Project2

(Marks 15)

The continuum Euclidean action of a scalar field theory is given by

$$S_E = \int d^d x \left[\frac{1}{2}\phi(-\partial^2 + m^2)\phi + \frac{\lambda}{4!}\phi^4\right].$$

We learned that this theory describes interacting relativistic scalar particles of mass M_{phys}. Note that $M_{\text{phys}} = m$ is true only in the free theory ($\lambda = 0$). In general one needs to compute the relationship between M_{phys} and m more carefully.
Here we assume that space-time $x = (\mathbf{r}, \tau)$ is a d-dimensional torus of size L_{phys} in each direction. We can regulate the theory on a lattice using the action

$$S_E^\ell = a^d \sum_x \left\{\frac{1}{2}\Big(\frac{2d}{a^2} + m_0^2\Big)\phi_x^2 - \frac{1}{a^2}\sum_\mu \phi_x\phi_{x+\hat{\mu}} + \frac{\lambda_0}{4!}\phi_x^4\right\},$$

where x is a point on a space-time lattice with lattice spacing a. We call m_0 the bare mass and λ_0 the bare coupling. Let L be the number of lattice sites so that $L_{\text{phys}} = La$. We can write the lattice action in the form

$$S_E^\ell = \sum_x \left\{\frac{1}{2}\chi_x^2 - \kappa\sum_\mu \chi_x\chi_{x+\hat{\mu}} + g\chi_x^4\right\},$$

which is more convenient for Monte Carlo calculations we developed in Project 1. Derive the relations

$$\kappa = \frac{1}{(2d + m_0^2 a^2)}, \quad g = \frac{\lambda_0 \kappa^2 a^{4-d}}{4!}, \quad \chi_x = \frac{a^{(d-2)/2}}{\sqrt{\kappa}}\phi_x.$$

1. **Computing the physical mass:** In the symmetric massive phase one can argue that

$$\langle\phi(x)\phi(x')\rangle = \int d^d p \; \frac{Z}{p^2 + M_{\text{phys}}^2} \; \Big(1 + \mathcal{O}(p^2)\Big)\mathrm{e}^{ip\cdot(x-x')}$$

where Z is a constant called the residue of the propagator. Assuming the same form holds in the lattice discretized theory if we define the momentum to be the lattice momentum and change the sum into an integration, show that $M(L)$ defined in the previous project is in fact given by

$$M(L) = M_{\text{phys}} a$$

when L becomes very large for a fixed value of a.

2. **Computing the lattice spacing:** In lattice regularization of a quantum field theory, one uses a physical mass to compute the lattice spacing. The value of $(m_0 a)^2$ in the definition of κ is just assumed to be a tuning parameter and not the mass of the particle. Assume $d = 2, g = 0.01$ and $(m_0 a)^2 = -0.5$ and that you are studying scalar particles of mass $M_{\text{phys}} = 1GeV$. Compute the lattice spacing in units of MeV^{-1}. Hint: Compute $M(L)$ at $L = 24, 32, 48$ and observe that it reaches a constant and then use the relation $M_{\text{phys}} a = M(L)$.

3. **Renormalization:**

   (a) Repeat part (2) above for $d = 2$, $L = 48$, and $(m_0 a)^2 = -0.55, -0.56, -0.57, -0.58, -0.59, -0.60, -0.61$ and compute how the lattice spacing changes. Assuming that $L = 48$ is large enough to approximate $M_{\text{phys}} a \approx M(L)$ show that your data fits well to the form

   $$M_{\text{phys}} a \ = \ f_0((m_0 a)^2 - \alpha_c)^{\nu},$$

   when $\nu = 1$. Make sure your errors for $M(L)$ are about 1-2%. Compute f_0 and α_c. Here ν is called a critical exponent. Note that this means m_0 can be tuned as a function of a for a fixed M_{phys} using the relation.

   $$m_0^2 = \ \alpha_c/a^2 + M_{\text{phys}}/(f_0 a).$$

   This tuning is called renormalization of the bare parameter m_0. Note that in the continuum limit (small values of a) the bare parameter is divergent although the physical mass remains fixed. The continuum limit is reached when $(m_0 a)^2 = \alpha_c$.

   (b) Repeat part 3 for $d = 3$, $L = 48$ and $(m_0 a)^2 = -0.68, -0.69, -0.70, -0.71, -0.72$. Assuming again that $L = 48$ is large enough to approximate $M_{\text{phys}} a \approx M(L)$ show that the data for $M_{\text{phys}} a$ fits to the same form given above with $\nu = 0.629971$.

   (c) Repeat part 3 for $d = 4$, $L = 12$ and $(m_0 a)^2 = -0.55, -0.6, -0.65, -0.7, -0.75$. Assuming again that $L = 12$ is large enough to approximate $M_{\text{phys}} a \approx M(L)$ show that the data for $M_{\text{phys}} a$ fits to the same form given above with $\nu = 0.5$.

4. **Spontaneous Symmetry Breaking:**

   Assume $d = 3$ and compute χ as a function L for $L = 16, 24, 32, 48$ for $(m_0 a)^2 = -0.7$ and $(m_0 a)^2 = -0.8$. What is the difference in the behavior of χ as a function of L and why? What happens when $(m_0 a)^2 = \alpha_c$ from this perspective.