Assignment
on
Multivariable Calculus

Student ID : 180041120
Name : Md Farhan Ishmam
Course Title : Multivariable Calculus and Complex Variables
Course Code : Math 4541
Date of Submission : 09-Sept-21

Exercise - 13.1

(9) $f(x,y) = 2x - y + 3$

(a) $f(0,2) = 2\times 0 - 2 + 3 = 1$

(b) $f(-1,0) = 2(-1) - 0 + 3 = 1$

(c) $f(5,30) = 2\times 5 - 30 + 3 = -17$

(d) $f(3,y) = 2\times 3 - y + 3 = 9 - y$

(e) $f(x,4) = 2x - 4 + 3 = 2x - 1$

(f) $f(5,t) = 2\times 5 - t + 3 = 13 - t$

(10) $f(x,y) = 4 - x^2 - 4y^2$

(a) $f(0,0) = 4 - 0 - 0 = 4$

(b) $f(0,1) = 4 - 0 - 4\times 1^2 = 0$

(c) $f(2,3) = 4 - 4 - 36 = -36$

(d) $f(1,y) = 4 - 1 - 4y^2 = 3 - 4y^2$

(e) $f(x,0) = 4 - x^2$

(f) $f(t,1) = 4 - t^2 - 4 = -t^2$

(14) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

(a) $f(2,2,5) = \sqrt{2^2+2^2+5^2} = \cancel{\sqrt{33}}\ \sqrt{9} = 3$

(b) $f(0,6,-2) = \sqrt{0+36\cancel{+4}\underset{-2}{}} = \cancel{\sqrt{40}}\ \sqrt{4} = 2$

(c) $f(8,-7,2) = \cancel{\sqrt{64+49+4} = 3\sqrt{13}}\ \sqrt{8-7+2} = \sqrt{3}$

(d) $f(0,1,-1) = \sqrt{0+1\cancel{+1}-1} = \cancel{\sqrt{2}}\ 0$

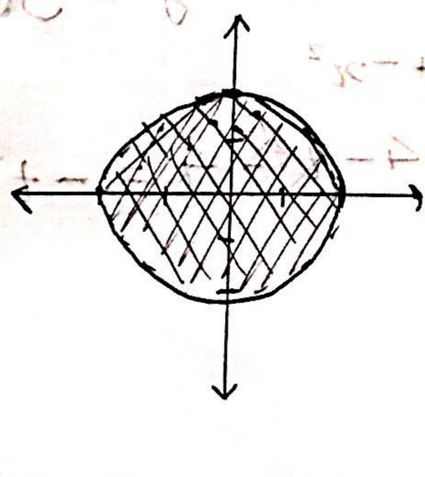
(27) $f(x,y) = \sqrt{4-x^2-y^2}$

$4 - x^2 - y^2 \geq 0$

$\Rightarrow x^2+y^2 \leq 4$

So, domain is the set of all points lying on or inside the ~~circ~~ circle $x^2+y^2=4$.

Domain



Range: $[0, 2]$

(28) $f(x,y)=\sqrt{9-6x^2+y^2}$

For domain, $9-6x^2+y^2 \geq 0$

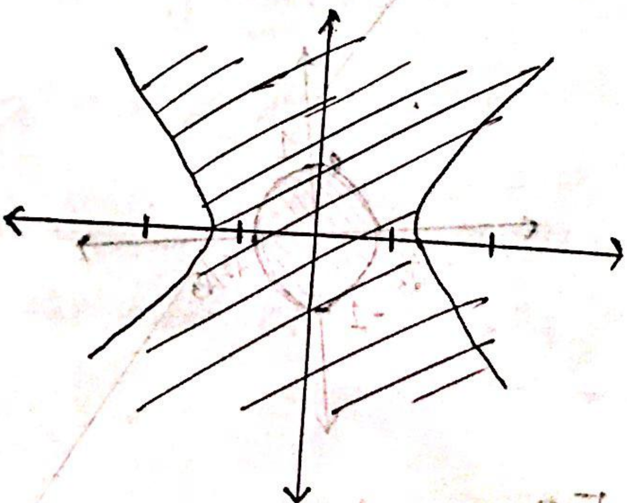
$$\Rightarrow -9+6x^2-y^2 \leq 0$$

$$\Rightarrow 6x^2-y^2 \leq 9$$

$$\Rightarrow \frac{x^2}{3/2} - y^2 \leq 1$$

The domain is the set of all points on or inside the hyperbola $\frac{x^2}{3/2}-y^2=1$. (Ans.)

Domain



The range is

$[0, \infty)$ (Ans.)

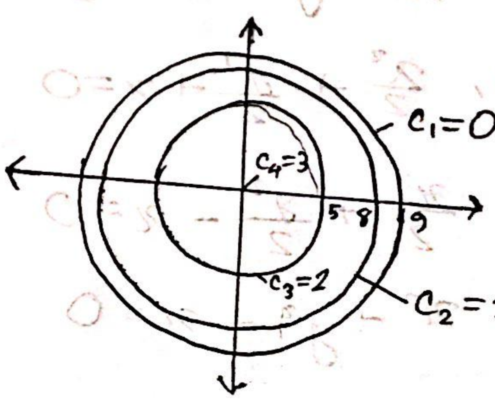
(43) Graph $z = y^2 - x^2 + 1$

The graph is attached.

(44) Graph $z = \frac{1}{12}\sqrt{144 - 16x^2 - 9y^2}$

The graph is attached.

(54) $f(x,y) = \sqrt{9 - x^2 - y^2}$, $c = 0, 1, 2, 3$



~~57 f(x,y)~~ When, $c_1 = 0$, $x^2+y^2 = 9$

$c_2 = 1$, $x^2+y^2 = 8$

$c_3 = 2$, $x^2+y^2 = 5$

$c_4 = 3$, $x^2+y^2 = 0$

57 $f(x,y) = \dfrac{x}{x^2+y^2}$ $c = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$

$c=-2$, then, $\dfrac{x}{x^2+y^2} = -2$

$\Rightarrow 2x^2+2y^2+x=0$

$c=-3/2$, then, $3/2\, x^2 + 3/2\, y^2 + x = 0$

$c=-1$, then, $x^2+y^2+x=0$

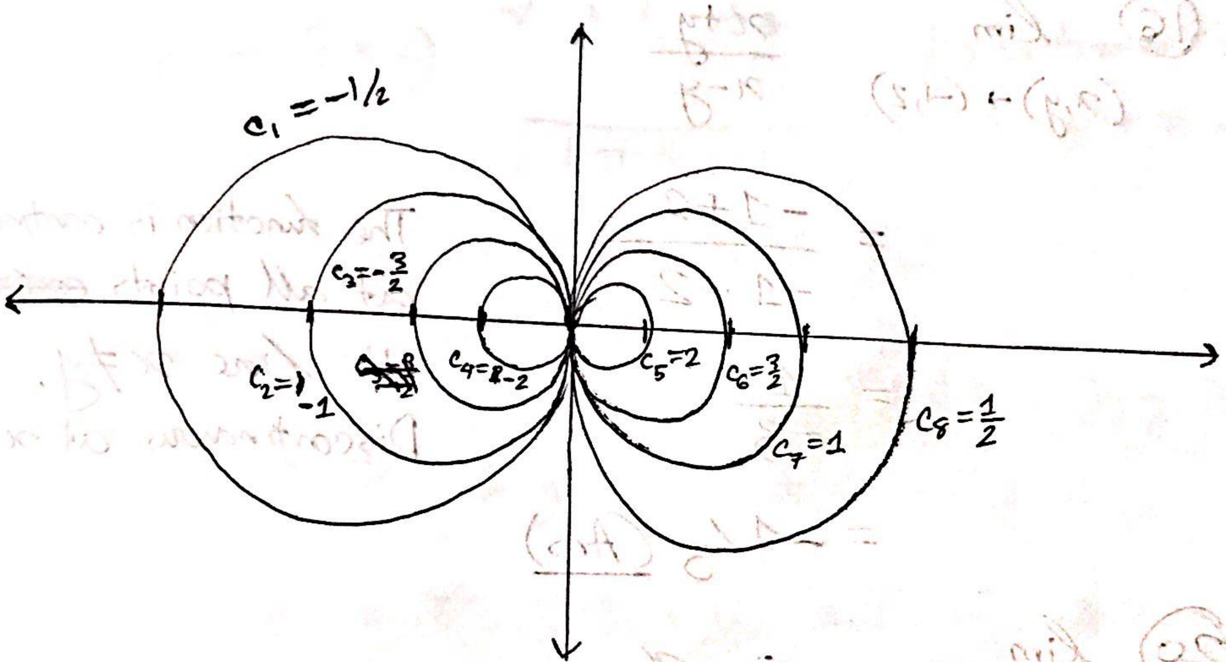
$c=-1/2$, then, $\dfrac{x^2}{2} + \dfrac{y^2}{2} + x = 0$

$c=1/2$ then, $\dfrac{x^2}{2} + \dfrac{y^2}{2} - x = 0$

$C=1$, then, $x^2+y^2-x=0$

$c=3/2$, then, $\frac{3}{2}x^2 + 3/2\, y^2 - x = 0$

$c=2$, then, $2x^2+2y^2-x=0$.



$c_1 = -1/2$
$c_2 = -1$
$c_3 = -\frac{3}{2}$
$c_4 = -2$
$c_5 = 2$
$c_6 = \frac{3}{2}$
$c_7 = 1$
$c_8 = \frac{1}{2}$

Exercise-13.2

(16) $\lim_{(x,y)\to(-1,2)} \dfrac{x+y}{x-y}$

$= \dfrac{-1+2}{-1-2}$

$= \dfrac{1}{-3}$

$= -1/3$ (Ans)

The function is continuous at all points ~~except~~ for the line $x \neq y$.
Discontinuous at $x=y$.

(20) $\lim_{(x,y)\to(\pi,-4)} \sin\dfrac{x}{y}$

$= \sin\dfrac{\pi}{-4}$

$= -\sin \pi/4$

$= -\dfrac{1}{\sqrt{2}}$ (Ans.)

The function is continuous at all points for ~~except~~ $y \neq 0$.
Discontinuous at $y=0$.

(22) $\lim_{(x,y)\to(0,1)} \dfrac{\arccos(x/y)}{1+xy}$

$= \dfrac{\arccos(0/1)}{1+0\times 1}$

$= \pi/2$ (Ans.)

Continous at all points ~~except~~ for $xy \neq -1$ and $y \neq 0$.
Discontinuous at $xy=-1$ and $y=0$.

(23) $\lim_{(x,y,z)\to(1,3,4)} \sqrt{x+y+z}$

Continuous for all $x+y+z>0$.

Discontinuous at $x+y+z<0$.

$= \sqrt{1+3+4}$

$= 2\sqrt{2}$ (Ans)

(30) $\lim_{(x,y)\to(2,1)} \dfrac{x-y-1}{\sqrt{x-y}-1} = \lim_{(x,y)\to(2,1)} \dfrac{(\sqrt{x-y}+1)(\sqrt{x-y}-1)}{\sqrt{x-y}-1}$

$= \cancel{\dfrac{2-1-1}{\sqrt{2-1}-1}} = \sqrt{x-y}+1$

$= \sqrt{2-1}+1 = 2$

The function will be not be continuous at

$x-y<0$ ~~and~~ or, $\sqrt{x-y}=1$.

So, the function is discontinous at $x-y<0$

and $\sqrt{x-y}=1$.

Since, $(x,y)\to(2,1)$ then $\sqrt{x-y}+1$.

$\cancel{\sqrt{x-1} = \sqrt{2-1} = 1}$ $= \sqrt{2-1}+1$

~~So, limit does not exist.~~ $= 2$

So, the limiting value is 2 (Ans.)

(31) $\lim\limits_{(x,y)\to(0,0)} \frac{x+y}{x^2+y} = \lim\limits_{(x,0)\to(0,0)} \frac{x}{x^2} = \lim\limits_{(x,0)\to(0,0)} \frac{1}{x}$

The function is not continuous at $(x,y)\to(0,0)$ as the value approaches to undefined. (Ans.)

(42) $f(x,y) = \lim\limits_{(x,y)\to(0,0)} \left[1 - \frac{\cos(x^2+y^2)}{x^2+y^2}\right] = -\infty$

The function is not continuous at $x=0$ and $y=0$.

So, the limit doesn't exist. (Ans.)

(46) $f(x,y) = \frac{2x-y^2}{2x^2+y}$

For $y\to 0$, $\lim\limits_{y\to 0} \frac{2x-y^2}{2x^2+y}$

$= \frac{2x}{2x^2} = \frac{1}{x}$

For points $(1,0)$, $1/x = 1$

$(0.25,0)$, $1/x = 4$

$(0.01,0)$, $1/x = 100$

$(0.001,0)$, $1/x = 1000$

$(0.000001,0)$, $1/x = 100000$

For path $y = x$,

$$\lim_{y \to x} \frac{2x - y^2}{2x^2 + y}$$

$$= \frac{2x - x^2}{2x^2 + x} = \frac{x(2 - x)}{x(2x + 1)}$$

$$= \frac{\cancel{2x+1}}{\cancel{2x+1}} = \frac{2 - x}{2x + 1}$$

For points,

$(1,1)$, $\frac{2-x}{2x+1} = \frac{2-1}{2+1} = \frac{1}{3}$

$(0.25, 0.25)$, $\frac{2-0.25}{2 \times 0.25 + 1} = \frac{1.75}{1.5} = \frac{7}{6}$

$(0.01, 0.01)$, $\frac{2-0.01}{0.02+1} = 1.951$

$(0.001, 0.001)$, $\frac{2-0.001}{0.002+1} = 1.995$

$(0.0001, 0.0001)$,

$$\frac{2 - 0.0001}{0.0002 + 1} = 1.9995$$

(45) $f(x,y) = \dfrac{y}{x^2+y^2}$

For path, $y \to 0$, $\underset{\lim y\to 0}{f(x,y)} = \dfrac{0}{x^2+0} = 0$

For points, $(1,0)$, $(0.5,0)$, $(0.1,0)$, $(0.01, 0)$ and $(0.001,0)$, the value will be 0.

For path $y=x$, $\underset{\lim (y\to x)}{f(x,y)} = \dfrac{x}{x^2+x} = \dfrac{1}{x+1}$

So, for the points,

$(1,1)$, $\dfrac{1}{1+x} = \dfrac{1}{1+1} = 1/2 = 0.5$

$(0.5,0.5)$, $\dfrac{1}{1+0.5} = \dfrac{2}{3} = 0.667$

$(0.01,0.01)$, $\dfrac{1}{1+0.01} = \dfrac{100}{101} = \cancel{0.9901}$ 0.9901

$(0.01,0.01)$, $\dfrac{1}{1+0.01} = \dfrac{100}{101} = 0.9901$

Exercise-13.3

(16) $z = 2y^2\sqrt{x}$

$$\frac{\partial z}{\partial x} = 2y^2 \frac{1}{2\sqrt{x}} = \frac{y^2}{\sqrt{x}}$$

$$\frac{\partial z}{\partial y} = 4y\sqrt{x}$$

(28) $g(x,y) = \ln\sqrt{x^2+y^2}$

$$\frac{\partial g}{\partial x} = \frac{1}{\sqrt{x^2+y}} \times \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x$$

$$= \frac{x}{x^2+y^2}$$

$$\frac{\partial g}{\partial y} = \frac{y}{x^2+y^2}$$

(37) $z = \sinh(2x+3y)$

$$\frac{\partial z}{\partial x} = \cosh(2x+3y)\cdot 2$$

$$\frac{\partial z}{\partial y} = \cosh(2x+3y)\cdot 3$$

(43) $f(x,y) = \sqrt{x+y}$

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+y} - \sqrt{x+y}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x+\Delta x+y-x-y}{\Delta x\left(\sqrt{x+\Delta x+y} + \sqrt{x+y}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x+y} + \sqrt{x+y}\right)}$$

$$= \frac{1}{\sqrt{x+y} + \sqrt{x+y}}$$

$$= \frac{1}{2\sqrt{x+y}} \quad \text{(Ans.)}$$

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{\sqrt{x+y+\Delta y} - \sqrt{x+y}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x+y+\Delta y-x-y}{\Delta y\left(\sqrt{x+\Delta y+y} + \sqrt{x+y}\right)}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y\left(\sqrt{x+\Delta y+y} + \sqrt{x+y}\right)}$$

$$= \frac{1}{2\sqrt{x+y}} \quad \text{(Ans.)}$$

(52) $f(x,y) = \dfrac{2xy}{\sqrt{4x^2+5y^2}}$

$= \dfrac{2y}{\sqrt{4+\frac{5y^2}{x^2}}}$

$f_x(x,y) = \dfrac{\partial}{\partial x}\left(\dfrac{2y}{\sqrt{4+\frac{5y^2}{x^2}}}\right)$

$= 2y \times \cancel{\left(-\frac{1}{2}\right)} \left(4+\frac{5y^2}{x^2}\right)^{-3/2} \cdot 5y^2 \cdot \cancel{(-2)}\, x^{-3}$

$= \dfrac{10y^3}{(4x^2+5y^2)^{3/2}}$

$\cancel{f_y}$ $f_x(1,1) = \dfrac{10}{(4+5)^{3/2}} = \dfrac{10}{27}$ (Ans)

$f_y(x,y) = \dfrac{\partial}{\partial y}\left(\dfrac{2x}{\sqrt{\frac{4x^2}{y^2}+5}}\right)$

$= 2x \cdot \left(\frac{4x^2}{y^2}+5\right)^{-3/2} \cancel{\left(-\frac{1}{2}\right)} \cdot 4x^2 \cancel{(-2)}\, y^{-3}$

$= \dfrac{8x^3}{(4x^2+5y^2)^{3/2}}$

$f(1,1) = \dfrac{8}{(4+5)^{3/2}} = \dfrac{8}{27}$ (Ans.)

55) $g(x,y) = 4 - x^2 - y^2$ at $(1,1,2)$

$f_x(x,y) = -2x$

$f(1,1) = -2$ (Ans.)

$f_y(x,y) = -2y$

$f(1,1) = -2$ (Ans.)

62) $G(x,y,z) = \dfrac{1}{\sqrt{1-x^2-y^2-z^2}}$

$F_x(x,y,z) = \dfrac{-1}{2}(1-x^2-y^2-z^2)^{-3/2} \cdot -2x$

$= \dfrac{x}{(1-x^2-y^2-z^2)^{3/2}}$

$f_y(x,y,z) = \dfrac{y}{(1-x^2-y^2-z^2)^{3/2}}$

$f_z(x,y,z) = \dfrac{z}{(1-x^2-y^2-z^2)^{3/2}}$

(68) $f(x,y,z) = \sqrt{3x^2+y^2-2z^2}$ $(1,-2,1)$

$$f_x = \frac{6x}{2\sqrt{3x^2+y^2-2z^2}}$$

$$f_x(1,-2,1) = \frac{+6}{2\sqrt{3+4-2}} = \frac{3}{\sqrt{5}} \text{ (Ans.)}$$

$$f_y = \frac{+y}{\sqrt{3x^2+y^2-2z^2}}$$

$$f_y(1,-2,1) = \frac{(-2)}{\sqrt{3+4-2}} = \frac{-2}{\sqrt{5}} \text{ (Ans.)}$$

$$f_z = \frac{-(+4z)}{2\sqrt{3x^2+y^2-2z^2}} = \frac{-2z}{\sqrt{3x^2+y^2-2z^2}}$$

$$f_z(1,-2,1) = -\frac{2}{\sqrt{5}} \text{ (Ans.)}$$

(71) $f(x,y) = x^2+4xy+y^2-4x+16y+3$

$f_x(x,y) = 2x+4y-4$

$\Rightarrow 2x+4y-4=0$

$\therefore x+2y-2=0$ ___ (i)

$f_y(x,y) = 4x + 2y + 16$

$\Rightarrow 4x + 2y + 16 = 0$

$\therefore 2x + y + 8 = 0$ — (ii)

Solving (i) and (ii), we get,

$x = -6$ and $y = 4$. (Ans.)

(88) $f(x,y) = \sqrt{25 - x^2 - y^2}$

$f_x(x,y) = \frac{\partial}{\partial x}\sqrt{25 - x^2 - y^2} = \frac{-x}{\sqrt{25 - x^2 - y^2}}$

$f_y(x,y) = \frac{\partial}{\partial y}\sqrt{25 - x^2 - y^2} = \frac{-y}{\sqrt{25 - x^2 - y^2}}$

$\frac{\partial^2}{\partial x^2}\left(\sqrt{25 - x^2 - y^2}\right) = \frac{y^2 - 25}{(25 - x^2 - y^2)^{3/2}}$

$\frac{\partial^2}{\partial y^2}\left(\sqrt{25 - x^2 - y^2}\right) = \frac{x^2 - 25}{(25 - x^2 - y^2)^{3/2}}$

The values of (x,y) if $f_x = 0$ and $f_y = 0$ is

$x = 0$ and $y = 0$ (Ans)

(92) $f(x,y,z) = x^2 - 3xy + 4yz + z^3$

$f_x = 2x - 3y$

$f_y = -3x + 4z$

$f_{xy} = -3$ $\quad$ $f_{xyy} = 0$

$f_{yx} = -3$ $\quad$ $f_{yxy} = 0$

$f_{yx} = 0$ $\quad$ $f_{yyx} = 0$

(Ans)

(128) (a) $z = -0.12x^2 + 0.657y^2 + 17.7x - 51.53y + 842.5$

$$\frac{\partial z}{\partial x} = -0.24x + 17.7$$

$$\frac{\partial^2 z}{\partial x^2} = -0.24 \text{ (Ans)}$$

$$\frac{\partial z}{\partial y} = 1.314y - \cancel{53}\,51.53$$

$$\frac{\partial^2 z}{\partial y^2} = 1.314 \text{ (Ans.)}$$

129 (a) $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$$f_x(x,y) = xy(x^2-y^2)(x^2+y^2)^{-1}$$

$$= \frac{y(x^2-y^2)\cancel{(x^2+y^2)}}{x^2+y^2} + \frac{2x \cdot xy(x^2-y^2)}{x^2+y^2}$$

$$- \frac{xy(x^2-y^2) \cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{(x^2-y^2)(2x^2y+y)}{x^2+y^2} - \frac{2x^2y(x^2-y^2)}{(x^2+y^2)^2}$$

$$f_y(x,y) = xy(x^2-y^2)(x^2+y^2)^{-1}$$

$$= \frac{x(x^2-y^2)}{(x^2+y^2)} - \frac{2y \cdot xy(x^2-y^2)}{(x^2+y^2)}$$

$$- \frac{xy(x^2-y^2) \cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{(x^2-y^2)(x-2xy^2)}{x^2+y^2} - \frac{2xy^2(x^2-y^2)}{(x^2+y^2)^2}$$

(b) $f_x(0,0) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0,0)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{0}{\Delta x^2} - 0}{\Delta x}$$

$$= 0 \quad \text{(Ans.)}$$

Similarly, $f_y(0,0) = \lim\limits_{\Delta y \to 0} \dfrac{f(\Delta y, 0) - f(0,0)}{\Delta y}$

$$= \lim_{\Delta y \to 0} \frac{\dfrac{0}{\Delta y^2} - 0}{\Delta y}$$

$$= 0 \quad \text{(Ans.)}$$

(c) $\cancel{f_x(0,0) = 0}$

$\cancel{\therefore f_{xy}(0,0) = 0}$

~~Similarly,~~ $\cancel{f_y(0,0) = 0}$

$\cancel{\therefore f_{yx}(0,0) = 0}$

$$f_{xy}(0,0) = \left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]_{0,0} = \lim_{\Delta y \to 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(-\Delta y)^2 \Delta y}{\Delta y^2} - \frac{2\Delta y(-\Delta y)^2}{\Delta y^2}$$

$$= -1$$

$$\therefore f_{xy}(0,0) = -1$$

Similarly, $f_{yx}(0,0) = 1$ (Ans.)

(d) $\cancel{f_{xy} = f_{x}}$ $\cancel{f_{xy} \neq f_{yx} \neq 0}$ $\quad f_{xy}(0,0) \neq f_{yx}(0,0)$

Both are not continious at $(0,0)$. (Ans.)

Exercise-13.4

(7) $w = x^2yz^2 + \sin yz$

$F_x = 2xyz^2$

$F_y = x^2z^2 + z\cos y$

$F_z = 2x^2yz + y\cos z$

So, total differential is

$$F_x\,dx + F_y\,dy + F_z\,dz$$
$$= 2xyz^2\,dx + (x^2z^2 + z\cos y)\,dy + (2x^2yz + y\cos z)\,dz \quad \text{(Ans.)}$$

(8) $w = \dfrac{x+y}{z-3y}$

$F_x = \dfrac{1}{z-3y}$

$F_y = \dfrac{1}{z-3y} + \dfrac{(x+y)\cdot 3}{(z-3y)^2}$

$= \dfrac{3(x+y) + 1(z-3y)}{(z-3y)^2}$

$$F_z = \frac{-(x+y)}{(z-3y)^2}$$

Total differenciation

$$\frac{dx}{z-3y} + \frac{3(x+y)+(z-3y)}{(z-3y)^2}\,dy - \frac{(x+y)}{(z-3y)^2}\,dz \quad \text{(Ans.)}$$

(11)(b) $f(x,y) = 16 - x^2 - y^2$

$F_x = -2x$

$F_y = -2y$

Total differenciation $= -2x\,dx - 2y\,dy$

Approximation is $\Delta z = -2x\,\Delta x - 2y\,\Delta y$

$\Delta x = 2.1 - 2 = 0.1$

$\Delta y = 1.05 - 1 = 0.05$

$\therefore \Delta z$ at $(2,1)$ is

$-2 \times 2 \times 0.1 - 2 \times 1 \times 0.05$

Linear Approximation $= -0.5$ (Ans.)

(a) $f(2,1) = 16 - 2^2 - 1^2 = 11$

$f(2.1, 1.05) = 16 - 2.1^2 - 1.05^2 = 10.4875$

$\therefore \Delta z = f(2.1, 1.05) - f(2,1)$

$= 10.4875 - 11$

Exact $\Delta z = -0.5125$ (Ans.)

(13) (a) $f(x, y) = ye^x$

$f(2,1) = e^2 = 7.389$

$f(2.1, 1.05) = 1.05e^{2.1}$

$= 8.5745$

Exact, $\Delta z = f(2.1, 1.05) - f(2,1)$

$= 1.1855$ (Ans.)

(b) $f_x = ye^x$

$f_y = e^x$

$\Delta z = ye^x \Delta x + e^x \Delta y$

$\Delta x = 0.1$ and $\Delta y = 0.05$

At $(2,1)$, $\Delta z = e^2 \times 0.1 + e^2 \times 0.05$

Linear Approximation $= 1.10836$ (Ans.)

(16) The corresponding equations,

$$\frac{1-x^2}{y^2} \text{ at } (3,6)$$

$$F_x = \frac{-2x}{y^2}$$

$$\Delta x = 3.05 - 3 = 0.05$$

$$F_y = -2\frac{(1-x^2)}{y^3}$$

$$\Delta y = 5.95 - 6 = -0.05$$

$$\Delta z = \frac{-2 \times 3}{6^2} \times 0.05 + \frac{2(1-3^2)}{6^3} \times 0.05$$

$$= -0.00833 - 0.003704$$

$$= -0.012$$

(23) The corresponding equation is

$$V = xyz$$

$$\therefore \Delta v = yz\,\Delta x + xz\,\Delta y + xy\,\Delta z$$

At $V(8,5,12)$ with $\Delta x = \Delta y = \Delta z = \pm 0.02$,

we get,

$$\Delta v = 60 \times (\pm 0.02) + 96(\pm 0.02) \pm 40(0.02)$$

$$= \pm 3.92 \text{ cubic inches (Ans)}$$

Relative error, $\frac{\Delta v}{v} = \pm \frac{3.92}{8 \times 5 \times 12} = 0.8167\%$ (Ans)

Exercise - 13.5

(11) $w = xy + xz + yz$

$x = t-1 \quad y = t^2-1 \quad z = t$

(a) $\cancel{\frac{dw}{dt} = \frac{d}{dt}(xy) + \frac{d}{dt}(xz) + \frac{d}{dt}yz}$

$\cancel{= \frac{dx}{d}}$

(a) $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$

$= \frac{\partial}{\partial x}(xy + xz + yz) \cdot \frac{d}{dt}(t-1) +$

$\frac{\partial}{\partial y}(xy + xz + yz) \cdot \frac{d}{dt}(t^2-1) +$

$\frac{\partial}{\partial z}(xy + xz + yz) \cdot \frac{d}{dt}(t)$

$= (y+z) \cdot 1 + (x+z) \cdot 2t + (x+y) \cdot 1$

$= (2t+1)x + 2y + (2t+1)z$

$= (2t+1)(t-1) + 2(t^2-1) + (2t+1)t$

$= 2t^2 - 2t + t - 1 + 2t^2 - 2 + 2t^2 + t$

$= 6t^2 - 2t + 2t - 3$

$= 6t^2 - 3$ (Ans.)

(b) $w = xy + xz + yz$

$= (t-1)(t^2-1) + (t-1)t + (t^2-1)t$

$= t^3 - t - t^2 + 1 + t^2 - t + t^3 - t$

$= 2t^3 - 3t + 1$

$\frac{dw}{dt} = \frac{d}{dt}(2t^3 - 3t + 1)$

$= 6t^2 - 3$ (Ans)

(17) $w = \sin(2x + 3y)$

$x = s + t,\ y = s - t \qquad s = 0,\ t = \pi/2$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$

$= 2\cos(2x+3y) \cdot 1 + 3\cos(2x+3y) \cdot 1$

$= 2\cos(2x+3y) + 3\cos(2x+3y)$

$= 5\cos(2x+3y)$ ~~(Ans)~~

$\frac{\partial w}{\partial s}$ at $[s = 0$ and $t = \pi/2]$ is $5\cos(2x+3y)$

$= 5\cos(2s + 2t + 3s - 3t)$

$= 5\cos(5s - t)$

At $s = 0$ and $t = \pi/2$, $5\cos(-\pi/2)$

$= 0$ (Ans.)

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= 2\cos(2x+3y) - 3\cos(2x+3y)$$

$$= -\cos(2x+3y)$$

$$= -\cos(2s+2t+3s-3t)$$

$$= -\cos(5s-t)$$

At $s=0$ and $t=\pi/2$, we get,

$$\frac{\partial w}{\partial t} = -\cos(-\pi/2)$$

$$= 0 \text{ (Ans)}$$

(22) $w = x\cos yz$

$x = s^2$, $y = t^2$, $z = s-2t$

(a) $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$

$$= (\cos yz)\cdot 2s + 0 \cdot - xy\sin yz$$

$$= 2s\cos yz - xy\sin yz$$

$$\cancel{= 2s\cos(st^2-2t^3) - s^2t^2\sin(s-2t)t^2}$$

$$= 2s\cos(st^2-2t^3) - s^2t^2\sin(st^2-2t^3) \text{ (Ans.)}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$= \cos(yz)\cdot 0 - \{xz\sin(yz)\}\,2t - \{xy\sin(yz)\}(-2)$$

$$= 2xy(\sin yz) - 2t\,xz\sin(yz)$$

$$= 2s^2t^2\sin(st^2-2t^3) - 2ts^2(s-2t)\sin(st^2-2t^3)$$

$$= 2s^2t(3t-3)\sin(st^2-2t^3) \quad \text{(Ans.)}$$

Again, $w = x\cos(yz)$

$$= s^2\cos(st^2-2t^3)$$

$$\frac{\partial w}{\partial s} = 2s\cos(st^2-2t^3) - s^2t^2\sin(st^2-2t^3) \quad \text{(Ans.)}$$

$$\frac{\partial w}{\partial t} = -s^2\{\sin(st^2-2t^3)\}\cdot(2ts-6t^2)$$

$$= -2s^2t(s-3t)\cdot\sin(st^2-2t^3) \quad \cancel{\text{Ans}}$$

$$= 2s^2t(3t-s)\cdot\sin(st^2-2t^3) \quad \text{(Ans.)}$$

(25) $\ln\sqrt{x^2+y^2} + x + y = 4$

~~Let,~~ $\Rightarrow \ln\sqrt{x^2+y^2} + x + y - 4 = 0$

$$F_x(x,y) = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{2x}{2\sqrt{x^2+y^2}} + 1$$

$$= \frac{x}{x^2+y^2} + 1$$

$$F_y(x,y) = \frac{2y}{2(x^2+y^2)} + 1$$

$$= \frac{y}{x^2+y^2} + 1$$

$$\therefore \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{x}{x^2+y^2}+1}{\frac{y}{x^2+y^2}+1}$$

$$= -\frac{x + x^2+y^2}{y + x^2+y^2} \text{ (Ans.)}$$

(29) $x^2 + 2yz + z^2 = 1$

$F_x = 2x$

$F_y = 2z$

$F_z = 2y + 2z$

$\therefore \dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{2x}{2y+2z} = -\dfrac{x}{y+z}$ (Ans.)

$\therefore \dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = \dfrac{-2z}{2y+2z} = \dfrac{-z}{y+z}$ (Ans.)

Exercise-13.6

(5) $f(x,y) = \sin(2x+y)$

$F_x = 2\cos(2x+y)$

$F_y = \cancel{2\sin}\cos(2x+y)$

$D_u f(x,y) = f_x\cos\theta + f_y\sin\theta$

$= 2\cos(2x+y)\cos\theta + \cancel{B}\cos(2x+y)(\sin\theta)$

At $P(0,\pi,)$ when $\theta = \cancel{\frac{3\pi}{}} - \frac{5\pi}{6}$

$D_u f(0,\pi) = 2\cos\pi\cdot\cos\left(-\frac{5\pi}{6}\right) + \cos\pi\sin\left(-\frac{5\pi}{6}\right)$

$= 2\times\frac{\sqrt{3}}{2} + \frac{1}{2}$

$= \sqrt{3} + \frac{1}{2}$ (Ans.)

(8) $f(x,y) = x^3 - y^3$

$F_x = 3x^2$

$F_y = -3y^2$

$\nabla f(x,y) = 3x^2\hat{i} - 3y^2\hat{j}$

$\nabla f(4,3) = 3\times 4^2\hat{i} - 3\times 3^2\hat{j}$

$= 48\hat{i} - 27\hat{j}$

$v = \frac{\sqrt{2}}{2}(\hat{i}+\hat{j})$

$$D_u f(x,y) = \nabla f(x,y) \cdot u$$

$$= (48\hat{i} - 27\hat{j}) \cdot \frac{\sqrt{2}}{2}(\hat{i}+\hat{j})$$

$$= \frac{48}{\sqrt{2}} - \frac{27}{\sqrt{2}}$$

$$= \frac{21}{\sqrt{2}} \text{ (Ans.)}$$

(12) $\overrightarrow{PQ} = \vec{q} - \vec{p}$

$$= \left(\frac{\pi}{2}\hat{i} + 0\hat{j}\right) - \left(0\hat{i} + \pi\hat{j}\right)$$

$$= \frac{\pi}{2}\hat{i} - \pi\hat{j}$$

$$\therefore u = \frac{\frac{\pi}{2}\hat{i} - \pi\hat{j}}{\sqrt{(\pi/2)^2 + \pi^2}} = \frac{\pi\left(\frac{\hat{i}}{2} - \hat{j}\right)}{\frac{\sqrt{5}\pi}{2}}$$

$$\cancel{= \frac{2}{\sqrt{5}\pi}\left(\frac{\pi}{2}\hat{i} - \pi\hat{j}\right)}$$

$$\cancel{= \frac{1}{\sqrt{5}}\hat{i}}$$

$$= \frac{2}{\sqrt{5}}\left(\frac{\hat{i}}{2} - \hat{j}\right)$$

$$= \frac{\hat{i}}{\sqrt{5}} - \frac{2\hat{j}}{\sqrt{5}}$$

$$f(x,y) = \cos(x+y)$$

$$F_x = -\sin(x+y)$$

$$F_y = -\sin(x+y)$$

$$\nabla f(x,y) = -\sin(x+y)\,\hat{i} - \sin(x+y)\,\hat{j}$$

$$\nabla f(0,\pi) = -\sin\pi\,\hat{i} - \sin\pi\,\hat{j}$$

$$= 0$$

$$\therefore D_u f(x,y) = 0 \cdot u$$

$$= 0 \quad \text{(Ans.)}$$

(18) $z = \cos(x^2+y^2)$ at $(3,-4)$.

$$f_x = -2x\sin(x^2+y^2)$$

$$f_y = -2y\sin(x^2+y^2)$$

$$\nabla f(x,y) = -2x\sin(x^2+y^2)\,\hat{i} - 2y\sin(x^2+y^2)\,\hat{j}$$

$$\nabla f(3,-4) = -2\cdot 3\sin 25\,\hat{i} - 2\cdot(-4)\sin 25\,\hat{j}$$

$$= -6\sin 25\,\hat{i} + 8\sin 25\,\hat{j} \quad \text{(Ans.)}$$

(23) $f(x,y,z) = x^2+y^2+z^2$

$f_x = 2x \quad ; \quad f_y = 2y \quad ; \quad f_z = 2z$

$\nabla f(x,y,z) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

At, $P(1,1,1)$, we get,

$$\nabla f(1,1,1) = 2\hat{i}+2\hat{j}+2\hat{k}$$

$$D_v f(1,1,1) = (2\hat{i}+2\hat{j}+2\hat{k}) \cdot \frac{\sqrt{3}}{3}(\hat{i}-\hat{j}+\hat{k})$$

$$= \frac{2\sqrt{3}}{3} - \frac{2\sqrt{3}}{3} + \frac{2\sqrt{3}}{3}$$

$$= \frac{2\sqrt{3}}{3} \text{ (Ans.)}$$

(28) $h(x,y,z) = \ln(x+y+z)$

$f_x = \frac{1}{x+y+z} \quad ; \quad f_y = \frac{1}{x+y+z} \quad ; \quad f_z = \frac{1}{x+y+z}$

$\nabla f(x,y,z) = \left(\frac{1}{x+y+z}\right)(\hat{i}+\hat{j}+\hat{k})$

$\cancel{\nabla f(1,0,0) =}$ ~~RO~~ $\quad \nabla f(1,0,0) = \hat{i}+\hat{j}+\hat{k}$

$\cancel{\overrightarrow{PQ} = \vec{q} - \vec{p}}$

$\cancel{= -2\hat{i} - 4\hat{j} + 0\hat{k}}$

$\cancel{\therefore u = \frac{-2\hat{i}-4\hat{j}}{\sqrt{2^2+4^2}} = -\frac{2}{2\sqrt{5}}\hat{i} - \frac{4}{2\sqrt{5}}\hat{j}}$

$\cancel{= -\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}}$

$\cancel{D_u f(x,y,z) = (}$

$$\overrightarrow{PQ} = \vec{q} - \vec{p} = (4\hat{i}+3\hat{j}+\hat{k}) - (\hat{i})$$

$$= 3\hat{i} + 3\hat{j} + \hat{k}$$

$$\therefore u = \frac{3\hat{i}+3\hat{j}+\hat{k}}{\sqrt{3^2+3^2+1}}$$

$$= \frac{1}{\sqrt{19}}(3\hat{i}+3\hat{j}+\hat{k})$$

$$D_u f(x,y,z) = \left(\frac{1}{x+y+z}\right)(\hat{i}+\hat{j}+\hat{k}) \cdot \frac{1}{\sqrt{19}}(3\hat{i}+3\hat{j}+\hat{k})$$

$$= \frac{7}{\sqrt{19}} \times \frac{1}{x+y+z} \quad \cancel{\text{(Ans)}}$$

$D_u f(1,0,0) = 7/\sqrt{19}$ (Ans)

35 $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ at $(1,4,2)$

$$f_x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$f_y = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$f_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla f(x,y,z) = \frac{x\hat{i}}{\sqrt{x^2+y^2+z^2}} + \frac{y\hat{j}}{\sqrt{x^2+y^2+z^2}} + \frac{z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla f(1,4,2) = \frac{1\cdot\hat{i}}{\sqrt{1^2+4^2+2^2}} + \frac{4\hat{j}}{\sqrt{1^2+4^2+2^2}} + \frac{2\hat{k}}{\sqrt{1^2+4^2+2^2}}$$

$$= \frac{1}{\sqrt{21}}(\hat{i}+4\hat{j}+2\hat{k})$$

The maximum value is

$$|\nabla f(1,4,2)|$$

$$= \frac{1}{\sqrt{21}} \times \sqrt{1^2+4^2+2^2}$$

$$= 1 \text{ (Ans.)}$$

Exercise-13.7

⑥ $F(x,y,z) = 16x^2 - 9y^2 + 36z$

Here, $F(x,y,z) = 0$

$\Rightarrow 16x^2 - 9y^2 + 36z = 0$

The level surface is a hyperbolic paraboloid. (Ans.)

⑪ $g(x,y) = x^2 + y^2$ at $(1,-1,2)$

Equation of tangent plane is

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

$\cancel{\Rightarrow 2x_0(x-x_0) + 2y_0(y-y_0)}$

$f_x(x_0,y_0) = 2x$

$f_x(1,-1) = 2$

$f_y(x,y) = 2y$

$f_y(1,-1) = -2$

Equation of tangent plane at $(1,-1,2)$ is

$$2(x-1) - 2(y+1) - (z-2) = 0$$

$\Rightarrow 2x - 2 - 2y - 2 - z + 2 = 0$

$\Rightarrow 2x - 2y - z - 2 = 0$ (Ans.)

(18) $x^2+y^2+z^2=9$ at $(1,2,2)$

$f_x(x,y,z)=2x$

$f_x(1,2,2)=2$

$f_y(x,y,z)=2y$

$f_y(1,2,2)=4$

$f_z(x,y,z)=2z$

$f_z(1,2,2)=4$

Equation of tangent plane at $(1,2,2)$ is

$$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0$$

$\Rightarrow 2(x-1)+4(y-2)+4(z-2)=0$

$\Rightarrow 2x-2+4y-8+4z-8=0$

$\Rightarrow x+2y+2z-9=0$ (Ans.)

For normal line to surface, ~~$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-2}{2}$ (Ans.)~~

~~$\nabla f(x,y,z)=2xi+2yj+2zk$~~

~~$\nabla f(1,2,2)=2i+4j+4k$ (Ans.)~~

$\nabla f(1,2,2)=2\hat{i}+4\hat{j}+4\hat{k}$

The corresponding sets of symmetric equations are:

$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-2}{4} \quad \text{(Ans)}$$

(22) $xy - z = 0$ at $(-2,-3,6)$

$F_x(x,y,z) = y \qquad f_x(-2,-3,6) = -3$

$f_y(x,y,z) = x \qquad f_y(-2,-3,6) = -2$

$f_z(x,y,z) = -1 \qquad f_z(-2,-3,6) = -1$

So, equation of tangent plane is

$$f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$$

$$\Rightarrow -3(x+2) - 2(y+3) - 1(z-6) = 0$$

$$\Rightarrow 3x + 2y + z + 6 = 0 \quad \text{(Ans)}$$

Now, $\nabla f(x,y,z) = y\hat{i} + x\hat{j} - \hat{k}$

$\nabla f(-2,-3,6) = -3\hat{i} - 2\hat{j} - \hat{k}$

So, normal line is $-3\hat{i} - 2\hat{j} - \hat{k}$.

Sets of symmetric equations are:

$$\frac{x+2}{-3} = \frac{y+3}{-2} = \frac{z-6}{-1}$$

$$\Rightarrow \frac{x+2}{3} = \frac{y+3}{2} = \frac{z-6}{1} \quad \text{(Ans.)}$$

(26) ~~y=l~~ $y \ln x z^2 = 2$ at $(e, 2, 1)$

$$\Rightarrow y \ln(xz^2) - 2 = 0$$

$f_x(x,y,z) = \dfrac{y \cdot z^2}{xz^2} = y/x \qquad f_x(e,2,1) = 2/e$

$f_y(x,y,z) = \ln(xz^2) \qquad f_y(e,2,1) = 1$

$f_z(x,y,z) = \dfrac{y \cdot x \cdot 2z}{xz^2} = 2y/z \qquad f_z(e,2,1) = 4$

Equation of tangent plane is

$$2/e\,(x-e) + 1(y-2) + 4(z-1) = 0$$

$$\Rightarrow \frac{2x}{e} - 2 + y - 2 + \cancel{2z - 1 = 0}\ 4z - 4 = 0$$

$$\cancel{\Rightarrow \frac{2}{e}x + y + 4z - 8 = 0}$$

$$\Rightarrow \frac{2}{e}x + y + 4z - 8 = 0 \quad \text{(Ans.)}$$

The equation of normal line is

$$\nabla f(e,2,1) = \tfrac{2}{e}\,\hat{i} + \hat{j} + 4\hat{k}$$

~~The equations of symmetrics~~

Sets of symmetric equations are

$$\frac{x-e}{2/e} = \frac{y-2}{1} = \frac{z-1}{4} \quad \text{(Ans.)}$$

(30) $f(x,y,z) = \sqrt{x^2+y^2} - z$

$$\nabla f(x,y,z) = f_x\,\hat{i} + f_y\,\hat{j} + f_z\,\hat{k}$$

$$= \frac{x}{\sqrt{x^2+y^2}}\,\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\,\hat{j} - \hat{k}$$

$$\nabla f(3,4,5) = \frac{3}{5}\,\hat{i} + \frac{4}{5}\,\hat{j} - \hat{k}$$

$$g(x,y,z) = 5x - 2y + 3z - 22$$

$$\nabla g(x,y,z) = 5\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\nabla g(3,4,5) = 5\hat{i} - 2\hat{j} + 3\hat{k}$$

$\nabla f(3,4,5) \times \nabla g(3,4,5)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3/5 & 4/5 & -1 \\ 5 & -2 & 3 \end{vmatrix}$$

$$= \left(\frac{12}{5} - 2\right)\hat{i} - \left(\frac{9}{5} + 5\right)\hat{j} + \left(-\frac{6}{5} - 4\right)\hat{k}$$

$$= \frac{2}{5}\hat{i} - \frac{34}{5}\hat{j} - \frac{26}{5}\hat{k}$$

$$= \frac{2}{5}\left(\hat{i} - 17\hat{j} - 13\hat{k}\right)$$

From ~~equation~~ sets of symmetric equations,

$x = x_0 + at$ $\qquad x = 3 + t$ (Ans.)

$y = y_0 + bt$ $\qquad y = 4 - 17t$ (Ans.)

$z = z_0 + ct$ $\qquad z = 5 - 13t$ (Ans.)

or, $\dfrac{x-3}{1} = \dfrac{y-4}{-17} = \dfrac{z-5}{-13}$ are sets of symmetric equations.

(b) $\cos\theta = \dfrac{\nabla F \cdot \nabla G}{\|\nabla F\| \|\nabla G\|} = \dfrac{\left(\frac{3}{5}i + \frac{4}{5}j - k\right)(5i - 2j + 3k)}{\sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + 1^2}\sqrt{5^2 + 2^2 + 3^2}}$

$$= \frac{3 - 8/5 - 3}{\sqrt{2} \times \sqrt{38}}$$

$$= -\frac{8}{5\sqrt{76}}$$

They are not orthogonal (Ans)

(36) $x^2+y^2=5$ with $(2,1,3)$

$f_x = 2x \qquad f_x(2,1,3) = 4$

$f_y = 2y \qquad f_y(2,1,3) = 2$

$\nabla f(x,y,z) = \cancel{2x\hat{i}}\ 2x\hat{i} + 2y\hat{j}$

$\therefore$ Angle of inclination is

$$\cos\theta = \frac{|\nabla f(2,1,3)\cdot k|}{\|\nabla f(2,1,3)\|}$$

$$= \frac{(4\hat{i}+2\hat{j})\hat{k}}{\sqrt{4^2+2^2}}$$

$$= 0$$

$\therefore \theta = \pi/2$ (Ans.)

$= 90^\circ$

(40) $z = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4$

$\Rightarrow 4x^2 + 4xy - 2y^2 + 8x - 5y - 4 - z = 0$

$\nabla f(x,y,z) = (8x + 4y + 8)\hat{i} + (4x - 4y - 5)\hat{j} - \hat{k} = 0$

As the tangent plane is horizontal,

Now, $8x + 4y + 8 = 0$

$\Rightarrow 2x + y + 2 = 0$ ——— (i)

And, $4x - 4y - 5 = 0$ ——— (ii)

Solving (i) and (ii),

$\therefore x = -1/4$ and $y - 3/2$

$f(x,y) = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4$

$f(-1/4, -3/2) = 4(-\frac{1}{4})^2 + 4(-\frac{1}{4})(-3/2) - 2(-\frac{3}{2})^2$
$+ 8(-1/4) - 5(-3/2) - 4$

$= -5/4$

~~So, the point is (0, 0, -5/4) (Ans.~~

So, the point is $(-1/4, -3/2, -5/4)$ (Ans.)

Exercise-13.8

(8) $f(x,y) = -x^2 - y^2 + 10x + 12y - 64$

$= -x^2 + 10x - 25 - y^2 + 12y - 36 - 3$

$= -(x-5)^2 - (y-6)^2 - 3$

The critical point ~~is (5,6)~~ has $x = 5, y = 6$

$f(5,6) = -5^2 - 6^2 + 10 \times 5 + 12 \times 6 - 64 = -3$

$f_x(x,y) = -2x + 10$

$f_y(x,y) = -2y + 12$

At critical point, $-2x + 10 = 0$

$\therefore x = 5$

and, $-2y + 12 = 0$

$\therefore y = 6$

The critical point is ~~(5,6)~~ $(5, 6, -3)$

~~When $y = 0$, $z =$~~ $d = f_{xx}(x,y) f_{yy}(x,y) - \{f_{xy}(x,y)\}^2$

$d = (-2)(-2) - 0$

$= 4$

$d > 0$ and, $f_{xx}(x,y) < 0$. So, the point is relative maximum.

Ans: Relative maximum ~~(5,6)~~ $(5, 6, -3)$

(13) $f(x,y) = -3x^2 - 2y^2 + 3x - 4y + 5$

$f_x = -6x + 3$

$f_y = -4y - 4$

$f_x = 0 \Rightarrow -6x + 3 = 0 \quad \therefore x = 1/2$

$f_y = 0 \Rightarrow -4y - 4 = 0 \quad \therefore y = -1$

$f_{xx} = -6$, $f_{xy} = 0$ and $f_{yy} = -4$

$$d = f_{xx} f_{yy} - (f_{xy})^2$$
$$= 24 - 0$$
$$= 24$$

$\therefore d > 0$ and $f_{xx} < 0$.

~~So, (1/2, -1) is relative maximum (Ans)~~

$f(1/2, -1) = -3\left(\frac{1}{2}\right)^2 - 2 \cdot 1^2 + \frac{3}{2} - 4 + 5$

$= -\frac{3}{4} - 2 + \frac{3}{2} + 4 + 5$

$= \cancel{1/4 - 7} = 31/4$

The relative maximum is at $(1/2, -1, 31/4)$ (Ans.)

(15) $f(x,y) = 7x^2 + 2y^2 - 7x + 16y - 13$

$f_x = 14x - 7$ $\qquad$ $f_y = 4y + 16$

$\Rightarrow 0 = 14x - 7$ $\qquad$ $0 = 4(y+4)$

$\therefore x = 1/2$ $\qquad$ $\therefore y = -4$

So, the critical point is $(1/2, -4)$

Now, $f_{xx} = 14$ and $f_{xy} = 0$

$f_{yy} = 4$

$\therefore d = f_{xx} f_{yy} - (f_{xy})^2$

$= 14 \times 4 - 0$

$= 56$

So, $d > 0$ and $f_{xx} > 0$.

~~So, the critical point $(1/2, -4)$ is relative minimum (Ans.)~~

$f(1/2, -4) = 7 \times \frac{1}{4} + 2 \times \overset{16}{\cancel{4}} - \frac{7}{2} - 16 \times 4 - 13$

$= \frac{7}{4} + \frac{\overset{32}{\cancel{8}}}{1} - \frac{7}{2} - \cancel{87}\ 77$

$\cancel{= \frac{229}{4} = 57.25}$

$= -46.75$

The relative minimum is at $(0.5, -4, -46.75)$

(Ans.)

(18) $z = -5x^2 + 4xy - y^2 + 16x + 10$

$f_x(x,y) = -10x + 4y + 16$

$\Rightarrow 0 = 10x - 4y - 16$

$\Rightarrow 5x - 2y - 8 = 0$ ——— (i)

$f_y(x,y) = 4x - 2y$

$\Rightarrow 2x - y = 0$ ——— (ii)

Solving (i) and (ii), we get, the critical point,

$$(x, y) \equiv (8, 16)$$

$f_{xx} = -10 \quad f_{xy} = +4 \quad f_{yy} = -2$

Now, $d = f_{xx} f_{yy} - (f_{xy})^2$

$= (-10) \times (-2) - 16$

$= 4$

$\therefore d > 0$ and $f_{xx} < 0$

~~So, the critical point (8,16) is a global maximum.~~

$f(x,y) = -5 \times 8^2 + 4 \times 8 \times 16 - 16^2 + 16 \times 8 + 10$

$= 74$

Relative maximum is $(8, 16, 74)$ (Ans)

36 (a) $f(x,y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

$f_x(x,y) = 3x^2 - 12x + 12$

$\Rightarrow 0 = 3(x^2 - 4x + 4)$

$\therefore x = 2$

And, $f_y(x,y) = 3y^2 + 18y + 27$

$\Rightarrow 0 = 3(y^2 + 6y + 9)$

$\therefore y = -3$ $\qquad f(2,-3) = -54$

So, critical point is $(2, -3, -54)$ (Ans.)

(b) $f_{xx} = 6x - 12$; $f_{xx}(2,-3) = 12 - 12 = 0$

$f_{xy} = 0$

$f_{yy} = 6y + 18$ $\qquad f_{yy}(2,-3) = -18 + 18 = 0$

$\therefore d = f_{xx} f_{yy} - (f_{xy})^2$

$= 0 - 0$

$= 0$

Since, $d = 0$, the result is inconclusive. (Ans.)

(c) The point for which second derivative test fails is

$\cancel{(2,-3)}\ (2, -3, -54)$

(40) $f(x,y) = x^2 + xy$

$f_x(x,y) = 2x + y$ $\qquad$ $f_y(x,y) = x$

$\Rightarrow 0 = 2x + y$ —— (i) $\qquad$ $\therefore x = 0$

$\therefore y = 0$

~~The~~ $f(0,0) = 0$

So, the critical point is $(0,0,0)$.

The other critical points are,

when $x = 2$, then $2x + y = 0$

$\Rightarrow 4 + y = 0$

$\therefore y = -4$.

$f(2,-4) = 4 - 8 = -4$

So, the critical point is $(2,-4,-4)$.

And, at $y = 1$, $2x + 1 = 0$

$\Rightarrow x = -\frac{1}{2}$

$f(-\frac{1}{2}, 1) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{2}$

So, critical point is $(-\frac{1}{2}, 1, -\frac{1}{2})$

At $(0,0,0)$, $f_{xx} = 2$ $f_{xy} = 1$
$f_{yy} = 0$

$\therefore d = f_{xx} f_{yy} - (f_{xy})^2$
$= \cancel{2-1}\ 0 - 1 = -1$

$\because d < 0$, $(0,0,0)$ is a saddle point. (Ans.)

At $(2,-4,-4)$, $d < 0$,
$(2,-4,-4)$ is a saddle point. (Ans.)

And, $(-1/2, 1, -1/2)$ is a saddle point. (Ans.)

Similarly, for $x = -2$, $y = 4$ and ~~f(x,y)~~
$f(-2,4) = -4$.

So, $(-2,4,-4)$ is also a saddle point. (Ans.)

And, for $y = -1$, $x = 1/2$, $f(\cancel{2}, -1, 1/2) = -1/2$

So, $(-1/2, -1, -1/2)$ is also saddle point. (Ans.)

Exercise – 13.9

(12) Let, x, y and z be the length, breadth and height of the open ~~box~~ box.

According to question,

$$\cancel{xy =}$$

$$C = 1.5xy + 2yz + 2xz$$

$$\Rightarrow C - 1.5xy = 2z(x+y)$$

$$\Rightarrow z = \frac{C - 1.5xy}{2(x+y)}$$

So, the volume is $V = xyz$

$$= xy \times \frac{C - 1.5xy}{2(x+y)}$$

$$V_x(x,y) = \frac{\partial}{\partial x}\left(xy \times \frac{C - 1.5xy}{2(x+y)}\right)$$

$$= y\left(\frac{C - 1.5xy}{2(x+y)}\right) + xy \times \frac{(x+y)(-1.5y) - (C - 1.5xy)}{2(x+y)^2}$$

$$= \frac{y(C - 1.5xy)}{2(x+y)} + \frac{xy(-1.5xy - 1.5y^2 - C + 1.5xy)}{2(x+y)^2}$$

$$= \frac{y(C - 1.5xy)(x+y) + xy(-1.5y^2 - C)}{2(x+y)^2}$$

$$= \frac{y\{(2C - 3xy)(x+y) + x(-3y^2 - 2C)\}}{4(x+y)^2}$$

$$= \frac{y}{4(x+y)^2}\left(2cx+2cy-3x^2y-3xy^2-3xy^2-2cx\right)$$

$$= \frac{y}{4(x+y)^2}\left(2cy-3x^2y-6xy^2\right)$$

$$= \frac{y^2\left(2c-3x^2-6xy\right)}{4(x+y)^2}$$

$$V_y(x,y) = \frac{\partial}{\partial y}\left\{xy \times \frac{c-1.5xy}{2(x+y)}\right\}$$

$$= \frac{\partial}{\partial y}\left\{\frac{cxy}{2(x+y)} - \frac{1.5x^2y^2}{2(x+y)}\right\}$$

$$= \cancel{\frac{\partial}{\partial y}}\ \frac{(x+y)cx - cxy\cdot 1}{2(x+y)^2} - \frac{1.5x^2\cdot 2y(x+y) - 1.5x^2y^2}{2(x+y)^2}$$

$$= \frac{cx^2+cxy-cxy-3x^3y-3x^2y^2+1.5x^2y^2}{2(x+y)^2}$$

$$= \frac{cx^2-3x^3y-3x^2y\ 1.5x^2y^2}{2(x+y)^2}$$

$$= \frac{x^2\left(2c-3y^2-6xy\right)}{4(x+y)^2}$$

Now, $V_x = 0$ and $V_y = 0$.

Since the equations are symmetric $x = y$.

Putting $x = y$ in $V_x = 0$, we get,

$$0 = \frac{x^2(2C - 3x^2 - 6x^2)}{4 \cdot 4x^2}$$

$$\Rightarrow 0 = \frac{2C - 9x^2}{16}$$

$$\Rightarrow x = \frac{1}{3}\sqrt{2C}$$

Since, $x = y$

$$\therefore y = \frac{1}{3}\sqrt{2C}$$

$$\therefore z = \frac{C - 1.5x^2}{2 \cdot 2x} = \frac{C - 1.5 \times \left(\frac{1}{3}\sqrt{2C}\right)^2}{2 \times 2 \times \frac{1}{3}\sqrt{2C}}$$

$$= \frac{C - \frac{1.5}{9}\left(\sqrt{2C}\right)^2}{\frac{4}{3}\sqrt{2C}}$$

$$= \frac{C - \frac{1}{3}C}{\frac{4}{3}\sqrt{2C}}$$

$$= \frac{2C}{4\sqrt{2C}}$$

$\therefore z = \frac{1}{4}\sqrt{2c}$

<u>Ans:</u> $x = \frac{1}{3}\sqrt{2c}$, $y = \frac{1}{3}\sqrt{2c}$ and $z = \frac{1}{4}\sqrt{2c}$

(15) Given,

$$R(x_1, x_2) = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$$

$R_{x_1} = -10x_1 - 2x_2 + 42$

$\Rightarrow 0 = -10x_1 - 2x_2 + 42$

$\therefore 5x_1 + x_2 = 21$ ——— (i)

$R_{x_2} = -16x_2 - 2x_1 + 102$

$\Rightarrow 0 = -16x_2 - 2x_1 + 102$

$\therefore x_1 + 8x_2 = 51$ ——— (ii)

Solving (i) and (ii), we get,

$x_1 = 3$ and $x_2 = 6$

$R_{x_1x_1} = -10$, $R_{x_1x_2} = -2$ + $R_{x_2x_2} = -16$

$d = R_{x_1x_1} R_{x_2x_2} - (R_{x_1x_2})^2$

$= (-10) \times (-16) - (-2)^2$

$= 156$

Since, $R_{x_1 x_2} < 0$ and $d > 0$

So, revenue is maximized when $x_1 = 3$

and $x_2 = 6$ (Ans.)

(17) Given, $P(p,q,r) = 2pq + 2pr + 2qr$

Since, the variables p, q and r are proportions,

$$p + q + r = 1$$

$$\Rightarrow r = 1 - p - q$$

$$\therefore P(p,q) = 2pq + 2p(1-p-q) + 2q(1-p-q)$$

$$= 2pq + 2p - 2p^2 - 2pq + 2q - 2pq - 2q^2$$

$$= -2pq + 2p + 2q - 2p^2 - 2q^2$$

So, $P_p(p,q) = \frac{\partial P}{\partial p} = -2q + 2 - 4p$

$$\Rightarrow 0 = -2q + 2 - 4p$$

$$\Rightarrow q + 2p - 1 = 0 \quad \text{---} \quad (i)$$

And, $P_q(p,q) = \frac{\partial P}{\partial q} = -2p + 2 - 4q$

$$\Rightarrow 0 = -2p + 2 - 4q$$

$$\Rightarrow p + 2q - 1 = 0 \quad \text{---} \quad (ii)$$

Solving (i) and (ii), we get,

$$p = 1/3 \text{ and } q = 1/3.$$

$$\therefore P(p,q) = 2pq + 2p(1-p-q) + 2q(1-p-q)$$

$$= \frac{2}{9} + \frac{2}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{1}{3}$$

$$= \frac{2}{9} + \frac{2}{9} + \frac{2}{9}$$

$$= 6/9 = 2/3$$

So, the maximum proportion of heterozygous individuals in any population is $2/3$. (proved).

(19) From the figure, we can say,

$$\text{distance } PQ = \sqrt{x^2+2^2} = \sqrt{x^2+4}$$

$$\text{distance } RQ = \sqrt{(x-y)^2+1^2} = \sqrt{(x-y)^2+1}$$

$$\text{distance } RS = 10-y$$

According to question,

$$C(x,y) = 3k\sqrt{x^2+4} + 2k\sqrt{(y-x)^2+1} + k(10-y)$$

The partial derivatives are:

$$C_x(x,y) = \frac{3k.x}{\sqrt{x^2+4}} + \frac{2k\{-(y-x)\}}{\sqrt{(y-x)^2+1}} = 0$$

$$\Rightarrow \frac{3x}{\sqrt{x^2+4}} - \frac{2(y-x)}{\sqrt{(y-x)^2+1}} = 0$$

$$\cancel{\Rightarrow 3x\sqrt{(y-x)^2+1} - (y-x)\sqrt{x^2+4} = 0}$$

$$\Rightarrow \frac{3x}{2\sqrt{x^2+4}} = \frac{2(y-x)}{\sqrt{(y-x)^2+1}} \quad \text{——— (i)}$$

Again, $C_y = \dfrac{2k(y-x)}{\sqrt{(y-x)^2+1}} - k$

$$\Rightarrow 0 = k\left\{\frac{2(y-x)}{\sqrt{(y-x)^2+1}} - 1\right\}$$

$$\Rightarrow \frac{y-x}{\sqrt{(y-x)^2+1}} = 1/2 \quad \text{——— (ii)}$$

$\cancel{\Rightarrow}$

Substituting $\dfrac{y-x}{\sqrt{(y-x)^2+1}}$ in eq (ii), we get,

$$\frac{3x}{2\sqrt{x^2+4}} = 1/2$$

$$\Rightarrow 6x = 2\sqrt{x^2+4}$$

$$\Rightarrow 3x = \sqrt{x^2+4}$$

$$\Rightarrow \cancel{36x^2 = x^2+4}$$

$$\Rightarrow 9x^2 = x^2+4$$

$$\Rightarrow x^2 = 1/2$$

$$\Rightarrow x = 1/\sqrt{2} = 0.7071 \text{ km}$$

Putting the value of x in eq (i), we get,

$$\frac{3/\sqrt{2}}{\sqrt{\frac{1}{2}+4}} = \frac{2(y-1/\sqrt{2})}{\sqrt{(y-1/\sqrt{2})^2+1}}$$

$$\Rightarrow \frac{3/\sqrt{2}}{3/\sqrt{2}} = \frac{2(y-1/\sqrt{2})}{\sqrt{(y-1/\sqrt{2})^2+1}}$$

$$\Rightarrow (y-1/\sqrt{2})^2+1 = 4(y-1/\sqrt{2})^2$$

$$\Rightarrow (y-1/\sqrt{2})^2 = 1/3$$

$$\Rightarrow y = 1/\sqrt{3} + 1/\sqrt{2}$$

$$= 1.2844 \text{ km}$$

Ans: For $x = 0.7071$ ~~ko~~ km and

$y = 1.2844$ km, the cost C will be minimized.

(28) (6,4), (1,2), (3,3), (8,6), (11,8) and (13,8)

Here, $n = 6$.

x	y	xy	x^2
6	4	24	36
1	2	2	1
3	3	9	9
8	6	48	64
11	8	88	121
13	8	104	169
$\sum_{i=1}^{n} x_i = 42$	$\sum_{i=1}^{n} y_i = 31$	$\sum_{i=1}^{n} x_i y_i = 275$	$\sum_{i=1}^{n} x_i^2 = 400$

We know,

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$= \frac{6\times 275 - (42\times 31)}{6\times 400 - 42^2} \cancel{= \frac{29}{53}}$$

$$= 29/53$$

$$\approx 0.5472$$

And $b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right)$

$$= \frac{1}{6}\left(31 - \frac{29}{53} \times 42\right)$$

$$= \frac{425}{318}$$

$$\approx 1.3365$$

So, the equation is $y = ax + b$

$$\Rightarrow y = 0.5472x + 1.3365 \text{ (Ans.)}$$

Ans: $y = \frac{29x}{53} + \frac{425}{318}$

$$= 0.5472x + 1.3365.$$

30 (a) The points are

(288, 72), (397, 77), (540, 110), (676, 174), (1137, 236), (1108, 307), (1164, 278), (1760, 390).

Here, $n = 8$.

x	y	xy	x^2
288	72	20736	82944
397	77	30569	157609
540	110	59400	291600
676	174	117624	456976
1137	236	268332	1292769
1108	307	340156	1227664
1164	278	323592	1354896
1760	390	686400	3097600
$\sum_{i=1}^{n} x_i = 7070$	$\sum_{i=1}^{n} y_i = 1644$	$\sum x_i y_i = 1846809$	$\sum x_i^2 = 7962058$

Now, $a = \dfrac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$

$= \dfrac{8 \times 1846809 - \cancel{7962058}\ 1644 \times 7070}{8 \times 7962058 - (7070)^2}$

$= \dfrac{3151392}{13711564}$

≈ 0.2298

And, $b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right)$

$= \frac{1}{8}(1644 - 0.2298 \times 7070)$

≈ 2.3837

So, the equation is $y = 0.2298x + 2.3837$ (Ans.)

(b) When $x = 1300$ (dollar),

$y = 0.2298 \times 1300 + 2.3837$

≈ 301.169 billion dollar (Ans.)

(c) By using this information, we can find percentage of taxes coming from individual and business income, and can also have seperate regression lines for individual and business income.

Exercise - 13.10

(6) Maximize $f(x,y) = x^2 - y^2$

Constraint ~~$2x+y=100$~~ $2y - x^2 = 0$

$$\nabla f = 2xi - 2yj$$

$$\lambda \nabla g = \lambda(-2xi + 2)$$

$$= -2x\lambda i + 2\lambda$$

Now, $\nabla f = \lambda \nabla g$.

~~$2xi - 2yj$~~

Hence, $2x = -2x\lambda$ $\quad [\nabla_x f_x = \lambda \nabla_x g_x]$

$$\Rightarrow 2x + 2x\lambda = 0$$

$$\Rightarrow 2x(1+\lambda) = 0$$

If $x = 0$, then,

$$-2y = 2\lambda \quad [\nabla_y f = \lambda \nabla_y g]$$

$$\therefore y = -\lambda$$

and, $2y - x^2 = 0$ [constraint-1]

$$\therefore y = 0.$$

And, $f(0,0) = 0$.

If $\lambda + 1 = 0$, then

$$\lambda = -1$$

Then, $y = -\lambda$

$= -(-1)$

$\therefore y = 1$.

And, $2y - x^2 = 0$ [constraint-1]

$\Rightarrow 2 - x^2 = 0$

$\therefore x = \sqrt{2}$

Then $f(\sqrt{2}, 1) = 2 - 1 = 1$.

So, the maximum value is 1.

Ans: $f(\sqrt{2}, 1) = 1$ is the maximum value.

(9) $f(x,y) = \sqrt{6-x^2-y^2}$

Constraint : $x+y-2=0$

Now, $\nabla f = \frac{-2x}{2\sqrt{6-x^2-y^2}} i - \frac{2y}{2\sqrt{6-x^2-y^2}} j$

$= -\frac{x}{\sqrt{6-x^2-y^2}}\hat{i} - \frac{y}{\sqrt{6-x^2-y^2}}\hat{j}$

$\lambda \nabla g = \lambda(i+j)$

$= \lambda i + \lambda j$

Now, $\nabla f = \lambda \Delta g$

So, $\frac{-\cancel{2}x}{\sqrt{6-x^2-y^2}} = \lambda$ —— (i)

and, $\frac{-y}{\sqrt{6-x^2-y^2}} = \lambda$ —— (ii)

From, (i) and (ii)

$\frac{-x}{\sqrt{6-x^2-y^2}} = \frac{-y}{\sqrt{6^2-x^2-y^2}}$

$\Rightarrow x = y$

Putting the value in constraint,

$$x + x - 2 = 0$$

$$\Rightarrow 2x = 2$$

$$\therefore x = 1 \quad [x = y \ \therefore y = 1]$$

So, the point is $(x, y) = (1, 1)$

$$f(1,1) = \sqrt{6-1-1} = 2$$

<u>Ans:</u> The maximum value is

$$f(1,1) = 2$$

(11) $f(x, y, z) = x^2 + y^2 + z^2$

Constraint: $x + y + z - 9 = 0$; $x > 0, y > 0, z > 0$

$$\nabla f = 2xi + 2yj + 2zk$$

$$\lambda \nabla g = \lambda (i + j + k)$$

$$= \lambda i + \lambda j + \lambda k$$

Now, $\nabla f = \lambda \Delta g$

So,

$$2x = \lambda$$

$$2y = \lambda$$

$$2z = \lambda$$

Equating, we get, $x = y = z$.

Putting the values in constraint, we get,

$$x+x+x-9=0$$
$$\Rightarrow 3x-9=0$$
$$\therefore x=3$$

So, the maximum point is (3, 3, 3)

$$f(3,3,3) = 3^2+3^2+3^2 = 27.$$

Ans: $f(3,3,3) = 27$

15 $\quad f(x,y) = x^2+3xy+y^2$

Constraint: $x^2+y^2 \leq 1$

We have 2 cases, on the circle and inside the circle.

On the circle,

$$f(x,y) = x^2+3xy+y^2$$
$$x^2+y^2 = 1$$

$$\nabla f = (2x+3y)i + (3x+2y)j$$
$$\lambda \nabla g = 2x\lambda i + 2y\lambda j$$

Now, $2x + 3y = 2x\lambda$ —— (i)

$3x + 2y = 2y\lambda$ —— (ii)

Dividing (i) by (ii),

$$\frac{2x+3y}{3x+2y} = \frac{x}{y}$$

$$\Rightarrow 2xy + 3y^2 = 3x^2 + 2xy$$

$$\Rightarrow x^2 = y^2$$

Putting the value in constraint on the circle,

$$x^2 + y^2 = 1$$

$$\Rightarrow 2x^2 = 1$$

$$\therefore x = \pm 1/\sqrt{2}$$

Similarly, $y = \pm 1/\sqrt{2}$

So, the point is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{3}{2} + \frac{1}{2} = \frac{5}{2}$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{3}{2} + \frac{1}{2} = -\frac{1}{2}$$

$$f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{3}{2} + \frac{1}{2} = -\frac{1}{2}$$

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{3}{2} + \frac{1}{2} = \frac{5}{2}$$

The maximum value is

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{5}{2} \quad \underline{\text{(Ans.)}}$$

The minimum value is

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2} \quad \underline{\text{(Ans)}}$$

For, the point inside the circle,

$$x^2 + y^2 = 0$$

$$\Rightarrow 2x^2 = 0$$

$$\therefore x = y = 0$$

$f(0,0) = 0$

This is a saddle point

Saddle point is

$$f(0,0) = 0 \quad \underline{\text{(Ans.)}}$$

<u>Ans:</u> Maximum:

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{5}{2}$$

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{5}{2}$$

Minima: $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$

$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$

(16) $f(x,y) = e^{-xy/4}$, constraint $x^2+y^2 \leq 1$

$\nabla f = -\frac{y}{4} e^{-xy/4} i - \frac{x}{4} e^{-xy/4} j$

$\lambda \nabla g = \lambda(2xi + 2yj)$

$= 2x\lambda i + 2y\lambda j$

Now, $-\frac{y}{4} e^{-xy/4} = 2x\lambda$

$-\frac{x}{4} e^{-xy/4} = 2y\lambda$

Solving, we get, $\cancel{x^2 = y^2}$ $\left(-\frac{y}{4}\right) \times \left(-\frac{4}{x}\right) = \frac{2x}{2y}$

$\Rightarrow \frac{y}{x} = \frac{x}{y}$

$\Rightarrow y^2 = x^2$

Now, $x^2+y^2 = 1$ (on the circle)

$2x^2 = 1$

$\therefore x = \pm \frac{1}{\sqrt{2}}$

and, $y = \pm \frac{1}{\sqrt{2}}$

So, the points are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),$

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{3}{2} + \frac{1}{2} = 5/2$$

$$f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{3}{2} - \frac{1}{2} = -\frac{1}{2}$$

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{3}{2} - \frac{1}{2} = -\frac{1}{2}$$

$$f\left(-1/\sqrt{2}, -1/\sqrt{2}\right) = \frac{1}{2} + 3/2 + 1/2 = 5/2$$

For points inside the circle,

$$x^2+y^2=0$$

$$\therefore x=y=0$$

$$f(0,0)=0.$$

This is a saddle point.

<u>Ans</u>: Maxima: $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{5}{2}$

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{5}{2}$$

Minima: $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$$

(18) $f(x,y,z) = x^2+y^2+z^2$

Constraints: $x+2z=6$

$x+y=12$

$\nabla f = 2xi + 2yj + 2zk$

$\lambda \nabla g = \lambda i + 2\lambda k$

$\mu \nabla h = \mu i + \mu j$

Now, $\nabla f = \lambda \Delta g + \mu \Delta h$

So, $2x = \lambda + \mu$ ——— (i)

$2y = \mu$ $\quad \therefore \mu = 2y$

$2z = 2\lambda$ $\quad \therefore \lambda = z$

So, putting in (i),

$$2x = 2y + z$$

$$\Rightarrow 2x - 2y - z = 0 \quad \text{———} \quad \text{(ii)}$$

Solving, (ii), constraint-1 and constraint-2, we get,

$$x=6, \; y=6 \text{ and } z=0$$

$f(6,6,0) = 36+36+0 = 72$

Ans: The minimum value is $f(6,6,0) = 72$.

Exercise - 14

1) (a) $\dfrac{2z+1}{z^2-z-2}$

The poles are when denominator is equal to zero.

$$z^2-z-2=0$$
$$\Rightarrow z^2-2z+z-2=0$$
$$\Rightarrow z(z-2)+1(z-2)=0$$
$$\Rightarrow (z-2)(z+1)=0$$
$$\therefore z=2,-1$$

So, the poles are $z=2$ and $z=-1$.

At $z=2$, residue is

$$\text{Res } f(z) = \lim_{z\to 2} (z-2)\times\frac{(2z+1)}{(z-2)(z+1)}$$
$$= \lim_{z\to 2} \frac{2z+1}{z+1}$$
$$\therefore \text{Res } f(2) = \frac{5}{3}$$

At $z=-1$, residue is

$$\text{Res}\cdot f(z) = \lim_{z\to -1} (z+1)\,\frac{(2z+1)}{(z-2)(z+1)}$$
$$= \lim_{z\to -1} \frac{2z+1}{z-2}$$
$$\therefore \text{Res } f(-1) = \frac{1}{3}$$

Ans: Res $f(2) = 5/3$ and Res $f(-1) = \frac{1}{3}$

(b) $\left(\frac{z+1}{z-1}\right)^2 = \frac{(z+1)^2}{(z-1)^2}$

So, there is only one pole at $z=1$ of order 2.

Res $f(z)$ at pole $z=1$ is

$$\text{Res} f(z) = \lim_{z \to 1} \frac{1}{1!} \frac{d}{dz} \left\{ (z-1)^2 \times \frac{(z+1)^2}{(z-1)^2} \right\}$$

$$= \lim_{z \to 1} \frac{d}{dz} (z+1)^2$$

$$= \lim_{z \to 1} 2(z+1)$$

$$= 2 \times (1+1)$$

$\therefore \text{Res} f(1) = 4$

Ans: Res $f(1) = 4$.

(c) $\frac{\sin z}{z^2}$

Here there is one pole $z=0$ of order 2.

$$\text{Res } f(z) = \frac{1}{1!} \lim_{z\to 0} \frac{d}{dz} z^2\left(\frac{\sin z}{z^2}\right)$$

$$= \lim_{z\to 0} \frac{d}{dz}(\sin z)$$

$$= \lim_{z\to 0} \cos z$$

$\therefore$ Res $f(0) = 1$ (Ans.)

Ex: 2 $f(z) = \frac{z^2+4}{z^3+2z^2+2z}$

$$= \frac{z^2+4}{z(z^2+2z+2)}$$

The poles are when denominator is zero.

$z=0$ or, $z^2+2z+2=0$

$\Rightarrow z^2+2z+1=-1$

$\Rightarrow (z+1)^2=-1$

$\Rightarrow z+1 = \pm i$

$\Rightarrow z = -1 \pm i$

So, the poles are $0, -1 \pm i$

At, $z=0$, we get,

$$\text{Res } f(z) = \lim_{z\to 0} z \times \frac{z^2+4}{z^3+2z^2+2z}$$

$$= \lim_{z\to 0} \frac{z^2+4}{z^2+2z+2}$$

$$= 4/2$$

$\therefore \text{Res } f(0) = 2$ (Ans.)

At $z=-1+i$, we get,

$$\text{Res } f(z) = \lim_{z\to -1+i} (z+1-i) \times \frac{z^2+4}{z(z+1-i)(z+1+i)}$$

$$= \lim_{z\to -1+i} \frac{z^2+4}{z(z+1+i)}$$

$$= \frac{(-1+i)^2+4}{(-1+i)\cdot 2i}$$

$$= \frac{4-2i}{-2-2i}$$

$\therefore \text{Res } f(-1+i) = -\frac{1}{2}+\frac{3}{2}i$ (Ans.)

At $z=-1-i$, we get,

$$\text{Res } f(z) = \lim_{z\to -1-i} \{z-(-1-i)\} \times \frac{z^2+4}{z(z+1-i)(z+1+i)}$$

$$= \lim_{z\to -1-i} \frac{z^2+4}{z(z+1-i)}$$

$$= \frac{(-1-i)^2+4}{(-1-i)(-2i)}$$

$$= \frac{4+2i}{-2+2i}$$

$$= -\frac{1}{2} - \frac{3}{2}i \text{ (Ans.)}$$

Ex-3 $\oint_c e^{-1/z} \sin(1/z)\, dz$ where $|z|=1$.

The pole is $z=0$, which is within the ~~circ~~ domain $|z|=1$, which is a circle at center at origin and radius $=1$.

Now, residue at $z=0$, is,

~~$\text{Res } f(z) = \oint_c e^{-1/z} \sin(1/z)\, dz$~~

$$\text{Res } f(z) = \lim_{z\to 0} \frac{1}{1!} \frac{d}{dz}\left\{z^2 \cdot e^{-1/z} \cdot \sin\left(\frac{1}{z}\right)\right\}$$

$$= \lim_{z\to 0} \frac{d}{dz}\left\{\frac{z^2 \sin(1/z)}{e^{1/z}}\right\}$$

$$= \lim_{z\to 0} \frac{d}{dz} \frac{z^2 \times \sin(1/z)}{e^{1/z}}$$

$$= \lim_{z\to 0} \frac{e^{1/z} \frac{d}{dz}\left(z^2 \sin\frac{1}{z}\right) - z^2 \sin\left(\frac{1}{z}\right) \cdot e^{1/z} \cdot \left(-\frac{1}{z^2}\right)}{e^{2/z}}$$

$$= \lim_{z\to 0} \frac{e^{1/z}\left\{\left(\sin\frac{1}{z}\right) 2z + z^2 \left(\cos\frac{1}{z}\right) \cdot \left(-\frac{1}{z^2}\right)\right\} + e^{1/z} \sin\left(\frac{1}{z}\right)}{e^{2/z}}$$

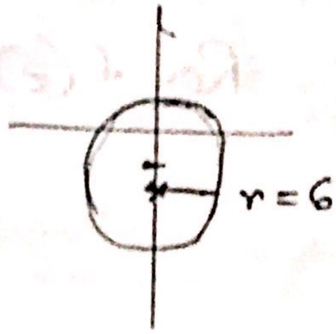
$$= \lim_{z\to 0} \frac{e^{1/z}\left\{\left(\sin\frac{1}{z}\right) 2z - \cos\left(\frac{1}{z}\right) + \sin\left(\frac{1}{z}\right)\right\}}{e^{2/z}}$$

$$= \lim_{z\to 0} \frac{\left(\sin\frac{1}{z}\right)(2z+1) - \cos\left(\frac{1}{z}\right)}{e^{1/z}}$$

Here two sided limit doesn't exist.

So, it is not possible to evaluate the given expression.

Ex-4 $\oint_c \frac{2z^2+5}{(z+2)^3(z^2+4)z^2}\,dz$



(a) $|z-2i| = 6$

The poles are $z = 0, -2, \pm 2i$

Now, $|z-2i| = 6$ $\Rightarrow |x+iy-2i| = 6$

$\cancel{\Rightarrow z-2i = \pm 6}$ $\Rightarrow x^2+(y-2)^2 = 36$

$\cancel{\Rightarrow z = \pm 6+2i}$

The poles within this limit is $0, -2, \pm 2i$

At pole, $z = 0$, the residue is

$$\text{Res } f(z) = \lim_{z\to 0} \frac{1}{1!}\frac{d}{dz}\, z^2 \times \frac{2z^2+5}{(z+2)^3(z^2+4)z^2}$$

$$= \lim_{z\to 0} \frac{d}{dz}\frac{(2z^2+5)}{(z+2)^3 \times (z^2+4)}$$

$$= \lim_{z\to 0} \frac{(z+2)^3(z^2+4)\cdot 4z - (2z^2+5)\frac{d}{dz}\{(z+2)^3(z^2+4)\}}{(z+2)^6(z^2+4)^2}$$

$$= \frac{2^3\cdot 4\cdot 0 - 5\{(z^2+4)\cdot 3(z+2)^2 + (z+2)^3\cdot 2z\}}{2^6\cdot 4^2}$$

$$= \frac{0-5(4\cdot 3\cdot 2^2)}{2^6\cdot 4^2}$$

$\therefore \text{Res } f(0) = -\frac{15}{256}$

At pole $z=-2$, we get,

$$\operatorname{Res} f(z) = \lim_{z\to -2} \frac{1}{2!}\frac{d^2}{dz^2}(z+2)^3 \times \frac{2z^2+5}{(z+2)^3(z^2+4)z^2}$$

$$= \lim_{z\to -2} \frac{1}{2}\frac{d^2}{dz^2}\frac{2z^2+5}{(z^2+4)z^2}$$

$$= \frac{1}{2}\times \lim_{z\to -2} \times \frac{12(x^6+7x^4+30x^2+40)}{x^4(x^2+4)^3}$$

$$= \frac{1}{2}\times \frac{12\times(2^6+7\cdot 2^4+30\cdot 2^2+40)}{2^4(4+4)^3}$$

$$= \frac{63}{256}$$

And at pole $z=2i$, we get,

$$\operatorname{Res} f(z) = \lim_{z\to 2i} (z-2i)\times\frac{2z^2+5}{(z+2)^3(z^2+4)z^2}$$

$$= \lim_{z\to 2i} \frac{2z^2+5}{(z+2)^3(z+2i)z^2}$$

$$= \frac{2(2i)^2+5}{(2i+2)^3\cdot 4i\cdot(2i)^2}$$

$$\operatorname{Res} f(2i) = -\frac{3}{512}+\frac{3}{512}i$$

And, $\operatorname{Res} f(-2i) = -\frac{3}{512}-\frac{3}{512}i$

~~The~~

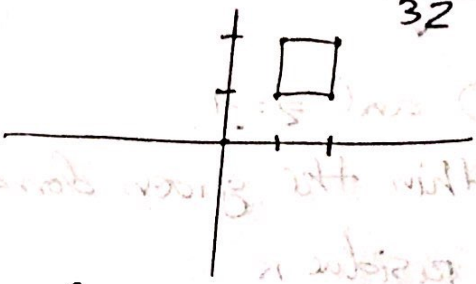
$$\oint_c \frac{2z^2+5}{(z+2)^3(z^2+4)z^2}\,dz = 2\pi i\left(-15+\frac{63}{256}\cancel{d}-\frac{3}{512}+\frac{3}{512}i-\frac{3}{512}-\frac{3}{512}i\right)$$

$$= 2\pi i\left(-15+\frac{63}{256}-\frac{3\times 2}{512}\right)\cancel{+\frac{3}{512}i}$$

$$\cancel{= 8\pi\left(-\frac{3}{256}-\frac{7557}{256}i\right)\ (Ans)}$$

$$= -\frac{945}{32}\pi i \quad \text{(Ans.)}$$

(b)



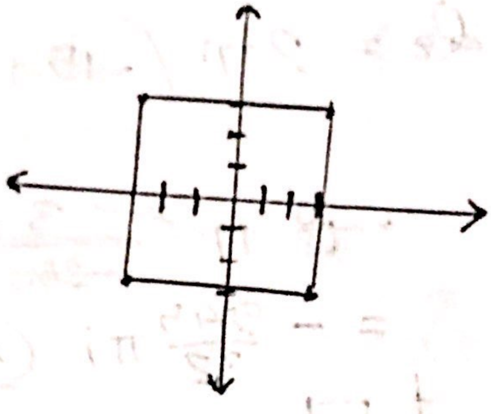
Within the domain, the poles are none.

$$\oint_c \frac{2z^2+5}{z^2(z+2)^3(z^2+4)}$$ in the given square is

$$\oint_c \frac{2z^2+5}{z^2(z+2)^3(z^2+4)} = 2\pi i \times 0$$

$$= 0 \quad \text{(Ans.)}$$

Ex-5 The given square is



The poles are $z=0$ and $z=1$.
Both poles lie within the given domain C.
At pole $z=0$, the residue is

$$\text{Res } f(z) = \lim_{z\to 0} \cancel{\oint} z \times \frac{2+3\sin\pi z}{z(z-1)^2}$$

$$= \lim_{z\to 0} \cancel{3+} \frac{2+3\sin\pi z}{(z-1)^2}$$

$$\therefore \text{Res } f(0) = \frac{2}{1^2} = 2$$

At pole $z=1$, the residue is

$$\cancel{\text{Res } f(z) = \lim_{z\to 1} 2+3\sin\pi z}$$

$$\text{Res } f(z) = \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz} (z-1)^2 \times \frac{2+3\sin\pi z}{z(z-1)^2}$$

$$= \lim_{z\to 1} \frac{d}{dz} \left(\frac{3+\sin\pi z}{z}\right)$$

$$= \lim_{z\to 1} \frac{z(\cos\pi z)\cdot\pi - (3+\sin\pi z)}{z^2}$$

$$= \frac{\pi\cos(\pi) - (3+\sin\pi)}{1^2}$$

$$= -\pi - 3$$

$$\therefore \oint_C \frac{2+3\sin\pi z}{z(z-1)^2}\,dz = 2\pi i \times [\text{sum of residue at poles}]$$

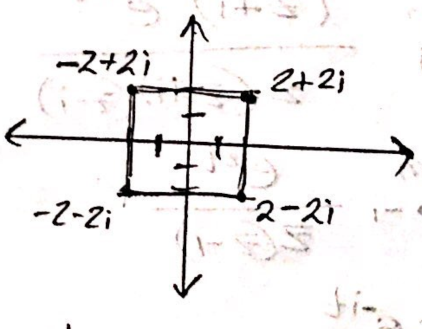
$$= 2\pi i \times (2-\pi-3)$$

$$= (-1-\pi)\times 2\pi i \quad \text{(Ans.)}$$

<u>Ex-6</u>

The poles are at $z=0$ and $z=\pm i$

The poles lie within the given domain



At, $z=0$, we get,

$$\text{Res } f(z) = \lim_{z\to 0} z \times \frac{e^{zt}}{z(z^2+1)}$$

$$= \lim_{z\to 0} \frac{e^{zt}}{z^2+1}$$

$$= \frac{e^0}{1}$$

$$= 1$$

And, at $z = i$, we get,

$$\text{Res } f(z) = \lim_{z \to i} \frac{(z-i)\, e^{zt}}{z(z+i)(z-i)}$$

$$= \lim_{z \to i} \frac{e^{zt}}{z(z+i)}$$

$$= \frac{e^{it}}{i \cdot 2i}$$

$$\therefore \text{Res } f(i) = -\frac{1}{2} e^{it}$$

~~Res~~ At $z = -i$, we get,

$$\text{Res } f(z) = \lim_{z \to -i} \frac{(z+i)\, e^{zt}}{z(z+i)(z-i)}$$

$$= \lim_{z \to -i} \frac{e^{zt}}{z(z-i)}$$

$$= \frac{1}{2} e^{-it}$$

Then, $\frac{1}{2\pi i} \oint_c \frac{e^{zt}}{z(z^2+1)} dz = \frac{1}{2\pi i} \times 2\pi i \times (\text{Sum of residue at poles})$

$$= 1 - \frac{1}{2} e^{it} + \frac{1}{2} e^{-it} \quad \text{(Ans.)}$$