

MATH100: Differential Calculus with Application to Physical Sciences and Engineering

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Chapter 1

Limits

What does this mean

$$\lim_{x \rightarrow a} f(x) = L?$$

The "limit" appears when we want to

- find the tangent to a curve; or
- find the velocity of an object.

1.1 Tangent line



The **tangent line to a curve** $y = f(x)$ at a point P (if exists) is a line L that there is a neighborhood for P such that in that neighborhood the line L touches (does not cross) the curve $y = f(x)$ only at P (and not other points in that neighborhood).

The equation of a line

- The formula for a line that passes through (x_1, y_1) with slope m is

$$y = y_1 + m(x - x_1).$$

- Given two points (x_1, y_1) and (x_2, y_2) on a line, then the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

and the formula for the line then is

$$y = y_1 + m(x - x_1).$$

Example 1.1.1. Find the equation of the line with slope -3 that passes through $(1, 2)$.

Solution. The equation of the line is

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

Example 1.1.2. Find the equation of the line that passes through $(1, 2)$ and $(2, -1)$.

Solution. First we find the slope which is

$$\frac{-1 - 2}{2 - 1} = -3.$$

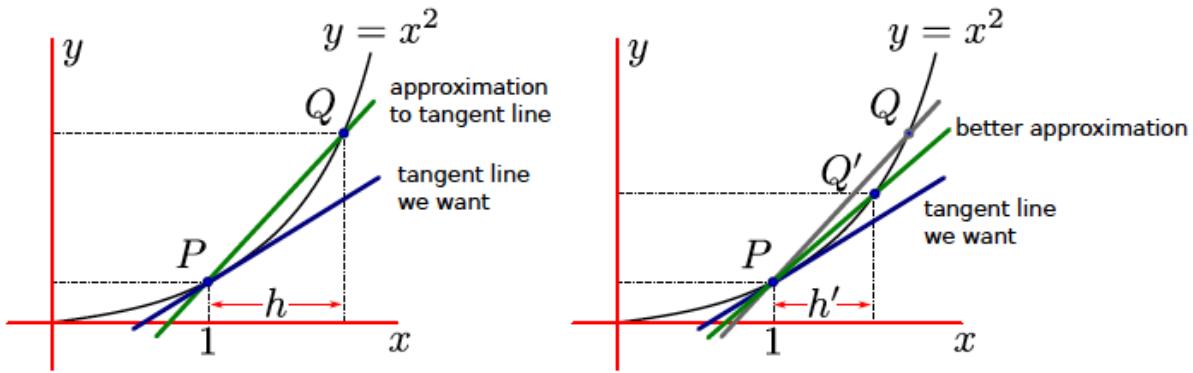
Then the equation of the line is

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

The equation of a tangent line: Given a curve $y = f(x)$ and a point P on the curve, how to find the slope of the tangent to a curve at P : let do this through an example.

Example 1.1.3. Find the tangent line to the curve $y = x^2$ that passes through $P = (1, 1)$.





So we want to find the slope the line that passes through the points $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (1 + h, (1 + h)^2)$. The slope then is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1 + h)^2 - 1^2}{(1 + h) - 1} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(h + 2)}{h} = 2 + h$$

h	$m = \frac{(1+h)^2 - 1^2}{(1+h) - 1}$
0.1	2.1
0.01	2.01
0.001	2.001

When h gets smaller and smaller, the slope will be closer and closer to the slope of the tangent line to $y = x^2$ at $(1, 1)$, which the slope will be closer and closer to 2, we can write this more mathematically as

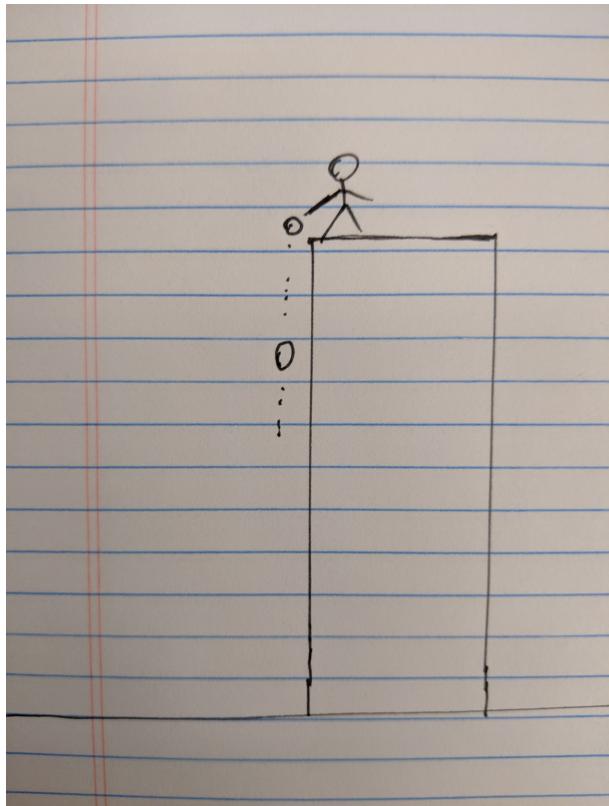
$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1^2}{(1 + h) - 1} = 2$$

Read: the limit of $\frac{(1+h)^2 - 1^2}{(1+h) - 1}$ as h approaches 0 is 2.
Tangent line is

$$y = 1 + 2(x - 1) = 2x - 1.$$

1.2 Velocity

- Let t be elapsed time measured in second
- $S(t)$ be the distance the ball has fallen in meters
- What is $S(0)$? $S(0) = 0$.
- (**Galileo**) $S(t) = 4.9t^2$.



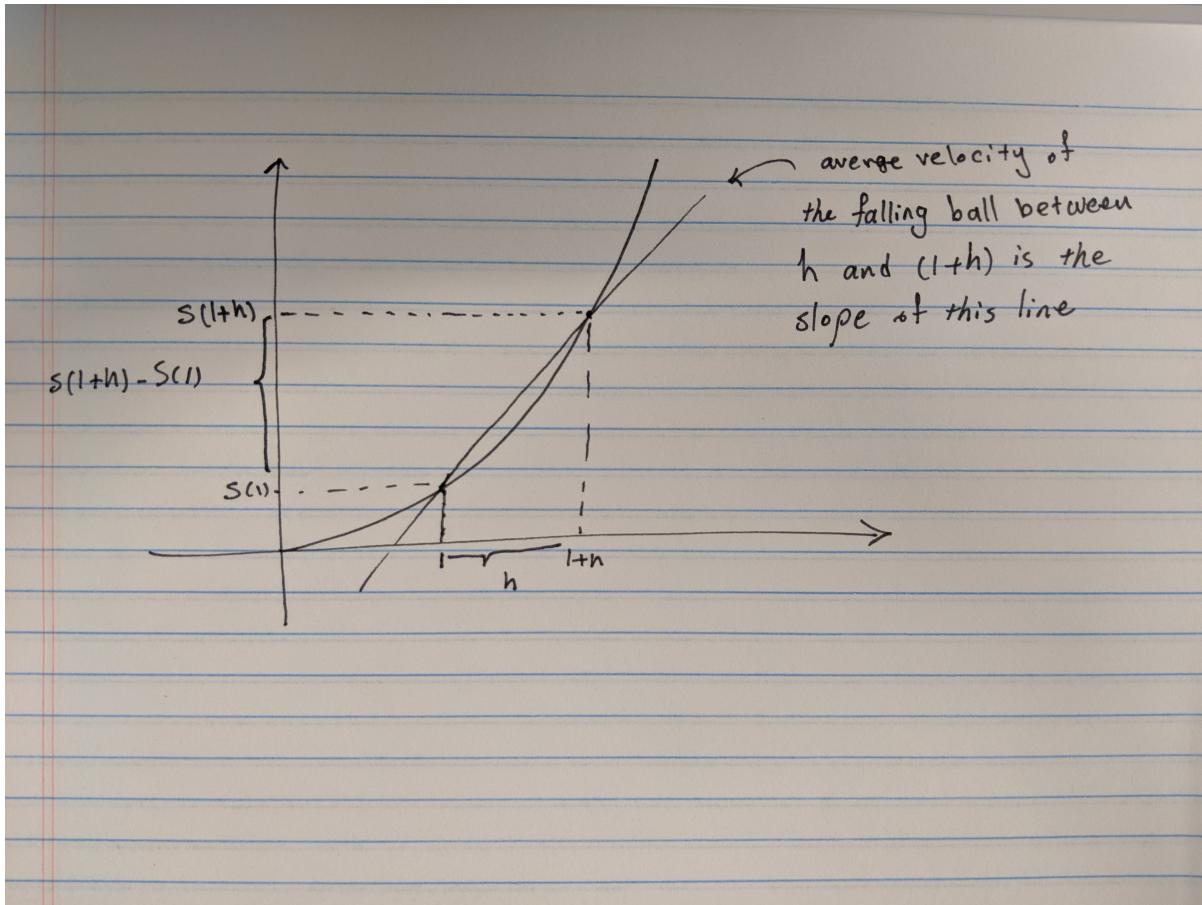
Question: How fast the ball is fallen after 1 second, that is, find $v(1)$, the velocity at $t = 1$?

$$\text{average velocity} = \frac{\text{difference in position}}{\text{difference in time}} = \frac{S(t_2) - S(t_1)}{t_2 - t_1}.$$

To answer the question we should find the average velocity of the falling ball between $(1 + h)$ and 1. So,

average velocity when $(t_2 = 1 + h)$ and $(t_1 = 1)$

$$= \frac{S(1 + h) - S(1)}{h} = \frac{4.9(1 + h)^2 - 4.9}{h} = 4.9(2 + h).$$



time window	average velocity
$1 \leq t \leq 1.1$	10.29
$1 \leq t \leq 1.01$	9.84
$1 \leq t \leq 1.01$	9.8049
$1 \leq t \leq 1.001$	9.80049

So we can write

$$v(1) = \lim_{h \rightarrow 0} \frac{S(1+h) - S(1)}{h} = 9.8.$$

More generally:

We define the instantaneous velocity at time $t = a$ to be the limit

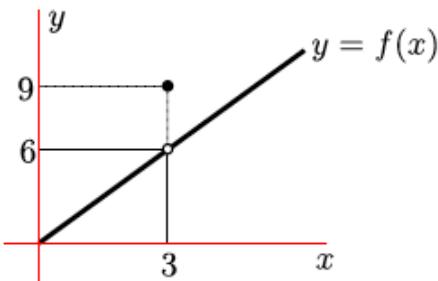
$$v(a) = \lim_{h \rightarrow 0} \frac{S(a+h) - S(a)}{h}$$

1.3 The limit of a function

To arrive at the definition of limit, we start with a very simple example.

Example 1.3.1. Consider the following function

$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



If we plug in some numbers very close to 3 (but not exactly 3) into the function we see

x	2.9	2.99	2.999	○	3.001	3.01	3.1
$f(x)$	5.8	5.98	5.998	○	6.002	6.02	6.2

So as x moves closer and closer to 3, without being exactly 3, we see that the function moves closer and closer to 6. We can then write this as

$$\lim_{x \rightarrow 3} f(x) = 6.$$

Definition. (Informal definition of limit) We write

$$\lim_{x \rightarrow a} f(x) = L.$$

if the value of the function $f(x)$ is sure to be arbitrary close to L whenever the value of x is close enough to a , without being exactly a .

Example 1.3.2. Let $f(x) = \frac{x-2}{x^2+x-6}$ and find its limit as $x \rightarrow 2$.

Solution. We want to find

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2+x-6}.$$

Important point: if we compute $f(2)$, then we have $\frac{0}{0}$ which is undefined.

Again we plug in numbers close to 2 and we have

x	1.9	1.99	1.999	○	2.001	2.01	2.1
$f(x)$	0.20408	0.20040	0.20004	○	0.19996	0.19960	0.19608

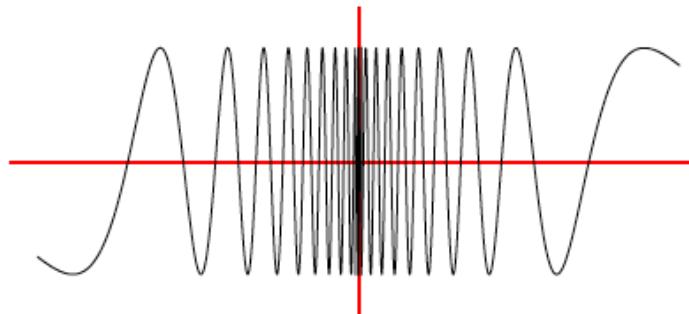
So

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2+x-6} = 2.$$

Example 1.3.3. Consider the following function $f(x) = \sin(\pi/x)$. Find the limit as $x \rightarrow 0$ of $f(x)$.

Solution. When x is getting closer and closer to 0, it oscillates faster and faster. Since the function does not approach a single number as we bring x closer and closer to zero, the limit does not exist. Thus,

$$\lim_{x \rightarrow 0} \sin(\pi/x) = \text{DNE}$$



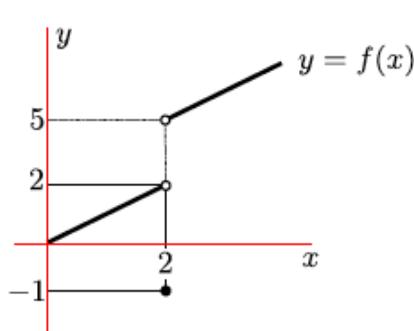
Example 1.3.4. Consider the function

$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$

Find

$$\lim_{x \rightarrow 2} f(x).$$

Solution.



Let again plug in some numbers close to 2 (but not exactly 2)

x	1.9	1.99	1.999	○	2.001	2.01	2.1
$f(x)$	1.9	1.99	1.999	○	5.001	5.01	5.1

Now when we approach from below (or left), we seem to be getting closer to 2 ($\lim_{x \rightarrow 2^-} f(x) = 2$), but when we approach from above (or right) we seem to be getting closer to 5 ($\lim_{x \rightarrow 2^+} f(x) = 5$). Since we are not approaching the same number the limit does not exist.

$$\lim_{x \rightarrow 2} f(x) = \text{DNE}$$

Definition. (Informal definition of one-sided limits) We write

$$\lim_{x \rightarrow a^-} f(x) = K$$

when the value of $f(x)$ gets closer and closer to K when $x < a$ and x moves closer and closer to a . Since the x -values are always less than a , we say that x approaches a from below (or left). This is also often called the left-hand limit since the x -values lie to the left of a on a sketch of the graph.

We similarly write

$$\lim_{x \rightarrow a^+} f(x) = L$$

when the values of $f(x)$ gets closer and closer to L when $x > a$ and x moves closer and closer to a . For similar reason we say that x approaches a from above, and sometimes to this as the the right-hand limit.

Theorem 1.3.5.

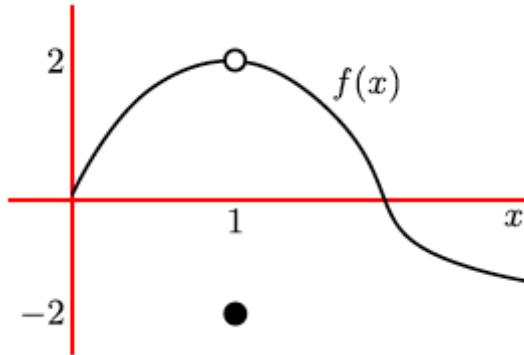
$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x) = L$$

- If the limit of $f(x)$ as x approaches a exists and is equal to L , then both the left-hand and right-hand limits exist and are equal to L .
- If the left-hand and right-hand limits as x approaches a exist and are equal, then the limit as x approaches a exists and is equal to the one-sided limits.

Contrapositive of the above argument says

- If either of the left-hand and right-hand limits as x approaches a fail to exist, or if they both exist but are different, then the limit as x approaches a does not exist. AND,
- If the limit as x approaches a does not exist, then the left-hand and right-hand limits are either different or at least one of them does not exist.

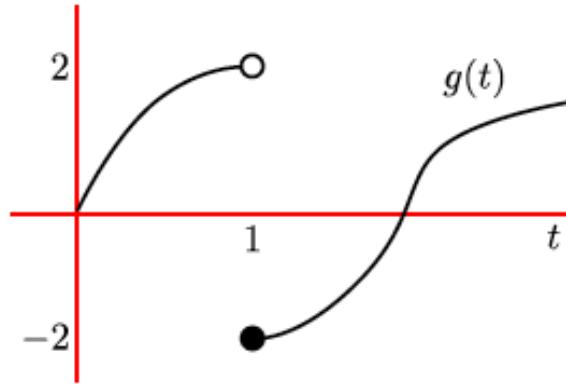
Example 1.3.6. Consider the graph of the function $f(x)$.



Then

$$\lim_{x \rightarrow 1^-} f(x) = 2 \quad \lim_{x \rightarrow 1^+} f(x) = 2 \quad \lim_{x \rightarrow 1} f(x) = 2$$

Example 1.3.7. Consider the graph of the function $g(t)$.

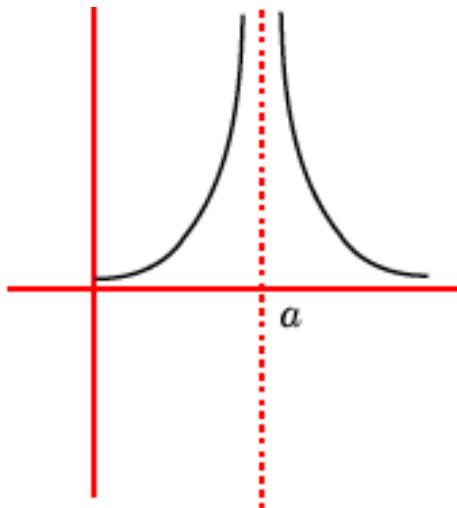


Then

$$\lim_{t \rightarrow 1^-} g(t) = 2 \quad \lim_{t \rightarrow 1^+} g(t) = -2 \quad \lim_{t \rightarrow 1} g(t) = DNE$$

In the following example even though the limit doesn't exist when x approaches a , we can say more.

Example 1.3.8. Consider the graph for the function $f(x)$.



$$\lim_{x \rightarrow a} f(x) = +\infty$$

Example 1.3.9. Consider the graph for the function $g(x)$.



$$\lim_{x \rightarrow a} g(x) = -\infty$$

Definition. We write

$$\lim_{x \rightarrow a} f(x) = +\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as x gets closer and closer to a , without being exactly a .

Similarly, we write

$$\lim_{x \rightarrow a} f(x) = -\infty$$

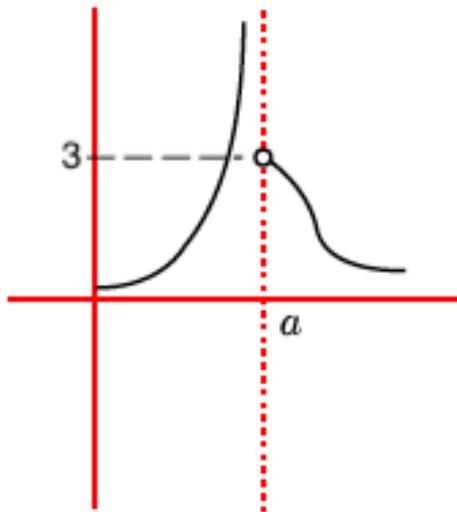
when the value of the function $f(x)$ becomes arbitrarily large and negative as x gets closer and closer to a , without being exactly a .

Example 1.3.10.

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \lim_{x \rightarrow 0} -\frac{1}{x^2} = -\infty$$

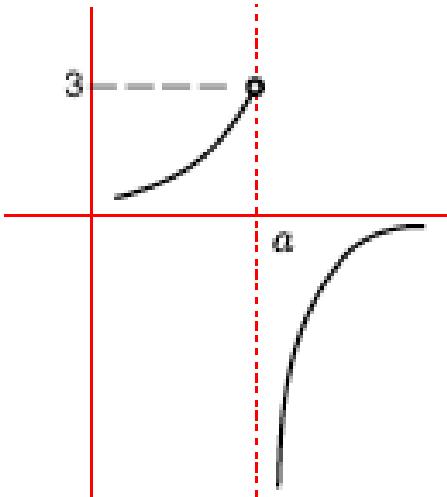
Important Point: Do not think of “ $+\infty$ ” and “ $-\infty$ ” in these statements as numbers. When we write $\lim_{x \rightarrow a} f(x) = +\infty$, it says “the function $f(x)$ becomes arbitrary large as x approaches a ”.

Example 1.3.11. Consider the graph for the function $h(x)$.



$$\lim_{x \rightarrow a^-} h(x) = +\infty \quad \lim_{x \rightarrow a^+} h(x) = 3 \quad \lim_{x \rightarrow a} h(x) = \text{DNE}$$

Example 1.3.12. Consider the graph for the function $s(x)$.



$$\lim_{x \rightarrow a^-} s(x) = 3 \quad \lim_{x \rightarrow a^+} s(x) = -\infty \quad \lim_{x \rightarrow a} s(x) = \text{DNE}$$

Definition. We write

$$\lim_{x \rightarrow a^+} f(x) = +\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as x gets closer and closer to a from above (equivalently, from right), without being exactly a . Similarly, we write

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

when the values of the function $f(x)$ becomes arbitrarily large and negative as x gets closer and closer to a from above (equivalently, from right), without being exactly a .

The notation

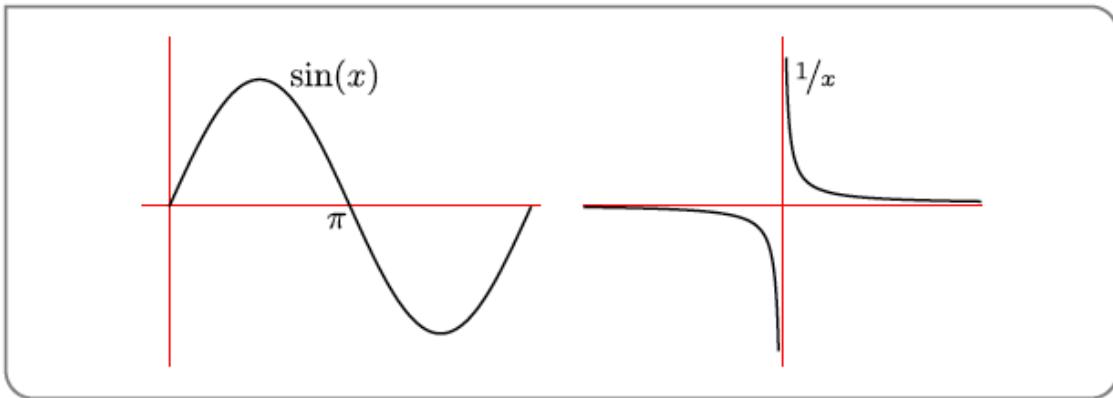
$$\lim_{x \rightarrow a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \rightarrow a^-} f(x) = -\infty$$

has a similar meaning except that limits are approached from below (from left).

Example 1.3.13. Consider the function

$$g(x) = \frac{1}{\sin(x)}.$$

Find the one-side limits of this function as $x \rightarrow \pi$.



- As $x \rightarrow \pi$ from the left, $\sin(x)$ is a small positive number that is getting closer and closer to zero. That is, as $x \rightarrow \pi^-$, we have that $\sin(x) \rightarrow 0$ through positive numbers (i.e. from above). Now look at the graph of $1/x$, and think what happens as we move $x \rightarrow 0^+$, the function is positive and becomes larger and larger.
So as $x \rightarrow \pi$ from the left, $\sin(x) \rightarrow 0$ from above, and so $1/\sin(x) \rightarrow +\infty$.
- By very similar reasoning, as $x \rightarrow \pi$ from the right, $\sin(x)$ is a small negative number that gets closer and closer to zero. So as $x \rightarrow \pi$ from the right, $\sin(x) \rightarrow 0$ through negative numbers (i.e. from below) and so $1/\sin(x) \rightarrow -\infty$.

Thus

$$\lim_{x \rightarrow \pi^-} \frac{1}{\sin(x)} = +\infty \qquad \lim_{x \rightarrow \pi^+} \frac{1}{\sin(x)} = -\infty$$

1.4 Calculating Limits with Limit Laws

Theorem 1.4.1. Let $a, c \in \mathbb{R}$. The following two limits hold

$$\lim_{x \rightarrow a} c = c \quad \lim_{x \rightarrow a} x = a$$

Theorem 1.4.2. (Arithmetic of Limits) Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x 's that lie in some interval about a (but f and g need not to be defined exactly at a).

$$\lim_{x \rightarrow a} f(x) = F \quad \lim_{x \rightarrow a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. Then the following limits hold

- $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$ – limit of the sum is the sum of the limits.
- $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$ – limit of the difference is the difference of the limits.
- $\lim_{x \rightarrow a} cf(x) = cF$.
- $\lim_{x \rightarrow a} (f(x).g(x)) = F.G$ – limit of the product is the product of the limits.
- If $G \neq 0$ then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}$.

Example 1.4.3. Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} g(x)} = \frac{3}{2} = 3.$$

Example 1.4.4.

$$\lim_{x \rightarrow 3} 4x^2 - 1 = 4 \times \lim_{x \rightarrow 3} x^2 - \lim_{x \rightarrow 3} 1 = 35.$$

$$\lim_{x \rightarrow 2} \frac{x}{x-1} = \frac{\lim_{x \rightarrow 2} x}{\lim_{x \rightarrow 2} x - \lim_{x \rightarrow 1} 1} = \frac{2}{2-1} = 2.$$

Consider that we apply the theorem Arithmetic of Limits to compute the limit of a ratio if the limit of denominator is not zero. **What will happen if the limit of denominator is zero:**

- the limit does not exist, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = DNE$$

– the limit is $\pm\infty$, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x^4} = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{-x^2}{x^4} = \lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty.$$

– the limit is 0, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

– the limit exists and it nonzero, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x} = 1.$$

Theorem 1.4.5. Let n be a positive integer, let $a \in R$ and let f be a function so that

$$\lim_{x \rightarrow a} f(x) = F$$

for some real number F . Then the following holds

$$\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit. Similarly, if

- n is an even number and $F > 0$, or
- n is an odd number and F is any real number

then

$$\lim_{x \rightarrow a} (f(x))^{1/n} = \left(\lim_{x \rightarrow a} f(x) \right)^{1/n} = F^{1/n}.$$

Example 1.4.6.

$$\lim_{x \rightarrow 4} x^{1/2} = 4^{1/2} = 2.$$

$$\lim_{x \rightarrow 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \rightarrow 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = 13^{1/3}$$

Example 1.4.7. Compute the following limits.

$$1. \lim_{x \rightarrow 2} \frac{x^3 - x^2}{x - 1}$$

$$2. \lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1}$$

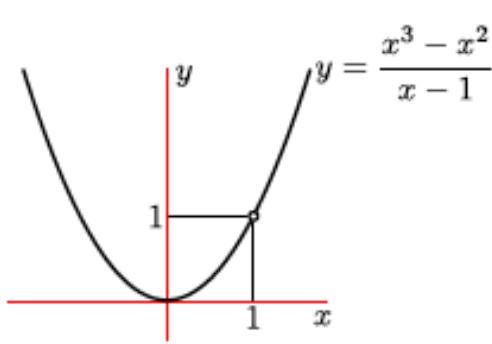
Solution. 1. $\lim_{x \rightarrow 2} \frac{x^3 - x^2}{x - 1} = 4$.

2. Consider that $\lim_{x \rightarrow 1} x^3 - x^2 = 0$ and $\lim_{x \rightarrow 1} x - 1 = 0$. However,

$$\frac{x^3 - x^2}{x - 1} = \frac{x^2(x - 1)}{x - 1},$$

thus

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases}$$



And so

$$\lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \rightarrow 1} x^2 = 1.$$

The reasoning in the above example can be made more general:

Theorem 1.4.8. If $f(x) = g(x)$ except when $x = a$ then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

provided the limit of g exists.

We mostly use the above theorem when we end up with $\frac{0}{0}$.

Example 1.4.9. Compute

$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1}{h}.$$

Solution. Note that

$$\frac{(1 + h)^2 - 1}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(2 + h)}{h}.$$

Thus,

$$\frac{(1 + h)^2 - 1}{h} = \begin{cases} 2 + h & h \neq 0 \\ \text{undefined} & h = 0. \end{cases}$$

And so

$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1}{h} = \lim_{h \rightarrow 0} 2 + h = 2.$$

We now present a slightly harder example.

Example 1.4.10. Compute the limit

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1 + x} - 1}.$$

Solution. Both the limits of the numerator and denominator as $x \rightarrow 0$ are 0, so we cannot use the Theorem Arithmetic of limits. We now can simply multiply the numerator and denominator by the conjugation of $\sqrt{1+x} - 1$, that is, $\sqrt{1+x} + 1$. We have

$$\begin{aligned}
 \frac{x}{\sqrt{1+x}-1} &= \frac{x}{\sqrt{1+x}-1} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} && \text{multiply by } \frac{\text{conjugate}}{\text{conjugate}} = 1 \\
 &= \frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)} && \text{bring things together} \\
 &= \frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x})^2 - 1 \cdot 1} && \text{since } (a-b)(a+b) = a^2 - b^2 \\
 &= \frac{x(\sqrt{1+x}+1)}{1+x-1} && \text{clean up a little} \\
 &= \frac{x(\sqrt{1+x}+1)}{x} && \\
 &= \sqrt{1+x}+1 && \text{cancel the } x
 \end{aligned}$$

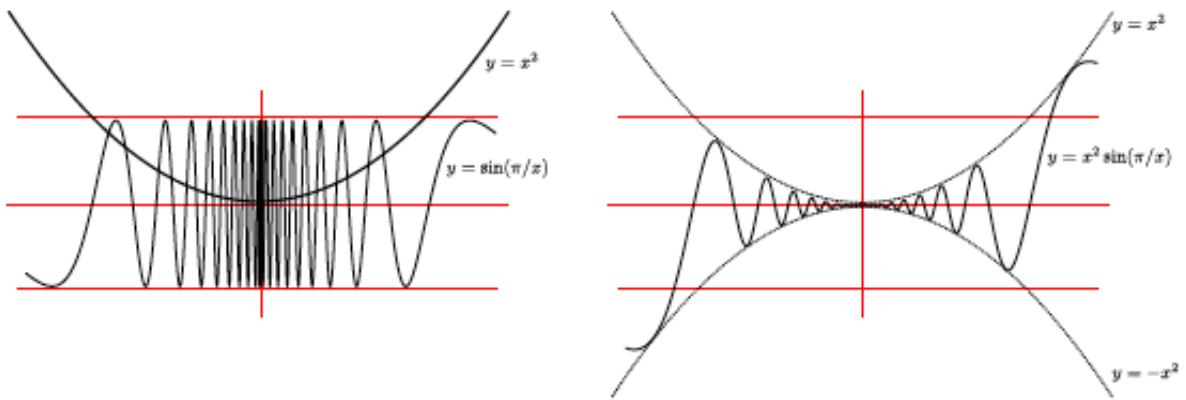
So now we have

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x}-1} &= \lim_{x \rightarrow 0} \sqrt{1+x}+1 \\
 &= \sqrt{1+0}+1 = 2
 \end{aligned}$$

Before we move to the next section and study the limits at infinity, we have one more theorem to state.

Example 1.4.11. Compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



Solution. It is not possible to simply use the theorem Arithmetic of Limits since the limit of $\sin\left(\frac{\pi}{x}\right)$ as $x \rightarrow 0$ does not exist. Since $-1 \leq \sin(\theta) \leq 1$ for all real numbers θ , we have

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1 \quad \text{for all } x \neq 0$$

Multiplying the above by x^2 we see that

$$-x^2 \leq x^2 \sin\left(\frac{\pi}{x}\right) \leq x^2 \quad \text{for all } x \neq 0.$$

Since

$$\lim_{x \rightarrow 0} x^2 = \lim_{x \rightarrow 0} (-x^2) = 0$$

by the sandwich (or squeeze or pinch) theorem (look at below for the sandwich theorem) we have

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0.$$

Theorem 1.4.12. (sandwich (or squeeze or pinch) theorem) Let $a \in \mathbb{R}$ and let f, g, h be three functions so that

$$f(x) \leq g(x) \leq h(x)$$

for all x in an interval around a , except possibly at $x = a$. Then if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then it is also the case that

$$\lim_{x \rightarrow a} g(x) = L.$$

Example 1.4.13. Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \rightarrow 1} f(x)?$$

Solution. Consider that

$$\lim_{x \rightarrow 1} x = 1 \quad \text{and} \quad \lim_{x \rightarrow 1} x^2 - 2x + 2 = 1.$$

Therefore, by the sandwich/pinch/squeeze theorem

$$\lim_{x \rightarrow 1} f(x) = 1.$$

1.5 Limits at Infinity

Example 1.5.1. We want to compute

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{x}$$

By plug in some large numbers into $\frac{1}{x}$ we have

-10000	-1000	-100	100	1000	10000
-0.0001	0.001	-0.01	0.01	0.001	0.0001

We see that as x is getting bigger and positive the function $\frac{1}{x}$ is getting closer to 0. Thus,

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0.$$

Moreover,

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

Definition. (Informal limit at infinity.) We write

$$\lim_{x \rightarrow \infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and positive.

Similarly, we write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and negative.

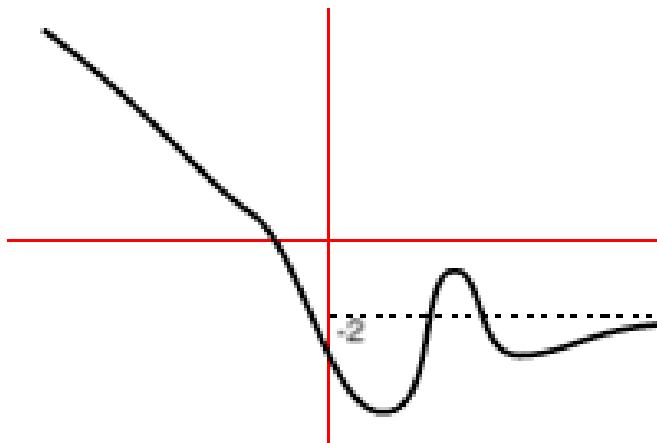
Example 1.5.2. Consider the graph of the function $f(x)$.



Then

$$\lim_{x \rightarrow \infty} f(x) = -2 \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = 2$$

Example 1.5.3. Consider the graph of the function $g(x)$.



Then

$$\lim_{x \rightarrow \infty} g(x) = -2 \quad \text{and} \quad \lim_{x \rightarrow -\infty} g(x) = +\infty$$

Same as usual we start with two very simple building blocks and build other limits from them.

Theorem 1.5.4. *Let $c \in \mathbb{R}$ then the following limits hold*

$$\lim_{x \rightarrow +\infty} c = c \quad \lim_{x \rightarrow -\infty} c = c$$

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0 \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

Theorem 1.5.5. *Let $f(x)$ and $g(x)$ be two functions for which the limits*

$$\lim_{x \rightarrow \infty} f(x) = F \quad \lim_{x \rightarrow \infty} g(x) = G$$

exist. Then the following limits hold

$$\lim_{x \rightarrow \infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x \rightarrow \infty} f(x)g(x) = FG$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \text{provided } G \neq 0$$

and for rational numbers r ,

$$\lim_{x \rightarrow \infty} (f(x))^r = F^r$$

provided that $f(x)^r$ is defined for all x .

The analogous results hold for limits to $-\infty$.

We need a little extra care with the powers of functions.

Warning: Consider that

$$\lim_{x \rightarrow +\infty} \frac{1}{x^{1/2}} = 0$$

However,

$$\lim_{x \rightarrow +\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because $x^{1/2}$ is not defined for $x < 0$.

Example 1.5.6. *Compute the following limit:*

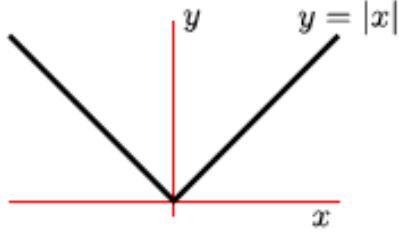
$$\lim_{x \rightarrow \infty} \frac{x^2 - 3x + 4}{3x^2 + 8x + 1}$$

Solution. By factoring x with largest exponent in the numerator and denominator we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2 - 3x + 4}{3x^2 + 8x + 1} &= \lim_{x \rightarrow \infty} \frac{x^2(1 + \frac{-3x}{x^2} + \frac{4}{x^2})}{x^2(3 + \frac{8x}{x^2} + \frac{1}{x^2})} = \lim_{x \rightarrow \infty} \frac{(1 + \frac{-3x}{x^2} + \frac{4}{x^2})}{(3 + \frac{8x}{x^2} + \frac{1}{x^2})} = \\ &\frac{\left(\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \frac{-3x}{x^2} + \lim_{x \rightarrow \infty} \frac{4}{x^2}\right)}{\left(\lim_{x \rightarrow \infty} 3 + \lim_{x \rightarrow \infty} \frac{8x}{x^2} + \lim_{x \rightarrow \infty} \frac{1}{x^2}\right)} = \frac{1}{3}. \end{aligned}$$

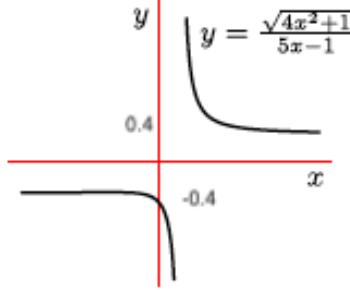
Remark. Note that

$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



Example 1.5.7. Compute the following limit:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{5x - 1}.$$



Solution. Factor the terms with the largest exponents in the numerator and denominator.

We have

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{5x - 1} = \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2(1 + \frac{1}{4x^2})}}{5x(1 - \frac{1}{5x})} = \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2} \sqrt{(1 + \frac{1}{4x^2})}}{5x(1 - \frac{1}{5x})} = \lim_{x \rightarrow \infty} \frac{2|x|}{5x} = \lim_{x \rightarrow \infty} \frac{2x}{5x} = \frac{2}{5}.$$

Example 1.5.8. Compute the following limit:

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{5x - 1}.$$

Solution. By the same kind of computation we have

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{5x - 1} = \lim_{x \rightarrow \infty} \frac{2|x|}{5x}.$$

Consider that since x is getting negative values, we have $|x| = -x$. Therefore,

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{5x - 1} = \lim_{x \rightarrow \infty} \frac{2|x|}{5x} = \lim_{x \rightarrow \infty} \frac{-2x}{5x} = \frac{-2}{5}.$$

Example 1.5.9. Compute the following limit:

$$\lim_{x \rightarrow \infty} (x^{7/5} - x).$$

Solution. We factor the term with the largest exponent, we have

$$\lim_{x \rightarrow \infty} (x^{7/5} - x) = \lim_{x \rightarrow \infty} x^{7/5} \left(1 - \frac{1}{x^{2/5}}\right) = \infty.$$

Theorem 1.5.10. Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that

$$\lim_{x \rightarrow a} f(x) = +\infty \quad \lim_{x \rightarrow a} g(x) = +\infty \quad \lim_{x \rightarrow a} h(x) = H$$

1.

$$\lim_{x \rightarrow a} (f(x) + g(x)) = +\infty.$$

2.

$$\lim_{x \rightarrow a} (f(x) + h(x)) = +\infty.$$

3.

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \text{undetermined}.$$

4.

$$\lim_{x \rightarrow a} (f(x) - h(x)) = +\infty.$$

5.

$$\lim_{x \rightarrow a} cf(x) = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$$

6.

$$\lim_{x \rightarrow a} (f(x).g(x)) = +\infty.$$

7.

$$\lim_{x \rightarrow a} (f(x).h(x)) = \begin{cases} +\infty & H > 0 \\ \text{undetermined} & H = 0 \\ -\infty & H < 0 \end{cases}$$

8.

$$\lim_{x \rightarrow a} \frac{h(x)}{f(x)} = 0.$$

Example 1.5.11. Consider the following three functions:

$$f(x) = x^{-2} \quad g(x) = 2x^{-2} \quad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \rightarrow 0} f(x) = +\infty \quad \lim_{x \rightarrow 0} g(x) = +\infty \quad \lim_{x \rightarrow 0} h(x) = +\infty.$$

Then

•

$$\lim_{x \rightarrow 0} (f(x) - g(x)) = \lim_{x \rightarrow 0} x^{-2} = -\infty$$

•

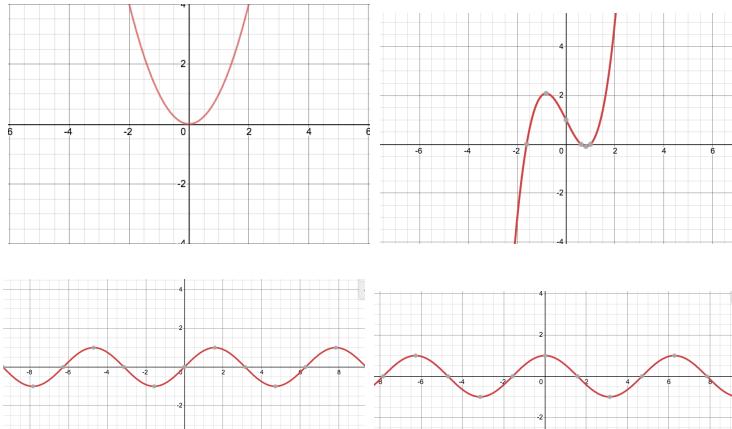
$$\lim_{x \rightarrow 0} (f(x) - h(x)) = \lim_{x \rightarrow 0} (1) = 1$$

•

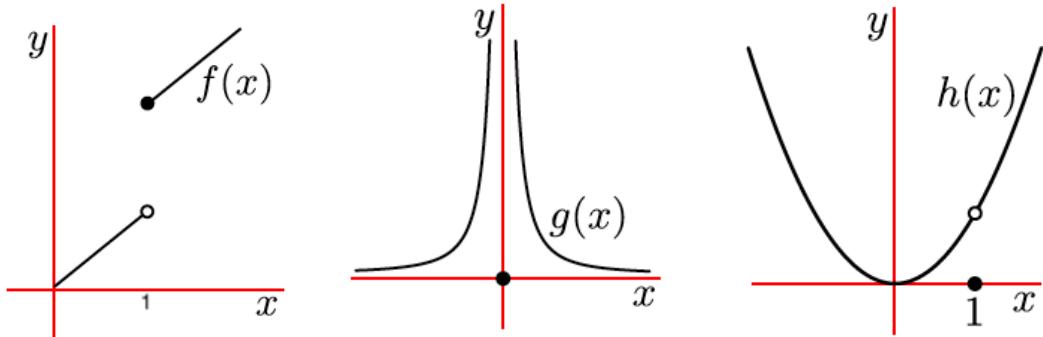
$$\lim_{x \rightarrow 0} (g(x) - h(x)) = \lim_{x \rightarrow 0} x^{-2} + 1 = \infty$$

1.6 Continuity

Look at all the following functions.



All of these functions are continuous. Roughly speaking, a function is continuous if it does not have any abrupt jumps. Now consider the following function.



These functions are not continuous. The function f , g , and h have abrupt jumps at $x = 2$, $x = 0$, and $x = 1$, respectively, so f is not continuous at a , g is not continuous at 0 , and h is not continuous at 1 .

Definition. A function $f(x)$ is continuous at a if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

If a function is not continuous at a then it is said to be discontinuous at a . When we write that f is continuous without specifying a point, then typically this means that f is continuous at a for all $a \in \mathbb{R}$. When we write that $f(x)$ is continuous on the open interval (a, b) then the function is continuous at every point c satisfying $a < c < b$.

From the above definition we immediately have that if f is continuous at a , then

1. $f(a)$ exists;
2. $\lim_{x \rightarrow a^-} f(x)$ exists and is equal to $f(a)$.
3. $\lim_{x \rightarrow a^+} f(x)$ exists and is equal to $f(a)$.

Definition. A function is continuous from the left at a if

$$\lim_{x \rightarrow a^-} = f(a).$$

And a function is continuous from the right at a if

$$\lim_{x \rightarrow a^+} = f(a).$$

Definition. A function $f(x)$ is continuous on an interval $[a, b]$ if

1. $f(x)$ continuous on (a, b) ,
2. $f(x)$ is continuous form the right at a ,
3. $f(x)$ is continuous form the left at b .

Definition. A function $f(x)$ is continuous on an interval $(a, b]$ (*on the interval $[a, b)$*) if

1. $f(x)$ continuous on (a, b) ,
2. $f(x)$ is continuous form the left at b (*from the right at a*).

Example 1.6.1. Consider the function

$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$



- $\lim_{x \rightarrow 1^-} f(x) = 1 \quad \lim_{x \rightarrow 1^+} f(x) = 3 \quad f(1) = 3.$

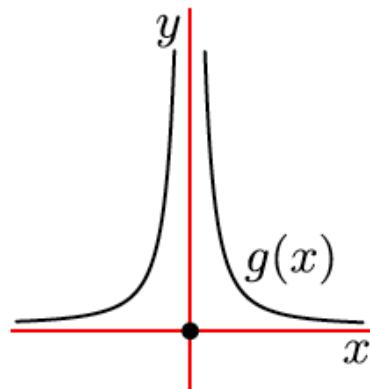
- The function $f(x)$, at $x = 1$ is not continuous because the limit does not exist; however, it is continuous form the right at 1 since

$$\lim_{x \rightarrow 1^+} f(x) = 3 = f(1).$$

- The function $f(x)$, on $[1, \infty)$ (for $x \geq 1$) is continuous.
- The function $f(x)$, on $(-\infty, -1)$ is continuous.

Example 1.6.2. Consider the function

$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



infinite discontinuity

- Consider that

$$\lim_{x \rightarrow 0^-} g(x) = \infty = \lim_{x \rightarrow 0^+} g(x) \quad g(0) = 0.$$

Thus the function $g(x)$ is not continuous at 0 because

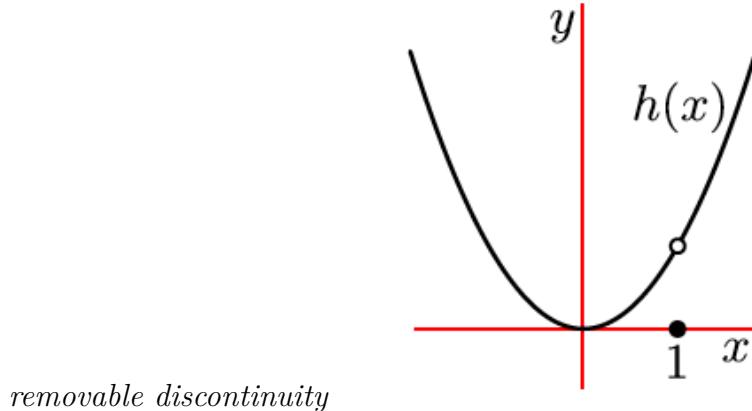
$$\lim_{x \rightarrow 0} g(x) = \infty \neq 0 = g(0).$$

It is not continuous at 0 from the left since $\lim_{x \rightarrow 0^-} g(x) = \infty \neq 0 = g(0)$ and not from the right since $\lim_{x \rightarrow 0^+} g(x) = \infty \neq 0 = g(0)$.

- the function $g(x)$ is continuous at all points in \mathbb{R} except 0.

Example 1.6.3. Consider the function

$$h(x) = \begin{cases} \frac{x^3 - x^2}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$



- $\lim_{x \rightarrow 1^-} h(x) = 1 = \lim_{x \rightarrow 1^+} h(x) \quad f(1) = 0.$
- $\lim_{x \rightarrow 1} h(x) = 1.$
- the function $h(x)$ is not continuous at 1 since

$$\lim_{x \rightarrow 1} h(x) = 1 \neq 0 = h(1).$$

It is not continuous from the left since

$$\lim_{x \rightarrow 1^-} h(x) = 1 \neq 0 = h(1)$$

and not from the right since

$$\lim_{x \rightarrow 1^+} h(x) = 1 \neq 0 = h(1).$$

- the function $h(x)$ is continuous at all points in \mathbb{R} except 1.

Lemma 1.6.4. Let $c \in \mathbb{R}$. The functions

$$f(x) = x \quad g(x) = c$$

are continuous everywhere on the real line.

Theorem 1.6.5. (Arithmetic of continuity) Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at a . Then the following functions are also continuous at $x = a$.

- $f(x) + g(x)$ and $f(x) - g(x)$,
- $cf(x)$ and $f(x)g(x)$, and
- $\frac{f(x)}{g(x)}$ provided $g(a) \neq 0$.

Theorem 1.6.6. The following functions are continuous everywhere in their domains

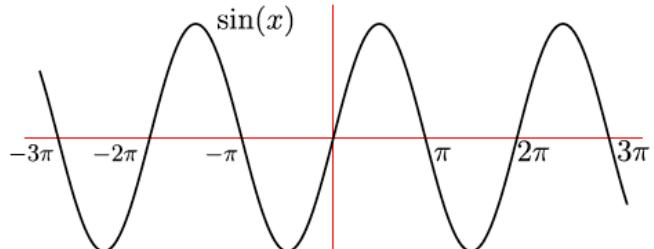
- polynomials and rational functions (for example $f(x) = x^5 + 4x^2 + 1$ and $g(x) = \frac{x^2+1}{x+1}$)
- roots and powers (for example $h(x) = \sqrt{x}$ and $r(x) = 2^x$)
- trig functions and their inverses (for example $k(x) = \sin(x)$ and $t(x) = \cos^{-1}(x)$)
- exponentials and logarithms (for example $s(x) = e^x$ and $q(x) = \ln x$).

Example 1.6.7. Determine when the function $f(x) = \frac{\sin(x)}{x^2-5x+6}$ is continuous? Since both $\sin(x)$ and $x^2 - 5x + 6$ are continuous by the above theorem we only need to check when $x^2 - 5x + 6 = 0$. Note that $x^2 - 5x + 6 = (x - 2)(x - 3)$, thus this polynomial is only zero at $x = 2$ and $x = 3$. Therefore, $f(x)$ is continuous at all points in \mathbb{R} except 2 and 3.

Theorem 1.6.8. If g is continuous at a and $f(x)$ is continuous at $g(a)$, then $(f \circ g)(x) = f(g(x))$ is continuous at $x = a$.

Example 1.6.9. Determine when the function $h(x) = \sqrt{\sin(x)}$ is continuous.

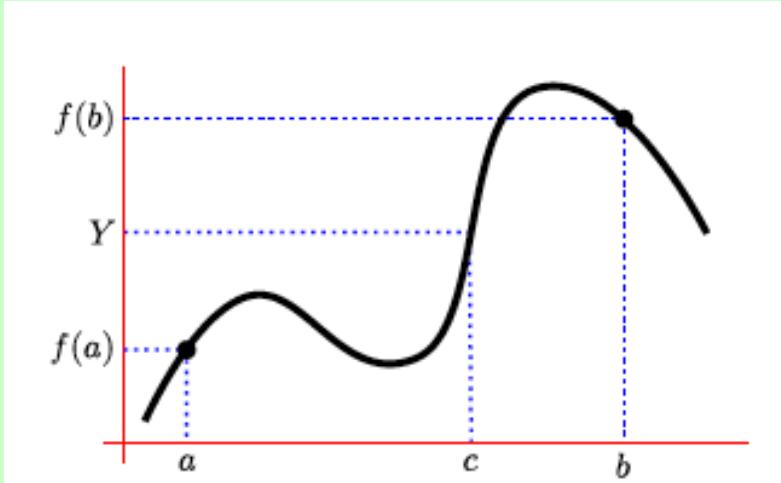
Solution. Let $f(x) = \sqrt{x}$ and $g(x) = \sin(x)$, then $h(x) = (f \circ g)(x)$. We only need to find out at what points $\sin(x)$ is positive.



The function $\sqrt{\sin(x)}$ is continuous if

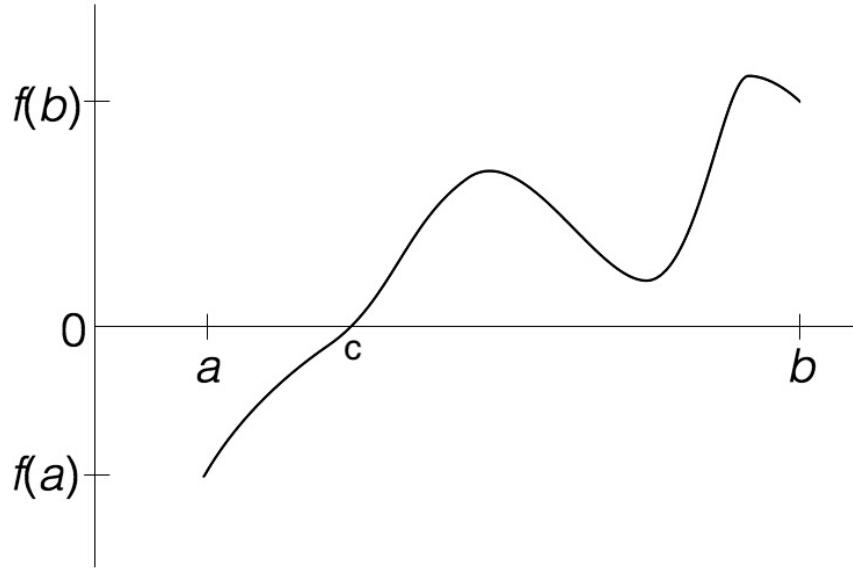
$$x \in [2n\pi, (2n+1)\pi] \quad \text{for all natural numbers } n.$$

Theorem 1.6.10. (Intermediate value theorem(IVT)) Let $a < b$ and let $f(x)$ be a function that is continuous at all points $a \leq x \leq b$. If Y is any number between $f(a)$ and $f(b)$ then there exists some number $c \in [a, b]$ so that $f(c) = Y$.



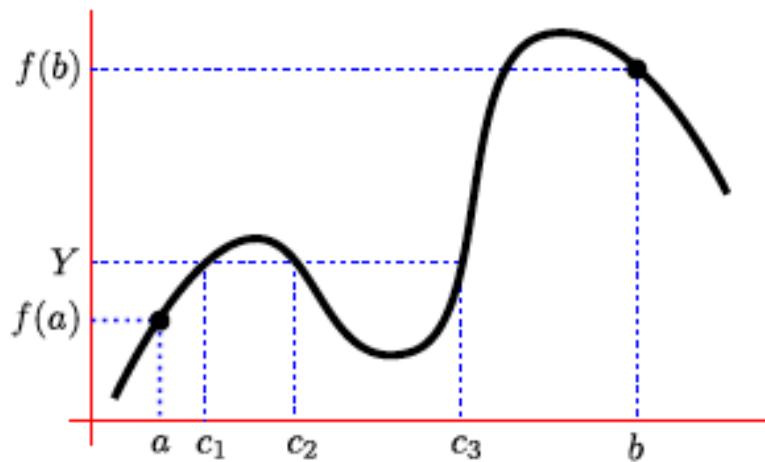
Remark. One of the main application of the IVT theorem is showing a function f has

a zero inside an interval. For example, in the following picture



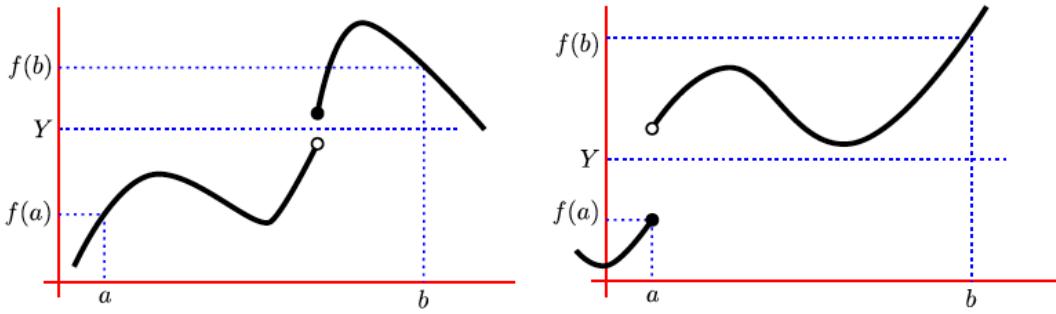
we can see that $f(a) < 0$ and $f(b) > 0$, therefore by IVT, there is a number c between a and b such that $f(c) = 0$.

Remark. If f is continuous and $f(a) \leq Y \leq f(b)$, the IVT merely shows that there is a $a \leq c \leq b$ such that $f(c) = Y$, but it doesn't show how many of them exist. For example, in the following picture, we can see $f(a) \leq Y \leq f(b)$, and there are three numbers c_1, c_2 , and c_3 such that $f(c_1) = f(c_2) = f(c_3) = Y$.



Remark. Consider that if the function f is not continuous at the interval $[a, b]$ then the IVT fails. In the following examples, even though $f(a) \leq Y \leq f(b)$, there is not a number

$a \leq c \leq b$ such that $f(c) = Y$.



Example 1.6.11. Show that the function $f(x) = x - 1 + \sin(\pi x/2)$ has a zero in $0 \leq x \leq 1$.

Solution. Consider that $f(x)$ is a continuous function such that $f(0) = -1$ and $f(1) = 1$. Therefore, by IVT, since $f(0) = -1 \leq 0 \leq 1 = f(1)$, we have $f(c) = 0$ for some $c \in [0, 1]$.

Example 1.6.12. Use the bisection method to find a zero of $f(x) = x - 1 + \sin(\pi x/2)$ that lies between 0 and 1.

Solution.

- Let $a = 0$ and $b = 1$. Then

$$f(0) = -1$$

$$f(1) = 1$$

- Test the point in the middle $x = \frac{1-0}{2} = 0.5$,

$$f(0.5) = 0.2071067813 > 0$$

- Let $a = 0$ and $b = 0.5$. Then

$$f(0) = -1$$

$$f(1) = 0.2071067813$$

So by IVT, there is a zero in $[0, 0.5]$.

- Test the point in the middle $x = \frac{0.5-0}{2} = 0.25$.

$$f(0.25) = -0.3673165675 < 0.$$

- Let $a = 0.25$, $b = 0.5$ where $f(0.25) < 0$ and $f(0.5) > 0$. By IVT there is a zero in the interval $[0.25, 0.5]$.

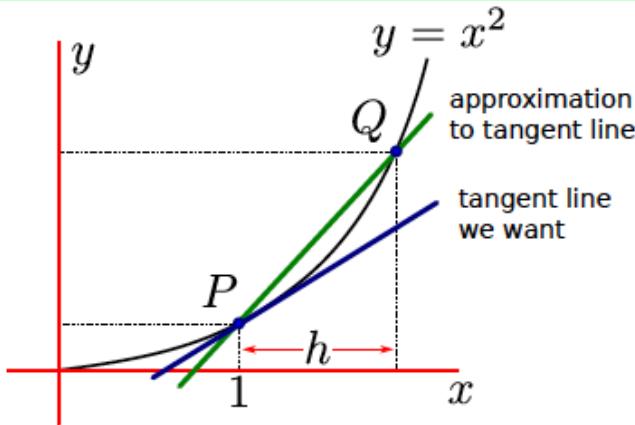
- So without much work we know the location of a zero inside a range of length $1/4$. Each iteration will halve the length of the range and we keep going until we reach the precision we need, though it is much easier to program a computer to do it.

Chapter 2

Derivatives

2.1 Revisiting Tangent Lines

Example 2.1.1. Find the slope of the tangent line to the curve $y = x^2$ that passes through $P = (1, 1)$.



Solution. Consider that the slope of the secant line is

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

And the slope of the tangent line is the same as

$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}.$$

Theorem 2.1.2. Given a function $f(x)$ the slope of the tangent line at $x = a$ (if exists) is

$$\lim_{x \rightarrow a} \frac{f(a+h) - f(a)}{h}.$$

2.2 Definition of the derivative

Definition. (*Derivative at a point*) Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined on an open interval that contains a .

- The derivative of $f(x)$ at $x = a$ is denoted $f'(a)$ and is defined by

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \quad (2.2.1)$$

if the limit exists.

- When the above limit exists, the function $f(x)$ is said to be differentiable at $x = a$. When the limit does not exist, the function $f(x)$ is said to be not differentiable at $x = a$.
- We can equivalently define the derivative $f'(a)$ by the limit

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

To see that these two definitions are the same, we set $x = a + h$ ($x - a = h$) and then when h approaches 0, we have x approaches a , and the limit in 2.2.1 becomes $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$.

Example 2.2.1. Let $a, c \in \mathbb{R}$ be constants. Compute the derivative of the function $f(x) = c$ at $x = a$.

Solution. By the definition of the derivative, we have

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c - c}{h} \\ &= \lim_{h \rightarrow 0} 0 \\ &= 0 \end{aligned}$$

Example 2.2.2. Let $a \in \mathbb{R}$. Compute the limit of the function $g(x) = x$ at $x = a$.

Solution. By the definition of the derivative we have

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a+h) - a}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} \\ &= \lim_{h \rightarrow 0} 1 \\ &= 1. \end{aligned}$$

We have so proved our first theorem which is the following.

Theorem 2.2.3. (easiest derivative) Let $a, c \in \mathbb{R}$ and let $f(x) = c$ and $g(x) = x$. Then

$$f'(a) = 0$$

and

$$g'(a) = 1.$$

Example 2.2.4. Compute the derivative of $f(t) = t^2$ at $t = a$.

Solution. We have that

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 + h^2 + 2ah - a^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 + 2ah}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(h+2a)}{h} \\ &= \lim_{h \rightarrow 0} h + 2a \\ &= 2a \end{aligned}$$

Homework:

Go to this link

<https://www.mooculus.osu.edu/textbook/mooculus.pdf> and download the book "MOOCULUS". Then do the following questions:

- all questions in page 35;
- in page 33 see why $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$. Then do Questions 1-8 page 38;
- in page 42, do questions 1-10.

Bibliography

- [1] CLP1: Differential Calculus by J. Feldman, A. Rechnitzer, and E. Yeager.