Ordinary Differential Equation :-

Solution of D.E. :-

When a function or the curve or the family of curve which satisfy the differential equation, then this family of curves is said to be a solution of that diff. equation.

Solution →
- General solution $\left[\frac{dy}{dx} = 2x,\ \text{Soln:- } y = x^2 + c\right]$
- Particular Solution $\left[\frac{dy}{dx} = 2x,\ \text{if } x=0, y=5 \Rightarrow y(0)=5,\ \text{Soln:- } y = x^2+5\ (c=5)\right]$

Formation of Differential Equation :- → [Eliminate arbitrary constants.]

$y = x+1 \rightarrow \frac{dy}{dx} = \frac{dx}{dx} = 1.$

(i) $y^2 = 4a(x+b)$ → [number of arbitrary constant = order of d.e.]

a, b are arbitrary constant.

$\frac{dy}{dx} \cdot 2y = 4a + 0$

or, $y_1 y = 2a$

or, $y \cdot y_1 = 2a$

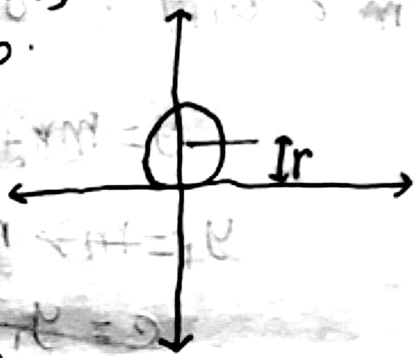
or, $y_1^2 + y_1 y_2 = 0$.

$\therefore (y_1)^2 + y_1 y_2 = 0 \Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$

Ex:-

Form a differential equation of family of circle which touches the x axis at the origin.

If a circle touchen x-axis at origin, then centre, (0,b) and radius b.



∴ Eqn of a circle,

$$(x-0)^2+(y-b)^2=b^2$$

or, $x^2+(y-b)^2=b^2$ ___ (i) Here 'b' is an arbitrary constant

Differentiate with respect to x,

$$2x+2(y-b)\cdot y_1=0$$

or, $(y-b)=-\frac{x}{y_1}$ and $b=y+\frac{x}{y_1}$

Now, putting (y-b) and b in (i)

$$x^2+\left(-\frac{x}{y_1}\right)^2=\left(y+\frac{x}{y_1}\right)^2$$

or, $x^2+\cancel{\frac{x^2}{y_1^2}}=y^2+2\cdot y\cdot\frac{x}{y_1}+\cancel{\frac{x^2}{y_1^2}}$

or, $x^2-y^2-2y\cdot\frac{x}{y_1}=0$

or, $(x^2-y^2)\frac{dy}{dx}-2xy=0$.

This is, 1st order, 1st degree non-linear Ordinary differential equation.

Form a diff. of family of st. line

$$y = mx + c$$

$$y_1 = \cancel{mx}\, m + \cancel{c} \qquad \frac{dy}{dx} = m$$

$$\cancel{c = y_1 - m} \qquad \frac{d^2 y}{dx^2} = 0$$

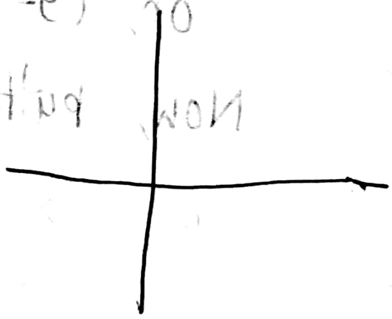
$$\cancel{y = mx + y_1 - m}$$

$$y\cancel{_{07}}\ y$$

$$0$$

at a unit distance from the origin

$$\frac{y - mx - c}{\sqrt{1^2 + m^2}} = 1$$



$$y - mx - c = \sqrt{1 + m^2}$$

From orgin,

$$\frac{0 - 0 - c}{\sqrt{1 + m^2}} = 1 \Rightarrow c = \sqrt{1 + m^2}$$

$$y = mx + \sqrt{1 + m^2}.$$

$$\frac{dy}{dx} = m + \frac{1}{2\sqrt{1+m^2}} \cdot 2m \cdot 0$$

$$\therefore \frac{dy}{dx} = m$$

$$\therefore m = y_1$$

$$y = y_1 x + \sqrt{1+y_1^2}$$

$$\text{or, } y - y_1 x - \sqrt{1+y_1^2} = 0.$$

\# form a D.E. from, $y = c_1 \cdot e^{-2x} + c_2 \cdot e^{3x}$ — (i) $\left.\begin{matrix} c_1 \\ c_2 \end{matrix}\right\} a.c.$

$$\frac{dy}{dx} = -2 \cdot c_1 e^{-2x} + 3 \cdot c_2 e^{3x} \text{ — (ii)}$$

$$\text{or, } \frac{d^2y}{dx^2} = 4 \cdot c_1 e^{-2x} + 9 \cdot c_3 \cdot e^{3x} \text{ — (iii)}$$

(i) X 2 + (ii),

$$y_1 + 2y = 5c_2 e^{3x} \text{ — (iv)}$$

$$y_2 + 2y_1 = 15 c_2 e^{3x} \text{ — (v)}$$

$$\text{or, } y_2 + 2y_1 = 3 \times 5 c_2 e^{3x}$$

$$= 3 \times (y + 2y)$$

$$\therefore \frac{d^2y}{dx} + 2 \cdot \frac{dy}{dx} - 3 \cdot \frac{dy}{dx} + 6y = 0$$

2nd order 1st degree linear ODE.

find the D.E. of the system of ellipse having their axes along the x axis and the y axis

$\boxed{Ex}$ Show that, $y = A\cos 2x + B\sin 2x$ is a soln of $y'' + 4y = 0$.

$y' = -2A \sin 2A + 2B\cos 2x$

$y'' = -4A \cos 2A - 4B\sin 2x$.

$L.H.S. = y'' + 4y$

$= -4A\cos 2A - 4B\sin 2x + 4A\cos 2A + 4B\sin 2x$

$= 0$.

$= R.H.S.$

(showed)

Therefore $y = - - -$ is a soln of

$y'' + 4y = 0$

Ex Show that, $y = x + 3e^{-x}$ is a soln of $\frac{dy}{dx} + y = x+1$

$\frac{dy}{dx} = 1 - 3.e^{-x}$.

L.H.S. $= \frac{dy}{dx} + y$

$= 1 - \cancel{3e^{-x}} + x + \cancel{3e^{-x}}$

$= 1 + x = R.H.S$

$\therefore y = x + 3e^{-x}$ is a soln of $y'' + 4y = 0$.

Ex a) Show that, $\left|\frac{dy}{dx}\right| + |y| + 1 = 0$ has no real soln.

(b) Show that, $\left(\frac{dy}{dx}\right)^2 - 4y = 0$ has one parameter family of soln of the form, $f(x) = (x+c)^2$, where c is an arb. cons.

a) Let, $y = f(x)$ be a soln of the given eqn.

L.H.S. equal to must zero for $y = f(x)$

Here,

$y = f(x)$ L.H.S.

$\frac{dy}{dx} = f'(x)$ $|f'(x)| + |f(x)| + 1$

$|f'(x)| \geq 0$.

$|f(x)| \geq 0$

i.e., $|f'(x)| + |f(x)| + 1 \geq 1$.

Initial Value Problem (IVP):-

A.D.E. that has given condition allows us to find the specific function that specifies the given DE rather than a family of functions. These type of problems are called IVP.

Such as, $y(x_0)=y_0$, $y'(x_0)=z_0$

Boundary Value problem (BVP):-

If the conditions are given more than one point of x and D.E. is of order 2 or greater, this is called BVP such as

$y(x_0)=y_0$, $y'(t_0)=m_0$

Ex Find the soln to the BVP $\frac{d^2y}{dx^2} - y = 0$; $y(0) = 0$, $y(1) = 1$ if we know, $y(x) = c_1e^{x} + c_2e^{-x}$ is the G.S. of the given D.E.

For first :-

$x = 0$.
$y = 0$.

$0 = c_1 + c_2$

$c_1 = -c_2$

For 2nd :

$x = 1$
$y = 1$

$1 = c_1 e + c_2 e^{-1}$

or, $c_1 e - c_1 e^{-1} = 1$

or, $c_1 (e - e^{-1}) = 1$

or, $c_1 = \frac{e}{e^2 - 1}$.

$e - e^{-1} \frac{1}{e}$
$= \frac{e^2 - 1}{e}$

$c_2 = \frac{-e}{e^2 - 1}$.

$\boxed{\text{Ex}}$ show that,

(a) $y = 4e^{2x} + 2e^{-3x}$

is a soln of the IVP $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$,

$y(0) = 6$, $y'(0) = 2$.

(b) If $y = 2e^{2x} + 4e^{-3x}$ also a soln of this problem? Explain why or why not?

First Order First Degree Ordinary D.E. :-

General form of 1st Order 1st Degree ODE,

$$\frac{dy}{dx} = f(x, y) \quad ; \quad \frac{dy}{dx} = \frac{x^2-1}{y^2-1}$$

or, $(y^2-1)dy - (x^2-1)dx = 0$

or, $M(x,y)dx + N(x,y)dy = 0$; or, $M dx + N \cdot dy = 0$

Solving Method :-

(i) Variable Separation

(ii) Reduce to "

(iii) Linear to

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)} = f(x,y)$$

If $M(x,y)$ is a fn. of x only or const. and $N(x,y)$ is a fn. of y only or const.

$$\frac{dy}{dx} = \frac{-M(x)}{N(y)}$$

separate the values,

$M(x)dx + N(y)dy = 0$.

Integrating, $\int M(x)dx + \int N(y)dy = c$.

Ex

Solve,

$$\frac{dy}{dx} = x^2 y + x^2$$

or, $\frac{dy}{dx} = x^2(y+1)$

or, $x^2(y+1)dx - dy = 0$; Now separate the variables,

or, $x^2 . dx - \frac{1}{y+1} dy = 0$; Integrating, $\int x^2 dx - \int \frac{1}{y+1} dy = 0$

or, $\frac{1}{3}x^3 - \ln|1+y| = c.$

$\therefore \frac{1}{3}x^3 - \ln|y+1| = c.$

Ex

Solve,

$$x^2(y+1)dx + y^2(x-1)dy = 0$$

Separating the variables,

$$\frac{x^2}{x-1}dx + \frac{y^2}{y+1}dy = 0$$

or, $(x+1+\frac{1}{x-1})dx + [(y-1) + \frac{1}{y+1}]dy = 0$

Integrating,

$$\frac{x^2}{2} + x + \ln|x-1| + \frac{y^2}{2} - y + \ln|y+1| = c.$$

$\therefore x^2 + 2x + 2\ln|x-1| + y^2 - 2y + 2\ln(y+1) = 2c$

$\therefore (x^2+y^2) + 2(x+y) + 2[\log|x-1| + \log|y+1|] = 2c.$

This is General soln of the given equation where c is an arbitrary const.

Ex

Solve,

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

or, $\frac{dy}{dx} = e^{x} \cdot e^{-y} + x^2 \cdot e^{-y}$

or, $\frac{dy}{dx} = e^{-y} [e^{x} + x^2)$

or, $\frac{dy}{dx} = \frac{e^{x}+x^2}{e^{y}}.$

or, $\cancel{dy\, e^{y} \cdot dx} \cdot (e^{x}+x^2)dx - e^{y}\, dy = 0.$

~~or~~ Inte.,

$\int (e^{x}+x^2)\, dx - \int e^{y}\, dy = 0$

or, $e^{x} + \frac{x^3}{3} - e^{y} = c.$

$\therefore\ 3e^{x} - 3e^{y} + x^3 = 3c.$

Reducible to Separable:-

Ex:-

Solve, $\frac{dy}{dx} = (x+y)^2$

Let, $z = x+y$

$\frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{dz}{dx} - 1$

So, $\frac{dz}{dx} - 1 = z^2$

or, $\frac{dz}{dx} = z^2 + 1$

separating the variable,

$\frac{1}{1+z^2} dz = dx$

Inte.,

$\cancel{\ln|1+z^2| - x +} \tan^{-1} z = x + c$

or, $\tan^{-1}[(x+y)] = x + c.$

$\therefore \cancel{\tan^{-1}} x - \tan^{-1}[(x+y)] + c = 0$

$\therefore \tan^{-1}(x+y) - x = c.$

Where 'c' is an arbitrary constant.

Ex

Solve,

$$\frac{dy}{dx} = \cos(x+y)$$

Let, $x+y = z$

or, $1+\frac{dy}{dx} = \frac{dz}{dx}$

or, $\frac{dy}{dx} = \frac{dz}{dx} - 1.$

Now,

$$\frac{dz}{dx} - 1 = \cos z$$

or, $\frac{dz}{dx} = \cos z + 1$

or, $\frac{1}{1+\cos z} \cdot dz = dx$

Integrating, $\int \frac{1}{1+\cos z} dz = \int dx$

$\ln|1+\cos z| = x+c.$ X

$\therefore \ln|1+\cos z| - x = c.$

Where 'c' is an arbitrary constant.

or, $\frac{1}{2\cos^2\frac{z}{2}} dz = dx$

or, $\frac{1}{2}\sec^2\frac{z}{2} dz = dx$

Integration,

$\frac{1}{2} \cdot 2\tan\frac{z}{2} = x+c.$

$\therefore \tan\frac{z}{2} - x = c$

$\therefore \tan\left(\frac{x+y}{2}\right) - x = c$

Where 'c' is an arbitrary constant.

Ex.

Find the particular solution of $\cos y\, dx + (1+e^{-x}) \sin y\, dy = 0$ where $y(0) = \frac{\pi}{4}$.

$$\cos y\, dx + (1+e^{-x}) \sin y\, dy = 0$$

$1+e^{-x} = z$

or, $-e^{-x}\, dx = dz$

or,

$$\text{or, } \frac{dx}{(1+e^{x})} + \tan y\, dy = 0$$

$$\text{or, } \frac{e^{-x}\, dx}{(1+e^{-x})} + \tan y\, dy = 0$$

$$\text{or, } \ln(1+e^{x}) - \ln|\cos y| = \ln c$$

$$\cancel{y(0)=\frac{\pi}{4}} \qquad (1+e^{x}) \sec y = c$$

$$\text{or, } y(0) = \frac{\pi}{4}$$

$$(1+1) \sec \frac{\pi}{4} = c$$

$$\text{or, } 2\sqrt{2} = c$$

$$\therefore c = 2\sqrt{2}$$

$$\therefore (1+e^{-x}) \sec y = 2\sqrt{2}.$$

Ex:

Suppose that, the derivative $\frac{dx}{dt}$ is proportional to x, that $x=5$ when $t=0$ and that $x=10$ when $t=5$. What is the value of x?

Soln:- Given that,

$$\frac{dx}{dt} \propto x$$

$$\text{or, } \frac{dx}{dt} = kx \quad \text{--- (i)}$$

$$\text{or, } dt \cdot k = dx \cdot \frac{1}{x}$$

Inte.,

$$c + \ k \cdot t = \ln x \cancel{+c} \quad \text{--- (ii)}$$

$$\therefore k \cdot t + \ln x = c. \quad \therefore \ln x - kt = c.$$

Now Here, $x(0)=5$

$$\text{or, } \ln 5 = 0 + c$$

$$\therefore c = \ln 5.$$

From (ii), $\ln x = kt + \ln 5$

$$\text{or, } \ln x - \ln 5 = kt.$$

$$\text{or, } \ln\left(\frac{x}{5}\right) = kt \quad \text{--- (iii)}$$

If $t=5$, $x=10$.

$$\therefore \ln\left(\frac{10}{5}\right) = 5k.$$

$$\text{or, } \frac{1}{5}\ln 2 = k. \quad \therefore k = \frac{1}{5}\ln 2.$$

Putting the value of 'k' in (iii),

$$\ln\left(\frac{x}{5}\right) = \frac{1}{5}\ln 2 \cdot t.$$

$$\therefore \ln\left(\frac{x}{5}\right) = \frac{\ln 2}{5} t.$$

$$\therefore \ln\left(\frac{x}{5}\right) = 0.138\, t.$$

$$\text{or, } \frac{x}{5} = e^{0.138t}.$$

$$\therefore x = 5 \cdot e^{0.138t}.$$

The 1st order, 1st deg. ODE :-

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

If M and N are both homogeneous fn. with same degree, then this type of ODE is called Homogeneous ODE.

$$\frac{dy}{dx} = f(x,y)$$

Degree of $f(x,y)$ is zero.

$$f(x,y) = x^0 \phi\left(\frac{y}{x}\right).$$

Solving Procedure :-

Step-1 :-

Putting, $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Step-2 :-

After replacing $y = vx$, the equation will be separable in v and x.

Step-3 :- Separable. Separate the Variables and Integrate.

Homogeneous fn :-

If $f(x,y)$ can be expressed by $x^n \phi(y/x)$ or $y^n \phi\left(\frac{x}{y}\right)$ form, then $f(x,y)$ is called homogeneous fn. of degree n.

$$f(x,y) = x^2 + 2xy + y^2$$
$$= x^2\left(1 + \frac{y^2}{x^2} + 2\cdot\frac{y}{x}\right)$$
$$= x^2 \phi\left(\frac{y}{x}\right)$$

$$f(kx,ky)$$
$$= k^2x^2 + k^2y^2 + k^2 2xy$$
$$= k^2(x^2 + y^2 + 2xy)$$
$$= k^2 f(x,y).$$

$$f(x,y) = \sin\frac{x}{y} + \log\frac{x}{y} + \cos\frac{x}{y}.$$

$$f(kx,ky) = \sin\frac{kx}{kx} + \log\frac{kx}{ky} + \cos\frac{kx}{ky}.$$

Replace x with kx and y with ky

$$= k^0\left(\sin\frac{x}{y} + \log\frac{x}{y} + \cos\frac{x}{y}\right).$$

Homogeneous function of degree zero.

Ex.
Solve, $\frac{dy}{dx} = \frac{y(y-2x)}{x(x-2y)}$

or, $y(y-2x)\,dx - x(x-2y)\,dy = 0$

This is a homogeneous diff. equation.

Putting $y = vx$.

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

or, $v + x\cdot\frac{dv}{dx} = \frac{vx(vx-2x)}{x(x-2vx)}$

or, $x\cdot\frac{dv}{dx} = \frac{v^2-2v}{1-2v} - v$

or, $x\cdot\frac{dv}{dx} = \frac{v^2-2v-v+2v^2}{1-2v}$

or, $x\frac{dv}{dx} = \frac{3v^2-3v}{1-2v}$

separating the variable,

$\frac{1-2v}{3v^2-3v}\cdot dv = \frac{1}{x}\,dx$

or, $\frac{1-2v}{v^2-v}\,dv = \frac{1}{x}\cdot 3\,dx$

Int., $-\int\frac{2v-1}{v^2-v}\,dv = \int\frac{3}{x}\,dx$

or, $-\log(v^2-v) = 3\log(x) + c$

or, $3\log x + \log\left(\frac{y^2}{x^2} - \frac{y}{x}\right) + c = 0$

$\therefore 3\log x + \log\left(\frac{y^2}{x^2} - \frac{y}{x}\right) + c = 0$

Quiz-1

23-07-19

Syllabus- From 1st lecture to Homogeneous D.E.

Time:- 1:45 pm.

Ex: Solve,

$$\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$$

This is a homogeneous diff. equation.

Let, $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or, $\cancel{\frac{y}{x} + \tan\frac{y}{x} = v + x}$

or, $v + x\cdot\frac{dv}{dx} = v + \tan v$

or, $x\frac{dv}{dx} = \tan v$

This is separable D.E. of v and x.

Now, separating the variables,

$$\frac{1}{\tan v}dv = \frac{dx}{x}$$

or, $\cot v \, dv = \frac{1}{x}\cdot dx$

Integrating, $\int \cot v \, dv = \int \frac{1}{x} dx$

or, $\int \frac{\cos v}{\sin v} dv = \int \frac{1}{x} dx$

or, $\log(\sin v) = \log x + \log c$.

$\therefore \sin v = cx$

replacing, $v = \frac{y}{x}$, $\sin\frac{y}{x} = cx$. This is the G.S. of the given D.E. where c is an arbitrary constant.

Ex:-

Solve,

$$\underbrace{\left(x\sin\frac{y}{x} - y\cos\frac{y}{x}\right)}_{M}dx + \underbrace{x\cos\frac{y}{x}}_{N}dy = 0$$

$$\frac{dy}{dx} = \frac{-x\sin\frac{y}{x} + y\cos\frac{y}{x}}{x\cos\frac{y}{x}\cancel{dy}}$$

Putting $y = vx$,

$$\frac{dy}{dx} = \frac{-x\sin v + vx\cos v}{x\cos v\,\cancel{dy}}$$

Ex:- Solve,

$$\left(x+\sqrt{y^2-xy}\right)dy-y\,dx=0$$

$$\frac{dy}{dx}=\frac{y}{x+\sqrt{y^2-xy}}$$

This is homogeneous diff. equations.

Putting $y=vx$,

$$\frac{dy}{dx}=\frac{vx}{x+\sqrt{v^2x^2-x^2v}}=\frac{vx}{x\left[1+\sqrt{v^2-v}\right]}=\frac{v}{1+\sqrt{v^2-v}}$$

Now,

$$v+x\frac{dv}{dx}=\frac{v}{1+\sqrt{v^2-v}}$$

$$\therefore x\frac{dv}{dx}=\frac{v}{1+\sqrt{v^2-v}}-v=\frac{v-v\left(1+\sqrt{v^2-v}\right)}{1+\sqrt{v^2-v}}$$

$$\frac{dx}{dy}=\frac{x+\sqrt{\cancel{x}\,y^2-xy}}{y}$$

Let, $x=vy$.,

$$\frac{dx}{dy}=v+y\frac{dv}{dy}$$

$$v+y\frac{dv}{dy}=\frac{vy+\sqrt{y^2-vy^2}}{y}$$

or, $v+y\dfrac{dv}{dy}=v+\sqrt{1-v}$

$\therefore y\dfrac{dv}{dy}=\sqrt{1-v}$.

This is separable differential eqn of v and y.

$$\frac{1}{\sqrt{1-v}}dv=\frac{dy}{y}.$$

Integrating,

$$\int \frac{1}{\sqrt{1-v}}\, dv = \int \frac{dy}{y}$$

or, $-2\int \frac{1}{z}\, dz = \frac{1}{4}\int \frac{dy}{y}$

or, $-2\sqrt{1-v} = \log y + A$

$\therefore -2\sqrt{1-\frac{x}{y}} = \log y + A$

Let,

$1-v = z^2$

or, $-\frac{dv}{\cancel{dy}} = 2z \cdot dz$

or, $dv = -2z\,dz$

Ex 8-

Find the particular soln of :-

$x y^2 dy - (x^3+y^3)dx = 0$. where $y(1)=0$

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$$

$$v + x\frac{dv}{dx} = \frac{x^3+v^3x^3}{x \cdot v^2 x^2} = \frac{1+v^3}{v^2}$$

$$x \cdot \frac{dv}{dx} = \frac{1+v^3-v^3}{v^2} = \frac{1}{v^2}$$

$$v^2 dv = \frac{1}{x}dx$$

$$\frac{v^3}{3} = \log x + \log c$$

$$\frac{y^3}{3x^3} = \log x + \log c$$

using the initial value term,

$y(1) = 0$

or, $0 = \log 1 + \log c$

$\therefore c = 0$

$$\frac{y^3}{3x^3} = \log x$$

$\therefore$ P.S.,

$y^3 = 3x^3 \log x$

<u>Ex:</u>- Find the equation of the curve passing through the point (1,1) whose differential equation is,

$x \cdot dy = (2x^2+1)\,dx$

or, $\frac{dy}{dx} = \frac{2x^2+1}{x}$

~~replacing $y = vx$,~~

$\cancel{v + x\frac{dv}{dy} = \frac{2x^2+1}{x}}$

$\cancel{\text{or, } \frac{y}{x} + \frac{dv}{dy} =}$

or, $dy = \int\left(2x + \frac{1}{x}\right)dx$

or, $y = x^2 + \log x + c$

$\cancel{y(1,1)}$

$y(1) = 1$

or, $1^2 + \log 1 + c = 1$

or, $c = 0$

$\therefore y = x^2 + \log x$

In other words, the expression $Mdx + Ndy$ is called exact differential, if here, exists a function u for which the expression the total differential du.

Solution of Exact D.E. :-

$Mdx + N.dy = 0$

$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$.

$du = 0$

$u = const.$

$\therefore u(x, y) = const.$ This is the general solⁿ.

Necessary and sufficient condition for a differential equation to be exact:-

or, consider the differential equation,

$$Mdx + Ndy = 0 \quad \text{------ (i)}$$

where M and N have continuous partial derivatives then-

a) If the D.E. (i) is exact, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

[Necessary condition]

b) If $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$, then the D.E. is exact.

[Sufficient condition]

Total Differential:-

$u = u(x,y)$

Differentiate partially with respect to x,

$\frac{\partial y}{\partial x}$

Total Derivative,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$u = f(x,y,z)$$

$$du = \frac{\partial y}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$y = f(x)$

$\frac{dy}{dx} = f'(x)$

$\therefore \boxed{dy = f'(x)dx}$

Exact D.E :-

The first order first degree ODE $Mdx + Ndy = 0$ is called exact differential equation, if Left hand side is the total differential of some function $u(x,y)$.

$$Mdx + N \cdot dy = 0$$

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$du = 0$$

$$u = \text{constant.}$$

Ex.:-

$(2xy+1)dx + (x^2+4y)dy = 0$ —— (1)

Solve,

$M(x,y) = 2xy+1$ $\qquad$ $N(x,y) = x^2+4y$

$\frac{\partial M}{\partial y} = 2x$ $\qquad$ $\frac{\partial N}{\partial x} = 2x.$

Here, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$

∴ The differential eqn is exact.

$\therefore \cancel{\partial y}\ \frac{\partial u}{\partial x} = M(x,y)$

$\frac{\partial u}{\partial x} = 2xy+1$

$u = \int (2xy+1)\, \partial x + Q(y)$ $\cancel{(2)}$

$= x^2 y + \cancel{b} + \phi(y) + x$ —— (2)

$\therefore \frac{\partial u}{\partial y} = x^2 + \cancel{1} + \phi'(y)$

Now, $\frac{\partial u}{\partial y} = N = x^2 + 4y$

$\therefore \cancel{x^2} + 4y = \cancel{x^2} + \cancel{1} + \phi'(y)$

$\therefore \phi'(y) = 4y.$

$\therefore \phi(y) = 2y^2 + c_0$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,\partial x\right] + \frac{d\phi}{dy}$$

let; $\int M(x,y)\,dx = g(x,y)$

$$\frac{\partial y}{\partial y} = \frac{\partial g}{\partial y} + \frac{d\phi}{dy}$$

$$\text{or, } N(x,y) = \frac{\partial g}{\partial y} + \frac{d\phi}{dy}$$

$$\frac{d\phi}{dy} = \left[N\cancel{(x,y)} - \frac{\partial g}{\partial y}\right]$$

$$\therefore \phi(y) = \int\left[N - \frac{\partial g}{\partial y}\right]dy$$

Now, from (2)

$$u = \int M(x,y)\,\partial x + \int\left[N - \frac{\partial g}{\partial y}\right]dy$$

$$= g(x,y) + \int\left[N - \frac{\partial g}{\partial y}\right]dy$$

$\therefore$ Total derivative of u,

$$du = dg + N - \left(\frac{\partial g}{\partial y}\right)dy$$

$$du = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + Nd\cdot g - \frac{\partial g}{\partial y}dy$$

$$\therefore du = \frac{\partial g}{\partial x}dx + N\cdot dy = M\cdot dx + N\cdot dy$$

(Showed)

Proof:-

(a) If the given D.E. (i) is exact, then $Mdx+Ndy$ is the total differential of a function : $u(x,y)$. such that -

$$M=\frac{\partial u}{\partial x} \quad N=\frac{\partial u}{\partial y}.$$

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \qquad \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$=\frac{\partial^2 u}{\partial y\partial x} \qquad =\frac{\partial^2 u}{\partial x\partial y}$$

for partial derivative,

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$$

Therefore, $\boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$ (Necessary Condition).

(b) Here, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

We have to show that $Mdx+Ndy=0$ is exact. This means that, we must prove that there exists a function $u(x,y)$, such that, $\frac{\partial u}{\partial x}=M(x,y)$, $\frac{\partial u}{\partial y}=N(x,y)$.

Let us assume that, u satisfies,

$$\frac{\partial u}{\partial x}=M(x,y).$$

$$u=\int M(x,y)\,\partial x+\phi(y) \text{———} (2)$$

Put the value of $\phi(y)$ in (2),

$$u = x^2y + x + 2y^2 + c_0$$

Hense, the G.S.,

$$u(x,y) = c_1$$

$$\therefore x^2y + x + 2y^2 + (c_0 - c_1) = 0$$

$$\therefore x^2y + x + 2y^2 = c_1 - c_0$$

$$\therefore x^2y + x + 2y^2 = c.$$

$$\therefore x^2y + 2y^2 + x = c.$$

Standard Method :-

Ex

Solve

$(y\sec^2 x + \sec x \tan x)dx + (\tan x + 2y)dy = 0$ —— (1)

$M = y\sec^2 x + \sec x \tan x$, $N = \tan x + 2y$

$\frac{\partial M}{\partial y} = \sec^2 x$ $\qquad$ $\frac{\partial N}{\partial x} = \sec^2 x$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence, the given differential equation is exact.

Therefore, there have a fn. $u(x,y)$, such that,

$\frac{\partial u}{\partial x} = M$ and $\cancel{\frac{\partial M}{}}$ $\frac{\partial u}{\partial y} = N$.

$\frac{\partial u}{\partial x} = y\sec^2 x + \sec x \tan x$.

Integrating partially w.r.t. x,

$u(x,y) = y\tan x + \sec x + \phi(y)$. —— (2)

Diffe. with r.t. y,

$\frac{du}{dy} = \cancel{\tan y}\ \tan x + \frac{d\phi}{dy}$.

$\therefore N = \tan x + \frac{d\phi}{dy}$

or, $\tan x + 2y = \tan x + \frac{d\phi}{dy}$.

$\therefore \frac{d\phi}{dy} = 2y$.

$\therefore \cancel{\phi = y^2 + c}\ \phi(y) = y^2 + c$.

Putting $\phi(y)$ in (2),

$$u(x,y) = y\tan x + \sec x + y^2 + c_0$$

$\therefore$ the solution of D.E. is,

$$u(x,y) = \text{const.}$$

$$y\tan x + \sec x + y^2 + c_0 = c_1$$

$$\text{or, } y\tan x + \sec x + y^2 = c \quad [c = c_1 - c_0]$$

where c is an arbitrary constant.

Grouping Method:

Solve, $(3x^2+4xy)dx+(2x^2+2y)dy=0$

$M=3x^2+4xy$ $\quad$ $N=2x^2+2y$

$M_y=\frac{\partial M}{\partial y}=4x$, $N_x=\frac{\partial N}{\partial x}=4x$

$\therefore$ Exact.

$d(2x^2y)$
$=4xy\,dx+2x^2dy$

The equation maybe written,

$3x^2dx+4xy\,dx+2x^2dy+2y\,dy=0$

or, $d(x^3)+d(2x^2y)+d(y^2)=d(c)$

Integrating,

$x^3+2x^2y+y^2=c$

Ex

$(2x^3 - xy^2 - 2y + 3)dx - (x^2y + 2x)dy = 0$

$2x^3dx - xy^2dx - 2ydx + 3dx - x^2ydy - 2xdy = 0$

or, ~~d(xy~~

or, $d\left(\frac{x^4}{2}\right) - d\left(\frac{1}{2}x^2y^2\right) - d(2xy) + d(3x) = d(c)$

$\therefore \frac{x^4}{2} - \frac{1}{2}x^2y^2 - 2xy + 3x = c_1$

or, $x^4 - x^2y^2 - 4xy + 6x = c$.

$\therefore x^4 - x^2y^2 - 4xy + 6x = c$.

If $Mdx + Ndy = 0$ is not exact then the expression whose multiplication with an inexact differential equation makes exact, then this expression is called Integrating factor.

$\cancel{y\,dx =}$

$$y\,dx + 2x\,dy = 0.$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 2 \quad \therefore \text{Not Exact}$$

multipleing with $\boxed{y}$

$$y^2 dx + 2xy\,dy = 0.$$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = 2y.$$

$\therefore$ y is the integrating factor here.

Finding I.F. :-

(i) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, function of x only,
then $I.F. = e^{\int f(x)dx}$

(ii) If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$, fn. of y only,
$I.F. = e^{-\int g(y)dy}$

(iii) If $Mx + Ny \neq 0$ and homogeneous,
$I.F. = \frac{1}{Mx+Ny}$.

(iv) If $Mx - Ny \neq 0$, $I.F. \frac{1}{Mx-Ny}$

$y\,dx + 2x\,dy = 0$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 2$

* $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y}(1-2) = -\frac{1}{y}$ fn of y.

$I.F. = e^{-\int -\frac{1}{y}} = e^{\int \frac{1}{y}dy} = e^{\ln y} = y$.

* $Mx + Ny = xy + 2xy = 3xy \neq 0$.

$I.F. = \frac{1}{3xy}$

$\therefore \frac{1}{3x}dx + \frac{2}{3}y\,dy = 0$

Solve,

$$(4xy + 3y^2 - x)dx + x(x+2y)dy = 0$$

$$\frac{\partial M}{\partial y} = 4x + 6y \qquad \frac{\partial N}{\partial x} = 2x + 2y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4x - 6y - 2x - 2y}{x(x+2y)} = \frac{2x+4}{x(x+2y)}$$

$$= \frac{2(x+2y)}{x(x+2y)} = \frac{2}{x}$$

$$\therefore \text{I.F.} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2.$$

H.W.

Solve

i) $(2xy^4 e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

ii) $(2x^2 + y)dx + (x^2y - x)dy = 0$

iii) $(2x^3y^2 + 4x^2y + 2xy^2 + xy^4 + 2y)dx$
$+ 2(y^3 + x^2y + x)dy = 0$

iv) $y(x+y+1)dx + x(x+3y+2)dy = 0$

The linean diff. equation of order One :-

Linear

(i) No product of dependent variable
(ii) " " of " " and it's derivative.
(iii) No transeentdental function of depentent variable.

Def^n of Linear D.E.
and
non-linear D.E.

The standard Form of Linear D.E. of order one is,

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

where P and Q are fn. of x or constant.

How to solve Linear D.E.:–

$$\frac{dy}{dx} + P(x) y = Q$$

or, ~~dy + P y dx = Q~~ $dy + Py\,dx = Q\,dx$

or, $(Py - Q)\,dx + dy = 0$ ——(i)

$M = Py - Q \quad N = 1$.

Let, $V(x)$ is a I.F.

Multiply $v(x)$ with (i)

$V(x)(Py - Q)\,dx + V(x)\,dy = 0$.

Now, this is exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$vPy - vQ$.

$M = Pyv - Qv$.

$$\frac{\partial M}{\partial y} = Pv$$

$$\frac{\partial N}{\partial x} = \frac{dv}{dx}$$

Therefore, $\frac{dv}{dx} = Pv$

$\frac{1}{v} \cdot dv = P \cdot dx$

integrating,

$$\log v = \int P(x)\,dx$$

$$\boxed{\therefore V = e^{\int P(x)\,dx}}$$

→ Must for linear d.e.

$\therefore$ I.F. for linear d.e. $= e^{\int P(x)\,dx}$

Ex.

$x\,dy = \cancel{2(x^4+4)}\ 2(x^4+y)dx$ — (i)

$\frac{dy}{dx} = \frac{2(x^4+y)}{x}$

$\frac{dy}{dx} = 2x^3 + \frac{2y}{x}$

or, $\frac{dy}{dx} - \frac{2}{x}\cdot y = 2x^3$

or, $\frac{dy}{dx} + \left(-\frac{2}{x}\right)\cdot y = 2x^3$. — (ii)

$P(x) = -\frac{2}{x}$. $Q(x) = 2x^3$

This is linear d.e.

$\therefore$ I.F. $= e^{\int P(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$.

$\therefore \cancel{\frac{1}{x}\frac{dy}{}} =$ multiplying with (ii),

$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x}\cdot\frac{1}{x^2}\cdot y = 2x^3\cdot\frac{1}{x^2}$

or, $d\left(y\cdot\frac{1}{x^2}\right) = \int 2x\,dx$

or, $\frac{y}{x^2} = x^2 + c$.

Ex. Solve,

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

or, $\frac{dy}{dx} + \frac{1}{\cos^2 x} y = \frac{\tan x}{\cos^2 x}$

or, $\frac{dy}{dx} + \sec^2 x \cdot y = \sec^2 x \cdot \tan x$ ——— (i) $\left(\frac{dy}{dx} + P(x)y = Q(x)\right)$

This is linear differential equation.

$$I.F. = e^{\int P(x)dx} = e^{\int \sec^2 x dx} = e^{\tan x}$$

Multiply (i) by I.F.,

$$e^{\tan x} \frac{dy}{dx} + \sec^2 x\, e^{\tan x} \cdot y = \sec^2 x \cdot \tan x \cdot e^{\tan x}.$$

or, $d(y e^{\tan x}) = \int \sec^2 x \cdot \tan x \cdot e^{\tan x} dx$

Let, $z = \tan x$

$dz = \sec^2 x dx$

or, $d(y e^{\tan x}) = \int z \cdot e^z dz$

Integrating,

$y \cdot e^{\tan x} = e^z (z-1) + A$

or, $y \cdot e^{\tan x} = e^{\tan x}(\tan x - 1) + A.$

This is the G.S.

Ex :- Solve,

$$(y\sin 2x-\cos x)dx+(1+\sin^2 x)\cancel{dx}\,dy=0$$

or. $\frac{dy}{dx}=\frac{-(y\sin 2x-\cos x)}{1+\sin^2 x}$

or, $\frac{dy}{dy}=\frac{-y\sin 2x}{1+\sin^2 x}+\frac{\cos x}{1+\sin^2 x}$

or, $\frac{dy}{dx}+\frac{\sin 2x}{1+\sin^2 x}\cdot y=\frac{\cos x}{1+\sin x}$ This is ~~te~~ linear — (i)

$e^{\int\frac{\sin 2x}{1+\sin x}}$

$\therefore$ I.F. $=1+\sin^2 x$

Multiply by I.F.,

Date

Ex. Solve,

$$(x^2+x-2)\frac{dy}{dx}+3(x+1)y=x-1$$

$$\frac{dy}{dx}+\frac{3(x+1)}{(x^2+x-2)}y=\frac{(x-1)}{(x^2+x-2)}$$

$$\text{or, } \frac{dy}{dx}+\frac{3(x+1)}{(x^2+x-2)}y=\frac{x-1}{(x+2)(x-1)}=\frac{1}{x+2} \text{ ---- (i)}$$

$$I.F. = e^{\int P(x)dx} = e^{\int \cancel{\frac{1}{x+2}}\frac{3}{}dx} = \boxed{3\cdot\frac{1}{x+2}} = \frac{3}{x+2}$$

$$= e^{\int\frac{3(x+1)}{(x^2+x-2)}} = e^{\int\frac{\cancel{6}}{(x+2)(x-1)}} +$$

$$= e^{\int\left(\frac{1}{x+2}+\frac{2}{x-1}\right)dx}$$

$$\frac{3(x+1)}{(x+2)(x-1)}=\frac{A}{(x+2)}+\frac{B}{(x-1)}$$

$$= e^{(\ln|x+2|+2\ln|x-1|)}$$

$$= \ln|x+2|+2\ln|x-1|$$

$$3(x+1)=A$$

$A=2$

$B=1$.

Bernoulli DE
(Reducable to Linear)

The standard form of Benouli D.E. is;

$$\frac{dy}{dx} + P(x)\, y = Q(x) \cdot y^n, \quad n \neq 0, 1$$

Method:

Divide both side by y^n

$$y^{-n} \cdot \frac{dy}{dx} + P(x) \cdot y^{1-n} = Q(x) \quad \text{--- (i)}$$

Let, $v = y^{1-n}$

$$\frac{dv}{dx} = (1-n) \cdot y^{1-n-1} \frac{dy}{dx}$$

$$\therefore (1-n) \cdot y^{-n} \cdot \frac{dy}{dx} = \frac{dv}{dx}$$

$$\text{or, } y^{-n} \cdot \frac{dy}{dx} = \frac{1}{1-n} \frac{dv}{dx}$$

from (i)

$$\frac{dv}{dx} + (1-n)\, P(x) \cdot v = (1-n) Q(x)$$

This is linear d.e. in v and x.

I.F. =

Ex :-

$$\frac{dy}{dx} + y = xy^3$$

This is the Bernoulli D.E.

$$y^{-3}\frac{dy}{dx} + \frac{y}{y^3} = x$$

or, $y^{-3}\frac{dy}{dx} + y^{-2} = x$ —— (i)

Let, $v = y^{-2}$,

or, $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$

$\therefore y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$

from (i),

$$-\frac{1}{2}\frac{dv}{dx} + v = x$$

or, $\frac{dv}{dx} - 2v = -2x$ —— (ii)

This is linear d.e. in v and x.

$$I.F. = e^{\int -2dx} = e^{-2x}$$

from (ii) $\times$ I.F.,

$$e^{-2x}\frac{dv}{dx} - 2\cdot e^{-2x}\cdot v = -2\cdot e^{-2x} x$$

or, $e^{-2x}\cdot\frac{dv}{dx} - 2v e^{-2x}dx = -2x\cdot e^{-2x}dx$

or, $d(v\cdot e^{-2x}) = \int -2x\cdot e^{-2x}\,dx$

Integrating,

or, $v \cdot e^{-2x} = \cancel{-2}\int\left(-\frac{z}{2}\right) \cdot e^{z}\, dz$

Let,

$z = -2x$

$dz = -2dx$

$\therefore -2dx = dz$

$x = -\frac{z}{2}$

$v \cdot e^{-2x} = \frac{1}{2} e^{-2x}(2x+1) + c$

Now, $v = y^{-2}$

$\therefore e^{-2x} = \frac{1}{2} e^{-2x}(2x+1) + c$

$\frac{1}{y^2} = \frac{1}{2}(2x+1) + c e^{-2x}$

Ex :-

$(x+2)\frac{dy}{dx} + y = f(x)$ where $f(x) = \begin{cases} 2x & 0 \le x < 2 \\ 4 & x \ge 2 \end{cases}$

and $y(0) = 4$.

$$\frac{dy}{dx} + \frac{1}{x+2}y = \frac{f(x)}{x+2} \quad \text{—(i)}$$

$$I.F. = e^{\int \frac{1}{x+2}dx} = e^{\ln(x+2)} = x+2$$

Multiply (i) $\times$ I.E. ,

$$(x+2)\cdot\frac{dy}{dx} + y = f(x)$$

or, $d[(x+2)\cdot y] = \int f(x) = \int f(x)$

~~or,~~ Integrating,

$$y(x+2) = \begin{cases} x^2 + c_1 & ; 0 \le x < 2 \\ 4x + c_2 & ; x \ge 2 \end{cases}$$

$$\therefore y = \begin{cases} \frac{x^2 + c_1}{(x+2)} & ; 0 \le x < 2 \\ \frac{4x}{(x+2)} & ; x \ge 2. \end{cases}$$

Here, $y(0)=4$, from first piece,

$x=0, y=4$, $\quad 4=\frac{0+c_1}{2}$. $c_1=8$.

$$f(x)=\begin{cases}\frac{x^2+8}{x+2} ; & 0\le x\le 2\\ \frac{4x+c_2}{x+2} ; & x\ge 2\end{cases}$$

Putting $x=2$ in both pieces and equating,

$$\frac{4+8}{\cancel{4}}=\frac{8+c_2}{\cancel{4}}$$

or, $c_2=4$

$\therefore c_2=4$

Therefore, Particular soln:-

$$f(x)=\begin{cases}\frac{x^2+8}{x+2} ; & 0\le x\le 2\\ \frac{4x+4}{x+2} ; & x\ge 2.\end{cases}$$ Ans.

Solve, $\frac{dy}{dx}+\frac{y}{2x}=\frac{x}{y^3}$, $y(1)=2$

$\frac{dy}{dx}+\frac{1}{2x}y=x\cdot y^{-3}$

or, $y^3\cdot\frac{dy}{dx}+\frac{1}{2x}y^4=x$

Replacing $v=y^4$

or, $\frac{dv}{dx}=4y^3\frac{dy}{dx}$

or, $y^3\frac{dy}{dx}=\frac{1}{4}\frac{dv}{dx}$

$\frac{1}{4}\frac{dv}{dx}+\frac{v}{2x}=x$

or, $\frac{dv}{dx}+\frac{2v}{x}=4x$ — (i)

I.E. $=e^{\int\frac{2}{x}dx}=\cancel{2\ln x}\ x^2$

$\frac{dv}{dx}x^2+2v\cdot x=4x^3$

or, $d(x^2\cdot v)=\int 4x^3$

or, $x^2\cdot v=x^4+c$

$\therefore y^4x^2=x^4+c$

$y(1)=2$

~~or, 1·4=1+c~~

~~∴ c=3~~

~~$\therefore y^4x^2=x^4+3$~~ (Ans.)

or, $16\cdot 1=1+c$

$\therefore c=15$.

$\therefore y^4x^2=x^4+15$ (Ans.)

Application of First Or. First De. Eqⁿ:

Ex.

The Population of town was 60,000 in 1990 and had increased to 63,000 by 2000. Assuming that the population is increasing at a rate Proportional to its size at any time. Estamite the Population in 2010.

Let,
the size of the population at any time is n.

By the condition,
rate of change,

$$\frac{dn}{dt} \propto n$$

$$\text{or, } \frac{dn}{dt} = kn \quad \text{This is separable.}$$

$$\text{or, } \cancel{\pm\frac{1}{k}}\ \frac{1}{n}dn = k \cdot dt$$

$$\text{or, } \cancel{\left(-\frac{4}{2}n^{-2} = \frac{1}{2}k^{4}\right)}$$

$$\text{or, } \ln n = kt + \ln A$$

$$\text{or, } \ln n = \ln e^{kt} + \ln A$$

$$\therefore n(t) = A \cdot e^{kt}.$$

Here, At 1990, i.e. $t=0$, $n = 60,000$
At 2000, i.e. $t=10$, $n = 63,000$

$$60,000 = A \cdot e^{0} = A.$$

$$n(t) = 60000e^{kt}.$$

if $t = 10$, $n = 63,000$

$$63,000 = 60,000\ e^{10k}.$$

$k = 0.00988.$

$$\therefore f n(t) = 60{,}000\, e^{0.00488t}$$

if in 2010, $t = 20$.

$$\therefore n(20) = 60{,}000\, e^{0.00488 \times 20}.$$

$$= 66{,}150.$$

Ex. A patient is recieving drug treatment when 1st measured, there is 0.5 mg of the drug per litre of blood. After 4 hours there is only 0.1 mg per litre. Assume that amount of drug in blood at times t. Find how long it takes for there to be only 0.05 mg is decreasing

let, $\cancel{\frac{dn}{dt}}$ amount of drug in any time is n

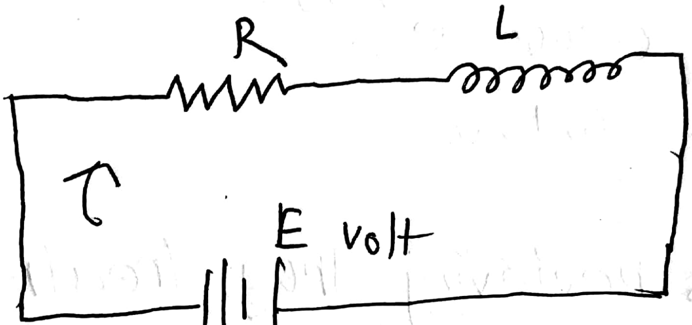
Here,

$$\frac{dn}{dn} \propto -n$$

$$\frac{dn}{dt} = -kn$$

Ex

A circuit consisting of resistance of R ohms and an inductance L henries is connected to a battery of constant voltage E volt. Find the current 1 amp. at time t after the circuit is closed.



An inductance and resistance each cause a drop of voltage. The drop due to resistance is iR and that due to inductance is $L \cdot \frac{di}{dt}$

Therefore ~~the~~ voltage supplied by the battery is equal to the voltage drop,

$$iR + L\frac{di}{dt} = E \Rightarrow \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}.$$

91. $\frac{dy}{dx} = 2x - 7$.

$dy = (2x-7)dx$

$\int dy = y = \int (2x-7)dx$

or, $y = 2 \cdot \frac{x^2}{2} - 7x + c$

$\therefore y = x^2 - 7x + c$.

Now, $y(2) = 0$

or, $4 - 14 + c = 0$

$\therefore c = 10$. $\therefore y = x^2 - 7x + 10$

92. $\frac{dy}{dx} = 10 - x$.

$dy = (10-x)dx$

$\int dy = \int (10-x)dx$

or, $y = 10x - \frac{x^2}{2} + c$.

$\therefore y = 10x - \frac{x^2}{2} + c$

Now, $y(0) = -1$

or, $10(0) - \frac{0}{2} + c = -1$

$\therefore c = -1$. $\therefore y = 10x - \frac{x^2}{2} - 1$.

93. $\frac{dy}{dx} = \frac{1}{x^2} + x$; $x > 0$

$dy = (\frac{1}{x^2} + x)dx$

$\int dy = \int (\frac{1}{x^2} + x)dx$

or, $y = -\frac{1}{x} + \frac{1}{2}x^2 + c$.

$\therefore y = -\frac{1}{x} + \frac{1}{2}x^2 + c$

Now, $y(2) = 1$.

or, $-\frac{1}{2} + \frac{1}{2} \cdot 4 + c = 1$

or, $c = -\frac{1}{2}$ $\therefore y = -\frac{1}{x} + \frac{x^2}{2} - \frac{1}{2}$

94. $\frac{dy}{dx} = 9x^2 - 4x + 5$

$\int dy = \int (9x^2 - 4x + 5)dx$

or, $y = \frac{9}{3}x^3 - \frac{4}{2}x^2 + 5x + c$

$\therefore y = 3x^3 - 2x^2 + 5x + c$.

Now, $y(-1) = 0$

or, $3(-1)^3 - 2(-1)^2 + 5(-1) + c = 0$

or, $-3 - 2 - 5 + c = 0$

$\therefore c = 10$.

$\therefore y = 3x^3 - 2x^2 + 5x + 10$.

Ex: A stone weighting 4 lb falls from the rest toward the earth from a great height. As it falls it is acted upon by air resistance that is numerically equaled to $\frac{1}{2}v$.

(a) Find the velocity and distance fallen at time t sec.

(b) " -- at $t = 5$ sec.

$F = ma$

$F = F_1 + F_2$

$=$ weigth + air resistance.

$= 4 - \frac{1}{2}v$

$m = \frac{4}{32}$

$= \frac{1}{8}$ ft/s.

$4 - \frac{1}{2}v = \frac{1}{8} \cdot \frac{dv}{dt}$

or, $\frac{dv}{dt} = 32 - 4v$

or, $\frac{dv}{dt} + 4v = 32$.

$v(t)$

Date ....

99. $\frac{dr}{d\theta} = -\pi \sin \pi\theta$

$\int dr = \int (-\pi \sin\theta\pi)\, d\theta$

or, $r = -(-1)\cos\pi\theta + c.$

$\therefore r = \cos\pi\theta + c.$

Now, $r(0) = 0$

or, $\cos 0 + c = 0$

$\therefore c = -1.$

$\therefore r = \cos\pi\theta - 1.$

100. $\frac{dr}{d\theta} = \cos\pi\theta$

$\int dr = \int \cos\pi\theta$

or, $r = (\sin\pi\theta)\cdot\frac{1}{\pi} + c$

$\therefore r = \frac{1}{\pi}\sin\pi\theta + c.$

Now, $r(0) = 1$

or, $\frac{1}{\pi}\sin 0 + c = 1.$

$\therefore c = 1.$

$\therefore r = \frac{1}{\pi}\sin\pi\theta + 1.$

101. $\frac{dv}{dt} = \frac{1}{2}\sec t \tan t$

$\int dv = \int \frac{1}{2}\sec t \cdot \tan t.$

or, $v = \frac{1}{2}\sec t + c$

$\therefore v = \frac{1}{2}\sec t + c.$

Now, $v(0) = 1$

or, $\frac{1}{2}\sec t + c = 1$

$\therefore c = \frac{1}{2}.$

$\therefore v = \frac{1}{2}\sec t + \frac{1}{2}.$

102. $\frac{dv}{dt} = 8t + \operatorname{cosec}^2 t$

$\int dv = \int (8t + \operatorname{cosec}^2 t)\, dt$

or, $v = \frac{8}{2}t^2 - \cot t + c.$

$\therefore v = 4t^2 - \cot t + c.$

Now, $v(\frac{\pi}{2}) = -7$

or, $\pi^2 - 0 + c = -7.$

$\therefore c = -7 - \pi^2.$

$\therefore v = 4t^2 - \cot t - 7 - \pi^2.$

95. $\frac{dy}{dx} = 3x^{-\frac{2}{3}}$

or, $\int dy = \int (3x^{-\frac{2}{3}})dx$

or, $y = 3 \cdot x^{\frac{1}{3}} + c$

$\therefore y = 3 \cdot x^{1/3} + c$

Now, $y(-1) = 0$

or, $3(-1)^{\frac{1}{3}} + c = 0$

$\therefore c = 3$

$\therefore y = 3x^{1/3} + 3$

96. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

or, $\int dy = \left(\frac{1}{2\sqrt{x}}\right)dx$

$\therefore y = 2 \cdot \frac{1}{2} \cdot \sqrt{x} + c$

Now, $y(4) = 0$

or, $\sqrt{4} + c = 0$

$\therefore c = \pm 2$

$\therefore y = \sqrt{x} \pm 2$

97. $\frac{ds}{dt} = 1 + \cos t$

$\int ds = \int (1 + \cos t)dt$

or, $s = x + \sin t + c$

$\therefore s = x + \sin t + t$

Now, $s(0) = 4$

or, $c = 4$

$\therefore c = 4$

$\therefore s = x + \sin t + 4$

98. $\frac{ds}{dt} = \cos t + \sin t$

$\int ds = \int (\cos t + \sin t)$

or, $s = \sin t - \cos t + c$

$\therefore s = \sin t - \cos t + c$

Now, $s(\pi) = 1$

or, $\sin \pi - \cos \pi + c = 1$

$\therefore c = 0$

$\therefore s = \sin t - \cos t + 1$

Date

106. $\frac{d^{\vee}y}{dx^{\vee}} = 0$

$\int d^{\vee}y = \int (0) dx^{\vee}$

or, $dy = C_1 dx$

$\therefore \frac{dy}{dx} = y' = C_1.$

Now, $y'(0) = 4/2$

$\therefore C_1 = 2$

$\therefore y' = \frac{dy}{dx} = 2$

or, $\int dy = \int 2dx$

$\therefore y = 2x + c.$

Now, $y(0) = 0$

or, $c = 0$

$\therefore c = 0$

$\therefore y = 2x + 0$

107. $\frac{d^{\vee}p}{dt^2} = \frac{2}{t^3}$

or, $\int d^{\vee}p = \int \frac{2}{t^3} dt^{\vee}$

or, $dp = -\frac{1}{t^2} dt + c_1$

$\therefore \frac{dp}{dt} = -\frac{1}{t^{\vee}} + C_1.$

Now, $\left.\frac{dp}{dt}\right|_{t=1} = 1$

or, $\frac{-1}{1} + C_1 = 1.$

$\therefore C_1 = 2$

$\therefore \frac{dp}{dt} = -\frac{1}{t^{\vee}} + 2$

or, $\int dp = \int \left(-\frac{1}{t^{\vee}} + 2\right) dt$

or, $p = \frac{1}{t} + 2t + c.$

$\therefore p = 1/t + 2t + c.$

Now, $p(1) = 1$

or, $1 + 2 + c = 1.$

$\therefore c = -2.$

$\therefore p = \frac{1}{t} + 2t - 2.$

103. $\frac{dv}{dt} = \frac{3}{t\sqrt{t^{v}-1}}, t>1$

$\int dv = \int \frac{3\,dt}{t\sqrt{t^{v}-1}}$

or, $v = 3\sec t + c$.

$\therefore v = 3\sec^{-1}t + c$

Now, $v(2) = 0$

or, $3\sec^{-1}2 + c = 0$

$\therefore c = -\pi$.

$\therefore v = 3\sec^{-1}t - \pi$.

104. $\frac{dv}{dt} = \frac{8}{1+t^{v}} + \sec^{2}t$

$\int dv = \int \left(\frac{8}{1+t^{v}} + \sec^{v}t\right)dt$

or, $v = 8\tan^{-1}t + \tan t + c$

$\therefore v = 8\tan^{-1}t + \tan t + c$.

Now, $v(0) = 1$

or, $c = 1$.

$\therefore c = 1$.

$\therefore v = 8\tan^{-1}t + \tan t + 1$.

105. $\frac{d^{2}y}{dx^{v}} = 2-6x$

$\int d^{v}y = \int (2-6x)\,dx^{v}$

or, $dy = (2x - 3x^{v} + c_1)\,dx$

$\therefore \frac{dy}{dx} = y' = 2x - 3x^{v} + c_1$

Now, $y'(0) = 4$

or, $c_1 = 4$.

$\therefore c_1 = 4$

$\therefore \frac{dy}{dx} = 2x - 3x^{v} + 4$

$\int dy = \int (2x - 3x^{v} + 4)\,dx$

or, $y = x^{v} - x^{3} + 4x + c$.

$\therefore y = x^{v} - x^{3} + 4x + c$.

Now, $y(0) = 1$

$\therefore c = 1$.

$\therefore y = x^{v} - x^{3} + 4x + 1$.

108. $\frac{d^2 s}{dt^2} = \frac{3t}{8}$

$\int d^2 s = \int \frac{dt^2 \cdot 3t}{8}$

or, $ds = \frac{3t^2}{16} dt + c_1$

$\therefore ds/dt = \frac{3t^2}{16} + c_1.$

Now, $\left.\frac{ds}{dt}\right|_{t=4} = 3$

or, $\frac{3 \cdot 16}{16} + c_1 = 3$

$\therefore c_1 = 0$

$\therefore \frac{ds}{dt} = \frac{3t^2}{16}$

or, $\int ds = \int \frac{3t^2}{16} dt$

or, $s = \frac{t^3}{16} + c.$

$\therefore s = \frac{t^3}{16} + c.$

Now, $s(4) = 4$

or, $4 + c = 4$

$\therefore c = 0$

$\therefore s = t^3/16.$

109. $\frac{d^3 y}{dx^3} = 6$

or, $\int d^3 y = \int dx^3 \cdot 6$

or, $d^2 y = (6x + c_2) dx^2$

$\therefore \frac{d^2 y}{dx^2} = 6x + 62$

Now, $y''(0) = -8$

or, $c_2 = -8.$

$\therefore c_2 = -8.$

$\therefore \frac{d^2 y}{dx^2} = 6x - 8.$

or, $\int d^2 y = (6x - 8) dx^2$

or, $dy = (3x^2 - 8x + c_1) dx$

$\therefore \frac{dy}{dx} = 3x^2 - 8x + c_1.$

Now, $y'(0) = 0$

$\therefore c_1 = 0$

$\therefore \int dy = \int (3x^2 - 8x + 0) dx$

$\therefore y = x^3 - 4x^2 + c$

Now, $y(0) = 5.$

or, $c = 5.$

$\therefore y = x^3 - 4x^2 + 5.$

Date..........

110. $\frac{d^3\theta}{dt^3} = 0$

or, $\int d^3\theta = \int 0 \cdot dt^3$

$\therefore d^2\theta = c_2 dt^2$.

$c_2 = -2$.

$\therefore d^2\theta = -2dt^2$

or, $\int d^2\theta = \int -2dt^2 dt^2$

or, $d\theta = (2t + c_1)dt$

$\therefore d\theta = (-2t + c_1)dt$

Now, $\theta'(0) = \frac{-1}{2}$

$c_1 = -\frac{1}{2}$

$\therefore d\theta = (-2t - \frac{1}{2})dt$

or, $\int d\theta = \int(-2t - \frac{1}{2})dt$

or, $\theta = -t^2 - \frac{1}{2}t + c$.

$\therefore \theta = -t^2 - \frac{1}{2}t + c$.

Now, $c = \sqrt{2}$

$\therefore \theta = -t^2 - \frac{1}{2}t + \sqrt{2}$.

111. $y^4 = -\sin t + \cos t$

or, $y^3 = [(\cos t + \sin t) + c_3] dt^3$

Now, $y''' = 0$

$\therefore c_3 = -1$.

$\therefore y^3 = \cos t + \sin t - 1$.

or, $y^2 = \sin t - \cos t - t + c_2$

Now, $y'' = -1$

or, $-1 + c_2 = -1$

$\therefore c_2 = 0$

$\therefore y^2 = \sin t - \cos t - t + 0$

or, $y = -\cos t - \sin t - t^2/2 + c$.

$\therefore y = -\cos t - \sin t - \frac{t^2}{2} + c$.

Now,

$y(0) = 0$

or, $-1 + c = 0$

$\therefore c = 1$.

$\therefore y = -\cos t - \sin t - \frac{t^2}{2} + 1$.

Chapter 5.5

2. $\int 7\sqrt{7x-1}\, dx$

Let, $7x-1=x^2$

or, $7dx=2x\,dz$

or, $2\int\sqrt{z^2}\,dz\cdot z$

$=2\int z^2$

$=\frac{2}{3}z^3+c$

$=\frac{2}{3}\sqrt{(7x-1)}^3+c$

$=\frac{2}{3}\left(\sqrt{7x-1}\right)^3+c.$

7. $\int \sin 3x\,dx$

$=\frac{1}{3}-\cos 3x+c$

$=-\frac{1}{3}\cos 3x+c$

14. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$

Let, $-\frac{1}{x}=z$

or, $+\frac{1}{x^2}dx=dz.$

$=\int \cos^2 z\,dz$

$=\frac{1}{2}\int(1+\cos 2z)dz$

$=\frac{1}{2}z+\frac{1}{4}\sin 2z+c$

$=\frac{1}{2}\left(-\frac{1}{x}\right)+\frac{1}{4}\sin\left(\frac{-2}{x}\right)+c.$

16. $\int \frac{dx}{\sqrt{5x+8}}$

Let,
$5x+8 = z^2$
or, $5dx = 2z\,dz$
or, $dx = \frac{2}{5} z \cdot dz$

$= \int \frac{2}{5} \cdot \frac{z\,dz}{\sqrt{z^2}}$

$= \frac{2}{5} \int dz$

$= \frac{2}{5} z + c$

$= \frac{2}{5} \sqrt{5x+8} + c.$

19. $\int \theta \sqrt[4]{1-\theta^2}\, d\theta$

Let,
$1-\theta^2 = z^4$
or, $-2\theta\, d\theta = z^3 . 4 . dz$
$\therefore \theta\, d\theta = 2z^3 dz\ (-1)$

$= -2\int z^3 . z\, dz$

$= -2\int z^4 dz$

$= -2 \cdot \frac{1}{5} z^5 + c$

$= -\frac{2}{5}\left(\sqrt[4]{1-\theta^2}\right)^5 + c.$

38. $\int \sqrt{\frac{x-1}{x^5}}\, dx$

$= \int \frac{\sqrt{x-1}}{x^2\sqrt{x}}\, dx$

$= \int \frac{1}{x^2} \cdot \sqrt{\frac{x-1}{x}}\, dx$

$= \int \frac{1}{x^2} \cdot \sqrt{1-\frac{1}{x}}\, dx$

$= 2\int z \cdot z\, dz$

$= 2\int z^2 dz$

$= \frac{2}{3} z^3 + c$

$= \frac{2}{3}\left(\sqrt{1-\frac{1}{x}}\right)^3 + c.$

Let,

$1-\frac{1}{x} = z^2$

or, $0+\frac{1}{x^2}dx = dz \cdot z \cdot 2$

57. $\int \frac{dz}{1+e^z}$

$= \int \frac{dz}{e^z(1+e^{-z})}$

$= \int \frac{e^{-z}dz}{1+e^{-z}}$

$= -\int \frac{dx}{x}$

$= -\ln|x| + c$

$= -\ln|1+e^{-z}| + c.$

Let,

$1+e^{-z} = x$

or, $-e^{-z}dz = dx$

Date

58. $\int \frac{dx}{x\sqrt{x^4-1}}$

$= \int \frac{dx}{x \cdot x^2 \sqrt{1-x^{-4}}}$

$= \int \frac{x^{-3}\,dx}{\sqrt{1-x^{-4}}}$

$= \frac{1}{2}\int \frac{z \cdot dz}{z}$

$= \frac{1}{2}\int dz$

$= \frac{1}{2} z + c$

$= \frac{1}{2}\sqrt{1-x^{-4}} + c.$

Let,

$\sqrt{1-x^{-4}} = z$

or, $0 + 4 \cdot x^{-3} dx = 2z\,dz$

or, $x^{-3} dx = \frac{1}{2} z\,dz$

$\int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{x\,dx}{x^2\sqrt{(x^2)^2-1}}$

$= \frac{1}{2}\int \frac{dz}{z\sqrt{z^2-1}}$

$= \frac{1}{2}\sec^{-1}|z| + c$

$= \frac{1}{2}\sec^{-1}(x^2) + c.$

Let,

$x^2 = z$

or, $2x\,dx = dz$

$\therefore x\,dx = \frac{1}{2} dz$

Ch. 8.1 (Integration by Parts)

Random:-

(i) $\int x \ln x \, dx$

$= \frac{1}{x}\int x\,dx - \int\left(\frac{d}{dx}\ln x\right)\int x\,dx\,)dx$

$= \ln x \cdot \frac{x^2}{2} - \frac{1}{2}\int x\,dx$

$= \ln x \cdot \frac{x^2}{2} - \frac{1}{4}x^2 + C$

$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + C.$

(ii) $\int x \sin^{-1}x \, dx$

$= \sin^{-1}x \int x\,dx - \int\left(\frac{d}{dx}\sin^{-1}x \int x\,dx\right)dx$

$= \frac{x^2}{2}\sin^{-1}x - \int\left(\frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2}\right)dx$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{(\sqrt{1-x^2})^2}{\sqrt{1-x^2}}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \sqrt{1-x^2}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\frac{x\sqrt{1-x^2}}{2} - \frac{1}{2}\sin^{-1}x\right] - \frac{1}{2}\sin^{-1}x + C$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + C$

$\left[\int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C\right]$

(iii) $\int x\tan^{-1}x\,dx$

$= \cancel{x}\int \tan^{-1}x\,\cancel{dx}\int x\,dx - \int\left(\frac{d}{dx}\tan^{-1}x\int x\,dx\right)dx$

$= \tan^{-1}x\cdot\frac{x^2}{2} - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$

$= \tan^{-1}x\cdot\frac{x^2}{2} - \frac{1}{2}\int\frac{1+x^2-1}{1+x^2}\,dx$

$= \tan^{-1}x\cdot\frac{x^2}{2} - \frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + c.$

(iv) $\int\frac{x^3}{1+x^2}\,dx$

Let,
$1+x^2 = z$
or, $2x\,dx = dz$
or, $x\,dx = \frac{1}{2}dz$

$= \frac{1}{2}\int\frac{(z-1)}{z}\,dz$

$= \frac{1}{2}\int\left(1-\frac{1}{z}\right)dz$

$= \frac{1}{2}z - \frac{1}{2}\ln|z| + c$

$= \frac{1}{2}(1+x^2) - \frac{1}{2}\ln|1+x^2| + c.$

Ch 8.2

(v) 28. $\int \ln(x+x^2)\,dx$

$$= \ln(x+x^2)\cdot\int dx - \int\left(\frac{d}{dx}\ln(x+x^2)\int dx\right)dx$$

$$= \ln(x+x^2)\cdot x - \int\left(\frac{2x+1}{x+x^2}\cdot x\right)dx$$

$$= \ln(x+x^2)\cdot x - \int \frac{2x^2+x}{x^2+x}\,dx$$

$$= x\ln(x+x^2) - \int \frac{2x+1}{1+x}\,dx$$

$$= x\ln(x+x^2) - \int \frac{1+x+x}{1+x}\,dx$$

$$= x\ln(x+x^2) - \int \frac{1+x}{1+x}\,dx - \int \frac{x}{1+x}\,dx$$

Let, ~~1+x=z~~ or,

$$= x\ln(x+x^2) - x - \int \frac{1+x-1}{1+x}\,dx$$

$$= x\ln(x+x^2) - x - x + \ln|1+x| + c.$$

$$= x\ln(x+x^2) - 2x + \ln|1+x| + c.$$

(i) $\int e^{ax} \sin bx \, dx$

Let, $I = \int e^{ax} \sin bx \, dx$

$$= \sin bx \int e^{ax} dx - \int \left[\left(\frac{d}{dx} \sin bx \right) \int e^{ax} dx \right] dx$$

$$= \frac{1}{a} e^{ax} \sin bx \cancel{+} - \int \left(\frac{1}{b} \cos bx \cdot \frac{1}{a} e^{ax} \right) dx$$

$$= \frac{1}{a} e^{ax} \sin bx + \frac{b}{ab} \int \cos bx \, e^{ax} dx$$

$$= \frac{1}{a} e^{ax} . \sin bx + \frac{b}{ab} \left[\cos bx \cdot \frac{1}{a} e^{ax} - \int \left(\frac{d}{dx} \cos bx \int e^{ax} dx \right) dx \right]$$

$$= \frac{1}{a} e^{ax} . \sin bx - \frac{b}{ab} \left[\cos bx \cdot \frac{1}{a} e^{ax} + \frac{b}{ab} \int \sin bx \, e^{ax} \right]$$

$$= \frac{1}{a} e^{ax} . \sin bx - \frac{b}{a^2} e^{ax} . \cos bx - \frac{b^2}{a^2} I + c_1$$

$$\therefore I = \frac{a^2}{a^2+b^2} \frac{e^{ax}}{a^2} \left[a \sin bx - b \cos bx \right] + c$$

$$\therefore I = \frac{a^2}{a^2} \cdot \frac{e^{ax}}{a^2+b^2} \left(a \sin bx - b \cos bx \right) + c .$$

(ii) $\int e^{ax} \cos bx \, dx$

Let, $I = \int e^{ax} \cos bx \, dx$

$= \cos bx \cdot \frac{1}{a} \cdot e^{ax} - \int \left(\frac{d}{dx} \cos bx \int e^{ax} dx \right) dx$

$= \frac{1}{a} \cdot e^{ax} \cos bx + \frac{b}{a} \int \sin bx \cdot e^{ax} dx$

$= \frac{1}{a} \cdot e^{ax} \cos bx + \frac{b}{a} \left[\sin bx \cdot \frac{1}{a} \cdot e^{ax} - \int \left(\frac{d}{dx} \sin bx \int e^{ax} dx \right) dx \right]$

$= \frac{1}{a} \cdot e^{ax} \cos bx + \frac{b}{a} \left[\frac{e^{ax}}{a} \sin bx - \frac{b}{a} I \right] + c_1$

$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} I + c_1$

or, $\left(1 + \frac{b^2}{a^2}\right) I = \frac{1}{a^2} e^{ax} \cdot \left(a \cos bx + \cancel{e^{ax}} b \sin bx\right) + c_1$

$\therefore I = \frac{e^{ax}}{a^2 + b^2} \left(a \cos bx + b \sin bx\right) + c$

Date

(iii) $\int e^{-ax} \sin bx \, dx$

Let, $I = \int e^{-ax} \sin bx \, dx$

$= \sin bx \, (-1) \cdot \frac{1}{a} \cdot e^{-ax} - \int \left(\frac{d}{dx} \sin bx \int e^{-ax} dx \right) dx$

$= -\frac{1}{a} e^{-ax} \sin bx + \frac{b}{a} \int \cos bx \cdot e^{-ax} \, dx$

$= -\frac{1}{a} e^{-ax} \sin bx + \frac{b}{a} \left[-\frac{1}{a} e^{-ax} . \cos bx - \int \left(\frac{d}{dx} \cos bx \int e^{-ax} dx \right) dx \right]$

$= -\frac{1}{a} e^{-ax} \sin bx + \frac{b}{a} \left[-\frac{1}{a} e^{-ax} \cos bx + \frac{(-b)}{a} I \right] + c_1$

$= -\frac{1}{a} e^{-ax} \sin bx - \frac{b}{a^2} e^{-ax} \cos bx - \frac{b^2}{a^2} I + c_1$

$\therefore$

$\left(1 + \frac{b^2}{a^2}\right) I = \frac{-1}{a^2} \cdot e^{-ax} (a \sin bx + b \cos bx)$

$\therefore I = \frac{-e^{-ax} \cancel{a^2}}{a^2 + b^2} (a \sin bx + b \cos bx).$

(iv) $\int e^{-ax} \cos bx \, dx$

Let, $I = \int e^{-ax} \cos bx \, dx$

$= \cos bx \, (-1) \frac{1}{a} \cdot e^{-ax} - \int \left(\frac{d}{dx} \cos bx \int e^{-ax} dx \right) dx$

$= -\cos bx \cdot \frac{1}{a} \cdot e^{-ax} - \frac{b}{a} \left[\int \sin bx \, e^{-ax} \, dx \right]$

$= -\cos bx \, \frac{1}{a} e^{-ax} - \frac{b}{a} \left[-\frac{1}{a} e^{-ax} \sin bx - \int \left(\frac{d}{dx} \sin bx \int e^{-ax} dx \right) dx \right.$

$= -\cos bx \, \frac{1}{a} e^{-ax} - \frac{b}{a} \left[-\frac{1}{a} e^{-ax} \sin bx + \frac{b}{a} \int I \right] + c_1$

$= -\cos bx \cdot \frac{1}{a} e^{-ax} + \frac{b}{a^2} e^{-ax} \sin bx - \frac{b^2}{a^2} I + c_1$

$\therefore \left(1 + \frac{b^2}{a^2}\right) I = \frac{-e^{-ax}}{a^2} (a \cos bx - b \sin bx) + c$

$\therefore I = \frac{-e^{-ax}}{a^2 + b^2} (a \cos bx - b \sin bx) + c.$

Reduction Formula

Date

(i) $I_n=\int \cos^n x\,dx = \dfrac{\cos^{n-1}x\,\sin x}{n} + \dfrac{n-1}{n}\int \cos^{n-2}x\,dx$

(ii) $I_n=\int \sin^n x\,dx = \dfrac{-\sin^{n-1}x\,\cos x}{n} + \dfrac{n-1}{n}\int \sin^{n-2}x\,dx$

Proof:-

(ii) $I_n=\int \sin^n dx$

$$=\int \sin^{n-1}x\,\sin x\,dx$$

$$=\sin^{n-1}x\int \sin x\,dx-\int\left(\frac{d}{dx}\sin^{n-1}x\int \sin x\,dx\right)dx$$

$$=\cancel{-\sin^{n-1}x\cos x+(n-1)\int(\cos^{n-1}x\cos x\,dx)}$$

$$=\cancel{-\sin^{n-1}x\cos x+(n-1)\left[\cos^{n-1}x\int\cos x\,dx-\int\frac{d}{dx}\cos^{n-1}\right.}$$

$$=-\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x\,.\cos x\cdot\cos x\,dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}.\cos^2 x\,dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int\left[\sin^{n-2}\left(1-\cancel{\cos^2 x}\,\sin^2 x\right)\right]dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int\left[\sin^{n-2}x-\sin^n x\right]dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x\,dx-(n-1)I_n+c_1$$

$$\therefore\ [1+(n-1)]I_n=-\sin^{n-1}x\cos x+(n-1)I_{n-2}$$

$$\therefore\ I_n=\frac{-\sin^{n-1}x\cos x}{n}+\frac{(n-1)}{n}I_{n-2}$$

(ii) $\int \cos^n x \, dx$

$$I_n = \int \cos^n x \, dx$$

$$= \int \cos^{n-1} x \cos x \, dx$$

$$= \cos^{n-1} x \int \cos x \, dx - \int \left[\frac{d}{dx} \cos^{n-1} x \int \cos x \, dx \right] dx$$

$$= \cos^{n-1} x \, (+1) \sin x + (n-1) \int \left[\cos^{n-2} x \cdot \sin x \cdot \sin x \right] dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x \, (1 - \cos^2 x) \, dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \left[\cos^{n-2} x - \cos^n x \right] dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) I_{n-2} - (n-1) I_n + c_1$$

$$\therefore (1+n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} + c_1.$$

$$\therefore I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} I_{n-2} + c.$$

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Application:-

$$I_4 = \int \sin^4 x \, dx$$

$$= \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} \int \sin^2 x \, dx$$

$$= \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} \left[\frac{-\sin x \cos x}{2} + \frac{1}{2} \int dx \right]$$

$$= \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} \left[\frac{-\sin x \cos x}{2} + \frac{1}{2} x \right] + c$$

$$= \frac{-\sin^3 x \cos x}{4} - \frac{3}{8} \sin x \cos x + \frac{1}{2} x + c.$$