Matematik baskurs, med diskret matematik

SF1671

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Föreläsning 4

Lös olikheten $\ln(1+2e^x) - \ln(1-e^x) = \ln(1+5e^x)$. $(*)$

$e^x < 1$ så $\underline{\underline{x < 0}}$

$$\boxed{e^{\ln z} = z}$$

Sätt $y = e^x$ och använd log-lagarna:

$$(*) \iff \ln(1+2y) = \ln(1-y) + \ln(1+5y) = \ln\big((1-y)(1+5y)\big)$$

$e^{VL} = e^{HL}$:

$$e^{VL} = 1+2y = e^{HL} = (1-y)(1+5y) = 1+4y-5y^2$$

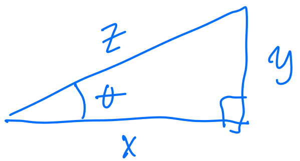
$$\iff 5y^2 - 2y = 0 \iff y(5y-2) = 0 \iff y = 0, \tfrac{2}{5}$$

falsk

$y = e^x > 0$.

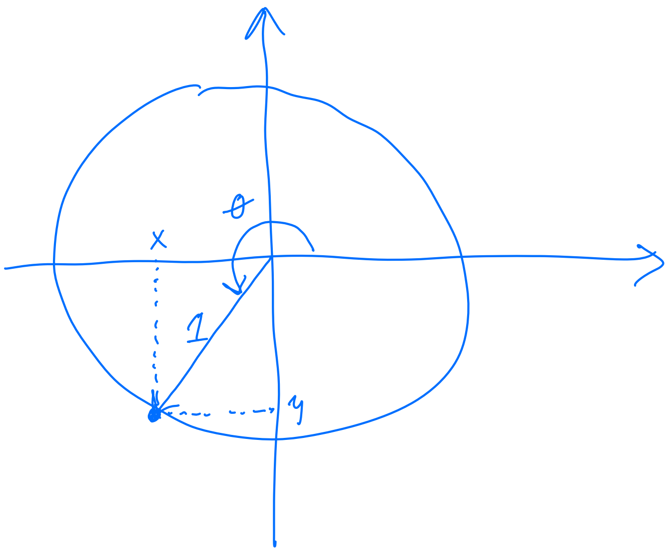
$$e^x = \frac{2}{5} \text{ dvs } \boxed{x = \ln\left(\frac{2}{5}\right)}$$

Trigonometri



Pythagoras sats $x^2+y^2=z^2$

$\sin(\theta) = \dfrac{y}{z}$, $\cos(\theta) = \dfrac{x}{z}$

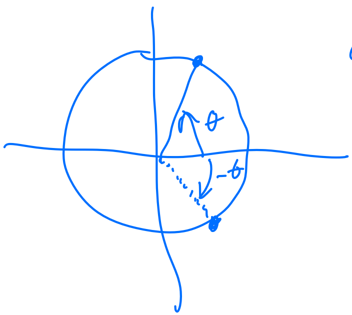


- θ kan vara vilket reellt tal som helst.
- θ mäts i radianer. Ett varv är 2π radianer.

$(x,y) = (\cos\theta, \sin\theta)$

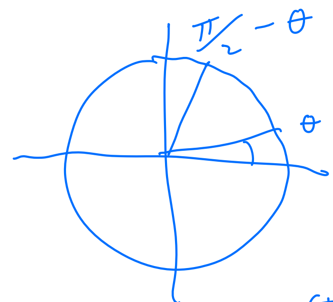
Trigonometri

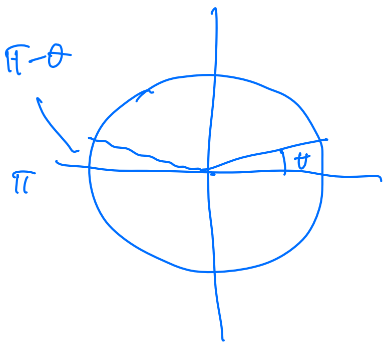
- De trigonometriska funktionerna defineras periodvis, så $\cos x : \mathbb{R} \to [-1, 1]$ och $\sin x : \mathbb{R} \to [-1, 1]$.
- Pythagoras sats: $\cos^2(\cancel{x}\,\theta) + \sin^2(\cancel{x}\,\theta) = 1.$ $\quad x^2 + y^2 = z^2 = 1$
- Symmetrier:



$\cos(\theta) = \cos(-\theta)$ "jämn funktion"

$\sin(-\theta) = -\sin(\theta)$ "udda funktion"





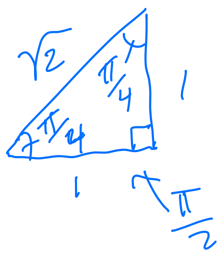
$\sin(\pi - \theta) = \sin(\theta)$

$\cos(\pi - \theta) = -\cos(\theta)$

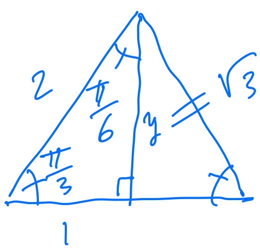
$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$

Speciella vinklar



$$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right)$$



Pyth: $1^2 + y^2 = 2^2 = 4$

$y^2 = 3, \; y = \sqrt{3}$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$$

Additionsformlerna

Subtraktionsformeln för cos:

$$\boxed{\cos(t-s) = \cos(s)\cos(t) + \sin(s)\cdot\sin(t)}$$

$\sqrt{x_1^2+y_1^2}$

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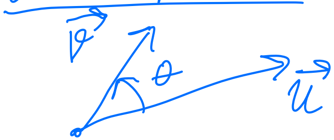
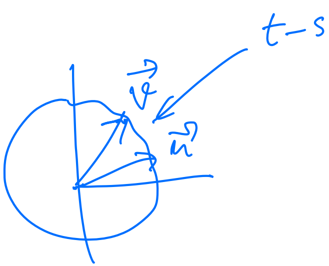
Skalärprodukt:



$$\vec{u} = (x_1, y_1)$$
$$\vec{v} = (x_2, y_2)$$

$$\begin{cases} \vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cdot\cos(\theta) & (*) \\ \vec{u}\cdot\vec{v} = x_1x_2 + y_1y_2 & (**) \end{cases}$$

Specialfall: $\vec{u} = (\cos(s), \sin(s))$, $\vec{v} = (\cos(t), \sin(t))$



$$\vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cdot\cos(t-s) = \cos(t-s)$$

$$\vec{u}\cdot\vec{v} = x_1x_2 + y_1y_2 = \cos(s)\cdot\cos(t) + \sin(s)\cdot\sin(t)$$

$\square$

Additionsformlerna.

Fås från subtraktions-formeln mha. symmetrierna

$$\cos(s+t) = \cos(s)\cos(t) - \sin(s)\sin(t)$$
$$\sin(s+t) = \sin(s)\cos(t) + \cos(s)\sin(t)$$

$$\cos(2t) = \cos^2(t) - \sin^2(t) = \cos^2(t) - (1-\cos^2(t)) = 2\cos^2(t) - 1$$
$$\sin(2t) = 2\sin(t)\cos(t)$$

trig-ettan

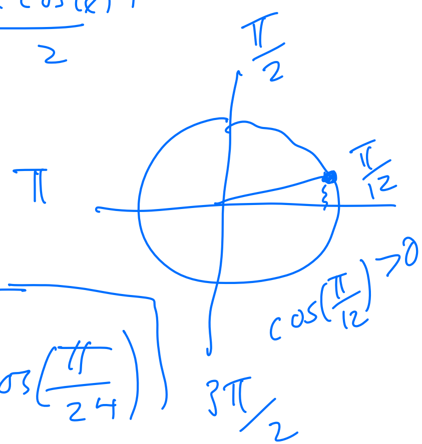
Exempel. Beräkna $\cos(\pi/12)$.

$\cos(2t) = 2\cos^2(t) - 1$, $\quad \cos^2(t) = \dfrac{1+\cos(2t)}{2}$. sätt $t = \dfrac{x}{2}$

$\cos^2\left(\dfrac{x}{2}\right) = \dfrac{1+\cos(x)}{2}$ dvs $\cos\left(\dfrac{x}{2}\right) = \pm\sqrt{\dfrac{1+\cos(x)}{2}}$

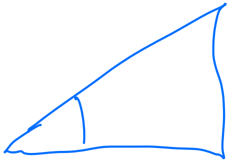
$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\overbrace{\pi/6}^{=x}}{2}\right) = (\pm)\sqrt{\frac{1+\cos(\frac{\pi}{6})}{2}}$$

$$= \sqrt{\frac{1+\sqrt{3}/2}{2}} = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}$$



$\boxed{\text{Träna: Beräkna } \cos\left(\frac{\pi}{24}\right)}$

Tangens och co-tangens.

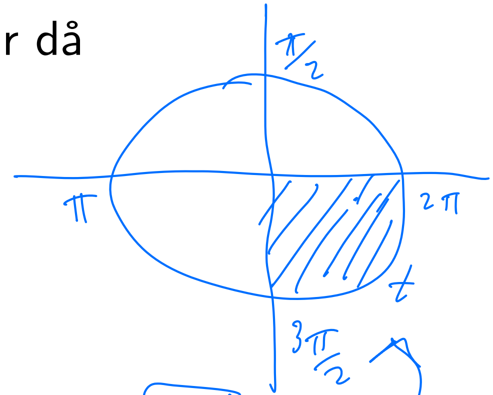


$$\tan x = \frac{\sin x}{\cos x}, \qquad \text{för } x \neq \pi/2 + n\pi, n \in \mathbb{Z}$$

$$\cot x = \frac{\cos x}{\sin x}, \qquad \text{för } x \neq n\pi, n \in \mathbb{Z}$$

Additionsformel. $\tan(s+t) = \frac{\tan(s)+\tan(t)}{1-\tan(s)\tan(t)}$

Exempel. Om $\cos t = x$, $3\pi/2 < t < 2\pi$, vad är då $\tan t/(1+\cot t)$?



$$\tan(t) = \frac{\sin(t)}{\cos(t)}, \quad \cot(t) = \frac{\cos(t)}{\sin(t)}$$

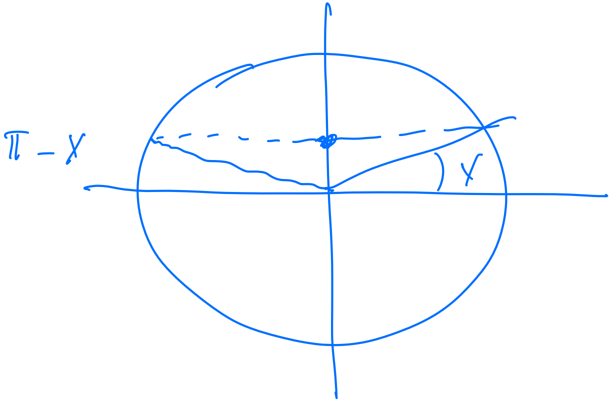
trig-ettan: $\cos^2 t + \sin^2 t = 1$

$$\sin^2 t = 1 - x^2 \Rightarrow \sin(t) = \pm\sqrt{1-x^2}$$

$$= -\sqrt{1-x^2}$$

$$\frac{\tan(t)}{1+\cot(t)} = \frac{\frac{\sin t}{\cos t}}{1+\frac{\cos t}{\sin t}} = \frac{\frac{-\sqrt{1-x^2}}{x}}{1+\frac{x}{-\sqrt{1-x^2}}} = \dots$$

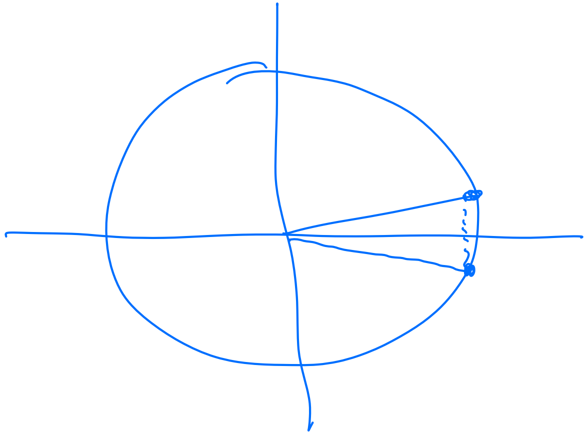
Exempel. Lös $\sin x = \sin y$.



$y = x + 2\pi k \quad , \quad k \in \mathbb{Z}$

$y = \pi - x + 2\pi \cdot k \quad , \quad k \in \mathbb{Z}$

Exempel. Lös $\cos x = \cos y$.



$y = x + 2\pi \cdot k \quad , \quad k \in \mathbb{Z}$

$y = -x + 2\pi \cdot k \quad , \quad k \in \mathbb{Z}$

Exempel. Lös $\sin(2x) = \cos(3x - \pi/4)$.

Gör om sin till cos. $\sin(t) = \cos\left(\frac{\pi}{2} - t\right)$

$t = 2x$

Få ekvationen: $\sin(2x) = \boxed{\cos\left(\frac{\pi}{2} - 2x\right) = \cos\left(3x - \frac{\pi}{4}\right)}$

Enligt förra sidan:

$$\frac{\pi}{2} - 2x = \pm\left(3x - \frac{\pi}{4}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

Två fall:

(+) $\frac{\pi}{2} - 2x = 3x - \frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}$

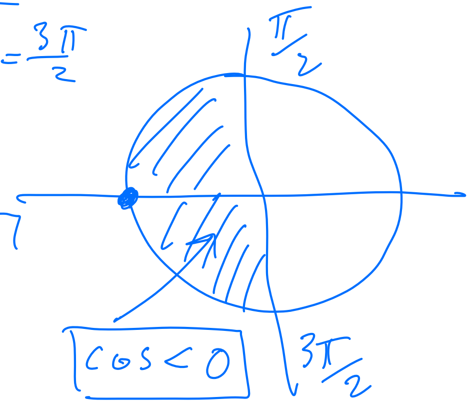
$5x = \frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}$ dvs $x = \frac{\pi}{20} + \frac{2}{5}k\pi, \quad k \in \mathbb{Z}$

(−) $\frac{\pi}{2} - 2x = -\left(3x - \frac{\pi}{4}\right) + 2k\pi, \quad k \in \mathbb{Z}$. På samma sätt

$\Rightarrow \quad x = -\frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}$

Exempel. Antag $\sin x = 1/3$ och $|x-\pi| \le \pi/2$. Vad är $\tan x$?

$$\frac{\pi}{2} = \pi - \frac{\pi}{2} \le x \le \pi + \frac{\pi}{2} = \frac{3\pi}{2}$$



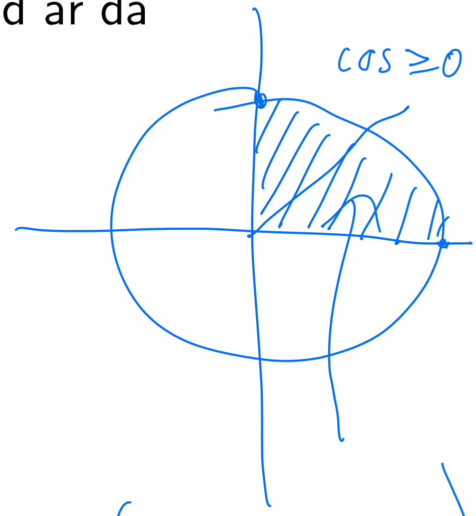
Trig-ettan: $\cos x = \pm\sqrt{1-\sin^2 x} = -\sqrt{1-\frac{1}{9}}$

$$= -\sqrt{\frac{9-1}{9}} = \frac{-1}{3}\sqrt{8} = \frac{-\sqrt{8}}{3}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{1/3}{-\sqrt{8}/3} = \frac{-1}{\sqrt{8}}$$

Exempel. Om $\sin(x) = 3/5$ och $0 \leq x \leq \pi/2$, vad är då

$$\frac{\tan(x) + \sin(x)}{\cot(x)}?$$



trig. ettan

$$\cos x = \pm\sqrt{1 - \sin^2 x} = \pm\sqrt{1 - \frac{9}{25}} = \pm\sqrt{\frac{25-9}{25}}$$

$$= \pm\frac{4}{5}. \quad \text{Men } \cos x \geq 0, \text{ så}$$

$$\cos x = \frac{4}{5}$$

$$\left(|x - \frac{\pi}{4}| \leq \frac{\pi}{4} \right)$$

$$\frac{\tan x + \sin x}{\cot x} = \frac{\frac{\sin x}{\cos x} + \sin x}{\cos x / \sin x} = \frac{3/4 + 3/5}{4/3} =$$

$$= \ldots = \frac{81}{80}$$

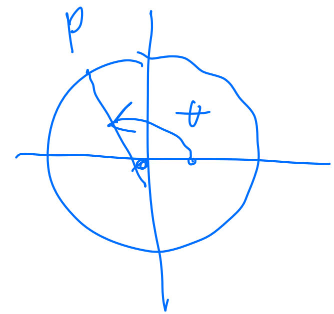
Exempel. I fysiken behöver man ibland addera sinus- och cosinusfunktioner:

$$f(x) = a\sin(x) + b\cos(x), \quad \text{där } a, b \in \mathbb{R}.$$

Hur ser sådana funktioner ut? *Använd additionsformel.*

Skriv $f(x) = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cdot\sin(x) + \frac{b}{\sqrt{a^2+b^2}}\cdot\cos(x)\right)$

$$P = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right) = (\cos(\theta), \sin(\theta))$$



$$f(x) = \sqrt{a^2+b^2}\left(\cos(\theta)\sin(x) + \sin(\theta)\cos(x)\right)$$

$$= \sqrt{a^2+b^2}\,\sin(x+\theta)$$

fasförskjutning

amplituden.

$f(x) = \sqrt{3}$

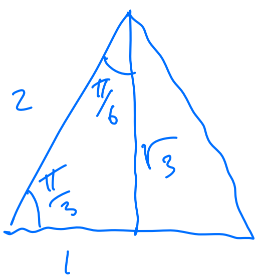
Exempel. Lös $\sqrt{3}\sin(2x) - \cos(2x) = \sqrt{3}$.

$\overset{\parallel}{a}$ $\qquad b = -1$

Enligt förra sidan $\quad f(x) = \sqrt{a^2+b^2}\,\sin(2x+\theta)$

$= 2\cdot\sin(2x+\theta)$

$$P = \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$$



$$\begin{cases}\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\\ \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}\end{cases}$$

$$\theta = -\frac{\pi}{6}$$

$$f(x) = 2\cdot\sin\left(2x - \frac{\pi}{6}\right) = \sqrt{3}$$

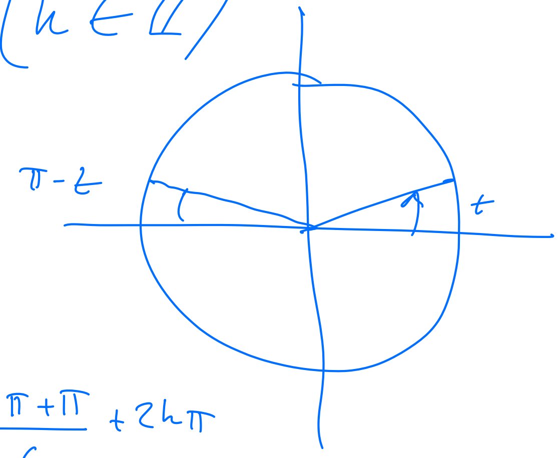
dvs

$$\sin\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$= \sin\left(\frac{\pi}{3}\right)$$

Exempel. Lös $\sqrt{3}\sin(2x) - \cos(2x) = \sqrt{3}$. $(k \in \mathbb{Z})$

$$\sin\left(2x - \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right)$$



Två fall: I: $2x - \frac{\pi}{6} = \frac{\pi}{3} + 2k\pi$

$$2x = \frac{\pi}{3} + \frac{\pi}{6} + 2k\pi = \frac{2\pi + \pi}{6} + 2k\pi$$
$$= \frac{\pi}{2} + 2k\pi$$

Så $x = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$

II: $2x - \frac{\pi}{6} = \pi - \frac{\pi}{3} + 2k\pi$ ....

... $x = \frac{5\pi}{12} + k\pi, \quad k \in \mathbb{Z}$