MATH 100

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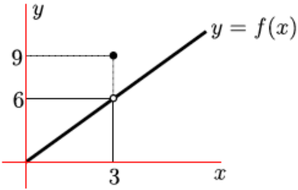
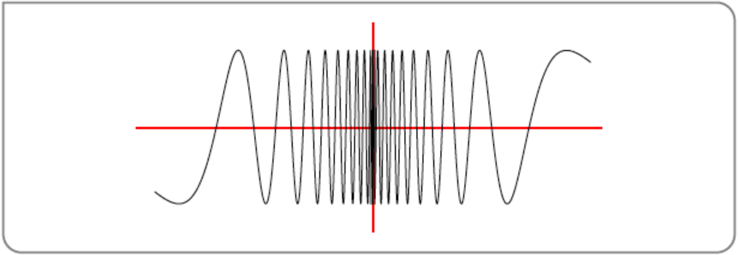
2019

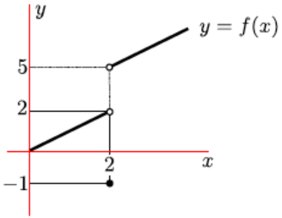
$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



$f(x) = \sin(\frac{\pi}{x})$

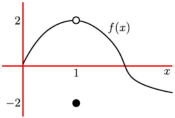


$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$



Example

Consider the graph of the function $f(x)$.



Then

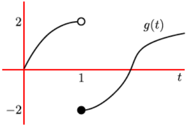
$$\lim_{x \to 1^-} f(x) =$$

$$\lim_{x \to 1^+} f(x) =$$

$$\lim_{x \to 1} f(x) =$$

Example

Consider the graph of the function $g(t)$.



Then

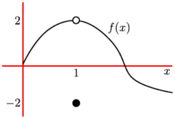
$$\lim_{t \to 1^-} g(t) =$$

$$\lim_{t \to 1^+} g(t) =$$

$$\lim_{t \to 1} g(t) =$$

Example

Consider the graph of the function $f(x)$.



Then

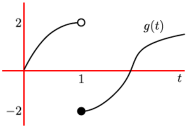
$$\lim_{x\to 1^-} f(x) = 2$$

$$\lim_{x\to 1^+} f(x) = 2$$

$$\lim_{x\to 1} f(x) = 2$$

Example

Consider the graph of the function $g(t)$.



Then

$$\lim_{t\to 1^-} g(t) = 2$$

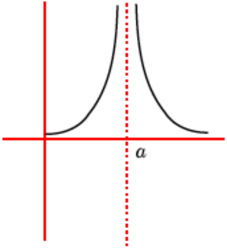
$$\lim_{t\to 1^+} g(t) = -2$$

$$\lim_{t\to 1} g(t) = DNE$$

When the limit goes to infinity

Example

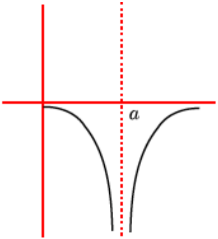
Consider the graph for the function $f(x)$.



$$\lim_{x \to a} f(x) = +\infty$$

Example

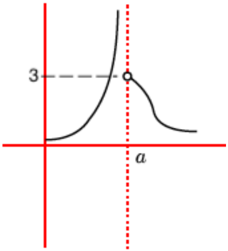
Consider the graph for the function $g(x)$.



$$\lim_{x \to a} g(x) = -\infty$$

Example

Consider the graph for the function $h(x)$.

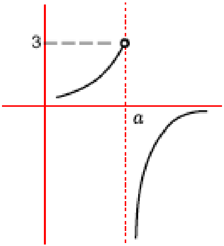


$$\lim_{x \to a^-} h(x) =$$

$$\lim_{x \to a^+} h(x) =$$

Example

Consider the graph for the function $s(x)$.

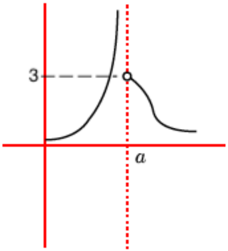


$$\lim_{x \to a^-} s(x) =$$

$$\lim_{x \to a^+} s(x) =$$

Example

Consider the graph for the function $h(x)$.

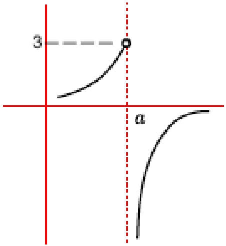


$$\lim_{x \to a^-} h(x) = +\infty$$

$$\lim_{x \to a^+} h(x) = 3$$

Example

Consider the graph for the function $s(x)$.



$$\lim_{x \to a^-} s(x) = 3$$

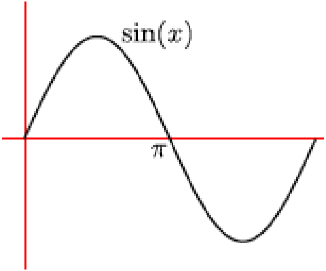
$$\lim_{x \to a^+} s(x) = -\infty$$

Example

Consider the function

$$g(x) = \frac{1}{sin(x)}.$$

Find the one-side limits of this function as $x \to \pi$.



$$\lim_{x \to \pi^-} \frac{1}{sin(x)} = +\infty$$

$$\lim_{x \to \pi^+} \frac{1}{sin(x)} = -\infty$$

Second Session Outline

- Arithmetic of the Limits
- Limit of a ratio: what will happen if the limit of the denominator is zero. For example,

$$\lim_{x\to 0} \frac{1}{x^2}? \quad \text{and} \quad \lim_{x\to 1} \frac{x^3 - x^2}{x - 1} =?$$

- Sandwich/ Squeeze/Pinch Theorem
- limit at infinity

Arithmetic of the Limits

Theorem

Let $a, c \in \mathbb{R}$. The following two limits hold

$$\lim_{x \to a} c = c \qquad \lim_{x \to a} x = a$$

Example

$$\lim_{x \to 3} -2 = -2 \qquad \lim_{x \to -1} x = -1$$

Theorem

(**Arithmetic of Limits**) *Let* $a, c \in \mathbb{R}$, *let* $f(x)$ *and* $g(x)$ *be defined for all* x*'s that lie in some interval about* a *(but* f *and* g *need not to be defined exactly at* a*).*

$$\lim_{x \to a} f(x) = F \qquad \lim_{x \to a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. *Then the following limits hold*

- $\lim_{x \to a}(f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*

Theorem

(**Arithmetic of Limits**) *Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f and g need not to be defined exactly at a).*

$$\lim_{x \to a} f(x) = F \qquad \lim_{x \to a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. Then the following limits hold

- $\lim_{x \to a}(f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*
- $\lim_{x \to a}(f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*

Theorem

(**Arithmetic of Limits**) *Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f and g need not to be defined exactly at a).*

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- $\lim_{x \to a}(f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*
- $\lim_{x \to a}(f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*
- $\lim_{x \to a} cf(x) = cF$.

Theorem

(**Arithmetic of Limits**) *Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f and g need not to be defined exactly at a).*

$$\lim_{x \to a} f(x) = F \qquad \lim_{x \to a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. Then the following limits hold

- $\lim_{x \to a} (f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*
- $\lim_{x \to a} (f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*
- $\lim_{x \to a} cf(x) = cF$.
- $\lim_{x \to a} (f(x).g(x)) = F.G$*–limit of the product is the product of the limits.*

If $G \neq 0$ then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) =$$

Example

Given

$$\lim_{x \to 1} f(x) = 3 \quad \text{and} \quad \lim_{x \to 1} g(x) = 2$$

We have

$$\lim_{x \to 1} 3f(x) = 3 \times \lim_{x \to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \to 1} 3f(x) - g(x) =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} f(x). \lim_{x\to 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x\to 1} \frac{f(x)}{f(x) - g(x)} =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} f(x). \lim_{x\to 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x\to 1} \frac{f(x)}{f(x) - g(x)} = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} f(x) - \lim_{x\to 1} g(x)} = \frac{3}{3-2} = 3.$$

Example

$$\lim_{x \to 3} 4x^2 - 1 =$$

$$\lim_{x \to 2} \frac{x}{x - 1} =$$

Example

$$\lim_{x \to 3} 4x^2 - 1 = 4 \times \lim_{x \to 3} x^2 - \lim_{x \to 3} 1 = 35.$$

$$\lim_{x \to 2} \frac{x}{x-1} = \frac{\lim_{x \to 2} x}{\lim_{x \to 2} x - \lim_{x \to 1} 1} = \frac{2}{2-1} = 2.$$

Limit of a ratio: what will happen if the limit of the denominator is zero.

Limit of a ratio: what will happen if the limit of denominator is zero:

– the limit does **not exist**, eg.

$$\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} = DNE$$

Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x\to 0}\frac{x}{x^2} = \lim_{x\to 0}\frac{1}{x} = DNE$$

- the **limit is** $\pm\infty$, eg.

$$\lim_{x\to 0}\frac{x^2}{x^4} = \lim_{x\to 0}\frac{1}{x^2} = +\infty \qquad \text{or} \qquad \lim_{x\to 0}\frac{-x^2}{x^4} = \lim_{x\to 0}\frac{-1}{x^2} = -\infty.$$

Limit of a ratio: what will happen if the limit of denominator is zero:

– the limit does **not exist**, eg.

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– the **limit is** 0, eg.

$$\lim_{x\to 0}\frac{x^2}{x}=\lim_{x\to 0}x=0.$$

Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x\to 0}\frac{x}{x^2}=\lim_{x\to 0}\frac{1}{x}=DNE$$

- the **limit is** $\pm\infty$, eg.

$$\lim_{x\to 0}\frac{x^2}{x^4}=\lim_{x\to 0}\frac{1}{x^2}=+\infty \qquad \text{or} \qquad \lim_{x\to 0}\frac{-x^2}{x^4}=\lim_{x\to 0}\frac{-1}{x^2}=-\infty.$$

- the **limit is** 0, eg.

$$\lim_{x\to 0}\frac{x^2}{x}=\lim_{x\to 0}x=0.$$

- the **limit exists and it nonzero**, eg.

$$\lim_{x\to 0}\frac{x}{x}=1.$$

Theorem

Let n be a positive integer, let $a \in R$ and let f be a function so that

$$\lim_{x \to a} f(x) = F$$

for some real number F. Then the following holds

$$\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit.

Theorem

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$$\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit. Similarly, if

- *n is an even number and $F > 0$, or*
- *n is an odd number and F is any real number*

then

$$\lim_{x \to a} (f(x))^{1/n} = \left(\lim_{x \to a} f(x) \right)^{1/n} = F^{1/n}.$$

Example

$$\lim_{x\to 4} x^{1/2} =$$

$$\lim_{x\to 4} (-x)^{1/2} =$$

$$\lim_{x\to 2} (4x^2 - 3)^{1/3} =$$

Example

$$\lim_{x \to 4} x^{1/2} = 4^{1/2} = 2.$$

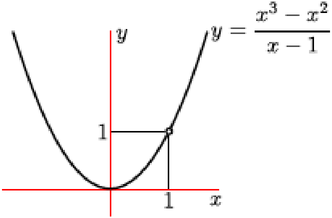
$$\lim_{x \to 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \to 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = (13)^{1/3}.$$

Limit of a ratio: what will happen if the limit of the numerator and denominator are zero, for example,

$$\lim_{x \to 1} \frac{x^3 - x^2}{x - 1} = ?$$

$$\lim_{x \to 1} \frac{x^3 - x^2}{x - 1} = ?$$



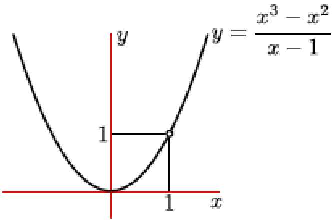
Theorem

If $f(x) = g(x)$ except when $x = a$ then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

provided the limit of g exists.

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases} \Rightarrow \lim_{x\to 1} \frac{x^3 - x^2}{x - 1} = \lim_{x\to 1} x^2 = 1.$$

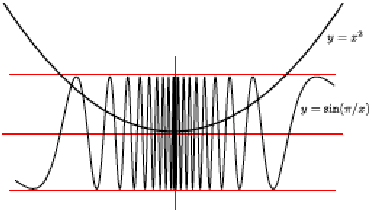


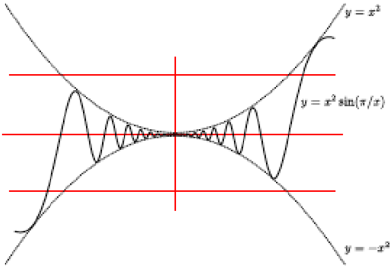
Sandwich/ Squeeze/Pinch Theorem

Example

Compute

$$\lim_{x \to 0} x^2 \sin\left(\frac{\pi}{x}\right)$$





Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \to 1} f(x)?$$

Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \to 1} f(x)?$$

Solution

Consider that

$$\lim_{x \to 1} x = 1 \qquad \text{and} \qquad \lim_{x \to 1} x^2 - 2x + 2 = 1.$$

Therefore, by the sandwich/pinch/squeeze theorem

$$\lim_{x \to 1} f(x) = 1.$$

Example

We want to compute

$$\lim_{x\to+\infty}\frac{1}{x} \qquad \text{and} \qquad \lim_{x\to-\infty}\frac{1}{x}$$

By plug in some large numbers into $\frac{1}{x}$ we have

-10000	-1000	$-100 \| \circ \| 100$	1000	10000
-0.0001	-0.001	$-0.01 \| \circ \| 0.01$	0.001	0.0001

We see that as x is getting bigger and positive the function $\frac{1}{x}$ is getting closer to 0. Thus,

$$\lim_{x\to+\infty}\frac{1}{x}=0.$$

Moreover,

$$\lim_{x\to-\infty}\frac{1}{x}=0.$$

Limit at Infinity

Definition

(**Informal limit at infinity.**) We write

$$\lim_{x \to \infty} f(x) = L$$

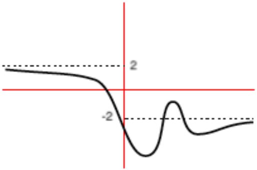
when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and positive.
Similarly, we write

$$\lim_{x \to -\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and negative.

Example

Consider the graph of the function $f(x)$.



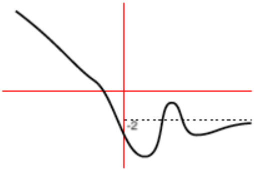
Then

$$\lim_{x\to\infty} f(x) =$$

$$\lim_{x\to-\infty} f(x) =$$

Example

Consider the graph of the function $g(x)$.



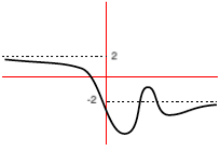
Then

$$\lim_{x\to\infty} g(x) =$$

$$\lim_{x\to-\infty} g(x) =$$

Example

Consider the graph of the function $f(x)$.



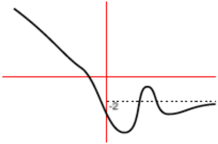
Then

$$\lim_{x\to\infty} f(x) = -2$$

$$\lim_{x\to-\infty} f(x) = 2$$

Example

Consider the graph of the function $g(x)$.



Then

$$\lim_{x\to\infty} g(x) = -2$$

$$\lim_{x\to-\infty} g(x) = +\infty$$

Review of the third session

Review

Theorem

sandwich (or squeeze or pinch) *Let* $a \in \mathbb{R}$ *and let* f, g, h *be three functions so that*

$$f(x) \leq g(x) \leq h(x)$$

for all x *in an interval around* a*, except possibly at* $x = a$*. Then if*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

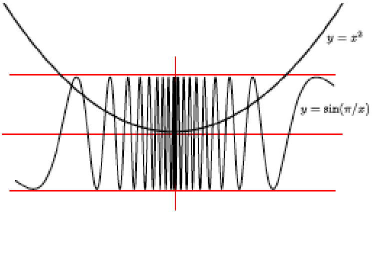
then it is also the case that

$$\lim_{x \to a} g(x) = L.$$

Example

Compute

$$\lim_{x \to 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



Theorem

Let $c \in \mathbb{R}$ then the following limits hold

$$\lim_{x \to +\infty} c = c \qquad \lim_{x \to -\infty} c = c$$

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

Outline For the Fourth Session

- Limit at Infinity

Limit at Infinity

Theorem

Let $f(x)$ and $g(x)$ be two functions for which the limits

$$\lim_{x\to\infty} f(x) = F \qquad\qquad \lim_{x\to\infty} = G$$

exist. Then the following limits hold

$$\lim_{x\to\infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x\to\infty} f(x)g(x) = FG$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \textit{provided } G \neq 0$$

and for rational numbers r,

$$\lim_{x \to \infty} (f(x))^r = F^r$$

provided that $f(x)^r$ is defined for all x.
The analogous results hold for limits to $-\infty$.

Warning: Consider that

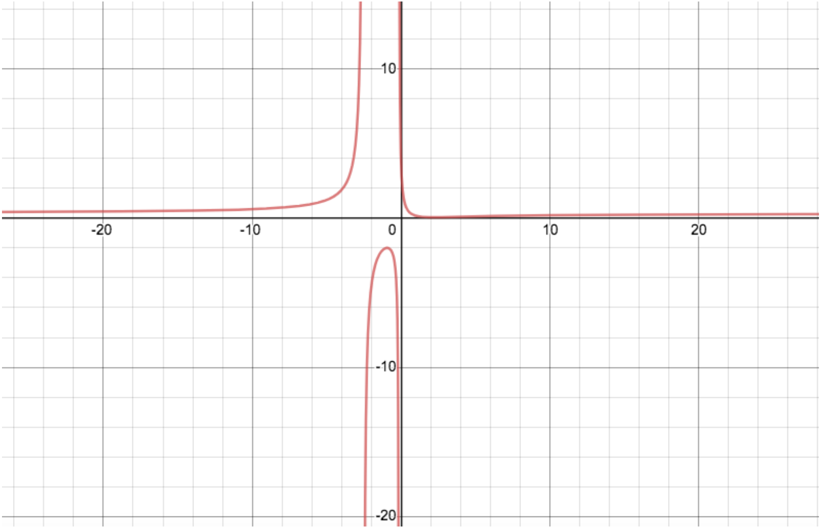
$$\lim_{x \to +\infty} \frac{1}{x^{1/2}} = 0$$

However,

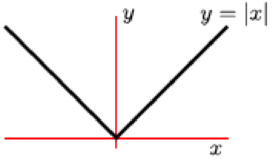
$$\lim_{x \to +\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because $x^{1/2}$ is not defined for $x < 0$.

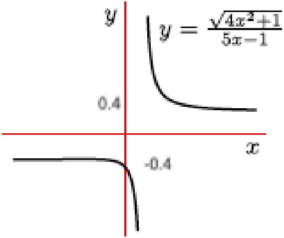
$$f(x) = \frac{x^2 - 3x + 4}{3x^2 + 8x + 1}$$



$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



$$y = \frac{\sqrt{4x^2+1}}{5x-1}$$



Theorem

Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that

$$\lim_{x\to a} f(x) = +\infty \qquad \lim_{x\to a} g(x) = +\infty \qquad \lim_{x\to a} h(x) = H$$

1. $$\lim_{x\to a}(f(x) + g(x)) =$$
2. $$\lim_{x\to a}(f(x) + h(x)) =$$
3. $$\lim_{x\to a}(f(x) - g(x)) =$$
4. $$\lim_{x\to a}(f(x) - h(x)) =$$

Theorem

Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that

$$\lim_{x\to a} f(x) = +\infty \qquad \lim_{x\to a} g(x) = +\infty \qquad \lim_{x\to a} h(x) = H$$

1. $$\lim_{x\to a}(f(x) + g(x)) = +\infty.$$
2. $$\lim_{x\to a}(f(x) + h(x)) = +\infty.$$
3. $$\lim_{x\to a}(f(x) - g(x)) = \textit{undetermined}.$$
4. $$\lim_{x\to a}(f(x) - h(x)) = +\infty.$$

Theorem

5.

$$\lim_{x \to a} cf(x) = \begin{cases} & c > 0 \\ & c = 0 \\ & c < 0 \end{cases}$$

6.

$$\lim(f(x).g(x)) =$$

7.

$$\lim_{x \to a}(f(x).h(x)) = \begin{cases} & H > 0 \\ & H = 0 \\ & H < 0 \end{cases}$$

8.

$$\lim_{x \to a} \frac{h(x)}{f(x)} =$$

Theorem

5.
$$\lim_{x\to a} cf(x) = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$$

6.
$$\lim(f(x).g(x)) = +\infty.$$

7.
$$\lim_{x\to a}(f(x).h(x)) = \begin{cases} +\infty & H > 0 \\ \textit{undetermined} & H = 0 \\ -\infty & H < 0 \end{cases}$$

8.
$$\lim_{x\to a} \frac{h(x)}{f(x)} = 0.$$

Example

Consider the following three functions:

$$f(x) = x^{-2} \qquad g(x) = 2x^{-2} \qquad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x\to 0} f(x) = +\infty \qquad \lim_{x\to 0} g(x) = +\infty \qquad \lim_{x\to 0} h(x) = +\infty.$$

Then

1. $$\lim_{x\to 0}(f(x) - g(x)) =$$
2. $$\lim_{x\to 0}(f(x) - h(x)) =$$
3. $$\lim_{x\to 0}(g(x) - h(x)) =$$

Example

Consider the following three functions:

$$f(x) = x^{-2} \qquad g(x) = 2x^{-2} \qquad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x\to 0} f(x) = +\infty \qquad \lim_{x\to 0} g(x) = +\infty \qquad \lim_{x\to 0} h(x) = +\infty.$$

Then

1. $$\lim_{x\to 0}(f(x) - g(x)) = \lim_{x\to 0} x^{-2} = \infty$$
2. $$\lim_{x\to 0}(f(x) - h(x)) = \lim_{x\to 0}(1) = 1$$
3. $$\lim_{x\to 0}(g(x) - h(x)) = \lim_{x\to 0} x^{-2} + 1 = \infty$$

Outline For the Session Five

- Limit at Infinity
- Continuity
- Continuous from the left and from the right
- Arithmetic of continuity
- continuity of composites
- Intermediate Value Theorem

Example

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{1}{x^2}$	100	10000	10^6		10^6	10000	100

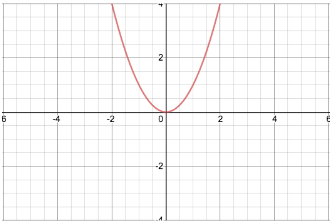
Consider that if

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a} g(x) = \infty$$

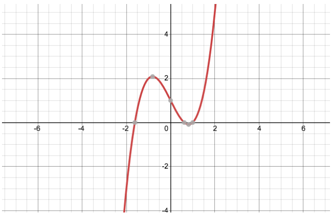
Then

$$\lim_{x \to a} (f(x) - g(x)) = \text{undetermined}$$

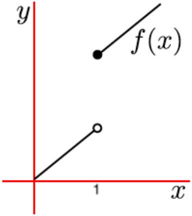
Continuity



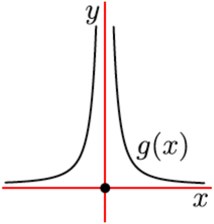
-6
-4
-2
0
2
4
6
2
-2



-6
-4
-2
0
2
4
6
4
2
-2
-4



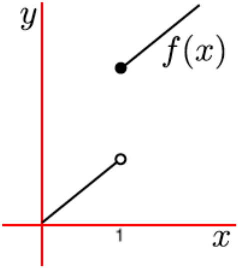
y
$f(x)$
1
x



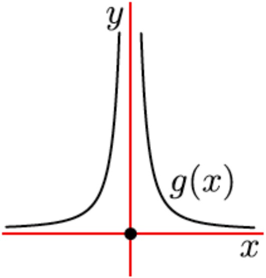
y
$g(x)$
x

$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$

jump discontinuity



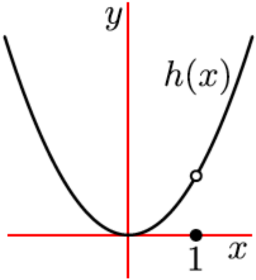
$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



infinite discontinuity

$$h(x) = \begin{cases} \frac{x^3 - x^2}{x - 1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$

removable discontinuity



Outline - September 16, 2019

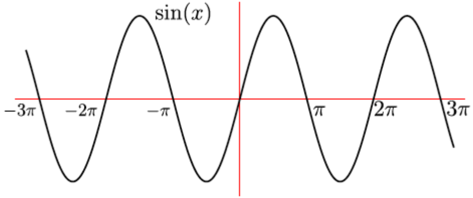
- **Section 1.6:**
  - Arithmetic of continuity
  - Continuity of composites
  - Intermediate Value Theorem
- **Section 2.1:**
  - Revisiting tangent lines

Arithmetic of continuity

Theorem

(**Arithmetic of continuity**) *Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at a. Then the following functions are also continuous at $x = a$.*

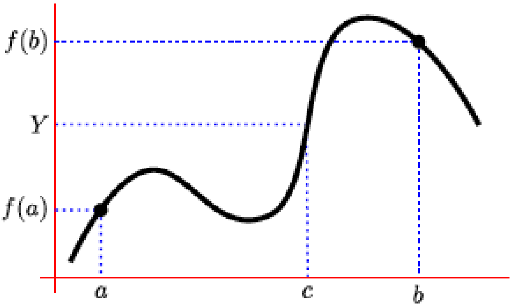
- $f(x) + g(x)$ *and* $f(x) - g(x)$,
- $cf(x)$ *and* $f(x)g(x)$, *and*
- $\frac{f(x)}{g(x)}$ *provided* $g(a) \neq 0$.


$\sin(x)$
-3π
-2π
$-\pi$
π
2π
3π

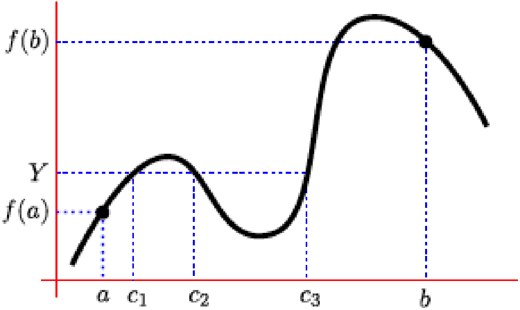
Intermediate value theorem(IVT)

Theorem

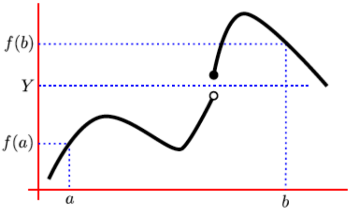
(Intermediate value theorem(IVT))

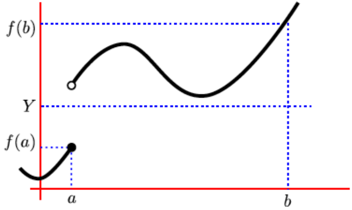


The existence not the uniqueness of c in IVT



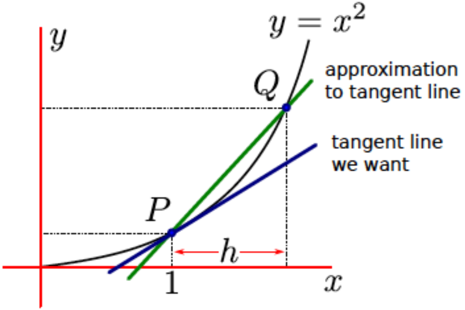
Not continuous functions at $[a, b]$ do not satisfy IVT





Revisiting tangent lines

Revisiting tangent lines



$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \leftarrow \quad \text{slope of the tangent line at } x = 1$$

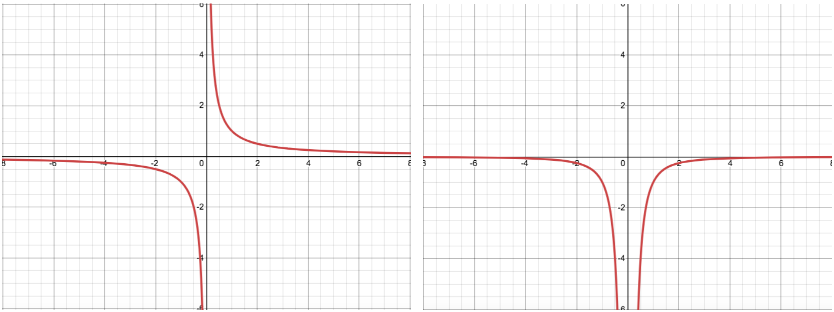
Definition of the derivative

- $$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
- $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Examples

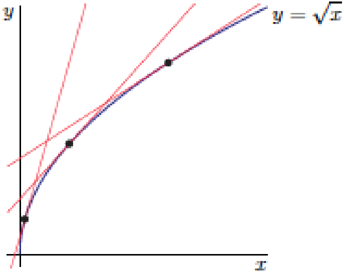
- $f(x) = c$
- $f(x) = x$
- $f(x) = x^2$
- $f(x) = \frac{1}{x}$
- $f(x) = \sqrt{x}$
- $f(x) = |x|$

$y = \frac{1}{x}$ and its derivative $-\frac{1}{x^2}$

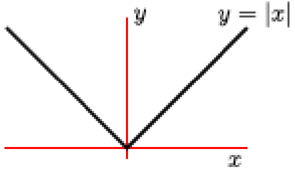


$$y = \frac{1}{x} \qquad\qquad y = \frac{-1}{x^2}$$

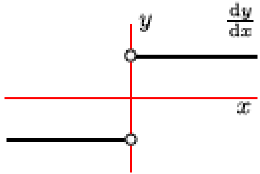
Tangent lines to $y = \sqrt{x}$



The derivative of the function $f(x) = |x|$: not differentiable at $x = 0$

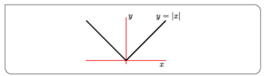


The derivative of the function $f(x) = |x|$

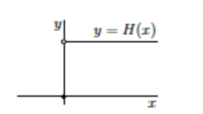


Where a function is not differentiable at $x = a$?

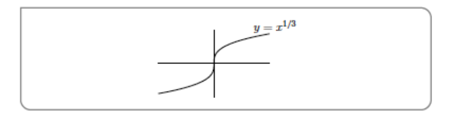
- Having a Sharp Corner at $x = a$



- The function is not continuous at $x = a$



- Having a tangent line, but the slope of the tangent line at $x = a$ is infinity

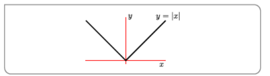


Outline - September 20, 2019

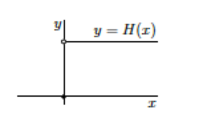
- **Section 2.2:**
  - Not differentiable examples
  - The relation between continuous and differentiable functions
- **Section 2.3:**
  - Interpretations of the derivative

Where a function is not differentiable at $x = a$?

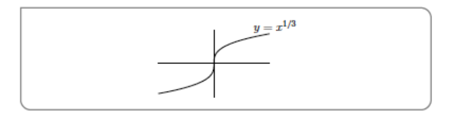
- Having a Sharp Corner at $x = a$



- The function is not continuous at $x = a$

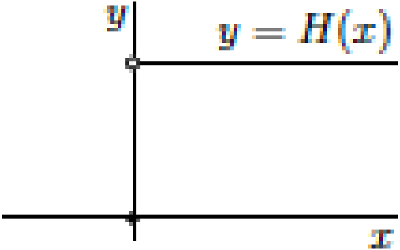


- Having a tangent line, but the slope of the tangent line at $x = a$ is infinity



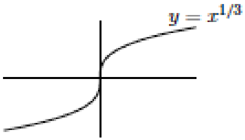
An example of a discontinuous and not differentiable function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$



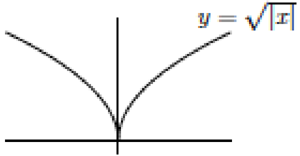
An example of a function with a tangent line with slope infinity at $x = 0$

$f(x) = x^{1/3}$

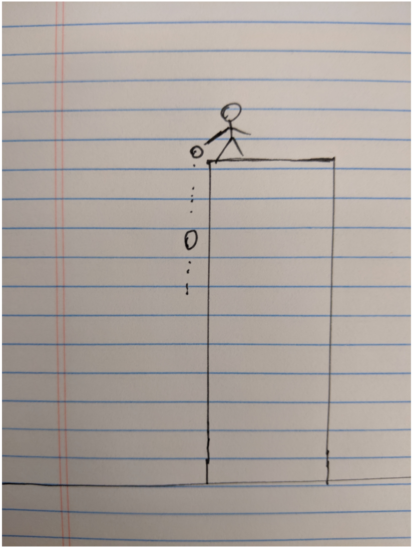


An example of a continuous and **not** differentiable function

$y = \sqrt{|x|}$



Instantaneous rate of change



average rate of change of $f(t)$ from $t=a$ to $t=a+h$ is

$$\frac{\text{change in } f(t) \text{ from } t=a \text{ to } t=a+h}{\text{length of time from } t=a \text{ to } t=a+h}$$

$$= \frac{f(a+h)-f(a)}{h}.$$

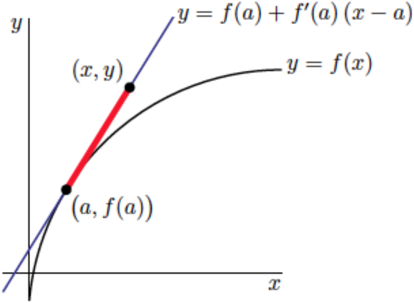
And so

instantaneous rate of change of $f(t)$ at $t=a$

$$= \lim_{h\to 0} \left[\text{average rate of change of } f(t) \text{ from } t=a \text{ to } t=a+h\right]$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a).$$

Finding tangent line to a curve at $x = a$



A line segment of a tangent line

$$y = f(a) + f'(a)(x - a)$$

Outline - September 23, 2019

- **Section 2.4 and 2.5:**
  - Derivative of some simple functions
  - Tools
  - Examples

A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \qquad \frac{d}{dx}x = 1 \qquad \frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \qquad \frac{d}{dx}x = 1 \qquad \frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Tools

Let $f(x)$ and $g(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- $\frac{d}{dx}\{f(x) + g(x)\} = f'(x) + g'(x)$
- $\frac{d}{dx}\{f(x) - g(x)\} = f'(x) - g'(x)$
- $\frac{d}{dx}\{cf(x)\} = cf'(x)$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- $\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- $\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$
- $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- $\frac{d}{dx}\{\frac{1}{g(x)}\} = \frac{-g'(x)}{g(x)^2} \qquad g(x) \neq 0$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- $\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$
- $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- $\frac{d}{dx}\{\frac{1}{g(x)}\} = \frac{-g'(x)}{g(x)^2} \qquad g(x) \neq 0$
- $\frac{d}{dx}\{f(x)g(x)h(x)\} =$ $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- $\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x)$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

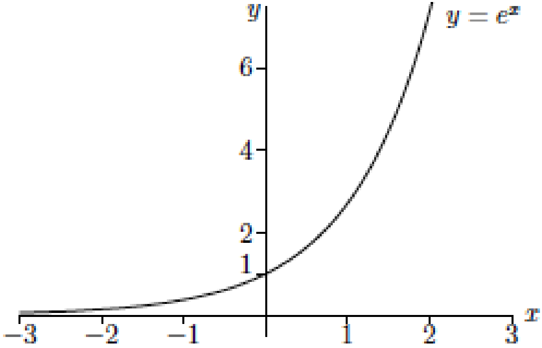
- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- $\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x)-g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$
- $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- $\frac{d}{dx}\{\frac{1}{g(x)}\} = \frac{-g'(x)}{g(x)^2} \qquad\qquad g(x) \neq 0$
- $\frac{d}{dx}\{f(x)g(x)h(x)\} =$
  $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- $\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x)$
- Let a be a rational number, then

$$\frac{d}{dx}x^a = ax^{a-1}.$$

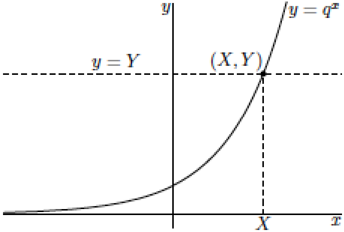
Outline - September 25, 2019

- **Section 2.7 and 2.8:**
  - Derivative of exponential functions
  - Derivative of trigonometric functions

The graph of e^x



The graph of q^x where $q > 1$



YOUR TURN!

Example

Find a such that the following function is continuous.

$$f(x) = \begin{cases} e^{x+a} & x < 0 \\ \sqrt{x+1} & x \geq 0 \end{cases}$$

Example

We have

1. $\log_q(xy) =$
   (a) $\log_q(x) + \log_q(y)$
   (b) $\log_q(x)\log_q(y)$
2. $\log_q(x/y) =$
3. $\log_q(x^r) =$

Example

We have

1. $\log_q(xy) = \log_q(x) + \log_q(y)$.
   The reason for this is that

$$q^{\log_q(xy)} = xy = q^{\log_q(x)} q^{\log_q(y)} = q^{\log_q(x)+\log_q(y)}$$

   Therefore, $\log_q(xy) = \log(x) + \log(y)$.
2. $\log_q(x/y) = \log_q(x) - \log_q(y)$
3. $\log_q(x^r) = r \log_q(x)$

TOOLS:

$$\frac{d}{dx}(f \circ g)(x) = g'(x)f'(g(x))$$

A list of derivative of some simple functions:

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = (\log_e a)a^x$$

Example

Find the derivative of $2^{\sqrt{x}}$.

Example

Find the derivative of $2^{\sqrt{x}}$.

Example

Find a and b such that the following function is differentiable.

$$f(x) = \begin{cases} x^3 + a & x < 1 \\ e^{x-1} + bx & x \geq 1 \end{cases}$$

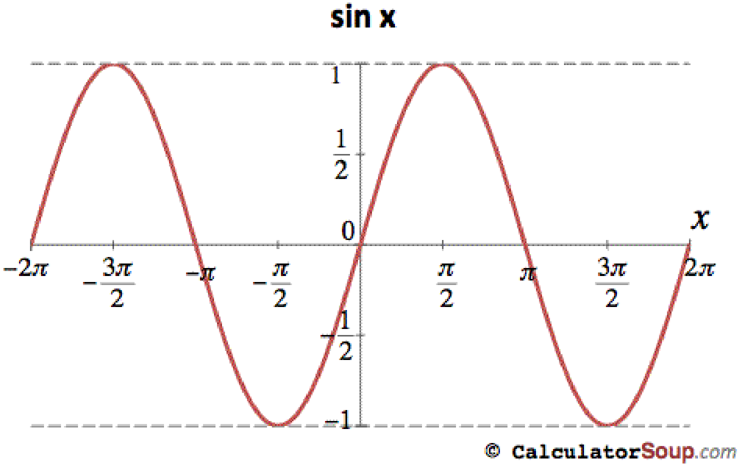
Outline - September 30, 2019

- **Section 2.8, 2.9, 0.6:**
  - Derivative of trigonometric functions
  - The chain rule
  - inverse of a function

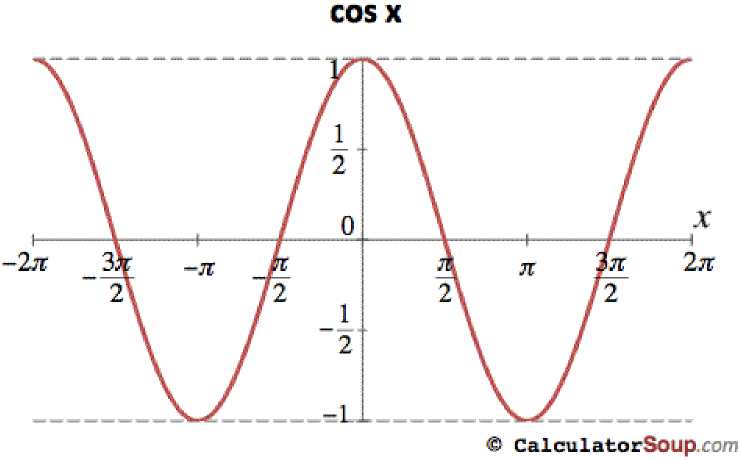
A list of derivative of some simple functions:

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = (\log_e a)a^x$$

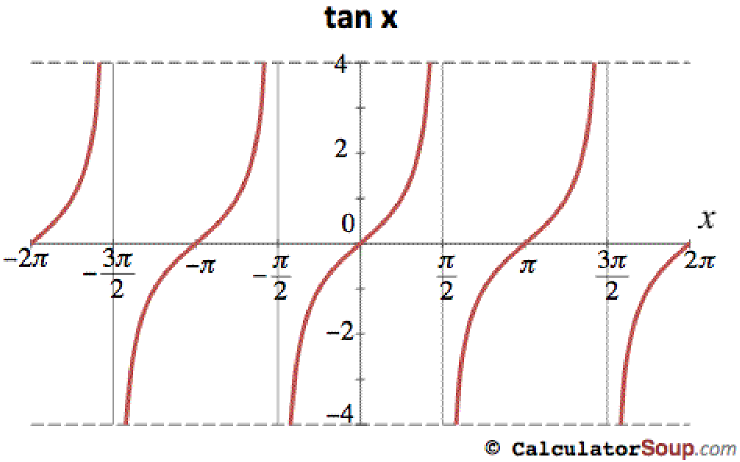
$\sin(x)$ $\quad$ domain $= \mathbb{R}$ $\quad$ range $= [-1, 1]$



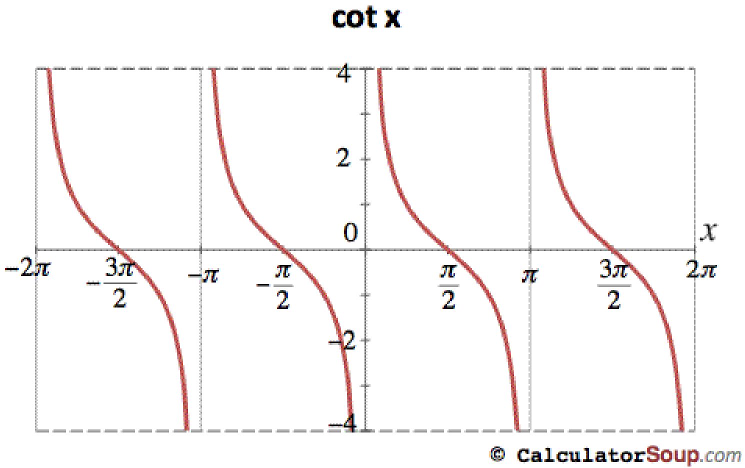
$\cos(x) \qquad \text{domain} = \mathbb{R} \qquad \text{range} = [-1, 1]$



$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \text{domain} = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R}$$

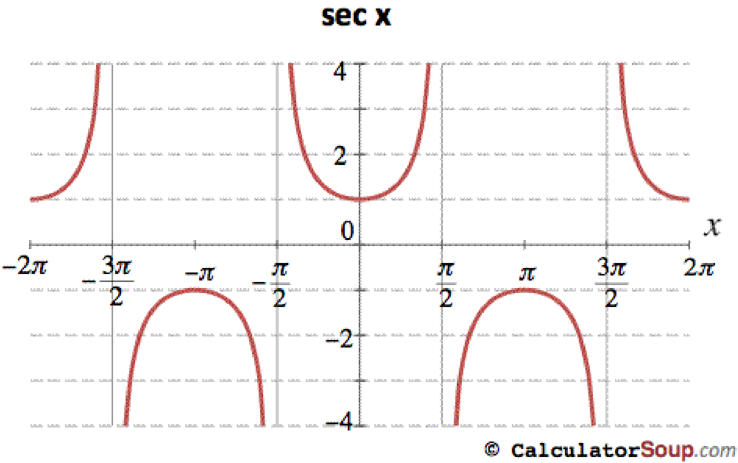


$$\cot(x) = \frac{\cos(x)}{\sin(x)} \quad \text{domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R}$$

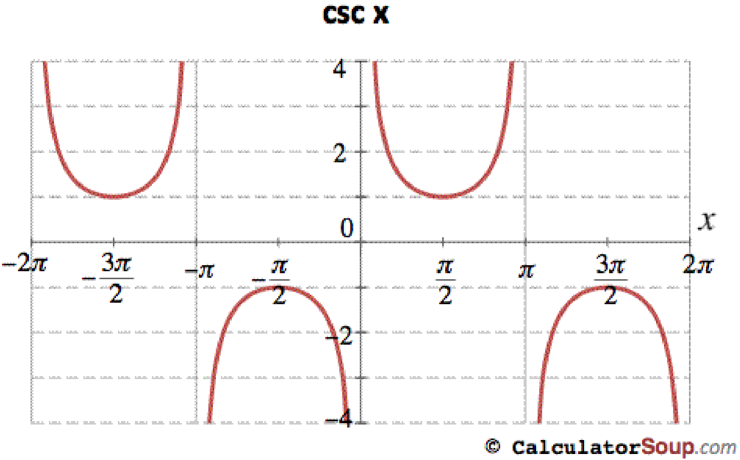


$$\sec(x) = \frac{1}{\cos(x)} \qquad \text{domain} = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$$

$$\text{range} = \mathbb{R} - (-1, 1)$$



$$\csc(x) = \frac{1}{\sin(x)} \quad \text{domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R} - (-1, 1)$$



Derivative of $\sin(x)$

Question: Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of $\sin(x)$ at $x = 0$.

Derivative of $\sin(x)$

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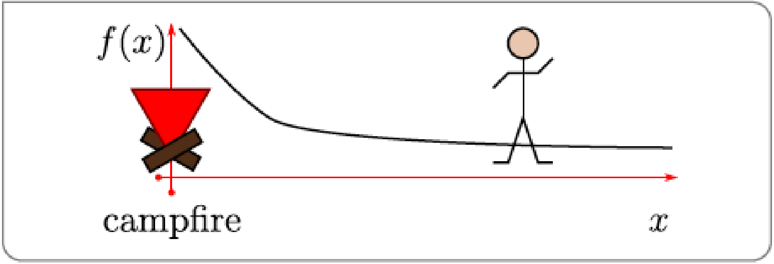
(**sandwich (or squeeze or pinch) theorem**) Let $a \in \mathbb{R}$ and let f, g, h be three functions so that $f(x) \leq g(x) \leq h(x)$ for all x in an interval around a, except possibly at $x = a$. Then if

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then it is also the case that

$$\lim_{x \to a} g(x) = L.$$

An example of the application of the chain rule



- Your position at time t is $x(t)$.
- The temperature of the air at position x is $f(x)$.
- The temperature that you feel at time t is $F(t) = f(x(t))$.
- The instantaneous rate of change of temperature that you feel is $F'(t)$.

The chain rule

Theorem

Let f and g be differentiable functions then

$$\frac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$$

The chain rule

Theorem

Let $y = f(u)$ and $u = g(x)$ be differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

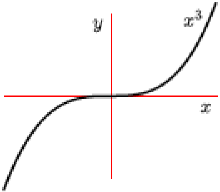
Outline - October 2, 2019

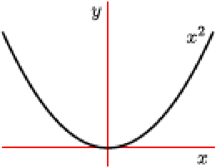
- **Section 0.6, 2.10:**
  - Inverse of a function
  - Natural logarithm

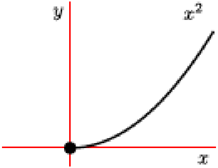
input number $x \mapsto f$ does "stuff" to $x \mapsto$ return number y

take output $y \mapsto$ do "stuff" to $y \mapsto$ return the original number x

One-to-one functions

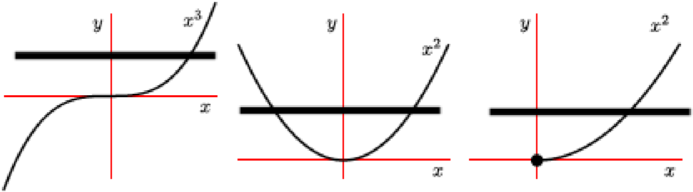






$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array} \qquad \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array} \qquad \begin{array}{ccc} [0, \infty] & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

One-to-one functions



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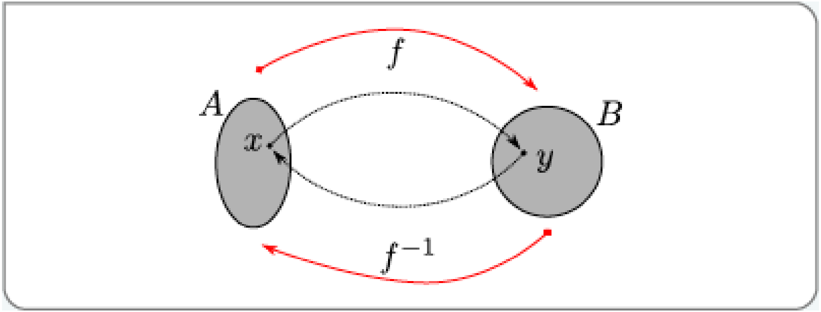
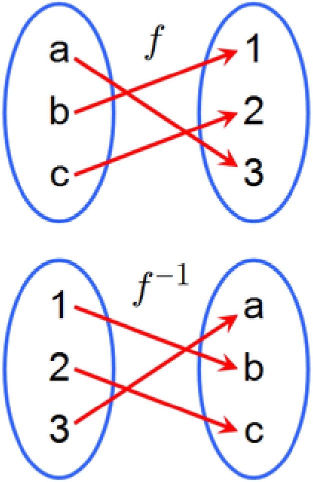
is one-to-one   is not one-to-one   is one-to-one

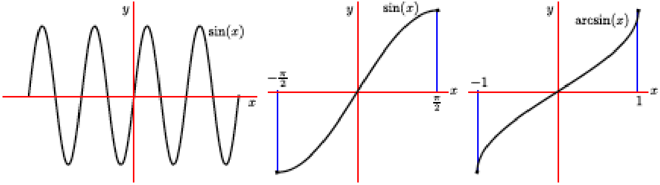
Inverse of a functions



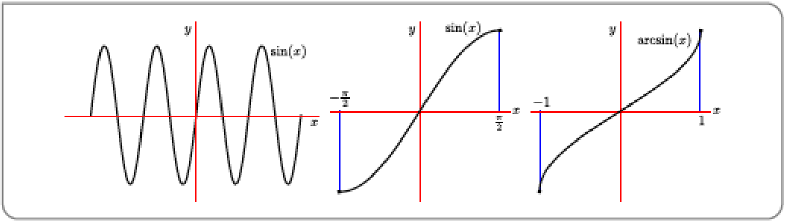
Inverse of a functions



Inverse of $\sin(x)$

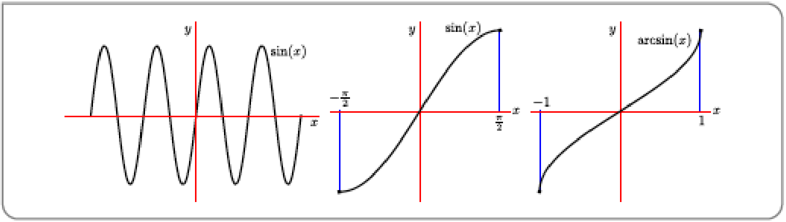


Inverse of $\sin(x)$



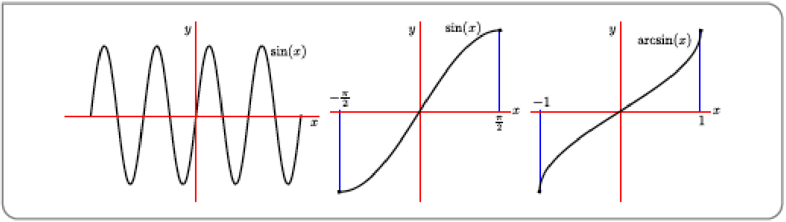
- $\sin(x)$ is not invertible on the domain $\mathbb{R}$ because it is not one-to-one.

Inverse of sin(x)



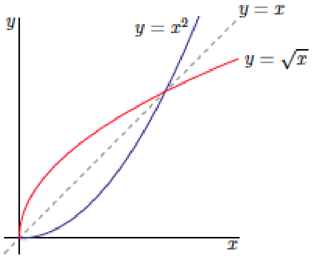
- sin(x) is not invertible on the domain $\mathbb{R}$ because it is not one-to-one.
- If we look at sin(x) on the domain $[-\pi/2, \pi/2]$, then it is one-to-one, and so it is has an inverse.

Inverse of $\sin(x)$



- $\sin(x)$ is not invertible on the domain $\mathbb{R}$ because it is not one-to-one.
- If we look at $\sin(x)$ on the domain $[-\pi/2, \pi/2]$, then it is one-to-one, and so it is has an inverse.
- The inverse of $\sin(x)$ is $\arcsin(x)$ on the domain $[-1, 1]$ and with the range $[-\pi/2, \pi/2]$.

How to find the inverse of a function by its graph



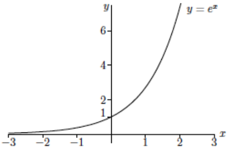
$a^{\log_a x} = x$

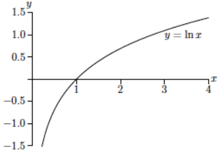
Remember that for $a > 1$,

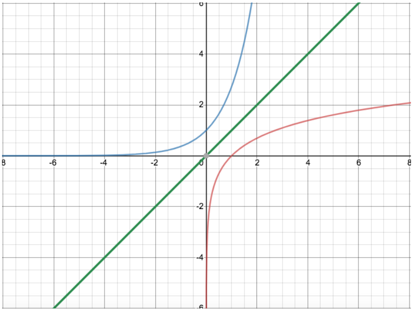
$$a^{\log_a x} = x,$$

$$\log_a x = \frac{\log_e x}{\log_e a}.$$

The inverse of e^x







Outline - October 4, 2019

- **Section 2.10 and 2.11:**
  - Natural logarithm
  - Implicit derivative

Useful facts!

- $\frac{d}{dx}a^x = (\ln a)a^x$.
- $\log_a x = \frac{\ln x}{\ln a} \qquad \ln x = \frac{\log_a x}{\log_a e} \qquad a > 1.$
- $\ln(xy) = \ln x + \ln y$.
- $\ln(x/y) = \ln x - \ln y$.
- $\ln x^r = r \ln x$.

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- $\frac{d}{dx} \ln x = \frac{1}{x}$.

Useful facts!

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- $\frac{d}{dx} \ln |x| = \frac{1}{x}$.

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- $\ln(x/y) = \ln x - \ln y$.
- $\ln x^r = r \ln x$.
- $\frac{d}{dx}\ln x = \frac{1}{x}$.
- $\frac{d}{dx}\ln |x| = \frac{1}{x}$.
- $\frac{d}{dx}\log_a x = \frac{1}{x . \ln a}$.

Useful facts!

- $\frac{d}{dx}a^x = (\ln a)a^x$.
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- $\frac{d}{dx} \ln |x| = \frac{1}{x}$.
- $\frac{d}{dx} \log_a x = \frac{1}{x. \ln a}$.
- $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$

Useful facts!

- $\frac{d}{dx} a^x = (\ln a) a^x$.
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- $\frac{d}{dx} \ln x = \frac{1}{x}$.
- $\frac{d}{dx} \ln |x| = \frac{1}{x}$.
- $\frac{d}{dx} \log_a x = \frac{1}{x.\ln a}$.
- $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$
- $\frac{d}{dx} |f(x)| = \frac{f'(x)}{f(x)}$.

Implicit derivative

$$\frac{d}{dx}x = \frac{d}{dx}e^{\ln x} \qquad (\frac{d}{dx}x = \frac{d}{dx}e^{y})$$

which is the same as

$$1 = (\frac{d}{dx}\ln x).e^{\ln x} \qquad (1 = y'e^{y}).$$

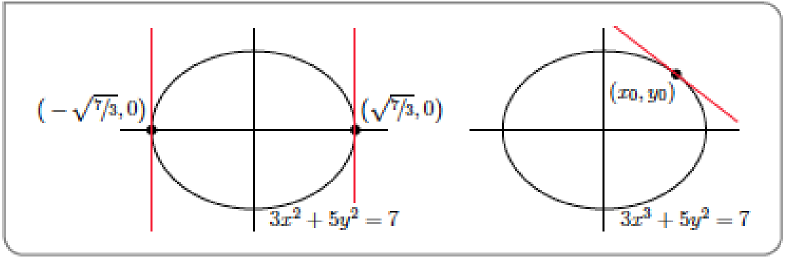
Note that $e^{\ln x} = x(e^{y} = x)$, thus

$$1 = (\frac{d}{dx}\ln x).x \qquad (1 = y'x)$$

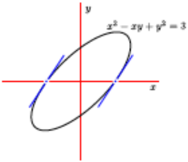
and so

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad (y' = \frac{1}{x}).$$

$3x^3 + 5y^2 = 7$



$x^2 - xy + y^2 = 3$



$x^{2/3} + y^{2/3} = 1$