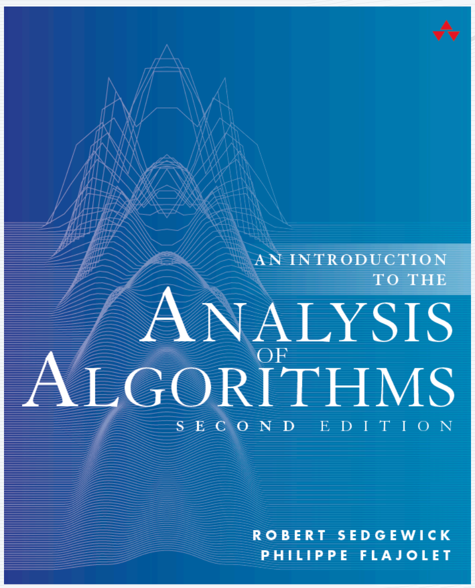
ANALYTIC COMBINATORICS
PART ONE



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4. Asymptotic Approximations

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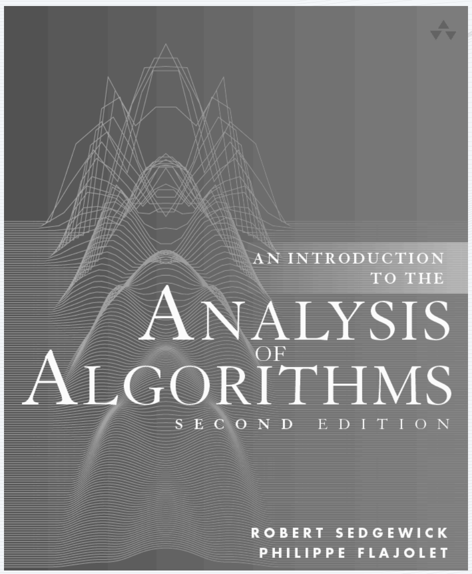

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4. Asymptotic Approximations

- **Standard scale**
- Manipulating expansions
- Asymptotics of finite sums
- Bivariate asymptotics

Asymptotic approximations

Goal: Develop *accurate* and *concise* estimates of quantities of interest

$$O(\log N)$$ ✗ *not accurate*

$$H_N = \sum_{0 \le k \le N} \frac{1}{k}$$ ✗ *not concise*

$$\ln N + \gamma + O(\frac{1}{N})$$ ✓

Informal definition of *concise*:

"easy to compute with constants and standard functions"

Notation (revisited)

"Big-Oh" notation for upper bounds

$$g(N) = O(f(N)) \text{ iff } |g(N)/f(N)| \text{ is bounded from above as } N \to \infty$$

"Little-oh" notation for lower bounds

$$g(N) = o(f(N)) \text{ iff } g(N)/f(N) \to 0 \text{ as } N \to \infty$$

"Tilde" notation for asymptotic equivalence

$$g(N) \sim f(N) \text{ iff } g(N)/f(N) \to 1 \text{ as } N \to \infty$$

Notation for approximations

"Big-Oh" approximation

$$g(N) = f(N) + O(h(N))$$

Error will be at most within a constant factor of $h(N)$ as N increases.

"Little-oh" approximation

$$g(N) = f(N) + o(h(N))$$

Error will decrease relative to $h(N)$ as N increases.

"Tilde" approximation

$$g(N) \sim f(N)$$

Weakest nontrivial o-approximation.

Standard asymptotic scale

sequence

Definition. A decreasing series $g_k(N)$ with $g_{k+1}(N) = o(g_k(N))$ is called an *asymptotic scale.* The series

$$f(N) \sim c_0 g_0(N) + c_1 g_1(N) + c_2 g_2(N) + \ldots$$

is called an *asymptotic expansion* of f. The expansion represents the collection of formulae

$$\begin{aligned}
f(N) &= O(g_0(N)) \\
f(N) &= c_0 g_0(N) + O(g_1(N)) \\
f(N) &= c_0 g_0(N) + c_1 g_1(N) + O(g_2(N)) \\
f(N) &= c_0 g_0(N) + c_1 g_1(N) + c_2 g_2(N) + O(g_3(N)) \\
&\ \ \vdots
\end{aligned}$$

The standard scale is products of powers of N, log N, iterated logs and exponentials.

Typically, we:

- use only 2, 3, or 4 terms (continuing until unused terms are extremely small)
- use ~-notation to *drop* information on unused terms.
- use O-notation or o-notation to *specify* information on unused terms.

Methods extend in principle to any desired precision.

Example: Asymptotics of linear recurrences

≠

Theorem. Assume that a rational GF $f(z)/g(z)$ with $f(z)$ and $g(z)$ relatively prime and $g(0)=0$ has a unique pole $1/\beta$ of smallest modulus and that the multiplicity of β *is* ν. Then

$$[z^n]\frac{f(z)}{g(z)} \sim C\beta^n n^{\nu-1} \quad \text{where} \quad C = \nu\frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)}$$

Proof sketch

$$\sum_{0\le j<m_1} c_{1j}n^j\beta_1^n + \sum_{0\le j<m_2} c_{2j}n^j\beta_2^n + \ldots + \sum_{0\le j<m_r} c_{rj}n^j\beta_r^n$$

Largest term dominates.

Ex.

N	3^N	3^N+2^N
7	2187	275
8	6561	17811
9	19683	20195
10	59049	60073
11	177147	179195

Notes:

- Pole of smallest modulus usually dominates.
- Easy to extend to cover multiple poles in neighborhood of pole of smallest modulus.

Example 1.
$3^N + 2^N \sim 3^N$ ✓

Example 2.
$2^N + 1.99999^N \sim 2^N$ ✗

Asymptotics of linear recurrences

≠

Theorem. Assume that a rational GF $f(z)/g(z)$ with $f(z)$ and $g(z)$ relatively prime and $g(0)$=0 has a unique pole $1/\beta$ of smallest modulus and that the multiplicity of β *is* ν. Then

$$[z^n]\frac{f(z)}{g(z)} \sim C\beta^n n^{\nu-1} \quad \text{where} \quad C = \nu\frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)}$$

Example from earlier lectures.

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2 \text{ with } a_0 = 0 \text{ and } a_1 = 1$$

Make recurrence valid for all *n*.

$$a_n = 5a_{n-1} - 6a_{n-2} + \delta_{n1}$$

Multiply by z^n and sum on *n*.

$$A(z) = 5zA(z) - 6z^2A(z) + z$$

Solve.

$$A(z) = \frac{z}{1 - 5z + 6z^2}$$

Smallest root of denominator is 1/3.

$$a_n \sim 3^n \qquad C = 1\frac{(-3)(1/3)}{-5 + 12/3} = 1$$

Fundamental asymptotic expansions

are immediate from Taylor's theorem.

exponential	$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)$
logarithmic	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$
binomial	$(1+x)^k = 1 + kx + \binom{k}{2}x^2 + \binom{k}{3}x^3 + O(x^4)$
geometric	$\frac{1}{1-x} = 1 + x + x^2 + x^3 + O(x^4)$

as x ➛ 0.

Fundamental asymptotic expansions

are immediate from Taylor's theorem.

Substitute x = 1/N to get expansions as N ➛ ∞.

exponential	$e^{1/N} = 1 + \frac{1}{N} + \frac{1}{2N^2} + \frac{1}{6N^3} + O(\frac{1}{N^4})$
logarithmic	$\ln(1 + \frac{1}{N}) = \frac{1}{N} - \frac{1}{2N^2} + \frac{1}{3N^3} + O(\frac{1}{N^4})$
binomial	$(1 + \frac{1}{N})^k = 1 + \frac{k}{N} + \binom{k}{2}\frac{1}{N^2} + \binom{k}{3}\frac{1}{N^3} + O(\frac{1}{N^4})$
geometric	$\frac{1}{N-1} = \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + O(\frac{1}{N^4})$

as N ➛ ∞.

Inclass exercise

Develop the following asymptotic approximations

$$\ln(1+\frac{1}{N})+\ln(1-\frac{1}{N}) \quad \text{to} \quad O(1/N^3)$$

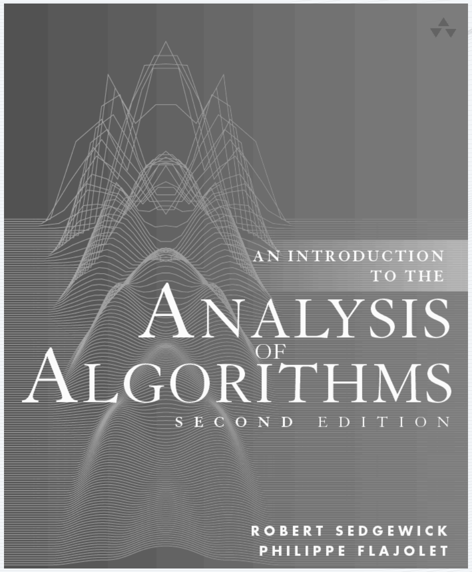
$$= \frac{1}{N} - \frac{1}{2N^2} + O(\frac{1}{N^3}) - \frac{1}{N} - \frac{1}{2N^2} + O(\frac{1}{N^3})$$

$$= -\frac{1}{N^2} + O(\frac{1}{N^3})$$

$$\ln(1+\frac{1}{N})-\ln(1-\frac{1}{N}) \quad \text{to} \quad O(1/N^3)$$

$$= \frac{1}{N} - \frac{1}{2N^2} + O(\frac{1}{N^3}) + \frac{1}{N} + \frac{1}{2N^2} + O(\frac{1}{N^3})$$

$$= \frac{2}{N} + O(\frac{1}{N^3})$$



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4. Asymptotic Approximations

- **Standard scale**
- Manipulating expansions
- Asymptotics of finite sums
- Bivariate asymptotics

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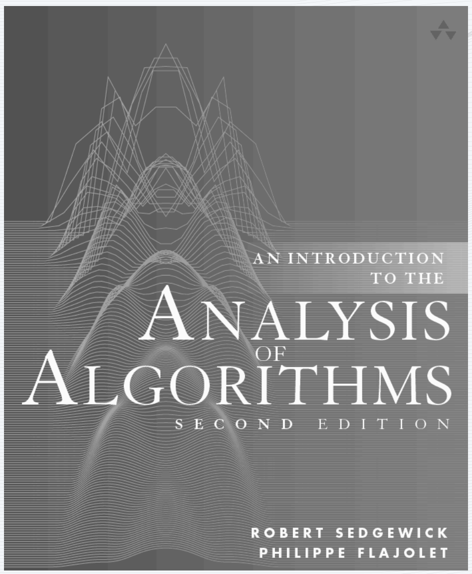

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Manipulating asymptotic expansions

Goal.

Develop expansion on the standard scale for any given expression.

$$\frac{1}{N^2+N} \qquad \frac{H_N}{\ln(N+1)} \qquad e^{H_N} \qquad \left(1-\frac{1}{N}\right)^N \qquad (H_N)^2 \qquad \binom{2N}{N}$$

Techniques.

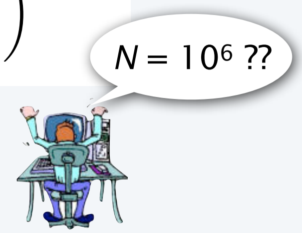
- simplification
- substitution
- factoring
- multiplication
- division
- composition
- exp/log

Why?

Facilitate comparisons of different quantities.

Simplify computations.

Ex. $\frac{1}{4^N}\binom{2N}{N}$



Manipulating asymptotic expansions

Simplification. An asymptotic series is only as good as its O-term.
Discard smaller terms.

$$\ln N + \gamma + O(1)$$ ✗

$$\ln N + O(1)$$ ✓

Substitution. Change variables in a known expansion.

Taylor series $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4) \quad \text{as } x \to 0$$

Substitute $x = 1/N$ $$\ln(1+\frac{1}{N}) = \frac{1}{N} - \frac{1}{2N^2} + \frac{1}{3N^3} + O(\frac{1}{N^4})) \quad \text{as } N \to \infty$$

Manipulating asymptotic expansions

Factoring. Estimate the leading term, factor it out, expand the rest.

$$\frac{1}{N^2 + N}$$

Factor out $1/N^2$.

$$= \frac{1}{N^2}\frac{1}{1 + 1/N}$$

Expand the rest.

$$= \frac{1}{N^2}\Big(1 - \frac{1}{N} + O(\frac{1}{N^2})\Big)$$

Distribute.

$$= \frac{1}{N^2} - \frac{1}{N^3} + O(\frac{1}{N^4})\Big)$$

Manipulating asymptotic expansions

Multiplication. Do term-by-term multiplication, simplify, collect terms.

Ex.
$$(H_N)^2 = \Big(\ln N + \gamma + O(\frac{1}{N})\Big)\Big(\ln N + \gamma + O(\frac{1}{N})\Big)$$

Term-by-term multiplication.
$$= \Big((\ln N)^2 + \gamma \ln N + O(\frac{\log N}{N})\Big)$$
$$+ \Big(\gamma \ln N + \gamma^2 + O(\frac{1}{N})\Big)$$
$$+ \Big(O(\frac{\log N}{N}) + O(\frac{1}{N}) + O(\frac{1}{N^2})\Big)$$

slight improvement in precision

big improvement in precision

Collect terms.
$$= (\ln N)^2 + 2\gamma \ln N + \gamma^2 + O(\frac{\log N}{N})$$

	$(H_N)^2$	$(\ln N)^2$	$+2\gamma \ln N$	$+\gamma^2$
1000	56.032	47.717	55.692	56.025
10000	95.797	84.830	95.463	95.796
100000	146.172	132.547	145.838	146.172

May need trial-and-error to get desired precision.

✓

Manipulating asymptotic expansions

Division. Expand, factor denominator, expand $1/(1-x)$, multiply.

Ex. $$\frac{H_N}{\ln(N+1)}$$

Expand. $$= \frac{\ln N + \gamma + O(\frac{1}{N})}{\ln N + O(\frac{1}{N})}$$

Factor denominator. $$= \frac{1 + \frac{\gamma}{\ln N} + O(\frac{1}{N})}{1 + O(\frac{1}{N})}$$

OK to simplify by replacing $O(1/N \log N)$ by $O(1/N)$

Expand $1/(1-x)$. $$= \Big(1 + \frac{\gamma}{\ln N} + O(\frac{1}{N})\Big)\Big(1 + O(\frac{1}{N})\Big)$$

Multiply. $$= 1 + \frac{\gamma}{\ln N} + O(\frac{1}{N})$$

Manipulating asymptotic expansions

Composition. Substitute an expansion.

$$e^{H_N}$$

Substitute H_N expansion. $$= e^{\ln N + \gamma + O(1/N)}$$

$$= Ne^{\gamma}e^{O(1/N)}$$

Lemma.
$$e^{O(1/N)} = 1 + O(\frac{1}{N})$$

Expand e^x. $$= Ne^{\gamma}(1 + O(\frac{1}{N}) + O(O(\frac{1}{N})^2)$$

Simplify. $$= Ne^{\gamma}(1 + O(\frac{1}{N}))$$

Distribute. $$= Ne^{\gamma} + O(1)$$

big improvement in precision

N	e^{H_N}	Ne^{γ}
1000	1782	1781
10000	17812	17811
100000	178108	178107
1000000	1781073	1781072

√

Manipulating asymptotic expansions

Exp/log. Start by writing $f(x) = \exp(\ln(f(x))$.

Q. How would you compute values for large N?

$$\left(1-\frac{1}{N}\right)^N$$

Exp/log.

$$= \exp\left\{\ln\left(\left(1-\frac{1}{N}\right)^N\right)\right\}$$

$$= \exp\left\{N\ln\left(1-\frac{1}{N}\right)\right\}$$

Expand ln(1+x)

$$= \exp\left\{N\left(-\frac{1}{N}+O\left(\frac{1}{N^2}\right)\right)\right\}$$

Distribute.

$$= \exp\left\{-1+O\left(\frac{1}{N}\right)\right\}$$

$$= 1/e+O\left(\frac{1}{N}\right)$$

big improvement in precision

Lemma.

$$e^{O(1/N)} = 1+O\left(\frac{1}{N}\right)$$

N	$\left(1-\frac{1}{N}\right)^N$	$1/e$
10000	0.367861	0.367879
100000	0.367878	0.367879
1000000	0.367879	0.367879 ✓

Inclass exercises

Develop asymptotic approximations for

$\ln(N-2)$ to $O(1/N^2)$

$$= \ln N + \ln(1 - \frac{2}{N})$$

Factor out ln N.

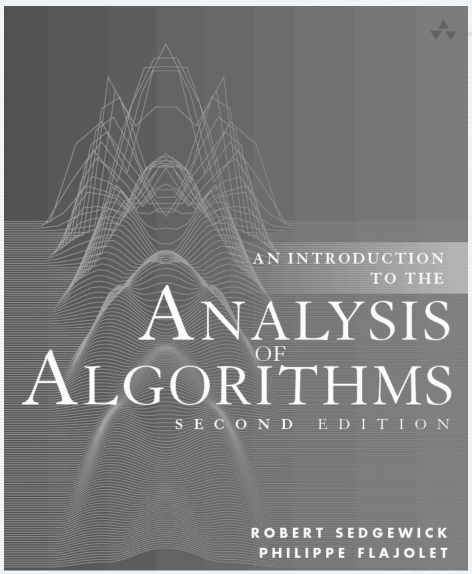
$$= \ln N - \frac{2}{N} + O(\frac{1}{N^2})$$

Expand the rest.

$(H_N)^2$ to $O(1/N)$

$$(H_N)^2 = \Big(\ln N + \gamma + \frac{1}{2N} + O(\frac{1}{N^2})\Big)\Big(\ln N + \gamma + \frac{1}{2N} + O(\frac{1}{N^2})\Big)$$

$$= (\ln N)^2 + 2\gamma \ln N + \gamma^2 + \frac{\ln N}{N} + O(\frac{1}{N})$$



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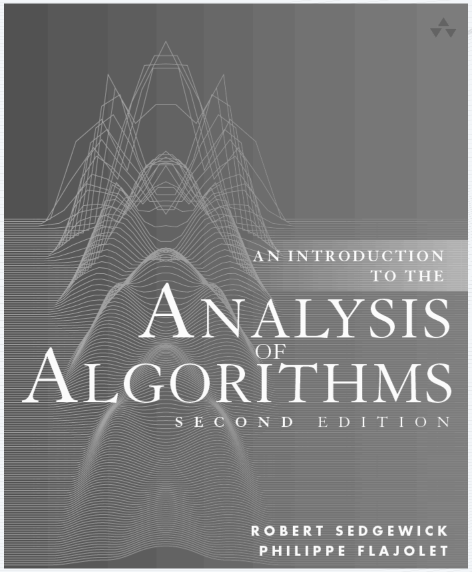

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Asymptotics of finite sums

Bounding the tail. Make a rapidly decreasing sum infinite.

$$N! \sum_{0\le k\le N} \frac{(-1)^k}{k!} = N!e^{-1} - R_N \quad \text{where} \quad R_N = N! \sum_{k>N} \frac{(-1)^k}{k!}$$

$$= \frac{N!}{e} + O(\frac{1}{N})$$

$$|R_N| < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \frac{1}{(N+1)^3} + \ldots = \frac{1}{N}$$

Using the tail. The last term of a rapidly increasing sum may dominate.

$$\sum_{0\le k\le N} k! = N!\Big(1 + \frac{1}{N} + \sum_{0\le k\le N-2} \frac{k!}{N!}\Big) = N!\Big(1 + O(\frac{1}{N})\Big)$$

$N-1$ terms, each less than $1/N(N-1)$

Approximating with an integral.

$$H_N = \sum_{1\le k\le N} \frac{1}{k} \sim \int_1^N \frac{1}{x} dx = \ln N$$

$$\ln N! = \sum_{1\le k\le N} \ln k \sim \int_1^N \ln x \, dx = N \ln N - N + 1$$

see text for proofs;
stay tuned for
better approximations

Euler-Maclaurin Summation

is a classic formula for estimating sums with integrals.

Theorem. (Euler-Maclaurin summation). Let f be a function defined on [1, ∞) whose derivatives exist and are absolutely integrable. Then

$$\sum_{1 \le k \le N} f(k) = \int_1^N f(x)dx + \frac{1}{2}f(N) + C_f + \frac{1}{12}f'(N) - \frac{1}{720}f'''(N) + \ldots$$

Asymptotic series diverges; need to check bound on last term (see text for many details).

BUT this form is useful for many applications.

Classic example 1. $$H_N = \ln N + \gamma + \frac{1}{2N} - \frac{1}{12N^2} + O(\frac{1}{N^4})$$

Classic example 2. $$\ln N! = N \ln N - N + \ln \sqrt{2\pi N} + \frac{1}{12N} + O(\frac{1}{N^3})$$

Inclass exercise

Given Stirling's approximation $\ln N! = N \ln N - N + \ln \sqrt{2\pi N} + O(\frac{1}{N})$

Develop an asymptotic approximation for $\binom{2N}{N}$ to $O(1/N)$ (relative error)

$$\begin{aligned}
\binom{2N}{N} &= \exp\big(\ln(2N!) - 2\ln N!\big) \\
&= \exp\big(2N\ln(2N) - 2N + \ln\sqrt{4\pi N} + O(1/N) \\
&\qquad - 2(N\ln(N) - N + \ln\sqrt{2\pi N} + O(1/N))\big) \\
&= \exp\big(2N\ln 2 - \ln\sqrt{\pi N} + O(1/N)\big) \\
&= \frac{4^N}{\sqrt{\pi N}}(1 + O(\frac{1}{N}))
\end{aligned}$$

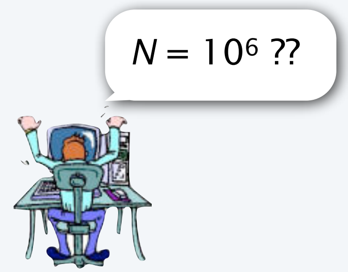
$$\begin{aligned}
\ln\sqrt{4\pi N} - 2\ln\sqrt{2\pi N} &= \ln 2 - 2\ln\sqrt{2} - \ln\sqrt{\pi N} \\
&= -\ln\sqrt{\pi N}
\end{aligned}$$

Ex. $\frac{1}{4^N}\binom{2N}{N} \sim \frac{1}{\sqrt{\pi N}}$

Asymptotics of the Catalan numbers: an application

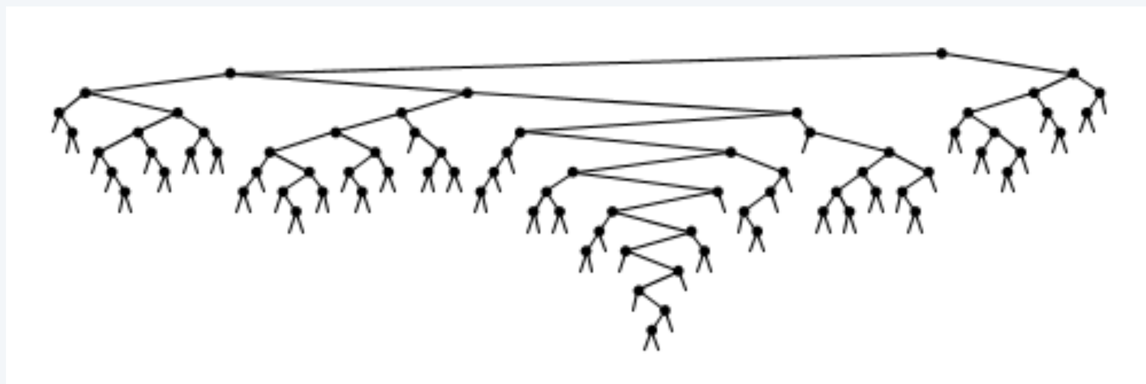
Q. How many bits to represent a binary tree with N internal nodes?

A. At least $\lg \frac{1}{N+1}\binom{2N}{N}$ ✗ *not concise*



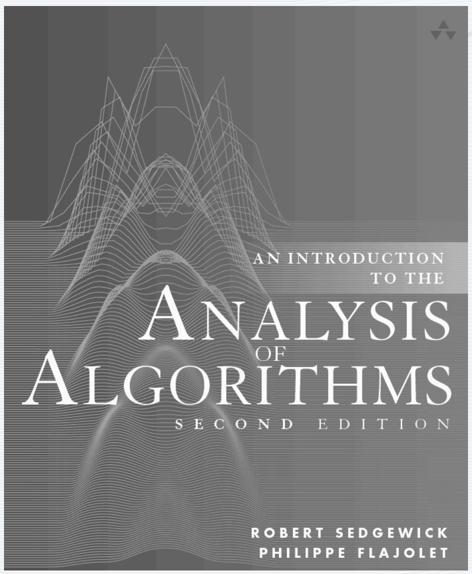
A. At least $\sim \lg \frac{4^N}{\sqrt{\pi N^3}} \sim 2N - 1.5 \lg N$ ✓

Note: Can do it with $2N$ bits



preorder traversal
0 for internal nodes
1 for external nodes

00001011000101011010110011011000001110010110110010110110100110110000011110000110110000111001001001111011000101111010000110110110010110000110011001110101100111



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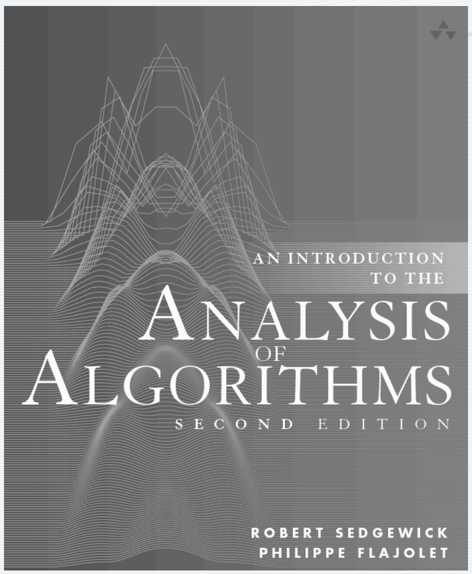

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Bivariate asymptotics

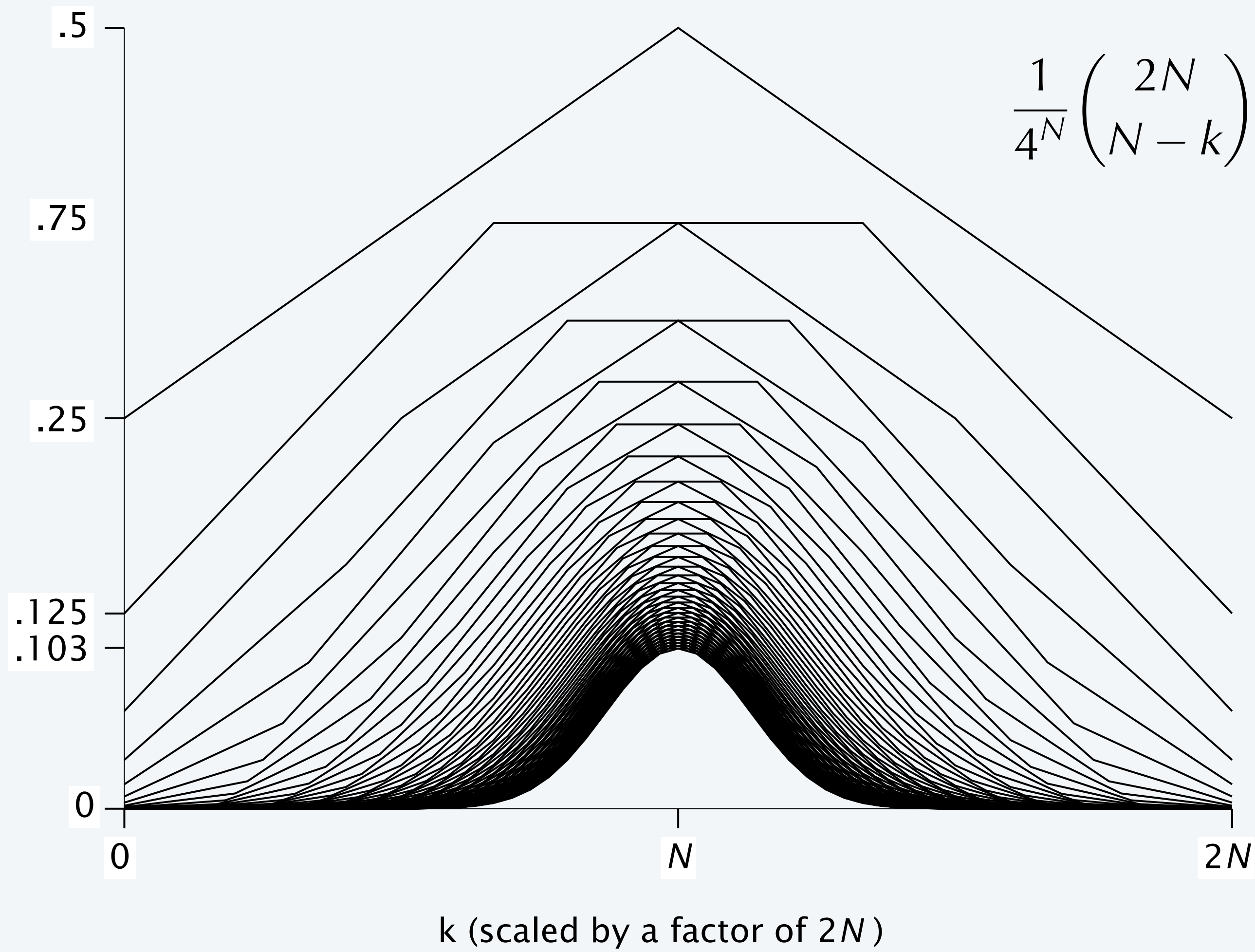
is often required to analyze functions of two variables.

Ex. applications in analysis of algorithms involve
- N (size)
- k (cost)

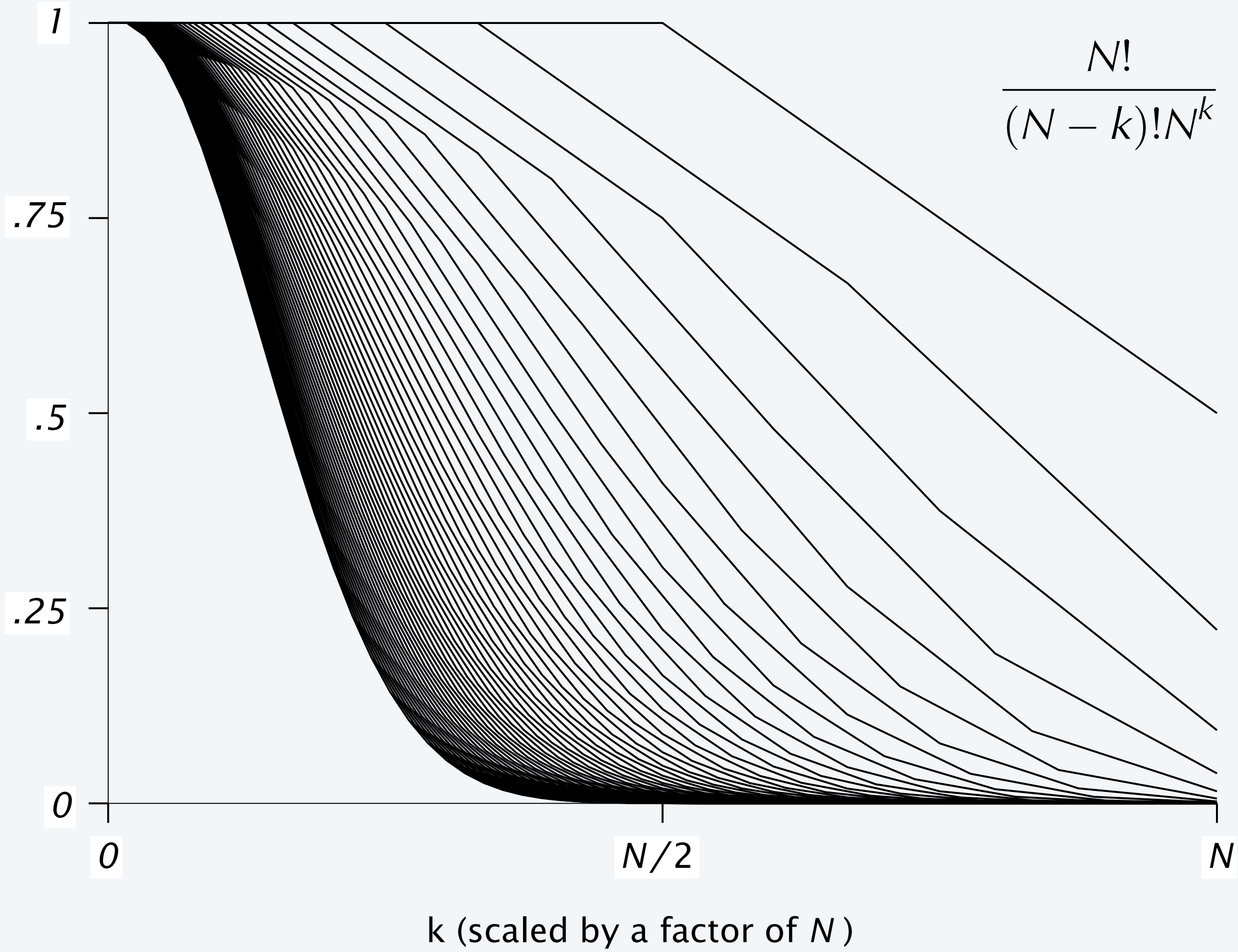
Challenges:
- asymptotics depends on relative values of variables
- may need to approximate sums over whole range of relative values.

Example 1: Binomial distribution	$$\frac{1}{4^N}\binom{2N}{N-k}$$	$\sim \frac{1}{\sqrt{\pi N}}$ for $k = 0$ exponentially small for k close to N
Example 2: Ramanujan Q-distribution	$$\frac{N!}{(N-k)!N^k}$$	1 for $k = 0$ exponentially small for k close to N

Binomial distribution



Ramanujan Q-distribution



Ramanujan Q-distribution

$$\frac{N!}{(N-k)!N^k} = \exp\big(\ln N! - \ln(N-k)! - k\ln N\big)$$

$$\ln N! = (N+\frac{1}{2})\ln N - N + \ln\sqrt{2\pi} + O(\frac{1}{N})$$

Stirling's approximation.

$$= \exp\big((N+\frac{1}{2})\ln N - N + \ln\sqrt{2\pi}$$
$$- (N-k+\frac{1}{2})\ln(N-k) + N - k - \ln\sqrt{2\pi} - k\ln N + O(\frac{1}{N})\big)$$

Collect terms

$$= \exp\big(-(N-k+\frac{1}{2})\ln(1-\frac{k}{N}) - k + O(\frac{1}{N})\big)$$

$$\ln(1-\frac{k}{N}) = -\frac{k}{N} - \frac{k^2}{2N^2} + O(\frac{k^3}{N^3})$$

Expand ln(1 – *k/N*)

$$= \exp\big(k + \frac{k^2}{2N} - \frac{k^2}{N} - k + O(\frac{k^3}{N^2}) + O(\frac{k}{N})\big)$$

Simplify.

$$= e^{-k^2/2N}\big(1 + O(\frac{k^3}{N^2}) + O(\frac{k}{N})\big)$$

k	k/N	k^3/N^2
$N^{2/5}$	$1/N^{3/5}$	$1/N^{4/5}$
$N^{1/2}$	$1/N^{1/2}$	$1/N^{1/2}$
$N^{3/5}$	$1/N^{2/5}$	$1/N^{1/5}$

Normal approximation to the binomial distribution

$$\binom{2N}{N-k} = \exp\big(\ln(2N!) - \ln(N-k)! - \ln(N+k)!\big)$$

$$\ln N! = (N+\frac{1}{2})\ln N - N + \ln\sqrt{2\pi} + O(\frac{1}{N})$$

Stirling's approximation.

$$= \exp\big((2N+\frac{1}{2})\ln(2N) - 2N + \ln\sqrt{2\pi} + O(1/N)$$
$$-(N-k+\frac{1}{2})\ln(N-k) - N + k - \ln\sqrt{2\pi} + O(1/N)$$
$$-(N+k+\frac{1}{2})\ln(N+k) - N - k - \ln\sqrt{2\pi} + O(1/N)\big)$$

Collect terms

$$= \exp\big((2N)\ln 2 - \ln\sqrt{\pi N} - (N-k+\frac{1}{2})\ln(1-\frac{k}{N}) - (N+k+\frac{1}{2})\ln(1+\frac{k}{N}) + O(1/N)\big)$$

Rearrange terms

$$= \exp\big(2N)\ln 2 - \ln\sqrt{\pi N}$$
$$-(N+\frac{1}{2})(\ln(1-\frac{k}{N}) + \ln(1+\frac{k}{N})) \quad +k(\ln(1-\frac{k}{N}) - \ln(1+\frac{k}{N})) + O(1/N)\big)$$

Expand ln(1 – *k/N*) and ln(1 + *k/N*).

$$= \exp\big((2N)\ln 2 - \ln\sqrt{\pi N} - \frac{k^2}{N} + O(\frac{k^4}{N^3}) + O(\frac{1}{N})\big)$$

$$\ln(1-\frac{k}{N}) + \ln(1+\frac{k}{N}) = -\frac{k^2}{N^2} + O(\frac{k^3}{N^3})$$
$$\ln(1-\frac{k}{N}) - \ln(1+\frac{k}{N}) = -\frac{2}{N} + O(\frac{k^3}{N^3})$$

$$\frac{1}{4^N}\binom{2N}{N-k} = \frac{e^{-k^2/N}}{\sqrt{\pi N}}\big(1 + O(\frac{k^4}{N^3}) + O(\frac{1}{N})\big)$$

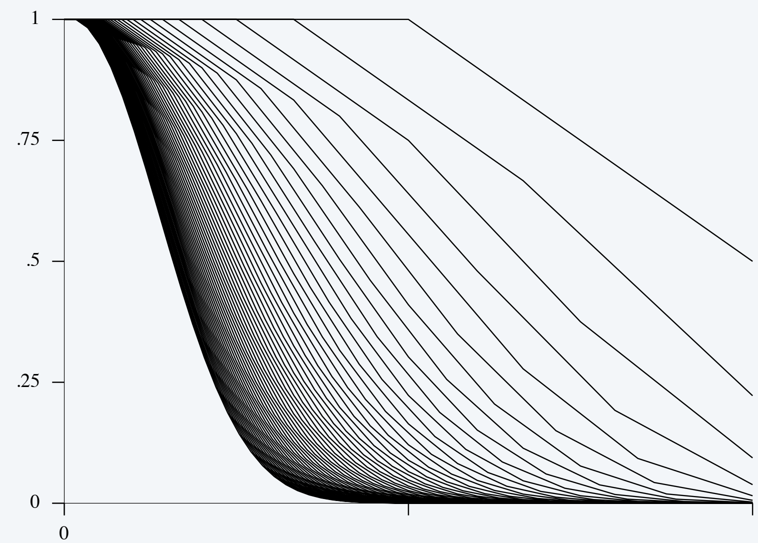
Fundamental bivariate approximations

		uniform	central
normal	$\binom{2N}{N-k}$	$\frac{e^{-k^2/N}}{\sqrt{\pi N}} + O(\frac{1}{N^{3/2}})$	$\frac{e^{-k^2/N}}{\sqrt{\pi N}}\Big(1 + O(\frac{1}{N}) + O(\frac{k^4}{N^3})\Big)$
Poisson	$\binom{N}{k}\Big(\frac{\lambda}{N}\Big)^k\Big(1-\frac{\lambda}{N}\Big)^{N-k}$	$\frac{\lambda^k e^{-\lambda}}{k!} + o(1)$	$\frac{\lambda^k e^{-\lambda}}{k!}\Big(1 + O(\frac{1}{N}) + O(\frac{k}{N})\Big)$
Q	$\frac{N!}{(N-k)!N^k}$	$e^{-k^2/(2N)} + O(\frac{1}{\sqrt{N}})$	$e^{-k^2/(2N)}\Big(1 + O(\frac{k}{N}) + O(\frac{k^3}{N^2})\Big)$

Next challenge: Approximating sums via bivariate asymptotics

Example: Ramanujan Q-function

$$Q(N) \equiv \sum_{1 \leq k \leq N} \frac{N!}{(N-k)!N^k}$$



What is the area under this curve?

Observations:

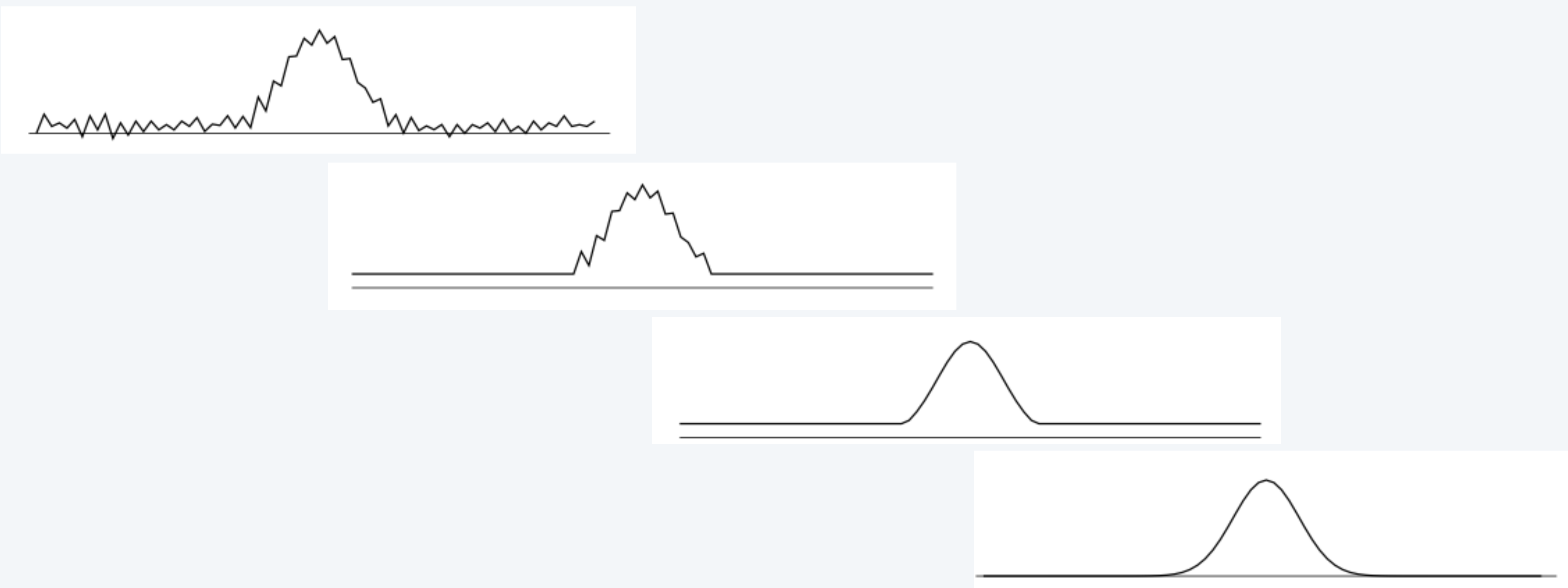
- nearly 1 for small k
- negligable for large k
- bivariate asymptotics needed to give different estimates in different ranges.

negligible

Laplace method

To approximate a sum:

- Restrict the range to an area that contains the largest summands.
- Approximate the summand.
- Extend the range by bounding the tails to get a simpler sum.
- Approximate the new sum with an integral.



Laplace method for Ramanujan Q-function

$$Q(N) \equiv \sum_{1\le k\le N} \frac{N!}{(N-k)!N^k}$$

Restrict the range to an area that contains the largest summands.

$$Q(N) = \sum_{1\le k\le k_0} \frac{N!}{(N-k)!N^k} + \sum_{k_0<k\le N} \frac{N!}{(N-k)!N^k}$$

Take $k_0 = o(N^{2/3})$ to make tail exponentially small.

Approximate the summand.

$$\sum_{1\le k\le k_0} \frac{N!}{(N-k)!N^k} \sim \sum_{1\le k\le k_0} e^{-k^2/2N}$$

Q-distribution approximation.

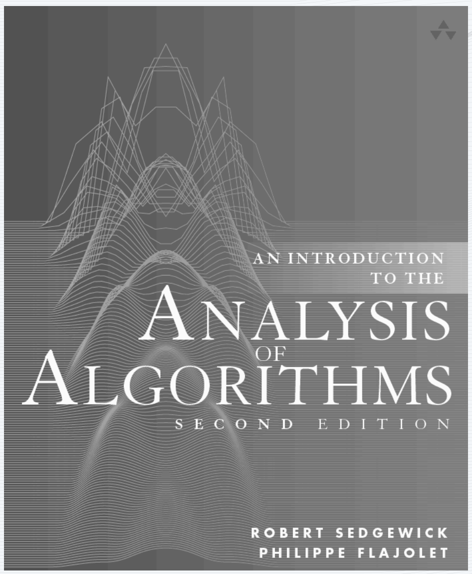
Extend the range by bounding the tails to get a simpler sum.

$$Q(N) \sim \sum_{k\ge 1} e^{-k^2/2N}$$

Tail is also exponentially small for this sum.

Approximate the new sum with an integral.

$$Q(N) \sim \sqrt{N}\int_0^\infty e^{-x^2/2}dx = \sqrt{\pi N/2}$$



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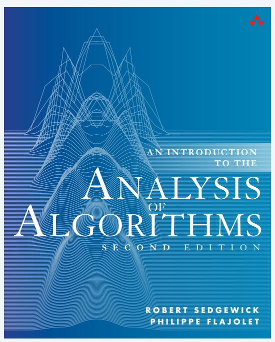
4. Asymptotic Approximations

- Standard scale
- Manipulating expansions
- Asymptotics of finite sums
- **Bivariate asymptotics**

Exercise 4.9

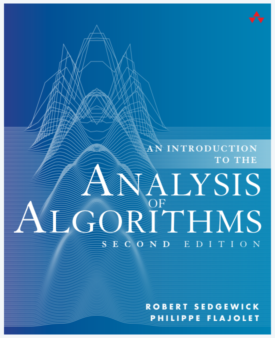
How small is "exponentially small"?



Exercise 4.9 If $\alpha < \beta$, show that α^N is exponentially small relative to β^N. For $\beta = 1.2$ and $\alpha = 1.1$, find the absolute and relative errors when $\alpha^N + \beta^N$ is approximated by β^N, for $N = 10$ and $N = 100$.

Exercise 4.71

Asymptotics of another Ramanujan function.



Exercise 4.71 Show that

$$P(N) = \sum_{k \geq 0} \frac{(N-k)^k (N-k)!}{N!} = \sqrt{\pi N/2} + O(1)$$

Assignments for next lecture

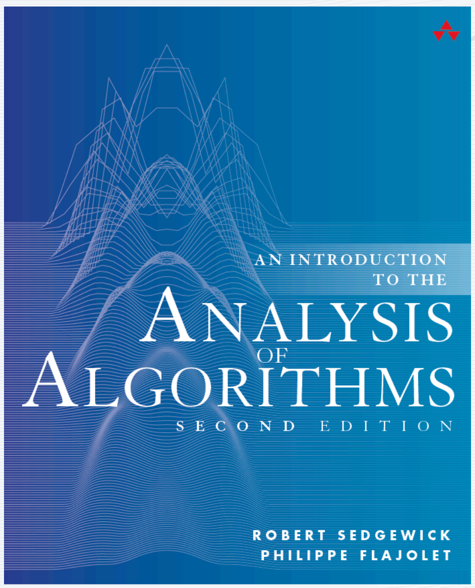
1. Write a program that takes N and k from the command line and prints

$$\lg \binom{N}{k}$$

2. Write up solutions to Exercises 4.9 and 4.70.

3. Read pages 149-215 (Asymptotic Approximations) in text.

ANALYTIC COMBINATORICS
PART ONE



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4. Asymptotic Approximations