MATH100: Differential Calculus with Application to Physical Sciences and Engineering

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Contents

Chapter 1

Limits

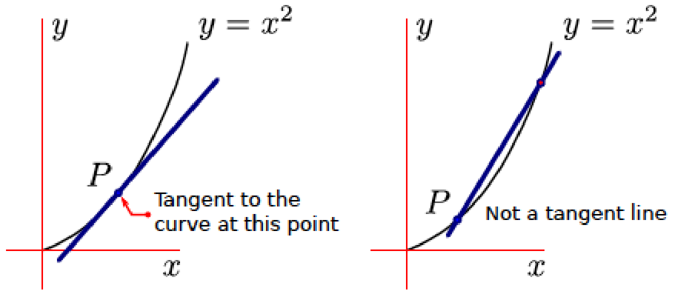
What does this mean

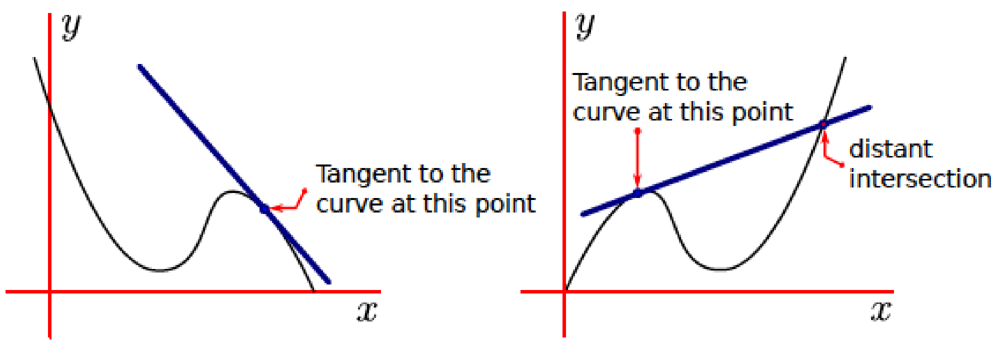
$$\lim_{x \to a} f(x) = L?$$

The "limit" appears when we want to

- find the tangent to a curve; or
- find the velocity of an object.

1.1 Tangent line





The **tangent line to a curve** $y = f(x)$ **at a point** P (if exists) is a line L that there is a neighborhood for P such that in that neighborhood the line L touches (does not cross) the curve $y = f(x)$ only at P (and not other points in that neighborhood).

The equation of a line

- The formula for a line that passes though (x_1, y_1) with slope m is

$$y = y_1 + m(x - x_1).$$

- Given two points (x_1, y_1) and (x_2, y_2) on a line, then the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

  and the formula for the line then is

$$y = y_1 + m(x - x_1).$$

Example 1.1.1. *Find the equation of the line with slope* -3 *that passes through* $(1, 2)$.

Solution. *The equation of the line is*

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

Example 1.1.2. *Find the equation of the line that passes through* $(1, 2)$ *and* $(2, -1)$.

Solution. *First we find the slope which is*

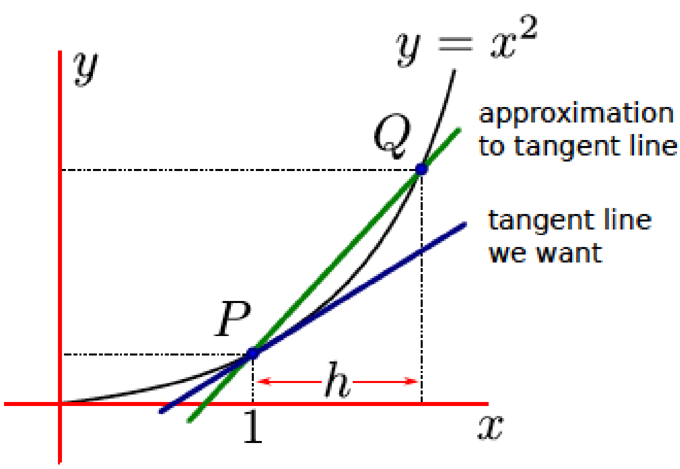
$$\frac{-1 - 2}{2 - 1} = -3.$$

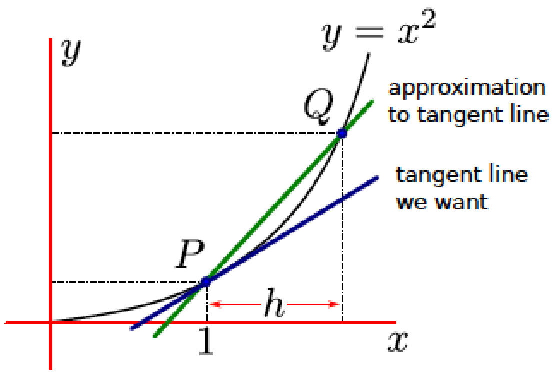
Then the equation of the line is

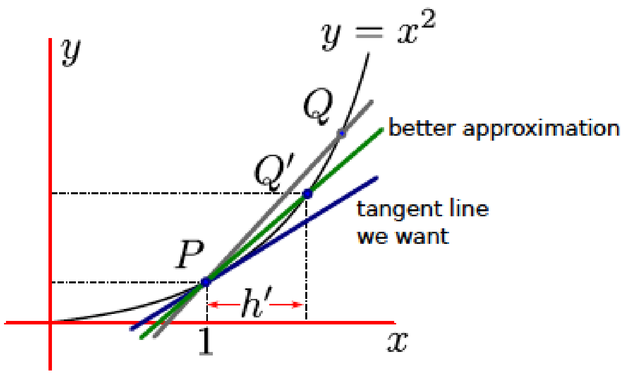
$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

Tangent line: Given a curve $y = f(x)$ and a point P on the curve, how to find the slope of the tangent to a curve at P: let do this through an example.

Example 1.1.3. *Find the tangent line to the curve* $y = x^2$ *that passes through* $P = (1, 1)$.







So we want to find the slope the line that passes through the points $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (1 + h, (1 + h)^2)$. The slope then is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1+h)^2 - 1^2}{(1+h) - 1} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(h+2)}{h} = 2 + h$$

h	$m = \frac{(1+h)^2 - 1^2}{(1+h)-1}$
0.1	2.1
0.01	2.01
0.001	2.001

When h gets smaller and smaller, the slope will be closer and closer to the slope of the tangent line to $y = x^2$ at $(1, 1)$, which the slope will be closer and closer to 2, we can write this more mathematically as

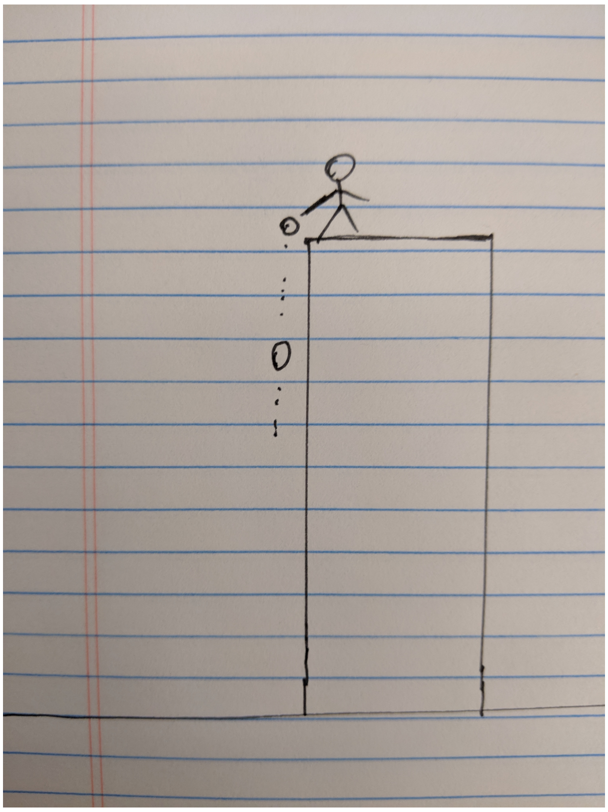
$$\lim_{h \to 0} \frac{(1+h)^2 - 1}{(1+h) - 1} = 2$$

Read: *the limit of* $\frac{(1+h)^2-1}{(1+h)-1}$ *as* h *approaches* 0 *is* 2.
Tangent line is

$$y = 1 + 2(x - 1) = 2x - 1.$$

1.2 Velocity

- Let t be elapsed time measured in second
- $S(t)$ be the distance the ball has fallen in meters
- What is $S(0)$? $S(0) = 0$.
- (**Galileo**) $S(t) = 4.9t^2$.



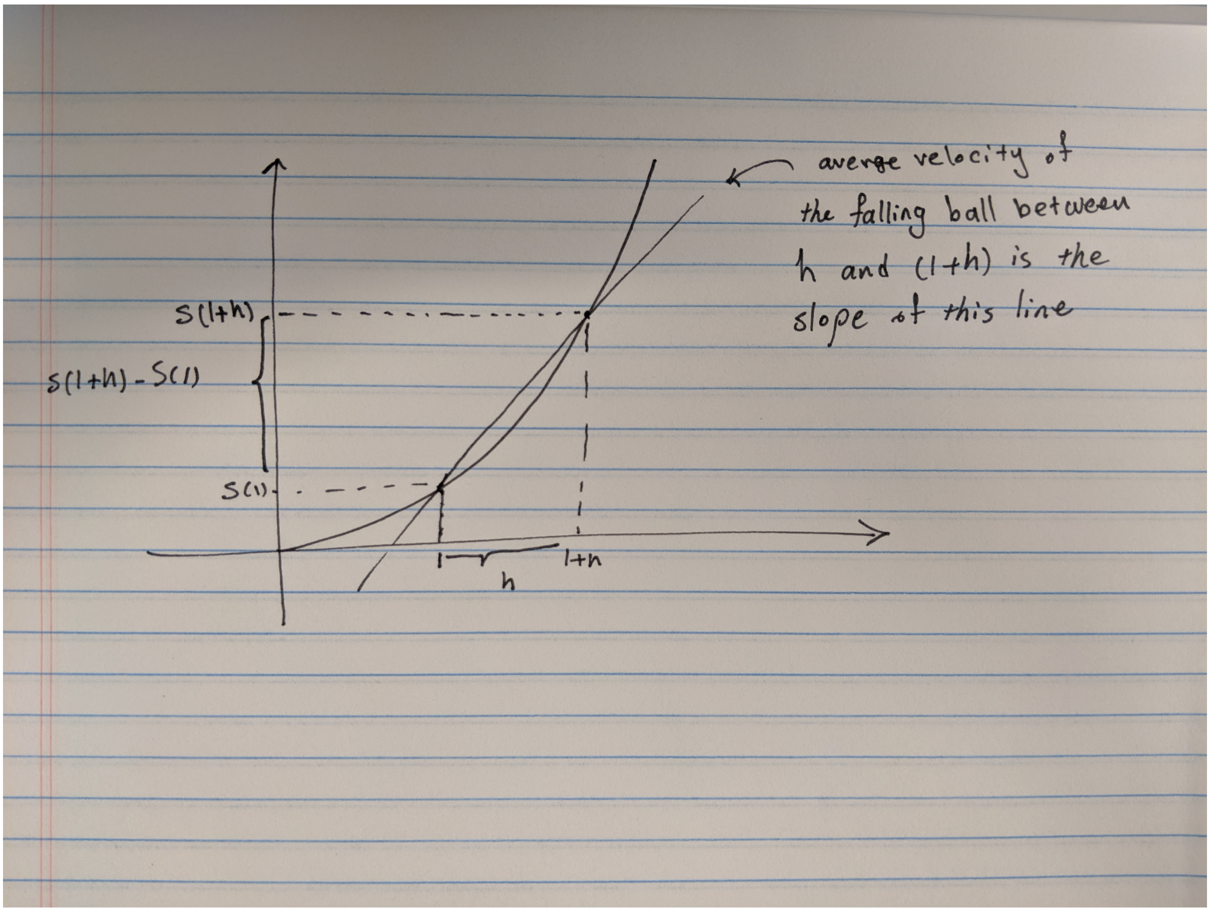
Question: How fast the ball is fallen after 1 second, that is, find $v(1)$, the velocity at $t = 1$?

$$\text{average velocity} = \frac{\text{difference in position}}{\text{difference in time}} = \frac{S(t_2) - S(t_1)}{t_2 - t_1}.$$

To answer the question we should find the average velocity of the falling ball between $(1 + h)$ and 1. So,

$$\text{average velocity when } (t_2 = 1 + h) \text{ and } (t_1 = 1)$$

$$= \frac{S(1+h) - S(1)}{h} = \frac{4.9(1+h)^2 - 4.9}{h} = 4.9(2+h).$$



time window	average velocity
$1 \leq t \leq 1.1$	10.29
$1 \leq t \leq 1.01$	9.84
$1 \leq t \leq 1.01$	9.8049
$1 \leq t \leq 1.001$	9.80049

So we can write

$$v(1) = \lim_{h \to 0} \frac{S(1+h) - S(1)}{h} = 9.8.$$

More generally:

We define the instantaneous velocity at time $t = a$ to be the limit

$$v(a) = \lim_{h \to 0} \frac{S(a+h) - S(a)}{h}$$

Bibliography

[1] CLP1: Differential Calculus by J. Feldman, A. Rechnitzer, and E. Yeager.