PHY 781: Final Project

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Chapter 1

Project 1

Define the lattice action

$$S_E^l = \sum_x \{\frac{1}{2}\chi_x^2 - \frac{\kappa}{2}\sum_\mu \chi_x\chi_{x+\hat{\mu}} + g\chi_x^4\}$$

$$\kappa^{-1} := 2d + \alpha$$
$$\alpha := (m_0 a)^2$$

$$\sigma = \frac{1}{L^d}\sum_x \langle\chi_x^2\rangle$$
$$\chi = \frac{1}{L^d}\sum_{x,y} \langle\chi_x\chi_y\rangle$$
$$F = \frac{1}{L^d}\sum_{x,y} \langle\chi_x\chi_y\rangle\cos(2\pi\Delta\tau/L)$$

and furthermore

$$M(L) = \frac{2\sin(\pi/L)}{\sqrt{(\chi/F) - 1}}$$

By definition,

$$\langle\varphi(x)\varphi(y)\rangle = \frac{1}{Z}\int [d\varphi]e^{-S_E(\varphi)}\varphi(x)\varphi(y)$$

where

$$Z = \int [d\varphi]e^{-S_E(\varphi)}$$

1.1 Exact Computation

1.1.1 $g = 0$ and $\kappa \neq 0$

We can construct S as a matrix-vector product in terms of M

$$S_E^l = \frac{1}{2}\chi_i M_{ij}\chi_j$$
$$M_{ij} = -\kappa \sum_\mu (\delta_{i+\mu,j} + \delta_{i-\mu,j}) + \delta_{ij}$$

where $i + \mu$ translates i to all forward neighbors in d dimensions i.e. $\mu = (0, \cdots, a, \cdots 0)$. Hence, $\sum_y M(x-y)M^{-1}(y-z) = \delta_{x,z}$ implies that

$$\sum_y [-\kappa \sum_\mu (\delta_{x+\mu,y} + \delta_{x-\mu,y}) + \delta_{xy}]M^{-1}(y-z) = \delta_{x,z}$$

In momentum space, we have

$$\begin{aligned}
\sum_y [-\kappa \sum_\mu (\delta_{x+\mu,y} + \delta_{x-\mu,y}) + \delta_{xy}]M^{-1}(y-z) &\to \sum_y \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} e^{ik\cdot(y-z)} \tilde{M}^{-1}(k)[-\kappa \sum_\mu (\delta_{x+\mu,y} + \delta_{x-\mu,y}) + \delta_{xy}] \\
&= \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} \tilde{M}^{-1}(k)[-\kappa \sum_\mu (e^{ik\cdot(x+\mu-z)} + e^{ik\cdot(x-\mu-z)}) + e^{ik\cdot(x-z)}] \\
&= \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} \tilde{M}^{-1}(k)[-\kappa \sum_\mu (e^{ik\cdot\mu} + e^{-ik\cdot\mu}) + 1]e^{ik\cdot(x-z)} \\
\delta_{x,z} &\to \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} e^{ik\cdot(x-z)}
\end{aligned}$$

Therefore, we can conclude that

$$\tilde{M}^{-1}(k)[-\kappa \sum_\mu (e^{ik\cdot\mu} + e^{-ik\cdot\mu}) + 1] = 1$$
$$\begin{aligned}
\tilde{M}^{-1}(k) &= \frac{1}{1 - \kappa \sum_\mu (e^{ik\cdot\mu} + e^{-ik\cdot\mu})} \\
&= \frac{1}{1 - 2\kappa \sum_\mu \cos(k\cdot\mu)}
\end{aligned}$$

So, we can write that the momentum in the μ direction as $k_\mu := k \cdot \mu$. Note that momentum is restricted to the Brillouin zone: $|k_\mu| \leq \pi$. Because the lattice is discretized and each dimension is of size L, k_μ takes on values

$$k_\mu = \frac{2\pi n_\mu}{L}, \qquad n_\mu \in \{0, \cdots, L-1\}$$

which gives

$$\tilde{M}^{-1}(k) = \frac{1}{1 - 2\kappa \sum_\mu \cos\left(\frac{2\pi n_\mu}{L}\right)}$$

Now, note that

$$\begin{aligned}
\sum_x \chi_x^2 &= \sum_x \langle\chi|x\rangle\langle x|\chi\rangle \\
&= \sum_k \langle\chi|k\rangle\langle k|\chi\rangle \\
\sigma &= \frac{1}{L^d}\sum_k \langle\chi_k^2\rangle \\
&= \sum_{n_1,\cdots,n_d} \frac{1}{1-2\kappa\sum_\mu \cos\left(\frac{2\pi n_\mu}{L}\right)}
\end{aligned}$$

Furthermore, since $|\langle k=0|x\rangle|^2 = \frac{e^0}{L^d}$

$$\begin{aligned}
\sum_{x,y} \chi_x\chi_y &= \frac{1}{L^d}\sum_{x,y}\langle\chi|x\rangle\langle y|\chi\rangle \\
&= \sum_{x,y,k,k'} \langle\chi|k\rangle\langle k|x\rangle\langle x|0\rangle\langle 0|y\rangle\langle y|k'\rangle\langle k'|\chi\rangle \\
&= |\langle k=0|\chi\rangle|^2 \\
\chi &= \langle\chi^2_{k=(0,\cdots,0)}\rangle \\
&= \frac{1}{1-2\kappa d}
\end{aligned}$$

Finally, using $\cos(2\pi(x_0-y_0)/L) = (e^{2\pi i(x_0-y_0)/L} + e^{-2\pi i(x_0-y_0)/L})/2$ where x_0 is the component of x along the time dimension,

$$\sum_{x_0,y_0} \chi_x\chi_y e^{2\pi i(x_0-y_0)/L} = \sum_{x_0,y_0}\langle\chi|x_0\rangle\langle x_0|k=1\rangle\langle k=1|y_0\rangle\langle y_0|\chi\rangle$$

and the same follows for complex conjugate $\sum_{x_0,y_0}\chi_x\chi_y e^{-2\pi i(x_0-y_0)/L}$. Hence, using this result and χ above,

$$\begin{aligned}
F &= \langle\chi^2_{k=(1,0,\cdots,0)}\rangle \\
&= \frac{1}{1-2\kappa[\cos\left(\frac{2\pi}{L}\right)+d-1]}
\end{aligned}$$

Using these derived relations, we find the following results for $L=16, g=0, \alpha=0.25$,

d	σ	χ	F	$M(L)$
2	1.6021	17.0000	10.5659	0.5000
3	1.3198	25.0000	15.5380	0.5000
4	1.1995	33.0000	20.5101	0.5000

Table 1.1: Exact Computation Results for $g=0$ and $\kappa \neq 0$

1.1.2 $g \neq 0$ and $\kappa = 0$

1.2 Monte Carlo for $g = 0$ or $\kappa = 0$

1.2.1 $g = 0$ and $\kappa \neq 0$

We used the Monte Carlo algorithm with 10,000 spin and regular updates. We found the following results for $L = 16, g = 0, \alpha = 0.25$.

d	σ	χ	F	$M(L)$
2	1.5984	16.9759	10.5659	0.5104
3	1.3201	24.5100	15.3275	0.5041
4	1.1995	32.8455	20.3969	0.4994

Table 1.2: Monte Carlo Results for $g = 0$ and $\kappa \neq 0$

which aligns closely with Table 1.1.1

1.2.2 $g \neq 0$ and $\kappa = 0$

We used the Monte Carlo algorithm with 10,000 spin and regular updates. We found the following results for $L = 16, g = 0.1, \kappa = 0, \alpha = 0.25$.

d	σ	χ	F	$M(L)$
2	0.6150	0.6151	0.6151	∞
3	0.6156	0.6241	0.6241	∞
4	0.6155	0.6211	0.6211	∞

Table 1.3: Monte Carlo Results for $g \neq 0$ and $\kappa = 0$

which aligns closely with Table **??**

1.3 Monte Carlo for $g \neq 0$ and $\kappa \neq 0$

We used the Monte Carlo algorithm with 10,000 spin and regular updates. We found the following results for $L = 8, g = 0.1, \alpha = -1.5$.

d	σ	χ	F	$M(L)$
2	0.8295	6.0336	2.4549	0.6339
3	0.6855	2.8406	2.0802	1.2659
4	0.6562	2.3761	1.9910	1.7403

Table 1.4: Monte Carlo Results for $g \neq 0$ and $\kappa = 0$

Chapter 2

Project 2

2.1 Computing the Physical Mass

2.2 Computing the Lattice Spacing

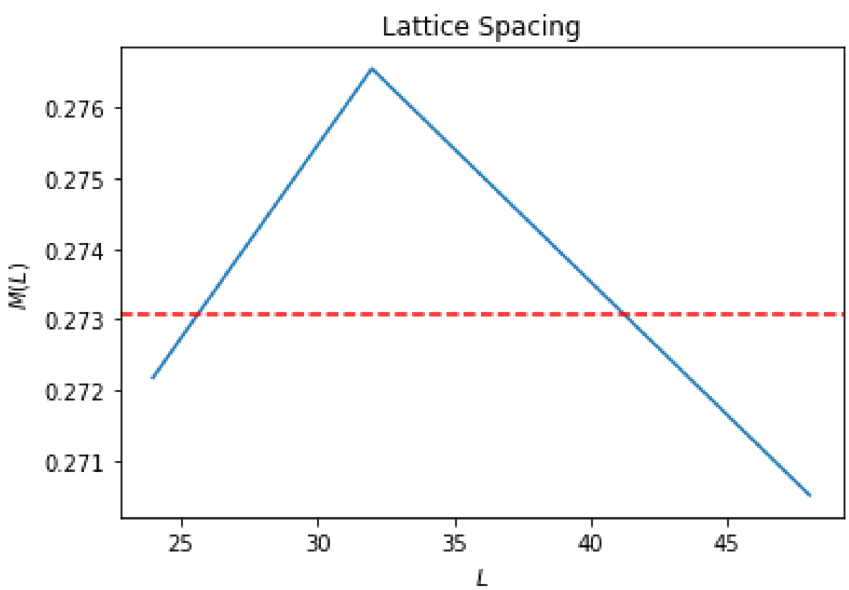

Figure 2.1: We see that $M(L)$ approaches a constant near 0.273 as L increases as seen by plotting $L = 24, 32, 48$.

2.3 Renormalization

2.3.1 $d = 2$

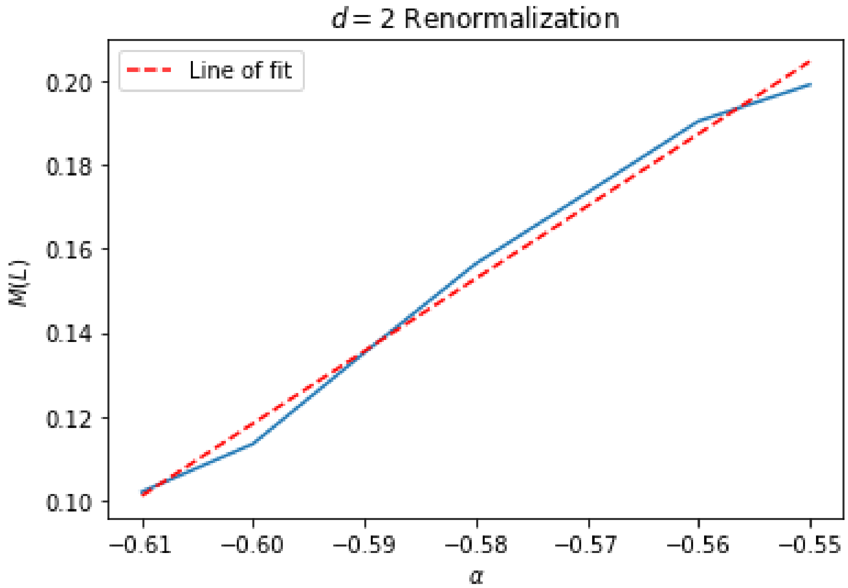

Figure 2.2: $M_{phys}a$ can be given in terms of the tuning parameter roughly by $1.724(\alpha + 0.669)$

2.3.2 $d = 3$

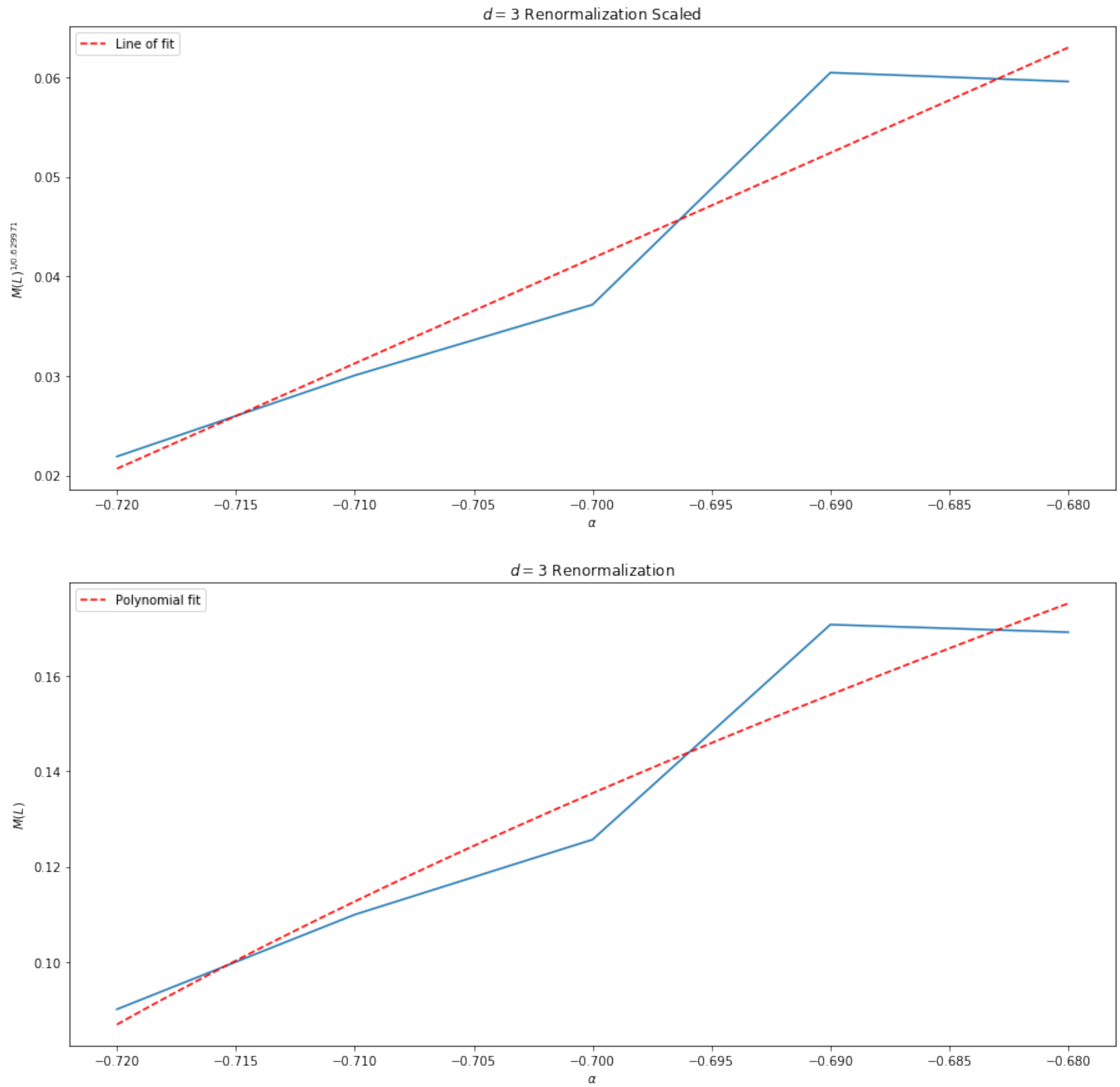

Figure 2.3: $M_{phys}a$ can be given in terms of the tuning parameter roughly by $1.036(\alpha + 0.740)^{0.629971}$

2.3.3 $d = 4$

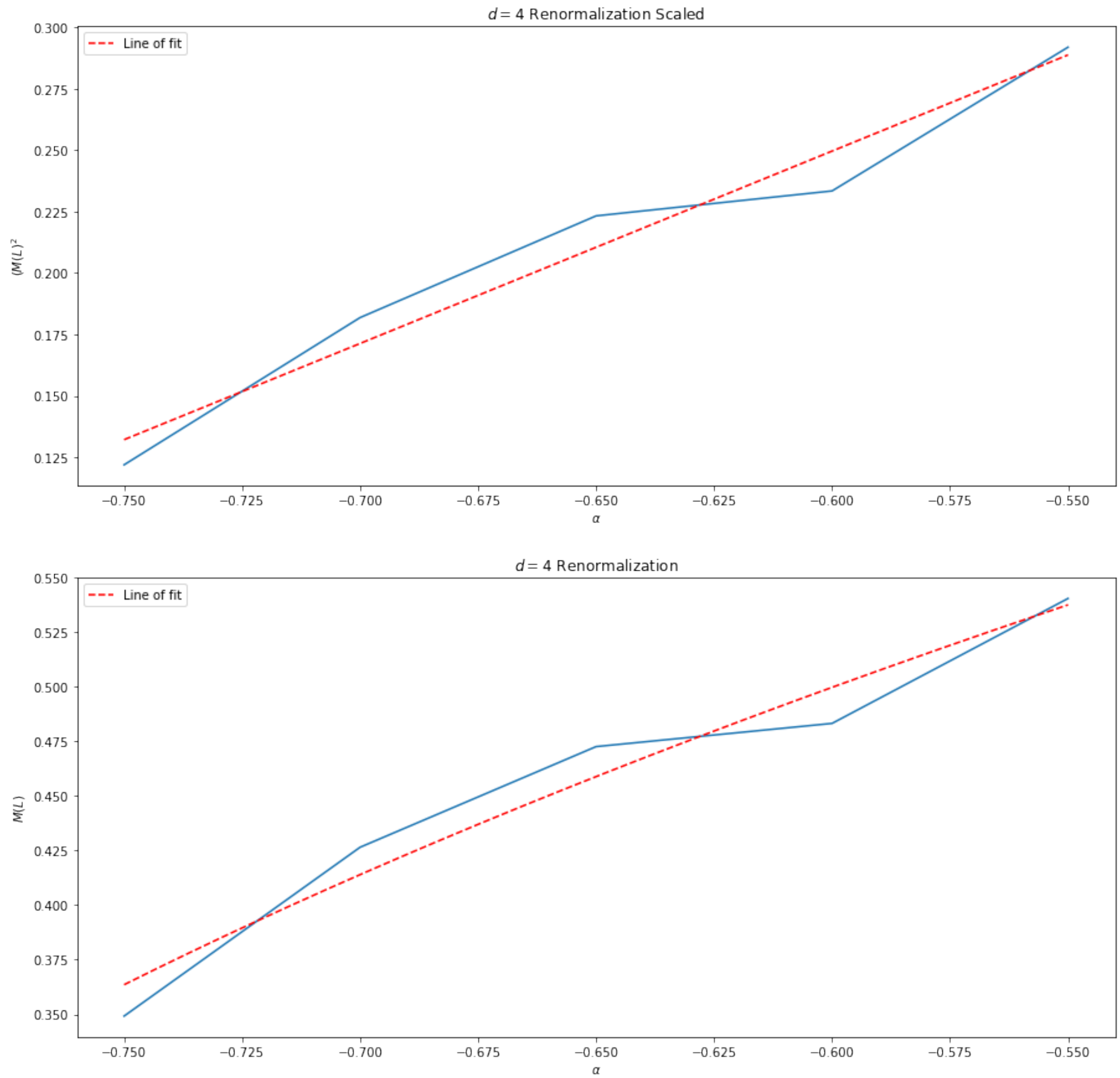

Figure 2.4: $M_{phys}a$ can be given in terms of the tuning parameter roughly by $0.885(\alpha + 0.919)^{0.5}$

2.4 Spontaneous Symmetry Breaking