A. Solid to Surface mesh:

We get centroidal deformation gradient, F, and Cauchy stress, S, in global frame from Abaqus for each tetrahedron of a solid mesh. Then, for each face of the mesh, we take the average F and S of the tets attached to it. Each internal face has two tets and each boundary face has one tet attached to it.

Then, we create the surface mesh from this solid mesh with all boundary triangles. Each triangle of this surface mesh has F and S values.

Steps of First Piola calculation at vertex in English:

B. How to get First Piola, P, on each **vertex's** tangent plane from deformation gradient, F, and Cauchy stress, S, of each triangle:

1. Deformation gradient (F) and Cauchy stress (S) of each triangle in the global coordinate system.

2. First, we subdivide[1] the mesh and copy each original triangle's F and S to its subdivided ones.

3. After that, we calculate First Piola, P, for each Triangle using following equation:

First Piola, $P = J * S * (F^{-1})^T$, where $J = det(F)$

4. Now that we have P for each triangle of the mesh,

5. To get P at each vertex, V,

6. We compute area weighted average of P of all triangles that are incident to V.

7. P is also in global basis.

7. Then, we project (details in D) P at each vertex onto vertex's tangent plane (details in C) to get surface tensor there.

C. How to get tangent plane at vertices of our surface mesh:

1. To get surface normal of each vertex, V,
2. We take average of the normals of all triangles incident to V.
3. Tangent plane at V is the plane that goes through V and perpendicular to the vertex normal at V.
4. To pick a X and Y-axis on the tangent plane,
5. Among the edges that are incident to V,
6. We find the one that's closest to V's tangent plane and
7. We take this edge's corresponding line on the tangent plane. This our X-axis.
8. We get Y-axis by taking cross product of this X-axis and vertex normal.

D. Projection:

1. To project P at a vertex V with normal N, onto V's tangent plane(X, Y),
   a. We form a matrix M = [X ; Y ; N]
   b. Then, we calculate $P_{proj} = M^T * P * M$ and take upper left 2x2 matrix of P_proj.

[1]we need to subdivide to make sure none of the triangle has more than one angle deficit vertex.

Steps of First Piola calculation at vertex with mathematical notaions:

B. How to get First Piola, P, on each **vertex's** tangent plane from deformation gradient, F, and Cauchy stress, S, of each triangle:

1. Suppose each triangle, T, of the surface triangular mesh, M, has deformation gradient $F(T)$ and Cauchy stress $S(T)$. Both F and S are 3x3 tensors.
2. We subdivide our mesh M to get mesh M' by subdividing each T of M into four triangles T_i' for $i = 1,2,3,4$
3. F and S values for subdivided triangles are same as the original triangles as they are same for the whole triangle

$$F(T_i') = F(T), \qquad i = 1,2,3,4$$
$$S(T_i') = S(T), \qquad i = 1,2,3,4$$

1. Now, for each triangle, T', of subdivided mesh, M'
2. We calculate First Piola, $P(T')$ as

$$P(T') = J * S(T') * (F(T'))^{-T}, \qquad \text{where } J = det(F)$$

3. Then, for each vertex, V', of subdivided mesh M',
4. We extrapolate First Piola, $P(V')$ at V' as below,

$$P(V') = \frac{\sum P(T'_{inc}), \quad for\ all\ triangles\ T'_{inc}\ that\ are\ incident\ to\ V'}{A},$$
$$where\ A = sum\ of\ the\ areas\ of\ all\ triangles\ T'_{inc}\ \text{that are incident to } V'$$

5. At each V', $P(V')$ is a $3x3$ tensor in global basis.
6. To get surface tensor at V', we project (details in D) $P(V')$ on to the tangent plane (details in C) at V'. [Now, we are taking projection onto vertex's tangent plane to go from 3D to 2D but this might change]

C. How to get tangent plane at vertices of our surface mesh:

1. For each triangle $T'(v0, v1, v2)$ of mesh M'
2. normal of T', $n(T')$, is calculated as

$$\overrightarrow{n(T')} = \overrightarrow{v_1 v_0}\ X\ \overrightarrow{v_1 v_2}, \qquad X\ denotes\ the\ cross\ product$$

3. For each vertex V' of mesh M',
4. Surface normal, $\overrightarrow{n(V')}$, at V' is calculated as

$$\overrightarrow{n(V')} = \frac{\sum \overrightarrow{n(T'_{inc})}, \quad for\ all\ triangles\ T'_{inc}\ that\ are\ incident\ to\ V'}{\#\ of\ triangles\ that\ are\ incident\ to\ V'}$$

5. Now, suppose E_{inc} is the set of edges that are incident to V'.
6. Find the edge $e(V', V'')$ in E_{inc} that has the smallest dot product with $n(V')$ [closest to the tangent plane[2] at V']
7. To get X-direction, get e's projection on the tangent plane at V'.

$$\vec{X} = \overrightarrow{V'V''} - dot(\overrightarrow{V'V''}, n(V')) * n(V')$$

8. Y-direction is then,

$$\vec{Y} = \overrightarrow{n(V')}\ X\ \vec{X}, \quad X\ denotes\ cross\ product$$

D. Projection:

2. To project $P(V')$ at a vertex V' with normal $\overrightarrow{n(V')}$ and tangent plane ($\vec{X}$, $\vec{Y}$)
   a. We form a matrix $M = [\vec{X}\ ; \vec{Y}\ ; \overrightarrow{n(V')}]$
   b. Then, we calculate

$$P_{proj} = M^T * P * M$$

   c. upper left 2x2 matrix of P_proj is our surface tensor at V'