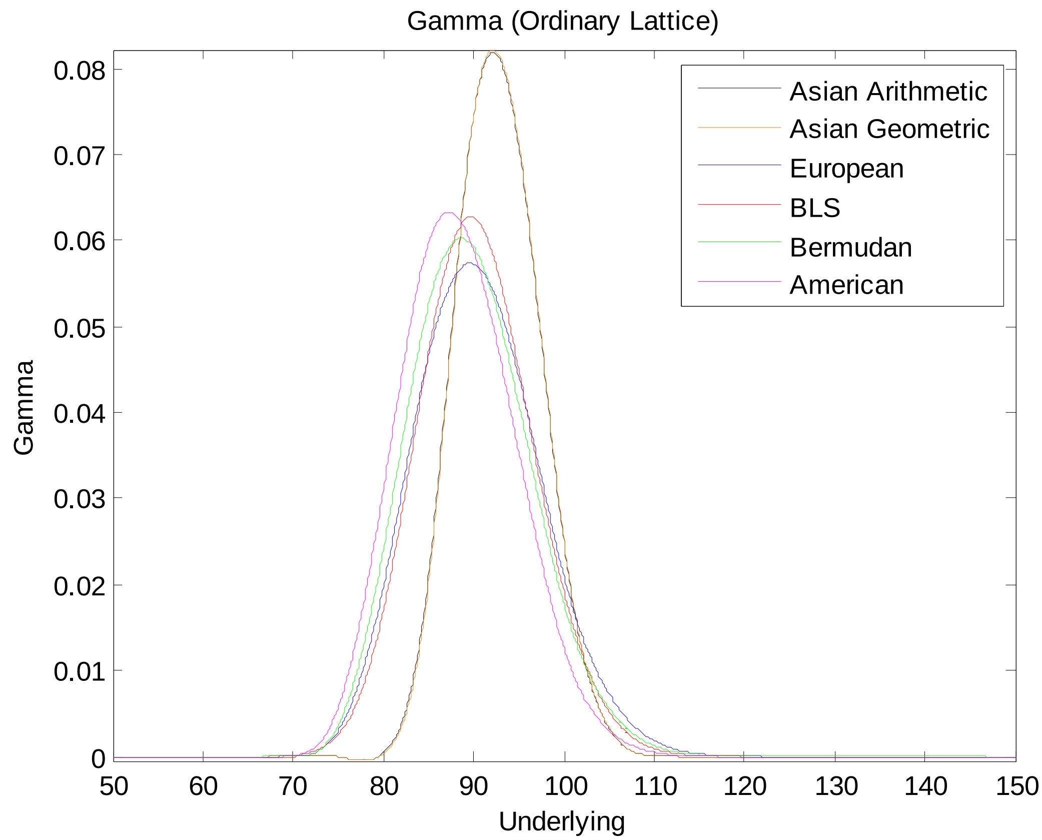


Figure 4.2 Gamma via the ordinary Trinomial Lattice.

In both figure, we graphically represent the Deltas and Gammas for Asian Arithmetic Average, Asian Geometric Average, European, Bermudan and American option prices.

4.1.2 Delta and Gamma through the Auxiliary Trinomial Lattice

We can also derive the Delta and Gamma via auxiliary lattice approach. In this case, we will first form the trinomial lattice from where we can have option prices.

For that, we construct a 2 −time steps auxiliary trinomial tree and have a general idea for N −time steps.

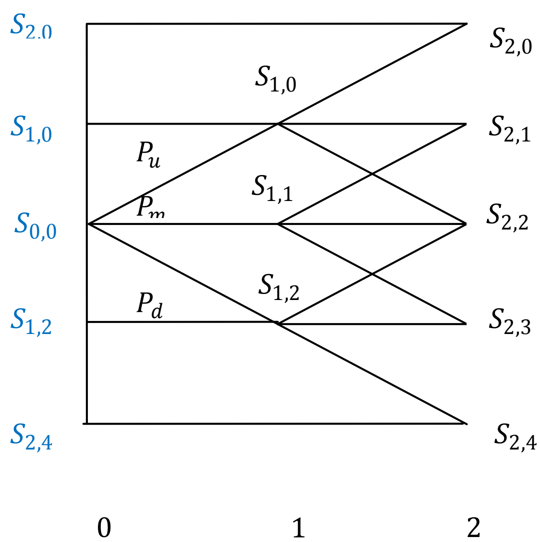


Figure 4.3 A 2 −time steps Auxiliary Trinomial Tree.

Now from the lattice we derive the auxiliary lattice prices $C(S_{i,j}, N, K, \sigma, T, r, \alpha)$ for $S_{2,0}, S_{1,0}, S_{0,0}, S_{1,2}$ and $S_{2,4}$ (Blue colored).

Then using these prices we will derive the deltas from auxiliary lattice.

$$\delta_{A_{1,0}} = \frac{C(S_{2,0}, N, K, \sigma, T, r, \alpha) - C(S_{0,0}, N, K, \sigma, T, r, \alpha)}{S_{2,0} - S_{0,0}}$$

$$\delta_{A_{0,0}} = \frac{C(S_{1,0}, N, K, \sigma, T, r, \alpha) - C(S_{1,2}, N, K, \sigma, T, r, \alpha)}{S_{1,0} - S_{1,2}}$$

$$\delta_{A_{1,2}} = \frac{C(S_{0,0}, N, K, \sigma, T, r, \alpha) - C(S_{2,4}, N, K, \sigma, T, r, \alpha)}{S_{0,0} - S_{2,4}}$$

From these calculated deltas we calculate the gammas,

$$\gamma_1 = \frac{\delta_{A_{1,0}} - \delta_{A_{0,0}}}{S_{1,0} - S_{0,0}}$$

$$\gamma_2 = \frac{\delta_{A_{0,0}} - \delta_{A_{1,2}}}{S_{0,0} - S_{1,2}}$$

In this way, we can derive deltas and gammas from N −time steps Auxiliary Trinomial Lattice. After that, we use interpolation to get deltas and gammas. This approach saves a lot of time than the earlier ordinary lattice approach, which is a great advantage in calculating Greeks from trinomial lattice under CEV.

Graphically, the deltas can be shown as follows:

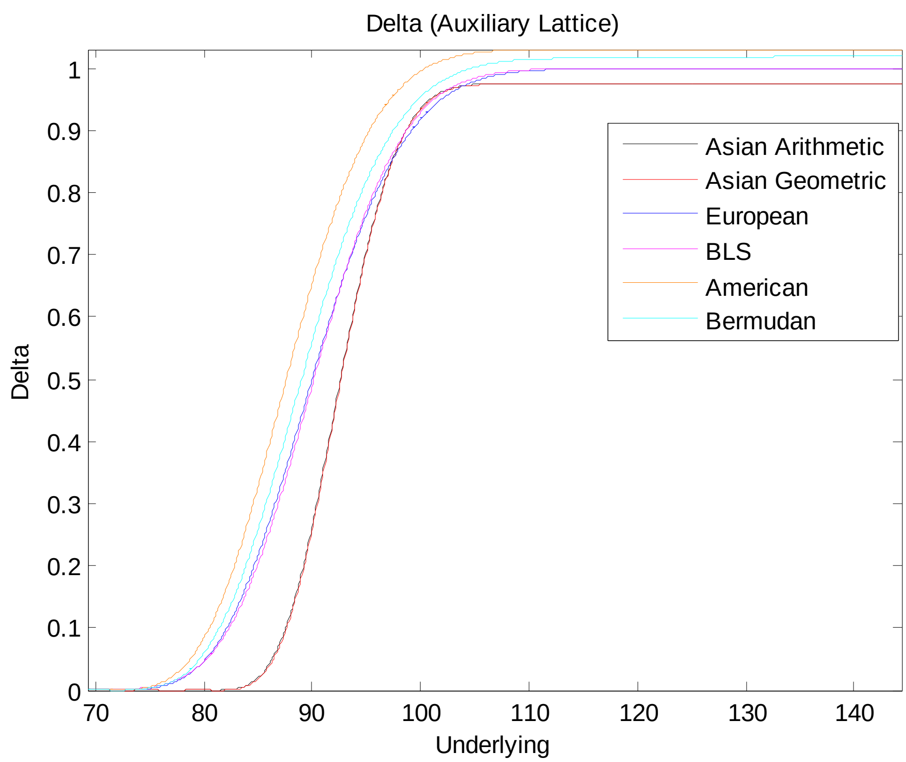


Figure 4.4 Delta via the Auxiliary Trinomial Lattice

Graphically, the gammas can be shown as follows:

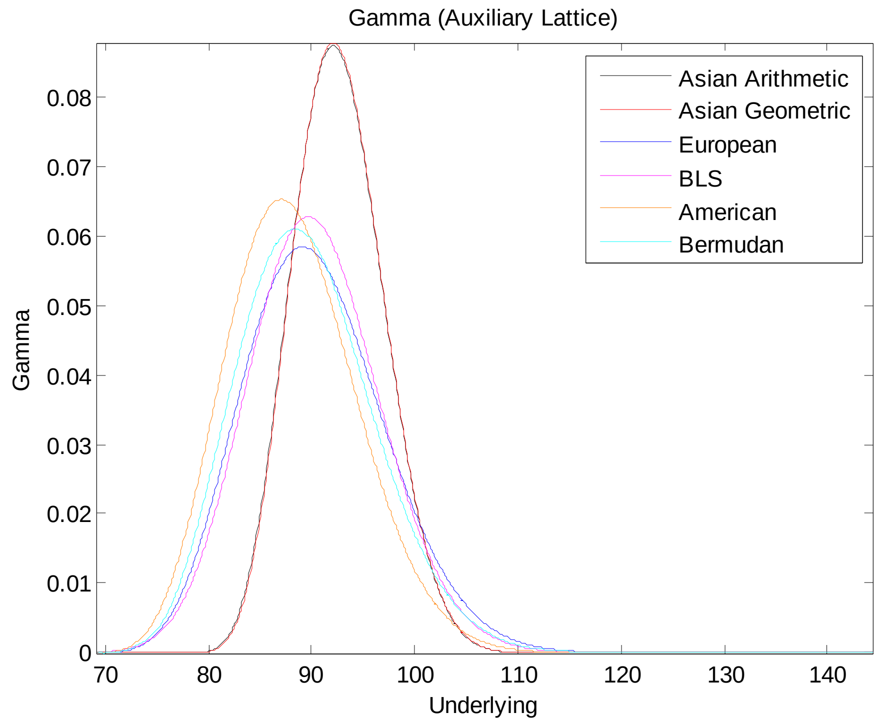


Figure 4.5 Gamma via the Auxiliary Trinomial Lattice

In Figure 4.4 and Figure 4.5, we plotted our derived delta, gammas for each corresponding option auxiliary lattice under CEV model.

The Greeks plays an important role in the financial literature. With the proper knowledge of Greeks, their efficient implementations as risk management tool impact the Greek-approximation to option portfolios under all four option styles (Chapter 5). From the figures of both ordinary and auxiliary lattice we can clearly see the auxiliary trinomial lattice approximately follows the same values as the ordinary trinomial lattice except for European under CEV model and European under BSM model. In Figure 4.1 and Figure 4.4, we plotted deltas through Ordinary lattice and auxiliary lattice respectively. It is evident from these figures that auxiliary lattice gives more accurate delta for European option as we can see that European delta under CEV (for elasticity 1.9999) similar to

BLS delta which was calculated by using MATLAB built in function *blsdelta*. Also, we have mentioned earlier in the subsection 4.2.2 that the second one between these two approaches gives our desired Greeks under all four contract styles more quickly using interpolation which lucratively solves a big issue of execution time in hedging option portfolio (Chapter 5).

Chapter 5

Hedging with Greeks from Auxiliary Lattice across Styles under CEV Dynamic

Mitigating the risk of option portfolio is the main objective of hedgers. The change in the value of option portfolio is subject to option sensitivities summarized in Greeks. In this chapter, we discuss Greek-based approximations to option portfolios and then show graphical representation of their accuracy across four contract styles.

5.1 Greek-based approximations to option portfolio values

We now consider a portfolio similar to one used in Britten-Jones and Schaefer (1999) [3]. This portfolio is constructed from our data set subject to the proviso that, while the call options in the portfolio are traded in the market, the put option is priced using put-call parity [16]. The option portfolio is described in Table 5.1.

Table 5.1 Illustrative option portfolio constructed using options traded on January 23, 2008

Type of option	Put	Call_1	Call_2
Strike (K)	1,200	1,200	1,550
Maturity (days)	23	23	23
Option price	4.2251	146.60	0.1750
Position (m_j)	-1	-1.5	2.5

However, whilst Britten-Jones and Schaefer (1999) and Christoffersen (2003) only considered the valuation of similar portfolio under the BS model [3] [5], we wish to examine portfolio valuation under CEV model for four types of options namely European, American, Bermudan, and Asian (Arithmetic average and Geometric average).

We first find the BS approximation [16]. Assume a risk-management horizon of five trading days (seven calendar days), which corresponds to the sampling interval for our weekly data. As in Christoffersen (2003) we consider the complete pay-off profile of the portfolio, under CEV model, for different future values of the underlying asset prices S_{t+5}. Let P_t and $P(S_{t+5})$ denote the portfolio value today and at the end of five trading days respectively.

We have:

$$P_t = m_1 put + m_2 call_1 + m_3 call_2$$
$$P^{\delta}(S_{t+5}) = P_t + \delta^p (S_{t+5} - S_t)$$
$$P^{\delta\gamma}(S_{t+5}) = P_t + \delta^p (S_{t+5} - S_t) + \frac{1}{2}\gamma^p (S_{t+5} - S_t)^2$$

Here δ^p and γ^p are model-dependent portfolio hedge factors defines as:

$$\delta^p = m_1 \delta^{BS}_{put} + m_2 \delta^{BS}_{call_1} + m_3 \delta^{BS}_{call_2}$$
$$\gamma^p = m_1 \gamma^{BS}_{put} + m_2 \gamma^{BS}_{call_1} + m_3 \gamma^{BS}_{call_2}$$

where m_1 is the number of puts and m_2 and m_3 are the numbers of calls respectively in the portfolio.

The true value of the option portfolio is then obtained through full-valuation using model-based option prices:

$$P^{exact}\,(S_{t+5}) = m_1 \times put^{BS}(K = 1200, T = 23 - 7)$$
$$+ m_2 \times call_1{}^{BS}(K = 1200, T = 23 - 7)$$
$$+ m_3 \times call_2{}^{BS}(K = 1200, T = 23 - 7)$$

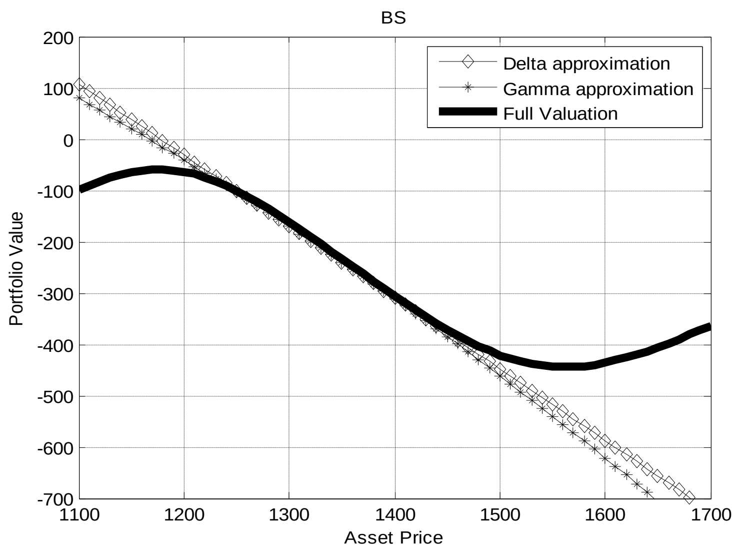


Figure 5.1 Portfolio valuation: full valuation versus Greek approximation for BS model (current asset price 1338.6)

5.2 Delta and Delta-Gamma approximation under CEV model

Now we will use the same approach as above for delta and delta-gamma approximation of Asian arithmetic average, Asian geometric average, European, American and Bermudan options but under CEV model. In each case we will consider the same contracts as in the Table 5.1 and give proxy to the market option prices by option prices calculated from model.

Table 5.2 Illustrative option portfolio using options of four styles

Type of option	Put	Call_1	Call_2
Strike (K)	1,200	1,200	1,550
Maturity (days)	23	23	23
Option price (European)	0.8498	146.9843	0.2915
Option price (American)	0.8498	182.1760	0.7042
Option price (Bermudan)	0.8498	171.0525	0.4068
Option price (Asian Arithmetic average)	0	141.9281	0
Option price (Asian Geometric average)	0	141.4394	0
Position (m_j)	-1	-1.5	2.5

Now we can graphically illustrate portfolio valuation for each option in Figure 5.2.

We can see the accuracy of these Greek approximations in Figure 5.2. The most clearly visible feature of these plots is the way the Greek-based approximations for all models diverge away from the full valuations when there are large swings in the underlying asset price. But most importantly we find that the Greek-based approximations are accurate for all four contract types for relatively small underlying asset price movements.

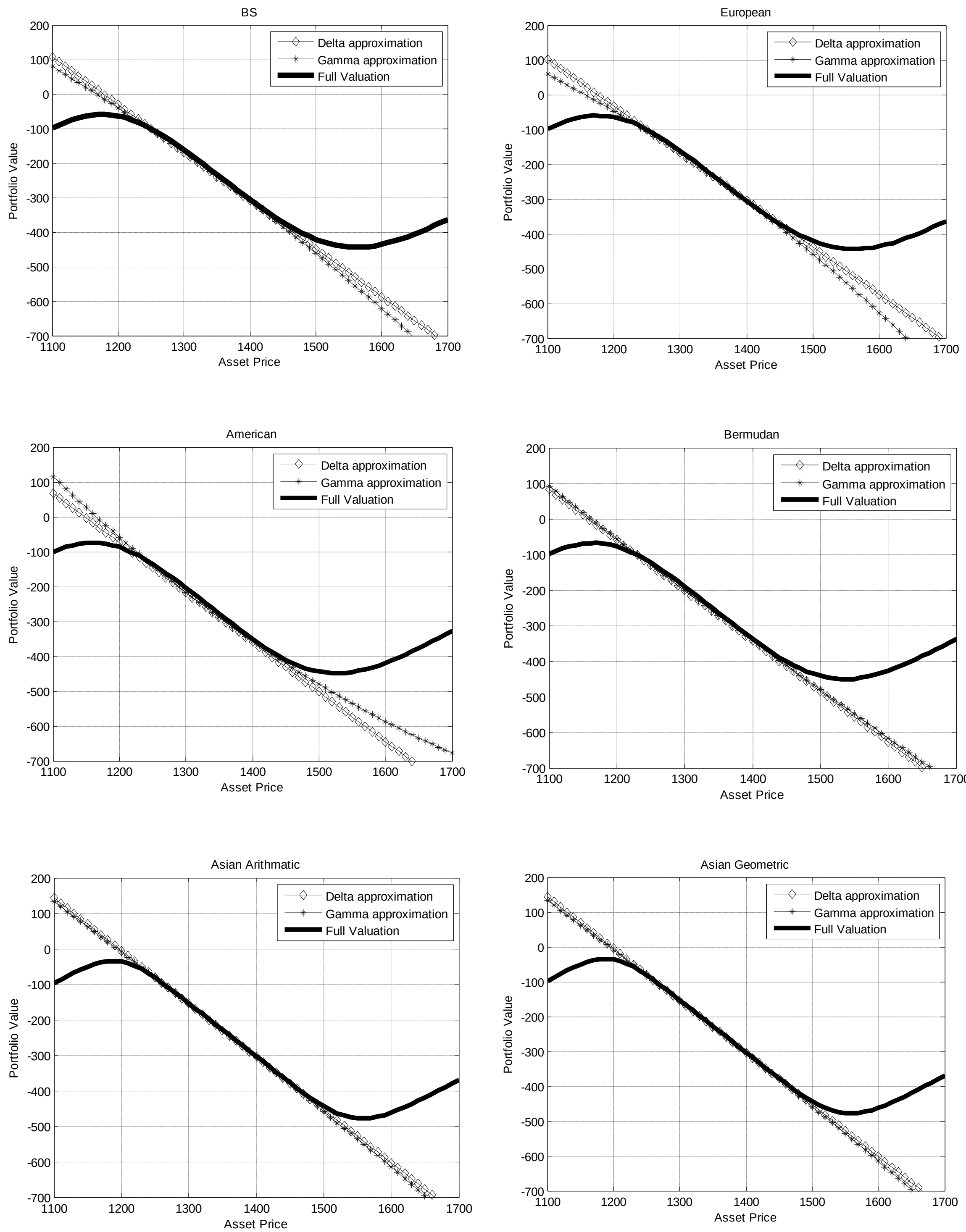


Figure 5.2 Portfolio valuation: full valuations versus Greek-based approximations (current asset price 1338.6)

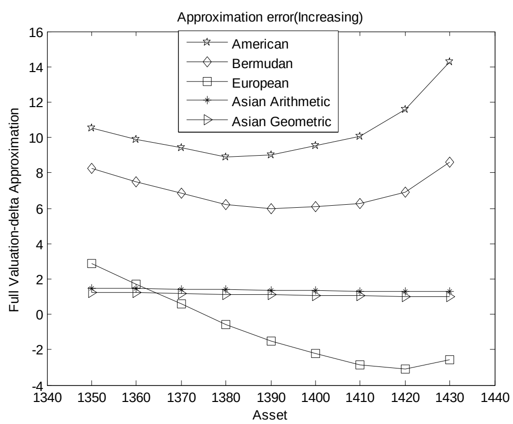
In Figure 5.2, we notice that Greek-based approximations to option portfolios for all four contract types. We also see that delta and delta-gamma approximation have differences across styles and these differences varies a little across the styles, being least for approximating the Bermudan option portfolio and greatest for approximating for American option. For Asian (Arithmetic and Geometric) average options the delta approximations and the delta-gamma approximations also show little variations over the underlying asset price movements.

It is clearly evident (see Figure 5.2) that delta and delta-gamma based approximations to European option portfolio under the CEV dynamics are better compare to the one under the Black-Scholes and Merton model. The Greek-based approximations to American option portfolio and Bermudan option portfolio look similar for relatively small underlying asset price movements. For Asian Geometric average option portfolio, delta and delta-gamma based approximations seem to perform better than that in the case of Asian Arithmetic average option.

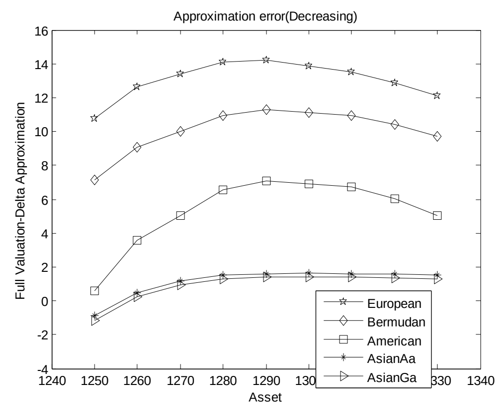
5.3 Delta and Delta-Gamma approximation Error

From the delta and delta-gamma approximation and the full valuation we compute the error (Figure 5.3).

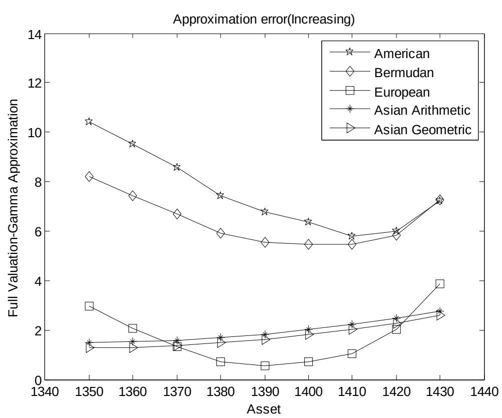
(a)



(b)



(c)



(d)

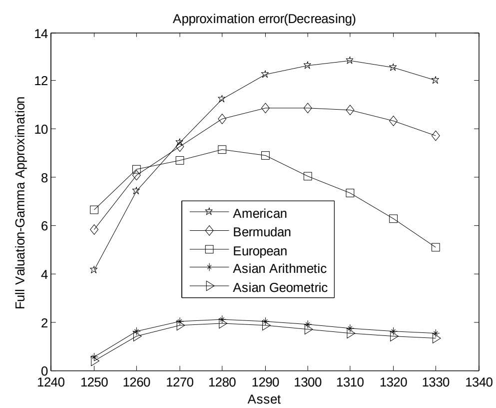


Figure 5.3 (a) Approximate delta Error (increasing) (b) Approximate delta Error (decreasing)

(c) Approximate delta-gamma Error (increasing) (d) Approximate delta-gamma Error (decreasing)

Option trading can be taken to next level with the help of effective Greek-based hedging techniques. In Greek-based hedging techniques, delta hedging mainly intends to reduce the risk with price movements in the underlying asset and hence provides protection against small changes in stock prices; for larger movements in the underlying price, effective risk management requires the use of both first order and second order hedging or delta-gamma hedging. We observe that these Greek-based approximations under all four contract styles are approximately similar to respective option portfolio values for relatively small underlying asset price movements (see Figure 5.3). However, the performance of delta hedging and delta-gamma hedging also clearly demonstrate how delta-gamma hedging preserves values for much larger swings in the underlying asset price as compared to delta hedging. So, we can conclude that the delta-gamma approximation to the option portfolio is among the more important concepts in option portfolio management.

From figure 5.3, we can also turn a clear conclusion about effective hedging performance across styles in the face of upward asset price movements as well as in the downward asset price movements.

The approximate delta error (see Figure 5.3(a)) for increasing asset price suggests that the delta approximation error to American option portfolio is very high compare to other option portfolios. Delta approximation to Asian Geometric option portfolio is better than all other option portfolios under different contract types.

In Figure 5.3(b), we see that in the face of downward asset price movements delta approximation to Asian Geometric average option portfolio a little more accurate than all other styles of contact. However, delta approximation to American option portfolio is very poor in this case as well.

Now we observe the accuracy of non-linear delta-gamma based approximation to option portfolios under all four contract types in Figure 5.3(c) and Figure 5.3(d).

In the face of upward asset price movements, it is evident that delta-gamma approximation to European option portfolio for asset prices in the range $[1370,1420]$ better than delta-gamma approximation to Asian (Arithmetic and Geometric average) option portfolios, which was seen nowhere in our considered earlier cases. The delta-gamma approximation to Bermudan option portfolio has more accuracy than approximation to American option portfolio.

For decreasing asset price (see Figure 5.3(d)) non-linear delta-gamma based approximation to Asian (Arithmetic and Geometric average) option portfolios have the highest accuracy among option portfolios of all other styles. But unlike the earlier cases delta-gamma approximation to American option portfolio is not always poor compare the other styles. For some asset prices (1250 to 1260), approximation to American option portfolio better than approximation to European as well as Bermudan option portfolios.

Chapter 6

Empirical Analysis of Option Portfolios

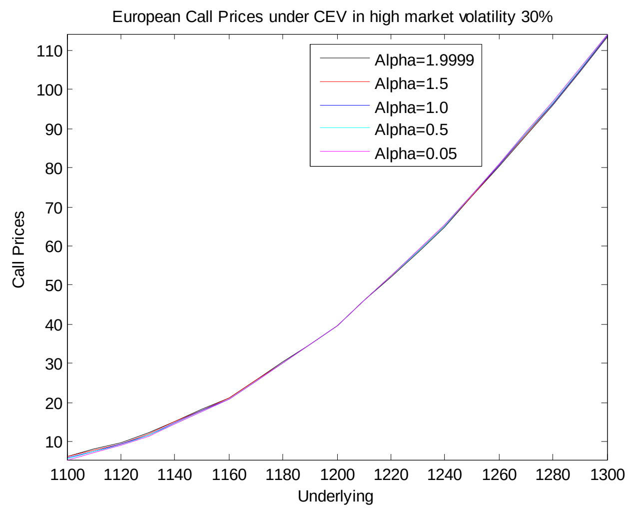
In chapter 5, we have seen how Greeks can be effectively used for hedging option portfolio. Chapter 6 will be completely devoted to show our analysis of the hedging error of the European option, American option, Bermudan option, Asian Arithmetic Average option and Asian Geometric Average option. We will use different values of elasticity ranging from 0 to 2 in both the high market volatility and low market volatility and observe the performance of portfolio hedging across four styles. For this we will first illustrate our option prices under all four option types in different market volatility to see their behavior.

6.1 Effects of Market Volatility on option prices

In this section, we observe the behavior of the option prices for different elasticity or alphas resulted from the effects of market volatility. Volatility, a measure of how fast and how much prices of the underlying asset move, is the key to understanding why option prices fluctuate and act the way they do. We will consider both high (30%) and low (10%) market volatility to show our results graphically (see Figure 6.1- 6.5). We first demonstrate European option prices in Figure 6.1.

American Option Prices

(a)



(b)

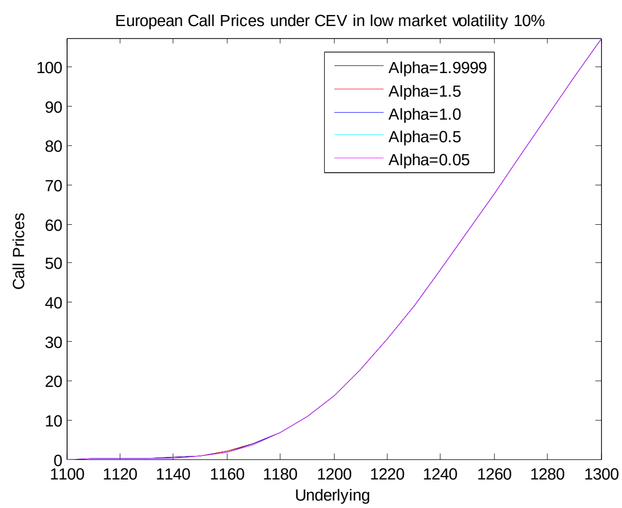
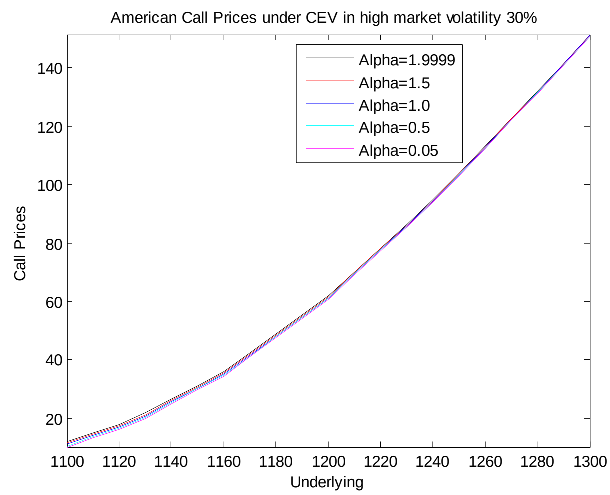


Figure 6.1 European option prices under CEV model in (a) high market volatility 30% and (b) low market volatility 10%

Note that, here we used time to maturity, $T = 23/365$ year, Strike, K= 1200. We will use these same specifications for the following Figures as well. For market volatility 30%, we see that European option prices are different for different alpha or elasticity even if there are large swings in the underlying asset price. However, when volatility is only 10%, they differ from each other for very small asset price movements.

American Option Prices

(a)



(b)

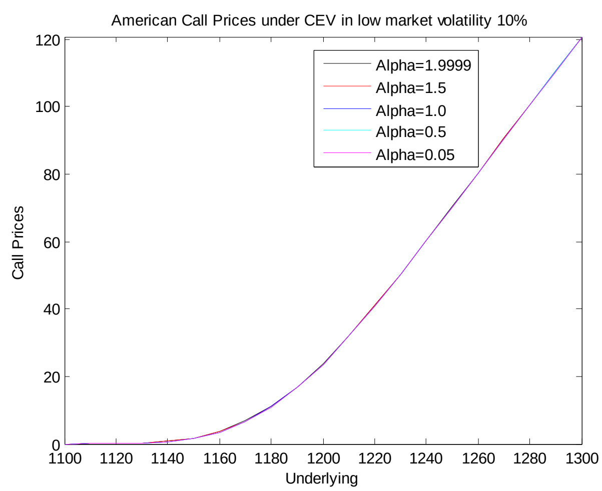
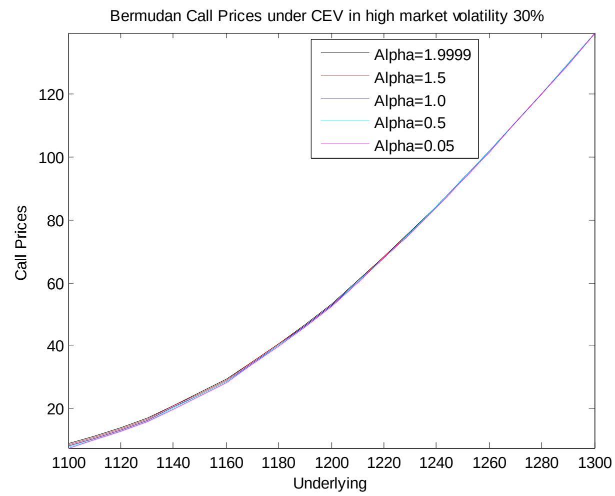


Figure 6.2 American option prices under CEV model in (a) high market volatility 30% and (b) low market volatility 10%

Unlike European option prices American option prices vary in price in the both type of market volatility, despite the fact that in the case of 10% the variation is relatively small.

Bermudan Option Prices

(a)



(b)

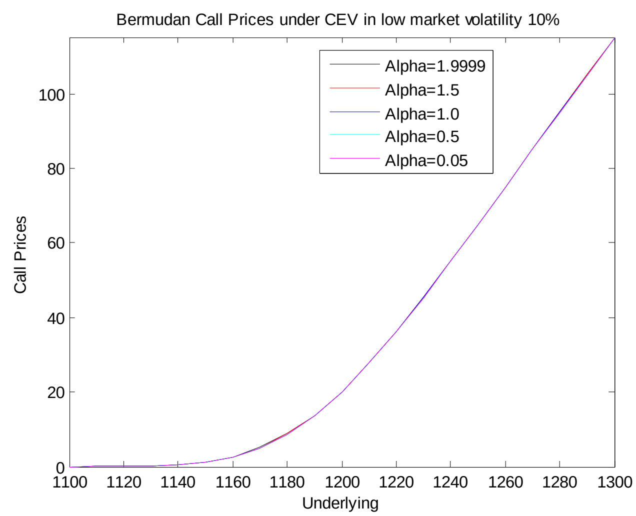
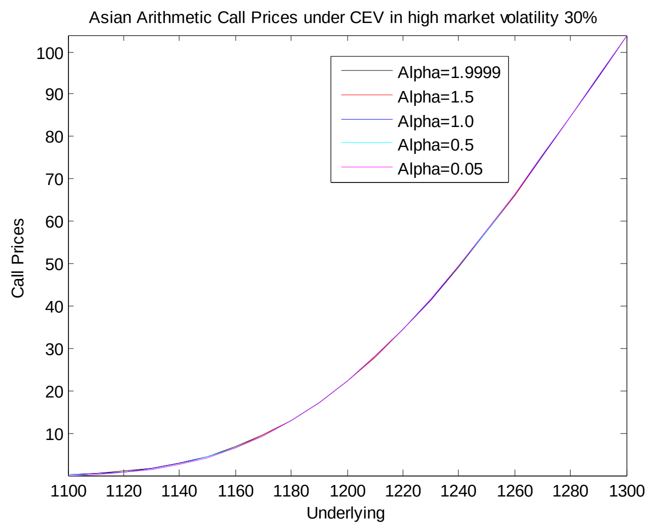


Figure 6.3 Bermudan option prices under CEV model in (a) high market volatility 30% and (b) low market volatility 10%

As we already know that Bermudan options allow a trader to exercise his option on any of several specified dates before the option expires; thus, Bermudan options are sort of a middle ground between American and European options. For the same reason, Bermudan option prices look very similar to the graphs in Figure 6.1 and Figure 6.2.

Asian Arithmetic Option Prices

(a)



(b)

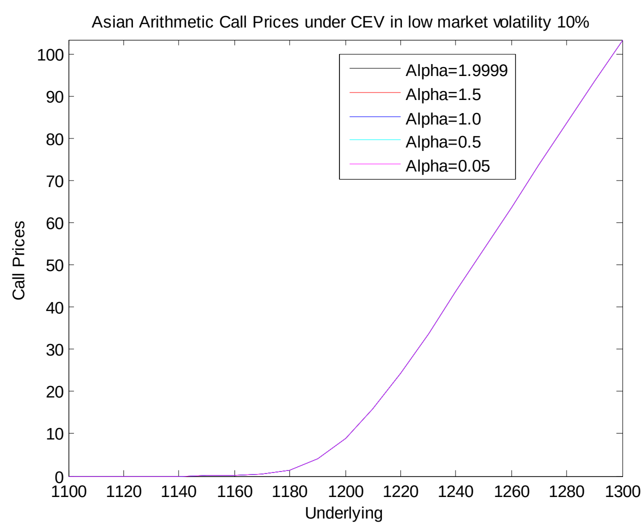
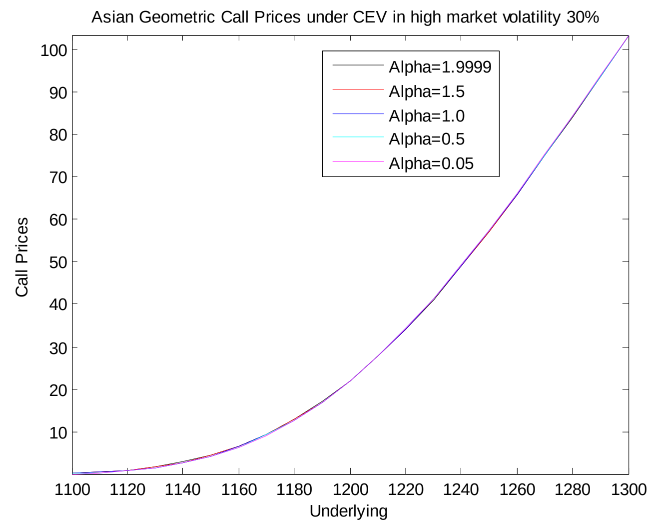


Figure 6.4 Asian Arithmetic option prices under CEV model in (a) high market volatility 30% and (b) low market volatility 10%

We clearly see that Asian Arithmetic option prices show no changes in low volatility market.

Asian Geometric Option Prices

(a)



(b)

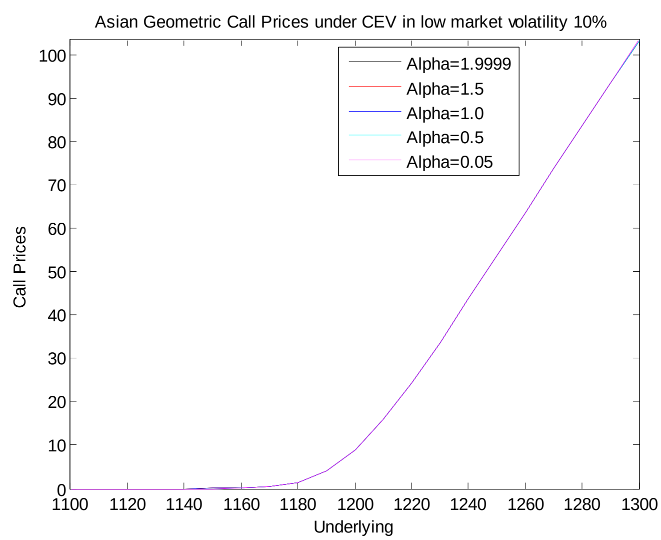


Figure 6.5 Asian Geometric option prices under CEV model in (a) high market volatility 30% and (b) low market volatility 10%

Asian Geometric option prices show differences when volatility is 30%, but no changes in the low volatility market like Asian Arithmetic option prices.

Volatility significantly affects the option prices. The higher the volatility of the underlying asset, the higher is the price for both call options and put options. This happens because higher volatility increases both the up potential and down potential. The upside helps calls and downside helps put options. We here considered only the call options but we could derive an overall idea. And the idea suggests that option prices do not behave alike for altered elasticity in the all market volatility variations.

Now, in the following section (6.2) we will see how the above results affect in hedging option portfolios across styles in market.

6.2 Empirical Analysis

We will now carry out our empirical analysis for all option portfolios. We will see the hedging accuracy for different elasticity values that is alphas 1.9999, 1.5, 1, 0.5 and 0.5. We plot delta approximation error (increasing (a) and decreasing (b)) and delta-gamma approximation error (increasing (c) and decreasing (d)).

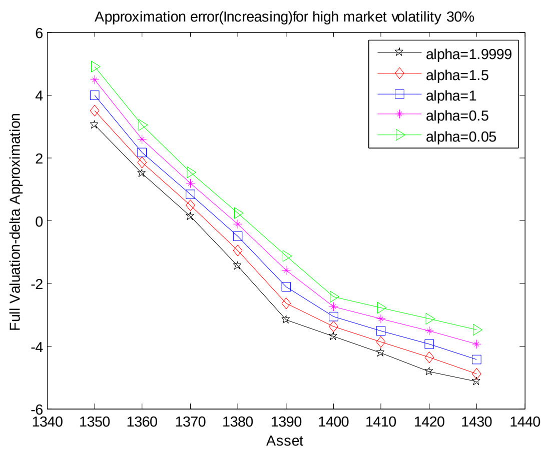
In all cases, we will first see the results in high market volatility and then in low market volatility. For this purpose we will take 30% and 10% market volatility respectively.

6.2.1 European option

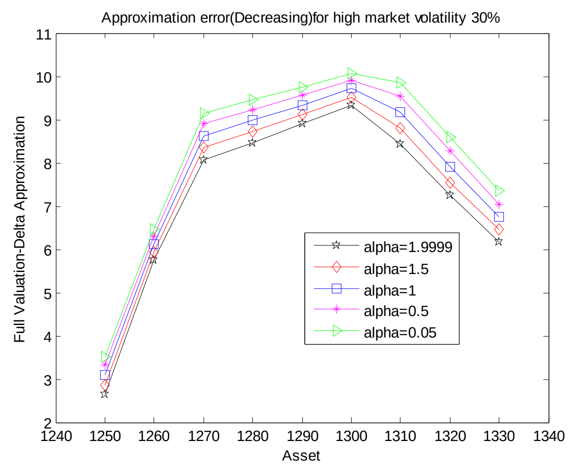
Approximation delta and delta-gamma error for European option portfolio are represented in Figure 6.6 and Figure 6.7.

For high market volatility 30%

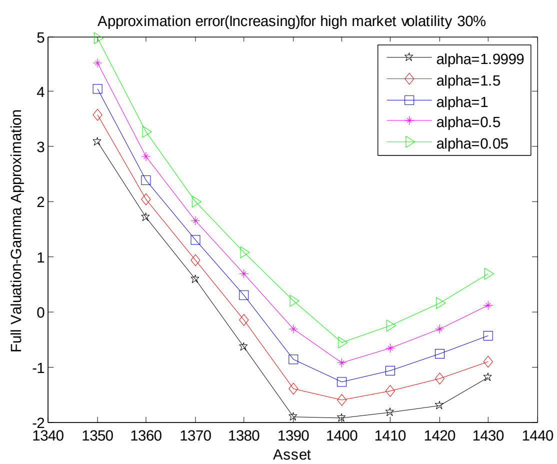
(a)



(b)



(c)



(d)

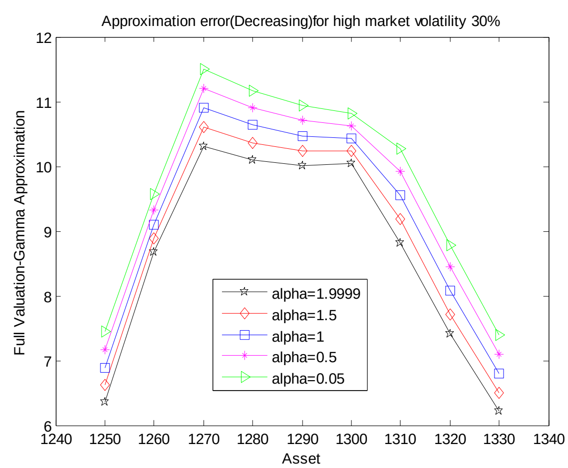
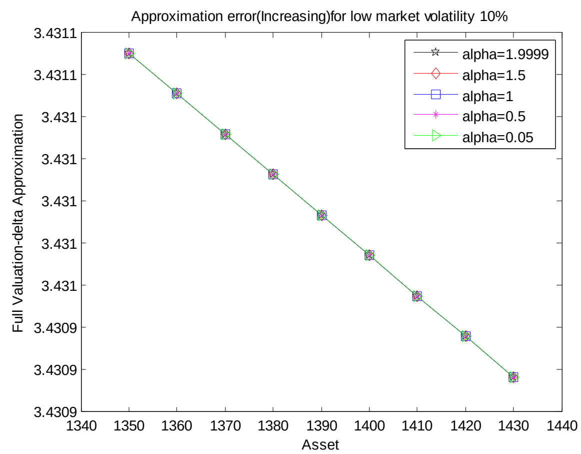


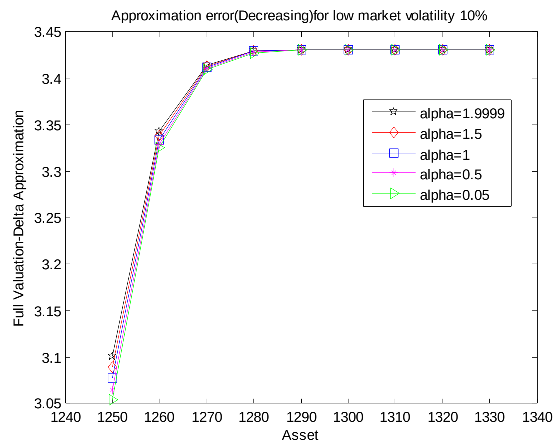
Figure 6.6 Approximation delta (increasing (a) and decreasing (b)) and delta-gamma error (increasing (c) and decreasing (d)) for European option portfolio in high market volatility

For low market volatility 10%

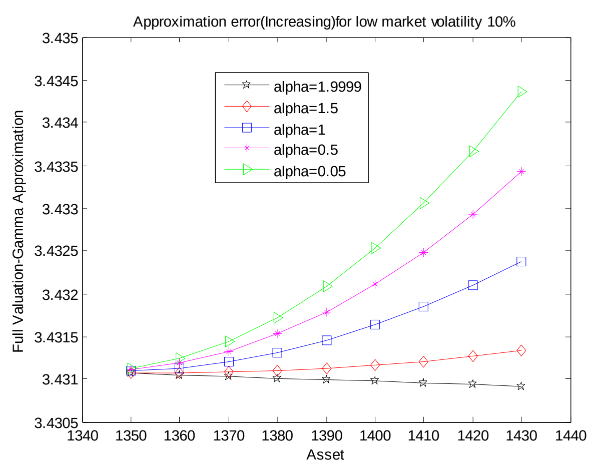
(a)



(b)



(c)



(d)

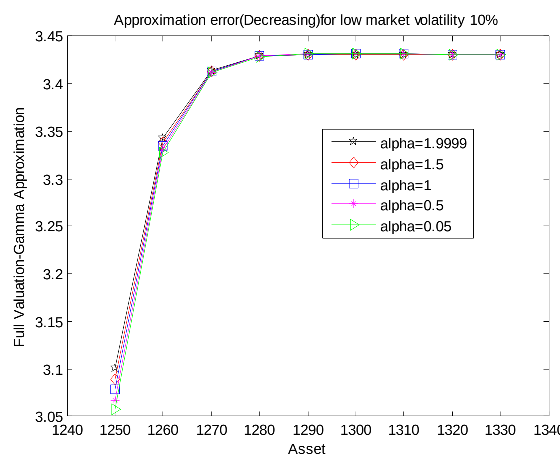


Figure 6.7 Approximation delta (increasing (a) and decreasing (b)) and delta-gamma error (increasing (c) and decreasing (d)) for European option portfolio in low market volatility

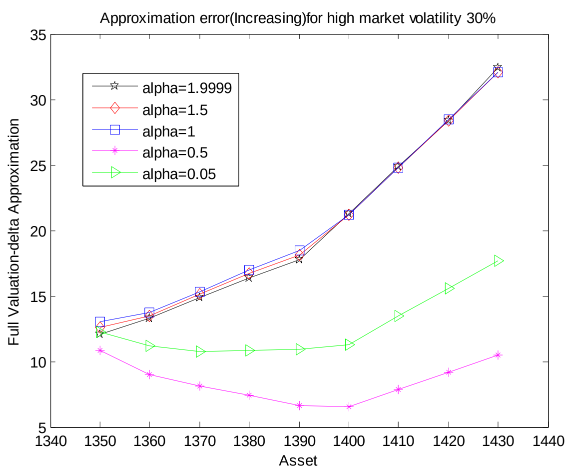
We observe that in high volatility market (see Figure 6.6) European option portfolio gives least error when alpha is very close to 2, and gives least accuracy when alpha is very close to 0. In low volatility market (see Figure 6.7) we cannot have a general conclusion but we can definitely comment that valuations of European option portfolio have quite similar accuracy in the face of downward asset price movements for alpha variations.

6.2.2 American option

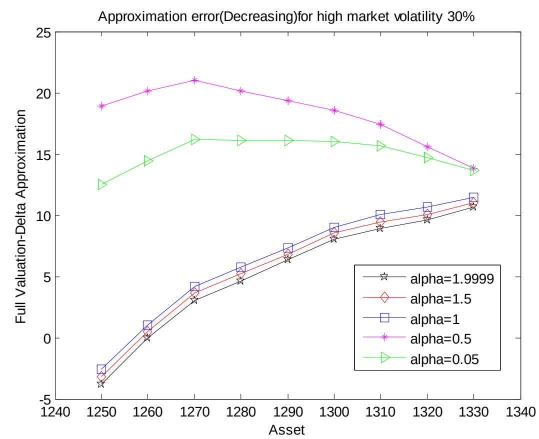
Approximation delta and delta-gamma error for American option portfolio are represented in Figure 6.8 and Figure 6.9.

For High market volatility 30%

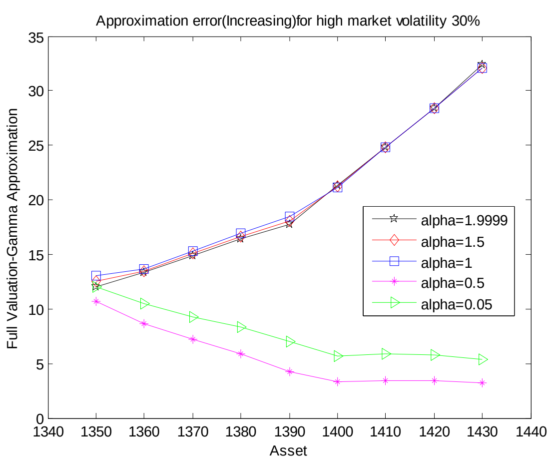
(a)



(b)



(c)



(d)

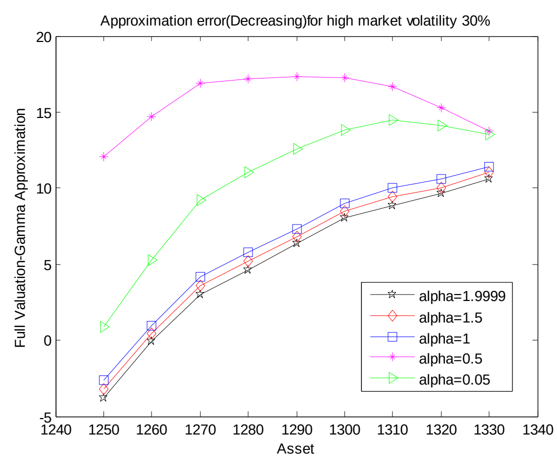
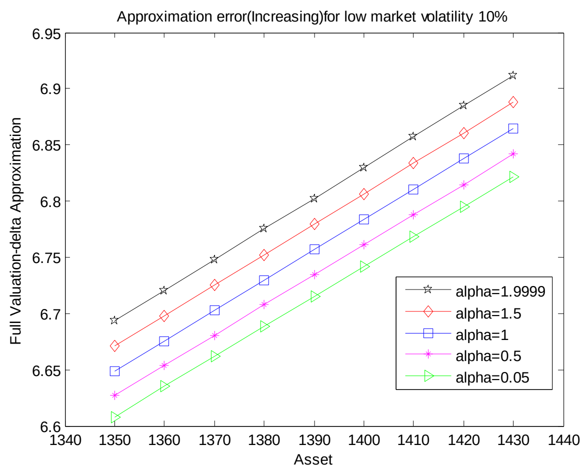


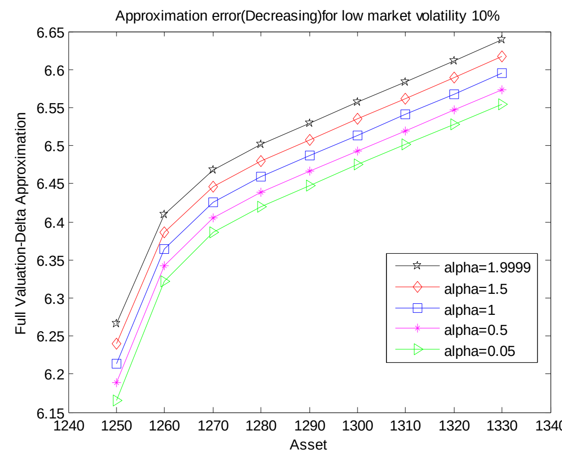
Figure 6.8 Approximation delta (increasing (a) and decreasing (b)) and delta-gamma error (increasing (c) and decreasing (d)) for American option portfolio in high market volatility

For Low market Volatility 10%

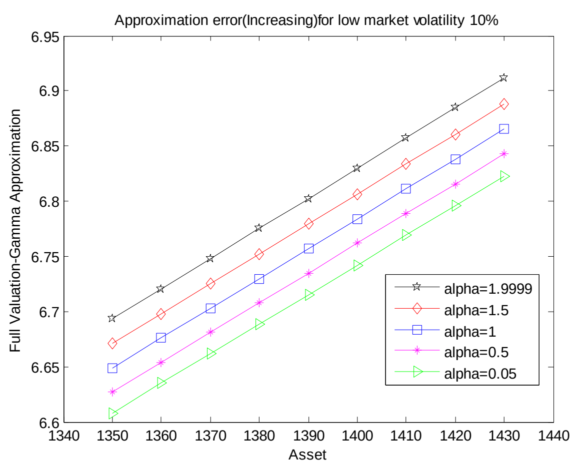
(a)



(b)



(c)



(d)

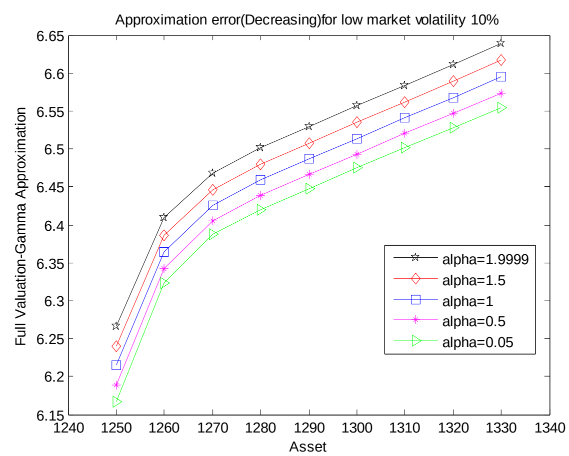


Figure 6.9 Approximation delta (increasing (a) and decreasing (b)) and delta-gamma error (increasing (c) and decreasing (d)) for American option portfolio in low market volatility

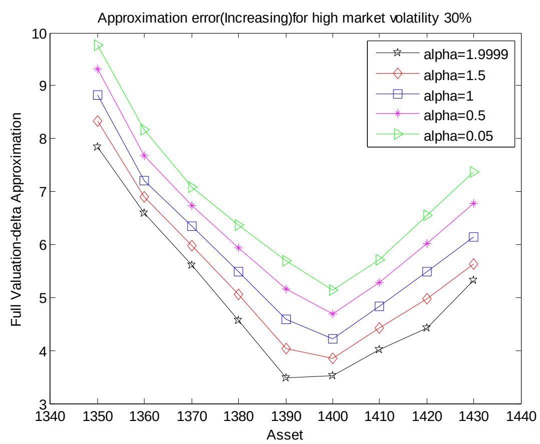
In high volatility market (see Figure 6.8) valuations of American option portfolio for alpha close to 2 give least accuracy in the face of upward asset price movements and give highest accuracy in the downward. We observe that in low volatility market (see Figure 6.9) American option portfolio gives least error when alpha is very close to 0, and gives least accuracy when alpha is very close to 2.

6.2.3 Bermudan option

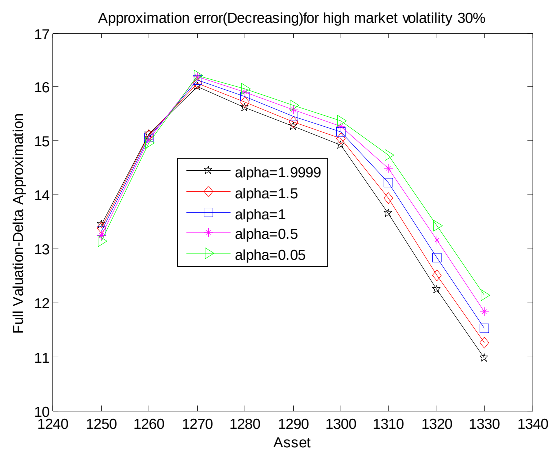
Approximation delta and delta-gamma error for Bermudan option portfolio are represented in Figure 6.10 and Figure 6.11.

For High Market Volatility 30%

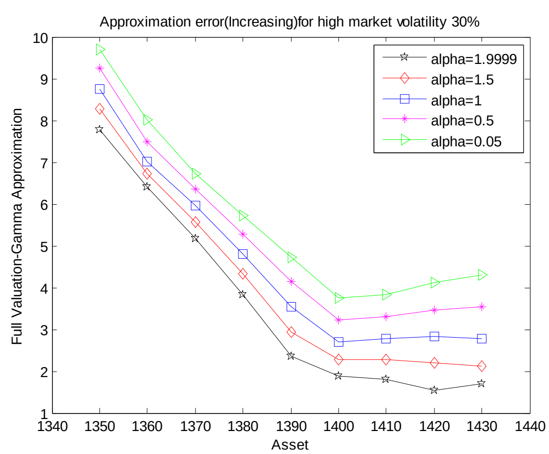
(a)



(b)



(c)



(d)

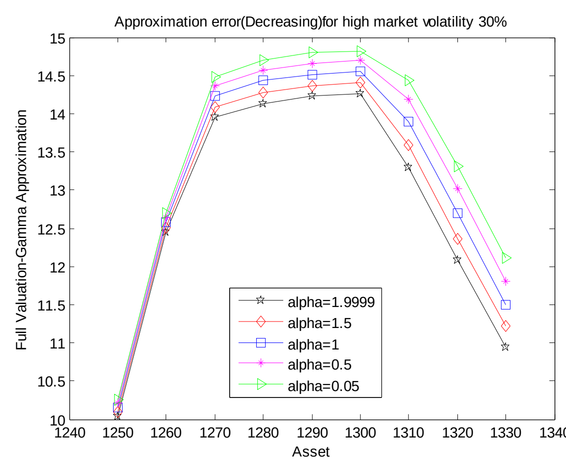


Figure 6.10 Approximation delta (increasing (a) and decreasing (b)) and delta-gamma error (increasing (c) and decreasing (d)) for Bermudan option portfolio in high market volatility

For Low market volatility 10%

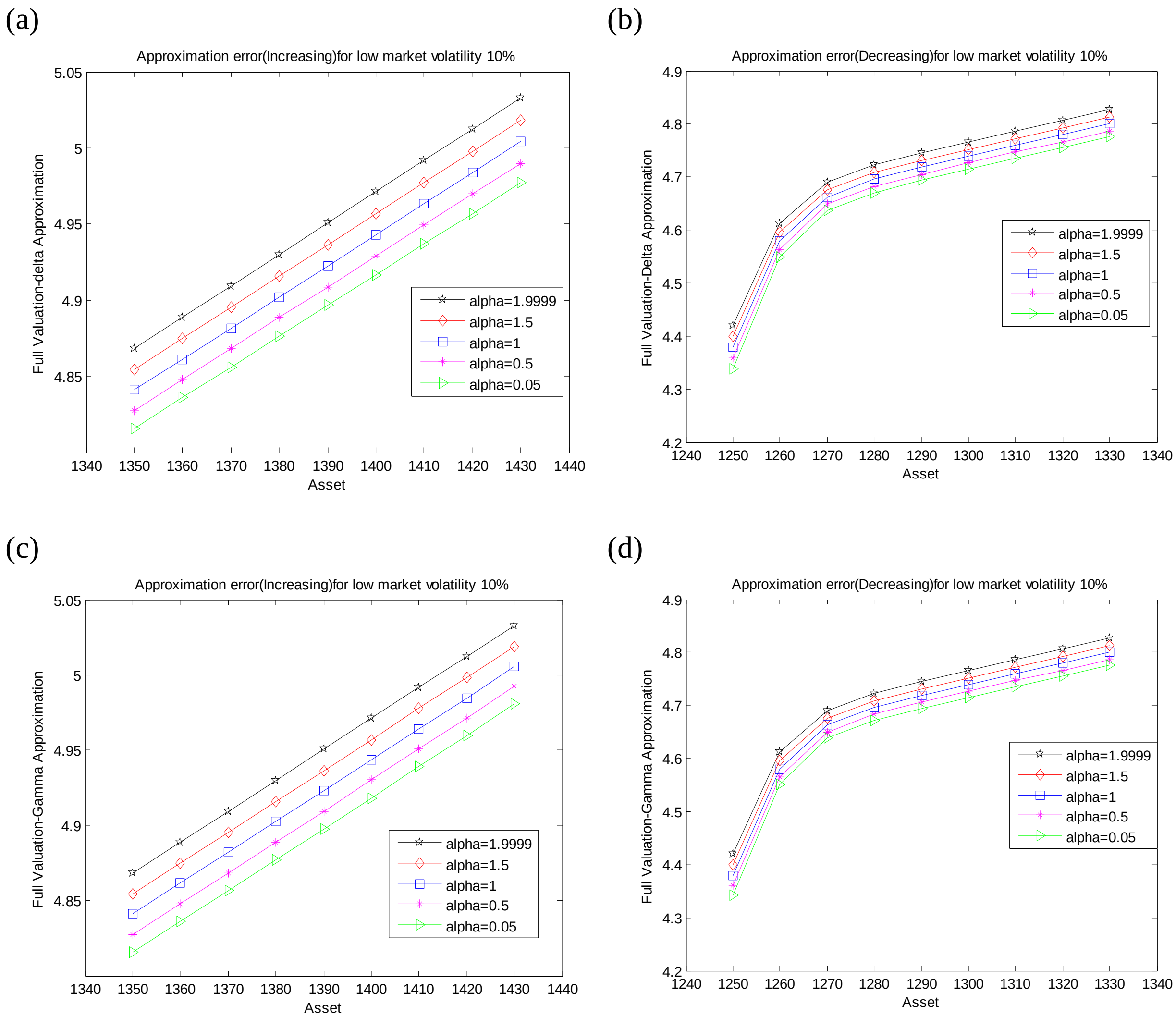


Figure 6.11 Approximation delta (increasing (a) and decreasing (b)) and delta-gamma error (increasing (c) and decreasing (d)) for Bermudan option portfolio in low market volatility

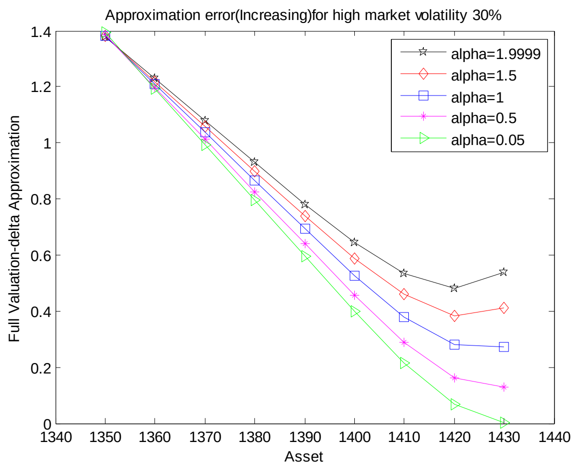
We observe that in high volatility market (see Figure 6.10) Bermudan option portfolio gives least error when alpha is very close to 2, and gives least accuracy when alpha is very close to 0. In low volatility market (see Figure 6.11) Bermudan option portfolio gives least error when alpha is very close to 0, and gives least accuracy when alpha is close to 2.

6.2.4 Asian Arithmetic Average option

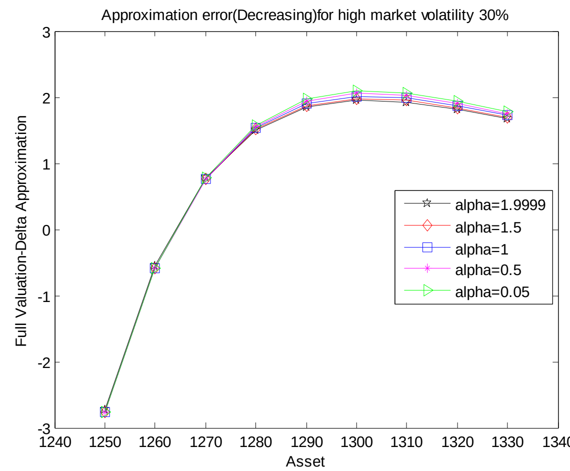
Approximation delta and delta-gamma error for Asian Arithmetic Average option portfolio are represented in Figure 6.12 and Figure 6.13.

For High market Volatility 30%

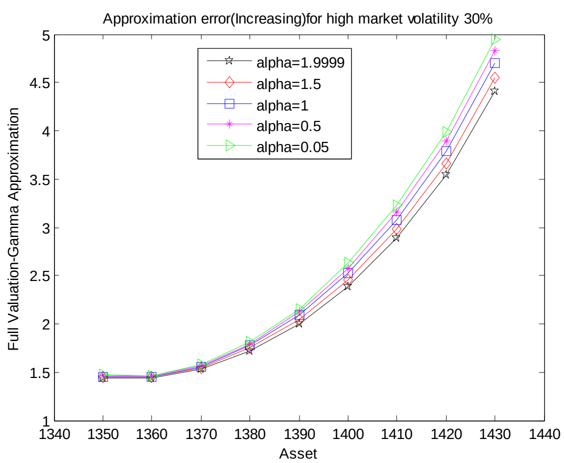
(a)



(b)



(c)



(d)

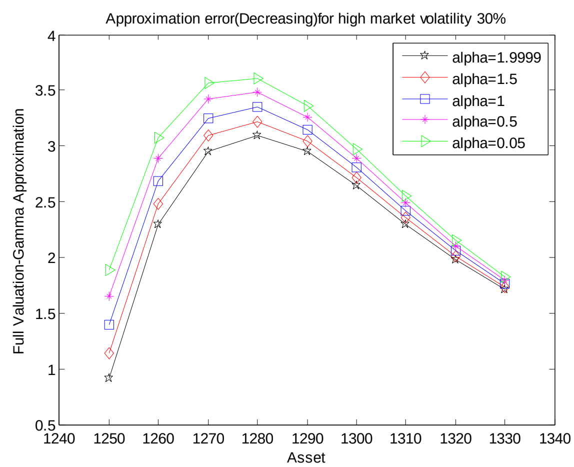


Figure 6.12 Approximation delta (increasing (a) and decreasing (b)) and delta-gamma error (increasing (c) and decreasing (d)) for Asian Arithmetic Average option portfolio in high market volatility

For low market volatility 10%

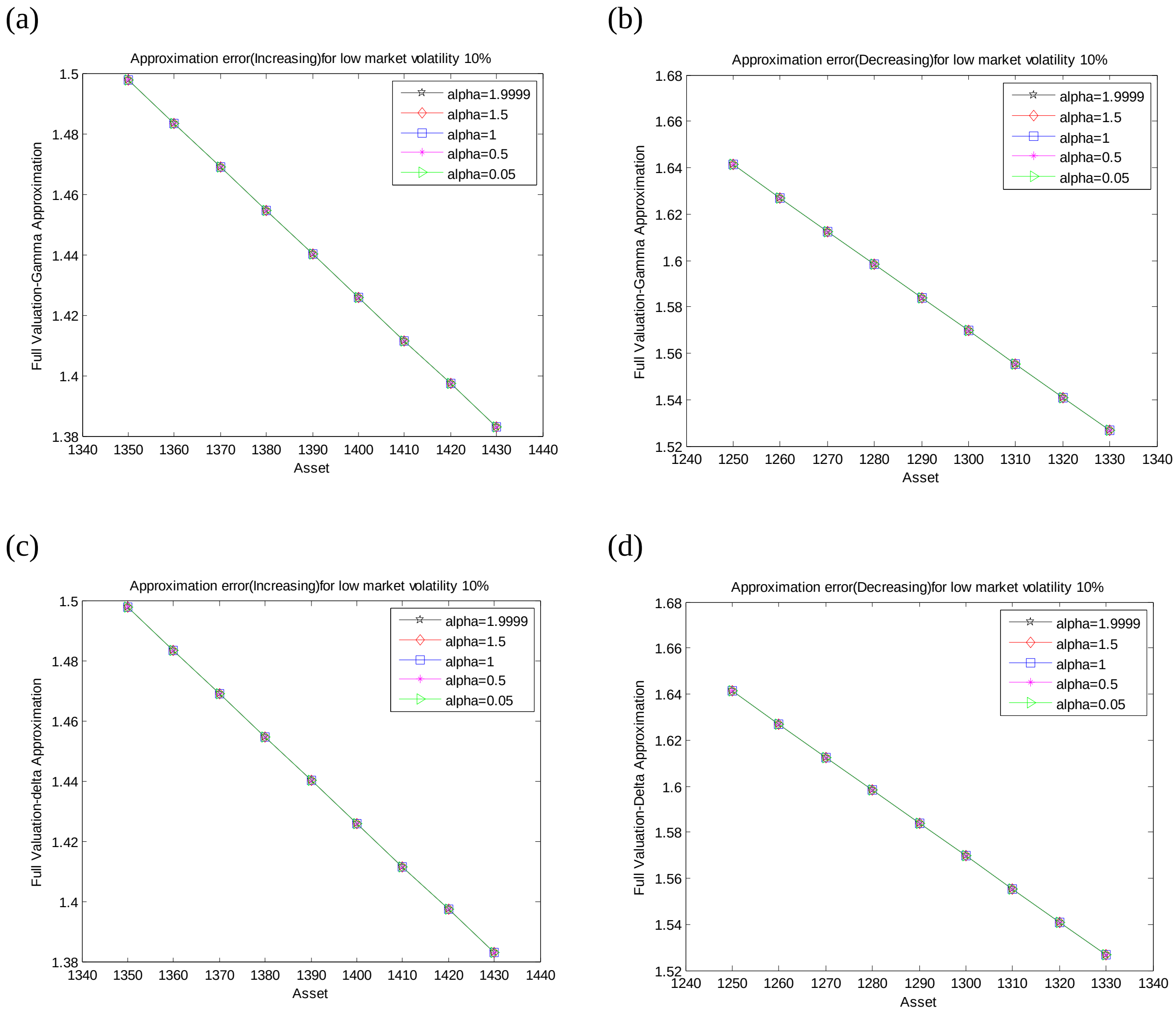


Figure 6.13 Approximation delta (increasing (a) and decreasing (b)) and delta-gamma error (increasing (c) and decreasing (d)) for Asian Arithmetic Average option portfolio in low market volatility

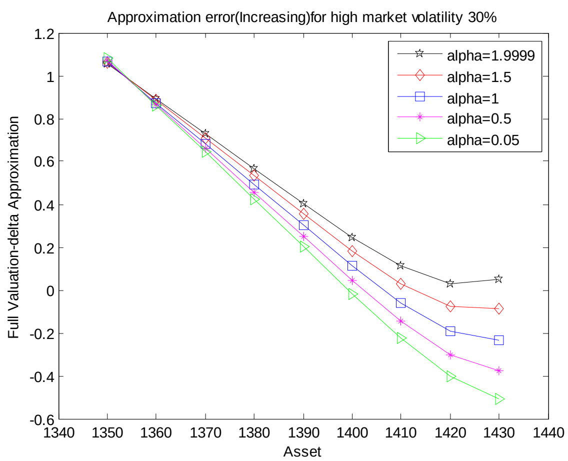
We observe differences in the high volatility market (see Figure 6.12) for alpha variations. However, in the low volatility market (see Figure 6.13) there is no change in the error plot for different elasticity. This is because in the low volatility market, Asian Arithmetic option prices do not show significant changes in values (see section 6.1) for different alphas. So, the elasticity of CEV model has very little effect on Asian Arithmetic Average option portfolio in the low volatility market.

6.2.5 Asian Geometric Average

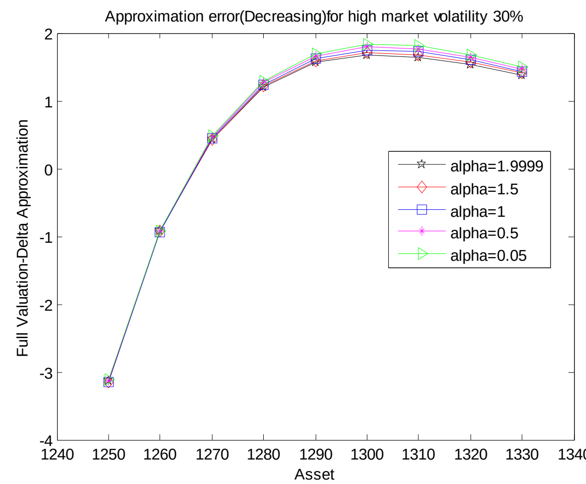
Approximation delta and delta-gamma error for Asian Geometric Average option portfolio are represented in Figure 6.14 and Figure 6.15.

For High Market Volatility 30%

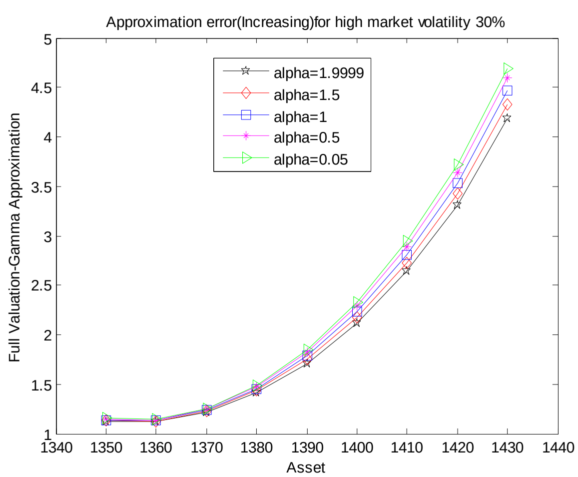
(a)



(b)



(c)



(d)

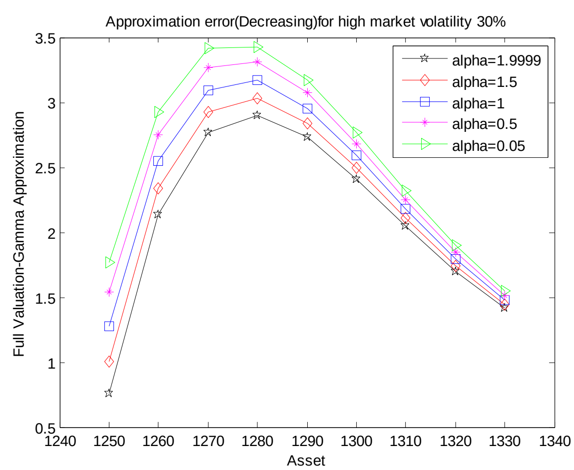


Figure 6.14 Approximation delta (increasing (a) and decreasing (b)) and delta-gamma error (increasing (c) and decreasing (d)) for Asian Geometric Average option portfolio in high market volatility

For Low Market Volatility 10%

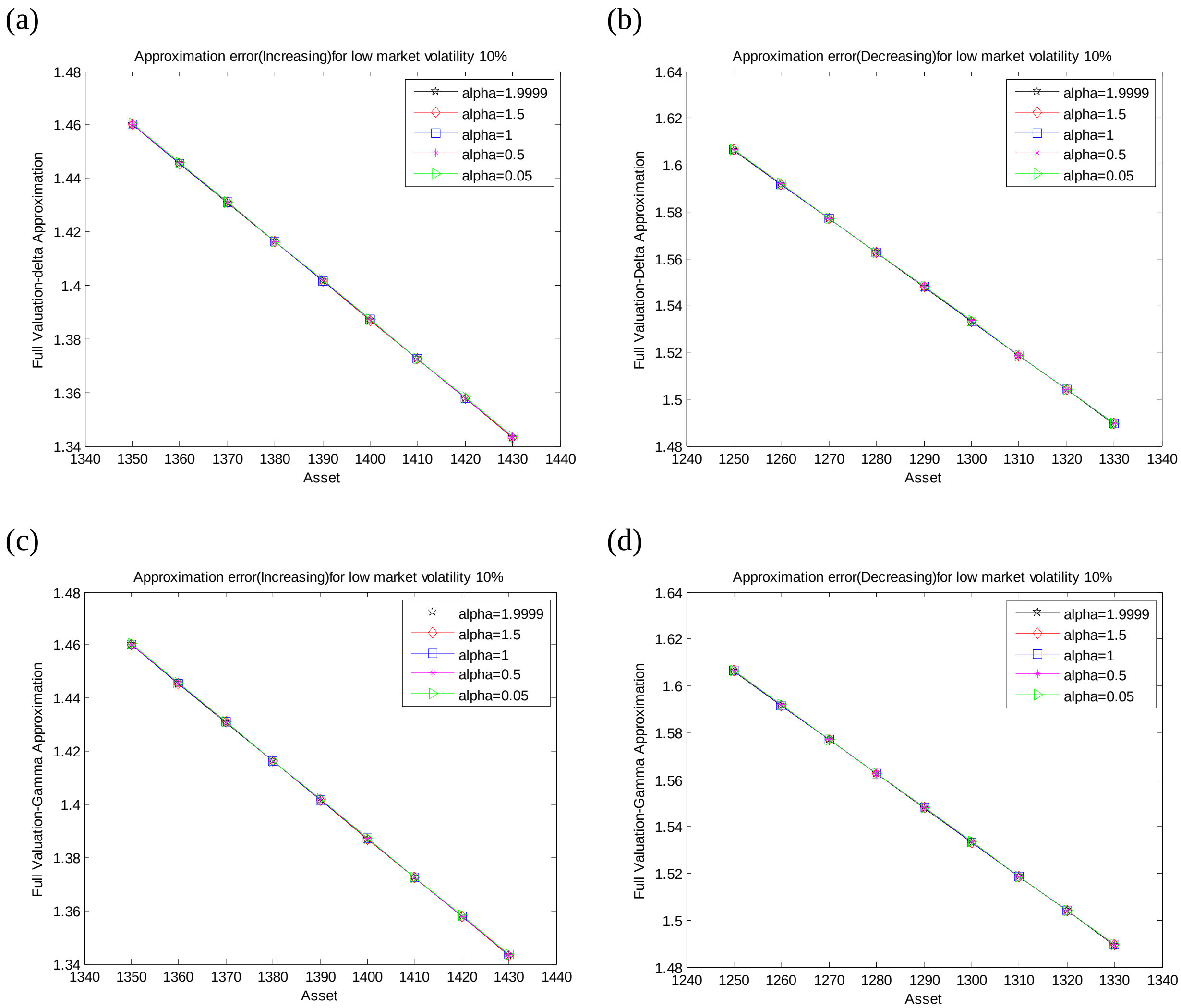


Figure 6.15 Approximation delta (increasing (a) and decreasing (b)) and delta-gamma error (increasing (c) and decreasing (d)) for Asian Geometric Average option portfolio in low market volatility

We discussed in the section 6.2.5 about the elasticity affect on Asian Arithmetic Average option portfolio in the low volatility market. For the similar reason, we observe differences in the high volatility market (see Figure 6.14) for alpha variations. But in the low volatility market (see Figure 6.15) there is no change in the error plot of Asian Geometric Average option portfolio for different elasticity.

These phenomena can be explained easily using the empirical observation we discussed in the section 6.1. With high market volatility influence on option, we obtain different option prices for all four contract types. Because of the changes in the price, the sensitivity measures of option price that is, Greeks also differ from each other for elasticity variations. This results a huge distinction in the error plot. But in the case of low volatility market, the changes are relatively very small (e.g. European option, American and Bermudan) or there is no change at all (e.g. Asian Arithmetic and Asian Geometric option). As a result, empirically we can see the above patterns in error plots of hedging.

Also, from a general view, we decide that Greek-based approximations to European, American and Bermudan option portfolios under Constant Elasticity of Variance (CEV) model better perform in the low volatility market. However, these three types of contracts under Black-Scholes and Merton model (when elasticity is very close to 2, i.e. 1.9999) carry out better results otherwise. Greek-based approximations to Asian (Arithmetic and Geometric average) option portfolios in high market volatility perform differently in different face of asset price movements. For this contract type, the delta approximation (increasing) to option portfolio under CEV has little more accuracy than that of under Black-Scholes and Merton model. However, delta approximation (decreasing) and delta-gamma approximations (increasing and decreasing) to Asian option portfolios under CEV model are slightly less accurate.

Chapter 7

Conclusion

The inability of the Black-Sholes-Merton (BSM) option pricing model to incorporate skewness and excess kurtosis has inspired us to work with an alternative model Constant Elasticity of Variance (CEV) model. In this research, we have studied hedging performance across styles under CEV model. We used recursive trinomial lattice approaches for pricing options of four different contract styles i.e. European, American, Bermudan, and Asian (Arithmetic average and Geometric average). We derived the option prices of each style under this model through recursive trinomial lattice approach and compared the results with actual option prices. We find that CEV model approximately follows same option prices as the BSM model (for elasticity 1.9999) under all four contract styles.

Then we have shown that the satisfactory estimates of CEV dynamics' hedge ratios, deltas and gammas can be obtained using the ordinary trinomial lattice and using the auxiliary trinomial lattice as well. However, we chose the effective one which is the one based on auxiliary lattice for hedging option portfolios under all four contract types. We have discussed the approximation errors of delta and delta-gamma hedging for each option style. We observed the corresponding approximation errors which have shown notable differences among the four types of contracts. Finally, we want analyze the approximation errors for different elasticity in high market volatility and low market volatility setup under all four contract styles. We have provided graphical illustrations explaining how hedging perform for different contract styles over different market volatility.

From this study, we have made some important conclusions. At first, we found that for recursive trinomial option pricing of all four contract types under CEV model, the number of time step is very crucial to choose. As option prices of all styles converge to true values for number of time steps 13, we have used number of time steps 13 to obtain all our results of this study. Then we have generated Greeks from trinomial lattice and compared them to BSM model which has been used as benchmark. We decided to use auxiliary lattice approach to obtain better and effective Greeks. We have discussed Greek-based approximations to option portfolios across styles. We carried out both the linear and the non-linear approximations that is, delta approximation and delta-gamma approximation respectively. We observed that their performance differ over the underlying asset price movements under all four contract styles. Under all contract styles, the delta approximation and delta-gamma approximation to option portfolios have very little difference for relatively small price movements. However, for larger swings in the movement our delta-gamma hedging is much better than linear delta hedging across styles. We have also found that both delta approximation and delta-gamma approximation perform better to approximate Asian (Arithmetic and Geometric average) option portfolios compare to the approximations to other three types of option portfolios in general.

Finally, we have studied the accuracy of delta approximations and delta-gamma approximations to option portfolios for various non-normalities induced by different values of elasticity parameters in both high and low volatility market for all four option styles. And we concluded that delta and delta-gamma based approximations to European, American and Bermudan option portfolios under Constant Elasticity of Variance (CEV) model have a little more accuracy than that of under Black-Scholes and Merton (BSM) (for elasticity very close to 2, i.e. 1.9999) in the low volatility market.

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Appendix

We produced MATLAB coding to get all numerical and graphical results, some of them are given here.

MATLAB code for simulated Black-Scholes Call and Put price

```
function y=MCEuropeanBS_Call(S0,K,T,r,sigma,NT,NSim)
simpath=AssetPaths(S0,r,sigma,T,NT,NSim);
ST=simpath(:,end);
payoffs=max(ST-K,0);
pricecall=exp(-r*T).*(1/NSim).*sum(payoffs);
y=pricecall;
end

%%------------------------%%---------------------%%
function y=MCEuropeanBS_Put(S0,K,T,r,sigma,NT,NSim)
simpath=AssetPaths(S0,r,sigma,T,NT,NSim);
ST=simpath(:,end);
payoffs=max(K-ST,0);
pricecall=exp(-r*T).*(1/NSim).*sum(payoffs);
y=pricecall;
end
%%------------------------%%---------------------%%
function sim_paths = AssetPaths(S0,r,sigma,T,NT,NSim)
sim_paths = zeros(NSim,NT+1);% Each row is a simulated path
sim_paths(:,1) = S0;% fill first col w/ spot prices
dt = T/NT;% time step
for i=1:NSim
    for j=2:NT+1
        wt = randn
       sim_paths(i,j) = sim_paths(i,j-1)*exp((r-0.5*sigma^2)...
            *dt+sigma*sqrt(dt)*wt);
    end
```

```
end
end


%%------------------------%%---------------------%%
%Script file for calculating error of simulated European prices under
Black-Scholes and Merton model
clf;clc;
S0=100;
K=95;
T=0.5;
r=0.05;
sigma=0.2;
NT=500;
    i=0;
for NSim=50:10:2000
[Call, Put] = blsprice(S0, K, r, T, sigma, 0);
c=MCEuropeanBS(S0,K,T,r,sigma,NT,NSim);
absolute_value=abs(Call-c);
i=i+1;
x(i)=NSim;
y(i)=absolute_value;

end
plot(x,y,'k')
axis tight
saveas(gcf,'plot_bls_european.png')


%%------------------------%%---------------------%%
function [C,P] = BlackScholesFcn(Underlying,Strike,Rate,T,Sigma)

d1 = (log(Underlying./Strike)+(Rate-Sigma.^2/2).*T)./(Sigma.*sqrt(T));
d2 = d1-Sigma.*sqrt(T);

C = Underlying.*normcdf(d1)-Strike*exp(-Rate*T)*normcdf(d2);
P = Strike*exp(-Rate*T)*normcdf(-d2)-Underlying.*normcdf(-d1);
%%------------------------%%---------------------%%
```

MATLAB code for simulated Black-Scholes Asian Call price

```
function Price = Asian(S0,K,r,sigma,T,m,n)
Payoff=zeros(n,1);
for i=1:n
paths=GBM(S0,r,sigma,T,m,1);
Payoff(i)=max(0,mean(paths(2:(m+1)))-K);
end
Price=exp(-r*T)*Payoff;
Price=mean(Price);
end

%%------------------------%%---------------------%%
function paths = GBM(S0,mu,sigma,T,m,n)
dt = T/m;
drift = (mu-0.5*sigma^2)*dt; % Calculation of the drift term.
diffusion = sigma*sqrt(dt); % Calculation of the diffusion term.
incr = drift + diffusion*randn(n,m);
logpaths = cumsum([log(S0)*ones(n,1),incr],2);
paths = exp(logpaths);
end


%%------------------------%%---------------------%%
%Asian pricing script
tic
%Initial Stock price
S0=100;
K=95;
r=.05;
T=.5;
m=1000;
dt=T/m;
n=10000;
%Initial volatility
sigma=0.20;
Price = Asian(S0,K,r,sigma,T,m,n)
```

```
toc
%%------------------------%%---------------------%%
```

MATLAB code for deriving Greeks from ordinary lattice

```
function [ delta ] = TrinDeltaEuropeanp(Underlying, Strike, r, T, ...
                NumOfTimeSteps, Sigma,alpha,lambda,M, StrCallPut)
          S   = Underlying;

dS=S.*(Sigma.*S.^(alpha/2-1).*sqrt(T));


delta=((priceOptionTrinCEVEuropean(S+dS, Strike, r, T, ...
                NumOfTimeSteps, Sigma,alpha,lambda,M, StrCallPut)-...
                priceOptionTrinCEVEuropean(S, Strike, r, T, ...
               NumOfTimeSteps, Sigma,alpha,lambda,M, StrCallPut))./dS);
End

%%------------------------%%---------------------%%


function [ gamma ] = TrinGammaEuropeanp(Underlying, Strike, r, T, ...
                NumOfTimeSteps, Sigma,alpha,lambda,M, StrCallPut)
          S   = Underlying;

dS=S.*(Sigma.*S.^(alpha/2-1).*sqrt(T));

gamma=((priceOptionTrinCEVEuropean(S+dS, Strike, r, T, ...
                NumOfTimeSteps, Sigma,alpha,lambda,M, StrCallPut)-...
                2.*priceOptionTrinCEVEuropean(S, Strike, r, T, ...
               NumOfTimeSteps, Sigma,alpha,lambda,M, StrCallPut)...
              + priceOptionTrinCEVEuropean(S-dS, Strike, r, T, ...
               NumOfTimeSteps, Sigma,alpha,lambda,M,
StrCallPut))./(dS^2));

end


%%------------------------%%---------------------%%


%%Script Files for plotting delta and gamma
clear;
clc;
tic
alpha = 1.9999;
r = 0.1;

T = 0.5;
Strike = 95;
NumOfTimeSteps=13;
 Sigma=0.10;
 lambda=sqrt(1.5);
```

```
M=2*NumOfTimeSteps+1;
StrCallPut='Call';

M_Bermudan=2;

Underlying=50:150;

for i=1:length( Underlying)


[delta_Asian(i)] = TrinDeltaAsianp(Underlying(i),Strike,r,T, ...
                NumOfTimeSteps,Sigma,alpha,lambda,M,StrCallPut) ;


[delta_AsianG(i)] = TrinDeltaAsianGp(Underlying(i), Strike, r, T, ...
                NumOfTimeSteps,Sigma,alpha,lambda,M,StrCallPut) ;

[delta_European(i)] = TrinDeltaEuropeanp(Underlying(i),Strike,r,T, ...
                NumOfTimeSteps,Sigma,alpha,lambda,M,StrCallPut);

[delta_Bermudan(i)]=TrinDeltaBermudanp(Underlying(i),Strike,r,T, ...

NumOfTimeSteps,M_Bermudan,Sigma,alpha,lambda,M,StrCallPut);

[delta_American(i)] = TrinDeltaAmericanp(Underlying(i),Strike,r,T, ...
                NumOfTimeSteps, Sigma,alpha,lambda,M,StrCallPut)  ;
end
[CallDelta, PutDelta] = blsdelta(Underlying, Strike, r, T,Sigma);
figure
plot(Underlying,delta_Asian,'k');
hold on

 plot(Underlying,delta_AsianG,'Color',[1 0.5 0]);
hold on


plot(Underlying,delta_European,'b');
hold on

plot(Underlying,CallDelta,'r');
hold on

plot(Underlying,delta_Bermudan,'g');
hold on

plot(Underlying,delta_American,'m');

title(' Call Delta (Ordinary Lattice) ')
axis tight
legend('Asian Arithmetic','Asian
Geometric','deltaEuropean','BLS','deltaBermudan','deltaAmerican')
xlabel('Underlying')
ylabel('Delta')
```

```
saveas(gcf,'TrinDeltap_All.png')
hold off
toc

%%------------------------%%---------------------%%

clear;
clc;
tic
alpha = 1.9999;
r = 0.1;

T = 0.5;
Strike = 95;
NumOfTimeSteps=13;
Sigma=0.15;
lambda=sqrt(1.5);
M=2*NumOfTimeSteps+1;
StrCallPut='Call';
M_Bermudan=2;

Underlying=60:10:130;

for i=1:length( Underlying)


[ Gamma_Asian(i)] = TrinGammaAsianp(Underlying(i),Strike,r,T, ...
                NumOfTimeSteps, Sigma,alpha,lambda,M, StrCallPut) ;

[ Gamma_European(i)] = TrinGammaEuropeanp(Underlying(i),Strike,r,T, ...
                NumOfTimeSteps, Sigma,alpha,lambda,M, StrCallPut);

[ Gamma_Bermudan(i)] = TrinGammaBermudanp(Underlying(i),Strike,r,T, ...
                NumOfTimeSteps,M_Bermudan, Sigma,alpha,lambda,M, ...
StrCallPut)  ;

[ Gamma_American(i)] = TrinGammaAmericanp(Underlying(i),Strike,r,T, ...
                NumOfTimeSteps, Sigma,alpha,lambda,M, StrCallPut)  ;
end

Gamma_bls = blsgamma(Underlying, Strike, r, T,Sigma);

figure
plot(Underlying,Gamma_Asian,'k');
hold on


plot(Underlying,Gamma_European,'b');
hold on

plot(Underlying, Gamma_bls,'r');
hold on
```

```
 plot(Underlying,Gamma_Bermudan,'g');
 hold on

 plot(Underlying,Gamma_American,'m');


legend('gammaAsian','gammaEuropean','gammabls','gammaBermudan','gammaAm
erican')

 saveas(gcf,'TrinGammap_All.png')
 hold off
 toc

%%------------------------%%---------------------%%
```