Course Code:- 4241
Course Name:- Integral Calculus and Differential Calculus.
Book:- (i) Differential Equation by S.L. Ross.
(ii) Engineering Mathematics by H.K. Das.
(iii) Partial Differential Equation by Bernard Epstain.
(iv) An Introduction to Differential Equation by Dr. S.M. Farid.

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Differential Equation:-
→ (If) an equation containing/envolving derivative is called Differential Equation.
→ An equation involving derivatives of one or more dependent variables with respect to one or more independent Variables is called a differential equation.

[Partial Derivative :→ More than one independent variable.
[Ordinary Derivative:→ Single independent variable.

$$\frac{\partial A}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \rightarrow A = f(x,y) \rightarrow \text{Partial Diff.}$$

$$\frac{\partial A}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Differential Equation → Ordinary Differential Equation. (ODE)
Differential Equation → Partial Differential Equation. (PDE)

(i) ODE:-

An equation involving derivatives of one or more dependent variables with respect to single independent variable. Ex:- $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=1$; $\frac{d^2s}{dt^2}+2\frac{ds}{dt}=3$.

(ii) PDE:-

An equation involving derivatives of one or more dependent variables with respect to more than one independent variables. Ex:- $\frac{\partial v}{\partial s}+\frac{\partial v}{\partial t}=v$.

(i) $\frac{dy}{dx}=\cos x \rightarrow$ Order=1, Deg=1

(ii) $\frac{d^2y}{dx^2}+k^2y=0 \rightarrow$ Order=2, Deg=1

(iii) $\frac{\partial v}{\partial s}+\frac{\partial v}{\partial t}=v \rightarrow$ Order=1, Deg=1

(iv) $\frac{\partial z}{\partial x}=z+x\frac{\partial z}{\partial y} \rightarrow$ Order=1

(v) $L\frac{d^2 I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=E\omega\cos\omega t \rightarrow$ Order=2

(vi) $\frac{\partial^2 x}{\partial y^2}+\frac{\partial^2 x}{\partial z^2}=1 \rightarrow$ Order=2

Order of a differential equation:-

Order of a differential equation is the order of the highest derivative involved.

Degree of a differential equation:-

"Exponent of the highest differential co-efficient" -when the differential equation is a polynomial in all differential co-efficient.

(i) $y''+(y')^3+3y=e^x \rightarrow$ order=2, degree=1

(ii) $\frac{d^3y}{dx^3}+2\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}=\cos x \rightarrow$ Order=3, deg=1.

(iii) $\sqrt{y'}+2y=0 \rightarrow$ order=1, degree=1.

or, $y'=(-2y)^2 \therefore y'-4y^2=0$.

(iv) $\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}} = \frac{dy}{dx} + 2 \rightarrow$ order $= 3$, deg $= 2$

or, $\left(\frac{d^3y}{dx^3}\right)^2 = \left(\frac{dy}{dx} + 2\right)^3$

(v) $\frac{d^2y}{dx^2} = \sin\left(\frac{dy}{dx}\right) \rightarrow$ Order $= 2$, degree is not defined.

(vi) $\frac{dy}{dx} = \sqrt{3x^2+5}$

or, $\underbrace{\left(\frac{dy}{dx}\right)^2 = 3x^2+5}_{\text{No need}} \rightarrow$ Order $= 1$, degree $= 1$

Ordinary Differential Equation :-

Solution of D.E. :-

When a function or the curve or the family of curve which satisfy the differential equation, then this family of curves is said to be a solution of that diff. equation.

Solution → General solution $\left[\frac{dy}{dx} = 2x, \text{ Soln:- } y = x^2 + c\right]$

Solution → Particular Solution $\left[\frac{dy}{dx} = 2x, \text{ if } x=0, y=5 \Rightarrow y(0)=5, \text{ Soln:- } y = x^2+5 \ (c=5)\right]$

Formation of Differential Equation :- → [Eliminate arbitrary constants.]

$$y = x+1 \to \frac{dy}{dx} = \frac{dx}{dx} = 1.$$

(i) $y^2 = 4a(x+b)$ → [number of arbitrary constant = order of d.e.]

a, b are arbitrary constant.

$\frac{dy}{dx} \cdot 2y = 4a + 0$

or, $y_1 y = 2a$

or, $y \cdot y_1 = 2a$

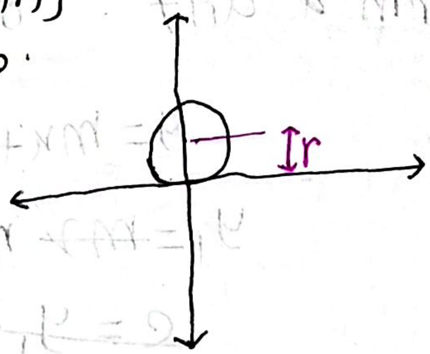
or, $y_1^2 + y_1 y_2 = 20$.

$$\therefore (y_1)^2 + y_0 y_2 = 0 \Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$$

Ex:-

Form a differential equation of family of circle which touches the x axis at the origin.

If a circle touches x-axis at origin, then centre, $(0,b)$ and radius b.



$\therefore$ Eqn of a circle,

$$(x-0)^2+(y-b)^2=b^2$$

or, $x^2+(y-b)^2=b^2$, ——— (i) Here 'b' is an arbitrary constant

Differentiate with respect to x,

$$2x+2(y-b)\cdot y_1=0$$

or, $(y-b)=-\dfrac{x}{y_1}$ and $b=y+\dfrac{x}{y_1}$

Now, putting $(y-b)$ and b in (i)

$$x^2+\left(-\frac{x}{y_1}\right)^2=\left(y+\frac{x}{y_1}\right)^2$$

or, $x^2+\cancel{\dfrac{x^2}{y_1^2}}=y^2+2\cdot y\cdot\dfrac{x}{y_1}+\cancel{\dfrac{x^2}{y_1^2}}$

or, $x^2-y^2-2y\cdot\dfrac{x}{y_1}=0$

or, $(x^2-y^2)\dfrac{dy}{dx}-2xy=0.$

This is, 1st order, 1st degree non-linear Ordinary differential equation.

Form a diff. of family of st. line

$y = mx + c$

$\frac{dy}{dx} = m$

$y_1 = \cancel{mx}\ m + \cancel{c}$

$\cancel{c = y_1 - m}$

$\frac{d^2y}{dx^2} = 0$

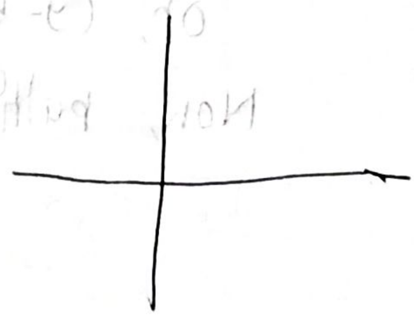
$\cancel{y = mx + y_1 - m}$

$y \cancel{\ \ } y$

0

at a unit distance from the origin.

$$\frac{y-mx-c}{\sqrt{1^2+m^2}} = 1$$



$$y-mx-c = \sqrt{1+m^2}$$

From orgin,

$$\frac{0-0-c}{\sqrt{1+m^2}} = 1 \Rightarrow c = \sqrt{1+m^2}$$

$$y = mx + \sqrt{1+m^2}.$$

$$\frac{dy}{dx} = m + \frac{1}{2\sqrt{1+m^2}} \cdot 2m \cdot 0$$

$$\therefore \frac{dy}{dx} = m$$

$$\therefore m = y_1$$

$$y = y_1 x + \sqrt{1+y_1^2}$$

$$\text{or, } y - y_1 x - \sqrt{1+y_1^2} = 0.$$

form a D.E. from, $y = c_1 \cdot e^{-2x} + c_2 \cdot e^{3x}$ — (i)

c_1, c_2 — a.c.

$$\frac{dy}{dx} = -2 \cdot c_1 e^{-2x} + 3 \cdot c_2 e^{3x} \text{ — (ii)}$$

$$\text{or, } \frac{d^2y}{dx^2} = 4 \cdot c_1 e^{-2x} + 9 \cdot c_3 \cdot e^{3x} \text{ — (iii)}$$

(i) $\times 2$ + (ii),

$$y_1 + 2y = 5 c_2 e^{3x} \text{ — (iv)}$$

$$y_2 + 2y_1 = 15 c_2 e^{3x} \text{ — (v)}$$

$$\text{or, } y_2 + 2y_1 = 3 \times 5 c_2 e^{3x}$$

$$= 3 \times (y + 2y)$$

$$\therefore \frac{d^2y}{dx^2} + 2 \cdot \frac{dy}{dx} - 3 \cdot \frac{dy}{dx} + 6y = 0$$

2nd order 1st degree linear ODE.

find the D.E. of the system of ellipse having their axes along the x axis and the y axis

$\boxed{Ex}$ Show that, $y = A\cos 2x + B\sin 2x$ is a soln of $y'' + 4y = 0$.

$y' = 2A \sin 2A + 2B\cos 2x$

$y'' = -4A\cos 2A - 4B\sin 2x$.

L.H.S. $= y'' + 4y$

$= -4A\cos 2A - 4B\sin 2x + 4A\cos 2A + 4B\sin 2x$

$= 0$.

$=$ R.H.S.

(showed)

Therefore, $y = - - -$ is a soln of

$y'' + 4y = 0$

Ex Show that, $y = x + 3e^{-x}$ is a soln of $\frac{dy}{dx} + y = x+1$

$\frac{dy}{dx} = 1 - 3.e^{-x}$

$L.H.S. = \frac{dy}{dx} + y$

$= 1 - \cancel{3e^{-x}} + x + \cancel{3e^{-x}}$

$= 1 + x = R.H.S.$

$\therefore y = x + 3e^{-x}$ is a soln of $y'' + 4y = 0$.

Ex a) Show that, $\left|\frac{dy}{dx}\right| + |y| + 1 = 0$ has no real soln.

(b) Show that, $\left(\frac{dy}{dx}\right)^2 - 4y = 0$ has one parameter family of soln of the form, $f(x) = (x+c)^2$, where c is an arb. cons.

a Let, $y = f(x)$ be a soln of the given eqn. L.H.S. equal to must zero for $y = f(x)$

$y = f(x)$

$\frac{dy}{dx} = f'(x)$

L.H.S.

$|f'(x)| + |f(x)| + 1$

i.e., $|f'(x)| + |f(x)| + 1 \geq 1$.

Here

$|f'(x)| \geq 0$.

$|f(x)| \geq 0$

Initial Value Problem (IVP) :-

A D.E. that has given condition allows us to find the specific function that specifies the given DE rather than a family of functions. These type of problems are called IVP.

Such as, $y(x_0) = y_0$, $y'(x_0) = z_0$

Boundary Value problem (BVP) :-

If the conditions are given more than one point of x and D.E. is of order 2 or greater, this is called BVP. such as

$y(x_0) = y_0$, $y'(t_0) = m_0$

$\boxed{\text{Ex}}$ Find the soln to the BVP $\frac{d^2y}{dx^2} - y = 0$, $y(0) = 0$, $y(1) = 1$ if we know, $y(x) = c_1 e^{x} + c_2 e^{-x}$ is the G.S. of the given D.E.

<u>For first :-</u>

$x = 0.$

$y = 0.$

$0 = c_1 + c_2$

$c_1 = -c_2$

<u>For 2nd :-</u>

$x = 1$

$y = 1$

$1 = c_1 e + c_2 e^{-1}$

or, $c_1 e - c_1 e^{-1} = 1$

or, $c_1 (e - e^{-1}) = 1$

or, $c_1 = \frac{e}{e^2 - 1}.$

$c_2 = \frac{-e}{e^2 - 1}.$

$e - e^{-1} \frac{1}{e}$

$= \frac{e^2 - 1}{e}$

$\boxed{Ex}$ show that,

(a) $y = 4e^{2x} + 2e^{-3x}$

is a soln of the IVP $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$,

$y(0) = 6$, $y'(0) = 2$.

(b) If $y = 2e^{2x} + 4e^{-3x}$ also a soln of this Problem? Explain why or why not?

First Order First Degree Ordinary D.E. :-

General Form of 1st Order 1st Degree ODE,

$$\frac{dy}{dx} = f(x, y) \quad ; \quad \frac{dy}{dx} = \frac{x^2-1}{y^2-1}$$

or, $(y^2-1)dy - (x^2-1)dx = 0$

or, $M(x,y)dx + N(x,y)dy = 0$

or, $Mdx + N \cdot dy = 0$

Solving Method :-

(i) Variable Separation
(ii) Reduce to "
(iii) Linear to

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)} = f(x,y)$$

If $M(x,y)$ is a fn. of x only on const.
and $N(x,y)$ is a fn. of y only or const.

$$\frac{dy}{dx} = \frac{-M(x)}{N(y)}$$

separate the values,

$M(x)dx + N(y)dy = 0$.

Integrating, $\int M(x)dx + \int N(y)dy = c$.

Ex

Solve,

$$\frac{dy}{dx} = x^2 y + x^2$$

or, $\frac{dy}{dx} = x^2(y+1)$

or, $x^2(y+1)dx - dy = 0$; Now separate the variables,

or, $x^2 . dx - \frac{1}{y+1} dy = 0$; Integrating, $\int x^2 dx - \int \frac{1}{y+1} dy = 0$

or, $\frac{1}{3}x^3 - \ln|1+y| = C$.

$\therefore \frac{1}{3}x^3 - \ln|y+1| = C$.

Ex

Solve,

$$x^2(y+1)\, dx + y^2(x-1)\, dy = 0$$

Separating the variables,

$$\frac{x^2}{x-1} dx + \frac{y^2}{y+1} dy = 0$$

or, $\left(x+1+\frac{1}{x-1}\right)dx + \left[(y-1) + \frac{1}{y+1}\right]dy = 0$

Integrating,

$$\frac{x^2}{2} + x + \ln|x-1| + \frac{y^2}{2} - y + \ln|y+1| = c.$$

$\therefore x^2 + 2x + 2\ln|x-1| + y^2 - 2y + 2\ln|y+1| = 2c$

$\therefore (x^2+y^2) + 2(x+y) + 2[\log|x-1| + \log|y+1|] = 2c$.

This is General Soln of the given equation where c is an arbitrary const.

Ex

Solve,

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

or, $\frac{dy}{dx} = e^{x} \cdot e^{-y} + x^2 \cdot e^{-y}$

or, $\frac{dy}{dx} = e^{-y}(e^{x} + x^2)$

or, $\frac{dy}{dx} = \frac{e^{x} + x^2}{e^{y}}$

or, $\cancel{dy\, e^{y}} \cdot \cancel{dx} \cdot (e^{x} + x^2)dx - e^{y} \cdot dy = 0$.

~~or~~ Inte.,

$$\int (e^{x} + x^2)dx - \int e^{y}\, dy = 0$$

or, $e^{x} + \frac{x^3}{3} - e^{y} = c$.

$\therefore 3e^{x} - 3e^{y} + x^3 = 3c$.

Reducible to Separable :-

Ex :-

Solve, $\frac{dy}{dx} = (x+y)^2$

Let, $z = x+y$

$$\frac{dz}{dx} = 1 + \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{dz}{dx} - 1$$

So, $\frac{dz}{dx} - 1 = z^2$

or, $\frac{dz}{dx} = z^2 + 1$

separating the variable,

$$\frac{1}{1+z^2} dz = dx$$

Inte.,

$\cancel{\ln|1+z^2| - x +}$ $\tan^{-1} z = x + c$

or, $\tan^{-1}[(x+y)^{\cancel{2}}] = x + c$.

$\therefore \cancel{\tan^{-1}}\ x - \tan^{-1}[(x+y)^{\cancel{2}}] + c = 0$

$\therefore \tan^{-1}(x+y) - x = c$.

where 'c' is an arbitrary constant.

Ex

Solve,

$$\frac{dy}{dx} = \cos(x+y)$$

Let, $x+y=z$

or, $1+\frac{dy}{dx} = \frac{dz}{dx}$

or, $\frac{dy}{dx} = \frac{dz}{dx} - 1.$

Now,

$$\frac{dz}{dx} - 1 = \cos z$$

or, $\frac{dz}{dx} = \cos z + 1$

or, $\frac{1}{1+\cos z} \cdot dz = dx$

Integrating, $\int \frac{1}{1+\cos z} dz = \int dx$

$\ln|1+\cos z| = x + c.$ X

$\therefore \ln|1+\cos z| - x = c.$

Where 'c' is an arbitrary constant.

or, $\frac{1}{2\cos^2\frac{z}{2}} dz = dx$

or, $\frac{1}{2}\sec^2\frac{z}{2} dz = dx$

Integration,

$\frac{1}{2} \cdot 2\tan\frac{z}{2} = x + c.$

$\therefore \tan\frac{z}{2} - x \cancel{\tan} = c$

$\therefore \tan\left(\frac{x+y}{2}\right) - x = c$

Where 'c' is an arbitrary constant.

Ex.
Find the Particular solution of $\cos y\, dx + (1+e^{-x}) \sin y\, dy = 0$ where $y(0) = \frac{\pi}{4}$.

$\cos y\, dx + (1+e^{-x}) \sin y\, dy = 0$

or, $\frac{dx}{(1+e^{x})} + \tan y\, dy = 0$

or, $\frac{e^{-x}\, dx}{(1+e^{-x})} + \tan y\, dy = 0$

or, $\ln(1+e^{x}) - \ln|\cos y| = \ln c$

$1 + e^{-x} = z$

or, $-e^{-x} dx = dz$

or,

~~$y(0) = \frac{\pi}{4}$~~ $(1+e^{x}) \sec y = c$

~~or~~ $y(0) = \frac{\pi}{4}$

$(1+1) \sec \frac{\pi}{4} = c$

or, $2\sqrt{2} = c$

$\therefore c = 2\sqrt{2}$

$\therefore (1+e^{-x}) \sec y = 2\sqrt{2}$.

Ex:

Suppose that the derivative $\frac{dx}{dt}$ is proportional to x, that $x=5$ when $t=0$ and that $x=10$ when $t=5$. What is the value of x?

Soln:- Given that,

$$\frac{dx}{dt} \propto x$$

$$\text{or, } \frac{dx}{dt} = kx \text{ ---- (i)}$$

$$\text{or, } dt.k = dx.\frac{1}{x}$$

Inte.,

$$k.t = \ln x + \cancel{c} \text{ ---- (ii)}$$

$$\therefore k.t + \ln x = c. \quad \therefore \ln x - kt = c.$$

Now Here, $x(0)=5$

$$\text{or, } \ln 5 = 0 + c$$

$$\therefore c = \ln 5.$$

From (ii), $\ln x = kt + \ln 5$

$$\text{or, } \ln x - \ln 5 = kt.$$

$$\text{or, } \ln\left(\frac{x}{5}\right) = kt \text{ ---- (iii)}$$

If $t=5$, $x=10$.

$$\therefore \ln\left(\frac{10}{5}\right) = 5k.$$

$$\text{or, } \frac{1}{5}\ln 2 = k. \quad \therefore k = \frac{1}{5}\ln 2.$$

Putting the value of 'k' in (iii),

$\ln\left(\frac{x}{5}\right) = \frac{1}{5}\ln 2 \cdot t.$

$\therefore \ln\left(\frac{x}{5}\right) = \frac{\ln 2}{5} t.$

$\therefore \ln\left(\frac{x}{5}\right) = 0.138\, t.$

or, $\frac{x}{5} = e^{0.138t}.$

$\therefore x = 5 \cdot e^{0.138t}.$

Homogeneous D.E. :-

The 1st order, 1st deg. ODE :-

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

If M and N are both homogeneous fn. with same degree, then this type of ODE is called Homogeneous ODE.

$$\frac{dy}{dx} = f(x,y)$$

Degree of $f(x,y)$ is zero.

$$f(x,y) = x^0 \phi\left(\frac{y}{x}\right).$$

Solving Procedure :-

Step-1 :-

Putting, $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Step-2 :-

After replacing $y = vx$, the equation will be separable in v and x.

Step-3 :- Separable. Separate the variables and integrate.

Homogeneous fn :-

If $f(x,y)$ can be expressed by $x^n \phi(y/x)$ or $y^n \phi\left(\frac{x}{y}\right)$ form, then $f(x,y)$ is called homogeneous fn. of degree n.

$$f(x,y) = x^2 + 2xy + y^2$$
$$= x^2\left(1 + \frac{y^2}{x^2} + 2\cdot\frac{y}{x}\right)$$
$$= x^2 \phi\left(\frac{y}{x}\right)$$

$$f(kx,ky)$$
$$= k^2x^2 + k^2y^2 + k^2 2xy$$
$$= k^2(x^2 + y^2 + 2xy)$$
$$= k^2 f(x,y).$$

$$f(x,y) = \sin\frac{x}{y} + \log\frac{x}{y} + \cos\frac{x}{y}.$$

$$f(kx,ky) = \sin\frac{kx}{kx} + \log\frac{kx}{ky} + \cos\frac{kx}{ky}.$$

[Replace x with kx and y with ky]

$$= k^0\left(\sin\frac{x}{y} + \log\frac{x}{y} + \cos\frac{x}{y}\right).$$

Homogeneous function of degree zero.

Ex:-

Solve,

$$\underbrace{\left(x\sin\frac{y}{x} - y\cos\frac{y}{x}\right)}_{M}dx + \underbrace{x\cos\frac{y}{x}}_{N}dy = 0$$

$$\frac{dy}{dx} = \frac{-x\sin\frac{y}{x} + y\cos\frac{y}{x}}{x\cos\frac{y}{x}\cancel{dy}}$$

Putting $y = vx$,

$$\frac{dy}{dx} = \frac{-x\sin v + vx\cos v}{x\cos v\,\cancel{dy}}$$

Ex:- Solve,

$$\left(x+\sqrt{y^2-xy}\right)dy-y\,dx=0$$

$$\frac{dy}{dx}=\frac{y}{x+\sqrt{y^2-xy}}$$

This is homogeneous diff. equations.

Putting $y=vx$,

$$\frac{dy}{dx}=\frac{vx}{x+\sqrt{v^2x^2-x^2v}}=\frac{vx}{x\left[1+\sqrt{v^2-v}\right]}=\frac{v}{1+\sqrt{v^2-v}}$$

Now,

$$v+x\,dx=\frac{v}{1+\sqrt{v^2-v}}$$

$$\therefore x\frac{dv}{dx}=\frac{v}{1+\sqrt{v^2-v}}-v=\frac{v-v\left(1+\sqrt{v^2-v}\right)}{1+\sqrt{v^2-v}}$$

$$\frac{dx}{dy}=\frac{x+\sqrt{\cancel{x}\,y^2-xy}}{y}$$

Let, $x=vy$,

$$\frac{dx}{dy}=v+y\cdot\frac{dv}{dy}$$

$$v+y\frac{dv}{dy}=\frac{vy+\sqrt{y^2-vy^2}}{y}$$

or, $v+y\frac{dv}{dy}=v+\sqrt{1-v}$

$\therefore y\frac{dv}{dy}=\sqrt{1-v}$

This is separable differential eq^n of v and y.

$$\frac{1}{\sqrt{1-v}}dv=\frac{dy}{y}.$$

Integrating,

$$\int \frac{1}{\sqrt{1-v}}\,dv = \int \frac{dy}{y}$$

or, $-2\int \frac{1}{z}\,dz = \frac{1}{4}\int \frac{dy}{y}$

or, $-2\sqrt{1-v} = \log y + A$

$\therefore -2\sqrt{1-\frac{x}{y}} = \log y + A$

Let,

$1-v = z^2$

or, $-\frac{dv}{dy} = 2z\,.\,dz$

or, $dv = -2z\,dz$

Ex:-

Find the Particular soln of :-

$x y^2\, dy - (x^3+y^3)dx = 0$. where $y(1)=0$

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$$

$$v + x\frac{dv}{dx} = \frac{x^3+v^3x^3}{x \cdot v^2 x^2} = \frac{1+v^3}{v^2}$$

$$x\cdot\frac{dv}{dx} = \frac{1+v^3-v^3}{v^2} = \frac{1}{v^2}$$

$$v^2 dv = \frac{1}{x}dx$$

$$\frac{v^3}{3} = \log x + \log c$$

$$\frac{y^3}{3x^3} = \log x + \log c$$

using the initial value term,

$y(1) = 0$

or, $0 = \log 1 + \log c$

$\therefore c = 0$

$$\frac{y^3}{3x^3} = \log x$$

∴ P.S.,

$y^3 = 3x^3 \log x$

Ex:- Find the equation of the curve passing through the point (1,1) whose differential equation is,

$$x \cdot dy = (2x^2+1)\,dx$$

or, $\dfrac{dy}{dx} = \dfrac{2x^2+1}{x}$

replacing $y = vx$,

$v + x\dfrac{dv}{dy} = \dfrac{2x^2+1}{x}$

or, $\cancel{\left(\dfrac{y}{x} + \dfrac{dv}{dy}\right)} =$

or, $y = \int\left(2x + \dfrac{1}{x}\right)dx$

or, $y = x^2 + \log x + c$

$\cancel{y(1,1)}$

$y(1) = 1$

or, $1^2 + \log 1 + c = 1$

or, $c = 0$

$\therefore y = x^2 + \log x$.

Exact Differential Equation:-

Total Differential:-

$u = u(x, y)$

Differentiate partially with respect to x,

$$\frac{\partial y}{\partial x}$$

Total Derivative,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$u = f(x, y, z)$$

$$du = \frac{\partial y}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz -$$

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x)$$

$$\therefore \boxed{dy = f'(x) dx}$$

Exact D.E:-

The first order first degree ODE $Mdx + Ndy = 0$ is called exact differential equation, if Left hand side is the total differential of some function $u(x, y)$.

$$M dx + N \cdot dy = 0$$

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$du = 0$$

$$u = \text{constant.}$$

In other words, the expression $Mdx + Ndy$ is called exact differential, if here, exists a function u for which the expression the total differential du.

Solution of Exact D.E. :-

$$M\,dx + N.\,dy = 0$$

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0.$$

$$du = 0$$

$$u = \text{const.}$$

$\therefore u(x, y) = \text{const.}$ This is the general soln.

Necessary and sufficient condition for a differential equation to be exact:-

or, consider the differential equation,

$$Mdx + Ndy = 0. \text{ ——— (i)}$$

where M and N have continuous Partial derivatives then -

a) If the D.E. (i) is exact, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

[Necessary condition]

b) If $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$, then the D.E. is exact.

[Sufficient condition]

Proof:-

(a) If the given D.E. (i) is exact, then $Mdx+Ndy$ is the total differential of a function : $u(x,y)$. such that -

$$M=\frac{\partial u}{\partial x} \quad N=\frac{\partial u}{\partial y}.$$

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \qquad \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$=\frac{\partial^2 u}{\partial y \partial x} \qquad =\frac{\partial^2 u}{\partial x \partial y}$$

For partial derivative,

$$\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}$$

Therefore, $\boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$ (Necessary Condition).

(b) Here, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

We have to show that $Mdx+Ndy=0$ is exact. This means that, we must prove that there exists a function $u(x,y)$, such that, $\frac{\partial u}{\partial x}=M(x,y)$, $\frac{\partial u}{\partial y}=N(x,y)$.

Let us assume that, u satisfies,

$$\frac{\partial u}{\partial x}=M(x,y).$$

$$u=\int M(x,y)\,\partial x+\phi(y) \quad \text{——— (2)}$$

Let, $\int M(x,y)\,\partial x = g(x,y)$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,\partial x\right] + \frac{d\phi}{dy}$$

$$\frac{\partial u}{\partial y} = \frac{\partial g}{\partial y} + \frac{d\phi}{dy}$$

or, $N(x,y) = \dfrac{\partial g}{\partial y} + \dfrac{d\phi}{dy}$

$$\frac{d\phi}{dy} = \left[N\cancel{(x,y)} - \frac{\partial g}{\partial y}\right]$$

$$\therefore \phi(y) = \int\left[N - \frac{\partial g}{\partial y}\right]dy$$

Now, from (2)

$$u = \int M(x,y)\,\partial x + \int\left[N - \frac{\partial g}{\partial y}\right]dy$$

$$= g(x,y) + \int\left[N - \frac{\partial g}{\partial y}\right]dy$$

$\therefore$ Total derivative of u,

$$du = dg + N - \left(\frac{\partial g}{\partial y}\right)dy$$

$$du = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + N d g - \frac{\partial g}{\partial y}dy$$

$$\therefore du = \frac{\partial g}{\partial x}dx + N\cdot dy = M\cdot dx + N\cdot dy$$

(Showed)

Ex.:-

$$(2xy+1)dx + (x^2+4y)dy = 0 \quad \text{——(1)}$$

Solve,

$M(x,y) = 2xy+1$

$\frac{\partial M}{\partial y} = 2x$

$N(x,y) = x^2+4y$

$\frac{\partial N}{\partial x} = 2x.$

Here, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$

$\therefore$ The differential eqn is exact.

$\therefore \cancel{\partial y} \; \frac{\partial u}{\partial x} = M(x,y)$

$\frac{\partial u}{\partial x} = 2xy+1$

$u = \int (2xy+1)\,\partial x + Q(y)$

$= x^2 y + \phi(y) + x \quad \text{——(2)}$

$\therefore \frac{\partial u}{\partial y} = x^2 + \phi'(y)$

Now, $\frac{\partial u}{\partial y} = N = x^2+4y$

$\therefore \cancel{x^2} + 4y = \cancel{x^2} + \phi'(y)$

$\therefore \phi'(y) = 4y.$

$\therefore \phi(y) = 2y^2 + c_0$

Put the value of $\phi(y)=0$ in (2),

$$u = x^2y + x + 2y^2 + c_0$$

Hense, the G.S.,

$$u(x,y) = c_1$$

$$\therefore x^2y + x + 2y^2 + (c_0 - c_1) = 0$$

$$\therefore x^2y + x + 2y^2 = c_1 - c_0$$

$$\therefore x^2y + x + 2y^2 = c.$$

$$\therefore x^2y + 2y^2 + x = c.$$