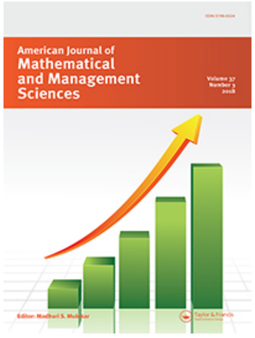

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Parameter Estimation for the Kumaraswamy Distribution Based on Hybrid Censoring

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SYNOPTIC ABSTRACT
We consider estimation of unknown parameters of a two-parameter Kumaraswamy distribution with hybrid censored samples. We obtain maximum likelihood estimates using an expectation-maximization algorithm. Bayes estimates are derived under the squared error loss function using different approximation methods. In addition, an importance sampling technique is also discussed. Interval estimation is considered as well. We conduct a simulation study to compare the performance of different estimates, and based on this study, recommendations are made. A real data set and a simulated data set are analyzed for illustration purposes.

KEY WORDS AND PHRASES
Bayes estimates; EM algorithm; importance sampling; Lindley method; Tierney and Kadane method

1. Introduction

Life testing experiments are concerned with studies of reliability data governed by some probabilistic models. Many applications of life tests occur in biology, clinical trials, financial, agricultural, and other fields of studies, including survival analysis. Commonly, life testing experiments are conducted under several constraints, such as time and cost limits. Given such restrictions, usually, failure times of all items put on a test are not recorded. Such experiments give rise to data which are known as censored data. In the literature, different types of censoring schemes have been discussed. Probably, Type I and Type II censoring are the most widely used schemes in this respect. Several authors have analyzed different lifetime models using property of these two censoring methods. One may refer to Meeker and Escobar (1998) for different interesting applications of Type I and Type II censoring schemes in reliability analysis.

Various modifications of these two schemes are also suggested in the literature, and we mention that a hybrid censoring scheme, which is a mixture of Type I and Type II censoring, has received considerable attention among researchers. In practice, this censoring can be applied in the following manner. Consider an experiment where n test units with common lifetime distribution are subjected to a life test, and the test stops when a prescribed number $r\ (\leq n)$ of items failed or a prescribed time T is reached. Hybrid censoring yields data which

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can be described in one of the following ways.

$$\text{Case I: } \{X_{1:n}, X_{2:n}, \ldots, X_{r:n}\}, \quad \text{if } X_{r:n} < T,$$
$$\text{Case II: } \{X_{1:n}, X_{2:n}, \ldots, X_{m:n}\}, \quad \text{if } m < r, \quad X_{m+1:n} > T.$$

Epstein (1954) first discussed the basic concept of hybrid censoring using a one-parameter exponential distribution with unknown mean and a constructed two-sided confidence interval for the unknown mean. Since then, many lifetime models have been analyzed using this scheme by several researchers from both classical and Bayesian perspectives. Among others, Chen and Bhattacharyya (1988) also constructed a confidence interval for the unknown mean θ using distribution of the corresponding maximum likelihood estimate (MLE). Gupta and Kundu (1998) also analyzed a one-parameter exponential distribution and proposed both classical and Bayesian intervals for the unknown mean. Kundu (2007) comprehensively studied a two-parameter Weibull distribution using hybrid censoring. He obtained both point and interval estimates for the unknown parameters using classical and Bayes methods. MLEs and approximate MLEs for two-parameter lognormal distribution under hybrid censoring are derived by Dube, Pradhan, and Kundu (2011). The authors studied the behavior of suggested estimates using Monte Carlo simulations, and obtained useful comments based on this numerical study. Singh and Tripathi (2016) also analyzed this distribution, and obtained various Bayes estimates for unknown parameters based on hybrid censored data. The problem of Bayesian prediction is also explored and analyzed. They compared performance of proposed point and interval estimates with corresponding classical estimates using a simulation study. Examples are discussed for illustration purposes. Asgharzadeh, Valiollahi, and Kundu (2015) studied the problem of classical and Bayesian prediction for Weibull distribution using hybrid censoring. They obtained Bayesian predictors under squared error and linear-exponential loss functions. Different point and interval predictive estimates are compared using a simulation study and based on this, comments are obtained. Rastogi and Tripathi (2013a, 2013b) studied the Burr XII and a bathtub-shaped distribution under hybrid censoring, and obtained various estimates of unknown model parameters. One may refer to Balakrishnan and Kundu (2013) for a comprehensive review of the various important results on hybrid censoring and also for a list of updated references on this topic.

Kumaraswamy (1980) proposed a two-parameter distribution supported on (0, 1) with probability density function given by,

$$f_X(x; \alpha, \beta) = \alpha\, \beta\, x^{\alpha-1}\, (1 - x^{\alpha})^{(\beta-1)}, \quad 0 \leq x \leq 1, \tag{1}$$

where parameters $\alpha > 0$ and $\beta > 0$ govern the shape of the distribution. We denote this distribution by $K(\alpha, \beta)$. Its distribution function is of the form,

$$F_X(x; \alpha, \beta) = 1 - (1 - x^{\alpha})^{\beta}, \quad 0 \leq x \leq 1.$$

The distribution shares many probabilistic features with the beta distribution. For instance its density function, like beta distribution, can be unimodal, uniantimodal, increasing, decreasing, or constant depending on the different values of its shape parameters. One advantage the Kumaraswamy distribution has over the beta distribution is that its distribution function is in closed form, and so, corresponding quantiles can easily be computed. We notice that distribution function of the beta distribution exists in the form of integral only. The Kumaraswamy distribution has found its natural applications in many areas of practical studies, such as average height of individuals in a certain population, scores obtained on a test, meteorological inference, etc. (see Sindhu, Feroze, & Aslam, 2013). The choices of parameters α and β can be used to transform Kumaraswamy distribution to several distributions, such as uniform, beta, exponential, and generalized beta distributions. Jones (2009) derived various important distributional properties of the Kumaraswamy distribution and mentioned that this distribution can be used to model reliability data arising from different life tests. Cordeiro, Ortega, and Nadarajah (2010) introduced and studied some mathematical properties of the Kumaraswamy Weibull distribution that is a flexible model in analyzing failure time data. Reyad and Ahmed (2016) obtained Bayes estimates for the shape parameter β with known α. They obtained these estimates under symmetric and asymmetric loss functions. Further, using Monte Carlo simulation study, they presented interesting comparison results among proposed methods. Jones (2009) also mentioned that the Kumaraswamy distribution has not gained enough interest among researchers despite being very flexible in its probabilistic structure. In particular, the distribution function of the Kumaraswamy distribution exists in a very simple form. Gilchrist (1997) described that the models having closed-from distribution functions have an advantage in modeling reliability data, as their parameters often directly relate to the shape of the corresponding distribution. Censoring can occur due to several reasons and the corresponding data may characterize different real life situations. Many physical data are often bounded in nature, such as heights of individuals, scores in an examination, time taken by sprinters to complete a race, average rain fall in a particular geographical area, etc. It is natural to incorporate additional information while fitting such data using different distributions. This may lead to better inference about unknown quantities of interest. One may refer to Ponnambalam, Seifi, and Vlach (2001) for further discussion. Approaches developed for Kumaraswamy distribution are also useful in studying several generalizations of this model as well (see for instance, Cordeiro et al., 2010).

To the best of our knowledge, a detailed comparison of different estimators of unknown parameters of a Kumaraswamy distribution has not been done in the literature, particularly under hybrid censoring. We will use the expectation-maximization (EM) algorithm to obtain the maximum likelihood estimates (MLEs) of parameters α and β. This procedure has not been used for computing MLEs in similar works on Kumaraswamy distribution. We further apply this algorithm to compute the observed Fisher information matrix from the missing value principle. From the corresponding asymptotic variance-covariance matrix, the approximate confidence intervals of unknown parameters can be constructed. We will also consider bootstrap confidence intervals for the unknown parameters. Further, Bayes estimates are computed using different methods, such as Lindley approximation (Lindley, 1980), Tierney and Kadane method (Tierney & Kadane, 1986) and importance sampling procedure. We mention that Bayes approaches discussed here have not been considered before for a Kumaraswamy distribution. The importance sampling procedure is also utilized in the construction of highest posterior density intervals for unknown parameters. It is quite important for reliability practitioners to notice related results for this distribution when underlying samples are hybrid censored.

The rest of the content here is organized as follows. In Section 2, we derive the MLEs of unknown parameters α and β of the Kumaraswamy distribution under hybrid censoring. Different confidence intervals, namely, the asymptotic confidence interval, Bootstrap-p, and Bootstrap-t confidence intervals, are obtained in Section 3. In Section 4, we derive different Bayes estimates of unknown parameters under the squared error loss function. We apply approximation methods, such as Lindley and Tierney and Kadane methods, for this purpose along with importance sampling. Highest posterior density intervals are constructed using samples generated from importance sampling. We conduct a simulation study in Section 5 to compare the performance of suggested estimates. A real life example is discussed for illustration purposes in Section 6. Some conclusions are made in Section 7.

2. Maximum Likelihood Estimation

In this section, we derive the maximum likelihood estimates of unknown model parameters. We also obtain the observed Fisher information matrix. Suppose that $\boldsymbol{X} = (X_{1:n}, X_{2:n}, \ldots, X_{n:n})$ is a random sample of size n taken from a Kumaraswamy distribution with parameters α and β. We obtain the MLEs of α and β under an additional constraint that samples are hybrid censored in nature. The corresponding likelihood function is given by,

$$\begin{cases} L(\alpha, \beta) \propto \prod_{i=1}^{r} f_X(x_{i:n})[1 - F_X(x_{r:n})]^{(n-r)}, & \text{for Case I,} \\ L(\alpha, \beta) \propto \prod_{i=1}^{m} f_X(x_{i:n})[1 - F_X(T)]^{(n-m)}, & \text{for Case II.} \end{cases}$$

The above two expressions can be represented as follows.

$$L(\alpha, \beta) \propto \alpha^d \beta^d \prod_{i=1}^{d} x_{i:n}^{(\alpha-1)} \left(1 - x_{i:n}^{\alpha}\right)^{(\beta-1)} (1 - c^{\alpha})^{\beta\,(n-d)}$$

with c and d defined as,

$$c = \begin{cases} x_{r:n}, & \text{for Case I,} \\ T, & \text{for Case II,} \end{cases} \qquad \text{and} \qquad d = \begin{cases} r, & \text{for Case I,} \\ m, & \text{for Case II.} \end{cases}$$

The log-likelihood function is,

$$\log L(\alpha, \beta) \propto d \log \alpha + d \log \beta + (\alpha - 1) \sum_{i=1}^{d} \log x_{i:n} + (\beta - 1) \sum_{i=1}^{d} \left(1 - x_{i:n}^{\alpha}\right) + \beta\,(n-d)\,\log(1 - c^{\alpha}).$$

Then, the likelihood equations are obtained as follows.

$$\frac{d}{\alpha} + \sum_{i=1}^{d} \log x_{i:n} - (\beta - 1) \sum_{i=1}^{d} \frac{x_{i:n}^{\alpha}\,\log x_{i:n}}{(1 - x_{i:n}^{\alpha})} - \beta\,(n-d)\,\frac{c^{\alpha}\,\log c}{(1 - c^{\alpha})} = 0,$$

and

$$\frac{d}{\beta} + \sum_{i=1}^{d} \log\left(1 - x_{i:n}^{\alpha}\right) + (n-d)\,\log(1 - c^{\alpha}) = 0.$$

The above system of likelihood equations is nonlinear, which can be solved using some numerical methods. For instance, the MLEs $(\hat{\alpha}, \hat{\beta})$ of (α, β) can be computed by applying the Newton-Raphson method. This procedure requires the second derivatives of the likelihood function at each step of the iteration. Such computations are sometimes complicated in nature under progressive censoring. In this regard, we suggest using the EM algorithm which was initially discussed by Dempster, Laird, and Rubin (1977). Pradhan and Kundu (2009) mentioned that the Newton-Raphson method may not always converge in such situations. Further, Little and Rubin (1983) discussed that although the EM algorithm converges slowly, it is more reliable compared to the Newton-Raphson method, particularly for censored data. Thus, we compute the MLEs of unknown parameters α and β using the EM algorithm. In this method, at each iteration, two steps are implemented; namely, the expectation step (E-step) and the maximization step (M-step). In this regard, let $\boldsymbol{X} = (X_{1:n}, X_{2:n}, \ldots, X_{d:n})$ be the observed data and $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_{n-d})$ be the censored data. Then, the complete data set is of the form $\boldsymbol{W} = (\boldsymbol{X}, \boldsymbol{Z})$ and the associated log-likelihood function is given by,

$$L_c(\alpha, \beta) = n \log \alpha + n \log \beta + (\alpha - 1)\left(\sum_{i=1}^{d} \log x_{i:n} + \sum_{i=1}^{n-d} \log z_i\right) + (\beta - 1)\left(\sum_{i=1}^{d} \log\left(1 - x_{i:n}^{\alpha}\right) + \sum_{i=1}^{n-d} \log\left(1 - z_i^{\alpha}\right)\right).$$

In the E-step, we obtain pseudo log-likelihood function $L_s(\alpha, \beta)$ which turns out to be,

$$L_s(\alpha, \beta) = n \log \alpha + n \log \beta + (\alpha - 1)\sum_{i=1}^{d} \log x_{i:n} + (\beta - 1)\sum_{i=1}^{d} \log\left(1 - x_{i:n}^{\alpha}\right) + (\alpha - 1)\sum_{i=1}^{n-d} E(\log Z_i \mid Z_i > c) + (\beta - 1)\sum_{i=1}^{n-d} E\big(\log\left(1 - Z_i^{\alpha}\right) \mid Z_i > c\big), \quad (2)$$

where,

$$\begin{aligned} E(\log Z_i \mid Z_i > c) &= \frac{\alpha\beta}{1 - F_X(c; \alpha, \beta)} \int_c^1 x^{\alpha-1} \log x \; (1 - x^{\alpha})^{\beta-1}\, dx \\ &= \frac{\beta}{\alpha\,(1 - c^{\alpha})^{\beta}} \int_0^{1-c^{\alpha}} u^{\beta-1} \log(1 - u)\, du \\ &= A(c, \alpha, \beta), \end{aligned}$$

and

$$\begin{aligned} E\big(\log\left(1 - Z_i^{\alpha}\right) \mid Z_i > c\big) &= \frac{\alpha\beta}{1 - F_X(c; \alpha, \beta)} \int_c^1 \log(1 - x^{\alpha})\, x^{\alpha-1}\, (1 - x^{\alpha})^{\beta-1}\, dx \\ &= \frac{\beta}{(1 - c^{\alpha})^{\beta}} \int_0^{1-c^{\alpha}} u^{\beta-1} \log(u)\, du \\ &= B(c, \alpha, \beta). \end{aligned}$$

The M-step deals with maximizing Equation (2) with respect to the model parameters α and β. In this regard, let $(\alpha^{(k)}, \beta^{(k)})$ denote an estimate of (α, β) at the k-th stage of the iteration. Then, $(\alpha^{(k+1)}, \beta^{(k+1)})$ can be computed by maximizing (see Kundu and Pradhan, 2009),

$$g(\alpha, \beta) = n \log \alpha + n \log \beta + (\alpha - 1)\sum_{i=1}^{d} \log x_{i:n} + (\beta - 1)\sum_{i=1}^{d} \log\left(1 - x_{i:n}^{\alpha}\right) + (\alpha - 1)(n - d)A(c, \alpha^{(k)}, \beta^{(k)}) + (\beta - 1)(n - d)B(c, \alpha^{(k)}, \beta^{(k)}).$$

We further observe that $\alpha^{(k+1)}$ can be computed from the equation,

$$h(\alpha) = \alpha,$$

where,

$$h(\alpha) = \frac{n}{-(n - d)A - \sum_{i=1}^{d} \log x_{i:n} + (\hat{\beta} - 1)\sum_{i=1}^{d} \frac{x_{i:n}^{\alpha} \log x_{i:n}}{(1 - x_{i:n}^{\alpha})}},$$

$$A = A(c, \alpha^{(k)}, \beta^{(k)}), \qquad B = B(c, \alpha^{(k)}, \beta^{(k)}),$$

and

$$\hat{\beta}(\alpha) = -\frac{n}{\sum_{i=1}^{d}\log(1 - x_{i:n}^{\alpha}) + (n-d)B}.$$

The updated estimate $\beta^{(k+1)}$ of the unknown parameter β is now given by $\beta^{(k+1)} = \hat{\beta}(\alpha^{(k+1)})$. This iterative process can be repeated until desired convergence is achieved.

3. Confidence Intervals

In this section, we construct confidence intervals for the unknown parameters. We use the asymptotic property of the MLEs to obtain interval estimates of unknown parameters. For comparison purposes, we also obtain bootstrap-p (Boot-p) and bootstrap-t (Boot-t) intervals.

3.1. Asymptotic Confidence Intervals

We apply the missing information method of Louis (1982) and derive the observed Fisher information matrix. Assume that $\theta = (\alpha, \beta)$, $\boldsymbol{I}_W(\theta)$ represents the complete information, $\boldsymbol{I}_X(\theta)$ represents the observed information and $\boldsymbol{I}_{W|X}(\theta)$ represents the missing information. Then, we have,

$$\boldsymbol{I}_X(\theta) = \boldsymbol{I}_W(\theta) - \boldsymbol{I}_{W|X}(\theta),$$

where,

$$\boldsymbol{I}_W(\theta) = -E\left[\frac{\partial^2 L_c(\boldsymbol{W};\theta)}{\partial\theta^2}\right] \quad \text{and} \quad \boldsymbol{I}_{W|X}(\theta) = -(n-d)E_{Z|X}\left[\frac{\partial^2 \log f_Z(\boldsymbol{z} \mid \boldsymbol{X}, \theta)}{\partial\theta^2}\right].$$

We now compute elements of these matrices. In this regard, let $a_{ij}(\alpha, \beta)$ denote the (i, j)-th element of the complete information matrix $\boldsymbol{I}_W(\theta)$, where $i, j = 1, 2$. We have,

$$a_{11}(\alpha, \beta) = \frac{n}{\alpha^2} + (\beta - 1)\, n\,\alpha\,\beta \int_0^1 \frac{x^{\alpha}\,(\log x)^2\, x^{\alpha-1}(1 - x^{\alpha})^{\beta-1}}{(1 - x^{\alpha})^2} dx,$$

$$a_{22}(\alpha, \beta) = \frac{n}{\beta^2}, \quad \text{and} \quad a_{12}(\alpha, \beta) = a_{21}(\alpha, \beta) = n\alpha\beta \int_0^1 \frac{x^{\alpha}\,\log x\, x^{\alpha-1}(1 - x^{\alpha})^{\beta-1}}{(1 - x^{\alpha})} dx.$$

Similarly, we have,

$$\boldsymbol{I}_{W|X}(\theta) = (n-d)\begin{bmatrix} b_{11}(c;\alpha,\beta) & b_{12}(c;\alpha,\beta) \\ b_{21}(c;\alpha,\beta) & b_{22}(c;\alpha,\beta) \end{bmatrix},$$

where,

$$b_{11}(c;\alpha,\beta) = \frac{1}{\alpha^2} + \frac{(\beta-1)\,\beta}{\alpha^2}\, h_1(c;\alpha,\beta) + \frac{\beta\, c^{\alpha}\,(\log c)^2}{(1 - c^{\alpha})^2},$$

$$b_{12}(c;\alpha,\beta) = \frac{\beta}{\alpha}\, h_2(c;\alpha,\beta) + \frac{c^{\alpha}\,\log c}{(1 - c^{\alpha})} = b_{21}(c;\alpha,\beta), \qquad b_{22}(c;\alpha,\beta) = \frac{1}{\beta^2},$$

$$h_1(c;\alpha,\beta) = \int_0^1 (1-u)\,(\log(1-u))^2\, u^{\beta-3} du,$$

and

$$h_2(c;\alpha,\beta) = \int_0^1 (1-u)\,\log(1-u)\, u^{\beta-2} du.$$

The asymptotic variance-covariance matrix of the MLEs is now given by $\boldsymbol{I}_{\boldsymbol{X}}^{-1}(\theta)$, which can be used to construct the asymptotic confidence intervals for unknown parameters α and β. In fact, the two-sided $100(1-\tau)\%$ symmetric confidence intervals for α and β turn out to be $\hat{\alpha} \pm z_{\tau/2}\sqrt{Var(\hat{\alpha})}$ and $\hat{\beta} \pm z_{\tau/2}\sqrt{Var(\hat{\beta})}$, respectively. Here, $z_{\tau/2}$ is the upper $\frac{\tau}{2}$th percentile of the standard normal distribution.

3.2. Bootstrap Confidence Intervals

In this section, we discuss bootstrap intervals for unknown model parameters α and β. A rich literature exists on this topic, and one may refer to Kundu and Joarder (2006) for applications of bootstrapping in life testing experiments. The Boot-p intervals are constructed as follows.

Step 1. Generate a hybrid censored sample $(X_{1:n}, X_{2:n}, \ldots, X_{d:n})$ from the distribution as give in (1). Then compute the estimate $\hat{\theta}$ of the unknown model parameter θ.

Step 2. Generate a bootstrap sample $(X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{d:n})$ using $\hat{\theta}$ and then compute the bootstrap estimate of $\hat{\theta}^*$ of θ.

Step 3. Repeat Step 2, B times.

Step 4. The approximate $100(1-\tau)\%$ confidence interval of unknown model parameter θ can be obtained as $(\hat{\theta}_{\text{Boot-}p}(\frac{\tau}{2}),\ \hat{\theta}_{\text{Boot-}p}(1-\frac{\tau}{2}))$, where $\hat{\theta}_{\text{Boot-}p}(x) = \widehat{F}^{-1}(x)$ and $\widehat{F}(x)$ is the distribution function of $\hat{\theta}^*$.

Moreover, the Boot-t intervals are constructed as follows.

Step 1. Generate a hybrid censored samples $(X_{1:n}, X_{2:n}, \ldots, X_{d:n})$ from the distribution as give in (1). Then compute the estimate $\hat{\theta}$.

Step 2. Generate a bootstrap samples $(X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{d:n})$ using $\hat{\theta}$ and compute the updated estimates $\hat{\theta}^*$ and $\hat{V}(\hat{\theta}^*)$.

Step 3. Compute the statistic $T^* = \frac{\hat{\theta}^* - \hat{\theta}}{\sqrt{\hat{V}(\hat{\theta}^*)}}$.

Step 4. Repeat Steps 2 and 3, B times.

Step 5. Let $\hat{\theta}_{\text{Boot-}t}(x) = \hat{\theta} + \sqrt{\hat{V}(\hat{\theta}^*)}\widehat{F}^{-1}(x)$, where $\widehat{F}(x)$ is the distribution function of T^*. The approximate $100(1-\tau)\%$ confidence interval of θ turns out to be $(\hat{\theta}_{\text{Boot-}t}(\frac{\tau}{2}),$ $\hat{\theta}_{\text{Boot-}t}(1-\frac{\tau}{2}))$.

4. Bayes Estimation

In this section, we consider Bayes estimation of the unknown parameters of a Kumaraswamy distribution. Let $\boldsymbol{X} = (X_{1:n}, X_{2:n}, \ldots, X_{d:n})$ be a hybrid censored sample from a Kumaraswamy distribution. Estimation of α and β is considered under the squared error loss function $L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2$, where $\hat{\theta}$ denotes an estimate of the unknown parameter θ. We assume that the prior distributions of α and β are gamma $G(a, b)$ and $G(p, q)$ distributions, respectively. We further assume that α and β are independent. Thus, the corresponding joint prior distribution turns out to be,

$$\pi(\alpha, \beta) \propto \alpha^{a-1} e^{-b\alpha} \beta^{p-1} e^{-q\beta}, \qquad \alpha > 0,\ \beta > 0,$$

where $a > 0$, $b > 0$, $p > 0$, and $q > 0$ reflect the prior knowledge about the unknown quantities α and β. It is seen that the joint posterior distribution can be written as follows.

$$\pi(\alpha, \beta \,|\, \boldsymbol{x}) = k\,\alpha^{d+a-1} e^{-\alpha(b - \sum_{i=1}^{d} \log x_{i:n})} \times \beta^{d+p-1} e^{-\beta(q - \sum_{i=1}^{d} \log(1 - x_{i:n}^{\alpha}) - (n-d)\log(1-c^{\alpha}))} e^{-\sum_{i=1}^{d} \log(1 - x_{i:n}^{\alpha})},$$

where,

$$k^{-1}=\int_0^\infty\int_0^\infty \alpha^{d+a-1}e^{-\alpha(b-\sum_{i=1}^d \log x_{i:n})}$$
$$\times\,\beta^{d+p-1}e^{-\beta(q-\sum_{i=1}^d \log(1-x_{i:n}^\alpha)-(n-d)\log(1-c^\alpha))}e^{-\sum_{i=1}^d \log(1-x_{i:n}^\alpha)}d\alpha\, d\beta,$$

and $\boldsymbol{x}=(x_{1:n},x_{2:n},\ldots,x_{d:n})$ denotes a realization of $\boldsymbol{X}=(X_{1:n},X_{2:n},\ldots,X_{d:n})$. The Bayes estimates $\tilde{\alpha}_B$ and $\tilde{\beta}_B$ of α and β are given by,

$$\tilde{\alpha}_B=k\int_0^\infty\int_0^\infty \alpha^{d+a}e^{-\alpha(b-\sum_{i=1}^d \log x_{i:n})}$$
$$\times\,\beta^{d+p-1}e^{-\beta(q-\sum_{i=1}^d \log(1-x_{i:n}^\alpha)-(n-d)\log(1-c^\alpha))}e^{-\sum_{i=1}^d \log(1-x_{i:n}^\alpha)}\,d\alpha\, d\beta,$$

and

$$\tilde{\beta}_B=k\int_0^\infty\int_0^\infty \alpha^{d+a-1}e^{-\alpha(b-\sum_{i=1}^d \log x_{i:n})}$$
$$\times\,\beta^{d+p}e^{-\beta(q-\sum_{i=1}^d \log(1-x_{i:n}^\alpha)-(n-d)\log(1-c^\alpha))}e^{-\sum_{i=1}^d \log(1-x_{i:n}^\alpha)}\,d\alpha\, d\beta.$$

It is quite difficult to solve these integrals analytically, so we use approximation methods to evaluate them. Here, we propose three approximation methods; namely, the Lindley method, Tierney and Kadane method, and importance sampling method. We mention that the first two methods can provide only point estimates of unknown parameters. They cannot be used to construct confidence intervals. In contrast, importance sampling can be used to derive both point and interval estimates of parameters. However, the first two methods are computationally less intensive compared to the importance sampling procedure.

4.1. Lindley Approximation Method

The expressions for estimates $\tilde{\alpha}_B$ and $\tilde{\beta}_B$ appear as the ratio of two integrals, and it is quite difficult to evaluate them analytically. We apply the Lindley method and obtain explicit expressions for these estimates. Let $I(\boldsymbol{x})$ denotes the posterior expectation of the parametric function $g(\alpha,\beta)$. We have,

$$I(\boldsymbol{x})=\frac{\int_0^\infty\int_0^\infty g(\alpha,\ \beta)e^{l(\alpha,\ \beta|\boldsymbol{x})+\rho(\alpha,\ \beta)}d\alpha d\beta}{\int_0^\infty\int_0^\infty e^{l(\alpha,\ \beta|\boldsymbol{x})+\rho(\alpha,\ \beta)}d\alpha d\beta}\ , \tag{3}$$

where $l(\alpha,\beta\mid\boldsymbol{x})$ denotes the log-likelihood and $\rho(\alpha,\beta)$ denotes the logarithm of the prior $\pi(\alpha,\beta)$. We approximate this expectation as,

$$\begin{aligned}I(\boldsymbol{x})&=g(\hat{\alpha},\hat{\beta})+\frac{1}{2}[(\hat{g}_{\alpha\alpha}+2\hat{g}_\alpha\hat{\rho}_\alpha)\hat{\sigma}_{\alpha\alpha}+(\hat{g}_{\beta\alpha}+2\hat{g}_\beta\hat{\rho}_\alpha)\hat{\sigma}_{\beta\alpha}+(\hat{g}_{\alpha\beta}+2\hat{g}_\alpha\hat{\rho}_\beta)\hat{\sigma}_{\alpha\beta}\\&+(\hat{g}_{\beta\beta}+2\hat{g}_\beta\,\hat{\rho}_\beta)\hat{\sigma}_{\beta\beta}]+\frac{1}{2}[(\hat{g}_\alpha\hat{\sigma}_{\alpha\alpha}+\hat{g}_\beta\hat{\sigma}_{\alpha\beta})(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha}\\&+\hat{l}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta})+(\hat{g}_\alpha\hat{\sigma}_{\beta\alpha}+\hat{g}_\beta\hat{\sigma}_{\beta\beta})(\hat{l}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{l}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta}+\hat{l}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha}+\hat{l}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta})],\end{aligned}$$

where $\sigma_{i,j}$ denotes the $(i,\,j)$-th element of the matrix $[-\frac{\partial^2 l(\alpha,\beta|\boldsymbol{x})}{\partial\alpha\partial\beta}]^{-1}$. We also have,

$$\hat{l}_{\alpha\alpha} = \left.\frac{\partial^2 l}{\partial \alpha^2}\right|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = -\frac{d}{\hat{\alpha}^2} - (\hat{\beta}-1)\sum_{i=1}^{d} \frac{x_{i:n}^{\hat{\alpha}} (\log x_{i:n})^2}{(1-x_{i:n}^{\hat{\alpha}})^2} - \hat{\beta}\,(n-d)\,\frac{c^{\hat{\alpha}}(\log c)^2}{(1-c^{\hat{\alpha}})^2},$$

$$\hat{l}_{\beta\beta} = \left.\frac{\partial^2 l}{\partial \beta^2}\right|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = -\frac{d}{\hat{\beta}^2},$$

$$\hat{l}_{\beta\alpha} = \left.\frac{\partial^2 l}{\partial \beta \partial \alpha}\right|_{\alpha=\hat{\alpha},\,\beta=\hat{\beta}} = \hat{l}_{\alpha\beta} = \left.\frac{\partial^2 l}{\partial \alpha \partial \beta}\right|_{\alpha=\hat{\alpha},\,\beta=\hat{\beta}} = -\sum_{i=1}^{d} \frac{x_{i:n}^{\hat{\alpha}} \log x_{i:n}}{(1-x_{i:n}^{\hat{\alpha}})} - (n-d)\,\frac{c^{\hat{\alpha}}\log c}{(1-c^{\hat{\alpha}})}$$

$$\hat{l}_{\beta\beta\alpha} = \left.\frac{\partial^3 l}{\partial \beta^2 \partial \alpha}\right|_{\alpha=\hat{\alpha},\,\beta=\hat{\beta}} = 0 = \hat{l}_{\beta\alpha\beta} = \hat{l}_{\alpha\beta\beta},$$

$$\hat{l}_{\beta\beta\beta} = \frac{2\,d}{\beta^3},$$

$$\hat{l}_{\alpha\alpha\alpha} = \left.\frac{\partial^3 l}{\partial \alpha^3}\right|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = \frac{2d}{\hat{\alpha}^3} - (\hat{\beta}-1)\sum_{i=1}^{d} \frac{x_{i:n}^{\hat{\alpha}}\,(\log x_{i:n})^3\,(1+x_{i:n}^{\hat{\alpha}})}{(1-x_{i:n}^{\hat{\alpha}})^3}$$
$$-\,\hat{\beta}\,(n-d)\frac{c^{\hat{\alpha}}\,(\log c)^3\,(1+c^{\hat{\alpha}})}{(1-c^{\hat{\alpha}})^3}$$

$$\hat{l}_{\beta\alpha\alpha} = \left.\frac{\partial^3 l}{\partial \beta \partial \alpha^2}\right|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = \hat{l}_{\alpha\beta\alpha} = -\sum_{i=1}^{d} \frac{x_{i:n}^{\hat{\alpha}}(\log x_{i:n})^2}{(1-x_{i:n}^{\hat{\alpha}})^2} - (n-d)\,\frac{c^{\hat{\alpha}}(\log c)^2}{(1-c^{\hat{\alpha}})^2},$$

$$\hat{\rho}_{\alpha} = \frac{(a-1)}{\hat{\alpha}} - b, \quad \text{and} \quad \hat{\rho}_{\beta} = \frac{(p-1)}{\hat{\beta}} - q.$$

For estimating the unknown parameter α, we first note that,

$$g(\alpha,\beta) = \alpha, \quad g_{\alpha} = 1, \quad \text{and} \quad g_{\alpha\alpha} = g_{\beta} = g_{\beta\beta} = g_{\beta\alpha} = g_{\alpha\beta} = 0.$$

The corresponding Bayes estimate of α is then given by,

$$\tilde{\alpha}_B = \hat{\alpha} + 0.5[2\,\hat{\rho}_{\alpha}\,\hat{\sigma}_{\alpha\alpha} + 2\,\hat{\rho}_{\beta}\,\hat{\sigma}_{\alpha\beta} + \hat{\sigma}_{\alpha\alpha}^2\,\hat{l}_{\alpha\alpha\alpha} + 2\hat{\sigma}_{\alpha\alpha}\,\hat{\sigma}_{\alpha\beta}\,\hat{l}_{\beta\alpha\alpha} + \hat{\sigma}_{\beta\alpha}\,\hat{\sigma}_{\alpha\alpha}\,\hat{l}_{\beta\alpha\alpha} + \hat{\sigma}_{\beta\alpha}\,\hat{\sigma}_{\beta\beta}\,\hat{l}_{\beta\beta\beta}].$$

Similarly, for β with,

$$g(\alpha,\beta) = \beta, \quad g_{\beta} = 1, \quad \text{and} \quad g_{\alpha} = g_{\alpha\alpha} = g_{\beta\beta} = g_{\beta\alpha} = g_{\alpha\beta} = 0,$$

the Bayes estimate of β is given by,

$$\tilde{\beta}_B = \hat{\beta} + 0.5[2\,\hat{\rho}_{\beta}\,\hat{\sigma}_{\beta\beta} + 2\,\hat{\rho}_{\alpha}\,\hat{\sigma}_{\beta\alpha} + \hat{\sigma}_{\beta\beta}^2\,\hat{l}_{\beta\beta\beta} + 2\,\hat{\sigma}_{\alpha\beta}^2\,\hat{l}_{\beta\alpha\alpha} + \hat{\sigma}_{\alpha\alpha}\,\hat{\sigma}_{\beta\beta}\,\hat{l}_{\beta\alpha\alpha} + \hat{\sigma}_{\alpha\alpha}\,\hat{\sigma}_{\alpha\beta}\,\hat{l}_{\alpha\alpha\alpha}].$$

4.2. Tierney and Kadane Method

For comparison purposes, we also obtain Bayes estimates of α and β using the method proposed by Tierney and Kadane (1986). We first consider the following functions.

$$\delta(\alpha,\beta) = \frac{l(\alpha,\beta|\boldsymbol{x}) + \rho(\alpha,\beta)}{n}, \qquad \delta_{\theta}^{*}(\alpha,\beta) = \delta(\alpha,\beta) + \frac{\log g(\alpha,\beta)}{n},$$

and also assume that $(\hat{\alpha}_{\delta}, \hat{\beta}_{\delta})$ and $(\hat{\alpha}_{\delta*}, \hat{\beta}_{\delta*})$ maximize functions $\delta(\alpha,\beta)$ and $\delta_{\theta}^{*}(\alpha,\beta)$, respectively. Then, $I(\boldsymbol{x})$, defined in Equation (3), can be approximated in to the form,

$$I(\boldsymbol{x}) = \sqrt{\frac{|\Sigma_{\theta}^{*}|}{|\Sigma|}}\,\exp\left[n\left\{\delta_{\theta}^{*}(\hat{\alpha}_{\delta*},\hat{\beta}_{\delta*}) - \delta(\hat{\alpha}_{\delta},\hat{\beta}_{\delta})\right\}\right],$$

where $|\Sigma|$ and $|\Sigma^*_\theta|$ are the negatives of inverse hessians of $\delta(\alpha,\beta)$ and $\delta^*_\theta(\alpha,\beta)$, respectively. These expressions are evaluated at $(\hat{\alpha}_\delta, \hat{\beta}_\delta)$ and $(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*})$. We have,

$$\begin{aligned}\delta(\alpha,\beta) = \frac{1}{n}\Bigg\{ &(d+a-1)\log\alpha - \alpha\left(b - \sum_{i=1}^{d}\log x_{i:m}\right) + (d+p-1)\log\beta \\ &- \sum_{i=1}^{d}\log\left(1-x^{\alpha}_{i:m}\right) - \sum_{i=1}^{d}\log x_{i:m} \\ &- \beta\left(q - \sum_{i=1}^{d}\log\left(1-x^{\alpha}_{i:m}\right) - (n-d)\,\log(1-c^{\alpha})\right)\Bigg\}\end{aligned}$$

and $(\hat{\alpha}_\delta, \hat{\beta}_\delta)$ satisfies equations,

$$\begin{aligned}\frac{\partial\delta}{\partial\alpha} = &\frac{(d+a-1)}{n\,\alpha} \\ &- \frac{1}{n}\left\{ b - \sum_{i=1}^{d}\log x_{i:m} + (\beta-1)\sum_{i=1}^{d}\frac{x^{\alpha}_{i:m}\,\log x_{i:m}}{(1-x^{\alpha}_{i:m})} + (n-d)\,\beta\frac{c^{\alpha}\log c}{(1-c^{\alpha})}\right\} = 0,\end{aligned}$$

and

$$\frac{\partial\delta}{\partial\beta} = \frac{(d+p-1)}{n\,\beta} + \frac{1}{n}\left\{\sum_{i=1}^{d}\log\left(1-x^{\alpha}_{i:n}\right) + (n-d)\,\log(1-c^{\alpha}) - q\right\} = 0.$$

We further obtain

$$\frac{\partial^2\delta}{\partial\alpha^2} = -\frac{(d+a-1)}{n\,\alpha^2} - \frac{1}{n}\left\{(\beta-1)\sum_{i=1}^{d}\frac{x^{\alpha}_{i:n}\,(\log x_{i:n})^2}{(1-x^{\alpha}_{i:n})^2} + (n-d)\beta\,\frac{c^{\alpha}\,(\log c)^2}{(1-c^{\alpha})}\right\},$$

and

$$\frac{\partial^2\delta}{\partial\beta\partial\alpha} = \frac{\partial^2\delta}{\partial\alpha\partial\beta} = -\frac{1}{n}\left\{\sum_{i=1}^{d}\frac{x^{\alpha}_{i:n}\,\log x_{i:n}}{(1-x^{\alpha}_{i:n})} + (n-d)\,\frac{c^{\alpha}\,\log c}{(1-c^{\alpha})}\right\},\ \frac{\partial^2\delta}{\partial\beta^2} = -\frac{(d+p-1)}{n\,\beta^2}.$$

Then, we have,

$$|\Sigma| = \left[\left(\frac{\partial^2\delta}{\partial\alpha^2}\right)\left(\frac{\partial^2\delta}{\partial\beta^2}\right) - \left(\frac{\partial^2\delta}{\partial\alpha\partial\beta}\right)\left(\frac{\partial^2\delta}{\partial\beta\partial\alpha}\right)\right]^{-1}.$$

For estimating α, we take $g(\alpha,\beta)=\alpha$ and then we obtain $(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*})$. In this regard, we have,

$$\delta^*_\alpha(\alpha,\beta) = \delta(\alpha,\beta) + \frac{1}{n}\,\log\alpha,$$

and $(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*})$ satisfies,

$$\frac{\partial\delta^*_\alpha}{\partial\alpha} = \frac{\partial\delta}{\partial\alpha} + \frac{1}{n\,\alpha} = 0, \quad \text{and} \quad \frac{\partial\delta^*_\alpha}{\partial\beta} = \frac{\partial\delta}{\partial\beta} = 0.$$

Using the following expressions,

$$\frac{\partial^2\delta^*_\alpha}{\partial\alpha^2} = \frac{\partial^2\delta}{\partial\alpha^2} - \frac{1}{n\,\alpha^2}, \qquad \frac{\partial^2\delta^*_\alpha}{\partial\beta\partial\alpha} = \frac{\partial^2\delta^*_\alpha}{\partial\alpha\partial\beta} = \frac{\partial^2\delta}{\partial\beta\partial\alpha}, \quad \text{and} \quad \frac{\partial^2\delta^*_\alpha}{\partial\beta^2} = \frac{\partial^2\delta}{\partial\beta^2},$$

we are able to find,

$$|\Sigma_\alpha^*| = \left[\left(\frac{\partial^2 \delta_\alpha^*}{\partial \alpha^2} \right) \left(\frac{\partial^2 \delta_\alpha^*}{\partial \beta^2} \right) - \left(\frac{\partial^2 \delta_\alpha^*}{\partial \alpha \partial \beta} \right) \left(\frac{\partial^2 \delta_\alpha^*}{\partial \beta \partial \alpha} \right) \right]^{-1}.$$

Using the above calculations, the Bayes estimate of α is now given by,

$$\tilde{\alpha}_{TK} = \sqrt{\frac{|\Sigma_\alpha^*|}{|\Sigma|}} \exp[n\{\delta_\alpha^*(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}) - \delta(\hat{\alpha}_\delta, \hat{\beta}_\delta)\}].$$

For estimating β, we take $g(\alpha, \beta) = \beta$ and observe that,

$$\delta_\beta^*(\alpha, \beta) = \delta(\alpha, \beta) + \frac{1}{n} \log \beta.$$

We also note that $(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*})$ satisfies equations,

$$\frac{\partial \delta_\beta^*}{\partial \alpha} = \frac{\partial \delta}{\partial \alpha} = 0, \quad \text{and} \quad \frac{\partial \delta_\beta^*}{\partial \beta} = \frac{\partial \delta}{\partial \beta} + \frac{1}{n\beta} = 0.$$

Using the following expressions,

$$\frac{\partial^2 \delta_\beta^*}{\partial \alpha^2} = \frac{\partial^2 \delta}{\partial \alpha^2}, \qquad \frac{\partial^2 \delta_\beta^*}{\partial \beta \partial \alpha} = \frac{\partial^2 \delta_\beta^*}{\partial \alpha \partial \beta} = \frac{\partial^2 \delta}{\partial \beta \partial \alpha}, \quad \text{and} \quad \frac{\partial^2 \delta_\beta^*}{\partial \beta^2} = \frac{\partial^2 \delta}{\partial \beta^2} - \frac{1}{n\beta^2},$$

we obtain,

$$|\Sigma_\beta^*| = \left[\left(\frac{\partial^2 \delta_\beta^*}{\partial \alpha^2} \right) \left(\frac{\partial^2 \delta_\beta^*}{\partial \beta^2} \right) - \left(\frac{\partial^2 \delta_\beta^*}{\partial \alpha \partial \beta} \right) \left(\frac{\partial^2 \delta_\beta^*}{\partial \beta \partial \alpha} \right) \right]^{-1}.$$

The Bayes estimate of β is now given by,

$$\tilde{\beta}_{TK} = \sqrt{\frac{|\Sigma_\beta^*|}{|\Sigma|}} \exp[n\{\delta_\beta^*(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}) - \delta(\hat{\alpha}_\delta, \hat{\beta}_\delta)\}].$$

The previous methods of Bayes estimation cannot be used to construct Bayes intervals of unknown parameters. In this connection, we next discuss importance sampling which can be used to compute both point and interval Bayes estimates of unknown model parameters.

4.3. Importance Sampling

We now suggest an importance sampling method for computing Bayes estimates of unknown parameters α and β of a Kumaraswamy distribution under squared error loss function. Samples generated from this method can also be used to construct highest posterior density (HPD) intervals. We rewrite the posterior distribution of α and β as,

$$\pi(\alpha, \beta \mid \boldsymbol{x}) \propto G_{\beta|\alpha} \left(d + p,\ q - \sum_{i=1}^{d} \log\left(1 - x_{i:n}^\alpha\right) - (n-d)(1 - c^\alpha) \right)$$
$$\times G_\alpha \left(d + a,\ b - \sum_{i=1}^{d} \log x_{i:n} \right) h(\alpha, \beta),$$

where $h(\alpha, \beta) = e^{-\sum_{i=1}^{d} \log(1 - x_{i:n}^\alpha)} [q - \sum_{i=1}^{d} \log(1 - x_{i:n}^\alpha) - (n-d)\log(1 - c^\alpha)]^{-d-p}$. The procedure to obtain the Bayes estimates using importance sampling is as follows.

Step 1. Generate α_1 from $G_\alpha(.,.)$.
Step 2. Generate β_1 from $G_{\beta|\alpha}(.,.)$.
Step 3. Repeat the above two steps s times to generate $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_s, \beta_s)$.
Step 4. The Bayes estimate of a parametric function $g(\alpha, \beta)$ is given by,

$$\tilde{g}_{IS}(\alpha, \beta) = \frac{\sum_{i=1}^{s} g(\alpha_i, \beta_i)h(\alpha_i, \beta_i)}{\sum_{i=1}^{s} h(\alpha_i, \beta_i)}.$$

The Bayes estimate of α is obtained by considering $g(\alpha, \beta) = \alpha$ in the above computation. Similarly, the Bayes estimate of β can be computed. The $100(1-\tau)\%$ HPD intervals for parameters α and β can be obtained by applying the method proposed by Chen and Shao (1999). Among others, some further applications of this method can also be found in Kundu and Pradhan (2009) and Rastogi and Tripathi (2013a).

5. Simulation Results

In this section, we conduct a Monte Carlo simulation study to assess the behavior of suggested estimation methods in terms of their mean squared error (MSE) and bias values. These values are obtained based on 5,000 replications drawn from a Kumaraswamy distribution with $\alpha = 2$ and $\beta = 3$ for different sample sizes. The *nleqslv* package of the statistical software *R* has been used for various numerical computations. We compute maximum likelihood estimates of unknown model parameters using the EM algorithm. We observe that the initial guess of the unknown parameters α and β is required to perform the computations. Here, we choose the values of true parameters as the initial guess. We compute Bayes estimates of unknown parameters with respect to the squared error loss function. These estimates are derived using three different methods; namely, Lindley method, Tierney and Kadane method, and importance sampling method. The informative Bayes estimates are obtained with respect to the prior where hyperparameters take values as $a = 9$, $b = 6$, $p = 5$, and $q = 10$.

The values for different hyperparameters can be chosen by utilizing the prior information from the past data. Suppose that N number of past data are available from the Kumaraswamy distribution. Let $\hat{\alpha}_j$ and $\hat{\beta}_j$, $j = 1, 2, \dots, N$, denote the corresponding maximum likelihood estimates of unknown parameters α and β. The selection of hyperparameters can be made by equating mean and variance of $\hat{\alpha}_j$ and $\hat{\beta}_j$ with the corresponding mean and variance of the prior distribution. For an instance, let the prior distribution of θ is given by $h(\theta) \propto \theta^{u_0-1}e^{-v_0\theta}$, then we can estimate u_0 and v_0 from the equations $\frac{1}{N}\sum_{j=1}^{N}\hat{\theta}^j = u_0 v_0^{-1}$ and $\frac{1}{N-1}\sum_{j=1}^{N}(\hat{\theta}^j - \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}^i)^2 = u_0 v_0^{-2}$.

We compute MSEs and biases for various combinations of n, T, and r values. All the estimates are presented in Tables 1 and 2. These tables contain four values in each cell for the case of MLEs. In this case, the first value denotes the estimated value of α, and the second value denotes the corresponding MSE value. The last two values denote similar values of β. All other cells contain eight values each for Lindley, Tierney and Kadane, and importance sampling methods. Among these, the first two values denote the Bayes estimates of α obtained using informative and noninformative prior distributions, respectively. The next two values denote the associated MSEs, respectively. The last four values have similar interpretations for the parameter β. From tabulated estimates, we observe that the MLEs of both unknown parameters α and β perform reasonably well compared to the corresponding noninformative Bayes estimates in terms of bias and MSE. However, informative Bayes estimates show superior behavior compared to these two estimates both in terms of bias and MSE.

Table 1. Average estimates and MSE values of MLEs and Bayes estimators of α and β for different choices of T and r when $n = 20$.

		$T = 0.5$				$T = 0.8$			
Method	Parameter	$r = 10$	$r = 12$	$r = 16$	$r = 18$	$r = 10$	$r = 12$	$r = 16$	$r = 18$
MLE	α	2.4494	2.5943	2.2251	2.4264	2.8116	2.3476	2.1980	2.1569
		0.2019	0.7757	0.2620	0.6145	0.9070	0.2705	0.0935	0.0778
	β	3.8177	3.3097	3.1553	3.2971	3.4259	3.2711	3.0940	3.0981
		0.6687	0.2562	0.0546	0.2325	0.3974	0.3254	0.3991	0.5752
Lindley	α	2.2514	2.2141	2.1034	2.2134	2.3457	2.1013	2.0113	2.1003
		2.7675	2.9290	2.6104	2.5849	2.8149	2.3763	1.9848	2.1840
		0.0632	0.0458	0.0106	0.0181	0.1195	0.0102	0.0001	0.0100
		0.5890	0.8630	0.3726	0.3421	0.6641	0.1416	0.0002	0.0338
	β	3.5673	3.2412	3.1471	3.2716	3.3126	3.2134	3.0524	3.0312
		3.8353	3.7513	3.1591	3.1386	3.2726	3.2700	3.0956	3.2079
		0.3218	0.0582	0.0246	0.0135	0.0977	0.0455	0.0012	0.0106
		0.6978	0.5644	0.0253	0.0192	0.0743	0.0729	0.0091	0.0432
Tierney & Kadane	α	2.0682	2.0543	1.9769	2.0113	1.9822	2.0604	2.0180	1.9663
		1.3235	2.4035	2.5829	1.7915	2.4134	2.1608	1.6226	2.2508
		0.0046	0.0029	0.0005	0.0001	0.0003	0.0036	0.0003	0.0011
		0.4575	0.1628	0.3398	0.0434	0.1709	0.0258	0.1423	0.0629
	β	3.0486	3.0370	3.0689	3.0635	2.9686	3.0209	3.0205	2.9424
		2.5617	2.9585	2.8087	3.1517	2.4307	2.7007	2.8577	3.2848
		0.0023	0.0013	0.0047	0.0040	0.0009	0.0004	0.0004	0.0033
		0.1920	0.0017	0.0365	0.0230	0.3240	0.0895	0.0202	0.0811
Importance sampling	α	1.8505	1.8335	1.8309	1.8408	1.8428	1.8122	1.8685	1.8664
		1.7651	1.8193	1.7581	1.6412	1.7401	1.7740	2.1207	2.2134
		0.0877	0.0985	0.0959	0.0930	0.0865	0.0982	0.0830	0.0821
		0.3790	0.4151	0.7821	0.4585	0.4198	0.3174	0.3335	0.4927
	β	2.7526	2.7532	2.7560	2.7175	2.8163	2.7483	2.8291	2.8552
		2.4839	2.6847	2.6960	2.6921	2.7103	2.7150	2.7806	2.7744
		0.3007	0.3125	0.3713	0.3173	0.2821	0.2589	0.2474	0.2407
		0.7376	0.3376	0.3981	0.3836	0.3860	0.3856	0.3570	0.3360

Among Bayes estimates, we find that in the case we estimate α results obtained from Tierney and Kadane method are quite good compared to the other two methods. This holds in general for all tabulated values. The performance of Lindley estimates is also good, as can be seen from the tabulated values. In this regard, we mention that importance sampling estimates show steady behavior in terms of bias and MSE values. A similar behavior is observed in the case we estimate the shape parameter β. We have observed that, in general, MSE values tend to decrease with the increase in effective sample size. In Tables 3–6, we have tabulated confidence intervals of both the unknown parameters α and β for different sampling situations. In these tables, in each cell the first interval belongs to the unknown model parameter α and the second interval belongs to β. These intervals are constructed based on hybrid censored samples for different combinations of n, r, and T values. We compute asymptotic confidence intervals of both the shape parameters using the observed information matrix obtained from the Louis method. In addition, we also present both bootstrap intervals along with informative and noninformative highest posterior density intervals. All the intervals contain the true unknown parameter values. In general, noninformative and Boot-p intervals are wider among all the tabulated intervals. Boot-t intervals compete quite well with asymptotic intervals. Tabulated estimates indicate that the highest posterior intervals obtained using the proper gamma prior distribution perform really well in terms of average length.

6. Illustrative Examples

In this section, we discuss one real life example and simulated data to illustrate the suggested methods of estimation.

Table 2. Average estimates and MSE values of MLEs and Bayes estimators of α and β for different choices of T and r when $n = 40$.

		$T = 0.5$				$T = 0.8$			
Method	Parameter	$r = 20$	$r = 25$	$r = 30$	$r = 35$	$r = 20$	$r = 25$	$r = 30$	$r = 35$
MLE	α	2.4289	2.5089	2.3772	2.3867	2.7736	2.2413	2.1864	2.1448
		0.6812	0.5090	0.3253	0.3754	0.7304	0.1212	0.1499	0.1482
	β	3.3501	3.2050	3.2580	3.2008	3.3420	3.2018	3.0321	2.9648
		0.1972	0.1182	0.1399	0.1192	0.2181	0.1513	0.1547	0.2111
Lindley	α	2.3412	2.2156	2.1856	2.1823	2.3014	2.1532	1.9442	1.9643
		2.7698	2.4672	2.4404	2.4066	2.8013	2.1839	1.9721	1.8830
		0.1164	0.0464	0.0344	0.0332	0.0908	0.0234	0.0031	0.0012
		0.5926	0.2183	0.1939	0.1653	0.6421	0.0338	0.0077	0.0136
	β	3.2314	3.2175	3.1639	3.1105	3.2171	3.1911	3.0167	3.0134
		3.2377	3.2047	3.1791	3.1947	3.2495	3.2035	3.1015	2.9045
		0.0535	0.0473	0.0126	0.0121	0.0471	0.0365	0.0012	0.0001
		0.0565	0.0419	0.0321	0.0379	0.0622	0.0414	0.0103	0.0091
Tierney & Kadane	α	2.0645	2.0519	1.9339	2.0671	1.9766	1.9752	2.0660	1.9853
		2.2561	2.2198	1.8825	1.9192	1.6224	2.0817	1.8808	1.6308
		0.0041	0.0026	0.0043	0.0045	0.0005	0.0006	0.0043	0.0003
		0.0655	0.0483	0.0137	0.0065	0.1425	0.0066	0.0141	0.1362
	β	3.2177	3.2017	3.1553	3.1971	3.3259	3.1711	3.0140	3.0281
		2.5871	3.7442	2.6654	2.4927	3.6021	3.2181	3.2016	2.8300
		0.1687	0.1062	0.0546	0.1025	0.1974	0.1254	0.1391	0.0175
		0.1704	0.5538	0.1119	0.2573	0.3626	0.1478	0.1467	0.0286
Importance sampling	α	1.8718	1.8848	1.917	1.9262	1.9281	1.9484	1.9785	1.9314
		1.3462	1.4741	1.4362	1.4538	1.3347	1.5414	1.7245	1.9177
		0.2520	0.2564	0.2613	0.2572	0.2565	0.2362	0.2097	0.1667
		0.5371	0.4137	0.4880	0.4317	0.5636	0.3169	0.2077	0.1703
	β	2.7385	2.7816	2.7626	2.8229	2.7650	2.8394	2.9838	2.9782
		2.6964	2.6848	2.7297	2.7808	2.7369	2.7426	2.8533	2.8916
		0.1005	0.1039	0.1152	0.1157	0.1700	0.1684	0.1420	0.1383
		0.1818	0.1645	0.1285	0.1288	0.1762	0.1720	0.1592	0.1470

Table 3. Interval estimates of α and β for different choices of r when $n = 20$ and $T = 0.5$.

r	Boot-p	Boot-t	Asymptotic	HPD	Non-informative
10	(1.1689, 3.7801)	(1.1918, 3.7585)	(1.2554, 3.7377)	(1.2943, 3.7107)	(1.1189, 3.7631)
	(2.7792, 4.7895)	(2.7810, 4.6904)	(2.7495, 4.6818)	(2.8493, 4.5768)	(1.8664, 4.5300)
12	(0.7356, 3.5705)	(1.0065, 3.4519)	(1.1496, 3.4355)	(1.2468, 3.4213)	(1.1165, 3.6743)
	(2.8348, 4.9858)	(2.8681, 4.9506)	(2.8408, 4.9313)	(2.8392, 4.9067)	(1.8960, 4.9562)
16	(1.1123, 3.5510)	(1.0828, 2.5367)	(1.1464, 2.5061)	(1.1672, 2.4961)	(1.1176, 3.5643)
	(2.4386, 4.2966)	(2.4584, 4.0582)	(2.5007, 4.0790)	(2.5912, 4.0878)	(1.9530, 4.2107)
18	(1.1031, 3.0225)	(1.0665, 2.9679)	(1.1496, 2.9855)	(1.1268, 2.9213)	(0.9599, 2.9141)
	(2.7644, 4.0736)	(2.4481, 4.0162)	(2.7808, 4.1183)	(2.8092, 4.0356)	(1.8449, 4.5300)

Table 4. Interval estimates of α and β for different choices of r when $n = 20$ and $T = 0.8$.

r	Boot-p	Boot-t	Asymptotic	HPD	Non-informative
10	(1.4109, 3.3811)	(1.3618, 3.2889)	(1.4051, 3.3377)	(1.4894, 3.3087)	(1.0837, 3.4307)
	(2.7815, 4.7778)	(2.7348, 4.6904)	(2.8109, 4.6818)	(2.8493, 4.5768)	(1.9325, 4,5701)
12	(1.2959, 3.0905)	(1.2265, 2.9679)	(1.2064, 2.9415)	(1.2968, 2.9213)	(1.1110, 2.8621)
	(2.5966, 4.5711)	(2.4481, 4.4923)	(2.5608, 4.5183)	(2.5339, 4.4956)	(1.9785, 4.6024)
16	(1.3883, 3.7217)	(1.3281, 3.6721)	(1.3464, 3.6813)	(1.3672, 3.5961)	(1.4964, 3.9175)
	(2.7829, 4.8684)	(2.7584, 4.7582)	(2.7071, 4.7390)	(2.6912, 4.6878)	(1.0791, 3.6510)
18	(1.1120, 2.8615)	(1.0965, 2.8379)	(1.0496, 2.8155)	(1.1268, 2.7823)	(1.1540, 2.9906)
	(2.7495, 4.1025)	(2.6813, 4.0662)	(2.6608, 4.0183)	(2.6933, 3.9357)	(1.1224, 3.7524)

Table 5. Interval estimates of α and β for different choices of r when $n = 40$ and $T = 0.5$.

r	Boot-p	Boot-t	Asymptotic	HPD	Non-informative
20	(1.5904, 3.1658)	(1.5618, 3.1189)	(1.5541, 3.0778)	(1.6143, 3.0237)	(1.0925, 2.7006)
	(2.9580, 4.3810)	(2.9348, 4.3046)	(2.8935, 4.2988)	(2.9061, 4.2768)	(1.9758, 4.2727)
25	(1.8565, 2.1926)	(1.8065, 2.1679)	(1.9496, 2.1155)	(1.9618, 2.0213)	(1.2321, 2.8044)
	(2.8816, 4.0805)	(2.8163, 4.0652)	(2.8608, 3.9833)	(2.8339, 3.9357)	(2.0385, 3.3156)
30	(1.7539, 2.1626)	(1.7248, 2.1387)	(1.7614, 2.1061)	(1.7712, 2.0961)	(1.1911, 2.8000)
	(2.8776, 3.8905)	(2.8488, 3.7582)	(2.8571, 3.7390)	(2.8929, 3.6878)	(1.9970, 3.2735)
35	(1.9310, 2.3619)	(1.8925, 2.3179)	(1.9494, 2.3055)	(1.9468, 2.2513)	(1.1804, 2.9217)
	(2.8288, 3.6886)	(2.8133, 3.6462)	(2.8608, 3.6183)	(2.8339, 3.5675)	(1.9660, 3.2735)

Table 6. Interval estimates of α and β for different choices of r when $n = 40$ and $T = 0.8$.

r	Boot-p	Boot-t	Asymptotic	HPD	Non-informative
20	(1.3762, 2.5261)	(1.3982, 2.5089)	(1.3554, 2.4777)	(1.3093, 2.4097)	(1.0940, 2.7087)
	(2.8158, 4.2762)	(2.8348, 4.2204)	(2.8309, 4.2885)	(2.8093, 4.1768)	(1.7532, 3.2509)
25	(1.8721, 2.2594)	(1.9065, 2.2279)	(1.9494, 2.2155)	(1.9468, 2.1937)	(1.2969, 2.9270)
	(2.6138, 3.9778)	(2.6113, 3.9062)	(2.6208, 3.9183)	(2.6339, 3.8967)	(2.0223, 3.3049)
30	(1.9213, 2.3086)	(1.9528, 2.2667)	(1.9464, 2.2066)	(1.9667, 2.1961)	(1.5158, 2.1423)
	(2.5533, 3.7762)	(2.5584, 3.7582)	(2.5714, 3.7708)	(2.5912, 3.6878)	(1.1194, 3.3894)
35	(1.8879, 2.2372)	(1.9065, 2.2279)	(1.9496, 2.2155)	(1.9681, 2.1983)	(1.7360, 2.3963)
	(2.4218, 3.4080)	(2.4481, 3.3629)	(2.4308, 3.2833)	(2.4339, 3.2357)	(1.1948, 3.4667)

Example 1. (**Real Data**) In this example, we analyze a real data set which represents the monthly water capacity from the Shasta reservoir in California, USA, and data are recorded for the month of February from 1991 to 2010. For further details about the data, one may visit http://cdec.water.ca.gov/reservoir_map.html. The maximum capacity of the reservoir is 4552000 AF. The data points are listed below as follows.

```
0.338936, 0.431915, 0.759932, 0.724626, 0.757583, 0.811556,
0.785339, 0.783660, 0.815627, 0.847413, 0.768007, 0.843485,
0.787408, 0.849868, 0.695970, 0.842316, 0.828689, 0.580194,
0.430681, 0.742563
```

A goodness of fit test is conducted to verify whether a Kumaraswamy distribution can be used to make adequate inference from the considered data set. We fit Kumaraswamy distribution to this data set along with three other distributions; namely, generalized exponential (GE), Poisson exponential (PE), and exponential (EX) distributions. The MLEs of unknown parameters of all competing models are reported in Table 7 along with corresponding estimates of the negative log-likelihood criterion (NLC), Akaike's information criterion (AIC), the associated second order criterion (AICc), and Bayesian information criterion (BIC). From these estimates, we conclude that the Kumaraswamy distribution fits the data set well compared to the other models. Thus, we analyze the given data set using this distribution under hybrid censoring. We generate hybrid censored samples for arbitrarily selected values of T, such as 0.78, 0.8, and 0.84 when $r = 15$. We obtain the MLEs of unknown parameters α and β using the EM algorithm. Since in the case of real data, usually, no prior information is available, we compute

Table 7. Goodness of fit tests for different distributions for real data.

	$\hat{\alpha}$	$\hat{\beta}$	Negative log-likelihood	AIC	AICc	BIC
$K(\alpha, \beta)$	2.7084	0.9917	0.0321	4.0643	4.7702	6.0558
GE	7.9999	3.4697	3.1209	10.2419	10.9478	12.2334
PED	3.1099	5.9998	5.0573	14.1147	14.8206	16.1062
EX	1.0495		14.2748	30.5497	30.7719	31.5455

Table 8. Estimates of α and β for different choices of T for real data.

Method	$T = 0.78$	$T = 0.8$	$T = 0.84$
MLE	2.2604	2.4022	2.7084
	3.5631	3.7802	3.8917
Lindley	1.7915	1.9335	1.9847
	2.6128	2.9095	2.8366
Tierney & Kadane	1.9778	1.8952	2.0281
	3.2713	3.3877	3.4650
Importance Sampling	1.9843	1.9984	2.0045
	3.4135	3.4171	3.4185

Bayes estimates using a noninformative prior in which case hyperparameters a, b, p, and q are assigned a value as zero. In Table 8, we have reported various estimates of α and β. The upper entry in each cell denotes the estimated values of α, and the lower entry denotes estimated values of β. In Table 9, we present the corresponding 95% intervals of the unknown parameters. Specifically Bootstrap, asymptotic, and the noninformative HPD intervals are tabulated. In general, Boot-p intervals are wider compared to the other intervals. Boot-t intervals compete well with both asymptotic and HPD intervals. We further observe that the noninformative HPD intervals of α and β are marginally better than the other intervals in terms of interval lengths. Overall, given the computation simplifications, we suggest using Boot-t intervals.

Example 2. (**Simulated Data**) We now analyze a simulated data set of size 20 drawn from the Kumaraswamy distribution when $\alpha = 2$ and $\beta = 3$. The simulated values are as follows.
`0.20446759, 0.14697062, 0.69175148, 0.07079152, 0.56719003, 0.65389413, 0.41438717, 0.74917045, 0.19148513, 0.24160137, 0.17997061, 0.38487179,0.44530844, 0.35384861, 0.25371203, 0.35644221, 0.46396004, 0.54320944,0.40983151, 0.45815914`

We compute the estimates of α and β with different hybrid censored samples obtained using various combinations of r and T. We arbitrarily choose r as 15 and 18, and then two different sets of values for T, like 0.45, 0.65, 0.78 and 0.5, 0.56, 0.7, are taken into consideration. Bayes estimates are obtained against the squared error loss function using the gamma prior with hyperparameters chosen as $a = 10$, $b = 5$, $p = 9$, and $q = 3$. We also compute noninformative Bayes estimates for both unknown parameters. In Table 10, we present the estimated values of α and β for different sampling combinations. In the first cell, respective MLEs of α and β are presented. In all other cells, four estimates are given. Among these, the first two values, respectively, denote informative and noninformative Bayes estimates of α. Similarly, the last two values indicate corresponding estimates of β. It is seen that Bayes estimates of parameters α and β are quite close to the unknown parameters compared to the

Table 9. Interval estimates of α and β for different choices of T for real data.

T	Boot-p	Boot-t	Asymptotic	Noninformative
0.78	(1.6648, 2.3200)	(1.6821, 2.3589)	(1.7552, 2.3177)	(1.8250, 2.2780)
	(3.0963, 4.1333)	(3.1248, 4.1646)	(3.1095, 4.1188)	(3.1336, 3.9954)
0.8	(1.7641, 2.4619)	(1.8065, 2.4791)	(1.8496, 2.4551)	(1.8651, 2.4160)
	(3.0646, 3.9936)	(3.0813, 3.9762)	(3.1082, 3.9583)	(3.1339, 3.9356)
0.84	(1.8177, 2.6313)	(1.8218, 2.6147)	(1.8844, 2.5688)	(1.9002, 2.5564)
	(3.0389, 4.1206)	(3.0848, 4.1082)	(3.1071, 4.0885)	(3.1492, 3.9878)

Table 10. Estimates of α and β under simulated data for different choices of T and r.

	$r = 15$			$r = 18$		
Method	$T = 0.45$	$T = 0.65$	$T = 0.78$	$T = 0.5$	$T = 0.56$	$T = 0.7$
MLE	1.1825	1.3189	1.3189	1.2670	1.2924	1.4726
	1.7585	2.2449	2.2449	2.0961	2.1704	2.6433
Lindley	1.8632	1.7597	1.7597	1.9353	2.0398	1.8864
	2.3465	2.3267	2.3267	2.3104	2.3014	1.7856
	3.2433	3.1501	3.1501	3.7841	4.1324	3.5732
	3.4202	3.1792	3.1792	3.7109	4.2357	4.017
Tierney & Kadane	1.8244	1.8332	1.8332	1.8267	1.8218	1.8225
	2.2939	2.3214	2.321	2.3341	2.3039	1.7502
	3.2180	3.4271	3.4282	3.3186	3.2563	3.5661
	3.9925	3.7162	3.7162	3.3284	3.7179	3.5777
Importance sampling	1.5320	1.7793	1.8209	1.7200	1.7404	1.8704
	1.4609	2.5291	2.5113	2.2963	2.3073	2.5398
	2.8312	2.9490	2.9490	3.1886	3.1366	3.0961
	2.6568	2.6912	2.8735	3.2788	3.2857	3.3589

Table 11. Interval estimates of α and β under simulated data for different choices of T when $r = 15$ and 18.

T	Boot-p	Boot-t	Asymptotic	HPD	Non-informative
0.45	(0.9941, 2.8677)	(0.9490, 2.6858)	(1.7728, 2.6255)	(1.0105, 2.4599)	(1.6108, 2.9178)
	(1.1008, 3.7950)	(1.0576, 3.7386)	(0.9590, 3.5289)	(1.9380, 3.4952)	(0.9629, 3.3707)
0.65	(0.9053, 2.7258)	(0.8858, 2.5719)	(0.7475, 2.4791)	(1.2416, 2.7056)	(1.9893, 3.2906)
	(1.1863, 3.4423)	(1.1488, 3.7681)	(0.9769, 3.5821)	(1.9880, 3.3445)	(1.0187, 3.3795)
0.78	(1.0159, 2.8736)	(0.9258, 2.6019)	(0.8349, 2.6124)	(1.2209, 2.8309)	(1.9907, 3.2709)
	(1.1694, 3.9784)	(1.1398, 3.7589)	(1.0137, 3.6056)	(1.9766, 3.1551)	(1.0232, 3.3756)
0.5	(1.1565, 2.7013)	(1.1753, 2.6984)	(1.1786, 2.6412)	(1.1985, 2.6348)	(1.8270, 3.0393)
	(3.1133, 3.7121)	(2.8915, 3.6990)	(2.9713, 3.6582)	(2.9807, 3.4447)	(0.9973, 3.3772)
0.56	(1.1177, 2.7632)	(1.1637, 2.7518)	(1.1813, 2.7132)	(1.2054, 2.6872)	(1.8018, 3.0696)
	(2.8916, 3.6916)	(2.9389, 3.6796)	(2.9578, 3.6396)	(2.9754, 3.4243)	(0.9858, 3.3896)
0.7	(1.2459, 2.8810)	(1.2590, 2.8317)	(1.3140, 2.8045)	(1.3206, 2.7880)	(2.0405, 3.3849)
	(2.7041, 3.6913)	(2.7613, 3.6730)	(2.8076, 3.6590)	(2.8912, 3.4538)	(1.0601, 3.4601)

MLEs. Importance sampling estimates indicate steady behavior for different sampling situations. In Table 11, we present the 95% bootstrap, asymptotic, noninformative, and informative HPD confidence intervals for unknown parameters. In this case, we find that informative HPD intervals are quite superior than all the reported intervals. This holds true for all discussed sampling schemes.

7. Conclusions

In this article, we have considered estimation of unknown parameters α and β of a Kumaraswamy distribution when the data are collected under a hybrid censoring scheme. We have derived the MLEs of these unknown parameters. Bayes estimates are computed by using the Lindley method, Tierney and Kadane method, and importance sampling under squared error loss. The asymptotic confidence intervals and HPD intervals are also obtained. Two numerical examples are also analyzed using the proposed methods of estimation.

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