Introduction to Functions of Several Variables

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Objectives

- Understand the notation for a function of several variables.
- Sketch the graph of a function of two variables.
- Sketch level curves for a function of two variables.
- Sketch level surfaces for a function of three variables.
- Use computer graphics to graph a function of two variables.

Functions of Several Variables

- We know only the functions of a single (independent) variable. Many familiar quantities, however, are functions of two or more variables. Here are three examples.

✓ The work done by a force, $W = FD$, is a function of two variables.

✓ The volume of a right circular cylinder, is a function of two variables.

✓ The volume of a rectangular solid, $V = lwh$, is a function of three variables.

Functions of Several Variables

- The notation for a function of two or more variables is similar to that for a function of a single variable. Here are two examples.

$$z = f\underbrace{(x, y)}_{\text{2 variables}} = x^2 + xy \qquad \text{Function of two variables}$$

- and

$$w = f\underbrace{(x, y, z)}_{\text{3 variables}} = x + 2y - 3z \qquad \text{Function of three variables}$$

Definition of a Function of Two Variables

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is a **function of** $\boldsymbol{x}$ **and** $\boldsymbol{y}$. The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f. For the function

$$z = f(x, y)$$

x and y are called the **independent variables** and z is called the **dependent variable.**

Example 1:Domains of Functions of Several Variables

- Find the domain of each function.

a. $f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 9}}{x}$ **b.** $g(x, y, z) = \dfrac{x}{\sqrt{9 - x^2 - y^2 - z^2}}$

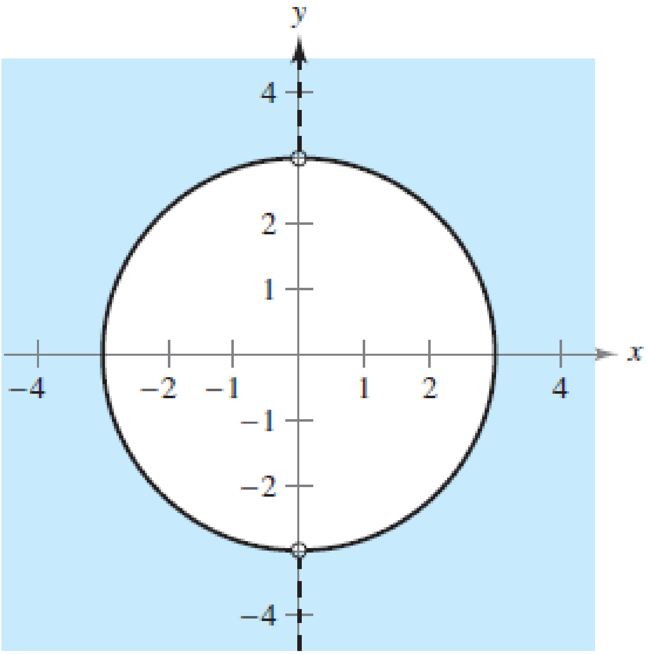
Solution:

a. The function *f* is defined for all points (x, y) such that

$x \neq 0$ and $x^2 + y^2 \geq 9.$

Example 1:Solution

- So, the domain is the set of all points lying on or outside the circle $x^2 + y^2 = 9$ *except* those points on the *y*-axis, as shown in following Figure.



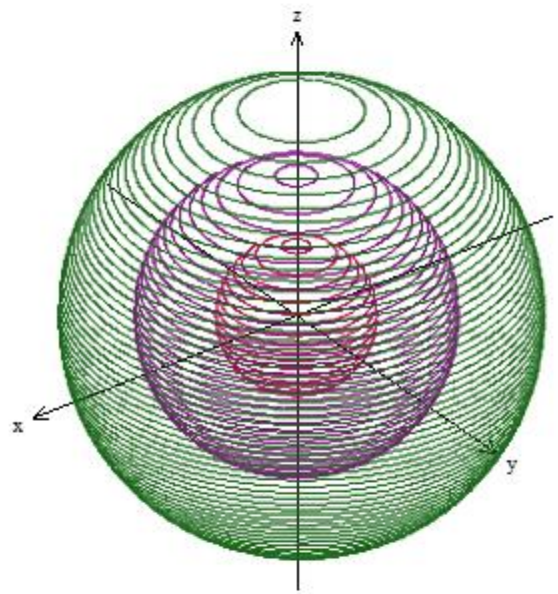
Domain of
$f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 9}}{x}$

Example 1:Solution

b. The function g is defined for all points (x, y, z) such that

$$x^2 + y^2 + z^2 < 9.$$

Consequently, the domain is the set of all points (x, y, z) lying inside a sphere of radius 3 that is centered at the origin.



- Functions of several variables can be combined in the same ways as functions of single variables. For instance, you can form the sum, difference, product, and quotient of two functions of two variables as follows.

$$(f \pm g)(x, y) = f(x, y) \pm g(x, y) \qquad \text{Sum or difference}$$

$$(fg)(x, y) = f(x, y)g(x, y) \qquad \text{Product}$$

$$\frac{f}{g}(x, y) = \frac{f(x, y)}{g(x, y)}, \quad g(x, y) \neq 0 \qquad \text{Quotient}$$

- You cannot form the composite of two functions of several variables. You can, however, form the **composite** function $(g \circ h)(x, y)$, where *g* is a function of a single variable and *h* is a function of two variables.

$$(g \circ h)(x, y) = g(h(x, y)) \qquad \text{Composition}$$

Y'

- A function that can be written as a sum of functions of the form $cx^m y^n$ (where c is a real number m and n are nonnegative integers) is called a **polynomial function** of two variables.

✓ For instance, the functions

$$f(x, y) = x^2 + y^2 - 2xy + x + 2 \quad \text{and} \quad g(x, y) = 3xy^2 + x - 2$$

are polynomial functions of two variables.

- A **rational function** is the quotient of two polynomial functions.

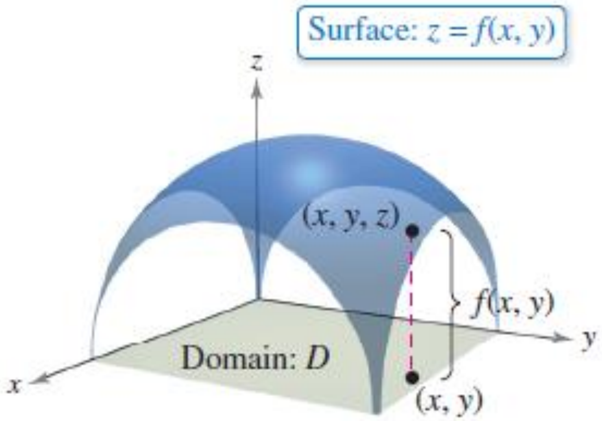
- Similar terminology is used for functions of more than two variables.

The Graph of a Function of Two Variables

The Graph of a Function of Two Variables

- The **graph** of a function *f* of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of *f*.

- This graph can be interpreted geometrically as a *surface in space*. In Figure note that the graph of $z = f(x, y)$ is a surface whose projection onto the *xy*-plane is *D*, the domain of *f*.



The Graph of a Function of Two Variables

- To each point (x, y) in D there corresponds a point (x, y, z) on the surface, and, conversely, to each point (x, y, z) on the surface there corresponds a point (x, y) in D.

Example 2: *Describing the Graph of a Function of Two Variables*

- Consider the function given by

$$f(x, y) = \sqrt{16 - 4x^2 - y^2}.$$

a. Find the domain and range of the function.

b. Describe the graph of *f*.

- Solution:

a. The domain *D* implied by the equation of *f* is the set of all points (x, y) such that

$$16 - 4x^2 - y^2 \geq 0.$$

So, *D* is the set of all points lying on or inside the ellipse

$$\frac{x^2}{4} + \frac{y^2}{16} = 1. \qquad \text{Ellipse in the } xy\text{-plane}$$

Example 2 – *Solution*

- The range of f is all values $z = f(x, y)$ such that $0 \le z \le \sqrt{16}$, or

$$0 \le z \le 4. \qquad \text{Range of } f$$

b. A point (x, y, z) is on the graph of f if and only if

$$z = \sqrt{16 - 4x^2 - y^2}$$

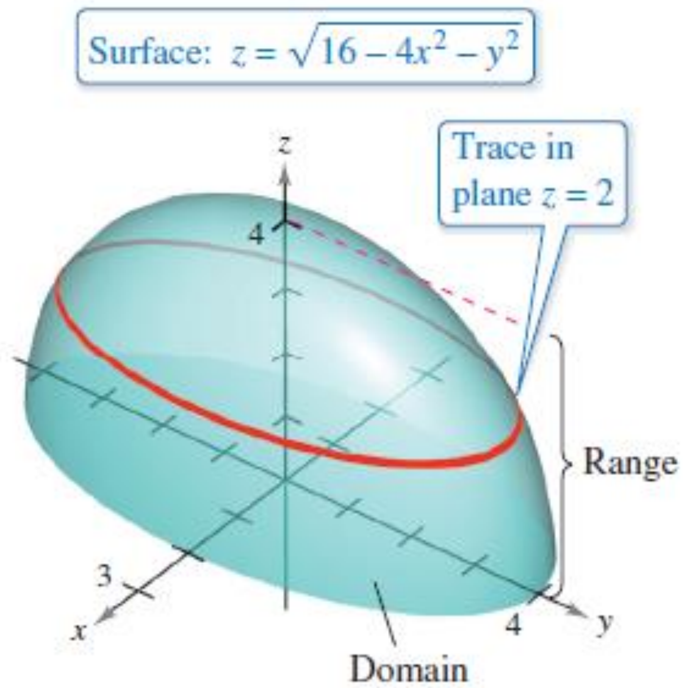
$$z^2 = 16 - 4x^2 - y^2$$

$$4x^2 + y^2 + z^2 = 16$$

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1, \qquad 0 \le z \le 4.$$

Example 2 – *Solution*

- You know that the graph of f is the upper half of an ellipsoid, as shown in below.



The graph of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$ is the upper half of an ellipsoid.

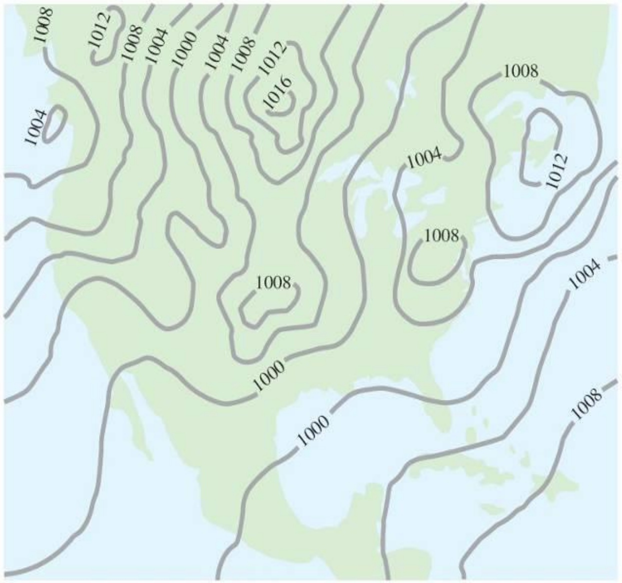
Level Curves

Level Curves

- A second way to visualize a function of two variables is to use a **scalar field** in which the scalar $z = f(x, y)$ is assigned to the point (x, y).

- A scalar field can be characterized by **level curves** (or **contour lines**) along which the value of $f(x, y)$ is constant.

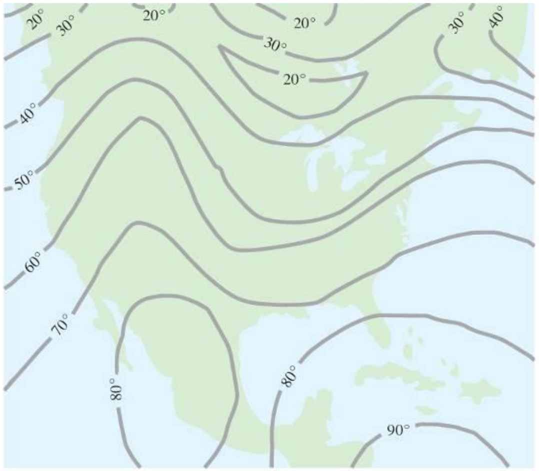
- For instance, the weather map in Figure shows level curves of equal pressure called **isobars.**



Level curves show the lines of equal pressure (isobars) measured in millibars.

Level Curves

- In weather maps for which the level curves represent points of equal temperature, the level curves are called **isotherms,** as shown in Figure.



Level curves show the lines of equal temperature (isotherms) measured in degrees Fahrenheit.

- Another common use of level curves is in representing electric potential fields.

- In this type of map, the level curves are called **equipotential lines.**

Level Curves

- Contour maps are commonly used to show regions on Earth's surface, with the level curves representing the height above sea level. This type of map is called a **topographic map.** For example, the mountain shown in Figure 13.7 is represented by the topographic map in Figure 13.8.

Figure 13.7

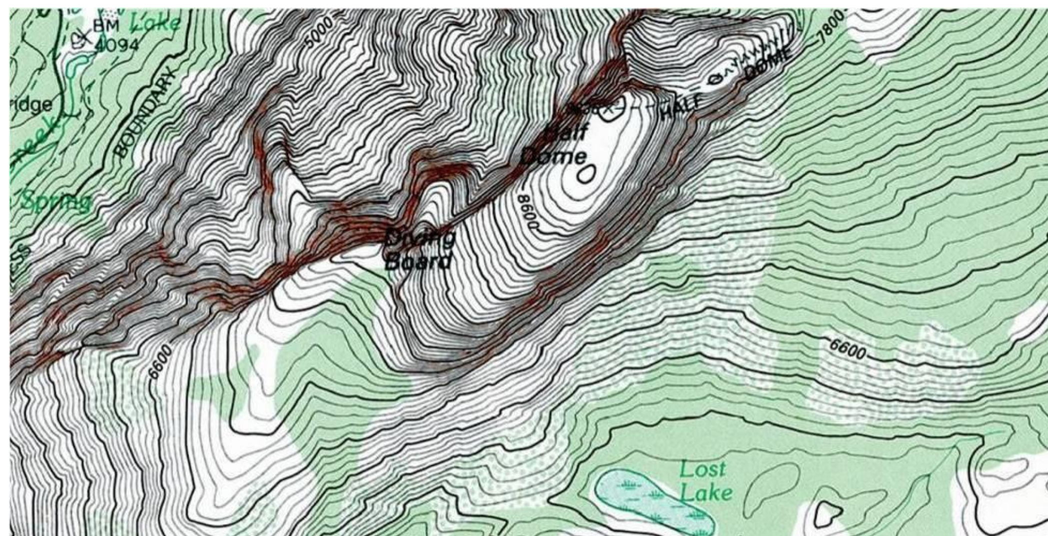


Figure 13.8

Level Curves

- A contour map depicts the variation of z with respect to x and y by the spacing between level curves.

- Much space between level curves indicates that z is changing slowly, whereas little space indicates a rapid change in z.

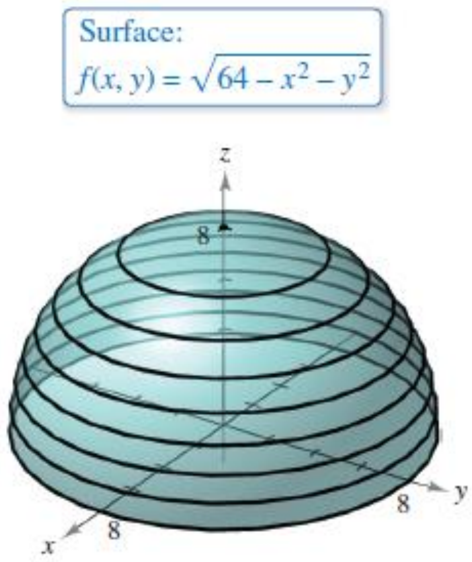
- Furthermore, to produce a good three-dimensional illusion in a contour map, it is important to choose c-values that are *evenly spaced*.

Example 3: *Sketching a Contour Map*

- The hemisphere

$$f(x, y) = \sqrt{64 - x^2 - y^2}$$

is shown in Figure. Sketch a contour map of this surface using level curves corresponding to $c = 0, 1, 2, \ldots, 8$.



Hemisphere

Example 3 – *Solution*

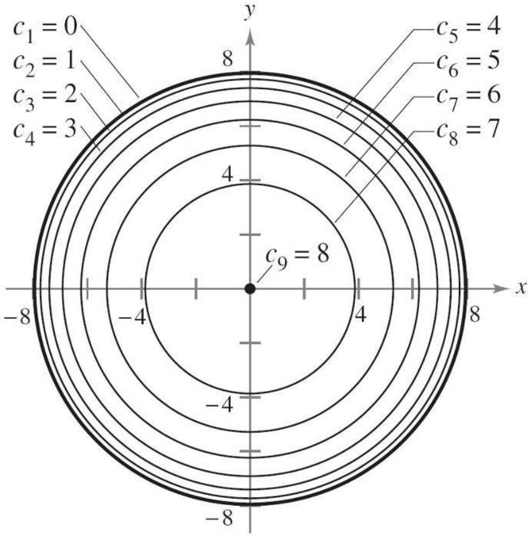
- For each value of c, the equation $f(x, y) = c$ is a circle (or point) in the xy-plane.

- For example, when $c_1 = 0$, the level curve is

$$x^2 + y^2 = 64$$ Circle of radius 8

which is a circle of radius 8.

Figure shows the nine level curves for the hemisphere.

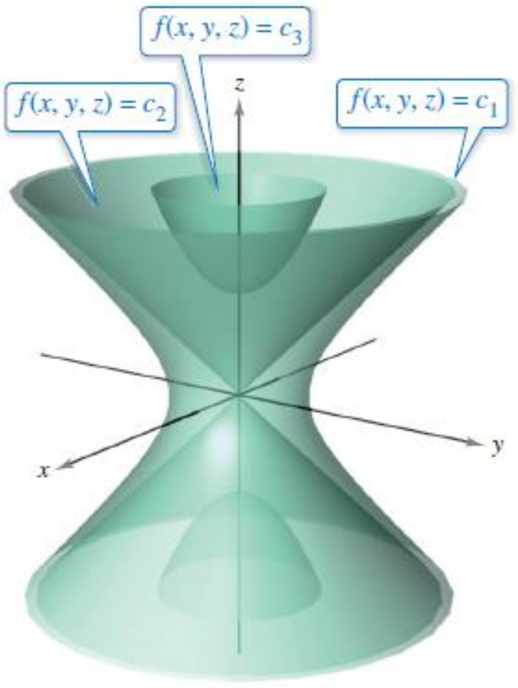


Contour map

Level Surfaces

- The concept of a level curve can be extended by one dimension to define a **level surface.**

- If f is a function of three variables and c is a constant, then the graph of the equation $f(x, y, z) = c$ is a **level surface** of f, as shown in Figure.



Level surfaces of f

Example 6 – *Level Surfaces*

- Describe the level surfaces of

$$f(x, y, z) = 4x^2 + y^2 + z^2.$$

- Solution:

Each level surface has an equation of the form

$$4x^2 + y^2 + z^2 = c. \qquad \text{Equation of level surface}$$

So, the level surfaces are ellipsoids (whose cross sections parallel to the *yz*-plane are circles).

As *c* increases, the radii of the circular cross sections increase according to the square root of *c*.

Example 6 – *Solution*

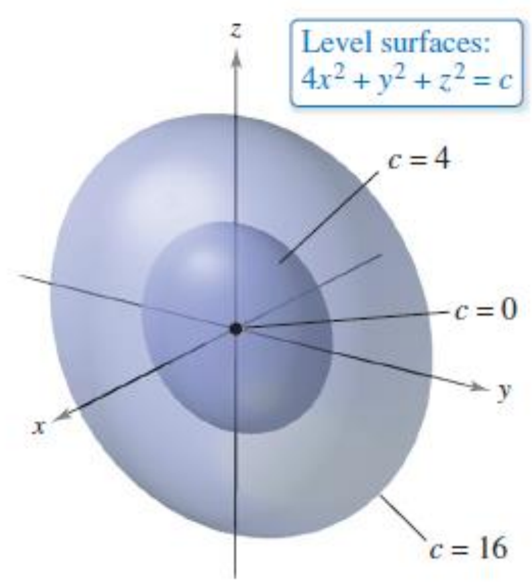
- For example, the level surfaces corresponding to the values $c = 0$, $c = 4$, and $c = 16$ are as follows.

$$4x^2 + y^2 + z^2 = 0$$ Level surface for $c = 0$ (single point)

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{4} = 1$$ Level surface for $c = 4$ (ellipsoid)

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1$$ Level surface for $c = 16$ (ellipsoid)

- These level surfaces are shown in Figure.



Level Surfaces

- If the function represented the *temperature* at the point (x, y, z), then the level surfaces shown in previous Figure would be called **isothermal surfaces.**

Computer Graphics

- The problem of sketching the graph of a surface in space can be simplified by using a computer.
- Although there are several types of three-dimensional graphing utilities, most use some form of trace analysis to give the illusion of three dimensions.
- To use such a graphing utility, you usually need to enter the equation of the surface and the region in the *xy*-plane over which the surface is to be plotted. (You might also need to enter the number of traces to be taken.)

- For instance, to graph the surface

$$f(x, y) = (x^2 + y^2)e^{1-x^2-y^2}$$

- you might choose the following bounds for *x*, *y*, and *z*.

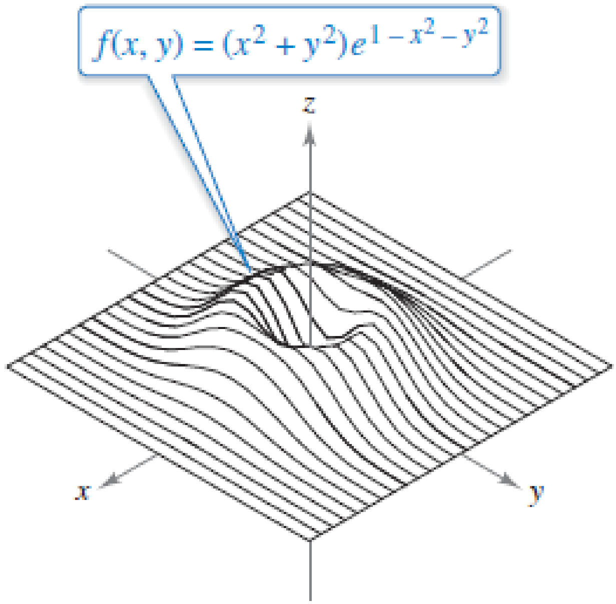
$$-3 \leq x \leq 3 \qquad \text{Bounds for } x$$

$$-3 \leq y \leq 3 \qquad \text{Bounds for } y$$

$$0 \leq z \leq 3 \qquad \text{Bounds for } z$$

- Figure shows a computer-generated graph of this surface using 26 traces taken parallel to the *yz*-plane.

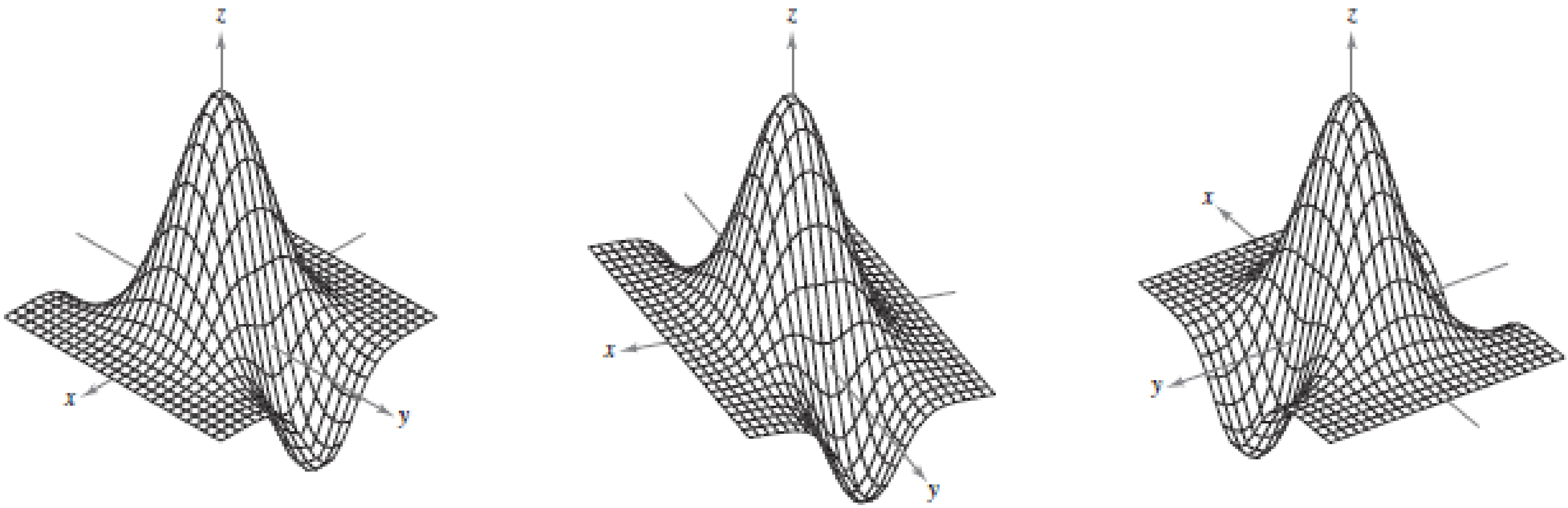
- To heighten the three-dimensional effect, the program uses a "hidden line" routine.



- That is, it begins by plotting the traces in the foreground (those corresponding to the largest *x*-values), and then, as each new trace is plotted, the program determines whether all or only part of the next trace should be shown.

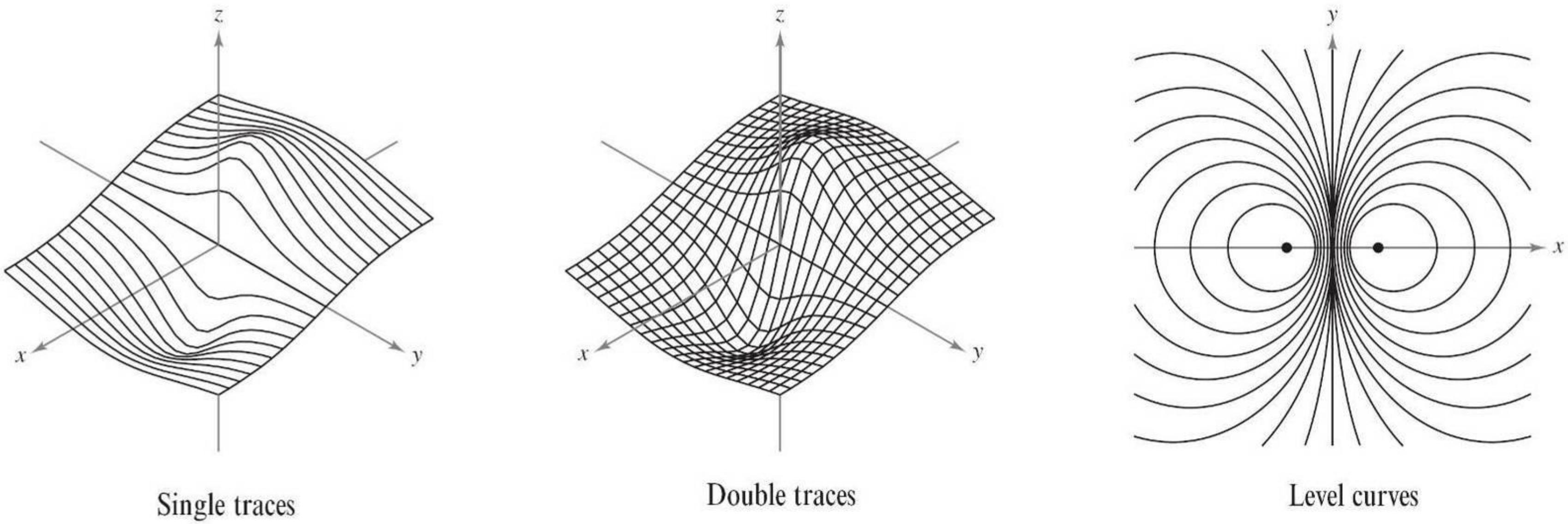
- The graphs on the next slide show a variety of surfaces that were plotted by computer.

- If you have access to a computer drawing program, use it to reproduce these surfaces.



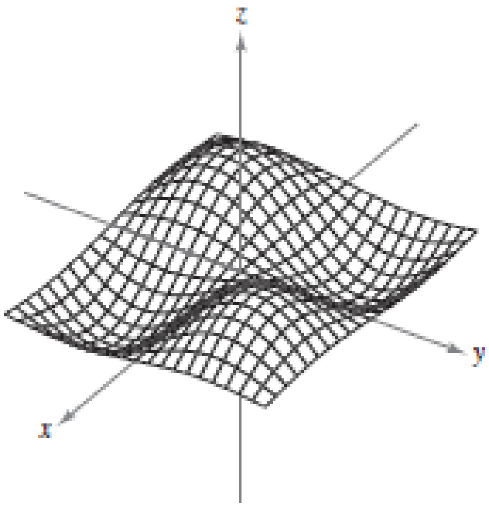
Three different views of the graph of $f(x, y) = (2 - y^2 + x^2)e^{1-x^2-(y^2/4)}$

Computer Graphics

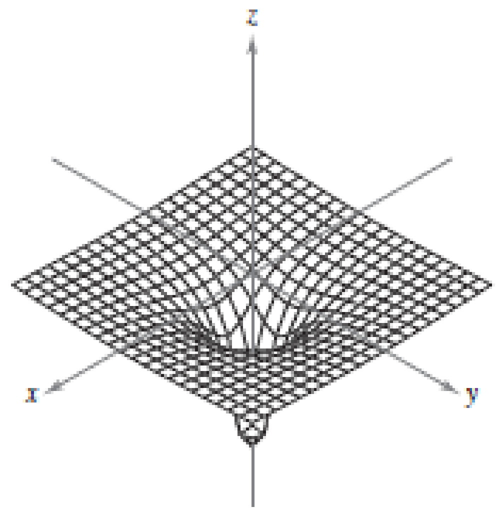


Traces and level curves of the graph of $f(x, y) = \dfrac{-4x}{x^2 + y^2 + 1}$

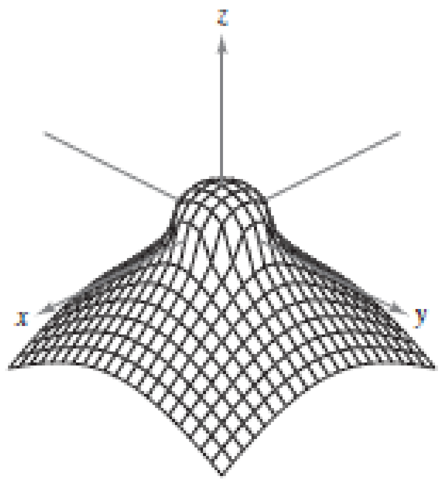
Computer Graphics



$f(x, y) = \sin x \sin y$



$$f(x, y) = -\frac{1}{\sqrt{x^2 + y^2}}$$



$$f(x, y) = \frac{1 - x^2 - y^2}{\sqrt{|1 - x^2 - y^2|}}$$

Suggested Problems

Exercise: 13.1-9,10,14,27,28,43,44,54,57

Thanks a lot …

13:2-Limits and Continuity

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Objectives

- Understand the definition of a neighborhood in the plane.

- Understand and use the definition of the limit of a function of two variables.

- Extend the concept of continuity to a function of two variables.

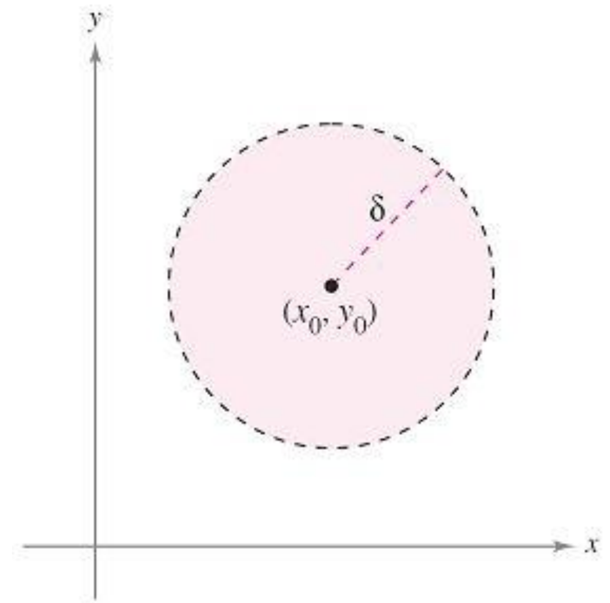
- Extend the concept of continuity to a function of three variables.

Neighborhoods in the Plane

- Using the formula for the distance between two points (x, y) and (x_0, y_0) in the plane, you can define the **δ-neighborhood** about (x_0, y_0) to be the **disk** centered at (x_0, y_0) with radius $\delta > 0$

$$\{(x, y): \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\}$$

Open disk



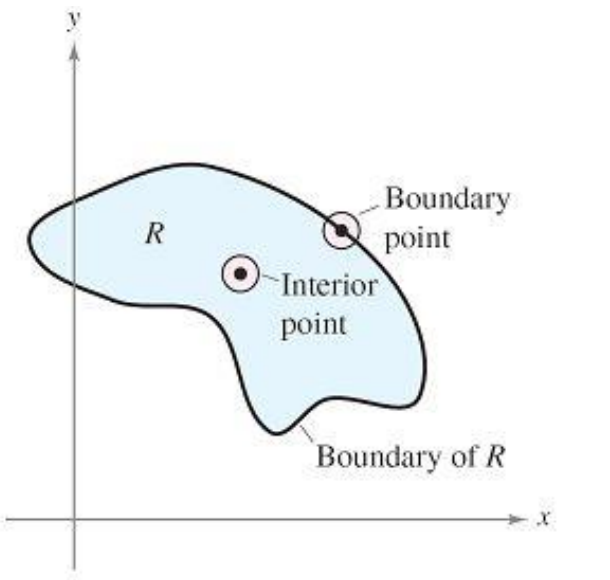
An open disk

as shown in Figure.

Neighborhoods in the Plane

- When this formula contains the *less than* inequality sign, <, the disk is called **open,** and when it contains the *less than or equal to* inequality sign, ≤, the disk is called **closed.** This corresponds to the use of < and ≤ to define open and closed intervals.

- Let the region R be a set of points in the plane. A point (x_0, y_0) in R is an **interior point** of R if there exists a δ-neighborhood about (x_0, y_0) that lies entirely in R, as shown in Figure.



The boundary and interior points of a region R

Neighborhoods in the Plane

- If every point in R is an interior point, then R is an **open region.** A point (x_0, y_0) is a **boundary point** of R if every open disk centered at (x_0, y_0) contains points inside R *and* points outside R. If R contains all its boundary points, then R is a **closed region.**

Limit of a Function of Two Variables

Definition of the Limit of a Function of Two Variables

Let f be a function of two variables defined, except possibly at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

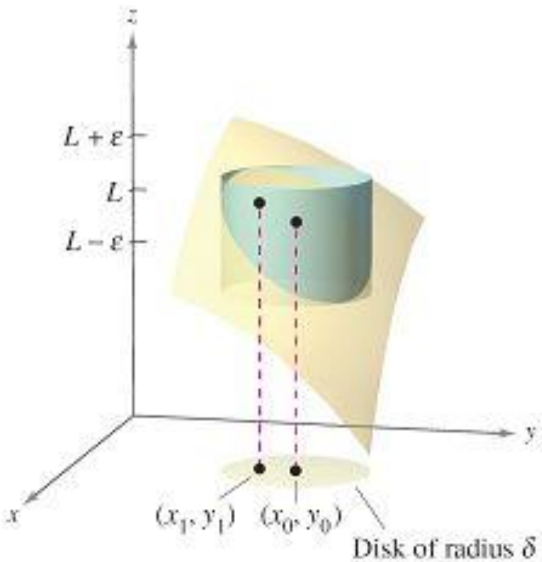
$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

if for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Limit of a Function of Two Variables

- Graphically, the definition of the limit of a function of two variables implies that for any point $(x, y) \neq (x_0, y_0)$ in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$, as shown in Figure.



For any (x, y) in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$.

Limit of a Function of Two Variables

- The definition of the limit of a function of two variables is similar to the definition of the limit of a function of a single variable, yet there is a critical difference.

- To determine whether a function of a single variable has a limit, you need only test the approach from two directions—from the right and from the left.

- When the function approaches the same limit from the right and from the left, you can conclude that the limit exists.

Limit of a Function of Two Variables

- For a function of two variables, however, the statement

$$(x, y) \to (x_0, y_0)$$

means that the point (x, y) is allowed to approach (x_0, y_0) from any direction.

- If the value of

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y)$$

is not the same for all possible approaches, or **paths,** to (x_0, y_0), then the limit does not exist.

Example 1 – *Verifying a Limit by the Definition*

- Show that $\lim_{(x,\, y)\to(a,\, b)} x = a.$

- Solution:

Let $f(x, y) = x$ and $L = a$.

You need to show that for each $\varepsilon > 0$, there exists a δ-neighborhood about (a, b) such that

$$|f(x, y) - L| = |x - a| < \varepsilon$$

whenever $(x, y) \neq (a, b)$ lies in the neighborhood.

Example 1 – *Solution*

- You can first observe that from

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

it follows that

$$\begin{aligned} |f(x,y) - L| &= |x - a| \\ &= \sqrt{(x-a)^2} \\ &\leq \sqrt{(x-a)^2 + (y-b)^2} \\ &< \delta. \end{aligned}$$

- So, you can choose $\delta = \varepsilon$, and the limit is verified.

Limit of a Function of Two Variables

- Limits of functions of several variables have the same properties regarding sums, differences, products, and quotients as do limits of functions of single variables.

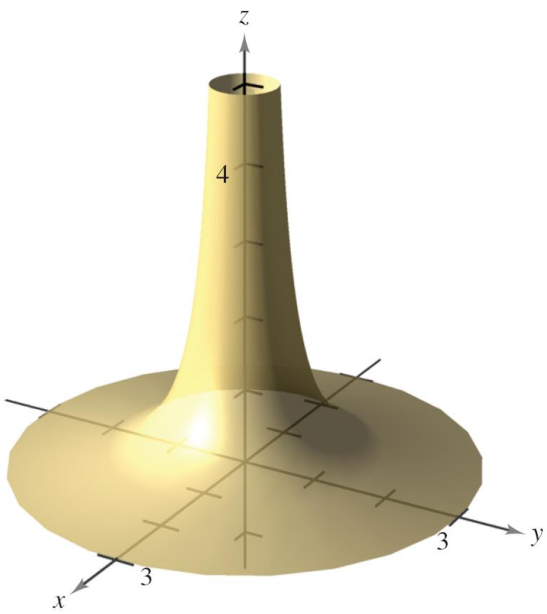
Limit of a Function of Two Variables

- For some functions, it is easy to recognize that a limit does not exist.

- For instance, it is clear that the limit

$$\lim_{(x, y)\to(0, 0)} \frac{1}{x^2 + y^2}$$

does not exist because the values of $f(x, y)$ increase without bound as (x, y) approaches $(0, 0)$ along *any* path (see Figure).



$\lim_{(x, y)\to(0, 0)} \frac{1}{x^2 + y^2}$ does not exist.

Continuity of a Function of Two Variables

- The limit of $f(x, y) = 5x^2y/(x^2 + y^2)$ as $(x, y) \to (1, 2)$ can be evaluated by direct substitution.

- That is, the limit is $f(1, 2) = 2$.

- In such cases, the function f is said to be **continuous** at the point (1, 2).

Continuity of a Function of Two Variables

Definition of Continuity of a Function of Two Variables

A function f of two variables is **continuous at a point (x_0, y_0)** in an open region R if $f(x_0, y_0)$ is defined and is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0). That is,

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = f(x_0, y_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

Continuity of a Function of Two Variables

- The function

$$f(x, y) = \frac{5x^2y}{x^2 + y^2}$$

is not continuous at (0, 0). Because the limit at this point exists, however, you can remove the discontinuity by defining *f* at (0, 0) as being equal to its limit there. Such a discontinuity is called **removable.**

- The function

$$f(x, y) = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

is not continuous at (0, 0), and this discontinuity is **nonremovable.**

Continuity of a Function of Two Variables

THEOREM 13.1 Continuous Functions of Two Variables

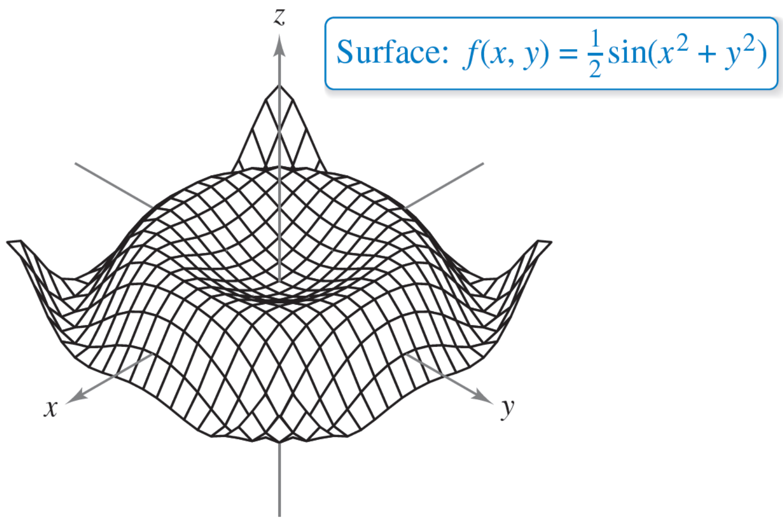
If k is a real number and $f(x, y)$ and $g(x, y)$ are continuous at (x_0, y_0), then the following functions are also continuous at (x_0, y_0).

1. Scalar multiple: kf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: f/g, $g(x_0, y_0) \neq 0$

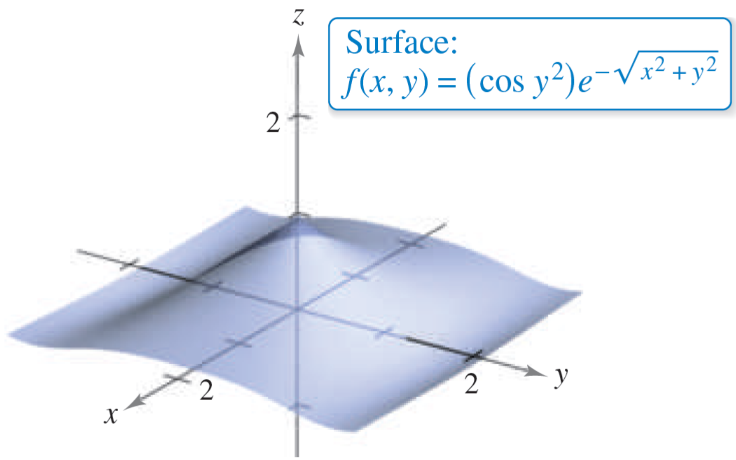
- Theorem 13.1 establishes the continuity of *polynomial* and *rational* functions at every point in their domains. Furthermore, the continuity of other types of functions can be extended naturally from one to two variables.

Continuity of a Function of Two Variables

- For instance, the functions whose graphs are shown in Figures are continuous at every point in the plane.



The function f is continuous at every point in the plane.



The function f is continuous at every point in the plane.

Continuity of a Function of Two Variables

THEOREM 13.2 Continuity of a Composite Function

If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the composite function given by $(g \circ h)(x, y) = g(h(x, y))$ is continuous at (x_0, y_0). That is,

$$\lim_{(x,y)\to(x_0,y_0)} g(h(x, y)) = g(h(x_0, y_0)).$$

- Note in Theorem 13.2 that h is a function of two variables and g is a function of one variable.

Example 5 – *Testing for Continuity*

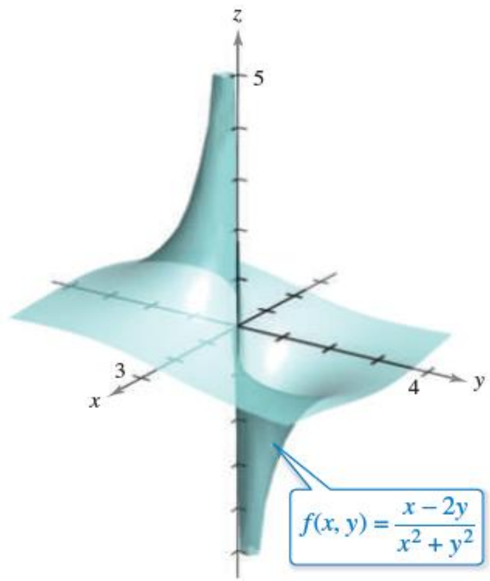
- Discuss the continuity of each function.

$$\textbf{a. } f(x, y) = \frac{x - 2y}{x^2 + y^2}$$

$$\textbf{b. } g(x, y) = \frac{2}{y - x^2}$$

Example 5(a) – *Solution*

- Because a rational function is continuous at every point in its domain, you can conclude that *f* is continuous at each point in the *xy*-plane except at (0, 0), as shown in Figure.



The function f is not continuous at (0, 0).

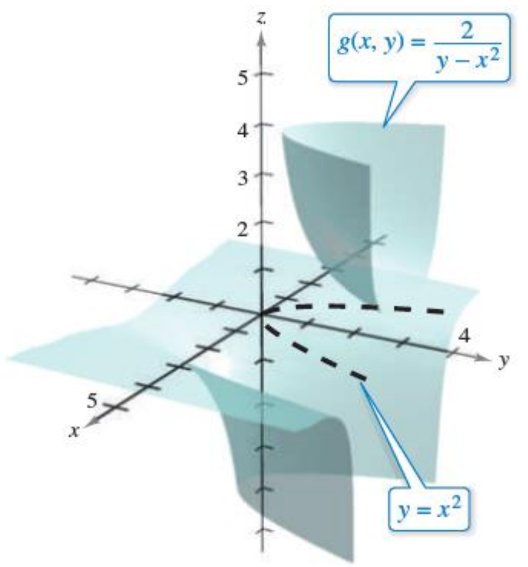
Example 5(b) – *Solution*

- The function $g(x, y) = 2/(y - x^2)$ is continuous except at the points at which the denominator is 0. These points are given by the equation
  $y - x^2 = 0.$

- So, you can conclude that the function is continuous at all points except those lying on the parabola $y = x^2$.

Example 5(b) – *Solution*

- Inside this parabola, you have $y > x^2$, and the surface represented by the function lies above the *xy*-plane, as shown in Figure 13.26.



The function g is not continuous on the parabola $y = x^2$.

- Outside the parabola, $y < x^2$, and the surface lies below the *xy*-plane.

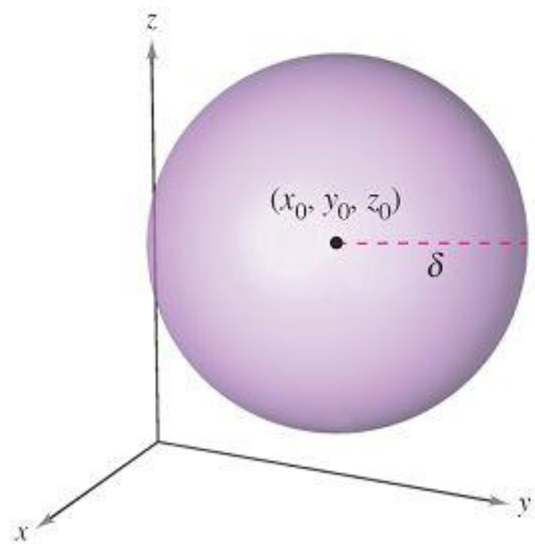
Continuity of a Function of Three Variables

- The definitions of limits and continuity can be extended to functions of three variables by considering points (x, y, z) within the *open sphere*

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2.$$

Open sphere

- The radius of this sphere is δ, and the sphere is centered at (x_0, y_0, z_0) as shown in Figure.



Open sphere in space

Continuity of a Function of Three Variables

- A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if there exists a δ-sphere about (x_0, y_0, z_0) that lies entirely in R. If every point in R is an interior point, then R is called **open.**

Definition of Continuity of a Function of Three Variables

A function f of three variables is **continuous at a point (x_0, y_0, z_0)** in an open region R if $f(x_0, y_0, z_0)$ is defined and is equal to the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0). That is,

$$\lim_{(x, y, z)\to(x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

Example 6 – *Testing Continuity of a Function of Three Variables*

- Discuss the continuity of

$$f(x, y, z) = \frac{1}{x^2 + y^2 - z}.$$

- Solution:

The function *f* is continuous except at the points at which the denominator is 0, which are given by the equation

$$x^2 + y^2 - z = 0.$$

- So, *f* is continuous at each point in space except at the points on the paraboloid

$$z = x^2 + y^2.$$

Suggested Problems

Exercise 13.2: 16,20,22,23,30,31,42,45,46

Thanks a lot …

13:3-Partial Derivatives

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Objectives

- Find and use partial derivatives of a function of two variables.
- Find and use partial derivatives of a function of three or more variables.
- Find higher-order partial derivatives of a function of two or three variables.

Partial Derivatives of a Function of Two Variables

- You can determine the rate of change of a function f with respect to one of its several independent variables.

- This process is called **partial differentiation,** and the result is referred to as the **partial derivative** of f with respect to the chosen independent variable.

Partial Derivatives of a Function of Two Variables

Definition of Partial Derivatives of a Function of Two Variables

If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$ Partial derivative with respect to x

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$ Partial derivative with respect to y

provided the limits exist.

- This definition indicates that if $z = f(x, y)$, then to find f_x, you *consider y constant* and differentiate with respect to x.
- Similarly, to find f_y, you *consider x constant* and differentiate with respect to y.

Example 1 – Finding Partial Derivatives

- **a**. To find f_x for $f(x, y) = 3x - x^2y^2 + 2x^3y$, consider y to be constant and differentiate with respect to x.

$$f_x(x, y) = 3 - 2xy^2 + 6x^2y \qquad \text{Partial derivative with respect to } x$$

- To find f_y, consider x to be constant and differentiate with respect to y.

$$f_y(x, y) = -2x^2y + 2x^3 \qquad \text{Partial derivative with respect to } y$$

Example 1 – Finding Partial Derivatives cont'd

- **b**. To find f_x for $(x, y) = (\ln x)(\sin x^2y)$, consider y to be constant and differentiate with respect to x.

$$f_x(x, y) = (\ln x)(\cos x^2y)(2xy) + \frac{\sin x^2y}{x}$$ Partial derivative with respect to x

- To find f_y, consider x to be constant and differentiate with respect to y.

$$f_y(x, y) = (\ln x)(\cos x^2y)(x^2)$$ Partial derivative with respect to y

Partial Derivatives of a Function of Two Variables

Notation for First Partial Derivatives

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$

Partial derivative with respect to x

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$

Partial derivative with respect to y

The first partials evaluated at the point (a, b) are denoted by

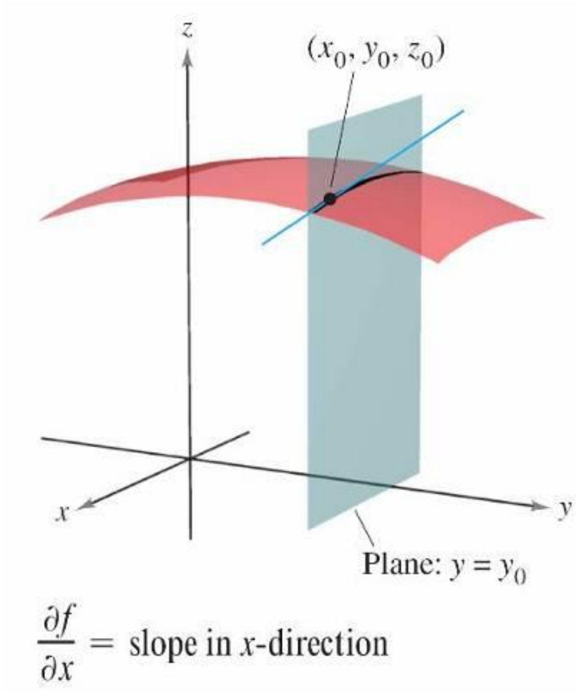
$$\left.\frac{\partial z}{\partial x}\right|_{(a, b)} = f_x(a, b)$$

and

$$\left.\frac{\partial z}{\partial y}\right|_{(a, b)} = f_y(a, b).$$

Partial Derivatives of a Function of Two Variables

- The partial derivatives of a function of two variables, $z = f(x, y)$, have a useful geometric interpretation.
- If $y = y_0$, then $z = f(x, y_0)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$, as shown in Figure.



Partial Derivatives of a Function of Two Variables

- Therefore,

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

represents the slope of this curve at the point $(x_0, y_0, f(x_0, y_0))$.

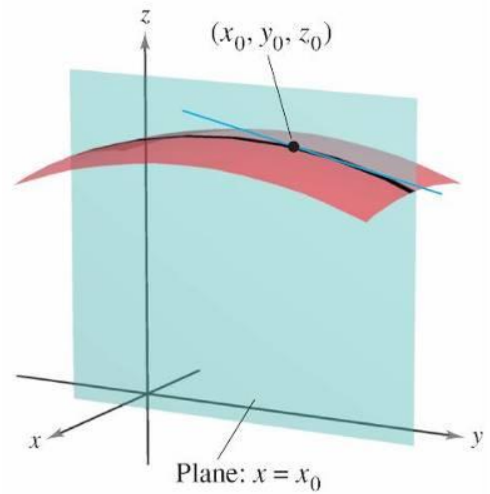
- Note that both the curve and the tangent line lie in the plane $y = y_0$.

Partial Derivatives of a Function of Two Variables

- Similarly,

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

represents the slope of the curve given by the intersection of $z = f(x, y)$ and the plane $x = x_0$ at $(x_0, y_0, f(x_0, y_0))$, as shown in Figure.



$\frac{\partial f}{\partial y}$ = slope in y-direction

Partial Derivatives of a Function of Two Variables

- Informally, the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point (x_0, y_0, z_0) denote the **slopes of the surface in the *x*- and *y*-directions**, respectively.

Example 3 – Finding the Slopes of a Surface

- Find the slopes in the *x*-direction and in the *y*-direction of the surface

$$f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$$

at the point $\left(\frac{1}{2}, 1, 2\right)$.

- Solution:

The partial derivatives of *f* with respect to *x* and *y* are

$$f_x(x, y) = -x \qquad \text{and} \qquad f_y(x, y) = -2y.$$

Partial derivatives

Example 3 – Solution

cont'd

- So, in the x-direction, the slope is

$$f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2}$$

Figure 13.30

- and in the y-direction, the slope is

$$f_y\left(\frac{1}{2}, 1\right) = -2.$$

Figure 13.31

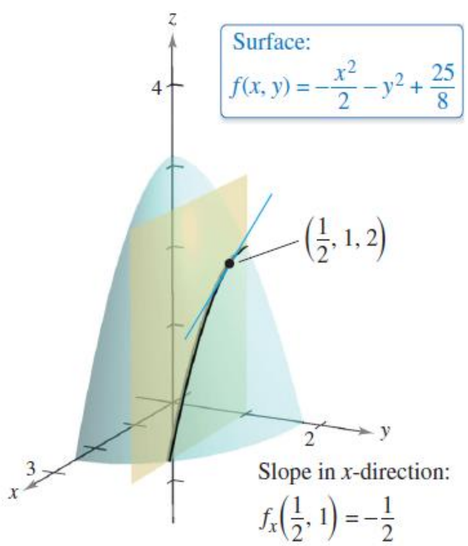


Figure 13.30

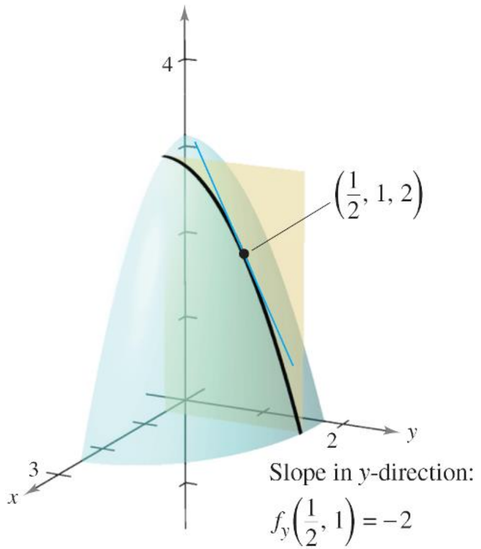


Figure 13.31

Partial Derivatives of a Function of Three or More Variables

- The concept of a partial derivative can be extended naturally to functions of three or more variables. For instance, if $w = f(x, y, z)$, then there are three partial derivatives, each of which is formed by holding two of the variables constant.

- That is, to define the partial derivative of w with respect to x, consider y and z to be constant and differentiate with respect to x.

- A similar process is used to find the derivatives of w with respect to y and with respect to z.

Partial Derivatives of a Function of Three or More Variables

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

- In general, if $w = f(x_1, x_2, \ldots, x_n)$, then there are n partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n.$$

- To find the partial derivative with respect to one of the variables, hold the other variables constant and differentiate with respect to the given variable.

Example 6 – Finding Partial Derivatives

- **a.** To find the partial derivative of $f(x, y, z) = xy + yz^2 + xz$ with respect to z, consider x and y to be constant and obtain

$$\frac{\partial}{\partial z}[xy + yz^2 + xz] = 2yz + x.$$

- **b.** To find the partial derivative of $f(x, y, z) = z \sin(xy^2 + 2z)$ with respect to z, consider x and y to be constant. Then, using the Product Rule, you obtain

$$\begin{aligned}\frac{\partial}{\partial z}[z \sin(xy^2 + 2z)] &= (z)\frac{\partial}{\partial z}[\sin(xy^2 + 2z)] + \sin(xy^2 + 2z)\frac{\partial}{\partial z}[z] \\ &= (z)[\cos(xy^2 + 2z)](2) + \sin(xy^2 + 2z) \\ &= 2z \cos(xy^2 + 2z) + \sin(xy^2 + 2z).\end{aligned}$$

Example 6 – Finding Partial Derivatives

cont'd

- **c.** To find the partial derivative of

$$f(x, y, z, w) = \frac{x + y + z}{w}$$

with respect to *w*, consider *x*, *y*, and *z* to be constant obtain

$$\frac{\partial}{\partial w}\left[\frac{x + y + z}{w}\right] = -\frac{x + y + z}{w^2}.$$

Higher-Order Partial Derivatives

- The function $z = f(x, y)$ has the following second partial derivatives.
  - **1.** Differentiate twice with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

  - **2.** Differentiate twice with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

  - **3.** Differentiate first with respect to x and then with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$

Higher-Order Partial Derivatives

✓ **4.** Differentiate first with respect to *y* and then with respect to *x*:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

The third and fourth cases are called **mixed partial derivatives**.

Example 7 – Finding Second Partial Derivatives

- Find the second partial derivatives of $f(x, y) = 3xy^2 - 2y + 5x^2y^2$ and determine the value of $f_{xy}(-1, 2)$.

- Solution:

  Begin by finding the first partial derivatives with respect to x and y.

  $$f_x(x, y) = 3y^2 + 10xy^2 \quad \text{and} \quad f_y(x, y) = 6xy - 2 + 10x^2y$$

- Then, differentiate each of these with respect to x and y.

  $$f_{xx}(x, y) = 10y^2 \quad \text{and} \quad f_{yy}(x, y) = 6x + 10x^2$$

Example 7 – Solution

cont'd

$f_{xy}(x, y) = 6y + 20xy$ and $f_{yx}(x, y) = 6y + 20xy$

At (–1, 2), the value of f_{xy} is $f_{xy}(-1, 2) = 12 - 40 = -28$.

Suggested Problems

Exercise 13.3:16,28,37,43,52,55,62,68,88,92,71,129,128

Thanks a lot …

13:4-Differentials

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Objectives

- Understand the concepts of increments and differentials.
- Extend the concept of differentiability to a function of two variables.
- Use a differential as an approximation

Increments and Differentials

- For $y = f(x)$, the differential of y was defined as $dy = f'(x)dx$.

- Similar terminology is used for a function of two variables, $z = f(x, y)$. That is, Δx and Δy are the **increments of *x* and *y***, and the **increment of *z*** is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

Increment of z

Increments and Differentials

Definition of Total Differential

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the **differentials** of the independent variables x and y are

$$dx = \Delta x \quad \text{and} \quad dy = \Delta y$$

and the **total differential** of the dependent variable z is

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = f_x(x, y)\,dx + f_y(x, y)\,dy.$$

- This definition can be extended to a function of three or more variables. For instance, if $w = f(x, y, z, u)$, then $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$, $du = \Delta u$, and the total differential of w is

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz + \frac{\partial w}{\partial u}\,du.$$

Example 1 – *Finding the Total Differential*

- Find the total differential for each function.

  a. $z = 2x \sin y - 3x^2y^2$ **b.** $w = x^2 + y^2 + z^2$

- Solution:

  a. The total differential *dz* for $z = 2x \sin y - 3x^2y^2$ is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad \text{Total differential } dz$$

$$= (2 \sin y - 6xy^2)\,dx + (2x \cos y - 6x^2y)\,dy.$$

b. The total differential *dw* for $w = x^2 + y^2 + z^2$ is

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \qquad \text{Total differential } dw$$

$$= 2x\,dx + 2y\,dy + 2z\,dz.$$

Differentiability

- For a *differentiable* function given by $y = f(x)$, you can use the differential $dy = f'(x)dx$ as an approximation (for small Δx) of the value $\Delta y = f(x + \Delta x) - f(x)$.
- When a similar approximation is possible for a function of two variables, the function is said to be **differentiable**. This is stated explicitly in the following definition.

Definition of Differentiability

A function f given by $z = f(x, y)$ is **differentiable** at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both ε_1 and $\varepsilon_2 \to 0$ as

$$(\Delta x, \Delta y) \to (0, 0).$$

The function f is **differentiable in a region** $\boldsymbol{R}$ if it is differentiable at each point in R.

Example 2 – Showing That a Function Is Differentiable

- Show that the function

  $$f(x, y) = x^2 + 3y$$

  is differentiable at every point in the plane.
- Solution:

  Letting $z = f(x, y)$, the increment of z at an arbitrary point (x, y) in the plane is

  $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$ Increment of z

  $$= (x + \Delta x)^2 + 3(y + \Delta y) - (x^2 + 3y)$$
  $$= x^2 + 2x\Delta x + (\Delta x^2) + 3y + 3\Delta y - x^2 - 3y$$
  $$= 2x\Delta x + (\Delta x)^2 + 3\Delta y$$

Example 2 – *Solution* cont'd

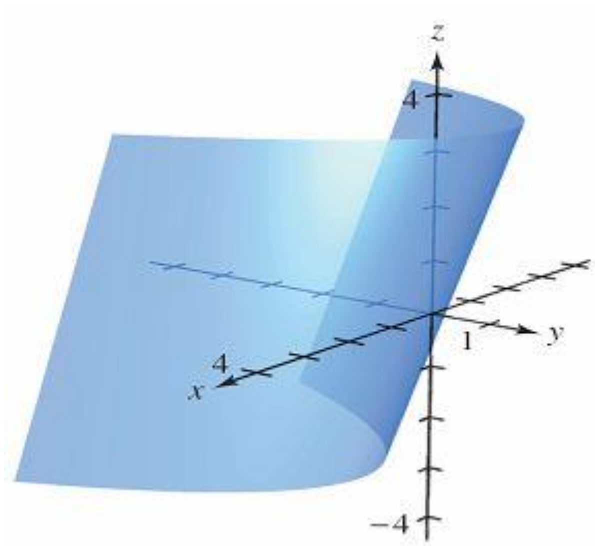
$$= 2x(\Delta x) + 3(\Delta y) + \Delta x(\Delta x) + 0(\Delta y)$$

$$= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

where $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = 0$.

✓ Because $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$, it follows that f is differentiable at every point in the plane.

The graph of f is shown in Figure.



Differentiability

THEOREM 13.4 Sufficient Condition for Differentiability

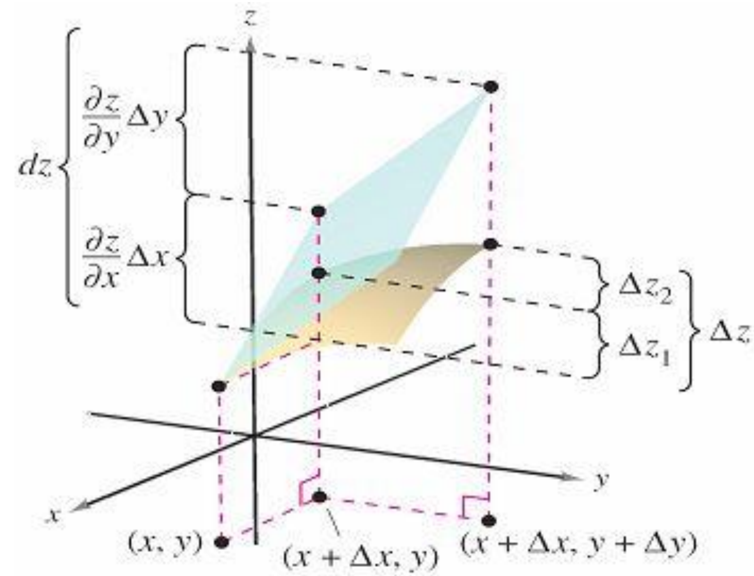
If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

Approximation by Differentials

- Theorem 13.4 tells you that you can choose $(x + \Delta x, y + \Delta y)$ close enough to (x, y) to make $\varepsilon_1 \Delta x$ and $\varepsilon_2 \Delta y$ insignificant. In other words, for small Δx and Δy, you can use the approximation

✓ $\Delta z \approx dz.$ Approximate change in z

✓ This approximation is illustrated graphically in Figure 13.35.



The exact change in z is Δz. This change can be approximated by the differential dz.

Approximation by Differentials

- The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ can be interpreted as the slopes of the surface in the *x*- and *y*-directions.

- This means that

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

represents the change in height of a plane that is tangent to the surface at the point $(x, y, f(x, y))$.

- Because a plane in space is represented by a linear equation in the variables *x*, *y*, and *z*, the approximation of Δz by dz is called a **linear approximation**.

Example 3 – *Using a Differential as an Approximation*

- Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x, y) moves from the point (1, 1) to the point (1.01, 0.97). Compare this approximation with the exact change in z.

- Solution:

Letting $(x, y) = (1, 1)$ and $(x + \Delta x, y + \Delta y) = (1.01, 0.97)$ produces

$$dx = \Delta x = 0.01 \text{ and } dy = \Delta y = -0.03.$$

Example 3 – *Solution*

cont'd

- So, the change in z can be approximated by

$$\Delta z \approx dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= \frac{-x}{\sqrt{4 - x^2 - y^2}}\Delta x + \frac{-y}{\sqrt{4 - x^2 - y^2}}\Delta y.$$

- When $x = 1$ and $y = 1$, you have

$$\Delta z \approx -\frac{1}{\sqrt{2}}(0.01) - \frac{1}{\sqrt{2}}(-0.03)$$

$$= \frac{0.02}{\sqrt{2}}$$

$$= \sqrt{2}(0.01) \approx 0.0141.$$

Example 3 – *Solution*

cont'd

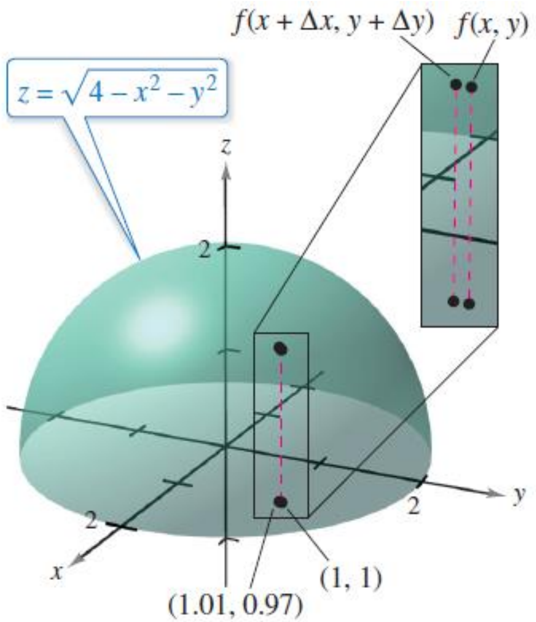
- In Figure, you can see that the exact change corresponds to the difference in the heights of two points on the surface of a hemisphere.

- This difference is given by

$\Delta z = f(1.01, 0.97) - f(1, 1)$

$$= \sqrt{4 - (1.01)^2 - (0.97)^2} - \sqrt{4 - 1^2 - 1^2}$$

$$\approx 0.0137.$$



As (x, y) moves from the point (1, 1) to the point (1.01, 0.97), the value of $f(x, y)$ changes by about 0.0137.

Approximation by Differentials

- A function of three variables $w = f(x, y, z)$ is **differentiable** at (x, y, z) provided that
  $\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$
  can be written in the form
  $\Delta w = f_x\Delta x + f_y\Delta y + f_z\Delta z + \varepsilon_1\Delta x + \varepsilon_2\Delta y + \varepsilon_3\Delta z$
  ✓where ε_1, ε_2, and $\varepsilon_3 \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0, 0, 0)$.

- With this definition of differentiability, Theorem 13.4 has the following extension for functions of three variables: If f is a function of x, y, and z, where f_x, f_y, and f_z are continuous in an open region R, then f is differentiable on R.

Approximation by Differentials

- As is true for a function of a single variable, when a function in two or more variables is differentiable at a point, it is also continuous there.

THEOREM 13.5 Differentiability Implies Continuity

If a function of x and y is differentiable at (x_0, y_0), then it is continuous at (x_0, y_0).

Example 5 – *A Function That Is Not Differentiable*

- For the function,

$$f(x, y) = \begin{cases} \dfrac{-3xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

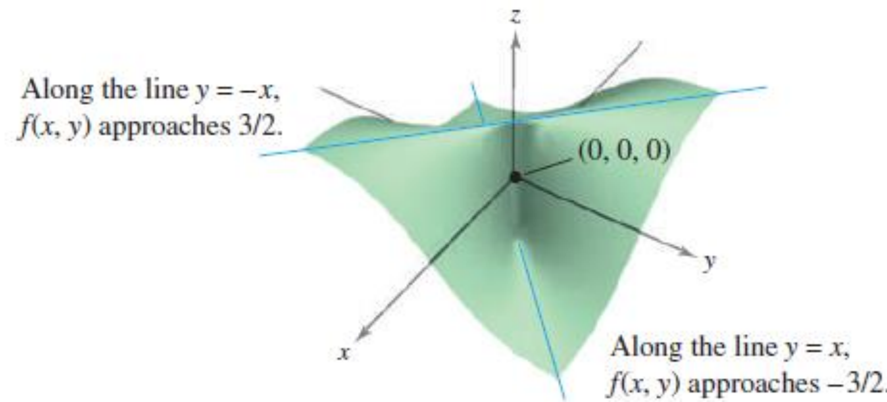
show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but that f is not differentiable at (0, 0).

Example 5 – *Solution*

- You can show that f is not differentiable at (0, 0) by showing that it is not continuous at this point.

- To see that f is not continu at (0, 0), look at the values $f(x, y)$ along two different approaches to (0, 0), as shown in Figure.

$$f(x, y) = \begin{cases} \dfrac{-3xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$



Example 5 – *Solution* cont'd

- Along the line $y = x$, the limit is

$$\lim_{(x,\,x)\to(0,\,0)} f(x, y) = \lim_{(x,\,x)\to(0,\,0)} \frac{-3x^2}{2x^2}$$

whereas along $y = -x$ you have

$$\lim_{(x,\,-x)\to(0,\,0)} f(x, y) = \lim_{(x,\,-x)\to(0,\,0)} \frac{3x^2}{2x^2} = \frac{3}{2}.$$

- So, the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist, and you can conclude that f is not continuous at (0, 0).

- Therefore, by Theorem 13.5, you know that f is not differentiable at (0, 0).

Example 5 – *Solution* cont'd

- On the other hand, by the definition of the partial derivatives f_x and f_y you have

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

and

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0.$$

✓ So, the partial derivatives at (0, 0) exist.

Suggested Problems

Exercise 13.4:7,8,11,13,16,23,32.

Thanks a lot …

13:5-Chain Rules for Functions of Several Variables

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Objectives

- Use the Chain Rules for functions of several variables.
- Find partial derivatives implicitly.

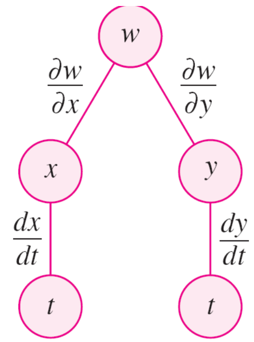
Chain Rules for Functions of Several Variables

THEOREM 13.6 Chain Rule: One Independent Variable

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

The Chain Rule is shown schematically in Figure 13.39.



Chain Rule: one independent variable w is a function of x and y, which are each functions of t. This diagram represents the derivative of w with respect to t.

Figure 13.39

Example 1 – ***Chain Rule: One Independent Variable***

- Let $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Find dw/dt when $t = 0$.

- Solution:

By the Chain Rule for one independent variable, you have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= 2xy(\cos t) + (x^2 - 2y)e^t$$

Example 1 – *Solution*

$$= 2(\sin t)(e^t)(\cos t) + (\sin^2 t - 2e^t)e^t$$

$$= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}.$$

When $t = 0$, it follows that

$$\frac{dw}{dt} = -2.$$

Chain Rules for Functions of Several Variables

- The Chain Rule in Theorem 13.6 can provide alternative techniques for solving many problems in single-variable calculus. For instance, in Example 1, you could have used single-variable techniques to find *dw*/*dt* by first writing *w* as a function of *t*,

$$\begin{aligned} w &= x^2y - y^2 \\ &= (\sin t)^2(e^t) - (e^t)^2 \\ &= e^t \sin^2 t - e^{2t} \end{aligned}$$

and then $\dfrac{dw}{dt} = 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}$

Chain Rules for Functions of Several Variables

- The Chain Rule in Theorem 13.6 can be extended to any number of variables. For example, if each x_i is a differentiable function of a single variable t, then for

$$w = f(x_1, x_2, \ldots, x_n)$$

- you have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial w}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}.$$

Chain Rules for Functions of Several Variables

THEOREM 13.7 Chain Rule: Two Independent Variables

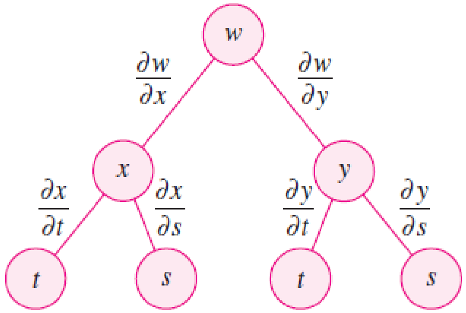
Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(s, t)$ and $y = h(s, t)$ such that the first partials $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, and $\partial y/\partial t$ all exist, then $\partial w/\partial s$ and $\partial w/\partial t$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

The Chain Rule is shown schematically in Figure 13.41.



Chain Rule: two independent variables

Figure 13.41

Example 4 – *The Chain Rule with Two Independent Variables*

- Use the Chain Rule to find $\partial w/\partial s$ and $\partial w/\partial t$ for $w = 2xy$ where $x = s^2 + t^2$ and $y = s/t$.

- Solution:

  Using Theorem 13.7, you can hold t constant and differentiate with respect to s to obtain

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$= 2y(2s) + 2x\left(\frac{1}{t}\right)$$

Example 4 – *Solution*

cont'd

$$= 2\left(\frac{s}{t}\right)(2s) + 2(s^2 + t^2)\left(\frac{1}{t}\right) \qquad \text{Substitute } (s/t) \text{ for } y \text{ and } s^2 + t^2 \text{ for } x.$$

$$= \frac{4s^2}{t} + \frac{2s^2 + 2t^2}{t}$$

$$= \frac{6s^2 + 2t^2}{t}.$$

Similarly, holding *s* constant gives

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

Example 4 – *Solution* cont'd

$$= 2y(2t) + 2x\left(\frac{-s}{t^2}\right)$$

$$= 2\left(\frac{s}{t}\right)(2t) + 2(s^2 + t^2)\left(\frac{-s}{t^2}\right)$$ Substitute (s/t) for y and $s^2 + t^2$ for x.

$$= 4s - \frac{2s^3 + 2st^2}{t^2}$$

$$= \frac{4st^2 - 2s^3 - 2st^2}{t^2}$$

$$= \frac{2st^2 - 2s^3}{t^2}.$$

Chain Rules for Functions of Several Variables

- The Chain Rule in Theorem 13.7 can also be extended to any number of variables. For example, if w is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ where each x_i is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$, then for $w = f(x_1, x_2, \ldots, x_n)$ you obtain the following.

$$\frac{\partial w}{\partial t_1} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$

$$\frac{\partial w}{\partial t_2} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2}$$

$$\vdots$$

$$\frac{\partial w}{\partial t_m} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

Implicit Partial Differentiation

- This section concludes with an application of the Chain Rule to determine the derivative of a function defined *implicitly*.

✓Let x and y be related by the equation $F(x, y) = 0$, where $y = f(x)$ is a differentiable function of x. To find dy/dx, you could use the techniques discussed in Section 2.5. You will see, however, that the Chain Rule provides a convenient alternative. Consider the function $w = F(x, y) = F(x, f(x))$.

- You can apply Theorem 13.6 to obtain

$$\frac{dw}{dx} = F_x(x, y)\frac{dx}{dx} + F_y(x, y)\frac{dy}{dx}.$$

Implicit Partial Differentiation

THEOREM 13.8 Chain Rule: Implicit Differentiation

If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0.$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

Example 6 – *Finding a Derivative Implicitly*

- Find dy/dx for $y^3 + y^2 - 5y - x^2 + 4 = 0$.
- Solution:

  Begin by letting

  $F(x, y) = y^3 + y^2 - 5y - x^2 + 4.$

  Then

  $F_x(x, y) = -2x$ and $F_y(x, y) = 3y^2 + 2y - 5.$

- Using Theorem 13.8, you have

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = \frac{-(-2x)}{3y^2 + 2y - 5} = \frac{2x}{3y^2 + 2y - 5}.$$

Suggested Problems

Exercise 13.5: 11,17,22,25,29.

Thanks a lot …

13:6-Directional Derivatives and Gradients

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Objectives

- Find and use directional derivatives of a function of two variables.
- Find the gradient of a function of two variables.
- Use the gradient of a function of two variables in applications.
- Find directional derivatives and gradients of functions of three variables.

Directional Derivative

- You are standing on the hillside represented by $z = f(x, y)$ in Figure 13.42 and want to determine the hill's incline toward the z-axis. You already know how to determine the slopes in two different directions—the slope in the y-direction is given by the partial derivative $f_y(x, y)$, and the slope in the x-direction is given by the partial derivative $f_x(x, y)$.
- In this section, you will see that these two partial derivatives can be used to find the slope in *any* direction.

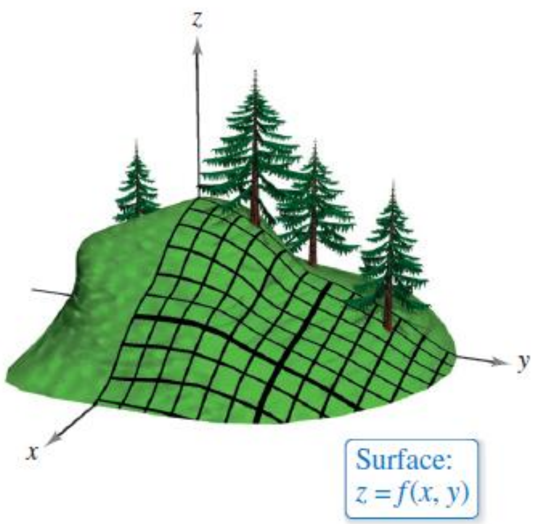


Figure 13.42

Directional Derivative

- To determine the slope at a point on a surface, you will define a new type of derivative called a **directional derivative.**
- Begin by letting $z = f(x, y)$ be a *surface* and $P(x_0, y_0)$ be a *point* in the domain of f, as shown in Figure 13.43.
- The "direction" of the directional derivative is given by a unit vector

  $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$

  where θ is the angle the vector makes with the positive x-axis.

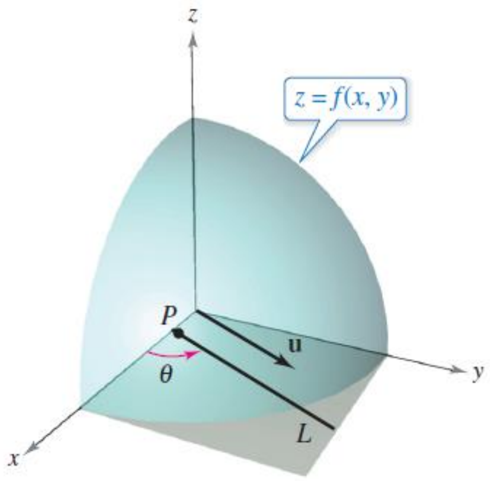


Figure 13.43

Directional Derivative

- To find the desired slope, reduce the problem to two dimensions by intersecting the surface with a vertical plane passing through the point *P* and parallel to **u**, as shown in Figure 13.44.
- This vertical plane intersects the surface to form a curve *C*.
- The slope of the surface at $(x_0, y_0, f(x_0, y_0))$ in the direction of **u** is defined as the slope of the curve *C* at that point.

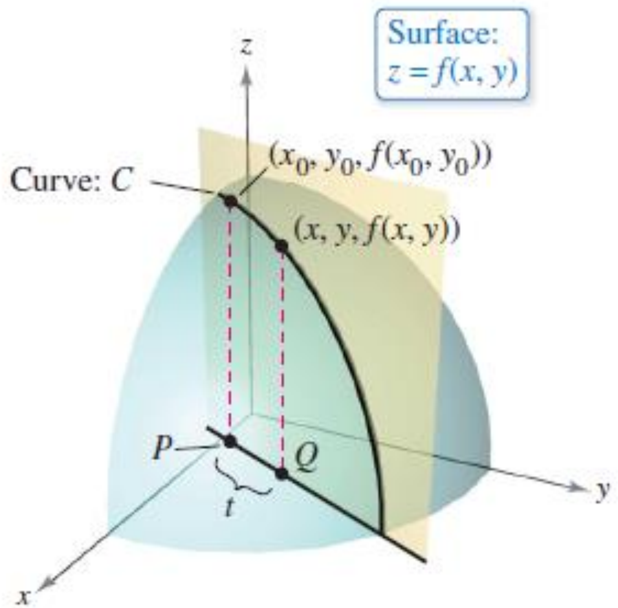


Figure 13.44

Directional Derivative

- Informally, you can write the slope of the curve *C* as a limit that looks much like those used in single-variable calculus.
- The vertical plane used to form *C* intersects the *xy*-plane in a line *L*, represented by the parametric equations

$$x = x_0 + t\cos\theta \qquad \text{and} \qquad y = y_0 + t\sin\theta$$

so that for any value of *t*, the point $Q(x, y)$ lies on the line *L*.
- For each of the points *P* and *Q*, there is a corresponding point on the surface.

$(x_0, y_0, f(x_0, y_0))$ Point above *P*

$(x, y, f(x, y))$ Point above *Q*

Directional Derivative

- Moreover, because the distance between *P* and *Q* is

$$\begin{aligned}\sqrt{(x-x_0)^2+(y-y_0)^2} &= \sqrt{(t\cos\theta)^2+(t\sin\theta)^2}\\ &= |t|\end{aligned}$$

- you can write the slope of the secant line through $(x_0, y_0, f(x_0, y_0))$ and $(x, y, f(x, y))$ as

$$\frac{f(x, y)-f(x_0, y_0)}{t}=\frac{f(x_0+t\cos\theta, y_0+t\sin\theta)-f(x_0, y_0)}{t}.$$

- Finally, by letting *t* approach 0, you arrive at the following definition.

Directional Derivative

Definition of Directional Derivative

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ be a unit vector. Then the **directional derivative of f in the direction of u**, denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t\to 0}\frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

provided this limit exists.

- Calculating directional derivatives by this definition is similar to finding the derivative of a function of one variable by the limit process. A simpler formula for finding directional derivatives involves the partial derivatives f_x and f_y.

Directional Derivative

THEOREM 13.9 Directional Derivative

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

- There are infinitely many directional derivatives of a surface at a given point—one for each direction specified by **u**, as shown in Figure

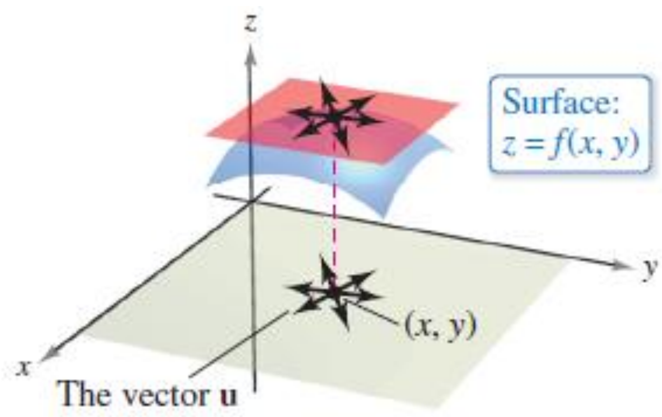


Figure 13.45

Directional Derivative

- Two of these are the partial derivatives f_x and f_y.

1. Direction of positive x-axis ($\theta = 0$): $\mathbf{u} = \cos 0\,\mathbf{i} + \sin 0\,\mathbf{j} = \mathbf{i}$

$$D_{\mathbf{i}} f(x, y) = f_x(x, y) \cos 0 + f_y(x, y) \sin 0 = f_x(x, y)$$

2. Direction of positive y-axis ($\theta = \pi/2$): $\mathbf{u} = \cos \dfrac{\pi}{2}\mathbf{i} + \sin \dfrac{\pi}{2}\mathbf{j} = \mathbf{j}$

$$D_{\mathbf{j}} f(x, y) = f_x(x, y) \cos \frac{\pi}{2} + f_y(x, y) \sin \frac{\pi}{2} = f_y(x, y)$$

Example 1 – *Finding a Directional Derivative*

- Find the directional derivative of

$$f(x, y) = 4 - x^2 - \tfrac{1}{4}y^2 \qquad \text{Surface}$$

at (1, 2) in the direction of

$$\mathbf{u} = \left(\cos \frac{\pi}{3}\right)\mathbf{i} + \left(\sin \frac{\pi}{3}\right)\mathbf{j}. \qquad \text{Direction}$$

- Solution:

Because $f_x(x,y) = -2x$ and $f_y(x,y) = -y/2$ are continuous, f is differentiable, and you can apply Theorem 13.9.

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

Example 1 – *Solution*

$$= (-2x)\cos\theta + \left(-\frac{y}{2}\right)\sin\theta$$

Evaluating at $\theta = \pi/3$, $x = 1$, and $y = 2$ produces

$$D_{\mathbf{u}}f(1, 2) = (-2)\left(\frac{1}{2}\right) + (-1)\left(\frac{\sqrt{3}}{2}\right)$$

$$= -1 - \frac{\sqrt{3}}{2}$$

$$\approx -1.866.$$

See Figure 13.46.

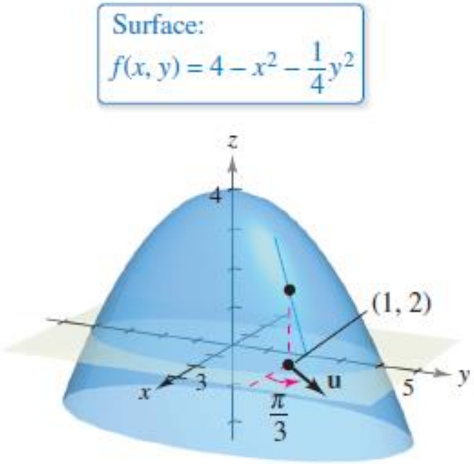


Figure 13.46

The Gradient of a Function of Two Variables

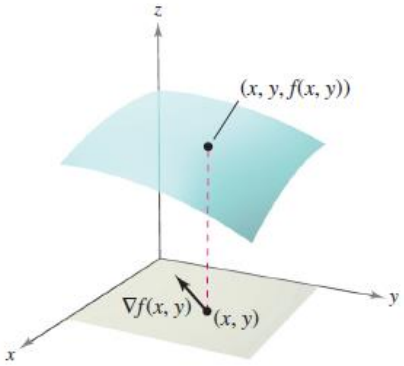
- The **gradient** of a function of two variables is a vector-valued function of two variables.

Definition of Gradient of a Function of Two Variables

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient of f,** denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

(The symbol ∇f is read as "del f.") Another notation for the gradient is given by **grad** $f(x, y)$. In Figure 13.48, note that for each (x, y), the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).



The gradient of f is a vector in the xy-plane.

Figure 13.48

Example 3 – *Finding the Gradient of a Function*

- Find the gradient of $f(x, y) = y \ln x + xy^2$ at the point (1, 2).
- Solution:

  Using $f_x(x, y) = \frac{y}{x} + y^2$ and $f_y(x, y) = \ln x + 2xy$

  you have

$$\begin{aligned}\nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \\ &= \left(\frac{y}{x} + y^2\right)\mathbf{i} + (\ln x + 2xy)\mathbf{j}.\end{aligned}$$

- At the point (1, 2), the gradient is

$$\begin{aligned}\nabla f(1, 2) &= \left(\frac{2}{1} + 2^2\right)\mathbf{i} + [\ln 1 + 2(1)(2)]\mathbf{j} \\ &= 6\mathbf{i} + 4\mathbf{j}.\end{aligned}$$

The Gradient of a Function of Two Variables

- Because the gradient of *f* is a vector, you can write the directional derivative of *f* in the direction of **u** as

$$D_{\mathbf{u}}f(x, y) = [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot [\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}].$$

- In other words, the directional derivative is the dot product of the gradient and the direction vector.

THEOREM 13.10 Alternative Form of the Directional Derivative

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

Example 4 – *Using $\nabla f(x, y)$ to Find a Directional Derivative*

- Find the directional derivative of

$$f(x, y) = 3x^2 - 2y^2$$

at $\left(-\frac{3}{4}, 0\right)$ in the direction from $P\left(-\frac{3}{4}, 0\right)$ to $Q(0, 1)$.

- Solution:

Because the partials of f are continuous, f is differentiable and you can apply Theorem 13.10.

✓ A vector in the specified direction is

$$\overrightarrow{PQ} = \left(0 + \frac{3}{4}\right)\mathbf{i} + (1 - 0)\mathbf{j}$$

$$= \frac{3}{4}\mathbf{i} + \mathbf{j}$$

Example 4 – *Solution*

cont'd

- And a unit vector in this direction is

$$\mathbf{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}. \qquad \text{Unit vector in direction of } \overrightarrow{PQ}$$

- Because $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 6x\mathbf{i} - 4y\mathbf{j}$, the gradient at $\left(-\frac{3}{4}, 0\right)$ is

$$\nabla f\left(-\frac{3}{4}, 0\right) = -\frac{9}{2}\mathbf{i} + 0\mathbf{j}. \qquad \text{Gradient at } \left(-\tfrac{3}{4}, 0\right)$$

Example 4 – *Solution*

cont'd

- Consequently, at $\left(-\frac{3}{4}, 0\right)$, the directional derivative is

$$D_{\mathbf{u}} f\left(-\frac{3}{4}, 0\right) = \nabla f\left(-\frac{3}{4}, 0\right) \cdot \mathbf{u}$$

$$= \left(-\frac{9}{2}\mathbf{i} + 0\mathbf{j}\right) \cdot \left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right)$$

$$= -\frac{27}{10}.$$

Directional derivative at $\left(-\frac{3}{4}, 0\right)$

- See Figure 13.49.

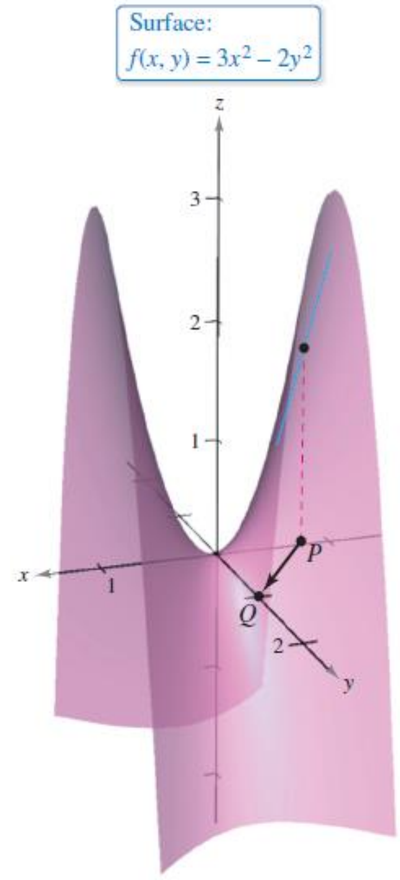


Figure 13.49

Applications of the Gradient

THEOREM 13.11 Properties of the Gradient

Let f be differentiable at the point (x, y).

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$.
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}} f(x, y)$ is

   $\|\nabla f(x, y)\|$. Maximum value of $D_{\mathbf{u}} f(x, y)$

3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}} f(x, y)$ is

   $-\|\nabla f(x, y)\|$. Minimum value of $D_{\mathbf{u}} f(x, y)$

Example 5 – *Finding the Direction of Maximum Increase*

- The temperature in degrees Celsius on the surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

✓ where x and y are measured in centimeters. In what direction from $(2, -3)$ does the temperature increase most rapidly? What is this rate of increase?

- Solution:

The gradient is

$$\nabla T(x, y) = T_x(x, y)\mathbf{i} + T_y(x, y)\mathbf{j}$$
$$= -8x\mathbf{i} - 2y\mathbf{j}.$$

Example 5 – *Solution*

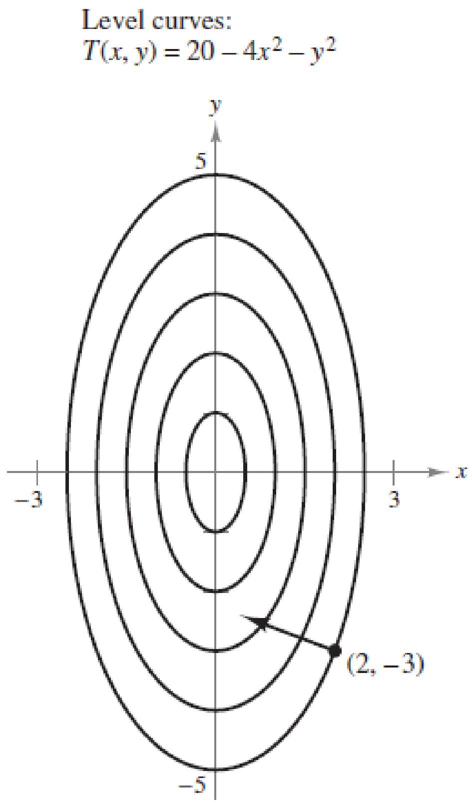
- It follows that the direction of maximum increase is given by $\nabla T(2, -3) = -16\mathbf{i} + 6\mathbf{j}$ as shown in Figure 13.51, and the rate of increase is

$\|\nabla T(2, -3)\|$

$$= \sqrt{256 + 36}$$

$$= \sqrt{292}$$

$$\approx 17.09^\circ \text{ per centimeter.}$$



The direction of most rapid increase in temperature at $(2, -3)$ is given by $-16\mathbf{i} + 6\mathbf{j}$.

Figure 13.51

Applications of the Gradient

THEOREM 13.12 Gradient Is Normal to Level Curves

If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

Example 7 – *Finding a Normal Vector to a Level Curve*

- Sketch the level curve corresponding to $c = 0$ for the function given by $f(x, y) = y - \sin x$ and find a normal vector at several points on the curve.
- Solution:

The level curve for $c = 0$ is given by

$$0 = y - \sin x \quad \Rightarrow \quad y = \sin x$$

as shown in Figure 13.53(a).

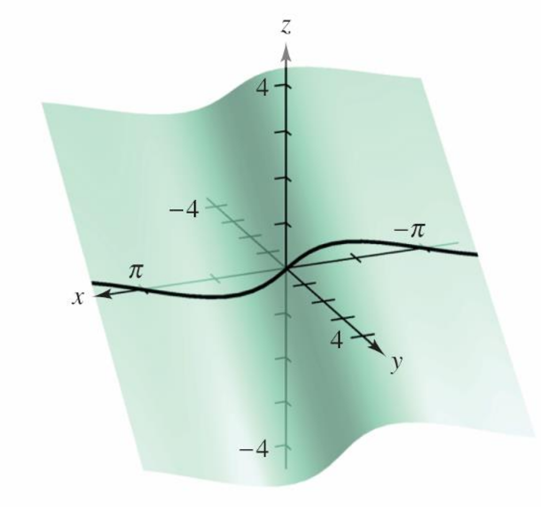


Figure 13.53(a)

Example 7 – *Solution*

cont'd

- Because the gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$
$$= -\cos x\mathbf{i} + \mathbf{j}$$

- you can use Theorem 13.12 to conclude that $\nabla f(x, y)$ is normal to the level curve at the point (x, y).
- Some gradient are

$$\nabla f(-\pi, 0) = \mathbf{i} + \mathbf{j}$$

$$\nabla f\left(-\frac{2\pi}{3}, -\frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f\left(-\frac{\pi}{2}, -1\right) = \mathbf{j}$$

Example 7 – *Solution*

cont'd

$$\nabla f\left(-\frac{\pi}{3}, -\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f(0, 0) = -\mathbf{i} + \mathbf{j}$$

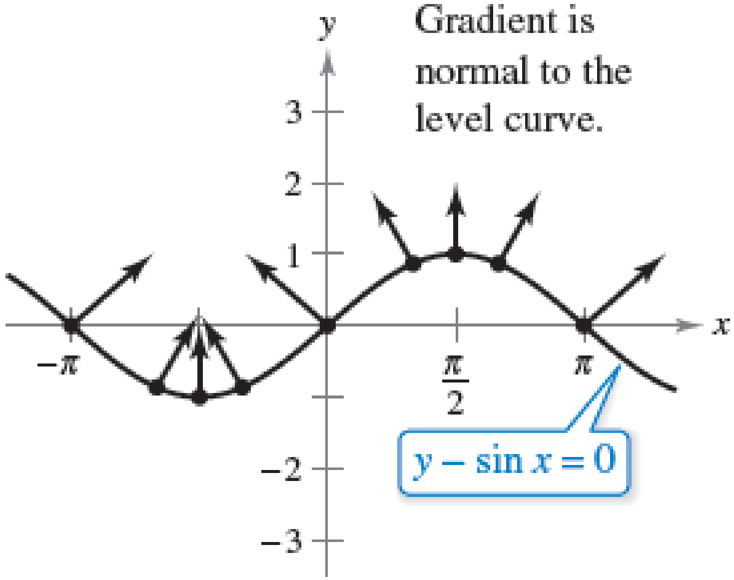
$$\nabla f\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f\left(\frac{\pi}{2}, 1\right) = \mathbf{j}.$$

$$\nabla f\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

and

$$\nabla f(\pi, 0) = \mathbf{i} + \mathbf{j}.$$



The level curve is given by $f(x, y) = 0$.

Figure 13.53(b)

These are shown in Figure 13.53(b).

Functions of Three Variables

Directional Derivative and Gradient for Three Variables

Let f be a function of x, y, and z with continuous first partial derivatives. The **directional derivative of f** in the direction of a unit vector

$$\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

is given by

$$D_{\mathbf{u}} f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient of f** is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all $\mathbf{u}$.
3. The direction of *maximum* increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is

   $$\|\nabla f(x, y, z)\|.$$ Maximum value of $D_{\mathbf{u}} f(x, y, z)$

4. The direction of *minimum* increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is

   $$-\|\nabla f(x, y, z)\|.$$ Minimum value of $D_{\mathbf{u}} f(x, y, z)$

Example 8 – *Finding the Gradient of a Function*

- Find $\nabla f(x, y, z)$ for the function

  $$f(x, y, z) = x^2 + y^2 - 4z$$

  and find the direction of maximum increase of f at the point $(2, -1, 1)$.
- Solution:

  The gradient is

  $$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

  $$= 2x\mathbf{i} + 2y\mathbf{j} - 4\mathbf{k}.$$

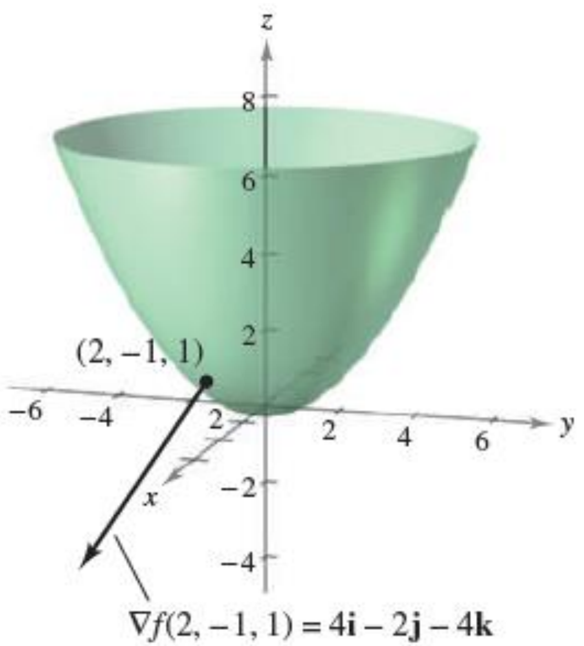
Example 8 – *Solution*

cont'd

✓ So, it follows that the direction of maximum increase at (2, –1, 1) is

$\nabla f(2, -1, 1) = 4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$.

See Figure 13.54.



Level surface and gradient at (2, −1, 1) for $f(x, y, z) = x^2 + y^2 - 4z$

Figure 13.54

Suggested Problems

Exercise 13.6:5,8,12,18,23,28,35.

Thanks a lot …

13:7-Tangent Planes and Normal Lines

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Objectives

- Find equations of tangent planes and normal lines to surfaces.

- Find the angle of inclination of a plane in space.

- Compare the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$.

Tangent Plane and Normal Line to a Surface

- You have represented the surfaces in space primarily by equations of the form
  $z = f(x, y)$. Equation of a surface S
- In the development to follow, however, it is convenient to use the more general representation
  $F(x, y, z) = 0$.
- For a surface S given by $z = f(x, y)$, you can convert to the general form by defining F as
  $F(x, y, z) = f(x, y) - z$.
- Because $f(x, y) - z = 0$, you can consider S to be the level surface of F given by
- $F(x, y, z) = 0$. Alternative equation of surface S

Example 1 – *Writing an Equation of a Surface*

- For the function

$$F(x, y, z) = x^2 + y^2 + z^2 - 4$$

describe the level surface given by $F(x, y, z) = 0$.

- Solution:

The level surface given by $F(x, y, z) = 0$ can be written

as $$x^2 + y^2 + z^2 = 4$$

which is a sphere of radius 2 whose center is at the origin.

Tangent Plane and Normal Line to a Surface

- Normal lines are equally important in analyzing surfaces and solids. For example, consider the collision of two billiard balls. When a stationary ball is struck at a point *P* on its surface, it moves along the **line of impact** determined by *P* and the center of the ball.
- The impact can occur in *two* ways. When the cue ball is moving along the line of impact, it stops dead and imparts all of its momentum to the stationary ball, as shown in Figure 13.55.

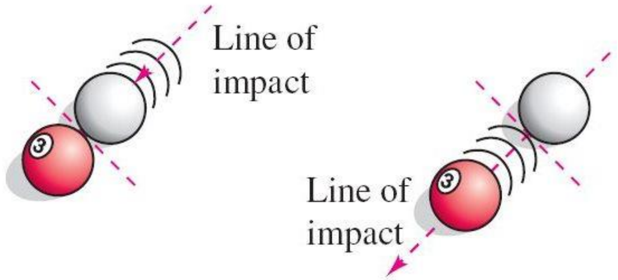


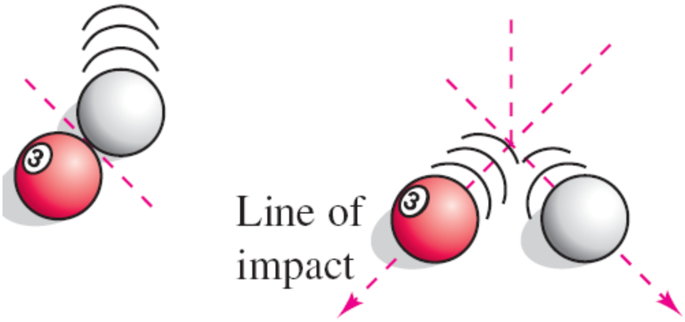
Figure 13.55

Tangent Plane and Normal Line to a Surface

- When the cue ball is not moving along the line of impact, it is deflected to one side or the other and retains part of its momentum.

- The part of the momentum that is transferred to the stationary ball occurs along the line of impact, *regardless* of the direction of the cue ball, as shown in Figure 13.56.

- This line of impact is called the **normal line** to the surface of the ball at the point *P*.



- Figure 13.56

Tangent Plane and Normal Line to a Surface

- In the process of finding a normal line to a surface, you are also able to solve the problem of finding a **tangent plane** to the surface.
- Let S be a surface given by $F(x, y, z) = 0$ and let $P(x_0, y_0, z_0)$ be a point on S.
- Let C be a curve on S through P that is defined by the vector-valued function $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

✓ Then, for all t, $F(x(t), y(t), z(t)) = 0$.

Tangent Plane and Normal Line to a Surface

- If F is differentiable and $x'(t)$, $y'(t)$, and $z'(t)$ all exist, then it follows from the Chain Rule that

$$0 = F'(t)$$

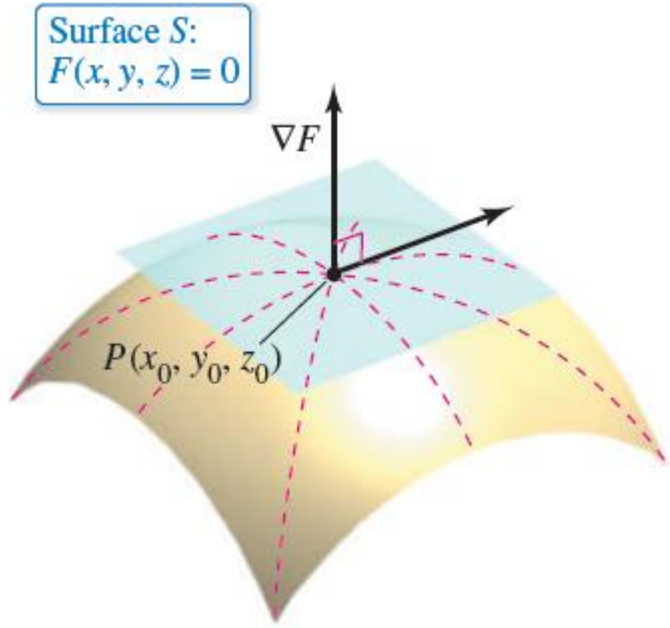
$$= F_x(x, y, z)x'(t) + F_y(x, y, z)y'(t) + F_z(x, y, z)z'(t)$$

At (x_0, y_0, z_0), the equivalent vector form is

$$0 = \underbrace{\nabla F(x_0, y_0, z_0)}_{\text{Gradient}} \cdot \underbrace{\mathbf{r}'(t_0)}_{\text{Tangent vector}}.$$

Tangent Plane and Normal Line to a Surface

- This result means that the gradient at P is orthogonal to the tangent vector of every curve on S through P. So, all tangent lines on S lie in a plane that is normal to $\nabla F(x_0, y_0, z_0)$ and contains P, as shown in Figure 13.57.



Tangent plane to surface S at P

Figure 13.57

Tangent Plane and Normal Line to a Surface

Definitions of Tangent Plane and Normal Line

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface S given by $F(x, y, z) = 0$ such that

$$\nabla F(x_0, y_0, z_0) \neq \mathbf{0}.$$

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane to S at P.**
2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line to S at P.**

- To find an equation for the tangent plane to S at (x_0, y_0, z_0), let (x, y, z) be an arbitrary point in the tangent plane. Then the vector

$$\mathbf{v} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

lies in the tangent plane.

Tangent Plane and Normal Line to a Surface

- Because $\nabla F(x_0, y_0, z_0)$ is normal to the tangent plane at (x_0, y_0, z_0), it must be orthogonal to every vector in the tangent plane, and you have
  $\nabla F(x_0, y_0, z_0) \cdot \mathbf{v} = 0$
  which leads to the next theorem.

THEOREM 13.13 Equation of Tangent Plane

If F is differentiable at (x_0, y_0, z_0), then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Example 2 – *Finding an Equation of a Tangent Plane*

- Find an equation of the tangent plane to the hyperboloid $z^2 - 2x^2 - 2y^2 = 12$ at the point (1, –1, 4).
- Solution:

  Begin by writing the equation of the surface as

  $z^2 - 2x^2 - 2y^2 - 12 = 0.$

  Then, considering

$$F(x, y, z) = z^2 - 2x^2 - 2y^2 - 12$$

  you have

$$F_x(x, y, z) = -4x, \quad F_y(x, y, z) = -4y \quad \textit{and} \quad F_z(x, y, z) = 2z.$$

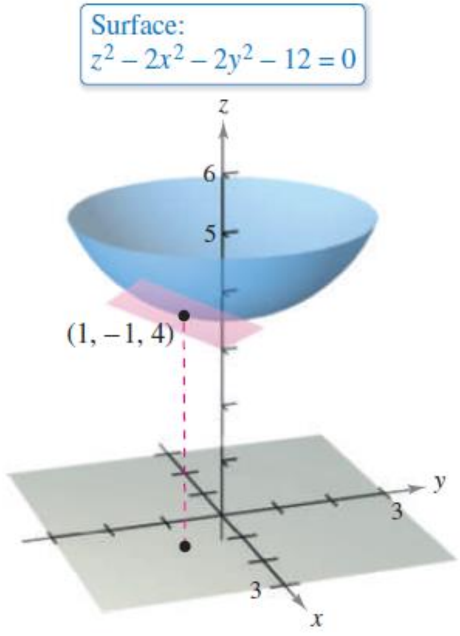
Example 2 – *Solution*

- At the point (1, –1, 4), the partial derivatives are $F_x(1, -1, 4) = -4$, $F_y(1, -1, 4) = 4$, and $F_z(1, -1, 4) = 8$.

✓ So, an equation of the tangent plane at (1, –1, 4) is

$$-4(x - 1) + 4(y + 1) + 8(z - 4) = 0$$
$$-4x + 4 + 4y + 4 + 8z - 32 = 0$$
$$-4x + 4y + 8z - 24 = 0$$
$$x - y - 2z + 6 = 0.$$

- Figure 13.58 shows a portion of the hyperboloid and tangent plane.



- Figure 13.58

Tangent Plane and Normal Line to a Surface

- To find the equation of the tangent plane at a point on a surface given by $z = f(x, y)$, you can define the function F by

$$F(x, y, z) = f(x, y) - z.$$

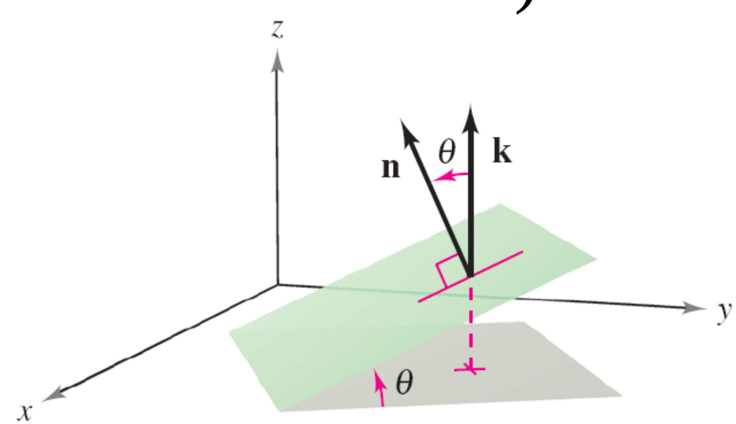
- Then S is given by the level surface $F(x, y, z) = 0$, and by Theorem 13.13, an equation of the tangent plane to S at the point (x_0, y_0, z_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

The Angle of Inclination of a Plane

- Another use of the gradient $\nabla F(x, y, z)$ is to determine the angle of inclination of the tangent plane to a surface.

- The **angle of inclination** of a plane is defined as the angle θ $(0 \le \theta \le \pi/2)$ between the given plane and the *xy*-plane, as shown in Figure 13.62. (The angle of inclination of a horizontal plane is defined as zero.)



The angle of inclination

Figure 13.62

The Angle of Inclination of a Plane

- Because the vector **k** is normal to the *xy*-plane, you can use the formula for the cosine of the angle between two planes to conclude that the angle of inclination of a plane with normal vector **n** is

$$\cos\theta = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|\,\|\mathbf{k}\|} = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|}.$$

Angle of inclination of a plane

Example 6 – *Finding the Angle of Inclination of a Tangent Plane*

- Find the angle of inclination of the tangent plane to the ellipsoid

$$\frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{3} = 1$$

at the point (2, 2, 1).

- Solution:

Begin bv letting

$$F(x, y, z) = \frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{3} - 1$$

✓ Then, the gradient of *F* at the point (2, 2, 1) is

$$\nabla F(x, y, z) = \frac{x}{6}\mathbf{i} + \frac{y}{6}\mathbf{j} + \frac{2z}{3}\mathbf{k}$$

Example 6 – *Solution* cont'd

$$\nabla F(2, 2, 1) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

- Because $\nabla F(2, 2, 1)$ is normal to the tangent plane and **k** is normal to the *xy*-plane, it follows that the angle of inclination of the tangent plane is

$$\cos\theta = \frac{|\nabla F(2, 2, 1) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 1)\|}$$

$$= \frac{2/3}{\sqrt{(1/3)^2 + (1/3)^2 + (2/3)^2}} \qquad = \sqrt{\frac{2}{3}}$$

Example 6 – *Solution*

cont'd

- which implies that

$$\theta = \arccos \sqrt{\frac{2}{3}} \approx 35.3^\circ$$

✓ as shown in Figure 13.63.

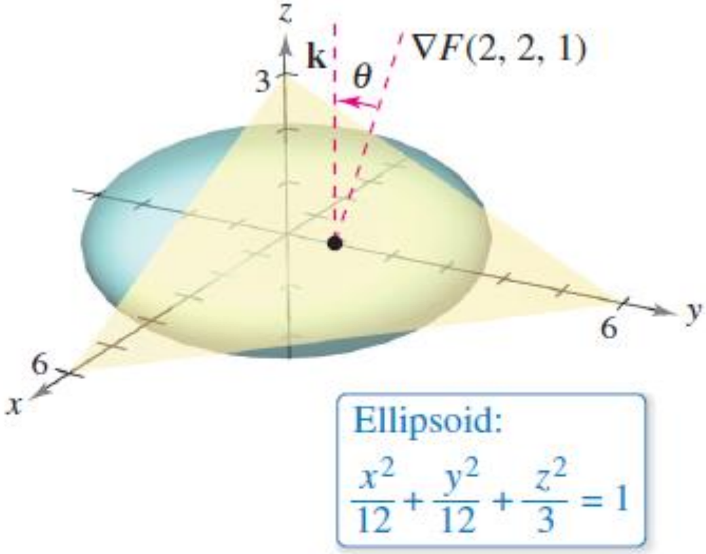


Figure 13.63

A Comparison of the Gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$

- This section concludes with a comparison of the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$.
- We know that the gradient of a function f of two variables is normal to the level curves of f.
- Specifically, Theorem 13.12 states that if f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

THEOREM 13.12 Gradient Is Normal to Level Curves

If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

A Comparison of the Gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$

- Having developed normal lines to surfaces, we can now extend this result to a function of three variables.

THEOREM 13.14 Gradient Is Normal to Level Surfaces

If F is differentiable at (x_0, y_0, z_0) and

$$\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$$

then $\nabla F(x_0, y_0, z_0)$ is normal to the level surface through (x_0, y_0, z_0).

Suggested Problems

Exercise 13.7:6,11,18,22,26,30,36,40.

Thanks a lot …

13:8-Extrema of Functions of Two Variables

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Objectives

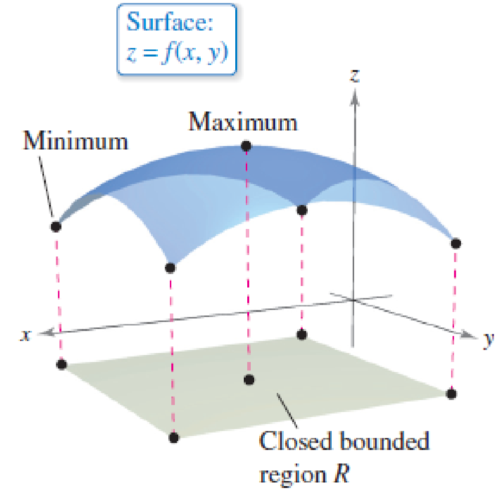
- Find absolute and relative extrema of a function of two variables.
- Use the Second Partials Test to find relative extrema of a function of two variables.

Absolute Extrema and Relative Extrema

- Consider the continuous function f of two variables, defined on a closed bounded region R in the xy-plane. The values $f(a, b)$ and $f(c, d)$ such that

$$f(a, b) \le f(x, y) \le f(c, d) \qquad (a, b) \text{ and } (c, d) \text{ are in } R.$$

for all (x, y) in R are called the **minimum** and **maximum** of f in the region R, as shown in Figure 13.64.



R contains point(s) at which $f(x, y)$ is a minimum and point(s) at which $f(x, y)$ is a maximum.

Figure 13.64

Absolute Extrema and Relative Extrema

- A region in the plane is *closed* when it contains all of its boundary points.
- The Extreme Value Theorem deals with a region in the plane that is both closed and *bounded*.
- A region in the plane is called **bounded** when it is a subregion of a closed disk in the plane.

THEOREM 13.15 Extreme Value Theorem

Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.

1. There is at least one point in R at which f takes on a minimum value.
2. There is at least one point in R at which f takes on a maximum value.

Absolute Extrema and Relative Extrema

- A minimum is also called an **absolute minimum** and a maximum is also called an **absolute maximum.** As in single-variable calculus, there is a distinction made between absolute extrema and **relative extrema.**

Definition of Relative Extrema

Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a **relative minimum** at (x_0, y_0) if

$$f(x, y) \geq f(x_0, y_0)$$

for all (x, y) in an *open* disk containing (x_0, y_0).

2. The function f has a **relative maximum** at (x_0, y_0) if

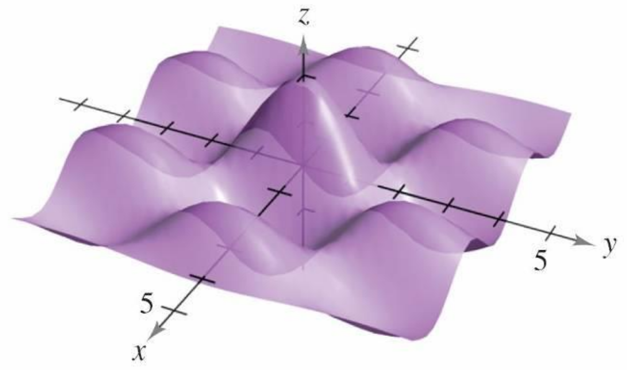
$$f(x, y) \leq f(x_0, y_0)$$

for all (x, y) in an *open* disk containing (x_0, y_0).

Absolute Extrema and Relative Extrema

- To say that f has a relative maximum at (x_0, y_0) means that the point (x_0, y_0, z_0) is at least as high as all nearby points on the graph of $z = f(x, y)$.

- Similarly, f has a relative minimum at (x_0, y_0) when (x_0, y_0, z_0) is at least as low as all nearby points on the graph. (See Figure 13.65.)



Relative extrema

Figure 13.65

Absolute Extrema and Relative Extrema

- To locate relative extrema of f, you can investigate the points at which the gradient of f is **0** or the points at which one of the partial derivatives does not exist. Such points are called **critical points** of f.

Definition of Critical Point

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Absolute Extrema and Relative Extrema

- If f is differentiable and

$$\nabla f(x_0, y_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$$
$$= 0\mathbf{i} + 0\mathbf{j}$$

then every directional derivative at (x_0, y_0) must be 0. This implies that the function has a horizontal tangent plane at the point (x_0, y_0), as shown in Figure 13.66.

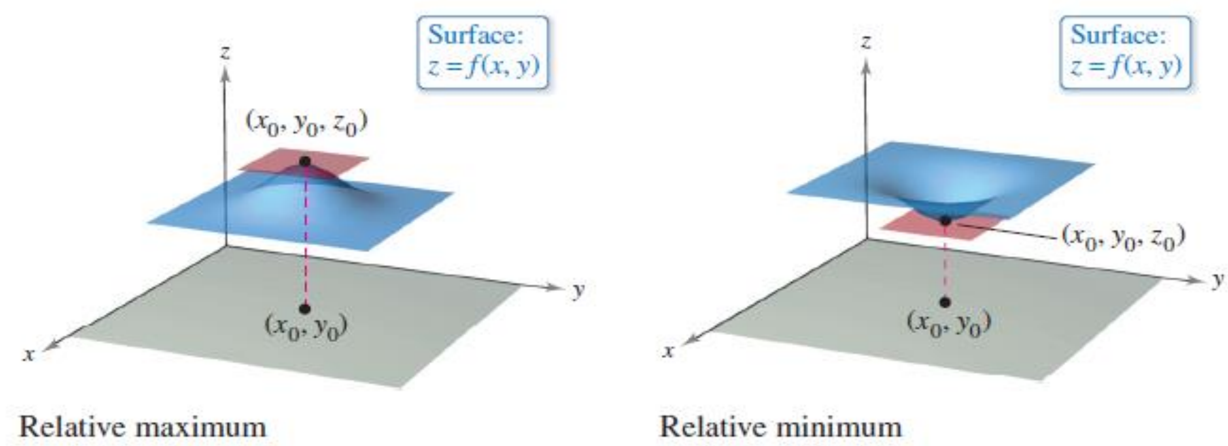


Figure 13.66

Absolute Extrema and Relative Extrema

- It appears that such a point is a likely location of a relative extremum.
- This is confirmed by Theorem 13.16.

THEOREM 13.16 Relative Extrema Occur Only at Critical Points

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

Example 1 – *Finding a Relative Extremum*

- Determine the relative extrema of

$$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20.$$

- Solution:

Begin by finding the critical points of f.

Because

$f_x(x, y) = 4x + 8$ Partial with respect to x

and

$f_y(x, y) = 2y - 6$ Partial with respect to y

are defined for all x and y, the only critical points are those for which both first partial derivatives are 0.

Example 1 – *Solution*

cont'd

- To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0, and solve the equations
  $4x + 8 = 0$ and $2y - 6 = 0$
  to obtain the critical point (–2, 3).

✓ By completing the square for f, you can see that for all $(x, y) \neq (-2, 3)$

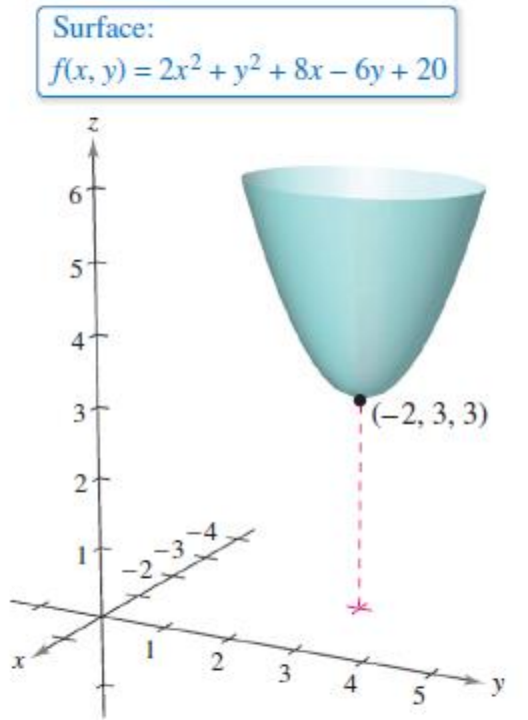
$$f(x, y) = 2(x + 2)^2 + (y - 3)^2 + 3 > 3.$$

✓ So, a relative *minimum* of f occurs at (–2, 3).

Example 1 – *Solution*

cont'd

- The value of the relative minimum is $f(-2, 3) = 3$, as shown in Figure 13.67.



The function $z = f(x, y)$ has a relative minimum at $(-2, 3)$.

Figure 13.67

The Second Partials Test

- To find relative extrema, you need only examine values of $f(x, y)$ at critical points. However, as is true for a function of one variable, the critical points of a function of two variables do not always yield relative maxima or minima.

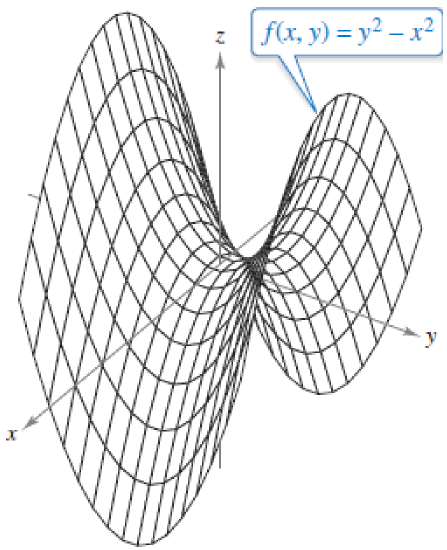
- Some critical points yield **saddle points,** which are neither relative maxima nor relative minima.

The Second Partials Test

- As an example of a critical point that does not yield a relative extremum, consider the hyperbolic paraboloid

$$f(x, y) = y^2 - x^2$$

as shown in Figure 13.69.



Saddle point at (0, 0, 0):
$f_x(0, 0) = f_y(0, 0) = 0$

Figure 13.69

The Second Partials Test

- At the point (0, 0), both partial derivatives

  $$f_x(x, y) = -2x \quad \text{and} \quad f_y(x, y) = 2y$$

  are 0.
- The function f does not, however, have a relative extremum at this point because in any open disk centered at (0, 0), the function takes on both negative values (along the x-axis) *and* positive values (along the y-axis).

- So, the point (0, 0, 0) is a saddle point of the surface.

The Second Partials Test

THEOREM 13.17 Second Partials Test

Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

To test for relative extrema of f, consider the quantity

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a **relative minimum** at (a, b).
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a **relative maximum** at (a, b).
3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point.**
4. The test is inconclusive if $d = 0$.

Example 3 – *Using the Second Partials Test*

- Find the relative extrema of

$$f(x, y) = -x^3 + 4xy - 2y^2 + 1.$$

- Solution:

Begin by finding the critical points of f.

✓ Because

$$f_x(x, y) = -3x^2 + 4y$$

and

$$f_y(x, y) = 4x - 4y$$

exist for all x and y, the only critical points are those for which both first partial derivatives are 0.

Example 3 – *Solution*

cont'd

- To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0 to obtain

$$-3x^2 + 4y = 0 \qquad \text{and} \qquad 4x - 4y = 0.$$

- From the second equation, you know that $x = y$, and, by substitution into the first equation, you obtain two solutions:

$y = x = 0$ and $y = x = \frac{4}{3}$.

Example 3 – *Solution*

cont'd

- Because

$$f_{xx}(x, y) = -6x, \quad f_{yy}(x, y) = -4, \quad \text{and} \quad f_{xy}(x, y) = 4$$

it follows that, for the critical point (0, 0),

$$d = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0 - 16 < 0$$

and, by the Second Partials Test, you can conclude that (0, 0, 1) is a saddle point of f.

Example 3 – *Solution*

cont'd

- Furthermore, for the critical p $\left(\frac{4}{3}, \frac{4}{3}\right)$,

$$d = f_{xx}\left(\tfrac{4}{3}, \tfrac{4}{3}\right) f_{yy}\left(\tfrac{4}{3}, \tfrac{4}{3}\right) - \left[f_{xy}\left(\tfrac{4}{3}, \tfrac{4}{3}\right)\right]^2$$

$$= -8(-4) - 16$$

$$= 16$$

$$> 0$$

and because $f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) = -8 < 0$, you can conclude that *f* has a relative maximum a $\left(\frac{4}{3}, \frac{4}{3}\right)$, as shown in Figure 13.70.

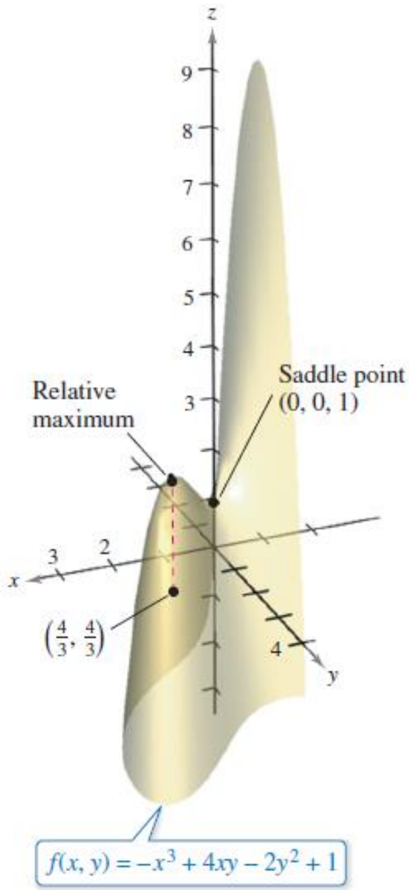


Figure 13.70

The Second Partials Test

- Absolute extrema of a function can occur in two ways.
- First, some relative extrema also happen to be absolute extrema. For instance, in Example 1, $f(-2, 3)$ is an absolute minimum of the function. (On the other hand, the relative maximum found in Example 3 is not an absolute maximum of the function.)
- Second, absolute extrema can occur at a boundary point of the domain.

Suggested Problems

Exercise 13.8:8,13,15,18,36,40.

Thanks a lot …

13:9-Applications of Extrema

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Objectives

- Solve optimization problems involving functions of several variables.

- Use the method of least squares.

Example 1 – *Finding Maximum Volume*

- A rectangular box is resting on the *xy*-plane with one vertex at the origin. The opposite vertex lies in the plane

$$6x + 4y + 3z = 24$$

as shown in Figure 13.73.

✓ Find the maximum volume of the box.

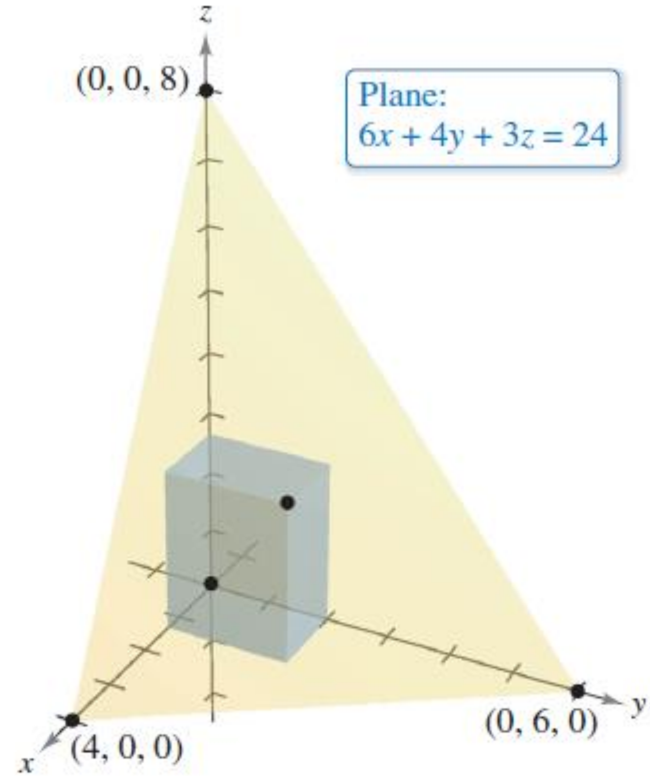


Figure 13.73

Example 1 – *Solution*

- Let *x*, *y*, and *z* represent the length, width, and height of the box.

✓ Because one vertex of the box lies in the plane $6x + 4y + 3z = 24$, you know that $z = \frac{1}{3}(24 - 6x - 4y)$. So, you can write the volume *xyz* of the box as a function of two variables.

$$V(x, y) = (x)(y)\left[\tfrac{1}{3}(24 - 6x - 4y)\right]$$

$$= \tfrac{1}{3}(24xy - 6x^2y - 4xy^2)$$

Example 1 – *Solution*

cont'd

- Next, find the first partial derivatives of V.

$$V_x(x, y) = \frac{1}{3}(24y - 12xy - 4y^2) = \frac{y}{3}(24 - 12x - 4y)$$

$$V_y(x, y) = \frac{1}{3}(24x - 6x^2 - 8xy) = \frac{x}{3}(24 - 6x - 8y)$$

- Note that the first partial derivatives are defined for all x and y. So, by setting $V_x(x, y)$ and $V_y(x, y)$ equal to 0 and solving the equations $y\,(24 - 12x - 4y) = 0$ and $\frac{1}{3}x\,(24 - 6x - 8y) = 0$, you obtain the critical points (0, 0), (4, 0), (0, 6), and $\left(\frac{4}{3}, 2\right)$.

- At (0, 0), (4, 0), and (0, 6), the volume is 0, so these points do not yield a maximum volume.

Example 1 – *Solution*

cont'd

- At the point $\left(\frac{4}{3}, 2\right)$, you can apply the Second Partials Test.

$$V_{xx}(x, y) = -4y$$

$$V_{yy}(x, y) = \frac{-8x}{3}$$

$$V_{xy}(x, y) = \frac{1}{3}(24 - 12x - 8y)$$

- Because

$$V_{xx}\left(\tfrac{4}{3}, 2\right)V_{yy}\left(\tfrac{4}{3}, 2\right) - \left[V_{xy}\left(\tfrac{4}{3}, 2\right)\right]^2 = (-8)\left(-\tfrac{32}{9}\right) - \left(-\tfrac{8}{3}\right)^2 = \frac{64}{3} > 0$$

and

$$V_{xx}\left(\tfrac{4}{3}, 2\right) = -8 < 0$$

Example 1 – *Solution*

- You can conclude from the Second Partials Test that the maximum volume is

$$V\left(\tfrac{4}{3}, 2\right) = \tfrac{1}{3}\left[24\left(\tfrac{4}{3}\right)(2) - 6\left(\tfrac{4}{3}\right)^2(2) - 4\left(\tfrac{4}{3}\right)(2^2)\right]$$

$$= \tfrac{64}{9} \text{ cubic units.}$$

- Note that the volume is 0 at the boundary points of the triangular domain of *V*.

The Method of Least Squares

- Many examples involved **mathematical models.** For example, a quadratic model for profit.
- There are several ways to develop such models; one is called the **method of least squares.**
- In constructing a model to represent a particular phenomenon, the goals are simplicity and accuracy.
- Of course, these goals often conflict.

The Method of Least Squares

- For instance, a simple linear model for the points in Figure 13.74 is

$$y = 1.9x - 5.$$

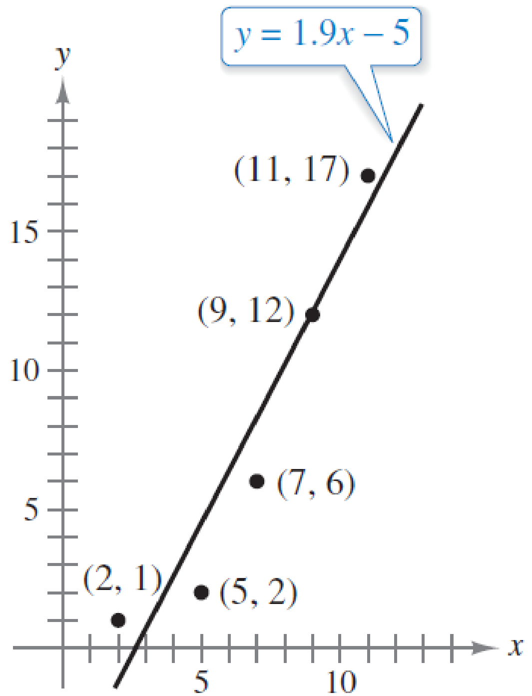


Figure 13.74

The Method of Least Squares

- However, Figure 13.75 shows that by choosing the slightly more complicated quadratic model

$$y = 0.20x^2 - 0.7x + 1$$

✓ you can achieve greater accuracy.

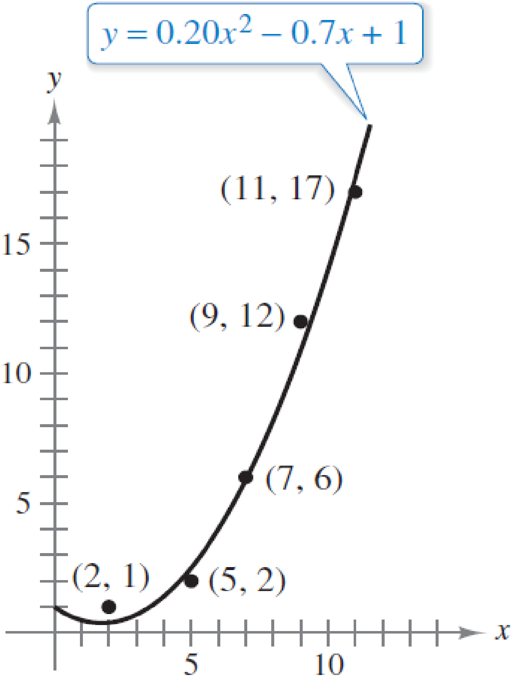


Figure 13.75

The Method of Least Squares

- As a measure of how well the model $y = f(x)$ fits the collection of points
$\{(x_1, y_1), (x_2, y_2), (x_3, y_3),. . . , (x_n, y_n)\}$

- you can add the squares of the differences between the actual y-values and the values given by the model to obtain the **sum of the squared errors**

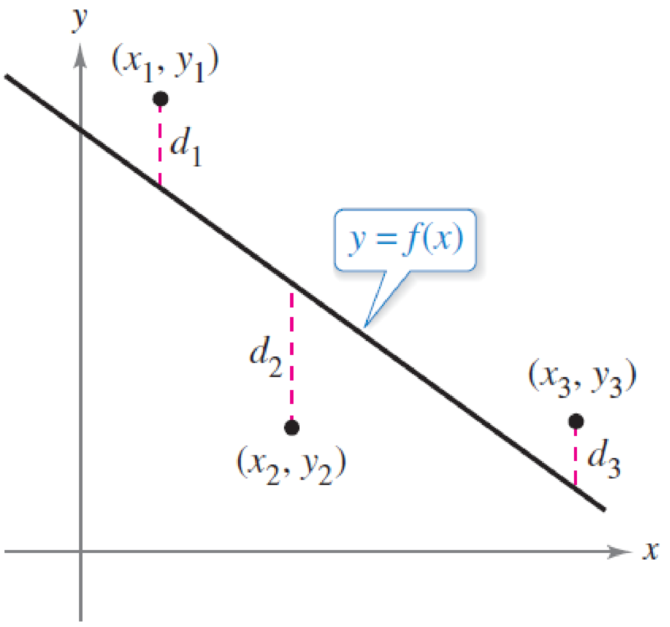
$$S = \sum_{i=1}^{n} [f(x_i) - y_i]^2.$$

Sum of the squared errors

The Method of Least Squares

- Graphically, S can be interpreted as the sum of the squares of the vertical distances between the graph of f and the given points in the plane, as shown in Figure 13.76.

- If the model is perfect, then $S = 0$. However, when perfection is not feasible, you can settle for a model that minimizes S.



Sum of the squared errors:
$S = d_1^2 + d_2^2 + d_3^2$

Figure 13.76

The Method of Least Squares

- For instance, the sum of the squared errors for the linear model in Figure 13.74 is $S = 17.6$.

- Statisticians call the *linear model* that minimizes S the **least squares regression line.**

- The proof that this line actually minimizes S involves the minimizing of a function of two variables.

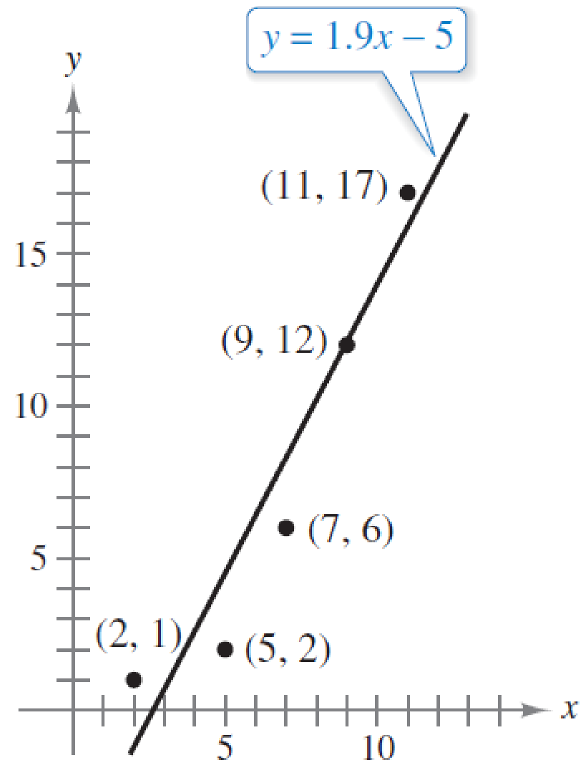


Figure 13.74

The Method of Least Squares

THEOREM 13.18 Least Squares Regression Line

The **least squares regression line** for $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ is given by $f(x) = ax + b$, where

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \quad \text{and} \quad b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right).$$

The Method of Least Squares

- If the x-values are symmetrically spaced about the y-axis, then $\Sigma\, x_i = 0$ and the formulas for a and b simplify to

$$a = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} \qquad \text{and} \qquad b = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

- This simplification is often possible with a translation of the x-values.

- For instance, given that the x-values in a data collection consist of the values 9, 10, 11, 12, and 13, you could let 11 be represented by 0.

Example 3 – *Finding the Least Squares Regression Line*

- Find the least squares regression line for the points (–3, 0), (–1, 1), (0, 2), and (2, 3).
- Solution:

✓ The table shows the calculations involved in finding the least squares regression line using $n = 4$.

x	y	xy	x^2
-3	0	0	9
-1	1	-1	1
0	2	0	0
2	3	6	4
$\sum_{i=1}^{n} x_i = -2$	$\sum_{i=1}^{n} y_i = 6$	$\sum_{i=1}^{n} x_i y_i = 5$	$\sum_{i=1}^{n} x_i^2 = 14$

Example 3 – *Solution*

cont'd

✓ Applying Theorem 13.18 produces

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{4(5) - (-2)(6)}{4(14) - (-2)^2} = \frac{8}{13}$$

and

$$b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right) = \frac{1}{4}\left[6 - \frac{8}{13}(-2)\right] = \frac{47}{26}.$$

Suggested Problems

Exercise 13.9: 12,15,17,19,28,29.

Thanks a lot …

13:10-Lagrange Multipliers

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Objectives

- Understand the Method of Lagrange Multipliers.
- Use Lagrange multipliers to solve constrained optimization problems.
- Use the Method of Lagrange Multipliers with two constraints.

Lagrange Multipliers

- Many optimization problems have restrictions, or **constraints,** on the values that can be used to produce the optimal solution. Such constraints tend to complicate optimization problems because the optimal solution can occur at a boundary point of the domain. In this section, you will study an ingenious technique for solving such problems. It is called the **Method of Lagrange Multipliers.**

- To see how this technique works, consider the problem of finding the rectangle of maximum area that can be inscribed in the ellipse $\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$

Lagrange Multipliers

- Let (x, y) be the vertex of the rectangle in the first quadrant, as shown in Figure 13.78.

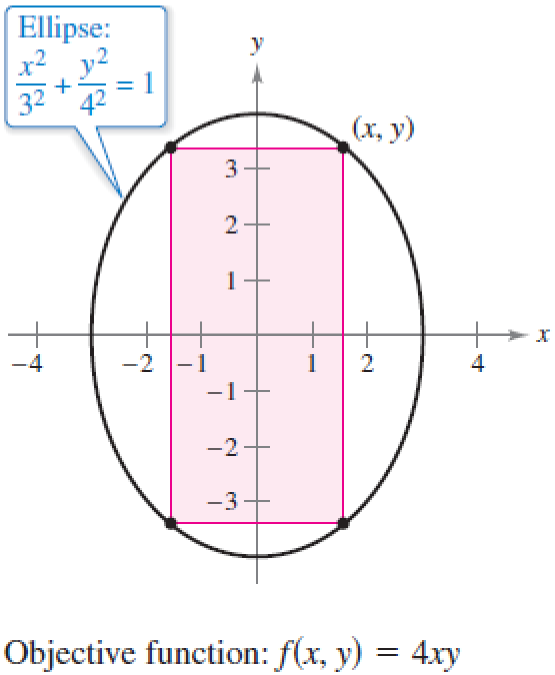

Figure 13.78

Lagrange Multipliers

- Because the rectangle has sides of lengths $2x$ and $2y$, its area is given by

$$f(x, y) = 4xy.$$

Objective function

- You want to find x and y such that $f(x, y)$ is a maximum.

- Your choice of (x, y) is restricted to first-quadrant points that lie on the ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

Constraint

Lagrange Multipliers

- Now, consider the constraint equation to be a fixed level curve of

$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2}.$$

- The level curves of f represent a family of hyperbolas $f(x, y) = 4xy = k$.

- In this family, the level curves that meet the constraint correspond to the hyperbolas that intersect the ellipse. Moreover, to maximize $f(x, y)$, you want to find the hyperbola that just barely satisfies the constraint.

Lagrange Multipliers

- The level curve that does this is the one that is *tangent* to the ellipse, as shown in Figure 13.79.

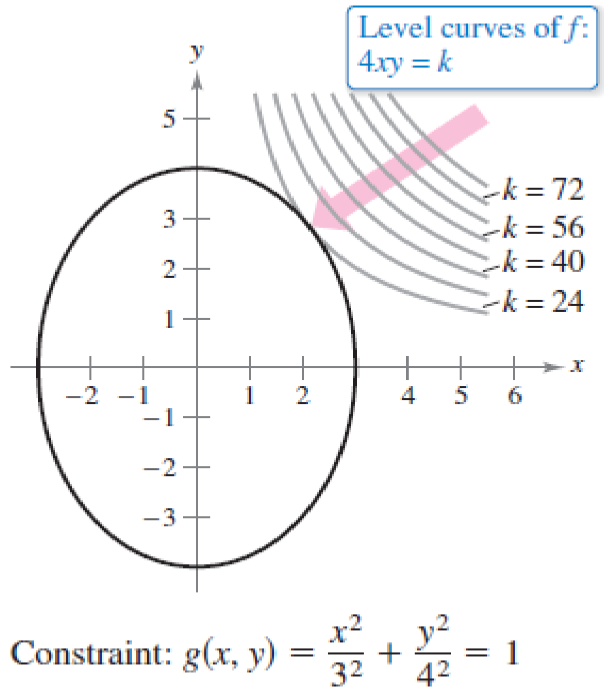

Figure 13.79

Lagrange Multipliers

- To find the appropriate hyperbola, use the fact that two curves are tangent at a point if and only if their gradients are parallel.
- This means that $\nabla f(x, y)$ must be a scalar multiple of $\nabla g(x, y)$ at the point of tangency.

- In the context of constrained optimization problems, this scalar is denoted by λ (the lowercase Greek letter lambda).

Lagrange Multipliers

- $\nabla f(x, y) = \lambda \nabla g(x, y)$

- The scalar λ is called a **Lagrange multiplier.** Theorem 13.19 gives the necessary conditions for the existence of

THEOREM 13.19 Lagrange's Theorem

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Lagrange Multipliers

Method of Lagrange Multipliers

Let f and g satisfy the hypothesis of Lagrange's Theorem, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of f, use these steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = c$ by solving the following system of equations.

$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$
$$g(x, y) = c$$

2. Evaluate f at each solution point obtained in the first step. The greatest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the least value yields the minimum of f subject to the constraint $g(x, y) = c$.

Constrained Optimization Problems

Example 1 – *Using a Lagrange Multiplier with One Constraint*

- Find the maximum value of $f(x, y) = 4xy$, where $x > 0$ and $y > 0$, subject to the constraint $(x^2/3^2) + (y^2/4^2) = 1$.
- Solution:

To begin, let

$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

By equating $\nabla f(x, y) = 4y\mathbf{i} + 4x\mathbf{j}$ and $\lambda\nabla g(x, y) = (2\lambda x/9)\mathbf{i} + (\lambda y/8)\mathbf{j}$, you obtain the following system of equations.

$$4y = \frac{2}{9}\lambda x \qquad f_x(x, y) = \lambda g_x(x, y)$$

$$4x = \frac{1}{8}\lambda y \qquad f_y(x, y) = \lambda g_y(x, y)$$

Example 1 – *Solution*

cont'd

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$

Constraint

- From the first equation, you obtain $\lambda = 18y/x$, and substitution into the second equation produces

$$4x = \frac{1}{8}\left(\frac{18y}{x}\right)y \quad \Rightarrow \quad x^2 = \frac{9}{16}y^2.$$

✓ Substituting this value for x^2 into the third equation produces

$$\frac{1}{9}\left(\frac{9}{16}y^2\right) + \frac{1}{16}y^2 = 1 \quad \Rightarrow \quad y^2 = 8 \quad \Rightarrow \quad y = \pm 2\sqrt{2}.$$

Example 1 – *Solution*

cont'd

- Because $y > 0$, choose the positive value and find that

$$x^2 = \frac{9}{16}y^2$$

$$= \frac{9}{16}(8) \quad = \frac{9}{2}$$

$$x = \pm\frac{3}{\sqrt{2}}.$$

- Because $x > 0$, choose the positive value. So, the maximum value of f is

$$f\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right) = 4xy = 4\left(\frac{3}{\sqrt{2}}\right)\left(2\sqrt{2}\right) \quad = 24.$$

Constrained Optimization Problems

- Economists call the Lagrange multiplier obtained in a production function the **marginal productivity of money.**
- For instance, if the marginal productivity of money is λ and *x* represents the units of labor and *y* represents the units of capital then the marginal productivity of money at $x = 250$ and $y = 50$ is

$$\lambda = \frac{x^{-1/4}y^{1/4}}{2} = \frac{(250)^{-1/4}(50)^{1/4}}{2} \approx 0.334$$

- which means that for each additional dollar spent on production, an additional 0.334 unit of the product can be produced.

The Method of Lagrange Multipliers with Two Constraints

- For optimization problems involving *two* constraint functions g and h, you can introduce a second Lagrange multiplier, μ (the lowercase Greek letter mu), and then solve the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

✓ where the gradient vectors are not parallel, as illustrated in

Example 5 – *Optimization with Two Constraints*

- Let $T(x, y, z) = 20 + 2x + 2y + z^2$ represent the temperature at each point on the sphere $x^2 + y^2 + z^2 = 11$.

✓ Find the extreme temperatures on the curve formed by the intersection of the plane $x + y + z = 3$ and the sphere.

- Solution:

The two constraints are

$$g(x, y, z) = x^2 + y^2 + z^2 = 11$$

and

$$h(x, y, z) = x + y + z = 3.$$

Example 5 – *Solution*

cont'd

- Using

$$\nabla T(x, y, z) = 2\mathbf{i} + 2\mathbf{j} + 2z\mathbf{k}$$

$$\lambda \nabla g(x, y, z) = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j} + 2\lambda z\mathbf{k}$$

and

$$\mu \nabla h(x, y, z) = \mu\mathbf{i} + \mu\mathbf{j} + \mu\mathbf{k}$$

- you can write the following system of equations.

$$2 = 2\lambda x + \mu \qquad T_x(x, y, z) = \lambda g_x(x, y, z) + \mu h_x(x, y, z)$$

$$2 = 2\lambda y + \mu \qquad T_y(x, y, z) = \lambda g_y(x, y, z) + \mu h_y(x, y, z)$$

$$2z = 2\lambda z + \mu \qquad T_z(x, y, z) = \lambda g_z(x, y, z) + \mu h_z(x, y, z)$$

$$x^2 + y^2 + z^2 = 11 \qquad \text{Constraint 1}$$

$$x + y + z = 3 \qquad \text{Constraint 2}$$

Example 5 – *Solution*

cont'd

- By subtracting the second equation from the first, you can obtain the following system.

$$\lambda(x - y) = 0$$
$$2z(1 - \lambda) - \mu = 0$$
$$x^2 + y^2 + z^2 = 11$$
$$x + y + z = 3$$

✓ From the first equation, you can conclude that $\lambda = 0$ or $x = y$.

✓ If $\lambda = 0$, you can show that the critical points are $(3, -1, 1)$ and $(-1, 3, 1)$.

Example 5 – *Solution*

cont'd

- For $\lambda \neq 0$, then $x = y$ and you can show that the critical points occur when $x = y = \left(3 \pm 2\sqrt{3}\right)/3$ and $z = \left(3 \mp 4\sqrt{3}\right)/3$.

- Finally, to find the optimal solutions, compare the temperatures at the four critical points.

$$T(3, -1, 1) = T(-1, 3, 1) = 25$$

$$T\left(\frac{3 - 2\sqrt{3}}{3}, \frac{3 - 2\sqrt{3}}{3}, \frac{3 + 4\sqrt{3}}{3}\right) = \frac{91}{3} \approx 30.33$$

$$T\left(\frac{3 + 2\sqrt{3}}{3}, \frac{3 + 2\sqrt{3}}{3}, \frac{3 - 4\sqrt{3}}{3}\right) = \frac{91}{3} \approx 30.33$$

- So, $T = 25$ is the minimum temperature and $T = \frac{91}{3}$ is the maximum temperature on the curve.

Suggested Problems

Exercise 13.10:6,9,11,18,15,16.

Thanks a lot …

Laurent's Theorem and Cauchy's Residue Theorem

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Laurent's Theorem (LS)

If we are required to expand *f(z)* about a point where *f(z)* is not analytic, then it is expanded by Laurent's series.

Statement: If *f(z)* is analytic on C_1 and C_2 and on the annular region R bounded by the two concentric circles C_1 and C_2 of radii r_1 and r_2 $(r_1 > r_2)$ having centers at a, then for all z in R,

$$f(z) = \left. \begin{aligned} & a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots\ldots\ldots + \\ & + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots\ldots\ldots\ldots\ldots\ldots + \end{aligned} \right\} \ldots\ldots\ldots\ldots(A)$$

where

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz, \quad n = 0, 1, 2, \ldots..$$

$$b_n = \frac{1}{2\pi i} \oint_{C_2} (z-a)^{n-1} f(z)\, dz, \quad n = 1, 2, \ldots..$$

\# Here the part $a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots.$ is called the analytic part of Laurent's series, while the remaining part, that is $\frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots\ldots\ldots\ldots$ is called the principal part. If the principal part is zero, then the LS reduces to Taylor's series.

Example 1: Expand f(z) = $f(z)=\dfrac{1}{(z-1)(z-2)}$ for $1<|z|<2$ by LS.

Solution: Given that

$$1<|z|<2$$

$$\Rightarrow \frac{1}{|z|}<1 \text{ and } \frac{|z|}{2}<1$$

To ensure that the LS converses, we have to expand f(z) in terms of $\dfrac{1}{|z|}$ or $\dfrac{|z|}{2}$. Therefore, we should write f(z) as

$$f(z)=\frac{1}{(z-1)(z-2)}$$

$$=\frac{1}{(z-2)}-\frac{1}{(z-1)}$$

$$=\frac{1}{(-2)\left(1-z/2\right)}-\frac{1}{z\left(1-1/z\right)}$$

$$= -\frac{1}{2}\left(1 - z/2\right)^{-1} - \frac{1}{z}\left(1 - 1/z\right)^{-1}$$

$$= -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} \ldots\ldots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} \ldots\ldots\right]$$

$$= -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \ldots\ldots\ldots - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} \ldots\ldots$$

which is required LS where $\frac{1}{|z|} < 1$ and $\frac{|z|}{2} < 1$ QED.

Exercises : Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a LS vaild for (a) $|z| < 1$, (b) $|z| > 2$, (c) $1 < |z| < 2$, (d) $|z-1| > 1$, (e) $0 < |z-2| < 1$

Residue in Complex Number

In complex analysis we can represent a function as a series called the Laurent series. Now if we want to integrate this function we can just integrate the series term by term. Well thanks to some theorems about integrals a lot of these integrals will be zero. What isn't zero is called a **Residue** and represents the value of the integral in question.

The coefficient of $\frac{1}{z-a}$, i.e., b_1 in LS

$$f(z) = \left. \begin{array}{l} a_0 + a_1(z-a) + a_2(z-a)^2 + \dots\dots\dots + \\ + \frac{b_1}{(z-a)} + \frac{a_2}{(z-a)^2} + \dots\dots\dots\dots\dots + \end{array} \right\} \dots\dots\dots\dots(A)$$

about an isolated singular point z =a is called the **residue** of *f(z)* at z = a.

Method of finding residue:

1. **Rule 1:** Residue at a simple pole is given by

$$\text{Res } f(z) = \underset{z\to a}{Lt}(z-a)f(z)$$

2. **Rule 2:** Residue at a pole of order n is given by

$$\text{Res } f(z) = \frac{1}{\lfloor (n-1)}\left[\frac{d^{n-1}}{dz^{n-1}}\left\{(z-a)^n f(z)\right\}\right]_{z=a}$$

Calculation of Residues

To obtain the residue of a function $f(z)$ at $z = a$, it may appear from (7.1) that the Laurent expansion of $f(z)$ about $z = a$ must be obtained. However, in the case where $z = a$ is a pole of order k, there is a simple formula for a_{-1} given by

$$a_{-1} = \lim_{z \to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \{(z-a)^k f(z)\}$$

If $k = 1$ (simple pole), then the result is especially simple and is given by

$$a_{-1} = \lim_{z \to a} (z-a)\, f(z)$$

which is a special case of (7.5) with $k = 1$ if we define $0! = 1$.

EXAMPLE 1: If $f(z) = z/(z-1)(z+1)^2$, then $z = 1$ and $z = -1$ are poles of orders one and two, respectively.

We have, using (7.6) and (7.5) with $k = 2$,

$$\text{Residue at } z = 1 \text{ is } \lim_{z \to 1} (z-1) \left\{ \frac{z}{(z-1)(z+1)^2} \right\} = \frac{1}{4}$$

$$\text{Residue at } z = -1 \text{ is } \lim_{z \to -1} \frac{1}{1!} \frac{d}{dz} \left\{ (z+1)^2 \left(\frac{z}{(z-1)(z+1)^2} \right) \right\} = -\frac{1}{4}$$

If $z = a$ is an essential singularity, the residue can sometimes be found by using known series expansions.

The Residue Theorem

Let $f(z)$ be single-valued and analytic inside and on a simple closed curve C except at the singularities $a, b, c, \ldots$ inside C, which have residues given by $a_{-1}, b_{-1}, c_{-1}, \ldots$ [see Fig. 7-1]. Then, the *residue theorem* states that

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) \tag{7.7}$$

i.e., the integral of $f(z)$ around C is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities enclosed by C. Note that (7.7) is a generalization of (7.3). Cauchy's theorem and integral formulas are special cases of this theorem

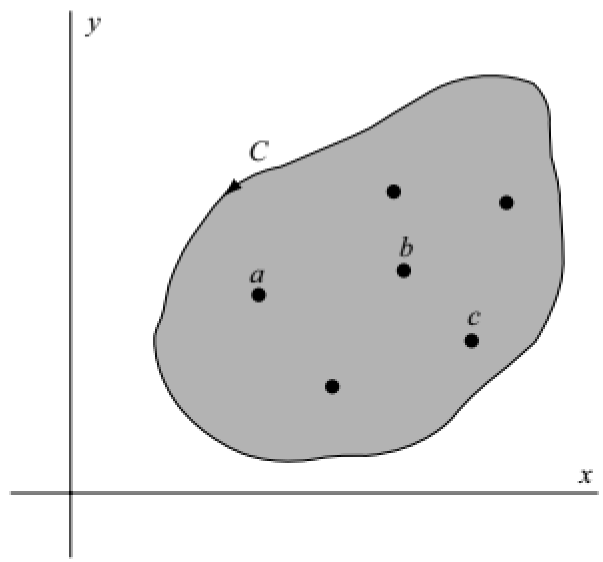

Fig. 7-1

Residue Theorem (RT)

If a function *f(z)* is analytic in a closed curve C except at a finite poles within C, then

$$\int_C f(z)\,dz = 2\pi i\left(\text{sum of resideues at all the poles within C}\right)$$

Example 1: Determine the poles of $\dfrac{4-3z}{z(z-1)(z-2)}$, over $C:|z|=\dfrac{3}{2}$ and residue at each pole.

Solution: The poles of the integrand are given by putting the denominator to zero. That is,

$$z(z-1)(z-2)=0$$
$$\Rightarrow z=0, z=1, z=2$$

The given circle $|z|=\dfrac{3}{2}$ with centre at the origin (z = 0) and radius $r=\dfrac{3}{2}$ encloses two simple poles z = 0, and z = =1 only and the other pole lies out side of the circle (domain). Now using Rule 1 above, we have

Residue of f(z) at the simple pole z = 0 is

$$\text{Res } f(z) = \underset{z\to 0}{Lt}(z-0) f(z)$$

$$= \underset{z\to 0}{Lt}\, z \frac{4-3z}{z(z-1)(z-2)}$$

$$= \underset{z\to 0}{Lt} \frac{4-3z}{(z-1)(z-2)}$$

$$= \frac{4-0}{(0-1)(0-2)} = 2$$

$$\therefore \text{Res } f(0) = 2 ----------------(1)$$

Next, residue of f(z) at the simple pole z = 1 is

$$\text{Res } f(z) = \underset{z\to 1}{Lt}(z-1) f(z)$$

$$= \underset{z\to 1}{Lt}(z-1) \frac{4-3z}{z(z-1)(z-2)}$$

$$= \underset{z \to 1}{Lt} \frac{4-3z}{z(z-2)}$$

$$= \frac{4-3}{(1)(1-2)} = -1$$

$$\therefore \operatorname{Re} s\, f(1) = -1 ----------------------(2)$$

Example 2: Determine the poles of $\dfrac{4-3z}{z(z-1)(z-2)}$, over $C:|z|=\dfrac{3}{2}$ and residue at each pole. Hence evaluate $\displaystyle\int_C \frac{4-3z}{z(z-1)(z-2)}dz$, over $C:|z|=\dfrac{3}{2}$.

Solution: The poles of the integrand and the residues have already been evaluated in example 1. Now by RT we have

$$\int_C \frac{4-3z}{z(z-1)(z-2)}dz, \text{ over } C:|z|=\frac{3}{2}$$

$$=2\pi i\left(\text{sum of resideues at all the poles within } |z|=\frac{3}{2}\right)$$

$$=2\pi i(2-1) \quad [\text{By eqn (1) and (2)}]$$

$$=2\pi i$$

$$\therefore \int_C \frac{4-3z}{z(z-1)(z-2)}dz=2\pi i \square$$

One can verify the result by comparing the results obtained earlier by CIF.

Example 3:

Find the residues of (a) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2+4)}$ and (b) $f(z) = e^z \csc^2 z$ at all its poles in the finite plane.

Solution

(a) $f(z)$ has a double pole at $z = -1$ and simple poles at $z = \pm 2i$.

Residue at $z = -1$ is

$$\lim_{z\to -1} \frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2 \cdot \frac{z^2-2z}{(z+1)^2(z^2+4)}\right\} = \lim_{z\to -1} \frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2} = -\frac{14}{25}$$

Residue at $z = 2i$ is

$$\lim_{z\to 2i}\left\{(z-2i)\cdot\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\} = \frac{-4-4i}{(2i+1)^2(4i)} = \frac{7+i}{25}$$

Residue at $z = -2i$ is

$$\lim_{z\to -2i}\left\{(z+2i)\cdot\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\} = \frac{-4+4i}{(-2i+1)^2(-4i)} = \frac{7-i}{25}$$

(b) $f(z)=e^z\csc^2 z = e^z/\sin^2 z$ has double poles at $z=0,\ \pm\pi,\ \pm 2\pi,\ \ldots,$ i.e., $z=m\pi$ where $m=0,$ $\pm 1,\ \pm 2, \ldots$.

Residue at $z=m\pi$ is

$$\lim_{z\to m\pi}\frac{1}{1!}\frac{d}{dz}\left\{(z-m\pi)^2\frac{e^z}{\sin^2 z}\right\} = \lim_{z\to m\pi}\frac{e^z[(z-m\pi)^2\sin z+2(z-m\pi)\sin z-2(z-m\pi)^2\cos z]}{\sin^3 z}$$

Letting $z-m\pi=u$ or $z=u+m\pi$, this limit can be written

$$\lim_{u\to 0} e^{u+m\pi}\left\{\frac{u^2\sin u+2u\sin u-2u^2\cos u}{\sin^3 u}\right\} = e^{m\pi}\left\{\lim_{u\to 0}\frac{u^2\sin u+2u\sin u-2u^2\cos u}{\sin^3 u}\right\}$$

The limit in braces can be obtained using L'Hospital's rule. However, it is easier to first note that

$$\lim_{u\to 0}\frac{u^3}{\sin^3 u}=\lim_{u\to 0}\left(\frac{u}{\sin u}\right)^3=1$$

and thus write the limit as

$$e^{m\pi}\lim_{u\to 0}\left(\frac{u^2\sin u+2u\sin u-2u^2\cos u}{u^3}\cdot\frac{u^3}{\sin^3 u}\right)=e^{m\pi}\lim_{u\to 0}\frac{u^2\sin u+2u\sin u-2u^2\cos u}{u^3}=e^{m\pi}$$

using L'Hospital's rule several times. In evaluating this limit, we can instead use the series expansions $\sin u = u-u^3/3!+\cdots,\ \cos u = 1-u^2/2!+\cdots$.

Example 4: Evaluate $\dfrac{1}{2\pi i}\oint\limits_C \dfrac{e^{zt}}{z^2(z^2+2z+2)}\,dz$ around the circle C with equation $|z| = 3$.

Solution

The integrand $e^{zt}/\{z^2(z^2+2z+2)\}$ has a double pole at $z = 0$ and two simple poles at $z = -1 \pm i$ [roots of $z^2 + 2z + 2 = 0$]. All these poles are inside C.

Residue at $z = 0$ is

$$\lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\left\{z^2\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \lim_{z\to 0}\frac{(z^2+2z+2)(te^{zt})-(e^{zt})(2z+2)}{(z^2+2z+2)^2} = \frac{t-1}{2}$$

Residue at $z = -1 + i$ is

$$\lim_{z\to -1+i}\left\{[z-(-1+i)]\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \lim_{z\to -1+i}\left\{\frac{e^{zt}}{z^2}\right\}\lim_{z\to -1+i}\left\{\frac{z+1-i}{z^2+2z+2}\right\}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2}\cdot\frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

Residue at $z = -1 - i$ is

$$\lim_{z\to -1-i}\left\{[z-(-1-i)]\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \frac{e^{(-1-i)t}}{4}$$

Then, by the residue theorem

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = 2\pi i \text{ (sum of residues)} = 2\pi i\left\{\frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}\right\}$$

$$= 2\pi i\left\{\frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t\right\}$$

that is,

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = \frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t$$

Example 5: Evaluate $\int_0^{\infty} \frac{dx}{x^6+1}$.

Solution

Consider $\oint_C dz/(z^6+1)$, where C is the closed contour of Fig. 7-5 consisting of the line from $-R$ to R and the semicircle Γ, traversed in the positive (counterclockwise) sense.

Since $z^6+1=0$ when $z=e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$, these are simple poles of $1/(z^6+1)$. Only the poles $e^{\pi i/6}$, $e^{3\pi i/6}$, and $e^{5\pi i/6}$ lie within C. Then, using L'Hospital's rule,

$$\text{Residue at } e^{\pi i/6} = \lim_{z\to e^{\pi i/6}} \left\{ (z-e^{\pi i/6}) \frac{1}{z^6+1} \right\} = \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-5\pi i/6}$$

$$\text{Residue at } e^{3\pi i/6} = \lim_{z\to e^{3\pi i/6}} \left\{ (z-e^{3\pi i/6}) \frac{1}{z^6+1} \right\} = \lim_{z\to e^{3\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-5\pi i/2}$$

$$\text{Residue at } e^{5\pi i/6} = \lim_{z\to e^{5\pi i/6}} \left\{ (z-e^{5\pi i/6}) \frac{1}{z^6+1} \right\} = \lim_{z\to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-25\pi i/6}$$

Thus $$\oint_C \frac{dz}{z^6+1} = 2\pi i \left\{ \frac{1}{6} e^{-5\pi i/6} + \frac{1}{6} e^{-5\pi i/2} + \frac{1}{6} e^{-25\pi i/6} \right\} = \frac{2\pi}{3}$$

that is, $$\int_{-R}^{R} \frac{dx}{x^6+1} + \int_{\Gamma} \frac{dz}{z^6+1} = \frac{2\pi}{3} \qquad (1)$$

Taking the limit of both sides of (1) as $R \to \infty$ and using Problems 7.7 and 7.8, we have

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^6+1} = \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3} \tag{2}$$

Since

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\int_{0}^{\infty} \frac{dx}{x^6+1}$$

the required integral has the value $\pi/3$.

Example 6: Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}.$

Solution

The poles of $z^2/(z^2+1)^2(z^2+2z+2)$ enclosed by the contour C of Fig. 7-5 are $z = i$ of order 2 and $z = -1+i$ of order 1.

Residue at $z = i$ is

$$\lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2 \frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\} = \frac{9i-12}{100}$$

Residue at $z = -1+i$ is

$$\lim_{z\to -1+i} (z+1-i)\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}$$

Then

$$\oint_C \frac{z^2\, dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i\left\{\frac{9i-12}{100} + \frac{3-4i}{25}\right\} = \frac{7\pi}{50}$$

or

$$\int_{-R}^{R} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} + \int_{\Gamma} \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50}$$

Taking the limit as $R \to \infty$ and noting that the second integral approaches zero by Problem **5**. , we obtain the required result.

Example 7: Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$.

Solution

Let $z = e^{i\theta}$. Then $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i = (z - z^{-1})/2i$, $\cos\theta = (e^{i\theta} + e^{-i\theta})/2 = (z + z^{-1})/2$, $dz = iz\, d\theta$ so that

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \oint_C \frac{dz/iz}{3 - 2(z + z^{-1})/2 + (z - z^{-1})/2i} = \oint_C \frac{2\,dz}{(1-2i)z^2 + 6iz - 1 - 2i}$$

where C is the circle of unit radius with center at the origin (Fig. 7-6).

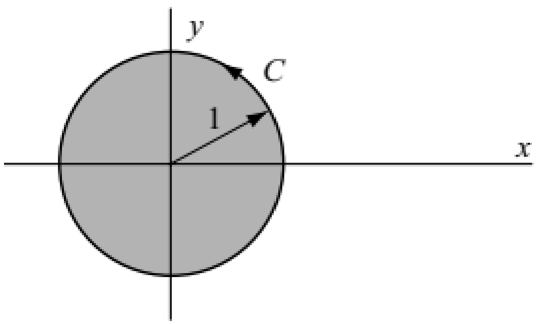

Fig. 7-6

The poles of $2/\left\{(1-2i)z^2+6iz-1-2i\right\}$ are the simple poles

$$z=\frac{-6i\pm\sqrt{(6i)^2-4(1-2i)(-1-2i)}}{2(1-2i)}$$
$$=\frac{-6i\pm 4i}{2(1-2i)}=2-i,\ (2-i)/5$$

Only $(2-i)/5$ lies inside C.

Residue at

$$(2-i)/5=\lim_{z\to(2-i)/5}\{z-(2-i)/5\}\left\{\frac{2}{(1-2i)z^2+6iz-1-2i}\right\}$$
$$=\lim_{z\to(2-i)/5}\frac{2}{2(1-2i)z+6i}=\frac{1}{2i}$$

by L'Hospital's rule.

Then

$$\oint_C \frac{2\,dz}{(1-2i)z^2+6iz-1-2i}=2\pi i\left(\frac{1}{2i}\right)=\pi,$$

the required value.

Example 8: Show that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta = \frac{\pi}{12}$.

Solution

Let $z = e^{i\theta}$. Then $\cos\theta = (z + z^{-1})/2$, $\cos 3\theta = (e^{3i\theta} + e^{-3i\theta})/2 = (z^3 + z^{-3})/2$, $dz = iz\, d\theta$ so that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta = \oint_C \frac{(z^3 + z^{-3})/2}{5 - 4(z + z^{-1})/2} \frac{dz}{iz} = -\frac{1}{2i} \oint_C \frac{z^6 + 1}{z^3(2z - 1)(z - 2)} dz$$

where C is the contour of Fig. 7-6.

The integrand has a pole of order 3 at $z = 0$ and a simple pole $z = \frac{1}{2}$ inside C.

Residue at $z = 0$ is

$$\lim_{z \to 0} \frac{1}{2!} \frac{d^2}{dz^2} \left\{ z^3 \cdot \frac{z^6 + 1}{z^3(2z - 1)(z - 2)} \right\} = \frac{21}{8}$$

Residue at $z = \frac{1}{2}$ is

$$\lim_{z \to 1/2} \left\{ \left(z - \frac{1}{2} \right) \cdot \frac{z^6 + 1}{z^3(2z - 1)(z - 2)} \right\} = -\frac{65}{24}$$

Then

$$-\frac{1}{2i} \oint_C \frac{z^6 + 1}{z^3(2z - 1)(z - 2)} dz = -\frac{1}{2i}(2\pi i) \left\{ \frac{21}{8} - \frac{65}{24} \right\} = \frac{\pi}{12} \text{ as required.}$$

SUPPLEMENTARY PROBLEMS

Ex 1: For each of the following functions, determine the poles and the residues at the poles:

$$\text{(a)}\ \frac{2z+1}{z^2-z-2}, \quad \text{(b)}\ \left(\frac{z+1}{z-1}\right)^2, \quad \text{(c)}\ \frac{\sin z}{z^2}$$

Ex 2: Find the zeros and poles of $f(z) = \dfrac{z^2+4}{z^3+2z^2+2z}$ and determine the residues at the poles.

Ex 3: Evaluate $\displaystyle\oint_C e^{-1/z}\sin(1/z)\,dz$ where C is the circle $|z| = 1$.

Ex 4: Evaluate $\displaystyle\oint_C \frac{2z^2+5}{(z+2)^3(z^2+4)z^2}dz$ where C is (a) $|z-2i| = 6$, (b) the square with vertices at $1+i$, $2+i$, $2+2i$, $1+2i$.

Ex 5: Evaluate $\displaystyle\oint_C \frac{2+3\sin \pi z}{z(z-1)^2}dz$ where C is a square having vertices at $3+3i$, $3-3i$, $-3+3i$, $-3-3i$.

Ex 6: Evaluate $\displaystyle\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z(z^2+1)}dz$, $t > 0$ around the square with vertices at $2+2i$, $-2+2i$, $-2-2i$, $2-2i$.

Thanks a lot …