

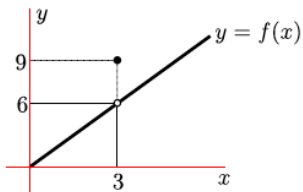
MATH 100

Farid AliniaEIFARD

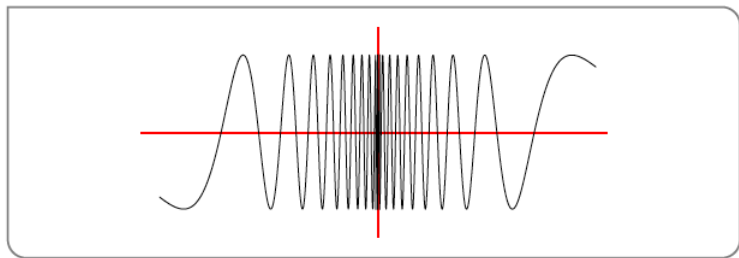
University of British Columbia

2019

$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



$$f(x) = \sin\left(\frac{\pi}{x}\right)$$

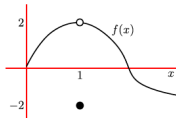


$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$



Example

Consider the graph of the function $f(x)$.



Then

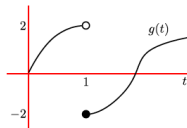
$$\lim_{x \rightarrow 1^-} f(x) =$$

$$\lim_{x \rightarrow 1^+} f(x) =$$

$$\lim_{x \rightarrow 1} f(x) =$$

Example

Consider the graph of the function $g(t)$.



Then

$$\lim_{t \rightarrow 1^-} g(t) =$$

$$\lim_{t \rightarrow 1^+} g(t) =$$

$$\lim_{t \rightarrow 1} g(t) =$$

Example

Consider the graph of the function $f(x)$.



Then

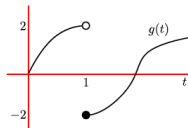
$$\lim_{x \rightarrow 1^-} f(x) = 2$$

$$\lim_{x \rightarrow 1^+} f(x) = 2$$

$$\lim_{x \rightarrow 1} f(x) = 2$$

Example

Consider the graph of the function $g(t)$.



Then

$$\lim_{t \rightarrow 1^-} g(t) = 2$$

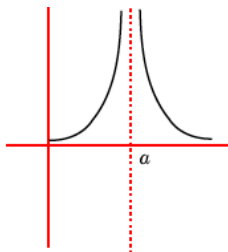
$$\lim_{t \rightarrow 1^+} g(t) = -2$$

$$\lim_{t \rightarrow 1} g(t) = DNE$$

When the limit goes to infinity

Example

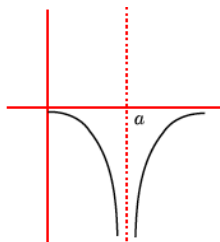
Consider the graph for the function $f(x)$.



$$\lim_{x \rightarrow a} f(x) = +\infty$$

Example

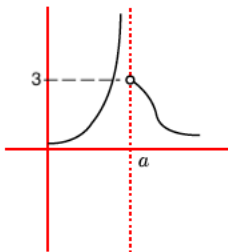
Consider the graph for the function $g(x)$.



$$\lim_{x \rightarrow a} g(x) = -\infty$$

Example

Consider the graph for the function $h(x)$.

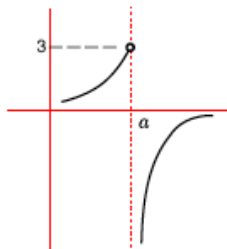


$$\lim_{x \rightarrow a^-} h(x) =$$

$$\lim_{x \rightarrow a^+} h(x) =$$

Example

Consider the graph for the function $s(x)$.

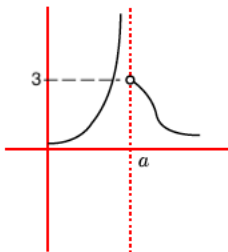


$$\lim_{x \rightarrow a^-} s(x) =$$

$$\lim_{x \rightarrow a^+} s(x) =$$

Example

Consider the graph for the function $h(x)$.

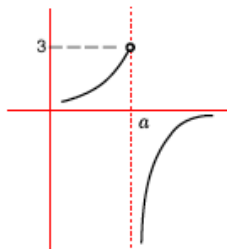


$$\lim_{x \rightarrow a^-} h(x) = +\infty$$

$$\lim_{x \rightarrow a^+} h(x) = 3$$

Example

Consider the graph for the function $s(x)$.



$$\lim_{x \rightarrow a^-} s(x) = 3$$

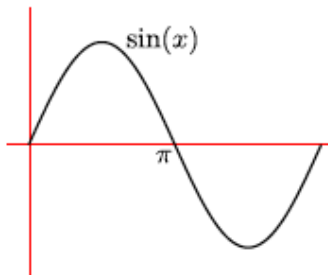
$$\lim_{x \rightarrow a^+} s(x) = -\infty$$

Example

Consider the function

$$g(x) = \frac{1}{\sin(x)}.$$

Find the one-side limits of this function as $x \rightarrow \pi$.



$$\lim_{x \rightarrow \pi^-} \frac{1}{\sin(x)} = +\infty$$

$$\lim_{x \rightarrow \pi^+} \frac{1}{\sin(x)} = -\infty$$

Second Session Outline

- ▶ Arithmetic of the Limits
- ▶ Limit of a ratio: what will happen if the limit of the denominator is zero. For example,

$$\lim_{x \rightarrow 0} \frac{1}{x^2} ? \quad \text{and} \quad \lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = ?$$

- ▶ Sandwich/ Squeeze/Pinch Theorem
- ▶ limit at infinity

Arithmetic of the Limits

Theorem

Let $a, c \in \mathbb{R}$. The following two limits hold

$$\lim_{x \rightarrow a} c = c \qquad \lim_{x \rightarrow a} x = a$$

Example

$$\lim_{x \rightarrow 3} -2 = -2 \qquad \lim_{x \rightarrow -1} x = -1$$

Theorem

(Arithmetic of Limits) Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x 's that lie in some interval about a (but f and g need not to be defined exactly at a).

$$\lim_{x \rightarrow a} f(x) = F \quad \lim_{x \rightarrow a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. Then the following limits hold

- ▶ $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$ —limit of the sum is the sum of the limits.

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- ▶ $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$ —limit of the sum is the sum of the limits.
- ▶ $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$ —limit of the difference is the difference of the limits.

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- ▶ $\lim_{x \rightarrow a} cf(x) = cF$.

Theorem

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exists with $F, G \in \mathbb{R}$. Then the following limits hold

- ▶ $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$ —limit of the sum is the sum of the limits.
- ▶ $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$ —limit of the difference is the difference of the limits.
- ▶ $\lim_{x \rightarrow a} cf(x) = cF$.
- ▶ $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = F \cdot G$ —limit of the product is the product of the limits.

If $G \neq 0$ then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}$$

Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) =$$

Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) =$$

Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) =$$

Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

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$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x \rightarrow 1} \frac{f(x)}{f(x) - g(x)} =$$

Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

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$$\lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x \rightarrow 1} \frac{f(x)}{f(x) - g(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x)} = \frac{3}{3 - 2} = 3.$$

Example

$$\lim_{x \rightarrow 3} 4x^2 - 1 =$$

$$\lim_{x \rightarrow 2} \frac{x}{x - 1} =$$

Example

$$\lim_{x \rightarrow 3} 4x^2 - 1 = 4 \times \lim_{x \rightarrow 3} x^2 - \lim_{x \rightarrow 3} 1 = 35.$$

$$\lim_{x \rightarrow 2} \frac{x}{x-1} = \frac{\lim_{x \rightarrow 2} x}{\lim_{x \rightarrow 2} x - \lim_{x \rightarrow 1} 1} = \frac{2}{2-1} = 2.$$

Limit of a ratio: what will happen if the limit of the denominator is zero.

Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = DNE$$

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- the limit does **not exist**, eg.

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- the **limit is** $\pm\infty$, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x^4} = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{-x^2}{x^4} = \lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty.$$

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- the **limit is** 0, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

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- the **limit is** 0, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

- the **limit exists and it nonzero**, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x} = 1.$$

Theorem

Let n be a positive integer, let $a \in \mathbb{R}$ and let f be a function so that

$$\lim_{x \rightarrow a} f(x) = F$$

for some real number F . Then the following holds

$$\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit.

Theorem

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so that the limit of a power is the power of the limit.

Similarly, if

- ▶ n is an even number and $F > 0$, or
- ▶ n is an odd number and F is any real number

then

$$\lim_{x \rightarrow a} (f(x))^{1/n} = \left(\lim_{x \rightarrow a} f(x) \right)^{1/n} = F^{1/n}.$$

Example

$$\lim_{x \rightarrow 4} x^{1/2} =$$

$$\lim_{x \rightarrow 4} (-x)^{1/2} =$$

$$\lim_{x \rightarrow 2} (4x^2 - 3)^{1/3} =$$

Example

$$\lim_{x \rightarrow 4} x^{1/2} = 4^{1/2} = 2.$$

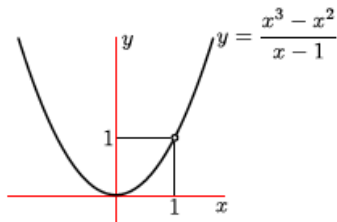
$$\lim_{x \rightarrow 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \rightarrow 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = (13)^{1/3}.$$

**Limit of a ratio: what will happen if the limit
of the numerator and denominator are zero,
for example,**

$$\lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = ?$$

$$\lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = ?$$



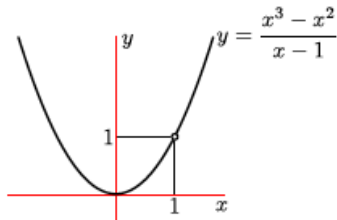
Theorem

If $f(x) = g(x)$ except when $x = a$ then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

provided the limit of g exists.

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases} \Rightarrow \lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \rightarrow 1} x^2 = 1.$$

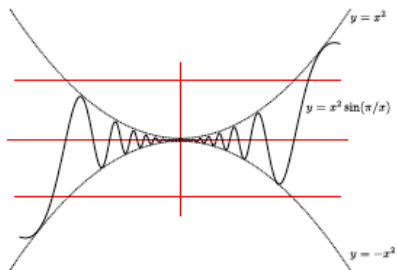
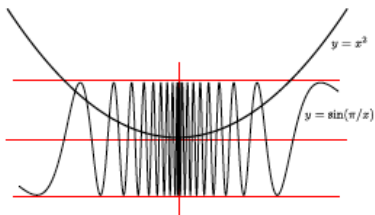


Sandwich/ Squeeze/Pinch Theorem

Example

Compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \rightarrow 1} f(x)?$$

Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \rightarrow 1} f(x)?$$

Solution

Consider that

$$\lim_{x \rightarrow 1} x = 1 \quad \text{and} \quad \lim_{x \rightarrow 1} x^2 - 2x + 2 = 1.$$

Therefore, by the sandwich/pinch/squeeze theorem

$$\lim_{x \rightarrow 1} f(x) = 1.$$

Example

We want to compute

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{x}$$

By plug in some large numbers into $\frac{1}{x}$ we have

-10000	-1000	-100	o	100	1000	10000
-0.0001	-0.001	-0.01	o	0.01	0.001	0.0001

We see that as x is getting bigger and positive the function $\frac{1}{x}$ is getting closer to 0. Thus,

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0.$$

Moreover,

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

Limit at Infinity

Definition

(**Informal limit at infinity.**) We write

$$\lim_{x \rightarrow \infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and positive.

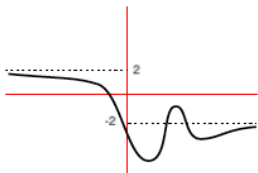
Similarly, we write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and negative.

Example

Consider the graph of the function $f(x)$.



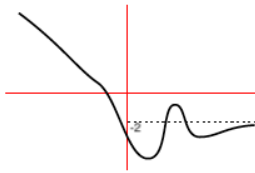
Then

$$\lim_{x \rightarrow \infty} f(x) =$$

$$\lim_{x \rightarrow -\infty} f(x) =$$

Example

Consider the graph of the function $g(x)$.



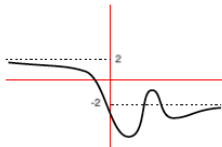
Then

$$\lim_{x \rightarrow \infty} g(x) =$$

$$\lim_{x \rightarrow -\infty} g(x) =$$

Example

Consider the graph of the function $f(x)$.



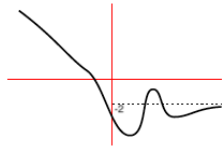
Then

$$\lim_{x \rightarrow \infty} f(x) = -2$$

$$\lim_{x \rightarrow -\infty} f(x) = 2$$

Example

Consider the graph of the function $g(x)$.



Then

$$\lim_{x \rightarrow \infty} g(x) = -2$$

$$\lim_{x \rightarrow -\infty} g(x) = +\infty$$

Review of the third session

Review

Theorem

sandwich (or squeeze or pinch) *Let $a \in \mathbb{R}$ and let f, g, h be three functions so that*

$$f(x) \leq g(x) \leq h(x)$$

for all x in an interval around a , except possibly at $x = a$. Then if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

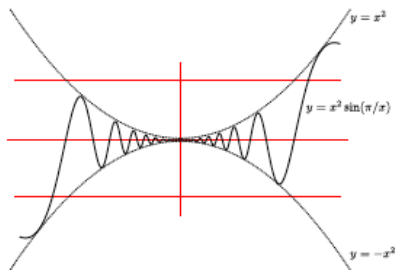
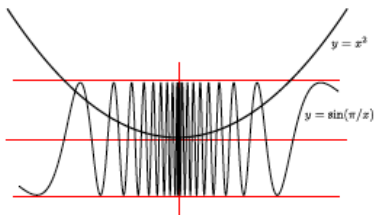
then it is also the case that

$$\lim_{x \rightarrow a} g(x) = L.$$

Example

Compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



Theorem

Let $c \in \mathbb{R}$ then the following limits hold

$$\lim_{x \rightarrow +\infty} c = c$$

$$\lim_{x \rightarrow -\infty} c = c$$

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

Outline For the Fourth Session

- ▶ Limit at Infinity

Limit at Infinity

Theorem

Let $f(x)$ and $g(x)$ be two functions for which the limits

$$\lim_{x \rightarrow \infty} f(x) = F \qquad \lim_{x \rightarrow \infty} g(x) = G$$

exist. Then the following limits hold

$$\lim_{x \rightarrow \infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x \rightarrow \infty} f(x)g(x) = FG$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \text{provided } G \neq 0$$

and for rational numbers r ,

$$\lim_{x \rightarrow \infty} (f(x))^r = F^r$$

provided that $f(x)^r$ is defined for all x .

The analogous results hold for limits to $-\infty$.



Warning: Consider that

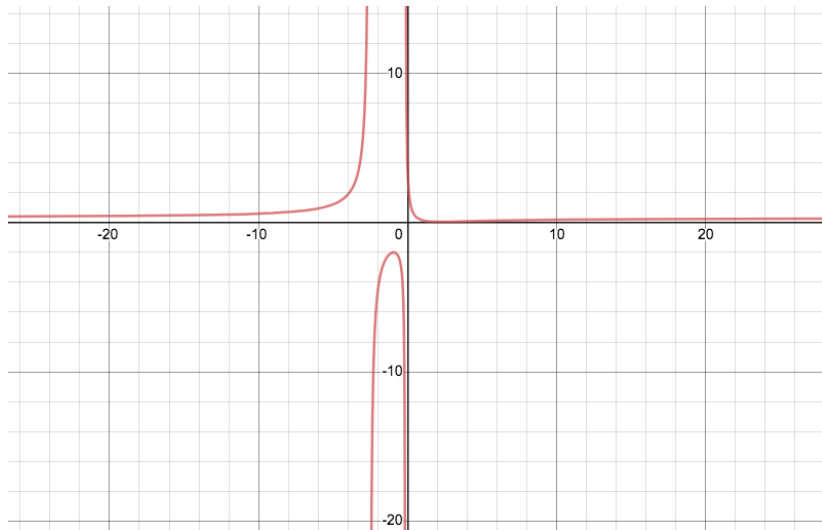
$$\lim_{x \rightarrow +\infty} \frac{1}{x^{1/2}} = 0$$

However,

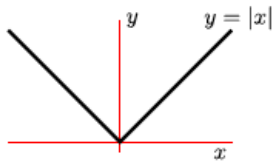
$$\lim_{x \rightarrow +\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because $x^{1/2}$ is not defined for $x < 0$.

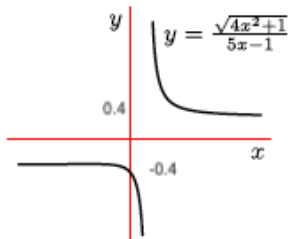
$$f(x) = \frac{x^2 - 3x + 4}{3x^2 + 8x + 1}$$



$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



$$y = \frac{\sqrt{4x^2 + 1}}{5x - 1}$$



Theorem

Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that

$$\lim_{x \rightarrow a} f(x) = +\infty \quad \lim_{x \rightarrow a} g(x) = +\infty \quad \lim_{x \rightarrow a} h(x) = H$$

1.

$$\lim_{x \rightarrow a} (f(x) + g(x)) =$$

2.

$$\lim_{x \rightarrow a} (f(x) + h(x)) =$$

3.

$$\lim_{x \rightarrow a} (f(x) - g(x)) =$$

4.

$$\lim_{x \rightarrow a} (f(x) - h(x)) =$$

Theorem

Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that

$$\lim_{x \rightarrow a} f(x) = +\infty \quad \lim_{x \rightarrow a} g(x) = +\infty \quad \lim_{x \rightarrow a} h(x) = H$$

1.

$$\lim_{x \rightarrow a} (f(x) + g(x)) = +\infty.$$

2.

$$\lim_{x \rightarrow a} (f(x) + h(x)) = +\infty.$$

3.

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \text{undetermined}.$$

4.

$$\lim_{x \rightarrow a} (f(x) - h(x)) = +\infty.$$

Theorem

5.

$$\lim_{x \rightarrow a} cf(x) = \begin{cases} c > 0 \\ c = 0 \\ c < 0 \end{cases}$$

6.

$$\lim(f(x).g(x)) =$$

7.

$$\lim_{x \rightarrow a} (f(x).h(x)) = \begin{cases} H > 0 \\ H = 0 \\ H < 0 \end{cases}$$

8.

$$\lim_{x \rightarrow a} \frac{h(x)}{f(x)} =$$

Theorem

5.

$$\lim_{x \rightarrow a} cf(x) = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$$

6.

$$\lim(f(x).g(x)) = +\infty.$$

7.

$$\lim_{x \rightarrow a} (f(x).h(x)) = \begin{cases} +\infty & H > 0 \\ \text{undetermined} & H = 0 \\ -\infty & H < 0 \end{cases}$$

8.

$$\lim_{x \rightarrow a} \frac{h(x)}{f(x)} = 0.$$

Example

Consider the following three functions:

$$f(x) = x^{-2} \quad g(x) = 2x^{-2} \quad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \rightarrow 0} f(x) = +\infty \quad \lim_{x \rightarrow 0} g(x) = +\infty \quad \lim_{x \rightarrow 0} h(x) = +\infty.$$

Then

1.

$$\lim_{x \rightarrow 0} (f(x) - g(x)) =$$

2.

$$\lim_{x \rightarrow 0} (f(x) - h(x)) =$$

3.

$$\lim_{x \rightarrow 0} (g(x) - h(x)) =$$

Example

Consider the following three functions:

$$f(x) = x^{-2} \quad g(x) = 2x^{-2} \quad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \rightarrow 0} f(x) = +\infty \quad \lim_{x \rightarrow 0} g(x) = +\infty \quad \lim_{x \rightarrow 0} h(x) = +\infty.$$

Then

1.

$$\lim_{x \rightarrow 0} (f(x) - g(x)) = \lim_{x \rightarrow 0} x^{-2} = \infty$$

2.

$$\lim_{x \rightarrow 0} (f(x) - h(x)) = \lim_{x \rightarrow 0} (1) = 1$$

3.

$$\lim_{x \rightarrow 0} (g(x) - h(x)) = \lim_{x \rightarrow 0} x^{-2} + 1 = \infty$$

Outline For the Session Five

- ▶ Limit at Infinity
- ▶ Continuity
- ▶ Continuous from the left and from the right
- ▶ Arithmetic of continuity
- ▶ continuity of composites
- ▶ Intermediate Value Theorem

Example

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{1}{x^2}$	100	10000	10^6		10^6	10000	100

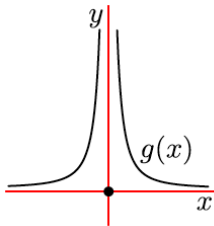
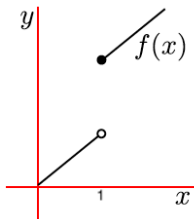
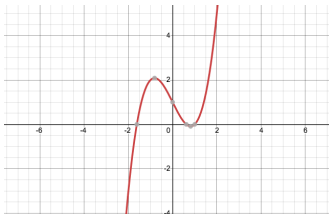
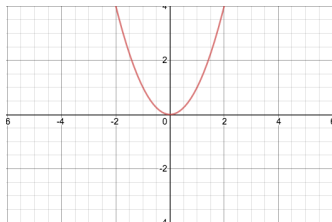
Consider that if

$$\lim_{x \rightarrow a} f(x) = \infty \quad \lim_{x \rightarrow a} g(x) = \infty$$

Then

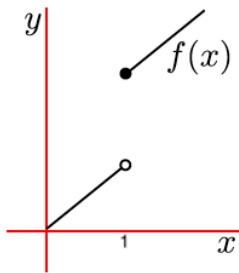
$$\lim_{x \rightarrow a} (f(x) - g(x)) = \text{undetermined}$$

Continuity

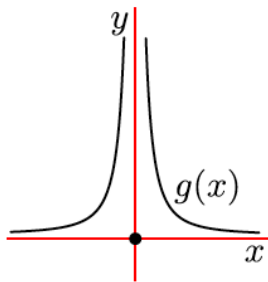


$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$

jump discontinuity

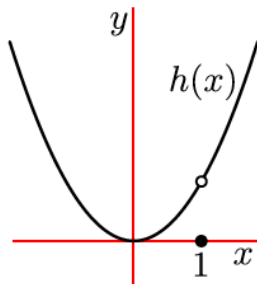


$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



infinite discontinuity

$$h(x) = \begin{cases} \frac{x^3 - x^2}{x - 1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$



removable discontinuity

Outline - September 16, 2019

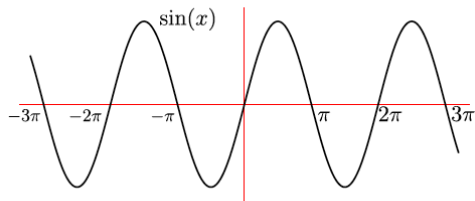
- ▶ **Section 1.6:**
 - ▶ Arithmetic of continuity
 - ▶ Continuity of composites
 - ▶ Intermediate Value Theorem
- ▶ **Section 2.1:**
 - ▶ Revisiting tangent lines

Arithmetic of continuity

Theorem

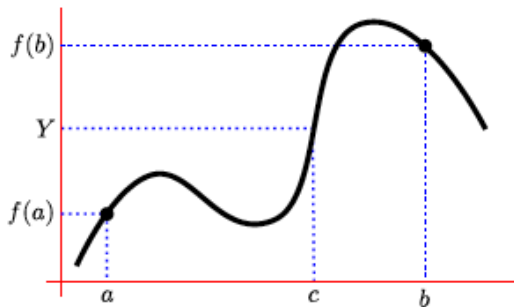
(Arithmetic of continuity) Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at a . Then the following functions are also continuous at $x = a$.

- ▶ $f(x) + g(x)$ and $f(x) - g(x)$,
- ▶ $cf(x)$ and $f(x)g(x)$, and
- ▶ $\frac{f(x)}{g(x)}$ provided $g(a) \neq 0$.

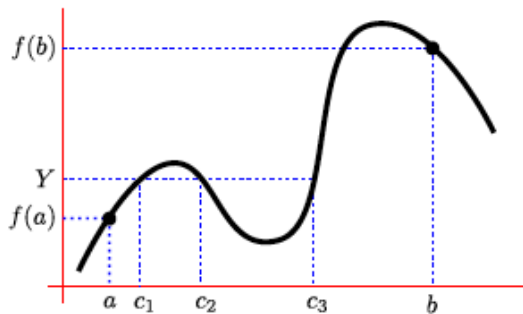


Intermediate value theorem(IVT)

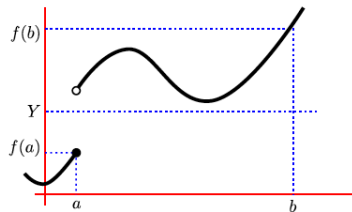
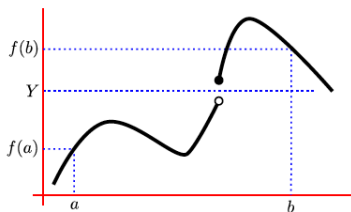
Theorem
(Intermediate value theorem(IVT))



The existence not the uniqueness of c in IVT

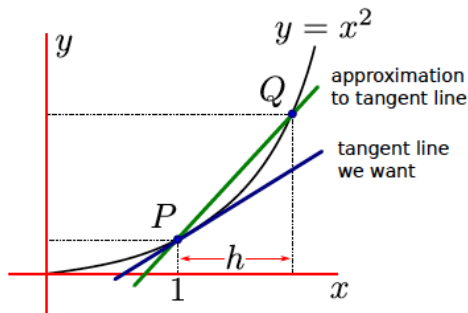


Not continuous functions at $[a, b]$ do not satisfy IVT



Revisiting tangent lines

Revisiting tangent lines



$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \leftarrow \text{slope of the tangent line at } x = 1$$

Definition of the derivative

▶

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

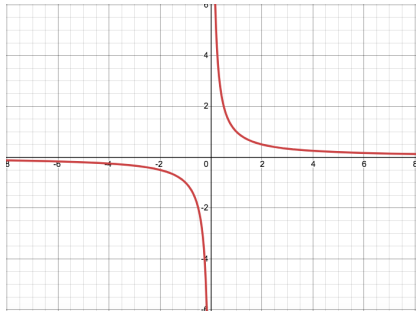
▶

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

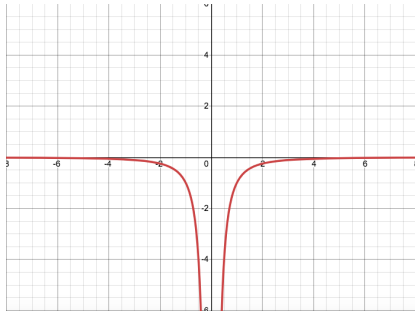
Examples

- ▶ $f(x) = c$
- ▶ $f(x) = x$
- ▶ $f(x) = x^2$
- ▶ $f(x) = \frac{1}{x}$
- ▶ $f(x) = \sqrt{x}$
- ▶ $f(x) = |x|$

$y = \frac{1}{x}$ and its derivative $-\frac{1}{x^2}$

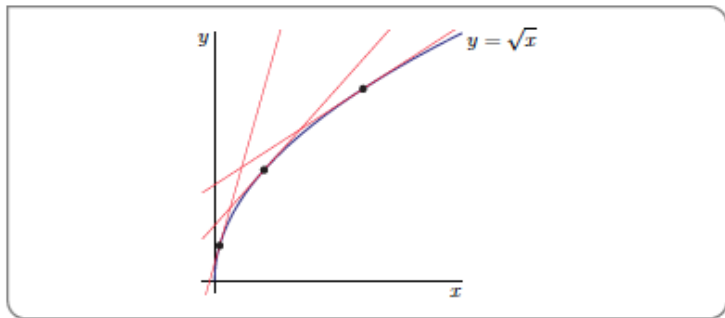


$$y = \frac{1}{x}$$

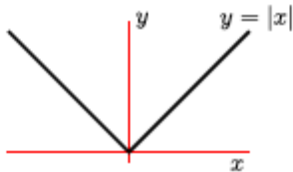


$$y = -\frac{1}{x^2}$$

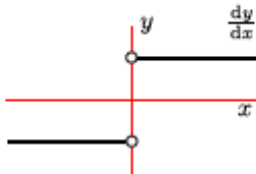
Tangent lines to $y = \sqrt{x}$



The derivative of the function $f(x) = |x|$: not differentiable at $x = 0$

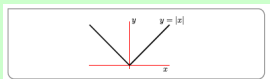


The derivative of the function $f(x) = |x|$

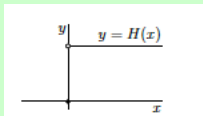


Where a function is not differentiable at $x = a$

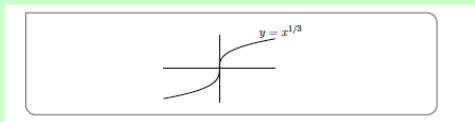
- ▶ Having a Sharp Corner at $x = a$



- ▶ The function is not continuous at $x = a$



- ▶ Having a tangent line, but the slope of the tangent line at $x = a$ is infinity

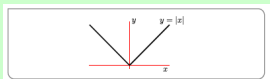


Outline - September 20, 2019

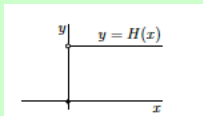
- ▶ **Section 2.2:**
 - ▶ Not differentiable examples
 - ▶ The relation between continuous and differentiable functions
- ▶ **Section 2.3:**
 - ▶ Interpretations of the derivative

Where a function is not differentiable at $x = a$

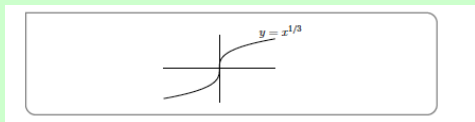
- ▶ Having a Sharp Corner at $x = a$



- ▶ The function is not continuous at $x = a$

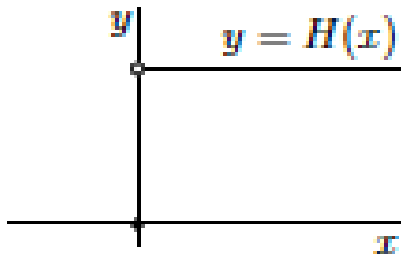


- ▶ Having a tangent line, but the slope of the tangent line at $x = a$ is infinity



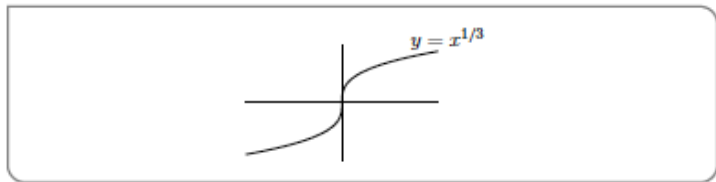
An example of a discontinuous and not differentiable function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$



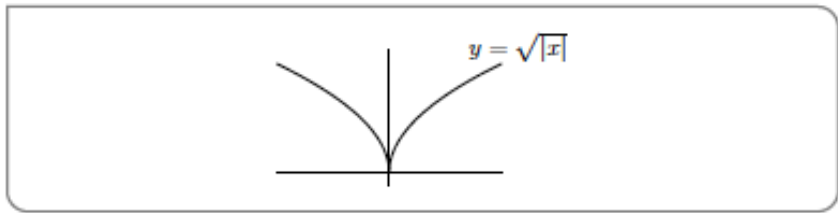
An example of a function with a tangent line with slope infinity at $x = 0$

$$f(x) = x^{1/3}$$

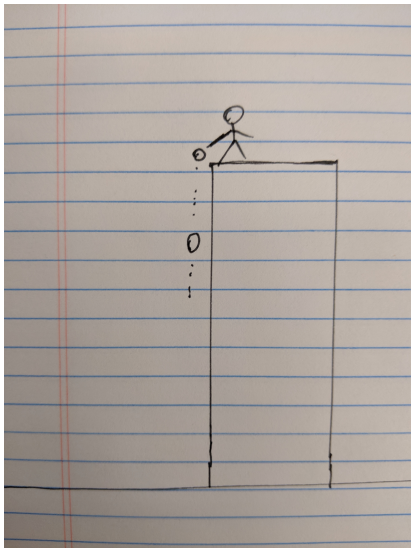


An example of a continuous and **not** differentiable function

$$y = \sqrt{|x|}$$



Instantaneous rate of change



average rate of change of $f(t)$ from $t = a$ to $t = a + h$ is

$$\frac{\text{change in } f(t) \text{ from } t = a \text{ to } t = a + h}{\text{length of time from } t = a \text{ to } t = a + h}$$

$$= \frac{f(a + h) - f(a)}{h}.$$

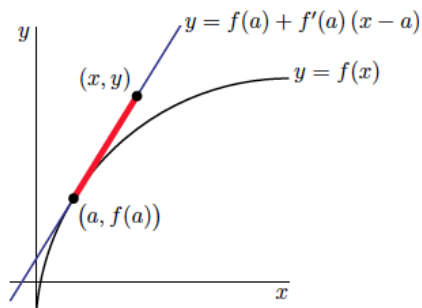
And so

instantaneous rate of change of $f(t)$ at $t = a$

$$= \lim_{h \rightarrow 0} [\text{average rate of change of } f(t) \text{ from } t = a \text{ to } t = a + h]$$

$$= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} = f'(a).$$

Finding tangent line to a curve at $x = a$



A line segment of a tangent line

$$y = f(a) + f'(a)(x - a)$$

Outline - September 23, 2019

- ▶ **Section 2.4 and 2.5:**
 - ▶ Derivative of some simple functions
 - ▶ Tools
 - ▶ Examples

A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0$$

$$\frac{d}{dx}x = 1$$

$$\frac{d}{dx}x^2 = 2x$$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \quad \frac{d}{dx}x = 1 \quad \frac{d}{dx}x^2 = 2x \quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Tools

Let $f(x)$ and $g(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- ▶ $\frac{d}{dx}\{f(x) + g(x)\} = f'(x) + g'(x)$
- ▶ $\frac{d}{dx}\{f(x) - g(x)\} = f'(x) - g'(x)$
- ▶ $\frac{d}{dx}\{cf(x)\} = cf'(x)$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- ▶ $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- ▶ $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- ▶ $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- ▶ $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶ $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- ▶ $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- ▶ $\frac{d}{dx}\left\{\frac{1}{g(x)}\right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- ▶ $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- ▶ $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶ $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- ▶ $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- ▶ $\frac{d}{dx}\left\{\frac{1}{g(x)}\right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶ $\frac{d}{dx}\{f(x)g(x)h(x)\} = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- ▶ $\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x)$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

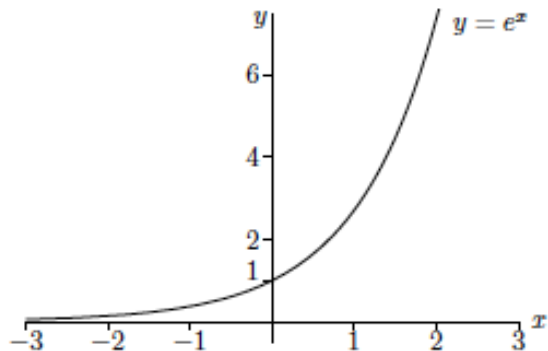
- ▶ $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- ▶ $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶ $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- ▶ $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- ▶ $\frac{d}{dx}\left\{\frac{1}{g(x)}\right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶ $\frac{d}{dx}\{f(x)g(x)h(x)\} = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- ▶ $\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x)$
- ▶ Let a be a rational number, then

$$\frac{d}{dx}x^a = ax^{a-1}.$$

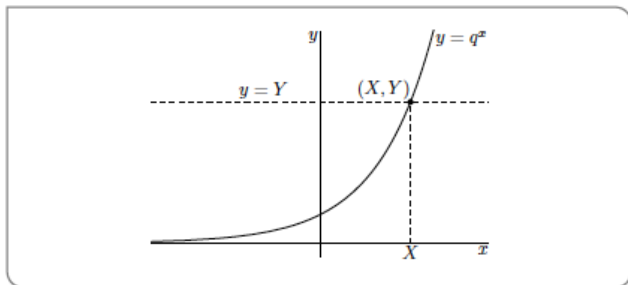
Outline - September 25, 2019

- ▶ **Section 2.7 and 2.8:**
 - ▶ Derivative of exponential functions
 - ▶ Derivative of trigonometric functions

The graph of e^x



The graph of q^x where $q > 1$



YOUR TURN!

Example

Find a such that the following function is continuous.

$$f(x) = \begin{cases} e^{x+a} & x < 0 \\ \sqrt{x+1} & x \geq 0 \end{cases}$$

Example

We have

1. $\log_q(xy) =$
 - (a) $\log_q(x) + \log_q(y)$
 - (b) $\log_q(x) \log_q(y)$
2. $\log_q(x/y) =$
3. $\log_q(x^r) =$

Example

We have

1. $\log_q(xy) = \log_q(x) + \log_q(y)$.

The reason for this is that

$$q^{\log_q(xy)} = xy = q^{\log_q(x)} q^{\log_q(y)} = q^{\log_q(x) + \log_q(y)}$$

Therefore, $\log_q(xy) = \log(x) + \log(y)$.

2. $\log_q(x/y) = \log_q(x) - \log_q(y)$

3. $\log_q(x^r) = r \log_q(x)$

TOOLS:

$$\frac{d}{dx}(f \circ g)(x) = g'(x)f'(g(x))$$

A list of derivative of some simple functions:

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = (\log_e a)a^x$$

Example

Find the derivative of $2^{\sqrt{x}}$.

Example

Find the derivative of $2^{\sqrt{x}}$.

Example

Find a and b such that the following function is differentiable.

$$f(x) = \begin{cases} x^3 + a & x < 1 \\ e^{x-1} + bx & x \geq 1 \end{cases}$$

Outline - September 30, 2019

► Section 2.8, 2.9, 0.6:

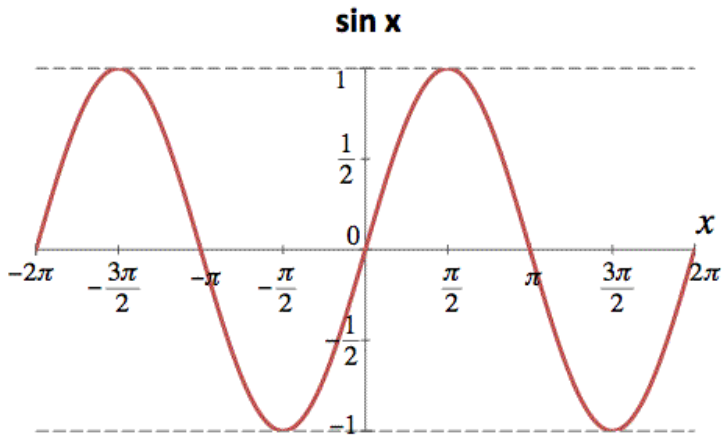
- Derivative of trigonometric functions
- The chain rule
- inverse of a function

A list of derivative of some simple functions:

$$\frac{d}{dx}e^x = e^x$$

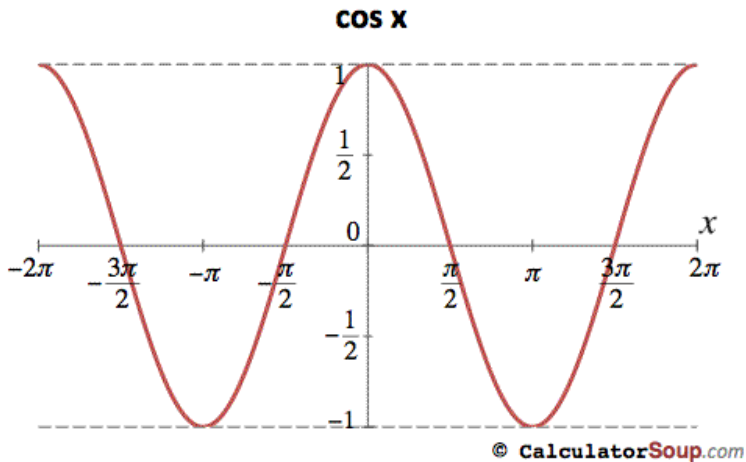
$$\frac{d}{dx}a^x = (\log_e a)a^x$$

$\sin(x)$ domain = \mathbb{R} range = $[-1, 1]$

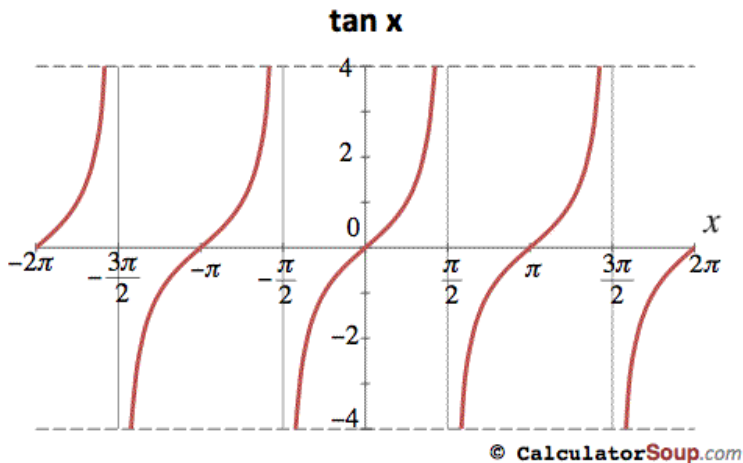


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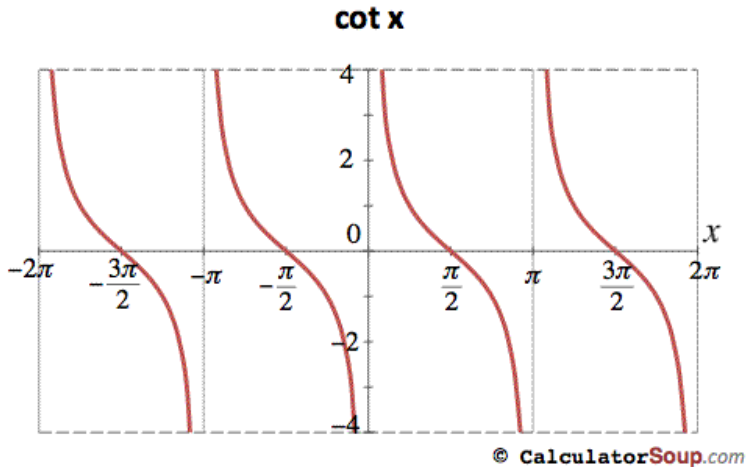
$\cos(x)$ domain = \mathbb{R} range = $[-1, 1]$



$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \text{domain} = \mathbb{R} - \left\{ (2n+1)\frac{\pi}{2} : n \in \mathbb{Z} \right\} \quad \text{range} = \mathbb{R}$$



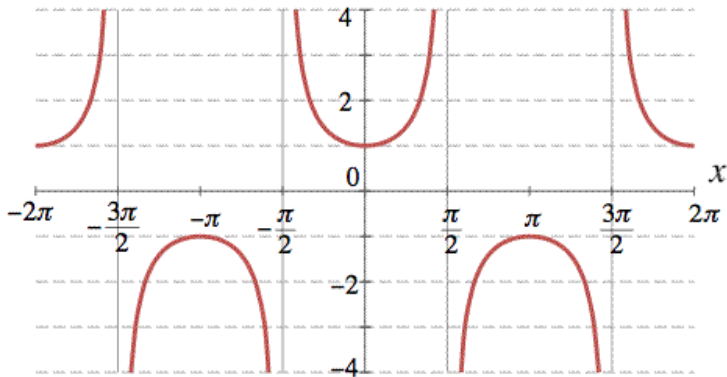
$$\cot(x) = \frac{\cos(x)}{\sin(x)} \quad \text{domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R}$$



$$\sec(x) = \frac{1}{\cos(x)} \quad \text{domain} = \mathbb{R} - \left\{ (2n+1)\frac{\pi}{2} : n \in \mathbb{Z} \right\}$$

$$\text{range} = \mathbb{R} - (-1, 1)$$

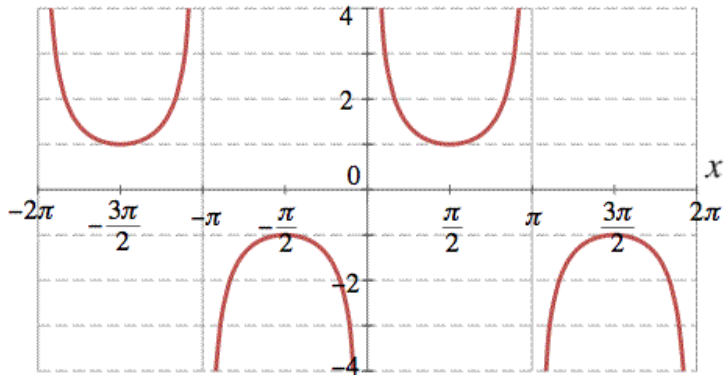
sec x



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$$\csc(x) = \frac{1}{\sin(x)} \quad \text{domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R} - (-1, 1)$$

CSC X



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Derivative of $\sin(x)$

Question: Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of $\sin(x)$ at $x = 0$.

Derivative of $\sin(x)$

Question: Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of $\sin(x)$ at $x = 0$.

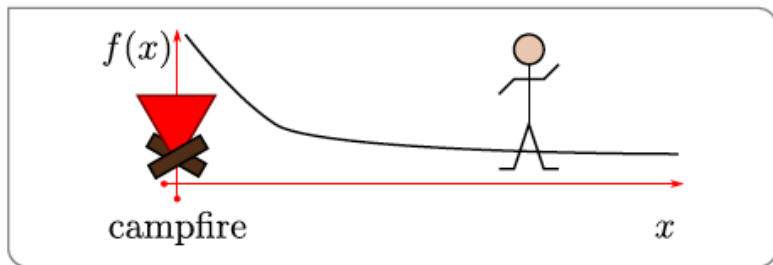
(**sandwich (or squeeze or pinch) theorem**) Let $a \in \mathbb{R}$ and let f, g, h be three functions so that $f(x) \leq g(x) \leq h(x)$ for all x in an interval around a , except possibly at $x = a$. Then if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then it is also the case that

$$\lim_{x \rightarrow a} g(x) = L.$$

An example of the application of the chain rule



- ▶ Your position at time t is $x(t)$.
- ▶ The temperature of the air at position x is $f(x)$.
- ▶ The temperature that you feel at time t is $F(t) = f(x(t))$.
- ▶ The instantaneous rate of change of temperature that you feel is $F'(t)$.

The chain rule

Theorem

Let f and g be differentiable functions then

$$\frac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$$

The chain rule

Theorem

Let $y = f(u)$ and $u = g(x)$ be differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

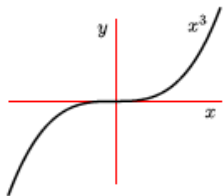
Outline - October 2, 2019

- ▶ **Section 0.6, 2.10:**
 - ▶ inverse of a function
 - ▶ Natural logarithm

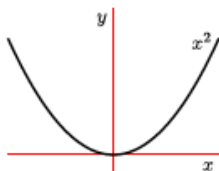
input number $x \mapsto f$ does “stuff” to $x \mapsto$ return number y

take output $y \mapsto$ do “stuff” to $y \mapsto$ return the original
number x

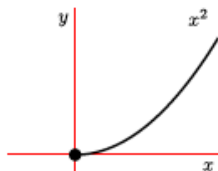
One-to-one functions



$$\begin{array}{lcl} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

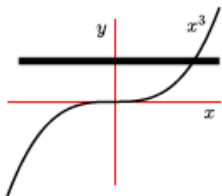


$$\begin{array}{lcl} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$



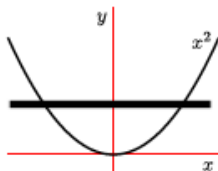
$$\begin{array}{lcl} [0, \infty] & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

One-to-one functions



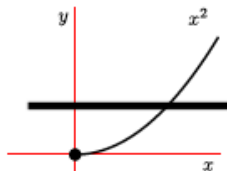
$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

is one-to-one



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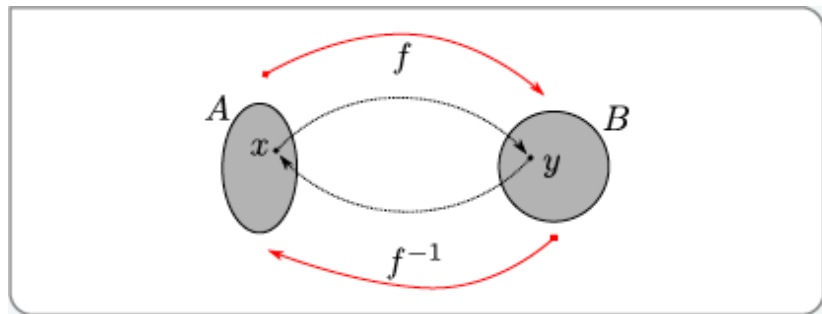
is not one-to-one



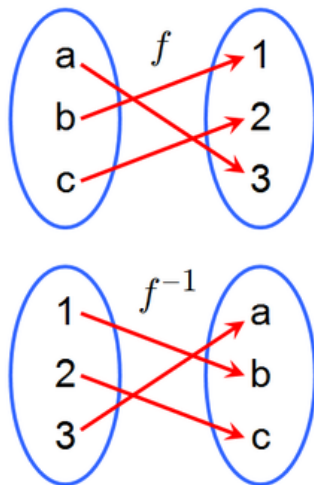
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is one-to-one

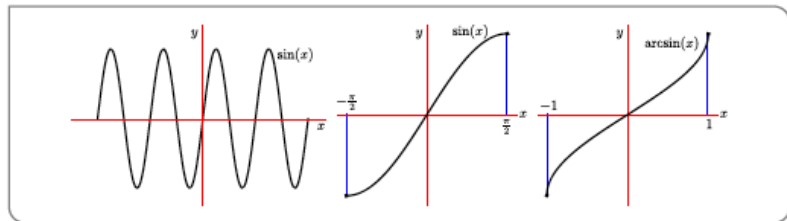
Inverse of a functions



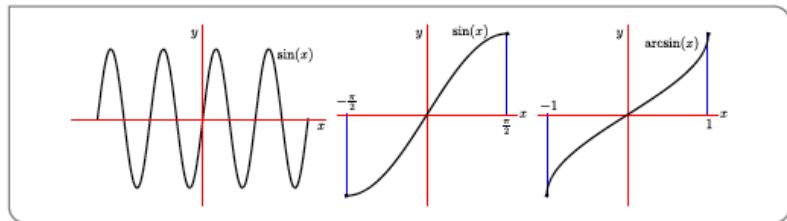
Inverse of a functions



Inverse of $\sin(x)$

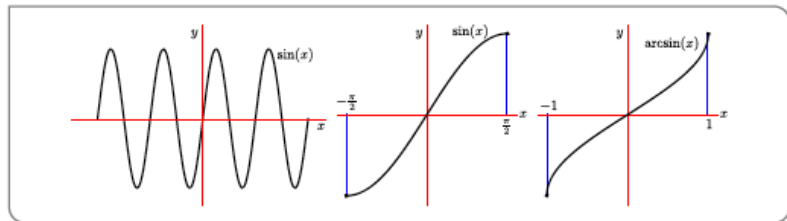


Inverse of $\sin(x)$



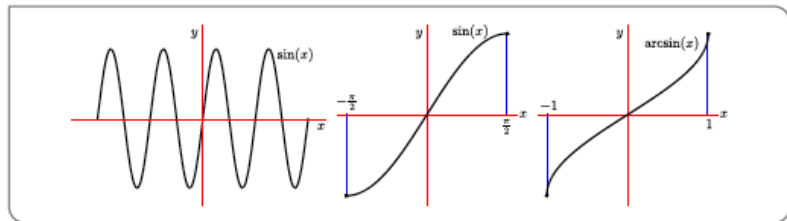
- ▶ $\sin(x)$ is not invertible on the domain \mathbb{R} because it is not one-to-one.

Inverse of $\sin(x)$



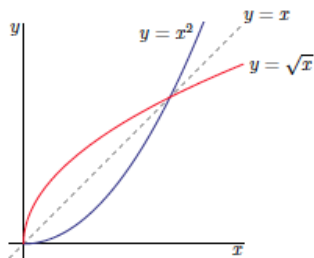
- ▶ $\sin(x)$ is not invertible on the domain \mathbb{R} because it is not one-to-one.
- ▶ If we look at $\sin(x)$ on the domain $[-\pi/2, \pi/2]$, then it is one-to-one, and so it has an inverse.

Inverse of $\sin(x)$



- ▶ $\sin(x)$ is not invertible on the domain \mathbb{R} because it is not one-to-one.
- ▶ If we look at $\sin(x)$ on the domain $[-\pi/2, \pi/2]$, then it is one-to-one, and so it has an inverse.
- ▶ The inverse of $\sin(x)$ is $\arcsin(x)$ on the domain $[-1, 1]$ and with the range $[-\pi/2, \pi/2]$.

How to find the inverse of a function by its graph



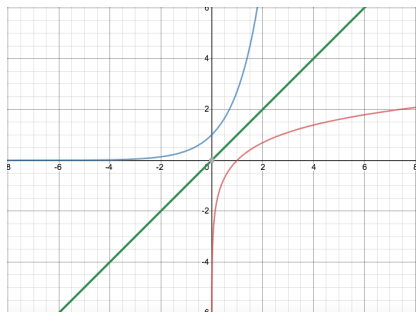
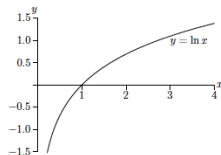
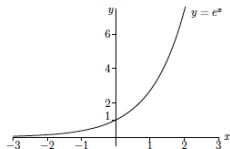
$$a^{\log_a x} = x$$

Remember that for $a > 1$,

$$a^{\log_a x} = x,$$

$$\log_a x = \frac{\log_e x}{\log_e a}.$$

The inverse of e^x



Useful facts!

- ▶ $\frac{d}{dx} a^x = (\ln a) a^x.$
- ▶ $\log_a x = \frac{\ln x}{\ln a} \quad \ln x = \frac{\log_a x}{\log_a e} \quad a > 1.$
- ▶ $\ln(xy) = \ln x + \ln y.$
- ▶ $\ln(x/y) = \ln x - \ln y.$
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- ▶ $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$

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