GENERALIZATIONS OF PAULI CHANNELS

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Abstract. The Pauli channel acting on 2×2 matrices is generalized to an n-level quantum system. When the full matrix algebra M_n is decomposed into pairwise complementary subalgebras, then trace-preserving linear mappings $M_n \to M_n$ are constructed such that the restriction to the subalgebras are depolarizing channels. The result is the necessary and sufficient condition of complete positivity. The main examples appear on bipartite systems.

In this paper we consider particular subalgebras of a full matrix algebra $M_n = M_n(\mathbb{C})$. (By a subalgebra we mean $*$-subalgebra with unit.) An F-subalgebra is a subalgebra isomorphic to a full matrix algebra M_k. ("F" is the abbreviation of "factor", the center of such a subalgebra is minimal, $\mathbb{C}I$.) An M-subalgebra is a maximal Abelian subalgebra, equivalently, it is isomorphic to $\mathbb{C}^n$. ("M" is an abbreviation of "MASA", the center is maximal, it is the whole subalgebra.) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are subalgebras, then they are called quasi-orthogonal or complementary if the subspaces $\mathcal{A}_1 \ominus \mathbb{C}I$ and $\mathcal{A}_2 \ominus \mathbb{C}I$ are orthogonal (with respect to the Hilbert–Schmidt inner product $\langle A, B \rangle = \operatorname{Tr} A^* B$). Concerning complementary subalgebras we refer to [8], see also [6, 7, 9].

Complementary M-subalgebras can be given by mutually unbiased bases. Assume that $\xi_1, \xi_2, \ldots, \xi_n$ and $\eta_1, \eta_2, \ldots, \eta_n$ are orthonormal bases such that

$$\left| \langle \xi_i, \eta_j \rangle \right| = \frac{1}{\sqrt{n}} \qquad (1 \leqq i, j \leqq n).$$

If $\mathcal{A}_1$ is the algebra of all operators with diagonal matrix in the first basis and $\mathcal{A}_2$ is defined similarly with respect to the second basis, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are complementary M-subalgebras.

There are examples such that M_n is the linear span of pairwise complementary subalgebras in the case when n is a power of a prime number. If M_n is decomposed into complementary subalgebras, then we construct trace-preserving mappings $M_n \to M_n$ which are completely positive under some conditions.

1. Introduction

If the pairwise complementary subalgebras $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r$ of M_n are given and they linearly span the whole algebra M_n, then any operator is the sum of the components in the subspaces $\mathcal{A}_i \ominus \mathbb{C}I$ $(1 \leqq i \leqq r)$ and $\mathbb{C}I$:

$$A = -\frac{(r-1)\operatorname{Tr} A}{n} I + \sum_{i=1}^{r} E_i(A),$$

where $E_i : M_n \to \mathcal{A}_i$ is the trace-preserving conditional expectation (which is nothing else but the orthogonal projection with respect to the Hilbert–Schmidt inner product, see [10] about details). It is easier to formulate things for matrices of trace 0. If $\operatorname{Tr} B = 0$, then it has an orthogonal decomposition

$$B = \sum_{i=1}^{r} E_i(B).$$

As a generalization of the Pauli channel on a qubit, we define a linear mapping $\alpha : M_n \to M_n$ such that

$$\alpha(B) = \sum_{i=1}^{r} \lambda_i E_i(B)$$

or for an arbitrary A

$$(1) \qquad \alpha(A) = \left(1 - \sum_{i=1}^{r} \lambda_i \right) \frac{\operatorname{Tr} A}{n} I + \sum_{i=1}^{r} \lambda_i E_i(A),$$

where $\lambda_i \in \mathbb{R}$, $1 \leqq i \leqq r$. We want to find the condition for complete positivity. The motivation is the following well-known example in which the complementary subalgebras are generated by the Pauli matrices [10].

EXAMPLE 1. Let $\sigma_0 = I$ and σ_1, σ_2, σ_3 be Pauli matrices, i.e.,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and let $\mathcal{E} : M_2 \to M_2$ be defined as

$$(2) \qquad \mathcal{E}\big(w_0\sigma_0 + (w_1, w_2, w_3) \cdot \sigma\big) = w_0\sigma_0 + (\lambda_1 w_1, \lambda_2 w_2, \lambda_3 w_3) \cdot \sigma$$

for $\omega_i \in \mathbb{C}$, where $\lambda_i \in \mathbb{R}$ and

$$(w_1, w_2, w_3) \cdot \sigma = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3.$$

Density matrices are sent to density matrices if and only if $-1 \leqq \lambda_i \leqq 1$. It is not difficult to compute the representing block matrix $X := \sum_{i,j} \mathcal{E}(E_{ij}) \otimes E_{ij}$. We have

$$X = \begin{bmatrix} \frac{1+\lambda_3}{2} & 0 & 0 & \frac{\lambda_1+\lambda_2}{2} \\ 0 & \frac{1-\lambda_3}{2} & \frac{\lambda_1-\lambda_2}{2} & 0 \\ 0 & \frac{\lambda_1-\lambda_2}{2} & \frac{1-\lambda_3}{2} & 0 \\ \frac{\lambda_1+\lambda_2}{2} & 0 & 0 & \frac{1+\lambda_3}{2} \end{bmatrix}.$$

According to Choi's theorem [2] the positivity of this matrix is equivalent to the complete positivity of $\mathcal{E}$. X is unitarily equivalent to the matrix

$$\begin{bmatrix} \frac{1+\lambda_3}{2} & \frac{\lambda_1+\lambda_2}{2} & 0 & 0 \\ \frac{\lambda_1+\lambda_2}{2} & \frac{1+\lambda_3}{2} & 0 & 0 \\ 0 & 0 & \frac{1-\lambda_3}{2} & \frac{\lambda_1-\lambda_2}{2} \\ 0 & 0 & \frac{\lambda_1-\lambda_2}{2} & \frac{1-\lambda_3}{2} \end{bmatrix}.$$

This matrix is obviously positive if and only if

$$(3) \qquad\qquad\qquad 1 \pm \lambda_3 \geqq |\lambda_1 \pm \lambda_2|.$$

This is necessary and sufficient condition of complete positivity. $\square$

It is not obvious that condition (3) is symmetric in the three variables λ_1, λ_2, λ_3. Condition (3) actually determines the tetrahedron which is the convex hull of the points $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$ and $(-1, -1, 1)$.

Now we show the idea leading to the generalization. The mapping $\mathcal{E}$ in (2) has the form

$$\mathcal{E}(\,\cdot\,) = \sum_{i=0}^{3} \mu_i \sigma_i(\,\cdot\,)\sigma_i.$$

From the expansion of $\mathcal{E}(\sigma_j)$ we can get equations and the solution is the following:

$$\mu_0 = \frac{1}{4}(1 + \lambda_1 + \lambda_2 + \lambda_3), \qquad \mu_1 = \frac{1}{4}(1 + \lambda_1 - \lambda_2 - \lambda_3),$$

$$\mu_2 = \frac{1}{4}(1 - \lambda_1 + \lambda_2 - \lambda_3), \qquad \mu_3 = \frac{1}{4}(1 - \lambda_1 - \lambda_2 + \lambda_3).$$

If $\mu_i \geqq 0$ for every i, then $\mathcal{E}$ is a completely positive mapping. Therefore,

$$1 + \lambda_3 \geqq \pm(\lambda_1 + \lambda_2), \qquad 1 - \lambda_3 \geqq \pm(\lambda_1 - \lambda_2)$$

or together this is (3). (Actually, this argument gives that (3) is a sufficient condition for the complete positivity.)

Pauli channels form an important and popular subject in quantum information theory [1, 3, 4]. The mappings (1) were studied in the paper [5] in the case when the subalgebras are maximal Abelian and pairwise complementary. Our method is different and we allow non-commutative subalgebras as well.

The mapping (1) restricted to $\mathcal{A}_i$ has the form

$$D \mapsto \lambda_i D + (1 - \lambda_i)\frac{I}{n}$$

on density matrices D. If $0 \leqq \lambda_i \leqq 1$, then we can say that D does not change with probability λ_i and with probability $1 - \lambda_i$ it is sent to the tracial state. Such mappings are usually called as depolarizing channels [10].

A simple example including non-commutative subalgebras is the following.

EXAMPLE 2. Consider $M_4 = M_2 \otimes M_2$ and the complementary F-subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_4$ generated by the following triplets of unitaries:

$$\begin{array}{ccc}
\sigma_0 \otimes \sigma_1 & \sigma_0 \otimes \sigma_2 & \sigma_0 \otimes \sigma_3, \\
\sigma_1 \otimes \sigma_0 & \sigma_2 \otimes \sigma_1 & \sigma_3 \otimes \sigma_1, \\
\sigma_2 \otimes \sigma_0 & \sigma_3 \otimes \sigma_2 & \sigma_1 \otimes \sigma_2, \\
\sigma_3 \otimes \sigma_0 & \sigma_1 \otimes \sigma_3 & \sigma_2 \otimes \sigma_3.
\end{array}$$

We take also the M-subalgebra $\mathcal{A}_5$ generated by $\sigma_1 \otimes \sigma_1$, $\sigma_2 \otimes \sigma_2$, $\sigma_3 \otimes \sigma_3$. The conditional expectations $E_j : M_4 \to \mathcal{A}_j$ are convex combinations of the automorphisms

$$(4) \qquad E_j(A) = \frac{1}{4} \sum_{i=1}^{4} U_{ji}^* A U_{ji},$$

where $U_{j1} = I$ and U_{ji}'s are orthogonal unitaries from $\mathcal{A}_j'$. Since $\mathcal{A}_5$ is an M-subalgebra, $\mathcal{A}_5' = \mathcal{A}_5$. The subalgebras $\mathcal{A}_1', \dots, \mathcal{A}_4'$ are F-subalgebras generated by the following unitaries:

$$
\begin{array}{ccc}
\sigma_1 \otimes \sigma_0 & \sigma_2 \otimes \sigma_0 & \sigma_3 \otimes \sigma_0, \\
\sigma_0 \otimes \sigma_1 & \sigma_1 \otimes \sigma_2 & \sigma_1 \otimes \sigma_3, \\
\sigma_2 \otimes \sigma_1 & \sigma_0 \otimes \sigma_2 & \sigma_2 \otimes \sigma_3, \\
\sigma_3 \otimes \sigma_1 & \sigma_3 \otimes \sigma_2 & \sigma_0 \otimes \sigma_3.
\end{array}
$$

(The above triplets generating $\mathcal{A}_j$ and $\mathcal{A}_j'$ ($1 \leqq j \leqq 4$) are Pauli triplets, see [7] for details.) Moreover,

$$(\mathrm{Tr}\, A)I = \frac{1}{4}\left(A + \sum_{j=1}^{5} \sum_{k=2}^{4} U_{jk}^* A U_{jk} \right).$$

The linear mapping (1) has the concrete form

$$\alpha(A) = \left(1 - \sum_{i=1}^{5} \lambda_i \right) \frac{\mathrm{Tr}\, A}{4} I + \sum_{i=1}^{5} \lambda_i E_i(A),$$

where the conditional expectations E_j is expressed by the commutant, see (4). (The condition for complete positivity of α is in Theorem 4.) $\quad\square$

Our main result is the necessary and sufficient condition for the complete positivity of mappings like (1) which can be called generalized Pauli channel.

2. Generalized Pauli channels

Let $\mathcal{A}$ be a (unital $*$-) subalgebra of M_n. Our aim is to describe the conditional expectation onto $\mathcal{A}$ by means of an orthogonal system in the commutant.

Up to unitary equivalence, a subalgebra $\mathcal{A}$ of M_n can be written as

$$\mathcal{A} = \bigoplus_{i=1}^{k} M_{n_i} \otimes I_{m_i}.$$

The commutant $\mathcal{A}'$ in M_n is

$$\mathcal{A}' = \bigoplus_{i=1}^{k} I_{n_i} \otimes M_{m_i}.$$

Let $N = \sum_{i=1}^{k} n_i^2$ and let P_i be a minimal central projection of $\mathcal{A}$, that is, $P_i = I_{n_i} \otimes I_{m_i}$.

PROPOSITION 1. *Let $\{U_i\}_{i=1}^{N}$ be an orthonormal basis of $\mathcal{A}$. Then the completely positive map F from M_n onto $\mathcal{A}'$ given by*

$$F(X) = \sum_{i=1}^{N} U_i^* X U_i \qquad (X \in M_n)$$

is equal to

$$F(X) = \bigoplus_{i=1}^{k} \frac{n_i}{m_i} \operatorname{Tr}_{n_i}(P_i X P_i),$$

where Tr_{n_i} is a partial trace from $M_{n_i} \otimes M_{m_i}$ onto M_{m_i}.

In particular, if all n_i/m_i are equal, then $\frac{n}{\dim \mathcal{A}} F$ is the trace-preserving conditional expectation from M_n onto $\mathcal{A}'$.

PROOF. If all n_i/m_i are equal, then their ratio is equal to $\frac{\dim \mathcal{A}}{n}$. Therefore it is sufficient to prove the first assertion.

Let $\left\{e_{ij}^{(l)}\right\}_{i,j=1}^{n_l}$ and $\left\{f_{ij}^{(l)}\right\}_{i,j=1}^{m_l}$ be matrix units of M_{n_l} and M_{m_l}, respectively. Then U_i is written as

$$U_i = \sum_{l=1}^{k} \sum_{s,t=1}^{n_l} U_{i,st}^{(l)} e_{st}^{(l)}$$

for some $U_{i,st}^{(l)} \in \mathbb{C}$. The operator $W \in M_N$ given in terms of its matrix entries by the formula

$$W_{i,(l,s,t)} = \sqrt{m_l}\, U_{i,st}^{(l)}$$

for $1 \leqq i \leqq N$, $1 \leqq l \leqq k$ and $1 \leqq s,t \leqq n_l$ is unitary. Indeed, W can be considered as the matrix which takes the orthonormal basis $\left\{\frac{1}{\sqrt{m_l}} e_{st}^{(l)}\right\}$ of $\mathcal{A}$ into the orthonormal basis $\{U_i\}$. Hence we have

$$\sum_{i=1}^{N} \overline{W_{i,(l,s,t)}} W_{i,(l',s',t')} = \delta_{ll'} \delta_{ss'} \delta_{tt'}.$$

and therefore

$$\sum_{i=1}^{N} \overline{U_{i,st}^{(l)}} U_{i,s't'}^{(l')} = \delta_{ll'}\delta_{ss'}\delta_{tt'}\frac{1}{m_l}. \tag{5}$$

Let T be a partial isometry with $T^*T = e_{s_1 s_1}^{(l_1)} \otimes f_{t_1 t_1}^{(l_1)}$ and $TT^* = e_{s_2 s_2}^{(l_2)} \otimes f_{t_2 t_2}^{(l_2)}$. Then we obtain

$$F(T) = \sum_{i=1}^{N} U_i^* T U_i = \sum_{i=1}^{N} \sum_{p=1}^{n_{l_2}} \sum_{q=1}^{n_{l_1}} \overline{U_{i,s_2 p}^{(l_2)}} U_{i,s_1 q}^{(l_1)} e_{ps_2}^{(l_2)} T e_{s_1 q}^{(l_1)}$$

$$= \delta_{l_1 l_2}\delta_{s_1 s_2}\delta_{pq} \sum_{p=1}^{n_{l_1}} \frac{1}{m_{l_1}} e_{ps_1}^{(l_1)} T e_{s_1 p}^{(l_1)}$$

by (5) so that F maps the off-diagonal part to 0, that is, if $l_1 \neq l_2$ then $F(T) = 0$.

Now let $T = e_{s_2 s_1}^{(l)} \otimes f_{t_2 t_1}^{(l)}$. Then we obtain

$$F(T) = \delta_{s_1 s_2} \sum_{p=1}^{n_l} \frac{1}{m_l} e_{pp}^{(l)} \otimes f_{t_2 t_1}^{(l)} = \delta_{s_1 s_2} \frac{1}{m_l} I_{n_l} \otimes f_{t_2 t_1} = \frac{n_l}{m_l} \operatorname{Tr}_{n_l}(T)$$

which shows the first assertion. $\quad\square$

The commutant of M-subalgebras and F-subalgebras are again M-subalgebras and F-subalgebras, and in both types it is possible to choose an orthogonal basis consisting of unitaries, only. Thus by an application of the previous proposition, for such a subalgebra $\mathcal{A}$, the trace-preserving conditional expectation is the convex combination of the automorphisms

$$X \mapsto \frac{1}{\dim \mathcal{A}'} \sum_{i=1}^{m} U_i'^* X U_i' \qquad (X \in M_n),$$

where $\{U_i'\}$ is an orthogonal basis of $\mathcal{A}'$ consisting of unitaries. Bases consisting of unitaries are important also in quantum state teleportation [13].

THEOREM 1. *Let $\{U_i : 1 \leq i \leq m\}$ be an orthonormal system in M_n. Then the linear mapping*

$$\alpha(A) = \sum_{i=1} \mu_i U_i^* A U_i$$

is completely positive if and only if $\mu_i \geq 0$ for every $1 \leq i \leq m$.

PROOF. If $\mu_i \geqq 0$ for every $1 \leqq i \leqq m$, it is clear that α is completely positive. To prove the converse, we first show that

$$\sum_{i,j} W^* E_{ij} W \otimes E_{ij}$$

is a projection if $\operatorname{Tr} WW^* = 1$. This is obviously self-adjoint and we can compute that it is idempotent:

$$\left(\sum_{i,j} W^* E_{ij} W \otimes E_{ij} \right) \left(\sum_{k,l} W^* E_{kl} W \otimes E_{kl} \right)$$

$$= \sum_{i,j,l} W^* E_{ij} WW^* E_{jl} W \otimes E_{il} = \operatorname{Tr} WW^* \left(\sum_{i,l} W^* E_{il} W \otimes E_{il} \right).$$

It follows that

$$P_k := \sum_{i,j} U_k^* E_{ij} U_k \otimes E_{ij}$$

is a projection for every $1 \leqq k \leqq m$. To show that they are pairwise orthogonal, we compute the trace of $P_k P_l$:

$$\operatorname{Tr} P_k P_l = \operatorname{Tr} \sum_{i,j,u,v} U_k^* E_{ij} U_k U_l^* E_{uv} U_l \otimes E_{ij} E_{uv}$$

$$= \sum_{i,j} \operatorname{Tr} U_k^* E_{ij} U_k U_l^* E_{ji} U_l = \sum_{i,j} \operatorname{Tr} E_{ij} U_k U_l^* E_{ji} U_l U_k^*.$$

Due to Lemma 1 below this equals $\operatorname{Tr} U_k U_l^* \operatorname{Tr} U_l U_k^* = 0$ when $k \neq l$.

The complete positivity implies that

$$\sum_{i,j} \left(\sum_k \mu_k U_k^* E_{ij} U_k \otimes E_{ij} \right) = \sum_k \mu_k \left(\sum_{i,j} U_k^* E_{ij} U_k \otimes E_{ij} \right) = \sum_k \mu_k P_k$$

is positive, therefore $\mu_k \geqq 0$. $\quad\square$

LEMMA 1. $\sum_{i,j} \operatorname{Tr} E_{ij} X E_{ji} Y = (\operatorname{Tr} X)(\operatorname{Tr} Y)$.

PROOF. Since both sides are bilinear in the variables X and Y, it is enough to check the case $X = E_{ab}$ and $Y = E_{cd}$. Simple computation gives that the left hand side is $\delta_{ab}\delta_{cd}$. A physicist might make a different proof of the lemma:

$$\sum_{i,j} \operatorname{Tr} E_{ij} X E_{ji} Y = \sum_{i,j} \operatorname{Tr} |e_i\rangle\langle e_j| X |e_j\rangle\langle e_i| Y = \sum_{i,j} \langle e_j|X|e_j\rangle \langle e_i|Y|e_i\rangle$$

and the right hand side is $(\operatorname{Tr} X)(\operatorname{Tr} Y)$. $\quad\square$

We also need the next lemma; the proof can be found in [13].

LEMMA 2. *Let $V_1, V_2, \ldots, V_{n^2}$ be matrices in M_n. Then the following conditions are equivalent:*

1. $\operatorname{Tr} V_i^* V_j = \delta_{ij} \qquad (1 \leqq i, j \leqq n^2)$,
2. $\sum_{i=1}^{n^2} V_i^* A V_i = (\operatorname{Tr} A) I$ *for every $A \in M_n$.*

The next result includes important particular cases which are formulated afterwards.

THEOREM 2. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_r$ be pairwise complementary subalgebras of M_n such that their commutants $\mathcal{A}_1', \ldots, \mathcal{A}_r'$ are pairwise complementary as well. Then the trace-preserving conditional expectations $E_j : M_n \to \mathcal{A}_j$ can be expressed by the orthonormal bases $U_{j1}, U_{j2}, \ldots, U_{jn(j)} \in \mathcal{A}_j'$, where $U_{j1} = \frac{1}{\sqrt{n}} I$, via the formula*

$$(6) \qquad E_j(A) = \frac{n}{\dim \mathcal{A}_j'} \sum_{i=1}^{n(j)} U_{ji}^* A U_{ji}$$

and the generalized Pauli channel (1) is completely positive if and only if

$$1 + \frac{n^2 \lambda_i}{\dim \mathcal{A}_i'} \geqq \sum_j \lambda_j$$

for every $1 \leqq i \leqq r$ and

$$\sum_j \lambda_j \left(\frac{n^2}{\dim \mathcal{A}_j'} - 1 \right) \geqq -1.$$

To prove this theorem we prepare the following proposition.

PROPOSITION 2. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be complementary subalgebras of M_n. Then $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are complementary if and only if $\mathcal{A}_1 \mathcal{A}_2$ linearly spans M_n. Moreover, in this case the trace-preserving conditional expectation $E_1 : M_n \to \mathcal{A}_1'$ can be expressed as*

$$E_1(X) = \frac{n}{\dim \mathcal{A}_1} \sum_i U_i^* X U_i \quad (X \in M_n),$$

where $\{U_i\}$ is an orthonormal basis of $\mathcal{A}_1$.

PROOF. Assume $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are complementary. Let $\{U_i'\}$ and $\{V_j'\}$ be orthonormal bases of $\mathcal{A}_1'$ and $\mathcal{A}_2'$, respectively, which consist of scalar multiple of their matrix units. Then the trace-preserving conditional expectations onto

$\mathcal{A}_1$ and $\mathcal{A}_2$ are given by the linear combinations of $U_i'^*(\,\cdot\,)U_i'$ and $V_j'^*(\,\cdot\,)V_j'$, respectively, thanks to Proposition 1.

Since $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are complementary subalgebras, $\{V_j'U_i'\}_{i,j}$ is an orthogonal system. Moreover the trace is written by the linear combination of $U_i'^*V_j'^*(\,\cdot\,)V_j'U_i'$, because $\mathcal{A}_1$ and $\mathcal{A}_2$ are complementary subalgebras if and only if the composition of two conditional expectations equals $\frac{1}{n}\operatorname{Tr}$. But this shows that $\{V_j'U_i'\}_{i,j}$ linearly spans the whole M_n thanks to Lemma 2.

Conversely assume $\mathcal{A}_1\mathcal{A}_2$ linearly spans the whole space M_n. Since $\mathcal{A}_1$ is a subalgebra of M_n, $\mathcal{A}_1$ can be written as

$$\mathcal{A}_1 = \bigotimes_{l=1}^{k} M_{n_l} \otimes I_{m_l}.$$

Let Q be a minimal central projection in $\mathcal{A}_2$ and let $\{U_i^{(s)}\}$ and $\{V_j\}$ be orthonormal bases of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, with the assumption $U_i^{(s)} \in M_{n_s} \otimes I_{m_s}$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are complementary and $\operatorname{span}\{\mathcal{A}_1\mathcal{A}_2\} = M_n$, $\{\sqrt{n}\,U_i^{(s)}V_j\}$ is an orthonormal basis of M_n. Therefore by Lemma 2 and Proposition 1, we have

$$\sum_{s,i,j} U_i^{(s)*} V_j^* Q V_j U_i^{(s)} = \frac{1}{n} \operatorname{Tr} Q \cdot I$$

and

$$\sum_{j} V_j^* Q V_j = cQ$$

for some $c > 0$. These equations imply, for $1 \leqq s \leqq k$,

$$\sum_{i} U_i^{(s)*} Q U_i^{(s)} = \frac{\operatorname{Tr} Q}{cn} P_s,$$

where P_s is a central projection $I_{n_s} \otimes I_{m_s}$. Now we take the trace to the above equation. Then we have

$$\operatorname{Tr}\left(\sum_{i}^{n_s^2} U_i^{(s)*} Q U_i^{(s)} \right) = \sum_{i}^{n_s^2} \frac{1}{n} \operatorname{Tr}\left(U_i^{(s)} U_i^{(s)*} \right) \operatorname{Tr} Q = \frac{n_s^2}{n} \operatorname{Tr} Q$$

and

$$\frac{\operatorname{Tr} Q}{cn} \operatorname{Tr} P_s = \frac{\operatorname{Tr} Q}{cn} n_s m_s.$$

so that $\frac{n_s}{m_s} = \frac{1}{c}$. Hence n_s/m_s is equal to $1/c = \frac{\dim \mathcal{A}_1}{n}$ for all $1 \leqq s \leqq k$ and so

$$E_1 = \frac{n}{\dim \mathcal{A}_1} \sum_{s,i} U_i^{(s)*}(\,\cdot\,)U_i^{(s)}$$

is the trace-preserving conditional expectation onto $\mathcal{A}_1'$ by Proposition 1. Similarly,

$$E_2 = \frac{n}{\dim \mathcal{A}_2} \sum_j V_j^*(\,\cdot\,)V_j.$$

is the trace-preserving conditional expectation onto $\mathcal{A}_2'$. Since

$$\sum_{s,i,j} U_i^{(s)*}V_j^*(\,\cdot\,)V_j U_i^{(s)} = \sum_{s,i,j} V_j^* U_i^{(s)*}(\,\cdot\,)U_i^{(s)}V_j$$

is the normalized trace on M_n by Lemma 2, we obtain $\frac{n^2}{\dim \mathcal{A}_1 \dim \mathcal{A}_2} = 1$ and so the composition $E_1 \circ E_2$ equals $\frac{1}{n}\operatorname{Tr}$. $\square$

PROOF OF THEOREM 2. The first assertion is already proven in the above proposition. Due to Lemma 2, we have

$$(\operatorname{Tr} A)I = \frac{A}{n} + \sum_{j=0}^{n} \sum_{k=2}^{n} U_{jk}^* A U_{jk} + \sum_{t=1}^{\ell} W_t^* A W_t,$$

where the orthonormal system W_t extends the orthonormal system U_{jk} to a complete system in the linear space M_n. In formula (1) we use this expression for $(\operatorname{Tr} A)I$ and the assumed decomposition of the conditional expectations. So in the expansion of $\alpha(A)$ the coefficient of $U_{jk}^* A U_{jk}$ is

$$\frac{1}{n}\left(1 - \sum_i \lambda_i\right) + \frac{n\lambda_j}{\dim \mathcal{A}_j'}$$

and the coefficient of $\frac{A}{n} = \left(\frac{1}{\sqrt{n}}I\right)A\left(\frac{1}{\sqrt{n}}I\right)$ is

$$\frac{1}{n}\left(1 - \sum_j \lambda_j\right) + \sum_j \frac{n\lambda_j}{\dim \mathcal{A}_j'}.$$

Theorem 1 tells us that complete positivity holds if and only if both are positive. $\square$

COROLLARY 1. *Assume that M_n contains pairwise complementary M-subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_r$. Then the generalized Pauli channel is completely positive if and only if*

$$1 + n\lambda_i \geqq \sum_j \lambda_j \geqq -\frac{1}{n-1}$$

for every $1 \leqq i \leqq r$.

This result appeared also in [5].

3. Bipartite channels

In this section we consider subalgebras of $M_n \otimes M_n$. A subalgebra isomorphic to M_n will be called F-subalgebra. An M-subalgebra is a maximal Abelian subalgebra. Both kinds of subalgebras are subspaces of dimension n^2.

THEOREM 3. *Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are F- or M-subalgebras of $M_n \otimes M_n$. If they are complementary, then the commutants $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are complementary as well.*

PROOF. Since both kinds of subalgebras are subspaces of dimension n^2, the dimension of $\mathcal{A}_1\mathcal{A}_2$ is n^4 so that $\mathcal{A}_1\mathcal{A}_2 = M_n \otimes M_n$. Therefore the commutants $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are complementary by Proposition 2. □

THEOREM 4. *Assume that $M_n \otimes M_n$ is decomposed to pairwise complementary F- and M-subalgebras $\mathcal{A}_i$ ($1 \leqq i \leqq n^2 + 1$). The trace-preserving conditional expectation of $M_n \otimes M_n$ onto $\mathcal{A}_i$ is denoted by E_i. The linear trace-preserving mapping acting as*

$$\alpha(B) = \sum_{i=1}^{n^2+1} \lambda_i E_i(B) \qquad (B \in M_n \otimes M_n, \ \operatorname{Tr} B = 0)$$

is completely positive if and only if

$$1 + n^2\lambda_i \geqq \sum_j \lambda_j \geqq -\frac{1}{n^2-1}$$

for every $1 \leqq i \leqq n^2 + 1$.

PROOF. Theorem 3 allows to use Theorem 2 and the result follows. □

The theorem can be applied in Example 2. Note that decompositions of $M_2 \otimes M_2$ into F- and M-subalgebras are discussed in [9], while decomposition of $M_n \otimes M_n$ into F-subalgebras is constructed in [6] if $n = p^k$ with a prime number $p > 2$.

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