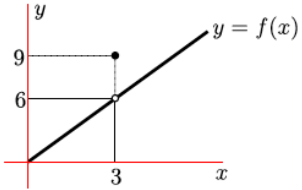
# MATH 100

Farid Aliniaeifard

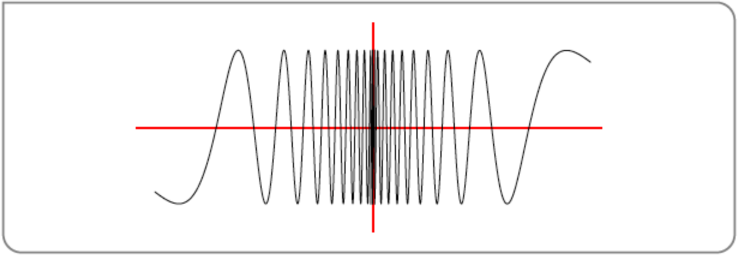
**University of British Columbia**

2019

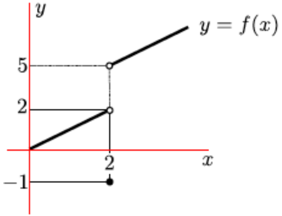
$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



## $f(x) = \sin(\frac{\pi}{x})$

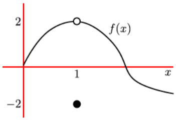


$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$



## Example

Consider the graph of the function $f(x)$.




Then

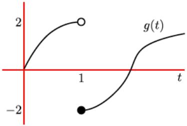
$$\lim_{x \to 1^-} f(x) =$$

$$\lim_{x \to 1^+} f(x) =$$

$$\lim_{x \to 1} f(x) =$$

## Example

Consider the graph of the function $g(t)$.




Then

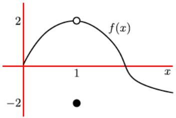
$$\lim_{t \to 1^-} g(t) =$$

$$\lim_{t \to 1^+} g(t) =$$

$$\lim_{t \to 1} g(t) =$$

### Example

Consider the graph of the function $f(x)$.




Then

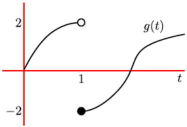
$$\lim_{x\to 1^-} f(x) = 2$$

$$\lim_{x\to 1^+} f(x) = 2$$

$$\lim_{x\to 1} f(x) = 2$$

### Example

Consider the graph of the function $g(t)$.




Then

$$\lim_{t\to 1^-} g(t) = 2$$

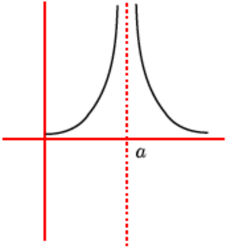
$$\lim_{t\to 1^+} g(t) = -2$$

$$\lim_{t\to 1} g(t) = DNE$$

# When the limit goes to infinity

### Example

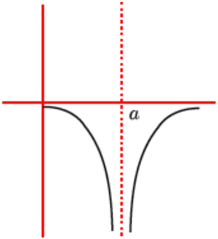
Consider the graph for the function $f(x)$.




$$\lim_{x \to a} f(x) = +\infty$$

### Example

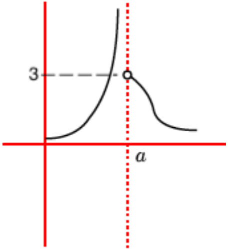
Consider the graph for the function $g(x)$.




$$\lim_{x \to a} g(x) = -\infty$$

## Example

Consider the graph for the function $h(x)$.

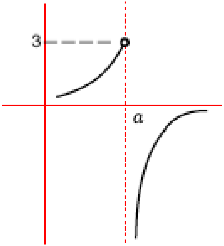



$$\lim_{x \to a^-} h(x) =$$

$$\lim_{x \to a^+} h(x) =$$

## Example

Consider the graph for the function $s(x)$.

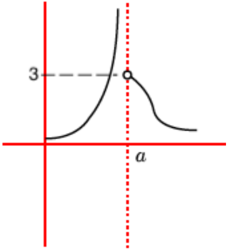



$$\lim_{x \to a^-} s(x) =$$

$$\lim_{x \to a^+} s(x) =$$

## Example

Consider the graph for the function $h(x)$.

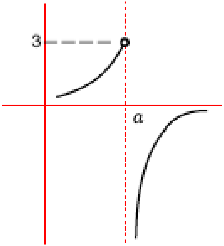



$$\lim_{x \to a^-} h(x) = +\infty$$

$$\lim_{x \to a^+} h(x) = 3$$

## Example

Consider the graph for the function $s(x)$.




$$\lim_{x \to a^-} s(x) = 3$$

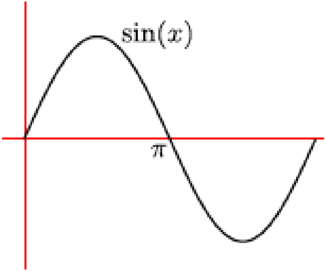
$$\lim_{x \to a^+} s(x) = -\infty$$

## Example

Consider the function

$$g(x) = \frac{1}{sin(x)}.$$

Find the one-side limits of this function as $x \to \pi$.




$$\lim_{x \to \pi^-} \frac{1}{sin(x)} = +\infty$$

$$\lim_{x \to \pi^+} \frac{1}{sin(x)} = -\infty$$

# Second Session Outline

- Arithmetic of the Limits
- Limit of a ratio: what will happen if the limit of the denominator is zero. For example,

$$\lim_{x\to 0} \frac{1}{x^2}? \quad \text{and} \quad \lim_{x\to 1} \frac{x^3 - x^2}{x - 1} =?$$

- Sandwich/ Squeeze/Pinch Theorem
- limit at infinity

# Arithmetic of the Limits

## Theorem

*Let $a, c \in \mathbb{R}$. The following two limits hold*

$$\lim_{x \to a} c = c \qquad \lim_{x \to a} x = a$$

## Example

$$\lim_{x \to 3} -2 = -2 \qquad \lim_{x \to -1} x = -1$$

### Theorem

(**Arithmetic of Limits**) *Let* $a, c \in \mathbb{R}$, *let* $f(x)$ *and* $g(x)$ *be defined for all* $x$*'s that lie in some interval about* $a$ *(but* $f$ *and* $g$ *need not to be defined exactly at* $a$*).*

$$\lim_{x \to a} f(x) = F \qquad \lim_{x \to a} g(x) = G$$

*exists with* $F, G \in \mathbb{R}$. *Then the following limits hold*

- $\lim_{x \to a}(f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*

### Theorem

(**Arithmetic of Limits**) *Let* $a, c \in \mathbb{R}$, *let* $f(x)$ *and* $g(x)$ *be defined for all* $x$*'s that lie in some interval about* $a$ *(but* $f$ *and* $g$ *need not to be defined exactly at* $a$*).*

$$\lim_{x\to a} f(x) = F \qquad \lim_{x\to a} g(x) = G$$

*exists with* $F, G \in \mathbb{R}$. *Then the following limits hold*

- $\lim_{x\to a}(f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*
- $\lim_{x\to a}(f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*

## Theorem

(**Arithmetic of Limits**) *Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all $x$'s that lie in some interval about $a$ (but $f$ and $g$ need not to be defined exactly at $a$).*

$$\lim_{x \to a} f(x) = F \qquad \lim_{x \to a} g(x) = G$$

*exists with $F, G \in \mathbb{R}$. Then the following limits hold*

- $\lim_{x \to a}(f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*
- $\lim_{x \to a}(f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*
- $\lim_{x \to a} cf(x) = cF$.

### Theorem

(**Arithmetic of Limits**) *Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all $x$'s that lie in some interval about $a$ (but $f$ and $g$ need not to be defined exactly at $a$).*

$$\lim_{x \to a} f(x) = F \qquad \lim_{x \to a} g(x) = G$$

*exists with $F, G \in \mathbb{R}$. Then the following limits hold*

- $\lim_{x \to a} (f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*
- $\lim_{x \to a} (f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*
- $\lim_{x \to a} cf(x) = cF$.
- $\lim_{x \to a} (f(x).g(x)) = F.G$*–limit of the product is the product of the limits.*

If $G \neq 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}$$

### Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) =$$

## Example

Given

$$\lim_{x \to 1} f(x) = 3 \quad \text{and} \quad \lim_{x \to 1} g(x) = 2$$

We have

$$\lim_{x \to 1} 3f(x) = 3 \times \lim_{x \to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \to 1} 3f(x) - g(x) =$$

### Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) =$$

## Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} f(x). \lim_{x\to 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x\to 1} \frac{f(x)}{f(x) - g(x)} =$$

## Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} f(x). \lim_{x\to 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x\to 1} \frac{f(x)}{f(x) - g(x)} = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} f(x) - \lim_{x\to 1} g(x)} = \frac{3}{3-2} = 3.$$

# Example

$$\lim_{x \to 3} 4x^2 - 1 =$$

$$\lim_{x \to 2} \frac{x}{x-1} =$$

# Example

$$\lim_{x \to 3} 4x^2 - 1 = 4 \times \lim_{x \to 3} x^2 - \lim_{x \to 3} 1 = 35.$$

$$\lim_{x \to 2} \frac{x}{x-1} = \frac{\lim_{x \to 2} x}{\lim_{x \to 2} x - \lim_{x \to 1} 1} = \frac{2}{2-1} = 2.$$

## Limit of a ratio: what will happen if the limit of the denominator is zero.

## Limit of a ratio: what will happen if the limit of denominator is zero:

– the limit does **not exist**, eg.

$$\lim_{x \to 0} \frac{x}{x^2} = \lim_{x \to 0} \frac{1}{x} = DNE$$

# Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} = DNE$$

- the **limit is** $\pm\infty$, eg.

$$\lim_{x\to 0} \frac{x^2}{x^4} = \lim_{x\to 0} \frac{1}{x^2} = +\infty \qquad \text{or} \qquad \lim_{x\to 0} \frac{-x^2}{x^4} = \lim_{x\to 0} \frac{-1}{x^2} = -\infty.$$

# Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} = DNE$$

- the **limit is** $\pm\infty$, eg.

$$\lim_{x\to 0} \frac{x^2}{x^4} = \lim_{x\to 0} \frac{1}{x^2} = +\infty \quad \text{or} \quad \lim_{x\to 0} \frac{-x^2}{x^4} = \lim_{x\to 0} \frac{-1}{x^2} = -\infty.$$

- the **limit is** 0, eg.

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0.$$

## Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} = DNE$$

- the **limit is** $\pm\infty$, eg.

$$\lim_{x\to 0} \frac{x^2}{x^4} = \lim_{x\to 0} \frac{1}{x^2} = +\infty \qquad \text{or} \qquad \lim_{x\to 0} \frac{-x^2}{x^4} = \lim_{x\to 0} \frac{-1}{x^2} = -\infty.$$

- the **limit is** 0, eg.

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0.$$

- the **limit exists and it nonzero**, eg.

$$\lim_{x\to 0} \frac{x}{x} = 1.$$

### Theorem

*Let $n$ be a positive integer, let $a \in R$ and let $f$ be a function so that*

$$\lim_{x \to a} f(x) = F$$

*for some real number $F$. Then the following holds*

$$\lim_{x \to a} (f(x))^n = \left( \lim_{x \to a} f(x) \right)^n = F^n$$

*so that the limit of a power is the power of the limit.*

### Theorem

*Let $n$ be a positive integer, let $a \in R$ and let $f$ be a function so that*

$$\lim_{x \to a} f(x) = F$$

*for some real number $F$. Then the following holds*

$$\lim_{x \to a} (f(x))^n = \left( \lim_{x \to a} f(x) \right)^n = F^n$$

*so that the limit of a power is the power of the limit. Similarly, if*

- *$n$ is an even number and $F > 0$, or*
- *$n$ is an odd number and $F$ is any real number*

*then*

$$\lim_{x \to a} (f(x))^{1/n} = \left( \lim_{x \to a} f(x) \right)^{1/n} = F^{1/n}.$$

# Example

$$\lim_{x \to 4} x^{1/2} =$$

$$\lim_{x \to 4} (-x)^{1/2} =$$

$$\lim_{x \to 2} (4x^2 - 3)^{1/3} =$$

# Example

$$\lim_{x \to 4} x^{1/2} = 4^{1/2} = 2.$$

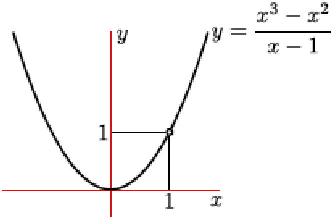
$$\lim_{x \to 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \to 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = (13)^{1/3}.$$

**Limit of a ratio: what will happen if the limit of the numerator and denominator are zero, for example,**

$$\lim_{x \to 1} \frac{x^3 - x^2}{x - 1} = ?$$

$$\lim_{x \to 1} \frac{x^3 - x^2}{x - 1} = ?$$



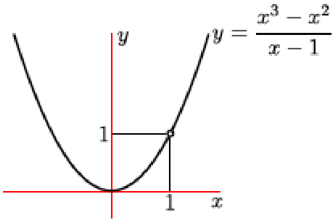
**Theorem**

*If $f(x) = g(x)$ except when $x = a$ then*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

*provided the limit of $g$ exists.*

$$\frac{x^3 - x^2}{x-1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases} \Rightarrow \lim_{x\to 1} \frac{x^3 - x^2}{x-1} = \lim_{x\to 1} x^2 = 1.$$

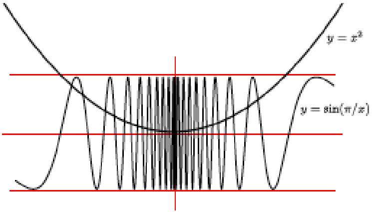


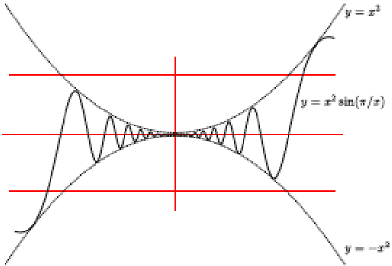
# Sandwich/ Squeeze/Pinch Theorem

## Example

Compute

$$\lim_{x \to 0} x^2 \sin(\frac{\pi}{x})$$



## Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \to 1} f(x)?$$

### Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \to 1} f(x)?$$

### Solution

*Consider that*

$$\lim_{x \to 1} x = 1 \qquad \textit{and} \qquad \lim_{x \to 1} x^2 - 2x + 2 = 1.$$

*Therefore, by the sandwich/pinch/squeeze theorem*

$$\lim_{x \to 1} f(x) = 1.$$

## Example

We want to compute

$$\lim_{x\to+\infty} \frac{1}{x} \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x}$$

By plug in some large numbers into $\frac{1}{x}$ we have

| $-10000$ | $-1000$ | $-100$ $\|\| \circ \|\|$ $100$ | $1000$ | $10000$ |
|---|---|---|---|---|
| $-0.0001$ | $-0.001$ | $-0.01$ $\|\| \circ \|\|$ $0.01$ | $0.001$ | $0.0001$ |

We see that as $x$ is getting bigger and positive the function $\frac{1}{x}$ is getting closer to 0. Thus,

$$\lim_{x\to+\infty} \frac{1}{x} = 0.$$

Moreover,

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

# Limit at Infinity

## Definition

(**Informal limit at infinity.**) We write

$$\lim_{x \to \infty} f(x) = L$$

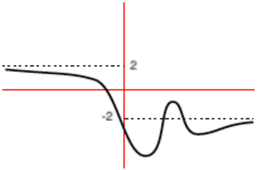
when the value of the function $f(x)$ gets closer and closer to $L$ as we make $x$ larger and larger and positive.
Similarly, we write

$$\lim_{x \to -\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to $L$ as we make $x$ larger and larger and negative.

## Example

Consider the graph of the function $f(x)$.



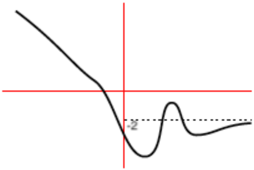

Then

$$\lim_{x\to\infty} f(x) =$$

$$\lim_{x\to-\infty} f(x) =$$

## Example

Consider the graph of the function $g(x)$.



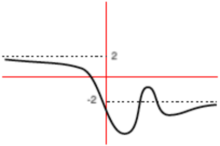

Then

$$\lim_{x\to\infty} g(x) =$$

$$\lim_{x\to-\infty} g(x) =$$

## Example

Consider the graph of the function $f(x)$.



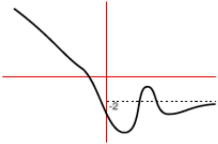

Then

$$\lim_{x\to\infty} f(x) = -2$$

$$\lim_{x\to-\infty} f(x) = 2$$

## Example

Consider the graph of the function $g(x)$.




Then

$$\lim_{x\to\infty} g(x) = -2$$

$$\lim_{x\to-\infty} g(x) = +\infty$$

# Review of the third session

# Review

## Theorem

**sandwich (or squeeze or pinch)** *Let $a \in \mathbb{R}$ and let $f, g, h$ be three functions so that*

$$f(x) \leq g(x) \leq h(x)$$

*for all $x$ in an interval around $a$, except possibly at $x = a$. Then if*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

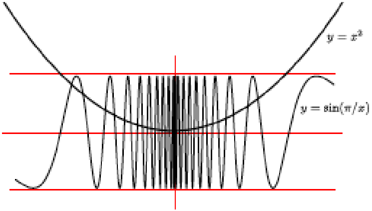
*then it is also the case that*

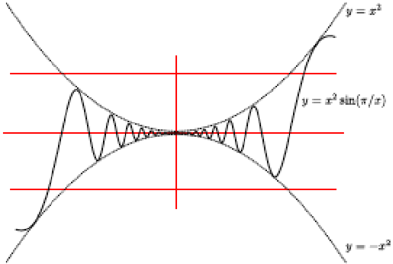
$$\lim_{x \to a} g(x) = L.$$

## Example

Compute

$$\lim_{x \to 0} x^2 \sin(\frac{\pi}{x})$$



## Theorem

*Let $c \in \mathbb{R}$ then the following limits hold*

$$\lim_{x \to +\infty} c = c \qquad \lim_{x \to -\infty} c = c$$

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

# Outline For the Fourth Session

- Limit at Infinity

# Limit at Infinity

## Theorem

*Let $f(x)$ and $g(x)$ be two functions for which the limits*

$$\lim_{x\to\infty} f(x) = F \qquad \lim_{x\to\infty} = G$$

*exist. Then the following limits hold*

$$\lim_{x\to\infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x\to\infty} f(x)g(x) = FG$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \textit{provided } G \neq 0$$

and for rational numbers $r$,

$$\lim_{x\to\infty}(f(x))^r = F^r$$

provided that $f(x)^r$ is defined for all $x$.
The analogous results hold for limits to $-\infty$.

**Warning:** Consider that

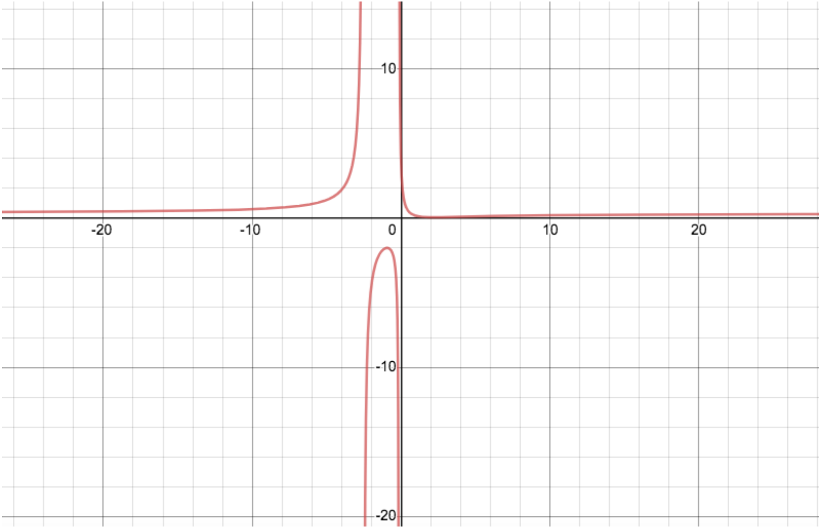
$$\lim_{x \to +\infty} \frac{1}{x^{1/2}} = 0$$

However,

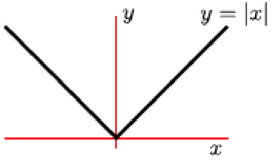
$$\lim_{x \to +\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because $x^{1/2}$ is not defined for $x < 0$.

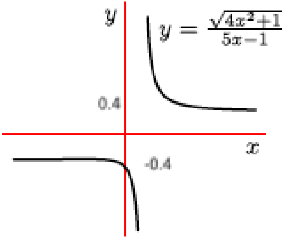
$f(x) = \frac{x^2-3x+4}{3x^2+8x+1}$



$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



$$y = \frac{\sqrt{4x^2+1}}{5x-1}$$



## Theorem

*Let* $a, c, H \in \mathbb{R}$ *and let* $f, g, h$ *be functions defined in an interval around* $a$ *(but they need not be defined at* $x = a$*), so that*

$$\lim_{x\to a} f(x) = +\infty \qquad \lim_{x\to a} g(x) = +\infty \qquad \lim_{x\to a} h(x) = H$$

1. $$\lim_{x\to a}(f(x) + g(x)) =$$
2. $$\lim_{x\to a}(f(x) + h(x)) =$$
3. $$\lim_{x\to a}(f(x) - g(x)) =$$
4. $$\lim_{x\to a}(f(x) - h(x)) =$$

## Theorem

*Let $a, c, H \in \mathbb{R}$ and let $f, g, h$ be functions defined in an interval around $a$ (but they need not be defined at $x = a$), so that*

$$\lim_{x \to a} f(x) = +\infty \qquad \lim_{x \to a} g(x) = +\infty \qquad \lim_{x \to a} h(x) = H$$

1. $$\lim_{x \to a}(f(x) + g(x)) = +\infty.$$
2. $$\lim_{x \to a}(f(x) + h(x)) = +\infty.$$
3. $$\lim_{x \to a}(f(x) - g(x)) = \textit{undetermined.}$$
4. $$\lim_{x \to a}(f(x) - h(x)) = +\infty.$$

## Theorem

5.

$$\lim_{x \to a} cf(x) = \begin{cases} & c > 0 \\ & c = 0 \\ & c < 0 \end{cases}$$

6.

$$\lim(f(x).g(x)) =$$

7.

$$\lim_{x \to a}(f(x).h(x)) = \begin{cases} & H > 0 \\ & H = 0 \\ & H < 0 \end{cases}$$

8.

$$\lim_{x \to a} \frac{h(x)}{f(x)} =$$

## Theorem

5.
$$\lim_{x\to a} cf(x) = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$$

6.
$$\lim(f(x).g(x)) = +\infty.$$

7.
$$\lim_{x\to a}(f(x).h(x)) = \begin{cases} +\infty & H > 0 \\ undetermined & H = 0 \\ -\infty & H < 0 \end{cases}$$

8.
$$\lim_{x\to a} \frac{h(x)}{f(x)} = 0.$$

## Example

Consider the following three functions:

$$f(x) = x^{-2} \qquad g(x) = 2x^{-2} \qquad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x\to 0} f(x) = +\infty \qquad \lim_{x\to 0} g(x) = +\infty \qquad \lim_{x\to 0} h(x) = +\infty.$$

Then

1. $$\lim_{x\to 0}(f(x) - g(x)) =$$
2. $$\lim_{x\to 0}(f(x) - h(x)) =$$
3. $$\lim_{x\to 0}(g(x) - h(x)) =$$

## Example

Consider the following three functions:

$$f(x) = x^{-2} \qquad g(x) = 2x^{-2} \qquad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x\to 0} f(x) = +\infty \qquad \lim_{x\to 0} g(x) = +\infty \qquad \lim_{x\to 0} h(x) = +\infty.$$

Then

1. $$\lim_{x\to 0}(f(x) - g(x)) = \lim_{x\to 0} x^{-2} = \infty$$

2. $$\lim_{x\to 0}(f(x) - h(x)) = \lim_{x\to 0}(1) = 1$$

3. $$\lim_{x\to 0}(g(x) - h(x)) = \lim_{x\to 0} x^{-2} + 1 = \infty$$

# Outline For the Session Five

- Limit at Infinity
- Continuity
- Continuous from the left and from the right
- Arithmetic of continuity
- continuity of composites
- Intermediate Value Theorem

### Example

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|---|
| $\frac{1}{x^2}$ | 100 | 10000 | $10^6$ | | $10^6$ | 10000 | 100 |

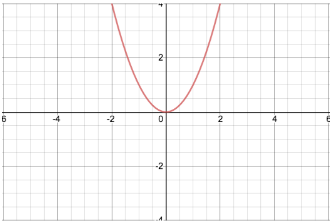
Consider that if

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a} g(x) = \infty$$

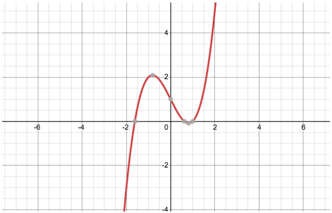
Then

$$\lim_{x \to a} (f(x) - g(x)) = \text{undetermined}$$

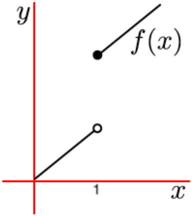
# Continuity



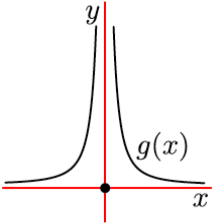
6
-4
-2
0
2
4
6
2
-2

-6
-4
-2
0
2
4
6
4
2
-2
-4

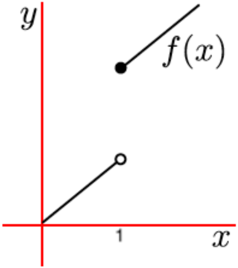



y
f(x)
1
x

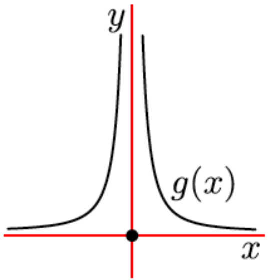
y
g(x)
x

$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$




jump discontinuity

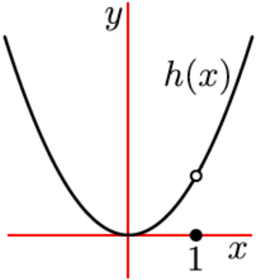
$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$




infinite discontinuity

$$h(x) = \begin{cases} \frac{x^3 - x^2}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$

removable discontinuity



# Outline - September 16, 2019

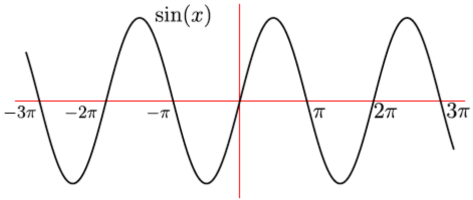
- **Section 1.6:**
  - Arithmetic of continuity
  - Continuity of composites
  - Intermediate Value Theorem
- **Section 2.1:**
  - Revisiting tangent lines

# Arithmetic of continuity

### Theorem

(**Arithmetic of continuity**) *Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at $a$. Then the following functions are also continuous at $x = a$.*

- $f(x) + g(x)$ *and* $f(x) - g(x)$,
- $cf(x)$ *and* $f(x)g(x)$, *and*
- $\frac{f(x)}{g(x)}$ *provided* $g(a) \neq 0$.

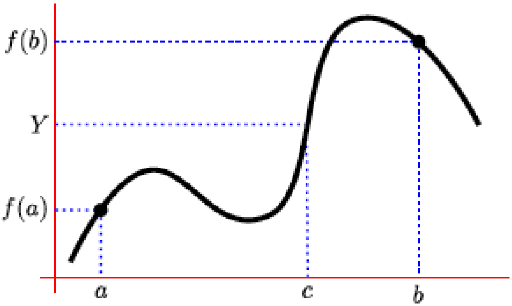


sin(x)
−3π
−2π
−π
π
2π
3π

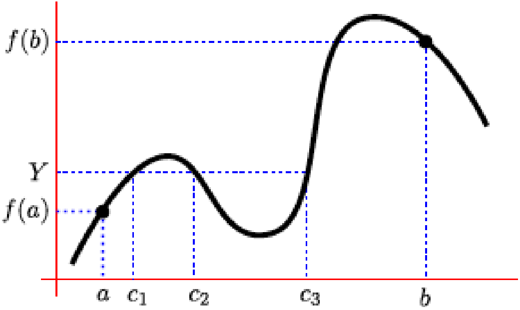
## Intermediate value theorem(IVT)

## Theorem

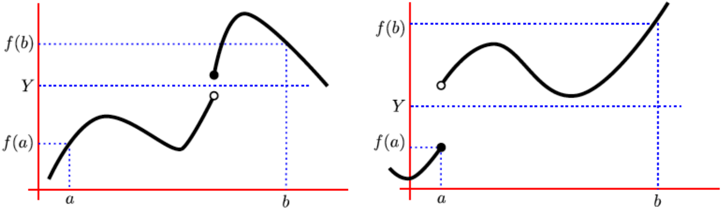
**(Intermediate value theorem(IVT))**



# The existence not the uniqueness of $c$ in IVT

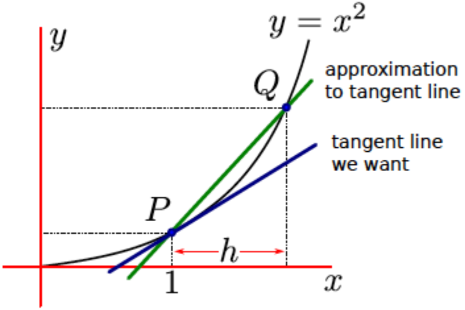


# Not continuous functions at $[a, b]$ do not satisfy IVT



## Revisiting tangent lines

# Revisiting tangent lines




$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \leftarrow \quad \text{slope of the tangent line at } x = 1$$

# Outline - September 18, 2019

- **Section 2.2:**
  - Definition of the derivative
  - Examples
  - Where is the derivative undefined

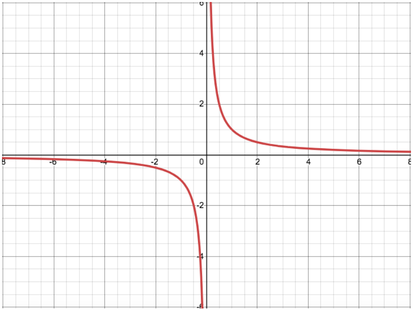
## Definition of the derivative

- $$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$
- $$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

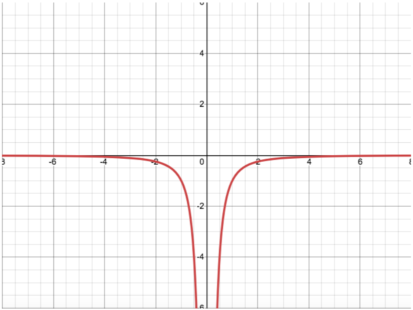
## Examples

- $f(x) = c$
- $f(x) = x$
- $f(x) = x^2$
- $f(x) = \frac{1}{x}$
- $f(x) = \sqrt{x}$
- $f(x) = |x|$

# $y = \frac{1}{x}$ and its derivative $-\frac{1}{x^2}$

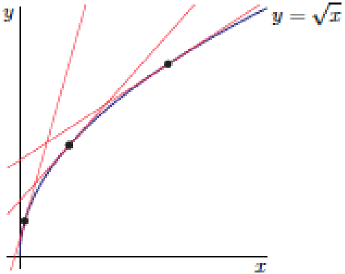



$$y = \frac{1}{x}$$

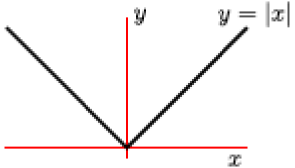



$$y = \frac{-1}{x^2}$$

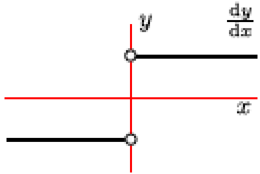
# Tangent lines to $y = \sqrt{x}$



# The derivative of the function $f(x) = |x|$: not differentiable at $x = 0$

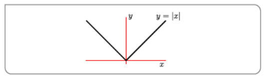



The derivative of the function $f(x) = |x|$

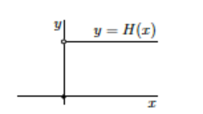


## Where a function is not differentiable at $x = a$?

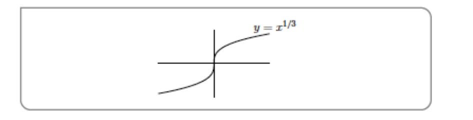
- Having a Sharp Corner at $x = a$




- The function is not continuous at $x = a$




- Having a tangent line, but the slope of the tangent line at $x = a$ is infinity

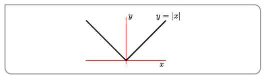


# Outline - September 20, 2019

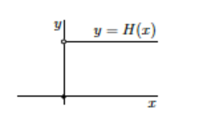
- **Section 2.2:**
  - Not differentiable examples
  - The relation between continuous and differentiable functions
- **Section 2.3:**
  - Interpretations of the derivative

## Where a function is not differentiable at $x = a$?

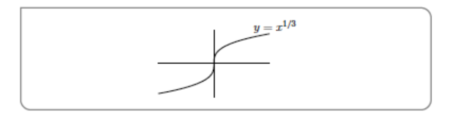
- Having a Sharp Corner at $x = a$




- The function is not continuous at $x = a$

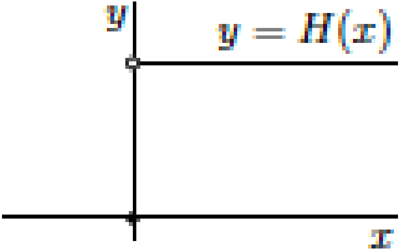



- Having a tangent line, but the slope of the tangent line at $x = a$ is infinity



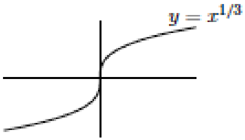
# An example of a discontinuous and not differentiable function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$



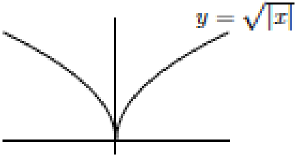
# An example of a function with a tangent line with slope infinity at $x = 0$

$f(x) = x^{1/3}$

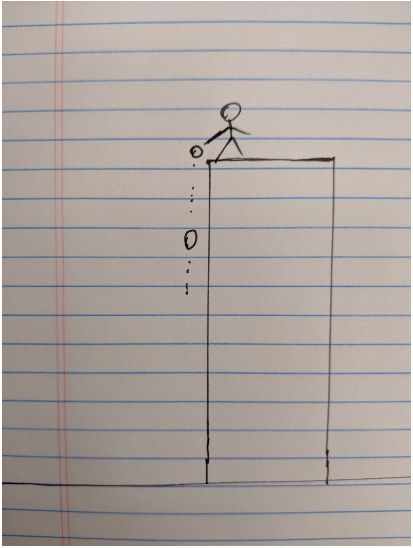


# An example of a continuous and **not** differentiable function

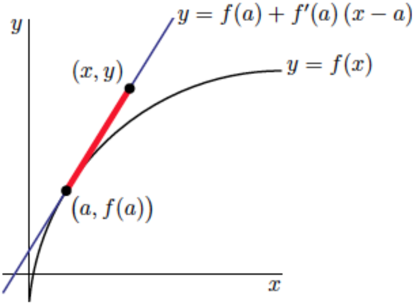
$y = \sqrt{|x|}$



# Instantaneous rate of change



# Finding tangent line to a curve at $x = a$




A line segment of a tangent line

$$y = f(a) + f'(a)(x - a)$$