

# MATH100: Differential Calculus with Application to Physical Sciences and Engineering

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# Contents

<b>1</b>	<b>Limits</b>	<b>5</b>
1.1	Tangent line . . . . .	5
1.2	Velocity . . . . .	8
1.3	The limit of a function . . . . .	10
1.4	Calculating Limits with Limit Laws . . . . .	17
1.5	Limits at Infinity . . . . .	21
1.6	Continuity . . . . .	27
<b>2</b>	<b>Derivatives</b>	<b>35</b>
2.1	Revisiting Tangent Lines . . . . .	35
2.2	Definition of the derivative . . . . .	36
2.3	Interpretations of the Derivative . . . . .	45
2.4	Arithmetic of Derivatives - a Differentiation Toolbox . . . . .	48
2.5	Using the Arithmetic of Derivatives - Examples . . . . .	48



# Chapter 1

## Limits

What does this mean

$$\lim_{x \rightarrow a} f(x) = L?$$

The "limit" appears when we want to

- find the tangent to a curve; or
- find the velocity of an object.

### 1.1 Tangent line



The **tangent line to a curve**  $y = f(x)$  at a point  $P$  (if exists) is a line  $L$  that there is a neighborhood for  $P$  such that in that neighborhood the line  $L$  touches (does not cross) the curve  $y = f(x)$  only at  $P$  (and not other points in that neighborhood).

## The equation of a line

- The formula for a line that passes through  $(x_1, y_1)$  with slope  $m$  is

$$y = y_1 + m(x - x_1).$$

- Given two points  $(x_1, y_1)$  and  $(x_2, y_2)$  on a line, then the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

and the formula for the line then is

$$y = y_1 + m(x - x_1).$$

**Example 1.1.1.** Find the equation of the line with slope  $-3$  that passes through  $(1, 2)$ .

**Solution.** The equation of the line is

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

**Example 1.1.2.** Find the equation of the line that passes through  $(1, 2)$  and  $(2, -1)$ .

**Solution.** First we find the slope which is

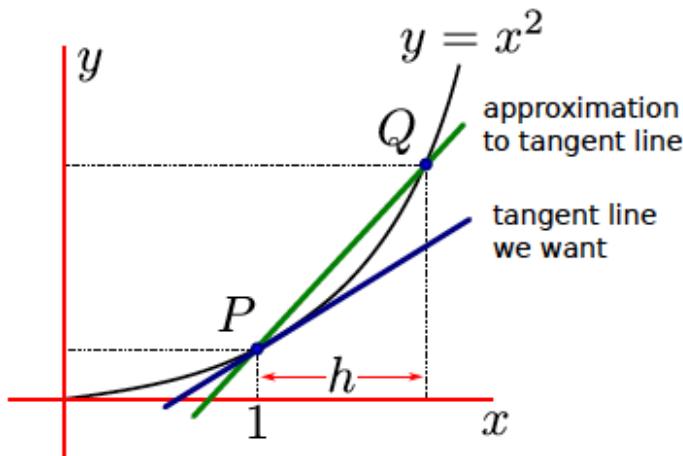
$$\frac{-1 - 2}{2 - 1} = -3.$$

Then the equation of the line is

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

**The equation of a tangent line:** Given a curve  $y = f(x)$  and a point  $P$  on the curve, how to find the slope of the tangent to a curve at  $P$ : let do this through an example.

**Example 1.1.3.** Find the tangent line to the curve  $y = x^2$  that passes through  $P = (1, 1)$ .





So we want to find the slope the line that passes through the points  $(x_1, y_1) = (1, 1)$  and  $(x_2, y_2) = (1 + h, (1 + h)^2)$ . The slope then is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1 + h)^2 - 1^2}{(1 + h) - 1} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(h + 2)}{h} = 2 + h$$

$h$	$m = \frac{(1+h)^2 - 1^2}{(1+h) - 1}$
0.1	2.1
0.01	2.01
0.001	2.001

When  $h$  gets smaller and smaller, the slope will be closer and closer to the slope of the tangent line to  $y = x^2$  at  $(1, 1)$ , which the slope will be closer and closer to 2, we can write this more mathematically as

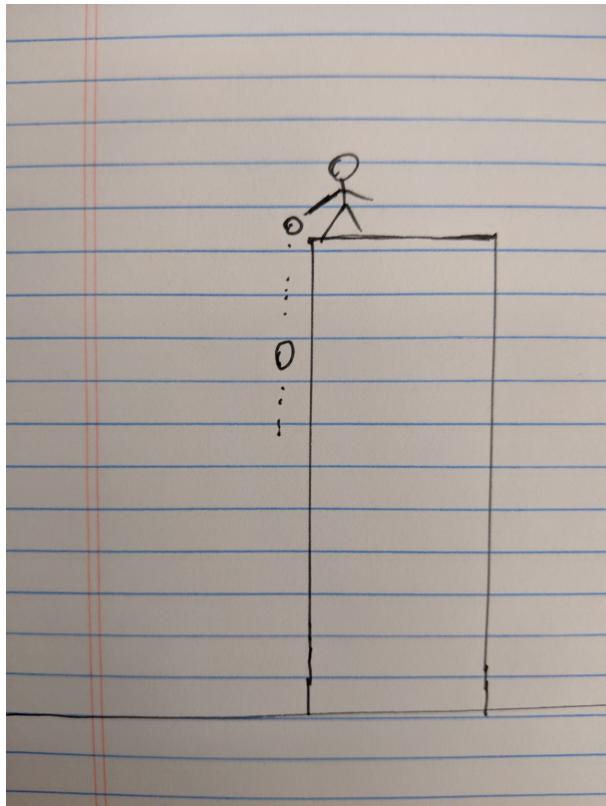
$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1^2}{(1 + h) - 1} = 2$$

**Read:** the limit of  $\frac{(1+h)^2 - 1^2}{(1+h) - 1}$  as  $h$  approaches 0 is 2.  
Tangent line is

$$y = 1 + 2(x - 1) = 2x - 1.$$

## 1.2 Velocity

- Let  $t$  be elapsed time measured in second
- $S(t)$  be the distance the ball has fallen in meters
- What is  $S(0)$ ?  $S(0) = 0$ .
- (**Galileo**)  $S(t) = 4.9t^2$ .



**Question:** How fast the ball is fallen after 1 second, that is, find  $v(1)$ , the velocity at  $t = 1$ ?

$$\text{average velocity} = \frac{\text{difference in position}}{\text{difference in time}} = \frac{S(t_2) - S(t_1)}{t_2 - t_1}.$$

To answer the question we should find the average velocity of the falling ball between  $(1 + h)$  and 1. So,

average velocity when  $(t_2 = 1 + h)$  and  $(t_1 = 1)$

$$= \frac{S(1 + h) - S(1)}{h} = \frac{4.9(1 + h)^2 - 4.9}{h} = 4.9(2 + h).$$



time window	average velocity
$1 \leq t \leq 1.1$	10.29
$1 \leq t \leq 1.01$	9.84
$1 \leq t \leq 1.01$	9.8049
$1 \leq t \leq 1.001$	9.80049

So we can write

$$v(1) = \lim_{h \rightarrow 0} \frac{S(1+h) - S(1)}{h} = 9.8.$$

More generally:

We define the instantaneous velocity at time  $t = a$  to be the limit

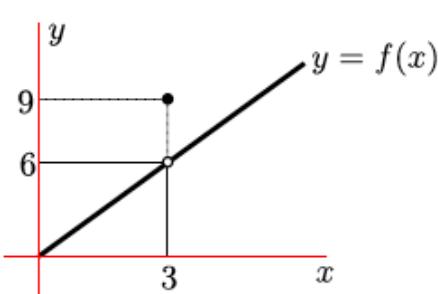
$$v(a) = \lim_{h \rightarrow 0} \frac{S(a+h) - S(a)}{h}$$

## 1.3 The limit of a function

To arrive at the definition of limit, we start with a very simple example.

**Example 1.3.1.** Consider the following function

$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



If we plug in some numbers very close to 3 (but not exactly 3) into the function we see

$x$	2.9	2.99	2.999	○	3.001	3.01	3.1
$f(x)$	5.8	5.98	5.998	○	6.002	6.02	6.2

So as  $x$  moves closer and closer to 3, without being exactly 3, we see that the function moves closer and closer to 6. We can then write this as

$$\lim_{x \rightarrow 3} f(x) = 6.$$

**Definition. (Informal definition of limit)** We write

$$\lim_{x \rightarrow a} f(x) = L.$$

if the value of the function  $f(x)$  is sure to be arbitrary close to  $L$  whenever the value of  $x$  is close enough to  $a$ , without being exactly  $a$ .

**Example 1.3.2.** Let  $f(x) = \frac{x-2}{x^2+x-6}$  and find its limit as  $x \rightarrow 2$ .

**Solution.** We want to find

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2+x-6}.$$

**Important point:** if we compute  $f(2)$ , then we have  $\frac{0}{0}$  which is undefined.

Again we plug in numbers close to 2 and we have

$x$	1.9	1.99	1.999	○	2.001	2.01	2.1
$f(x)$	0.20408	0.20040	0.20004	○	0.19996	0.19960	0.19608

So

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2+x-6} = 2.$$

**Example 1.3.3.** Consider the following function  $f(x) = \sin(\pi/x)$ . Find the limit as  $x \rightarrow 0$  of  $f(x)$ .

**Solution.** When  $x$  is getting closer and closer to 0, it oscillates faster and faster. Since the function does not approach a single number as we bring  $x$  closer and closer to zero, the limit does not exist. Thus,

$$\lim_{x \rightarrow 0} \sin(\pi/x) = \text{DNE}$$



**Example 1.3.4.** Consider the function

$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$

Find

$$\lim_{x \rightarrow 2} f(x).$$

**Solution.**



Let again plug in some numbers close to 2 (but not exactly 2)

$x$	1.9	1.99	1.999	○	2.001	2.01	2.1
$f(x)$	1.9	1.99	1.999	○	5.001	5.01	5.1

Now when we approach from below (or left), we seem to be getting closer to 2 ( $\lim_{x \rightarrow 2^-} f(x) = 2$ ), but when we approach from above (or right) we seem to be getting closer to 5 ( $\lim_{x \rightarrow 2^+} f(x) = 5$ ). Since we are not approaching the same number the limit does not exist.

$$\lim_{x \rightarrow 2} f(x) = \text{DNE}$$

**Definition. (Informal definition of one-sided limits)** We write

$$\lim_{x \rightarrow a^-} f(x) = K$$

when the value of  $f(x)$  gets closer and closer to  $K$  when  $x < a$  and  $x$  moves closer and closer to  $a$ . Since the  $x$ -values are always less than  $a$ , we say that  $x$  approaches  $a$  from below (or left). This is also often called the left-hand limit since the  $x$ -values lie to the left of  $a$  on a sketch of the graph.

We similarly write

$$\lim_{x \rightarrow a^+} f(x) = L$$

when the values of  $f(x)$  gets closer and closer to  $L$  when  $x > a$  and  $x$  moves closer and closer to  $a$ . For similar reason we say that  $x$  approaches  $a$  from above, and sometimes to this as the the right-hand limit.

### Theorem 1.3.5.

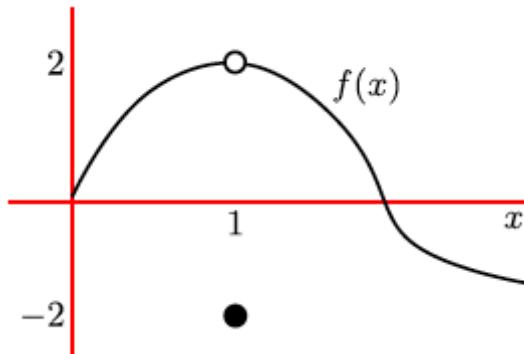
$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x) = L$$

- If the limit of  $f(x)$  as  $x$  approaches  $a$  exists and is equal to  $L$ , then both the left-hand and right-hand limits exist and are equal to  $L$ .
- If the left-hand and right-hand limits as  $x$  approaches  $a$  exist and are equal, then the limit as  $x$  approaches  $a$  exists and is equal to the one-sided limits.

Contrapositive of the above argument says

- If either of the left-hand and right-hand limits as  $x$  approaches  $a$  fail to exist, or if they both exist but are different, then the limit as  $x$  approaches  $a$  does not exist. AND,
- If the limit as  $x$  approaches  $a$  does not exist, then the left-hand and right-hand limits are either different or at least one of them does not exist.

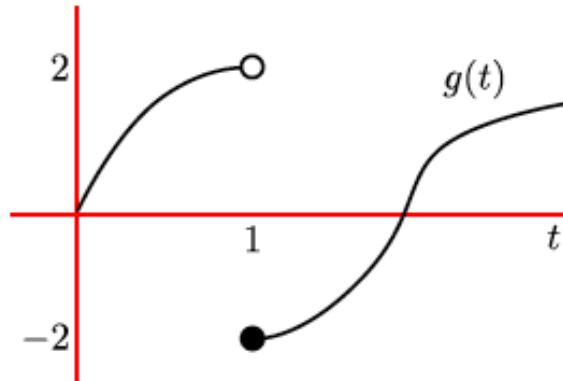
**Example 1.3.6.** Consider the graph of the function  $f(x)$ .



Then

$$\lim_{x \rightarrow 1^-} f(x) = 2 \quad \lim_{x \rightarrow 1^+} f(x) = 2 \quad \lim_{x \rightarrow 1} f(x) = 2$$

**Example 1.3.7.** Consider the graph of the function  $g(t)$ .



Then

$$\lim_{t \rightarrow 1^-} g(t) = 2 \quad \lim_{t \rightarrow 1^+} g(t) = -2 \quad \lim_{t \rightarrow 1} g(t) = DNE$$

In the following example even though the limit doesn't exists when  $x$  approaches  $a$ , we can say more.

**Example 1.3.8.** Consider the graph for the function  $f(x)$ .



$$\lim_{x \rightarrow a} f(x) = +\infty$$

**Example 1.3.9.** Consider the graph for the function  $g(x)$ .



$$\lim_{x \rightarrow a} g(x) = -\infty$$

**Definition.** We write

$$\lim_{x \rightarrow a} f(x) = +\infty$$

when the value of the function  $f(x)$  becomes arbitrarily large and positive as  $x$  gets closer and closer to  $a$ , without being exactly  $a$ .

Similarly, we write

$$\lim_{x \rightarrow a} f(x) = -\infty$$

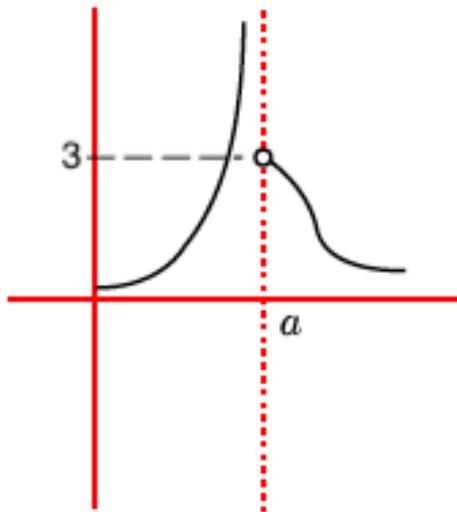
when the value of the function  $f(x)$  becomes arbitrarily large and negative as  $x$  gets closer and closer to  $a$ , without being exactly  $a$ .

**Example 1.3.10.**

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \lim_{x \rightarrow 0} -\frac{1}{x^2} = -\infty$$

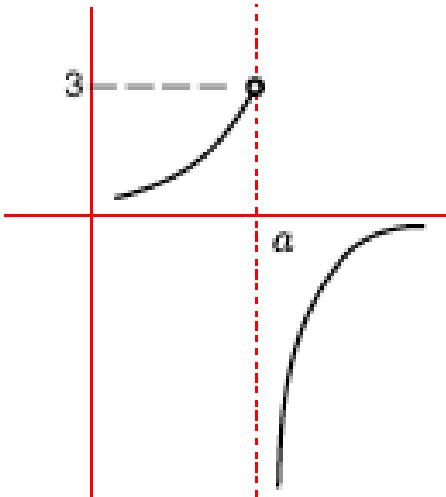
**Important Point:** Do not think of “ $+\infty$ ” and “ $-\infty$ ” in these statements as numbers. When we write  $\lim_{x \rightarrow a} f(x) = +\infty$ , it says “the function  $f(x)$  becomes arbitrary large as  $x$  approaches  $a$ ”.

**Example 1.3.11.** Consider the graph for the function  $h(x)$ .



$$\lim_{x \rightarrow a^-} h(x) = +\infty \quad \lim_{x \rightarrow a^+} h(x) = 3 \quad \lim_{x \rightarrow a} h(x) = \text{DNE}$$

**Example 1.3.12.** Consider the graph for the function  $s(x)$ .



$$\lim_{x \rightarrow a^-} s(x) = 3 \quad \lim_{x \rightarrow a^+} s(x) = -\infty \quad \lim_{x \rightarrow a} s(x) = \text{DNE}$$

**Definition.** We write

$$\lim_{x \rightarrow a^+} f(x) = +\infty$$

when the value of the function  $f(x)$  becomes arbitrarily large and positive as  $x$  gets closer and closer to  $a$  from above (equivalently, from right), without being exactly  $a$ . Similarly, we write

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

when the values of the function  $f(x)$  becomes arbitrarily large and negative as  $x$  gets closer and closer to  $a$  from above (equivalently, from right), without being exactly  $a$ .

The notation

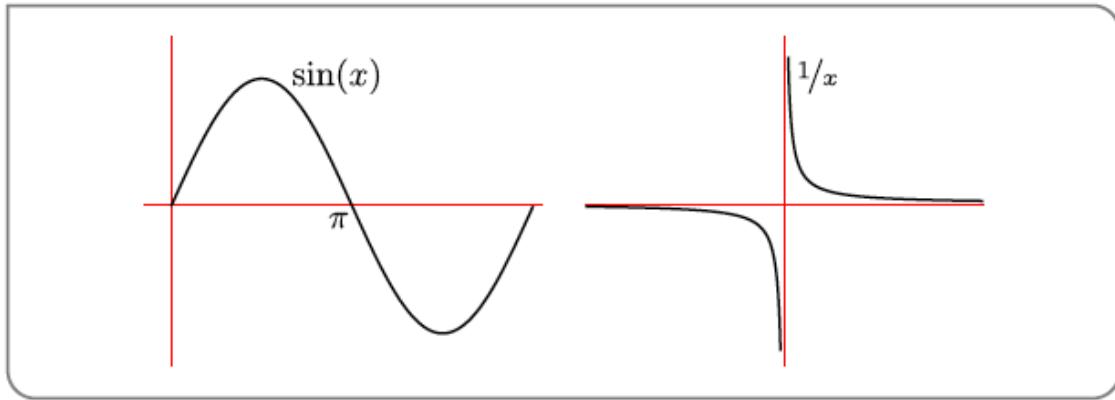
$$\lim_{x \rightarrow a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \rightarrow a^-} f(x) = -\infty$$

has a similar meaning except that limits are approached from below (from left).

**Example 1.3.13.** Consider the function

$$g(x) = \frac{1}{\sin(x)}.$$

Find the one-side limits of this function as  $x \rightarrow \pi$ .



- As  $x \rightarrow \pi$  from the left,  $\sin(x)$  is a small positive number that is getting closer and closer to zero. That is, as  $x \rightarrow \pi^-$ , we have that  $\sin(x) \rightarrow 0$  through positive numbers (i.e. from above). Now look at the graph of  $1/x$ , and think what happens as we move  $x \rightarrow 0^+$ , the function is positive and becomes larger and larger.  
So as  $x \rightarrow \pi$  from the left,  $\sin(x) \rightarrow 0$  from above, and so  $1/\sin(x) \rightarrow +\infty$ .
- By very similar reasoning, as  $x \rightarrow \pi$  from the right,  $\sin(x)$  is a small negative number that gets closer and closer to zero. So as  $x \rightarrow \pi$  from the right,  $\sin(x) \rightarrow 0$  through negative numbers (i.e. from below) and so  $1/\sin(x)$  to  $-\infty$ .

Thus

$$\lim_{x \rightarrow \pi^-} \frac{1}{\sin(x)} = +\infty \qquad \lim_{x \rightarrow \pi^+} \frac{1}{\sin(x)} = -\infty$$

## 1.4 Calculating Limits with Limit Laws

**Theorem 1.4.1.** Let  $a, c \in \mathbb{R}$ . The following two limits hold

$$\lim_{x \rightarrow a} c = c \quad \lim_{x \rightarrow a} x = a$$

**Theorem 1.4.2. (Arithmetic of Limits)** Let  $a, c \in \mathbb{R}$ , let  $f(x)$  and  $g(x)$  be defined for all  $x$ 's that lie in some interval about  $a$  (but  $f$  and  $g$  need not to be defined exactly at  $a$ ).

$$\lim_{x \rightarrow a} f(x) = F \quad \lim_{x \rightarrow a} g(x) = G$$

exists with  $F, G \in \mathbb{R}$ . Then the following limits hold

- $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$  – limit of the sum is the sum of the limits.
- $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$  – limit of the difference is the difference of the limits.
- $\lim_{x \rightarrow a} cf(x) = cF$ .
- $\lim_{x \rightarrow a} (f(x).g(x)) = F.G$  – limit of the product is the product of the limits.
- If  $G \neq 0$  then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}$ .

**Example 1.4.3.** Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} g(x)} = \frac{3}{2} = 3.$$

**Example 1.4.4.**

$$\lim_{x \rightarrow 3} 4x^2 - 1 = 4 \times \lim_{x \rightarrow 3} x^2 - \lim_{x \rightarrow 3} 1 = 35.$$

$$\lim_{x \rightarrow 2} \frac{x}{x-1} = \frac{\lim_{x \rightarrow 2} x}{\lim_{x \rightarrow 2} x - \lim_{x \rightarrow 1} 1} = \frac{2}{2-1} = 2.$$

Consider that we apply the theorem Arithmetic of Limits to compute the limit of a ratio if the limit of denominator is not zero. **What will happen if the limit of denominator is zero:**

- the limit does not exist, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = DNE$$

– the limit is  $\pm\infty$ , eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x^4} = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{-x^2}{x^4} = \lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty.$$

– the limit is 0, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

– the limit exists and it nonzero, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x} = 1.$$

**Theorem 1.4.5.** Let  $n$  be a positive integer, let  $a \in R$  and let  $f$  be a function so that

$$\lim_{x \rightarrow a} f(x) = F$$

for some real number  $F$ . Then the following holds

$$\lim_{x \rightarrow a} (f(x))^n = \left( \lim_{x \rightarrow a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit. Similarly, if

- $n$  is an even number and  $F > 0$ , or
- $n$  is an odd number and  $F$  is any real number

then

$$\lim_{x \rightarrow a} (f(x))^{1/n} = \left( \lim_{x \rightarrow a} f(x) \right)^{1/n} = F^{1/n}.$$

### Example 1.4.6.

$$\lim_{x \rightarrow 4} x^{1/2} = 4^{1/2} = 2.$$

$$\lim_{x \rightarrow 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \rightarrow 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = 13^{1/3}$$

### Example 1.4.7. Compute the following limits.

$$1. \lim_{x \rightarrow 2} \frac{x^3 - x^2}{x - 1}$$

$$2. \lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1}$$

**Solution.** 1.  $\lim_{x \rightarrow 2} \frac{x^3 - x^2}{x - 1} = 4$ .

2. Consider that  $\lim_{x \rightarrow 1} x^3 - x^2 = 0$  and  $\lim_{x \rightarrow 1} x - 1 = 0$ . However,

$$\frac{x^3 - x^2}{x - 1} = \frac{x^2(x - 1)}{x - 1},$$

thus

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases}$$



And so

$$\lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \rightarrow 1} x^2 = 1.$$

The reasoning in the above example can be made more general:

**Theorem 1.4.8.** If  $f(x) = g(x)$  except when  $x = a$  then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

provided the limit of  $g$  exists.

We mostly use the above theorem when we end up with  $\frac{0}{0}$ .

**Example 1.4.9.** Compute

$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1}{h}.$$

**Solution.** Note that

$$\frac{(1 + h)^2 - 1}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(2 + h)}{h}.$$

Thus,

$$\frac{(1 + h)^2 - 1}{h} = \begin{cases} 2 + h & h \neq 0 \\ \text{undefined} & h = 0. \end{cases}$$

And so

$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1}{h} = \lim_{h \rightarrow 0} 2 + h = 2.$$

We now present a slightly harder example.

**Example 1.4.10.** Compute the limit

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1 + x} - 1}.$$

**Solution.** Both the limits of the numerator and denominator as  $x \rightarrow 0$  are 0, so we cannot use the Theorem Arithmetic of limits. We now can simply multiply the numerator and denominator by the conjugation of  $\sqrt{1+x} - 1$ , that is,  $\sqrt{1+x} + 1$ . We have

$$\begin{aligned}
 \frac{x}{\sqrt{1+x}-1} &= \frac{x}{\sqrt{1+x}-1} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} && \text{multiply by } \frac{\text{conjugate}}{\text{conjugate}} = 1 \\
 &= \frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)} && \text{bring things together} \\
 &= \frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x})^2 - 1 \cdot 1} && \text{since } (a-b)(a+b) = a^2 - b^2 \\
 &= \frac{x(\sqrt{1+x}+1)}{1+x-1} && \text{clean up a little} \\
 &= \frac{x(\sqrt{1+x}+1)}{x} && \\
 &= \sqrt{1+x}+1 && \text{cancel the } x
 \end{aligned}$$

So now we have

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x}-1} &= \lim_{x \rightarrow 0} \sqrt{1+x}+1 \\
 &= \sqrt{1+0}+1 = 2
 \end{aligned}$$

Before we move to the next section and study the limits at infinity, we have one more theorem to state.

**Example 1.4.11.** Compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



**Solution.** It is not possible to simply use the theorem Arithmetic of Limits since the limit of  $\sin\left(\frac{\pi}{x}\right)$  as  $x \rightarrow 0$  does not exist. Since  $-1 \leq \sin(\theta) \leq 1$  for all real numbers  $\theta$ , we have

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1 \quad \text{for all } x \neq 0$$

Multiplying the above by  $x^2$  we see that

$$-x^2 \leq x^2 \sin\left(\frac{\pi}{x}\right) \leq x^2 \quad \text{for all } x \neq 0.$$

Since

$$\lim_{x \rightarrow 0} x^2 = \lim_{x \rightarrow 0} (-x^2) = 0$$

by the sandwich (or squeeze or pinch) theorem (look at below for the sandwich theorem) we have

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0.$$

**Theorem 1.4.12. (sandwich (or squeeze or pinch) theorem )** Let  $a \in \mathbb{R}$  and let  $f, g, h$  be three functions so that

$$f(x) \leq g(x) \leq h(x)$$

for all  $x$  in an interval around  $a$ , except possibly at  $x = a$ . Then if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then it is also the case that

$$\lim_{x \rightarrow a} g(x) = L.$$

**Example 1.4.13.** Let  $f(x)$  be a function such that  $1 \leq f(x) \leq x^2 - 2x + 2$ . What is

$$\lim_{x \rightarrow 1} f(x)?$$

**Solution.** Consider that

$$\lim_{x \rightarrow 1} x = 1 \quad \text{and} \quad \lim_{x \rightarrow 1} x^2 - 2x + 2 = 1.$$

Therefore, by the sandwich/pinch/squeeze theorem

$$\lim_{x \rightarrow 1} f(x) = 1.$$

## 1.5 Limits at Infinity

**Example 1.5.1.** We want to compute

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{x}$$

By plug in some large numbers into  $\frac{1}{x}$  we have

-10000	-1000	-100	100	1000	10000
-0.0001	0.001	-0.01	0.01	0.001	0.0001

We see that as  $x$  is getting bigger and positive the function  $\frac{1}{x}$  is getting closer to 0. Thus,

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0.$$

Moreover,

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

**Definition.** (Informal limit at infinity.) We write

$$\lim_{x \rightarrow \infty} f(x) = L$$

when the value of the function  $f(x)$  gets closer and closer to  $L$  as we make  $x$  larger and larger and positive.

Similarly, we write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

when the value of the function  $f(x)$  gets closer and closer to  $L$  as we make  $x$  larger and larger and negative.

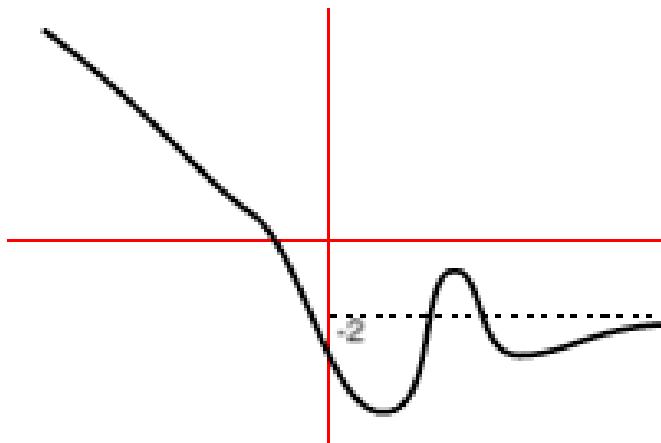
**Example 1.5.2.** Consider the graph of the function  $f(x)$ .



Then

$$\lim_{x \rightarrow \infty} f(x) = -2 \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = 2$$

**Example 1.5.3.** Consider the graph of the function  $g(x)$ .



Then

$$\lim_{x \rightarrow \infty} g(x) = -2 \quad \text{and} \quad \lim_{x \rightarrow -\infty} g(x) = +\infty$$

Same as usual we start with two very simple building blocks and build other limits from them.

**Theorem 1.5.4.** Let  $c \in \mathbb{R}$  then the following limits hold

$$\lim_{x \rightarrow +\infty} c = c \quad \lim_{x \rightarrow -\infty} c = c$$

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0 \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

**Theorem 1.5.5.** Let  $f(x)$  and  $g(x)$  be two functions for which the limits

$$\lim_{x \rightarrow \infty} f(x) = F \quad \lim_{x \rightarrow \infty} g(x) = G$$

exist. Then the following limits hold

$$\lim_{x \rightarrow \infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x \rightarrow \infty} f(x)g(x) = FG$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \text{provided } G \neq 0$$

and for rational numbers  $r$ ,

$$\lim_{x \rightarrow \infty} (f(x))^r = F^r$$

provided that  $f(x)^r$  is defined for all  $x$ .

The analogous results hold for limits to  $-\infty$ .

We need a little extra care with the posers of functions.

**Warning:** Consider that

$$\lim_{x \rightarrow +\infty} \frac{1}{x^{1/2}} = 0$$

However,

$$\lim_{x \rightarrow +\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because  $x^{1/2}$  is not defined for  $x < 0$ .

**Example 1.5.6.** Compute the following limit:

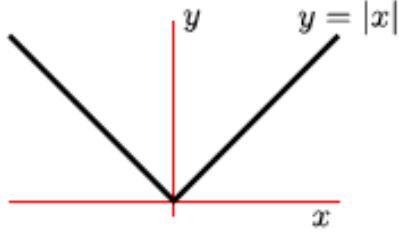
$$\lim_{x \rightarrow \infty} \frac{x^2 - 3x + 4}{3x^2 + 8x + 1}$$

**Solution.** By factoring  $x$  with largest exponent in the numerator and denominator we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2 - 3x + 4}{3x^2 + 8x + 1} &= \lim_{x \rightarrow \infty} \frac{x^2(1 + \frac{-3x}{x^2} + \frac{4}{x^2})}{x^2(3 + \frac{8x}{x^2} + \frac{1}{x^2})} = \lim_{x \rightarrow \infty} \frac{(1 + \frac{-3x}{x^2} + \frac{4}{x^2})}{(3 + \frac{8x}{x^2} + \frac{1}{x^2})} = \\ &\frac{\left(\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \frac{-3x}{x^2} + \lim_{x \rightarrow \infty} \frac{4}{x^2}\right)}{\left(\lim_{x \rightarrow \infty} 3 + \lim_{x \rightarrow \infty} \frac{8x}{x^2} + \lim_{x \rightarrow \infty} \frac{1}{x^2}\right)} = \frac{1}{3}. \end{aligned}$$

**Remark.** Note that

$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



**Example 1.5.7.** Compute the following limit:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{5x - 1}.$$



**Solution.** Factor the terms with the largest exponents in the numerator and denominator.

We have

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{5x - 1} = \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2(1 + \frac{1}{4x^2})}}{5x(1 - \frac{1}{5x})} = \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2} \sqrt{(1 + \frac{1}{4x^2})}}{5x(1 - \frac{1}{5x})} = \lim_{x \rightarrow \infty} \frac{2|x|}{5x} = \lim_{x \rightarrow \infty} \frac{2x}{5x} = \frac{2}{5}.$$

**Example 1.5.8.** Compute the following limit:

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{5x - 1}.$$

**Solution.** By the same kind of computation we have

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{5x - 1} = \lim_{x \rightarrow \infty} \frac{2|x|}{5x}.$$

Consider that since  $x$  is getting negative values, we have  $|x| = -x$ . Therefore,

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{5x - 1} = \lim_{x \rightarrow \infty} \frac{2|x|}{5x} = \lim_{x \rightarrow \infty} \frac{-2x}{5x} = \frac{-2}{5}.$$

**Example 1.5.9.** Compute the following limit:

$$\lim_{x \rightarrow \infty} (x^{7/5} - x).$$

**Solution.** We factor the term with the largest exponent, we have

$$\lim_{x \rightarrow \infty} (x^{7/5} - x) = \lim_{x \rightarrow \infty} x^{7/5} \left(1 - \frac{1}{x^{2/5}}\right) = \infty.$$

**Theorem 1.5.10.** Let  $a, c, H \in \mathbb{R}$  and let  $f, g, h$  be functions defined in an interval around  $a$  (but they need not be defined at  $x = a$ ), so that

$$\lim_{x \rightarrow a} f(x) = +\infty \quad \lim_{x \rightarrow a} g(x) = +\infty \quad \lim_{x \rightarrow a} h(x) = H$$

1.

$$\lim_{x \rightarrow a} (f(x) + g(x)) = +\infty.$$

2.

$$\lim_{x \rightarrow a} (f(x) + h(x)) = +\infty.$$

3.

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \text{undetermined}.$$

4.

$$\lim_{x \rightarrow a} (f(x) - h(x)) = +\infty.$$

5.

$$\lim_{x \rightarrow a} cf(x) = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$$

6.

$$\lim_{x \rightarrow a} (f(x).g(x)) = +\infty.$$

7.

$$\lim_{x \rightarrow a} (f(x).h(x)) = \begin{cases} +\infty & H > 0 \\ \text{undetermined} & H = 0 \\ -\infty & H < 0 \end{cases}$$

8.

$$\lim_{x \rightarrow a} \frac{h(x)}{f(x)} = 0.$$

**Example 1.5.11.** Consider the following three functions:

$$f(x) = x^{-2} \quad g(x) = 2x^{-2} \quad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \rightarrow 0} f(x) = +\infty \quad \lim_{x \rightarrow 0} g(x) = +\infty \quad \lim_{x \rightarrow 0} h(x) = +\infty.$$

Then

•

$$\lim_{x \rightarrow 0} (f(x) - g(x)) = \lim_{x \rightarrow 0} x^{-2} = -\infty$$

•

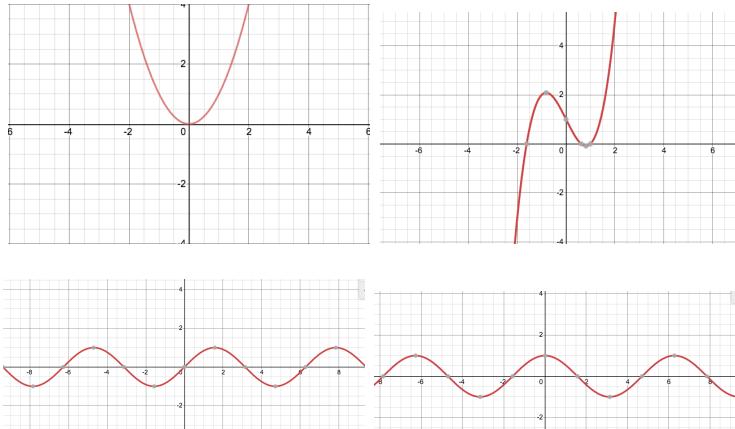
$$\lim_{x \rightarrow 0} (f(x) - h(x)) = \lim_{x \rightarrow 0} (1) = 1$$

•

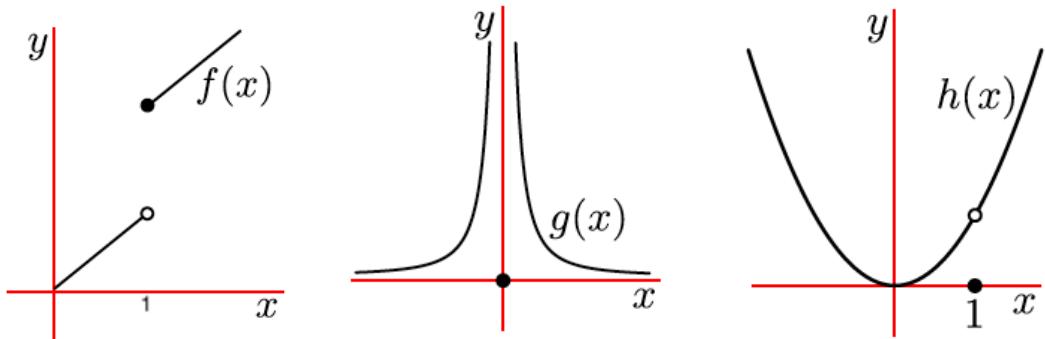
$$\lim_{x \rightarrow 0} (g(x) - h(x)) = \lim_{x \rightarrow 0} x^{-2} + 1 = \infty$$

## 1.6 Continuity

Look at all the following functions.



All of these functions are continuous. Roughly speaking, a function is continuous if it does not have any abrupt jumps. Now consider the following function.



These functions are not continuous. The function  $f$ ,  $g$ , and  $h$  have abrupt jumps at  $x = 2$ ,  $x = 0$ , and  $x = 1$ , respectively, so  $f$  is not continuous at  $a$ ,  $g$  is not continuous at  $0$ , and  $h$  is not continuous at  $1$ .

**Definition.** A function  $f(x)$  is continuous at  $a$  if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

If a function is not continuous at  $a$  then it is said to be discontinuous at  $a$ . When we write that  $f$  is continuous without specifying a point, then typically this means that  $f$  is continuous at  $a$  for all  $a \in \mathbb{R}$ . When we write that  $f(x)$  is continuous on the open interval  $(a, b)$  then the function is continuous at every point  $c$  satisfying  $a < c < b$ .

From the above definition we immediately have that if  $f$  is continuous at  $a$ , then

1.  $f(a)$  exists;
2.  $\lim_{x \rightarrow a^-} f(x)$  exists and is equal to  $f(a)$ .
3.  $\lim_{x \rightarrow a^+} f(x)$  exists and is equal to  $f(a)$ .

**Definition.** A function is continuous from the left at  $a$  if

$$\lim_{x \rightarrow a^-} = f(a).$$

And a function is continuous from the right at  $a$  if

$$\lim_{x \rightarrow a^+} = f(a).$$

**Definition.** A function  $f(x)$  is continuous on an interval  $[a, b]$  if

1.  $f(x)$  continuous on  $(a, b)$ ,
2.  $f(x)$  is continuous from the right at  $a$ ,
3.  $f(x)$  is continuous from the left at  $b$ .

**Definition.** A function  $f(x)$  is continuous on an interval  $(a, b]$  (*on the interval  $[a, b)$* ) if

1.  $f(x)$  continuous on  $(a, b)$ ,
2.  $f(x)$  is continuous from the left at  $b$  (*from the right at  $a$* ).

**Example 1.6.1.** Consider the function

$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$



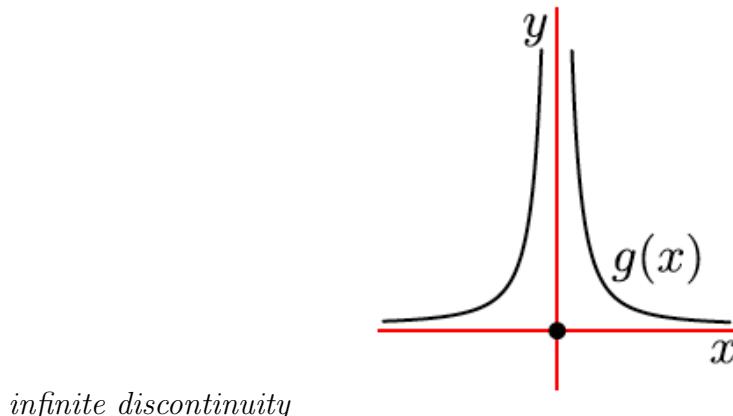
- $\lim_{x \rightarrow 1^-} f(x) = 1 \quad \lim_{x \rightarrow 1^+} f(x) = 3 \quad f(1) = 3.$
- The function  $f(x)$ , at  $x = 1$  is not continuous because the limit does not exist; however, it is continuous from the right at 1 since

$$\lim_{x \rightarrow 1^+} f(x) = 3 = f(1).$$

- The function  $f(x)$ , on  $[1, \infty)$  (for  $x \geq 1$ ) is continuous.
- The function  $f(x)$ , on  $(-\infty, -1)$  is continuous.

**Example 1.6.2.** Consider the function

$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



- Consider that

$$\lim_{x \rightarrow 0^-} g(x) = \infty = \lim_{x \rightarrow 0^+} g(x) \quad g(0) = 0.$$

Thus the function  $g(x)$  is not continuous at 0 because

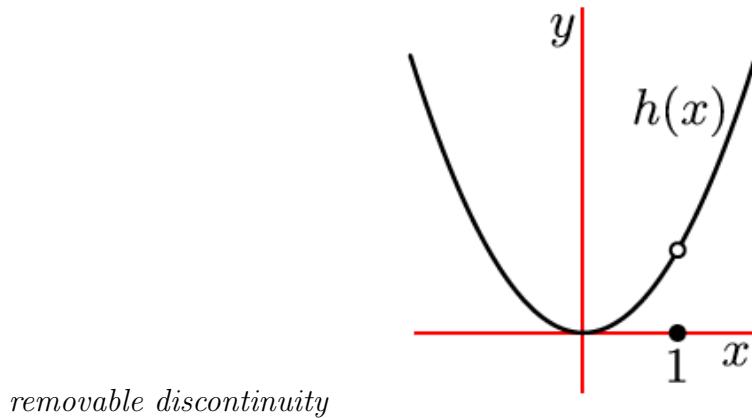
$$\lim_{x \rightarrow 0} g(x) = \infty \neq 0 = g(0).$$

It is not continuous at 0 from the left since  $\lim_{x \rightarrow 0^-} g(x) = \infty \neq 0 = g(0)$  and not from the right since  $\lim_{x \rightarrow 0^+} g(x) = \infty \neq 0 = g(0)$ .

- the function  $g(x)$  is continuous at all points in  $\mathbb{R}$  except 0.

**Example 1.6.3.** Consider the function

$$h(x) = \begin{cases} \frac{x^3 - x^2}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$



- $\lim_{x \rightarrow 1^-} h(x) = 1 = \lim_{x \rightarrow 1^+} h(x) \quad f(1) = 0.$
- $\lim_{x \rightarrow 1} h(x) = 1.$
- the function  $h(x)$  is not continuous at 1 since

$$\lim_{x \rightarrow 1} h(x) = 1 \neq 0 = h(1).$$

It is not continuous from the left since

$$\lim_{x \rightarrow 1^-} h(x) = 1 \neq 0 = h(1)$$

and not from the right since

$$\lim_{x \rightarrow 1^+} h(x) = 1 \neq 0 = h(1).$$

- the function  $h(x)$  is continuous at all points in  $\mathbb{R}$  except 1.

**Lemma 1.6.4.** Let  $c \in \mathbb{R}$ . The functions

$$f(x) = x \quad g(x) = c$$

are continuous everywhere on the real line.

**Theorem 1.6.5. (Arithmetic of continuity)** Let  $a, c \in \mathbb{R}$  and let  $f(x)$  and  $g(x)$  be functions that are continuous at  $a$ . Then the following functions are also continuous at  $x = a$ .

- $f(x) + g(x)$  and  $f(x) - g(x)$ ,
- $cf(x)$  and  $f(x)g(x)$ , and
- $\frac{f(x)}{g(x)}$  provided  $g(a) \neq 0$ .

**Theorem 1.6.6.** The following functions are continuous everywhere in their domains

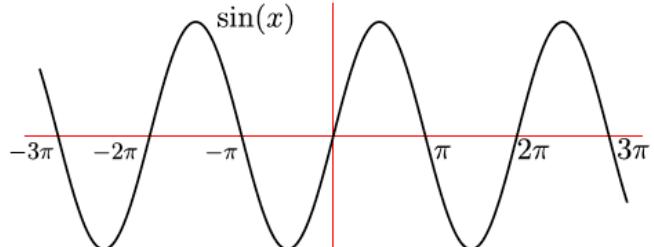
- polynomials and rational functions (for example  $f(x) = x^5 + 4x^2 + 1$  and  $g(x) = \frac{x^2+1}{x+1}$ )
- roots and powers (for example  $h(x) = \sqrt{x}$  and  $r(x) = 2^x$ )
- trig functions and their inverses (for example  $k(x) = \sin(x)$  and  $t(x) = \cos^{-1}(x)$ )
- exponentials and logarithms (for example  $s(x) = e^x$  and  $q(x) = \ln x$ ).

**Example 1.6.7.** Determine when the function  $f(x) = \frac{\sin(x)}{x^2 - 5x + 6}$  is continuous? Since both  $\sin(x)$  and  $x^2 - 5x + 6$  are continuous by the above theorem we only need to check when  $x^2 - 5x + 6 = 0$ . Note that  $x^2 - 5x + 6 = (x - 2)(x - 3)$ , thus this polynomial is only zero at  $x = 2$  and  $x = 3$ . Therefore,  $f(x)$  is continuous at all points in  $\mathbb{R}$  except 2 and 3.

**Theorem 1.6.8.** If  $g$  is continuous at  $a$  and  $f(x)$  is continuous at  $g(a)$ , then  $(f \circ g)(x) = f(g(x))$  is continuous at  $x = a$ .

**Example 1.6.9.** Determine when the function  $h(x) = \sqrt{\sin(x)}$  is continuous.

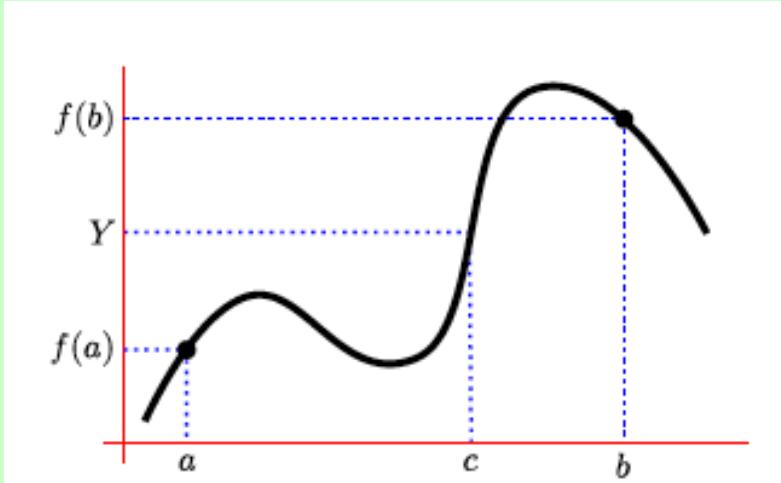
**Solution.** Let  $f(x) = \sqrt{x}$  and  $g(x) = \sin(x)$ , then  $h(x) = (f \circ g)(x)$ . We only need to find out at what points  $\sin(x)$  is positive.



The function  $\sqrt{\sin(x)}$  is continuous if

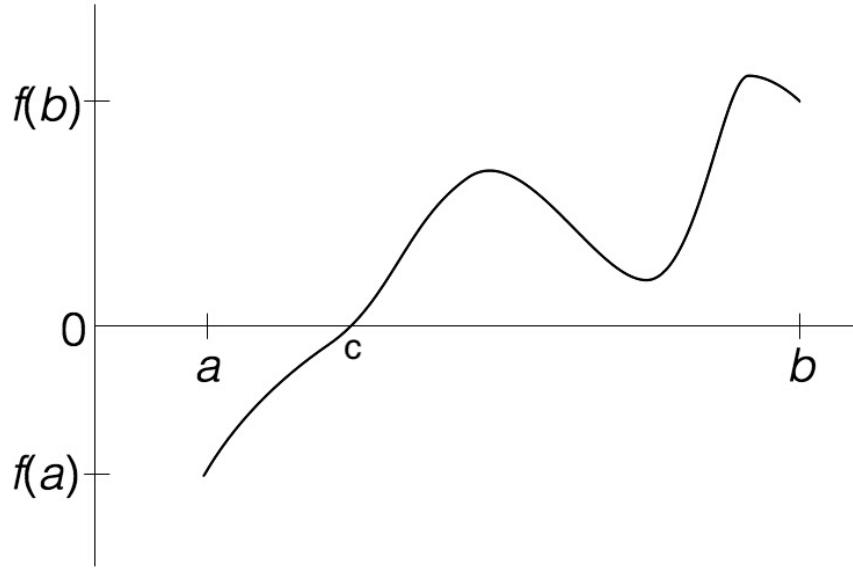
$$x \in [2n\pi, (2n+1)\pi] \quad \text{for all natural numbers } n.$$

**Theorem 1.6.10. (Intermediate value theorem(IVT))** Let  $a < b$  and let  $f(x)$  be a function that is continuous at all points  $a \leq x \leq b$ . If  $Y$  is any number between  $f(a)$  and  $f(b)$  then there exists some number  $c \in [a, b]$  so that  $f(c) = Y$ .



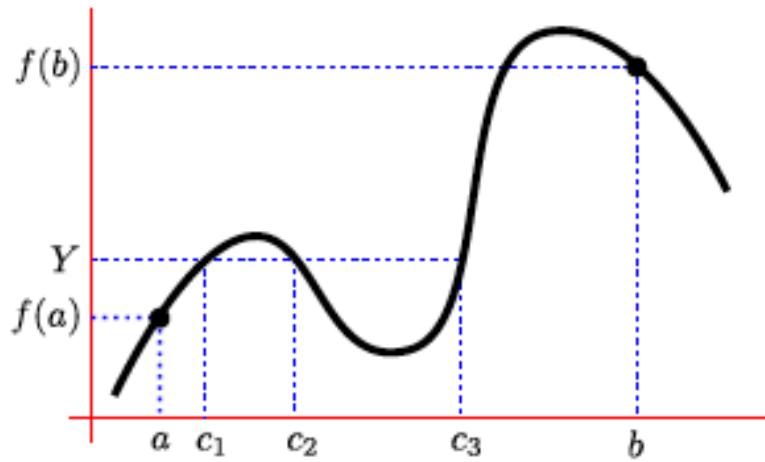
**Remark.** One of the main application of the IVT theorem is showing a function  $f$  has

a zero inside an interval. For example, in the following picture



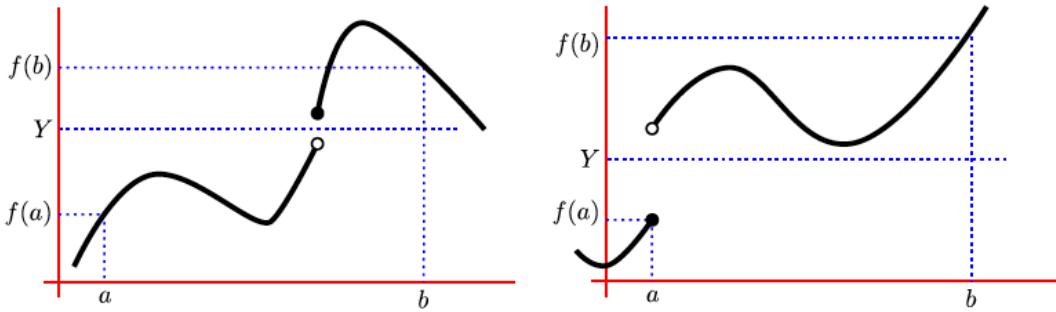
we can see that  $f(a) < 0$  and  $f(b) > 0$ , therefore by IVT, there is a number  $c$  between  $a$  and  $b$  such that  $f(c) = 0$ .

**Remark.** If  $f$  is continuous and  $f(a) \leq Y \leq f(b)$ , the IVT merely shows that there is a  $a \leq c \leq b$  such that  $f(c) = Y$ , but it doesn't show how many of them exist. For example, in the following picture, we can see  $f(a) \leq Y \leq f(b)$ , and there are three numbers  $c_1, c_2$ , and  $c_3$  such that  $f(c_1) = f(c_2) = f(c_3) = Y$ .



**Remark.** Consider that if the function  $f$  is not continuous at the interval  $[a, b]$  then the IVT fails. In the following examples, even though  $f(a) \leq Y \leq f(b)$ , there is not a number

$a \leq c \leq b$  such that  $f(c) = Y$ .



**Example 1.6.11.** Show that the function  $f(x) = x - 1 + \sin(\pi x/2)$  has a zero in  $0 \leq x \leq 1$ .

**Solution.** Consider that  $f(x)$  is a continuous function such that  $f(0) = -1$  and  $f(1) = 1$ . Therefore, by IVT, since  $f(0) = -1 \leq 0 \leq 1 = f(1)$ , we have  $f(c) = 0$  for some  $c \in [0, 1]$ .

**Example 1.6.12.** Use the bisection method to find a zero of  $f(x) = x - 1 + \sin(\pi x/2)$  that lies between 0 and 1.

**Solution.**

- Let  $a = 0$  and  $b = 1$ . Then

$$f(0) = -1$$

$$f(1) = 1$$

- Test the point in the middle  $x = \frac{1+0}{2} = 0.5$ ,

$$f(0.5) = 0.2071067813 > 0$$

- Let  $a = 0$  and  $b = 0.5$ . Then

$$f(0) = -1$$

$$f(1) = 0.2071067813$$

So by IVT, there is a zero in  $[0, 0.5]$ .

- Test the point in the middle  $x = \frac{0.5+0}{2} = 0.25$ .

$$f(0.25) = -0.3673165675 < 0.$$

- Let  $a = 0.25$ ,  $b = 0.5$  where  $f(0.25) < 0$  and  $f(0.5) > 0$ . By IVT there is a zero in the interval  $[0.25, 0.5]$ .

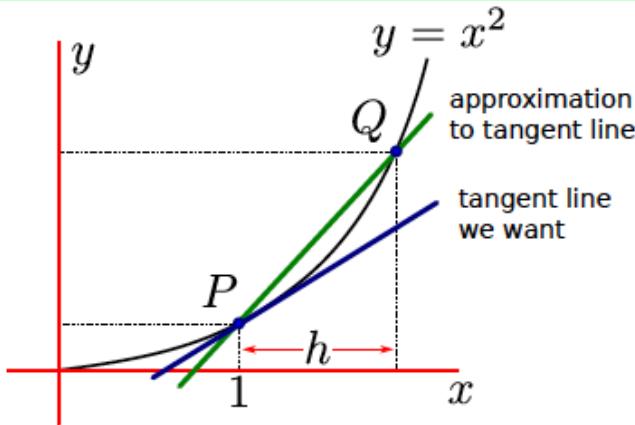
- So without much work we know the location of a zero inside a range of length  $1/4$ . Each iteration will halve the length of the range and we keep going until we reach the precision we need, though it is much easier to program a computer to do it.

# Chapter 2

## Derivatives

### 2.1 Revisiting Tangent Lines

**Example 2.1.1.** Find the slope of the tangent line to the curve  $y = x^2$  that passes through  $P = (1, 1)$ .



**Solution.** Consider that the slope of the secant line is

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

And the slope of the tangent line is the same as

$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}.$$

**Theorem 2.1.2.** Given a function  $f(x)$  the slope of the tangent line at  $x = a$  (if exists) is

$$\lim_{x \rightarrow a} \frac{f(a+h) - f(a)}{h}.$$

## 2.2 Definition of the derivative

**Definition.** (*Derivative at a point*) Let  $a \in \mathbb{R}$  and let  $f(x)$  be a function defined on an open interval that contains  $a$ .

- The derivative of  $f(x)$  at  $x = a$  is denoted  $f'(a)$  and is defined by

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \quad (2.2.1)$$

if the limit exists.

- When the above limit exists, the function  $f(x)$  is said to be differentiable at  $x = a$ . When the limit does not exist, the function  $f(x)$  is said to be not differentiable at  $x = a$ .
- We can equivalently define the derivative  $f'(a)$  by the limit

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

To see that these two definitions are the same, we set  $x = a + h$  ( $x - a = h$ ) and then when  $h$  approaches 0, we have  $x$  approaches  $a$ , and the limit in 2.2.1 becomes  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ .

**Example 2.2.1.** Let  $a, c \in \mathbb{R}$  be constants. Compute the derivative of the function  $f(x) = c$  at  $x = a$ .

**Solution.** By the definition of the derivative, we have

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c - c}{h} \\ &= \lim_{h \rightarrow 0} 0 \\ &= 0 \end{aligned}$$

**Example 2.2.2.** Let  $a \in \mathbb{R}$ . Compute the limit of the function  $g(x) = x$  at  $x = a$ .

**Solution.** By the definition of the derivative we have

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a+h) - a}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} \\ &= \lim_{h \rightarrow 0} 1 \\ &= 1. \end{aligned}$$

We have so proved our first theorem which is the following.

**Theorem 2.2.3. (easiest derivative)** Let  $a, c \in \mathbb{R}$  and let  $f(x) = c$  and  $g(x) = x$ . Then

$$f'(a) = 0$$

and

$$g'(a) = 1.$$

**Example 2.2.4.** Compute the derivative of  $f(t) = t^2$  at  $t = a$ .

**Solution.** We have that

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 + h^2 + 2ah - a^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 + 2ah}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(h+2a)}{h} \\ &= \lim_{h \rightarrow 0} h + 2a \\ &= 2a \end{aligned}$$

►► We can tweak the derivative at a specific point  $a$  to obtain the derivative as a function  $x$ . We replace

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

with

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

This gives us the following definition.

**Definition.** Let  $f(x)$  be a function

- The derivative of  $f(x)$  with respect to  $x$  is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

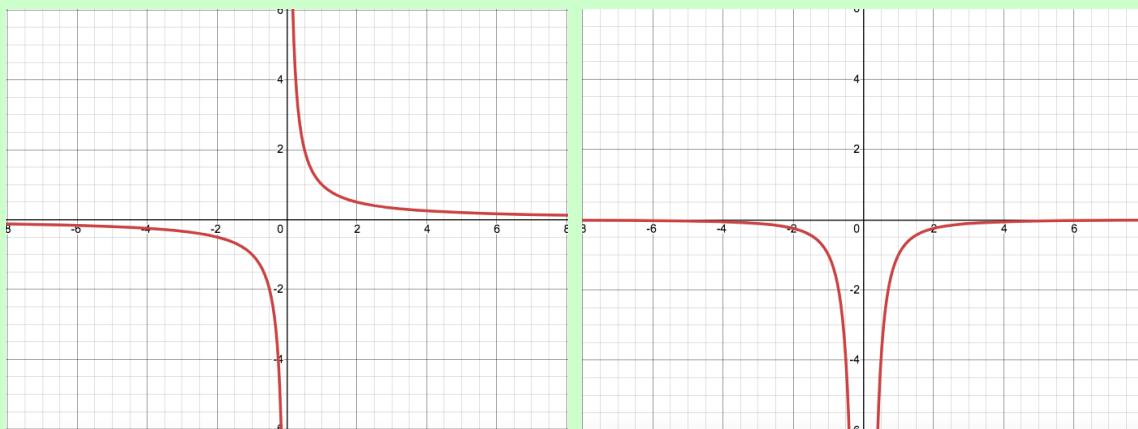
provided the limit exists.

- If the derivative  $f'(x)$  exists for all  $x \in (a, b)$  we say that  $f$  is differentiable on  $(a, b)$ .
- Note that we will sometimes be a little sloppy with our discussion and simply write “ $f$  is differentiable” to mean “ $f$  is differentiable on an interval we are interested in” or “ $f$  is differentiable everywhere.”

**Example 2.2.5.** Let  $f(x) = \frac{1}{x}$  and compute its derivative with respect to  $x$ .

**Solution.** We have that

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{1}{x+h} - \frac{1}{x} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{x - (x+h)}{x(x+h)} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{-h}{x(x+h)} \\ &= -\frac{1}{x^2}. \end{aligned}$$



$$y = \frac{1}{x}$$

$$y = \frac{-1}{x^2}$$

►► Notice that the original function  $f(x) = \frac{1}{x}$  was not defined at  $x = 0$ , and the deriva-

tive is also not defined at  $x = 0$ . This does happen more generally—if  $f(x)$  is not defined at a particular point  $x = a$ , then the derivative will not exist at that point either.

**Notation.** There are several notation all used for “the derivative of  $f(x)$  with respect to  $x$ ”; however,

in this course we generally use the following notations

1.  $f'(x)$ . This notation is due to Lagrange, and we read it as “ $f$ -prime of  $x$ ”.
2.  $\frac{df}{dx}$ . This notation is due to Leibniz, and we read it as “dee- $f$ -dee- $x$ ”.
3.  $\frac{d}{dx}f$ . We read this as dee-by-dee- $x$  of  $f$ .

**Example 2.2.6.** Compute the derivative,  $f'(a)$ , of the function  $f(x) = \sqrt{x}$  at the point  $x = a$  for any  $a > 0$ .

**Solution.** We have that

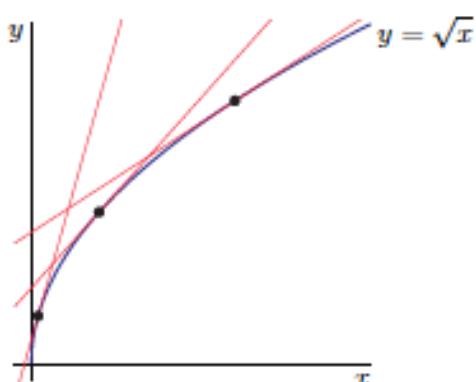
$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$

We now multiply the numerator and denominator by the conjugate of  $\sqrt{x} - \sqrt{a}$ , that is  $\sqrt{x} + \sqrt{a}$ . Then we have

$$\frac{\sqrt{x} - \sqrt{a}}{x - a} \times \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} = \frac{(x - a)}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}}.$$

Therefore,

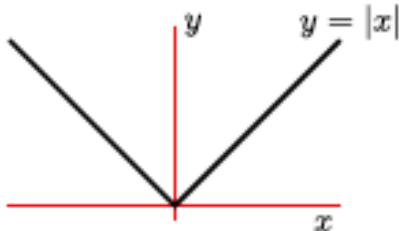
$$f'(a) = \lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \lim_{x \rightarrow a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}.$$



**Example 2.2.7.** Find the derivative,  $f'(a)$ , of the function  $f(x) = |x|$  at the point  $x = a$ .

**Solution.** Recall that

$$|x| = \begin{cases} -x & x < 0 \\ 0 & x = 0 \\ x & x > 0 \end{cases}$$



We should break our computation of the derivative into three cases depending on whether  $x$  is positive, negative, or zero.

- Assume  $x > 0$ . Then

$$\begin{aligned} \frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} \end{aligned}$$

Since  $x > 0$  and  $h$  is much more smaller than  $x$ , we have  $x+h > 0$  and so  $|x+h| = x+h$ , moreover, since  $x$  is positive,  $|x| = x$ .

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} = 1. \end{aligned}$$

- Assume  $x < 0$ . Then we have

$$\begin{aligned} \frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} \end{aligned}$$

Since  $x < 0$  and  $h$  is much more smaller than  $x$ , we have  $x+h < 0$  and so  $|x+h| = -(x+h)$ , moreover, since  $x < 0$  is positive,  $|x| = -x$ .

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h \rightarrow 0} \frac{-(x+h) - (-x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h} = -1. \end{aligned}$$

- Assume  $x = 0$ . Then we have

$$\begin{aligned}\frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|h|}{h}\end{aligned}$$

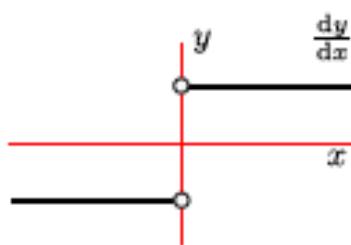
Consider that

$$\lim_{h \rightarrow 0^+} \frac{|h|}{h} = 1 \quad \text{and} \quad \lim_{h \rightarrow 0^-} \frac{|h|}{h} = -1$$

Therefore, this limit does not exist and so the function  $|x|$  is not derivative at  $x = 0$ .

In summary:

$$\frac{d}{dx}|x| = \begin{cases} -1 & x < 0 \\ DNE & x = 0 \\ 1 & x > 0 \end{cases}$$

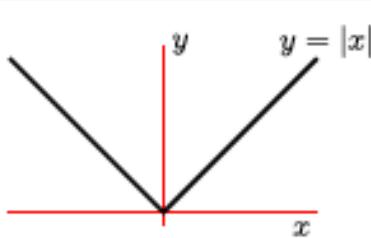


►► Where is the derivative undefined? The derivative  $f'(a)$  exists precisely when the limit

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

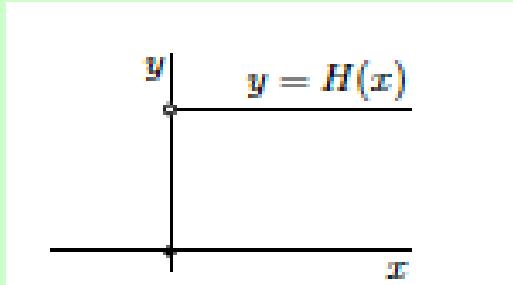
exists. That limit is the slope of the tangent line to the curve  $y = f(x)$  at  $x = a$ . Thus, that limit does not exist one of the following happens.

- ❶ The curve  $y = f(x)$  does not have a tangent line at  $x = a$  when it has a sharp corner at  $x = a$ , as an example  $f(x) = |x|$  is not differentiable at  $x = 0$  since it has a sharp corner at  $x = 0$ .

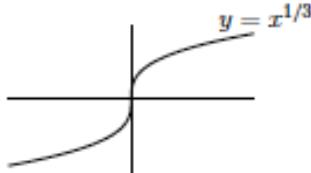


- ❷ When the curve does have a tangent line because it is not continuous at  $x = a$ .

As an example, we have seen that  $f(x) = H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$  does not have a tangent line at  $x = 0$  since it is not continuous at  $x = 0$ .

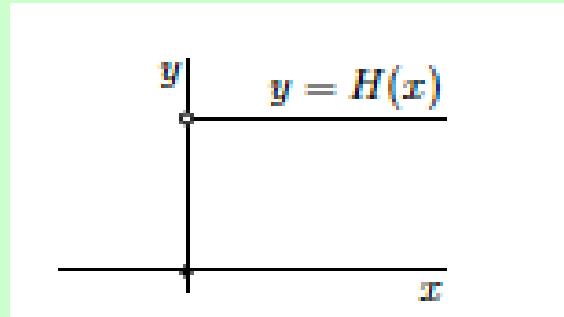


- ❸ When the curve has a tangent line at  $x = a$  but the slope of the tangent line at  $x = a$  is infinity. As an example,  $f(x) = x^{1/3}$  is not differentiable at  $x = 0$  since it has a tangent line with slope infinity.



**Example 2.2.8.** Verify that the function

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$



does not have a tangent line at  $x = 0$ .

**Solution.** Consider that if the tangent line exists then the following limit also must exists,

$$\lim_{h \rightarrow 0} \frac{H(0 + h) - H(0)}{h}.$$

Consider that

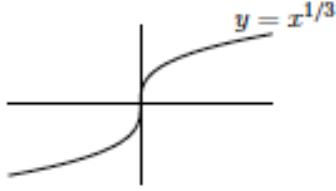
$$\lim_{h \rightarrow 0^+} \frac{H(0 + h) - H(0)}{h} = \lim_{h \rightarrow 0^+} \frac{1}{h} = +\infty$$

and

$$\lim_{h \rightarrow 0^-} \frac{H(0 + h) - H(0)}{h} = \lim_{h \rightarrow 0^-} \frac{0}{h} = 0.$$

Therefore, the limit does not exists.

**Example 2.2.9.** Verify that the derivative of  $f(x) = x^{1/3}$  at  $x = 0$  does not exist.



**Solution.** You can already see in the graph that the derivative at  $x = 0$  does not exist since the tangent line has infinite slope. However, we need a mathematical proof, and we should show that  $f'(0)$  which is the same as the following limit

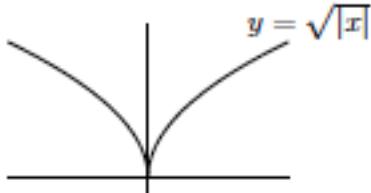
$$\lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h}$$

does not exist. We have

$$\lim_{h \rightarrow 0} \frac{(0 + h)^{1/3} - 0^{1/3}}{h} = \lim_{h \rightarrow 0} \frac{h^{1/3}}{h} = \lim_{h \rightarrow 0} \frac{1}{h^{2/3}} = +\infty$$

(or we can say DNE).

**Example 2.2.10.** Verify that the derivative of  $f(x) = \sqrt{|x|}$  at  $x = 0$  does not exist.



**Solution.** Even though you can see in the graph that at  $x = 0$ , the graph has a sharp corner, we also show that the following limit doesn't exist,

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{|h|} - 0}{h}.$$

Consider that

$$\lim_{h \rightarrow 0^+} \frac{\sqrt{|h|}}{h} = \lim_{h \rightarrow 0^+} \frac{\sqrt{1}}{\sqrt{h}} = +\infty$$

(or DNE).

**»» What is the relation between continuity and differentiability?**

**Theorem 2.2.11.** *If the function  $f(x)$  is differentiable at  $x = a$ , then  $f(x)$  is also continuous at  $x = a$ .*

**Theorem 2.2.12.** *If  $f(x)$  is not continuous at  $x = a$ , then it is not differentiable at  $x = a$ .*

## 2.3 Interpretations of the Derivative

Interpretation of the derivative:

- the instantaneous rate of change of a quality
- the slope of a curve.

**»» Instantaneous Rate of Change**

Assume that we have the function  $f(t)$  of the measuring of some quantity. Then

average rate of change of  $f(t)$  from  $t = a$  to  $t = a + h$  is

$$\begin{aligned} & \frac{\text{change in } f(t) \text{ from } t = a \text{ to } t = a + h}{\text{length of time from } t = a \text{ to } t = a + h} \\ &= \frac{f(a + h) - f(a)}{h}. \end{aligned}$$

And so

$$\begin{aligned} & \text{instantaneous rate of change of } f(t) \text{ at } t = a \\ &= \lim_{h \rightarrow 0} [\text{average rate of change of } f(t) \text{ from } t = a \text{ to } t = a + h] \\ &= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} = f'(a). \end{aligned}$$

**Example 2.3.1.** You drop a ball from a tall building. After  $t$  seconds the ball has fallen a distance of  $s(t) = 4.9t^2$  meters. What is the (instantaneous) velocity of the ball one second after it is dropped?

**Solution.** Since the instantaneous velocity is actually the instance rate of change if distance, we should compute  $f'(1)$ . Computing the limit of the derivative we have

$$f'(t) = 9.8t \quad \text{and} \quad f'(1) = 9.8.$$

**Example 2.3.2.** You are taking a walk and that as you walk, you are continuously measuring some quantity, like temperature, and that the measurement at time  $t$  is  $f(t) = \sqrt{t}$ . What is the average rate of  $f(t) = \sqrt{t}$  from  $t = 1$  to  $t = 2$ ? What is the instantaneous rate of change of  $f(t)$  at  $t = 1$ ?

**Solution.** Consider that the average rate of change is

$$\frac{f(a+h) - f(a)}{h}$$

The difference between times is  $h$ , so  $h = 2 - 1 = 1$ . Therefore, the average is

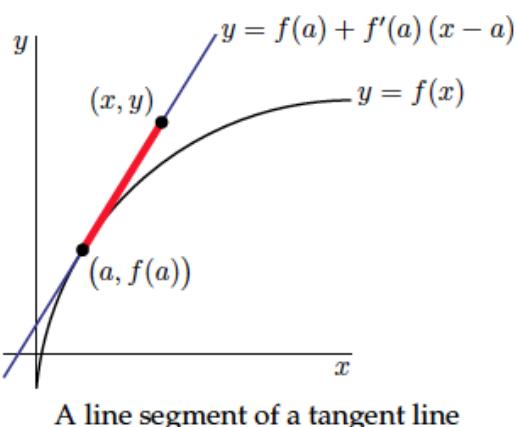
$$\frac{f(a+h) - f(a)}{h} = \frac{f(2) - f(1)}{1} = \sqrt{2} - 1.$$

The instantaneous rate of change is  $f'(1)$ . We already have seen that  $f'(t) = \frac{1}{2\sqrt{t}}$ , and so  $f'(1) = \frac{1}{2}$ .

### ►► Slope and the formula for the tangent line to a curve at $x = a$

We know that the slope of a function at  $x = a$  is  $f'(a)$ . To find the tangent line we need to find a point at the tangent line; however, for sure we know that  $(a, f(a))$  is on the tangent line. So we should find the equation for a line that passes through  $(a, f(a))$  with slope  $f'(a)$  which is

$$y = f(a) + f'(a)(x - a).$$



**Theorem 2.3.3.** The tangent line to the curve  $y = f(x)$  at  $x = a$  is given by the equation

$$y = f(a) + f'(a)(x - a)$$

provided the derivative  $f'(a)$  exists.

**Example 2.3.4.** Find the tangent line to the curve  $y = \sqrt{x}$  at  $x = 4$ .

**Solution.** By the above theorem we have the equation for the tangent line is

$$y = f(a) + f'(a)(x - a).$$

Note that  $a = 4$  and so  $f(a) = f(4) = \sqrt{4} = 2$ . We have already had  $f'(x) = \frac{1}{2\sqrt{x}}$ . Therefore,  $f'(a) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$ . And so the equation for the tangent line is

$$y = 2 + \frac{1}{4}(x - 4).$$

## 2.4 Arithmetic of Derivatives - a Differentiation Toolbox

It is more efficient to have access to

- a list of derivatives of some simple functions and
- a collection of rules for breaking down complicated derivative computations into sequence of simple derivative computations.

### ►► A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \quad \frac{d}{dx}x = 1 \quad \frac{d}{dx}x^2 = 2x \quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

### Tools

**Theorem 2.4.1.** Let  $f(x)$  and  $g(x)$  be differentiable functions and let  $c, d \in \mathbb{R}$ . Then

$$\frac{d}{dx}\{f(x) + g(x)\} = f'(x) + g'(x)$$

$$\frac{d}{dx}\{f(x) - g(x)\} = f'(x) - g'(x)$$

$$\frac{d}{dx}\{cf(x)\} = cf'(x)$$

$$\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$$

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$$

►  $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$

►  $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$

►  $\frac{d}{dx}\left\{\frac{1}{g(x)}\right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$

## 2.5 Using the Arithmetic of Derivatives - Examples

### Example 2.5.1.

$$\begin{aligned} \frac{d}{dx}4x + 6 &= 4 \cdot \frac{d}{dx}\{x\} + 7 \cdot \frac{d}{dx}\{1\} \\ &= 4 \cdot 1 + 7 \cdot 0 = 4. \end{aligned}$$

**Example 2.5.2.**

$$\begin{aligned}\frac{d}{dx}\{x(4x+7)\} &= \frac{d}{dx}\{x\}(4x+7) + x\frac{d}{dx}\{4x+7\} \\ &= 4x+7+4x = 8x+7\end{aligned}$$

**Example 2.5.3.**

$$\begin{aligned}\frac{d}{dx}\left\{\frac{x}{(4x+7)}\right\} &= \frac{\frac{d}{dx}\{x\}(4x+7) - x\frac{d}{dx}\{4x+7\}}{(4x+7)^2} \\ &= \frac{4x+7-x(4)}{(4x+7)^2} = \frac{7}{(4x+7)^2}.\end{aligned}$$

**Example 2.5.4.** Differentiate

$$f(x) = \frac{x}{2x + \frac{1}{3x+1}}$$

**Solution.** Let

$$f_1(x) = x \quad \text{and} \quad f_2(x) = 2x + \frac{1}{3x+1}$$

Let

$$f_3 = 2x \quad \text{and} \quad f_4(x) = \frac{1}{3x+1}.$$

Therefore,

$$f(x) = \frac{f_1(x)}{f_2(x)}$$

and so

$$f'(x) = \frac{f'_1(x)f_2(x) - f'_2(x)f_1(x)}{f_2(x)^2}.$$

Consider that  $f'_1(x) = 1$ . We have that  $f_2(x) = f_3(x) + f_4(x)$ , and thus

$$f'_2(x) = f'_3(x) + f'_4(x) = 2 + \frac{-3}{(3x+1)^2}.$$

Therefore,

$$f'(x) = \frac{1 \times 2x + \frac{1}{3x+1} - 2 + \frac{-3}{(3x+1)^2}x}{(2x + \frac{1}{3x+1})^2}.$$

If we clean up this formula we have

$$f'(x) = \frac{6x+1}{(6x^2+2x+1)^2}.$$

### Tools

**Corollary 2.5.5.** Let  $n$  be an integer.

$$\frac{d}{dx}\{f(x)g(x)h(x)\} = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x).$$

**Example 2.5.6.** Let  $n$  be an integer, differentiate

$$g(x) = x^n$$

**Solution.** Let  $f(x) = x$ . Consider that  $g(x) = f(x)^n$ , by the above theorem we have

$$g'(x) = nf(x)f(x)^{n-1} = n \times 1 \times x^{n-1} = nx^{n-1}.$$

**Example 2.5.7.** Differentiate

$$1. \quad f(x) = (3x + 9)(x^2 + 4x^3)$$

$$2. \quad \frac{4x^3 - 7x}{4x^2 + 1}$$

**Solution.** (1) Let  $f_1(x) = 3x + 9$  and  $f_2(x) = x^2 + 4x^3$ .

$$f(x) = f_1(x)f_2(x) \quad \text{and} \quad f'(x) = f'_1(x)f_2(x) + f'_2(x)f_1(x).$$

Note that

$$f'_1(x) = 3 \quad \text{and} \quad f'_2(x) = 2x + 12x^2.$$

Therefore,

$$f'(x) = 3 \times (x^2 + 4x^3) + (2x + 12x^2) \times (3x + 9) = 18x + 117x^2 + 48x^3.$$

(2) Let  $f_1(x) = 4x^3 - 7x$  and  $f_2(x) = 4x^2 + 1$ . Then

$$f(x) = \frac{f_1(x)}{f_2(x)} \quad \text{and} \quad f'(x) = \frac{f'_1(x)f_2(x) - f'_2(x)f_1(x)}{f_2(x)^2}.$$

Consider that

$$f'_1(x) = 12x^2 - 7 \quad \text{and} \quad f'_2(x) = 8x.$$

Therefore,

$$f'(x) = \frac{(12x^2 - 7)(4x^2 + 1) - (8x)(4x^3 - 7x)}{(4x^2 + 1)^2}.$$

After clean up,

$$f'(x) = \frac{16x^4 + 40x^2 - 7}{(4x^2 + 1)^2}.$$

### Tools

**Corollary 2.5.8.** Let  $a$  be a rational number, then

$$\frac{d}{dx}x^a = ax^{a-1}.$$

**Example 2.5.9.** The derivative of  $f(x) = x^{1/3}$  is  $f'(x) = (1/3)x^{1/3-1} = (1/3)x^{-2/3} = \frac{1}{x^{2/3}}$ .

**Example 2.5.10.** Find the derivative of

$$f(x) = \frac{(\sqrt{x}-1)(2-x)(1-x^2)}{\sqrt{x}(3+2x)}$$

**Solution.** Let  $f_1(x) = (\sqrt{x}-1)(2-x)(1-x^2)$  and  $f_2(x) = \sqrt{x}(3+2x)$ . Then

$$f'(x) = \frac{f'_1(x)f_2(x) - f'_2(x)f_1(x)}{f_2(x)^2}. \quad (2.5.1)$$

Let  $f_3(x) = (\sqrt{x}-1)$ ,  $f_4(x) = (2-x)$  and  $f_5(x) = (1-x^2)$ . Then

$$f_1(x) = f_3(x)f_4(x)f_5(x)$$

and

$$f'(x) = f'_3(x)f_4(x)f_5(x) + f_3(x)f'_4(x)f_5(x) + f_3(x)f_4(x)f'_5(x).$$

Consider that

$$f'_3(x) = \frac{1}{2\sqrt{x}}, \quad f'_4(x) = -1, \quad \text{and} \quad f'_5(x) = -2x.$$

Therefore,

$$f'_1(x) = \left(\frac{1}{2\sqrt{x}}\right)(2-x)(1-x^2) + (\sqrt{x}-1)(-1)(1-x^2) + (\sqrt{x}-1)(2-x)(-2x).$$

Also,

$$f'_2(x) = \frac{1}{2\sqrt{x}}(3+2x) + 2\sqrt{x}.$$

So we now substitute in 2.5.1 and we have the derivative of  $f(x)$ .

**Homework 1:**

Go to this link  
<https://www.mooculus.osu.edu/textbook/mooculus.pdf> and download the book "MOOCULUS". Then do the following questions:

- all questions in page 35;
- in page 33 see why  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$ . Then do Questions 1-8 page 38;
- in page 42, do questions 1-10.

**Homework 2:**

- First set of questions (CLICK) by Prof. DRAGOS GHIOCA.
- Second set of questions (CLICK) by Prof. DRAGOS GHIOCA.

# Bibliography

- [1] CLP1: Differential Calculus by J. Feldman, A. Rechnitzer, and E. Yeager.

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