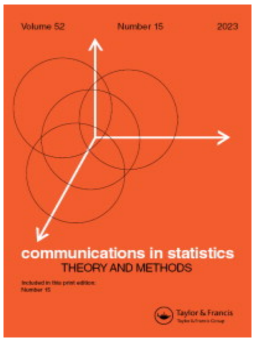

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Tampered random variable modeling for multiple step-stress life test

Farha Sultana[a] and Anup Dewanji[b]

[a]Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, India; [b]Applied Statistics Unit, Indian Statistical Institute, Kolkata, India

ABSTRACT

In this paper, we introduce the tampered random variable (TRV) modeling in multiple step-stress life testing experiments. Here $\tau_1 < \tau_2 < \ldots < \tau_{k-1}$ be $(k-1)$ prespecified time points and $s_1, s_2, \ldots, s_k$ be k prefixed stress levels with s_i being the stress level in force during the time interval $[\tau_{i-1}, \tau_i)$ for $i = 1, \ldots, k$ with $\tau_0 = 0$ and $\tau_k = \infty$. We define the tampered random variable $T_{TRV}^{(k)}$ in multiple step-stress scenario and calculate the PDF, CDF, and Hazard rate for the proposed tampered variable $T_{TRV}^{(k)}$. We derive a general expression for the expectation of $T_{TRV}^{(k)}$ under different number k of stress levels and also obtain some results on stochastic ordering for different k. All these results are obtained under arbitrary baseline (under normal stress condition with stress level s_1) life distribution. In particular, we consider exponential distribution with mean θ and Weibull distribution with scale parameter λ and shape parameter α for specific expressions. We also prove some results on equivalence of the TRV modeling with the two other existing models for step-stress life testing, namely, cumulative exposure and tampered failure rate. Finally, we consider some variations of the modeling approach for $T_{TRV}^{(k)}$ to include incorporation of the stress levels, discrete life time, bivariate or multivariate life times.

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Life testing experiments; tampering coefficients; tampering times; stress conditions; baseline lifetime; stochastic ordering

1. Introduction

Life testing experiments are in process of development over 60 years or so with the objective of estimating one or more reliability characteristics of a product under certain constraint. In such experiments, certain number of identical items are placed on test under specified operating condition and the failure time of the items are recorded. One of the main difficulties of life testing experiment is that it is very difficult to observe sufficient number of failures in an affordable time under normal operating condition. To overcome such situations, experimenters use an alternative technique, known as accelerated life testing (ALT), in which experimental units are put on test under some extreme operational conditions affecting the lifetime of the items adversely so that the items generally fail more quickly than under the normal condition and some meaningful inference can be drawn on the underlying lifetime distribution. The factors which

CONTACT Anup Dewanji dewanjia@isical.ac.in Applied Statistics Unit, Indian Statistical Institute, Kolkata 700108, India.

affect the lifetime of an item are called stress factors. For example, voltage, temperature, and humidity could be stress factors for an electronic equipment. In an ALT, the experimenter usually gets to observe more failure within shorter time period thereby cutting down the experimentation cost. Among many variations of ALT, step-stress life-testing (SSLT) is an important one that provides freedom to change the stress level in a sequential manner during the experiment and has been practiced in some experimental situations. From simple SSLT with only one change of stress level, there are examples of multiple SSLT with several changes in stress level (e.g. Nelson 1980; Samanta et al. 2019, etc.). So, it is imperative that some suitable modeling approaches are developed for the analysis of data arising from such experiments.

Let $s_1, s_2, \ldots, s_k$ be k pre-determined stress levels, with s_1 usually representing the normal stress condition and $s_2, \ldots, s_k$ usually representing successively accelerated stress conditions, although it need not be so in general. Suppose $0 < \tau_1 < \cdots < \tau_{k-1}$ are the $(k-1)$ prefixed time points when the stress levels are to be accelerated. In a basic form of SSLT, a certain number of units are initially put under test at the normal stress condition with stress level s_1. At time point τ_1, the stress level is changed to s_2 from s_1. Similarly, the stress level is changed to s_3 from s_2 at time point τ_2 and so on. Finally, at the time point τ_{k-1}, the stress level is changed from s_{k-1} to s_k. A SSLT is called a simple SSLT if $k=2$ (i.e. there are only two stress levels).

The most popular and commonly used modeling approach for a simple SSLT is known as the cumulative exposure (CE) model, first proposed by Sedyakin (1966) and then studied extensively by Nelson (1980). Denoting the two life distributions under the two stress conditions by $F_1(\cdot)$ and $F_2(\cdot)$, respectively, the overall distribution function of a lifetime is written as

$$F_{CE}^{(2)}(t) = \begin{cases} F_1(t), & \text{if } t \le \tau_1, \\ F_2(t - \tau_1 + h_1), & \text{if } t > \tau_1, \end{cases} \tag{1.1}$$

where $h_1 = F_2^{-1}\{F_1(\tau_1)\}$ which comes from the solution of the equation $F_2(h_1) = F_1(\tau_1)$ so that there is continuity at $t = \tau_1$. Here, the superscript "(2)" in $F_{CE}^{(2)}(t)$ in (1.1) indicates that the model is for a simple SSLT with $k=2$. A second modeling approach, known as the tampered failure rate (TFR) model, was first proposed by Bhattacharyya and Soejoeti (1989), which assumes that the effect of changing the stress is to multiply the initial failure rate function $\lambda_1(t)$ by an unknown factor α_1 (usually, greater than 1) after the change point τ_1. Denoting the failure rate function of the overall lifetime by $\lambda_{TFR}^{(2)}(t)$, the proposed TFR model is defined as

$$\lambda_{TFR}^{(2)}(t) = \begin{cases} \lambda_1(t), & \text{if } t \le \tau_1, \\ \alpha_1 \lambda_1(t), & \text{if } t > \tau_1, \end{cases} \tag{1.2}$$

where the factor α_1 (usually, greater than 1) depends on both the stress levels s_1 and s_2 and possibly on τ_1 as well. Goel (1971) first introduced the tampered random variable (TRV) modeling in the context of a simple SSLT (See also DeGroot and Goel 1979), which assumes that the effect of change of the stress level at time τ_1 is equivalent to changing (usually, scaling down) the remaining life of the experimental unit by an unknown positive factor, say β_1 (usually, less than 1). Let T be the random variable representing the baseline lifetime under normal stress condition. Then, the overall lifetime, denoted by the "tampered" random variable $T^{(2)}_{TRV}$, is defined as

$$T^{(2)}_{TRV} = \begin{cases} T, & \text{if } T \leq \tau_1, \\ \tau_1 + \beta_1(T - \tau_1), & \text{if } T > \tau_1, \end{cases} \tag{1.3}$$

where the scale factor β_1, called the tampering coefficient, depends on both s_1 and s_2 and possibly on τ_1 as well. The time point τ_1 is called the tampering time.

There have been natural generalizations of the CE and TFR modeling approaches to more than two stress levels. See, for example, Pan and Balakrishnan (2010), Samanta et al. (2019), Tang (2003) for generalization of the CE modeling and Madi (1993), Balakrishnan, Beutner, and Kateri (2009), Wang and Fei (2003) for generalization of the TFR modeling. See also Kundu and Ganguly (2017). However, although there have been a number of works on estimation and optimal design for a simple SSLT under TRV modeling (See Goel 1975; Bai and Chung 1992; Bai, Chung, and Chun 1993), there has not been any attempt to generalize the TRV modeling approach to more than two stress levels, possibly because the mechanism of tampering of the lifetime due to successive multiple stress levels was less understood. The present paper fills that void with a clearly explained mechanism of successive tampering incorporating multiple tampering coefficients in a natural way. The idea is an extension of the same modeling approach for simple SSLT with introduction of additional tampering coefficient parameters representing additional shrinking of lifetime over and above the already shrunk lifetime due to the previous stress levels.

Meanwhile, there have been some work establishing "equivalence" between CE and TRV modeling, and also TRV and TFR modeling, in the context of simple SSLT (See Wang and Fei 2004) in the sense that, for any fixed tampering time τ_1, the two distributions are from the same parametric family. In this work, we also make an attempt to establish such equivalence in the general SSLT framework with more than two stress levels.

The idea behind the proposed generalization of the TRV modeling approach for multiple step-stress life testing is based on a mechanism of the nature of tampering that a lifetime goes through as a result of application of successive and different stress levels. As the life time passes through different stress conditions, it becomes tampered over and above the already tampered life time due to the previous stress conditions. In other words, tampering of the remaining lifetime at a stress condition takes place on the successively tampered lifetime due to the previous stress conditions. So, there is a possibility that, in case some stress level is less severe than the previous one(s), tampering of lifetime will be reversed from its immediate previous condition(s). This physical phenomenon has been captured in the present TRV modeling approach, which can be readily extended to include the specific stress levels, as in Shaked and Singpurwalla (1983), and also to model discrete and multivariate life time.

In Section 2, we introduce the TRV modeling for general multiple step-stress life testing and derive the corresponding distribution function (CDF), density function (PDF) and hazard rate. Also, the corresponding expectation and moment generating function (MGF) are derived in most general form. In particular, algebraic expressions for the CDF and the mean, corresponding to the exponential and Weibull distributions for the baseline lifetime T, have been worked out. Section 3 derives some results on stochastic ordering in this framework of TRV modeling. In section 4, we investigate the equivalence between CE and TRV, and also between TFR and TRV, modeling approaches under the multiple SSLT. Some variations of this TRV modeling have been discussed in Section 5, while Section 6 ends with some concluding remarks.

2. Model description

We consider a general SSLT with k stress levels, denoted by k-SSLT, as described in the previous section. Let T be the baseline lifetime of an unit under normal stress condition with stress level s_1. Clearly, the new random variable $T_{TRV}^{(k)}$, representing the tampered lifetime under the k-SSLT, is same as T in $[0, \tau_1]$. Now, if the unit survives till time τ_1, and as the stress level is changed from s_1 to s_2 at time τ_1, it is assumed that the impact of such a change is to tamper the remaining lifetime by an unknown scale factor $\beta_1 < 1$, as in the simple SSLT. That is, after time τ_1, the tampered lifetime $T_{TRV}^{(k)}$ becomes $\tau_1 + \beta_1(T - \tau_1)$, if no further change of stress level is required (that is, the unit fails before time τ_2). Although it is not usually practiced, as remarked in the Introduction, in case the stress level s_2 is less severe than the normal stress s_1, the resultant remaining lifetime at time τ_1 will be stretched and such tampering will be modeled with $\beta_1 > 1$. We do not consider such cases in the following. As the stress condition is scheduled to change from stress level s_2 to s_3 at time τ_2, the once-tampered lifetime of the unit, if it survives till τ_2, goes through further tampering by another unknown tampering coefficient $\beta_2 < 1$ after time τ_2. Note that, as the once-tampered lifetime reaches τ_2, the baseline lifetime T under the initial normal stress condition would have reached $\tau_1 + \frac{(\tau_2 - \tau_1)}{\beta_1} = \tau_2^*$, say. Therefore, we can write the tampered lifetime $T_{TRV}^{(k)}$ as $\tau_1 + \beta_1(T - \tau_1)$ for T in $(\tau_1, \tau_2^*]$. By the similar argument, the twice-tampered lifetime $T_{TRV}^{(k)}$ after time τ_2 can be written as $\tau_2 + \beta_1\beta_2(T - \tau_2^*)$ for T in $(\tau_2^*, \tau_3^*]$, where $\tau_3^* = \tau_2^* + \frac{(\tau_3 - \tau_2)}{\beta_1\beta_2}$. Continuing this argument over the k scheduled stress conditions (for $k \geq 2$) with stress levels $s_1, s_2, \ldots, s_k$, we have the successively tampered lifetime given by

$$T_{TRV}^{(k)} = \begin{cases} T, & 0 \leq T \leq \tau_1 \\ \tau_1 + \beta_1(T - \tau_1^*), & \tau_1^* < T \leq \tau_2^* \\ \tau_2 + \beta_1\beta_2(T - \tau_2^*), & \tau_2^* < T \leq \tau_3^* \\ \vdots & \\ \tau_{k-1} + \prod_{i=1}^{k-1} \beta_i (T - \tau_{k-1}^*), & T > \tau_{k-1}^*, \end{cases} \tag{2.4}$$

where $\tau_1^* = \tau_1$ and $\tau_i^* = \tau_{i-1}^* + \frac{(\tau_i - \tau_{i-1})}{\prod_{j=1}^{i-1} \beta_j}$ for $i = 2, \ldots, k-1$. One can write $\tau_i^* = \tau_1 + \sum_{l=2}^{i} \left(\prod_{j=1}^{l-1} \beta_j\right)^{-1} (\tau_l - \tau_{l-1})$, for $i = 2, \ldots, k-1$. It can be verified that, as $T_{TRV}^{(k)}$ lies between τ_{i-1} and τ_i, the baseline lifetime T takes values between τ_{i-1}^* and τ_i^*, for $i = 1, \ldots, k+1$, with $\tau_0 = \tau_0^* = 0$ and $\tau_{k+1} = \tau_{k+1}^* = \infty$. Also, as $T_{TRV}^{(k)}$ takes the values τ_i, the corresponding value of T is τ_i^*, for $i = 1, \ldots, k-1$, with $\tau_1^* = \tau_1$ and $\tau_i^* > \tau_i$, for

$i = 2, ..., k-1,$ as expected. In the context of TRV modeling for k-SSLT, as remarked before, these τ_i's are called the tampering times and the β_i's are called the corresponding tampering coefficients. Note that the tampering effect in a stress condition acts on the successively tampered lifetime due to the previous stress conditions. Therefore, the distribution of the tampered lifetime $T_{TRV}^{(k)}$ depends on the entire history of the SSLT experiment including the pattern of change in stress conditions.

Let $F_T(t;\theta), f_T(t;\theta)$ and $\lambda_T(t;\theta)$ denote the cumulative distribution function (CDF), the probability density function (PDF) and hazard rate, respectively, for the baseline lifetime T, which may belong to any parametric or non-parametric family of distributions under normal stress condition, where θ denotes the associated model parameter(s). Clearly, $T_{TRV}^{(1)}$ represents the baseline lifetime T with distribution function $F_{T_{TRV}^{(1)}}$, or F_T. For notational ease, we suppress the dependence of the different CDF, PDF and hazard rate on θ in the following descriptions. Then, for a fixed value of k, it can be easily checked that, under the above TRV modeling assumption, the CDF, $F_{TRV}^{(k)}(\cdot)$, of $T_{TRV}^{(k)}$ is given by

$$F_{TRV}^{(k)}(t) = \begin{cases} F_T(t), & 0 < t \le \tau_1 \\ F_T\left(\tau_1^* + \dfrac{t-\tau_1}{\beta_1}\right), & \tau_1 < t \le \tau_2 \\ F_T\left(\tau_2^* + \dfrac{t-\tau_2}{\beta_1\beta_2}\right), & \tau_2 < t \le \tau_3 \\ \vdots & \\ F_T\left(\tau_{k-1}^* + \dfrac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right), & t > \tau_{k-1}. \end{cases}$$

Note that, although not explicitly mentioned, this distribution of $T_{TRV}^{(k)}$ given by the CDF above also depends on, besides θ, the unknown tampering coefficients $\beta_1, ..., \beta_{k-1}$. The corresponding PDF, $f_{TRV}^{(k)}(\cdot)$, is given by

$$f_{TRV}^{(k)}(t) = \begin{cases} f_T(t), & 0 \le t \le \tau_1 \\ \dfrac{1}{\beta_1} f_T\left(\tau_1^* + \dfrac{t-\tau_1}{\beta_1}\right), & \tau_1 < t \le \tau_2 \\ \dfrac{1}{\beta_1\beta_2} f_T\left(\tau_2^* + \dfrac{t-\tau_2}{\beta_1\beta_2}\right), & \tau_2 < t \le \tau_3 \\ \vdots & \\ \dfrac{1}{\prod\limits_{i=1}^{k-1}\beta_i} f_T\left(\tau_{k-1}^* + \dfrac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right), & t > \tau_{k-1}, \end{cases}$$

and the hazard rate, $\lambda_{TRV}^{(k)}(\cdot)$, is given by

$$\lambda_{TRV}^{(k)}(t) = \begin{cases} \lambda_T(t), & 0 \le t \le \tau_1 \\ \dfrac{1}{\beta_1}\lambda_T\left(\tau_1^* + \dfrac{t-\tau_1}{\beta_1}\right), & \tau_1 < t \le \tau_2 \\ \dfrac{1}{\beta_1\beta_2}\lambda_T\left(\tau_2^* + \dfrac{t-\tau_2}{\beta_1\beta_2}\right), & \tau_2 < t \le \tau_3 \\ \vdots & \\ \dfrac{1}{\prod_{i=1}^{k-1}\beta_i}\lambda_T\left(\tau_{k-1}^* + \dfrac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right), & t > \tau_{k-1}. \end{cases}$$

Let us now derive the expectation $E[T_{TRV}^{(k)}]$ of $T_{TRV}^{(k)}$, denoted by $E^{(k)}$, for different values of k, so that $E^{(1)} = E[T] = \int_0^\infty [1 - F_T(t)]dt$. For the simple SSLT with $k=2$, we have

$$\begin{aligned} E^{(2)} = E(T_{TRV}^{(2)}) &= \int_0^{\tau_1} (1 - F_{TRV}^{(2)}(t))dt + \int_{\tau_1}^\infty (1 - F_{TRV}^{(2)}(t))dt \\ &= \int_0^{\tau_1} (1 - F_T(t))dt + \int_{\tau_1}^\infty \left(1 - F_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1}\right)\right)dt \\ &= \int_0^\infty (1 - F_T(t))dt - \int_{\tau_1}^\infty (1 - F_T(t))dt + \int_{\tau_1}^\infty \left(1 - F_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1}\right)\right)dt \\ &= E^{(1)} - \int_{\tau_1}^\infty \left[F_T\left(\tau_1^* + \frac{t-\tau_1}{\beta_1}\right) - F_T(t)\right]dt. \end{aligned}$$

It can be easily checked that $\tau_1^* + \frac{t-\tau_1}{\beta_1} > t$ for $t > \tau_1$, since $0 < \beta_1 < 1$. Hence, $E^{(2)} \le E^{(1)}$. In general, the expectation for k-SSLT can be obtained similarly by partitioning the range of integral and is given by

$$E^{(k)} = E(T_{TRV}^{(k)}) = E^{(k-1)} - \int_{\tau_{k-1}}^\infty \left[F_T\left(\tau_{k-1}^* + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) - F_T\left(\tau_{k-2}^* + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right)\right]dt.$$

Again, it can be checked that $\tau_{k-1}^* + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i} > \tau_{k-2}^* + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}$ for $t > \tau_{k-1}$, since we have $0 < \beta_i < 1$ for $i = 1, \dots, k-1$. Hence, $E^{(k)} \le E^{(k-1)}$. Note that the condition $0 < \beta_i < 1$, for all i, is important for this inequality to hold. This means that, as the stress levels are more and more severe, the expected lifetime under a number $(k-1)$ of stress changes is more than that under an additional stress change, as expected.

In particular, suppose the baseline lifetime variable T follows $Weibull(\alpha, \lambda)$ with the CDF $F_T(t) = 1 - e^{-\lambda t^\alpha}$ and the expectation given by

$$E^{(1)} = E(T) = \lambda^{-\frac{1}{\alpha}}\Gamma\left(\frac{1}{\alpha} + 1\right).$$

Then, for the simple SSLT with $k=2$, the CDF and PDF for $T_{TRV}^{(2)}$ can be written as

$$F_{TRV}^{(2)}(t) = \begin{cases} 1 - e^{-\lambda t^{\alpha}}, & 0 \le t \le \tau_1 \\ 1 - e^{-\lambda\left(\tau_1 + \frac{t-\tau_1}{\beta_1}\right)^{\alpha}}, & t > \tau_1, \end{cases}$$

and

$$f_{TRV}^{(2)}(t) = \begin{cases} \lambda\alpha \ t^{\alpha-1} \ e^{-\lambda t^{\alpha}}, & 0 \le t \le \tau_1 \\ \dfrac{\lambda\alpha}{\beta_1} \ \left(\tau_1 + \dfrac{t-\tau_1}{\beta_1}\right)^{\alpha-1} \ e^{-\lambda\left(\tau_1 + \frac{t-\tau_1}{\beta_1}\right)^{\alpha}}, & t > \tau_1, \end{cases}$$

respectively. The corresponding hazard rate is given by

$$\lambda_{TRV}^{(2)}(t) = \begin{cases} \lambda\alpha \ t^{\alpha-1}, & 0 \le t \le \tau_1 \\ \dfrac{\lambda\alpha}{\beta_1} \ \left(\tau_1 + \dfrac{t-\tau_1}{\beta_1}\right)^{\alpha-1}, & t > \tau_1. \end{cases}$$

The expectation of $T_{TRV}^{(2)}$ is

$$\begin{aligned} E^{(2)} &= \lambda\alpha \int_0^{\tau_1} t \times \ t^{\alpha-1} e^{-\lambda t^{\alpha}} \ dt + \frac{\lambda\alpha}{\beta_1}\int_{\tau_1}^{\infty} t \times \ \left(\tau_1 + \frac{t-\tau_1}{\beta_1}\right)^{\alpha-1} \ e^{-\lambda\left(\tau_1 + \frac{t-\tau_1}{\beta_1}\right)^{\alpha}} \ dt \\ &= \int_0^{\lambda\tau_1^{\alpha}} (\frac{z}{\lambda})^{\frac{1}{\alpha}} e^{-z} \ dz + \int_{\lambda\tau_1^{\alpha}}^{\infty} \left(\tau_1 + \beta_1\left((\frac{z}{\lambda})^{\frac{1}{\alpha}} - \tau_1\right)\right) e^{-z} \ dz \\ &= E^{(1)} - \int_{\lambda\tau_1^{\alpha}}^{\infty} \left[(1-\beta_1)\left((\frac{z}{\lambda})^{\frac{1}{\alpha}} - \tau_1\right)\right] e^{-z} \ dz. \end{aligned}$$

Similarly, for the k-SSLT, we have

$$\begin{aligned} E^{(k)} = E(T_{TRV}^{(k)}) = E^{(k-1)} &- \int_{\lambda(\tau_{k-1}^*)^{\alpha}}^{\infty} \left(\prod_{i=1}^{k-2}\beta_i\right)(1-\beta_{k-1})(\frac{z}{\lambda})^{\frac{1}{\alpha}} e^{-z} \ dz \\ &- \int_{\lambda(\tau_{k-1}^*)^{\alpha}}^{\infty} \left[\left(1 - \prod_{i=1}^{k-1}\beta_i\right)\tau_{k-1} + \left(1 - \prod_{i=1}^{k-2}\beta_i\right)\tau_{k-2}\right] e^{-z} \ dz. \end{aligned}$$

Note that, putting $\alpha = 1$ in the above results for Weibull baseline lifetime distribution, we obtain the same for exponential baseline lifetime distribution. In general, the MGF of $T_{TRV}^{(k)}$, for a fixed k, can be derived by partitioning the range of integral as before and is given by

$$\begin{aligned} M^{(k)}(u) &= E(e^{uT_{TRV}^{(k)}}) \\ &= M^{(k-1)}(u) + \int_{\tau_{k-1}}^{\infty} \frac{e^{ut}}{\prod_{i=1}^{k-1}\beta_i} f_T\left(\tau_{k-1}^* + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) dt \\ &\quad - \int_{\tau_{k-1}}^{\infty} \frac{e^{ut}}{\prod_{i=1}^{k-2}\beta_i} f_T\left(\tau_{k-2}^* + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right) dt. \end{aligned}$$

After differentiating l times, for $l \ge 1$, and then evaluating at $u = 0$, we have

$$
\begin{aligned}
E\left[\left(T_{TRV}^{(k)}\right)^{l}\right] = & E\left[\left(T_{TRV}^{(k-1)}\right)^{l}\right] + \int_{\tau_{k-1}}^{\infty} t^{l} \frac{1}{\prod_{i=1}^{k-1}\beta_i} f_T\left(\tau_{k-1}^{*} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) dt \\
& - \int_{\tau_{k-1}}^{\infty} t^{l} \frac{1}{\prod_{i=1}^{k-2}\beta_i} f_T\left(\tau_{k-2}^{*} + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}\right) dt.
\end{aligned}
\tag{2.5}
$$

Through integration by parts, and noting that $\tau_{k-2}^{*} + \frac{\tau_{k-1}-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i} = \tau_{k-1}^{*}$ and $\tau_{k-1}^{*} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i} > \tau_{k-2}^{*} + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}$, for $t > \tau_{k-1}$ and $0 < \beta_{k-1} < 1$, as in the derivation of $E^{(k)}$, it can be checked that the second term in (2.5) is less than the third term there. Hence, $E\left[(T_{TRV}^{(k)})^{l}\right] < E\left[(T_{TRV}^{(k-1)})^{l}\right]$, for all $l = 1, 2, \ldots$. See Sultana and Dewanji (2020) for details.

3. Some results on stochastic ordering

In order to explore different properties of the proposed TRV model, we now consider the study of different stochastic orderings involving the associated random variables. Some common examples of such ordering between random variables are stochastic ordering, hazard rate ordering, likelihood ratio ordering and mean time to failure (MTTF) ordering. In this section, we investigate these orderings for the TRV at different number of stress conditions. In other words, we explore how the lifetime random variables $T_{TRV}^{(k)}$, for $k = 1, 2, \ldots$ are ordered.

3.1. Stochastic ordering

A random variable X is said to be stochastically larger than a random variable Y, (denoted by $\overset{st}{\geq}$) if $P(X > t) \geq P(Y > t)$ (i.e. $F_X(t) \leq F_Y(t)$), for all t. See Bergmann (1991), Dykstra, Kochar, and Robertson (1991), and Bäuerle and Müller (2006) for details. Considering the expressions of $F_{TRV}^{(k)}(\cdot)$ for different k in Section 2, we have $F_{TRV}^{(k)}(t) = F_{TRV}^{(k-1)}(t)$ for all $t \leq \tau_{k-1}$. For $t > \tau_{k-1}$, note that $\tau_{k-1}^{*} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i} > \tau_{k-2}^{*} + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2}\beta_i}$, since $0 < \beta_i < 1$ for all i, as in Section 2. Hence, we have $F_{TRV}^{(k)}(t) \geq F_{TRV}^{(k-1)}(t)$ for $t > \tau_{k-1}$. Therefore, the $T_{TRV}^{(k)}$'s are stochastically decreasing in k. The decreasing order of the $E^{(k)}$'s obtained in Section 2 is a consequence of this ordering.

3.2. Hazard rate ordering

Given two non-negative random variables X and Y with absolutely continuous CDFs, X is said to be greater than a random variable Y in the hazard rate ordering (written as $X \overset{hr}{\geq} Y$) if $\lambda_X(t) = \frac{f_X(t)}{S_X(t)} \leq \lambda_Y(t) = \frac{f_Y(t)}{S_Y(t)}$, for all $t \geq 0$. For details, one is referred to Boland, El-Neweihi, and Proschan (1994) and Nanda and Shaked (2001).

Comparing $\lambda_{TRV}^{(k)}(t)$ with $\lambda_{TRV}^{(k-1)}(t)$, we have $\lambda_{TRV}^{(k)}(t) = \lambda_{TRV}^{(k-1)}(t)$ for all $t \le \tau_{k-1}$. For $t > \tau_{k-1}$, recall that

$$\lambda_{TRV}^{(k)}(t) = \frac{1}{\prod_{i=1}^{k-1} \beta_i} \lambda_T \left(\tau_{k-1}^* + \frac{t - \tau_{k-1}}{\prod_{i=1}^{k-1} \beta_i} \right)$$

and

$$\lambda_{TRV}^{(k-1)}(t) = \frac{1}{\prod_{i=1}^{k-2} \beta_i} \lambda_T \left(\tau_{k-2}^* + \frac{t - \tau_{k-2}}{\prod_{i=1}^{k-2} \beta_i} \right).$$

Since $0 < \beta_i < 1$ for all i, we have $\tau_{k-1}^* + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1} \beta_i} > \tau_{k-2}^* + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2} \beta_i}$ (See Section 2). Hence, assuming that the distribution of T is IFR, we have $\lambda_{TRV}^{(k)}(t) \ge \lambda_{TRV}^{(k-1)}(t)$ for $t > \tau_{k-1}$. Hence, $T_{TRV}^{(k-1)}$ is larger than $T_{TRV}^{(k)}$ in the hazard rate ordering, for $k = 1, 2, \ldots,$ provided the baseline lifetime T is IFR.

3.3. Likelihood Ratio ordering

Given two random variables X and Y with PDFs (or PMFs) $f_X(\cdot)$ and $f_Y(\cdot)$, respectively, X is said to be larger than Y in the likelihood ratio order (written as $X \overset{lr}{\ge} Y$), if $\frac{f_X(t)}{f_Y(t)}$ is a non-decreasing function of t over the union of supports of X and Y. For more details, see Bapat and Kochar (1994), Ma (1998), and Yang and Zhuang (2014).

Comparing $f_{TRV}^{(k)}(t)$ with $f_{TRV}^{(k-1)}(t)$, we have $f_{TRV}^{(k)}(t) = f_{TRV}^{(k-1)}(t)$ for all $t \le \tau_{k-1}$. For $t > \tau_{k-1}$, one can easily see that

$$\frac{f_{TRV}^{(k-1)}(t)}{f_{TRV}^{(k)}(t)} = \frac{\beta_{k-1} f_T \left(\tau_{k-2}^* + \frac{t-\tau_{k-2}}{\prod_{i=1}^{k-2} \beta_i} \right)}{f_T \left(\tau_{k-1}^* + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1} \beta_i} \right)}.$$

It is difficult to identify all the $f_T(\cdot)$'s for which this ratio is non-decreasing in t. However, if $f_T(\cdot)$ follows an exponential distribution, then one can prove that this ratio is non-decreasing in t, since $1/\prod_{i=1}^{k-2} \beta_i < 1/\prod_{i=1}^{k-1} \beta_i$. Therefore, when $f_T(\cdot)$ follows an exponential distribution, the $T_{TRV}^{(k)}$'s for different k are decreasing in the likelihood ratio order.

3.4. Mean time to failure ordering

Consider the lifetime X of an item with CDF $F(\cdot)$ in an age replacement model so that the item is replaced by a new one, whose lifetime is equal to X in distribution, at failure or at age t, whichever is earlier. Then, according to Barlow and Proschan (1965), the MTTF is defined as

$$m_X(t) = \frac{\int_0^t [1 - F(x)] \, dx}{F(t)}.$$

A random variable X is said to be larger than another random variable Y in the MTTF ordering (written as $X \overset{MTTF}{\geq} Y$) if $m_X(t) \geq m_Y(t)$ for all $t \geq 0$. See Kayid et al. (2013) for details.

Asha and Unnikrishnan (2010) have proved that stochastic order implies MTTF order. Therefore, since the $T_{TRV}^{(k)}$'s are stochastically decreasing in k, these are also decreasing in MTTF order.

4. Equivalence results

In this section, we explore the relationship between the TRV model with the other two models, namely, CE and TFR, under the framework of k-SSLT. Wang and Fei (2004) studied the conditions for "equivalence" of the TRV model with the CE and TFR models under the simple SSLT (i.e. 2-SSLT). Note that the survival function of the overall lifetime for the TFR model under the 2-SSLT can be obtained as

$$\bar{F}_{TFR}^{(2)}(t) = \begin{cases} \bar{F}_T(t), & \text{if } t \leq \tau_1 \\ \bar{F}_T(\tau_1)\left[\frac{\bar{F}_T(t)}{\bar{F}_T(\tau_1)}\right]^{\alpha_1}, & \text{if } t > \tau_1, \end{cases} \tag{4.6}$$

where $\bar{F}_T(\cdot)$ is the survival function of the baseline lifetime T and $\alpha_1 > 1$. The CDF $F_{CE}^{(2)}(\cdot)$ of overall lifetime for the CE model under the 2-SSLT framework is given in Section 1. Note that the $F_1(\cdot)$ in the expression for $F_{CE}^{(2)}(\cdot)$ is the same as $F_T(\cdot)$.

Wang and Fei (2004) have proved that the survival function $\bar{F}_{TFR}^{(2)}(\cdot)$ and that of $T_{TRV}^{(2)}$ (See Section 2) are equivalent if and only if $F_T(\cdot)$ is an exponential distribution, in the sense that, for any given τ_1, there exist $0 < \beta_1 < 1$ and $\alpha_1 > 1$ such that $\bar{F}_{TFR}^{(2)}(t) = \bar{F}_{TRV}^{(2)}(t)$ for all $t > 0$. They have also proved that $F_{CE}^{(2)}(t) = F_{TRV}^{(2)}(t)$ for all $t > 0$ if and only if the two CDFs $F_1(\cdot)$ and $F_2(\cdot)$ of the CE modeling belong to the same scale parameter family of distributions. That is, $F_i(t) = F\left(\frac{t}{s_i}\right)$, $i = 1, 2$, for some CDF $F(\cdot)$ (Nelson 1980). In the following theorem, we provide the similar equivalence results under the framework of k-SSLT for $k \geq 2$.

Theorem 4.1. *In a k-SSLT framework, the TRV model is equivalent to the TFR model if and only if the baseline lifetime distribution $F_T(\cdot)$, is exponential.*

Proof. Let us first assume that $F_T(\cdot)$ is exponential with mean θ. Note that the survival function of TFR model under the k-SSLT framework is given as (Madi 1993)

$$\bar{F}_{TFR}^{(k)}(t) = \begin{cases} \bar{F}_T(t), & \text{if } t \leq \tau_1 \\ \bar{F}_T(\tau_1)\left[\frac{\bar{F}_T(t)}{\bar{F}_T(\tau_1)}\right]^{\alpha_1}, & \text{if } \tau_1 < t \leq \tau_2 \\ \bar{F}_T(\tau_1)\left[\frac{\bar{F}_T(\tau_2)}{\bar{F}_T(\tau_1)}\right]^{\alpha_1}\left[\frac{\bar{F}_T(t)}{\bar{F}_T(\tau_2)}\right]^{\alpha_1\alpha_2}, & \text{if } \tau_2 < t \leq \tau_3, \\ \vdots & \\ \bar{F}_T(\tau_1)\left[\frac{\bar{F}_T(\tau_2)}{\bar{F}_T(\tau_1)}\right]^{\alpha_1} \times \cdots \times \left[\frac{\bar{F}_T(t)}{\bar{F}_T(\tau_{k-1})}\right]^{\prod_{j=1}^{k-1}\alpha_j}, & t > \tau_{k-1}. \end{cases}$$

For exponential baseline lifetime distribution, the survival function, for $\tau_{i-1} < t \le \tau_i$, reduces to

$$\bar{F}^{(k)}_{TFR}(t) = exp\left(-\frac{1}{\theta}\left(\tau_1 + \alpha_1(\tau_2 - \tau_1) + \cdots + \prod_{j=1}^{i-1}\alpha_j(t - \tau_{i-1})\right)\right),$$

for $i = 2, ..., k,$ with $\tau_k = \infty$. Clearly, for $t \le \tau_1$, $\bar{F}^{(k)}_{TFR}(t) = \bar{F}_T(t) = e^{-\frac{t}{\theta}}$. From Section 2, the survival function of the TRV model uder the k-SSLT framework and exponential baseline lifetime distribution, when $\tau_{i-1} < t \le \tau_i$, is

$$\begin{aligned}\bar{F}^{(k)}_{TRV}(t) &= e^{-\frac{1}{\theta}\left(\tau^*_{i-1} + \frac{t-\tau_{i-1}}{\prod_{j=1}^{i-1}\beta_j}\right)} \\ &= e^{-\frac{1}{\theta}\left(\tau_1 + \frac{\tau_2-\tau_1}{\beta_1} + \cdots + \frac{t-\tau_{i-1}}{\prod_{j=1}^{i-1}\beta_j}\right)},\end{aligned}$$

for $i = 2, ..., k,$ with $\tau_k = \infty$. Again, for $t \le \tau_1$, $\bar{F}^{(k)}_{TRV}(t) = \bar{F}_T(t) = e^{-\frac{t}{\theta}}$. It can be easily checked that, with $\alpha_i = \frac{1}{\beta_i}$, for $i = 1, ..., k-1$, the two survival functions $\bar{F}^{(k)}_{TFR}(t)$ and $\bar{F}^{(k)}_{TRV}(t)$ are equal for all t. Hence, the TFR and TRV models are equivalent if the baseline lifetime distribution is exponential.

Conversely, let us suppose that $\bar{F}^{(k)}_{TFR}(t) = \bar{F}^{(k)}_{TRV}(t)$ for all $t > 0$. In particular, for $t > \tau_{k-1}$, we have

$$\bar{F}_T\left(\tau^*_{k-1} + \frac{t - \tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right) = \left[\bar{F}_T(\tau_1)\right]^{1-\alpha_1}\left(\prod_{i=2}^{k-1}\left[\bar{F}_T(\tau_i)\right]^{(1-\alpha_i)\prod_{j=1}^{i-1}\alpha_j}\right)\left[\bar{F}_T(t)\right]^{\prod_{j=1}^{k-1}\alpha_j}.$$

Taking logarithm on both sides, as in Wang and Fei (2004) for $k = 2$, and then taking derivative with respect to t, we get

$$\frac{1}{\prod_{i=1}^{k-1}\beta_i}\frac{f_T\left(\tau^*_{k-1} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right)}{\bar{F}_T\left(\tau^*_{k-1} + \frac{t-\tau_{k-1}}{\prod_{i=1}^{k-1}\beta_i}\right)} = \left[\prod_{i=1}^{k-1}\alpha_i\right]\left[\frac{f_T(t)}{\bar{F}_T(t)}\right]. \tag{4.7}$$

As in Wang and Fei (2004), since the τ_i's are arbitrarily fixed, letting $t \to \tau_{k-1}$ and $\tau_{k-1} \to \tau_{k-2}, ..., \tau_2 \to \tau_1$, we get

$$\frac{1}{\prod_{i=1}^{k-1}\beta_i}\frac{f_T(\tau_1)}{\bar{F}_T(\tau_1)} = \left[\prod_{i=1}^{k-1}\alpha_i\right]\frac{f_T(\tau_1)}{\bar{F}_T(\tau_1)}$$

Hence, because of the arbitrariness of τ_1, we have

$$\frac{1}{\prod_{i=1}^{k-1}\beta_i} = \prod_{i=1}^{k-1}\alpha_i.$$

Now, letting $\tau_{k-1} \to \tau_{k-2}, ..., \tau_2 \to \tau_1$ and $\tau_1 \to 0$ in (4.7) and using the result $\left[\prod_{i=1}^{k-1}\beta_i\right]^{-1} = \prod_{i=1}^{k-1}\alpha_i$, we have

$$\frac{f_T\left(\frac{t}{\prod_{i=1}^{k-1}\beta_i}\right)}{\bar{F}_T\left(\frac{t}{\prod_{i=1}^{k-1}\beta_i}\right)} = \frac{f_T(t)}{\bar{F}_T(t)}. \tag{4.8}$$

Assuming $\frac{t}{\prod_{i=1}^{k-1}\beta_i} = y,$ we have $t = \beta y$ with $\beta = \prod_{i=1}^{k-1}\beta_i$. So, from (4.8),

$$\frac{f_T(y)}{\bar{F}_T(y)} = \frac{f_T(\beta\ y)}{\bar{F}_T(\beta\ y)} \tag{4.9}$$

Using (4.9) n times repeatedly yields

$$\frac{f_T(y)}{\bar{F}_T(y)} = \frac{f_T(\beta\ y)}{\bar{F}_T(\beta\ y)} = \cdots = \frac{f_T(\beta^n\ y)}{\bar{F}_T(\beta^n\ y)} \tag{4.10}$$

Letting $n \to \infty$, (4.10) becomes $\frac{f_T(y)}{\bar{F}_T(y)} = \lim_{n\to\infty} \frac{f_T(\beta^n\ y)}{\bar{F}_T(\beta^n\ y)} = \frac{f_T(0)}{\bar{F}_T(0)} = f_T(0)$. That is, the failure rate of the distribution $F_T(\cdot)$ is the constant $f_T(0)$. Furthermore, $\bar{F}_T(y) = e^{-\int_0^y f_T(0)\ dx} = e^{-f_T(0)y}$; that is to say that the distribution function F_T is exponential.

Theorem 4.2. *In a k-SSLT framework, the TRV model is equivalent to the CE model if and only if the marginal lifetime distributions at the k stress levels belong to the same scale parametric family.*

Proof. Let us write $F_i(\cdot)$ for the marginal lifetime distribution under the i^{th} stress condition, for $i = 1, ..., k,$ under the k-SSLT framework. Clearly, $F_1(\cdot)$ is same as the CDF $F_T(\cdot)$. Note that the CDF of the CE model under the k-SSLT framework is given as (Balakrishnan, Beutner, and Kateri 2009; Samanta et al. 2019)

$$F_{CE}^{(k)}(t) = \begin{cases} F_1(t), & \text{if } t \le \tau_1 \\ F_2(t - \tau_1 + h_1), & \text{if } \tau_1 < t \le \tau_2 \\ F_3(t - \tau_2 + h_2), & \text{if } \tau_2 < t \le \tau_3, \\ \vdots & \\ F_k(t - \tau_{k-1} + h_{k-1}), & t > \tau_{k-1}, \end{cases}$$

where h_i, $i = 1, ..., k-1,$ can be successively obtained by solving

$$\begin{aligned} F_2(h_1) &= F_1(\tau_1), \\ F_3(h_2) &= F_2(\tau_2 - \tau_1 + h_1), \\ &\vdots \\ F_k(h_{k-1}) &= F_{k-1}(\tau_{k-1} - \tau_{k-2} + h_{k-2}). \end{aligned}$$

Let us first assume that the marginal lifetime distributions under the k stress levels belong to the same scale parametric family so that $F_i(t)$ can be written as $G\left(\frac{t}{\eta_i}\right)$, where $G(\cdot)$ is some distribution function and η_i is a scale parameter, for $i = 1, ..., k$. Let us define $\beta_1, \beta_2, ..., \beta_{k-1}$ in a way such that $\prod_{j=1}^{i} \beta_j = \frac{\eta_{i+1}}{\eta_1}$, for $i = 1, ..., k-1$; that is, $\beta_i = \frac{\eta_{i+1}}{\eta_i}$, for $i = 1, ..., k-1$.

By definition, we have $F_{CE}^{(k)}(t) = F_{TRV}^{(k)}(t)$ for $t \leq \tau_1$. Wang and Fei (2004) have proved this equality for $\tau_1 < t \leq \tau_2$; however, we reproduce the proof in the following for better understanding of the generalization. For $\tau_1 < t \leq \tau_2$, we have

$$\begin{aligned}
F_{TRV}^{(k)}(t) &= F_1\left(\tau_1 + \frac{\eta_1}{\eta_2}(t - \tau_1)\right) \\
&= G\left(\frac{\tau_1 + \frac{\eta_1}{\eta_2}(t - \tau_1)}{\eta_1}\right) \\
&= G\left(\frac{\frac{\eta_2}{\eta_1}\tau_1 + (t - \tau_1)}{\eta_2}\right) \\
&= F_2\left(t - \tau_1 + \frac{\eta_2}{\eta_1}\tau_1\right) \\
&= F_{CE}^{(k)}(t), \quad \text{with } h_1 = \frac{\eta_2}{\eta_1}\tau_1,
\end{aligned}$$

since h_1 satisfies $F_2(h_1) = F_1(\tau_1)$, or $G\left(\frac{\frac{\eta_2}{\eta_1}\tau_1}{\eta_2}\right) = G\left(\frac{\tau_1}{\eta_1}\right)$. Now, in general, when $\tau_{i-1} < t \leq \tau_i$, for $i = 2, ..., k$ with $\tau_0 = 0$ and $\tau_k = \infty$, we have

$$\begin{aligned}
F_{TRV}^{(k)}(t) &= F_1\left(\tau_{i-1}^* + \frac{\eta_1}{\eta_i}(t - \tau_{i-1})\right) \\
&= G\left(\frac{\tau_{i-1}^* + \frac{\eta_1}{\eta_i}(t - \tau_{i-1})}{\eta_1}\right) \\
&= G\left(\frac{\frac{\eta_i}{\eta_1}\tau_{i-1}^* + (t - \tau_{i-1})}{\eta_i}\right) \\
&= F_i\left(t - \tau_{i-1} + \frac{\eta_i}{\eta_1}\tau_{i-1}^*\right) \\
&= F_{CE}^{(k)}(t), \quad \text{with } h_{i-1} = \frac{\eta_i}{\eta_1}\tau_{i-1}^*.
\end{aligned}$$

We need to verify that h_{i-1} satisfies $F_i(h_{i-1}) = F_{i-1}(\tau_{i-1} - \tau_{i-2} + h_{i-2})$ with $h_{i-2} = \frac{\eta_{i-1}}{\eta_1}\tau_{i-2}^*$. That is, to verify

$$F_i\left(\frac{\eta_i}{\eta_1}\left(\tau_{i-2}^* + \frac{\eta_1}{\eta_{i-1}}(\tau_{i-1} - \tau_{i-2})\right)\right) = F_{i-1}\left(\frac{\eta_{i-1}}{\eta_1}\tau_{i-2}^* + (\tau_{i-1} - \tau_{i-2})\right),$$

or

$$G\left(\frac{\frac{\eta_i}{\eta_1}\left(\tau_{i-2}^* + \frac{\eta_1}{\eta_{i-1}}(\tau_{i-1} - \tau_{i-2})\right)}{\eta_i}\right) = G\left(\frac{\frac{\eta_{i-1}}{\eta_1}\tau_{i-2}^* + (\tau_{i-1} - \tau_{i-2})}{\eta_{i-1}}\right),$$

or

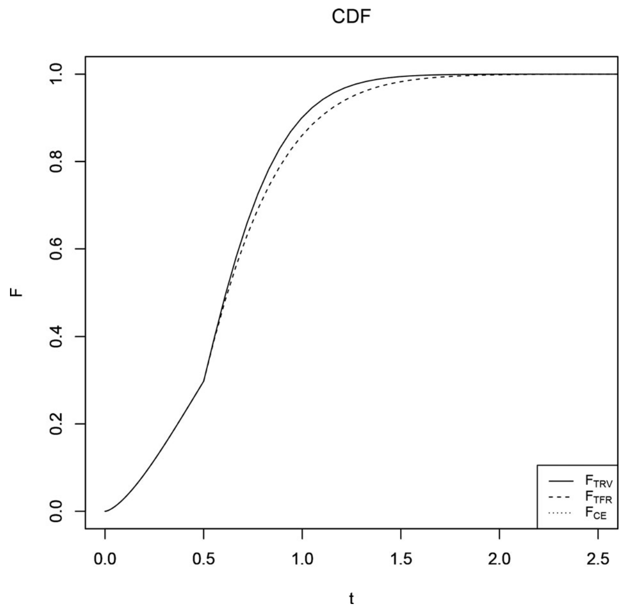

Figure 1. Plots for CDFs of $T^{(2)}_{TRV}$, $T^{(2)}_{TFR}$, and $T^{(2)}_{CE}$ having the tampering time point $\tau_1 = 0.5$ and the tampering coefficient $\beta_1 = 0.4$, when the baseline lifetime T under normal stress condition follows Weibull distribution with the scale parameter $\lambda = 1$ and the shape parameter $\alpha = 1.5$.

$$\frac{\tau^*_{i-2}}{\eta_1} + \frac{\tau_{i-1} - \tau_{i-2}}{\eta_{i-1}} = \frac{\tau^*_{i-2}}{\eta_1} + \frac{\tau_{i-1} - \tau_{i-2}}{\eta_{i-1}},$$

which is an identity. Hence, the TRV and the CE models are equivalent.

Conversely, suppose $F^{(k)}_{TRV}(t)$ and $F^{(k)}_{CE}(t)$ are equivalent for all $t > 0$. Then, when $\tau_{i-1} < t \le \tau_i$ for $i = 1, ..., k$, with $\tau_0 = 0$ and $\tau_k = \infty$, we have $F_1\left(\tau^*_{i-1} + \frac{t-\tau_{i-1}}{\prod_{j=1}^{i-1}\beta_j}\right) = F_i(t - \tau_{i-1} + h_{i-1})$. Since the τ_i's are arbitrary, letting $\tau_{i-1} \to \tau_{i-2}, ..., \tau_1 \to 0$, we have $\tau^*_{i-1} \to 0$ and $h_{i-1} \to 0$, using the definition of $F^{(k)}_{CE}(\cdot)$. Hence, $F_1\left(\frac{t}{\prod_{j=1}^{i-1}\beta_j}\right) = F_i(t)$. Therefore, $F_i(\cdot)$ and $F_1(\cdot)$ belong to the same scale parametric family of distributions, for $i = 2, ..., k$.

In the context of this equivalence result, it is relevant to mention the work of Shaked and Singpurwalla (1983) who have considered a CE model under k-SSLT with each $F_i(\cdot)$ belonging to the same scale parameter family, which can be verified to be the TRV model similar to the one described in Section 2. The results of Theorems 4.1 and 4.2 can be aptly illustrated through Figure 1 in which the three CDFs for TRV, TFR and CE modeling approaches are plotted for simple SSLT and Weibull baseline lifetime distribution with the scale parameter $\lambda = 1$ and shape parameter $\alpha = 1.5$. The single tampering time point τ_1 is taken as 0.5 with the corresponding tampering coefficient for TRV modeling as $\beta_1 = 0.40$. For the TFR and CE modeling, the stress parameters are chosen from the two equivalence results in Theorems 4.1 and 4.2. That is, for the TFR modeling, the constant multiplier α_1 is taken as $\beta_1^{-1} = 2.5$; for the CE modeling, the scale parameter λ_2 for the Weibull distribution under the second stress level, with the same shape parameter $\alpha = 1.5$, is solved from $F_1(\tau_1) = F_2(h_1)$ with $h_1 = \beta_1 \tau_1$ as $\lambda_2 = \lambda(\frac{\tau_1}{h_1})^\alpha$. As the shape parameter is kept the same, the two Weibull distributions under the two stress levels in the CE modeling belong to the same scale family of distributions. Therefore, following Theorem 4.2, the two CDFs for TRV and CE modeling coincide in Figure 1. However, since the baseline lifetime distribution is not exponential, by Theorem 4.1, the two CDFs for TRV and TFR modeling do not coincide in Figure 1. It is to be noted that all the three CDFs coincide in $[0, \tau_1 = 0.5]$, as expected.

5. Other modeling approaches

In this section, we discuss some variations of the TRV modeling approach of Section 2 with different domains of application.

5.1. Trend modeling

Note that the tampering coefficients β_i's of the TRV modeling of Section 2 implicitly depend on the relative magnitudes of the different stress conditions with respect to the normal stress condition. The objective of trend modeling is to explicitly model these β_i's as functions of the stress values $s_1, ..., s_k$ so that one can assess the extent of tampering for a particular stress value. For convenience, let us transform the stress values to $s'_i = s_i - s_1$, for $i = 1, ..., k$, so that s'_1 is equal to 0 representing the normal stress condition. The other transformed s'_i's now represent the additional stress values over the normal stress condition.

A simple choice for modeling the tampering coefficient $\beta(s')$ at the transformed stress value $s' = s - s_1$ is $\beta(s') = \exp(-\beta s')$, with $\beta > 0, s \geq s_1$, so that $\beta(s')$ lies between 0 and 1 with $\beta(0) = 1$, as desired. As a consequence of this trend modeling, there are now fewer parameters. Also note that, by this trend modeling, the tampering effect is more adverse (that is, $\beta(s')$ moves further away from 1) with accelerated stress conditions (that is, increasing values of s'), which is usually desirable. Nevertheless, the TRV model of Section 2 can now be written with $\beta_i = \beta(s'_i) = \exp(-\beta s'_i)$ involving only the single parameter β for the tampering coefficients. This trend modeling allows us to assess the tampering effect at any stress value s which is not in the domain of the stress conditions used for the k-SSLT under study (that is, not one of $s_1, ..., s_k$).

5.2. Discrete life time

Suppose the baseline lifetime T under normal stress condition follows a discrete distribution with mass points $0 \leq x_1 < x_2 < \cdots$ having mass $p_1, p_2, ...$ (such that $\sum_{i=1}^{\infty} p_i = 1$) and discrete hazards $\lambda_1, \lambda_2, ...$ (such that $0 \leq \lambda_i \leq 1$ for all i). For an example, one can think of once daily checking of lifetime of a system that results in observed lifetime measured in days. Consider the TRV modeling approach of Section 2 to model the distribution of $T_{TRV}^{(k)}$ when T is discrete. Note that the tampering time points τ_i's need not belong to the support $\{x_1, x_2 \cdots\}$ of T.

As in Section 2, we define

$$T_{TRV}^{(2)} = \begin{cases} T, & 0 \leq T \leq \tau_1 \\ \tau_1 + \beta_1(T - \tau_1^*), & T > \tau_1 \end{cases}$$

Clearly, the support of $T_{TRV}^{(2)}$ is not the same as that of T. Writing $i_1 = max\{i : x_i \leq \tau_1\}$, the support of $T_{TRV}^{(2)}$ becomes $\{x_i, i = 1, 2, ...i_1\} \cup \{\tau_1 + \beta_1(x_i - \tau_1), i = i_1 + 1, i_1 + 2, ...\}$. As before, $\tau_2^* = \tau_1 + \beta_1^{-1}(\tau_2 - \tau_1)$. Writing $i_2 = max\{i : x_i \leq \tau_2^*\}$, the support of $T_{TRV}^{(3)}$ becomes $\{x_i, i = 1, 2, ..., i_1\} \cup \{\tau_1 + \beta_1(x_i - \tau_1^*), i = i_1 + 1, ..., i_2\} \cup \{\tau_2 + \beta_1\beta_2(x_i - \tau_2^*), i = i_2 + 1, i_2 + 2, ...\}$, where $\tau_1^* = \tau_1$. Note that the masses at these mass points remain the same as those of T. For example, if $T \sim$ *Geometric* distribution with mass points $\{1, 2, ...\}$, then the mass points for $T_{TRV}^{(2)}$ under a simple SSLT plan with $\tau_1 = 5$ and $\beta_1 = 0.5$ are $\{1, 2, 3, 4, 5, 5.5, 6, 6.5, ...\}$, but the corresponding masses remain the same.

In general, for k stress conditions with $(k - 1)$ tampering time points $\tau_1 < \tau_2 \cdots < \tau_{k-1}$, define $\tau_l^* = \tau_{l-1}^* + (\beta_1 \cdots \beta_{l-1})^{-1}(\tau_l - \tau_{l-1})$ for $l = 2, ..., k - 1$, and $i_l = max\{i : x_i \leq \tau_l^*\}$, for $l = 1, ..., k - 1$. Then, the support of $T_{TRV}^{(k)}$ can be written as $\{x_i, i = 1, 2, ..., i_1\} \cup \{\tau_1 + \beta_1(x_i - \tau_1^*), i = i_1 + 1, ..., i_2\} \cup \cdots \cup \{\tau_{k-1} + (\prod_{i=1}^{k-1} \beta_i)(x_i - \tau_{k-1}^*), i = i_{k-1} + 1, i_{k-1} + 2, ...\}$. Note that the masses at these mass points remain the same as $p_1, p_2, ...$ with the same discrete hazards $\lambda_1, \lambda_2, ...$ The tampering effect is only on the mass points which are reduced from the original ones.

There may be some inherent difficulty in the CE and TFR modeling approach for discrete T. In the CE approach the non-uniqueness of $F_T^{-1}(\cdot)$ may be a problem. In the TFR approach, evaluating the discrete hazard will have to deal with the constraint that these lie between 0 and 1.

5.3. Bivariate life time

The TRV modeling of Section 2 can be used to model bivariate lifetime data arising from a SSLT experiment. The principle is to tamper each component of the bivariate lifetime, possibly differently, due to change in stress level while maintaining the dependence between the two. As for example, one can think of any parallel system with two dependent components (e.g. hearing impairment in the two ears of an individual). Let us denote the bivariate lifetime of a single unit under normal stress condition by $T = (T_1, T_2)$ with the corresponding joint CDF given by $F_T(t_1, t_2)$ for $t_1, t_2 \geq 0$. Consider first the simple SSLT with a single tampering time point τ_1 with the corresponding tampered life times denoted by $T_{TRV}^{(2)} = (T_{TRV,1}^{(2)}, T_{TRV,2}^{(2)})$. Note that, in this case, there are two different tampering coefficients, $0 < \beta_1 < 1$ and $0 < \gamma_1 < 1$, say, acting on T_1 and T_2 to result in the tampered random variables $T_{TRV,1}^{(2)}$ and $T_{TRV,2}^{(2)}$, respectively. This, therefore, allows different amounts of tampering in the two component lifetimes due to a single stress condition. Then, $T_{TRV,1}^{(2)}$ and $T_{TRV,2}^{(2)}$ are defined as

$$T_{TRV,1}^{(2)} = \begin{cases} T_1, & 0 \leq T_1 \leq \tau_1 \\ \tau_1 + \beta_1(T_1 - \tau_1), & T_1 > \tau_1 \end{cases}$$

and

$$T_{TRV,2}^{(2)} = \begin{cases} T_2, & 0 \le T_2 \le \tau_1 \\ \tau_1 + \gamma_1(T_2 - \tau_1), & T_2 > \tau_1, \end{cases}$$

respectively. Then, proceeding as in Section 2, the joint CDF of $T_{TRV}^{(2)} = (T_{TRV,1}^{(2)}, T_{TRV,2}^{(2)})$ can be obtained as

$$F_{TRV}^{(2)}(t_1, t_2) = \begin{cases} F_T(t_1, t_2), & t_1 \le \tau_1, t_2 \le \tau_1 \\ F_T\left(t_1, \tau_1 + \dfrac{t_2 - \tau_1}{\gamma_1}\right), & t_1 \le \tau_1, t_2 > \tau_1 \\ F_T\left(\tau_1 + \dfrac{t_1 - \tau_1}{\beta_1}, t_2\right), & t_1 > \tau_1, t_2 \le \tau_1 \\ F_T\left(\tau_1 + \dfrac{t_1 - \tau_1}{\beta_1}, \tau_1 + \dfrac{t_2 - \tau_1}{\gamma_1}\right), & t_1 > \tau_1, t_2 > \tau_1. \end{cases}$$

The corresponding joint PDF is given by

$$f_{TRV}^{(2)}(t_1, t_2) = \begin{cases} f_T(t_1, t_2), & t_1 \le \tau_1, t_2 \le \tau_1 \\ \dfrac{1}{\gamma_1} f_T\left(t_1, \tau_1 + \dfrac{t_2 - \tau_1}{\gamma_1}\right), & t_1 \le \tau_1, t_2 > \tau_1 \\ \dfrac{1}{\beta_1} f_T\left(\tau_1 + \dfrac{t_1 - \tau_1}{\beta_1}, t_2\right), & t_1 > \tau_1, t_2 \le \tau_1 \\ \dfrac{1}{\beta_1\gamma_1} f_T\left(\tau_1 + \dfrac{t_1 - \tau_1}{\beta_1}, \tau_1 + \dfrac{t_2 - \tau_1}{\gamma_1}\right), & t_1 > \tau_1, t_2 > \tau_1. \end{cases}$$

For example, suppose $T = (T_1, T_2)$ follows Marshall–Olkin bivariate exponential (MOBVE) distribution with parameters α_1, α_2 and α_3 and the bivariate survival function given by $S_T(t_1, t_2) = \exp[-(\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 \max(t_1, t_2)]$. Then, using TRV modeling on T_1 and T_2 under a simple SSLT plan with two different tampering coefficients, $0 < \beta_1 < 1$ and $0 < \gamma_1 < 1$, say, the joint survival function of $T_{TRV}^{(2)} = (T_{TRV,1}^{(2)}, T_{TRV,2}^{(2)})$ can be written as

$$S_{TRV}^{(2)}(t_1, t_2) = \begin{cases} \exp\left[-(\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 \max(t_1, t_2))\right], & t_1,\ t_2 \le \tau_1, \\ \exp\left[-\left(\alpha_1 t_1 + (\alpha_2 + \alpha_3)\dfrac{\tau_1 + (t_2 - \tau_1)}{\gamma_1}\right)\right], & t_1 \le \tau_1 < t_2, \\ \exp\left[-\left((\alpha_1 + \alpha_3)\dfrac{\tau_1 + (t_1 - \tau_1)}{\beta_1} + \alpha_2 t_2\right)\right], & t_2 \le \tau_1 < t_1, \\ \exp\left[-\left(\alpha_1 \dfrac{\tau_1 + (t_1 - \tau_1)}{\beta_1} + \alpha_2 \dfrac{\tau_1 + (t_2 - \tau_1)}{\gamma_1} \right.\right. & \\ \qquad \left.\left. + \alpha_3 \max\left(\dfrac{\tau_1 + (t_1 - \tau_1)}{\beta_1}, \dfrac{\tau_1 + (t_2 - \tau_1)}{\gamma_1}\right)\right)\right], & \tau_1 < t_1, t_2. \end{cases}$$

Similarly, in general, for k stress conditions with $(k-1)$ tampering time points $0 < \tau_1 < \cdots < \tau_{k-1}$ and the $(k-1)$ pairs of tampering coefficients $(\beta_1, \gamma_1), \ldots, (\beta_{k-1}, \gamma_{k-1})$, the joint CDF of $T_{TRV}^{(k)} = (T_{TRV,1}^{(k)}, T_{TRV,2}^{(k)})$ can be written as

$$F_{TRV}^{(k)}(t_1,t_2) = \begin{cases} F_T(t_1,t_2), & t_1 \le \tau_1, t_2 \le \tau_1 \\ F_T\left(t_1, \tau^*_{j-1,2} + \dfrac{t_2 - \tau_{j-1}}{\prod_{l=1}^{j-1} \gamma_l}\right), & t_1 \le \tau_1, \tau_{j-1} < t_2 \le \tau_j \\ F_T\left(t_1, \tau^*_{k-1,2} + \dfrac{t_2 - \tau_{k-1}}{\prod_{l=1}^{k-1} \gamma_l}\right), & t_1 \le \tau_1, t_2 > \tau_{k-1} \\ F_T\left(\tau^*_{i-1,1} + \dfrac{t_1 - \tau_{i-1}}{\prod_{l=1}^{i-1} \beta_l}, t_2\right), & \tau_{i-1} < t_1 \le \tau_i, t_2 \le \tau_1 \\ F_T\left(\tau^*_{k-1,1} + \dfrac{t_1 - \tau_{k-1}}{\prod_{l=1}^{k-1} \beta_l}, t_2\right), & t_1 > \tau_{k-1}, t_2 \le \tau_1 \\ F_T\left(\tau^*_{i-1,1} + \dfrac{t_1 - \tau_{i-1}}{\prod_{l=1}^{i-1} \beta_l}, \tau^*_{j-1,2} + \dfrac{t_2 - \tau_{j-1}}{\prod_{l=1}^{j-1} \gamma_l}\right), & \tau_{i-1} < t_1 \le \tau_i, \tau_{j-1} < t_2 \le \tau_j \\ F_T\left(\tau^*_{i-1,1} + \dfrac{t_1 - \tau_{i-1}}{\prod_{l=1}^{i-1} \beta_l}, \tau^*_{k-1,2} + \dfrac{t_2 - \tau_{k-1}}{\prod_{l=1}^{k-1} \gamma_l}\right), & \tau_{i-1} < t_1 \le \tau_i, t_2 > \tau_{k-1} \\ F_T\left(\tau^*_{k-1,1} + \dfrac{t_1 - \tau_{k-1}}{\prod_{l=1}^{k-1} \beta_l}, \tau^*_{j-1,2} + \dfrac{t_2 - \tau_{j-1}}{\prod_{l=1}^{j-1} \gamma_l}\right), & t_1 > \tau_{k-1}, \tau_{j-1} < t_2 \le \tau_j \\ F_T\left(\tau^*_{k-1,1} + \dfrac{t_1 - \tau_{k-1}}{\prod_{l=1}^{k-1} \beta_l}, \tau^*_{k-1,2} + \dfrac{t_2 - \tau_{k-1}}{\prod_{l=1}^{k-1} \gamma_l}\right), & t_1 > \tau_{k-1}, t_2 > \tau_{k-1}, \end{cases}$$

for $i,j = 2,...,k-1$, where $\tau^*_{i,1} = \tau^*_{i-1,1} + (\prod_{l=1}^{i-1} \beta_l)^{-1}(\tau_i - \tau_{i-1})$ and $\tau^*_{i,2} = \tau^*_{i-1,2} + (\prod_{l=1}^{i-1} \gamma_l)^{-1}(\tau_i - \tau_{i-1})$ are the tampered version of τ_i after passing through i number of stress conditions as manifested in the first and second component of $T_{TRV}^{(k)}$, for $i = 2,...,k-1$, with $\tau^*_{i,1} = \tau^*_{1,2} = \tau_1$.

This bivariate TRV modeling can be naturally extended to model multivariate lifetimes with more than two components. However, the notation becomes increasingly complicated. As for the discrete life, there are some inherent difficulties in generalizing the CE and TFR approaches to model multivariate life time as well. While for the CE approach, the inverse of the bivariate CDF $F_T(\cdot,\cdot)$ creates the difficulty, the TFR approach has to deal with the absence of a unique definition of multivariate hazard rate.

6. Conclusion

In this work, we have developed the TRV modeling approach for multiple stress conditions with the initial stress condition being the normal one and then the subsequent stress conditions being increasingly severe (i.e. accelerated in its true meaning). From the development of the model, it is clear that the initial stress condition does not need to be the normal one and also the subsequent stress conditions need not be increasingly severe. If, after any tampering time point, the changed stress condition is less severe (or, more favorable to life) than the previous stress condition, the corresponding tampering coefficient will be greater than unity instead of lying between 0 and 1. Therefore, in general, one does not have to restrict the tampering coefficients between 0 and 1 while analyzing data, letting it dictate the impact of different stress conditions.

The principle behind the TRV modeling approach is simple and appealing, possibly compared to the other two approaches, namely CE and TFR modeling. Moreover, this modeling seems to be more flexible to generalize for discrete and multivariate lifetime, as discussed at the end of Sections 5.2 and 5.3, respectively. Also, simulation of a tampered lifetime is very simple by its algebraic expression in terms of the baseline lifetime under normal stress condition using the tampering time points and the tampering coefficients. So, whenever the baseline lifetime under normal stress condition can be simulated, the corresponding simulated tampered lifetime can be immediately obtained. In general, there can be variability in the amount of tampering, or the impact of different stress conditions, over the different individuals. We plan to address this issue through covariate analysis or by considering the tampering coefficients as random effects.

As the three modeling approaches (namely, CE, TFR, and TRV) for SSLT appear as three competing models, one natural question arises regarding the suitability of a particular one over the others. Given a data set, there are many model selection methods to choose the one which gives the "best" fit, including the well-known AIC criterion. In case there is some information on the mechanism of changes in lifetime due to changes in stress levels, that can be used to choose the most suitable model.

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