Implementation of Sequential Estimator and Bayesian Linear Regression

Description:

1. Random Data Generator

- a. Univariate gaussian data generator
  - Input
    - Expectation value or mean: m
    - Variance: s
  - Output: A data point from $N(m, s)$
  - HINT
    - [Generating values from normal distribution]()
    - You have to handcraft your geneartor based on one of the approaches given in the hyperlink.
    - You can use uniform distribution function (Numpy)
- b. Polynomial basis linear model data generator
  - $y = W^T \phi(x) + e$
    - W is a $n \times 1$ vector
    - $e \sim N(0, a)$
  - Input: n (basis number), a, w
    - e.g. $n = 2 \rightarrow y = w_0 x^0 + w_1 x^1$,
  - Output: y (a number)
  - Internal constraint
    - $-1.0 < x < 1.0$
    - x is uniformly distributed.

2. Sequential Estimator

- Sequential estimate the mean and variance
  - Data is given from the univariate gaussian data generator (1.a).
- Input: m, s as in (1.a)
- Function:
  - Call (1.a) to get a new data point from $N(m, s)$
  - Use sequential estimation to find the current estimates to m and s

- Repeat steps above until the estimates converge.

- Output: Print the new data point and the current estimiates of m and s in each iteration.
- Notes
  - You should derive the recursive function of mean and variance based on the sequential esitmation.
  - Hint: **Online algorithm**
- Sample input & output (⚠️ ***for reference only*** ⚠️)

```
Data point source function: N(3.0, 5.0)

Add data point: 3.234685454257290
Mean = 3.408993960833291   Variance = 0.030383455464755956
Add data point: 0.519242879651157
Mean = 2.445743600439247   Variance = 1.875958150575018
Add data point: 1.347113997201991
Mean = 2.171086199629932   Variance = 1.633278676389248
Add data point: 8.979491998496083
Mean = 3.532767359403163   Variance = 8.723325264636875
Add data point: 3.603448448693051
Mean = 3.544547540951477   Variance = 7.270131583917285
Add data point: 4.127197937610908
Mean = 3.627783311902824   Variance = 6.273110519038578
Add data point: 4.992735798186870
Mean = 3.798402372688330   Variance = 5.692747751482052

...

Add data point: 4.233592159021013
Mean = 2.961576104513964   Variance = 5.045715437349161
Add data point: 3.529990930040463
Mean = 2.961883688294010   Variance = 5.043159812425648
Add data point: 1.125210345431449
Mean = 2.960890354955524   Variance = 5.042255747918937
```

3. Baysian Linear regression

- Input
  - The precision (i.e., b) for initial prior $w \sim N(0, b^{-1}I)$
  - All other required inputs for the polynomial basis linear model geneartor (1.b)
- Function
  - Call (1.b) to generate one data point
  - Update the prior, and calculate the parameters of predictive distribution
  - Repeat steps above until the posterior probability converges.
- Output
  - Print the new data point and the current paramters for posterior and predictive distribution.

- After probability converged, do the visualization
    - Ground truth function (from linear model generator)
    - Final predict result
    - At the time that have seen 10 data points
    - At the time that have seen 50 data points
    - Note
        - Except ground truth, you have to draw those data points which you have seen before
        - Draw a black line to represent the mean of function at each point
        - Draw two red lines to represent the variance of function at each point
            - In other words, distance between red line and mean is **ONE** variance

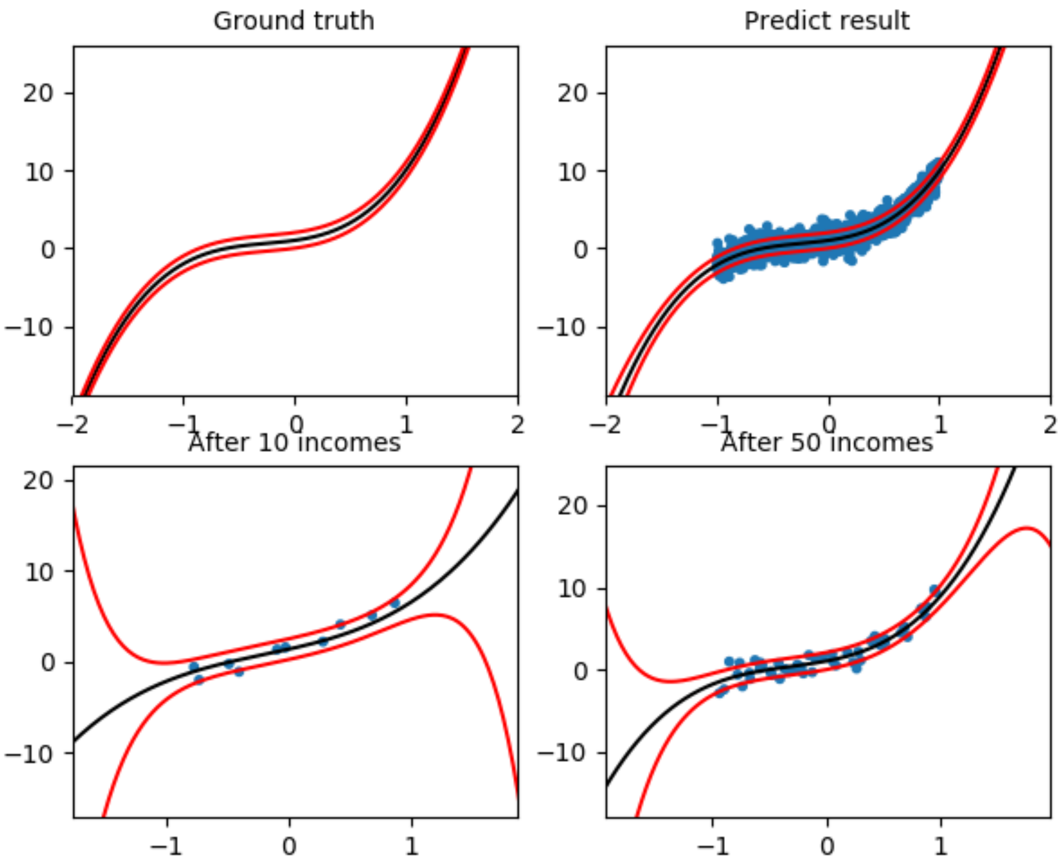
- Hint: Online learning
- Sample input & output (***for reference only***)

1. b = 1, n = 4, a = 1, w = [1, 2, 3, 4]

```
Add data point (-0.64152, 0.19039):

Postirior mean:
   0.0718294547
  -0.0460797888
   0.0295609502
  -0.0189638408

Posterior variance:
   0.6227289276,   0.2420256620,  -0.1552634839,   0.0996041049
   0.2420256620,   0.8447365161,   0.0996041049,  -0.0638976884
  -0.1552634839,   0.0996041049,   0.9361023116,   0.0409914289
   0.0996041049,  -0.0638976884,   0.0409914289,   0.9737033172

Predictive distribution ~ N(0.00000, 2.65061)
--------------------------------------------------
Add data point (0.07122, 1.63175):

Postirior mean:
   0.6736864869
   0.2388980107
  -0.1054659080
   0.0710615952

Posterior variance:
   0.3765992302,   0.1254838660,  -0.1000441911,   0.0627881634
   0.1254838660,   0.7895542671,   0.1257503020,  -0.0813299447
  -0.1000441911,   0.1257503020,   0.9237138418,   0.0492510997
   0.0627881634,  -0.0813299447,   0.0492510997,   0.9681964094
```

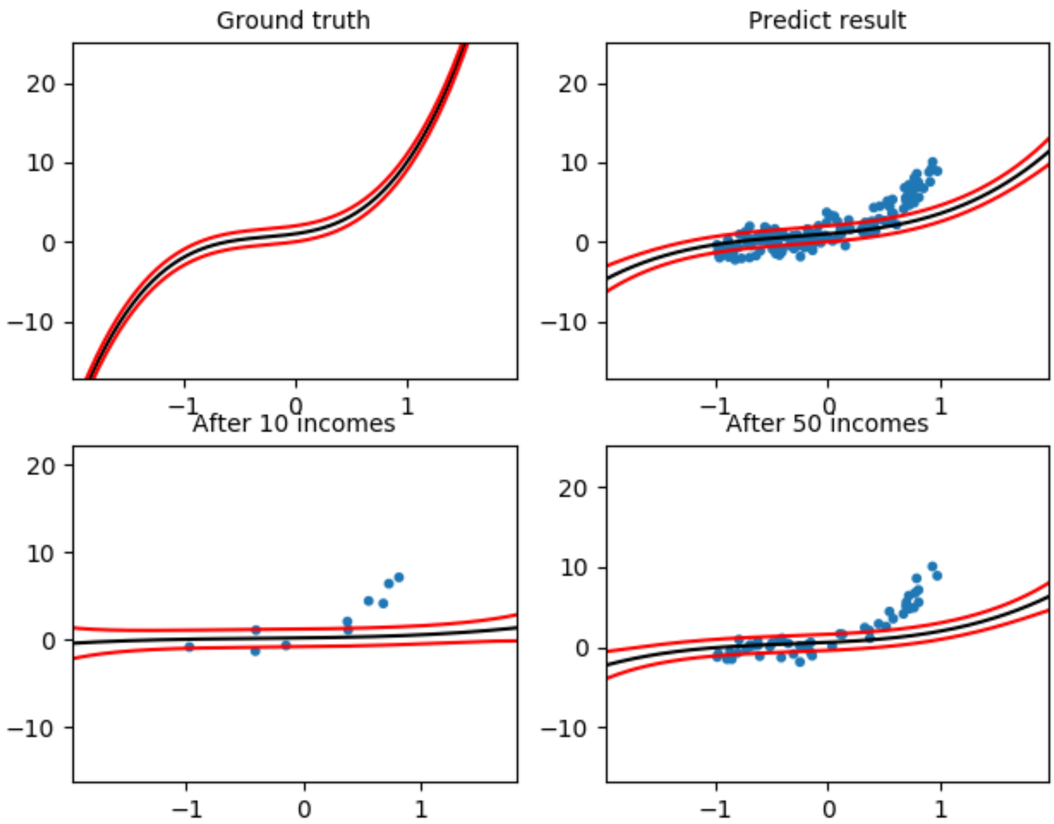
```

Predictive distribution ~ N(0.06869, 1.66008)
--------------------------------------------------
Add data point (-0.19330, 0.24507):

Postirior mean:
   0.5760972313
   0.2450231522
  -0.0801842453
   0.0504992402

Posterior variance:
   0.2867129751,   0.1311255325,  -0.0767580827,   0.0438488542
   0.1311255325,   0.7892001707,   0.1242887609,  -0.0801412282
  -0.0767580827,   0.1242887609,   0.9176812972,   0.0541575540
   0.0438488542,  -0.0801412282,   0.0541575540,   0.9642058389

Predictive distribution ~ N(0.62305, 1.34848)
--------------------------------------------------

...

--------------------------------------------------
Add data point (-0.76990, -0.34768):

Postirior mean:
   0.9107496675
   1.9265499885
   3.1119297129
   4.1312375189

Posterior variance:
   0.0051883836,  -0.0004416700,  -0.0086000319,   0.0008247001
  -0.0004416700,   0.0401966605,   0.0012708906,  -0.0554822477
  -0.0086000319,   0.0012708906,   0.0265353911,  -0.0031205875
   0.0008247001,  -0.0554822477,  -0.0031205875,   0.0937197255

Predictive distribution ~ N(-0.61566, 1.00921)
--------------------------------------------------
Add data point (0.36500, 2.22705):

Postirior mean:
   0.9107404583
   1.9265225090
   3.1119408740
   4.1312734131

Posterior variance:
   0.0051731092,  -0.0004872471,  -0.0085815201,   0.0008842340
```

```
  -0.0004872471,   0.0400606628,   0.0013261280,  -0.0553046044
  -0.0085815201,   0.0013261280,   0.0265129556,  -0.0031927398
   0.0008842340,  -0.0553046044,  -0.0031927398,   0.0934876838

Predictive distribution ~ N(2.22942, 1.00682)
--------------------------------------------------
```



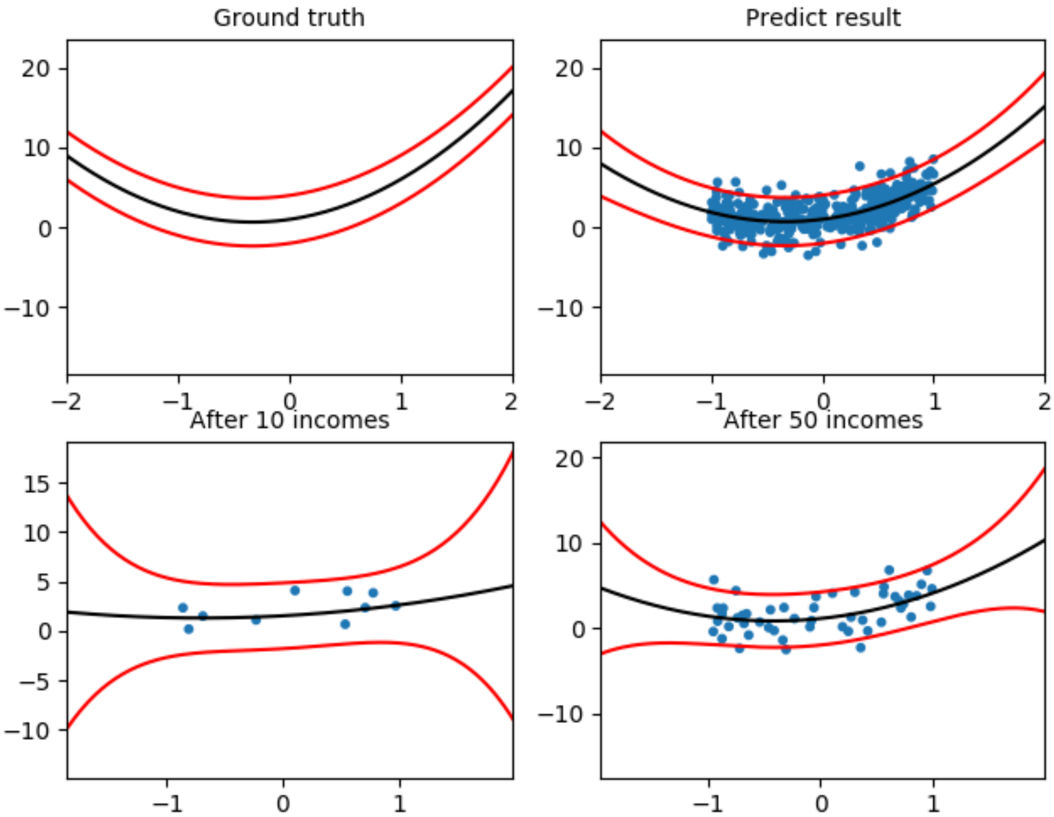
2. b = 100, n = 4, a = 1, w = [1, 2, 3, 4]

```
(Console output omitted)
```



3. b = 1, n = 3, a = 3, w = [1, 2, 3]

```
(Console output omitted)
```


Ground truth
Predict result
After 10 incomes
After 50 incomes