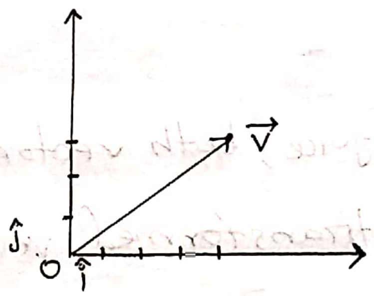
08-08-20

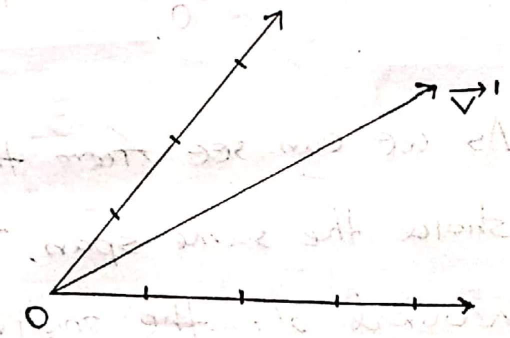
Lecture - 17(b)

Eigenvalues and Eigenvectors

When we perform linear transformation on any vector space (say, 2D space), the basis vectors are changed. After performing linear transformation we can compare the vectors plotted in our pevious grid (untransformed) vs. the vectors plotted in our transformed grid.

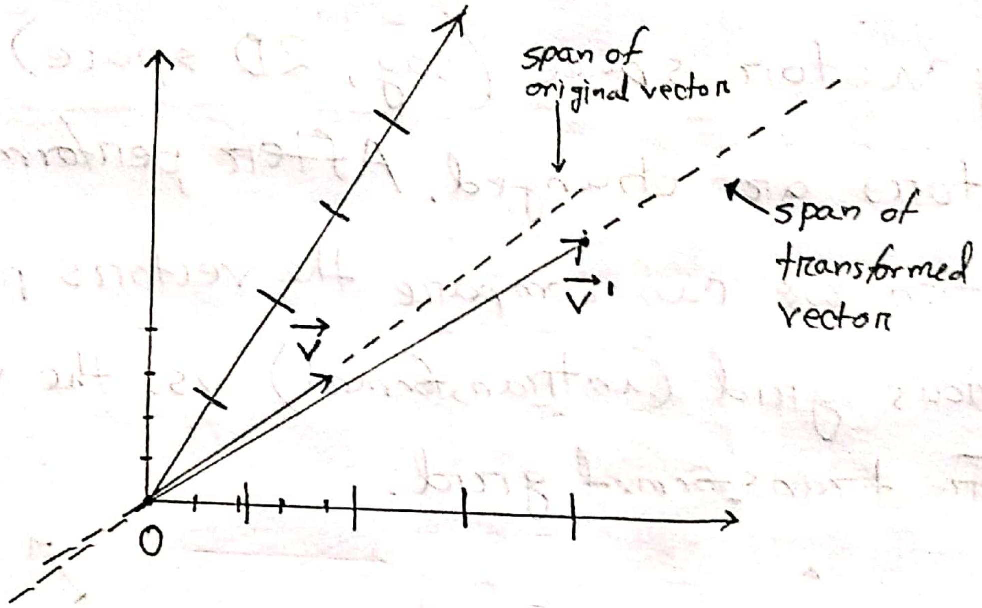


Original grid with $\hat{i}$ and $\hat{j}$



Transformed grid

As we can see in the transformed grid, the basis vectors are changed; both $\hat{i}$ and $\hat{j}$ from original grid are stretched and $\hat{j}$ is more inclined. Now, if we look at the vector $\vec{v}$, we can see that it's completely different — not only has the magnitude been changed (stratched) but also the direction has been changed. Let's consider the span of this vector $\vec{v}$. The span would be the imaginary line going through the vector $\vec{v}$ in both directions.



As we can see from the figure, both vectors don't share the same span. The transformed vector gets knocked off the original span.

But for ~~same~~ some special vectors, they remain in the original span even after the transformation of the basis vectors. These special vectors are called eigenvectors and the factor by which the vector gets stretched or squeezed is called the eigenvalue.

Let's take the ~~vector~~ matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$. In order to linearly transform the basis vectors, we can simply multiply this matrix to ~~the~~ any vector of our original grid.

Let's say the vector $\vec{v_1} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ is on our original grid.

To know the corresponding value in the transformed grid, we perform matrix multiplication.

$$v_1' = A \times v_1 = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 21+5 \\ 0+10 \end{bmatrix} = \begin{bmatrix} 28 \\ 10 \end{bmatrix}$$

Clearly, we can tell that the transformed vector is not on the span of original vector and hence, <u>NOT an eigenvector.</u>

Let's take another vector on x-axis, say, $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

Then,

$$v_2' = A \times v_2 = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \end{bmatrix}$$

Clearly, the transformed vector is in the span of the original vector, v_2. Hence, $v_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ is <u>an eigen vector of</u> the matrix A.

Similarly, any vector taken on the x-axis would give the same result since they are within span of v_2.

We conclude by the definition of eigen vectors that,

$$\boxed{\text{Eigenvectors DO NOT change span after performing Linear Transformation}}$$

Let's keep looking at our previous example. The original vector was $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$.
After linear transformation, the vector got stretched to ~~3 time~~ $\begin{bmatrix} 15 \\ 0 \end{bmatrix}$. So, it was stretched by a factor of 3. This factor is called the eigen value.

$$\boxed{\text{Eigenvalue is the factor by which Eigenvectors get stretched or squished.}}$$

Can eigen ~~values~~ values be factors or negative?
Yes, they can. Let's say, a vector has eigen value of $\lambda = -1/2$. This implies that the vectors will be squished to half. The negative sign means that the vector will be flipped.

Let's think of mathematically representing the eigenvectors. By definition, we know, eigen vectors after transformation will be transformed to the vector stretched by it's eigenvalue.

The linear transformation of a vector is $A \times \vec{v}$,

where, A is the Tranformation matrix
$\vec{v}$ is the Eigen vector

Again, the scaled up eigen vector is $\lambda \vec{v}$
where, λ is the eigen ~~valu~~ value.

Matrix Multi.

So, $\boxed{A\vec{v} = \lambda \vec{v}}$ — Scalar Multiplication

$\Rightarrow A\vec{v} = (\lambda \cdot I)\vec{v}$ [λ is scalar while A is a matrix. So, I is introduced which means λ will scale the identity matrix. Only then, comparism with A can be done]

$\Rightarrow A\vec{v} - (\lambda \cdot I)\vec{v} = \vec{0}$

$\Rightarrow \underbrace{(A - \lambda I)}\vec{v} = \vec{0}$

We need to find a matrix, which when multiplied to $\vec{v}$ will result in $\vec{0}$. We are assuming $\vec{v}$ to be a non-zero vector.

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The transformation matrix has to squish the vector $\vec{v}$ to $\vec{0}$ and so, it means, the determinant of the transformation matrix will be zero.

$$\therefore \boxed{\det(A-\lambda I)=0}$$

Steps of solving $Ax=\lambda x$:

(i) Find λ

(ii) Find nullspace of A, which is x.

Ex-1:

Let's take a matrix

$$A=\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

It is symmetric and constant along diagonals.

$$\therefore \det(A-\lambda I)=0$$

$$\Rightarrow \det\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}-\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)=0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}=0$$

$$\Rightarrow (3-\lambda)^2-1=0$$

$$\Rightarrow \lambda^2 - 6\lambda + 8 = 0$$

$$\Rightarrow (\lambda - 4)(\lambda - 2) = 0$$

$$\therefore \lambda = 2, 4.$$

These are the eigen values of the given matrix.

For the eigenvector, $(A - \lambda I)\vec{x} = \vec{0}$

Taking $\lambda_1 = 2$, we get,

$$\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\vec{x} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\vec{x} = \vec{0} \quad \text{——— (i)}$$

And, for $\lambda_2 = 4$, we get,

$$\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)\vec{x} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\vec{x} = \vec{0} \quad \text{——— (ii)}$$

Both these cases can be solved by solving the equation to calculate null-space $(Ax = 0)$.

From ~~we~~ (i), we can easily say,

$$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and, from (ii), $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So, for $\lambda_1 = 2$, $x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

and, $\lambda_2 = 4$, $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

<u>Ex-2:</u>

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow$$ also symmetric and consistant along diagonals

$$\det(A - \lambda I) = 0$$

For the λ, $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^2 - 1 = 0$$
$$\therefore \lambda = \pm 1$$

For, $\lambda_1 = 1$, $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = \vec{0}$

$$\therefore x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For, $\lambda_2 = -1$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \vec{0}$

$$\therefore x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Summary of the solution process:

(i) Compute the determinant of $A-\lambda I$

(ii) Solve the polynomial obtain from $\det(A-\lambda I)=0$

(iii) For each eigenvalue, solve $(A-\lambda I)x=0$ to find the eigenvectors.

Ex-3 (single eigen value)

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \longrightarrow \text{[Triangular matrix]}$$

$\therefore \det(A-\lambda I)=0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

We got a triangular matrix.

Usually, we get n eigenvalues for an $n \times n$ matrix.
But in this case we won't get n eigen values.

$$\Rightarrow (3-\lambda)^2 = 0$$

$$\therefore \lambda = 3.$$

We got only 1 eigen value.

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Ex - 4 (no eigen value)

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

trace $= \lambda_1 + \lambda_2 = 0$

det $= 0 - (-1) = 1 = \lambda_1 \lambda_2$

$\therefore \det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-\lambda)(-\lambda) - (-1)1 = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\therefore \lambda = \pm i$$

So, the matrix has no real eigen value.

If we have a closer look at the matrix, we can tell that it is a rotational-matrix; by multiplying it, our basis vectors will be rotated by 90°. So, every vector gets knocked off its span and thus, it has no real eigen value.

Facts related to eigen-values and eigen-vectors:

(i) For $(n \times n)$ matrix, the product of the n eigen values equals to the determinant.

⊛ For example in ex-1, the determinant of A is

$$\det(A) = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 9 - 1 = 8$$

And the product of eigenvalues are

$$\lambda_1 \times \lambda_2 = 2 \times 4 = 8.$$

⊛ In case of example-3, where it had a single eigen value,

$$\det(A) = \begin{vmatrix} 3 & 1 \\ 0 & 3 \end{vmatrix} = 9 - 0 = 9$$

$$\lambda \times \lambda = 3^2 = 9.$$

⊛ For example-4, where it had NO eigenvalue,

$$\det(A) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 0 - (-1) = 1.$$

$$\lambda_1 \times \lambda_2 = (+i) \times (-i) = -i^2 = 1.$$

(ii) The sum of the eigenvalues equal the sum of the diagonals. The sum of the diagonals is also called **trace**.

$$\lambda_1 + \lambda_2 + \lambda_3 + \dots \lambda_n = a_{11} + a_{22} + \dots a_{nn}$$

For example -

for ex-1, $\lambda_1 + \lambda_2 = 2 + 4 = 6$

$a_{11} + a_{22} = 3 + 3 = 6$

for ex-3, $\lambda_1 + \lambda_2 = 3 + 3 = 6$

$a_{11} + a_{22} = 3 + 3 = 6$

(iii) If 'KI', where K∈N is added to any matrix, the eigen value is increased by K, and the eigenvectors remain the same.

For example - in our previous matrices

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

+3

+3

$(\lambda_1, \lambda_2) = (4, 2)$ $\qquad$ $(\lambda_1, \lambda_2) = (1, -1)$

If we compare the eigenvalues, the first one has (4, 2) which is increased by (+3) from (1, -1).

~~So, $K\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ where $K=3$.~~

So, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + KI = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Here $K=3$.

That is why eigenvalues were increased by 3.

$$Ax = \lambda x$$

$$\Rightarrow (A+3I)x = Ax + 3Ix$$

$$= \lambda x + 3Ix$$

$$= \underbrace{(\lambda+3)}_{\text{Increased factor or } K.}x$$

What if $Ax = \lambda x$

and $Bx = \alpha x$

Can we conclude that

$$(A+B)x = (\lambda + \alpha)x$$

Actually, no, we can not come to this conclusion. The first equation has an eigen vector x but eigen value λ. In second equation, we have the same eigen vector x but with a different matrix B and different eigenvalue α. Hence, the second equation is not possible to be formed and is, therefore, incorrect.

From our previous observations, we can see that the more symmetric the matrix is, the more "proper" eigenvalues we get. If we think about it, the transformation that occurs by multiplying a symmetric matrix would ~~be~~ keep the vector in its span. So, proper eigen values can be obtained.

But if the matrix is less symmetric, like the rotational matrix from ex-4, the eigen values will be really uncommon, like imaginary number.

Home-task:

(i) Find the eigenvalues and eigenvectors ~~and eigenvectors~~ of all rotational matrices.

Ans: In 2D ~~matr~~ space, the matrices

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are used for 90°, 180° and 270° counter-clockwise rotation.

→ For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$\therefore \lambda = \pm i$ (Ans.) [No possible eigenvector]

→ For $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)^2 = 0$$

$$\Rightarrow \lambda = -1 \text{ (Ans.)}$$

~~→ For A = [~~

So, $\begin{bmatrix} -1-(-1) & 0 \\ 0 & -1-(-1) \end{bmatrix} \vec{x} = \vec{0}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$$

This equation has infinite solutions. Hence, the matrix has infinite eigenvectors (Ans.)

→ For $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\therefore \lambda = \pm (i)$$

So, it has no real eigen-value and hence, no eigenvectors (Ans.)

(ii) Find the eigenvalues and eigenvectors of all reflection matrices.

Ans: Typically used reflection matrices are

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$ for

$0°$, $45°$, $90°$ and $(-45°)$ respectively.

→ For $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(1-\lambda^2) = 0$$

$$\Rightarrow \lambda = \pm 1 \text{ (Ans.)}$$

The eigenvector will be,

for $\lambda_1 = 1$, $\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \vec{x} = \vec{0}$

$$\therefore \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for $\lambda_2 = -1$, $\begin{bmatrix} 1+1 & 0 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$

$$\therefore \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So, these are the eigenvectors.

→ For $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $\det(A-\lambda I)=0$

$$\Rightarrow \begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix}=0$$

$$\Rightarrow \lambda^2-1^2=0$$

$$\therefore \lambda=\pm 1 \text{ (Ans.)}$$

$\lambda_1=1$, $\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\vec{x}=\vec{0}$

$\therefore \vec{x}=\begin{bmatrix}1\\ 1\end{bmatrix}$

and, $\lambda_2=-1$, $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\vec{x}=\vec{0}$

$\therefore \vec{x}=\begin{bmatrix}-1\\ 1\end{bmatrix}$

The eigenvectors will be ~~same as previous one~~ $\begin{bmatrix}-1\\ 1\end{bmatrix}$ and $\begin{bmatrix}1\\ 1\end{bmatrix}$. (Ans.)

→ For $A=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$, $\det(A-\lambda I)=0$

$$\Rightarrow \begin{vmatrix}-1-\lambda & 0\\ 0 & 1-\lambda\end{vmatrix}=\vec{0}$$

$$\Rightarrow \lambda=\pm 1 \text{ (Ans.)}$$

For, $\lambda_1=1$,

$\begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix}\vec{x}=\vec{0}$

$\therefore \vec{x}=\begin{bmatrix}0\\ 1\end{bmatrix}$

For, $\lambda_2=-1$,

$\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix}\vec{x}=\vec{0}$

~~The eigenvectors will be same as previous one~~, $\therefore \vec{x}=\begin{bmatrix}1\\ 0\end{bmatrix}$

→ $A=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$, $\det(A-\lambda I)=0$

$$\Rightarrow \begin{vmatrix}-\lambda & -1\\ -1 & -\lambda\end{vmatrix}=0$$

$$\Rightarrow \lambda^2-1^2=0$$

$$\therefore \lambda=\pm 1 \text{ (Ans.)}$$

~~The eigenvectors will be same as previous one,~~

For, $\lambda_1=1$, $\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\vec{x}=\vec{0}$ $\therefore \vec{x}=\begin{bmatrix}-1\\ 1\end{bmatrix}$

$\lambda_2=-1$, $\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\vec{x}=\vec{0}$ $\therefore \vec{x}=\begin{bmatrix}1\\ 1\end{bmatrix}$ (Ans.)

(iii) Find the eigenvalues and eigenvectors of all projection matrices.

Ans: Let's consider a two projection matrices,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

→ For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda + \lambda^2 = 0$$

$$\therefore \lambda = 0, 1.$$

So, eigenvetors will be,

for $\lambda_1 = 0$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$

$$\therefore \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and, for $\lambda_2 = 1$, $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \vec{x} = \vec{0}$

$$\therefore \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We will see that, for any projection matrix, the eigenvalues will be 0 and 1 only.

$\rightarrow$ For $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda + \lambda^2 = 0$$

$$\therefore \lambda = 0, 1.$$

(Ans.)

$\lambda_1 = 0$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$

$\therefore \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$, $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} = \vec{0}$

$\therefore \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(Ans.)

~~Eigenvectors are~~

General eigen-values for rotational matrices

The general equation of a proper rotational matrix is

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \text{ where } 0 \leq \theta \leq 2\pi$$

which reprents a proper counterclockwise rotation by angle of θ in the x-y plane.

Let's consider the eigenvalue equation,

$$A(\theta)\vec{x} = \lambda \vec{x}$$

$$\Rightarrow \det(A(\theta) - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\Rightarrow \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

Solving the quadratic equation, we get,

$$\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$
$$= \cos\theta \pm i\sin\theta$$
$$\boxed{\lambda = e^{\pm i\theta}} \quad [\text{Euler's identity}]$$

What if we use an improper rotational matrix? An improper rotational matrix will rotate the matrix and also cause reflection. The general form is

$$A(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \text{ where } 0 \le \theta < 2\pi$$

this can also be expressed as the product of a proper ~~reflection~~ rotation and a reflection matrix.

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Now, let's consider the eigenvalue equation

$$A(\theta)\vec{x} = \lambda\vec{x}$$

So, $\det(A(\theta) - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\cos\theta - \lambda)(-\cos\theta - \lambda) - \sin^2\theta = 0$$

$$\Rightarrow \lambda^2 - 1 = 0$$

$$\boxed{\therefore \lambda = \pm 1.}$$

General eigenvalues for Reflection Matrices

The general equation for a reflection matrix is

$$A(\theta) = ref(\theta) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

where $0 \leq \theta \leq \pi$

Let's consider the eigen value problem,

$$A(\theta)\vec{x} = \lambda\vec{x}$$

$$\Rightarrow \det(A(\theta) - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} \cos 2\theta - \lambda & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\cos 2\theta - \lambda)(-\cos 2\theta - \lambda) - \sin^2 2\theta = 0$$

$$\Rightarrow \lambda^2 - \cos^2 2\theta - \sin^2 2\theta = 0$$

$$\Rightarrow \lambda^2 - 1 = 0$$

$$\therefore \lambda = \pm 1$$

So, the value of λ is (± 1) for any Reflection matrix.

General eigenvalues for Projection Matrices

An orthogonal projection matrix is what we call idempotent. ~~Then~~ This means, when squared the matrix would not change its value. If A is an idempotent matrix, then, $A^2 = A$.

Now, for our eigenvalue problem,

$$A\vec{x} = \lambda\vec{x}.$$

Again, $A^2\vec{x} = \lambda\vec{x}$ [Idempotent matrix]

Now, if, $A\vec{x} = \lambda\vec{x}$

then, $A^2\vec{x} = \lambda^2\vec{x}$ [Squaring both sides]

$$\Rightarrow \lambda\vec{x} = \lambda^2\vec{x}$$

$$\Rightarrow \lambda^2 = \lambda$$

$$\Rightarrow \lambda(\lambda - 1) = 0$$

$$\therefore \lambda = 0, 1.$$

So, a projection matrix can have eigenvalues 0 and 1.

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