Math-4241:-

Integral Calculus and differential equations:-

Book:- Thomas Calculus

$y = x^2 + 5 \rightarrow$ Anti-derivative / Main Function

$\frac{dy}{dx} = 2x \rightarrow$ derivative

Integration:-

Process to find the anti-derivatives.

$\int 2x \cdot dx = x^2 + c \leftarrow$ Anti-derivative

$\Rightarrow$ Inverse Process to derivation / differentiation

$\boxed{4.8}$

Anti-derivatives:-

Integral Calculus:-

(i) Indefinite Integral : Only represents the anti-der.

(ii) Definite Integral :- Represents the area.

Defn:- The process of recovering a function F(x) from it's derivative. The process of recovering a function F(x) from it's derivative f(x) is called anti-differentiation or integration.

$y = F(x)$

$\frac{dy}{dx} = F'(x) = f(x)$

(i) Indefinite Integral:

The collection of all anti-derivatives of $f(x)$ is called the indefinite integral, with respect to x and it is denoted by " $\int f(x)\,dx$ ".

Integral (or, anti-derivative)

sign of integration ← $\int f(x)\,dx = F(x) + c$ ← constant of integration (or, integral constant)

← variable of integration

↑ The function to be integrated is called "Integrant" (or derivative)

Q. Why it is necessary to add the integral constant (c) to the result? Explain with example.

We consider,

$$y = x^2 + 5$$

$$\frac{dy}{dx} = 2x$$

$$\int 2x\,dx = x^2 + c$$

$$y = x^2 - 2$$

$$\frac{dy}{dx} = 2x$$

$$\int 2x\,dx = x^2 + c$$

Same derivative → different anti-derivative

Power Law: $\frac{d}{dx}(x^n) = nx^{n-1}$

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$; $n \neq -1$.

Print sheet

Ex-91:- (91-112) → (71-90)

Solve the initial value problem.

$\frac{dy}{dx} = 2x - 7$. $y(2) = 0$

when, $x = 2$

$y = 0$

$dy = (2x-7)\,dx$

or, $\int dy = \int (2x-7)\,dx$

or, $y = \frac{2x^2}{2} - 7x + c$

or, $y = x^2 - 7x + c$

$\therefore y = x^2 - 7x + c$

Now, using the condition (given),

$y(2) = 0$

or, $(2)^2 - 7(2) + c = 0$

or, $4 - 14 + c = 0$

$\therefore c = 10$

$\therefore y = x^2 - 7x + 10$ (Ans.)

Ex-113:

Find the curve $y=f(x)$ in the xy-plane that passes through the point $(9,4)$ and whose slope at each point is $3\sqrt{x}$.

$$\frac{dy}{dx} = 3\sqrt{x} \qquad y(9)=4.$$

Ex:- that cuts the x-axis at 2.

$$y(2)=0 \quad \equiv (2,0)$$

(i) $\frac{dy}{dx} = 3\sqrt{x}$

or, $dy = 3\sqrt{x}\, dx$

or, $\int dy = \int 3\sqrt{x}\, dx$

or, $y = \cancel{3}\cdot\frac{2}{\cancel{3}}\cdot x^{3/2} + c$

$\therefore y = 2x^{\frac{3}{2}} + c$

$\therefore y = 2x^{\frac{3}{2}} + (-1)\cdot 2^{\frac{5}{2}}$

Now, using given cond.

$y(2) = 0$

or, $2\cdot x^{\frac{3}{2}} + c = 0$

or, $2^{\frac{5}{2}} + c = 0$

$\therefore c = -2^{\frac{5}{2}}$

chapter 5.5 :-

Indefinite Integrals of substitution method

$2x = z^2 - 1$

$\therefore x = \frac{z^2-1}{2}$

Ex-7 :-

$\int x\sqrt{2x+1}\, dx$

$= 2\int z\sqrt{z^2}\, dz$

$= 2\int z^2 dz$

$= 2 \cdot \frac{z^3}{3} + c$

$= \frac{2}{3} z^3 + c$

$= \frac{2}{3}(\sqrt{2x+1})^3 + c$

$2x+1 = z^2$

or, $x \cdot dx = 2z\, dz$

$2x+1 = z^2$

or, $2 \cancel{dx+}$

$2dx = 2z\,dz$

$\therefore dx = z\,dz$

$x = \frac{z^2-1}{2}$

$= \int \frac{z^2-1}{2} \cdot z^2 dz$

$= \frac{1}{2}\int \frac{z^4 - z^2}{\cancel{2}} dz$

$= \frac{1}{2}\int \frac{1}{2} \cdot \frac{1}{5} z^5 - \frac{1}{2} \cdot \frac{1}{3} \cdot z^3 + c$

$= \frac{1}{10} \cdot (\sqrt{2x+1})^5 - \frac{1}{6}(\sqrt{2x+1})^3 + c$

$$\int \sin^3 x \, dx = \int \sin^2 x \cdot \sin x \, dx \qquad \Big| \; z = \cos x$$

$$= \int (1 - \cos^2 x) \sin x \, dx$$

$$= - \quad \ldots$$

$$= -\cos x + \frac{1}{3}\cos^3 x + c.$$

Ex:

i) $\int \sin^4 x \, dx$

ii) $\int \cos^4 x \, dx$

$$\int \cos^2 2x \, dx$$

$$= \frac{1}{2} \int (2\cos^2 2x) \, dx$$

$$= \frac{1}{2} \int (1 + \cos 4x) \, dx$$

$$= \frac{1}{2} x + \frac{1}{8} \sin 4x + c.$$

$$= \frac{x}{2} + \frac{\sin 4x}{8} + c.$$

x^4

=

The Importance of the Constant of integration:

If there are no limits been provided with the integral, we have to add a constant to the integrated solution. That is because when we integrate in two different methods, we end up with two different solution which belongs to same family of function. When we add up a constant, it equates the difference between two answers, as the constant added up ~~to~~ the antiderivative gives the same answer.

24-06-19

Ex-14: $\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) dx$

Let, $z = \frac{1}{x}$

$dz = -\frac{1}{x^2} dx$

$= -\int \cos^2 z \, dz$

$= -\frac{1}{2}\int 2\cos^2 z \, dz$

$= -\frac{1}{2}\int (1+\cos 2z) dz$

$= -\frac{1}{2} z - \frac{1}{2\cdot 2} \sin 2z + c$

$= -\frac{1}{2x} - \sin\frac{2}{x} + c$

13. $\int \sqrt{x} \sin^2\left(x^{\frac{3}{2}} - 1\right) dx$

Let, $x^{3/2} - 1 = z$

or, $dz = \frac{3}{2} x^{\frac{1}{2}} dx$

or, $\frac{2}{3} dz = \sqrt{x}\, dx$

$= \frac{2}{3}\int \sin^2 z \, dz$

$= \frac{1}{3}\int (1 - \cos 2z) dz$

$= \frac{1}{3} z - \frac{1}{3}\cos 2z + c$

$= \frac{1}{3}\left(x^{3/2} - 1\right) - \frac{1}{3}\cos\left[2\cdot\left(x^{\frac{3}{2}} - 1\right)\right] + c$

Ex-57:- $\int \frac{dx}{1+e^x}$

$= \int \frac{dx}{e^x(1+e^{-x})}$

$= \int \frac{e^{-x}\,dx}{1+e^{-x}}$

$= -\int \frac{dz}{z}$

$= -\ln|z| + c$

$= -\ln|1+e^{-x}| + c.$

Let,

$1+e^{-x} = z$

or, $-e^{-x}\,dx = dz$

$e^{-x}\,dx = -dz$

Ex-38:- $\int \sqrt{\frac{x-1}{x^5}}\,dx$

$= \int \frac{\sqrt{x-1}}{x^2\sqrt{x}}\,dx$

$= \int \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\,dx$

$= 2\int z\cdot z\,dz$

$= 2\int z^2\,dz = 2\cdot\frac{1}{3}\cdot z^3 + c = \frac{2}{3}\left(1-\frac{1}{x}\right)^{\frac{3}{2}} + c.$

Let

$1-\frac{1}{x} = z^2$

or, $\frac{1}{x^2}\,dx = 2z\,dz$

Ex-58: $\displaystyle\int \frac{dx}{x\sqrt{x^4-1}}$

$\displaystyle = \int \frac{dx}{x^3\sqrt{1-\frac{1}{x^4}}}$

Let,

$1-\frac{1}{x^4} = z^2$

or, $4\cdot\frac{1}{x^3}\,dx = 2z\,dz$

or, $\frac{1}{x^3}\,dx = \frac{1}{2}z\,dz$

$\displaystyle = \frac{1}{2}\int \frac{z\,dz}{z}$

$\displaystyle = \frac{1}{2}\int dz$

$= \frac{1}{2}2z + c$

$= \sqrt{1-\frac{1}{x^4}} + c.$

It is estimated that x month from now the population of a certain will be changing at the rate of $2+6\sqrt{x}$ people per month. The current population is 5000. What will be the population 9 months from now?

Let, $P(x)$ be denoted the population of the city.

$$\frac{dp}{dx} = 2+6\sqrt{x}$$

$$P(x) = 2x + 6\cdot\frac{2}{3}\cdot x^{\frac{3}{2}} + c$$

$$= 2x + 4\cdot x^{\frac{3}{2}} + c.$$

When, $x=0$, $P=5000$

$$\therefore c = 5000.$$

Finally, The population 9 months from now,

$$P(9) = 2\cdot 9 + 4\cdot 9^{\frac{3}{2}} + 5000$$

$$= 18 + \cancel{8+5000}\ 4\cdot 27 + 5000$$

$$= \cancel{45+5000}\ 5000+126$$

$$= \cancel{5045}\ 5126.$$

8.1 :- Integration By Parts

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

LIATE

Ex :-

$\int x \ln x \, dx$

$\int x \sin^{-1} x \, dx$

$\int x \tan^{-1} x \, dx$

$\frac{N}{N^2+1}$ $\quad$ $\frac{x^3}{1}$ $\quad$ $\frac{x^2}{1+x^2}$

$1+x^2 = z$

or,

$\int x \tan^{-1} x \, dx$

$= \tan^{-1}x \int x\,dx - \int \left\{ \frac{d}{dx}(\tan^{-1}x) \int x\,dx \right\} dx$

$= \frac{x^2}{2} \tan^{-1}x - \int \frac{1}{1+x^2} \frac{x^2}{2} dx$

$= \frac{x^2}{2} \tan^{-1}x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$

$= \frac{x^2}{2} \tan^{-1}x - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2} dx$

$= \frac{x^2}{2} \tan^{-1}x - \frac{1}{2} \tan^{-1} \cancel{x} - \frac{1}{2}x + C$

$$\int \frac{x^3}{1+x^2}\,dx$$

$$\cancel{\frac{x^3}{1+x^2}}$$

$$= \cancel{\int \frac{x^3}{x^2(1+x^{-2})}\,dx}$$

$$= \cancel{\int \frac{x}{1+x^2}\,dx}$$

$$= \cancel{\int x} \quad \frac{1}{2}\int \frac{(z-1)\,dz}{z}$$

$1+x^2 = z$

or, $2x\,dx = dz$.

$$= \frac{1}{2}\int \left(1 - \frac{1}{z}\right) dz$$

$$= \frac{1}{2} z - \frac{1}{2}\ln|z| + c$$

$$= \frac{1}{2}(1+x^2) - \frac{1}{2}\ln|1+x^2| + c.$$

$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}$

$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}$

$\int e^{-ax} \sin bx \, dx$
$= \frac{-e^{-ax}}{a^2+b^2}$

$\int e^{-ax} \cos bx \, dx$
$= \frac{-e^{-ax}}{a^2+b^2}$

<u>Soln:-</u>

$\int e^{-ax} \sin bx \, dx$

Let, $I = \int e^{-ax} \sin bx \, dx$
$= \sin bx \int e^{-bx} dx - \int$

$= -\frac{1}{a} e^{-ax} \cdot \sin bx - \int$

$= -\frac{1}{a} e^{-ax} \sin bx + \frac{b}{a}$

$= \frac{e^{-ax} \sin bx}{a} + \frac{b}{a}$

$\int e^{-ax} dx \} dx]$

$= \quad " \quad + \frac{b}{a} [-\frac{1}{a} \cdot$
$\cdot (-\frac{1}{a} \cdot e^{-ax}) dx]$

$$I = \frac{-e^{-ax}\sin bx}{a} - \frac{b}{a^2} e^{-ax}\cos bx - \frac{b^2}{a^2} I$$

or, $\left(1 + \frac{b^2}{a^2}\right) I = \frac{-e^{-ax}}{a^2}\left[a\sin bx + b\cos bx\right] + c$

or, $(a^2+b^2) I = -e^{-ax}\left[a\sin bx + b\cos bx\right] + c_1$

$\therefore I = \frac{-e^{-ax}}{a^2+b^2}\left[a\sin bx + b\cos bx\right] + c_1.$

<u>Evaluate</u>: $\int e^{-x}\sin 2x\,dx$

$\hookrightarrow$ Derivation

Put the value a and b in the general method

Obtain the reduction formula for $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$

$$I_n = \int \cos^n x\,dx = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n}\underbrace{\int \cos^{n-2} x\,dx}_{I_{n-2}}$$

$$I_n = \int \sin^n x\,dx = \frac{-\sin^{n-1}x \cdot \cos x}{n} + \frac{n-1}{n}\underbrace{\int \sin^{n-2} x\,dx}_{I_{n-2}}$$

Exa-5

$\int \sin^n x\,dx$

Let, $I_n = \int \sin^n x\,dx$

$$= \int \sin^{n-1}x \cdot \sin x\,dx$$

$$= \sin^{n-1}x \int \sin x\,dx - \int \left[\frac{d}{dx}(\sin^{n-1}x)\int \sin x\,dx\right]dx$$

$$= -\sin^{n-1}x \cdot \cos x - \int (n-1)\sin^{n-2}x \cos x\,(-\cos x)\,dx$$

$$= -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2} \cdot (1-\sin^2 x)\cdot dx$$

$$= -\sin^{n-1}x \cos x + (n-1)\int (\sin^{n-2}x - \sin^n x)\,dx$$

$$= -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}dx - (n-1)\int \sin^n x\,dx$$

$$[1+(n-1)]\, I_n = -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x\,dx$$

$$\therefore I_n = \frac{-\sin^{n-1}x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$n=4$

$$I_4=\int \sin^4 x\,dx = -\frac{\sin^3 x\cos x}{4}+\frac{3}{4}\int \sin^2 x\,dx$$

$$= -\frac{\sin^3 x\cos x}{4}+\frac{3}{4}\left[\frac{-\sin x\cos x}{2}+\frac{1}{2}\int dx\right]$$

$$= -\frac{\sin^3 x\cos x}{4}+\frac{3}{8}\left(-\sin x\cos x\right)+\frac{3}{8}x+C$$

(35-48) → Two times substitution

Ex-38:

$$\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}}$$

$z = \ln x$

or, $dz = \frac{1}{x}dx$

$$= \int_0^1 \frac{dz}{\sqrt{1+z^2}}$$

$\cancel{z = \sec\theta}$

$\therefore \cancel{dz = \sec\theta \tan\theta\, d\theta}$

$z = \tan\theta$

$dz = \sec^2\theta\, d\theta$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\, d\theta}{\sec\theta}$$

$$= \int_0^{\frac{\pi}{4}} \sec\theta\, d\theta$$

$$= \Big[\ln(\sec\theta + \tan\theta)\Big]_0^{\frac{\pi}{4}}$$

$$= \ln(\sqrt{2}+1) - \cancel{\ln(0+0)}\ \ln(1)$$

$$= \ln(1+\sqrt{2})$$

$\sec^2\theta + \tan^2 = 1$

$\sqrt{1+\tan^2\theta} = z$

43. $\int \frac{x\,dx}{\sqrt{1+x^4}}$

Let, $x^2 = z$

or, $2x\,dx = dz$

$= \frac{1}{2}\int \frac{dz}{\sqrt{1+z^2}}$

$= \frac{1}{2}\ln(\sec\theta + \tan\theta) + c$

$= \frac{1}{2}\ln\left|\sqrt{1+z^2} + z\right| + c$

$= \frac{1}{2}\ln\left|\sqrt{1+x^4} + x^2\right| + c.$

$= \frac{1}{2}\ln\left|\sqrt{1+x^4} + x^2\right| + c.$

8.4- Integration of rational functions by partial fractions

(i) General Method
(ii) Heaviside Cover-up method
(iii) Differentiating Method

(i) $\int \frac{P(x)}{Q(x)}\,dx$

(i) $Q(x)$ is a linear factor :- $(x-r)$: $\frac{A}{x-r}$

$(x-r_1)(x-r_2)$: $\frac{A}{x-r_1}+\frac{B}{x-r_2}$

$(x-r)^2$: $\frac{A}{(x-r)}+\frac{B}{(x-r)^2}$

(ii) $Q(x)$ is (ax^2+bx+c) form : $\frac{Ax+B}{ax^2+bx+c}$

Exa-1 : $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}$$

or, $x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x+1)(x-1)$ ——(i)

$x=-1$, $\cancel{1+4}$

$1-4+1 = B(2)(-2)$

$B = \cancel{}\ \frac{1}{\cancel{}}/2$

$x=1$, $1+4+1 = A(2)\cdot 4$

or, $6 = 8A$.

$A = \frac{6}{8} = \frac{3}{4}$

$x=-3$, $9-36+1 = C\,\cancel{(-2)(-4)}\ 4$.

or, $\cancel{C = \frac{-26}{8}}$ $C = -\frac{1}{4}$.

Equating like terms on both sides, we have,

co-of x^2 : $1 = A+B+C$ — (i)

co-of x : $4 = 4A+2B$

or, $2A+B=2$ — (ii)

co-of cons : $1 = 3A-3B-C$ — (iii)

$A = \frac{3}{4}$

$B = \frac{1}{2}$

$C = -\frac{1}{4}$

$$\therefore \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \int \frac{3}{4}\frac{dx}{x-1} + \int \frac{dx}{2}\frac{1}{x+1} - \int \frac{1}{4}\frac{dx}{x+3}$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x+3)+c$$

$x(x-3)+1$

Exa-3:- $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$

$$\begin{array}{r|l|l} x^2-2x-3 & 2x^3-4x^2-x-3 & 2x \\ & 2x^3-4x^2-6x & \\ \hline & 5x-3 & \end{array}$$

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \cancel{5x+} \frac{5x-3}{x^2-2x-3}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{x^2-3x+x-3} = \frac{5x-3}{(x-3)(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x-3)}$$

$5x-3 = A(x-3)+B(x+1)$

co-off x :- $5 = A+B$

coeff x^0 :- $-3 = -3A+B$

$A = 2$

$B = 3$

Finally,

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{2}{x+1} + \frac{3}{x-3}$$

$$\int = \int 2x\,dx + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + c.$$

Exa-5 :- $\int \frac{dx}{x(x^2+1)^2}$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$1 = A(x^2+1)^2 + (Bx+C)\,x(x^2+1) + (Dx+E)x.$$

$$\text{or, } 1 = A(x^4+2x^2+1) + (Bx+C)(x^3+x) + (Dx+E)x.$$

$$1 = A(x^4+2x^2+1) + (Bx^4+Bx^2) + C(x^3+x) + (Dx+E)x$$

Co-off x^4 :- $0 = A + B$ $A = 1$

x^3 :- $0 = C$ $\therefore B = -1$

x^2 :- $0 = 2A + B + D$ $C = 0$

x^1 :- $0 = C + E$ $D = -2 + 1 = -1$

x^0 : $1 = A$ $E = 0.$

$$\int \frac{dx}{x(x^2+1)^2} = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \ln x - \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\int \frac{dz}{z^2}$$

$$= \ln x - \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\frac{1}{(x^2+1)} + c.$$

Method-2

Heaviside cover-up method:-

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n}$$

$$\text{or, } \frac{P(x)}{Q(x)}(x-r_1) = A_1 + A_2(x$$

Ex-5:

$$\int \frac{x^2+1}{(x-1)(x-2)(x-3)}\,dx$$

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{}{(}$$

$$\frac{(x^2+1)\cancel{(x-1)}}{\cancel{(x-1)}(x-2)(x-3)} = A + \frac{B(x-1)}{x-2}$$

$x = 1 \rightarrow$

$$\Rightarrow \frac{+1}{1} = A \quad \therefore A = 1.$$

$$\frac{x^2+1}{(x-1)(x-3)} = \frac{A(x-2)}{(x-1)} + B + \frac{C}{(x}$$

$x = 2,\ B = -5.$

$$\frac{x^2+1}{(x-1)(x-2)} = \frac{A(x-3)}{(x-1)} + \frac{B(x-3)}{x-2} + C$$

$x=3 \rightarrow C=5$

$$\int \frac{x^2+1}{(x-1)(x-2)(x-3)}\,dx$$

$=$

M-3:-

Differentiating Method :-

$$\int \frac{x-1}{(x+1)^3} dx$$

$$\frac{x-1}{(x+1)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

Or, $x-1 = A(x+1)^2 + B(x+1) + C$ ——(i)

Putting $x=-1$, in (i), we get $C=-2$.

Now, differentiating (i) with respect to x,

$$1 = 2A(x+1) + B \text{ —— (ii)}$$

Putting $x=-1$ in (ii) we get $B=1$.

Again, differentiating (ii) w.r.t. x,

$$0 = 2A$$

$$\therefore A = 0.$$

$\therefore$ The constants are $A=0$, $B=1$ and $C=-2$.

Now, $\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} + \frac{(-2)}{(x+1)^3}$

$$\int \frac{x-1}{(x+1)^3} dx = \int \frac{dx}{(x+1)^2} - 2\int \frac{dx}{(x+1)^3}$$

$z = x+1$

$dz = dx$

$$= \int \frac{dz}{z^2} - 2\int \frac{dz}{z^3} = -z^{-1} + 4z^{-2} + c$$

$$= -(x+1)^{-1} + 4(x+1)^{-2} + c.$$

Exc-2

$$\int \frac{5x-7}{x^2-3x+2}\,dx$$

$$\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{x^2-2x-x+2} = \frac{5x-7}{(x-2)(x-1)}$$

Now,

$$\frac{5x-7}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

or, $5x-7 = A(x-2)+B(x-1)$

co-eff of x,

$5 = A+B$. or, $-A-B = -5$.

co-eff of x^0,

$-7 = -2A-B$.

$\therefore 2A+B = 7$.

$A = 2$.

$-B = -5+2 = -3$

$\therefore B = 3$.

We have, $A = 2$, $B = 3$.

$$\int \frac{3}{x-2}\,dx + \int \frac{2}{x-1}\,dx$$

$$= 3\ln|x-2| + 2\ln|x-1| + c$$

$$= \ln(x-2)^3 + \ln(x-1)^2 + c$$

$$= \ln[(x-1)^2(x-2)^3] + c.$$

Exc-14

$$\int_{\frac{1}{2}}^{1} \frac{x+4}{x^2+x}\,dx$$

18. $$\int_{-1}^{0} \frac{x^3}{x^2-2x+1}\,dx$$

$$x^2-2x+1 \,\big)\, x^3 \,\big(\, x+2$$
$$x^3-2x^2+x$$
$$+2x^2-x$$
$$2x^2-4x+2$$
$$3x-2$$

$$\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{x^2-2x+1}$$

$$\therefore \frac{3x-2}{x^2-2x+1} = \frac{3x-2}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

$$\therefore 3x-2 = A(x-1) + B.$$

$x=1$, $3-2 = B = 1.$

$\therefore B = 1$

$\cancel{2=}$ $3 = A.$

$\therefore A = 3$

$$\therefore \int (x+2)\,dx + \int \frac{3}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

Ex(39-50):-

40. $\int \frac{e^{4t}+2\cdot e^{2t}-e^{t}}{e^{2t}+1}\,dk$

Let,
$x = e^{t}$
$dx = e^{t}\,dt$

$= \int \frac{e^{t}(e^{3t}+2e^{t}-1)}{e^{2t}+1}\,dx$

$= \int \frac{x^3+2x-1}{x^2+1}\,dx$

$x^2+1 \,|\, x^3+2x-1 \,|\, x$
x^3+x
$x-1$

$\therefore \frac{x^3+2x-1}{x^2+1} = x + \frac{x-1}{x^2+1}$

$\therefore \int \frac{x^3+2x-1}{x^2+1}\,dx = \int x\,dx + \int \frac{x-1}{x^2+1}\,dx$

Now, $\frac{x-1}{x^2+1} = \frac{Ax+B}{x^2+1}$

or, $(x-1) = Ax+B$

Now, $x=1$, $A+B=0$
$x=0$, $B=-1$
$\therefore A=1$

$\therefore \frac{x-1}{x^2+1} = \frac{x-1}{x^2+1}$

$\therefore \int \frac{(x^3+2x-1)}{x^2+1}dx = \int x\,dx + \int \frac{x-1}{x^2+1}dx$

$= \int x\,dx + \int \frac{x}{x^2+1}dx - \int \frac{dx}{x^2+1}$

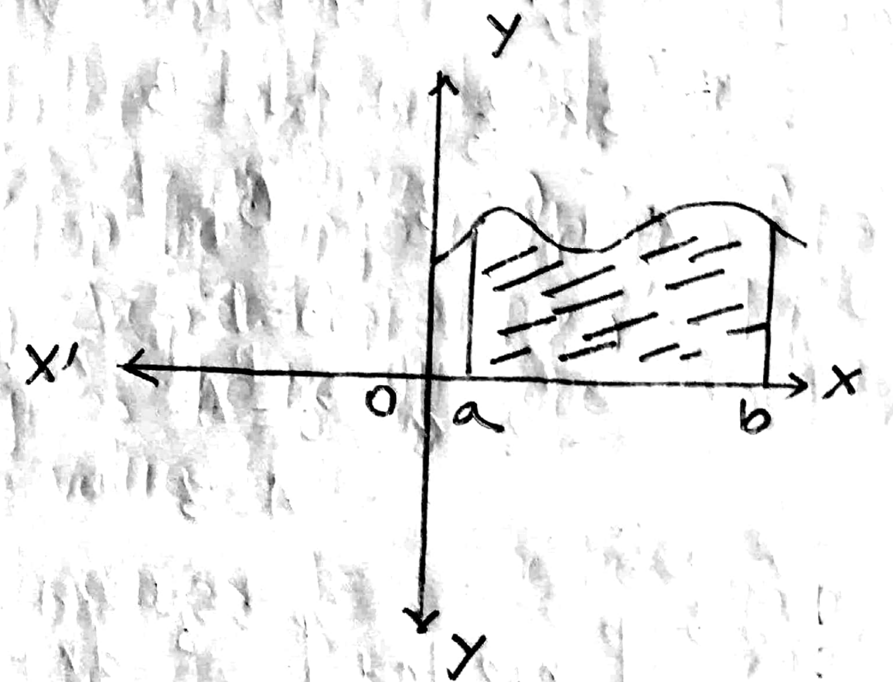
$= \frac{x^2}{2} + \frac{1}{2}\ln|x^2+1| + \tan^{-1}(x^2+1) + c$

$= \frac{(e^{t})^2}{2} + \frac{1}{2}\ln|(e^{t})^2+1| + \tan^{-1}[(e^{t})^2+1] + c.$

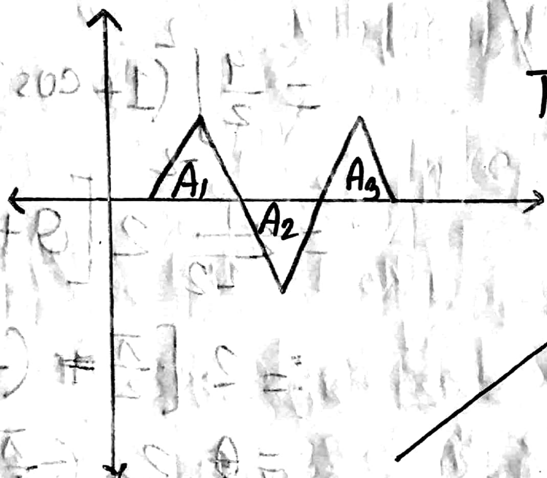
Chapter 5.1

Area and Estimating with Finite Sums

$\int_a^b f(x)\,dx$ represents the area of a space bounded by the curve $y=f(x)$ (upper boundary) with x-axis (lower boun.) and the ordinates $x=a$ (left boun.) and $x=b$ (right boun.).



□ $\int_1^2 x\,dx = \frac{3}{2}$ sq. unit, verify using appropiate formula from geometry.

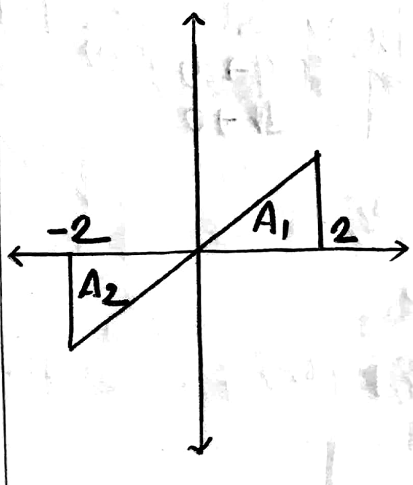


Total Area $= |A_1| + |A_2| + |A_3|$

Net signed area $= |A_1| - |A_2| + |A_3|$

$\int_{-2}^{2} x\,dx = \frac{1}{2}(4-4) = 0$

∴ Integrating will get us the net signed area only.



A_1 = area of ΔOAB

$= \frac{1}{2} OA \cdot AB = 2$ sq·unit.

$A_2 = 2$ sq. unit.

Evaluate:

$$\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$\cancel{=\int_{-2}^{0}\sqrt{4-x^2}\,dx+\int_{0}^{2}\sqrt{4-x^2}\,dx}$$

$$\cancel{=\int_{-2}^{0}\sqrt{4-x^2}\,dx-\int_{-2}^{0}\sqrt{4-x^2}\,dx}$$

$$\cancel{=0}$$

$$x=2\sin\theta$$

$$dx=2\cos\theta\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta$$

$$=2\cdot 2\int_{-2}^{2}\cos^2\theta\,d\theta$$

$$=\frac{4}{2}\int_{-2}^{2}(1+\cos 2\theta)\,d\theta$$

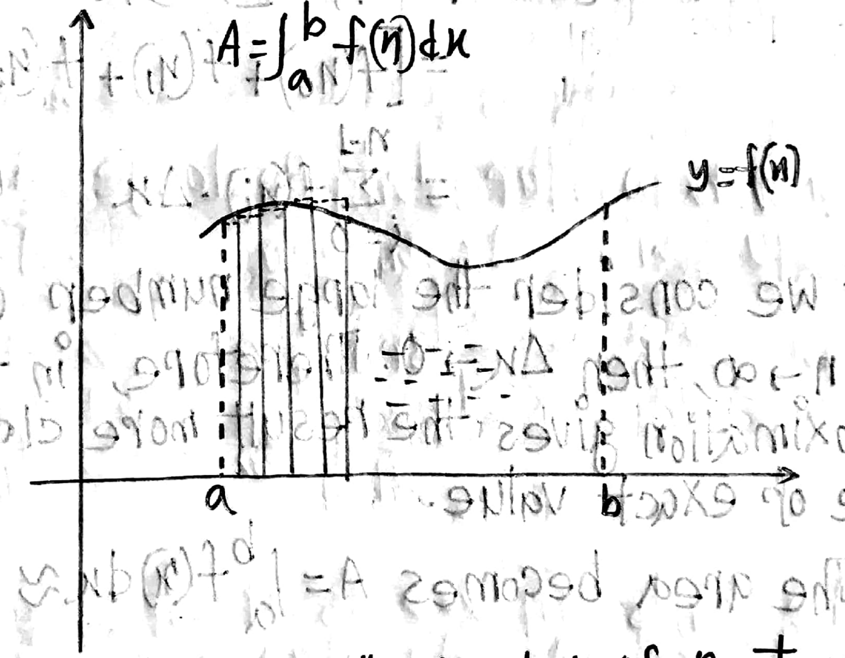
$$=\cancel{2\int_{-2}}\;2\left[\theta+\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=2\cdot\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]$$

$$=2\cdot\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$=2\pi.$$

1. Anti-derivative Method (Definite Method)
2. Approximation Method (Rectangle Method)
— (i) Left-end point approximation (Lower-sum)
— (ii) Right-endpoint " (Upper-sum)
— (iii) Midpoint - "



Let 'n' be the number of rectangles with equal width Δx, where $\Delta x = \frac{b-a}{n}$, so there are $(n+1)$ points on x-axis.

Let, $x_0 = a \longrightarrow$ Lower limit
$x_1 = x_0 + \Delta x$
$x_2 = x_1 + \Delta x$
$x_n = x_{n-1} + \Delta x = b \longrightarrow$ Upper limit

M:1 :- Left-end point approximation

From fig., the area of the first rectangle $A_1 = f(x_0)$

the area of the second rect $A_2 = f(x_1) \cdot \Delta x$

" " of " third " $A_3 = f(x_2) \Delta x$

$\therefore$ The area of the nth " " $A_n = f(x_{n-1}) \cdot \Delta x$

$\therefore$ The total area, $A = A_1 + A_2 + A_3 + \dots + A_n$

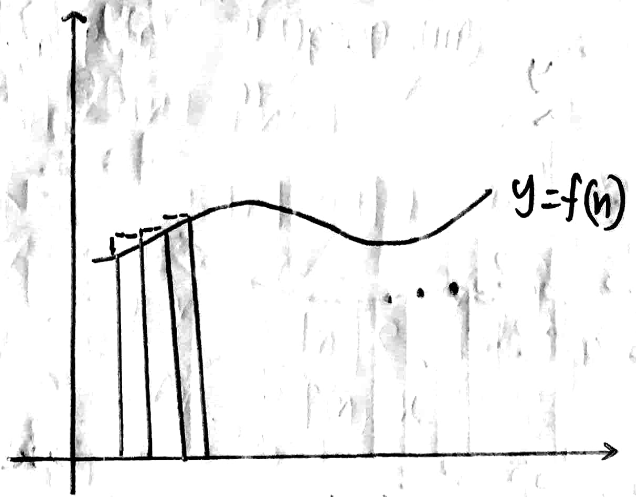
$$= [f(x_0) + f(x_1) + f(x_2) + \dots + f(x_{n-1})] \Delta x$$

$$= \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x$$

If we consider the large number of rectangles i.e. $n \to \infty$, then $\Delta x \to 0$. Therefore, in that case, our approximation gives the result more close to true vellue of exact value.

$\therefore$ The area becomes $A = \int_a^b f(x)\,dx \approx \underset{\substack{n \to \infty \\ \Delta x \to 0}}{\mathrm{Lt}} \sum_{i=0}^{n-1} f(x_i) \Delta x$

M:2 :- Right-endpoint Approximation



The area of the 1st rectangle $= A_1 = f(x_1)\cdot\Delta x$
" " " 2nd " $= A_2 = f(x_2)\Delta x$
" " " 3rd " $= A_3 = f(x_3)\Delta x$
" " " nth " $= A_n = f(x_n)\Delta x$

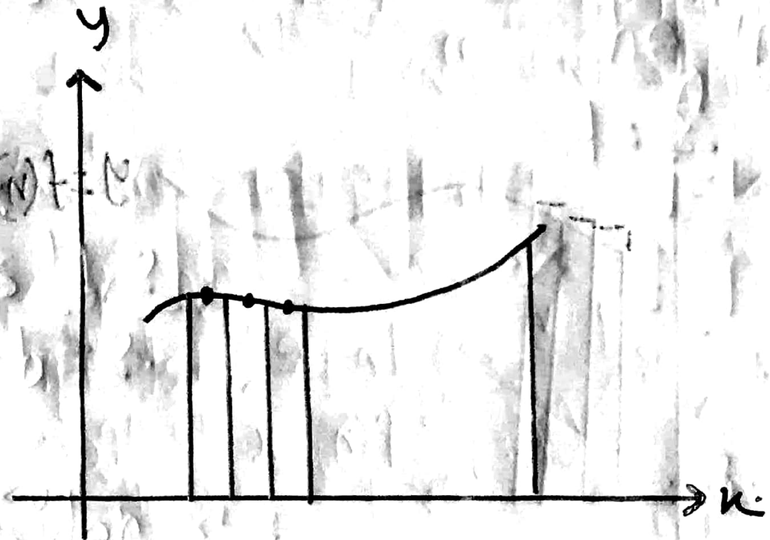
Total area, $A = A_1 + A_2 + A_3 + \cdots + A_n$

$$= [f_1 + f_2 + f_3 + \cdots + f(x_n)]\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

$\therefore$ The area becomes, $A = \int_a^b f(x)\,dx \approx \lim_{\substack{n\to\infty \\ \Delta x\to 0}} \sum_{i=1}^{n} f(x_i)\Delta x$

M:3 :- Mid point Approx. / Trapozidal rule



The area of the 1st rectange, $A_1 = \frac{f(x_0)+f(x_1)}{2}\Delta x$

2nd " , $A_2 = \frac{f(x_1)+f(x_2)}{2}\Delta x$

3rd " , $A_3 = \frac{f(x_2)+f(x_3)}{2}\Delta x$

nth " , $A_n = \frac{f(x_{n-1})+f(x_n)}{2}\Delta x$

$\therefore$ Total Area $= A_1 + A_2 + A_3 + \cdots + A_n$

$= \left[\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}\right] \cdot \Delta x$

$= \left[\frac{f(x_0)+f(x_n)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right]\Delta x$

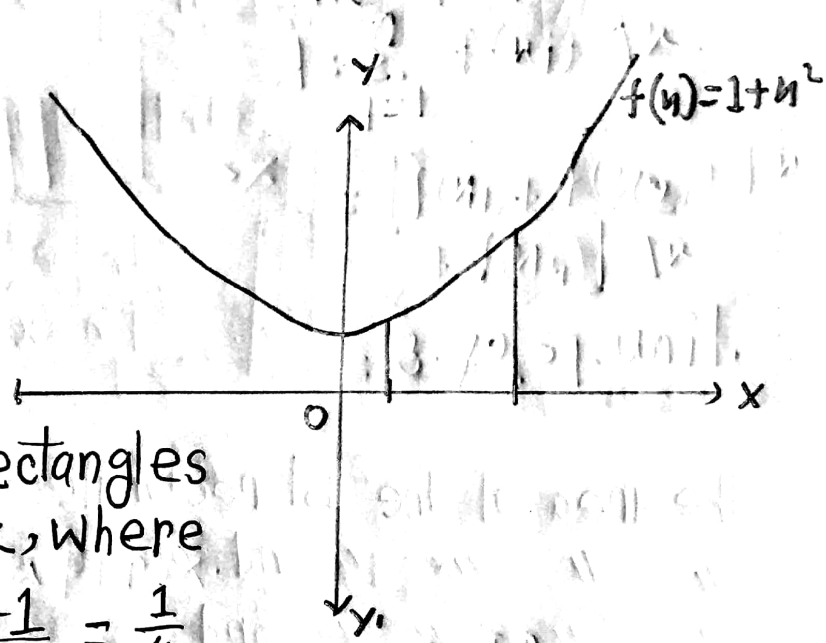
$= \left[\frac{f(x_0)+f(x_n)}{2} + \sum_{i=1}^{n-1} f(x_i)\right]\Delta x.$

The area becomes, $A = \int_a^b f(x)dx \approx \underset{\substack{n\to\infty \\ \Delta x \to 0}}{\mathrm{Lt}} [\quad] \Delta x$

Show that, the definite integral can be written as the limit of finite sum.

Find the area of the region bounded by the curve $f(x)=1+x^2$ with x-axis in the interval $[1,2]$ using three different approximations by considering $n=4$. Finally, compare your results with true value and comments.

Given fn,
$f(x)=1+x^2$
and intervals $[1,2]$.
i.e. $a=1$, $b=2$.
Here,
We can consider
$n=4$ number of rectangles
with equal width Δx, where

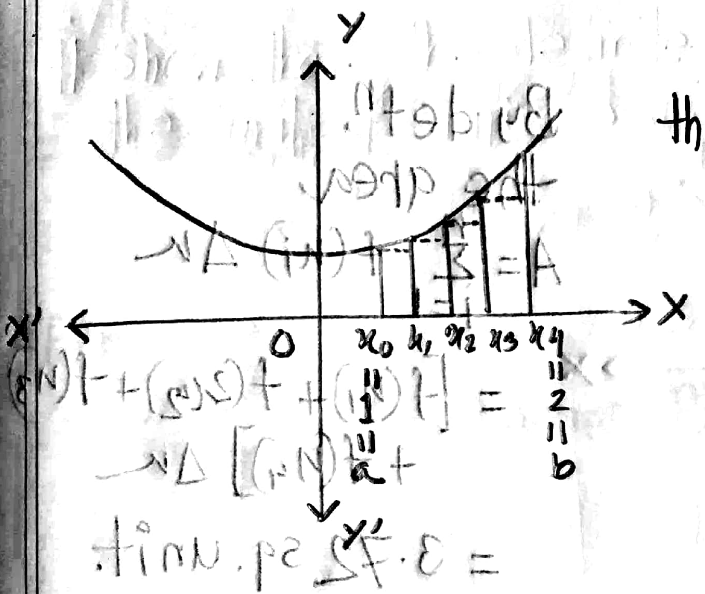


$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$
$$= 0.25$$

$\therefore$ The points on x-axis are:-

$x_0 = 1 = a$	$f(x_0) = 2$
$x_1 = x_0 + \Delta x = 1.25$	$f(x_1) = 2.5625$
$x_2 = 1.5$	$f(x_2) = 3.25$
$x_3 = 1.75$	$f(x_3) = 4.06$
$x_4 = 2$	$f(x_4) = 5$

M-1:- Left-endpoint approx.



By def^n, the area, $A=\sum_{i=0}^{n-1} f(x_i)\,\Delta x$

$=\sum_{i=0}^{3} f(x_i)\,\Delta x$

$=\Delta x[f(x_0)+f(x_1)+f(x_2)+f(x_3)]$

$=\cancel{47.48}$ 2.97 sq.unit

True value, $E=\int_1^2 (1+x^2)\,dx$

$= \left[x+\frac{x^3}{3}\right]_1^2 = 2+\frac{8}{3}-1-\frac{1}{3} = 3.33$ sq. unit.

$$= \left[\frac{2+5}{2} + 2.56 + 3.25 + 4.06\right] \Delta x$$

$= 3.34$ sq. unit.

For M-1:-

Percantage of error, $PE = \left|\frac{E-A}{E}\right| \times 100\%$

$= 10.81\%$

For M-2:-

$P.E. = 11.71\%$

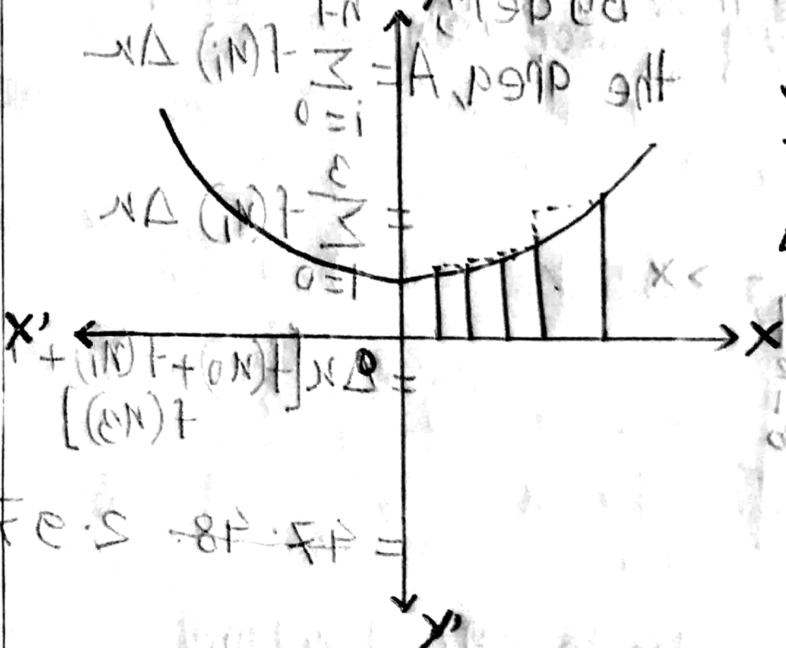
For M-3:-

$P.E = 0.3\%$

From the above discussion, we can observe, our result from midpoint approx. is the closest to the actual result and the p.e. is the least among all the three methods.

$\therefore$ This is the best method.

M-2 :- Right-endpoint approx.



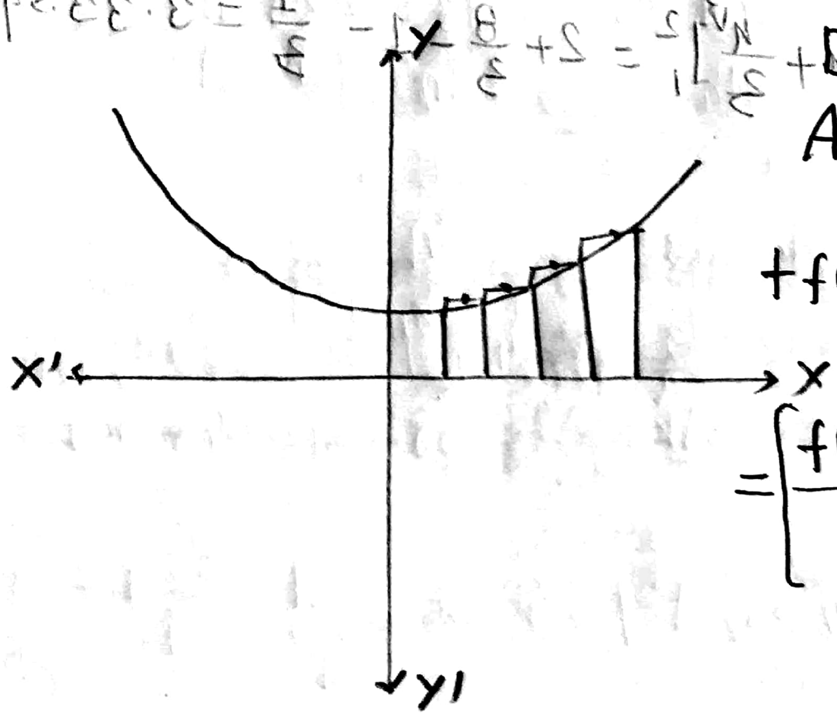
By def^n, the area

$$A = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\,\Delta x$$

$= 3.72$ sq. unit.

M-3 :- Mid-point approx.



By def^n,

$$A = \left[\frac{f(x_0) + f(x_n)}{2} + f(x_1) + f(x_2) + \dots + f(x_{n-1})\right]\Delta x$$

$$= \left[\frac{f(x_0) + f(x_4)}{2} + f(x_1) + f(x_2) + f(x_3)\right]\Delta x$$