13 Functions of Several Variables

Manufacturing (Example 2, p. 959)

Ocean Floor (Exercise 66, p. 930)

Wind Chill (Exercise 27, p. 910)

Marginal Costs (Exercise 118, p. 902)

Forestry (Exercise 77, p. 882)

13.1 Introduction to Functions of Several Variables

- Understand the notation for a function of several variables.
- Sketch the graph of a function of two variables.
- Sketch level curves for a function of two variables.
- Sketch level surfaces for a function of three variables.
- Use computer graphics to graph a function of two variables.

Functions of Several Variables

So far in this text, you have dealt only with functions of a single (independent) variable. Many familiar quantities, however, are functions of two or more variables. Here are three examples.

1. The work done by a force, $W = FD$, is a function of two variables.
2. The volume of a right circular cylinder, $V = \pi r^2 h$, is a function of two variables.
3. The volume of a rectangular solid, $V = lwh$, is a function of three variables.

The notation for a function of two or more variables is similar to that for a function of a single variable. Here are two examples.

$$z = f\underbrace{(x, y)}_{\text{2 variables}} = x^2 + xy$$ Function of two variables

and

$$w = f\underbrace{(x, y, z)}_{\text{3 variables}} = x + 2y - 3z$$ Function of three variables

Exploration

Without using a graphing utility, describe the graph of each function of two variables.

a. $z = x^2 + y^2$

b. $z = x + y$

c. $z = x^2 + y$

d. $z = \sqrt{x^2 + y^2}$

e. $z = \sqrt{1 - x^2 + y^2}$

Definition of a Function of Two Variables

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is a **function of x and y.** The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f. For the function

$$z = f(x, y)$$

x and y are called the **independent variables** and z is called the **dependent variable.**

MARY FAIRFAX SOMERVILLE (1780–1872)

Somerville was interested in the problem of creating geometric models for functions of several variables. Her most well-known book, *The Mechanics of the Heavens,* was published in 1831.

See LarsonCalculus.com to read more of this biography.

Similar definitions can be given for functions of three, four, or n variables, where the domains consist of ordered triples (x_1, x_2, x_3), quadruples (x_1, x_2, x_3, x_4), and n-tuples $(x_1, x_2, \ldots, x_n)$. In all cases, the range is a set of real numbers. In this chapter, you will study only functions of two or three variables.

As with functions of one variable, the most common way to describe a function of several variables is with an *equation*, and unless it is otherwise restricted, you can assume that the domain is the set of all points for which the equation is defined. For instance, the domain of the function

$$f(x, y) = x^2 + y^2$$

is the entire xy-plane. Similarly, the domain of

$$f(x, y) = \ln xy$$

is the set of all points (x, y) in the plane for which $xy > 0$. This consists of all points in the first and third quadrants.

Mary Evans Picture Library/The Image Works

EXAMPLE 1 Domains of Functions of Several Variables

Find the domain of each function.

a. $f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 9}}{x}$ **b.** $g(x, y, z) = \dfrac{x}{\sqrt{9 - x^2 - y^2 - z^2}}$

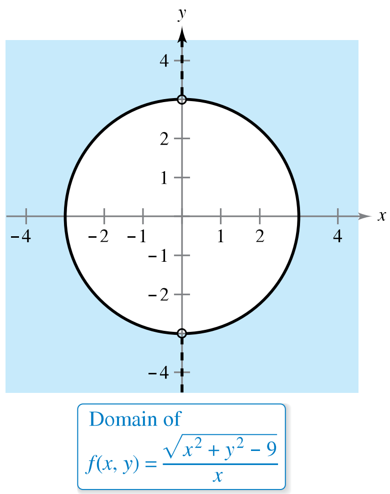


Figure 13.1

Solution

a. The function f is defined for all points (x, y) such that $x \neq 0$ and

$$x^2 + y^2 \geq 9.$$

So, the domain is the set of all points lying on or outside the circle $x^2 + y^2 = 9$ *except* those points on the y-axis, as shown in Figure 13.1.

b. The function g is defined for all points (x, y, z) such that

$$x^2 + y^2 + z^2 < 9.$$

Consequently, the domain is the set of all points (x, y, z) lying inside a sphere of radius 3 that is centered at the origin.

Functions of several variables can be combined in the same ways as functions of single variables. For instance, you can form the sum, difference, product, and quotient of two functions of two variables as follows.

$$(f \pm g)(x, y) = f(x, y) \pm g(x, y) \qquad \text{Sum or difference}$$

$$(fg)(x, y) = f(x, y)g(x, y) \qquad \text{Product}$$

$$\frac{f}{g}(x, y) = \frac{f(x, y)}{g(x, y)}, \quad g(x, y) \neq 0 \qquad \text{Quotient}$$

You cannot form the composite of two functions of several variables. You can, however, form the **composite** function $(g \circ h)(x, y)$, where g is a function of a single variable and h is a function of two variables.

$$(g \circ h)(x, y) = g(h(x, y)) \qquad \text{Composition}$$

The domain of this composite function consists of all (x, y) in the domain of h such that $h(x, y)$ is in the domain of g. For example, the function

$$f(x, y) = \sqrt{16 - 4x^2 - y^2}$$

can be viewed as the composite of the function of two variables given by

$$h(x, y) = 16 - 4x^2 - y^2$$

and the function of a single variable given by

$$g(u) = \sqrt{u}.$$

The domain of this function is the set of all points lying on or inside the ellipse $4x^2 + y^2 = 16$.

A function that can be written as a sum of functions of the form $cx^m y^n$ (where c is a real number and m and n are nonnegative integers) is called a **polynomial function** of two variables. For instance, the functions

$$f(x, y) = x^2 + y^2 - 2xy + x + 2 \quad \text{and} \quad g(x, y) = 3xy^2 + x - 2$$

are polynomial functions of two variables. A **rational function** is the quotient of two polynomial functions. Similar terminology is used for functions of more than two variables.

The Graph of a Function of Two Variables

As with functions of a single variable, you can learn a lot about the behavior of a function of two variables by sketching its graph. The **graph** of a function f of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f. This graph can be interpreted geometrically as a *surface in space,* as discussed in Sections 11.5 and 11.6. In Figure 13.2, note that the graph of $z = f(x, y)$ is a surface whose projection onto the xy-plane is D, the domain of f. To each point (x, y) in D there corresponds a point (x, y, z) on the surface, and, conversely, to each point (x, y, z) on the surface there corresponds a point (x, y) in D.

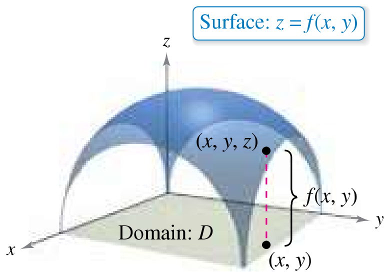


Figure 13.2

EXAMPLE 2 Describing the Graph of a Function of Two Variables

Consider the function given by

$$f(x, y) = \sqrt{16 - 4x^2 - y^2}.$$

a. Find the domain and range of the function.

b. Describe the graph of f.

Solution

a. The domain D implied by the equation of f is the set of all points (x, y) such that

$$16 - 4x^2 - y^2 \geq 0.$$

So, D is the set of all points lying on or inside the ellipse

$$\frac{x^2}{4} + \frac{y^2}{16} = 1. \qquad \text{Ellipse in the } xy\text{-plane}$$

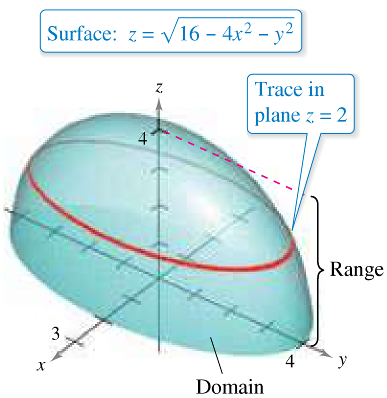
The range of f is all values $z = f(x, y)$ such that $0 \leq z \leq \sqrt{16}$, or

$$0 \leq z \leq 4. \qquad \text{Range of } f$$

b. A point (x, y, z) is on the graph of f if and only if

$$\begin{aligned} z &= \sqrt{16 - 4x^2 - y^2} \\ z^2 &= 16 - 4x^2 - y^2 \\ 4x^2 + y^2 + z^2 &= 16 \\ \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} &= 1, \quad 0 \leq z \leq 4. \end{aligned}$$

From Section 11.6, you know that the graph of f is the upper half of an ellipsoid, as shown in Figure 13.3.



The graph of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$ is the upper half of an ellipsoid.
Figure 13.3

To sketch a surface in space *by hand,* it helps to use traces in planes parallel to the coordinate planes, as shown in Figure 13.3. For example, to find the trace of the surface in the plane $z = 2$, substitute $z = 2$ in the equation $z = \sqrt{16 - 4x^2 - y^2}$ and obtain

$$2 = \sqrt{16 - 4x^2 - y^2} \implies \frac{x^2}{3} + \frac{y^2}{12} = 1.$$

So, the trace is an ellipse centered at the point $(0, 0, 2)$ with major and minor axes of lengths $4\sqrt{3}$ and $2\sqrt{3}$.

Traces are also used with most three-dimensional graphing utilities. For instance, Figure 13.4 shows a computer-generated version of the surface given in Example 2. For this graph, the computer took 25 traces parallel to the xy-plane and 12 traces in vertical planes.

If you have access to a three-dimensional graphing utility, use it to graph several surfaces.

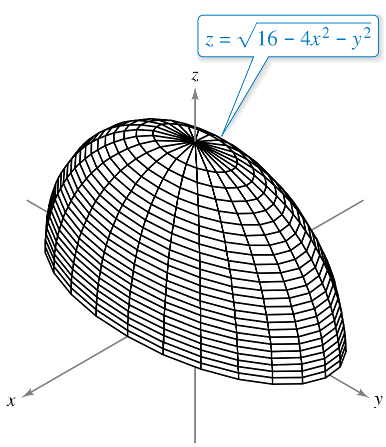


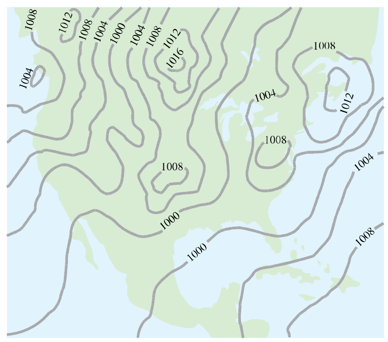
Figure 13.4

Level Curves

A second way to visualize a function of two variables is to use a **scalar field** in which the scalar

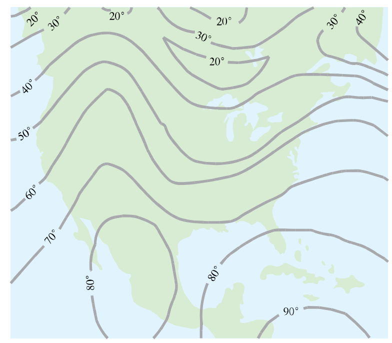
$$z = f(x, y)$$

is assigned to the point (x, y). A scalar field can be characterized by **level curves** (or **contour lines**) along which the value of $f(x, y)$ is constant. For instance, the weather map in Figure 13.5 shows level curves of equal pressure called **isobars.** In weather maps for which the level curves represent points of equal temperature, the level curves are called **isotherms,** as shown in Figure 13.6. Another common use of level curves is in representing electric potential fields. In this type of map, the level curves are called **equipotential lines.**



Level curves show the lines of equal pressure (isobars), measured in millibars.

Figure 13.5



Level curves show the lines of equal temperature (isotherms), measured in degrees Fahrenheit.

Figure 13.6

Contour maps are commonly used to show regions on Earth's surface, with the level curves representing the height above sea level. This type of map is called a **topographic map.** For example, the mountain shown in Figure 13.7 is represented by the topographic map in Figure 13.8.

Figure 13.7

Figure 13.8

A contour map depicts the variation of z with respect to x and y by the spacing between level curves. Much space between level curves indicates that z is changing slowly, whereas little space indicates a rapid change in z. Furthermore, to produce a good three-dimensional illusion in a contour map, it is important to choose c-values that are *evenly spaced.*

EXAMPLE 3 Sketching a Contour Map

The hemisphere

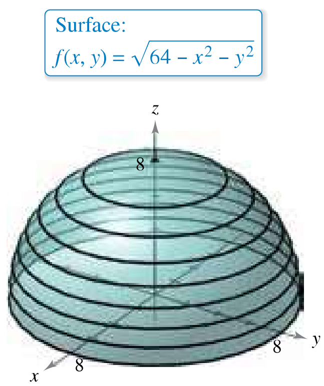
$$f(x, y) = \sqrt{64 - x^2 - y^2}$$

is shown in Figure 13.9. Sketch a contour map of this surface using level curves corresponding to $c = 0, 1, 2, \dots, 8$.

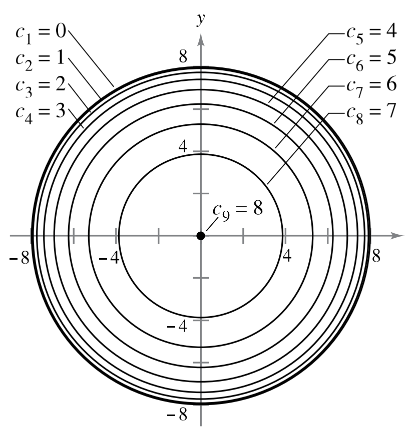
Solution For each value of c, the equation $f(x, y) = c$ is a circle (or point) in the xy-plane. For example, when $c_1 = 0$, the level curve is

$$x^2 + y^2 = 64 \qquad \text{Circle of radius 8}$$

which is a circle of radius 8. Figure 13.10 shows the nine level curves for the hemisphere.



Hemisphere
Figure 13.9



Contour map
Figure 13.10

EXAMPLE 4 Sketching a Contour Map

See LarsonCalculus.com for an interactive version of this type of example.

The hyperbolic paraboloid

$$z = y^2 - x^2$$

is shown in Figure 13.11. Sketch a contour map of this surface.

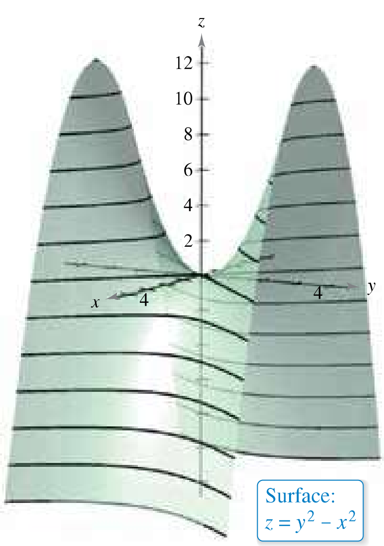
Solution For each value of c, let $f(x, y) = c$ and sketch the resulting level curve in the xy-plane. For this function, each of the level curves $(c \neq 0)$ is a hyperbola whose asymptotes are the lines $y = \pm x$. When $c < 0$, the transverse axis is horizontal. For instance, the level curve for $c = -4$ is

$$\frac{x^2}{2^2} - \frac{y^2}{2^2} = 1.$$

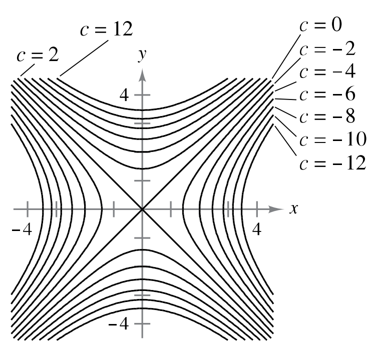
When $c > 0$, the transverse axis is vertical. For instance, the level curve for $c = 4$ is

$$\frac{y^2}{2^2} - \frac{x^2}{2^2} = 1.$$

When $c = 0$, the level curve is the degenerate conic representing the intersecting asymptotes, as shown in Figure 13.12.



Hyperbolic paraboloid
Figure 13.11



Hyperbolic level curves (at increments of 2)
Figure 13.12

One example of a function of two variables used in economics is the **Cobb-Douglas production function.** This function is used as a model to represent the numbers of units produced by varying amounts of labor and capital. If x measures the units of labor and y measures the units of capital, then the number of units produced is

$$f(x, y) = Cx^a y^{1-a}$$

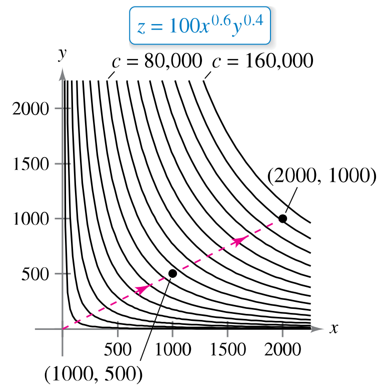
where C and a are constants with $0 < a < 1$.

EXAMPLE 5 The Cobb-Douglas Production Function

A manufacturer estimates a production function to be

$$f(x, y) = 100x^{0.6}y^{0.4}$$

where x is the number of units of labor and y is the number of units of capital. Compare the production level when $x = 1000$ and $y = 500$ with the production level when $x = 2000$ and $y = 1000$.



Level curves (at increments of 10,000)
Figure 13.13

Solution When $x = 1000$ and $y = 500$, the production level is

$$f(1000, 500) = 100(1000^{0.6})(500^{0.4}) \approx 75{,}786.$$

When $x = 2000$ and $y = 1000$, the production level is

$$f(2000, 1000) = 100(2000^{0.6})(1000^{0.4}) \approx 151{,}572.$$

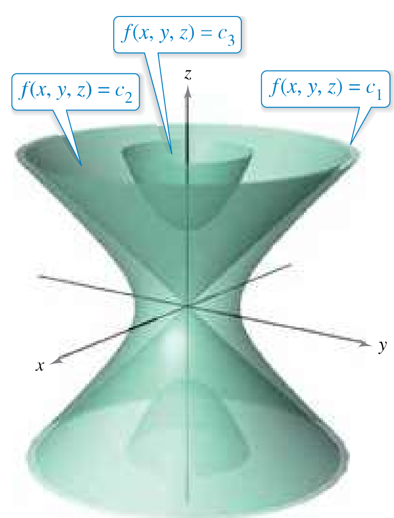
The level curves of $z = f(x, y)$ are shown in Figure 13.13. Note that by doubling both x and y, you double the production level (see Exercise 83).

Level Surfaces

The concept of a level curve can be extended by one dimension to define a **level surface.** If f is a function of three variables and c is a constant, then the graph of the equation

$$f(x, y, z) = c$$

is a **level surface** of f, as shown in Figure 13.14.



Level surfaces of f
Figure 13.14

EXAMPLE 6 Level Surfaces

Describe the level surfaces of

$$f(x, y, z) = 4x^2 + y^2 + z^2.$$

Solution Each level surface has an equation of the form

$$4x^2 + y^2 + z^2 = c. \qquad \text{Equation of level surface}$$

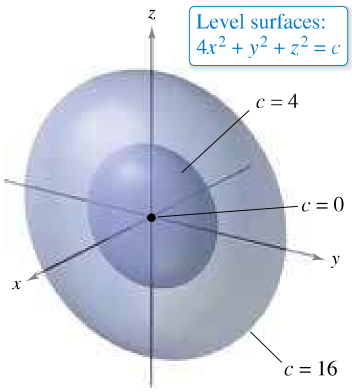


Figure 13.15

So, the level surfaces are ellipsoids (whose cross sections parallel to the yz-plane are circles). As c increases, the radii of the circular cross sections increase according to the square root of c. For example, the level surfaces corresponding to the values $c = 0$, $c = 4$, and $c = 16$ are as follows.

$$4x^2 + y^2 + z^2 = 0 \qquad \text{Level surface for } c = 0 \text{ (single point)}$$

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{4} = 1 \qquad \text{Level surface for } c = 4 \text{ (ellipsoid)}$$

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1 \qquad \text{Level surface for } c = 16 \text{ (ellipsoid)}$$

These level surfaces are shown in Figure 13.15.

If the function in Example 6 represented the *temperature* at the point (x, y, z), then the level surfaces shown in Figure 13.15 would be called **isothermal surfaces.**

Computer Graphics

The problem of sketching the graph of a surface in space can be simplified by using a computer. Although there are several types of three-dimensional graphing utilities, most use some form of trace analysis to give the illusion of three dimensions. To use such a graphing utility, you usually need to enter the equation of the surface and the region in the xy-plane over which the surface is to be plotted. (You might also need to enter the number of traces to be taken.) For instance, to graph the surface

$$f(x, y) = (x^2 + y^2)e^{1 - x^2 - y^2}$$

you might choose the following bounds for x, y, and z.

$$-3 \le x \le 3 \qquad \text{Bounds for } x$$

$$-3 \le y \le 3 \qquad \text{Bounds for } y$$

$$0 \le z \le 3 \qquad \text{Bounds for } z$$

Figure 13.16 shows a computer-generated graph of this surface using 26 traces taken parallel to the yz-plane. To heighten the three-dimensional effect, the program uses a "hidden line" routine. That is, it begins by plotting the traces in the foreground (those corresponding to the largest x-values), and then, as each new trace is plotted, the program determines whether all or only part of the next trace should be shown.

The graphs on the next page show a variety of surfaces that were plotted by computer. If you have access to a computer drawing program, use it to reproduce these surfaces. Remember also that the three-dimensional graphics in this text can be viewed and rotated. These rotatable graphs are available at *LarsonCalculus.com.*

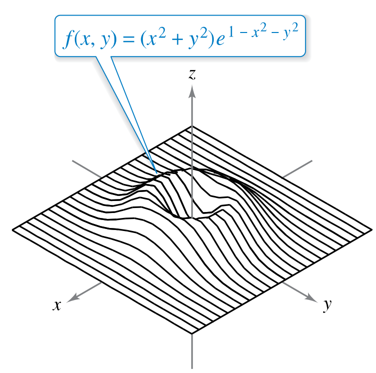
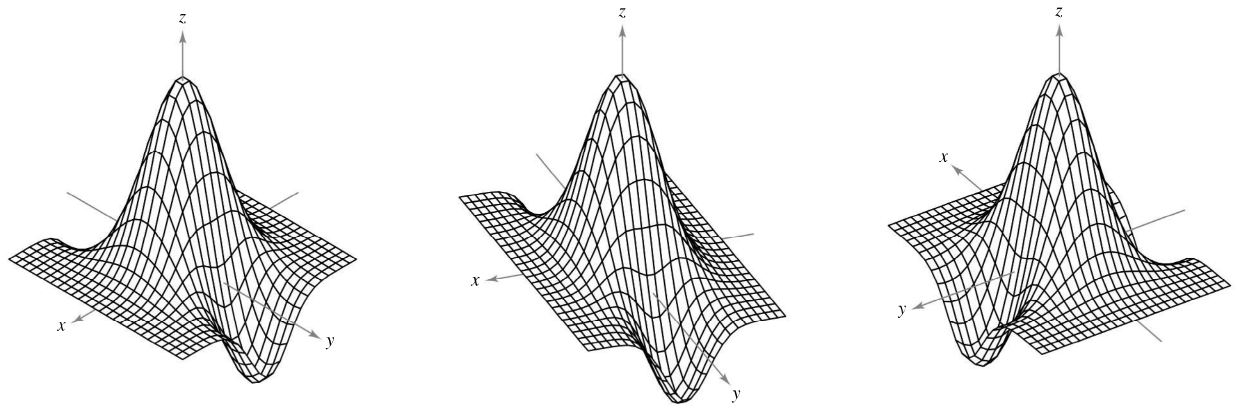
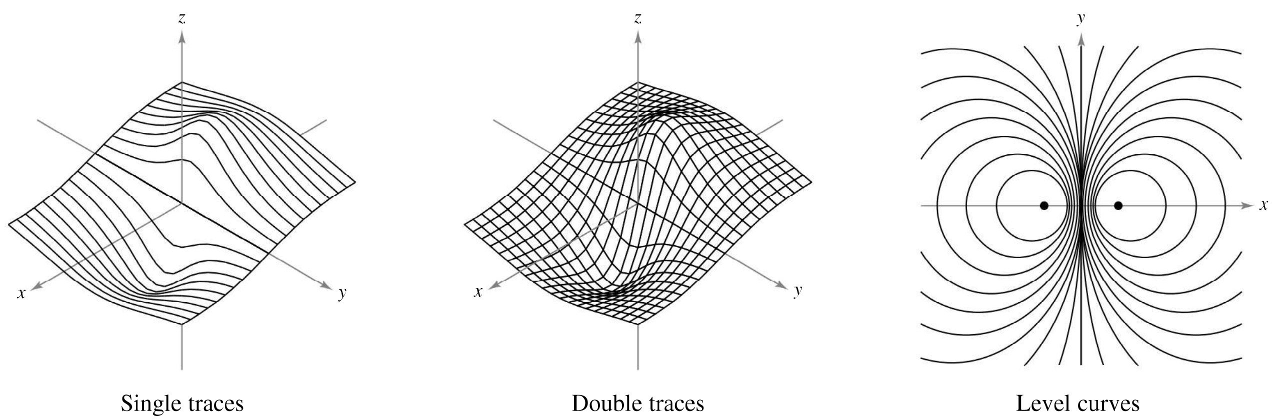


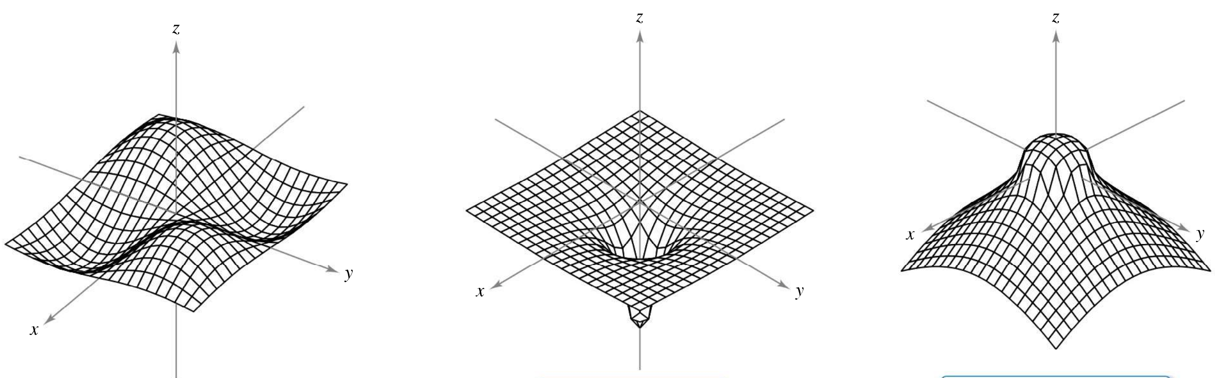
Figure 13.16



Three different views of the graph of $f(x, y) = (2 - y^2 + x^2)e^{1 - x^2 - (y^2/4)}$



Traces and level curves of the graph of $f(x, y) = \dfrac{-4x}{x^2 + y^2 + 1}$



$f(x, y) = \sin x \sin y$

$f(x, y) = -\dfrac{1}{\sqrt{x^2 + y^2}}$

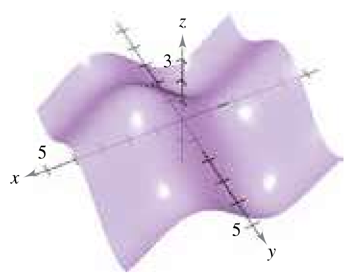
$f(x, y) = \dfrac{1 - x^2 - y^2}{\sqrt{|1 - x^2 - y^2|}}$

13.1 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Think About It** Explain why $z^2 = x + 3y$ is not a function of x and y.
2. **Function of Two Variables** What is a graph of a function of two variables? How is it interpreted geometrically?
3. **Determining Whether a Graph Is a Function** Use the graph to determine whether z is a function of x and y. Explain.



4. **Contour Map** Explain how to sketch a contour map of a function of x and y.

Determining Whether an Equation Is a Function **In Exercises 5–8, determine whether z is a function of x and y.**

5. $x^2z + 3y^2 - xy = 10$ **6.** $xz^2 + 2xy - y^2 = 4$

7. $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$ **8.** $z + x \ln y - 8yz = 0$

Evaluating a Function **In Exercises 9–20, evaluate the function at the given values of the independent variables. Simplify the results.**

9. $f(x, y) = 2x - y + 3$

(a) $f(0, 2)$ (b) $f(-1, 0)$ (c) $f(5, 30)$ (d) $f(3, y)$ (e) $f(x, 4)$ (f) $f(5, t)$

10. $f(x, y) = 4 - x^2 - 4y^2$

(a) $f(0, 0)$ (b) $f(0, 1)$ (c) $f(2, 3)$ (d) $f(1, y)$ (e) $f(x, 0)$ (f) $f(t, 1)$

11. $f(x, y) = xe^y$

(a) $f(-1, 0)$ (b) $f(0, 2)$ (c) $f(x, 3)$ (d) $f(t, -y)$

12. $g(x, y) = \ln|x + y|$

(a) $g(1, 0)$ (b) $g(0, -t^2)$ (c) $g(e, 0)$ (d) $g(e, e)$

13. $h(x, y, z) = \dfrac{xy}{z}$

(a) $h(-1, 3, -1)$ (b) $h(2, 2, 2)$ (c) $h(4, 4t, t^2)$ (d) $h(-3, 2, 5)$

14. $f(x, y, z) = \sqrt{x + y + z}$

(a) $f(2, 2, 5)$ (b) $f(0, 6, -2)$ (c) $f(8, -7, 2)$ (d) $f(0, 1, -1)$

15. $f(x, y) = x \sin y$

(a) $f(2, \pi/4)$ (b) $f(3, 1)$ (c) $f(-3, 0)$ (d) $f(4, \pi/2)$

16. $V(r, h) = \pi r^2 h$

(a) $V(3, 10)$ (b) $V(5, 2)$ (c) $V(4, 8)$ (d) $V(6, \pi)$

17. $g(x, y) = \displaystyle\int_x^y (2t - 3)\, dt$

(a) $g(4, 0)$ (b) $g(4, 1)$ (c) $g\left(4, \frac{3}{2}\right)$ (d) $g\left(\frac{3}{2}, 0\right)$

18. $g(x, y) = \displaystyle\int_x^y \frac{1}{t}\, dt$

(a) $g(4, 1)$ (b) $g(6, 3)$ (c) $g(2, 5)$ (d) $g\left(\frac{1}{2}, 7\right)$

19. $f(x, y) = 2x + y^2$

(a) $\dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ (b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

20. $f(x, y) = 3x^2 - 2y$

(a) $\dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ (b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

Finding the Domain and Range of a Function **In Exercises 21–32, find the domain and range of the function.**

21. $f(x, y) = 3x^2 - y$ **22.** $f(x, y) = e^{xy}$

23. $g(x, y) = x\sqrt{y}$ **24.** $g(x, y) = \dfrac{y}{\sqrt{x}}$

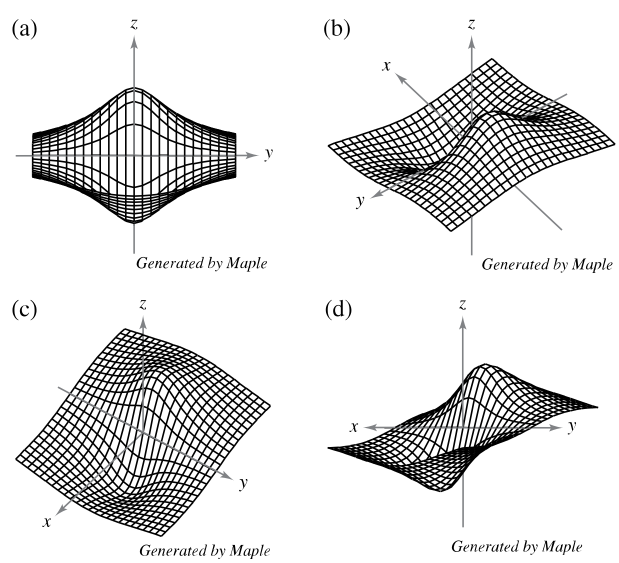
25. $z = \dfrac{x + y}{xy}$ **26.** $z = \dfrac{xy}{x + y}$

27. $f(x, y) = \sqrt{4 - x^2 - y^2}$ **28.** $f(x, y) = \sqrt{9 - 6x^2 + y^2}$

29. $f(x, y) = \arccos(x + y)$ **30.** $f(x, y) = \arcsin(y/x)$

31. $f(x, y) = \ln(5 - x - y)$ **32.** $f(x, y) = \ln(xy - 6)$

33. Think About It The graphs labeled (a), (b), (c), and (d) are graphs of the function $f(x, y) = -4x/(x^2 + y^2 + 1)$. Match each of the four graphs with the point in space from which the surface is viewed. The four points are $(20, 15, 25)$, $(-15, 10, 20)$, $(20, 20, 0)$, and $(20, 0, 0)$.



34. Think About It Use the function given in Exercise 33.

(a) Find the domain and range of the function.

(b) Identify the points in the xy-plane at which the function value is 0.

(c) Does the surface pass through all the octants of the rectangular coordinate system? Give reasons for your answer.

Sketching a Surface **In Exercises 35–42, describe and sketch the surface given by the function.**

35. $f(x, y) = 4$

36. $f(x, y) = 6 - 2x - 3y$

37. $f(x, y) = y^2$

38. $g(x, y) = \frac{1}{2}y$

39. $z = -x^2 - y^2$

40. $z = \frac{1}{2}\sqrt{x^2 + y^2}$

41. $f(x, y) = e^{-x}$

42. $f(x, y) = \begin{cases} xy, & x \geq 0,\ y \geq 0 \\ 0, & x < 0 \text{ or } y < 0 \end{cases}$

Graphing a Function Using Technology **In Exercises 43–46, use a computer algebra system to graph the function.**

43. $z = y^2 - x^2 + 1$

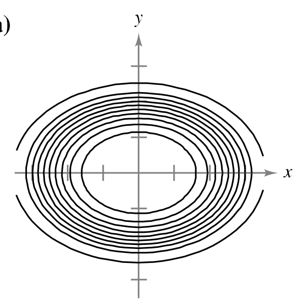
44. $z = \frac{1}{12}\sqrt{144 - 16x^2 - 9y^2}$

45. $f(x, y) = x^2 e^{(-xy/2)}$

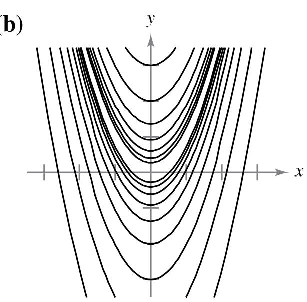
46. $f(x, y) = x \sin y$

Matching **In Exercises 47–50, match the graph of the surface with one of the contour maps. [The contour maps are labeled (a), (b), (c), and (d).]**

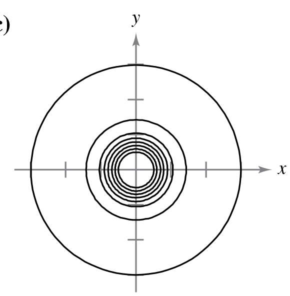
(a)



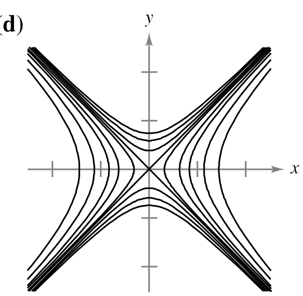
(b)



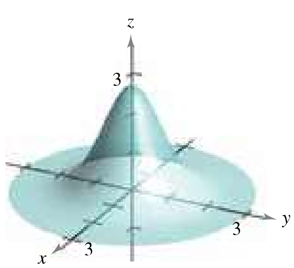
(c)



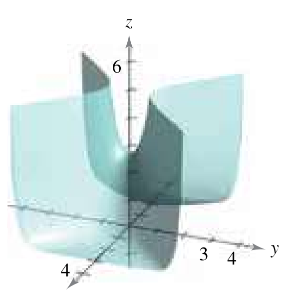
(d)



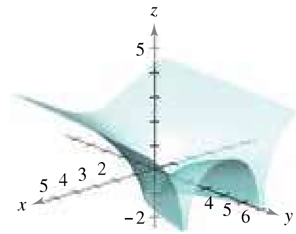
47. $f(x, y) = e^{1 - x^2 - y^2}$



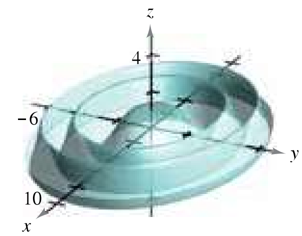
48. $f(x, y) = e^{1 - x^2 + y^2}$



49. $f(x, y) = \ln|y - x^2|$



50. $f(x, y) = \cos\left(\frac{x^2 + 2y^2}{4}\right)$



Sketching a Contour Map **In Exercises 51–58, describe the level curves of the function. Sketch a contour map of the surface using level curves for the given c-values.**

51. $z = x + y, \quad c = -1, 0, 2, 4$

52. $z = 6 - 2x - 3y, \quad c = 0, 2, 4, 6, 8, 10$

53. $z = x^2 + 4y^2, \quad c = 0, 1, 2, 3, 4$

54. $f(x, y) = \sqrt{9 - x^2 - y^2}, \quad c = 0, 1, 2, 3$

55. $f(x, y) = xy, \quad c = \pm 1, \pm 2, \ldots, \pm 6$

56. $f(x, y) = e^{xy/2}, \quad c = 2, 3, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

57. $f(x, y) = x/(x^2 + y^2), \quad c = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$

58. $f(x, y) = \ln(x - y), \quad c = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$

Graphing Level Curves Using Technology **In Exercises 59–62, use a graphing utility to graph six level curves of the function.**

59. $f(x, y) = x^2 - y^2 + 2$

60. $f(x, y) = |xy|$

61. $g(x, y) = \dfrac{8}{1 + x^2 + y^2}$

62. $h(x, y) = 3 \sin(|x| + |y|)$

EXPLORING CONCEPTS

63. Vertical Line Test Does the Vertical Line Test apply to functions of two variables? Explain your reasoning.

64. Using Level Curves All of the level curves of the surface given by $z = f(x, y)$ are concentric circles. Does this imply that the graph of f is a hemisphere? Illustrate your answer with an example.

65. Creating a Function Construct a function whose level curves are lines passing through the origin.

66. Conjecture Consider the function $f(x, y) = xy$, for $x \geq 0$ and $y \geq 0$.

(a) Sketch the graph of the surface given by f.

(b) Make a conjecture about the relationship between the graphs of f and $g(x, y) = f(x, y) - 3$. Explain your reasoning.

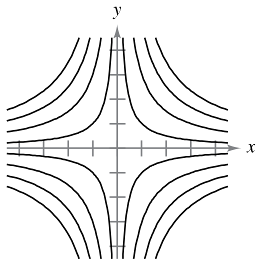
(c) Repeat part (b) for $g(x, y) = -f(x, y)$.

(d) Repeat part (b) for $g(x, y) = \frac{1}{2}f(x, y)$.

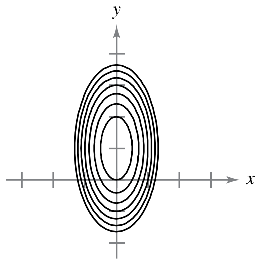
(e) On the surface in part (a), sketch the graph of $z = f(x, x)$.

Writing **In Exercises 67 and 68, use the graphs of the level curves (c-values evenly spaced) of the function f to write a description of a possible graph of f. Is the graph of f unique? Explain.**

67.



68.



69. Investment In 2016, an investment of \$1000 was made in a bond earning 6% compounded annually. Assume that the buyer pays tax at rate R and the annual rate of inflation is I. In the year 2026, the value V of the investment in constant 2016 dollars is

$$V(I, R) = 1000\left[\frac{1 + 0.06(1 - R)}{1 + I}\right]^{10}.$$

Use this function of two variables to complete the table.

	Inflation Rate		
Tax Rate	0	0.03	0.05
0			
0.28			
0.35			

70. Investment A principal of \$5000 is deposited in a savings account that earns interest at a rate of r (written as a decimal), compounded continuously. The amount $A(r, t)$ after t years is

$$A(r, t) = 5000e^{rt}.$$

Use this function of two variables to complete the table.

	Number of Years			
Rate	5	10	15	20
0.02				
0.03				
0.04				
0.05				

Sketching a Level Surface **In Exercises 71–76, describe and sketch the graph of the level surface $f(x, y, z) = c$ at the given value of c.**

71. $f(x, y, z) = x - y + z,\ c = 1$

72. $f(x, y, z) = 4x + y + 2z,\ c = 4$

73. $f(x, y, z) = x^2 + y^2 + z^2,\ c = 9$

74. $f(x, y, z) = x^2 + \frac{1}{4}y^2 - z,\ c = 1$

75. $f(x, y, z) = 4x^2 + 4y^2 - z^2,\ c = 0$

76. $f(x, y, z) = \sin x - z,\ c = 0$

77. Forestry

The *Doyle Log Rule* is one of several methods used to determine the lumber yield of a log (in board-feet) in terms of its diameter d (in inches) and its length L (in feet). The number of board-feet is

$$N(d, L) = \left(\frac{d - 4}{4}\right)^2 L.$$

(a) Find the number of board-feet of lumber in a log 22 inches in diameter and 12 feet in length.

(b) Find $N(30, 12)$.

78. Queuing Model The average length of time that a customer waits in line for service is

$$W(x, y) = \frac{1}{x - y}, \quad x > y$$

where y is the average arrival rate, written as the number of customers per unit of time, and x is the average service rate, written in the same units. Evaluate each of the following.

(a) $W(15, 9)$ (b) $W(15, 13)$

(c) $W(12, 7)$ (d) $W(5, 2)$

79. Temperature Distribution The temperature T (in degrees Celsius) at any point (x, y) on a circular steel plate of radius 10 meters is

$$T = 600 - 0.75x^2 - 0.75y^2$$

where x and y are measured in meters. Sketch the isothermal curves for $T = 0, 100, 200, \ldots, 600$.

80. Electric Potential The electric potential V at any point (x, y) is

$$V(x, y) = \frac{5}{\sqrt{25 + x^2 + y^2}}.$$

Sketch the equipotential curves for $V = \frac{1}{2}$, $V = \frac{1}{3}$, and $V = \frac{1}{4}$.

Cobb-Douglas Production Function **In Exercises 81 and 82, use the Cobb-Douglas production function to find the production level when $x = 600$ units of labor and $y = 350$ units of capital.**

81. $f(x, y) = 80x^{0.5}y^{0.5}$ **82.** $f(x, y) = 100x^{0.65}y^{0.35}$

83. Cobb-Douglas Production Function Use the Cobb-Douglas production function, $f(x, y) = Cx^a y^{1-a}$, to show that when the number of units of labor and the number of units of capital are doubled, the production level is also doubled.

84. Cobb-Douglas Production Function Show that the Cobb-Douglas production function $z = Cx^a y^{1-a}$ can be rewritten as

$$\ln \frac{z}{y} = \ln C + a \ln \frac{x}{y}.$$

85. Ideal Gas Law According to the Ideal Gas Law, $PV = kT$, where P is pressure, V is volume, T is temperature (in kelvins), and k is a constant of proportionality. A tank contains 2000 cubic inches of nitrogen at a pressure of 26 pounds per square inch and a temperature of 300 K.

(a) Determine k.

(b) Write P as a function of V and T and describe the level curves.

86. Modeling Data The table shows the net sales x (in billions of dollars), the total assets y (in billions of dollars), and the shareholder's equity z (in billions of dollars) for Walmart for the years 2010 through 2015. *(Source: Wal-Mart Stores, Inc.)*

Year	2010	2011	2012	2013	2014	2015
x	405.0	418.5	443.4	465.6	473.1	482.2
y	170.7	180.8	193.4	203.1	204.8	203.7
z	70.7	68.5	71.3	76.3	76.3	81.4

A model for the data is $z = f(x, y) = 0.428x - 0.653y + 8.172$.

(a) Complete a fourth row in the table using the model to approximate z for the given values of x and y. Compare the approximations with the actual values of z.

(b) Which of the two variables in this model has more influence on shareholder's equity? Explain.

(c) Simplify the expression for $f(x, 150)$ and interpret its meaning in the context of the problem.

87. Meteorology Meteorologists measure the atmospheric pressure in millibars. From these observations, they create weather maps on which the curves of equal atmospheric pressure (isobars) are drawn (see figure). On the map, the closer the isobars, the higher the wind speed. Match points A, B, and C with (a) highest pressure, (b) lowest pressure, and (c) highest wind velocity.

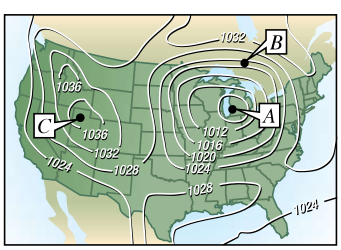


Figure for 87

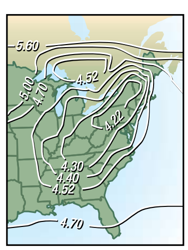
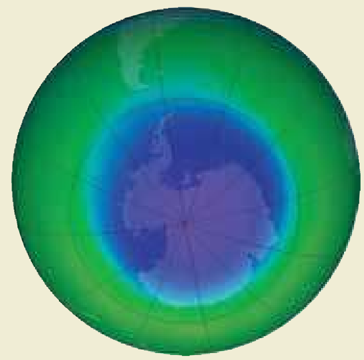


Figure for 88

88. Acid Rain The acidity of rainwater is measured in units called pH. A pH of 7 is neutral, smaller values are increasingly acidic, and larger values are increasingly alkaline. The map shows curves of equal pH and gives evidence that downwind of heavily industrialized areas, the acidity has been increasing. Using the level curves on the map, determine the direction of the prevailing winds in the northeastern United States.

89. Construction Cost A rectangular storage box with an open top has a length of x feet, a width of y feet, and a height of z feet. It costs \$4.50 per square foot to build the base and \$2.50 per square foot to build the sides. Write the cost C of constructing the box as a function of x, y, and z.

90. HOW DO YOU SEE IT? The contour map of the Southern Hemisphere shown in the figure was computer generated using data collected by satellite instrumentation. Color is used to show the "ozone hole" in Earth's atmosphere. The purple and blue areas represent the lowest levels of ozone, and the green areas represent the highest levels. *(Source: NASA)*



(a) Do the level curves correspond to equally spaced ozone levels? Explain.

(b) Describe how to obtain a more detailed contour map.

True or False? **In Exercises 91–94, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. If $f(x_0, y_0) = f(x_1, y_1)$, then $x_0 = x_1$ and $y_0 = y_1$.

92. If f is a function, then $f(ax, ay) = a^2 f(x, y)$.

93. The equation for a sphere is a function of three variables.

94. Two different level curves of the graph of $z = f(x, y)$ can intersect.

PUTNAM EXAM CHALLENGE

95. Let f: $\mathbb{R}^2$ $\mathbb{R}$ be a function such that

$$f(x, y) + f(y, z) + f(z, x) = 0$$

for all real numbers x, y, and z. Prove that there exists a function g: $\mathbb{R}$ $\mathbb{R}$ such that

$$f(x, y) = g(x) - g(y)$$

for all real numbers x and y.

NASA

13.2 Limits and Continuity

- Understand the definition of a neighborhood in the plane.
- Understand and use the definition of the limit of a function of two variables.
- Extend the concept of continuity to a function of two variables.
- Extend the concept of continuity to a function of three variables.

Neighborhoods in the Plane

SONYA KOVALEVSKY (1850–1891)

Much of the terminology used to define limits and continuity of a function of two or three variables was introduced by the German mathematician Karl Weierstrass (1815–1897). Weierstrass's rigorous approach to limits and other topics in calculus gained him the reputation as the "father of modern analysis." Weierstrass was a gifted teacher. One of his best-known students was the Russian mathematician Sonya Kovalevsky, who applied many of Weierstrass's techniques to problems in mathematical physics and became one of the first women to gain acceptance as a research mathematician.

In this section, you will study limits and continuity involving functions of two or three variables. The section begins with functions of two variables. At the end of the section, the concepts are extended to functions of three variables.

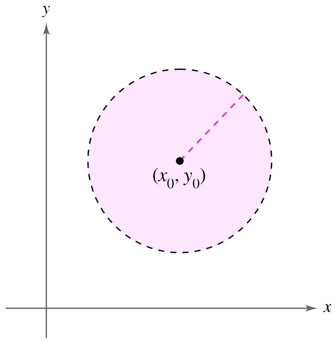
Your study of the limit of a function of two variables begins by defining a two-dimensional analog to an interval on the real number line. Using the formula for the distance between two points

(x, y) and (x_0, y_0)

in the plane, you can define the **δ-neighborhood** about (x_0, y_0) to be the **disk** centered at (x_0, y_0) with radius $\delta > 0$

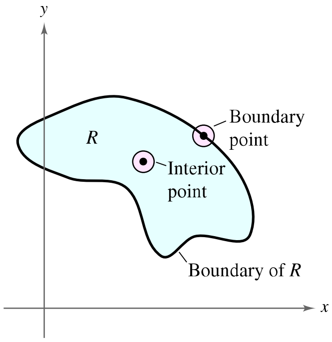
$$\{(x, y): \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\} \qquad \text{Open disk}$$

as shown in Figure 13.17. When this formula contains the *less than* inequality sign, $<$, the disk is called **open,** and when it contains the *less than or equal to* inequality sign, $\leq$, the disk is called **closed.** This corresponds to the use of $<$ and $\leq$ to define open and closed intervals.



An open disk

Figure 13.17



The boundary and interior points of a region R

Figure 13.18

Let the region R be a set of points in the plane. A point (x_0, y_0) in R is an **interior point** of R if there exists a δ-neighborhood about (x_0, y_0) that lies entirely in R, as shown in Figure 13.18. If every point in R is an interior point, then R is an **open region.** A point (x_0, y_0) is a **boundary point** of R if every open disk centered at (x_0, y_0) contains points inside R *and* points outside R. If R contains all its boundary points, then R is a **closed region.**

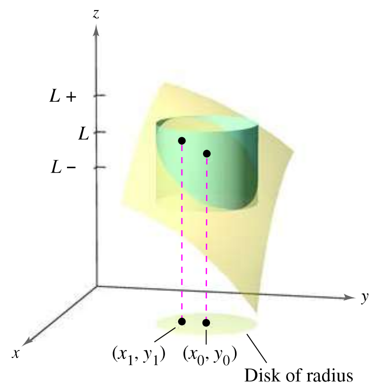
FOR FURTHER INFORMATION For more information on Sonya Kovalevsky, see the article "S. Kovalevsky: A Mathematical Lesson" by Karen D. Rappaport in *The American Mathematical Monthly*. To view this article, go to *MathArticles.com*.

Limit of a Function of Two Variables

> **Definition of the Limit of a Function of Two Variables**
>
> Let f be a function of two variables defined, except possibly at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then
>
> $$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$
>
> if for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that
>
> $$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Graphically, the definition of the limit of a function of two variables implies that for any point $(x, y) \neq (x_0, y_0)$ in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$, as shown in Figure 13.19.

The definition of the limit of a function of two variables is similar to the definition of the limit of a function of a single variable, yet there is a critical difference. To determine whether a function of a single variable has a limit, you need only test the approach from two directions—from the right and from the left. When the function approaches the same limit from the right and from the left, you can conclude that the limit exists. For a function of two variables, however, the statement



For any (x, y) in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$.
Figure 13.19

$$(x, y) \to (x_0, y_0)$$

means that the point (x, y) is allowed to approach (x_0, y_0) from any direction. If the value of

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y)$$

is not the same for all possible approaches, or **paths,** to (x_0, y_0), then the limit does not exist.

EXAMPLE 1 Verifying a Limit by the Definition

Show that $\lim_{(x, y) \to (a, b)} x = a$.

Solution Let $f(x, y) = x$ and $L = a$. You need to show that for each $\varepsilon > 0$, there exists a δ-neighborhood about (a, b) such that

$$|f(x, y) - L| = |x - a| < \varepsilon$$

whenever $(x, y) \neq (a, b)$ lies in the neighborhood. You can first observe that from

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

it follows that

$$\begin{aligned} |f(x, y) - L| &= |x - a| \\ &= \sqrt{(x - a)^2} \\ &\le \sqrt{(x - a)^2 + (y - b)^2} \\ &< \delta. \end{aligned}$$

So, you can choose $\delta = \varepsilon$, and the limit is verified. ■

Limits of functions of several variables have the same properties regarding sums, differences, products, and quotients as do limits of functions of single variables. (See Theorem 1.2 in Section 1.3.) Some of these properties are used in the next example.

EXAMPLE 2 Finding a Limit

Find the limit.

$$\lim_{(x, y) \to (1, 2)} \frac{5x^2y}{x^2 + y^2}$$

Solution By using the properties of limits of products and sums, you obtain

$$\lim_{(x, y) \to (1, 2)} 5x^2y = 5(1^2)(2) = 10$$

and

$$\lim_{(x, y) \to (1, 2)} (x^2 + y^2) = (1^2 + 2^2) = 5.$$

Because the limit of a quotient is equal to the quotient of the limits (and the denominator is not 0), you have

$$\lim_{(x, y) \to (1, 2)} \frac{5x^2y}{x^2 + y^2} = \frac{10}{5} = 2.$$

EXAMPLE 3 Finding a Limit

Find the limit: $\displaystyle \lim_{(x, y) \to (0, 0)} \frac{5x^2y}{x^2 + y^2}$.

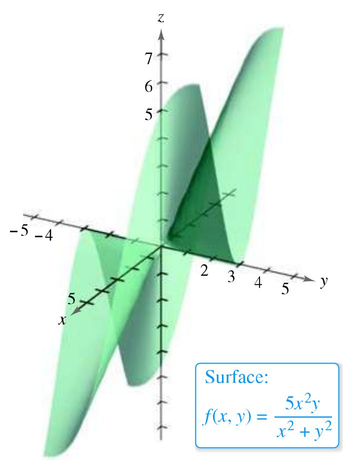


Figure 13.20

Solution In this case, the limits of the numerator and of the denominator are both 0, so you cannot determine the existence (or nonexistence) of a limit by taking the limits of the numerator and denominator separately and then dividing. From the graph of f in Figure 13.20, however, it seems reasonable that the limit might be 0. So, you can try applying the definition to $L = 0$. First, note that

$$|y| \le \sqrt{x^2 + y^2}$$

and

$$\frac{x^2}{x^2 + y^2} \le 1.$$

Then, in a δ-neighborhood about $(0, 0)$, you have

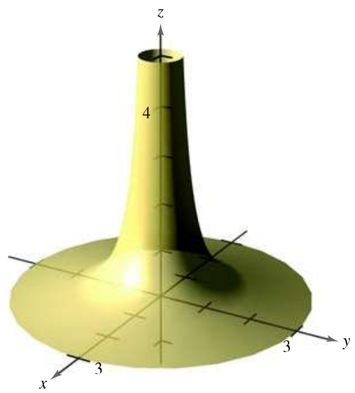
$$0 < \sqrt{x^2 + y^2} < \delta$$

and it follows that, for $(x, y) \ne (0, 0)$,

$$\begin{aligned} |f(x, y) - 0| &= \left|\frac{5x^2y}{x^2 + y^2}\right| \\ &= 5|y|\left(\frac{x^2}{x^2 + y^2}\right) \\ &\le 5|y| \\ &\le 5\sqrt{x^2 + y^2} \\ &< 5\delta. \end{aligned}$$

So, you can choose $\delta = \varepsilon/5$ and conclude that

$$\lim_{(x, y) \to (0, 0)} \frac{5x^2y}{x^2 + y^2} = 0.$$



$\lim_{(x, y) \to (0, 0)} \frac{1}{x^2 + y^2}$ does not exist.

Figure 13.21

For some functions, it is easy to recognize that a limit does not exist. For instance, it is clear that the limit

$$\lim_{(x, y) \to (0, 0)} \frac{1}{x^2 + y^2}$$

does not exist because the values of $f(x, y)$ increase without bound as (x, y) approaches $(0, 0)$ along *any* path (see Figure 13.21).

For other functions, it is not so easy to recognize that a limit does not exist. For instance, the next example describes a limit that does not exist because the function approaches different values along different paths.

EXAMPLE 4 A Limit That Does Not Exist

See LarsonCalculus.com for an interactive version of this type of example.

Show that the limit does not exist.

$$\lim_{(x, y) \to (0, 0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

Solution The domain of the function

$$f(x, y) = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

consists of all points in the xy-plane except for the point $(0, 0)$. To show that the limit as (x, y) approaches $(0, 0)$ does not exist, consider approaching $(0, 0)$ along two different "paths," as shown in Figure 13.22. Along the x-axis, every point is of the form

$$(x, 0)$$

and the limit along this approach is

$$\lim_{(x, 0) \to (0, 0)} \left(\frac{x^2 - 0^2}{x^2 + 0^2}\right)^2 = \lim_{(x, 0) \to (0, 0)} 1^2 = 1. \qquad \text{Limit along } x\text{-axis}$$

However, when (x, y) approaches $(0, 0)$ along the line $y = x$, you obtain

$$\lim_{(x, x) \to (0, 0)} \left(\frac{x^2 - x^2}{x^2 + x^2}\right)^2 = \lim_{(x, x) \to (0, 0)} \left(\frac{0}{2x^2}\right)^2 = 0. \qquad \text{Limit along line } y = x$$

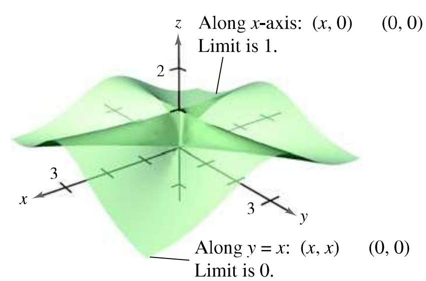
This means that in any open disk centered at $(0, 0)$, there are points (x, y) at which f takes on the value 1 and other points at which f takes on the value 0. For instance,

$$f(x, y) = 1$$

at $(1, 0)$, $(0.1, 0)$, $(0.01, 0)$, and $(0.001, 0)$, and

$$f(x, y) = 0$$

at $(1, 1)$, $(0.1, 0.1)$, $(0.01, 0.01)$, and $(0.001, 0.001)$. So, f does not have a limit as (x, y) approaches $(0, 0)$.



$\lim_{(x, y) \to (0, 0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$ does not exist.

Figure 13.22

In Example 4, you could conclude that the limit does not exist because you found two approaches that produced different limits. Be sure you understand that when two approaches produce the same limit, you *cannot* conclude that the limit exists. To form such a conclusion, you must show that the limit is the same along *all* possible approaches.

Continuity of a Function of Two Variables

Notice in Example 2 that the limit of $f(x, y) = 5x^2y/(x^2 + y^2)$ as $(x, y) \to (1, 2)$ can be evaluated by direct substitution. That is, the limit is $f(1, 2) = 2$. In such cases, the function f is said to be **continuous** at the point $(1, 2)$.

REMARK This definition of continuity can be extended to *boundary points* of the open region R by considering a special type of limit in which (x, y) is allowed to approach (x_0, y_0) along paths lying in the region R. This notion is similar to that of one-sided limits, as discussed in Chapter 1.

Definition of Continuity of a Function of Two Variables

A function f of two variables is **continuous at a point (x_0, y_0)** in an open region R if $f(x_0, y_0)$ is defined and is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0). That is,

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

In Example 3, it was shown that the function

$$f(x, y) = \frac{5x^2y}{x^2 + y^2}$$

is not continuous at $(0, 0)$. Because the limit at this point exists, however, you can remove the discontinuity by defining f at $(0, 0)$ as being equal to its limit there. Such a discontinuity is called **removable.** In Example 4, the function

$$f(x, y) = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

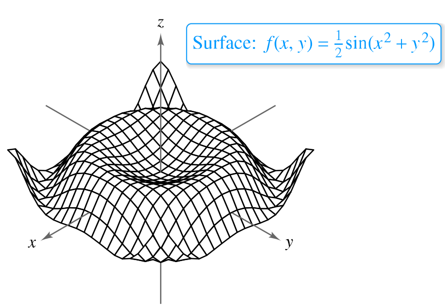
was also shown not to be continuous at $(0, 0)$, but this discontinuity is **nonremovable.**

THEOREM 13.1 Continuous Functions of Two Variables

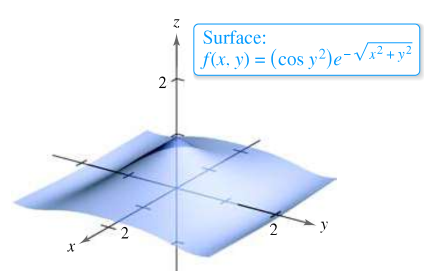
If k is a real number and $f(x, y)$ and $g(x, y)$ are continuous at (x_0, y_0), then the following functions are also continuous at (x_0, y_0).

1. Scalar multiple: kf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $f/g,\ g(x_0, y_0) \neq 0$

Theorem 13.1 establishes the continuity of *polynomial* and *rational* functions at every point in their domains. Furthermore, the continuity of other types of functions can be extended naturally from one to two variables. For instance, the functions whose graphs are shown in Figures 13.23 and 13.24 are continuous at every point in the plane.



The function f is continuous at every point in the plane.
Figure 13.23



The function f is continuous at every point in the plane.
Figure 13.24

The next theorem states conditions under which a composite function is continuous.

THEOREM 13.2 Continuity of a Composite Function

If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the composite function given by $(g \circ h)(x, y) = g(h(x, y))$ is continuous at (x_0, y_0). That is,

$$\lim_{(x, y) \to (x_0, y_0)} g(h(x, y)) = g(h(x_0, y_0)).$$

Note in Theorem 13.2 that h is a function of two variables and g is a function of one variable.

EXAMPLE 5 Testing for Continuity

Discuss the continuity of each function.

a. $f(x, y) = \dfrac{x - 2y}{x^2 + y^2}$ **b.** $g(x, y) = \dfrac{2}{y - x^2}$

Solution

a. Because a rational function is continuous at every point in its domain, you can conclude that f is continuous at each point in the xy-plane except at $(0, 0)$, as shown in Figure 13.25.

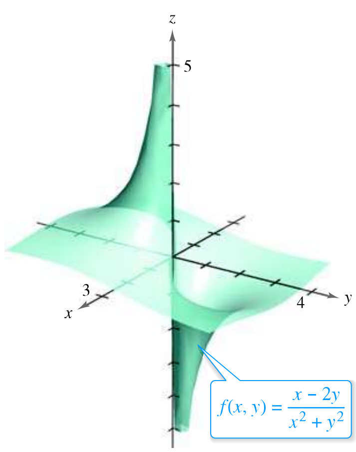
b. The function

$$g(x, y) = \frac{2}{y - x^2}$$

is continuous except at the points at which the denominator is 0. These points are given by the equation

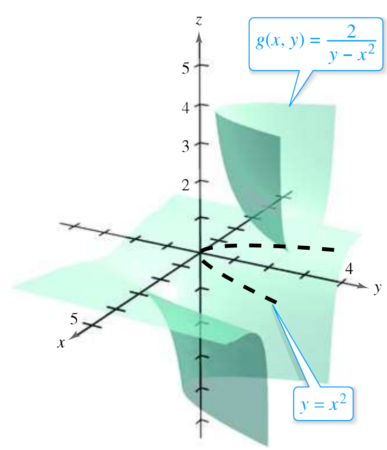
$$y - x^2 = 0.$$

So, you can conclude that the function is continuous at all points except those lying on the parabola $y = x^2$. Inside this parabola, you have $y > x^2$, and the surface represented by the function lies above the xy-plane, as shown in Figure 13.26. Outside the parabola, $y < x^2$, and the surface lies below the xy-plane.



The function f is not continuous at $(0, 0)$.

Figure 13.25



The function g is not continuous on the parabola $y = x^2$.

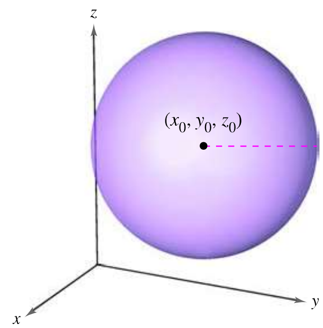
Figure 13.26

Continuity of a Function of Three Variables

The preceding definitions of limits and continuity can be extended to functions of three variables by considering points (x, y, z) within the *open sphere*

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2. \quad \text{Open sphere}$$

The radius of this sphere is δ, and the sphere is centered at (x_0, y_0, z_0), as shown in Figure 13.27.



Open sphere in space
Figure 13.27

A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if there exists a δ-sphere about (x_0, y_0, z_0) that lies entirely in R. If every point in R is an interior point, then R is called **open.**

Definition of Continuity of a Function of Three Variables

A function f of three variables is **continuous at a point (x_0, y_0, z_0)** in an open region R if $f(x_0, y_0, z_0)$ is defined and is equal to the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0). That is,

$$\lim_{(x, y, z) \to (x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

EXAMPLE 6 Testing Continuity of a Function of Three Variables

Discuss the continuity of

$$f(x, y, z) = \frac{1}{x^2 + y^2 - z}.$$

Solution The function f is continuous except at the points at which the denominator is 0, which are given by the equation

$$x^2 + y^2 - z = 0.$$

So, f is continuous at each point in space except at the points on the paraboloid

$$z = x^2 + y^2.$$

13.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. Describing Notation Write a brief description of the meaning of the notation $\lim_{(x, y)\to(-1, 3)} f(x, y) = 1.$

2. Limits Explain how examining limits along different paths might show that a limit does not exist. Does this type of examination show that a limit does exist? Explain.

Verifying a Limit by the Definition **In Exercises 3–6, use the definition of the limit of a function of two variables to verify the limit.**

3. $\lim_{(x, y)\to(1, 0)} x = 1$

4. $\lim_{(x, y)\to(4, -1)} x = 4$

5. $\lim_{(x, y)\to(1, -3)} y = -3$

6. $\lim_{(x, y)\to(a, b)} y = b$

Using Properties of Limits **In Exercises 7–10, find the indicated limit by using the limits**

$$\lim_{(x, y)\to(a, b)} f(x, y) = 4 \quad \text{and} \quad \lim_{(x, y)\to(a, b)} g(x, y) = -5.$$

7. $\lim_{(x, y)\to(a, b)} [f(x, y) - g(x, y)]$

8. $\lim_{(x, y)\to(a, b)} \left[\frac{3f(x, y)}{g(x, y)}\right]$

9. $\lim_{(x, y)\to(a, b)} [f(x, y)g(x, y)]$

10. $\lim_{(x, y)\to(a, b)} \left[\frac{f(x, y) + g(x, y)}{f(x, y)}\right]$

Limit and Continuity **In Exercises 11–24, find the limit and discuss the continuity of the function.**

11. $\lim_{(x, y)\to(3, 1)} (x^2 - 2y)$

12. $\lim_{(x, y)\to(-1, 1)} (x + 4y^2 + 5)$

13. $\lim_{(x, y)\to(1, 2)} e^{xy}$

14. $\lim_{(x, y)\to(2, 4)} \frac{x + y}{x^2 + 1}$

15. $\lim_{(x, y)\to(0, 2)} \frac{x}{y}$

16. $\lim_{(x, y)\to(-1, 2)} \frac{x + y}{x - y}$

17. $\lim_{(x, y)\to(1, 1)} \frac{xy}{x^2 + y^2}$

18. $\lim_{(x, y)\to(1, 1)} \frac{x}{\sqrt{x + y}}$

19. $\lim_{(x, y)\to(\pi/3, 2)} y \cos xy$

20. $\lim_{(x, y)\to(\pi, -4)} \sin \frac{x}{y}$

21. $\lim_{(x, y)\to(0, 1)} \frac{\arcsin xy}{1 - xy}$

22. $\lim_{(x, y)\to(0, 1)} \frac{\arccos(x/y)}{1 + xy}$

23. $\lim_{(x, y, z)\to(1, 3, 4)} \sqrt{x + y + z}$

24. $\lim_{(x, y, z)\to(-2, 1, 0)} xe^{yz}$

Finding a Limit **In Exercises 25–36, find the limit (if it exists). If the limit does not exist, explain why.**

25. $\lim_{(x, y)\to(1, 1)} \frac{xy - 1}{1 + xy}$

26. $\lim_{(x, y)\to(1, -1)} \frac{x^2 y}{1 + xy^2}$

27. $\lim_{(x, y)\to(0, 0)} \frac{1}{x + y}$

28. $\lim_{(x, y)\to(0, 0)} \frac{1}{x^2 y^2}$

29. $\lim_{(x, y)\to(0, 0)} \frac{x - y}{\sqrt{x} - \sqrt{y}}$

30. $\lim_{(x, y)\to(2, 1)} \frac{x - y - 1}{\sqrt{x - y} - 1}$

31. $\lim_{(x, y)\to(0, 0)} \frac{x + y}{x^2 + y}$

32. $\lim_{(x, y)\to(0, 0)} \frac{x}{x^2 - y^2}$

33. $\lim_{(x, y)\to(0, 0)} \frac{x^2}{(x^2 + 1)(y^2 + 1)}$

34. $\lim_{(x, y)\to(0, 0)} \ln(x^2 + y^2)$

35. $\lim_{(x, y, z)\to(0, 0, 0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2}$

36. $\lim_{(x, y, z)\to(0, 0, 0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^2}$

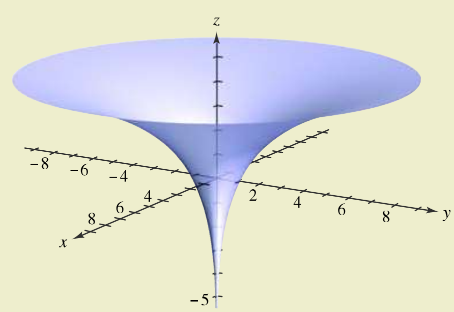
EXPLORING CONCEPTS

37. Limits If $f(2, 3) = 4$, can you conclude anything about $\lim_{(x, y)\to(2, 3)} f(x, y)$? Explain.

38. Limits If $\lim_{(x, y)\to(2, 3)} f(x, y) = 4$, can you conclude anything about $f(2, 3)$? Explain.

39. Think About It Given that $\lim_{(x, y)\to(0, 0)} f(x, y) = 0$, does $\lim_{(x, 0)\to(0, 0)} f(x, 0) = 0$? Explain.

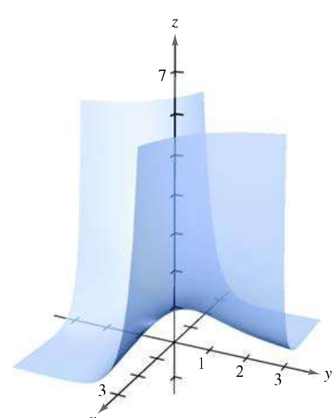
40. HOW DO YOU SEE IT? The figure shows the graph of $f(x, y) = \ln(x^2 + y^2)$. From the graph, does it appear that the limit at each point exists?



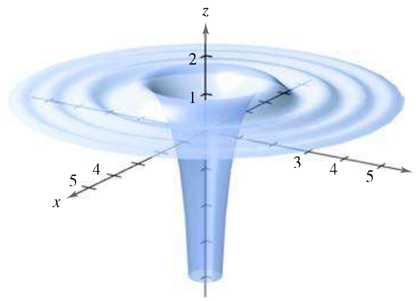
(a) $(-1, -1)$ (b) $(0, 3)$ (c) $(0, 0)$ (d) $(2, 0)$

Continuity **In Exercises 41 and 42, discuss the continuity of the function and evaluate the limit of $f(x, y)$ (if it exists) as $(x, y) \to (0, 0)$.**

41. $f(x, y) = e^{xy}$



42. $f(x, y) = 1 - \dfrac{\cos(x^2 + y^2)}{x^2 + y^2}$



Limit and Continuity **In Exercises 43–46, use a graphing utility to make a table showing the values of $f(x, y)$ at the given points for each path. Use the result to make a conjecture about the limit of $f(x, y)$ as $(x, y) \to (0, 0)$. Determine analytically whether the limit exists and discuss the continuity of the function.**

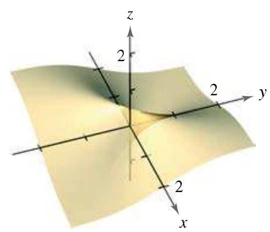
43. $f(x, y) = \dfrac{xy}{x^2 + y^2}$

Path: $y = 0$

Points: (1, 0), (0.5, 0), (0.1, 0), (0.01, 0), (0.001, 0)

Path: $y = x$

Points: (1, 1), (0.5, 0.5), (0.1, 0.1), (0.01, 0.01), (0.001, 0.001)



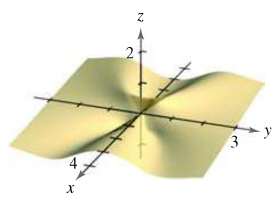
44. $f(x, y) = -\dfrac{xy^2}{x^2 + y^4}$

Path: $x = y^2$

Points: (1, 1), (0.25, 0.5), (0.01, 0.1), (0.0001, 0.01), (0.000001, 0.001)

Path: $x = -y^2$

Points: $(-1, 1)$, $(-0.25, 0.5)$, $(-0.01, 0.1)$, $(-0.0001, 0.01)$, $(-0.000001, 0.001)$



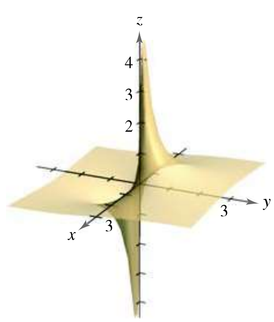
45. $f(x, y) = \dfrac{y}{x^2 + y^2}$

Path: $y = 0$

Points: (1, 0), (0.5, 0), (0.1, 0), (0.01, 0), (0.001, 0)

Path: $y = x$

Points: (1, 1), (0.5, 0.5), (0.1, 0.1), (0.01, 0.01), (0.001, 0.001)



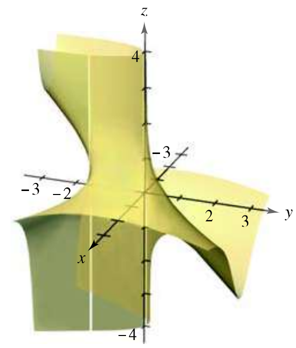
46. $f(x, y) = \dfrac{2x - y^2}{2x^2 + y}$

Path: $y = 0$

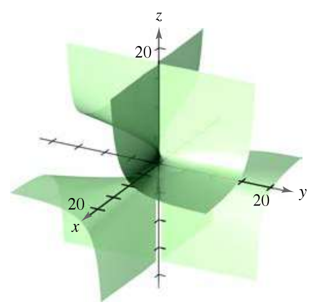
Points: (1, 0), (0.25, 0), (0.01, 0), (0.001, 0), (0.000001, 0)

Path: $y = x$

Points: (1, 1), (0.25, 0.25), (0.01, 0.01), (0.001, 0.001), (0.0001, 0.0001)



47. Limit Consider $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x^2 + y^2}{xy}$ (see figure).

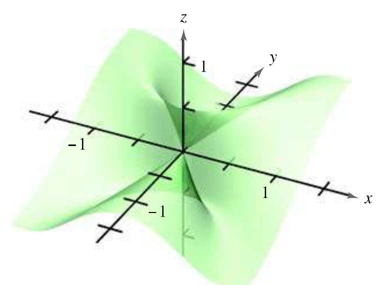


(a) Determine (if possible) the limit along any line of the form $y = ax$.

(b) Determine (if possible) the limit along the parabola $y = x^2$.

(c) Does the limit exist? Explain.

48. Limit Consider $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x^2y}{x^4 + y^2}$ (see figure).



Comparing Continuity **In Exercises 49 and 50, discuss the continuity of the functions f and g. Explain any differences.**

49. $f(x, y) = \begin{cases} \dfrac{x^4 - y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$g(x, y) = \begin{cases} \dfrac{x^4 - y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0) \end{cases}$

50. $f(x, y) = \begin{cases} \dfrac{x^2 + 2xy^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$g(x, y) = \begin{cases} \dfrac{x^2 + 2xy^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0) \end{cases}$

Finding a Limit Using Polar Coordinates **In Exercises 51–56, use polar coordinates to find the limit. [*Hint:* Let $x = r\cos\theta$ and $y = r\sin\theta$, and note that $(x, y) \to (0, 0)$ implies $r \to 0$.]**

51. $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{xy^2}{x^2 + y^2}$

52. $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x^3 + y^3}{x^2 + y^2}$

53. $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x^2y^2}{x^2 + y^2}$

54. $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$

55. $\displaystyle\lim_{(x, y) \to (0, 0)} \cos(x^2 + y^2)$

56. $\displaystyle\lim_{(x, y) \to (0, 0)} \sin\sqrt{x^2 + y^2}$

Finding a Limit Using Polar Coordinates **In Exercises 57–60, use polar coordinates and L'Hôpital's Rule to find the limit.**

57. $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{\sin\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}$

58. $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$

59. $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{1 - \cos(x^2 + y^2)}{x^2 + y^2}$

60. $\displaystyle\lim_{(x, y) \to (0, 0)} (x^2 + y^2)\ln(x^2 + y^2)$

Continuity **In Exercises 61–66, discuss the continuity of the function.**

61. $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

62. $f(x, y, z) = \dfrac{z}{x^2 + y^2 - 4}$

63. $f(x, y, z) = \dfrac{\sin z}{e^x + e^y}$

64. $f(x, y, z) = xy\sin z$

65. $f(x, y) = \begin{cases} \dfrac{\sin xy}{xy}, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

66. $f(x, y) = \begin{cases} \dfrac{\sin(x^2 - y^2)}{x^2 - y^2}, & x^2 \neq y^2 \\ 1, & x^2 = y^2 \end{cases}$

Continuity of a Composite Function **In Exercises 67–70, discuss the continuity of the composite function $f \circ g$.**

67. $f(t) = t^2$

$g(x, y) = 2x - 3y$

68. $f(t) = \dfrac{1}{t}$

$g(x, y) = x^2 + y^2$

69. $f(t) = \dfrac{1}{t}$

$g(x, y) = 2x - 3y$

70. $f(t) = \dfrac{1}{1 - t}$

$g(x, y) = x^2 + y^2$

Finding a Limit **In Exercises 71–76, find each limit.**

(a) $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$

(b) $\displaystyle\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

71. $f(x, y) = x^2 - 4y$

72. $f(x, y) = 3x^2 + y^2$

73. $f(x, y) = \dfrac{x}{y}$

74. $f(x, y) = \dfrac{1}{x + y}$

75. $f(x, y) = 3x + xy - 2y$

76. $f(x, y) = \sqrt{y}(y + 1)$

Finding a Limit Using Spherical Coordinates **In Exercises 77 and 78, use spherical coordinates to find the limit. [*Hint:* Let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, and note that $(x, y, z) \to (0, 0, 0)$ implies $\rho \to 0^+$.]**

77. $\displaystyle\lim_{(x, y, z) \to (0, 0, 0)} \frac{xyz}{x^2 + y^2 + z^2}$

78. $\displaystyle\lim_{(x, y, z) \to (0, 0, 0)} \tan^{-1}\left(\frac{1}{x^2 + y^2 + z^2}\right)$

True or False? **In Exercises 79–82, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

79. A closed region contains all of its boundary points.

80. Every point in an open region is an interior point.

81. If f is continuous for all nonzero x and y, and $f(0, 0) = 0$, then $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = 0$.

82. If g is a continuous function of x, h is a continuous function of y, and $f(x, y) = g(x) + h(y)$, then f is continuous.

83. **Finding a Limit** Find the following limit.

$$\lim_{(x, y) \to (0, 1)} \tan^{-1}\left[\frac{x^2 + 1}{x^2 + (y - 1)^2}\right]$$

84. **Continuity** For the function

$$f(x, y) = xy\left(\frac{x^2 - y^2}{x^2 + y^2}\right)$$

define $f(0, 0)$ such that f is continuous at the origin.

85. **Proof** Prove that

$$\lim_{(x, y) \to (a, b)} [f(x, y) + g(x, y)] = L_1 + L_2$$

where $f(x, y)$ approaches L_1 and $g(x, y)$ approaches L_2 as $(x, y) \to (a, b)$.

86. **Proof** Prove that if f is continuous and $f(a, b) < 0$, then there exists a δ-neighborhood about (a, b) such that $f(x, y) < 0$ for every point (x, y) in the neighborhood.

13.3 Partial Derivatives

- **Find and use partial derivatives of a function of two variables.**
- **Find and use partial derivatives of a function of three or more variables.**
- **Find higher-order partial derivatives of a function of two or three variables.**

Partial Derivatives of a Function of Two Variables

JEAN LE ROND D'ALEMBERT (1717–1783)

The introduction of partial derivatives followed Newton's and Leibniz's work in calculus by several years. Between 1730 and 1760, Leonhard Euler and Jean Le Rond d'Alembert separately published several papers on dynamics, in which they established much of the theory of partial derivatives. These papers used functions of two or more variables to study problems involving equilibrium, fluid motion, and vibrating strings.

See LarsonCalculus.com to read more of this biography.

In applications of functions of several variables, the question often arises, "How will the value of a function be affected by a change in one of its independent variables?" You can answer this by considering the independent variables one at a time. For example, to determine the effect of a catalyst in an experiment, a chemist could conduct the experiment several times using varying amounts of the catalyst while keeping constant other variables such as temperature and pressure. You can use a similar procedure to determine the rate of change of a function f with respect to one of its several independent variables. This process is called **partial differentiation,** and the result is referred to as the **partial derivative** of f with respect to the chosen independent variable.

Definition of Partial Derivatives of a Function of Two Variables

If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Partial derivative with respect to x

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Partial derivative with respect to y

provided the limits exist.

This definition indicates that if $z = f(x, y)$, then to find f_x, you *consider y constant* and differentiate with respect to x. Similarly, to find f_y, you *consider x constant* and differentiate with respect to y.

EXAMPLE 1 **Finding Partial Derivatives**

a. To find f_x for $f(x, y) = 3x - x^2y^2 + 2x^3y$, consider y to be constant and differentiate with respect to x.

$$f_x(x, y) = 3 - 2xy^2 + 6x^2y$$

Partial derivative with respect to x

To find f_y, consider x to be constant and differentiate with respect to y.

$$f_y(x, y) = -2x^2y + 2x^3$$

Partial derivative with respect to y

b. To find f_x for $f(x, y) = (\ln x)(\sin x^2y)$, consider y to be constant and differentiate with respect to x.

$$f_x(x, y) = (\ln x)(\cos x^2y)(2xy) + \frac{\sin x^2y}{x}$$

Partial derivative with respect to x

To find f_y, consider x to be constant and differentiate with respect to y.

$$f_y(x, y) = (\ln x)(\cos x^2y)(x^2)$$

Partial derivative with respect to y

REMARK The notation $\partial z/\partial x$ is read as "the partial derivative of z with respect to x," and $\partial z/\partial y$ is read as "the partial derivative of z with respect to y."

Notation for First Partial Derivatives

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$ Partial derivative with respect to x

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$ Partial derivative with respect to y

The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a,\, b)} = f_x(a, b)$$

and

$$\left.\frac{\partial z}{\partial y}\right|_{(a,\, b)} = f_y(a, b).$$

EXAMPLE 2 Finding and Evaluating Partial Derivatives

For $f(x, y) = xe^{x^2 y}$, find f_x and f_y, and evaluate each at the point $(1, \ln 2)$.

Solution Because

$$f_x(x, y) = xe^{x^2 y}(2xy) + e^{x^2 y}$$ Partial derivative with respect to x

the partial derivative of f with respect to x at $(1, \ln 2)$ is

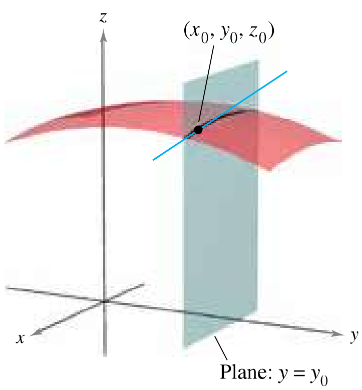
$$\begin{aligned} f_x(1, \ln 2) &= e^{\ln 2}(2 \ln 2) + e^{\ln 2} \\ &= 4 \ln 2 + 2. \end{aligned}$$

Because

$$\begin{aligned} f_y(x, y) &= xe^{x^2 y}(x^2) \\ &= x^3 e^{x^2 y} \end{aligned}$$ Partial derivative with respect to y

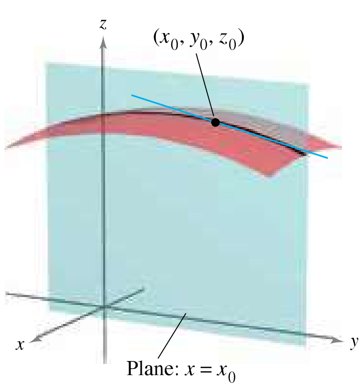
the partial derivative of f with respect to y at $(1, \ln 2)$ is

$$\begin{aligned} f_y(1, \ln 2) &= e^{\ln 2} \\ &= 2. \end{aligned}$$



$\dfrac{\partial f}{\partial x}$ = slope in x-direction

Figure 13.28



$\dfrac{\partial f}{\partial y}$ = slope in y-direction

Figure 13.29

The partial derivatives of a function of two variables, $z = f(x, y)$, have a useful geometric interpretation. If $y = y_0$, then $z = f(x, y_0)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$, as shown in Figure 13.28. Therefore,

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

represents the slope of this curve at the point $(x_0, y_0, f(x_0, y_0))$. Note that both the curve and the tangent line lie in the plane $y = y_0$. Similarly,

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

represents the slope of the curve given by the intersection of $z = f(x, y)$ and the plane $x = x_0$ at $(x_0, y_0, f(x_0, y_0))$, as shown in Figure 13.29.

Informally, the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point (x_0, y_0, z_0) denote the **slopes of the surface in the x- and y-directions,** respectively.

EXAMPLE 3 Finding the Slopes of a Surface

See LarsonCalculus.com for an interactive version of this type of example.

Find the slopes in the x-direction and in the y-direction of the surface

$$f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$$

at the point $\left(\frac{1}{2}, 1, 2\right)$.

Solution The partial derivatives of f with respect to x and y are

$$f_x(x, y) = -x \quad \text{and} \quad f_y(x, y) = -2y. \qquad \text{Partial derivatives}$$

So, in the x-direction, the slope is

$$f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2} \qquad \text{Figure 13.30}$$

and in the y-direction, the slope is

$$f_y\left(\frac{1}{2}, 1\right) = -2. \qquad \text{Figure 13.31}$$

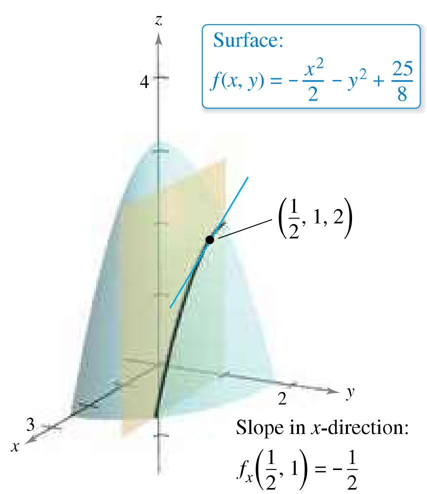


Figure 13.30

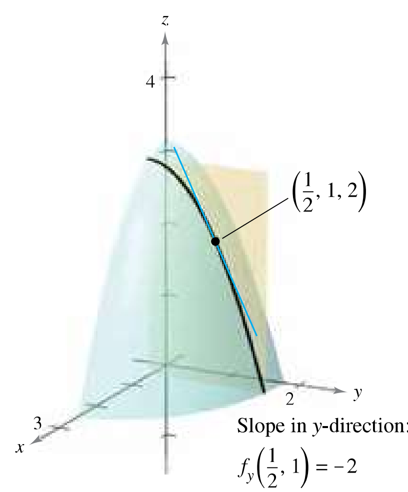


Figure 13.31

EXAMPLE 4 Finding the Slopes of a Surface

Find the slopes of the surface

$$f(x, y) = 1 - (x - 1)^2 - (y - 2)^2$$

at the point $(1, 2, 1)$ in the x-direction and in the y-direction.

Solution The partial derivatives of f with respect to x and y are

$$f_x(x, y) = -2(x - 1) \quad \text{and} \quad f_y(x, y) = -2(y - 2). \qquad \text{Partial derivatives}$$

So, at the point $(1, 2, 1)$, the slope in the x-direction is

$$f_x(1, 2) = -2(1 - 1) = 0$$

and the slope in the y-direction is

$$f_y(1, 2) = -2(2 - 2) = 0$$

as shown in Figure 13.32.

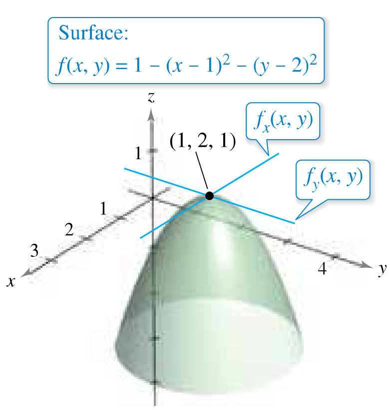
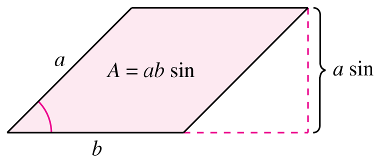


Figure 13.32

No matter how many variables are involved, partial derivatives can be interpreted as *rates of change.*

EXAMPLE 5 Using Partial Derivatives to Find Rates of Change

The area of a parallelogram with adjacent sides a and b and included angle θ is given by $A = ab \sin \theta$, as shown in Figure 13.33.



The area of the parallelogram is $ab \sin \theta$.
Figure 13.33

a. Find the rate of change of A with respect to a for $a = 10$, $b = 20$, and $\theta = \pi/6$.

b. Find the rate of change of A with respect to θ for $a = 10$, $b = 20$, and $\theta = \pi/6$.

Solution

a. To find the rate of change of the area with respect to a, hold b and θ constant and differentiate with respect to a to obtain

$$\frac{\partial A}{\partial a} = b \sin \theta. \qquad \text{Find partial derivative with respect to } a.$$

For $a = 10$, $b = 20$, and $\theta = \pi/6$, the rate of change of the area with respect to a is

$$\frac{\partial A}{\partial a} = 20 \sin \frac{\pi}{6} = 10. \qquad \text{Substitute for } b \text{ and } \theta.$$

b. To find the rate of change of the area with respect to θ, hold a and b constant and differentiate with respect to θ to obtain

$$\frac{\partial A}{\partial \theta} = ab \cos \theta. \qquad \text{Find partial derivative with respect to } \theta.$$

For $a = 10$, $b = 20$, and $\theta = \pi/6$, the rate of change of the area with respect to θ is

$$\frac{\partial A}{\partial \theta} = 200 \cos \frac{\pi}{6} = 100\sqrt{3}. \qquad \text{Substitute for } a, b, \text{ and } \theta.$$

Partial Derivatives of a Function of Three or More Variables

The concept of a partial derivative can be extended naturally to functions of three or more variables. For instance, if $w = f(x, y, z)$, then there are three partial derivatives, each of which is formed by holding two of the variables constant. That is, to define the partial derivative of w with respect to x, consider y and z to be constant and differentiate with respect to x. A similar process is used to find the derivatives of w with respect to y and with respect to z.

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

In general, if $w = f(x_1, x_2, \ldots, x_n)$, then there are n partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n.$$

To find the partial derivative with respect to one of the variables, hold the other variables constant and differentiate with respect to the given variable.

EXAMPLE 6 **Finding Partial Derivatives**

a. To find the partial derivative of $f(x, y, z) = xy + yz^2 + xz$ with respect to z, consider x and y to be constant and obtain

$$\frac{\partial}{\partial z}[xy + yz^2 + xz] = 2yz + x.$$

b. To find the partial derivative of $f(x, y, z) = z \sin(xy^2 + 2z)$ with respect to z, consider x and y to be constant. Then, using the Product Rule, you obtain

$$\begin{aligned}\frac{\partial}{\partial z}[z \sin(xy^2 + 2z)] &= (z)\frac{\partial}{\partial z}[\sin(xy^2 + 2z)] + \sin(xy^2 + 2z)\frac{\partial}{\partial z}[z]\\ &= (z)[\cos(xy^2 + 2z)](2) + \sin(xy^2 + 2z)\\ &= 2z\cos(xy^2 + 2z) + \sin(xy^2 + 2z).\end{aligned}$$

c. To find the partial derivative of

$$f(x, y, z, w) = \frac{x + y + z}{w}$$

with respect to w, consider x, y, and z to be constant and obtain

$$\frac{\partial}{\partial w}\left[\frac{x + y + z}{w}\right] = -\frac{x + y + z}{w^2}.$$

Higher-Order Partial Derivatives

As is true for ordinary derivatives, it is possible to take second, third, and higher-order partial derivatives of a function of several variables, provided such derivatives exist. Higher-order derivatives are denoted by the order in which the differentiation occurs. For instance, the function $z = f(x, y)$ has the following second partial derivatives.

1. Differentiate twice with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

2. Differentiate twice with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

3. Differentiate first with respect to x and then with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$

4. Differentiate first with respect to y and then with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

The third and fourth cases are called **mixed partial derivatives.**

REMARK Note that the two types of notation for mixed partials have different conventions for indicating the order of differentiation.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \quad \text{Right-to-left order}$$

$$(f_x)_y = f_{xy} \quad \text{Left-to-right order}$$

You can remember the order by observing that in both notations you differentiate first with respect to the variable "nearest" f.

EXAMPLE 7 Finding Second Partial Derivatives

Find the second partial derivatives of

$$f(x, y) = 3xy^2 - 2y + 5x^2y^2$$

and determine the value of $f_{xy}(-1, 2)$.

Solution Begin by finding the first partial derivatives with respect to x and y.

$$f_x(x, y) = 3y^2 + 10xy^2 \quad \text{and} \quad f_y(x, y) = 6xy - 2 + 10x^2y$$

Then, differentiate each of these with respect to x and y.

$$f_{xx}(x, y) = 10y^2 \quad \text{and} \quad f_{yy}(x, y) = 6x + 10x^2$$
$$f_{xy}(x, y) = 6y + 20xy \quad \text{and} \quad f_{yx}(x, y) = 6y + 20xy$$

At $(-1, 2)$, the value of f_{xy} is

$$f_{xy}(-1, 2) = 12 - 40 = -28.$$

Notice in Example 7 that the two mixed partials are equal. Sufficient conditions for this occurrence are given in Theorem 13.3.

> **THEOREM 13.3 Equality of Mixed Partial Derivatives**
>
> If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then, for every (x, y) in R,
>
> $$f_{xy}(x, y) = f_{yx}(x, y).$$

Theorem 13.3 also applies to a function f of *three or more variables* as long as all second partial derivatives are continuous. For example, if

$$w = f(x, y, z) \qquad \text{Function of three variables}$$

and all the second partial derivatives are continuous in an open region R, then at each point in R, the order of differentiation in the mixed second partial derivatives is irrelevant. If the third partial derivatives of f are also continuous, then the order of differentiation of the mixed third partial derivatives is irrelevant.

EXAMPLE 8 Finding Higher-Order Partial Derivatives

Show that $f_{xz} = f_{zx}$ and $f_{xzz} = f_{zxz} = f_{zzx}$ for the function

$$f(x, y, z) = ye^x + x \ln z.$$

Solution

First partials:

$$f_x(x, y, z) = ye^x + \ln z, \quad f_z(x, y, z) = \frac{x}{z}$$

Second partials (note that the first two are equal):

$$f_{xz}(x, y, z) = \frac{1}{z}, \quad f_{zx}(x, y, z) = \frac{1}{z}, \quad f_{zz}(x, y, z) = -\frac{x}{z^2}$$

Third partials (note that all three are equal):

$$f_{xzz}(x, y, z) = -\frac{1}{z^2}, \quad f_{zxz}(x, y, z) = -\frac{1}{z^2}, \quad f_{zzx}(x, y, z) = -\frac{1}{z^2}$$

13.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **First Partial Derivatives** List three ways of writing the first partial derivative with respect to x of $z = f(x, y)$.
2. **First Partial Derivatives** Sketch a surface representing a function f of two variables x and y. Use the sketch to give geometric interpretations of $\partial f/\partial x$ and $\partial f/\partial y$.
3. **Higher-Order Partial Derivatives** Describe the order in which the differentiation of $f(x, y, z)$ occurs for (a) f_{yxz} and (b) $\partial^2 f/\partial x \partial z$.
4. **Mixed Partial Derivatives** If f is a function of x and y such that f_{xy} and f_{yx} are continuous, what is the relationship between the mixed partial derivatives?

Examining a Partial Derivative **In Exercises 5–10, explain whether the Quotient Rule should be used to find the partial derivative. Do not differentiate.**

5. $\dfrac{\partial}{\partial x}\left(\dfrac{x^2y}{y^2-3}\right)$
6. $\dfrac{\partial}{\partial y}\left(\dfrac{x^2y}{y^2-3}\right)$
7. $\dfrac{\partial}{\partial y}\left(\dfrac{x-y}{x^2+1}\right)$
8. $\dfrac{\partial}{\partial x}\left(\dfrac{x-y}{x^2+1}\right)$
9. $\dfrac{\partial}{\partial x}\left(\dfrac{xy}{x^2+1}\right)$
10. $\dfrac{\partial}{\partial y}\left(\dfrac{xy}{x^2+1}\right)$

Finding Partial Derivatives **In Exercises 11–40, find both first partial derivatives.**

11. $f(x, y) = 2x - 5y + 3$
12. $f(x, y) = x^2 - 2y^2 + 4$
13. $z = 6x - x^2y + 8y^2$
14. $f(x, y) = 4x^3y^{-2}$
15. $z = x\sqrt{y}$
16. $z = 2y^2\sqrt{x}$
17. $z = e^{xy}$
18. $z = e^{x/y}$
19. $z = x^2e^{2y}$
20. $z = 7ye^{y/x}$
21. $z = \ln \dfrac{x}{y}$
22. $z = \ln\sqrt{xy}$
23. $z = \ln(x^2 + y^2)$
24. $z = \ln \dfrac{x+y}{x-y}$
25. $z = \dfrac{x^2}{2y} + \dfrac{3y^2}{x}$
26. $z = \dfrac{xy}{x^2 + y^2}$
27. $h(x, y) = e^{-(x^2+y^2)}$
28. $g(x, y) = \ln\sqrt{x^2 + y^2}$
29. $f(x, y) = \sqrt{x^2 + y^2}$
30. $f(x, y) = \sqrt{2x + y^3}$
31. $z = \cos xy$
32. $z = \sin(x + 2y)$
33. $z = \tan(2x - y)$
34. $z = \sin 5x \cos 5y$
35. $z = e^y \sin 8xy$
36. $z = \cos(x^2 + y^2)$
37. $z = \sinh(2x + 3y)$
38. $z = \cosh xy^2$
39. $f(x, y) = \displaystyle\int_x^y (t^2 - 1)\, dt$
40. $f(x, y) = \displaystyle\int_x^y (2t + 1)\, dt + \int_y^x (2t - 1)\, dt$

Finding Partial Derivatives **In Exercises 41–44, use the limit definition of partial derivatives to find $f_x(x, y)$ and $f_y(x, y)$.**

41. $f(x, y) = 3x + 2y$
42. $f(x, y) = x^2 - 2xy + y^2$
43. $f(x, y) = \sqrt{x + y}$
44. $f(x, y) = \dfrac{1}{x + y}$

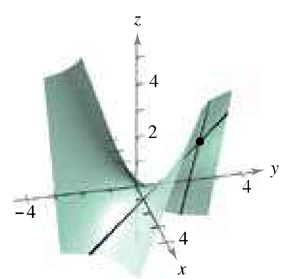
Finding and Evaluating Partial Derivatives **In Exercises 45–52, find f_x and f_y, and evaluate each at the given point.**

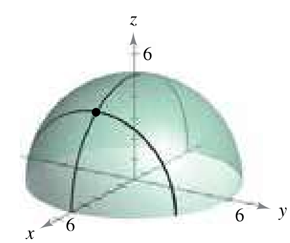
45. $f(x, y) = e^xy^2, \ (\ln 3, 2)$
46. $f(x, y) = x^3 \ln 5y, \ (1, 1)$
47. $f(x, y) = \cos(2x - y), \ \left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$
48. $f(x, y) = \sin xy, \ \left(2, \dfrac{\pi}{4}\right)$
49. $f(x, y) = \arctan \dfrac{y}{x}, \ (2, -2)$
50. $f(x, y) = \arccos xy, \ (1, 1)$
51. $f(x, y) = \dfrac{xy}{x - y}, \ (2, -2)$
52. $f(x, y) = \dfrac{2xy}{\sqrt{4x^2 + 5y^2}}, \ (1, 1)$

Finding the Slopes of a Surface **In Exercises 53–56, find the slopes of the surface in the x- and y-directions at the given point.**

53. $z = xy$
$(1, 2, 2)$

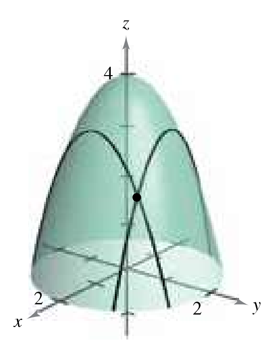
54. $z = \sqrt{25 - x^2 - y^2}$
$(3, 0, 4)$

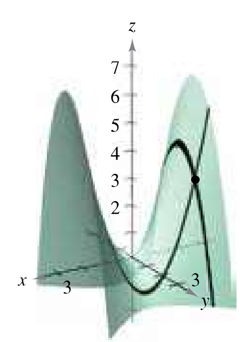




55. $g(x, y) = 4 - x^2 - y^2$
$(1, 1, 2)$

56. $h(x, y) = x^2 - y^2$
$(-2, 1, 3)$





Finding Partial Derivatives In Exercises 57–62, find the first partial derivatives with respect to x, y, and z.

57. $H(x, y, z) = \sin(x + 2y + 3z)$

58. $f(x, y, z) = 3x^2y - 5xyz + 10yz^2$

59. $w = \sqrt{x^2 + y^2 + z^2}$

60. $w = \dfrac{7xz}{x + y}$

61. $F(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$

62. $G(x, y, z) = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

Finding and Evaluating Partial Derivatives In Exercises 63–68, find f_x, f_y, and f_z, and evaluate each at the given point.

63. $f(x, y, z) = x^3yz^2, \ (1, 1, 1)$

64. $f(x, y, z) = x^2y^3 + 2xyz - 3yz, \ (-2, 1, 2)$

65. $f(x, y, z) = \dfrac{\ln x}{yz}, \ (1, -1, -1)$

66. $f(x, y, z) = \dfrac{xy}{x + y + z}, \ (3, 1, -1)$

67. $f(x, y, z) = z\sin(x + 6y), \ \left(0, \dfrac{\pi}{2}, -4\right)$

68. $f(x, y, z) = \sqrt{3x^2 + y^2 - 2z^2}, \ (1, -2, 1)$

Using First Partial Derivatives In Exercises 69–76, find all values of x and y such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously.

69. $f(x, y) = x^2 + xy + y^2 - 2x + 2y$

70. $f(x, y) = x^2 - xy + y^2 - 5x + y$

71. $f(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$

72. $f(x, y) = x^2 - xy + y^2$

73. $f(x, y) = \dfrac{1}{x} + \dfrac{1}{y} + xy$

74. $f(x, y) = 3x^3 - 12xy + y^3$

75. $f(x, y) = e^{x^2 + xy + y^2}$

76. $f(x, y) = \ln(x^2 + y^2 + 1)$

Finding Second Partial Derivatives In Exercises 77–86, find the four second partial derivatives. Observe that the second mixed partials are equal.

77. $z = 3xy^2$

78. $z = x^2 + 3y^2$

79. $z = x^4 - 2xy + 3y^3$

80. $z = x^4 - 3x^2y^2 + y^4$

81. $z = \sqrt{x^2 + y^2}$

82. $z = \ln(x - y)$

83. $z = e^x \tan y$

84. $z = 2xe^y - 3ye^{-x}$

85. $z = \cos xy$

86. $z = \arctan\dfrac{y}{x}$

Finding Partial Derivatives Using Technology In Exercises 87–90, use a computer algebra system to find the first and second partial derivatives of the function. Determine whether there exist values of x and y such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously.

87. $f(x, y) = x\sec y$

88. $f(x, y) = \sqrt{25 - x^2 - y^2}$

89. $f(x, y) = \ln\dfrac{x}{x^2 + y^2}$

90. $f(x, y) = \dfrac{xy}{x - y}$

Finding Higher-Order Partial Derivatives In Exercises 91–94, show that the mixed partial derivatives f_{xyy}, f_{yxy}, and f_{yyx} are equal.

91. $f(x, y, z) = xyz$

92. $f(x, y, z) = x^2 - 3xy + 4yz + z^3$

93. $f(x, y, z) = e^{-x}\sin yz$

94. $f(x, y, z) = \dfrac{2z}{x + y}$

Laplace's Equation In Exercises 95–98, show that the function satisfies Laplace's equation $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$.

95. $z = 5xy$

96. $z = \frac{1}{2}(e^y - e^{-y})\sin x$

97. $z = e^x \sin y$

98. $z = \arctan\dfrac{y}{x}$

Wave Equation In Exercises 99–102, show that the function satisfies the wave equation $\partial^2 z/\partial t^2 = c^2(\partial^2 z/\partial x^2)$.

99. $z = \sin(x - ct)$

100. $z = \cos(4x + 4ct)$

101. $z = \ln(x + ct)$

102. $z = \sin \omega ct \sin \omega x$

Heat Equation In Exercises 103 and 104, show that the function satisfies the heat equation $\partial z/\partial t = c^2(\partial^2 z/\partial x^2)$.

103. $z = e^{-t}\cos\dfrac{x}{c}$

104. $z = e^{-t}\sin\dfrac{x}{c}$

Cauchy-Riemann Equations In Exercises 105 and 106, show that the functions u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

105. $u = x^2 - y^2, \ v = 2xy$

106. $u = e^x\cos y, \ v = e^x \sin y$

Using First Partial Derivatives In Exercises 107 and 108, determine whether there exists a function $f(x, y)$ with the given partial derivatives. Explain your reasoning. If such a function exists, give an example.

107. $f_x(x, y) = -3\sin(3x - 2y), \ f_y(x, y) = 2\sin(3x - 2y)$

108. $f_x(x, y) = 2x + y, \ f_y(x, y) = x - 4y$

EXPLORING CONCEPTS

109. Think About It Consider $z = f(x, y)$ such that $z_x = z_y$. Does $z = c(x + y)$? Explain.

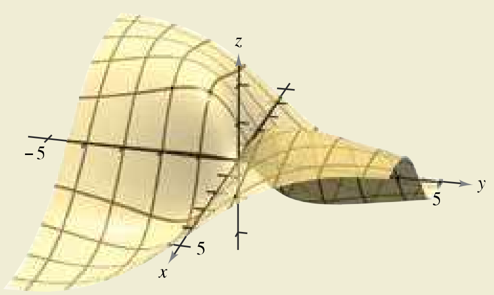
110. First Partial Derivatives Given $z = f(x)g(y)$, find $z_x + z_y$.

111. Sketching a Graph Sketch the graph of a function $z = f(x, y)$ whose derivative f_x is always negative and whose derivative f_y is always positive.

112. Sketching a Graph Sketch the graph of a function $z = f(x, y)$ whose derivatives f_x and f_y are always positive.

113. Think About It The price P (in dollars) of a used car is a function of its initial cost C (in dollars) and its age A (in years). What are the units of $\partial P/\partial A$? Is $\partial P/\partial A$ positive or negative? Explain.

114. HOW DO YOU SEE IT? Use the graph of the surface to determine the sign of each partial derivative. Explain your reasoning.



(a) $f_x(4, 1)$ (b) $f_y(4, 1)$

(c) $f_x(-1, -2)$ (d) $f_y(-1, -2)$

115. Area The area of a triangle is represented by $A = \frac{1}{2}ab \sin \theta$, where a and b are two of the side lengths and θ is the angle between a and b.

(a) Find the rate of change of A with respect to b for $a = 4$, $b = 1$, and $\theta = \pi/4$.

(b) Find the rate of change of A with respect to θ for $a = 2$, $b = 5$, and $\theta = \pi/3$.

116. Volume The volume of a right-circular cone of radius r and height h is represented by $V = \frac{1}{3}\pi r^2 h$.

(a) Find the rate of change of V with respect to r for $r = 2$ and $h = 2$.

(b) Find the rate of change of V with respect to h for $r = 2$ and $h = 2$.

117. Marginal Revenue A pharmaceutical corporation has two plants that produce the same over-the-counter medicine. If x_1 and x_2 are the numbers of units produced at plant 1 and plant 2, respectively, then the total revenue for the product is given by $R = 200x_1 + 200x_2 - 4x_1^2 - 8x_1x_2 - 4x_2^2$. When $x_1 = 4$ and $x_2 = 12$, find (a) the marginal revenue for plant 1, $\partial R/\partial x_1$, and (b) the marginal revenue for plant 2, $\partial R/\partial x_2$.

118. Marginal Costs

A company manufactures two types of wood-burning stoves: a freestanding model and a fireplace-insert model. The cost function for producing x freestanding and y fireplace-insert stoves is

$$C = 32\sqrt{xy} + 175x + 205y + 1050.$$

(a) Find the marginal costs ($\partial C/\partial x$ and $\partial C/\partial y$) when $x = 80$ and $y = 20$.

(b) When additional production is required, which model of stove results in the cost increasing at a higher rate? How can this be determined from the cost model?

119. Psychology Early in the twentieth century, an intelligence test called the *Stanford-Binet Test* (more commonly known as the IQ test) was developed. In this test, an individual's mental age M is divided by the individual's chronological age C and then the quotient is multiplied by 100. The result is the individual's IQ.

$$IQ(M, C) = \frac{M}{C} \times 100$$

Find the partial derivatives of IQ with respect to M and with respect to C. Evaluate the partial derivatives at the point $(12, 10)$ and interpret the result. *(Source: Adapted from Bernstein/Clark-Stewart/Roy/Wickens, Psychology, Fourth Edition)*

120. Marginal Productivity Consider the Cobb-Douglas production function $f(x, y) = 200x^{0.7}y^{0.3}$. When $x = 1000$ and $y = 500$, find (a) the marginal productivity of labor, $\partial f/\partial x$, and (b) the marginal productivity of capital, $\partial f/\partial y$.

121. Think About It Let N be the number of applicants to a university, p the charge for food and housing at the university, and t the tuition. Suppose that N is a function of p and t such that $\partial N/\partial p < 0$ and $\partial N/\partial t < 0$. What information is gained by noticing that both partials are negative?

122. Investment The value of an investment of \$1000 earning 6% compounded annually is

$$V(I, R) = 1000\left[\frac{1 + 0.06(1 - R)}{1 + I}\right]^{10}$$

where I is the annual rate of inflation and R is the tax rate for the person making the investment. Calculate $V_I(0.03, 0.28)$ and $V_R(0.03, 0.28)$. Determine whether the tax rate or the rate of inflation is the greater "negative" factor in the growth of the investment.

123. Temperature Distribution The temperature at any point (x, y) on a steel plate is $T = 500 - 0.6x^2 - 1.5y^2$, where x and y are measured in meters. At the point $(2, 3)$, find the rates of change of the temperature with respect to the distances moved along the plate in the directions of the x- and y-axes.

124. Apparent Temperature A measure of how hot weather feels to an average person is the Apparent Temperature Index. A model for this index is

$$A = 0.885t - 22.4h + 1.20th - 0.544$$

where A is the apparent temperature in degrees Celsius, t is the air temperature, and h is the relative humidity in decimal form. *(Source: The UMAP Journal)*

(a) Find $\dfrac{\partial A}{\partial t}$ and $\dfrac{\partial A}{\partial h}$ when $t = 30°$ and $h = 0.80$.

(b) Which has a greater effect on A, air temperature or humidity? Explain.

125. Ideal Gas Law The Ideal Gas Law states that

$$PV = nRT$$

where P is pressure, V is volume, n is the number of moles of gas, R is a fixed constant (the gas constant), and T is absolute temperature. Show that

$$\frac{\partial T}{\partial P} \cdot \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} = -1.$$

126. Marginal Utility The utility function $U = f(x, y)$ is a measure of the utility (or satisfaction) derived by a person from the consumption of two products x and y. The utility function for two products is

$$U = -5x^2 + xy - 3y^2.$$

(a) Determine the marginal utility of product x.

(b) Determine the marginal utility of product y.

(c) When $x = 2$ and $y = 3$, should a person consume one more unit of product x or one more unit of product y? Explain your reasoning.

(d) Use a computer algebra system to graph the function. Interpret the marginal utilities of products x and y graphically.

127. Modeling Data Personal consumption expenditures (in billions of dollars) for several types of recreation from 2009 through 2014 are shown in the table, where x is the expenditures on amusement parks and campgrounds, y is the expenditures on live entertainment (excluding sports), and z is the expenditures on spectator sports. *(Source: U.S. Bureau of Economic Analysis)*

Year	2009	2010	2011	2012	2013	2014
x	37.2	38.8	41.3	44.6	47.0	50.3
y	25.2	26.3	28.3	28.5	28.0	30.0
z	18.8	19.2	20.4	20.6	21.6	22.4

A model for the data is given by

$$z = 0.23x + 0.14y + 6.85.$$

(a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

(b) Interpret the partial derivatives in the context of the problem.

128. Modeling Data The table shows the national health expenditures (in billions of dollars) for the Department of Veterans Affairs x, workers' compensation y, and Medicaid z from 2009 through 2014. *(Source: Centers for Medicare and Medicaid Services)*

Year	2009	2010	2011	2012	2013	2014
x	42.5	45.7	48.2	49.8	52.8	57.2
y	36.0	36.1	39.1	41.7	44.1	47.3
z	374.5	397.2	406.4	422.0	446.7	495.8

A model for the data is given by

$$z = -0.120x^2 + 0.657y^2 + 17.70x - 51.53y + 842.5.$$

(a) Find $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$.

(b) Determine the concavity of traces parallel to the xz-plane. Interpret the result in the context of the problem.

(c) Determine the concavity of traces parallel to the yz-plane. Interpret the result in the context of the problem.

129. Using a Function Consider the function defined by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}.$$

(a) Find $f_x(x, y)$ and $f_y(x, y)$ for $(x, y) \neq (0, 0)$.

(b) Use the definition of partial derivatives to find $f_x(0, 0)$ and $f_y(0, 0)$.

$$\left[\textit{Hint}\text{: } f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} \right]$$

(c) Use the definition of partial derivatives to find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

(d) Using Theorem 13.3 and the result of part (c), what can be said about f_{xy} or f_{yx}?

130. Using a Function Consider the function

$$f(x, y) = (x^3 + y^3)^{1/3}.$$

(a) Find $f_x(0, 0)$ and $f_y(0, 0)$.

(b) Determine the points (if any) at which $f_x(x, y)$ or $f_y(x, y)$ fails to exist.

131. Using a Function Consider the function

$$f(x, y) = (x^2 + y^2)^{2/3}.$$

Show that

$$f_x(x, y) = \begin{cases} \dfrac{4x}{3(x^2 + y^2)^{1/3}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}.$$

FOR FURTHER INFORMATION For more information about this problem, see the article "A Classroom Note on a Naturally Occurring Piecewise Defined Function" by Don Cohen in *Mathematics and Computer Education.*

13.4 Differentials

- **Understand the concepts of increments and differentials.**
- **Extend the concept of differentiability to a function of two variables.**
- **Use a differential as an approximation.**

Increments and Differentials

In this section, the concepts of increments and differentials are generalized to functions of two or more variables. Recall from Section 3.9 that for $y = f(x)$, the differential of y was defined as

$$dy = f'(x)\,dx.$$

Similar terminology is used for a function of two variables, $z = f(x, y)$. That is, Δx and Δy are the **increments of x and y,** and the **increment of z** is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

Increment of z

Definition of Total Differential

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the **differentials** of the independent variables x and y are

$$dx = \Delta x \quad \text{and} \quad dy = \Delta y$$

and the **total differential** of the dependent variable z is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x, y)\,dx + f_y(x, y)\,dy.$$

This definition can be extended to a function of three or more variables. For instance, if $w = f(x, y, z, u)$, then $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$, $du = \Delta u$, and the total differential of w is

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial u}du.$$

EXAMPLE 1 **Finding the Total Differential**

Find the total differential for each function.

a. $z = 2x \sin y - 3x^2y^2$ **b.** $w = x^2 + y^2 + z^2$

Solution

a. The total differential dz for $z = 2x \sin y - 3x^2y^2$ is

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= (2 \sin y - 6xy^2)\,dx + (2x \cos y - 6x^2y)\,dy. \end{aligned}$$

Total differential dz

b. The total differential dw for $w = x^2 + y^2 + z^2$ is

$$\begin{aligned} dw &= \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \\ &= 2x\,dx + 2y\,dy + 2z\,dz. \end{aligned}$$

Total differential dw

Differentiability

In Section 3.9, you learned that for a *differentiable* function given by $y = f(x)$, you can use the differential $dy = f'(x)\,dx$ as an approximation (for small Δx) of the value $\Delta y = f(x + \Delta x) - f(x)$. When a similar approximation is possible for a function of two variables, the function is said to be **differentiable.** This is stated explicitly in the next definition.

Definition of Differentiability

A function f given by $z = f(x, y)$ is **differentiable** at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both ε_1 and $\varepsilon_2 \to 0$ as

$$(\Delta x, \Delta y) \to (0, 0).$$

The function f is **differentiable in a region *R*** if it is differentiable at each point in R.

EXAMPLE 2 Showing that a Function Is Differentiable

Show that the function

$$f(x, y) = x^2 + 3y$$

is differentiable at every point in the plane.

Solution Letting $z = f(x, y)$, the increment of z at an arbitrary point (x, y) in the plane is

$$\begin{aligned}
\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) && \text{Increment of } z\\
&= (x + \Delta x)^2 + 3(y + \Delta y) - (x^2 + 3y)\\
&= x^2 + 2x\Delta x + (\Delta x)^2 + 3y + 3\Delta y - x^2 - 3y\\
&= 2x\Delta x + (\Delta x)^2 + 3\Delta y\\
&= 2x(\Delta x) + 3(\Delta y) + \Delta x(\Delta x) + 0(\Delta y)\\
&= f_x(x, y)\Delta x + f_y(x, y)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\end{aligned}$$

where $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = 0$. Because $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$, it follows that f is differentiable at every point in the plane. The graph of f is shown in Figure 13.34.

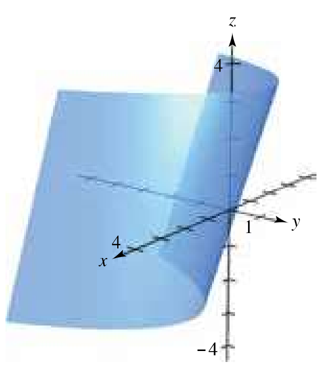


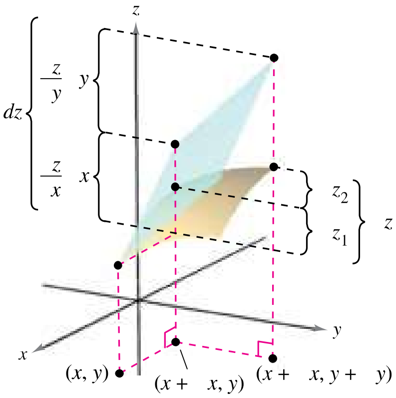
Figure 13.34

Be sure you see that the term "differentiable" is used differently for functions of two variables than for functions of one variable. A function of one variable is differentiable at a point when its derivative exists at the point. For a function of two variables, however, the existence of the partial derivatives f_x and f_y does not guarantee that the function is differentiable (see Example 5). The next theorem gives a *sufficient* condition for differentiability of a function of two variables.

THEOREM 13.4 Sufficient Condition for Differentiability

If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.
A proof of this theorem is given in Appendix A.

Approximation by Differentials



The exact change in z is Δz. This change can be approximated by the differential dz.
Figure 13.35

Theorem 13.4 tells you that you can choose $(x + \Delta x, y + \Delta y)$ close enough to (x, y) to make $\varepsilon_1 \Delta x$ and $\varepsilon_2 \Delta y$ insignificant. In other words, for small Δx and Δy, you can use the approximation

$$\Delta z \approx dz. \qquad \text{Approximate change in } z$$

This approximation is illustrated graphically in Figure 13.35. Recall that the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ can be interpreted as the slopes of the surface in the x- and y-directions. This means that

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

represents the change in height of a plane that is tangent to the surface at the point $(x, y, f(x, y))$. Because a plane in space is represented by a linear equation in the variables x, y, and z, the approximation of Δz by dz is called a **linear approximation.** You will learn more about this geometric interpretation in Section 13.7.

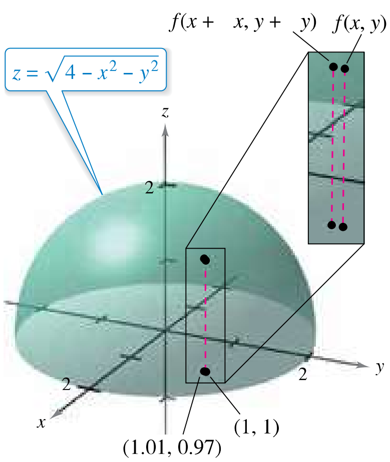
EXAMPLE 3 Using a Differential as an Approximation

See LarsonCalculus.com for an interactive version of this type of example.

Use the differential dz to approximate the change in

$$z = \sqrt{4 - x^2 - y^2}$$

as (x, y) moves from the point $(1, 1)$ to the point $(1.01, 0.97)$. Compare this approximation with the exact change in z.



As (x, y) moves from the point $(1, 1)$ to the point $(1.01, 0.97)$, the value of $f(x, y)$ changes by about 0.0137.
Figure 13.36

Solution Letting $(x, y) = (1, 1)$ and $(x + \Delta x, y + \Delta y) = (1.01, 0.97)$ produces

$$dx = \Delta x = 0.01 \quad \text{and} \quad dy = \Delta y = -0.03.$$

So, the change in z can be approximated by

$$\Delta z \approx dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{-x}{\sqrt{4 - x^2 - y^2}}\Delta x + \frac{-y}{\sqrt{4 - x^2 - y^2}}\Delta y.$$

When $x = 1$ and $y = 1$, you have

$$\Delta z \approx -\frac{1}{\sqrt{2}}(0.01) - \frac{1}{\sqrt{2}}(-0.03) = \frac{0.02}{\sqrt{2}} = \sqrt{2}(0.01) \approx 0.0141.$$

In Figure 13.36, you can see that the exact change corresponds to the difference in the heights of two points on the surface of a hemisphere. This difference is given by

$$\begin{aligned}\Delta z &= f(1.01, 0.97) - f(1, 1)\\ &= \sqrt{4 - (1.01)^2 - (0.97)^2} - \sqrt{4 - 1^2 - 1^2}\\ &\approx 0.0137.\end{aligned}$$

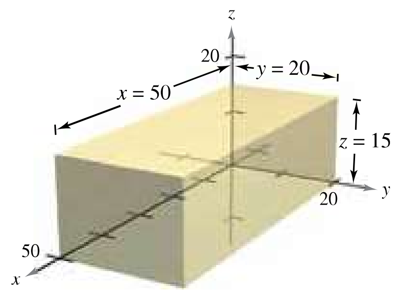
A function of three variables $w = f(x, y, z)$ is **differentiable** at (x, y, z) provided that $\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$ can be written in the form

$$\Delta w = f_x \Delta x + f_y \Delta y + f_z \Delta z + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z$$

where ε_1, ε_2, and $\varepsilon_3 \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0, 0, 0)$. With this definition of differentiability, Theorem 13.4 has the following extension for functions of three variables: If f is a function of x, y, and z, where f_x, f_y, and f_z are continuous in an open region R, then f is differentiable on R.

In Section 3.9, you used differentials to approximate the propagated error introduced by an error in measurement. This application of differentials is further illustrated in Example 4.

EXAMPLE 4 Error Analysis



Volume $= xyz$
Figure 13.37

The possible error involved in measuring each dimension of a rectangular box is ± 0.1 millimeter. The dimensions of the box are $x = 50$ centimeters, $y = 20$ centimeters, and $z = 15$ centimeters, as shown in Figure 13.37. Use dV to estimate the propagated error and the relative error in the calculated volume of the box.

Solution The volume of the box is $V = xyz$, so

$$\begin{aligned} dV &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \\ &= yz\,dx + xz\,dy + xy\,dz. \end{aligned}$$

Using 0.1 millimeter $=$ 0.01 centimeter, you have

$$dx = dy = dz = \pm 0.01$$

and the propagated error is approximately

$$\begin{aligned} dV &= (20)(15)(\pm 0.01) + (50)(15)(\pm 0.01) + (50)(20)(\pm 0.01) \\ &= 300(\pm 0.01) + 750(\pm 0.01) + 1000(\pm 0.01) \\ &= 2050(\pm 0.01) \\ &= \pm 20.5 \text{ cubic centimeters.} \end{aligned}$$

Because the measured volume is

$$V = (50)(20)(15) = 15{,}000 \text{ cubic centimeters}$$

the relative error, $\Delta V/V$, is approximately

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{\pm 20.5}{15{,}000} \approx \pm 0.0014$$

which is a percent error of about 0.14%.

As is true for a function of a single variable, when a function in two or more variables is differentiable at a point, it is also continuous there.

THEOREM 13.5 Differentiability Implies Continuity

If a function of x and y is differentiable at (x_0, y_0), then it is continuous at (x_0, y_0).

Proof Let f be differentiable at (x_0, y_0), where $z = f(x, y)$. Then

$$\Delta z = [f_x(x_0, y_0) + \varepsilon_1]\,\Delta x + [f_y(x_0, y_0) + \varepsilon_2]\,\Delta y$$

where both ε_1 and ε_2 0 as $(\Delta x, \Delta y)$ $(0, 0)$. However, by definition, you know that Δz is

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

Letting $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$ produces

$$\begin{aligned} f(x, y) - f(x_0, y_0) &= [f_x(x_0, y_0) + \varepsilon_1]\,\Delta x + [f_y(x_0, y_0) + \varepsilon_2]\,\Delta y \\ &= [f_x(x_0, y_0) + \varepsilon_1](x - x_0) + [f_y(x_0, y_0) + \varepsilon_2](y - y_0). \end{aligned}$$

Taking the limit as (x, y) (x_0, y_0), you have

$$\lim_{(x,y)\ \ (x_0, y_0)} f(x, y) = f(x_0, y_0)$$

which means that f is continuous at (x_0, y_0).

Remember that the existence of f_x and f_y is not sufficient to guarantee differentiability, as illustrated in the next example.

EXAMPLE 5 A Function That Is Not Differentiable

For the function

$$f(x, y) = \begin{cases} \dfrac{-3xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but that f is not differentiable at $(0, 0)$.

Solution You can show that f is not differentiable at $(0, 0)$ by showing that it is not continuous at this point. To see that f is not continuous at $(0, 0)$, look at the values of $f(x, y)$ along two different approaches to $(0, 0)$, as shown in Figure 13.38. Along the line $y = x$, the limit is

$$\lim_{(x, x) \to (0, 0)} f(x, y) = \lim_{(x, x) \to (0, 0)} \frac{-3x^2}{2x^2} = -\frac{3}{2}$$

whereas along $y = -x$, you have

$$\lim_{(x, -x) \to (0, 0)} f(x, y) = \lim_{(x, -x) \to (0, 0)} \frac{3x^2}{2x^2} = \frac{3}{2}.$$

So, the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist, and you can conclude that f is not continuous at $(0, 0)$. Therefore, by Theorem 13.5, you know that f is not differentiable at $(0, 0)$. On the other hand, by the definition of the partial derivatives f_x and f_y, you have

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

and

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0.$$

So, the partial derivatives at $(0, 0)$ exist.

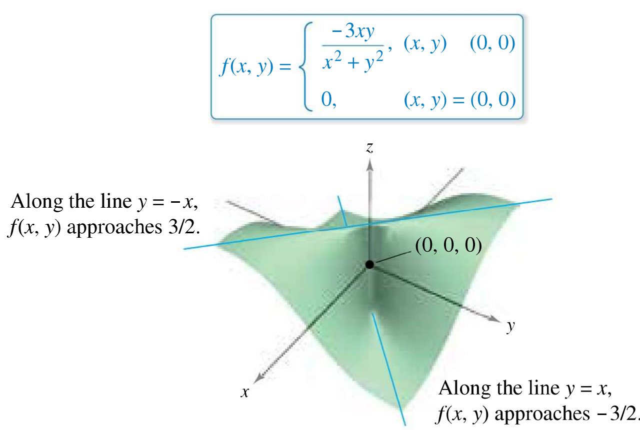
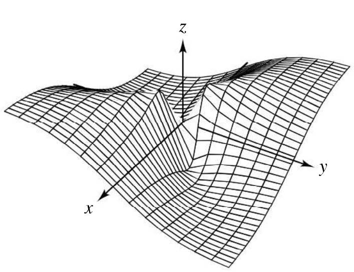


Figure 13.38



Generated by Mathematica

TECHNOLOGY A graphing utility can be used to graph piecewise-defined functions like the one given in Example 5. For instance, the graph shown at the left was generated by *Mathematica*.

13.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Approximation** Describe the change in accuracy of dz as an approximation of Δz as Δx and Δy increase.

2. **Linear Approximation** What is meant by a linear approximation of $z = f(x, y)$ at the point $P(x_0, y_0)$?

Finding a Total Differential **In Exercises 3–8, find the total differential.**

3. $z = 5x^3y^2$

4. $z = 2x^3y - 8xy^4$

5. $z = \frac{1}{2}\left(e^{x^2+y^2} - e^{-x^2-y^2}\right)$

6. $z = e^{-x} \tan y$

7. $w = x^2yz^2 + \sin yz$

8. $w = (x + y)/(z - 3y)$

Using a Differential as an Approximation **In Exercises 9–14, (a) find $f(2, 1)$ and $f(2.1, 1.05)$ and calculate Δz, and (b) use the total differential dz to approximate Δz.**

9. $f(x, y) = 2x - 3y$

10. $f(x, y) = x^2 + y^2$

11. $f(x, y) = 16 - x^2 - y^2$

12. $f(x, y) = y/x$

13. $f(x, y) = ye^x$

14. $f(x, y) = x \cos y$

Approximating an Expression **In Exercises 15–18, find $z = f(x, y)$ and use the total differential to approximate the quantity.**

15. $(2.01)^2(9.02) - 2^2 \cdot 9$

16. $\dfrac{1 - (3.05)^2}{(5.95)^2} - \dfrac{1 - 3^2}{6^2}$

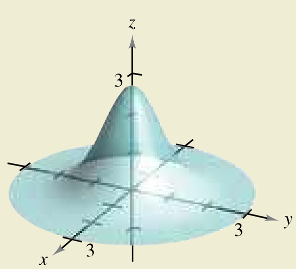
17. $\sin[(1.05)^2 + (0.95)^2] - \sin(1^2 + 1^2)$

18. $\sqrt{(4.03)^2 + (3.1)^2} - \sqrt{4^2 + 3^2}$

EXPLORING CONCEPTS

19. **Continuity** If f_x and f_y are each continuous in an open region R, is $f(x, y)$ continuous in R? Explain.

20. **HOW DO YOU SEE IT?** Which point has a greater differential, $(2, 2)$ or $\left(\frac{1}{2}, \frac{1}{2}\right)$? Explain. (Assume that dx and dy are the same for both points.)



21. **Area** The area of the shaded rectangle in the figure is $A = lh$. The possible errors in the length and height are Δl and Δh, respectively. Find dA and identify the regions in the figure whose areas are given by the terms of dA. What region represents the difference between ΔA and dA?

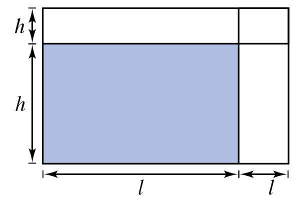


Figure for 21

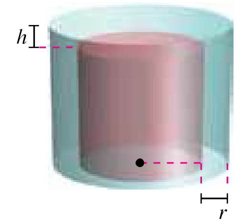


Figure for 22

22. **Volume** The volume of the red right circular cylinder in the figure is $V = \pi r^2 h$. The possible errors in the radius and the height are Δr and Δh, respectively. Find dV and identify the solids in the figure whose volumes are given by the terms of dV. What solid represents the difference between ΔV and dV?

23. **Volume** The possible error involved in measuring each dimension of a rectangular box is ± 0.02 inch. The dimensions of the box are 8 inches by 5 inches by 12 inches. Approximate the propagated error and the relative error in the calculated volume of the box.

24. **Volume** The possible error involved in measuring each dimension of a right circular cylinder is ± 0.05 centimeter. The radius is 3 centimeters and the height is 10 centimeters. Approximate the propagated error and the relative error in the calculated volume of the cylinder.

25. **Numerical Analysis** A right circular cone of height $h = 8$ meters and radius $r = 4$ meters is constructed, and in the process, errors Δr and Δh are made in the radius and height, respectively. Let V be the volume of the cone. Complete the table to show the relationship between ΔV and dV for the indicated errors.

Δr	Δh	dV or dS	ΔV or ΔS	$\Delta V - dV$ or $\Delta S - dS$
0.1	0.1			
0.1	-0.1			
0.001	0.002			
-0.0001	0.0002			

Table for Exercises 25 and 26

26. **Numerical Analysis** A right circular cone of height $h = 16$ meters and radius $r = 6$ meters is constructed, and in the process, errors of Δr and Δh are made in the radius and height, respectively. Let S be the lateral surface area of the cone. Complete the table above to show the relationship between ΔS and dS for the indicated errors.

27. Wind Chill The formula for wind chill C (in degrees Fahrenheit) is given by

$$C = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$$

where v is the wind speed in miles per hour and T is the temperature in degrees Fahrenheit. The wind speed is 23 ± 3 miles per hour and the temperature is $8° \pm 1°$. Use dC to estimate the maximum possible propagated error and relative error in calculating the wind chill. *(Source: National Oceanic and Atmospheric Administration)*

28. Resistance The total resistance R (in ohms) of two resistors connected in parallel is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Approximate the change in R as R_1 is increased from 10 ohms to 10.5 ohms and R_2 is decreased from 15 ohms to 13 ohms.

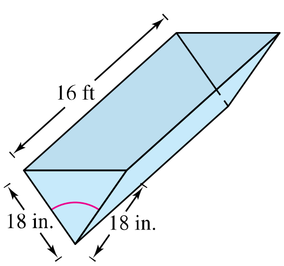
29. Power Electrical power P is given by

$$P = \frac{E^2}{R}$$

where E is voltage and R is resistance. Approximate the maximum percent error in calculating power when 120 volts is applied to a 2000-ohm resistor and the possible percent errors in measuring E and R are 3% and 4%, respectively.

30. Acceleration The centripetal acceleration of a particle moving in a circle is $a = v^2/r$, where v is the velocity and r is the radius of the circle. Approximate the maximum percent error in measuring the acceleration due to errors of 3% in v and 2% in r.

31. Volume A trough is 16 feet long (see figure). Its cross sections are isosceles triangles with each of the two equal sides measuring 18 inches. The angle between the two equal sides is θ.

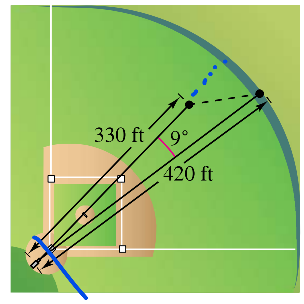


Not drawn to scale

(a) Write the volume of the trough as a function of θ and determine the value of θ such that the volume is a maximum.

(b) The maximum error in the linear measurements is one-half inch and the maximum error in the angle measure is 2°. Approximate the change in the maximum volume.

32. Sports A baseball player in center field is playing approximately 330 feet from a television camera that is behind home plate. A batter hits a fly ball that goes to the wall 420 feet from the camera (see figure).



(a) The camera turns 9° to follow the play. Approximate the number of feet that the center fielder has to run to make the catch.

(b) The position of the center fielder could be in error by as much as 6 feet and the maximum error in measuring the rotation of the camera is 1°. Approximate the maximum possible error in the result of part (a).

33. Inductance The inductance L (in microhenrys) of a straight nonmagnetic wire in free space is

$$L = 0.00021\left(\ln\frac{2h}{r} - 0.75\right)$$

where h is the length of the wire in millimeters and r is the radius of a circular cross section. Approximate L when $r = 2 \pm \frac{1}{16}$ millimeters and $h = 100 \pm \frac{1}{100}$ millimeters.

34. Pendulum The period T of a pendulum of length L is $T = \left(2\pi\sqrt{L}\right)/\sqrt{g}$, where g is the acceleration due to gravity. A pendulum is moved from the Canal Zone, where $g = 32.09$ feet per second per second, to Greenland, where $g = 32.23$ feet per second per second. Because of the change in temperature, the length of the pendulum changes from 2.5 feet to 2.48 feet. Approximate the change in the period of the pendulum.

Differentiability **In Exercises 35–38, show that the function is differentiable by finding values of ε_1 and ε_2 as designated in the definition of differentiability, and verify that both ε_1 and ε_2 approach 0 as $(\Delta x, \Delta y) \to (0, 0)$.**

35. $f(x, y) = x^2 - 2x + y$

36. $f(x, y) = x^2 + y^2$

37. $f(x, y) = x^2y$

38. $f(x, y) = 5x - 10y + y^3$

Differentiability **In Exercises 39 and 40, use the function to show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but that f is not differentiable at $(0, 0)$.**

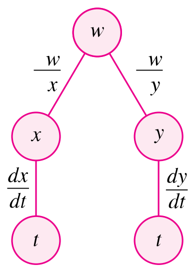
39. $f(x, y) = \begin{cases} \dfrac{3x^2y}{x^4 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

40. $f(x, y) = \begin{cases} \dfrac{5x^2y}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

13.5 Chain Rules for Functions of Several Variables

- **Use the Chain Rules for functions of several variables.**
- **Find partial derivatives implicitly.**

Chain Rules for Functions of Several Variables



Chain Rule: one independent variable w is a function of x and y, which are each functions of t. This diagram represents the derivative of w with respect to t.
Figure 13.39

Your work with differentials in the preceding section provides the basis for the extension of the Chain Rule to functions of two variables. There are two cases. The first case involves w as a function of x and y, where x and y are functions of a single independent variable t, as shown in Theorem 13.6.

THEOREM 13.6 Chain Rule: One Independent Variable

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

The Chain Rule is shown schematically in Figure 13.39.
A proof of this theorem is given in Appendix A.

EXAMPLE 1 Chain Rule: One Independent Variable

Let $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Find dw/dt when $t = 0$.

Solution By the Chain Rule for one independent variable, you have

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}\\ &= 2xy(\cos t) + (x^2 - 2y)e^t\\ &= 2(\sin t)(e^t)(\cos t) + (\sin^2 t - 2e^t)e^t\\ &= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}.\end{aligned}$$

When $t = 0$, it follows that

$$\frac{dw}{dt} = -2.$$

The Chain Rules presented in this section provide alternative techniques for solving many problems in single-variable calculus. For instance, in Example 1, you could have used single-variable techniques to find dw/dt by first writing w as a function of t,

$$\begin{aligned}w &= x^2y - y^2\\ &= (\sin t)^2(e^t) - (e^t)^2\\ &= e^t \sin^2 t - e^{2t}\end{aligned}$$

and then differentiating as usual.

$$\frac{dw}{dt} = 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}$$

The Chain Rule in Theorem 13.6 can be extended to any number of variables. For example, if each x_i is a differentiable function of a single variable t, then for

$$w = f(x_1, x_2, \ldots, x_n)$$

you have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial w}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}.$$

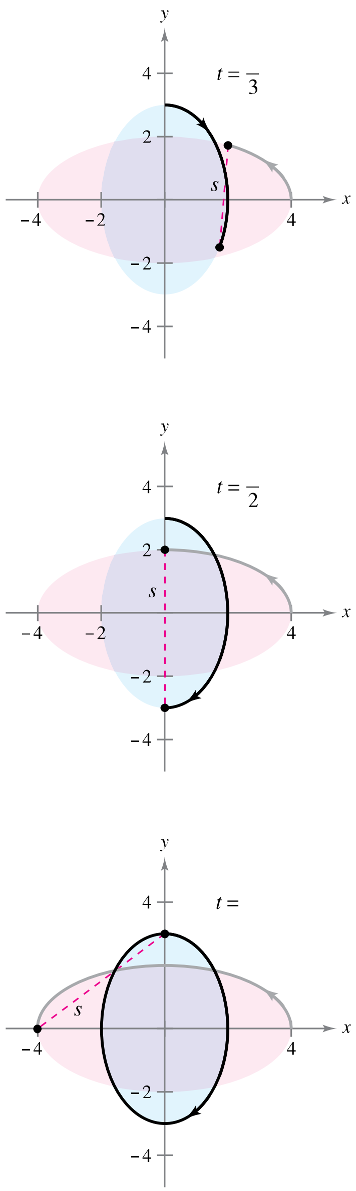
EXAMPLE 2 An Application of a Chain Rule to Related Rates

Two objects are traveling in elliptical paths given by the following parametric equations.

$$x_1 = 4\cos t \quad \text{and} \quad y_1 = 2\sin t \qquad \text{First object}$$

$$x_2 = 2\sin 2t \quad \text{and} \quad y_2 = 3\cos 2t \qquad \text{Second object}$$

At what rate is the distance between the two objects changing when $t = \pi$?



Paths of two objects traveling in elliptical orbits
Figure 13.40

Solution From Figure 13.40, you can see that the distance s between the two objects is given by

$$s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and that when $t = \pi$, you have $x_1 = -4$, $y_1 = 0$, $x_2 = 0$, $y_2 = 3$, and

$$s = \sqrt{(0 + 4)^2 + (3 + 0)^2} = 5.$$

When $t = \pi$, the partial derivatives of s are as follows.

$$\frac{\partial s}{\partial x_1} = \frac{-(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = -\frac{1}{5}(0 + 4) = -\frac{4}{5}$$

$$\frac{\partial s}{\partial y_1} = \frac{-(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = -\frac{1}{5}(3 - 0) = -\frac{3}{5}$$

$$\frac{\partial s}{\partial x_2} = \frac{(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = \frac{1}{5}(0 + 4) = \frac{4}{5}$$

$$\frac{\partial s}{\partial y_2} = \frac{(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = \frac{1}{5}(3 - 0) = \frac{3}{5}$$

When $t = \pi$, the derivatives of x_1, y_1, x_2, and y_2 are

$$\frac{dx_1}{dt} = -4\sin t = 0$$

$$\frac{dy_1}{dt} = 2\cos t = -2$$

$$\frac{dx_2}{dt} = 4\cos 2t = 4$$

$$\frac{dy_2}{dt} = -6\sin 2t = 0.$$

So, using the appropriate Chain Rule, you know that the distance is changing at a rate of

$$\begin{aligned}\frac{ds}{dt} &= \frac{\partial s}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial s}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial s}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial s}{\partial y_2}\frac{dy_2}{dt}\\ &= \left(-\frac{4}{5}\right)(0) + \left(-\frac{3}{5}\right)(-2) + \left(\frac{4}{5}\right)(4) + \left(\frac{3}{5}\right)(0)\\ &= \frac{22}{5}.\end{aligned}$$

In Example 2, note that s is the function of four *intermediate* variables, x_1, y_1, x_2, and y_2, each of which is a function of a single variable t. Another type of composite function is one in which the intermediate variables are themselves functions of more than one variable. For instance, for $w = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$, it follows that w is a function of s and t, and you can consider the partial derivatives of w with respect to s and t. One way to find these partial derivatives is to write w as a function of s and t explicitly by substituting the equations $x = g(s, t)$ and $y = h(s, t)$ into the equation $w = f(x, y)$. Then you can find the partial derivatives in the usual way, as demonstrated in the next example.

EXAMPLE 3 Finding Partial Derivatives by Substitution

Find $\partial w/\partial s$ and $\partial w/\partial t$ for $w = 2xy$, where $x = s^2 + t^2$ and $y = s/t$.

Solution Begin by substituting $x = s^2 + t^2$ and $y = s/t$ into the equation $w = 2xy$ to obtain

$$w = 2xy = 2(s^2 + t^2)\left(\frac{s}{t}\right) = 2\left(\frac{s^3}{t} + st\right).$$

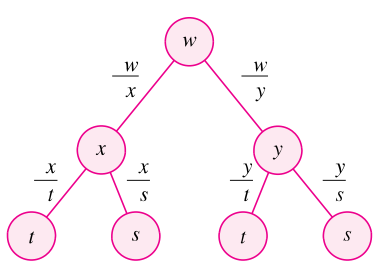
Then, to find $\partial w/\partial s$, hold t constant and differentiate with respect to s.

$$\begin{aligned}\frac{\partial w}{\partial s} &= 2\left(\frac{3s^2}{t} + t\right)\\ &= \frac{6s^2 + 2t^2}{t}\end{aligned}$$

Similarly, to find $\partial w/\partial t$, hold s constant and differentiate with respect to t to obtain

$$\begin{aligned}\frac{\partial w}{\partial t} &= 2\left(-\frac{s^3}{t^2} + s\right)\\ &= 2\left(\frac{-s^3 + st^2}{t^2}\right)\\ &= \frac{2st^2 - 2s^3}{t^2}.\end{aligned}$$

Theorem 13.7 gives an alternative method for finding the partial derivatives in Example 3 without explicitly writing w as a function of s and t.



Chain Rule: two independent variables
Figure 13.41

THEOREM 13.7 Chain Rule: Two Independent Variables

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(s, t)$ and $y = h(s, t)$ such that the first partials $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, and $\partial y/\partial t$ all exist, then $\partial w/\partial s$ and $\partial w/\partial t$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

The Chain Rule is shown schematically in Figure 13.41.

Proof To obtain $\partial w/\partial s$, hold t constant and apply Theorem 13.6 to obtain the desired result. Similarly, for $\partial w/\partial t$, hold s constant and apply Theorem 13.6.

EXAMPLE 4 **The Chain Rule with Two Independent Variables**

See LarsonCalculus.com for an interactive version of this type of example.

Use the Chain Rule to find $\partial w/\partial s$ and $\partial w/\partial t$ for

$$w = 2xy$$

where $x = s^2 + t^2$ and $y = s/t$.

Solution Note that these same partials were found in Example 3. This time, using Theorem 13.7, you can hold t constant and differentiate with respect to s to obtain

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}\\ &= 2y(2s) + 2x\left(\frac{1}{t}\right)\\ &= 2\left(\frac{s}{t}\right)(2s) + 2(s^2 + t^2)\left(\frac{1}{t}\right) && \text{Substitute } \tfrac{s}{t} \text{ for } y \text{ and } s^2 + t^2 \text{ for } x.\\ &= \frac{4s^2}{t} + \frac{2s^2 + 2t^2}{t}\\ &= \frac{6s^2 + 2t^2}{t}.\end{aligned}$$

Similarly, holding s constant gives

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}\\ &= 2y(2t) + 2x\left(\frac{-s}{t^2}\right)\\ &= 2\left(\frac{s}{t}\right)(2t) + 2(s^2 + t^2)\left(\frac{-s}{t^2}\right) && \text{Substitute } \tfrac{s}{t} \text{ for } y \text{ and } s^2 + t^2 \text{ for } x.\\ &= 4s - \frac{2s^3 + 2st^2}{t^2}\\ &= \frac{4st^2 - 2s^3 - 2st^2}{t^2}\\ &= \frac{2st^2 - 2s^3}{t^2}.\end{aligned}$$

The Chain Rule in Theorem 13.7 can also be extended to any number of variables. For example, if w is a differentiable function of the n variables

$$x_1, x_2, \ldots, x_n$$

where each x_i is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$, then for

$$w = f(x_1, x_2, \ldots, x_n)$$

you obtain the following.

$$\begin{aligned}\frac{\partial w}{\partial t_1} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1}\\ \frac{\partial w}{\partial t_2} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2}\\ &\vdots\\ \frac{\partial w}{\partial t_m} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}\end{aligned}$$

EXAMPLE 5 **The Chain Rule for a Function of Three Variables**

Find $\partial w/\partial s$ and $\partial w/\partial t$ when $s = 1$ and $t = 2\pi$ for

$$w = xy + yz + xz$$

where $x = s\cos t$, $y = s\sin t$, and $z = t$.

Solution By extending the result of Theorem 13.7, you have

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}\\ &= (y + z)(\cos t) + (x + z)(\sin t) + (y + x)(0)\\ &= (y + z)(\cos t) + (x + z)(\sin t).\end{aligned}$$

When $s = 1$ and $t = 2\pi$, you have $x = 1$, $y = 0$, and $z = 2\pi$. So,

$$\frac{\partial w}{\partial s} = (0 + 2\pi)(1) + (1 + 2\pi)(0) = 2\pi.$$

Furthermore,

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}\\ &= (y + z)(-s\sin t) + (x + z)(s\cos t) + (y + x)(1)\end{aligned}$$

and for $s = 1$ and $t = 2\pi$, it follows that

$$\begin{aligned}\frac{\partial w}{\partial t} &= (0 + 2\pi)(0) + (1 + 2\pi)(1) + (0 + 1)(1)\\ &= 2 + 2\pi.\end{aligned}$$

Implicit Partial Differentiation

This section concludes with an application of the Chain Rule to determine the derivative of a function defined *implicitly*. Let x and y be related by the equation $F(x, y) = 0$, where $y = f(x)$ is a differentiable function of x. To find dy/dx, you could use the techniques discussed in Section 2.5. You will see, however, that the Chain Rule provides a convenient alternative. Consider the function

$$w = F(x, y) = F(x, f(x)).$$

You can apply Theorem 13.6 to obtain

$$\frac{dw}{dx} = F_x(x, y)\frac{dx}{dx} + F_y(x, y)\frac{dy}{dx}.$$

Because $w = F(x, y) = 0$ for all x in the domain of f, you know that

$$\frac{dw}{dx} = 0$$

and you have

$$F_x(x, y)\frac{dx}{dx} + F_y(x, y)\frac{dy}{dx} = 0.$$

Now, if $F_y(x, y) \neq 0$, you can use the fact that $dx/dx = 1$ to conclude that

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.$$

A similar procedure can be used to find the partial derivatives of functions of several variables that are defined implicitly.

THEOREM 13.8 Chain Rule: Implicit Differentiation

If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0.$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

This theorem can be extended to differentiable functions defined implicitly with any number of variables.

REMARK Compare the solution to Example 6 with the solution to Example 2 in Section 2.5.

EXAMPLE 6 Finding a Derivative Implicitly

Find dy/dx for

$$y^3 + y^2 - 5y - x^2 + 4 = 0.$$

Solution Begin by letting

$$F(x, y) = y^3 + y^2 - 5y - x^2 + 4.$$

Then

$$F_x(x, y) = -2x \quad \text{and} \quad F_y(x, y) = 3y^2 + 2y - 5.$$

Using Theorem 13.8, you have

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = \frac{-(-2x)}{3y^2 + 2y - 5} = \frac{2x}{3y^2 + 2y - 5}.$$

EXAMPLE 7 Finding Partial Derivatives Implicitly

Find $\partial z/\partial x$ and $\partial z/\partial y$ for

$$3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0.$$

Solution Begin by letting

$$F(x, y, z) = 3x^2z - x^2y^2 + 2z^3 + 3yz - 5.$$

Then

$$F_x(x, y, z) = 6xz - 2xy^2$$
$$F_y(x, y, z) = -2x^2y + 3z$$

and

$$F_z(x, y, z) = 3x^2 + 6z^2 + 3y.$$

Using Theorem 13.8, you have

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = \frac{2xy^2 - 6xz}{3x^2 + 6z^2 + 3y}$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{2x^2y - 3z}{3x^2 + 6z^2 + 3y}.$$

13.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Chain Rule** Consider $w = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$. Describe two ways of finding the partial derivatives $\partial w/\partial s$ and $\partial w/\partial t$.
2. **Implicit Differentiation** Why is using the Chain Rule to determine the derivative of the equation $F(x, y) = 0$ implicitly easier than using the method you learned in Section 2.5?

Using the Chain Rule **In Exercises 3–6, find dw/dt using the appropriate Chain Rule. Evaluate dw/dt at the given value of t.**

Function	Value
3. $w = x^2 + 5y$	$t = 2$
$x = 2t,\ y = t$	
4. $w = \sqrt{x^2 + y^2}$	$t = 0$
$x = \cos t,\ y = e^t$	
5. $w = x \sin y$	$t = 0$
$x = e^t,\ y = \pi - t$	
6. $w = \ln \dfrac{y}{x}$	$t = \dfrac{\pi}{4}$
$x = \cos t,\ y = \sin t$	

Using Different Methods **In Exercises 7–12, find dw/dt (a) by using the appropriate Chain Rule and (b) by converting w to a function of t before differentiating.**

7. $w = x - \dfrac{1}{y}, \quad x = e^{2t}, \quad y = t^3$
8. $w = \cos(x - y), \quad x = t^2, \quad y = 1$
9. $w = x^2 + y^2 + z^2, \quad x = \cos t, \quad y = \sin t, \quad z = e^t$
10. $w = xy \cos z, \quad x = t, \quad y = t^2, \quad z = \arccos t$
11. $w = xy + xz + yz, \quad x = t - 1, \quad y = t^2 - 1, \quad z = t$
12. $w = xy^2 + x^2z + yz^2, \quad x = t^2, \quad y = 2t, \quad z = 2$

Projectile Motion **In Exercises 13 and 14, the parametric equations for the paths of two objects are given. At what rate is the distance between the two objects changing at the given value of t?**

13. $x_1 = 10 \cos 2t, \quad y_1 = 6 \sin 2t$ — First object
$x_2 = 7 \cos t, \quad y_2 = 4 \sin t$ — Second object
$t = \pi/2$

14. $x_1 = 48\sqrt{2}t, \quad y_1 = 48\sqrt{2}t - 16t^2$ — First object
$x_2 = 48\sqrt{3}t, \quad y_2 = 48t - 16t^2$ — Second object
$t = 1$

Finding Partial Derivatives **In Exercises 15–18, find $\partial w/\partial s$ and $\partial w/\partial t$ using the appropriate Chain Rule. Evaluate each partial derivative at the given values of s and t.**

Function	Values
15. $w = x^2 + y^2$	$s = 1,\ t = 3$
$x = s + t,\ y = s - t$	
16. $w = y^3 - 3x^2y$	$s = -1,\ t = 2$
$x = e^s,\ y = e^t$	
17. $w = \sin(2x + 3y)$	$s = 0,\ t = \dfrac{\pi}{2}$
$x = s + t,\ y = s - t$	
18. $w = x^2 - y^2$	$s = 3,\ t = \dfrac{\pi}{4}$
$x = s \cos t,\ y = s \sin t$	

Using Different Methods **In Exercises 19–22, find $\partial w/\partial s$ and $\partial w/\partial t$ (a) by using the appropriate Chain Rule and (b) by converting w to a function of s and t before differentiating.**

19. $w = xyz, \quad x = s + t, \quad y = s - t, \quad z = st^2$
20. $w = x^2 + y^2 + z^2, \quad x = t \sin s, \quad y = t \cos s, \quad z = st^2$
21. $w = ze^{xy}, \quad x = s - t, \quad y = s + t, \quad z = st$
22. $w = x \cos yz, \quad x = s^2, \quad y = t^2, \quad z = s - 2t$

Finding a Derivative Implicitly **In Exercises 23–26, differentiate implicitly to find dy/dx.**

23. $x^2 - xy + y^2 - x + y = 0$
24. $\sec xy + \tan xy + 5 = 0$
25. $\ln\sqrt{x^2 + y^2} + x + y = 4$
26. $\dfrac{x}{x^2 + y^2} - y^2 = 6$

Finding Partial Derivatives Implicitly **In Exercises 27–34, differentiate implicitly to find the first partial derivatives of z.**

27. $x^2 + y^2 + z^2 = 1$
28. $xz + yz + xy = 0$
29. $x^2 + 2yz + z^2 = 1$
30. $x + \sin(y + z) = 0$
31. $\tan(x + y) + \cos z = 2$
32. $z = e^x \sin(y + z)$
33. $e^{xz} + xy = 0$
34. $x \ln y + y^2z + z^2 = 8$

Finding Partial Derivatives Implicitly **In Exercises 35–38, differentiate implicitly to find the first partial derivatives of w.**

35. $7xy + yz^2 - 4wz + w^2z + w^2x - 6 = 0$
36. $x^2 + y^2 + z^2 - 5yw + 10w^2 = 2$
37. $\cos xy + \sin yz + wz = 20$
38. $w - \sqrt{x - y} - \sqrt{y - z} = 0$

Homogeneous Functions **A function f is *homogeneous of degree n* when $f(tx, ty) = t^n f(x, y)$. In Exercises 39–42, (a) show that the function is homogeneous and determine n, and (b) show that $xf_x(x, y) + yf_y(x, y) = nf(x, y)$.**

39. $f(x, y) = 2x^2 - 5xy$

40. $f(x, y) = x^3 - 3xy^2 + y^3$

41. $f(x, y) = e^{x/y}$

42. $f(x, y) = x \cos \dfrac{x + y}{y}$

43. Using a Table of Values Let $w = f(x, y)$, $x = g(t)$, and $y = h(t)$, where f, g, and h are differentiable. Use the appropriate Chain Rule and the table of values to find dw/dt when $t = 2$.

$g(2)$	$h(2)$	$g'(2)$	$h'(2)$	$f_x(4, 3)$	$f_y(4, 3)$
4	3	-1	6	-5	7

44. Using a Table of Values Let $w = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$, where f, g, and h are differentiable. Use the appropriate Chain Rule and the table of values to find $w_s(1, 2)$.

$g(1, 2)$	$h(1, 2)$	$g_s(1, 2)$	$h_s(1, 2)$	$f_x(4, 3)$	$f_y(4, 3)$
4	3	-3	5	-5	7

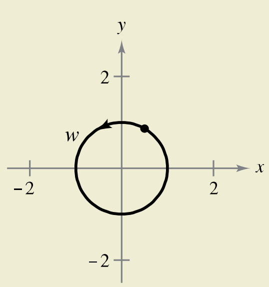
EXPLORING CONCEPTS

45. Using the Chain Rule Show that $\dfrac{\partial w}{\partial u} + \dfrac{\partial w}{\partial v} = 0$ for $w = f(x, y)$, $x = u - v$, and $y = v - u$.

46. Using the Chain Rule Demonstrate the result of Exercise 45 for $w = (x - y) \sin(y - x)$.

47. Using the Chain Rule Let $F(u, v)$ be a function of two variables. Find a formula for $f'(x)$ when (a) $f(x) = F(4x, 4)$ and (b) $f(x) = F(-2x, x^2)$.

48. HOW DO YOU SEE IT? The path of an object represented by $w = f(x, y)$ is shown, where x and y are functions of t. The point on the graph represents the position of the object.



Determine whether each of the following is positive, negative, or zero.

(a) $\dfrac{dx}{dt}$ (b) $\dfrac{dy}{dt}$

49. Volume and Surface Area The radius of a right circular cylinder is increasing at a rate of 6 inches per minute, and the height is decreasing at a rate of 4 inches per minute. What are the rates of change of the volume and surface area when the radius is 12 inches and the height is 36 inches?

50. Ideal Gas Law The Ideal Gas Law is

$$PV = mRT$$

where P is the pressure, V is the volume, m is the constant mass, R is a constant, T is the temperature, and P and V are functions of time. Find dT/dt, the rate at which the temperature changes with respect to time.

51. Moment of Inertia An annular cylinder has an inside radius of r_1 and an outside radius of r_2 (see figure). Its moment of inertia is

$$I = \tfrac{1}{2}m(r_1^2 + r_2^2)$$

where m is the mass. The two radii are increasing at a rate of 2 centimeters per second. Find the rate at which I is changing at the instant the radii are 6 centimeters and 8 centimeters. (Assume mass is a constant.)

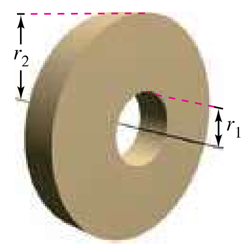


Figure for 51

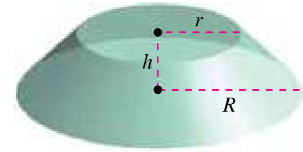


Figure for 52

52. Volume and Surface Area The two radii of the frustum of a right circular cone are increasing at a rate of 4 centimeters per minute, and the height is increasing at a rate of 12 centimeters per minute (see figure). Find the rates at which the volume and surface area are changing when the two radii are 15 centimeters and 25 centimeters and the height is 10 centimeters.

53. Cauchy-Riemann Equations Given the functions $u(x, y)$ and $v(x, y)$, verify that the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

can be written in polar coordinate form as

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial u}{\partial \theta}.$$

54. Cauchy-Riemann Equations Demonstrate the result of Exercise 53 for the functions

$$u = \ln \sqrt{x^2 + y^2} \quad \text{and} \quad v = \arctan \frac{y}{x}.$$

55. Homogeneous Function Show that if $f(x, y)$ is homogeneous of degree n, then

$$xf_x(x, y) + yf_y(x, y) = nf(x, y).$$

[*Hint:* Let $g(t) = f(tx, ty) = t^n f(x, y)$. Find $g'(t)$ and then let $t = 1$.]

13.6 Directional Derivatives and Gradients

- Find and use directional derivatives of a function of two variables.
- Find the gradient of a function of two variables.
- Use the gradient of a function of two variables in applications.
- Find directional derivatives and gradients of functions of three variables.

Directional Derivative

You are standing on the hillside represented by $z = f(x, y)$ in Figure 13.42 and want to determine the hill's incline toward the z-axis. You already know how to determine the slopes in two different directions—the slope in the y-direction is given by the partial derivative $f_y(x, y)$, and the slope in the x-direction is given by the partial derivative $f_x(x, y)$. In this section, you will see that these two partial derivatives can be used to find the slope in *any* direction.

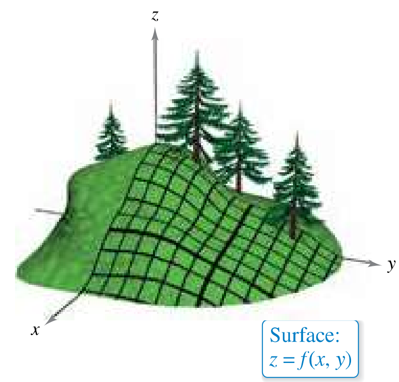


Figure 13.42

To determine the slope at a point on a surface, you will define a new type of derivative called a **directional derivative.** Begin by letting $z = f(x, y)$ be a *surface* and $P(x_0, y_0)$ be a *point* in the domain of f, as shown in Figure 13.43. The "direction" of the directional derivative is given by a unit vector

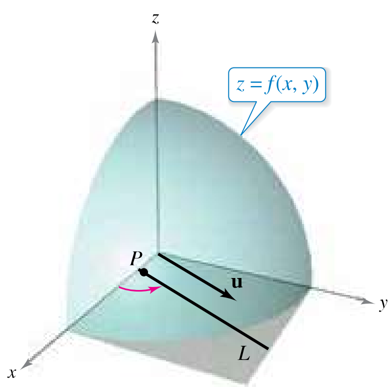


Figure 13.43

$$\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

where θ is the angle the vector makes with the positive x-axis. To find the desired slope, reduce the problem to two dimensions by intersecting the surface with a vertical plane passing through the point P and parallel to $\mathbf{u}$, as shown in Figure 13.44. This vertical plane intersects the surface to form a curve C. The slope of the surface at $(x_0, y_0, f(x_0, y_0))$ in the direction of $\mathbf{u}$ is defined as the slope of the curve C at that point.

Informally, you can write the slope of the curve C as a limit that looks much like those used in single-variable calculus. The vertical plane used to form C intersects the xy-plane in a line L, represented by the parametric equations

$$x = x_0 + t\cos\theta$$

and

$$y = y_0 + t\sin\theta$$

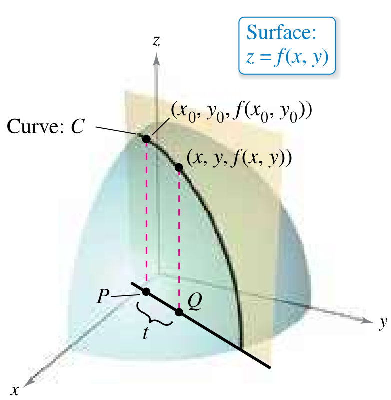


Figure 13.44

so that for any value of t, the point $Q(x, y)$ lies on the line L. For each of the points P and Q, there is a corresponding point on the surface.

$(x_0, y_0, f(x_0, y_0))$ Point above P

$(x, y, f(x, y))$ Point above Q

Moreover, because the distance between P and Q is

$$\begin{aligned}\sqrt{(x - x_0)^2 + (y - y_0)^2} &= \sqrt{(t\cos\theta)^2 + (t\sin\theta)^2}\\ &= |t|\end{aligned}$$

you can write the slope of the secant line through $(x_0, y_0, f(x_0, y_0))$ and $(x, y, f(x, y))$ as

$$\frac{f(x, y) - f(x_0, y_0)}{t} = \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}.$$

Finally, by letting t approach 0, you arrive at the definition on the next page.

REMARK Be sure you understand that the directional derivative represents the *rate of change of a function* in the direction of the unit vector $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$. Geometrically, you can interpret the directional derivative as giving the *slope of a surface* in the direction of $\mathbf{u}$ at a point on the surface. (See Figure 13.46.)

Definition of Directional Derivative

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ be a unit vector. Then the **directional derivative of f in the direction of u,** denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t \to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

provided this limit exists.

Calculating directional derivatives by this definition is similar to finding the derivative of a function of one variable by the limit process (see Section 2.1). A simpler formula for finding directional derivatives involves the partial derivatives f_x and f_y.

THEOREM 13.9 Directional Derivative

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

Proof For a fixed point (x_0, y_0), let

$$x = x_0 + t\cos\theta \quad \text{and} \quad y = y_0 + t\sin\theta.$$

Then, let $g(t) = f(x, y)$. Because f is differentiable, you can apply the Chain Rule given in Theorem 13.6 to obtain

$$\begin{aligned} g'(t) &= f_x(x, y)x'(t) + f_y(x, y)y'(t) && \text{Apply Chain Rule (Theorem 13.6).} \\ &= f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \end{aligned}$$

If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta.$$

By the definition of $g'(t)$, it is also true that

$$\begin{aligned} g'(0) &= \lim_{t \to 0} \frac{g(t) - g(0)}{t} \\ &= \lim_{t \to 0} \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}. \end{aligned}$$

Consequently, $D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$. ■

There are infinitely many directional derivatives of a surface at a given point—one for each direction specified by $\mathbf{u}$, as shown in Figure 13.45. Two of these are the partial derivatives f_x and f_y.

1. Direction of positive x-axis $(\theta = 0)$: $\mathbf{u} = \cos 0\mathbf{i} + \sin 0\mathbf{j} = \mathbf{i}$

$$D_{\mathbf{i}}f(x, y) = f_x(x, y)\cos 0 + f_y(x, y)\sin 0 = f_x(x, y)$$

2. Direction of positive y-axis $\left(\theta = \frac{\pi}{2}\right)$: $\mathbf{u} = \cos\frac{\pi}{2}\mathbf{i} + \sin\frac{\pi}{2}\mathbf{j} = \mathbf{j}$

$$D_{\mathbf{j}}f(x, y) = f_x(x, y)\cos\frac{\pi}{2} + f_y(x, y)\sin\frac{\pi}{2} = f_y(x, y)$$

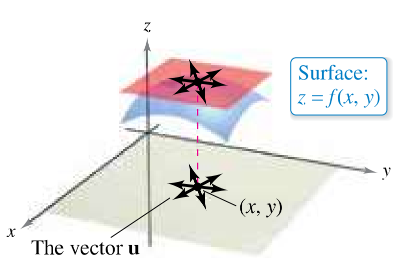


Figure 13.45

EXAMPLE 1 Finding a Directional Derivative

Find the directional derivative of

$$f(x, y) = 4 - x^2 - \frac{1}{4}y^2 \qquad \text{Surface}$$

at (1, 2) in the direction of

$$\mathbf{u} = \left(\cos\frac{\pi}{3}\right)\mathbf{i} + \left(\sin\frac{\pi}{3}\right)\mathbf{j}. \qquad \text{Direction}$$

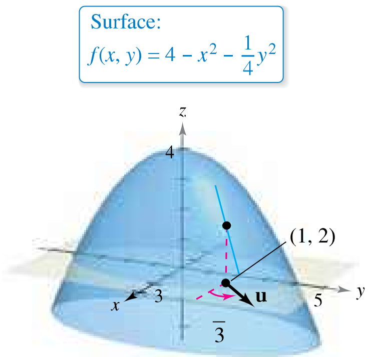


Figure 13.46

Solution Because $f_x(x, y) = -2x$ and $f_y(x, y) = -y/2$ are continuous, f is differentiable, and you can apply Theorem 13.9.

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta = (-2x)\cos\theta + \left(-\frac{y}{2}\right)\sin\theta$$

Evaluating at $\theta = \pi/3$, $x = 1$, and $y = 2$ produces

$$\begin{aligned} D_{\mathbf{u}} f(1, 2) &= (-2)\left(\frac{1}{2}\right) + (-1)\left(\frac{\sqrt{3}}{2}\right) \\ &= -1 - \frac{\sqrt{3}}{2} \\ &\approx -1.866. \end{aligned} \qquad \text{See Figure 13.46.}$$

Note in Figure 13.46 that you can interpret the directional derivative as giving the slope of the surface at the point (1, 2, 2) in the direction of the unit vector **u**.

You have been specifying direction by a unit vector **u**. When the direction is given by a vector whose length is not 1, you must normalize the vector before applying the formula in Theorem 13.9.

EXAMPLE 2 Finding a Directional Derivative

See LarsonCalculus.com for an interactive version of this type of example.

Find the directional derivative of

$$f(x, y) = x^2 \sin 2y \qquad \text{Surface}$$

at $(1, \pi/2)$ in the direction of

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}. \qquad \text{Direction}$$

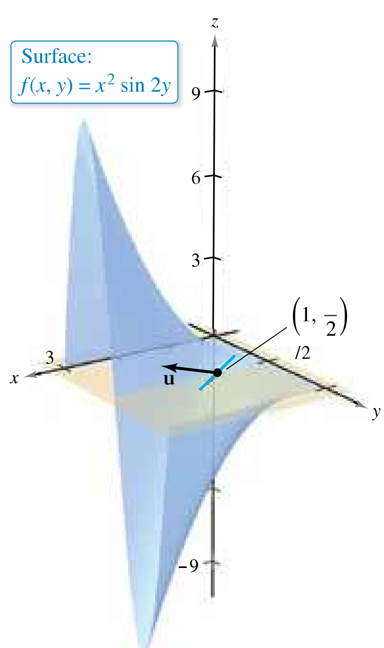


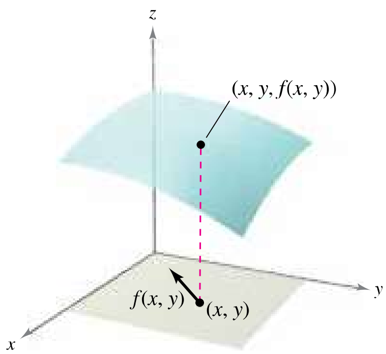
Figure 13.47

Solution Because $f_x(x, y) = 2x \sin 2y$ and $f_y(x, y) = 2x^2 \cos 2y$ are continuous, f is differentiable, and you can apply Theorem 13.9. Begin by finding a unit vector in the direction of **v**.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

Using this unit vector, you have

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= (2x \sin 2y)(\cos\theta) = (2x^2 \cos 2y)(\sin\theta) \\ D_{\mathbf{u}} f\left(1, \frac{\pi}{2}\right) &= (2\sin\pi)\left(\frac{3}{5}\right) + (2\cos\pi)\left(-\frac{4}{5}\right) \\ &= (0)\left(\frac{3}{5}\right) + (-2)\left(-\frac{4}{5}\right) \\ &= \frac{8}{5}. \end{aligned} \qquad \text{See Figure 13.47.}$$



The gradient of f is a vector in the xy-plane.
Figure 13.48

The Gradient of a Function of Two Variables

The **gradient** of a function of two variables is a vector-valued function of two variables. This function has many important uses, some of which are described later in this section.

> **Definition of Gradient of a Function of Two Variables**
>
> Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient of *f*,** denoted by $\nabla f(x, y)$, is the vector
>
> $$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$
>
> (The symbol ∇f is read as "del f.") Another notation for the gradient is given by **grad** $f(x, y)$. In Figure 13.48, note that for each (x, y), the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).

Notice that no value is assigned to the symbol ∇ by itself. It is an operator in the same sense that d/dx is an operator. When ∇ operates on $f(x, y)$, it produces the vector $\nabla f(x, y)$.

EXAMPLE 3 Finding the Gradient of a Function

Find the gradient of

$$f(x, y) = y \ln x + xy^2$$

at the point $(1, 2)$.

Solution Using

$$f_x(x, y) = \frac{y}{x} + y^2 \quad \text{and} \quad f_y(x, y) = \ln x + 2xy$$

you have

$$\begin{aligned}\nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}\\ &= \left(\frac{y}{x} + y^2\right)\mathbf{i} + (\ln x + 2xy)\mathbf{j}.\end{aligned}$$

At the point $(1, 2)$, the gradient is

$$\begin{aligned}\nabla f(1, 2) &= \left(\frac{2}{1} + 2^2\right)\mathbf{i} + [\ln 1 + 2(1)(2)]\mathbf{j}\\ &= 6\mathbf{i} + 4\mathbf{j}.\end{aligned}$$

Because the gradient of f is a vector, you can write the directional derivative of f in the direction of $\mathbf{u}$ as

$$D_{\mathbf{u}}f(x, y) = [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot (\cos\theta\mathbf{i} + \sin\theta\mathbf{j}).$$

In other words, the directional derivative is the dot product of the gradient and the direction vector. This useful result is summarized in the next theorem.

> **THEOREM 13.10 Alternative Form of the Directional Derivative**
>
> If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is
>
> $$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

EXAMPLE 4 Using $\nabla f(x, y)$ to Find a Directional Derivative

Find the directional derivative of $f(x, y) = 3x^2 - 2y^2$ at $\left(-\frac{3}{4}, 0\right)$ in the direction from $P\left(-\frac{3}{4}, 0\right)$ to $Q(0, 1)$.

Solution Because the partials of f are continuous, f is differentiable and you can apply Theorem 13.10. A vector in the specified direction is

$$\overrightarrow{PQ} = \left(0 + \frac{3}{4}\right)\mathbf{i} + (1 - 0)\mathbf{j} = \frac{3}{4}\mathbf{i} + \mathbf{j}$$

and a unit vector in this direction is

$$\mathbf{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}. \qquad \text{Unit vector in direction of } \overrightarrow{PQ}$$

Because $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 6x\mathbf{i} - 4y\mathbf{j}$, the gradient at $\left(-\frac{3}{4}, 0\right)$ is

$$\nabla f\left(-\frac{3}{4}, 0\right) = -\frac{9}{2}\mathbf{i} + 0\mathbf{j}. \qquad \text{Gradient at } \left(-\tfrac{3}{4}, 0\right)$$

Consequently, at $\left(-\frac{3}{4}, 0\right)$, the directional derivative is

$$\begin{aligned} D_{\mathbf{u}}f\left(-\frac{3}{4}, 0\right) &= \nabla f\left(-\frac{3}{4}, 0\right) \cdot \mathbf{u} \\ &= \left(-\frac{9}{2}\mathbf{i} + 0\mathbf{j}\right) \cdot \left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) \\ &= -\frac{27}{10}. \qquad \text{Directional derivative at } \left(-\tfrac{3}{4}, 0\right) \end{aligned}$$

See Figure 13.49.

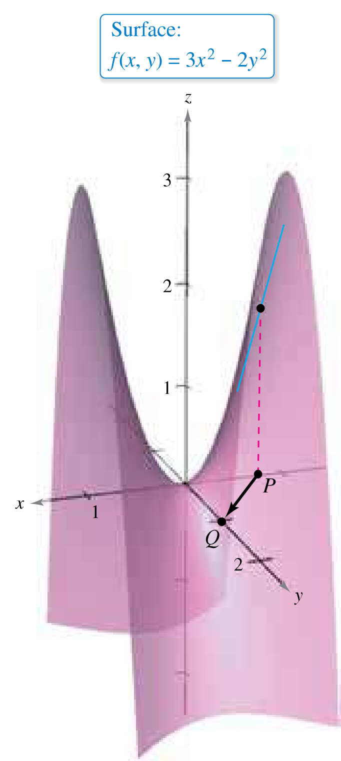


Figure 13.49

Applications of the Gradient

You have already seen that there are many directional derivatives at the point (x, y) on a surface. In many applications, you may want to know in which direction to move so that $f(x, y)$ increases most rapidly. This direction is called the direction of steepest ascent, and it is given by the gradient, as stated in the next theorem.

THEOREM 13.11 Properties of the Gradient

Let f be differentiable at the point (x, y).

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$.
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}} f(x, y)$ is

   $\|\nabla f(x, y)\|$. Maximum value of $D_{\mathbf{u}} f(x, y)$

3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}} f(x, y)$ is

   $-\|\nabla f(x, y)\|$. Minimum value of $D_{\mathbf{u}} f(x, y)$

REMARK Property 2 of Theorem 13.11 says that at the point (x, y), f increases most rapidly in the direction of the gradient, $\nabla f(x, y)$.

Proof If $\nabla f(x, y) = \mathbf{0}$, then for any direction (any $\mathbf{u}$), you have

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \nabla f(x, y) \cdot \mathbf{u} \\ &= (0\mathbf{i} + 0\mathbf{j}) \cdot (\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \\ &= 0. \end{aligned}$$

If $\nabla f(x, y) \neq \mathbf{0}$, then let ϕ be the angle between $\nabla f(x, y)$ and a unit vector $\mathbf{u}$. Using the dot product, you can apply Theorem 11.5 to conclude that

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \nabla f(x, y) \cdot \mathbf{u} \\ &= \|\nabla f(x, y)\| \, \|\mathbf{u}\| \cos\phi \\ &= \|\nabla f(x, y)\| \cos\phi \end{aligned}$$

and it follows that the maximum value of $D_{\mathbf{u}} f(x, y)$ will occur when

$$\cos\phi = 1.$$

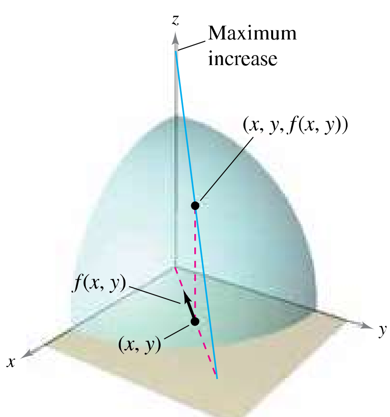
So, $\phi = 0$, and the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as $\nabla f(x, y)$. Moreover, this largest value of $D_{\mathbf{u}} f(x, y)$ is precisely

$$\|\nabla f(x, y)\| \cos\phi = \|\nabla f(x, y)\|.$$

Similarly, the minimum value of $D_{\mathbf{u}} f(x, y)$ can be obtained by letting

$$\phi = \pi$$

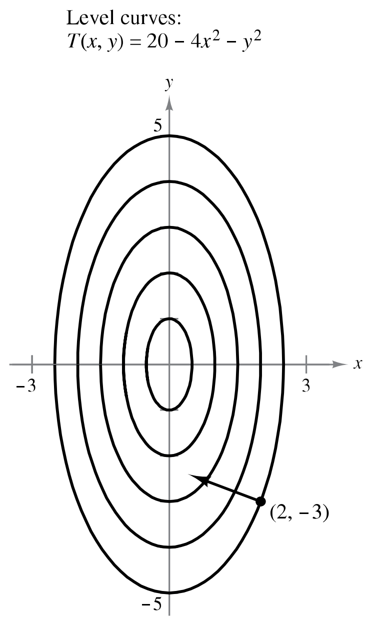
so that $\mathbf{u}$ points in the direction opposite that of $\nabla f(x, y)$, as shown in Figure 13.50. ∎



The gradient of f is a vector in the xy-plane that points in the direction of maximum increase on the surface given by $z = f(x, y)$.
Figure 13.50

To visualize one of the properties of the gradient, imagine a skier coming down a mountainside. If $f(x, y)$ denotes the altitude of the skier, then $-\nabla f(x, y)$ indicates the *compass direction* the skier should take to ski the path of steepest descent. (Remember that the gradient indicates direction in the xy-plane and does not itself point up or down the mountainside.)

As another illustration of the gradient, consider the temperature $T(x, y)$ at any point (x, y) on a flat metal plate. In this case, $\nabla T(x, y)$ gives the direction of greatest temperature increase at the point (x, y), as illustrated in the next example.



The direction of most rapid increase in temperature at $(2, -3)$ is given by $-16\mathbf{i} + 6\mathbf{j}$.

Figure 13.51

EXAMPLE 5 Finding the Direction of Maximum Increase

The temperature in degrees Celsius on the surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

where x and y are measured in centimeters. In what direction from $(2, -3)$ does the temperature increase most rapidly? What is this rate of increase?

Solution The gradient is

$$\nabla T(x, y) = T_x(x, y)\mathbf{i} + T_y(x, y)\mathbf{j} = -8x\mathbf{i} - 2y\mathbf{j}.$$

It follows that the direction of maximum increase is given by

$$\nabla T(2, -3) = -16\mathbf{i} + 6\mathbf{j}$$

as shown in Figure 13.51, and the rate of increase is

$$\|\nabla T(2, -3)\| = \sqrt{256 + 36} = \sqrt{292} \approx 17.09° \text{ per centimeter.}$$

The solution presented in Example 5 can be misleading. Although the gradient points in the direction of maximum temperature increase, it does not necessarily point toward the hottest spot on the plate. In other words, the gradient provides a local solution to finding an increase relative to the temperature at the point $(2, -3)$. *Once you leave that position, the direction of maximum increase may change.*

EXAMPLE 6 Finding the Path of a Heat-Seeking Particle

A heat-seeking particle is located at the point $(2, -3)$ on a metal plate whose temperature at (x, y) is

$$T(x, y) = 20 - 4x^2 - y^2.$$

Find the path of the particle as it continuously moves in the direction of maximum temperature increase.

Solution Let the path be represented by the position vector

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

A tangent vector at each point $(x(t), y(t))$ is given by

$$\mathbf{r}'(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}.$$

Because the particle seeks maximum temperature increase, the directions of $\mathbf{r}'(t)$ and $\nabla T(x, y) = -8x\mathbf{i} - 2y\mathbf{j}$ are the same at each point on the path. So,

$$-8x = k\frac{dx}{dt} \quad \text{and} \quad -2y = k\frac{dy}{dt}$$

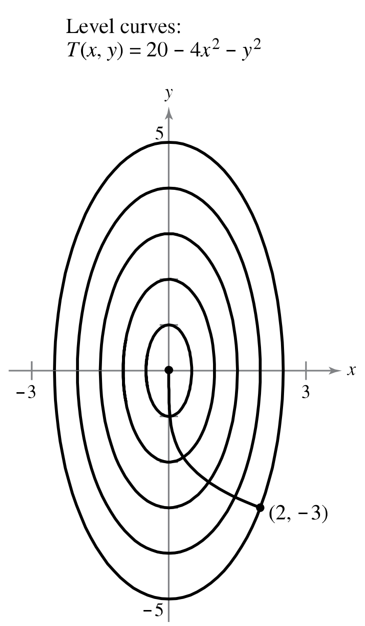
where k depends on t. By solving each equation for dt/k and equating the results, you obtain

$$\frac{dx}{-8x} = \frac{dy}{-2y}.$$

The solution of this differential equation is $x = Cy^4$. Because the particle starts at the point $(2, -3)$, you can determine that $C = 2/81$. So, the path of the heat-seeking particle is

$$x = \frac{2}{81}y^4.$$

The path is shown in Figure 13.52.



Path followed by a heat-seeking particle

Figure 13.52

In Figure 13.52, the path of the particle (determined by the gradient at each point) appears to be orthogonal to each of the level curves. This becomes clear when you consider that the temperature $T(x, y)$ is constant along a given level curve. So, at any point (x, y) on the curve, the rate of change of T in the direction of a unit tangent vector $\mathbf{u}$ is 0, and you can write

$$\nabla f(x, y) \cdot \mathbf{u} = D_{\mathbf{u}}T(x, y) = 0. \qquad \mathbf{u} \text{ is a unit tangent vector.}$$

Because the dot product of $\nabla f(x, y)$ and $\mathbf{u}$ is 0, you can conclude that they must be orthogonal. This result is stated in the next theorem.

THEOREM 13.12 Gradient Is Normal to Level Curves

If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

EXAMPLE 7 Finding a Normal Vector to a Level Curve

Sketch the level curve corresponding to $c = 0$ for the function given by

$$f(x, y) = y - \sin x$$

and find a normal vector at several points on the curve.

Solution The level curve for $c = 0$ is given by

$$0 = y - \sin x \quad \Longrightarrow \quad y = \sin x$$

as shown in Figure 13.53(a). Because the gradient of f at (x, y) is

$$\begin{aligned}\nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}\\ &= -\cos x\,\mathbf{i} + \mathbf{j}\end{aligned}$$

you can use Theorem 13.12 to conclude that $\nabla f(x, y)$ is normal to the level curve at the point (x, y). Some gradients are

$$\nabla f(-\pi, 0) = \mathbf{i} + \mathbf{j}$$

$$\nabla f\left(-\frac{2\pi}{3}, -\frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f\left(-\frac{\pi}{2}, -1\right) = \mathbf{j}$$

$$\nabla f\left(-\frac{\pi}{3}, -\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f(0, 0) = -\mathbf{i} + \mathbf{j}$$

$$\nabla f\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$

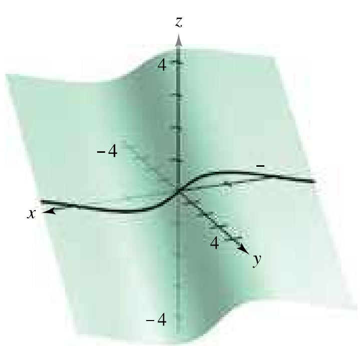
$$\nabla f\left(\frac{\pi}{2}, 1\right) = \mathbf{j}$$

$$\nabla f\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

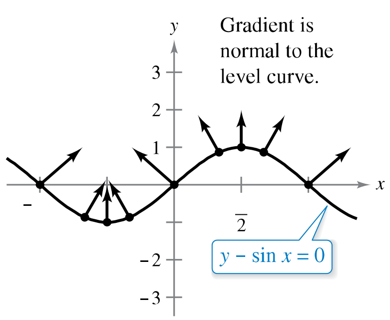
and

$$\nabla f(\pi, 0) = \mathbf{i} + \mathbf{j}.$$

These are shown in Figure 13.53(b).



(a) The surface is given by $f(x, y) = y - \sin x$.



(b) The level curve is given by $f(x, y) = 0$.

Figure 13.53

Functions of Three Variables

The definitions of the directional derivative and the gradient can be extended naturally to functions of three or more variables. As often happens, some of the geometric interpretation is lost in the generalization from functions of two variables to those of three variables. For example, you cannot interpret the directional derivative of a function of three variables as representing slope.

The definitions and properties of the directional derivative and the gradient of a function of three variables are listed below.

Directional Derivative and Gradient for Three Variables

Let f be a function of x, y, and z with continuous first partial derivatives. The **directional derivative of f** in the direction of a unit vector

$$\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

is given by

$$D_{\mathbf{u}} f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient of f** is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$

2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all $\mathbf{u}$.

3. The direction of *maximum* increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is

$$\|\nabla f(x, y, z)\|. \qquad \text{Maximum value of } D_{\mathbf{u}} f(x, y, z)$$

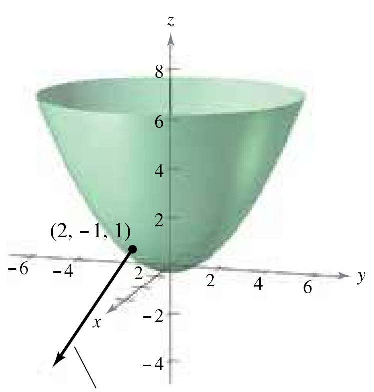
4. The direction of *minimum* increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is

$$-\|\nabla f(x, y, z)\|. \qquad \text{Minimum value of } D_{\mathbf{u}} f(x, y, z)$$

You can generalize Theorem 13.12 to functions of three variables. Under suitable hypotheses,

$$\nabla f(x_0, y_0, z_0)$$

is normal to the level surface through (x_0, y_0, z_0).



Level surface and gradient at $(2, -1, 1)$ for $f(x, y, z) = x^2 + y^2 - 4z$

Figure 13.54

EXAMPLE 8 Finding the Gradient of a Function

Find $\nabla f(x, y, z)$ for the function

$$f(x, y, z) = x^2 + y^2 - 4z$$

and find the direction of maximum increase of f at the point $(2, -1, 1)$.

Solution The gradient is

$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}\\ &= 2x\mathbf{i} + 2y\mathbf{j} - 4\mathbf{k}.\end{aligned}$$

So, it follows that the direction of maximum increase at $(2, -1, 1)$ is

$$\nabla f(2, -1, 1) = 4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}.$$ See Figure 13.54.

13.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Directional Derivative** For a function $f(x, y)$, when does the directional derivative at the point (x_0, y_0) equal the partial derivative with respect to x at the point (x_0, y_0)? What does this mean graphically?

2. **Gradient** What is the meaning of the gradient of a function f at a point (x, y)?

Finding a Directional Derivative In Exercises 3–6, use Theorem 13.9 to find the directional derivative of the function at P in the direction of the unit vector $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.

3. $f(x, y) = x^2 + y^2$, $P(1, -2)$, $\theta = \dfrac{\pi}{4}$

4. $f(x, y) = \dfrac{y}{x + y}$, $P(3, 0)$, $\theta = -\dfrac{\pi}{6}$

5. $f(x, y) = \sin(2x + y)$, $P(0, \pi)$, $\theta = -\dfrac{5\pi}{6}$

6. $g(x, y) = xe^y$, $P(0, 2)$, $\theta = \dfrac{2\pi}{3}$

Finding a Directional Derivative In Exercises 7–10, use Theorem 13.9 to find the directional derivative of the function at P in the direction of $\mathbf{v}$.

7. $f(x, y) = 3x - 4xy + 9y$, $P(1, 2)$, $\mathbf{v} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

8. $f(x, y) = x^3 - y^3$, $P(4, 3)$, $\mathbf{v} = \dfrac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

9. $g(x, y) = \sqrt{x^2 + y^2}$, $P(3, 4)$, $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

10. $h(x, y) = e^{-(x^2+y^2)}$, $P(0, 0)$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$

Finding a Directional Derivative In Exercises 11–14, use Theorem 13.9 to find the directional derivative of the function at P in the direction of $\overrightarrow{PQ}$.

11. $f(x, y) = x^2 + 3y^2$, $P(1, 1)$, $Q(4, 5)$

12. $f(x, y) = \cos(x + y)$, $P(0, \pi)$, $Q\left(\dfrac{\pi}{2}, 0\right)$

13. $f(x, y) = e^y \sin x$, $P(0, 0)$, $Q(2, 1)$

14. $f(x, y) = \sin 2x \cos y$, $P(\pi, 0)$, $Q\left(\dfrac{\pi}{2}, \pi\right)$

Finding the Gradient of a Function In Exercises 15–20, find the gradient of the function at the given point.

15. $f(x, y) = 3x + 5y^2 + 1$, $(2, 1)$

16. $g(x, y) = 2xe^{y/x}$, $(2, 0)$

17. $z = \dfrac{\ln(x^2 - y)}{x} - 4$, $(2, 3)$

18. $z = \cos(x^2 + y^2)$, $(3, -4)$

19. $w = 6xy - y^2 + 2xyz^3$, $(-1, 5, -1)$

20. $w = x\tan(y + z)$, $(4, 3, -1)$

Finding a Directional Derivative In Exercises 21–24, use the gradient to find the directional derivative of the function at P in the direction of $\mathbf{v}$.

21. $f(x, y) = xy$, $P(0, -2)$, $\mathbf{v} = \frac{1}{2}(\mathbf{i} + \sqrt{3}\mathbf{j})$

22. $h(x, y) = e^{-3x}\sin y$, $P\left(1, \dfrac{\pi}{2}\right)$, $\mathbf{v} = -\mathbf{i}$

23. $f(x, y, z) = x^2 + y^2 + z^2$, $P(1, 1, 1)$, $\mathbf{v} = \dfrac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$

24. $f(x, y, z) = xy + yz + xz$, $P(1, 2, -1)$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

Finding a Directional Derivative In Exercises 25–28, use the gradient to find the directional derivative of the function at P in the direction of $\overrightarrow{PQ}$.

25. $g(x, y) = x^2 + y^2 + 1$, $P(1, 2)$, $Q(2, 3)$

26. $f(x, y) = 3x^2 - y^2 + 4$, $P(-1, 4)$, $Q(3, 6)$

27. $g(x, y, z) = xye^z$, $P(2, 4, 0)$, $Q(0, 0, 0)$

28. $h(x, y, z) = \ln(x + y + z)$, $P(1, 0, 0)$, $Q(4, 3, 1)$

Using Properties of the Gradient In Exercises 29–38, find the gradient of the function and the maximum value of the directional derivative at the given point.

29. $f(x, y) = y^2 - x\sqrt{y}$, $(0, 3)$

30. $f(x, y) = \dfrac{x + y}{y + 1}$, $(0, 1)$

31. $h(x, y) = x\tan y$, $\left(2, \dfrac{\pi}{4}\right)$

32. $h(x, y) = y\cos(x - y)$, $\left(0, \dfrac{\pi}{3}\right)$

33. $f(x, y) = \sin x^2y^3$, $\left(\dfrac{1}{\pi}, \pi\right)$

34. $g(x, y) = \ln\sqrt[3]{x^2 + y^2}$, $(1, 2)$

35. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $(1, 4, 2)$

36. $w = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$, $(0, 0, 0)$

37. $w = xy^2z^2$, $(2, 1, 1)$

38. $f(x, y, z) = xe^{yz}$, $(2, 0, -4)$

Finding a Normal Vector to a Level Curve In Exercises 39–42, find a normal vector to the level curve $f(x, y) = c$ at P.

39. $f(x, y) = 6 - 2x - 3y$
$c = 6$, $P(0, 0)$

40. $f(x, y) = x^2 + y^2$
$c = 25$, $P(3, 4)$

41. $f(x, y) = xy$
$c = -3, \quad P(-1, 3)$

42. $f(x, y) = \dfrac{x}{x^2 + y^2}$
$c = \frac{1}{2}, \quad P(1, 1)$

Using a Function **In Exercises 43–46, (a) find the gradient of the function at P, (b) find a unit normal vector to the level curve $f(x, y) = c$ at P, (c) find the tangent line to the level curve $f(x, y) = c$ at P, and (d) sketch the level curve, the unit normal vector, and the tangent line in the xy-plane.**

43. $f(x, y) = 4x^2 - y$
$c = 6, \quad P(2, 10)$

44. $f(x, y) = x - y^2$
$c = 3, \quad P(4, -1)$

45. $f(x, y) = 3x^2 - 2y^2$
$c = 1, \quad P(1, 1)$

46. $f(x, y) = 9x^2 + 4y^2$
$c = 40, \quad P(2, -1)$

47. Using a Function Consider the function

$$f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}.$$

(a) Sketch the graph of f in the first octant and plot the point $(3, 2, 1)$ on the surface.

(b) Find $D_{\mathbf{u}} f(3, 2)$, where $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, using each given value of θ.

(i) $\theta = \dfrac{\pi}{4}$ (ii) $\theta = \dfrac{2\pi}{3}$ (iii) $\theta = \dfrac{4\pi}{3}$ (iv) $\theta = -\dfrac{\pi}{6}$

(c) Find $D_{\mathbf{u}} f(3, 2)$, where $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$, using each given vector $\mathbf{v}$.

(i) $\mathbf{v} = \mathbf{i} + \mathbf{j}$ (ii) $\mathbf{v} = -3\mathbf{i} - 4\mathbf{j}$

(iii) $\mathbf{v}$ is the vector from $(1, 2)$ to $(-2, 6)$.

(iv) $\mathbf{v}$ is the vector from $(3, 2)$ to $(4, 5)$.

(d) Find $\nabla f(x, y)$.

(e) Find the maximum value of the directional derivative at $(3, 2)$.

(f) Find a unit vector $\mathbf{u}$ orthogonal to $\nabla f(3, 2)$ and calculate $D_{\mathbf{u}} f(3, 2)$. Discuss the geometric meaning of the result.

48. Using a Function Consider the function

$$f(x, y) = 9 - x^2 - y^2.$$

(a) Sketch the graph of f in the first octant and plot the point $(1, 2, 4)$ on the surface.

(b) Find $D_{\mathbf{u}} f(1, 2)$, where $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, using each given value of θ.

(i) $\theta = -\dfrac{\pi}{4}$ (ii) $\theta = \dfrac{\pi}{3}$ (iii) $\theta = \dfrac{3\pi}{4}$ (iv) $\theta = -\dfrac{\pi}{2}$

(c) Find $D_{\mathbf{u}} f(1, 2)$, where $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$, using each given vector $\mathbf{v}$.

(i) $\mathbf{v} = 3\mathbf{i} + \mathbf{j}$ (ii) $\mathbf{v} = -8\mathbf{i} - 6\mathbf{j}$

(iii) $\mathbf{v}$ is the vector from $(-1, -1)$ to $(3, 5)$.

(iv) $\mathbf{v}$ is the vector from $(-2, 0)$ to $(1, 3)$.

(d) Find $\nabla f(1, 2)$.

(e) Find the maximum value of the directional derivative at $(1, 2)$.

(f) Find a unit vector $\mathbf{u}$ orthogonal to $\nabla f(1, 2)$ and calculate $D_{\mathbf{u}} f(1, 2)$. Discuss the geometric meaning of the result.

49. Investigation Consider the function

$$f(x, y) = x^2 - y^2$$

at the point $(4, -3, 7)$.

(a) Use a computer algebra system to graph the surface represented by the function.

(b) Determine the directional derivative $D_{\mathbf{u}} f(4, -3)$ as a function of θ, where $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$. Use a computer algebra system to graph the function on the interval $[0, 2\pi)$.

(c) Approximate the zeros of the function in part (b) and interpret each in the context of the problem.

(d) Approximate the critical numbers of the function in part (b) and interpret each in the context of the problem.

(e) Find $\|\nabla f(4, -3)\|$ and explain its relationship to your answers in part (d).

(f) Use a computer algebra system to graph the level curve of the function f at the level $c = 7$. On this curve, graph the vector in the direction of $\nabla f(4, -3)$ and state its relationship to the level curve.

50. Investigation Consider the function

$$f(x, y) = \frac{8y}{1 + x^2 + y^2}.$$

(a) Analytically verify that the level curve of $f(x, y)$ at the level $c = 2$ is a circle.

(b) At the point $(\sqrt{3}, 2)$ on the level curve for which $c = 2$, sketch the vector showing the direction of the greatest rate of increase of the function. To print a graph of the level curve, go to *MathGraphs.com.*

(c) At the point $(\sqrt{3}, 2)$ on the level curve for which $c = 2$, sketch a vector such that the directional derivative is 0.

(d) Use a computer algebra system to graph the surface to verify your answers in parts (a)–(c).

EXPLORING CONCEPTS

51. Think About It Consider $\mathbf{v} = 3\mathbf{u}$. Is the directional derivative of a differentiable function $f(x, y)$ in the direction of $\mathbf{v}$ at the point (x_0, y_0) three times the directional derivative of f in the direction of $\mathbf{u}$ at the point (x_0, y_0)? Explain.

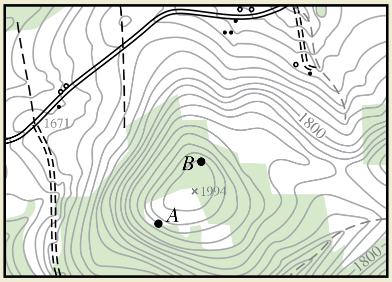
52. Sketching a Graph and a Vector Sketch the graph of a surface and select a point P on the surface. Sketch a vector in the xy-plane giving the direction of steepest ascent on the surface at P.

53. Topography The surface of a mountain is modeled by the equation

$$h(x, y) = 5000 - 0.001x^2 - 0.004y^2.$$

A mountain climber is at the point $(500, 300, 4390)$. In what direction should the climber move in order to ascend at the greatest rate?

54. HOW DO YOU SEE IT? The figure shows a topographic map carried by a group of hikers. Sketch the paths of steepest descent when the hikers start at point A and when they start at point B. (To print an enlarged copy of the graph, go to *MathGraphs.com*.)



55. Temperature The temperature at the point (x, y) on a metal plate is $T(x, y) = x/(x^2 + y^2)$. Find the direction of greatest increase in heat from the point $(3, 4)$.

56. Temperature The temperature at the point (x, y) on a metal plate is $T(x, y) = 400e^{-(x^2+y)/2}$, $x \geq 0$, $y \geq 0$.

(a) Use a computer algebra system to graph the temperature distribution function.

(b) Find the directions of no change in heat on the plate from the point $(3, 5)$.

(c) Find the direction of greatest increase in heat from the point $(3, 5)$.

Finding the Direction of Maximum Increase **In Exercises 57 and 58, the temperature in degrees Celsius on the surface of a metal plate is given by $T(x, y)$, where x and y are measured in centimeters. Find the direction from point P where the temperature increases most rapidly and this rate of increase.**

57. $T(x, y) = 80 - 3x^2 - y^2$, $\quad P(-1, 5)$

58. $T(x, y) = 50 - x^2 - 4y^2$, $\quad P(2, -1)$

Finding the Path of a Heat-Seeking Particle **In Exercises 59 and 60, find the path of a heat-seeking particle placed at point P on a metal plate whose temperature at (x, y) is $T(x, y)$.**

59. $T(x, y) = 400 - 2x^2 - y^2$, $\quad P(10, 10)$

60. $T(x, y) = 100 - x^2 - 2y^2$, $\quad P(4, 3)$

True or False? **In Exercises 61–64, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

61. If $f(x, y) = \sqrt{1 - x^2 - y^2}$, then $D_{\mathbf{u}} f(0, 0) = 0$ for any unit vector $\mathbf{u}$.

62. If $f(x, y) = x + y$, then $-1 \leq D_{\mathbf{u}} f(x, y) \leq 1$.

63. If $D_{\mathbf{u}} f(x, y)$ exists, then $D_{\mathbf{u}} f(x, y) = -D_{\mathbf{u}} f(x, y)$.

64. If $D_{\mathbf{u}} f(x_0, y_0) = c$ for any unit vector $\mathbf{u}$, then $c = 0$.

65. Finding a Function Find a function f such that

$$\nabla f = e^x \cos y\,\mathbf{i} - e^x \sin y\,\mathbf{j} + z\mathbf{k}.$$

66. Ocean Floor

A team of oceanographers is mapping the ocean floor to assist in the recovery of a sunken ship. Using sonar, they develop the model

$$D = 250 + 30x^2 + 50 \sin \frac{\pi y}{2}, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 2$$

where D is the depth in meters, and x and y are the distances in kilometers.

(a) Use a computer algebra system to graph D.

(b) Because the graph in part (a) is showing depth, it is not a map of the ocean floor. How could the model be changed so that the graph of the ocean floor could be obtained?

(c) What is the depth of the ship if it is located at the coordinates $x = 1$ and $y = 0.5$?

(d) Determine the steepness of the ocean floor in the positive x-direction from the position of the ship.

(e) Determine the steepness of the ocean floor in the positive y-direction from the position of the ship.

(f) Determine the direction of the greatest rate of change of depth from the position of the ship.

67. Using a Function Consider the function

$$f(x, y) = \sqrt[3]{xy}.$$

(a) Show that f is continuous at the origin.

(b) Show that f_x and f_y exist at the origin but that the directional derivatives at the origin in all other directions do not exist.

(c) Use a computer algebra system to graph f near the origin to verify your answers in parts (a) and (b). Explain.

68. Directional Derivative Consider the function

$$f(x, y) = \begin{cases} \dfrac{4xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) \neq (0, 0) \end{cases}$$

and the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}).$$

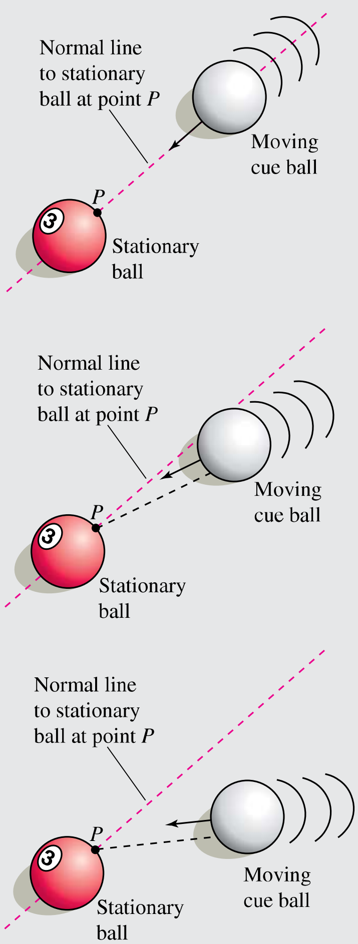
Does the directional derivative of f at $P(0, 0)$ in the direction of $\mathbf{u}$ exist? If $f(0, 0)$ were defined as 2 instead of 0, would the directional derivative exist? Explain.

13.7 Tangent Planes and Normal Lines

- **Find equations of tangent planes and normal lines to surfaces.**
- **Find the angle of inclination of a plane in space.**
- **Compare the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$.**

Exploration

Billiard Balls and Normal Lines In each of the three figures below, the cue ball is about to strike a stationary ball at point P. Explain how you can use the normal line to the stationary ball at point P to describe the resulting motion of each of the two balls. Assuming that each cue ball has the same speed, which stationary ball will acquire the greatest speed? Which will acquire the least? Explain your reasoning.



Tangent Plane and Normal Line to a Surface

So far, you have represented surfaces in space primarily by equations of the form

$$z = f(x, y). \qquad \text{Equation of a surface } S$$

In the development to follow, however, it is convenient to use the more general representation $F(x, y, z) = 0$. For a surface S given by $z = f(x, y)$, you can convert to the general form by defining F as

$$F(x, y, z) = f(x, y) - z.$$

Because $f(x, y) - z = 0$, you can consider S to be the level surface of F given by

$$F(x, y, z) = 0. \qquad \text{Alternative equation of surface } S$$

EXAMPLE 1 **Writing an Equation of a Surface**

For the function

$$F(x, y\, z) = x^2 + y^2 + z^2 - 4$$

describe the level surface given by

$$F(x, y, z) = 0.$$

Solution The level surface given by $F(x, y, z) = 0$ can be written as

$$x^2 + y^2 + z^2 = 4$$

which is a sphere of radius 2 whose center is at the origin.

You have seen many examples of the usefulness of normal lines in applications involving curves. Normal lines are equally important in analyzing surfaces and solids. For example, consider the collision of two billiard balls. When a stationary ball is struck at a point P on its surface, it moves along the **line of impact** determined by P and the center of the ball. The impact can occur in *two* ways. When the cue ball is moving along the line of impact, it stops dead and imparts all of its momentum to the stationary ball, as shown in Figure 13.55.

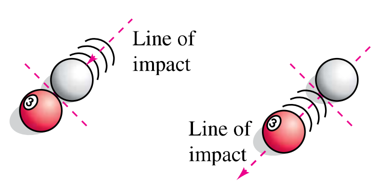


Figure 13.55

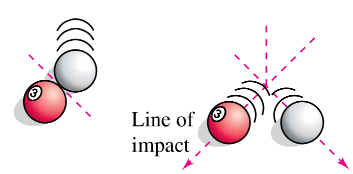


Figure 13.56

When the cue ball is not moving along the line of impact, it is deflected to one side or the other and retains part of its momentum. The part of the momentum that is transferred to the stationary ball occurs along the line of impact, *regardless* of the direction of the cue ball, as shown in Figure 13.56. This line of impact is called the **normal line** to the surface of the ball at the point P.

In the process of finding a normal line to a surface, you are also able to solve the problem of finding a **tangent plane** to the surface. Let S be a surface given by

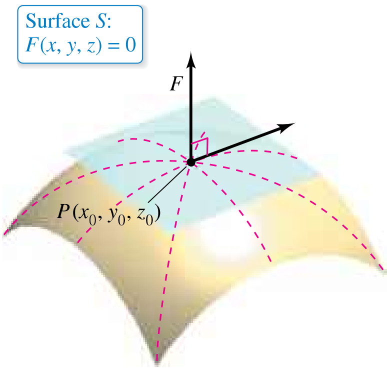
$$F(x, y, z) = 0$$

and let $P(x_0, y_0, z_0)$ be a point on S. Let C be a curve on S through P that is defined by the vector-valued function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

Then, for all t,

$$F(x(t), y(t), z(t)) = 0.$$



Tangent plane to surface S at P
Figure 13.57

If F is differentiable and $x'(t)$, $y'(t)$, and $z'(t)$ all exist, then it follows from the Chain Rule that

$$\begin{aligned} 0 &= F'(t) \\ &= F_x(x, y, z)x'(t) + F_y(x, y, z)y'(t) + F_z(x, y, z)z'(t). \end{aligned}$$

At (x_0, y_0, z_0), the equivalent vector form is

$$0 = \underbrace{\nabla F(x_0, y_0, z_0)}_{\text{Gradient}} \cdot \underbrace{\mathbf{r}'(t_0)}_{\text{Tangent vector}}.$$

This result means that the gradient at P is orthogonal to the tangent vector of every curve on S through P. So, all tangent lines on S lie in a plane that is normal to $\nabla F(x_0, y_0, z_0)$ and contains P, as shown in Figure 13.57.

REMARK In the remainder of this section, assume $\nabla F(x_0, y_0, z_0)$ to be nonzero unless stated otherwise.

Definitions of Tangent Plane and Normal Line

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface S given by $F(x, y, z) = 0$ such that

$$\nabla F(x_0, y_0, z_0) \neq \mathbf{0}.$$

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane to S at P.**
2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line to S at P.**

To find an equation for the tangent plane to S at (x_0, y_0, z_0), let (x, y, z) be an arbitrary point in the tangent plane. Then the vector

$$\mathbf{v} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

lies in the tangent plane. Because $\nabla F(x_0, y_0, z_0)$, is normal to the tangent plane at (x_0, y_0, z_0), it must be orthogonal to every vector in the tangent plane, and you have

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{v} = 0$$

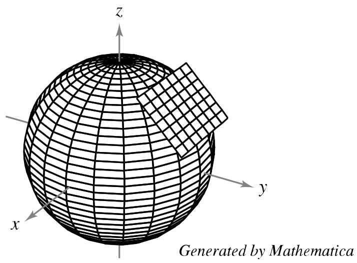
which leads to the next theorem.

THEOREM 13.13 Equation of Tangent Plane

If F is differentiable at (x_0, y_0, z_0), then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

TECHNOLOGY Some three-dimensional graphing utilities are capable of graphing tangent planes to surfaces. An example is shown below.



Sphere: $x^2 + y^2 + z^2 = 1$

EXAMPLE 2 Finding an Equation of a Tangent Plane

Find an equation of the tangent plane to the hyperboloid

$$z^2 - 2x^2 - 2y^2 = 12$$

at the point $(1, -1, 4)$.

Solution Begin by writing the equation of the surface as

$$z^2 - 2x^2 - 2y^2 - 12 = 0.$$

Then, considering

$$F(x, y, z) = z^2 - 2x^2 - 2y^2 - 12$$

you have

$$F_x(x, y, z) = -4x, \quad F_y(x, y, z) = -4y, \quad \text{and} \quad F_z(x, y, z) = 2z.$$

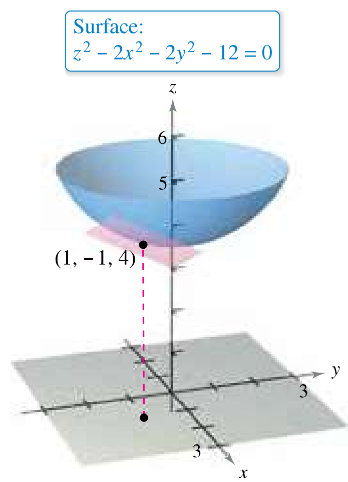
At the point $(1, -1, 4)$, the partial derivatives are

$$F_x(1, -1, 4) = -4, \quad F_y(1, -1, 4) = 4, \quad \text{and} \quad F_z(1, -1, 4) = 8.$$

So, an equation of the tangent plane at $(1, -1, 4)$ is

$$\begin{aligned} -4(x - 1) + 4(y + 1) + 8(z - 4) &= 0 \\ -4x + 4 + 4y + 4 + 8z - 32 &= 0 \\ -4x + 4y + 8z - 24 &= 0 \\ x - y - 2z + 6 &= 0. \end{aligned}$$

Figure 13.58 shows a portion of the hyperboloid and the tangent plane.



Tangent plane to surface
Figure 13.58

To find an equation of the tangent plane at a point on a surface given by $z = f(x, y)$, you can define the function F by

$$F(x, y, z) = f(x, y) - z.$$

Then S is given by the level surface $F(x, y, z) = 0$, and by Theorem 13.13, an equation of the tangent plane to S at the point (x_0, y_0, z_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

EXAMPLE 3 Finding an Equation of the Tangent Plane

Find an equation of the tangent plane to the paraboloid

$$z = 1 - \frac{1}{10}(x^2 + 4y^2)$$

at the point $\left(1, 1, \frac{1}{2}\right)$.

Solution From $z = f(x, y) = 1 - \frac{1}{10}(x^2 + 4y^2)$, you obtain

$$f_x(x, y) = -\frac{x}{5} \quad \Rightarrow \quad f_x(1, 1) = -\frac{1}{5}$$

and

$$f_y(x, y) = -\frac{4y}{5} \quad \Rightarrow \quad f_y(1, 1) = -\frac{4}{5}.$$

So, an equation of the tangent plane at $\left(1, 1, \frac{1}{2}\right)$ is

$$f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) - \left(z - \frac{1}{2}\right) = 0$$

$$-\frac{1}{5}(x - 1) - \frac{4}{5}(y - 1) - \left(z - \frac{1}{2}\right) = 0$$

$$-\frac{1}{5}x - \frac{4}{5}y - z + \frac{3}{2} = 0.$$

This tangent plane is shown in Figure 13.59.

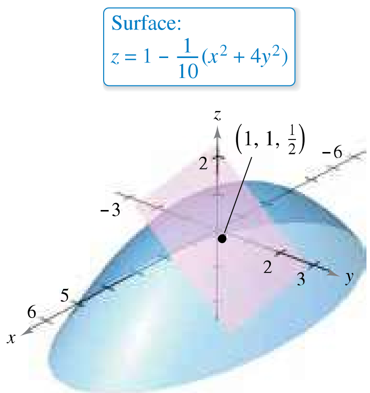


Figure 13.59

The gradient $\nabla F(x, y, z)$ provides a convenient way to find equations of normal lines, as shown in Example 4.

EXAMPLE 4 Finding an Equation of a Normal Line to a Surface

See LarsonCalculus.com for an interactive version of this type of example.

Find a set of symmetric equations for the normal line to the surface

$$xyz = 12$$

at the point $(2, -2, -3)$.

Solution Begin by letting

$$F(x, y, z) = xyz - 12.$$

Then, the gradient is given by

$$\begin{aligned}\nabla F(x, y, z) &= F_x(x, y, z)\mathbf{i} + F_y(x, y, z)\mathbf{j} + F_z(x, y, z)\mathbf{k}\\ &= yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}\end{aligned}$$

and at the point $(2, -2, -3)$, you have

$$\begin{aligned}\nabla F(2, -2, -3) &= (-2)(-3)\mathbf{i} + (2)(-3)\mathbf{j} + (2)(-2)\mathbf{k}\\ &= 6\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}.\end{aligned}$$

The normal line at $(2, -2, -3)$ has direction numbers 6, -6, and -4, and the corresponding set of symmetric equations is

$$\frac{x - 2}{6} = \frac{y + 2}{-6} = \frac{z + 3}{-4}.$$

See Figure 13.60.

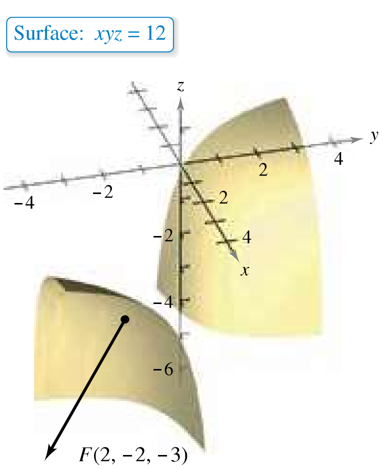


Figure 13.60

Knowing that the gradient $\nabla F(x, y, z)$ is normal to the surface given by $F(x, y, z) = 0$ allows you to solve a variety of problems dealing with surfaces and curves in space.

EXAMPLE 5 Finding the Equation of a Tangent Line to a Curve

Find a set of parametric equations for the tangent line to the curve of intersection of the ellipsoid

$$x^2 + 2y^2 + 2z^2 = 20 \qquad \text{Ellipsoid}$$

and the paraboloid

$$x^2 + y^2 + z = 4 \qquad \text{Paraboloid}$$

at the point (0, 1, 3), as shown in Figure 13.61.

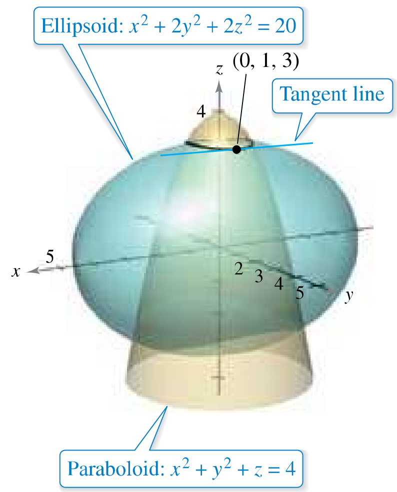


Figure 13.61

Solution Begin by finding the gradients to both surfaces at the point (0, 1, 3).

Ellipsoid	**Paraboloid**
$F(x, y, z) = x^2 + 2y^2 + 2z^2 - 20$	$G(x, y, z) = x^2 + y^2 + z - 4$
$\nabla F(x, y, z) = 2x\mathbf{i} + 4y\mathbf{j} + 4z\mathbf{k}$	$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$
$\nabla F(0, 1, 3) = 4\mathbf{j} + 12\mathbf{k}$	$\nabla G(0, 1, 3) = 2\mathbf{j} + \mathbf{k}$

The cross product of these two gradients is a vector that is tangent to both surfaces at the point (0, 1, 3).

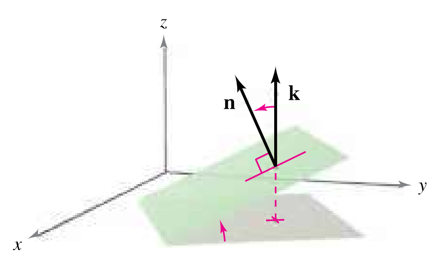
$$\nabla F(0, 1, 3) \times \nabla G(0, 1, 3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 12 \\ 0 & 2 & 1 \end{vmatrix} = -20\mathbf{i}$$

So, the tangent line to the curve of intersection of the two surfaces at the point (0, 1, 3) is a line that is parallel to the x-axis and passes through the point (0, 1, 3). Because $-20\mathbf{i} = -20(\mathbf{i} + 0\mathbf{j} + 0\mathbf{k})$, the direction numbers are 1, 0, and 0. So a set of parametric equations for the tangent line passing through the point (0, 1, 3) is $x = t$, $y = 1$, and $z = 3$.

The Angle of Inclination of a Plane

Another use of the gradient $\nabla F(x, y, z)$ is to determine the angle of inclination of the tangent plane to a surface. The **angle of inclination** of a plane is defined as the angle θ $(0 \le \theta \le \pi/2)$ between the given plane and the xy-plane, as shown in Figure 13.62. (The angle of inclination of a horizontal plane is defined as zero.) Because the vector $\mathbf{k}$ is normal to the xy-plane, you can use the formula for the cosine of the angle between two planes (given in Section 11.5) to conclude that the angle of inclination of a plane with normal vector $\mathbf{n}$ is

$$\cos\theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|\,\|\mathbf{k}\|} = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|}. \qquad \text{Angle of inclination of a plane}$$



The angle of inclination
Figure 13.62

EXAMPLE 6 Finding the Angle of Inclination of a Tangent Plane

Find the angle of inclination of the tangent plane to the ellipsoid

$$\frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{3} = 1$$

at the point $(2, 2, 1)$.

Solution Begin by letting

$$F(x, y, z) = \frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{3} - 1.$$

Then, the gradient of F at the point $(2, 2, 1)$ is

$$\nabla F(x, y, z) = \frac{x}{6}\mathbf{i} + \frac{y}{6}\mathbf{j} + \frac{2z}{3}\mathbf{k}$$

$$\nabla F(2, 2, 1) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

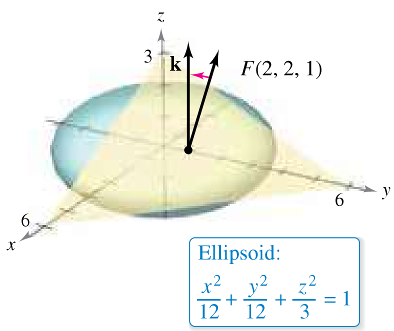


Figure 13.63

Because $\nabla F(2, 2, 1)$ is normal to the tangent plane and $\mathbf{k}$ is normal to the xy-plane, it follows that the angle of inclination of the tangent plane is

$$\cos\theta = \frac{|\nabla F(2, 2, 1) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 1)\|} = \frac{2/3}{\sqrt{(1/3)^2 + (1/3)^2 + (2/3)^2}} = \sqrt{\frac{2}{3}}$$

which implies that

$$\theta = \arccos\sqrt{\frac{2}{3}} \approx 35.3^\circ$$

as shown in Figure 13.63.

A special case of the procedure shown in Example 6 is worth noting. The angle of inclination θ of the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0, z_0) is

$$\cos\theta = \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}.$$

Alternative formula for angle of inclination (See Exercise 63.)

A Comparison of the Gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$

This section concludes with a comparison of the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$. In the preceding section, you saw that the gradient of a function f of two variables is normal to the level curves of f. Specifically, Theorem 13.12 states that if f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0). Having developed normal lines to surfaces, you can now extend this result to a function of three variables. The proof of Theorem 13.14 is left as an exercise (see Exercise 64).

THEOREM 13.14 Gradient Is Normal to Level Surfaces

If F is differentiable at (x_0, y_0, z_0) and

$$\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$$

then $\nabla F(x_0, y_0, z_0)$ is normal to the level surface through (x_0, y_0, z_0).

When working with the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$, be sure you remember that $\nabla f(x, y)$ is a vector in the xy-plane and $\nabla F(x, y, z)$ is a vector in space.

13.7 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Tangent Vector** Consider a point (x_0, y_0, z_0) on a surface given by $F(x, y, z) = 0$. What is the relationship between $\nabla F(x_0, y_0, z_0)$ and any tangent vector $\mathbf{v}$ at (x_0, y_0, z_0)? How do you represent this relationship mathematically?

2. **Normal Line** Consider a point (x_0, y_0, z_0) on a surface given by $F(x, y, z) = 0$. What is the relationship between $\nabla F(x_0, y_0, z_0)$ and the normal line through (x_0, y_0, z_0)?

Describing a Surface **In Exercises 3–6, describe the level surface $F(x, y, z) = 0$.**

3. $F(x, y, z) = 3x - 5y + 3z - 15$
4. $F(x, y, z) = 36 - x^2 - y^2 - z^2$
5. $F(x, y, z) = 4x^2 + 9y^2 - 4z^2$
6. $F(x, y, z) = 16x^2 - 9y^2 + 36z$

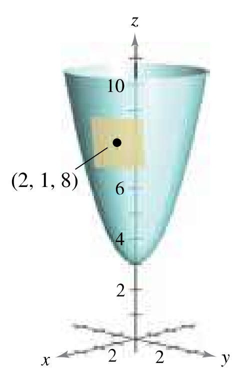
Finding an Equation of a Tangent Plane **In Exercises 7–16, find an equation of the tangent plane to the surface at the given point.**

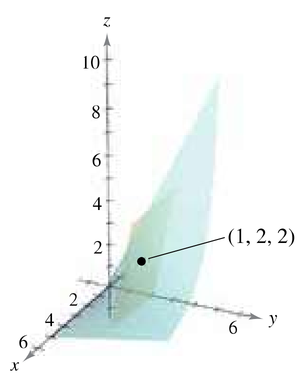
7. $z = x^2 + y^2 + 3$

   $(2, 1, 8)$

8. $f(x, y) = \dfrac{y}{x}$

   $(1, 2, 2)$





9. $z = \sqrt{x^2 + y^2}$, $(3, 4, 5)$
10. $g(x, y) = \arctan \dfrac{y}{x}$, $(1, 0, 0)$
11. $g(x, y) = x^2 + y^2$, $(1, -1, 2)$
12. $f(x, y) = x^2 - 2xy + y^2$, $(1, 2, 1)$
13. $h(x, y) = \ln\sqrt{x^2 + y^2}$, $(3, 4, \ln 5)$
14. $f(x, y) = \sin x \cos y$, $\left(\dfrac{\pi}{3}, \dfrac{\pi}{6}, \dfrac{3}{4}\right)$
15. $x^2 + y^2 - 5z^2 = 15$, $(-4, -2, 1)$
16. $x^2 + 2z^2 = y^2$, $(1, 3, -2)$

Finding an Equation of a Tangent Plane and a Normal Line **In Exercises 17–26, (a) find an equation of the tangent plane to the surface at the given point and (b) find a set of symmetric equations for the normal line to the surface at the given point.**

17. $x + y + z = 9$, $(3, 3, 3)$
18. $x^2 + y^2 + z^2 = 9$, $(1, 2, 2)$
19. $x^2 + 2y^2 + z^2 = 7$, $(1, -1, 2)$
20. $z = 16 - x^2 - y^2$, $(2, 2, 8)$
21. $z = x^2 - y^2$, $(3, 2, 5)$
22. $xy - z = 0$, $(-2, -3, 6)$
23. $xyz = 10$, $(1, 2, 5)$
24. $6xy = z$, $(-1, 1, -6)$
25. $z = ye^{2xy}$, $(0, 2, 2)$
26. $y \ln xz^2 = 2$, $(e, 2, 1)$

Finding the Equation of a Tangent Line to a Curve **In Exercises 27–32, find a set of parametric equations for the tangent line to the curve of intersection of the surfaces at the given point.**

27. $x^2 + y^2 = 2$, $z = x$, $(1, 1, 1)$
28. $z = x^2 + y^2$, $z = 4 - y$, $(2, -1, 5)$
29. $x^2 + z^2 = 25$, $y^2 + z^2 = 25$, $(3, 3, 4)$
30. $z = \sqrt{x^2 + y^2}$, $5x - 2y + 3z = 22$, $(3, 4, 5)$
31. $x^2 + y^2 + z^2 = 14$, $x - y - z = 0$, $(3, 1, 2)$
32. $z = x^2 + y^2$, $x + y + 6z = 33$, $(1, 2, 5)$

Finding the Angle of Inclination of a Tangent Plane **In Exercises 33–36, find the angle of inclination of the tangent plane to the surface at the given point.**

33. $3x^2 + 2y^2 - z = 15$, $(2, 2, 5)$
34. $2xy - z^3 = 0$, $(2, 2, 2)$
35. $x^2 - y^2 + z = 0$, $(1, 2, 3)$
36. $x^2 + y^2 = 5$, $(2, 1, 3)$

Horizontal Tangent Plane **In Exercises 37–42, find the point(s) on the surface at which the tangent plane is horizontal.**

37. $z = 3 - x^2 - y^2 + 6y$
38. $z = 3x^2 + 2y^2 - 3x + 4y - 5$
39. $z = x^2 - xy + y^2 - 2x - 2y$
40. $z = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4$
41. $z = 5xy$
42. $z = xy + \dfrac{1}{x} + \dfrac{1}{y}$

Tangent Surfaces **In Exercises 43 and 44, show that the surfaces are tangent to each other at the given point by showing that the surfaces have the same tangent plane at this point.**

43. $x^2 + 2y^2 + 3z^2 = 3,\ x^2 + y^2 + z^2 + 6x - 10y + 14 = 0,$
$(-1, 1, 0)$

44. $x^2 + y^2 + z^2 - 8x - 12y + 4z + 42 = 0,$
$x^2 + y^2 + 2z = 7,\ (2, 3, -3)$

Perpendicular Tangent Planes **In Exercises 45 and 46, (a) show that the surfaces intersect at the given point and (b) show that the surfaces have perpendicular tangent planes at this point.**

45. $z = 2xy^2,\ 8x^2 - 5y^2 - 8z = -13,\ (1, 1, 2)$

46. $x^2 + y^2 + z^2 + 2x - 4y - 4z - 12 = 0,$
$4x^2 + y^2 + 16z^2 = 24,\ (1, -2, 1)$

EXPLORING CONCEPTS

47. Tangent Plane The tangent plane to the surface represented by $F(x, y, z) = 0$ at a point P is also tangent to the surface represented by $G(x, y, z) = 0$ at P. Is $\nabla F(x, y, z) = \nabla G(x, y, z)$ at P? Explain.

48. Normal Lines For some surfaces, the normal lines at any point pass through the same geometric object. What is the common geometric object for a sphere? What is the common geometric object for a right circular cylinder? Explain.

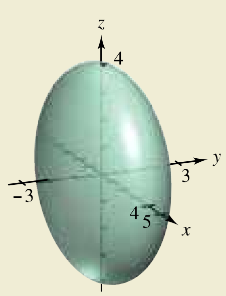
49. Using an Ellipsoid Find a point on the ellipsoid $3x^2 + y^2 + 3z^2 = 1$ where the tangent line is parallel to the plane $-12x + 2y + 6z = 0$.

50. Using a Hyperboloid Find a point on the hyperboloid $x^2 + 4y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $x + 4y - z = 0$.

51. Using an Ellipsoid Find a point on the ellipsoid $x^2 + 4y^2 + z^2 = 9$ where the tangent plane is perpendicular to the line with parametric equations

$x = 2 - 4t,\ y = 1 + 8t,$ and $z = 3 - 2t.$

52. HOW DO YOU SEE IT? The graph shows the ellipsoid $x^2 + 4y^2 + z^2 = 16$. Use the graph to determine the equation of the tangent plane at each of the given points.



(a) $(4, 0, 0)$ (b) $(0, -2, 0)$ (c) $(0, 0, -4)$

53. Investigation Consider the function

$$f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}$$

on the intervals $-2 \le x \le 2$ and $0 \le y \le 3$.

(a) Find a set of parametric equations of the normal line and an equation of the tangent plane to the surface at the point $(1, 1, 1)$.

(b) Repeat part (a) for the point $\left(-1, 2, -\frac{4}{5}\right)$.

(c) Use a computer algebra system to graph the surface, the normal lines, and the tangent planes found in parts (a) and (b).

54. Investigation Consider the function

$$f(x, y) = \frac{\sin y}{x}$$

on the intervals $-3 \le x \le 3$ and $0 \le y \le 2\pi$.

(a) Find a set of parametric equations of the normal line and an equation of the tangent plane to the surface at the point

$\left(2, \frac{\pi}{2}, \frac{1}{2}\right)$.

(b) Repeat part (a) for the point $\left(-\frac{2}{3}, \frac{3\pi}{2}, \frac{3}{2}\right)$.

(c) Use a computer algebra system to graph the surface, the normal lines, and the tangent planes found in parts (a) and (b).

55. Using Functions Consider the functions

$$f(x, y) = 6 - x^2 - \frac{y^2}{4} \quad \text{and} \quad g(x, y) = 2x + y.$$

(a) Find a set of parametric equations of the tangent line to the curve of intersection of the surfaces at the point $(1, 2, 4)$ and find the angle between the gradients of f and g.

(b) Use a computer algebra system to graph the surfaces and the tangent line found in part (a).

56. Using Functions Consider the functions

$$f(x, y) = \sqrt{16 - x^2 - y^2 + 2x - 4y}$$

and

$$g(x, y) = \frac{\sqrt{2}}{2}\sqrt{1 - 3x^2 + y^2 + 6x + 4y}.$$

(a) Use a computer algebra system to graph the first-octant portion of the surfaces represented by f and g.

(b) Find one first-octant point on the curve of intersection and show that the surfaces are orthogonal at this point.

(c) These surfaces are orthogonal along the curve of intersection. Does part (b) prove this fact? Explain.

Writing a Tangent Plane **In Exercises 57 and 58, show that the tangent plane to the quadric surface at the point (x_0, y_0, z_0) can be written in the given form.**

57. Ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Tangent plane: $\dfrac{x_0 x}{a^2} + \dfrac{y_0 y}{b^2} + \dfrac{z_0 z}{c^2} = 1$

58. Hyperboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Tangent plane: $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} = 1$

59. Tangent Planes of a Cone Show that any tangent plane to the cone

$$z^2 = a^2x^2 + b^2y^2$$

passes through the origin.

60. Tangent Planes Let f be a differentiable function and consider the surface

$$z = xf\left(\frac{y}{x}\right).$$

Show that the tangent plane at any point $P(x_0, y_0, z_0)$ on the surface passes through the origin.

61. Approximation Consider the following approximations for a function $f(x, y)$ centered at $(0, 0)$.

Linear Approximation:

$$P_1(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y$$

Quadratic Approximation:

$$P_2(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2$$

[Note that the linear approximation is the tangent plane to the surface at $(0, 0, f(0, 0))$.]

(a) Find the linear approximation of $f(x, y) = e^{x-y}$ centered at $(0, 0)$.

(b) Find the quadratic approximation of $f(x, y) = e^{x-y}$ centered at $(0, 0)$.

(c) When $x = 0$ in the quadratic approximation, you obtain the second-degree Taylor polynomial for what function? Answer the same question for $y = 0$.

(d) Complete the table.

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0			
0	0.1			
0.2	0.1			
0.2	0.5			
1	0.5			

(e) Use a computer algebra system to graph the surfaces $z = f(x, y)$, $z = P_1(x, y)$, and $z = P_2(x, y)$.

62. Approximation Repeat Exercise 61 for the function $f(x, y) = \cos(x + y)$.

63. Proof Prove that the angle of inclination θ of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is given by

$$\cos\theta = \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}.$$

64. Proof Prove Theorem 13.14.

SECTION PROJECT

Wildflowers

The diversity of wildflowers in a meadow can be measured by counting the numbers of daisies, buttercups, shooting stars, and so on. When there are n types of wildflowers, each with a proportion p_i of the total population, it follows that

$$p_1 + p_2 + \cdot \cdot \cdot + p_n = 1.$$

The measure of diversity of the population is defined as

$$H = -\sum_{i=1}^{n} p_i \log_2 p_i.$$

In this definition, it is understood that $p_i \log_2 p_i = 0$ when $p_i = 0$. The tables show proportions of wildflowers in a meadow in May, June, August, and September.

May

Flower type	1	2	3	4
Proportion	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{1}{16}$

June

Flower type	1	2	3	4
Proportion	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

August

Flower type	1	2	3	4
Proportion	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$

September

Flower type	1	2	3	4
Proportion	0	0	0	1

(a) Determine the wildflower diversity for each month. How would you interpret September's diversity? Which month had the greatest diversity?

(b) When the meadow contains 10 types of wildflowers in roughly equal proportions, is the diversity of the population greater than or less than the diversity of a similar distribution of 4 types of flowers? What type of distribution (of 10 types of wildflowers) would produce maximum diversity?

(c) Let H_n represent the maximum diversity of n types of wildflowers. Does H_n approach a limit as n approaches ∞?

FOR FURTHER INFORMATION Biologists use the concept of diversity to measure the proportions of different types of organisms within an environment. For more information on this technique, see the article "Information Theory and Biological Diversity" by Steven Kolmes and Kevin Mitchell in the *UMAP Modules.*

13.8 Extrema of Functions of Two Variables

- **Find absolute and relative extrema of a function of two variables.**
- **Use the Second Partials Test to find relative extrema of a function of two variables.**

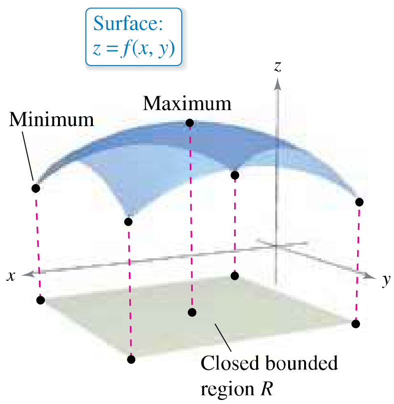
Absolute Extrema and Relative Extrema

In Chapter 3, you studied techniques for finding the extreme values of a function of a single variable. In this section, you will extend these techniques to functions of two variables. For example, in Theorem 13.15 below, the Extreme Value Theorem for a function of a single variable is extended to a function of two variables.

Consider the continuous function f of two variables, defined on a closed bounded region R in the xy-plane. The values $f(a, b)$ and $f(c, d)$ such that

$$f(a, b) \le f(x, y) \le f(c, d) \qquad (a, b) \text{ and } (c, d) \text{ are in } R.$$

for all (x, y) in R are called the **minimum** and **maximum** of f in the region R, as shown in Figure 13.64. Recall from Section 13.2 that a region in the plane is *closed* when it contains all of its boundary points. The Extreme Value Theorem deals with a region in the plane that is both closed and *bounded.* A region in the plane is **bounded** when it is a subregion of a closed disk in the plane.



R contains point(s) at which $f(x, y)$ is a minimum and point(s) at which $f(x, y)$ is a maximum.
Figure 13.64

THEOREM 13.15 Extreme Value Theorem

Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.

1. There is at least one point in R at which f takes on a minimum value.
2. There is at least one point in R at which f takes on a maximum value.

A minimum is also called an **absolute minimum** and a maximum is also called an **absolute maximum.** As in single-variable calculus, there is a distinction made between absolute extrema and **relative extrema.**

Definition of Relative Extrema

Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a **relative minimum** at (x_0, y_0) if

   $$f(x, y) \ge f(x_0, y_0)$$

   for all (x, y) in an *open* disk containing (x_0, y_0).
2. The function f has a **relative maximum** at (x_0, y_0) if

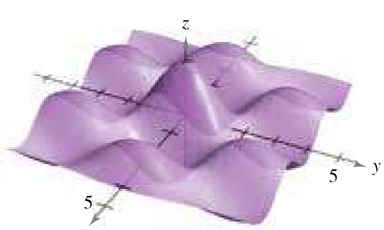
   $$f(x, y) \le f(x_0, y_0)$$

   for all (x, y) in an *open* disk containing (x_0, y_0).

To say that f has a relative maximum at (x_0, y_0) means that the point (x_0, y_0, z_0) is at least as high as all nearby points on the graph of

$$z = f(x, y).$$

Similarly, f has a relative minimum at (x_0, y_0) when (x_0, y_0, z_0) is at least as low as all nearby points on the graph. (See Figure 13.65.)



Relative extrema
Figure 13.65

KARL WEIERSTRASS (1815–1897)

Although the Extreme Value Theorem had been used by earlier mathematicians, the first to provide a rigorous proof was the German mathematician Karl Weierstrass. Weierstrass also provided rigorous justifications for many other mathematical results already in common use. We are indebted to him for much of the logical foundation on which modern calculus is built.

See LarsonCalculus.com to read more of this biography.

To locate relative extrema of f, you can investigate the points at which the gradient of f is $\mathbf{0}$ or the points at which one of the partial derivatives does not exist. Such points are called **critical points** of f.

Definition of Critical Point

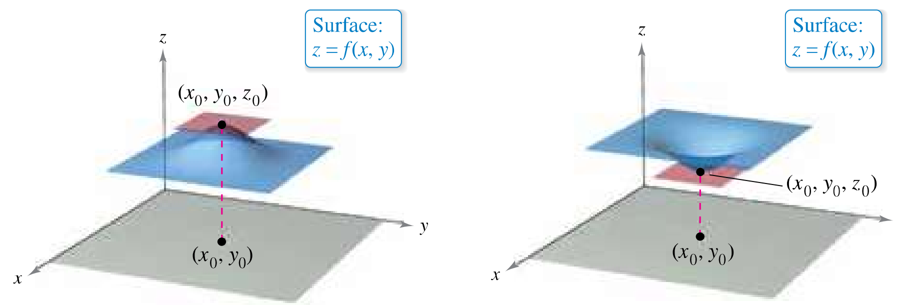
Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Recall from Theorem 13.11 that if f is differentiable and

$$\nabla f(x_0, y_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} = 0\mathbf{i} + 0\mathbf{j}$$

then every directional derivative at (x_0, y_0) must be 0. This implies that the function has a horizontal tangent plane at the point (x_0, y_0), as shown in Figure 13.66. It appears that such a point is a likely location of a relative extremum. This is confirmed by Theorem 13.16.



Relative maximum Relative minimum

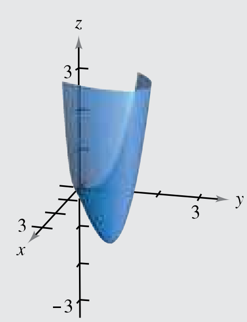
Figure 13.66

THEOREM 13.16 Relative Extrema Occur Only at Critical Points

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

Exploration

Use a graphing utility to graph $z = x^3 - 3xy + y^3$ using the bounds $0 \le x \le 3$, $0 \le y \le 3$, and $-3 \le z \le 3$. This view makes it appear as though the surface has an absolute minimum. Does the surface have an absolute minimum? Why or why not?



Jacques Boyer/Roger-Viollet/The Image Works

EXAMPLE 1 Finding a Relative Extremum

See LarsonCalculus.com for an interactive version of this type of example.

Determine the relative extrema of

$$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20.$$

Solution Begin by finding the critical points of f. Because

$$f_x(x, y) = 4x + 8 \qquad \text{Partial with respect to } x$$

and

$$f_y(x, y) = 2y - 6 \qquad \text{Partial with respect to } y$$

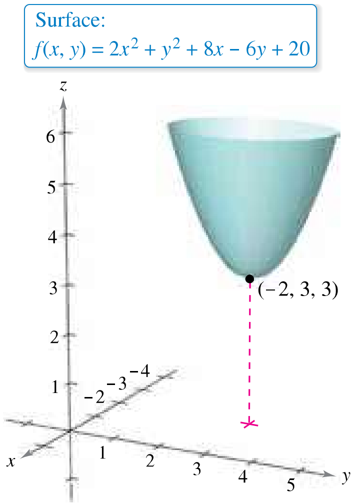
are defined for all x and y, the only critical points are those for which both first partial derivatives are 0. To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0, and solve the equations

$$4x + 8 = 0 \quad \text{and} \quad 2y - 6 = 0$$

to obtain the critical point $(-2, 3)$. By completing the square for f, you can see that for all $(x, y) \neq (-2, 3)$

$$f(x, y) = 2(x + 2)^2 + (y - 3)^2 + 3 > 3.$$

So, a relative *minimum* of f occurs at $(-2, 3)$. The value of the relative minimum is $f(-2, 3) = 3$, as shown in Figure 13.67.



The function $z = f(x, y)$ has a relative minimum at $(-2, 3)$.
Figure 13.67

Example 1 shows a relative minimum occurring at one type of critical point—the type for which both $f_x(x, y)$ and $f_y(x, y)$ are 0. The next example concerns a relative maximum that occurs at the other type of critical point—the type for which either $f_x(x, y)$ or $f_y(x, y)$ does not exist.

EXAMPLE 2 Finding a Relative Extremum

Determine the relative extrema of

$$f(x, y) = 1 - (x^2 + y^2)^{1/3}.$$

Solution Because

$$f_x(x, y) = -\frac{2x}{3(x^2 + y^2)^{2/3}} \qquad \text{Partial with respect to } x$$

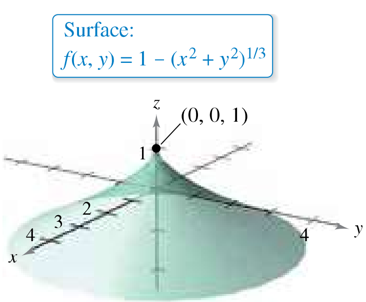
and

$$f_y(x, y) = -\frac{2y}{3(x^2 + y^2)^{2/3}} \qquad \text{Partial with respect to } y$$

it follows that both partial derivatives exist for all points in the xy-plane except for $(0, 0)$. Moreover, because the partial derivatives cannot both be 0 unless both x and y are 0, you can conclude that $(0, 0)$ is the only critical point. In Figure 13.68, note that $f(0, 0)$ is 1. For all other (x, y), it is clear that

$$f(x, y) = 1 - (x^2 + y^2)^{1/3} < 1.$$

So, f has a relative *maximum* at $(0, 0)$.



$f_x(x, y)$ and $f_y(x, y)$ are undefined at $(0, 0)$.
Figure 13.68

In Example 2, $f_x(x, y) = 0$ for every point on the y-axis other than $(0, 0)$. However, because $f_y(x, y)$ is nonzero, these are not critical points. Remember that *one* of the partials must not exist or *both* must be 0 in order to yield a critical point.

The Second Partials Test

Theorem 13.16 tells you that to find relative extrema, you need only examine values of $f(x, y)$ at critical points. However, as is true for a function of one variable, the critical points of a function of two variables do not always yield relative maxima or minima. Some critical points yield **saddle points,** which are neither relative maxima nor relative minima.

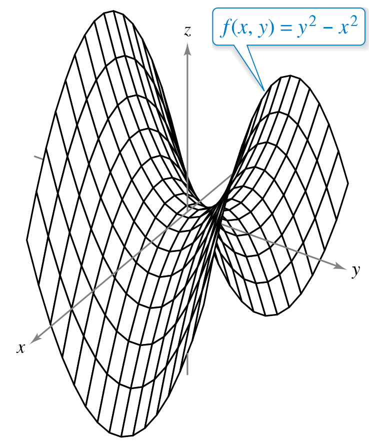
As an example of a critical point that does not yield a relative extremum, consider the hyperbolic paraboloid

$$f(x, y) = y^2 - x^2$$

as shown in Figure 13.69. At the point $(0, 0)$, both partial derivatives

$$f_x(x, y) = -2x \quad \text{and} \quad f_y(x, y) = 2y$$

are 0. The function f does not, however, have a relative extremum at this point because in any open disk centered at $(0, 0)$, the function takes on both negative values (along the x-axis) *and* positive values (along the y-axis), So, the point $(0, 0, 0)$ is a saddle point of the surface. (The term "saddle point" comes from the fact that surfaces such as the one shown in Figure 13.69 resemble saddles.)



Saddle point at $(0, 0, 0)$:
$f_x(0, 0) = f_y(0, 0) = 0$
Figure 13.69

For the functions in Examples 1 and 2, it was relatively easy to determine the relative extrema, because each function was either given, or able to be written, in completed square form. For more complicated functions, algebraic arguments are less convenient and it is better to rely on the analytic means presented in the following Second Partials Test. This is the two-variable counterpart of the Second Derivative Test for functions of one variable. The proof of this theorem is best left to a course in advanced calculus.

THEOREM 13.17 Second Partials Test

Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

To test for relative extrema of f, consider the quantity

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a **relative minimum** at (a, b).
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a **relative maximum** at (a, b).
3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point.**
4. The test is inconclusive if $d = 0$.

REMARK If $d > 0$, then $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have the same sign. This means that $f_{xx}(a, b)$ can be replaced by $f_{yy}(a, b)$ in the first two parts of the test.

A convenient device for remembering the formula for d in the Second Partials Test is given by the 2×2 determinant

$$d = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}$$

where $f_{xy}(a, b) = f_{yx}(a, b)$ by Theorem 13.3.

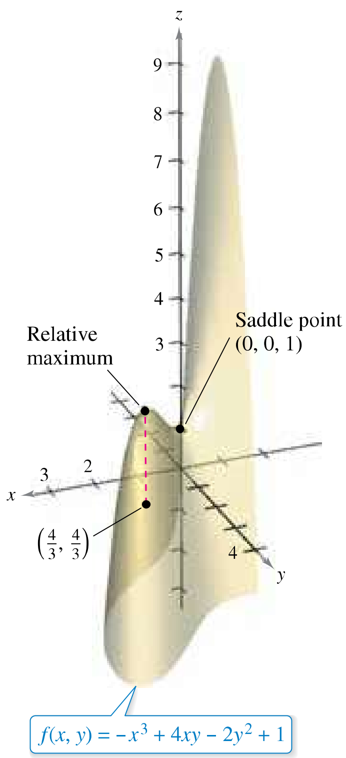


Figure 13.70

EXAMPLE 3 Using the Second Partials Test

Find the relative extrema of $f(x, y) = -x^3 + 4xy - 2y^2 + 1$.

Solution Begin by finding the critical points of f. Because

$$f_x(x, y) = -3x^2 + 4y \quad \text{and} \quad f_y(x, y) = 4x - 4y$$

exist for all x and y, the only critical points are those for which both first partial derivatives are 0. To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0 to obtain

$$-3x^2 + 4y = 0 \quad \text{and} \quad 4x - 4y = 0.$$

From the second equation, you know that $x = y$, and, by substitution into the first equation, you obtain two solutions: $y = x = 0$ and $y = x = \frac{4}{3}$. Because

$$f_{xx}(x, y) = -6x, \quad f_{yy}(x, y) = -4, \quad \text{and} \quad f_{xy}(x, y) = 4$$

it follows that, for the critical point $(0, 0)$,

$$d = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0 - 16 < 0$$

and, by the Second Partials Test, you can conclude that $(0, 0, 1)$ is a saddle point of f. Furthermore, for the critical point $\left(\frac{4}{3}, \frac{4}{3}\right)$,

$$\begin{aligned} d &= f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right)f_{yy}\left(\frac{4}{3}, \frac{4}{3}\right) - \left[f_{xy}\left(\frac{4}{3}, \frac{4}{3}\right)\right]^2 \\ &= -8(-4) - 16 \\ &= 16 \\ &> 0 \end{aligned}$$

and because $f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) = -8 < 0$, you can conclude that f has a relative maximum at $\left(\frac{4}{3}, \frac{4}{3}\right)$, as shown in Figure 13.70.

The Second Partials Test can fail to find relative extrema in two ways. If either of the first partial derivatives does not exist, you cannot use the test. Also, if

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$$

the test fails. In such cases, you can try a sketch or some other approach, as demonstrated in the next example.

EXAMPLE 4 Failure of the Second Partials Test

Find the relative extrema of $f(x, y) = x^2y^2$.

Solution Because $f_x(x, y) = 2xy^2$ and $f_y(x, y) = 2x^2y$, you know that both partial derivatives are 0 when $x = 0$ or $y = 0$. That is, every point along the x- or y-axis is a critical point. Moreover, because

$$f_{xx}(x, y) = 2y^2, \quad f_{yy}(x, y) = 2x^2, \quad \text{and} \quad f_{xy}(x, y) = 4xy$$

you know that

$$\begin{aligned} d &= f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 \\ &= 4x^2y^2 - 16x^2y^2 \\ &= -12x^2y^2 \end{aligned}$$

which is 0 when either $x = 0$ or $y = 0$. So, the Second Partials Test fails. However, because $f(x, y) = 0$ for every point along the x- or y-axis and $f(x, y) = x^2y^2 > 0$ for all other points, you can conclude that each of these critical points yields an absolute minimum, as shown in Figure 13.71.

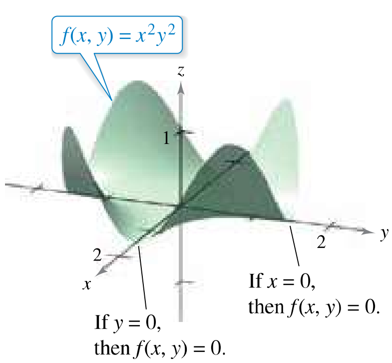


Figure 13.71

Absolute extrema of a function can occur in two ways. First, some relative extrema also happen to be absolute extrema. For instance, in Example 1, $f(-2, 3)$ is an absolute minimum of the function. (On the other hand, the relative maximum found in Example 3 is not an absolute maximum of the function.) Second, absolute extrema can occur at a boundary point of the domain. This is illustrated in Example 5.

EXAMPLE 5 Finding Absolute Extrema

Find the absolute extrema of the function

$$f(x, y) = \sin xy$$

on the closed region given by

$$0 \le x \le \pi \quad \text{and} \quad 0 \le y \le 1.$$

Solution From the partial derivatives

$$f_x(x, y) = y \cos xy \quad \text{and} \quad f_y(x, y) = x \cos xy$$

you can see that each point lying on the hyperbola $xy = \pi/2$ is a critical point. These points each yield the value

$$f(x, y) = \sin \frac{\pi}{2} = 1$$

which you know is the absolute maximum, as shown in Figure 13.72. The only other critical point of f *lying in the given region* is $(0, 0)$. It yields an absolute minimum of 0, because

$$0 \le xy \le \pi$$

implies that

$$0 \le \sin xy \le 1.$$

To locate other absolute extrema, you should consider the four boundaries of the region formed by taking traces with the vertical planes $x = 0$, $x = \pi$, $y = 0$, and $y = 1$. In doing this, you will find that $\sin xy = 0$ at all points on the x-axis, at all points on the y-axis, and at the point $(\pi, 1)$. Each of these points yields an absolute minimum for the surface, as shown in Figure 13.72.

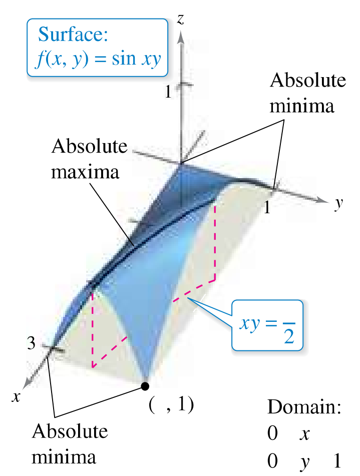


Figure 13.72

The concepts of relative extrema and critical points can be extended to functions of three or more variables. When all first partial derivatives of

$$w = f(x_1, x_2, x_3, \ldots, x_n)$$

exist, it can be shown that a relative maximum or minimum can occur at $(x_1, x_2, x_3, \ldots, x_n)$ only when every first partial derivative is 0 at that point. This means that the critical points are obtained by solving the following system of equations.

$$\begin{aligned} f_{x_1}(x_1, x_2, x_3, \ldots, x_n) &= 0 \\ f_{x_2}(x_1, x_2, x_3, \ldots, x_n) &= 0 \\ &\vdots \\ f_{x_n}(x_1, x_2, x_3, \ldots, x_n) &= 0 \end{aligned}$$

The extension of Theorem 13.17 to three or more variables is also possible, although you will not study such an extension in this text.

13.8 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Function of Two Variables** For a function of two variables, describe (a) relative minimum, (b) relative maximum, (c) critical point, and (d) saddle point.

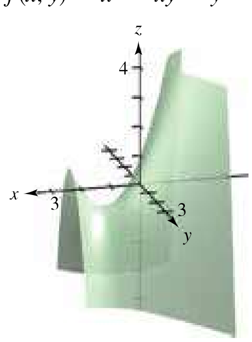
2. **Second Partials Test** Under what condition does the Second Partials Test fail?

Finding Relative Extrema **In Exercises 3–8, identify any extrema of the function by recognizing its given form or its form after completing the square. Verify your results by using the partial derivatives to locate any critical points and test for relative extrema.**

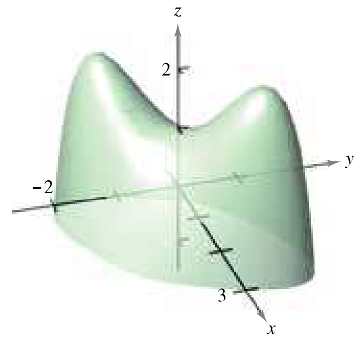
3. $g(x, y) = (x - 1)^2 + (y - 3)^2$
4. $g(x, y) = 5 - (x - 6)^2 - (y + 2)^2$
5. $f(x, y) = \sqrt{x^2 + y^2 + 1}$
6. $f(x, y) = \sqrt{49 - (x - 2)^2 - y^2}$
7. $f(x, y) = x^2 + y^2 + 2x - 6y + 6$
8. $f(x, y) = -x^2 - y^2 + 10x + 12y - 64$

Using the Second Partials Test **In Exercises 9–24, find all relative extrema and saddle points of the function. Use the Second Partials Test where applicable.**

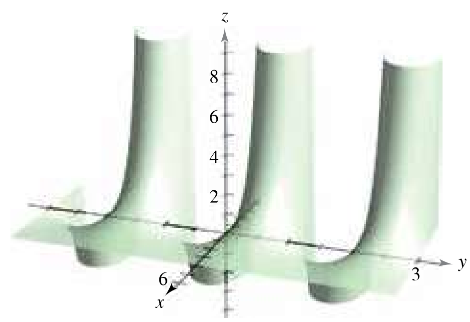
9. $f(x, y) = x^2 + y^2 + 8x - 12y - 3$
10. $g(x, y) = x^2 - y^2 - x - y$
11. $f(x, y) = -2x^4y^4$
12. $f(x, y) = \frac{1}{2}xy$
13. $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$
14. $h(x, y) = x^2 - 3xy - y^2$
15. $f(x, y) = 7x^2 + 2y^2 - 7x + 16y - 13$
16. $f(x, y) = x^5 + y^5$
17. $z = x^2 + xy + \frac{1}{2}y^2 - 2x + y$
18. $z = -5x^2 + 4xy - y^2 + 16x + 10$
19. $f(x, y) = -4(x^2 + y^2 + 81)^{1/4}$
20. $h(x, y) = (x^2 + y^2)^{1/3} + 2$
21. $f(x, y) = x^2 - xy - y^2 - 3x - y$



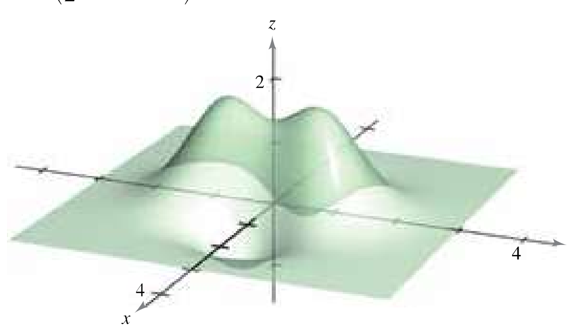
22. $f(x, y) = 2xy - \frac{1}{2}(x^4 + y^4) + 1$



23. $z = e^{-x} \sin y$



24. $z = \left(\frac{1}{2} - x^2 + y^2\right)e^{1-x^2-y^2}$



Finding Relative Extrema and Saddle Points Using Technology **In Exercises 25–28, use a computer algebra system to graph the surface and locate any relative extrema and saddle points.**

25. $z = \dfrac{-4x}{x^2 + y^2 + 1}$
26. $z = \cos x + \sin y, \ -\pi/2 < x < \pi/2, \ -\pi < y < \pi$
27. $z = (x^2 + 4y^2)e^{1-x^2-y^2}$
28. $z = e^{xy}$

Finding Relative Extrema **In Exercises 29 and 30, examine the function for extrema without using the derivative tests and use a computer algebra system to graph the surface and verify your answers. (*Hint:* By observation, determine whether it is possible for z to be negative. When is z equal to 0?)**

29. $z = \dfrac{(x - y)^4}{x^2 + y^2}$
30. $z = \dfrac{(x^2 - y^2)^2}{x^2 + y^2}$

Think About It **In Exercises 31–34, determine whether there is a relative maximum, a relative minimum, a saddle point, or insufficient information to determine the nature of the function $f(x, y)$ at the critical point (x_0, y_0).**

31. $f_{xx}(x_0, y_0) = 9, \quad f_{yy}(x_0, y_0) = 4, \quad f_{xy}(x_0, y_0) = 6$

32. $f_{xx}(x_0, y_0) = -3, \quad f_{yy}(x_0, y_0) = -8, \quad f_{xy}(x_0, y_0) = 2$

33. $f_{xx}(x_0, y_0) = -9, \quad f_{yy}(x_0, y_0) = 6, \quad f_{xy}(x_0, y_0) = 10$

34. $f_{xx}(x_0, y_0) = 25, \quad f_{yy}(x_0, y_0) = 8, \quad f_{xy}(x_0, y_0) = 10$

Finding Relative Extrema and Saddle Points **In Exercises 35–38, (a) find the critical points, (b) test for relative extrema, (c) list the critical points for which the Second Partials Test fails, and (d) use a computer algebra system to graph the function, labeling any extrema and saddle points.**

35. $f(x, y) = x^3 + y^3$

36. $f(x, y) = x^3 + y^3 - 6x^2 + 9x^2 + 12x + 27y + 19$

37. $f(x, y) = (x - 1)^2(y + 4)^2$

38. $f(x, y) = x^{2/3} + y^{2/3}$

Finding Absolute Extrema **In Exercises 39–46, find the absolute extrema of the function over the region R. (In each case, R contains the boundaries.) Use a computer algebra system to confirm your results.**

39. $f(x, y) = x^2 - 4xy + 5$

$R = \{(x, y): 1 \le x \le 4, 0 \le y \le 2\}$

40. $f(x, y) = x^2 + xy, \quad R = \{(x, y): |x| \le 2, |y| \le 1\}$

41. $f(x, y) = 12 - 3x - 2y$

R: The triangular region in the xy-plane with vertices $(2, 0)$, $(0, 1)$, and $(1, 2)$

42. $f(x, y) = (2x - y)^2$

R: The triangular region in the xy-plane with vertices $(2, 0)$, $(0, 1)$, and $(1, 2)$

43. $f(x, y) = 3x^2 + 2y^2 - 4y$

R: The region in the xy-plane bounded by the graphs of $y = x^2$ and $y = 4$

44. $f(x, y) = 2x - 2xy + y^2$

R: The region in the xy-plane bounded by the graphs of $y = x^2$ and $y = 1$

45. $f(x, y) = x^2 + 2xy + y^2, \quad R = \{(x, y): |x| \le 2, |y| \le 1\}$

46. $f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

Examining a Function **In Exercises 47 and 48, find the critical points of the function and, from the form of the function, determine whether a relative maximum or a relative minimum occurs at each point.**

47. $f(x, y, z) = x^2 + (y - 3)^2 + (z + 1)^2$

48. $f(x, y, z) = 9 - [x(y - 1)(z + 2)]^2$

EXPLORING CONCEPTS

49. Using the Second Partials Test A function f has continuous second partial derivatives on an open region containing the critical point $(3, 7)$. The function has a minimum at $(3, 7)$, and $d > 0$ for the Second Partials Test. Determine the interval for $f_{xy}(3, 7)$ when $f_{xx}(3, 7) = 2$ and $f_{yy}(3, 7) = 8$.

50. Using the Second Partials Test A function f has continuous second partial derivatives on an open region containing the critical point (a, b). If $f_{xx}(a, b)$ and $f_{yy}(a, b)$ have opposite signs, what is implied? Explain.

Sketching a Graph **In Exercises 51 and 52, sketch the graph of an arbitrary function f satisfying the given conditions. State whether the function has any extrema or saddle points. (There are many correct answers.)**

51. All of the first and second partial derivatives of f are 0.

52. $f_x(x, y) > 0$ and $f_y(x, y) < 0$ for all (x, y).

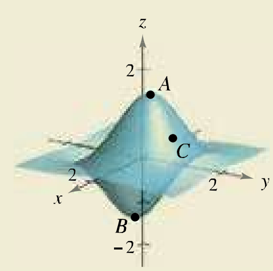
53. Comparing Functions Consider the functions

$$f(x, y) = x^2 - y^2 \quad \text{and} \quad g(x, y) = x^2 + y^2.$$

(a) Show that both functions have a critical point at $(0, 0)$.

(b) Explain how f and g behave differently at this critical point.

54. HOW DO YOU SEE IT? Determine whether each labeled point is an absolute maximum, an absolute minimum, or neither.



True or False? **In Exercises 55–58, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

55. If f has a relative maximum at (x_0, y_0, z_0), then

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

56. If $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, then f has a relative extremum at (x_0, y_0, z_0).

57. Between any two relative minima of f, there must be at least one relative maximum of f.

58. If f is continuous for all x and y and has two relative minima, then f must have at least one relative maximum.

13.9 Applications of Extrema

- **Solve optimization problems involving functions of several variables.**
- **Use the method of least squares.**

Applied Optimization Problems

In this section, you will study a few of the many applications of extrema of functions of two (or more) variables.

EXAMPLE 1 Finding Maximum Volume

See LarsonCalculus.com for an interactive version of this type of example.

A rectangular box is resting on the xy-plane with one vertex at the origin. The opposite vertex lies in the plane

$$6x + 4y + 3z = 24$$

as shown in Figure 13.73. Find the maximum volume of the box.

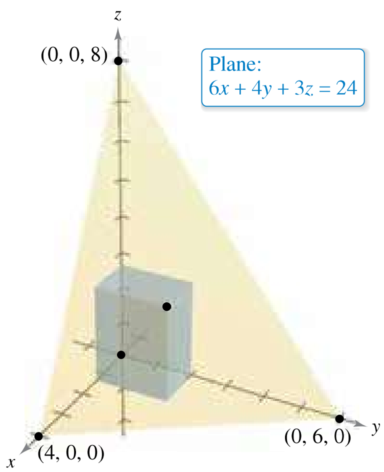


Figure 13.73

Solution Let x, y, and z represent the length, width, and height of the box. Because one vertex of the box lies in the plane $6x + 4y + 3z = 24$, you know that $z = \frac{1}{3}(24 - 6x - 4y)$. So, you can write the volume xyz of the box as a function of two variables.

$$\begin{aligned} V(x, y) &= (x)(y)\left[\tfrac{1}{3}(24 - 6x - 4y)\right] \\ &= \tfrac{1}{3}(24xy - 6x^2y - 4xy^2) \end{aligned}$$

Next, find the first partial derivatives of V.

$$V_x(x, y) = \frac{1}{3}(24y - 12xy - 4y^2) = \frac{y}{3}(24 - 12x - 4y)$$

$$V_y(x, y) = \frac{1}{3}(24x - 6x^2 - 8xy) = \frac{x}{3}(24 - 6x - 8y)$$

Note that the first partial derivatives are defined for all x and y. So, by setting $V_x(x, y)$ and $V_y(x, y)$ equal to 0 and solving the equations $\frac{1}{3}y(24 - 12x - 4y) = 0$ and $\frac{1}{3}x(24 - 6x - 8y) = 0$, you obtain the critical points $(0, 0)$, $(4, 0)$, $(0, 6)$, and $\left(\frac{4}{3}, 2\right)$. At $(0, 0)$, $(4, 0)$, and $(0, 6)$, the volume is 0, so these points do not yield a maximum volume. At the point $\left(\frac{4}{3}, 2\right)$, you can apply the Second Partials Test.

$$V_{xx}(x, y) = -4y$$

$$V_{yy}(x, y) = \frac{-8x}{3}$$

$$V_{xy}(x, y) = \frac{1}{3}(24 - 12x - 8y)$$

Because

$$V_{xx}\left(\tfrac{4}{3}, 2\right)V_{yy}\left(\tfrac{4}{3}, 2\right) - \left[V_{xy}\left(\tfrac{4}{3}, 2\right)\right]^2 = (-8)\left(-\tfrac{32}{9}\right) - \left(-\tfrac{8}{3}\right)^2 = \tfrac{64}{3} > 0$$

and

$$V_{xx}\left(\tfrac{4}{3}, 2\right) = -8 < 0$$

you can conclude from the Second Partials Test that the maximum volume is

$$V\left(\tfrac{4}{3}, 2\right) = \tfrac{1}{3}\left[24\left(\tfrac{4}{3}\right)(2) - 6\left(\tfrac{4}{3}\right)^2(2) - 4\left(\tfrac{4}{3}\right)(2^2)\right] = \tfrac{64}{9} \text{ cubic units.}$$

Note that the volume is 0 at the boundary points of the triangular domain of V.

REMARK In many applied problems, the domain of the function to be optimized is a closed bounded region. To find minimum or maximum points, you must not only test critical points, but also consider the values of the function at points on the boundary.

Applications of extrema in economics and business often involve more than one independent variable. For instance, a company may produce several models of one type of product. The price per unit and profit per unit are usually different for each model. Moreover, the demand for each model is often a function of the prices of the other models (as well as its own price). The next example illustrates an application involving two products.

EXAMPLE 2 Finding the Maximum Profit

A manufacturer determines that the profit P (in dollars) obtained by producing and selling x units of Product 1 and y units of Product 2 is approximated by the model

$$P(x, y) = 8x + 10y - (0.001)(x^2 + xy + y^2) - 10{,}000.$$

Find the production level that produces a maximum profit. What is the maximum profit?

Solution The partial derivatives of the profit function are

$$P_x(x, y) = 8 - (0.001)(2x + y)$$

and

$$P_y(x, y) = 10 - (0.001)(x + 2y).$$

By setting these partial derivatives equal to 0, you obtain the following system of equations.

$$\begin{aligned} 8 - (0.001)(2x + y) &= 0 \\ 10 - (0.001)(x + 2y) &= 0 \end{aligned}$$

After simplifying, this system of linear equations can be written as

$$\begin{aligned} 2x + y &= 8000 \\ x + 2y &= 10{,}000. \end{aligned}$$

Solving this system produces $x = 2000$ and $y = 4000$. The second partial derivatives of P are

$$\begin{aligned} P_{xx}(2000, 4000) &= -0.002 \\ P_{yy}(2000, 4000) &= -0.002 \\ P_{xy}(2000, 4000) &= -0.001. \end{aligned}$$

Because $P_{xx} < 0$ and

$$P_{xx}(2000, 4000)P_{yy}(2000, 4000) - [P_{xy}(2000, 4000)]^2 = (-0.002)^2 - (-0.001)^2$$

is greater than 0, you can conclude that the production level of $x = 2000$ units and $y = 4000$ units yields a *maximum* profit. The maximum profit is

$$\begin{aligned} &P(2000, 4000) \\ &\quad = 8(2000) + 10(4000) - (0.001)[2000^2 + 2000(4000) + 4000^2] - 10{,}000 \\ &\quad = \$18{,}000. \end{aligned}$$

In Example 2, it was assumed that the manufacturing plant is able to produce the required number of units to yield a maximum profit. In actual practice, the production would be bounded by physical constraints. You will study such constrained optimization problems in the next section.

FOR FURTHER INFORMATION For more information on the use of mathematics in economics, see the article "Mathematical Methods of Economics" by Joel Franklin in *The American Mathematical Monthly*. To view this article, go to *MathArticles.com*.

The Method of Least Squares

Many of the examples in this text have involved **mathematical models.** For instance, Example 2 involves a quadratic model for profit. There are several ways to develop such models; one is called the **method of least squares.**

In constructing a model to represent a particular phenomenon, the goals are simplicity and accuracy. Of course, these goals often conflict. For instance, a simple linear model for the points in Figure 13.74 is

$$y = 1.9x - 5.$$

However, Figure 13.75 shows that by choosing the slightly more complicated quadratic model

$$y = 0.20x^2 - 0.7x + 1$$

you can achieve greater accuracy.

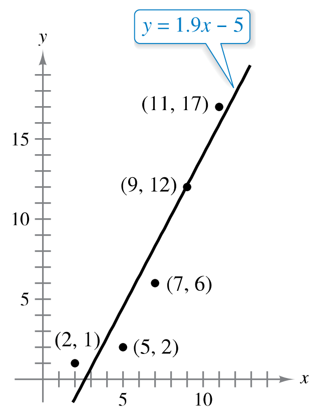


Figure 13.74

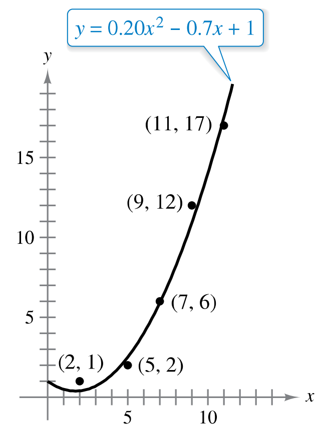


Figure 13.75

As a measure of how well the model $y = f(x)$ fits the collection of points

$$\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\}$$

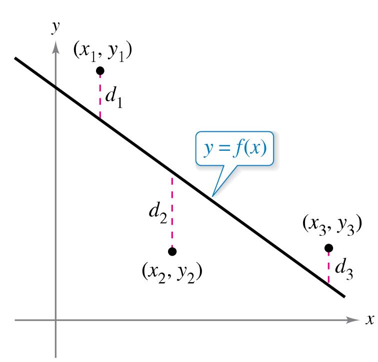
you can add the squares of the differences between the actual y-values and the values given by the model to obtain the **sum of the squared errors**

$$S = \sum_{i=1}^{n} [f(x_i) - y_i]^2. \qquad \text{Sum of the squared errors}$$

Graphically, S can be interpreted as the sum of the squares of the vertical distances between the graph of f and the given points in the plane, as shown in Figure 13.76. If the model is perfect, then $S = 0$. However, when perfection is not feasible, you can settle for a model that minimizes S. For instance, the sum of the squared errors for the linear model in Figure 13.74 is

$$S = 17.6.$$

Statisticians call the *linear model* that minimizes S the **least squares regression line.** The proof that this line actually minimizes S involves the minimizing of a function of two variables.



Sum of the squared errors:
$S = d_1^2 + d_2^2 + d_3^2$
Figure 13.76

REMARK A method for finding the least squares regression quadratic for a collection of data is described in Exercise 31.

ADRIEN-MARIE LEGENDRE (1752–1833)

The method of least squares was introduced by the French mathematician Adrien-Marie Legendre. Legendre is best known for his work in geometry. In fact, his text *Elements of Geometry* was so popular in the United States that it continued to be used for 33 editions, spanning a period of more than 100 years.

See LarsonCalculus.com to read more of this biography.

THEOREM 13.18 Least Squares Regression Line

The **least squares regression line** for $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ is given by $f(x) = ax + b$, where

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \quad \text{and} \quad b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right).$$

Proof Let $S(a, b)$ represent the sum of the squared errors for the model

$$f(x) = ax + b$$

and the given set of points. That is,

$$\begin{aligned} S(a, b) &= \sum_{i=1}^{n} [f(x_i) - y_i]^2 \\ &= \sum_{i=1}^{n} (ax_i + b - y_i)^2 \end{aligned}$$

where the points (x_i, y_i) represent constants. Because S is a function of a and b, you can use the methods discussed in the preceding section to find the minimum value of S. Specifically, the first partial derivatives of S are

$$\begin{aligned} S_a(a, b) &= \sum_{i=1}^{n} 2x_i(ax_i + b - y_i) \\ &= 2a\sum_{i=1}^{n} x_i^2 + 2b\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} x_i y_i \end{aligned}$$

and

$$\begin{aligned} S_b(a, b) &= \sum_{i=1}^{n} 2(ax_i + b - y_i) \\ &= 2a\sum_{i=1}^{n} x_i + 2nb - 2\sum_{i=1}^{n} y_i. \end{aligned}$$

By setting these two partial derivatives equal to 0, you obtain the values of a and b that are listed in the theorem. It is left to you to apply the Second Partials Test (see Exercise 41) to verify that these values of a and b yield a minimum.

If the x-values are symmetrically spaced about the y-axis, then $\Sigma\, x_i = 0$ and the formulas for a and b simplify to

$$a = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

and

$$b = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

This simplification is often possible with a translation of the x-values. For instance, given that the x-values in a data collection consist of the values 9, 10, 11, 12, and 13, you could let 11 be represented by 0.

EXAMPLE 3 Finding the Least Squares Regression Line

Find the least squares regression line for the points

$(-3, 0), \quad (-1, 1), \quad (0, 2), \quad \text{and} \quad (2, 3).$

Solution The table shows the calculations involved in finding the least squares regression line using $n = 4$.

x	y	xy	x^2
-3	0	0	9
-1	1	-1	1
0	2	0	0
2	3	6	4
$\sum_{i=1}^{n} x_i = -2$	$\sum_{i=1}^{n} y_i = 6$	$\sum_{i=1}^{n} x_i y_i = 5$	$\sum_{i=1}^{n} x_i^2 = 14$

Applying Theorem 13.18 produces

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{4(5) - (-2)(6)}{4(14) - (-2)^2} = \frac{8}{13}$$

and

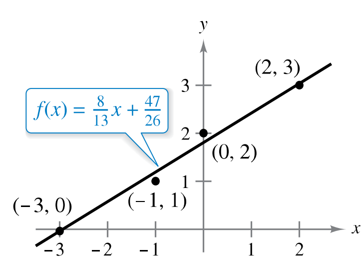
$$b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right) = \frac{1}{4}\left[6 - \frac{8}{13}(-2)\right] = \frac{47}{26}.$$

The least squares regression line is

$$f(x) = \frac{8}{13}x + \frac{47}{26}$$

as shown in Figure 13.77.

TECHNOLOGY Many calculators have "built-in" least squares regression programs. If your calculator has such a program, use it to duplicate the results of Example 3.



Least squares regression line
Figure 13.77

13.9 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Applied Optimization Problems** In your own words, state the problem-solving strategy for applied minimum and maximum problems.

2. **Method of Least Squares** In your own words, describe the method of least squares for finding mathematical models.

Finding Minimum Distance **In Exercises 3 and 4, find the minimum distance from the point to the plane $x - y + z = 3$. (*Hint:* To simplify the computations, minimize the square of the distance.)**

3. $(1, -3, 2)$
4. $(4, 0, 6)$

Finding Minimum Distance **In Exercises 5 and 6, find the minimum distance from the point to the surface $z = \sqrt{1 - 2x - 2y}$. (*Hint:* To simplify the computations, minimize the square of the distance.)**

5. $(-2, -2, 0)$
6. $(-4, 1, 0)$

Finding Positive Numbers **In Exercises 7–10, find three positive integers x, y, and z that satisfy the given conditions.**

7. The product is 27, and the sum is a minimum.
8. The sum is 32, and $P = xy^2z$ is a maximum.
9. The sum is 30, and the sum of the squares is a minimum.
10. The product is 1, and the sum of the squares is a minimum.

11. **Cost** A home improvement contractor is painting the walls and ceiling of a rectangular room. The volume of the room is 668.25 cubic feet. The cost of wall paint is \$0.06 per square foot and the cost of ceiling paint is \$0.11 per square foot. Find the room dimensions that result in a minimum cost for the paint. What is the minimum cost for the paint?

12. **Maximum Volume** The material for constructing the base of an open box costs 1.5 times as much per unit area as the material for constructing the sides. For a fixed amount of money C, find the dimensions of the box of largest volume that can be made.

13. **Volume and Surface Area** Show that a rectangular box of given volume and minimum surface area is a cube.

14. **Maximum Volume** Show that the rectangular box of maximum volume inscribed in a sphere of radius r is a cube.

15. **Maximum Revenue** A company manufactures running shoes and basketball shoes. The total revenue (in thousands of dollars) from x_1 units of running shoes and x_2 units of basketball shoes is

$$R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$$

where x_1 and x_2 are in thousands of units. Find x_1 and x_2 so as to maximize the revenue.

16. **Maximum Profit** A corporation manufactures candles at two locations. The cost of producing x_1 units at location 1 is $C_1 = 0.02x_1^2 + 4x_1 + 500$ and the cost of producing x_2 units at location 2 is $C_2 = 0.05x_2^2 + 4x_2 + 275$. The candles sell for \$15 per unit. Find the quantity that should be produced at each location to maximize the profit $P = 15(x_1 + x_2) - C_1 - C_2$.

17. **Hardy-Weinberg Law** Common blood types are determined genetically by three alleles A, B, and O. (An allele is any of a group of possible mutational forms of a gene.) A person whose blood type is AA, BB, or OO is homozygous. A person whose blood type is AB, AO, or BO is heterozygous. The Hardy-Weinberg Law states that the proportion P of heterozygous individuals in any given population is

$$P(p, q, r) = 2pq + 2pr + 2qr$$

where p represents the percent of allele A in the population, q represents the percent of allele B in the population, and r represents the percent of allele O in the population. Use the fact that

$$p + q + r = 1$$

to show that the maximum proportion of heterozygous individuals in any population is $\frac{2}{3}$.

18. **Shannon Diversity Index** One way to measure species diversity is to use the Shannon diversity index H. If a habitat consists of three species, A, B, and C, then its Shannon diversity index is

$$H = -x \ln x - y \ln y - z \ln z$$

where x is the percent of species A in the habitat, y is the percent of species B in the habitat, and z is the percent of species C in the habitat. Use the fact that

$$x + y + z = 1$$

to show that the maximum value of H occurs when $x = y = z = \frac{1}{3}$. What is the maximum value of H?

19. **Minimum Cost** A water line is to be built from point P to point S and must pass through regions where construction costs differ (see figure). The cost per kilometer (in dollars) is $3k$ from P to Q, $2k$ from Q to R, and k from R to S. Find x and y such that the total cost C will be minimized.

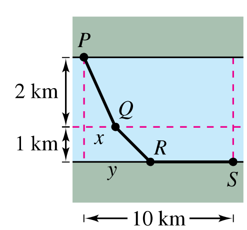


Figure for 19

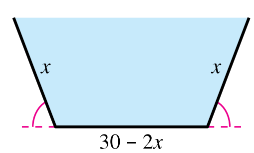
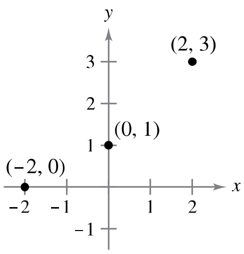


Figure for 20

20. **Area** A trough with trapezoidal cross sections is formed by turning up the edges of a 30-inch-wide sheet of aluminum (see figure). Find the cross section of maximum area.

Finding the Least Squares Regression Line **In Exercises 21–24, (a) find the least squares regression line and (b) calculate S, the sum of the squared errors. Use the regression capabilities of a graphing utility to verify your results.**

21.



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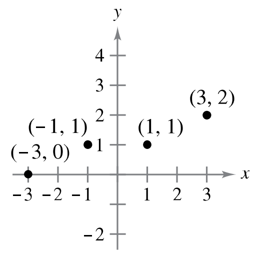
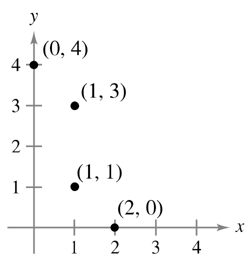
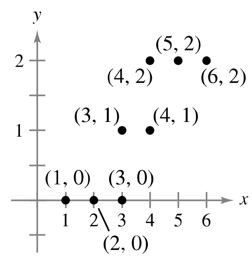


23.



24.



Finding the Least Squares Regression Line **In Exercises 25–28, find the least squares regression line for the points. Use the regression capabilities of a graphing utility to verify your results. Use the graphing utility to plot the points and graph the regression line.**

25. $(0, 0)$, $(1, 1)$, $(3, 6)$, $(4, 8)$, $(5, 9)$

26. $(0, 4)$, $(4, 1)$, $(7, -3)$

27. $(0, 6)$, $(4, 3)$, $(5, 0)$, $(8, -4)$, $(10, -5)$

28. $(6, 4)$, $(1, 2)$, $(3, 3)$, $(8, 6)$, $(11, 8)$, $(13, 8)$

29. Modeling Data The table shows the gross income tax collections (in billions of dollars) by the Internal Revenue Service for individuals x and businesses y for selected years. *(Source: U.S. Internal Revenue Service)*

Year	1980	1985	1990	1995
Individual, x	288	397	540	676
Business, y	72	77	110	174

Year	2000	2005	2010	2015
Individual, x	1137	1108	1164	1760
Business, y	236	307	278	390

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use the model to estimate the business income taxes collected when the individual income taxes collected is \$1300 billion.

(c) In 1975, the individual income taxes collected was \$156 billion and the business income taxes collected was \$46 billion. Describe how including this information would affect the model.

30. Modeling Data The ages x (in years) and systolic blood pressures y (in mmHg) of seven men are shown in the table.

Age, x	16	25	39	45	49	64	70
Systolic Blood Pressure, y	109	122	150	165	159	183	199

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model to approximate the change in systolic blood pressure for each one-year increase in age.

(d) A 30-year-old man has a systolic blood pressure of 180 mmHg. Describe how including this information would affect the model.

EXPLORING CONCEPTS

31. Method of Least Squares Find a system of equations whose solution yields the coefficients a, b, and c for the least squares regression quadratic

$$y = ax^2 + bx + c$$

for the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by minimizing the sum

$$S(a, b, c) = \sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c)^2.$$

32. HOW DO YOU SEE IT? Match the regression equation with the appropriate graph. Explain your reasoning. (Note that the x- and y-axes are broken.)

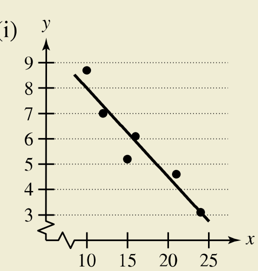
(a) $y = 0.22x - 7.5$

(b) $y = -0.35x + 11.5$

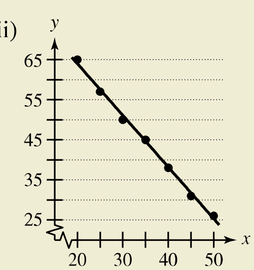
(c) $y = 0.09x + 19.8$

(d) $y = -1.29x + 89.8$

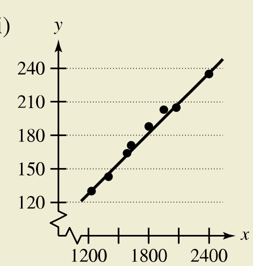
(i)



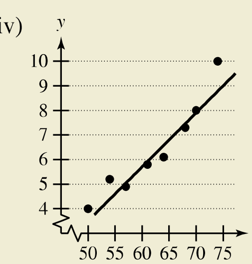
(ii)



(iii)



(iv)



Finding the Least Squares Regression Quadratic **In Exercises 33–36, use the result of Exercise 31 to find the least squares regression quadratic for the points. Use the regression capabilities of a graphing utility to verify your results. Use the graphing utility to plot the points and graph the least squares regression quadratic.**

33. $(-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$

34. $(-4, 5), (-2, 6), (2, 6), (4, 2)$

35. $(0, 0), (2, 2), (3, 6), (4, 12)$

36. $(0, 10), (1, 9), (2, 6), (3, 0)$

37. Modeling Data After a new turbocharger for an automobile engine was developed, the following experimental data were obtained for speed y in miles per hour at two-second time intervals x.

Time, x	0	2	4	6	8	10
Speed, y	0	15	30	50	65	70

(a) Use the result of Exercise 31 to find the least squares regression quadratic for the data.

(b) Use a graphing utility to plot the points and graph the model.

38. Modeling Data The table shows the total numbers of enrollees y (in millions) for the Veterans Health Administration for 2010 through 2014. Let $x = 0$ represent the year 2010. *(Source: U.S. Department of Veterans Affairs)*

Year, x	2010	2011	2012	2013	2014
Total Enrollees, y	8.3	8.6	8.8	8.9	9.1

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use the regression capabilities of a graphing utility to find the least squares regression quadratic for the data.

(c) Use a graphing utility to plot the data and graph the models.

(d) Use both models to forecast the total number of enrollees for the year 2025. How do the two models differ as you extrapolate into the future?

39. Modeling Data A meteorologist measures the atmospheric pressure P (in kilograms per square meter) at altitude h (in kilometers). The data are shown below.

Altitude, h	0	5	10	15	20
Pressure, P	10,332	5583	2376	1240	517

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the points $(h, \ln P)$.

(b) The result in part (a) is an equation of the form $\ln P = ah + b$. Write this logarithmic form in exponential form.

(c) Use a graphing utility to plot the original data and graph the exponential model in part (b).

40. Modeling Data The endpoints of the interval over which distinct vision is possible are called the near point and far point of the eye. With increasing age, these points normally change. The table shows the approximate near points y (in inches) for various ages x (in years). *(Source: Ophthalmology & Physiological Optics)*

Age, x	16	32	44	50	60
Near Point, y	3.0	4.7	9.8	19.7	39.4

(a) Find a rational model for the data by taking the reciprocals of the near points to generate the points $(x, 1/y)$. Use the regression capabilities of a graphing utility to find the least squares regression line for the revised data. The resulting line has the form $1/y = ax + b$. Solve for y.

(b) Use a graphing utility to plot the data and graph the model.

(c) Do you think the model can be used to predict the near point for a person who is 70 years old? Explain.

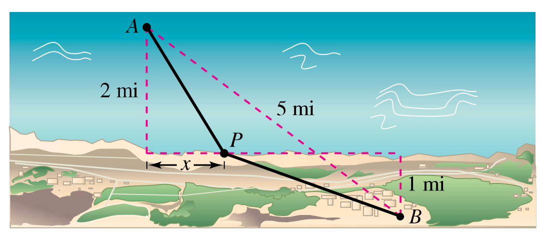
41. Using the Second Partials Test Use the Second Partials Test to verify that the formulas for a and b given in Theorem 13.18 yield a minimum.

$\left[\textit{Hint:} \text{ Use the fact that } n\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)^2.\right]$

SECTION PROJECT

Building a Pipeline

An oil company wishes to construct a pipeline from its offshore facility A to its refinery B. The offshore facility is 2 miles from shore, and the refinery is 1 mile inland. Furthermore, A and B are 5 miles apart, as shown in the figure.



The cost of building the pipeline is \$3 million per mile in the water and \$4 million per mile on land. So, the cost of the pipeline depends on the location of point P, where it meets the shore. What would be the most economical route of the pipeline?

Imagine that you are to write a report to the oil company about this problem. Let x be the distance shown in the figure. Determine the cost of building the pipeline from A to P and the cost of building it from P to B. Analyze some sample pipeline routes and their corresponding costs. For instance, what is the cost of the most direct route? Then use calculus to determine the route of the pipeline that minimizes the cost. Explain all steps of your development and include any relevant graphs.

13.10 Lagrange Multipliers

- **Understand the Method of Lagrange Multipliers.**
- **Use Lagrange multipliers to solve constrained optimization problems.**
- **Use the Method of Lagrange Multipliers with two constraints.**

Lagrange Multipliers

LAGRANGE MULTIPLIERS

The Method of Lagrange Multipliers is named after the French mathematician Joseph-Louis Lagrange. Lagrange first introduced the method in his famous paper on mechanics, written when he was just 19 years old.

Many optimization problems have restrictions, or **constraints,** on the values that can be used to produce the optimal solution. Such constraints tend to complicate optimization problems because the optimal solution can occur at a boundary point of the domain. In this section, you will study an ingenious technique for solving such problems. It is called the **Method of Lagrange Multipliers.**

To see how this technique works, consider the problem of finding the rectangle of maximum area that can be inscribed in the ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

Let (x, y) be the vertex of the rectangle in the first quadrant, as shown in Figure 13.78. Because the rectangle has sides of lengths $2x$ and $2y$, its area is given by

$$f(x, y) = 4xy. \qquad \text{Objective function}$$

You want to find x and y such that $f(x, y)$ is a maximum. Your choice of (x, y) is restricted to first-quadrant points that lie on the ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1. \qquad \text{Constraint}$$

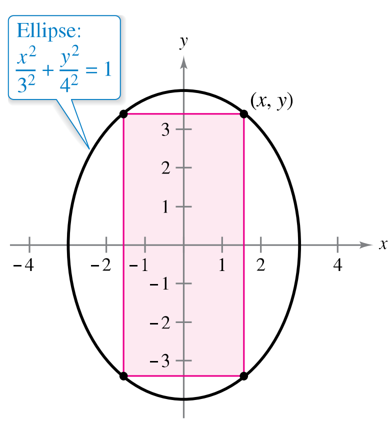
Now, consider the constraint equation to be a fixed level curve of

$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2}.$$

The level curves of f represent a family of hyperbolas

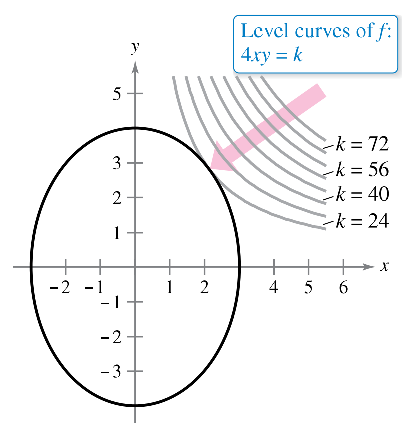
$$f(x, y) = 4xy = k.$$

In this family, the level curves that meet the constraint correspond to the hyperbolas that intersect the ellipse. Moreover, to maximize $f(x, y)$, you want to find the hyperbola that just barely satisfies the constraint. The level curve that does this is the one that is *tangent* to the ellipse, as shown in Figure 13.79.



Objective function: $f(x, y) = 4xy$

Figure 13.78



Constraint: $g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$

Figure 13.79

To find the appropriate hyperbola, use the fact that two curves are tangent at a point if and only if their gradients are parallel. This means that $\nabla f(x, y)$ must be a scalar multiple of $\nabla g(x, y)$ at the point of tangency. In the context of constrained optimization problems, this scalar is denoted by λ (the lowercase Greek letter lambda).

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

The scalar λ is called a **Lagrange multiplier.** Theorem 13.19 gives the necessary conditions for the existence of such multipliers.

REMARK Lagrange's Theorem can be shown to be true for functions of three variables, using a similar argument with level surfaces and Theorem 13.14.

THEOREM 13.19 Lagrange's Theorem

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Proof To begin, represent the smooth curve given by $g(x, y) = c$ by the vector-valued function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad \mathbf{r}'(t) \neq \mathbf{0}$$

where x' and y' are continuous on an open interval I. Define the function h as $h(t) = f(x(t), y(t))$. Then, because $f(x_0, y_0)$ is an extreme value of f, you know that

$$h(t_0) = f(x(t_0), y(t_0)) = f(x_0, y_0)$$

is an extreme value of h. This implies that $h'(t_0) = 0$, and, by the Chain Rule,

$$h'(t_0) = f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0) = \nabla f(x_0, y_0) \cdot \mathbf{r}'(t_0) = 0.$$

So, $\nabla f(x_0, y_0)$ is orthogonal to $\mathbf{r}'(t_0)$. Moreover, by Theorem 13.12, $\nabla g(x_0, y_0)$ is also orthogonal to $\mathbf{r}'(t_0)$. Consequently, the gradients $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel, and there must exist a scalar λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

The Method of Lagrange Multipliers uses Theorem 13.19 to find the extreme values of a function f subject to a constraint.

REMARK As you will see in Examples 1 and 2, the Method of Lagrange Multipliers requires solving systems of nonlinear equations. This often can require some tricky algebraic manipulation.

Method of Lagrange Multipliers

Let f and g satisfy the hypothesis of Lagrange's Theorem, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of f, use these steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = c$ by solving the following system of equations.

$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$
$$g(x, y) = c$$

2. Evaluate f at each solution point obtained in the first step. The greatest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the least value yields the minimum of f subject to the constraint $g(x, y) = c$.

Constrained Optimization Problems

In the problem at the beginning of this section, you wanted to maximize the area of a rectangle that is inscribed in an ellipse. Example 1 shows how to use Lagrange multipliers to solve this problem.

EXAMPLE 1 Using a Lagrange Multiplier with One Constraint

Find the maximum value of $f(x, y) = 4xy$, where $x > 0$ and $y > 0$, subject to the constraint $(x^2/3^2) + (y^2/4^2) = 1$.

Solution To begin, let

$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

By equating $\nabla f(x, y) = 4y\mathbf{i} + 4x\mathbf{j}$ and $\lambda\nabla g(x, y) = (2\lambda x/9)\mathbf{i} + (\lambda y/8)\mathbf{j}$, you obtain the following system of equations.

$$4y = \frac{2}{9}\lambda x \qquad f_x(x, y) = \lambda g_x(x, y)$$

$$4x = \frac{1}{8}\lambda y \qquad f_y(x, y) = \lambda g_y(x, y)$$

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1 \qquad \text{Constraint}$$

From the first equation, you obtain $\lambda = 18y/x$, and substitution into the second equation produces

$$4x = \frac{1}{8}\left(\frac{18y}{x}\right)y \quad \Longrightarrow \quad x^2 = \frac{9}{16}y^2.$$

Substituting this value for x^2 into the third equation produces

$$\frac{1}{9}\left(\frac{9}{16}y^2\right) + \frac{1}{16}y^2 = 1 \quad \Longrightarrow \quad y^2 = 8 \quad \Longrightarrow \quad y = \pm 2\sqrt{2}.$$

Because $y > 0$, choose the positive value and find that

$$\begin{aligned} x^2 &= \frac{9}{16}y^2 \\ &= \frac{9}{16}(8) \\ &= \frac{9}{2} \\ x &= \pm\frac{3}{\sqrt{2}}. \end{aligned}$$

Because $x > 0$, choose the positive value. So, the maximum value of f is

$$f\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right) = 4\left(\frac{3}{\sqrt{2}}\right)\left(2\sqrt{2}\right) = 24.$$

REMARK Note in Example 1 that writing the constraint as

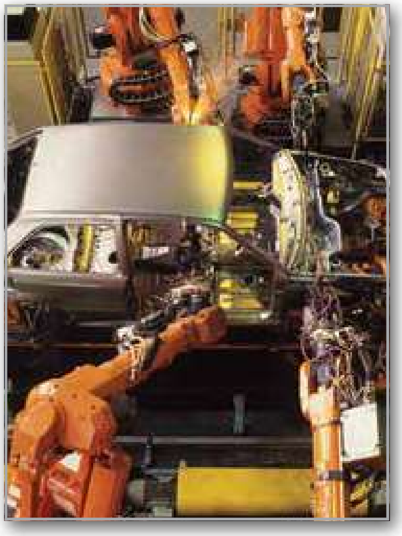
$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$

or

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} - 1 = 0$$

does not affect the solution—the constant is eliminated when you form ∇g.

Example 1 can also be solved using the techniques you learned in Chapter 3. To see how, try to find the maximum value of $A = 4xy$ given that $(x^2/3^2) + (y^2/4^2) = 1$. To begin, solve the second equation for y to obtain $y = \frac{4}{3}\sqrt{9 - x^2}$. Then substitute into the first equation to obtain $A = 4x\left(\frac{4}{3}\sqrt{9 - x^2}\right)$. Finally, use the techniques of Chapter 3 to maximize A.



For some industrial applications, a robot can cost more than the annual wages and benefits for one employee. So, manufacturers must carefully balance the amount of money spent on labor and capital.

EXAMPLE 2 A Business Application

The Cobb-Douglas production function (see Section 13.1) for a manufacturer is given by

$$f(x, y) = 100x^{3/4}y^{1/4} \qquad \text{Objective function}$$

where x represents the units of labor (at \$150 per unit) and y represents the units of capital (at \$250 per unit). The total cost of labor and capital is limited to \$50,000. Find the maximum production level for this manufacturer.

Solution The gradient of f is

$$\nabla f(x, y) = 75x^{-1/4}y^{1/4}\mathbf{i} + 25x^{3/4}y^{-3/4}\mathbf{j}.$$

The limit on the cost of labor and capital produces the constraint

$$g(x, y) = 150x + 250y = 50{,}000. \qquad \text{Constraint}$$

So, $\lambda\nabla g(x, y) = 150\lambda\mathbf{i} + 250\lambda\mathbf{j}$. This gives rise to the following system of equations.

$$\begin{aligned} 75x^{-1/4}y^{1/4} &= 150\lambda & & f_x(x, y) = \lambda g_x(x, y) \\ 25x^{3/4}y^{-3/4} &= 250\lambda & & f_y(x, y) = \lambda g_y(x, y) \\ 150x + 250y &= 50{,}000 & & \text{Constraint} \end{aligned}$$

By solving for λ in the first equation

$$\lambda = \frac{75x^{-1/4}y^{1/4}}{150} = \frac{x^{-1/4}y^{1/4}}{2}$$

and substituting into the second equation, you obtain

$$\begin{aligned} 25x^{3/4}y^{-3/4} &= 250\left(\frac{x^{-1/4}y^{1/4}}{2}\right) \\ 25x &= 125y & & \text{Multiply by } x^{1/4}y^{3/4}. \\ x &= 5y. \end{aligned}$$

By substituting this value for x in the third equation, you have

$$\begin{aligned} 150(5y) + 250y &= 50{,}000 \\ 1000y &= 50{,}000 \\ y &= 50 \text{ units of capital.} \end{aligned}$$

This means that the value of x is

$$\begin{aligned} x &= 5(50) \\ &= 250 \text{ units of labor.} \end{aligned}$$

So, the maximum production level is

$$\begin{aligned} f(250, 50) &= 100(250)^{3/4}(50)^{1/4} \\ &\approx 16{,}719 \text{ units of product.} \end{aligned}$$

FOR FURTHER INFORMATION For more information on the use of Lagrange multipliers in economics, see the article "Lagrange Multiplier Problems in Economics" by John V. Baxley and John C. Moorhouse in *The American Mathematical Monthly*. To view this article, go to *MathArticles.com*.

Economists call the Lagrange multiplier obtained in a production function the **marginal productivity of money.** For instance, in Example 2, the marginal productivity of money at $x = 250$ and $y = 50$ is

$$\lambda = \frac{x^{-1/4}y^{1/4}}{2} = \frac{(250)^{-1/4}(50)^{1/4}}{2} \approx 0.334$$

which means that for each additional dollar spent on production, an additional 0.334 unit of the product can be produced.

EXAMPLE 3 Lagrange Multipliers and Three Variables

See LarsonCalculus.com for an interactive version of this type of example.

Find the minimum value of

$$f(x, y, z) = 2x^2 + y^2 + 3z^2 \qquad \text{Objective function}$$

subject to the constraint $2x - 3y - 4z = 49$.

Solution Let $g(x, y, z) = 2x - 3y - 4z = 49$. Then, because

$$\nabla f(x, y, z) = 4x\mathbf{i} + 2y\mathbf{j} + 6z\mathbf{k}$$

and

$$\lambda \nabla g(x, y, z) = 2\lambda\mathbf{i} - 3\lambda\mathbf{j} - 4\lambda\mathbf{k}$$

you obtain the following system of equations.

$$\begin{aligned} 4x &= 2\lambda & f_x(x, y, z) &= \lambda g_x(x, y, z) \\ 2y &= -3\lambda & f_y(x, y, z) &= \lambda g_y(x, y, z) \\ 6z &= -4\lambda & f_z(x, y, z) &= \lambda g_z(x, y, z) \\ 2x - 3y - 4z &= 49 & &\text{Constraint} \end{aligned}$$

The solution of this system is $x = 3$, $y = -9$, and $z = -4$. So, the optimum value of f is

$$\begin{aligned} f(3, -9, -4) &= 2(3)^2 + (-9)^2 + 3(-4)^2 \\ &= 147. \end{aligned}$$

From the original function and constraint, it is clear that $f(x, y, z)$ has no maximum. So, the optimum value of f determined above is a minimum.

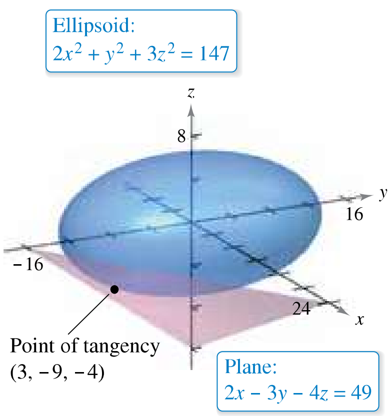


Figure 13.80

A graphical interpretation of constrained optimization problems in two variables was given at the beginning of this section. In three variables, the interpretation is similar, except that level surfaces are used instead of level curves. For instance, in Example 3, the level surfaces of f are ellipsoids centered at the origin, and the constraint

$$2x - 3y - 4z = 49$$

is a plane. The minimum value of f is represented by the ellipsoid that is tangent to the constraint plane, as shown in Figure 13.80.

EXAMPLE 4 Optimization Inside a Region

Find the extreme values of

$$f(x, y) = x^2 + 2y^2 - 2x + 3 \qquad \text{Objective function}$$

subject to the constraint $x^2 + y^2 \le 10$.

Solution To solve this problem, you can break the constraint into two cases.

a. For points *on the circle* $x^2 + y^2 = 10$, you can use Lagrange multipliers to find that the maximum value of $f(x, y)$ is 24—this value occurs at $(-1, 3)$ and at $(-1, -3)$. In a similar way, you can determine that the minimum value of $f(x, y)$ is approximately 6.675—this value occurs at $(\sqrt{10}, 0)$.

b. For points *inside the circle,* you can use the techniques discussed in Section 13.8 to conclude that the function has a relative minimum of 2 at the point $(1, 0)$.

By combining these two results, you can conclude that f has a maximum of 24 at $(-1, \pm 3)$ and a minimum of 2 at $(1, 0)$, as shown in Figure 13.81.

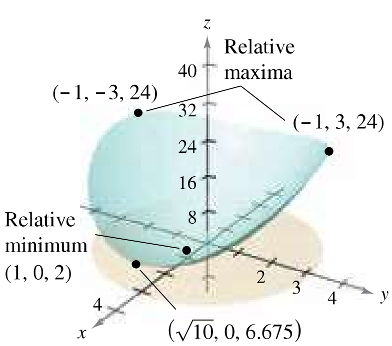


Figure 13.81

The Method of Lagrange Multipliers with Two Constraints

For optimization problems involving *two* constraint functions g and h, you can introduce a second Lagrange multiplier, μ (the lowercase Greek letter mu), and then solve the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

where the gradients are not parallel, as illustrated in Example 5.

EXAMPLE 5 Optimization with Two Constraints

Let $T(x, y, z) = 20 + 2x + 2y + z^2$ represent the temperature at each point on the sphere

$$x^2 + y^2 + z^2 = 11.$$

Find the extreme temperatures on the curve formed by the intersection of the plane $x + y + z = 3$ and the sphere.

Solution The two constraints are

$$g(x, y, z) = x^2 + y^2 + z^2 = 11 \quad \text{and} \quad h(x, y, z) = x + y + z = 3.$$

Using

$$\nabla T(x, y, z) = 2\mathbf{i} + 2\mathbf{j} + 2z\mathbf{k}$$
$$\lambda \nabla g(x, y, z) = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j} + 2\lambda z\mathbf{k}$$

and

$$\mu \nabla h(x, y, z) = \mu\mathbf{i} + \mu\mathbf{j} + \mu\mathbf{k}$$

you can write the following system of equations.

$$\begin{aligned}
2 &= 2\lambda x + \mu && T_x(x, y, z) = \lambda g_x(x, y, z) + \mu h_x(x, y, z)\\
2 &= 2\lambda y + \mu && T_y(x, y, z) = \lambda g_y(x, y, z) + \mu h_y(x, y, z)\\
2z &= 2\lambda z + \mu && T_z(x, y, z) = \lambda g_z(x, y, z) + \mu h_z(x, y, z)\\
x^2 + y^2 + z^2 &= 11 && \text{Constraint 1}\\
x + y + z &= 3 && \text{Constraint 2}
\end{aligned}$$

By subtracting the second equation from the first, you obtain the following system.

$$\begin{aligned}
\lambda(x - y) &= 0\\
2z(1 - \lambda) - \mu &= 0\\
x^2 + y^2 + z^2 &= 11\\
x + y + z &= 3
\end{aligned}$$

REMARK The systems of equations that arise when the Method of Lagrange Multipliers is used are not, in general, linear systems, and finding the solutions often requires ingenuity.

From the first equation, you can conclude that $\lambda = 0$ or $x = y$. For $\lambda = 0$, you can show that the critical points are $(3, -1, 1)$ and $(-1, 3, 1)$. (Try doing this—it takes a little work.) For $\lambda \neq 0$, then $x = y$ and you can show that the critical points occur when $x = y = \left(3 \pm 2\sqrt{3}\right)/3$ and $z = \left(3 \mp 4\sqrt{3}\right)/3$. Finally, to find the optimal solutions, compare the temperatures at the four critical points.

$$T(3, -1, 1) = T(-1, 3, 1) = 25$$
$$T\left(\frac{3 - 2\sqrt{3}}{3}, \frac{3 - 2\sqrt{3}}{3}, \frac{3 + 4\sqrt{3}}{3}\right) = \frac{91}{3} \approx 30.33$$
$$T\left(\frac{3 + 2\sqrt{3}}{3}, \frac{3 + 2\sqrt{3}}{3}, \frac{3 - 4\sqrt{3}}{3}\right) = \frac{91}{3} \approx 30.33$$

So, $T = 25$ is the minimum temperature and $T = \frac{91}{3}$ is the maximum temperature on the curve.

13.10 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Constrained Optimization Problems** Explain what is meant by constrained optimization problems.
2. **Method of Lagrange Multipliers** In your own words, describe the Method of Lagrange Multipliers for solving constrained optimization problems.

Using Lagrange Multipliers **In Exercises 3–10, use Lagrange multipliers to find the indicated extrema, assuming that x and y are positive.**

3. Maximize $f(x, y) = xy$

   Constraint: $x + y = 10$
4. Minimize $f(x, y) = 2x + y$

   Constraint: $xy = 32$
5. Minimize $f(x, y) = x^2 + y^2$

   Constraint: $x + 2y - 5 = 0$
6. Maximize $f(x, y) = x^2 - y^2$

   Constraint: $2y - x^2 = 0$
7. Maximize $f(x, y) = 2x + 2xy + y$

   Constraint: $2x + y = 100$
8. Minimize $f(x, y) = 3x + y + 10$

   Constraint: $x^2y = 6$
9. Maximize $f(x, y) = \sqrt{6 - x^2 - y^2}$

   Constraint: $x + y - 2 = 0$
10. Minimize $f(x, y) = \sqrt{x^2 + y^2}$

    Constraint: $2x + 4y - 15 = 0$

Using Lagrange Multipliers **In Exercises 11–14, use Lagrange multipliers to find the indicated extrema, assuming that x, y, and z are positive.**

11. Minimize $f(x, y, z) = x^2 + y^2 + z^2$

    Constraint: $x + y + z - 9 = 0$
12. Maximize $f(x, y, z) = xyz$

    Constraint: $x + y + z - 3 = 0$
13. Minimize $f(x, y, z) = x^2 + y^2 + z^2$

    Constraint: $x + y + z = 1$
14. Maximize $f(x, y, z) = x + y + z$

    Constraint: $x^2 + y^2 + z^2 = 1$

Using Lagrange Multipliers **In Exercises 15 and 16, use Lagrange multipliers to find any extrema of the function subject to the constraint $x^2 + y^2 \le 1$.**

15. $f(x, y) = x^2 + 3xy + y^2$
16. $f(x, y) = e^{-xy/4}$

Using Lagrange Multipliers **In Exercises 17 and 18, use Lagrange multipliers to find the indicated extrema of f subject to two constraints, assuming that x, y, and z are nonnegative.**

17. Maximize $f(x, y, z) = xyz$

    Constraints: $x + y + z = 32, \quad x - y + z = 0$
18. Minimize $f(x, y, z) = x^2 + y^2 + z^2$

    Constraints: $x + 2z = 6, \quad x + y = 12$

Finding Minimum Distance **In Exercises 19–28, use Lagrange multipliers to find the minimum distance from the curve or surface to the indicated point. (*Hint:* To simplify the computations, minimize the square of the distance.)**

	Curve	**Point**
19.	Line: $x + y = 1$	$(0, 0)$
20.	Line: $2x + 3y = -1$	$(0, 0)$
21.	Line: $x - y = 4$	$(0, 2)$
22.	Line: $x + 4y = 3$	$(1, 0)$
23.	Parabola: $y = x^2$	$(0, 3)$
24.	Parabola: $y = x^2$	$(-3, 0)$
25.	Circle: $x^2 + (y - 1)^2 = 9$	$(4, 4)$
26.	Circle: $(x - 4)^2 + y^2 = 4$	$(0, 10)$
	Surface	**Point**
27.	Plane: $x + y + z = 1$	$(2, 1, 1)$
28.	Cone: $z = \sqrt{x^2 + y^2}$	$(4, 0, 0)$

Intersection of Surfaces **In Exercises 29 and 30, use Lagrange multipliers to find the highest point on the curve of intersection of the surfaces.**

29. Cone: $x^2 + y^2 - z^2 = 0$

    Plane: $x + 2z = 4$
30. Sphere: $x^2 + y^2 + z^2 = 36$

    Plane: $2x + y - z = 2$

Using Lagrange Multipliers **In Exercises 31–38, use Lagrange multipliers to solve the indicated exercise in Section 13.9.**

31. Exercise 3
32. Exercise 4
33. Exercise 7
34. Exercise 8
35. Exercise 11
36. Exercise 12
37. Exercise 17
38. Exercise 18

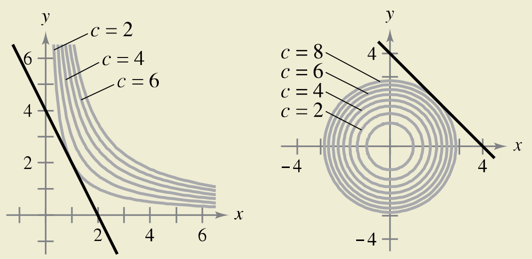
39. **Maximum Volume** Use Lagrange multipliers to find the dimensions of a rectangular box of maximum volume that can be inscribed (with edges parallel to the coordinate axes) in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

40. HOW DO YOU SEE IT? The graphs show the constraint and several level curves of the objective function. Use the graph to approximate the indicated extrema.

(a) Maximize $z = xy$
Constraint: $2x + y = 4$

(b) Minimize $z = x^2 + y^2$
Constraint: $x + y - 4 = 0$



EXPLORING CONCEPTS

41. Method of Lagrange Multipliers Explain why you cannot use Lagrange multipliers to find the minimum of the function $f(x, y) = x$ subject to the constraint $y^2 + x^4 - x^3 = 0$.

42. Method of Lagrange Multipliers Draw the level curves for $f(x, y) = x^2 + y^2 = c$ for $c = 1, 2, 3,$ and 4, and sketch the constraint $x + y = 2$. Explain analytically how you know that the extremum of $f(x, y) = x^2 + y^2$ at $(1, 1)$ is a minimum instead of a maximum.

43. Minimum Cost A cargo container (in the shape of a rectangular solid) must have a volume of 480 cubic feet. The bottom will cost \$5 per square foot to construct, and the sides and the top will cost \$3 per square foot to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has minimum cost.

44. Geometric and Arithmetic Means

(a) Use Lagrange multipliers to prove that the product of three positive numbers x, y, and z, whose sum has the constant value S, is a maximum when the three numbers are equal. Use this result to prove that

$$\sqrt[3]{xyz} \le \frac{x + y + z}{3}.$$

(b) Generalize the result of part (a) to prove that the product $x_1 x_2 x_3 \cdot \cdot \cdot x_n$ is a maximum when

$$x_1 = x_2 = x_3 = \cdot \cdot \cdot = x_n, \ \sum_{i=1}^{n} x_i = S, \text{ and all } x_i \ge 0.$$

Then prove that

$$\sqrt[n]{x_1 x_2 x_3 \cdot \cdot \cdot x_n} \le \frac{x_1 + x_2 + x_3 + \cdot \cdot \cdot + x_n}{n}.$$

This shows that the geometric mean is never greater than the arithmetic mean.

45. Minimum Surface Area Use Lagrange multipliers to find the dimensions of a right circular cylinder with volume V_0 and minimum surface area.

46. Temperature Let $T(x, y, z) = 100 + x^2 + y^2$ represent the temperature at each point on the sphere

$$x^2 + y^2 + z^2 = 50.$$

Use Lagrange multipliers to find the maximum temperature on the curve formed by the intersection of the sphere and the plane $x - z = 0$.

47. Refraction of Light When light waves traveling in a transparent medium strike the surface of a second transparent medium, they tend to "bend" in order to follow the path of minimum time. This tendency is called *refraction* and is described by **Snell's Law of Refraction,**

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the figure, and v_1 and v_2 are the velocities of light in the two media. Use Lagrange multipliers to derive this law using $x + y = a$.

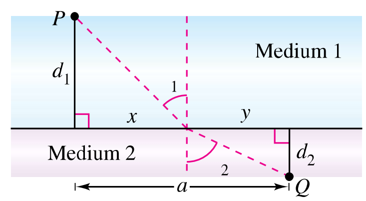


Figure for 47

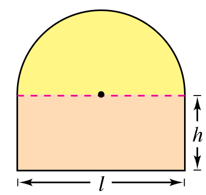


Figure for 48

48. Area and Perimeter A semicircle is on top of a rectangle (see figure). When the area is fixed and the perimeter is a minimum, or when the perimeter is fixed and the area is a maximum, use Lagrange multipliers to verify that the length of the rectangle is twice its height.

Production Level In Exercises 49 and 50, use Lagrange multipliers to find the maximum production level when the total cost of labor (at \$112 per unit) and capital (at \$60 per unit) is limited to \$250,000, where P is the production function, x is the number of units of labor, and y is the number of units of capital.

49. $P(x, y) = 100x^{0.25}y^{0.75}$ **50.** $P(x, y) = 100x^{0.4}y^{0.6}$

Cost In Exercises 51 and 52, use Lagrange multipliers to find the minimum cost of producing 50,000 units of a product, where P is the production function, x is the number of units of labor (at \$72 per unit), and y is the number of units of capital (at \$80 per unit).

51. $P(x, y) = 100x^{0.25}y^{0.75}$ **52.** $P(x, y) = 100x^{0.6}y^{0.4}$

PUTNAM EXAM CHALLENGE

53. A can buoy is to be made of three pieces, namely, a cylinder and two equal cones, the altitude of each cone being equal to the altitude of the cylinder. For a given area of surface, what shape will have the greatest volume?

This problem was composed by the Committee on the Putnam Prize Competition.

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Evaluating a Function In Exercises 1 and 2, evaluate the function at the given values of the independent variables. Simplify the results.

1. $f(x, y) = x^2y - 3$

(a) $f(0, 4)$ (b) $f(2, -1)$ (c) $f(-3, 2)$ (d) $f(x, 7)$

2. $f(x, y) = 6 - 4x - 2y^2$

(a) $f(0, 2)$ (b) $f(5, 0)$ (c) $f(-1, -2)$ (d) $f(-3, y)$

Finding the Domain and Range of a Function In Exercises 3 and 4, find the domain and range of the function.

3. $f(x, y) = \dfrac{\sqrt{x}}{y}$

4. $f(x, y) = \sqrt{36 - x^2 - y^2}$

Sketching a Surface In Exercises 5 and 6, describe and sketch the surface given by the function.

5. $f(x, y) = -2$ **6.** $g(x, y) = x$

Sketching a Contour Map In Exercises 7 and 8, describe the level curves of the function. Sketch a contour map of the surface using level curves for the given c-values.

7. $z = 3 - 2x + y$, $c = 0, 2, 4, 6, 8$

8. $z = 2x^2 + y^2$, $c = 1, 2, 3, 4, 5$

9. Conjecture Consider the function $f(x, y) = x^2 + y^2$.

(a) Sketch the graph of the surface given by f.

(b) Make a conjecture about the relationship between the graphs of f and $g(x, y) = f(x, y) + 2$. Explain your reasoning.

(c) Make a conjecture about the relationship between the graphs of f and $g(x, y) = f(x, y - 2)$. Explain your reasoning.

(d) On the surface in part (a), sketch the graphs of $z = f(1, y)$ and $z = f(x, 1)$.

10. Cobb-Douglas Production Function A manufacturer estimates that its production can be modeled by

$$f(x, y) = 100x^{0.8}y^{0.2}$$

where x is the number of units of labor and y is the number of units of capital.

(a) Find the production level when $x = 100$ and $y = 200$.

(b) Find the production level when $x = 500$ and $y = 1500$.

Sketching a Level Surface In Exercises 11 and 12, describe and sketch the graph of the level surface $f(x, y, z) = c$ at the given value of c.

11. $f(x, y, z) = x^2 - y + z^2$, $c = 2$

12. $f(x, y, z) = 4x^2 - y^2 + 4z^2$, $c = 0$

Limit and Continuity In Exercises 13–18, find the limit (if it exists) and discuss the continuity of the function.

13. $\displaystyle\lim_{(x, y)\to(1, 1)} \frac{xy}{x^2 + y^2}$ **14.** $\displaystyle\lim_{(x, y)\to(1, 1)} \frac{xy}{x^2 - y^2}$

15. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{y + xe^{-y^2}}{1 + x^2}$ **16.** $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{x^2y}{x^4 + y^2}$

17. $\displaystyle\lim_{(x, y, z)\to(-3, 1, 2)} \frac{\ln z}{xy - z}$ **18.** $\displaystyle\lim_{(x, y, z)\to(1, 3, \pi)} \sin\frac{xz}{2y}$

Finding Partial Derivatives In Exercises 19–26, find all first partial derivatives.

19. $f(x, y) = 5x^3 + 7y - 3$ **20.** $f(x, y) = 4x^2 - 2xy + y^2$

21. $f(x, y) = e^x \cos y$ **22.** $f(x, y) = \dfrac{xy}{x + y}$

23. $f(x, y) = y^3e^{y/x}$ **24.** $z = \ln(x^2 + y^2 + 1)$

25. $f(x, y, z) = 2xz^2 + 6xyz$ **26.** $w = \sqrt{x^2 - y^2 - z^2}$

Finding and Evaluating Partial Derivatives In Exercises 27–30, find all first partial derivatives, and evaluate each at the given point.

27. $f(x, y) = x^2 - y$, $(0, 2)$ **28.** $f(x, y) = xe^{2y}$, $(-1, 1)$

29. $f(x, y, z) = xy \cos xz$, $(2, 3, -\pi/3)$

30. $f(x, y, z) = \sqrt{x^2 + y - z^2}$, $(-3, -3, 1)$

Finding Second Partial Derivatives In Exercises 31–34, find the four second partial derivatives. Observe that the second mixed partials are equal.

31. $f(x, y) = 3x^2 - xy + 2y^3$ **32.** $h(x, y) = \dfrac{x}{x + y}$

33. $h(x, y) = x \sin y + y \cos x$ **34.** $g(x, y) = \cos(x - 2y)$

35. Finding the Slopes of a Surface Find the slopes of the surface $z = x^2 \ln(y + 1)$ in the x- and y-directions at the point $(2, 0, 0)$.

36. Marginal Revenue A company has two plants that produce the same lawn mower. If x_1 and x_2 are the numbers of units produced at plant 1 and plant 2, respectively, then the total revenue for the product is given by

$$R = 300x_1 + 300x_2 - 5x_1^2 - 10x_1x_2 - 5x_2^2.$$

When $x_1 = 5$ and $x_2 = 8$, find (a) the marginal revenue for plant 1, $\partial R/\partial x_1$, and (b) the marginal revenue for plant 2, $\partial R/\partial x_2$.

Finding a Total Differential In Exercises 37–40, find the total differential.

37. $z = x \sin xy$ **38.** $z = 5x^4y^3$

39. $w = 3xy^2 - 2x^3yz^2$ **40.** $w = \dfrac{3x + 4y}{y + 3z}$

Using a Differential as an Approximation In Exercises 41 and 42, (a) find $f(2, 1)$ and $f(2.1, 1.05)$ and calculate Δz, and (b) use the total differential dz to approximate Δz.

41. $f(x, y) = 4x + 2y$

42. $f(x, y) = 36 - x^2 - y^2$

43. **Volume** The possible error involved in measuring each dimension of a right circular cone is $\pm\frac{1}{8}$ inch. The radius is 2 inches and the height is 5 inches. Approximate the propagated error and the relative error in the calculated volume of the cone.

44. **Lateral Surface Area** Approximate the propagated error and the relative error in the calculated lateral surface area of the cone in Exercise 43. (The lateral surface area is given by $A = \pi r\sqrt{r^2 + h^2}$.)

Differentiability In Exercises 45 and 46, show that the function is differentiable by finding values of ε_1 and ε_2 as designated in the definition of differentiability, and verify that both ε_1 and ε_2 approach 0 as $(\Delta x, \Delta y) \quad (0, 0)$.

45. $f(x, y) = 6x - y^2$

46. $f(x, y) = xy^2$

Using Different Methods In Exercises 47–50, find dw/dt (a) by using the appropriate Chain Rule and (b) by converting w to a function of t before differentiating.

47. $w = \ln(x^2 + y), \quad x = 2t, \quad y = 4 - t$

48. $w = y^2 - x, \quad x = \cos t, \quad y = \sin t$

49. $w = x^2z + y + z, \quad x = e^t, \quad y = t, \quad z = t^2$

50. $w = \sin x + y^2z + 2z, \quad x = \arcsin(t - 1), \quad y = t^3, \quad z = 3$

Using Different Methods In Exercises 51 and 52, find $\partial w/\partial r$ and $\partial w/\partial t$ (a) by using the appropriate Chain Rule and (b) by converting w to a function of r and t before differentiating.

51. $w = \dfrac{xy}{z}, \quad x = 2r + t, \quad y = rt, \quad z = 2r - t$

52. $w = x^2 + y^2 + z^2, \quad x = r\cos t, \quad y = r\sin t, \quad z = t$

Finding a Derivative Implicitly In Exercises 53 and 54, differentiate implicitly to find dy/dx.

53. $x^3 - xy + 5y = 0$

54. $\dfrac{xy^2}{x + y} = 3$

Finding Partial Derivatives Implicitly In Exercises 55 and 56, differentiate implicitly to find the first partial derivatives of z.

55. $x^2 + xy + y^2 + yz + z^2 = 0$

56. $xz^2 - y\sin z = 0$

Finding a Directional Derivative In Exercises 57 and 58, use Theorem 13.9 to find the directional derivative of the function at P in the direction of $\mathbf{v}$.

57. $f(x, y) = x^2y, \quad P(-5, 5), \quad \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

58. $f(x, y) = \frac{1}{4}y^2 - x^2, \quad P(1, 4), \quad \mathbf{v} = 2\mathbf{i} + \mathbf{j}$

Finding a Directional Derivative In Exercises 59 and 60, use the gradient to find the directional derivative of the function at P in the direction of $\mathbf{v}$.

59. $w = y^2 + xz, \quad P(1, 2, 2), \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

60. $w = 5x^2 + 2xy - 3y^2z, \quad P(1, 0, 1), \quad \mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

Using Properties of the Gradient In Exercises 61–66, find the gradient of the function and the maximum value of the directional derivative at the given point.

61. $z = x^2y, \quad (2, 1)$

62. $z = e^{-x}\cos y, \quad \left(0, \dfrac{\pi}{4}\right)$

63. $z = \dfrac{y}{x^2 + y^2}, \quad (1, 1)$

64. $z = \dfrac{x^2}{x - y}, \quad (2, 1)$

65. $w = x^4y - y^2z^2, \quad \left(-1, \frac{1}{2}, 2\right)$

66. $w = e^{\sqrt{x+y+z^2}}, \quad (5, 0, 2)$

Using a Function In Exercises 67 and 68, (a) find the gradient of the function at P, (b) find a unit normal vector to the level curve $f(x, y) = c$ at P, (c) find the tangent line to the level curve $f(x, y) = c$ at P, and (d) sketch the level curve, the unit normal vector, and the tangent line in the xy-plane.

67. $f(x, y) = 9x^2 - 4y^2$

$c = 65, \quad P(3, 2)$

68. $f(x, y) = 4y\sin x - y$

$c = 3, \quad P\left(\dfrac{\pi}{2}, 1\right)$

Finding an Equation of a Tangent Plane In Exercises 69–72, find an equation of the tangent plane to the surface at the given point.

69. $z = x^2 + y^2 + 2, \quad (1, 3, 12)$

70. $9x^2 + y^2 + 4z^2 = 25, \quad (0, -3, 2)$

71. $z = -9 + 4x - 6y - x^2 - y^2, \quad (2, -3, 4)$

72. $f(x, y) = \sqrt{25 - y^2}, \quad (2, 3, 4)$

Finding an Equation of a Tangent Plane and a Normal Line In Exercises 73 and 74, (a) find an equation of the tangent plane to the surface at the given point and (b) find a set of symmetric equations for the normal line to the surface at the given point.

73. $f(x, y) = x^2y, \quad (2, 1, 4)$

74. $z = \sqrt{9 - x^2 - y^2}, \quad (1, 2, 2)$

Finding the Angle of Inclination of a Tangent Plane In Exercises 75 and 76, find the angle of inclination of the tangent plane to the surface at the given point.

75. $x^2 + y^2 + z^2 = 14, \quad (2, 1, 3)$

76. $xy + yz^2 = 32, \quad (-4, 1, 6)$

Horizontal Tangent Plane In Exercises 77 and 78, find the point(s) on the surface at which the tangent plane is horizontal.

77. $z = 9 - 2x^2 + y^3$

78. $z = 2xy + 3x + 5y$

Using the Second Partials Test **In Exercises 79–84, find all relative extrema and saddle points of the function. Use the Second Partials Test where applicable.**

79. $f(x, y) = -x^2 - 4y^2 + 8x - 8y - 11$

80. $f(x, y) = x^2 - y^2 - 16x - 16y$

81. $f(x, y) = 2x^2 + 6xy + 9y^2 + 8x + 14$

82. $f(x, y) = x^6y^6$

83. $f(x, y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$

84. $f(x, y) = -8x^2 + 4xy - y^2 + 12x + 7$

85. Finding Minimum Distance Find the minimum distance from the point $(2, 1, 4)$ to the surface $x + y + z = 4$. (*Hint:* To simplify the computations, minimize the square of the distance.)

86. Finding Positive Numbers Find three positive integers, x, y, and z, such that the product is 64 and the sum is a minimum.

87. Maximum Revenue A company manufactures two types of bicycles, a racing bicycle and a mountain bicycle. The total revenue (in thousands of dollars) from x_1 units of racing bicycles and x_2 units of mountain bicycles is

$$R = -6x_1^2 - 10x_2^2 - 2x_1x_2 + 32x_1 + 84x_2$$

where x_1 and x_2 are in thousands of units. Find x_1 and x_2 so as to maximize the revenue.

88. Maximum Profit A corporation manufactures digital cameras at two locations. The cost of producing x_1 units at location 1 is $C_1 = 0.05x_1^2 + 15x_1 + 5400$ and the cost of producing x_2 units at location 2 is $C_2 = 0.03x_2^2 + 15x_2 + 6100$. The digital cameras sell for \$180 per unit. Find the quantity that should be produced at each location to maximize the profit $P = 180(x_1 + x_2) - C_1 - C_2$.

Finding the Least Squares Regression Line **In Exercises 89 and 90, find the least squares regression line for the points. Use the regression capabilities of a graphing utility to verify your results. Use the graphing utility to plot the points and graph the regression line.**

89. $(0, 4), (1, 5), (3, 6), (6, 8), (8, 10)$

90. $(0, 10), (2, 8), (4, 7), (7, 5), (9, 3), (12, 0)$

91. Modeling Data An agronomist used four test plots to determine the relationship between the wheat yield y (in bushels per acre) and the amount of fertilizer x (in pounds per acre). The results are shown in the table.

Fertilizer, x	100	150	200	250
Yield, y	35	44	50	56

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use the model to approximate the wheat yield for a fertilizer application of 175 pounds per acre.

92. Modeling Data The table shows the yield y (in milligrams) of a chemical reaction after t minutes.

Minutes, t	1	2	3	4	5	6	7	8
Yield, y	1.2	7.1	9.9	13.1	15.5	16.0	17.9	18.0

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data. Then use the graphing utility to plot the data and graph the model.

(b) Use a graphing utility to plot the points $(\ln t, y)$. Do these points appear to follow a linear pattern more closely than the plot of the given data in part (a)?

(c) Use the regression capabilities of a graphing utility to find the least squares regression line for the points $(\ln t, y)$ and obtain the logarithmic model $y = a + b \ln t$.

(d) Use a graphing utility to plot the original data and graph the linear and logarithmic models. Which is a better model? Explain.

Using Lagrange Multipliers **In Exercises 93–98, use Lagrange multipliers to find the indicated extrema, assuming that x and y are positive.**

93. Minimize $f(x, y) = x^2 + y^2$

Constraint: $x + y - 8 = 0$

94. Maximize $f(x, y) = xy$

Constraint: $x + 3y - 6 = 0$

95. Maximize $f(x, y) = 2x + 3xy + y$

Constraint: $x + 2y = 29$

96. Minimize $f(x, y) = x^2 - y^2$

Constraint: $x - 2y + 6 = 0$

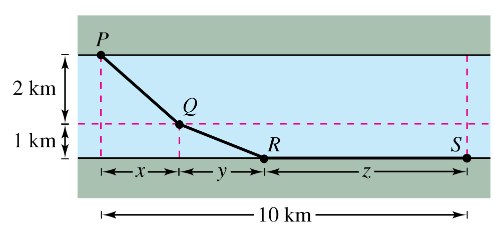
97. Maximize $f(x, y) = 2xy$

Constraint: $2x + y = 12$

98. Minimize $f(x, y) = 3x^2 - y^2$

Constraint: $2x - 2y + 5 = 0$

99. Minimum Cost A water line is to be built from point P to point S and must pass through regions where construction costs differ (see figure). The cost per kilometer (in dollars) is $3k$ from P to Q, $2k$ from Q to R, and k from R to S. For simplicity, let $k = 1$. Use Lagrange multipliers to find x, y, and z such that the total cost C will be minimized.



P.S. Problem Solving

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

1. Area **Heron's Formula** states that the area of a triangle with sides of lengths a, b, and c is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \dfrac{a+b+c}{2}$, as shown in the figure.

(a) Use Heron's Formula to find the area of the triangle with vertices $(0, 0)$, $(3, 4)$, and $(6, 0)$.

(b) Show that among all triangles having a fixed perimeter, the triangle with the largest area is an equilateral triangle.

(c) Show that among all triangles having a fixed area, the triangle with the smallest perimeter is an equilateral triangle.

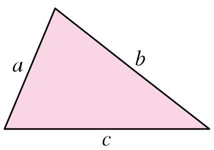


Figure for 1

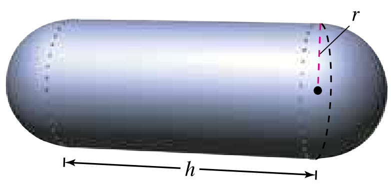


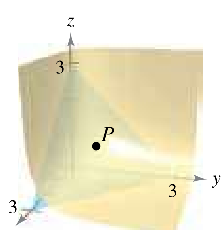
Figure for 2

2. Minimizing Material An industrial container is in the shape of a cylinder with hemispherical ends, as shown in the figure. The container must hold 1000 liters of fluid. Determine the radius r and length h that minimize the amount of material used in the construction of the tank.

3. Tangent Plane Let $P(x_0, y_0, z_0)$ be a point in the first octant on the surface $xyz = 1$, as shown in the figure.

(a) Find the equation of the tangent plane to the surface at the point P.

(b) Show that the volume of the tetrahedron formed by the three coordinate planes and the tangent plane is constant, independent of the point of tangency (see figure).



4. Using Functions Use a graphing utility to graph the functions

$$f(x) = \sqrt[3]{x^3 - 1} \quad \text{and} \quad g(x) = x$$

in the same viewing window.

(a) Show that

$$\lim_{x \to \infty} [f(x) - g(x)] = 0 \quad \text{and} \quad \lim_{x \to -\infty} [f(x) - g(x)] = 0.$$

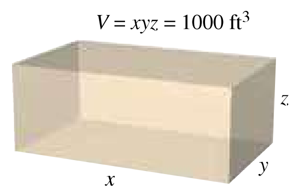
(b) Find the point on the graph of f that is farthest from the graph of g.

5. Finding Maximum and Minimum Values

(a) Let $f(x, y) = x - y$ and $g(x, y) = x^2 + y^2 = 4$. Graph various level curves of f and the constraint g in the xy-plane. Use the graph to determine the maximum value of f subject to the constraint $g = 4$. Then verify your answer using Lagrange multipliers.

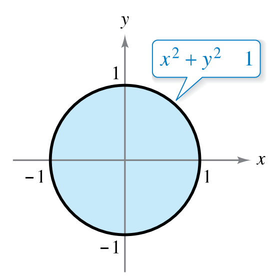
(b) Let $f(x, y) = x - y$ and $g(x, y) = x^2 + y^2 = 0$. Find the maximum and minimum values of f subject to the constraint $g = 0$. Does the Method of Lagrange Multipliers work in this case? Explain.

6. Minimizing Costs A heated storage room has the shape of a rectangular prism and has a volume of 1000 cubic feet, as shown in the figure. Because warm air rises, the heat loss per unit of area through the ceiling is five times as great as the heat loss through the floor. The heat loss through the four walls is three times as great as the heat loss through the floor. Determine the room dimensions that will minimize heat loss and therefore minimize heating costs.



7. Minimizing Costs Repeat Exercise 6 assuming that the heat loss through the walls and ceiling remain the same, but the floor is insulated so that there is no heat loss through the floor.

8. Temperature Consider a circular plate of radius 1 given by $x^2 + y^2 \le 1$, as shown in the figure. The temperature at any point $P(x, y)$ on the plate is $T(x, y) = 2x^2 + y^2 - y + 10$.



(a) Sketch the isotherm $T(x, y) = 10$. To print an enlarged copy of the graph, go to *MathGraphs.com.*

(b) Find the hottest and coldest points on the plate.

9. Cobb-Douglas Production Function Consider the Cobb-Douglas production function

$$f(x, y) = Cx^a y^{1-a}, \quad 0 < a < 1.$$

(a) Show that f satisfies the equation $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = f$.

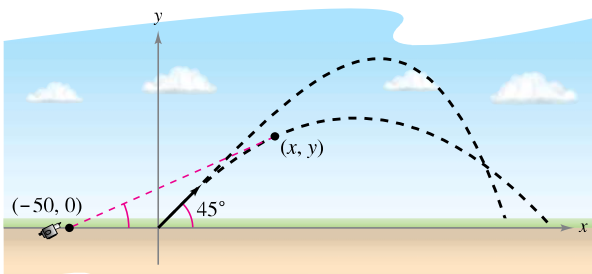
(b) Show that $f(tx, ty) = tf(x, y)$.

10. Minimizing Area Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

that encloses the circle $x^2 + y^2 = 2x$. Find values of a and b that minimize the area of the ellipse.

11. Projectile Motion A projectile is launched at an angle of 45° with the horizontal and with an initial velocity of 64 feet per second. A television camera is located in the plane of the path of the projectile 50 feet behind the launch site (see figure).



(a) Find parametric equations for the path of the projectile in terms of the parameter t representing time.

(b) Write the angle α that the camera makes with the horizontal in terms of x and y and in terms of t.

(c) Use the results of part (b) to find $\dfrac{d\alpha}{dt}$.

(d) Use a graphing utility to graph α in terms of t. Is the graph symmetric to the axis of the parabolic arch of the projectile? At what time is the rate of change of α greatest?

(e) At what time is the angle α maximum? Does this occur when the projectile is at its greatest height?

12. Distance Consider the distance d between the launch site and the projectile in Exercise 11.

(a) Write the distance d in terms of x and y and in terms of the parameter t.

(b) Use the results of part (a) to find the rate of change of d.

(c) Find the rate of change of the distance when $t = 2$.

(d) When is the rate of change of d minimum during the flight of the projectile? Does this occur at the time when the projectile reaches its maximum height?

13. Finding Extrema and Saddle Points Using Technology Consider the function

$$f(x, y) = (\alpha x^2 + \beta y^2)e^{-(x^2+y^2)}, \quad 0 < |\alpha| < \beta.$$

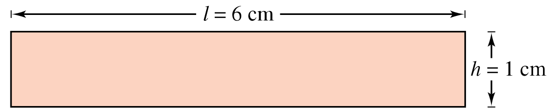
(a) Use a computer algebra system to graph the function for $\alpha = 1$ and $\beta = 2$, and identify any extrema or saddle points.

(b) Use a computer algebra system to graph the function for $\alpha = -1$ and $\beta = 2$, and identify any extrema or saddle points.

(c) Generalize the results in parts (a) and (b) for the function f.

14. Proof Prove that if f is a differentiable function such that $\nabla f(x_0, y_0) = \mathbf{0}$, then the tangent plane at (x_0, y_0) is horizontal.

15. Area The figure shows a rectangle that is approximately $l = 6$ centimeters long and $h = 1$ centimeter high.



(a) Draw a rectangular strip along the rectangular region showing a small increase in length.

(b) Draw a rectangular strip along the rectangular region showing a small increase in height.

(c) Use the results in parts (a) and (b) to identify the measurement that has more effect on the area A of the rectangle.

(d) Verify your answer in part (c) analytically by comparing the value of dA when $dl = 0.01$ and when $dh = 0.01$.

16. Tangent Planes Let f be a differentiable function of one variable. Show that all tangent planes to the surface $z = yf(x/y)$ intersect in a common point.

17. Wave Equation Show that

$$u(x, t) = \frac{1}{2}[\sin(x - t) + \sin(x + t)]$$

is a solution to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

18. Wave Equation Show that

$$u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)]$$

is a solution to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

(This equation describes the small transverse vibration of an elastic string such as those on certain musical instruments.)

19. Verifying Equations Consider the function $w = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$. Verify each of the following.

(a) $$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\sin\theta + \frac{\partial w}{\partial \theta}\frac{\cos\theta}{r}$$

(b) $$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{1}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2$$

20. Using a Function Demonstrate the result of Exercise 19(b) for

$$w = \arctan\frac{y}{x}.$$

21. Laplace's Equation Rewrite Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

in cylindrical coordinates.

25. (a) $3\mathbf{i} + \mathbf{j}$ (b) $-5\mathbf{i} + (2t - 2)\mathbf{j} + 2t^2\mathbf{k}$
(c) $18t\mathbf{i} + (6t - 3)\mathbf{j}$ (d) $4t + 3t^2$
(e) $\left(\frac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + (9t^2 - 2t + 1)\mathbf{k}$
(f) $2\mathbf{i} + 8t\mathbf{j} + 16t^2\mathbf{k}$

27. $\dfrac{1}{3}t^3\mathbf{i} + \dfrac{5}{2}t^2\mathbf{j} + 2t^4\mathbf{k} + \mathbf{C}$ **29.** $2t^{3/2}\mathbf{i} + 2\ln|t|\mathbf{j} + t\mathbf{k} + \mathbf{C}$

31. $\frac{32}{3}\mathbf{j}$ **33.** $2(e - 1)\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$

35. $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (e^t + 2)\mathbf{j} - (e^{-t} + 4)\mathbf{k}$

37. (a) $\mathbf{v}(t) = 4\mathbf{i} + 3t^2\mathbf{j} - \mathbf{k}$
$\|\mathbf{v}(t)\| = \sqrt{17 + 9t^4}$
$\mathbf{a}(t) = 6t\mathbf{j}$
(b) $\mathbf{v}(1) = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
$\mathbf{a}(1) = 6\mathbf{j}$

39. (a) $\mathbf{v}(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t, 3\rangle$
$\|\mathbf{v}(t)\| = 3\sqrt{\sin^2 t \cos^2 t + 1}$
$\mathbf{a}(t) = \langle 3\cos t(2\sin^2 t - \cos^2 t), 3\sin t(2\cos^2 t - \sin^2 t), 0\rangle$
(b) $\mathbf{v}(\pi) = \langle 0, 0. 3\rangle$
$\mathbf{a}(\pi) = \langle 3, 0, 0\rangle$

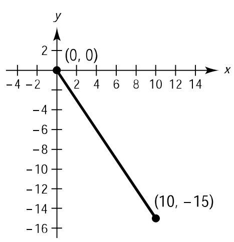
41. 11.67 ft; The ball will clear the fence.

43. $\mathbf{T}(2) = \dfrac{3\mathbf{i} - 2\mathbf{j}}{\sqrt{13}}$

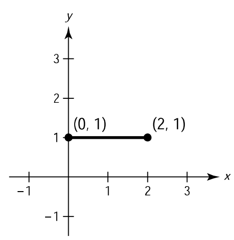
45. $\mathbf{T}(0) = \dfrac{2\mathbf{i} - 3\mathbf{k}}{\sqrt{13}}$; $x = 1 + 2t$, $y = 1$, $z = -3t$

47. $\mathbf{N}(1) = -\dfrac{3\sqrt{10}}{10}\mathbf{i} + \dfrac{\sqrt{10}}{10}\mathbf{j}$ **49.** $\mathbf{N}\left(\dfrac{\pi}{4}\right) = -\mathbf{j}$

51. $a_{\mathbf{T}} = -\dfrac{2\sqrt{13}}{585}$ **53.** $a_{\mathbf{T}} = 0$
$a_{\mathbf{N}} = \dfrac{4\sqrt{13}}{65}$ $a_{\mathbf{N}} = 1$

55.


$5\sqrt{13}$

57.


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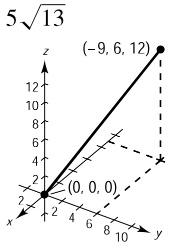
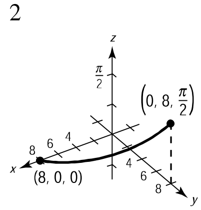
59.


$3\sqrt{29}$

61.


$\dfrac{\sqrt{65}\pi}{2}$

63. 0 **65.** $\dfrac{2\sqrt{5}}{(4 + 5t^2)^{3/2}}$ **67.** $\dfrac{\sqrt{2}}{3}$

69. $K = \dfrac{1}{26^{3/2}}, \dfrac{1}{K} = 26\sqrt{26}$ **71.** $K = \dfrac{\sqrt{2}}{4}, r = 2\sqrt{2}$

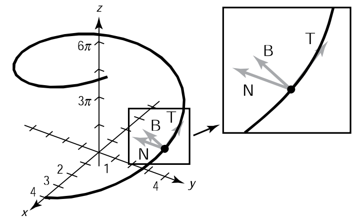
73. 2016.7 lb

P.S. Problem Solving *(page 869)*

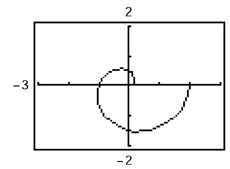
1. (a) a (b) πa (c) $K = \pi a$

3. Initial speed: 447.21 ft/sec; $\theta \approx 63.43°$ **5–7.** Proofs

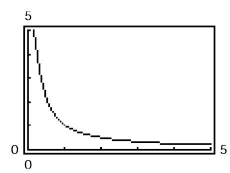
9. Unit tangent: $\left\langle -\frac{4}{5}, 0, \frac{3}{5}\right\rangle$
Principal unit normal: $\langle 0, -1, 0\rangle$
Binormal: $\left\langle \frac{3}{5}, 0, \frac{4}{5}\right\rangle$



11. (a) and (b) Proofs

13. (a)

(b) 6.766

(c) $K = \dfrac{\pi(\pi^2t^2 + 2)}{(\pi^2t^2 + 1)^{3/2}}$
$K(0) = 2\pi$
$K(1) = \dfrac{\pi(\pi^2 + 2)}{(\pi^2 + 1)^{3/2}} \approx 1.04$
$K(2) \approx 0.51$

(d)

(e) $\lim_{t \quad \infty} K = 0$

(f) As $t \quad \infty$, the graph spirals outward and the curvature decreases.

Chapter 13

Section 13.1 *(page 880)*

1. There is not a unique value of z for each ordered pair.

3. z is a function of x and y. **5.** z is a function of x and y.

7. z is not a function of x and y.

9. (a) 1 (b) 1 (c) -17
(d) $9 - y$ (e) $2x - 1$ (f) $13 - t$

11. (a) -1 (b) 0 (c) xe^3 (d) te^{-y}

13. (a) 3 (b) 2 (c) $\dfrac{16}{t}$ (d) $-\dfrac{6}{5}$

15. (a) $\sqrt{2}$ (b) $3\sin 1$ (c) 0 (d) 4

17. (a) -4 (b) -6 (c) $-\frac{25}{4}$ (d) $\frac{9}{4}$

19. (a) $2, \Delta x \neq 0$ (b) $2y + \Delta y, \Delta y \neq 0$

21. Domain: $\{(x, y)\colon x$ is any real number, y is any real number$\}$
Range: all real numbers

23. Domain: $\{(x, y)\colon y \geq 0\}$
Range: all real numbers

25. Domain: $\{(x, y)\colon x \neq 0, y \neq 0\}$
Range: all real numbers

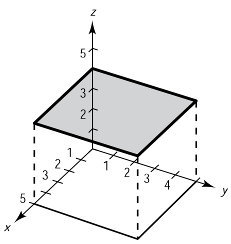
27. Domain: $\{(x, y)\colon x^2 + y^2 \leq 4\}$
Range: $0 \leq z \leq 2$

29. Domain: $\{(x, y)\colon -1 \leq x + y \leq 1\}$
Range: $0 \leq z \leq \pi$

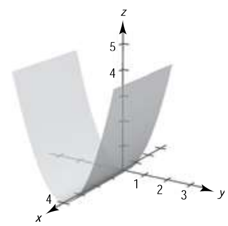
31. Domain: $\{(x, y):\ y < -x + 5\}$
Range: all real numbers

33. (a) $(20, 0, 0)$ (b) $(-15, 10, 20)$
(c) $(20, 15, 25)$ (d) $(20, 20, 0)$

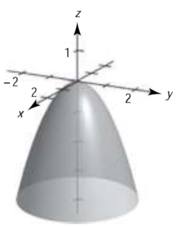
35. Plane



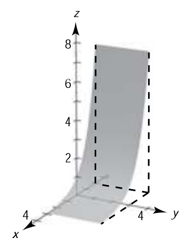
37. Cylinder with rulings parallel to the x-axis



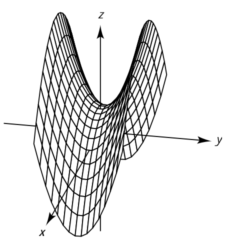
39. Paraboloid



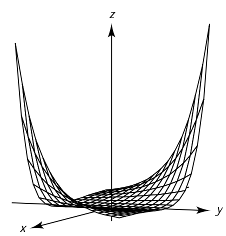
41. Cylinder with rulings parallel to the y-axis



43.

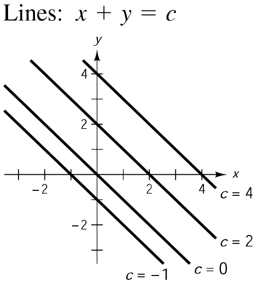


45.

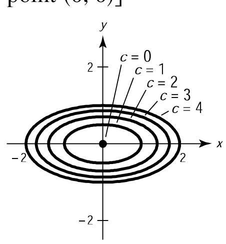


47. c **48.** d **49.** b **50.** a

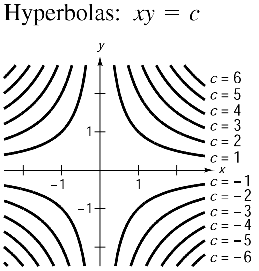
51. Lines: $x + y = c$



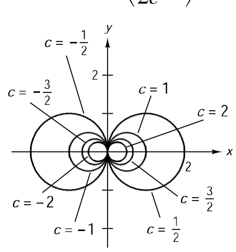
53. Ellipses: $x^2 + 4y^2 = c$
[except $x^2 + 4y^2 = 0$ is the point $(0, 0)$]



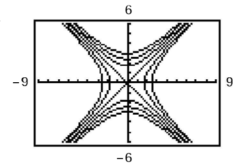
55. Hyperbolas: $xy = c$



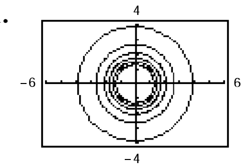
57. Circles passing through $(0, 0)$
Centered at $\left(\frac{1}{2c}, 0\right)$



59.



61.



63. Yes; The definition of a function of two variables requires that z be unique for each ordered pair (x, y) in the domain.

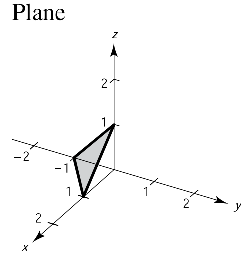
65. $f(x, y) = \dfrac{x}{y}$ $\left(\text{The level curves are the lines } y = \dfrac{x}{c}.\right)$

67. The surface may be shaped like a saddle. For example, let $f(x, y) = xy$. The graph is not unique because any vertical translation will produce the same level curves.

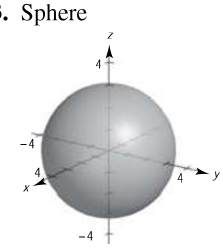
69.

	Inflation Rate		
Tax Rate	0	0.03	0.05
0	\$1790.85	\$1332.56	\$1099.43
0.28	\$1526.43	\$1135.80	\$937.09
0.35	\$1466.07	\$1090.90	\$900.04

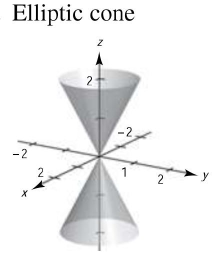
71. Plane



73. Sphere

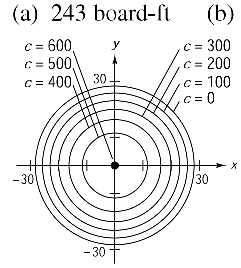


75. Elliptic cone



77. (a) 243 board-ft (b) 507 board-ft

79.



81. 36,661 units

83. Proof

85. (a) $k = \dfrac{520}{3}$

(b) $P = \dfrac{520T}{3V}$

The level curves are lines.

87. (a) C (b) A (c) B

89. $C = 4.50xy + 5.00(xz + yz)$

91. False. Let $f(x, y) = 4$.

93. False. The equation of a sphere is not a function.

95. Putnam Problem A1, 2008

Section 13.2 *(page 891)*

1. As x approaches -1 and y approaches 3, z approaches 1.
3–5. Proofs **7.** 9 **9.** -20 **11.** 7, continuous
13. e^2, continuous **15.** 0, continuous for $y \neq 0$
17. $\frac{1}{2}$, continuous except at $(0, 0)$ **19.** -1, continuous
21. 0, continuous for $xy \neq 1$, $|xy| \leq 1$
23. $2\sqrt{2}$, continuous for $x + y + z \geq 0$ **25.** 0
27. Limit does not exist. **29.** Limit does not exist.
31. Limit does not exist. **33.** 0
35. Limit does not exist.
37. No. The existence of $f(2, 3)$ has no bearing on the existence of the limit as $(x, y) \to (2, 3)$.
39. $\lim_{x \to 0} f(x, 0) = 0$ if $f(x, 0)$ exists. **41.** Continuous, 1

43.

(x, y)	$(1, 0)$	$(0.5, 0)$	$(0.1, 0)$	$(0.01, 0)$	$(0.001, 0)$
$f(x, y)$	0	0	0	0	0

$y = 0$: 0

(x, y)	$(1, 1)$	$(0.5, 0.5)$	$(0.1, 0.1)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

(x, y)	$(0.01, 0.01)$	$(0.001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$

$y = x$: $\frac{1}{2}$
Limit does not exist.
Continuous except at $(0, 0)$

45.

(x, y)	$(1, 0)$	$(0.5, 0)$	$(0.1, 0)$	$(0.01, 0)$	$(0.001, 0)$
$f(x, y)$	0	0	0	0	0

$y = 0$: 0

(x, y)	$(1, 1)$	$(0.5, 0.5)$	$(0.1, 0.1)$
$f(x, y)$	$\frac{1}{2}$	1	5

(x, y)	$(0.01, 0.01)$	$(0.001, 0.001)$
$f(x, y)$	50	500

$y = x$: ∞
The limit does not exist.
Continuous except at $(0, 0)$

47. (a) $\dfrac{1 + a^2}{a}$, $a \neq 0$ (b) Limit does not exist.
(c) No; Different paths result in different limits.
49. f is continuous. g is continuous except at $(0, 0)$. g has a removable discontinuity at $(0, 0)$.
51. 0 **53.** 0 **55.** 1 **57.** 1 **59.** 0
61. Continuous except at $(0, 0, 0)$ **63.** Continuous
65. Continuous **67.** Continuous
69. Continuous for $y \neq \dfrac{2x}{3}$ **71.** (a) $2x$ (b) -4
73. (a) $\dfrac{1}{y}$ (b) $-\dfrac{x}{y^2}$ **75.** (a) $3 + y$ (b) $x - 2$
77. 0
79. True
81. False. Let $f(x, y) = \begin{cases} \ln(x^2 + y^2), & x \neq 0, y \neq 0 \\ 0, & x = 0, y = 0 \end{cases}$.
83. $\dfrac{\pi}{2}$ **85.** Proof

Section 13.3 *(page 900)*

1. z_x, $f_x(x, y)$, $\dfrac{\partial z}{\partial x}$
3. (a) Differentiate first with respect to y, then with respect to x, and last with respect to z.
(b) Differentiate first with respect to z and then with respect to x.
5. No. Because you are finding the partial derivative with respect to x, you consider y to be constant. So, the denominator is considered a constant and does not contain any variables.
7. No. Because you are finding the partial derivative with respect to y, you consider x to be constant. So, the denominator is considered a constant and does not contain any variables.
9. Yes. Because you are finding the partial derivative with respect to x, you consider y to be constant. So, both the numerator and denominator contain variables.

11. $f_x(x, y) = 2$, $f_y(x, y) = -5$
13. $\dfrac{\partial z}{\partial x} = 6 - 2xy$, $\dfrac{\partial z}{\partial y} = -x^2 + 16y$
15. $\dfrac{\partial z}{\partial x} = \sqrt{y}$, $\dfrac{\partial z}{\partial y} = \dfrac{x}{2\sqrt{y}}$
17. $\dfrac{\partial z}{\partial x} = ye^{xy}$, $\dfrac{\partial z}{\partial y} = xe^{xy}$
19. $\dfrac{\partial z}{\partial x} = 2xe^{2y}$, $\dfrac{\partial z}{\partial y} = 2x^2e^{2y}$
21. $\dfrac{\partial z}{\partial x} = \dfrac{1}{x}$, $\dfrac{\partial z}{\partial y} = -\dfrac{1}{y}$
23. $\dfrac{\partial z}{\partial x} = \dfrac{2x}{x^2 + y^2}$, $\dfrac{\partial z}{\partial y} = \dfrac{2y}{x^2 + y^2}$
25. $\dfrac{\partial z}{\partial x} = \dfrac{x^3 - 3y^3}{x^2y}$, $\dfrac{\partial z}{\partial y} = \dfrac{-x^3 + 12y^3}{2xy^2}$
27. $h_x(x, y) = -2xe^{-(x^2 + y^2)}$, $h_y(x, y) = -2ye^{-(x^2 + y^2)}$
29. $f_x(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}$, $f_y(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}}$
31. $\dfrac{\partial z}{\partial x} = -y \sin xy$, $\dfrac{\partial z}{\partial y} = -x \sin xy$
33. $\dfrac{\partial z}{\partial x} = 2 \sec^2(2x - y)$, $\dfrac{\partial z}{\partial y} = -\sec^2(2x - y)$
35. $\dfrac{\partial z}{\partial x} = 8ye^y \cos 8xy$, $\dfrac{\partial z}{\partial y} = e^y(8x \cos 8xy + \sin 8xy)$
37. $\dfrac{\partial z}{\partial x} = 2 \cosh(2x + 3y)$, $\dfrac{\partial z}{\partial y} = 3 \cosh(2x + 3y)$
39. $f_x(x, y) = 1 - x^2$, $f_y(x, y) = y^2 - 1$
41. $f_x(x, y) = 3$, $f_y(x, y) = 2$

43. $f_x(x, y) = \dfrac{1}{2\sqrt{x+y}}$
$f_y(x, y) = \dfrac{1}{2\sqrt{x+y}}$

45. $f_x = 12$
$f_y = 12$

47. $f_x = -1$
$f_y = \frac{1}{2}$

49. $f_x = \frac{1}{4}$
$f_y = \frac{1}{4}$

51. $f_x = -\dfrac{1}{4}$
$f_y = \dfrac{1}{4}$

53. $\dfrac{\partial z}{\partial x}(1, 2) = 2$
$\dfrac{\partial z}{\partial y}(1, 2) = 1$

55. $g_x(1, 1) = -2$
$g_y(1, 1) = -2$

57. $H_x(x, y, z) = \cos(x + 2y + 3z)$
$H_y(x, y, z) = 2\cos(x + 2y + 3z)$
$H_z(x, y, z) = 3\cos(x + 2y + 3z)$

59. $\dfrac{\partial w}{\partial x} = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}$
$\dfrac{\partial w}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}$
$\dfrac{\partial w}{\partial z} = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$

61. $F_x(x, y, z) = \dfrac{x}{x^2 + y^2 + z^2}$
$F_y(x, y, z) = \dfrac{y}{x^2 + y^2 + z^2}$
$F_z(x, y, z) = \dfrac{z}{x^2 + y^2 + z^2}$

63. $f_x = 3,\ f_y = 1,\ f_z = 2$ **65.** $f_x = 1,\ f_y = 0,\ f_z = 0$

67. $f_x = 4,\ f_y = 24,\ f_z = 0$ **69.** $x = 2,\ y = -2$

71. $x = -6,\ y = 4$ **73.** $x = 1,\ y = 1$ **75.** $x = 0,\ y = 0$

77. $\dfrac{\partial^2 z}{\partial x^2} = 0$
$\dfrac{\partial^2 z}{\partial y^2} = 6x$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = 6y$

79. $\dfrac{\partial^2 z}{\partial x^2} = 12x^2$
$\dfrac{\partial^2 z}{\partial y^2} = 18y$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = -2$

81. $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{y^2}{(x^2 + y^2)^{3/2}}$
$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{x^2}{(x^2 + y^2)^{3/2}}$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{-xy}{(x^2 + y^2)^{3/2}}$

83. $\dfrac{\partial^2 z}{\partial x^2} = e^x \tan y$
$\dfrac{\partial^2 z}{\partial y^2} = 2e^x \sec^2 y \tan y$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = e^x \sec^2 y$

85. $\dfrac{\partial^2 z}{\partial x^2} = -y^2 \cos xy$
$\dfrac{\partial^2 z}{\partial y^2} = -x^2 \cos xy$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = -xy \cos xy - \sin xy$

87. $\dfrac{\partial z}{\partial x} = \sec y$
$\dfrac{\partial z}{\partial y} = x \sec y \tan y$
$\dfrac{\partial^2 z}{\partial x^2} = 0$
$\dfrac{\partial^2 z}{\partial y^2} = x \sec y(\sec^2 y + \tan^2 y)$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = \sec y \tan y$
No values of x and y exist such that $f_x(x, y) = f_y(x, y) = 0$.

89. $\dfrac{\partial z}{\partial x} = \dfrac{y^2 - x^2}{x(x^2 + y^2)}$
$\dfrac{\partial z}{\partial y} = \dfrac{-2y}{x^2 + y^2}$
$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{x^4 - 4x^2y^2 - y^4}{x^2(x^2 + y^2)^2}$
$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{2(y^2 - x^2)}{(x^2 + y^2)^2}$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{4xy}{(x^2 + y^2)^2}$
No values of x and y exist such that $f_x(x, y) = f_y(x, y) = 0$.

91. $f_{xyy}(x, y, z) = f_{yxy}(x, y, z) = f_{yyx}(x, y, z) = 0$

93. $f_{xyy}(x, y, z) = f_{yxy}(x, y, z) = f_{yyx}(x, y, z) = z^2 e^{-x} \sin yz$

95. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0 + 0 = 0$

97. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0$

99. $\dfrac{\partial^2 z}{\partial t^2} = -c^2 \sin(x - ct) = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right)$

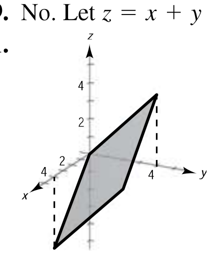
101. $\dfrac{\partial^2 z}{\partial t^2} = \dfrac{-c^2}{(x + ct)^2} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right)$

103. $\dfrac{\partial z}{\partial t} = \dfrac{-e^{-t} \cos x}{c} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right)$ **105.** Proof

107. Yes; $f(x, y) = \cos(3x - 2y)$

109. No. Let $z = x + y + 1$.

111.



113. Dollars/yr; negative; You expect the influence that age has on the cost of the car to be negative.

115. (a) $\sqrt{2}$ (b) $\frac{5}{2}$ **117.** (a) 72 (b) 72

119. $IQ_M = \dfrac{100}{C},\ IQ_M(12, 10) = 10$
IQ increases at a rate of 10 points per year of mental age when the mental age is 12 and the chronological age is 10.
$IQ_C = -\dfrac{100M}{C^2},\ IQ_C(12, 10) = -12$
IQ decreases at a rate of 12 points per year of chronological age when the mental age is 12 and the chronological age is 10.

121. An increase in either the charge for food and housing or the tuition will cause a decrease in the number of applicants.

123. $\dfrac{\partial T}{\partial x} = -2.4°/\text{m},\ \dfrac{\partial T}{\partial y} = -9°/\text{m}$

125. $T = \dfrac{PV}{nR}$ $\quad \dfrac{\partial T}{\partial P} = \dfrac{V}{nR}$
$P = \dfrac{nRT}{V}$ $\quad \dfrac{\partial P}{\partial V} = \dfrac{-nRT}{V^2}$
$V = \dfrac{nRT}{P}$ $\quad \dfrac{\partial V}{\partial T} = \dfrac{nR}{P}$
$\dfrac{\partial T}{\partial P} \cdot \dfrac{\partial P}{\partial V} \cdot \dfrac{\partial V}{\partial T} = -\dfrac{nRT}{VP} = -\dfrac{nRT}{nRT} = -1$

127. (a) $\dfrac{\partial z}{\partial x} = 0.23, \dfrac{\partial z}{\partial y} = 0.14$

(b) As the expenditures on amusement parks and campgrounds (x) increase, the expenditures on spectator sports (z) increase. As the expenditures on live entertainment (y) increase, the expenditures on spectator sports (z) also increase.

129. (a) $f_x(x, y) = \dfrac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

$f_y(x, y) = \dfrac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

(b) $f_x(0, 0) = 0, f_y(0, 0) = 0$

(c) $f_{xy}(0, 0) = -1, f_{yx}(0, 0) = 1$

(d) f_{xy} or f_{yx} or both are not continuous at $(0, 0)$.

131. Proof

Section 13.4 *(page 909)*

1. In general, the accuracy worsens as Δx and Δy increase.

3. $dz = 15x^2y^2\,dx + 10x^3y\,dy$

5. $dz = (e^{x^2+y^2} + e^{-x^2-y^2})(x\,dx + y\,dy)$

7. $dw = 2xyz^2\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$

9. (a) $f(2, 1) = 1, f(2.1, 1.05) = 1.05, \Delta z = 0.05$

(b) $dz = 0.05$

11. (a) $f(2, 1) = 11, f(2.1, 1.05) = 10.4875, \Delta z = -0.5125$

(b) $dz = -0.5$

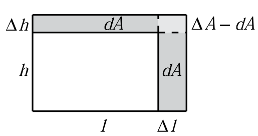
13. (a) $f(2, 1) = e^2 \approx 7.3891, f(2.1, 1.05) = 1.05e^{2.1} \approx 8.5745,$ $\Delta z \approx 1.1854$

(b) $dz \approx 1.1084$

15. 0.44 **17.** 0

19. Yes. Because f_x and f_y are continuous on R, you know that f is differentiable on R. Because f is differentiable on R, you know that f is continuous on R.

21. $dA = h\,dl + l\,dh$



$\Delta A - dA = dl\,dh$

23. $dV = \pm 3.92$ in.^{3}, $\dfrac{dV}{V} = 0.82\%$

25.

Δr	Δh	dV	ΔV	$\Delta V - dV$
0.1	0.1	8.3776	8.5462	0.1686
0.1	-0.1	5.0265	5.0255	-0.0010
0.001	0.002	0.1005	0.1006	0.0001
-0.0001	0.0002	-0.0034	-0.0034	0.0000

27. $dC = \pm 2.4418, \dfrac{dC}{C} = 19\%$ **29.** 10%

31. (a) $V = 18\sin\theta$ ft^3, $\theta = \dfrac{\pi}{2}$ (b) 1.047 ft^3

33. $L \approx 8.096 \times 10^{-4} \pm 6.6 \times 10^{-6}$ microhenrys

35. Answers will vary.
Sample answer:
$\varepsilon_1 = \Delta x$
$\varepsilon_2 = 0$

37. Answers will vary.
Sample answer:
$\varepsilon_1 = y\,\Delta x$
$\varepsilon_2 = 2x\,\Delta x + (\Delta x)^2$

39. Proof

Section 13.5 *(page 917)*

1. You can convert w into a function of s and t, or you can use the Chain Rule given in Theorem 13.7.

3. $8t + 5$; 21 **5.** $e^t(\sin t + \cos t)$; 1

7. (a) and (b) $2e^{2t} + \dfrac{3}{t^4}$ **9.** (a) and (b) $2e^{2t}$

11. (a) and (b) $3(2t^2 - 1)$ **13.** $\dfrac{-11\sqrt{29}}{29} \approx -2.04$

15. $\dfrac{\partial w}{\partial s} = 4s, 4$

$\dfrac{\partial w}{\partial t} = 4t, 12$

17. $\dfrac{\partial w}{\partial s} = 5\cos(5s - t), 0$

$\dfrac{\partial w}{\partial t} = -\cos(5s - t), 0$

19. (a) and (b)

$\dfrac{\partial w}{\partial s} = t^2(3s^2 - t^2)$

$\dfrac{\partial w}{\partial t} = 2st(s^2 - 2t^2)$

21. (a) and (b)

$\dfrac{\partial w}{\partial s} = te^{s^2-t^2}(2s^2 + 1)$

$\dfrac{\partial w}{\partial t} = se^{s^2-t^2}(1 - 2t^2)$

23. $\dfrac{y - 2x + 1}{2y - x + 1}$ **25.** $-\dfrac{x^2 + y^2 + x}{x^2 + y^2 + y}$

27. $\dfrac{\partial z}{\partial x} = -\dfrac{x}{z}$

$\dfrac{\partial z}{\partial y} = -\dfrac{y}{z}$

29. $\dfrac{\partial z}{\partial x} = -\dfrac{x}{y + z}$

$\dfrac{\partial z}{\partial y} = -\dfrac{z}{y + z}$

31. $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = \dfrac{\sec^2(x + y)}{\sin z}$

33. $\dfrac{\partial z}{\partial x} = -\dfrac{(ze^{xz} + y)}{xe^{xz}}$

$\dfrac{\partial z}{\partial y} = -e^{-xz}$

35. $\dfrac{\partial w}{\partial x} = \dfrac{7y + w^2}{4z - 2wz - 2wx}$

$\dfrac{\partial w}{\partial y} = \dfrac{7x + z^2}{4z - 2wz - 2wx}$

$\dfrac{\partial w}{\partial z} = \dfrac{2yz - 4w + w^2}{4z - 2wz - 2wx}$

37. $\dfrac{\partial w}{\partial x} = \dfrac{y\sin xy}{z}$

$\dfrac{\partial w}{\partial y} = \dfrac{x\sin xy - z\cos yz}{z}$

$\dfrac{\partial w}{\partial z} = -\dfrac{y\cos yz + w}{z}$

39. (a) $f(tx, ty) = 2(tx)^2 - 5(tx)(ty)$
$= t^2(2x^2 - 5xy) = t^2 f(x, y); n = 2$

(b) $xf_x(x, y) + yf_y(x, y) = 4x^2 - 10xy = 2f(x, y)$

41. (a) $f(tx, ty) = e^{tx/ty} = e^{x/y} = f(x, y);\ n = 0$

(b) $xf_x(x, y) + yf_y(x, y) = \dfrac{xe^{x/y}}{y} - \dfrac{xe^{x/y}}{y} = 0$

43. 47 **45.** Proof

47. (a) $\dfrac{\partial F}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial F}{\partial v}\dfrac{\partial v}{\partial x} = 4\dfrac{\partial F}{\partial u}$

(b) $\dfrac{\partial F}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial F}{\partial v}\dfrac{\partial v}{\partial x} = -2\dfrac{\partial F}{\partial u} + 2x\dfrac{\partial F}{\partial v}$

49. 4608π in.^{3}/min, 624π in.^{2}/min **51.** $28m$ cm^2/sec

53–55. Proofs

Section 13.6 *(page 928)*

1. The partial derivative with respect to x is the directional derivative in the direction of the positive x-axis. That is, the directional derivative for $\theta = 0$.

3. $-\sqrt{2}$ **5.** $\frac{1}{2} + \sqrt{3}$ **7.** 1 **9.** $-\frac{7}{25}$ **11.** 6

13. $\dfrac{2\sqrt{5}}{5}$ **15.** $3\mathbf{i} + 10\mathbf{j}$ **17.** $2\mathbf{i} - \dfrac{1}{2}\mathbf{j}$

19. $20\mathbf{i} - 14\mathbf{j} - 30\mathbf{k}$ **21.** -1 **23.** $\dfrac{2\sqrt{3}}{3}$ **25.** $3\sqrt{2}$

27. $-\dfrac{8}{\sqrt{5}}$ **29.** $-\sqrt{y}\,\mathbf{i} + \left(2y - \dfrac{x}{2\sqrt{y}}\right)\mathbf{j};\ \sqrt{39}$

31. $\tan y\,\mathbf{i} + x\sec^2 y\,\mathbf{j};\ \sqrt{17}$

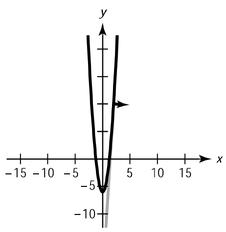
33. $\cos x^2y^3(2x\mathbf{i} + 3y^2\mathbf{j});\ \dfrac{1}{\pi}\sqrt{4 + 9\pi^6}$

35. $\dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}};\ 1$ **37.** $yz(yz\mathbf{i} + 2xz\mathbf{j} + 2xy\mathbf{k});\ \sqrt{33}$

39. $-2\mathbf{i} - 3\mathbf{j}$ **41.** $3\mathbf{i} - \mathbf{j}$

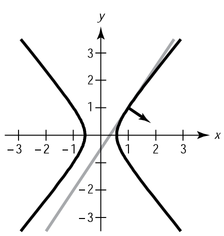
43. (a) $16\mathbf{i} - \mathbf{j}$ (b) $\dfrac{\sqrt{257}}{257}(16\mathbf{i} - \mathbf{j})$ (c) $y = 16x - 22$

(d)

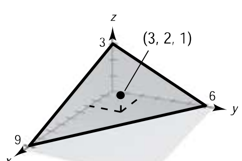


45. (a) $6\mathbf{i} - 4\mathbf{j}$ (b) $\dfrac{\sqrt{13}}{13}(3\mathbf{i} - 2\mathbf{j})$ (c) $y = \dfrac{3}{2}x - \dfrac{1}{2}$

(d)



47. (a)



(b) (i) $-\dfrac{5\sqrt{2}}{12}$ (ii) $\dfrac{2 - 3\sqrt{3}}{12}$

(iii) $\dfrac{2 + 3\sqrt{3}}{12}$ (iv) $\dfrac{3 - 2\sqrt{3}}{12}$

(c) (i) $-\dfrac{5\sqrt{2}}{12}$ (ii) $\dfrac{3}{5}$ (iii) $-\dfrac{1}{5}$ (iv) $-\dfrac{11\sqrt{10}}{60}$

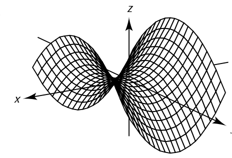
(d) $-\dfrac{1}{3}\mathbf{i} - \dfrac{1}{2}\mathbf{j}$ (e) $\dfrac{\sqrt{13}}{6}$

(f) $\mathbf{u} = \dfrac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$

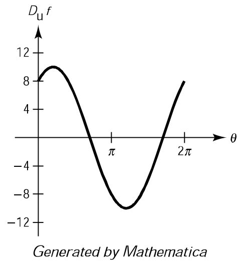
$D_{\mathbf{u}}f(3, 2) = \nabla f \cdot \mathbf{u} = 0$

∇f is the direction of the greatest rate of change of f. So, in a direction orthogonal to ∇f, the rate of change of f is 0.

49. (a)



(b) $D_{\mathbf{u}}f(4, -3) = 8\cos\theta + 6\sin\theta$



(c) $\theta \approx 2.21,\ \theta \approx 5.36$

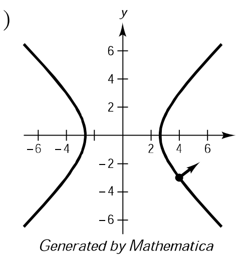
Directions in which there is no change in f

(d) $\theta \approx 0.64,\ \theta \approx 3.79$

Directions of greatest rate of change in f

(e) 10; Magnitude of the greatest rate of change

(f)



Orthogonal to the level curve

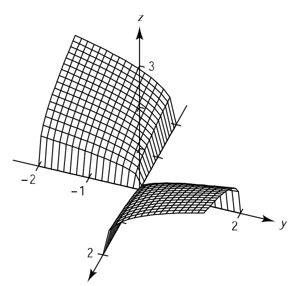
51. No; Answers will vary. **53.** $5\nabla h = -(5\mathbf{i} + 12\mathbf{j})$

55. $\frac{1}{625}(7\mathbf{i} - 24\mathbf{j})$ **57.** $6\mathbf{i} - 10\mathbf{j}$; 11.66°/cm **59.** $y^2 = 10x$

61. True **63.** True **65.** $f(x, y, z) = e^x \cos y + \frac{1}{2}z^2 + C$

67. (a) and (b) Proofs

(c)



Section 13.7 *(page 937)*

1. $\nabla F(x_0, y_0, z_0)$ and any tangent vector $\mathbf{v}$ at (x_0, y_0, z_0) are orthogonal. So, $\nabla F(x_0, y_0, z_0) \cdot \mathbf{v} = 0$.

3. The level surface can be written as $3x - 5y + 3z = 15$, which is an equation of a plane in space.

5. The level surface can be written as $4x^2 + 9y^2 - 4z^2 = 0$, which is an elliptic cone that lies on the z-axis.

7. $4x + 2y - z = 2$ **9.** $3x + 4y - 5z = 0$

11. $2x - 2y - z = 2$ **13.** $3x + 4y - 25z = 25(1 - \ln 5)$

15. $4x + 2y + 5z = -15$

17. (a) $x + y + z = 9$ (b) $x - 3 = y - 3 = z - 3$

19. (a) $x - 2y + 2z = 7$ (b) $x - 1 = \dfrac{y + 1}{-2} = \dfrac{z - 2}{2}$

21. (a) $6x - 4y - z = 5$ (b) $\dfrac{x - 3}{6} = \dfrac{y - 2}{-4} = \dfrac{z - 5}{-1}$

23. (a) $10x + 5y + 2z = 30$ (b) $\dfrac{x - 1}{10} = \dfrac{y - 2}{5} = \dfrac{z - 5}{2}$

25. (a) $8x + y - z = 0$ (b) $\dfrac{x}{8} = \dfrac{y - 2}{1} = \dfrac{z - 2}{-1}$

27. $x = t + 1,\ y = 1 - t,\ z = t + 1$

29. $x = 4t + 3,\ y = 4t + 3,\ z = 4 - 3t$

31. $x = t + 3,\ y = 5t + 1,\ z = 2 - 4t$

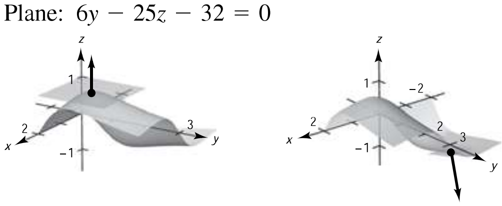
33. $86.0°$ **35.** $77.4°$ **37.** $(0, 3, 12)$ **39.** $(2, 2, -4)$

41. $(0, 0, 0)$ **43.** Proof **45.** (a) and (b) Proofs

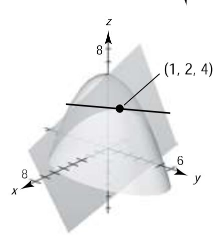
47. Not necessarily; They only need to be parallel.

49. $\left(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ or $\left(\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}\right)$ **51.** $(-2, 1, -1)$ or $(2, -1, 1)$

53. (a) Line: $x = 1,\ y = 1,\ z = 1 - t$
Plane: $z = 1$
(b) Line: $x = -1,\ y = 2 + \frac{6}{25}t,\ z = -\frac{4}{5} - t$
Plane: $6y - 25z - 32 = 0$
(c)



55. (a) $x = 1 + t$
$y = 2 - 2t$
$z = 4$
$\theta \approx 48.2°$
(b)



57. $F(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1$

$F_x(x, y, z) = \dfrac{2x}{a^2}$

$F_y(x, y, z) = \dfrac{2y}{b^2}$

$F_z(x, y, z) = \dfrac{2z}{c^2}$

Plane: $\dfrac{2x_0}{a^2}(x - x_0) + \dfrac{2y_0}{b^2}(y - y_0) + \dfrac{2z_0}{c^2}(z - z_0) = 0$

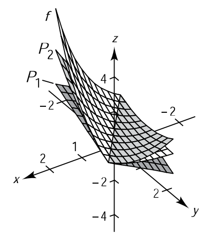
$\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} + \dfrac{z_0z}{c^2} = 1$

59. $F(x, y, z) = a^2x^2 + b^2y^2 - z^2$
$F_x(x, y, z) = 2a^2x$
$F_y(x, y, z) = 2b^2y$
$F_z(x, y, z) = -2z$
Plane: $2a^2x_0(x - x_0) + 2b^2y_0(y - y_0) - 2z_0(z - z_0) = 0$
$a^2x_0x + b^2y_0y - z_0z = 0$
Therefore, the plane passes through the origin.

61. (a) $P_1(x, y) = 1 + x - y$
(b) $P_2(x, y) = 1 + x - y + \frac{1}{2}x^2 - xy + \frac{1}{2}y^2$
(c) If $x = 0$, $P_2(0, y) = 1 - y + \frac{1}{2}y^2$.
This is the second-degree Taylor polynomial for e^{-y}.
If $y = 0$, $P_2(x, 0) = 1 + x + \frac{1}{2}x^2$.
This is the second-degree Taylor polynomial for e^x.
(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0.1	0.9048	0.9000	0.9050
0.2	0.1	1.1052	1.1000	1.1050
0.2	0.5	0.7408	0.7000	0.7450
1	0.5	1.6487	1.5000	1.6250

(e)



63. Proof

Section 13.8 *(page 946)*

1. (a) To say that f has a relative minimum at (x_0, y_0) means that the point (x_0, y_0, z_0) is at least as low as all nearby points on the graph of $z = f(x, y)$.
(b) To say that f has a relative maximum at (x_0, y_0) means that the point (x_0, y_0, z_0) is at least as high as all nearby points in the graph of $z = f(x, y)$.
(c) Critical points of f are the points at which the gradient of f is 0 or the points at which one of the partial derivatives does not exist.
(d) A critical point is a saddle point if it is neither a relative minimum nor a relative maximum.

3. Relative minimum: $(1, 3, 0)$ **5.** Relative minimum: $(0, 0, 1)$

7. Relative minimum: $(-1, 3, -4)$ **9.** Relative minimum: $(-4, 6, -55)$

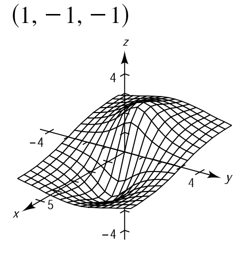
11. Every point along the x- or y-axis is a critical point. Each of the critical points yields an absolute maximum.

13. Relative maximum: $\left(\frac{1}{2}, -1, \frac{31}{4}\right)$ **15.** Relative minimum: $\left(\frac{1}{2}, -4, -\frac{187}{4}\right)$

17. Relative minimum: $(3, -4, -5)$ **19.** Relative maximum: $(0, 0, -12)$

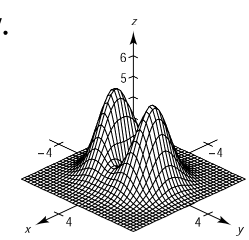
21. Saddle point: $(1, -1, -1)$ **23.** No critical numbers

25.



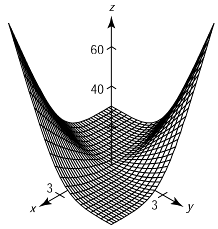
Relative maximum: $(-1, 0, 2)$
Relative minimum: $(1, 0, -2)$

27.



Relative minimum: $(0, 0, 0)$
Relative maxima: $(0, \pm 1, 4)$
Saddle points: $(\pm 1, 0, 1)$

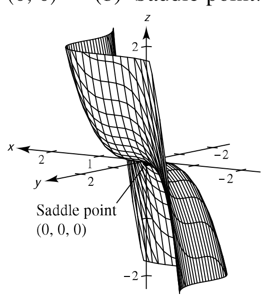
29. z is never negative. Minimum: $z = 0$ when $x = y \neq 0$.



31. Insufficient information **33.** Saddle point

35. (a) $(0, 0)$ (b) Saddle point: $(0, 0, 0)$ (c) $(0, 0)$

(d)

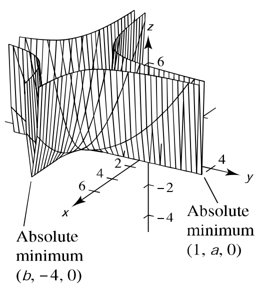


37. (a) $(1, a), (b, -4)$

(b) Absolute minima: $(1, a, 0), (b, -4, 0)$

(c) $(1, a), (b, -4)$

(d)



39. Absolute maximum: $(4, 0, 21)$
Absolute minimum: $(4, 2, -11)$

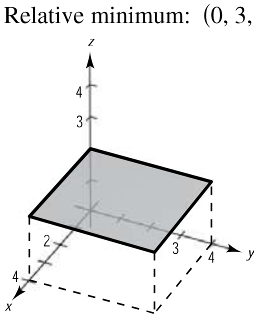
41. Absolute maximum: $(0, 1, 10)$
Absolute minimum: $(1, 2, 5)$

43. Absolute maxima: $(\pm 2, 4, 28)$
Absolute minimum: $(0, 1, -2)$

45. Absolute maxima: $(-2, -1, 9), (2, 1, 9)$
Absolute minima: $(x, -x, 0), |x| \leq 1$

47. Relative minimum: $(0, 3, -1)$ **49.** $-4 < f_{xy}(3, 7) < 4$

51.



Extrema at all (x, y)

53. (a) $f_x = 2x = 0, f_y = -2y = 0$ $(0, 0)$ is a critical point.

$g_x = 2x = 0, g_y = 2y = 0$ $(0, 0)$ is a critical point.

(b) $d = 2(-2) - 0 < 0$ $(0, 0)$ is a saddle point.

$d = 2(2) - 0 > 0$ $(0, 0)$ is a relative minimum.

55. False. Let $f(x, y) = 1 - |x| - |y|$ at the point $(0, 0, 1)$.

57. False. Let $f(x, y) = x^2y^2$ (see Example 4 on page 944).

Section 13.9 *(page 953)*

1. Write the equation to be maximized or minimized as a function of two variables. Take the partial derivatives and set them equal to zero or undefined to obtain the critical points. Use the Second Partials Test to test for relative extrema using the critical points. Check the boundary points.

3. $\sqrt{3}$ **5.** $\sqrt{7}$ **7.** $x = y = z = 3$

9. $x = y = z = 10$ **11.** 9 ft × 9 ft × 8.25 ft; \$26.73

13. Let x, y, and z be the length, width, and height, respectively, and let V_0 be the given volume. Then $V_0 = xyz$ and $z = \dfrac{V_0}{xy}$.

The surface area is

$$S = 2xy + 2yz + 2xz = 2\left(xy + \frac{V_0}{x} + \frac{V_0}{y}\right).$$

$$\left.\begin{aligned} S_x &= 2\left(y - \frac{V_0}{x^2}\right) = 0 \\ S_y &= 2\left(x - \frac{V_0}{y^2}\right) = 0 \end{aligned}\right\} \begin{aligned} x^2y - V_0 &= 0 \\ xy^2 - V_0 &= 0 \end{aligned}$$

So, $x = \sqrt[3]{V_0}$, $y = \sqrt[3]{V_0}$, and $z = \sqrt[3]{V_0}$.

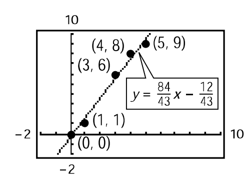
15. $x_1 = 3, x_2 = 6$ **17.** Proof

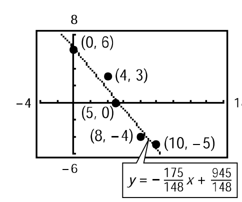
19. $x = \dfrac{\sqrt{2}}{2} \approx 0.707$ km

$y = \dfrac{3\sqrt{2} + 2\sqrt{3}}{6} \approx 1.284$ km

21. (a) $y = \frac{3}{4}x + \frac{4}{3}$ (b) $\frac{1}{6}$ **23.** (a) $y = -2x + 4$ (b) 2

25. $y = \frac{84}{43}x - \frac{12}{43}$ **27.** $y = -\frac{175}{148}x + \frac{945}{148}$





29. (a) $y = 0.23x + 2.38$ (b) \$301.4 billion

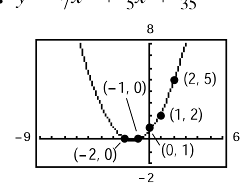
(c) The new model is $y = 0.23x + 5.09$, so the constant increases.

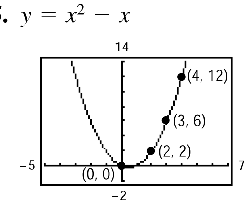
31. $$a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 y_i$$

$$a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

$$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i + cn = \sum_{i=1}^{n} y_i$$

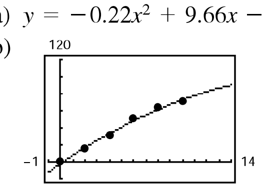
33. $y = \frac{3}{7}x^2 + \frac{6}{5}x + \frac{26}{35}$ **35.** $y = x^2 - x$



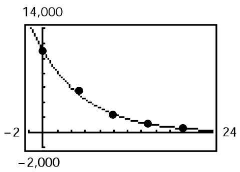


37. (a) $y = -0.22x^2 + 9.66x - 1.79$

(b)



39. (a) $\ln P = -0.1499h + 9.3018$ (b) $P = 10{,}957.7e^{-0.1499h}$
(c)



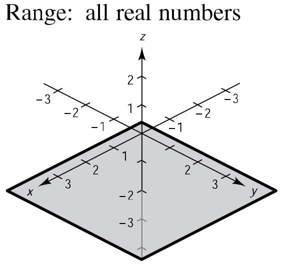
41. Proof

Section 13.10 *(page 962)*

1. Optimization problems that have restrictions or constraints on the values that can be used to produce the optimal solutions are called constrained optimization problems.
3. $f(5, 5) = 25$ **5.** $f(1, 2) = 5$ **7.** $f(25, 50) = 2600$
9. $f(1, 1) = 2$ **11.** $f(3, 3, 3) = 27$ **13.** $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$
15. Maxima: $f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{5}{2}$
$f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{5}{2}$
Minima: $f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$
$f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$
17. $f(8, 16, 8) = 1024$ **19.** $\frac{\sqrt{2}}{2}$ **21.** $3\sqrt{2}$ **23.** $\frac{\sqrt{11}}{2}$
25. 2 **27.** $\sqrt{3}$ **29.** $(-4, 0, 4)$ **31.** $\sqrt{3}$
33. $x = y = z = 3$ **35.** 9 ft × 9 ft × 8.25 ft; \$26.73
37. Proof **39.** $\frac{2\sqrt{3}a}{3} \times \frac{2\sqrt{3}b}{3} \times \frac{2\sqrt{3}c}{3}$
41. At $(0, 0)$, the Lagrange equations are inconsistent.
43. $\sqrt[3]{360} \times \sqrt[3]{360} \times \frac{4}{3}\sqrt[3]{360}$ ft
45. $r = \sqrt[3]{\frac{v_0}{2\pi}}$ and $h = 2\sqrt[3]{\frac{v_0}{2\pi}}$ **47.** Proof
49. $P\left(\frac{15{,}625}{28}, 3125\right) \approx 203{,}144$
51. $x \approx 237.4$
$y \approx 640.9$
Cost ≈ \$68,364.80
53. Putnam Problem 2, morning session, 1938

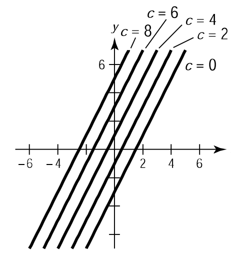
Review Exercises for Chapter 13 *(page 964)*

1. (a) -3 (b) -7 (c) 15 (d) $7x^2 - 3$
3. Domain: $\{(x, y)\colon x \ge 0 \text{ and } y \ne 0\}$
Range: all real numbers
5.

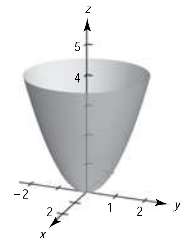


Plane

7. Lines: $y = 2x - 3 + c$

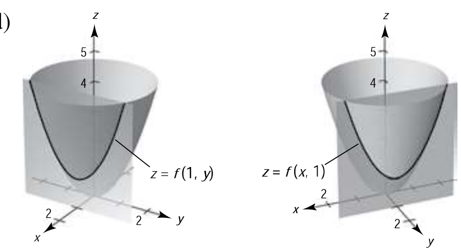


9. (a)

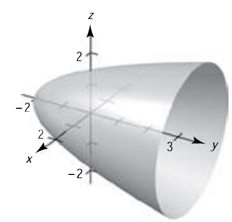


(b) g is a vertical translation of f two units upward.
(c) g is a horizontal translation of f two units to the right.

(d)



11. Elliptic paraboloid



13. Limit: $\frac{1}{2}$
Continuous except at $(0, 0)$
15. Limit: 0
Continuous
17. Limit: $-\frac{\ln 2}{5}$
Continuous for $x \ne \frac{z}{y}$
19. $f_x(x, y) = 15x^2$
$f_y(x, y) = 7$
21. $f_x(x, y) = e^x \cos y$
$f_y(x, y) = -e^x \sin y$
23. $f_x(x, y) = -\frac{y^4}{x^2}e^{y/x}$
$f_y(x, y) = \frac{y^3}{x}e^{y/x} + 3y^2e^{y/x}$
25. $f_x(x, y, z) = 2z^2 + 6yz$
$f_y(x, y, z) = 6xz$
$f_z(x, y, z) = 4xz + 6xy$
27. $f_x(0, 2) = 0$
$f_y(0, 2) = -1$
29. $f_x\left(2, 3, -\frac{\pi}{3}\right) = -\sqrt{3}\pi - \frac{3}{2}$
$f_y\left(2, 3, -\frac{\pi}{3}\right) = -1$
$f_z\left(2, 3, -\frac{\pi}{3}\right) = 6\sqrt{3}$
31. $f_{xx}(x, y) = 6$
$f_{yy}(x, y) = 12y$
$f_{xy}(x, y) = f_{yx}(x, y) = -1$

33. $h_{xx}(x, y) = -y \cos x$
$h_{yy}(x, y) = -x \sin y$
$h_{xy}(x, y) = h_{yx}(x, y) = \cos y - \sin x$

35. Slope in x-direction: 0
Slope in y-direction: 4

37. $(xy \cos xy + \sin xy)\, dx + (x^2 \cos xy)\, dy$

39. $dw = (3y^2 - 6x^2yz^2)\, dx + (6xy - 2x^3z^2)\, dy + (-4x^3yz)\, dz$

41. (a) $f(2, 1) = 10$
$f(2.1, 1.05) = 10.5$
$\Delta z = 0.5$
(b) $dz = 0.5$

43. $dV = \pm\pi$ in.^{3}, $\dfrac{dV}{V} = 15\%$ **45.** Proof

47. (a) and (b) $\dfrac{dw}{dt} = \dfrac{8t - 1}{4t^2 - t + 4}$

49. (a) and (b) $\dfrac{dw}{dt} = 2t^2e^{2t} + 2te^{2t} + 2t + 1$

51. (a) and (b) $\dfrac{\partial w}{\partial r} = \dfrac{4r^2t - 4rt^2 - t^3}{(2r - t)^2}$
$\dfrac{\partial w}{\partial t} = \dfrac{4r^2t - rt^2 - 4r^3}{(2r - t)^2}$

53. $\dfrac{-3x^2 + y}{-x + 5}$

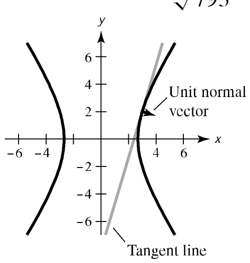
55. $\dfrac{\partial z}{\partial x} = \dfrac{-2x - y}{y + 2z}$
$\dfrac{\partial z}{\partial y} = \dfrac{-x - 2y - z}{y + 2z}$

57. -50 **59.** $\frac{2}{3}$ **61.** $\langle 4, 4 \rangle$, $4\sqrt{2}$ **63.** $\left\langle -\frac{1}{2}, 0 \right\rangle$, $\frac{1}{2}$

65. $\langle -2, -3, -1 \rangle$, $\sqrt{14}$

67. (a) $54\mathbf{i} - 16\mathbf{j}$ (b) $\dfrac{27}{\sqrt{793}}\mathbf{i} - \dfrac{8}{\sqrt{793}}\mathbf{j}$ (c) $y = \dfrac{27}{8}x - \dfrac{65}{8}$

(d)



69. $2x + 6y - z = 8$ **71.** $z = 4$

73. (a) $4x + 4y - z = 8$
(b) $x = 2 + 4t$, $y = 1 + 4t$, $z = 4 - t$

75. 36.7° **77.** $(0, 0, 9)$

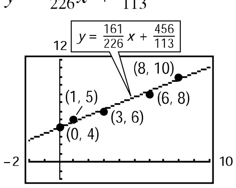
79. Relative maximum: $(4, -1, 9)$

81. Relative minimum: $\left(-4, \frac{4}{3}, -2\right)$

83. Relative minimum: $(1, 1, 3)$ **85.** $\sqrt{3}$

87. $x_1 = 2$, $x_2 = 4$

89. $y = \frac{161}{226}x + \frac{456}{113}$



91. (a) $y = 0.138x + 22.1$ (b) 46.25 bushels/acre

93. $f(4, 4) = 32$ **95.** $f(15, 7) = 352$ **97.** $f(3, 6) = 36$

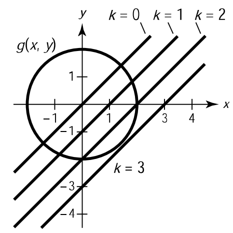
99. $x = \dfrac{\sqrt{2}}{2} \approx 0.707$ km, $y = \dfrac{\sqrt{3}}{3} \approx 0.577$ km,
$z = \left(60 - 3\sqrt{2} - 2\sqrt{3}\right)6 \approx 8.716$ km

P.S. Problem Solving *(page 967)*

1. (a) 12 square units (b) and (c) Proofs

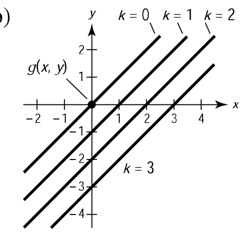
3. (a) $y_0z_0(x - x_0) + x_0z_0(y - y_0) + x_0y_0(z - z_0) = 0$
(b) $x_0y_0z_0 = 1$ $z_0 = \dfrac{1}{x_0y_0}$
Then the tangent plane is
$$y_0\left(\frac{1}{x_0y_0}\right)(x - x_0) + x_0\left(\frac{1}{x_0y_0}\right)(y - y_0) + x_0y_0\left(z - \frac{1}{x_0y_0}\right) = 0.$$
Intercepts: $(3x_0, 0, 0)$, $(0, 3y_0, 0)$, $\left(0, 0, \dfrac{3}{x_0y_0}\right)$

5. (a)



Maximum value: $2\sqrt{2}$

(b)

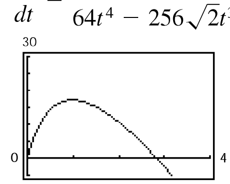


Maximum and minimum value: 0
The method of Lagrange multipliers does not work because $\nabla g(x_0, y_0) = \mathbf{0}$.

7. $2\sqrt[3]{150}$ ft $\times$ $2\sqrt[3]{150}$ ft $\times$ $\dfrac{5\sqrt[3]{150}}{3}$ ft

9. (a) $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = xCy^{1-a}ax^{a-1} + yCx^a(1 - a)y^{1-a-1}$
$= ax^aCy^{1-a} + (1 - a)x^aC(y^{1-a})$
$= Cx^ay^{1-a}[a + (1 - a)]$
$= Cx^ay^{1-a}$
$= f(x, y)$
(b) $f(tx, ty) = C(tx)^a(ty)^{1-a}$
$= Ctx^ay^{1-a}$
$= tCx^ay^{1-a}$
$= tf(x, y)$

11. (a) $x = 32\sqrt{2}t$
$y = 32\sqrt{2}t - 16t^2$
(b) $\alpha = \arctan\left(\dfrac{y}{x + 50}\right) = \arctan\left(\dfrac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)$
(c) $\dfrac{d\alpha}{dt} = \dfrac{-16\left(8\sqrt{2}t^2 + 25t - 25\sqrt{2}\right)}{64t^4 - 256\sqrt{2}t^3 + 1024t^2 + 800\sqrt{2}t + 625}$
(d)



No; The rate of change of α is greatest when the projectile is closest to the camera.
(e) α is maximum when $t = 0.98$ second.
No, the projectile is at its maximum height when $t = \sqrt{2} \approx 1.41$ seconds.