Task 5

A company will face the following cash requirements in the next eight quarters (positive entries represent cash needs while negative entries represent cash surpluses).

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
100	500	100	-600	-500	200	600	-900

The company has three borrowing possibilities.

- a 2-year loan available at the beginning of Q1, with a 1% interest per quarter.
- The other two borrowing opportunities are available at the beginning of every quarter: a 6-month loan with a 1.8% interest per quarter, and a quarterly loan with a 2.5% interest for the quarter.

Any surplus can be invested at a 0.5% interest per quarter. Formulate a linear program that maximises the wealth of the company at the beginning of Q9. Solve the problem using linprog and interpret the solution.

Approach: We first need to define the decision variables and, then, present the model. Consider the following variables:

a_1: the amount that we borrow at the beginning of Q1 for the 2-year loan

x_i: the amount that we borrow at the beginning of each quarter for the 6-month load

y_i: the amount that we borrow at the beginning of each quarter for the quarterly loan

z_i: the amount that we invest at each quarter

v: the wealth at the beginning of Q9

Observe that there may be different modelling approaches with respect to the timing.

Let us assume that the wealth at the beginning of Q9 is actually calculated as the wealth at the end of Q8 after repaying all loans and collecting any investment interest. This implies that during Q8 we cannot borrow or invest. **Note also that it is always beneficial to invest what remains in each quarter**.

Remark: An alternative modelling choice would be to calculate the wealth at the beginning of Q9 by allowing borrowing or investing at Q8 and repaying/collecting at the beginning of Q9. This would slightly modify the following LP.

We obtain the following LP

Maximize v

Subject to

$$a_1 + x_1 + y_1 - z_1 = 100$$
$$-0.01a_1 - 0.018x_1 - 1.025y_1 + 1.005z_1 + x_2 + y_2 - z_2 = 500$$
$$-0.01a_1 - 1.018x_1 - 0.018x_2 - 1.025y_2 + 1.005z_2 + x_3 + y_3 - z_3 = 100$$
$$-0.01a_1 - 1.018x_2 - 0.018x_3 - 1.025y_3 + 1.005z_3 + x_4 + y_4 - z_4 = -600$$
$$-0.01a_1 - 1.018x_3 - 0.018x_4 - 1.025y_4 + 1.005z_4 + x_5 + y_5 - z_5 = -500$$
$$-0.01a_1 - 1.018x_4 - 0.018x_5 - 1.025y_5 + 1.005z_5 + x_6 + y_6 - z_6 = 200$$
$$-0.01a_1 - 1.018x_5 - 0.018x_6 - 1.025y_6 + 1.005z_6 + y_7 - z_7 = 600$$

$$-1.01a_1 - 1.018x_6 - 1.025y_7 + 1.005z_7 - v \qquad = -900$$

The first constraint captures the fact that the amounts we borrow from the 2-year loan, the 6-month loan and the 3-month loan minus the amount we invest should equal 100.

The second constraint captures the fact that we pay interest for any borrowing during Q1 as well as we fully repay the 3-month loan during Q1 (hence, the $-0.01a_1$, $-0.018x_1$ and $-1.025y_1$ terms). We also collect interest from the investment in Q1 (hence, the $1.005z_1$ term) and we may borrow again for a 6-month or a 3-month period (hence, the x_2 and y_2 terms). Finally, we can invest again (the z_2 term).

The remaining constraints follow a similar logic. Note that in Q8, we make the assumption that the 2-year loan is fully repaid at the end of Q8 (hence, the $-1.01a_1$ term) and recall that no more borrowing is possible. Variable **v** captures the wealth and is also in the objective function.

Task 6

Consider a restaurant that is open seven days a week. Based on past experience, the number of workers needed on a particular day is given as follows:

Mon	Tue	Wed	Thu	Fri	Sat	Sun
14	13	15	16	19	18	11

Every worker works five days in a week and has two days off in the following pattern: three days work, one day off, two days work, one day off. So, there are workers working on Mon-Tue-Wed-Fri-Sat, other workers on Tue-Wed-Thu-Sat-Sun, etc. How can we minimize the number of workers that staff the restaurant?

Approach: Again, we need to begin by selecting the decision variables involved. Let's follow the following notation: x_1 denotes number of workers on the shift with days off on Sunday and Wednesday, x_2 denotes the number of workers on the shift with days off on Monday and Thursday, and so on, that is x_7 denotes the number of workers on the shift with days off on Tuesdays and Saturdays.

So, we need to ask ourselves: **Which workers work on Mondays?** Based on how we named the variables, these are the workers in x_1, x_3, x_4, x_5, and x_7. Note that x_2 corresponds to the shift with days off on Mondays and Thursdays, while x_6 corresponds to the shift with days off on Mondays and Fridays.

Therefore, we get constraints such as

$$x_1 + x_3 + x_4 + x_5 + x_7 \geq 14$$

We follow a similar reasoning for all days in the week to get our set of constraints.