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Course Code : Math 4641
Course Name : Numerical Methods
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Ans.to Q.no. 1(a)

Advantages of bisection method:

i) Bisection method is guaranteed to converge

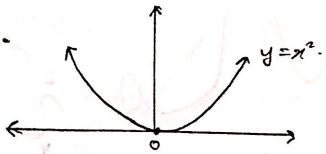
ii) As the intervals get halved in each iteration, the error is ~~guara~~ guaranteed in each step.

Disadvantages of bisection ~~me~~ method:

i) Bisection takes a long time, or large no. of iterations to converge.

ii) If the initial guess is closer to the root, then bisection method will take more iterations.

iii) Fails to detect a root that touches the x-axis.



This is because $f(x_1)\cdot f(x_2)>0$ here, hence bisection method can not find this root as the sign doesn't change.

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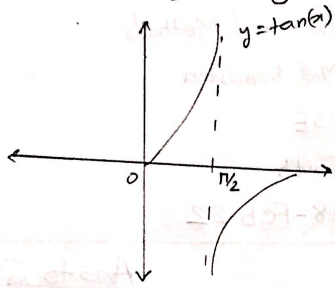
iv) At ~~sig~~ singularities (points where value is infinite) and the function reverses sign, bisection method converges at the ~~sig~~ singularity.

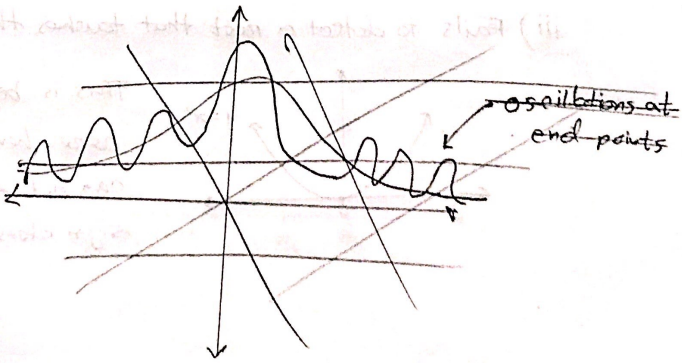


However, no root exists at singularity and hence, bisection method gives inaccurate results.

Ans. to Q.no. 1 (b)

(i)

Unlike the three previous methods of interpolation, spline method is a piece-wise interpolation. For Direct, NDD and Lagrangian, the interpolated equation oscillates at end points for higher order equations.



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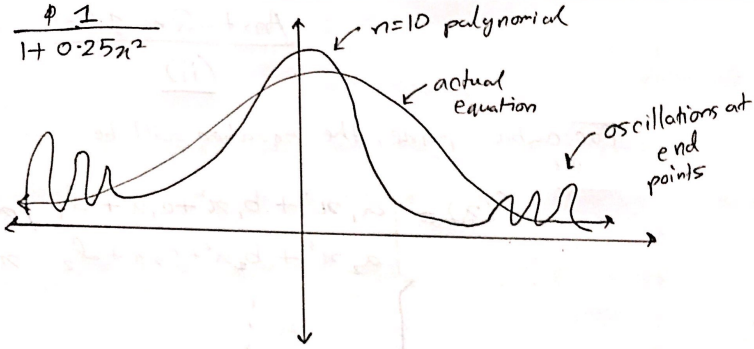
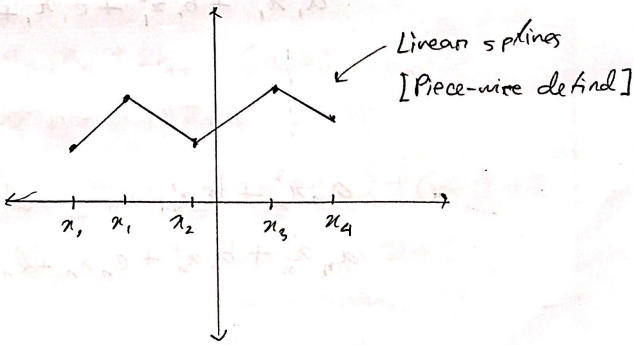
$$y = \frac{1}{1+0.25x^2}$$



As we can see in graph for a high order interpolated equation, the ~~no of~~ oscillations at the end points give inaccurate results. [I couldn't draw a good graph because I can't use pencil to draw graphs, Pencils don't show up in scanning]

However, if we use spline interpolation, then the interpolated equation will be piece-wise defined. That is for a fixed interval, there will be a fixed equation. Hence, the lower-order piece-wise polynomial will give us a similar if not better result if we use spline.



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Ans to Q.no. 1(b)

(ii)

For cubic spline, the equation will be

$$f(x) = \begin{cases} a_1 x^3 + b_1 x^2 + c_1 x + d_1 & x_0 \le x < x_1 \\ a_2 x^3 + b_2 x^2 + c_2 x + d_2 & x_1 \le x < x_2 \\ \vdots & \\ a_n x^3 + b_n x^2 + c_n x + d_n & x_{n-1} \le x < x_n \end{cases}$$

To get, the co-efficients we need three steps.

There are 4 unknowns in each equations and n equations.

In total there's 4n unknowns.

We need 4n ~~unknowns to solve this~~ equations to get these.

(i) Firstly, the equations through 2 points.

$a_1 x^3 + b_1 x^2 + c_1 x + d_1$ passes through x_0 and x_1

Similarly, $a_n x^3 + b_n x^2 + c_n x + d_n$ passes through x_{n-1} and x_n

Therefore,

$$a_1 x_0^3 + b_1 x_0^2 + c_1 x_0 + d_1 = y_0$$

$$a_1 x_1^3 + b_1 x_1^2 + c_1 x_1 + d_1 = y_1$$

$$\vdots$$

$$a_n x_{n-1}^3 + b_n x_{n-1}^2 + c_n x_{n-1} + d_n = y_{n-1}$$

$$a_n x_n^3 + b_n x_n^2 + c_n x_n + d_n = y_n$$

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For each interval, we get, 2 equations

We have, ~~4n~~ 2n equations for n intervals.

(ii) The first order derivate can be found as the interior points are continuous. First derivative will be zero.

$$3a_1x_1^2 + 2b_1x_1 + c_1 \cancel{=0} - 3a_2x_1^2 - 2b_2x_1 - c_2 = 0$$

$$\vdots \qquad\qquad \vdots$$

$$3a_{n-1}x_n^2 + 2b_{n-1}x_n + c_{n-1} - 3a_nx_n^2 - 2b_nx_n - c_n = 0$$

There are (n-1) equations.

(iii) Assuming first spline to be linear,

$$\left.\begin{aligned} a_1 &= 0 \\ \text{and } b_1 &= 0 \end{aligned}\right\} \text{ 2 equations.}$$

(iv) Second order derivate at interior points are continuous. Derivative will be zero.

$$6a_1x_1 + 2b_1 - 6a_2x_1 - 2b_2 = 0$$

$$\vdots \qquad\qquad \vdots$$

$$6a_{n-1}x_n + 2b_{n-1} - 6a_nx_n^2 - 2b_n = 0$$

There are (n-1) equations.

In total we have $2n + (n-1) + (n-1) + 2$

$$= 2n + 2n - 2 + 2$$

$$= 4n \text{ equations.}$$

We solve 4n equations to find the 4n unknowns.

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Ans. to Q.no. 1(c)

(i) 3.141590 — 7 significant digits

(ii) 1.0790×10^3 — 5 significant digits

(iii) 0.0098 — 2 significant digits

(iv) 0.00980 — 3 significant digits.

Ans. to Q.no. 2(a)

Given, $x = 1/a$

$$\Rightarrow a = 1/x$$

$$\Rightarrow a - 1/x = 0$$

Let, $f(x) = a - 1/x$

$$f'(x) = \frac{1}{x^2}$$

Now, $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

$$= x_i - \frac{a - 1/x_i}{1/x_i^2}$$

$$\cancel{= x_i - (a x_i - 1)}$$

$$= x_i - \frac{x_i^2 (a - 1/x_i)}{x_i^2 / x_i^2}$$

$$= x_i - a x_i^2 + x_i$$

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$$\therefore x_{i+1} = 2x_i - ax_i^2 \quad \underline{(Ans.)}$$

The Newton Rapson expression for x_{i+1} is this.

<u>Ans. to Q no. 2(b)</u>

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$\therefore a = 2.0.$

Given, the N.R. equation is $x_{i+1} = 2x_i - ax_i^2$

<u>Iteration-1</u>: Initial guess be ~~10~~ $x_0 = 0.1$

$$x_{i+1} = 2x_i - ax_i^2$$

$$a = 2.0 \text{ and } x_0 = 10$$

$$\therefore x_1 = 2x_0 - 2x_0^2$$

$$\cancel{= 20 - 2 \times 10^2}$$

$$\cancel{= 0}$$

~~Iteration 2~~:

$$= 2 \times 0.1 - 2 \times 0.1^2$$

$$= 0.18$$

Absolute Relative approx error $= \left| \frac{x_{new} - x_{old}}{x_{new}} \right| \%$

$$= \left| \frac{0.18 - 0.1}{0.18} \right| \%$$

$$|\epsilon_a| = 44.44\%$$

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Iteration-2: $x_1 = 0.18$

$$x_2 = 2x_1 - 2x_1^2$$

$$= 2 \times 0.18 - 2 \times 0.18^2$$

$$= 0.2952$$

$$|\epsilon_a| = \left| \frac{x_{new} - x_{old}}{x_{new}} \right| \%$$

$$= \frac{0.2952 - 0.18}{0.2952} \%$$

$$= \cancel{0.39} \; 39.02 \%$$

Iteration-3: $x_2 = 0.2952$

$$x_3 = 2x_2 - 2x_2^2$$

$$\cancel{= 2 \times 2.952 - 2 \times}$$

$$= 2 \times 0.2952 - 2 \times (0.2952)^2$$

$$= 0.41611$$

$$|\epsilon_a| = \left| \frac{0.41611 - 0.2952}{0.41611} \right| \%$$

$$= 29.06 \%$$

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Iteration-4:

$x_3 = 0.41611$

$x_4 = 2x_3 - 2x_3^2 = 2\times(0.41611) - 2\times(0.41611)^2$

$= 0.48592$

Error, $|\epsilon_a| = \left|\frac{0.48592 - 0.41611}{0.48592}\right| \%$

$= 14.37\%$.

The result from first 4 iteration is 0.48592 (Ans.) which is close to the actual result ($1/2 = 0.5$).

Ans. to Q.no. 2(c)

Given, a fixed set of discrete points $(x_0, y_0), (x_1, y_1) \ldots (x_n, y_n)$, and we have to find the functional value at a point x within the range of points, then it is called interpolation and if we need to find functional value outside the range of points, then it is extrapolation.

If the points are arranged in an ascending order then if $x \in [x_0, x_n]$, it is interpolation and if $x \notin [x_0, x_n]$, it is extrapolation.

Interpolation is needed to find find functional values within a dataset.

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x	y
0	25
10	35
20	47

To find $f(5)$, we need interpolation.

To find $f(25)$, we need extrapolation.

Ans to Q.no. ~~3(a)~~ 3(b)

The N.D.D. equation is

$$f(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + \dots$$

When $x = x_0$,

$$f(x_0) = b_0 + 0 + 0 \dots 0$$

$$\therefore f(x_0) = b_0$$

When $x = x_1$

$$f(x_1) = b_0 + b_1(x_1 - x_0) + 0$$

$$\Rightarrow f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

$$\Rightarrow b_1(x_1 - x_0) = f(x_1) - f(x_0)$$

$$\therefore b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So, the ~~quadratic~~ linear equation is,

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

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~~Following the pattern $b_3 = f(x$~~

When, $x = x_2$,

$$f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$\Rightarrow f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$\Rightarrow f(x_2) - f(x_0) = (x_2 - x_0)\left\{\frac{f(x_1) - f(x_0)}{x_1 - x_0} + b_2(x_2 - x_1)\right\}$$

$$\Rightarrow b_2(x_2 - x_0) = \frac{f(x_2) - f(x_0)}{x_2 - x_0} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\Rightarrow b_2 = \frac{\frac{f(x_2) - f(x_0)}{x_2 - x_0} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_1}$$

$$= \frac{\frac{f(x_2) - f(x_1) + f(x_1) - f(x_0)}{x_2 - x_0} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_1}$$

$$= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_0} + \frac{f(x_1) - f(x_0)}{x_2 - x_0} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_1}$$

$$= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_0} + \{f(x_1) - f(x_0)\}\left\{\frac{x_1 - x_0 - x_2 + x_0}{(x_2 - x_0)(x_1 - x_0)}\right\}}{x_2 - x_1}$$

$$\cancel{= \frac{f(x_2) - f(x_1)}{x_2 - x_0} + [f(x_1) - f(x_0)]\ x_1 =}$$

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$$= -(x_2 - x_1)$$

$$= \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} + \left[f(x_1)-f(x_0)\right] \times \dfrac{(x_1-x_2)}{(x_2-x_0)(x_1-x_0)}}{x_2-x_1}$$

$$= \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} + \left[f(x_1)-f(x_0)\right] \dfrac{-1\,\cancel{(x_2-x_1)}}{\cancel{(x_2-x_1)}(x_1-x_0)}}{x_2-x_0}$$

$$= \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} - \dfrac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}$$

$$\therefore b_2 = \frac{F(x_2,x_1) - F(x_1,x_0)}{x_2-x_0}.$$

So, $f(x) = f(x_0) + \dfrac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0) +$
(quadratic)

$$\frac{F(x_2,x_1) - F(x_1,x_0)}{x_2-x_0}(x-x_1)(x-x_0)$$

(Ans.)

For cubic it is,

$$f(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0) + \frac{F(x_2,x_1)-F(x_1,x_0)}{x_2-x_0}(x_0-x_1)(x-x_0)$$

$$+ \frac{F(x_3,x_2,x_1) - F(x_2,x_1,x_0)}{x_3-x_0}(x-x_0)(x-x_1)(x-x_2)$$

(Ans.)

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Ans. to Q. no. ~~1(e)~~ 3(c)

$$R_n^{x+h}(e) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(e) \quad x < e < x+h$$

Here, $x = 0$ and $h = 2, n = 3$.

$$\cancel{R_n(2) = \frac{2^{n+1}}{}}$$

$$R_3(2) = \frac{2^4}{4!} f^{(4)}(e)$$

As, $|\sin(x)| < 1$ and $|\cos(x)| < 1$ ie

$$\cancel{-\frac{2^3}{3!} \leq R_3(2) \leq \frac{2^3}{3!}} \quad -1 \leq \sin x \leq 1$$

$$\cancel{-\frac{4}{3} \leq}$$

$$-\frac{2^4}{4!} \leq R_3(2) \leq \frac{2^4}{4!}$$

$$\Rightarrow -2/3 \leq R_3(2) \leq \frac{2}{3}$$

So, the bounds are these (Ans)

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Ans. to Qno. 3(a)

$$x_{i+1} = x_i - f(x_i)\,\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

Iteration-1 Initial guess is $x_{-1} = 1$, $x_0 = 10$

$f(x_{-1}) = 34165$

$f(x_0) = 27514.911$

$$x_1 = 10 - 27514.911 \times \frac{10-1}{\cancel{34165 - 27514.911}\ -34165}$$

$= 17.7628.$

Iteration-2

$f(x_0) = 27514.911$

$f(x_1) = 22451.36$

$$x_2 = 17.7628 - 2245.36 \times \frac{17.7628-1}{22451.36 - 27514.911}$$

$= 25.1960$

Iteration-3:

$f(x_1) = \cancel{25.196}\ 2245.36$

$f(x_2)$

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