

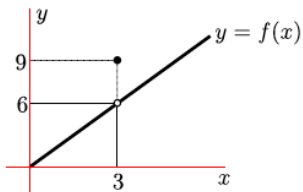
# MATH 100

Farid AliniaEIFARD

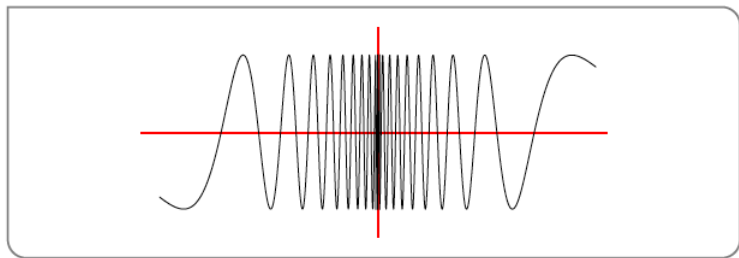
**University of British Columbia**

2019

$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



$$f(x) = \sin\left(\frac{\pi}{x}\right)$$

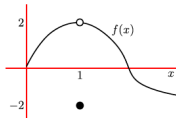


$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$



## Example

Consider the graph of the function  $f(x)$ .



Then

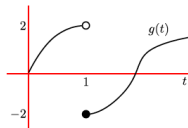
$$\lim_{x \rightarrow 1^-} f(x) =$$

$$\lim_{x \rightarrow 1^+} f(x) =$$

$$\lim_{x \rightarrow 1} f(x) =$$

## Example

Consider the graph of the function  $g(t)$ .



Then

$$\lim_{t \rightarrow 1^-} g(t) =$$

$$\lim_{t \rightarrow 1^+} g(t) =$$

$$\lim_{t \rightarrow 1} g(t) =$$

## Example

Consider the graph of the function  $f(x)$ .



Then

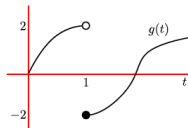
$$\lim_{x \rightarrow 1^-} f(x) = 2$$

$$\lim_{x \rightarrow 1^+} f(x) = 2$$

$$\lim_{x \rightarrow 1} f(x) = 2$$

## Example

Consider the graph of the function  $g(t)$ .



Then

$$\lim_{t \rightarrow 1^-} g(t) = 2$$

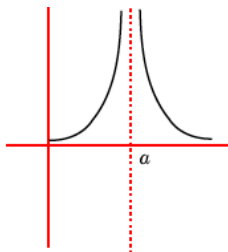
$$\lim_{t \rightarrow 1^+} g(t) = -2$$

$$\lim_{t \rightarrow 1} g(t) = DNE$$

# When the limit goes to infinity

## Example

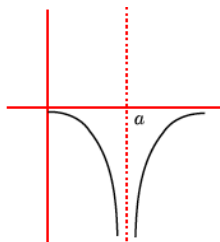
Consider the graph for the function  $f(x)$ .



$$\lim_{x \rightarrow a} f(x) = +\infty$$

## Example

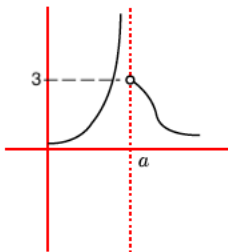
Consider the graph for the function  $g(x)$ .



$$\lim_{x \rightarrow a} g(x) = -\infty$$

### Example

Consider the graph for the function  $h(x)$ .

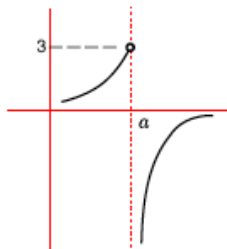


$$\lim_{x \rightarrow a^-} h(x) =$$

$$\lim_{x \rightarrow a^+} h(x) =$$

### Example

Consider the graph for the function  $s(x)$ .



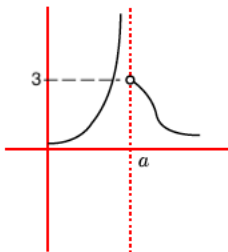
$$\lim_{x \rightarrow a^-} s(x) =$$

$$\lim_{x \rightarrow a^+} s(x) =$$



### Example

Consider the graph for the function  $h(x)$ .

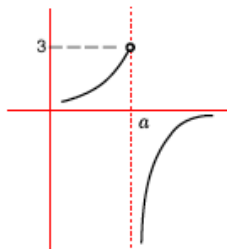


$$\lim_{x \rightarrow a^-} h(x) = +\infty$$

$$\lim_{x \rightarrow a^+} h(x) = 3$$

### Example

Consider the graph for the function  $s(x)$ .



$$\lim_{x \rightarrow a^-} s(x) = 3$$

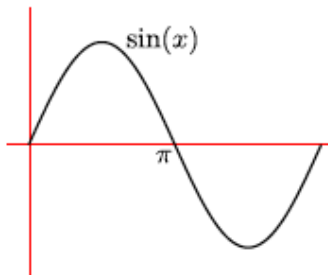
$$\lim_{x \rightarrow a^+} s(x) = -\infty$$

## Example

Consider the function

$$g(x) = \frac{1}{\sin(x)}.$$

Find the one-side limits of this function as  $x \rightarrow \pi$ .



$$\lim_{x \rightarrow \pi^-} \frac{1}{\sin(x)} = +\infty$$

$$\lim_{x \rightarrow \pi^+} \frac{1}{\sin(x)} = -\infty$$

## Second Session Outline

- ▶ Arithmetic of the Limits
- ▶ Limit of a ratio: what will happen if the limit of the denominator is zero. For example,

$$\lim_{x \rightarrow 0} \frac{1}{x^2} ? \quad \text{and} \quad \lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = ?$$

- ▶ Sandwich/ Squeeze/Pinch Theorem
- ▶ limit at infinity

# Arithmetic of the Limits

## Theorem

Let  $a, c \in \mathbb{R}$ . The following two limits hold

$$\lim_{x \rightarrow a} c = c \qquad \lim_{x \rightarrow a} x = a$$

## Example

$$\lim_{x \rightarrow 3} -2 = -2 \qquad \lim_{x \rightarrow -1} x = -1$$

## Theorem

**(Arithmetic of Limits)** Let  $a, c \in \mathbb{R}$ , let  $f(x)$  and  $g(x)$  be defined for all  $x$ 's that lie in some interval about  $a$  (but  $f$  and  $g$  need not to be defined exactly at  $a$ ).

$$\lim_{x \rightarrow a} f(x) = F \quad \lim_{x \rightarrow a} g(x) = G$$

exists with  $F, G \in \mathbb{R}$ . Then the following limits hold

- ▶  $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$ —limit of the sum is the sum of the limits.

## Theorem

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- ▶  $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$ —limit of the sum is the sum of the limits.
- ▶  $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$ —limit of the difference is the difference of the limits.

## Theorem

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- ▶  $\lim_{x \rightarrow a} cf(x) = cF$ .



## Theorem

**(Arithmetic of Limits)** Let  $a, c \in \mathbb{R}$ , let  $f(x)$  and  $g(x)$  be defined for all  $x$ 's that lie in some interval about  $a$  (but  $f$  and  $g$  need not to be defined exactly at  $a$ ).

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exists with  $F, G \in \mathbb{R}$ . Then the following limits hold

- ▶  $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$ —limit of the sum is the sum of the limits.
- ▶  $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$ —limit of the difference is the difference of the limits.
- ▶  $\lim_{x \rightarrow a} cf(x) = cF$ .
- ▶  $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = F \cdot G$ —limit of the product is the product of the limits.

If  $G \neq 0$  then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}$$

## Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) =$$

## Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) =$$

## Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) =$$

## Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

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$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x \rightarrow 1} \frac{f(x)}{f(x) - g(x)} =$$

## Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

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$$\lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x \rightarrow 1} \frac{f(x)}{f(x) - g(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x)} = \frac{3}{3 - 2} = 3.$$

## Example

$$\lim_{x \rightarrow 3} 4x^2 - 1 =$$

$$\lim_{x \rightarrow 2} \frac{x}{x - 1} =$$



## Example

$$\lim_{x \rightarrow 3} 4x^2 - 1 = 4 \times \lim_{x \rightarrow 3} x^2 - \lim_{x \rightarrow 3} 1 = 35.$$

$$\lim_{x \rightarrow 2} \frac{x}{x-1} = \frac{\lim_{x \rightarrow 2} x}{\lim_{x \rightarrow 2} x - \lim_{x \rightarrow 1} 1} = \frac{2}{2-1} = 2.$$

**Limit of a ratio: what will happen if the limit of the denominator is zero.**

## Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = DNE$$

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- the **limit is**  $\pm\infty$ , eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x^4} = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{-x^2}{x^4} = \lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty.$$

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- the **limit is** 0, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

# Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

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- the **limit is** 0, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

- the **limit exists and it nonzero**, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x} = 1.$$

## Theorem

Let  $n$  be a positive integer, let  $a \in \mathbb{R}$  and let  $f$  be a function so that

$$\lim_{x \rightarrow a} f(x) = F$$

for some real number  $F$ . Then the following holds

$$\lim_{x \rightarrow a} (f(x))^n = \left( \lim_{x \rightarrow a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit.

## Theorem

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so that the limit of a power is the power of the limit.

Similarly, if

- ▶  $n$  is an even number and  $F > 0$ , or
- ▶  $n$  is an odd number and  $F$  is any real number

then

$$\lim_{x \rightarrow a} (f(x))^{1/n} = \left( \lim_{x \rightarrow a} f(x) \right)^{1/n} = F^{1/n}.$$



## Example

$$\lim_{x \rightarrow 4} x^{1/2} =$$

$$\lim_{x \rightarrow 4} (-x)^{1/2} =$$

$$\lim_{x \rightarrow 2} (4x^2 - 3)^{1/3} =$$

## Example

$$\lim_{x \rightarrow 4} x^{1/2} = 4^{1/2} = 2.$$

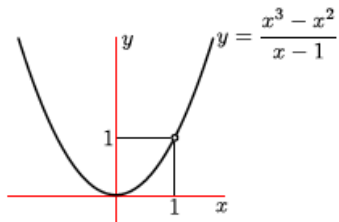
$$\lim_{x \rightarrow 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \rightarrow 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = (13)^{1/3}.$$

**Limit of a ratio: what will happen if the limit  
of the numerator and denominator are zero,  
for example,**

$$\lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = ?$$

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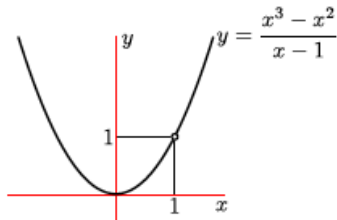
## Theorem

If  $f(x) = g(x)$  except when  $x = a$  then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

provided the limit of  $g$  exists.

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases} \Rightarrow \lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \rightarrow 1} x^2 = 1.$$

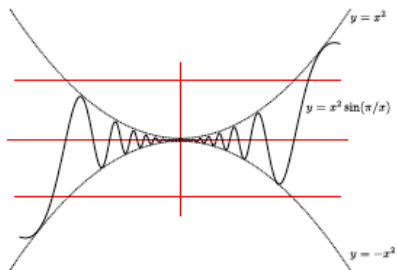
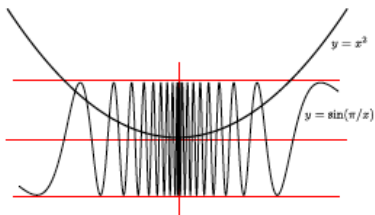


# Sandwich/ Squeeze/Pinch Theorem

## Example

Compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



### Example

Let  $f(x)$  be a function such that  $1 \leq f(x) \leq x^2 - 2x + 2$ . What is

$$\lim_{x \rightarrow 1} f(x)?$$



### Example

Let  $f(x)$  be a function such that  $1 \leq f(x) \leq x^2 - 2x + 2$ . What is

$$\lim_{x \rightarrow 1} f(x)?$$

### Solution

*Consider that*

$$\lim_{x \rightarrow 1} x = 1 \quad \text{and} \quad \lim_{x \rightarrow 1} x^2 - 2x + 2 = 1.$$

*Therefore, by the sandwich/pinch/squeeze theorem*

$$\lim_{x \rightarrow 1} f(x) = 1.$$

## Example

We want to compute

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{x}$$

By plug in some large numbers into  $\frac{1}{x}$  we have

-10000	-1000	-100	o	100	1000	10000
-0.0001	-0.001	-0.01	o	0.01	0.001	0.0001

We see that as  $x$  is getting bigger and positive the function  $\frac{1}{x}$  is getting closer to 0. Thus,

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0.$$

Moreover,

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

## Limit at Infinity

## Definition

(**Informal limit at infinity.**) We write

$$\lim_{x \rightarrow \infty} f(x) = L$$

when the value of the function  $f(x)$  gets closer and closer to  $L$  as we make  $x$  larger and larger and positive.

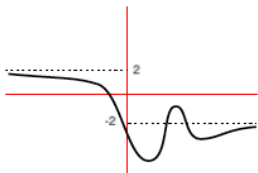
Similarly, we write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

when the value of the function  $f(x)$  gets closer and closer to  $L$  as we make  $x$  larger and larger and negative.

### Example

Consider the graph of the function  $f(x)$ .



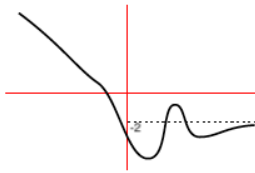
Then

$$\lim_{x \rightarrow \infty} f(x) =$$

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### Example

Consider the graph of the function  $g(x)$ .



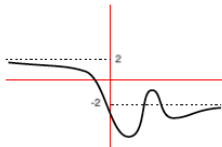
Then

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$$\lim_{x \rightarrow -\infty} g(x) =$$

## Example

Consider the graph of the function  $f(x)$ .



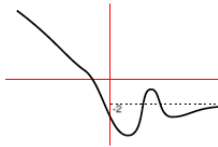
Then

$$\lim_{x \rightarrow \infty} f(x) = -2$$

$$\lim_{x \rightarrow -\infty} f(x) = 2$$

## Example

Consider the graph of the function  $g(x)$ .



Then

$$\lim_{x \rightarrow \infty} g(x) = -2$$

$$\lim_{x \rightarrow -\infty} g(x) = +\infty$$

## Review of the third session

# Review

## Theorem

**sandwich (or squeeze or pinch)** *Let  $a \in \mathbb{R}$  and let  $f, g, h$  be three functions so that*

$$f(x) \leq g(x) \leq h(x)$$

*for all  $x$  in an interval around  $a$ , except possibly at  $x = a$ . Then if*

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

*then it is also the case that*

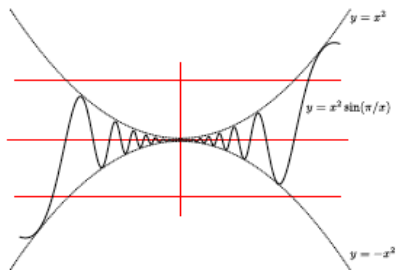
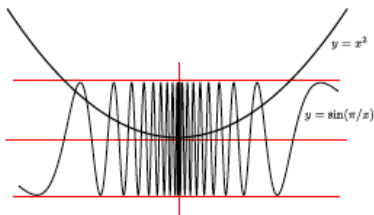
$$\lim_{x \rightarrow a} g(x) = L.$$



## Example

Compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



## Theorem

Let  $c \in \mathbb{R}$  then the following limits hold

$$\lim_{x \rightarrow +\infty} c = c$$

$$\lim_{x \rightarrow -\infty} c = c$$

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

# Outline For the Fourth Session

- ▶ Limit at Infinity

## Limit at Infinity

## Theorem

Let  $f(x)$  and  $g(x)$  be two functions for which the limits

$$\lim_{x \rightarrow \infty} f(x) = F \qquad \lim_{x \rightarrow \infty} g(x) = G$$

exist. Then the following limits hold

$$\lim_{x \rightarrow \infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x \rightarrow \infty} f(x)g(x) = FG$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \text{provided } G \neq 0$$

and for rational numbers  $r$ ,

$$\lim_{x \rightarrow \infty} (f(x))^r = F^r$$

provided that  $f(x)^r$  is defined for all  $x$ .

The analogous results hold for limits to  $-\infty$ .



**Warning:** Consider that

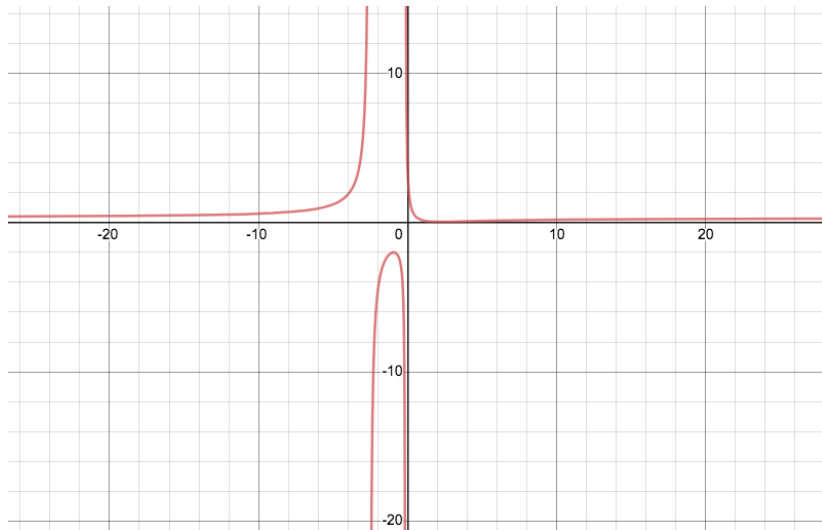
$$\lim_{x \rightarrow +\infty} \frac{1}{x^{1/2}} = 0$$

However,

$$\lim_{x \rightarrow +\infty} \frac{1}{(-x)^{1/2}}$$

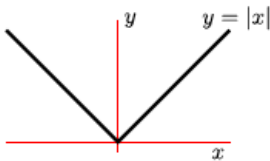
does not exist because  $x^{1/2}$  is not defined for  $x < 0$ .

$$f(x) = \frac{x^2 - 3x + 4}{3x^2 + 8x + 1}$$

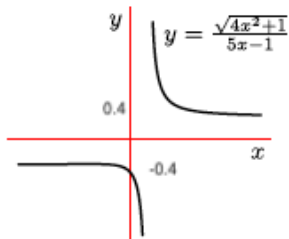




$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



$$y = \frac{\sqrt{4x^2 + 1}}{5x - 1}$$



## Theorem

Let  $a, c, H \in \mathbb{R}$  and let  $f, g, h$  be functions defined in an interval around  $a$  (but they need not be defined at  $x = a$ ), so that

$$\lim_{x \rightarrow a} f(x) = +\infty \quad \lim_{x \rightarrow a} g(x) = +\infty \quad \lim_{x \rightarrow a} h(x) = H$$

1.

$$\lim_{x \rightarrow a} (f(x) + g(x)) =$$

2.

$$\lim_{x \rightarrow a} (f(x) + h(x)) =$$

3.

$$\lim_{x \rightarrow a} (f(x) - g(x)) =$$

4.

$$\lim_{x \rightarrow a} (f(x) - h(x)) =$$

## Theorem

Let  $a, c, H \in \mathbb{R}$  and let  $f, g, h$  be functions defined in an interval around  $a$  (but they need not be defined at  $x = a$ ), so that

$$\lim_{x \rightarrow a} f(x) = +\infty \quad \lim_{x \rightarrow a} g(x) = +\infty \quad \lim_{x \rightarrow a} h(x) = H$$

1.

$$\lim_{x \rightarrow a} (f(x) + g(x)) = +\infty.$$

2.

$$\lim_{x \rightarrow a} (f(x) + h(x)) = +\infty.$$

3.

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \text{undetermined}.$$

4.

$$\lim_{x \rightarrow a} (f(x) - h(x)) = +\infty.$$

## Theorem

5.

$$\lim_{x \rightarrow a} cf(x) = \begin{cases} c > 0 \\ c = 0 \\ c < 0 \end{cases}$$

6.

$$\lim(f(x).g(x)) =$$

7.

$$\lim_{x \rightarrow a} (f(x).h(x)) = \begin{cases} H > 0 \\ H = 0 \\ H < 0 \end{cases}$$

8.

$$\lim_{x \rightarrow a} \frac{h(x)}{f(x)} =$$

## Theorem

5.

$$\lim_{x \rightarrow a} cf(x) = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$$

6.

$$\lim(f(x).g(x)) = +\infty.$$

7.

$$\lim_{x \rightarrow a} (f(x).h(x)) = \begin{cases} +\infty & H > 0 \\ \text{undetermined} & H = 0 \\ -\infty & H < 0 \end{cases}$$

8.

$$\lim_{x \rightarrow a} \frac{h(x)}{f(x)} = 0.$$

## Example

Consider the following three functions:

$$f(x) = x^{-2} \quad g(x) = 2x^{-2} \quad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \rightarrow 0} f(x) = +\infty \quad \lim_{x \rightarrow 0} g(x) = +\infty \quad \lim_{x \rightarrow 0} h(x) = +\infty.$$

Then

1.

$$\lim_{x \rightarrow 0} (f(x) - g(x)) =$$

2.

$$\lim_{x \rightarrow 0} (f(x) - h(x)) =$$

3.

$$\lim_{x \rightarrow 0} (g(x) - h(x)) =$$

## Example

Consider the following three functions:

$$f(x) = x^{-2} \quad g(x) = 2x^{-2} \quad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \rightarrow 0} f(x) = +\infty \quad \lim_{x \rightarrow 0} g(x) = +\infty \quad \lim_{x \rightarrow 0} h(x) = +\infty.$$

Then

1.

$$\lim_{x \rightarrow 0} (f(x) - g(x)) = \lim_{x \rightarrow 0} x^{-2} = \infty$$

2.

$$\lim_{x \rightarrow 0} (f(x) - h(x)) = \lim_{x \rightarrow 0} (1) = 1$$

3.

$$\lim_{x \rightarrow 0} (g(x) - h(x)) = \lim_{x \rightarrow 0} x^{-2} + 1 = \infty$$



# Outline For the Session Five

- ▶ Limit at Infinity
- ▶ Continuity
- ▶ Continuous from the left and from the right
- ▶ Arithmetic of continuity
- ▶ continuity of composites
- ▶ Intermediate Value Theorem

## Example

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

$x$	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{1}{x^2}$	100	10000	$10^6$		$10^6$	10000	100

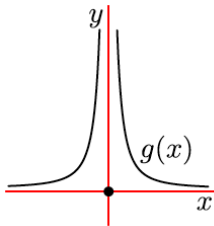
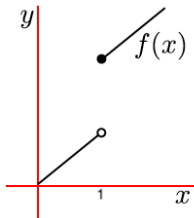
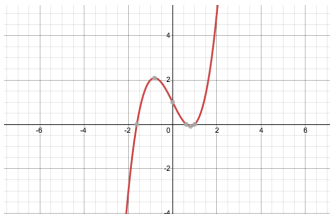
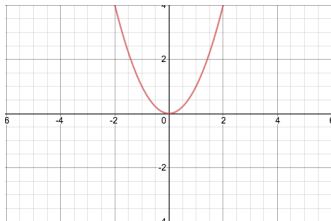
Consider that if

$$\lim_{x \rightarrow a} f(x) = \infty \quad \lim_{x \rightarrow a} g(x) = \infty$$

Then

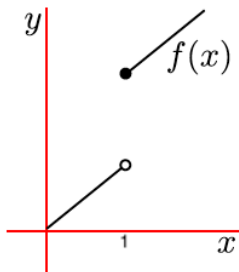
$$\lim_{x \rightarrow a} (f(x) - g(x)) = \text{undetermined}$$

# Continuity

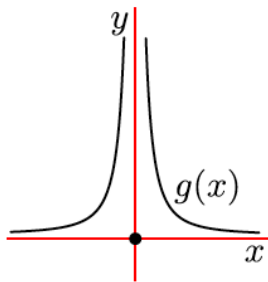


$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$

jump discontinuity

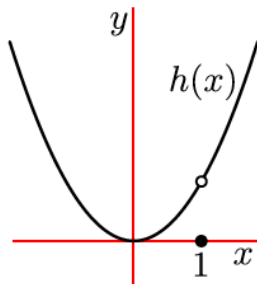


$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



infinite discontinuity

$$h(x) = \begin{cases} \frac{x^3 - x^2}{x - 1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$



removable discontinuity

# Outline - September 16, 2019

- ▶ **Section 1.6:**
  - ▶ Arithmetic of continuity
  - ▶ Continuity of composites
  - ▶ Intermediate Value Theorem
- ▶ **Section 2.1:**
  - ▶ Revisiting tangent lines

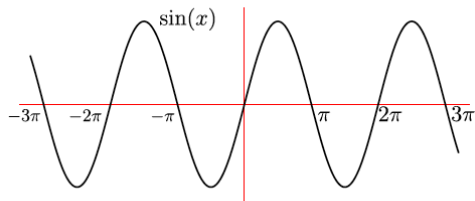


## Arithmetic of continuity

## Theorem

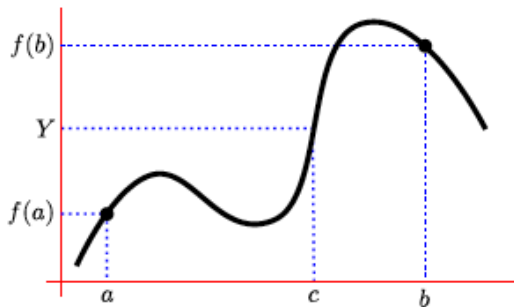
**(Arithmetic of continuity)** Let  $a, c \in \mathbb{R}$  and let  $f(x)$  and  $g(x)$  be functions that are continuous at  $a$ . Then the following functions are also continuous at  $x = a$ .

- ▶  $f(x) + g(x)$  and  $f(x) - g(x)$ ,
- ▶  $cf(x)$  and  $f(x)g(x)$ , and
- ▶  $\frac{f(x)}{g(x)}$  provided  $g(a) \neq 0$ .

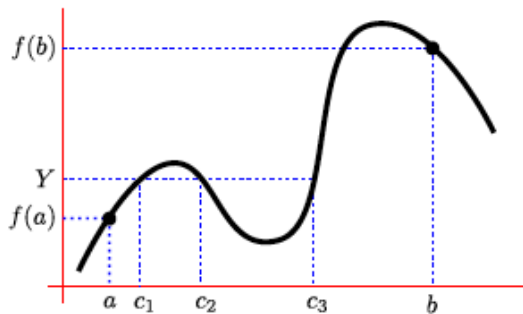


## Intermediate value theorem(IVT)

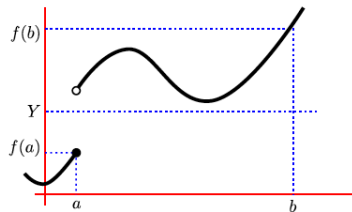
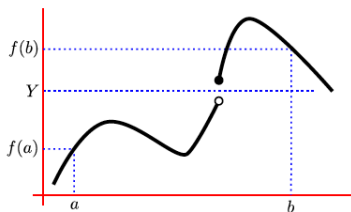
Theorem  
(Intermediate value theorem(IVT))



## The existence not the uniqueness of $c$ in IVT



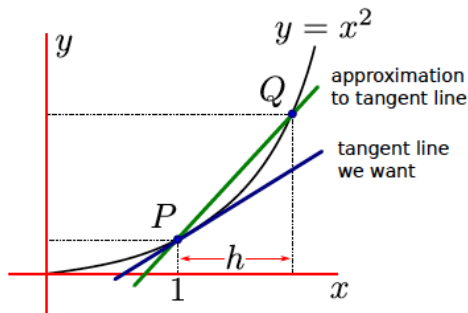
# Not continuous functions at $[a, b]$ do not satisfy IVT



## Revisiting tangent lines



# Revisiting tangent lines



$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \leftarrow \text{slope of the tangent line at } x = 1$$

## Definition of the derivative

▶

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

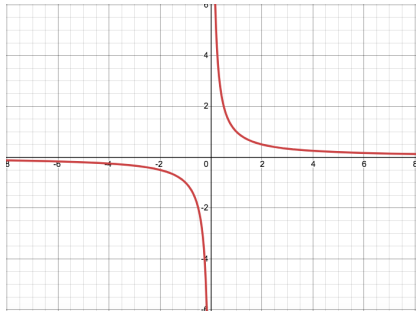
▶

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

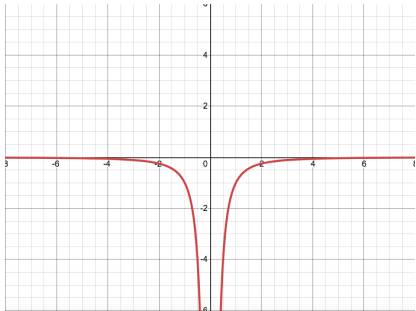
## Examples

- ▶  $f(x) = c$
- ▶  $f(x) = x$
- ▶  $f(x) = x^2$
- ▶  $f(x) = \frac{1}{x}$
- ▶  $f(x) = \sqrt{x}$
- ▶  $f(x) = |x|$

$y = \frac{1}{x}$  and its derivative  $-\frac{1}{x^2}$

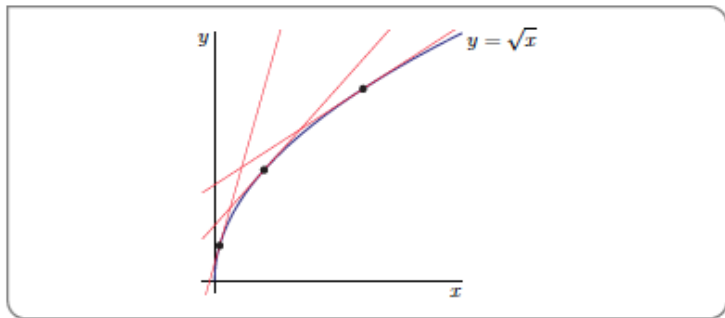


$$y = \frac{1}{x}$$

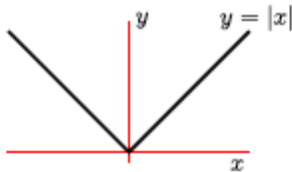


$$y = -\frac{1}{x^2}$$

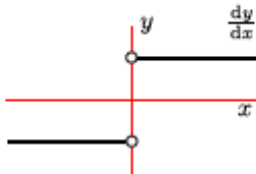
# Tangent lines to $y = \sqrt{x}$



The derivative of the function  $f(x) = |x|$ : not differentiable at  $x = 0$

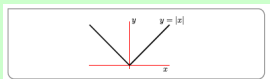


The derivative of the function  $f(x) = |x|$

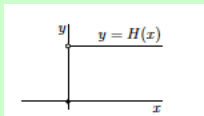


## Where a function is not differentiable at $x = a$ ?

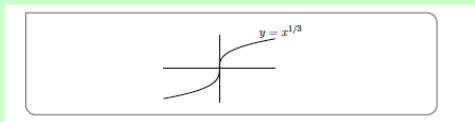
- ▶ Having a Sharp Corner at  $x = a$



- ▶ The function is not continuous at  $x = a$



- ▶ Having a tangent line, but the slope of the tangent line at  $x = a$  is infinity



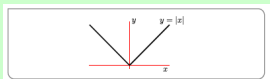
# Outline - September 20, 2019

- ▶ **Section 2.2:**
  - ▶ Not differentiable examples
  - ▶ The relation between continuous and differentiable functions
- ▶ **Section 2.3:**
  - ▶ Interpretations of the derivative

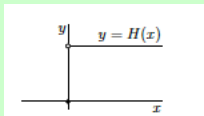


## Where a function is not differentiable at $x = a$

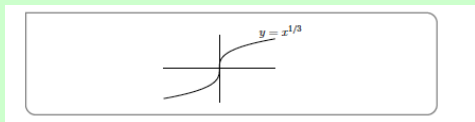
- ▶ Having a Sharp Corner at  $x = a$



- ▶ The function is not continuous at  $x = a$

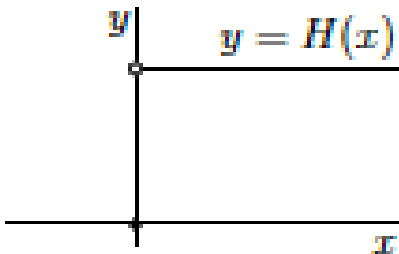


- ▶ Having a tangent line, but the slope of the tangent line at  $x = a$  is infinity



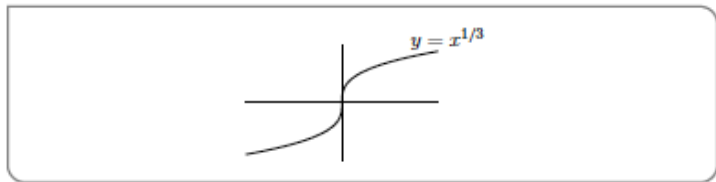
An example of a discontinuous and not differentiable function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$



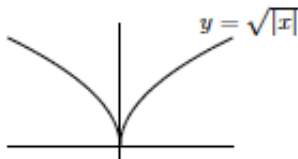
An example of a function with a tangent line with slope infinity at  $x = 0$

$$f(x) = x^{1/3}$$

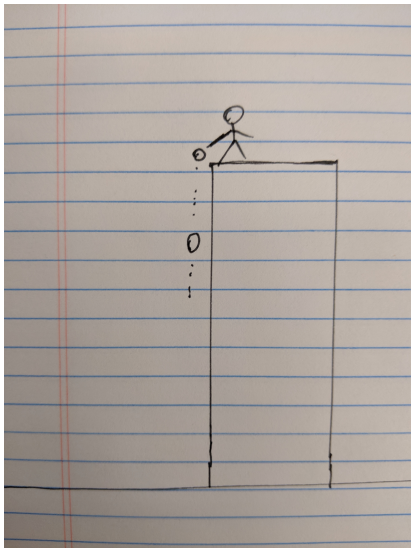


An example of a continuous and **not** differentiable function

$$y = \sqrt{|x|}$$



# Instantaneous rate of change



**average** rate of change of  $f(t)$  from  $t = a$  to  $t = a + h$  is

$$\frac{\text{change in } f(t) \text{ from } t = a \text{ to } t = a + h}{\text{length of time from } t = a \text{ to } t = a + h}$$

$$= \frac{f(a + h) - f(a)}{h}.$$

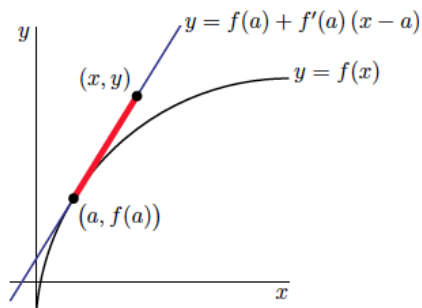
And so

**instantaneous** rate of change of  $f(t)$  at  $t = a$

$$= \lim_{h \rightarrow 0} [\text{average rate of change of } f(t) \text{ from } t = a \text{ to } t = a + h]$$

$$= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} = f'(a).$$

## Finding tangent line to a curve at $x = a$



A line segment of a tangent line

$$y = f(a) + f'(a)(x - a)$$

# Outline - September 23, 2019

- ▶ **Section 2.4 and 2.5:**
  - ▶ Derivative of some simple functions
  - ▶ Tools
  - ▶ Examples



## A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0$$

$$\frac{d}{dx}x = 1$$

$$\frac{d}{dx}x^2 = 2x$$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

## A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \quad \frac{d}{dx}x = 1 \quad \frac{d}{dx}x^2 = 2x \quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

## Tools

Let  $f(x)$  and  $g(x)$  be differentiable functions and let  $c, d \in \mathbb{R}$ .

- ▶  $\frac{d}{dx}\{f(x) + g(x)\} = f'(x) + g'(x)$
- ▶  $\frac{d}{dx}\{f(x) - g(x)\} = f'(x) - g'(x)$
- ▶  $\frac{d}{dx}\{cf(x)\} = cf'(x)$

## Tools

Let  $f(x)$ ,  $g(x)$ , and  $h(x)$  be differentiable functions and let  $c, d \in \mathbb{R}$ .

- ▶  $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- ▶  $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$

## Tools

Let  $f(x)$ ,  $g(x)$ , and  $h(x)$  be differentiable functions and let  $c, d \in \mathbb{R}$ .

- ▶  $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- ▶  $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶  $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- ▶  $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- ▶  $\frac{d}{dx}\left\{\frac{1}{g(x)}\right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$

## Tools

Let  $f(x)$ ,  $g(x)$ , and  $h(x)$  be differentiable functions and let  $c, d \in \mathbb{R}$ .

- ▶  $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- ▶  $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶  $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- ▶  $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- ▶  $\frac{d}{dx}\left\{\frac{1}{g(x)}\right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶  $\frac{d}{dx}\{f(x)g(x)h(x)\} = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- ▶  $\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x)$

## Tools

Let  $f(x)$ ,  $g(x)$ , and  $h(x)$  be differentiable functions and let  $c, d \in \mathbb{R}$ .

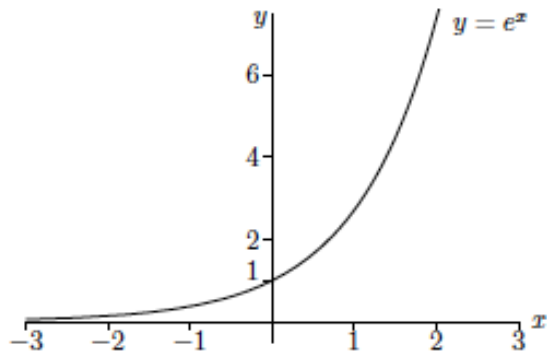
- ▶  $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- ▶  $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶  $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- ▶  $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- ▶  $\frac{d}{dx}\left\{\frac{1}{g(x)}\right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶  $\frac{d}{dx}\{f(x)g(x)h(x)\} = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- ▶  $\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x)$
- ▶ Let  $a$  be a rational number, then

$$\frac{d}{dx}x^a = ax^{a-1}.$$

# Outline - September 25, 2019

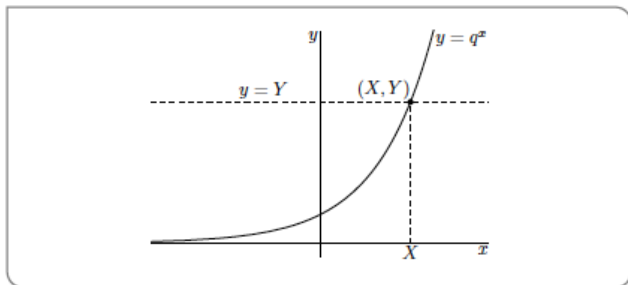
- ▶ **Section 2.7 and 2.8:**
  - ▶ Derivative of exponential functions
  - ▶ Derivative of trigonometric functions

# The graph of $e^x$





The graph of  $q^x$  where  $q > 1$



# YOUR TURN!

## Example

Find  $a$  such that the following function is continuous.

$$f(x) = \begin{cases} e^{x+a} & x < 0 \\ \sqrt{x+1} & x \geq 0 \end{cases}$$

## Example

We have

1.  $\log_q(xy) =$ 
  - (a)  $\log_q(x) + \log_q(y)$
  - (b)  $\log_q(x) \log_q(y)$
2.  $\log_q(x/y) =$
3.  $\log_q(x^r) =$

## Example

We have

1.  $\log_q(xy) = \log_q(x) + \log_q(y)$ .

The reason for this is that

$$q^{\log_q(xy)} = xy = q^{\log_q(x)} q^{\log_q(y)} = q^{\log_q(x) + \log_q(y)}$$

Therefore,  $\log_q(xy) = \log(x) + \log(y)$ .

2.  $\log_q(x/y) = \log_q(x) - \log_q(y)$

3.  $\log_q(x^r) = r \log_q(x)$

## TOOLS:

$$\frac{d}{dx}(f \circ g)(x) = g'(x)f'(g(x))$$

## A list of derivative of some simple functions:

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = (\log_e a)a^x$$

## Example

Find the derivative of  $2^{\sqrt{x}}$ .

### Example

Find the derivative of  $2^{\sqrt{x}}$ .

### Example

Find  $a$  and  $b$  such that the following function is differentiable.

$$f(x) = \begin{cases} x^3 + a & x < 1 \\ e^{x-1} + bx & x \geq 1 \end{cases}$$

# Outline - September 30, 2019

## ► Section 2.8, 2.9, 0.6:

- Derivative of trigonometric functions
- The chain rule
- inverse of a function

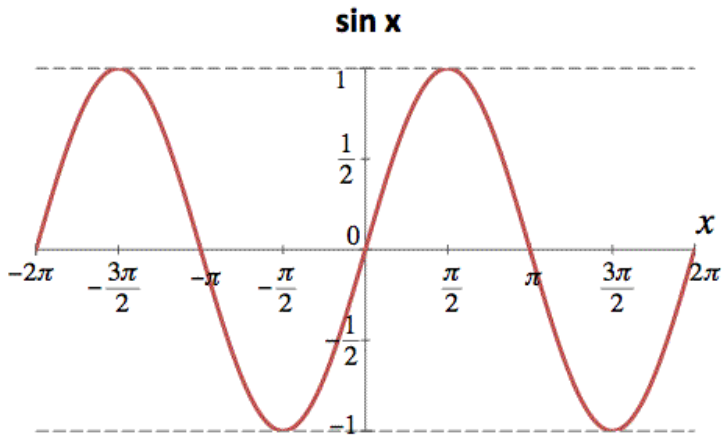
### A list of derivative of some simple functions:

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = (\log_e a)a^x$$

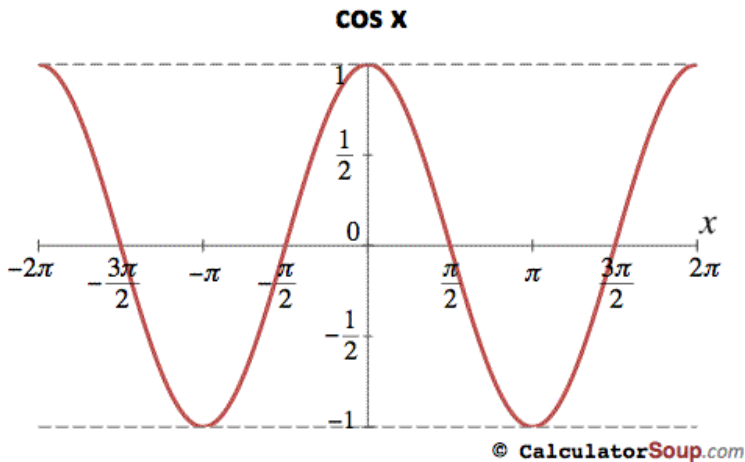


$\sin(x)$     domain =  $\mathbb{R}$     range =  $[-1, 1]$

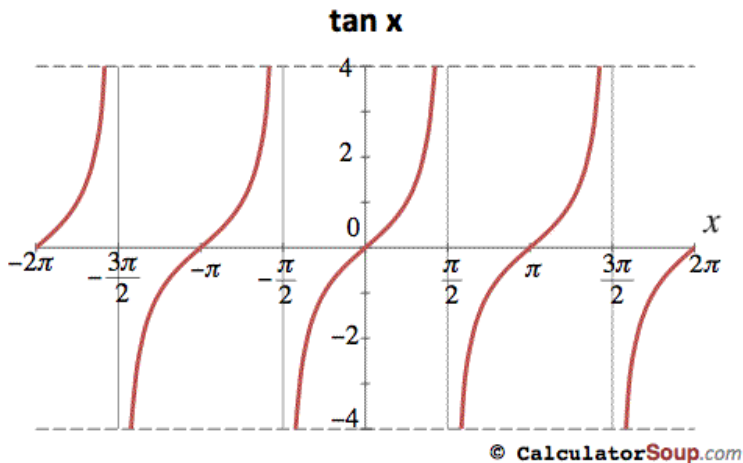


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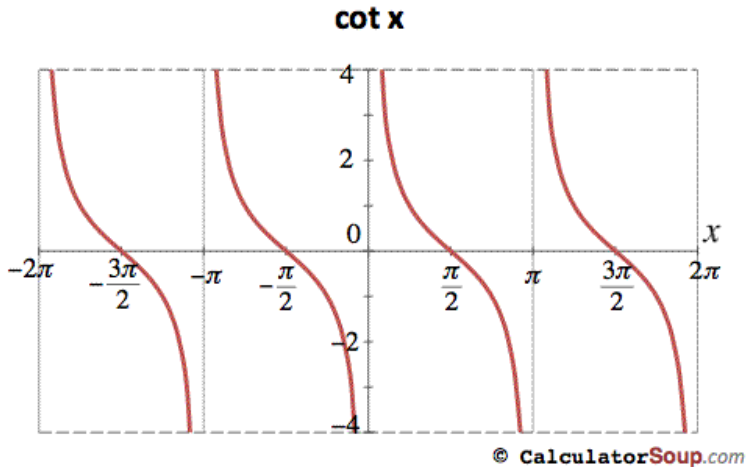
$$\cos(x) \quad \text{domain} = \mathbb{R} \quad \text{range} = [-1, 1]$$



$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \text{domain} = \mathbb{R} - \left\{ (2n+1)\frac{\pi}{2} : n \in \mathbb{Z} \right\} \quad \text{range} = \mathbb{R}$$



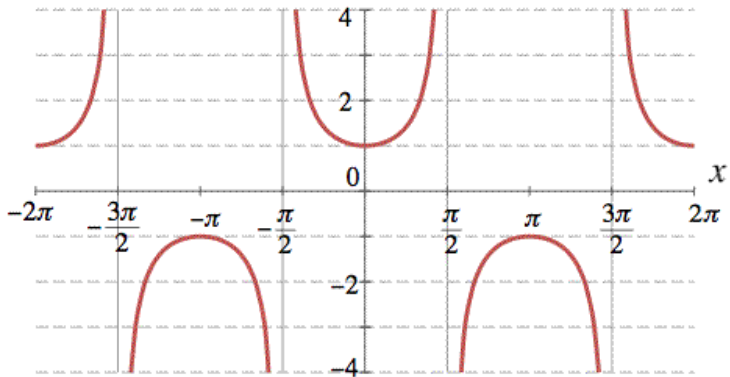
$$\cot(x) = \frac{\cos(x)}{\sin(x)} \quad \text{domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R}$$



$$\sec(x) = \frac{1}{\cos(x)} \quad \text{domain} = \mathbb{R} - \left\{ (2n+1)\frac{\pi}{2} : n \in \mathbb{Z} \right\}$$

$$\text{range} = \mathbb{R} - (-1, 1)$$

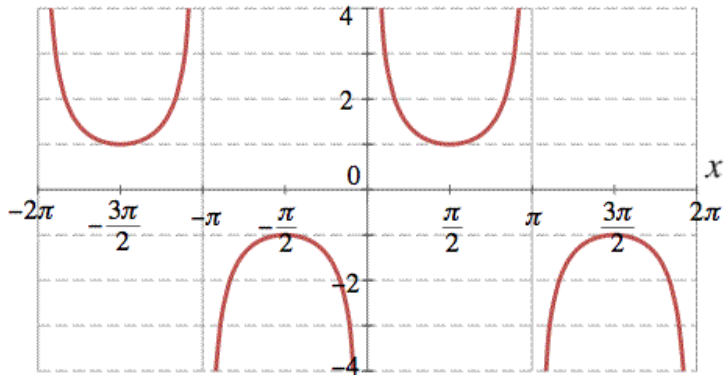
**sec x**



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$$\csc(x) = \frac{1}{\sin(x)} \quad \text{domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R} - (-1, 1)$$

### CSC X



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# Derivative of $\sin(x)$

**Question:** Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of  $\sin(x)$  at  $x = 0$ .

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compute the derivative of  $\sin(x)$  at  $x = 0$ .

(**sandwich (or squeeze or pinch) theorem**) Let  $a \in \mathbb{R}$  and let  $f, g, h$  be three functions so that  $f(x) \leq g(x) \leq h(x)$  for all  $x$  in an interval around  $a$ , except possibly at  $x = a$ . Then if

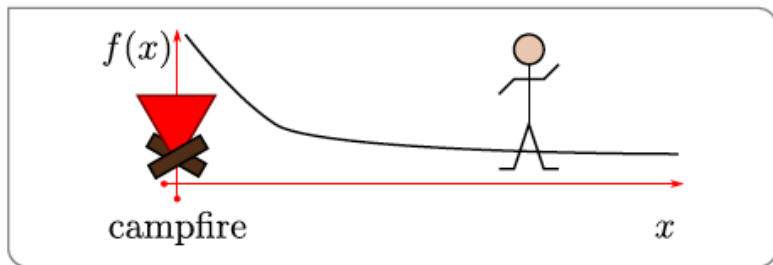
$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then it is also the case that

$$\lim_{x \rightarrow a} g(x) = L.$$



## An example of the application of the chain rule



- ▶ Your position at time  $t$  is  $x(t)$ .
- ▶ The temperature of the air at position  $x$  is  $f(x)$ .
- ▶ The temperature that you feel at time  $t$  is  $F(t) = f(x(t))$ .
- ▶ The instantaneous rate of change of temperature that you feel is  $F'(t)$ .

## The chain rule

### Theorem

*Let  $f$  and  $g$  be differentiable functions then*

$$\frac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$$

## The chain rule

### Theorem

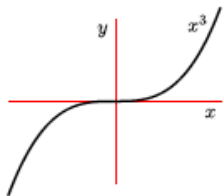
*Let  $y = f(u)$  and  $u = g(x)$  be differentiable functions, then*

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

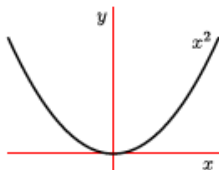
input number  $x \mapsto f$  does “stuff” to  $x \mapsto$  return number  $y$

take output  $y \mapsto$  do “stuff” to  $y \mapsto$  return the original  
number  $x$

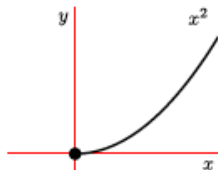
# One-to-one functions



$$\begin{array}{lcl} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

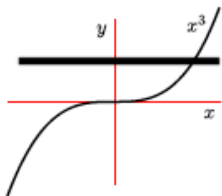


$$\begin{array}{lcl} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$



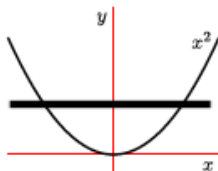
$$\begin{array}{lcl} [0, \infty] & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

# One-to-one functions



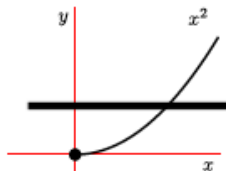
$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

is one-to-one



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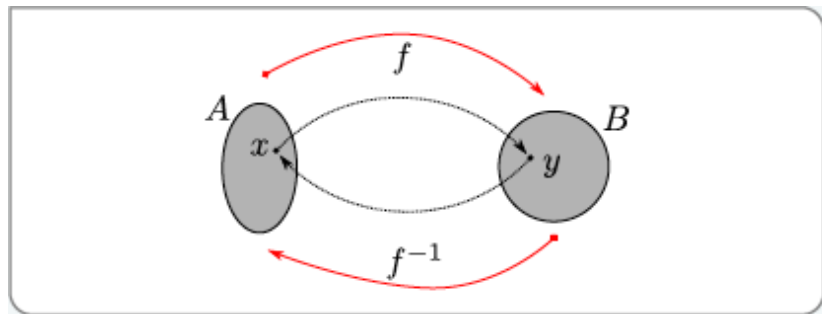
is not one-to-one



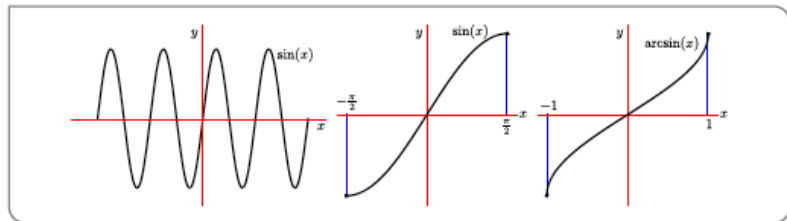
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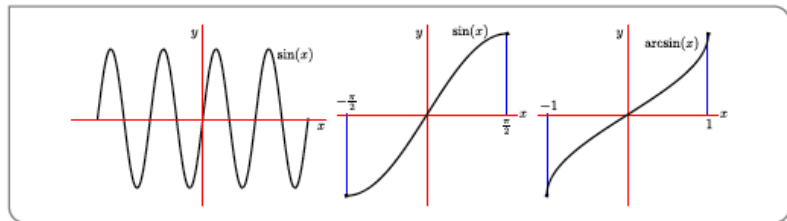
# Inverse of a functions



# Inverse of $\sin(x)$



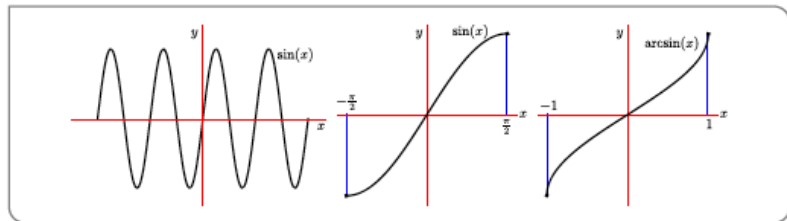
# Inverse of $\sin(x)$



- ▶  $\sin(x)$  is not invertible on the domain  $\mathbb{R}$  because it is not one-to-one.

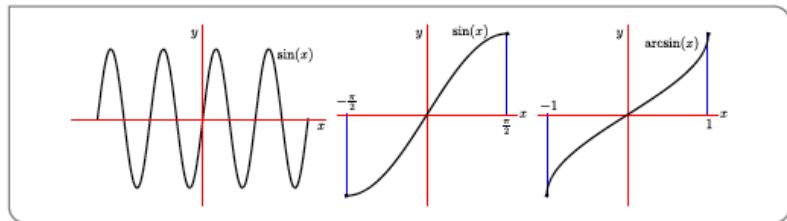


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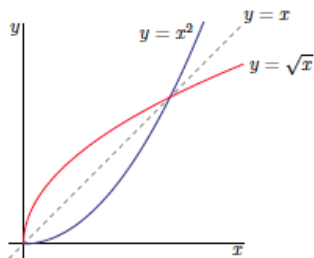
- ▶  $\sin(x)$  is not invertible on the domain  $\mathbb{R}$  because it is not one-to-one.
- ▶ If we look at  $\sin(x)$  on the domain  $[-\pi/2, \pi/2]$ , then it is one-to-one, and so it has an inverse.

# Inverse of $\sin(x)$

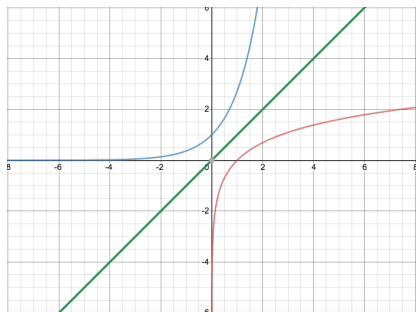
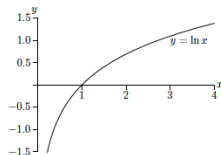
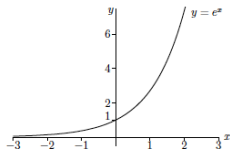


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- ▶ The inverse of  $\sin(x)$  is  $\arcsin(x)$  on the domain  $[-1, 1]$  and with the range  $[-\pi/2, \pi/2]$ .

# How to find the inverse of a function by its graph



# The inverse of $e^x$



# Useful facts!

- ▶  $\frac{d}{dx} a^x = (\ln a) a^x.$
- ▶  $\log_a x = \frac{\ln x}{\ln a} \quad \ln x = \frac{\log_a x}{\log_a e} \quad a > 1.$
- ▶  $\ln(xy) = \ln x + \ln y.$
- ▶  $\ln(x/y) = \ln x - \ln y.$
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