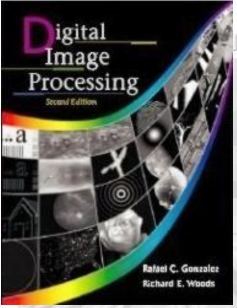


Chapter 4

Image Enhancement in the Frequency Domain

Md. Hasanul Kabir, Ph.D.

Professor

Department of CSE, IUT

Chapter 4
Image Enhancement in the Frequency Domain

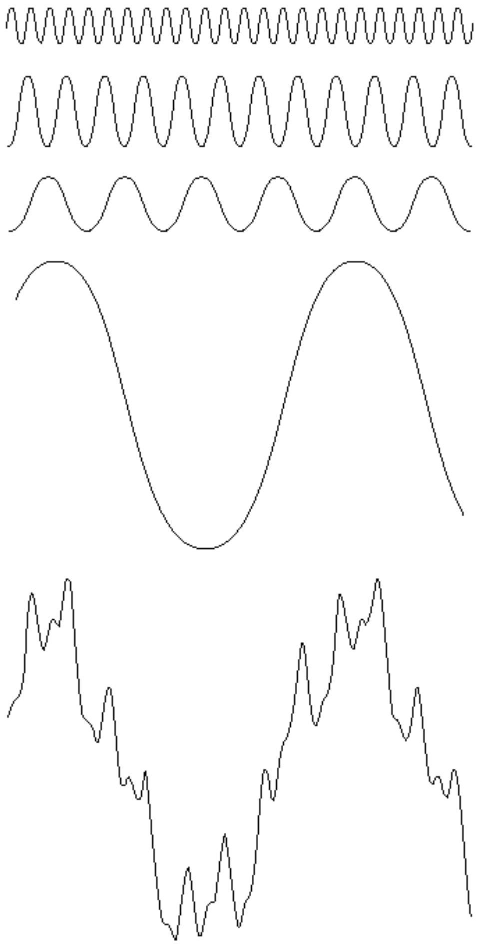
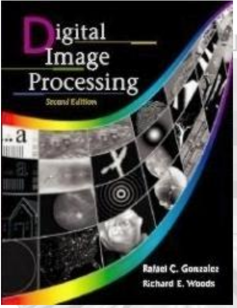


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.



Fundamentals

- Fourier: a periodic function can be represented by the sum of sines/cosines of different frequencies,multiplied by a different coefficient (Fourier series)
- Non-periodic functions can also be represented as the integral of sines/cosines multiplied by weighing function (Fourier transform)

Introduction to the Fourier Transform

- f(x): continuous function of a real variable x

- Fourier transform of f(x):

$$\Im\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x) \exp[-j2\pi ux]\,dx \qquad \text{Eq. 1}$$

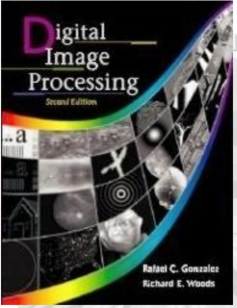
where $j = \sqrt{-1}$

Using Euler's formula,

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Introduction to the Fourier Transform

- (u) is the frequency variable.
- The integral of Eq. 1 shows that F(u) is composed of an infinite sum of sine and cosine terms and…
- Each value of u determines the frequency of its corresponding sine-cosine pair.

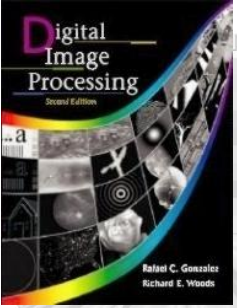


Introduction to the Fourier Transform

- Given F(u), f(x) can be obtained by the inverse Fourier transform:

$$\Im^{-1}\{F(u)\} = f(x)$$
$$= \int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]du$$

- The above two equations are the Fourier transform pair.



Introduction to the Fourier Transform

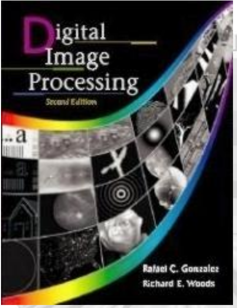
- Fourier transform pair for a function f(x,y) of two variables:

$$\Im\{f(x,y)\} = F(u,v) = \int_{-\infty}^{\infty}\int f(x,y)\exp[-j2\pi(ux+vy)]dxdy$$

and

$$\Im^{-1}\{F(u,v)\} = f(x,y) = \int_{-\infty}^{\infty}\int F(u,v)\exp[j2\pi(ux+vy)]dudv$$

where u,v are the frequency variables.



Introduction to the Fourier Transform

- The Fourier transform of a real function is generally complex and we use polar coordinates:

$$F(u) = R(u) + jI(u)$$

$$F(u) = |F(u)|e^{j\phi(u)}$$

$$|F(u)| = [R^2(u) + I^2(u)]^{1/2}$$

$$\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$

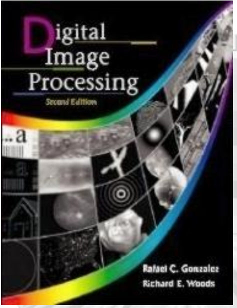
Introduction to the Fourier Transform

- |F(u)| (magnitude function) is the Fourier spectrum of f(x) and φ(u) its phase angle.

- The square of the spectrum

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

is referred to as the power spectrum of f(x) (spectral density).

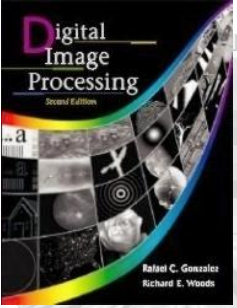


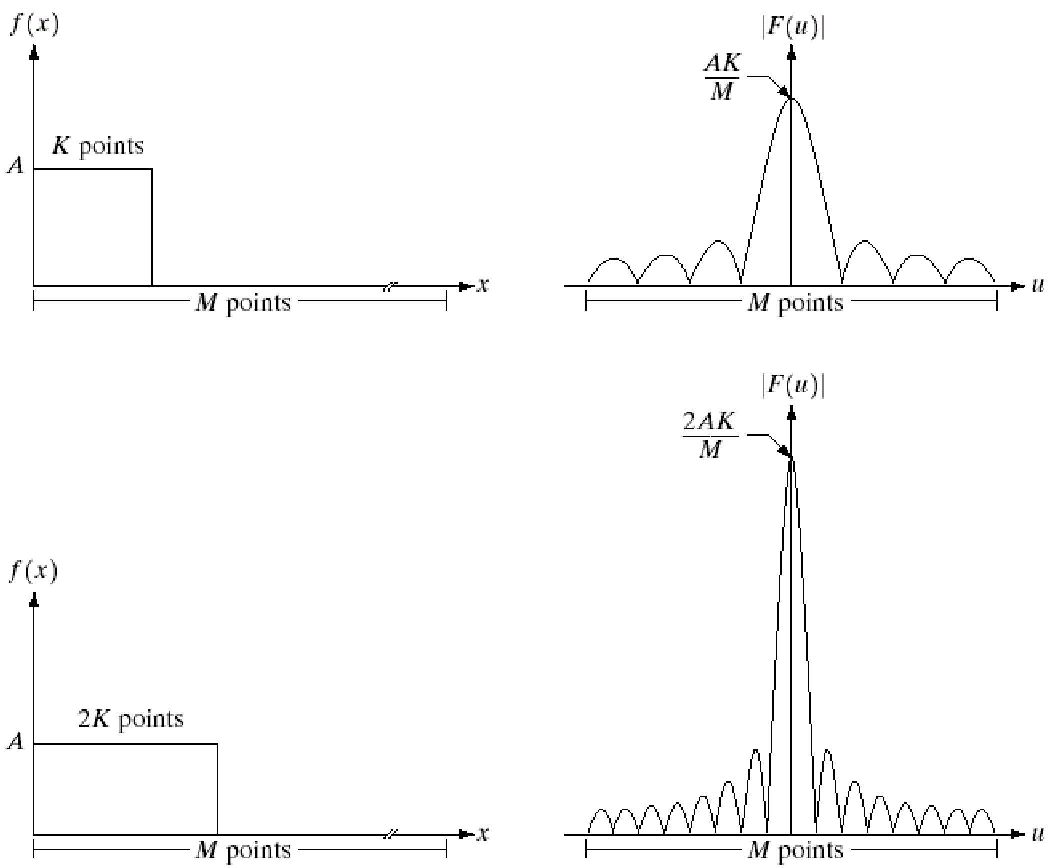
Introduction to the Fourier Transform

- Fourier spectrum: $|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$

- Phase: $\phi(u,v) = \tan^{-1}\left[\dfrac{I(u,v)}{R(u,v)}\right]$

- Power spectrum: $P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$





a b
c d

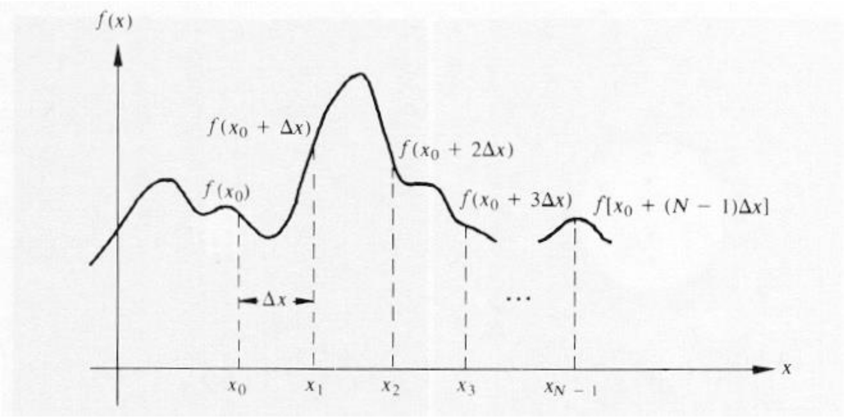
FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

Discrete Fourier Transform

- A continuous function f(x) is discretized into a sequence:

$$\{f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x), \ldots, f(x_0+[N-1]\Delta x)\}$$

by taking N or M samples Δx units apart.

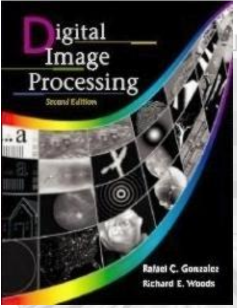


Discrete Fourier Transform

- Where x assumes the discrete values (0,1,2,3,…,M-1) then

$$f(x) = f(x_0 + x\Delta x)$$

- The sequence {f(0),f(1),f(2),…f(M-1)} denotes any M uniformly spaced samples from a corresponding continuous function.



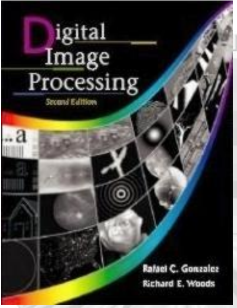
Discrete Fourier Transform

- The discrete Fourier transform pair that applies to sampled functions is given by:

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\exp[-j2\pi ux/M] \qquad \text{For } u=0,1,2,\ldots,M-1$$

and

$$f(x) = \sum_{u=0}^{M-1} \mathrm{F}(u)\exp[j2\pi ux/M] \qquad \text{For } x=0,1,2,\ldots,M-1$$



Discrete Fourier Transform

- In a 2-variable case, the discrete FT pair is:

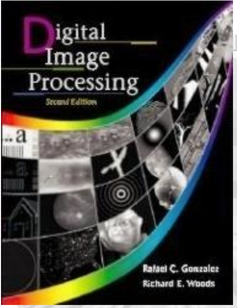
$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \exp[-j2\pi(ux/M + vy/N)]$$

For u=0,1,2,…,M-1 and v=0,1,2,…,N-1

AND:

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \exp[j2\pi(ux/M + vy/N)]$$

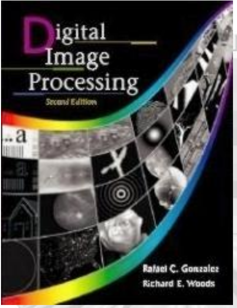
For x=0,1,2,…,M-1 and y=0,1,2,…,N-1



Periodicity & Conjugate Symmetry

- The discrete FT and its inverse are periodic with period N:

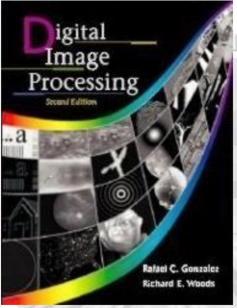
$$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$$



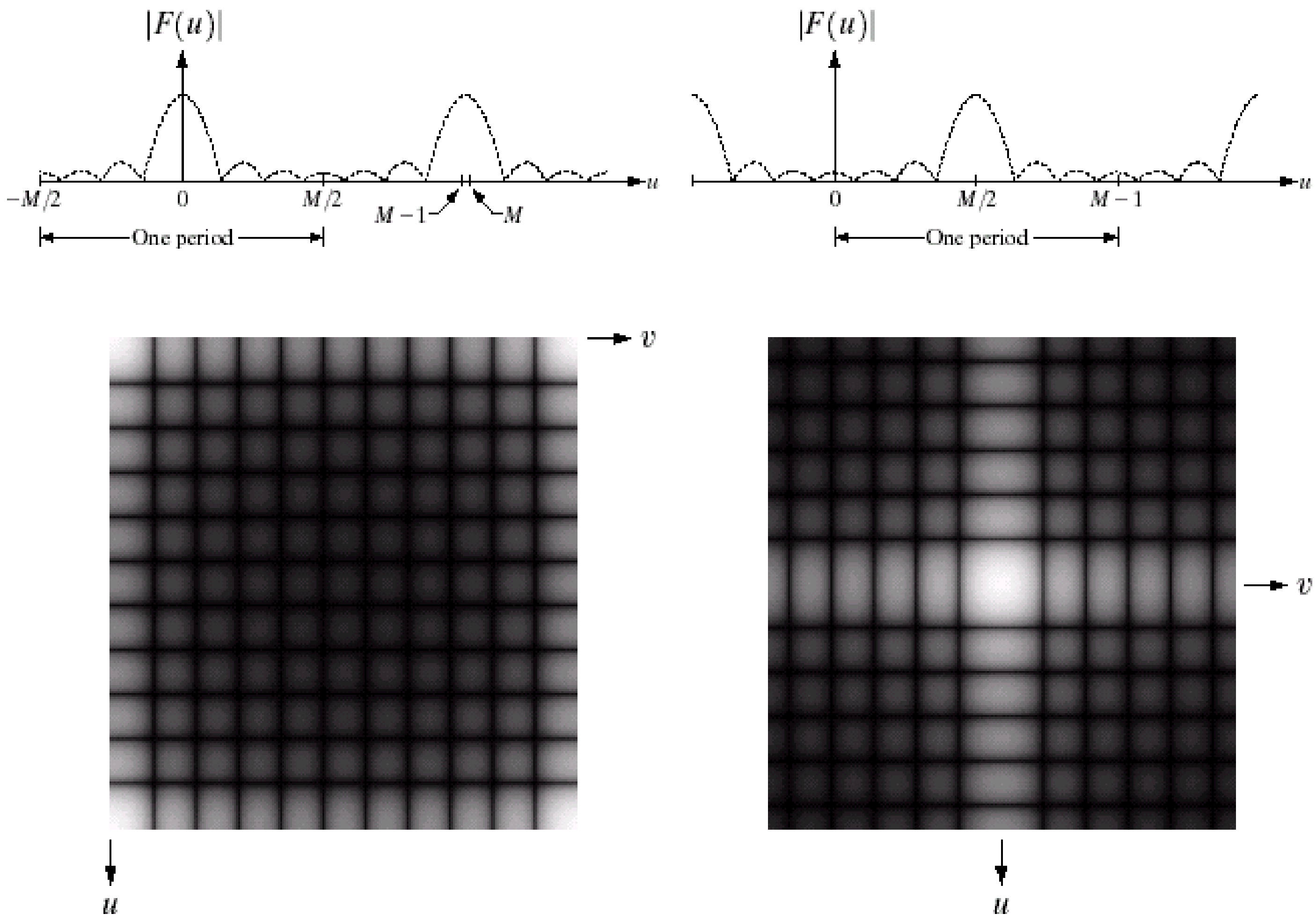
Periodicity & Conjugate Symmetry

- Although F(u,v) repeats itself for infinitely many values of u and v, only the M,N values of each variable in any one period are required to obtain f(x,y) from F(u,v).

- This means that only one period of the transform is necessary to specify F(u,v) completely in the frequency domain (and similarly f(x,y) in the spatial domain).



Periodicity & Conjugate Symmetry



a b
c d

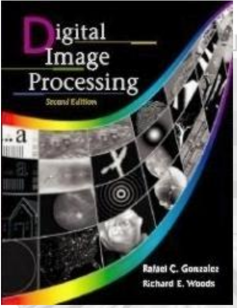
FIGURE 4.34
(a) Fourier spectrum showing back-to-back half periods in the interval $[0, M - 1]$.
(b) Shifted spectrum showing a full period in the same interval.
(c) Fourier spectrum of an image, showing the same back-to-back properties as (a), but in two dimensions.
(d) Centered Fourier spectrum.

Periodicity & Conjugate Symmetry

- For real f(x,y), FT also exhibits conjugate symmetry:

$$F(u,v) = F^*(-u,-v)$$

or $$|F(u,v)| = |F(-u,-v)|$$



Periodicity & Conjugate Symmetry

- In essence:

$$F(u) = F(u+N)$$

$$|F(u)| = |F(-u)|$$

- i.e. F(u) has a period of length N and the magnitude of the transform is centered on the origin.

Discrete Fourier Transform

- In a 2-variable case, the discrete FT pair is:

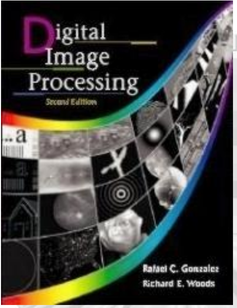
$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux/M + vy/N)]$$

For u=0,1,2,...,M-1 and v=0,1,2,...,N-1

AND:

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\exp[j2\pi(ux/M + vy/N)]$$

For x=0,1,2,...,M-1 and y=0,1,2,...,N-1



Basic Properties

- Common practice:

$$\Im\left[f(x,y)(-1)^{x+y}\right] = F(u - M/2, v - N/2)$$

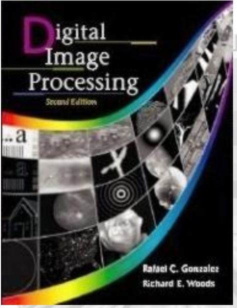
- F(0,0) is at u=M/2 and v=N/2
- Shifts the origin of F(u,v) to (M/2, N/2), i.e. the center of MxN of the 2-D DFT (frequency rectangle)
- Frequency rectangle:

  from u=0 to u=M-1, and v=0 to v=N-1

  (u,v integers, M,N even numbers)
- In computers:

  summations are from u=1 to M and v=1 to N

  center of transform: u=(M/2) +1 and v=(N/2) +1



Basic Properties

- Value of transform at (u,v)=(0,0):

$$F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

which means that the value of FT at the origin = the average gray level of the image

- FT is also conjugate summetric:

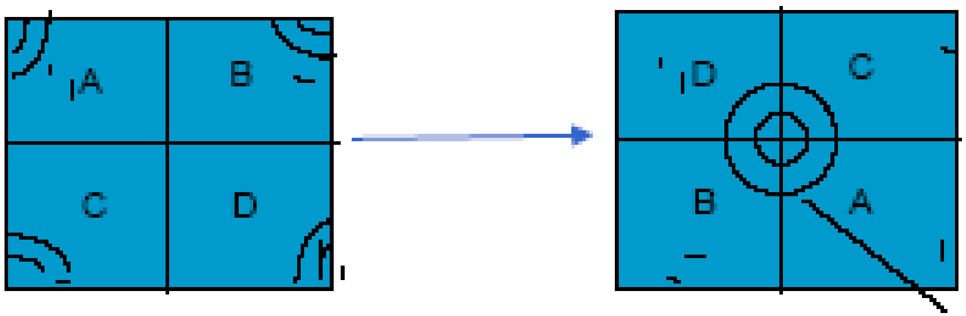
$$F(u,v)=F^*(-u,-v)$$

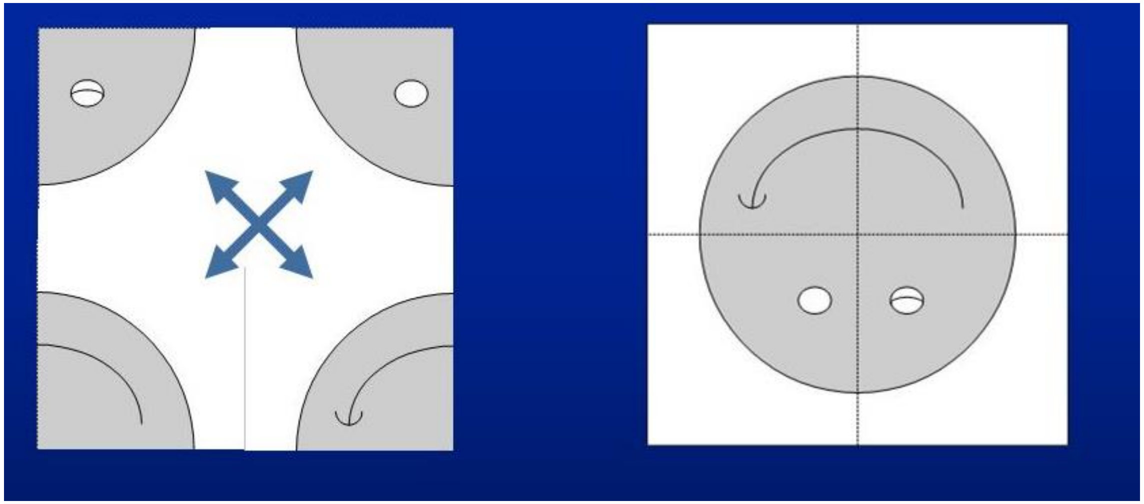
$$\text{so } |F(u,v)|=|F(-u,-v)|$$

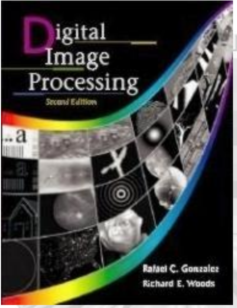
which means that the FT spectrum is symmetric.

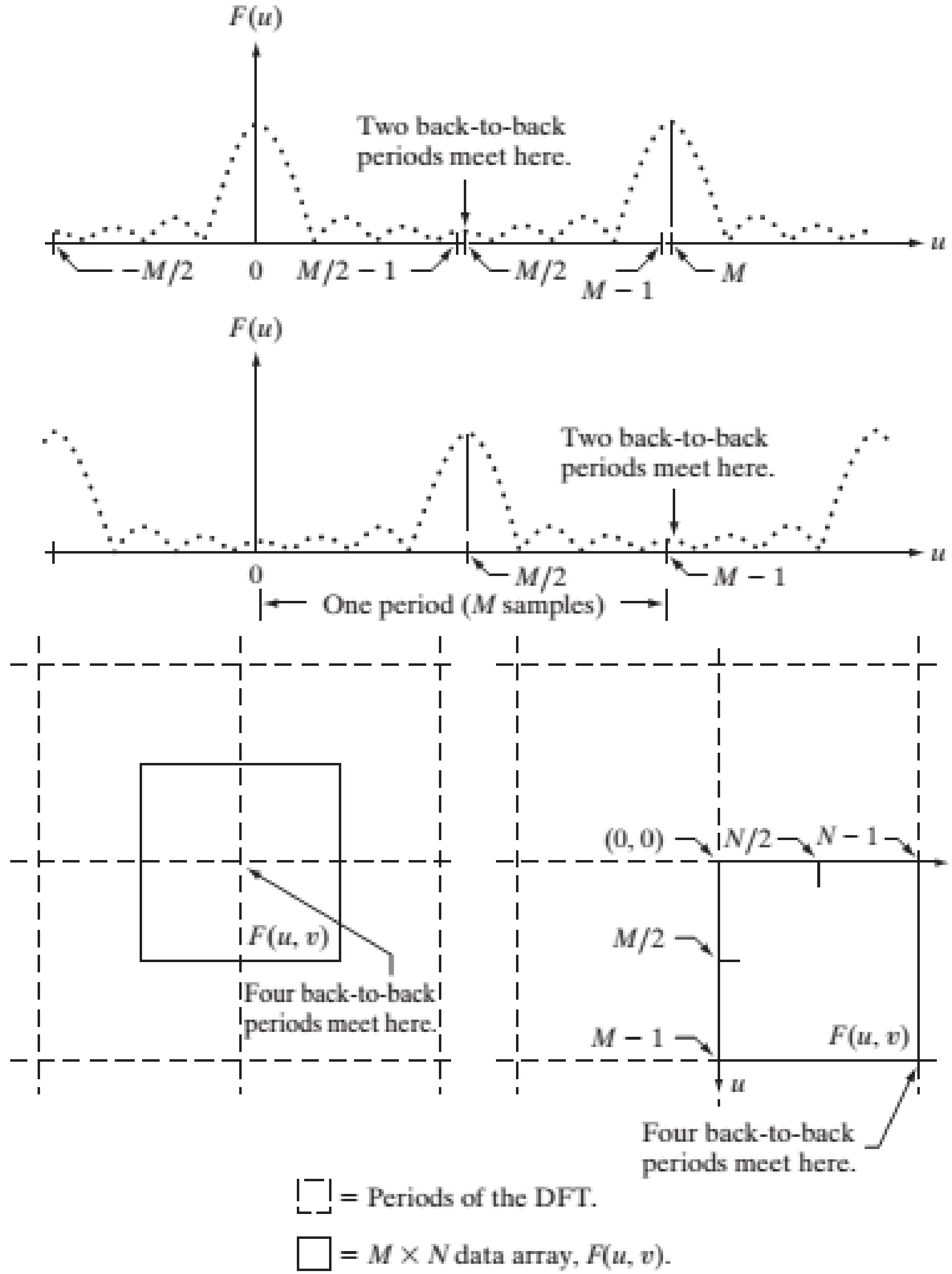
Displaying & Shifting

The origin of the F\{u(m,n)\} can be moved to the center of the array (N x N square) by first multiplying u(m,n) by $(-1)^{m+n}$ and then taking the Fourier transform.
Note: Shifting does not affect the magnitude of the Fourier transform.



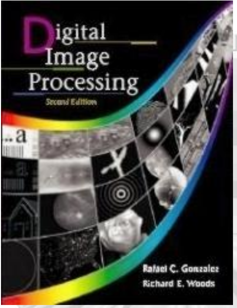






a
b
c d

FIGURE 4.23 Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The solid area is the $M \times N$ data array, $F(u, v)$, obtained with Eq. (4.5-15). This array consists of four quarter periods. (d) A Shifted DFT obtained by multiplying $f(x, y)$ by $(-1)^{x+y}$ before computing $F(u, v)$. The data now contains one complete, centered period, as in (b).



a b

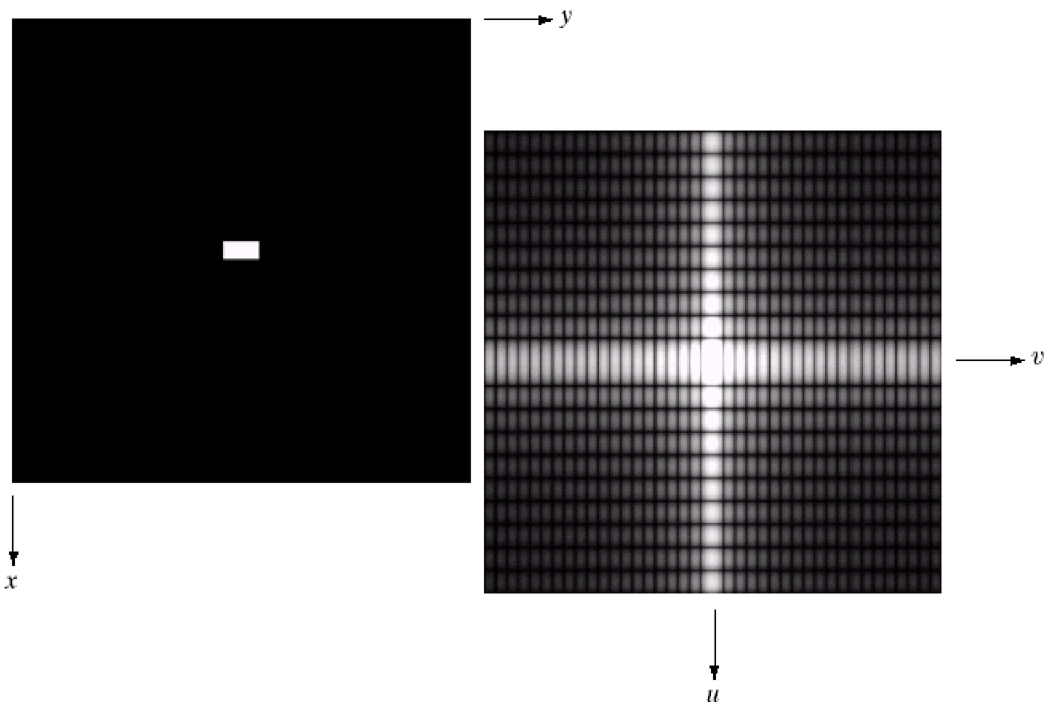
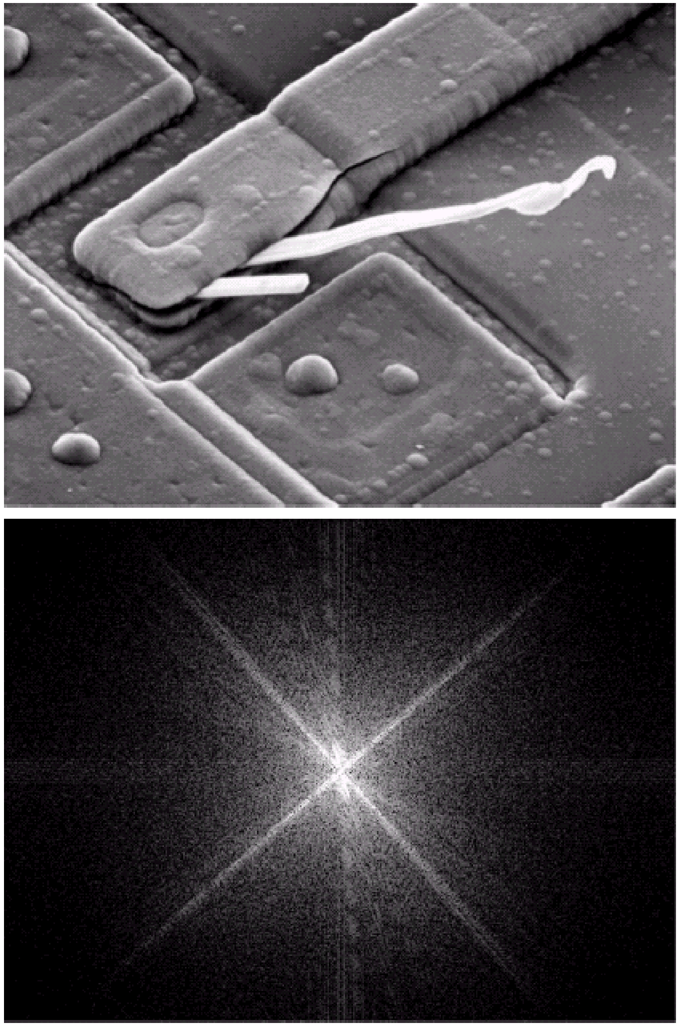
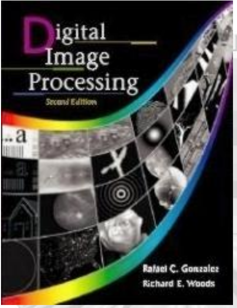


FIGURE 4.3 (a) Image of a 20×40 white rectangle on a black background of size 512×512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.



a
b

FIGURE 4.4
(a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)



Basic steps for filtering in the frequency domain

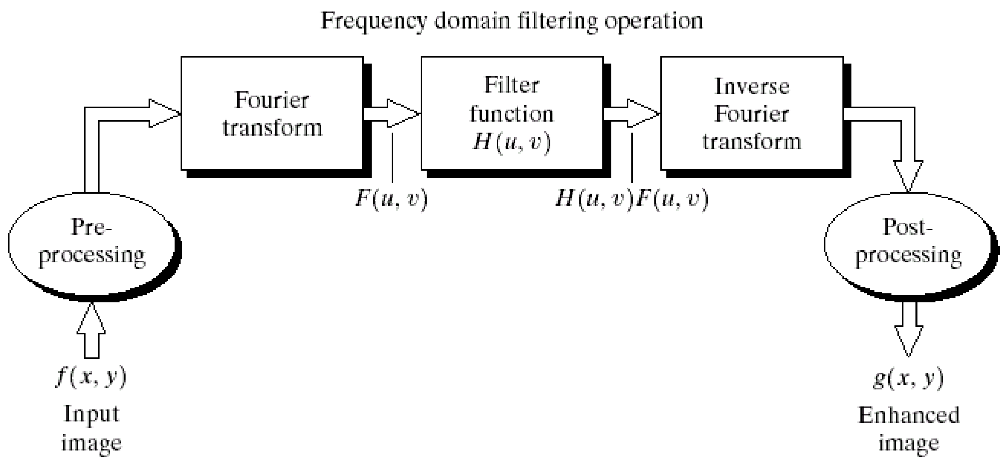


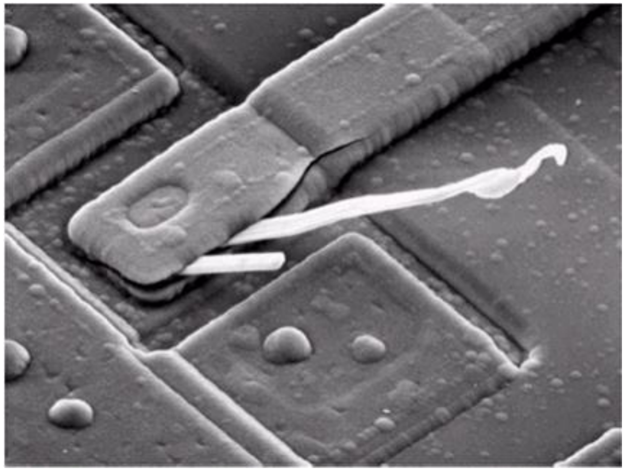
FIGURE 4.5 Basic steps for filtering in the frequency domain.

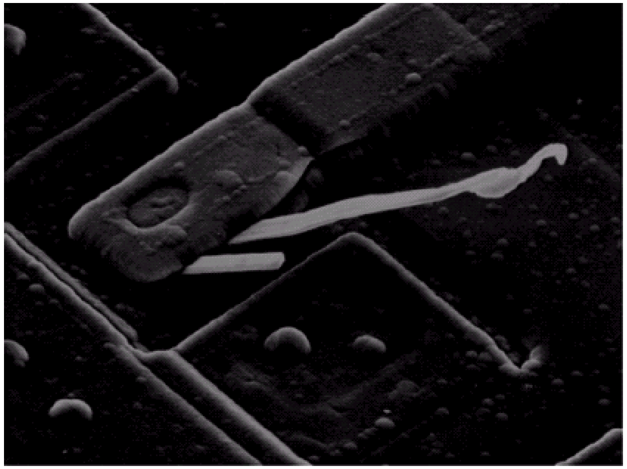
1. Multiply the input image by $(-1)^{x+y}$ to center the transform to
   u=M/2 and v=N/2 (M and N are even numbers)
2. Computer F(u,v), the DFT of the image form (1)
3. Multiply F(u,v) by a filter function H(u,v)
4. Compute the inverse DFT of the result in (3)
5. Obtain the real part of the result in (4)
6. Multiply the result in (5) by $(-1)^{x+y}$ to cancel the multiplication of the input image

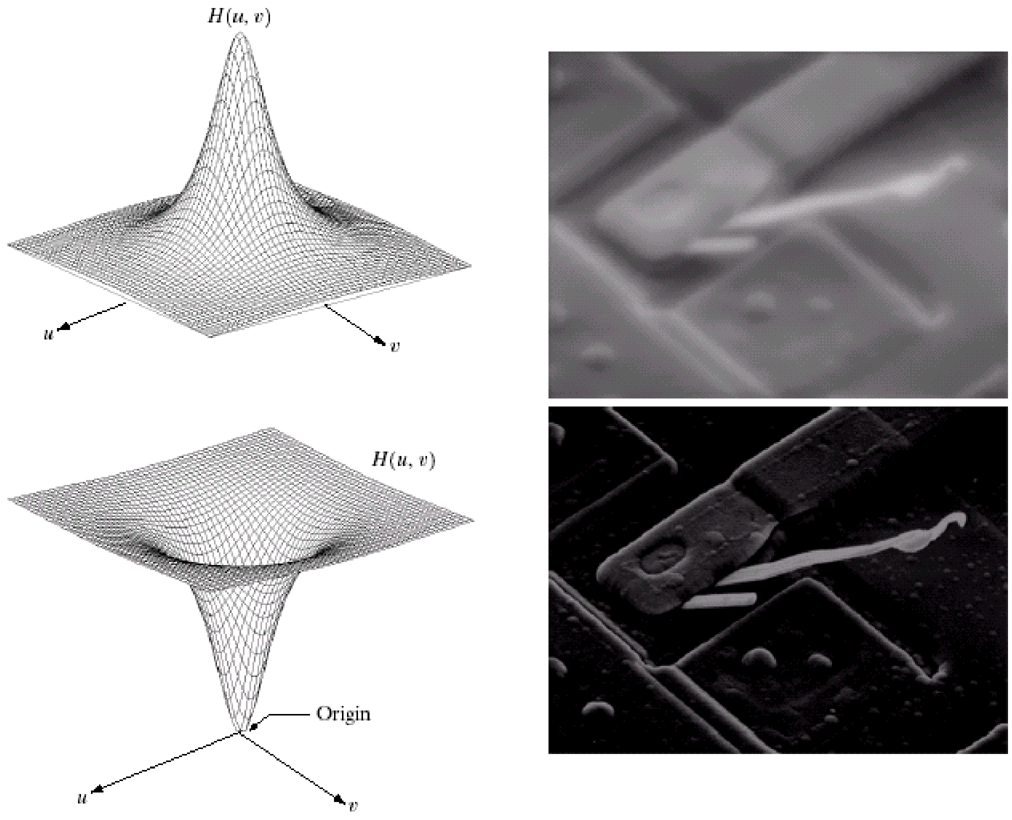
Notch filter

- this filter is to force the F(0,0) which is the average value of an image (dc component of the spectrum)
- the output has prominent edges
- in reality the average of the displayed image can't be zero as it needs to have negative gray levels. The output image needs to scale the gray level

$$H(u,v)=\begin{cases}0 & \text{if } (u,v)=(M/2,N/2)\\ 1 & \text{otherwise}\end{cases}$$







a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Lowpass filtering: reduce the high-frequency content

- blurring or smoothing

Highpass filtering: increase the magnitude of high-frequency components

- sharpening.

Add the ½ of filter height to F(0,0) of the high pass filter

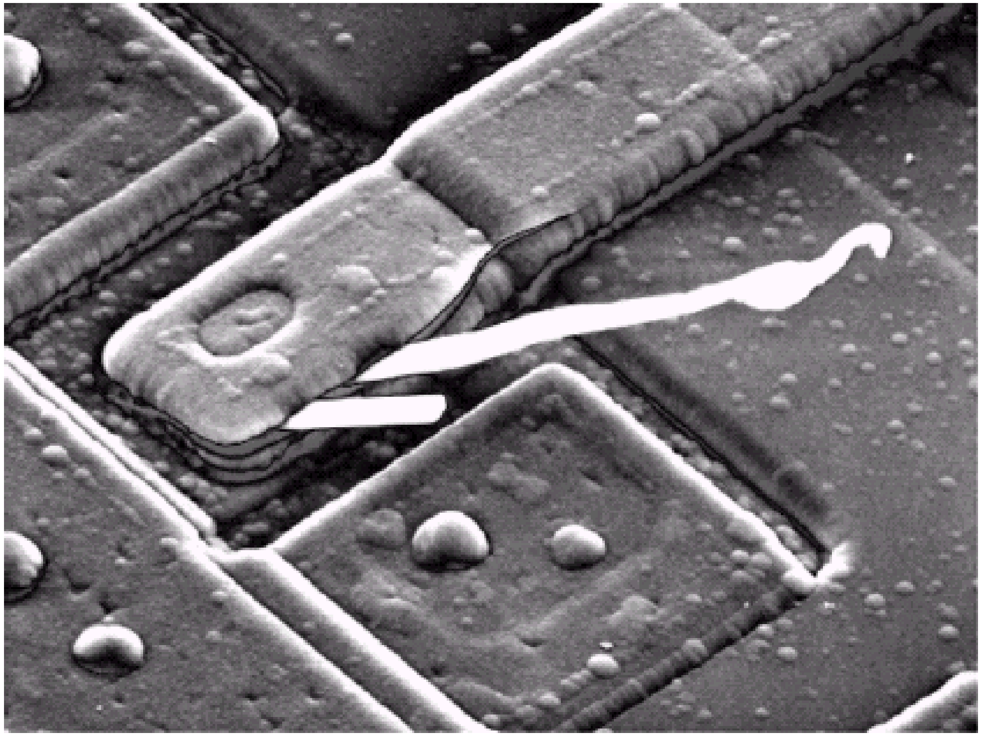
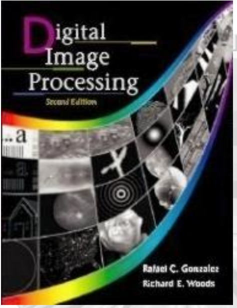


FIGURE 4.8 Result of highpass filtering the image in Fig. 4.4(a) with the filter in Fig. 4.7(c), modified by adding a constant of one-half the filter height to the filter function. Compare with Fig. 4.4(a).



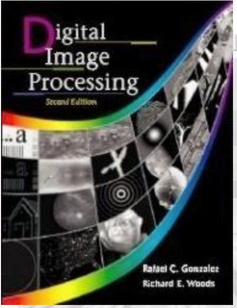
Spatial & Frequency Domain

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)\, H(u,v)$$

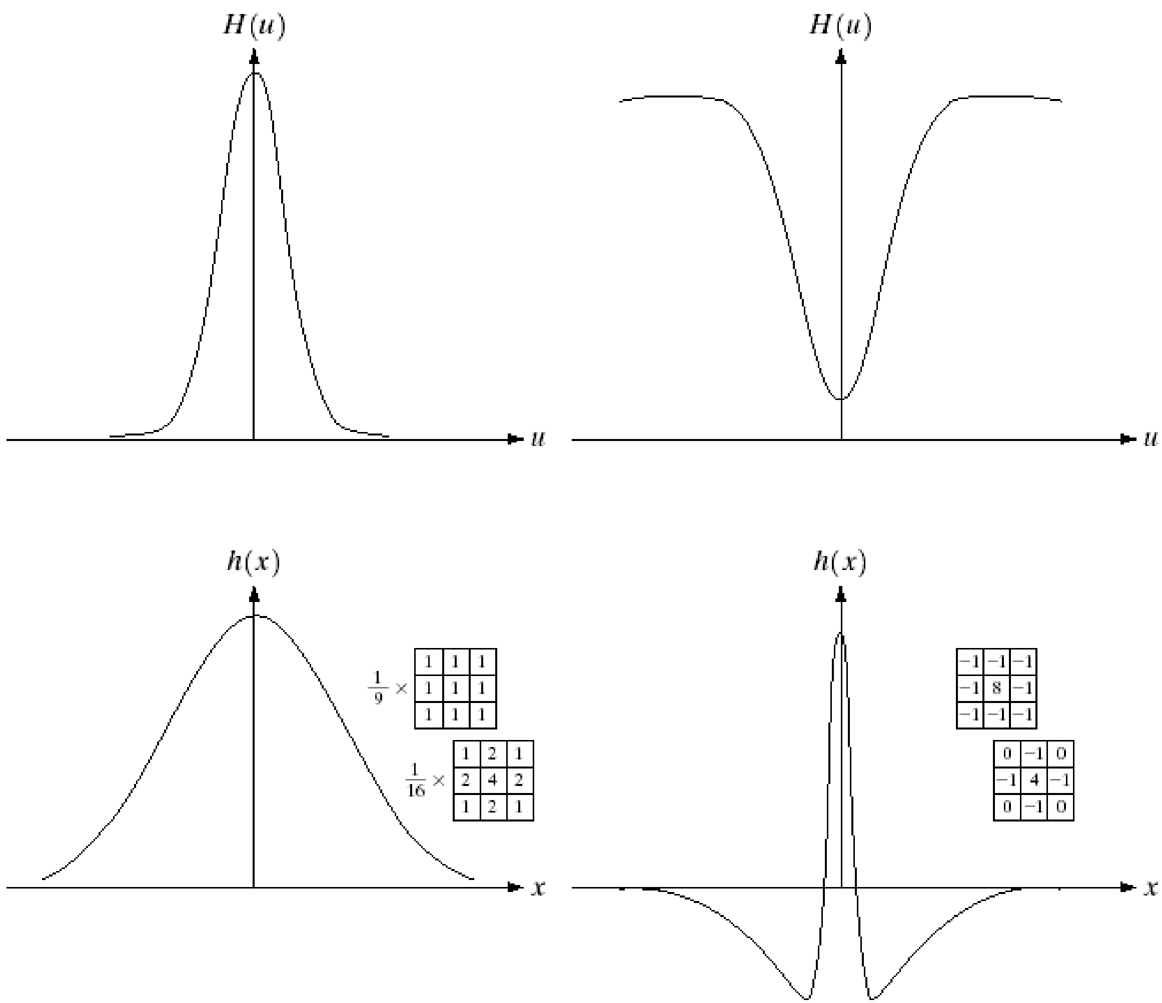
$$\delta(x,y) * h(x,y) \Leftrightarrow \Im[\delta(x,y)]\, H(u,v)$$

$$h(x,y) \Leftrightarrow H(u,v)$$

Filters in the spatial and frequency domain form a FT pair, i.e. given a filter in the frequency domain we can get the corresponding one in the spatial domain by taking its inverse FT

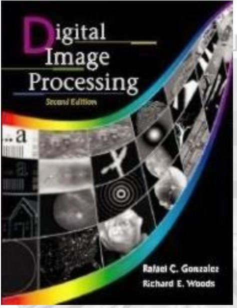


Correspondence between filter in spatial and frequency domains



| a | b |
| c | d |

FIGURE 4.9 (a) Gaussian frequency domain lowpass filter. (b) Gaussian frequency domain highpass filter. (c) Corresponding lowpass spatial filter. (d) Corresponding highpass spatial filter. The masks shown are used in Chapter 3 for lowpass and highpass filtering.



Lowpass Filtering in the Frequency Domain

- Edges, noise contribute significantly to the high-frequency content of the FT of an image.

- Blurring/smoothing is achieved by reducing a specified range of high-frequency components:

$$G(u,v) = H(u,v)F(u,v)$$

Smoothing in the Frequency Domain

$$G(u,v) = H(u,v)\, F(u,v)$$

- Ideal
- Butterworth (parameter: *filter order*)
- Gaussian

Ideal Filter (Lowpass)

- A 2-D ideal low-pass filter:

$$H(u,v) = \begin{cases} 1 \text{ if } & D(u,v) \le D_0 \\ 0 \text{ if } & D(u,v) > D_0 \end{cases}$$

where D_0 is a specified nonnegative quantity and D(u,v) is the distance from point (u,v) to the center of the frequency rectangle.

- Center of frequency rectangle: (u,v)=(M/2,N/2)
- Distance from any point to the center (origin) of the FT:

$$D(u,v) = (u^2 + v^2)^{1/2}$$

Smoothing Frequency-domain filters : Ideal Low pass filter

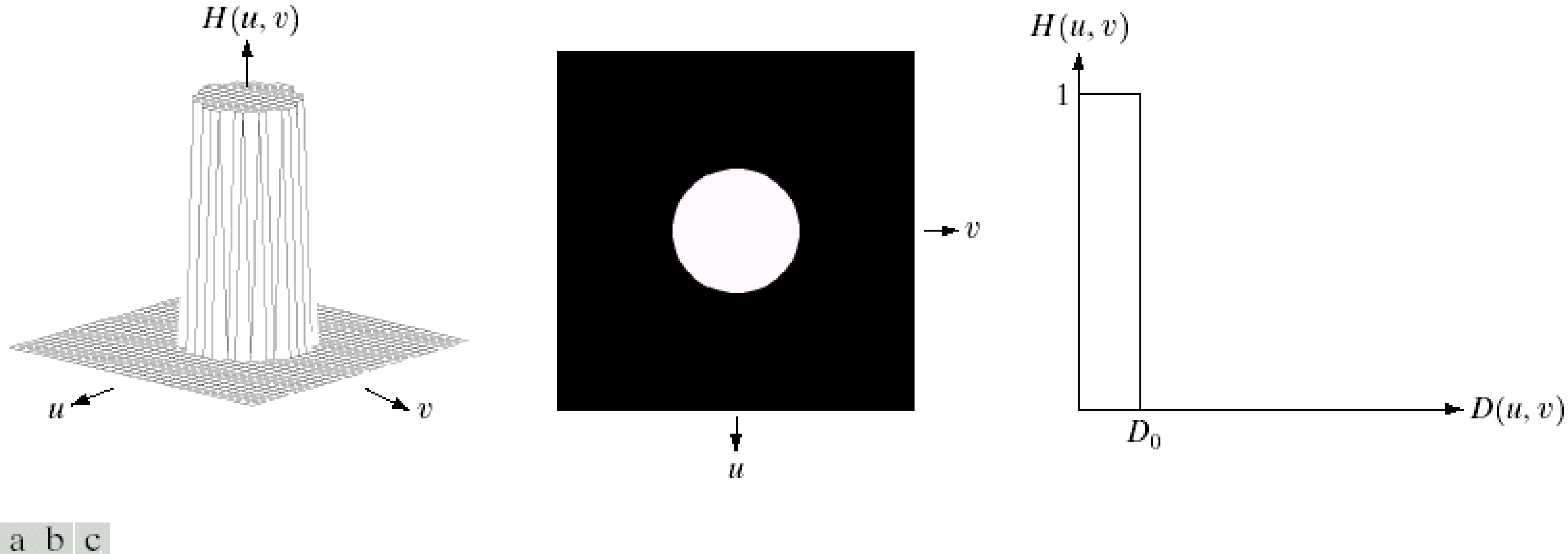


FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Ideal Filter (Lowpass)

- Ideal:
  - all frequencies inside a circle of radius D_0 are passed with no attenuation
  - all frequencies outside this circle are completely attenuated.

Ideal Filter (Lowpass)

- Cutoff-frequency: the point of transition between H(u,v)=1 and H(u,v)=0 (D_0)
- To establish cutoff frequency loci, we typically compute circles that enclose specified amounts of total image power P_T.

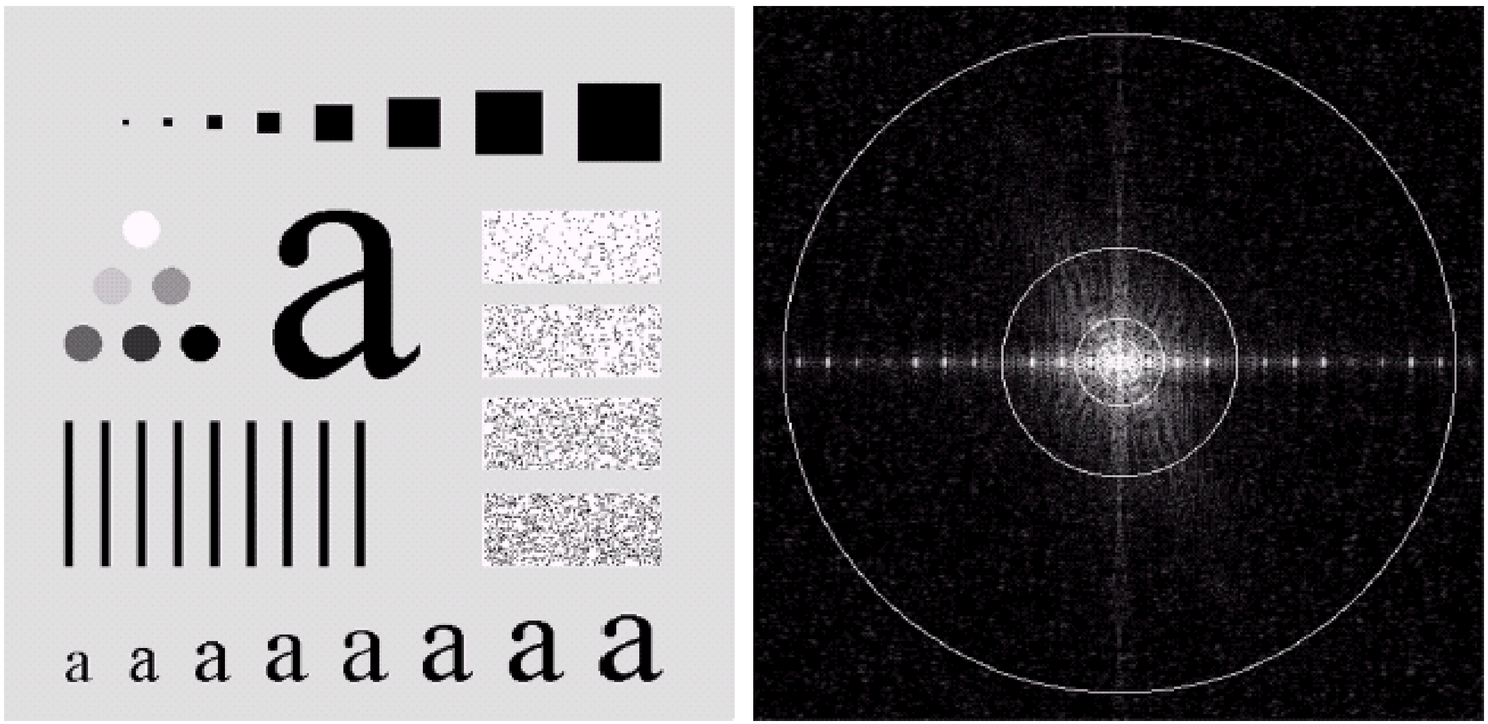
Ideal Filter *(cont.)*

- P_T is obtain by summing the components of power spectrum P(u,v) at each point for u up to M-1 and v up to N-1.
- A circle with radius *r*, origin at the center of the frequency rectangle encloses a percentage of the power which is given by the expression

$$100[\sum_{u}\sum_{v} P(u,v)/P_T$$

- The summation is taken within the circle *r*

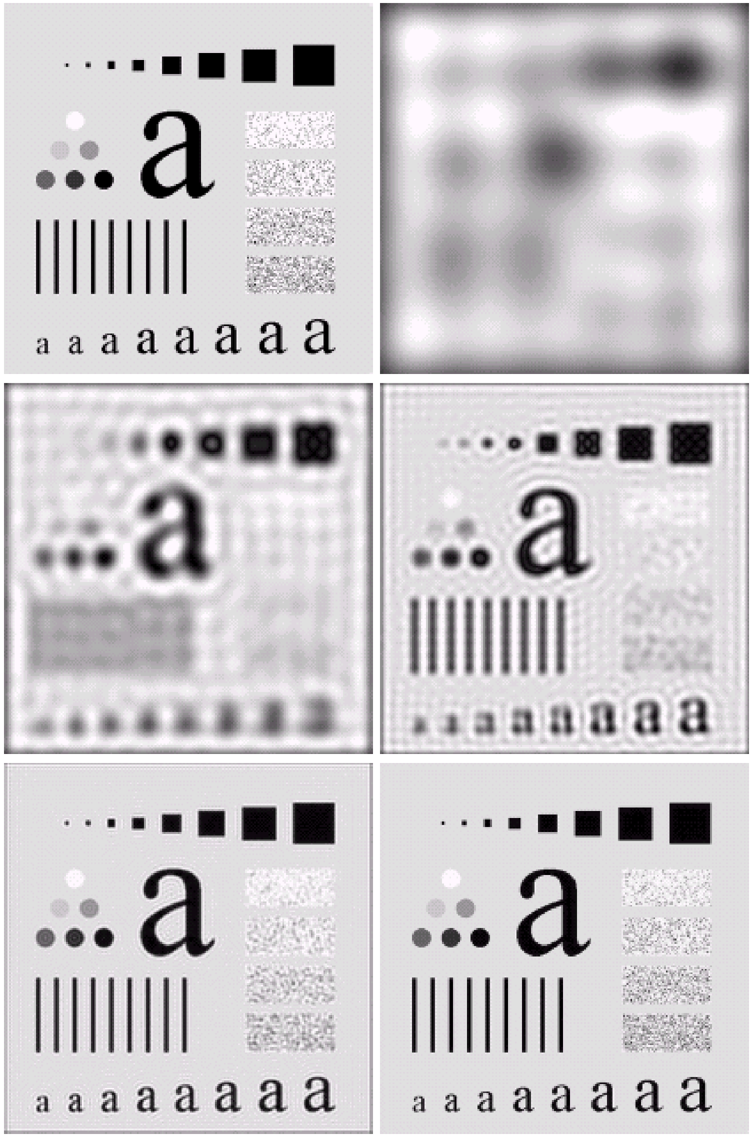
Image power circles



a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

Result of ILPF



a b
c d
e f

FIGURE 4.12 (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

Example

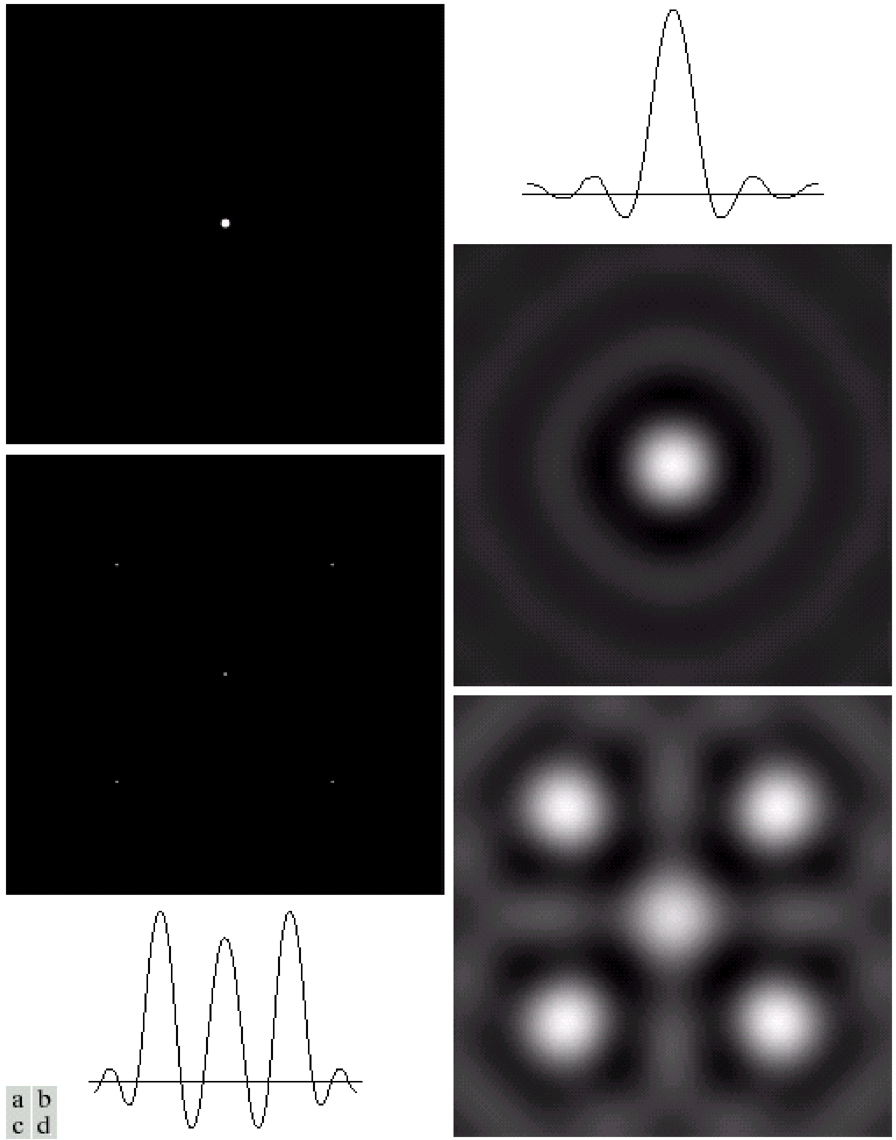


FIGURE 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

Butterworth Low pass Filter : BLPF

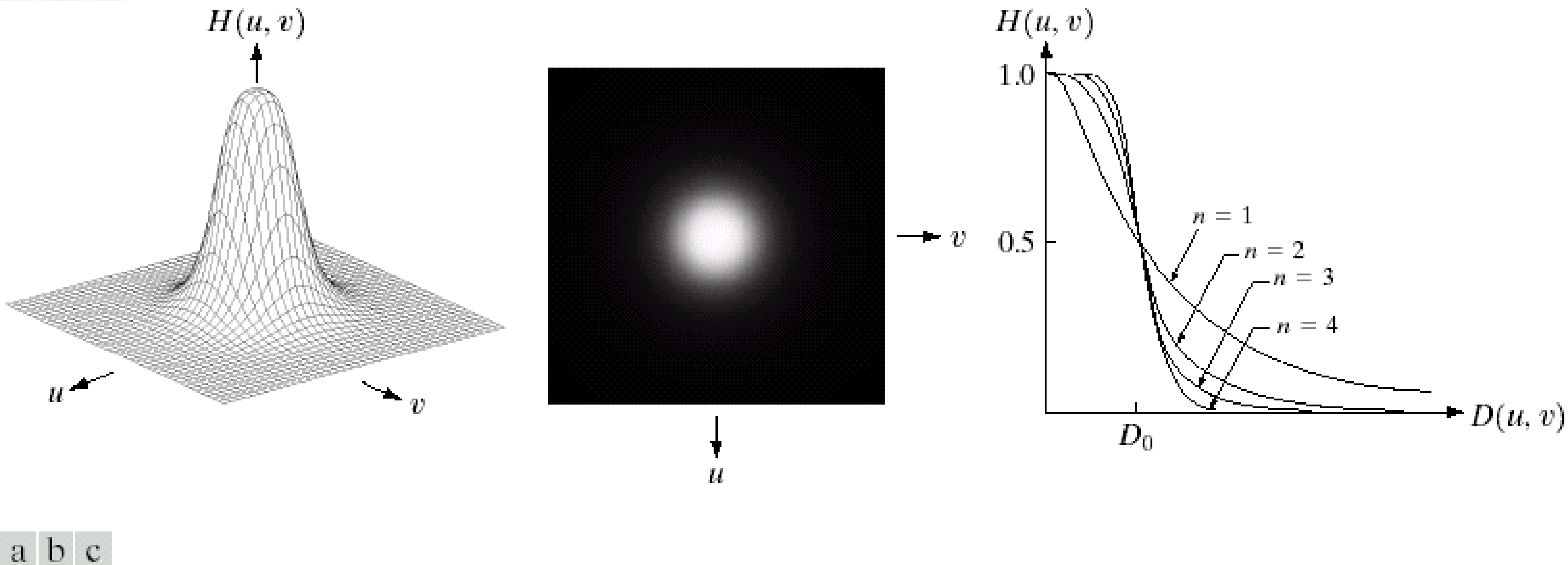
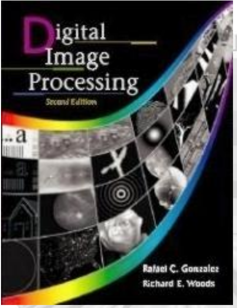


FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Butterworth Filter (Lowpass)

- This filter does not have a sharp discontinuity establishing a clear cutoff between passed and filtered frequencies.

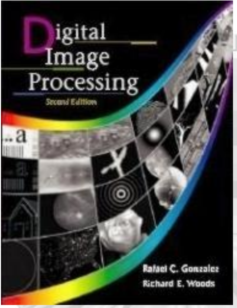
$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$$



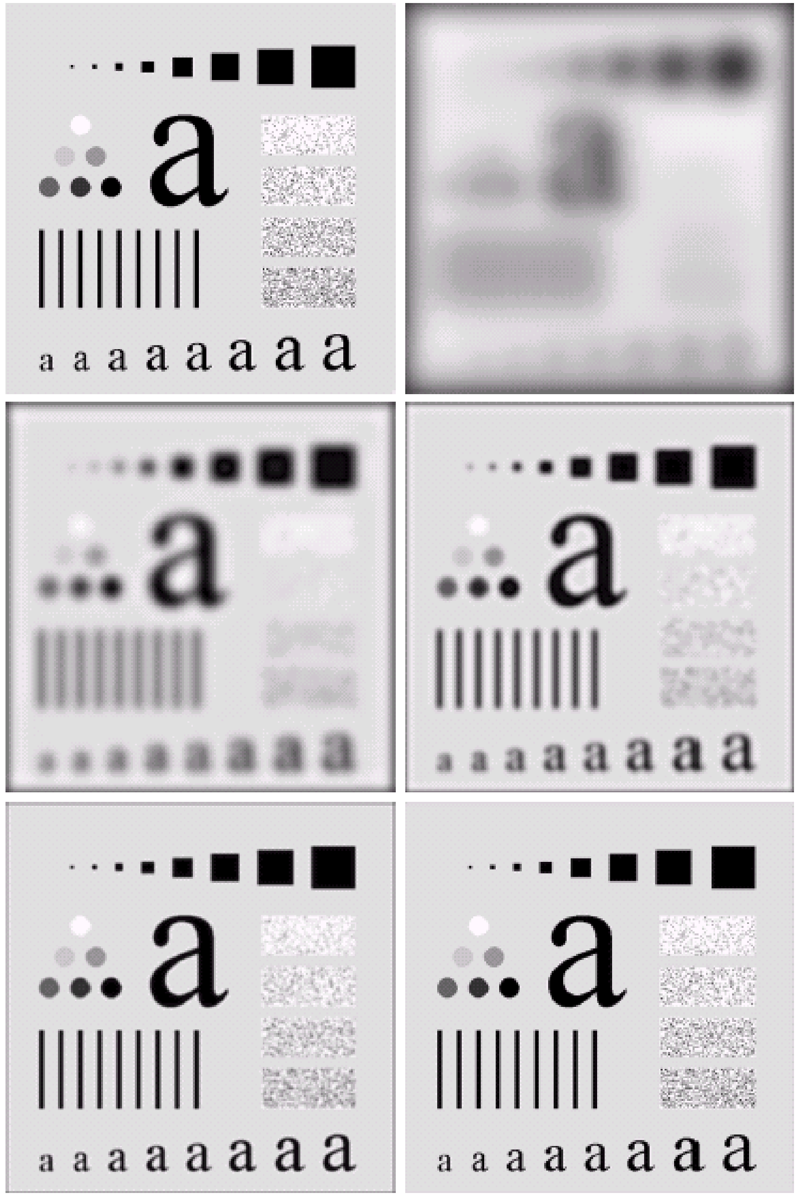
Butterworth Filter (Lowpass)

- To define a cutoff frequency locus: at points for which H(u,v) is down to a certain fraction of its maximum value.

- When $D(u,v) = D_0$, $H(u,v) = 0.5$
  - i.e. down 50% from its maximum value of 1.

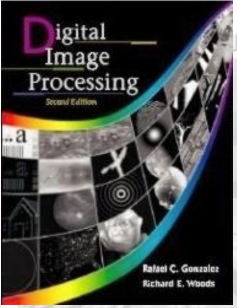


Example

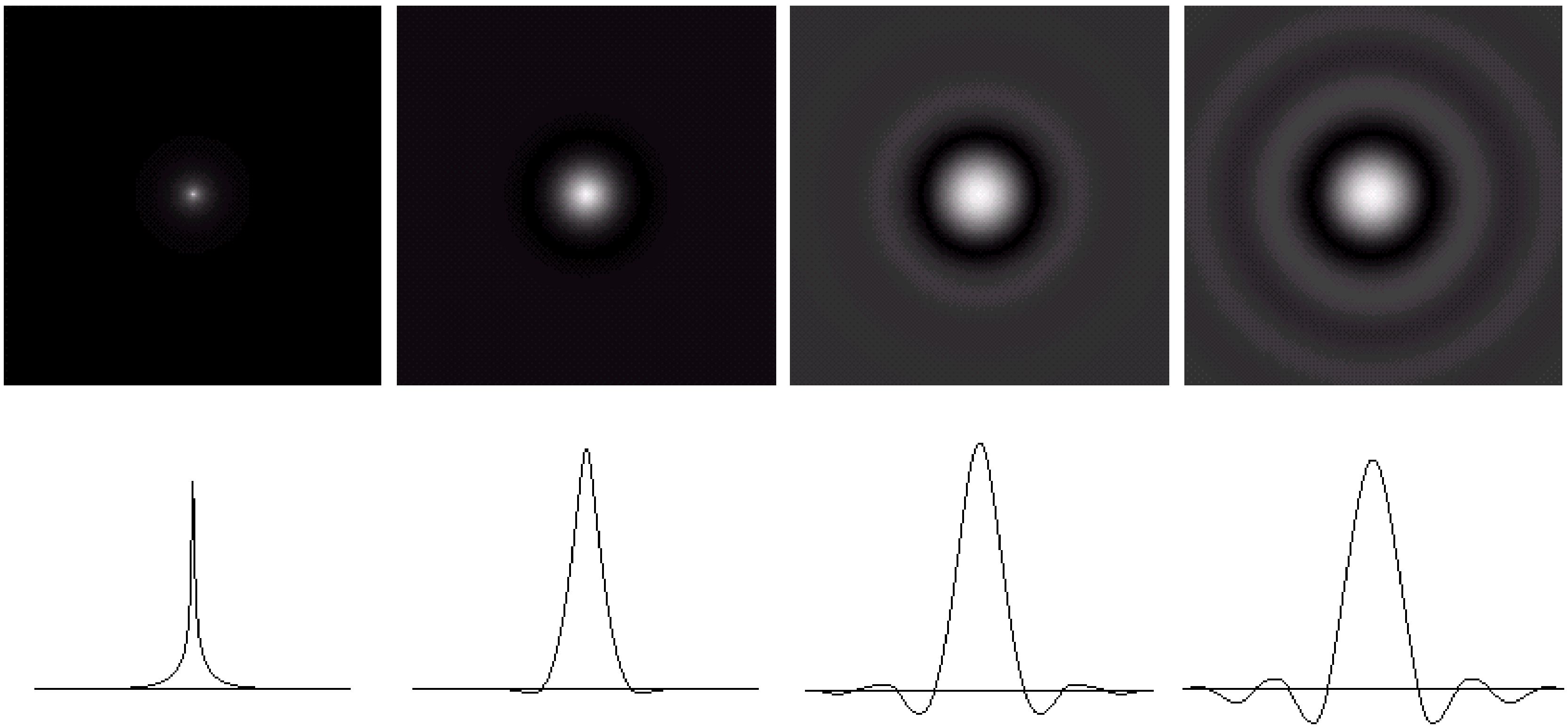


a b
c d
e f

FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.



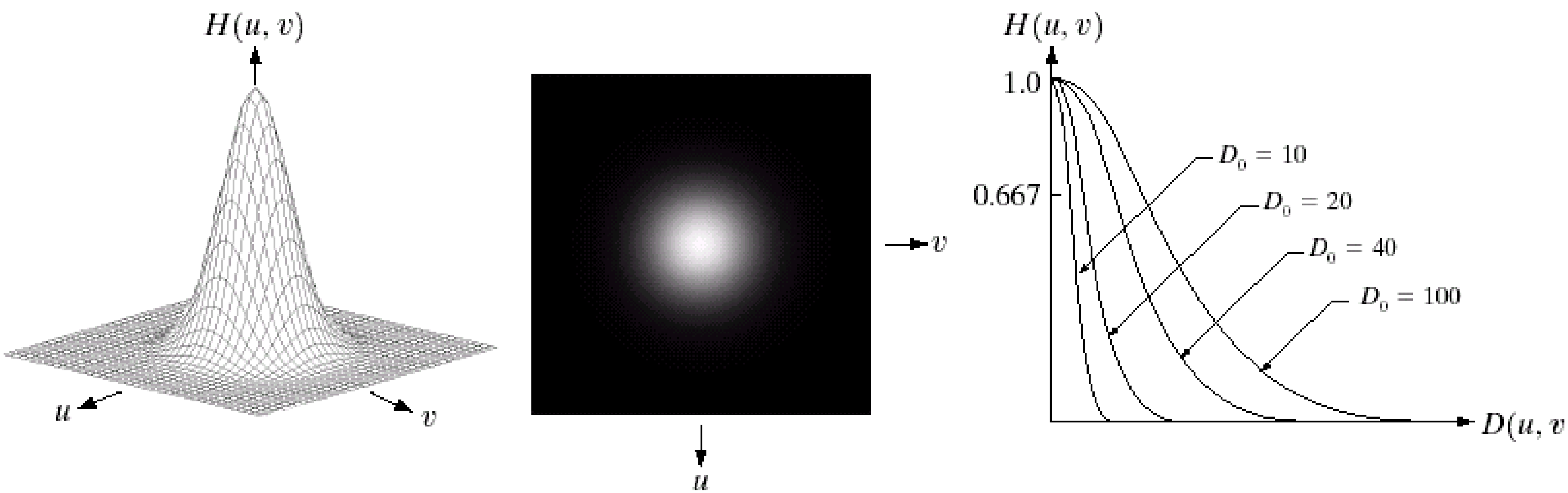
Spatial representation of BLPFs



a b c d

FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

Gaussian Low pass Filter : GLPF



a b c

FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

Gaussian Lowpass Filter

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

- D(u,v): distance from the origin of FT
- Parameter: $\sigma = D_0$ (cutoff frequency)
- The inverse FT of the Gaussian filter is also a Gaussian

Example

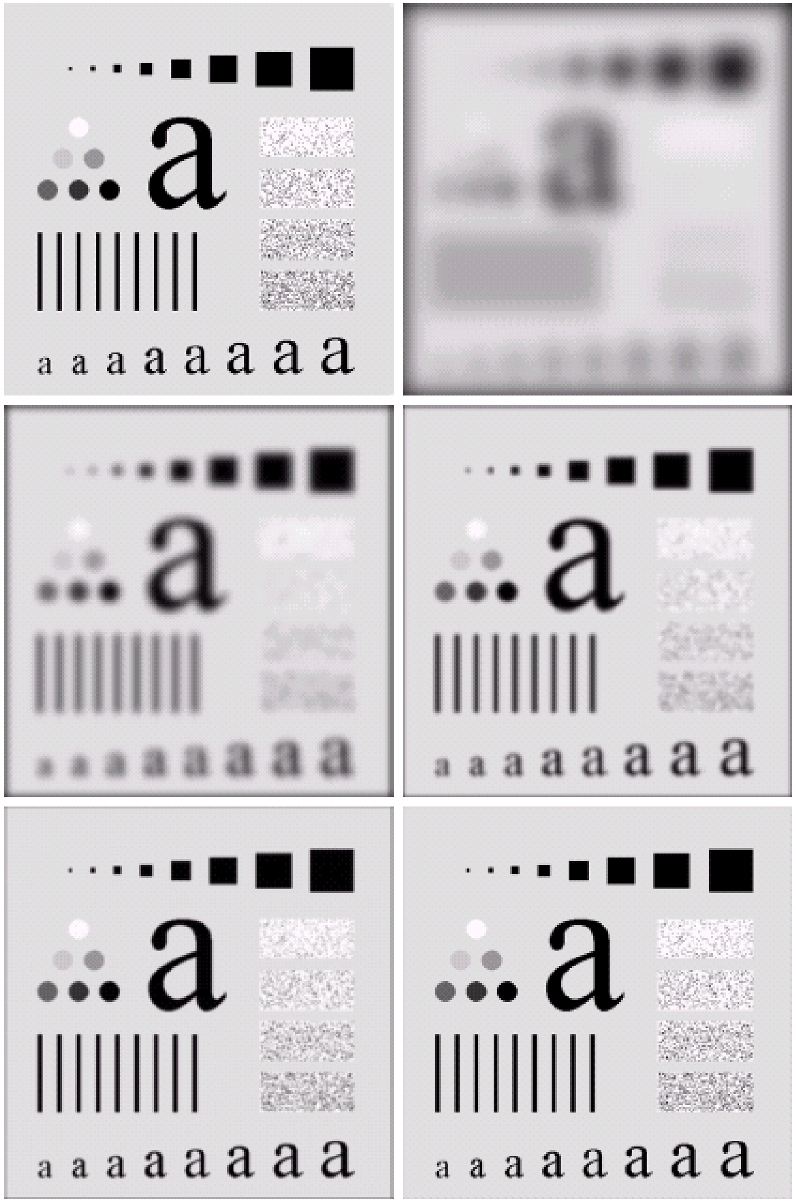
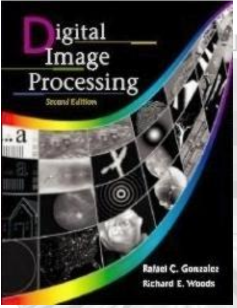


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

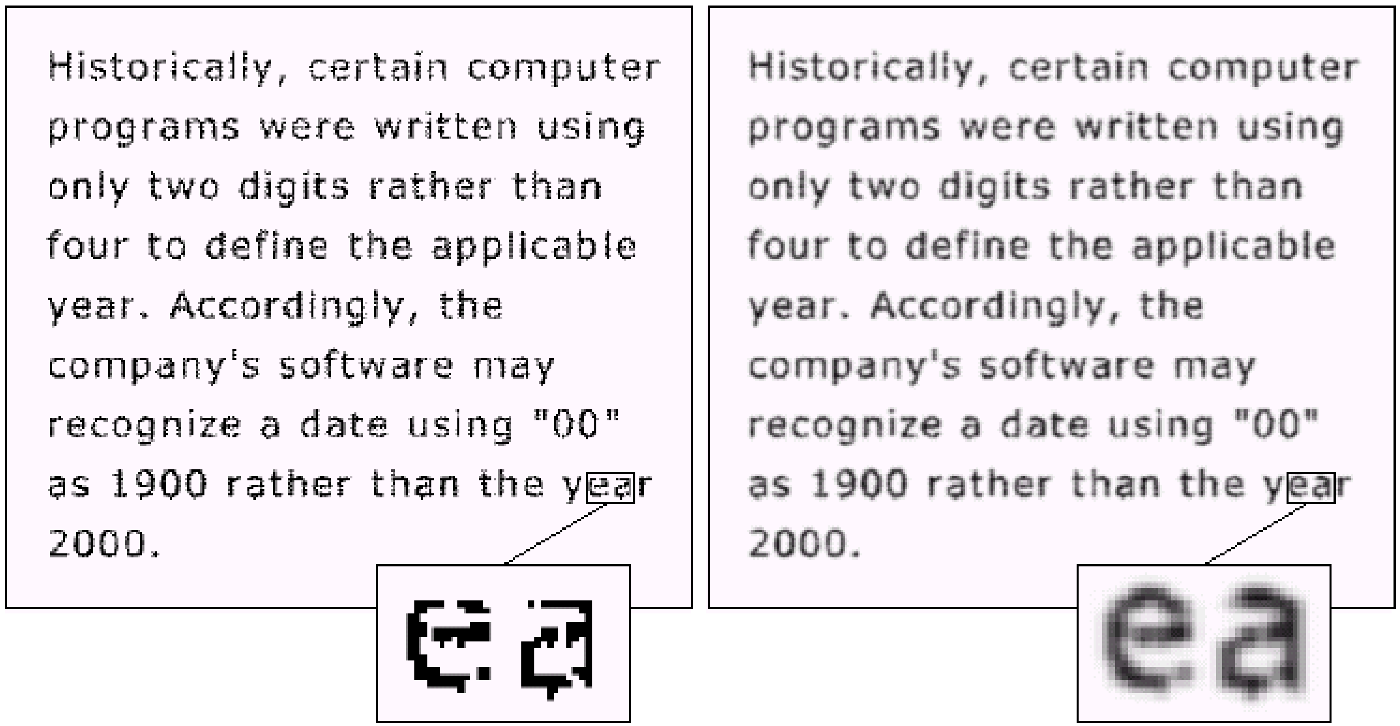
a b
c d
e f



Example

a b

FIGURE 4.19
(a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

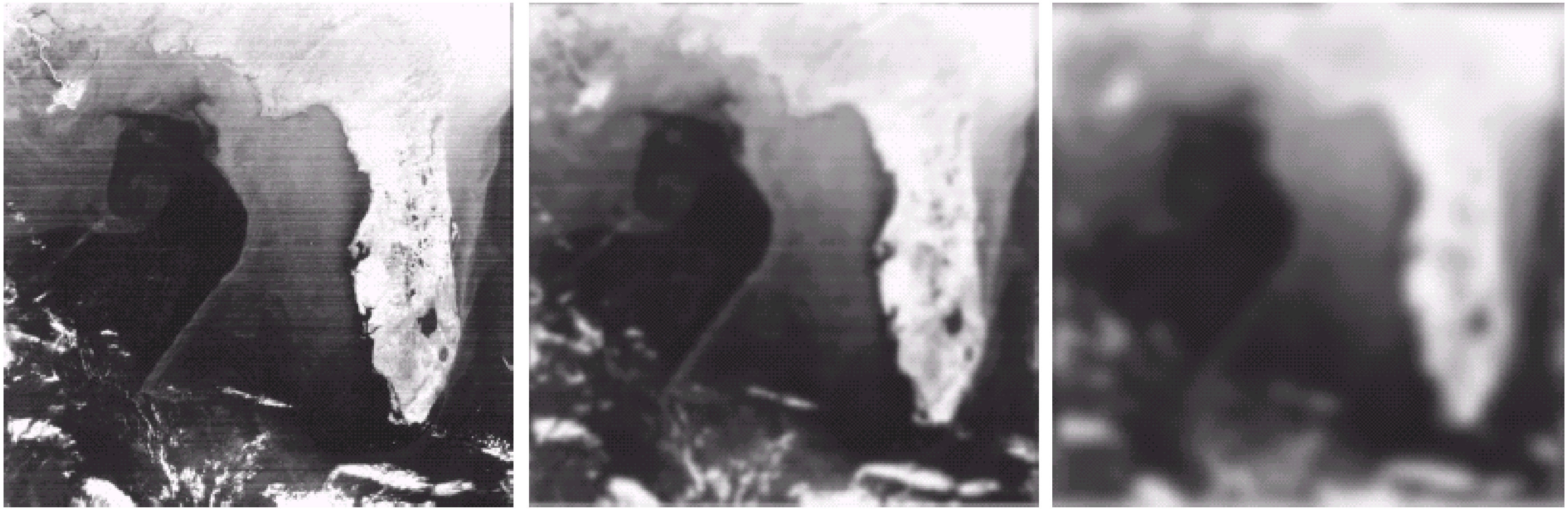


Example

a b c

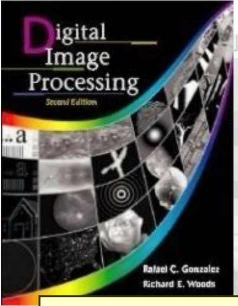
FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

Example



a b c

FIGURE 4.21 (a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)



Sharpening Frequency Domain Filter

- **Ideal high pass filter**

$$H(u,v)=\begin{cases}0 & \text{if} \quad D(u,v)\le D_0\\ 1 & \text{if} \quad D(u,v)>D_0\end{cases}$$

- **Butterworth high pass filter**

$$H(u,v)=\frac{1}{1+[D_0/D(u,v)]^{2n}}$$

- **Gaussian high pass filter**

$$H(u,v)=1-e^{-D^2(u,v)/2D_0^2}$$

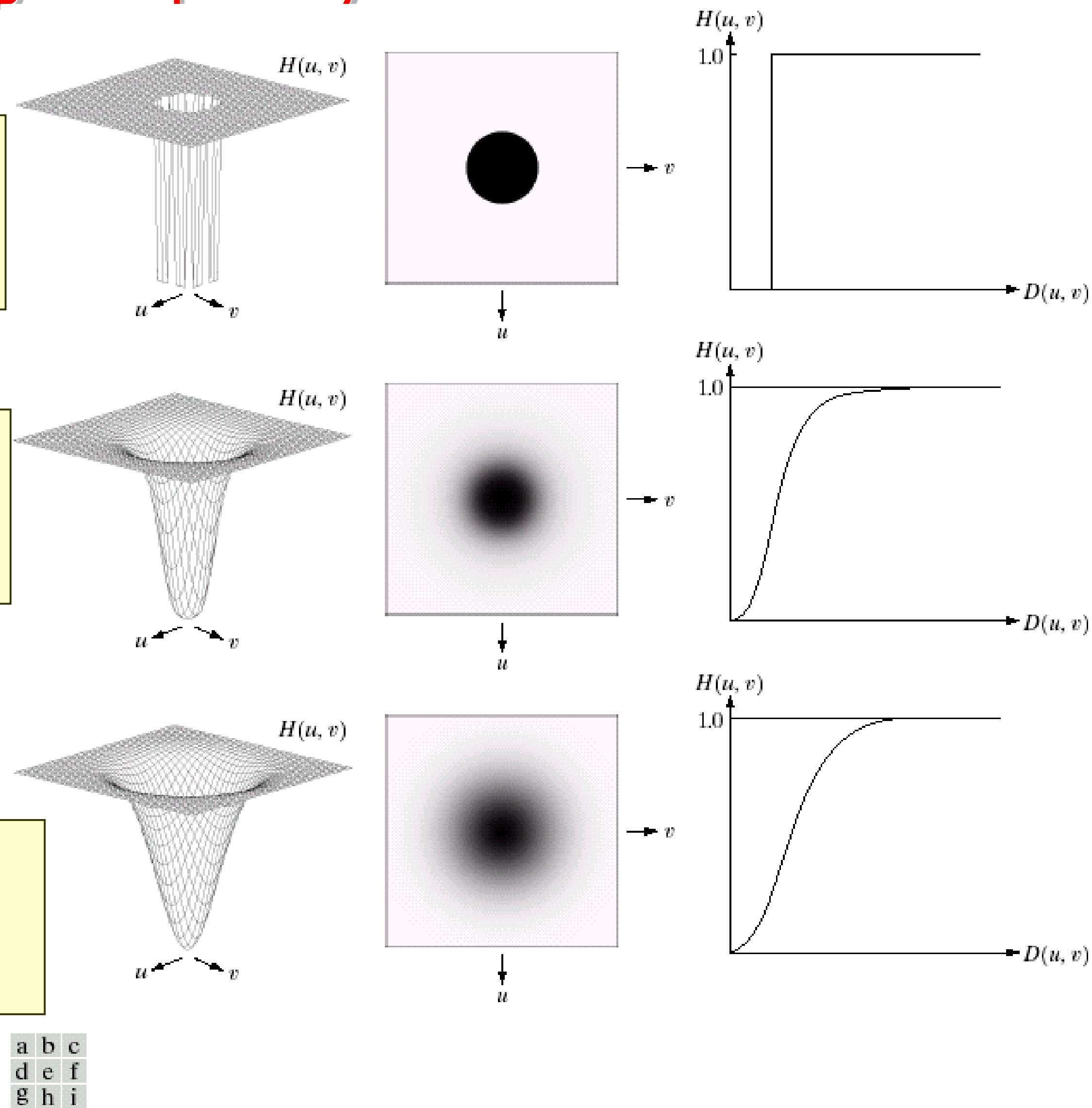


FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Spatial representation of Ideal, Butterworth and Gaussian high pass filter

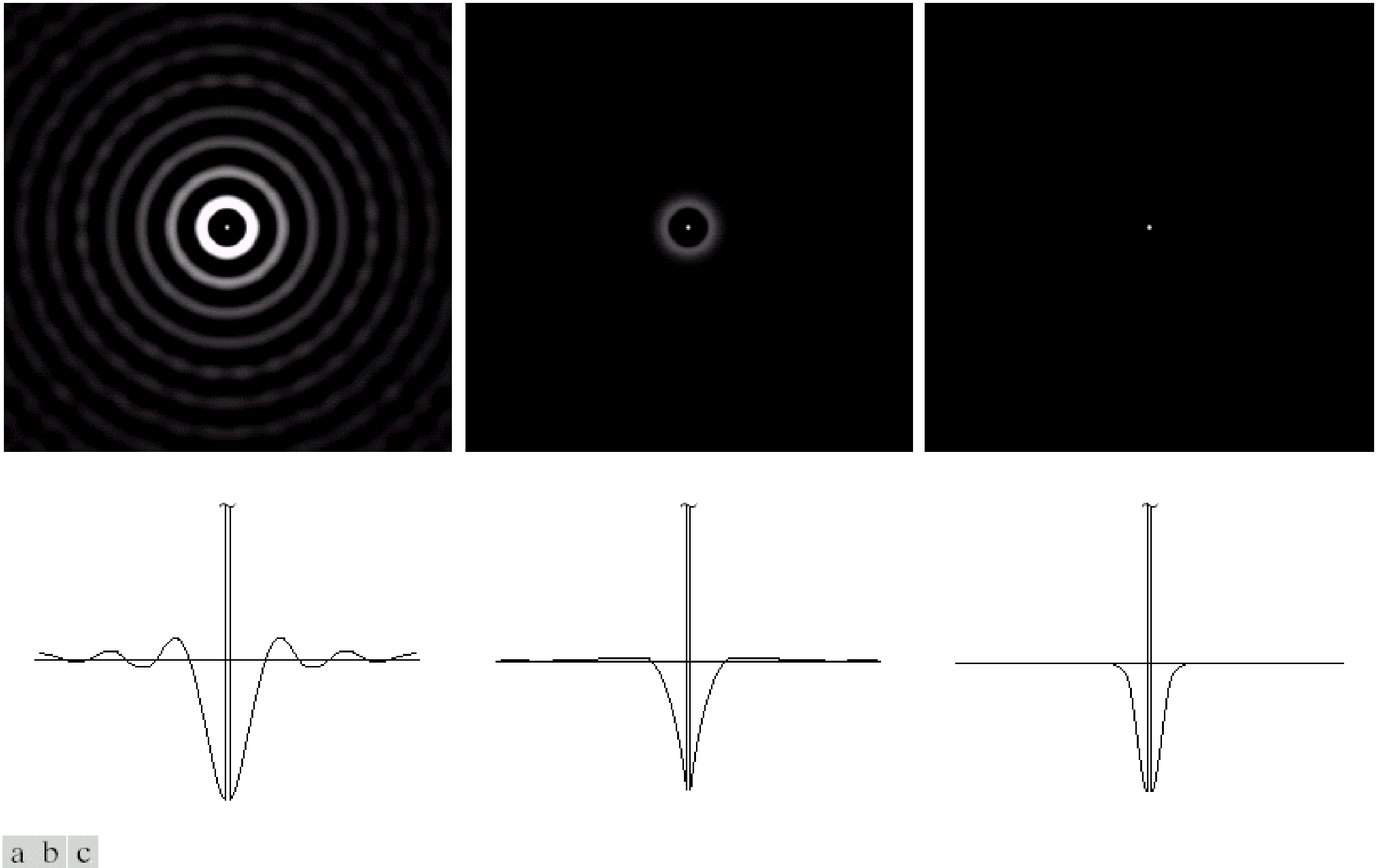
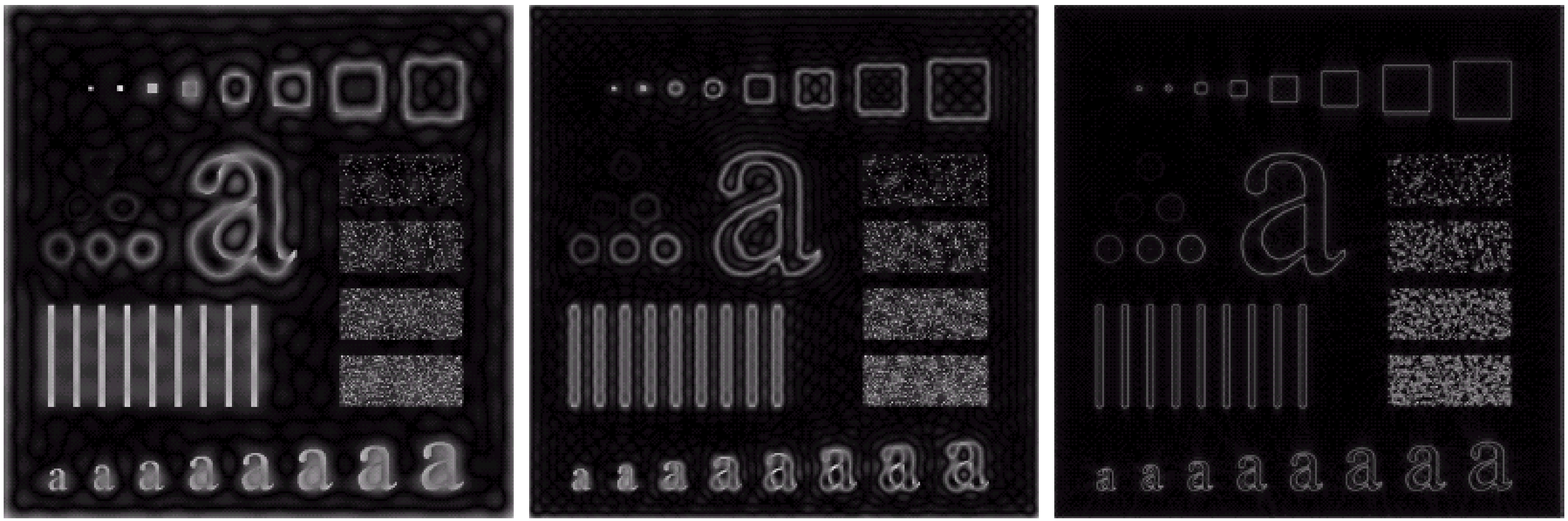


FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

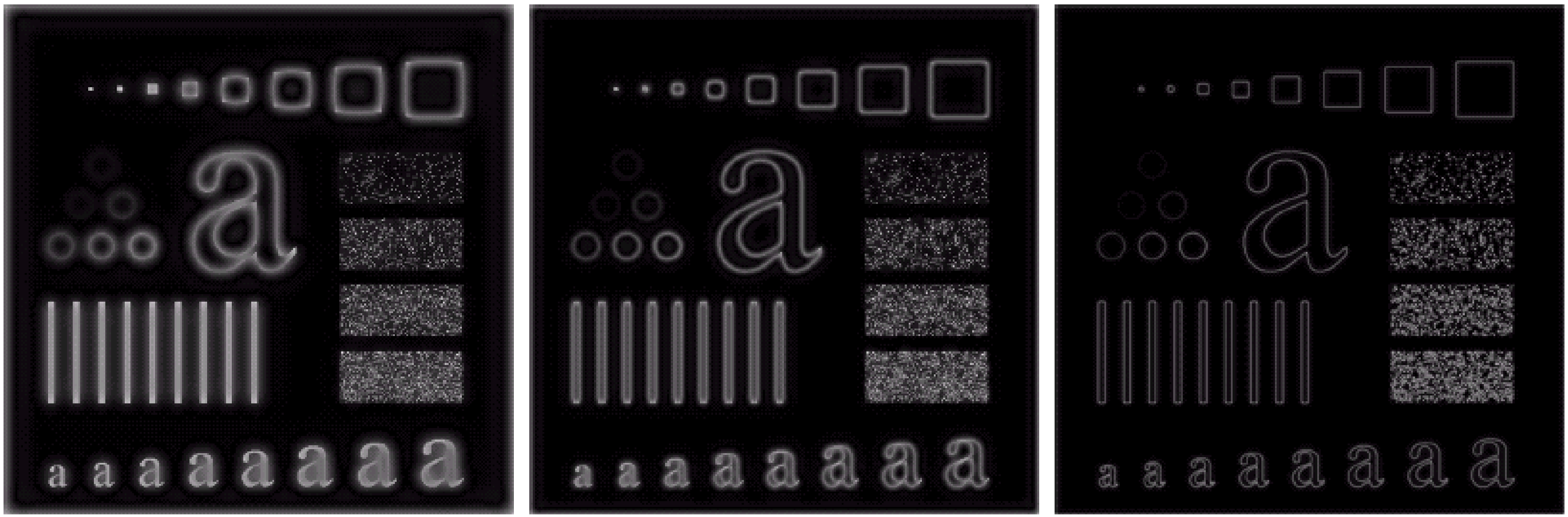
Example : result of IHPF



a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

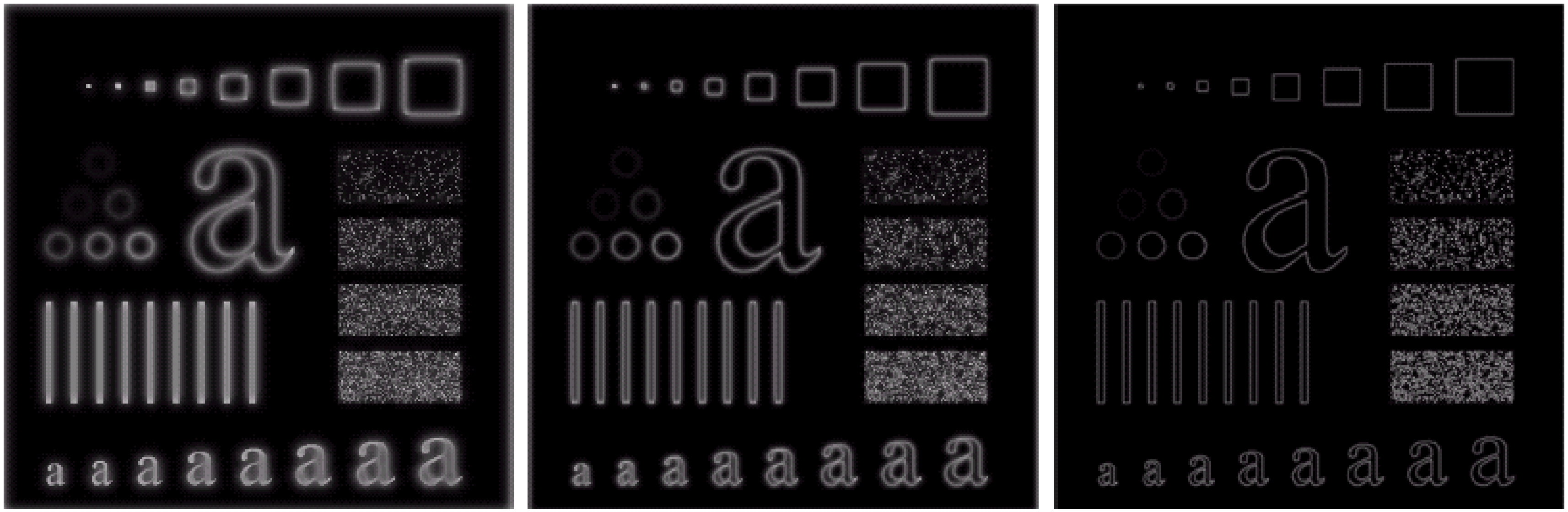
Example : result of BHPF



a b c

FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

Example : result of GHPF



a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.