MATH 100

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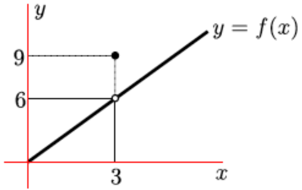
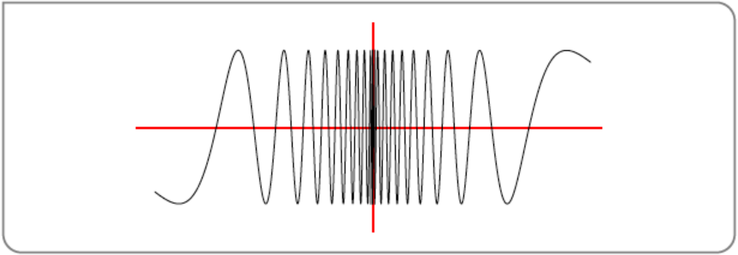
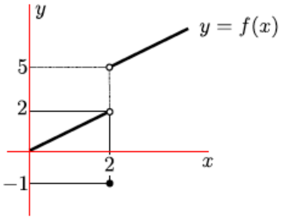
$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



$f(x) = \sin(\frac{\pi}{x})$

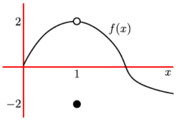


$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$



Example

Consider the graph of the function $f(x)$.



Then

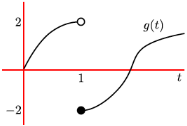
$$\lim_{x\to 1^-} f(x) =$$

$$\lim_{x\to 1^+} f(x) =$$

$$\lim_{x\to 1} f(x) =$$

Example

Consider the graph of the function $g(t)$.



Then

$$\lim_{t\to 1^-} g(t) =$$

$$\lim_{t\to 1^+} g(t) =$$

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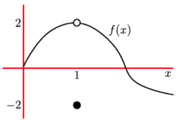
Example

Consider the graph of the function $f(x)$.



Then

$$\lim_{x\to 1^-} f(x) = 2$$

$$\lim_{x\to 1^+} f(x) = 2$$

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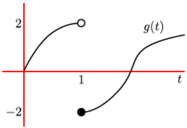
Example

Consider the graph of the function $g(t)$.



Then

$$\lim_{t\to 1^-} g(t) = 2$$

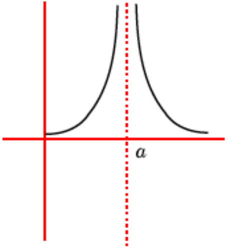
$$\lim_{t\to 1^+} g(t) = -2$$

$$\lim_{t\to 1} g(t) = DNE$$

When the limit goes to infinity

Example

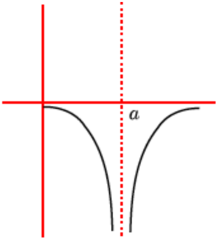
Consider the graph for the function $f(x)$.



$$\lim_{x \to a} f(x) = +\infty$$

Example

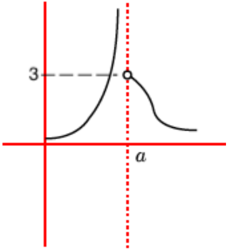
Consider the graph for the function $g(x)$.



$$\lim_{x \to a} g(x) = -\infty$$

Example

Consider the graph for the function $h(x)$.

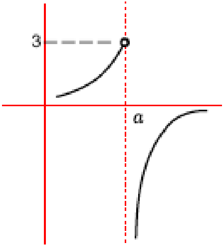


$$\lim_{x \to a^-} h(x) =$$

$$\lim_{x \to a^+} h(x) =$$

Example

Consider the graph for the function $s(x)$.

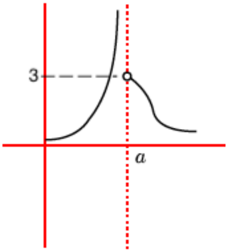


$$\lim_{x \to a^-} s(x) =$$

$$\lim_{x \to a^+} s(x) =$$

Example

Consider the graph for the function $h(x)$.

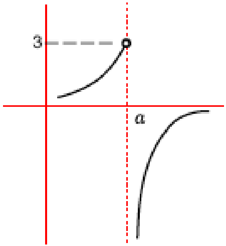


$$\lim_{x \to a^-} h(x) = +\infty$$

$$\lim_{x \to a^+} h(x) = 3$$

Example

Consider the graph for the function $s(x)$.



$$\lim_{x \to a^-} s(x) = 3$$

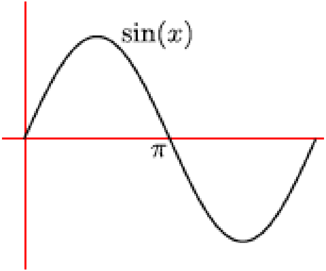
$$\lim_{x \to a^+} s(x) = -\infty$$

Example

Consider the function

$$g(x) = \frac{1}{sin(x)}.$$

Find the one-side limits of this function as $x \to \pi$.



$$\lim_{x \to \pi^-} \frac{1}{sin(x)} = +\infty$$

$$\lim_{x \to \pi^+} \frac{1}{sin(x)} = -\infty$$

Second Session Outline

- Arithmetic of the Limits
- Limit of a ratio: what will happen if the limit of the denominator is zero. For example,

$$\lim_{x\to 0} \frac{1}{x^2}? \quad \text{and} \quad \lim_{x\to 1} \frac{x^3 - x^2}{x - 1} =?$$

- Sandwich/ Squeeze/Pinch Theorem
- limit at infinity

Arithmetic of the Limits

Theorem

Let $a, c \in \mathbb{R}$. The following two limits hold

$$\lim_{x \to a} c = c \qquad \lim_{x \to a} x = a$$

Example

$$\lim_{x \to 3} -2 = -2 \qquad \lim_{x \to -1} x = -1$$

Theorem

(**Arithmetic of Limits**) *Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f and g need not to be defined exactly at a).*

$$\lim_{x \to a} f(x) = F \qquad \lim_{x \to a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. Then the following limits hold

- $\lim_{x \to a} (f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*

Theorem

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- $\lim_{x \to a}(f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*
- $\lim_{x \to a}(f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*

Theorem

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- $\lim_{x \to a} cf(x) = cF$.

Theorem

(**Arithmetic of Limits**) *Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f and g need not to be defined exactly at a).*

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exists with $F, G \in \mathbb{R}$. Then the following limits hold

- $\lim_{x \to a}(f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*
- $\lim_{x \to a}(f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*
- $\lim_{x \to a} cf(x) = cF$.
- $\lim_{x \to a}(f(x).g(x)) = F.G$*–limit of the product is the product of the limits.*

If $G \neq 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}$$

Example

Given

$$\lim_{x \to 1} f(x) = 3 \quad \text{and} \quad \lim_{x \to 1} g(x) = 2$$

We have

$$\lim_{x \to 1} 3f(x) =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} f(x). \lim_{x\to 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x\to 1} \frac{f(x)}{f(x) - g(x)} =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} f(x) . \lim_{x\to 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x\to 1} \frac{f(x)}{f(x) - g(x)} = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} f(x) - \lim_{x\to 1} g(x)} = \frac{3}{3-2} = 3.$$

Example

$$\lim_{x \to 3} 4x^2 - 1 =$$

$$\lim_{x \to 2} \frac{x}{x-1} =$$

Example

$$\lim_{x \to 3} 4x^2 - 1 = 4 \times \lim_{x \to 3} x^2 - \lim_{x \to 3} 1 = 35.$$

$$\lim_{x \to 2} \frac{x}{x - 1} = \frac{\lim_{x \to 2} x}{\lim_{x \to 2} x - \lim_{x \to 1} 1} = \frac{2}{2 - 1} = 2.$$

Limit of a ratio: what will happen if the limit of the denominator is zero.

Limit of a ratio: what will happen if the limit of denominator is zero:

– the limit does **not exist**, eg.

$$\lim_{x \to 0} \frac{x}{x^2} = \lim_{x \to 0} \frac{1}{x} = DNE$$

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- the limit does **not exist**, eg.

$$\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} = DNE$$

- the **limit is** $\pm\infty$, eg.

$$\lim_{x\to 0} \frac{x^2}{x^4} = \lim_{x\to 0} \frac{1}{x^2} = +\infty \qquad \text{or} \qquad \lim_{x\to 0} \frac{-x^2}{x^4} = \lim_{x\to 0} \frac{-1}{x^2} = -\infty.$$

Limit of a ratio: what will happen if the limit of denominator is zero:

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- the **limit is** 0, eg.

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0.$$

Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x\to 0}\frac{x}{x^2}=\lim_{x\to 0}\frac{1}{x}=DNE$$

- the **limit is** $\pm\infty$, eg.

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- the **limit is** 0, eg.

$$\lim_{x\to 0}\frac{x^2}{x}=\lim_{x\to 0}x=0.$$

- the **limit exists and it nonzero**, eg.

$$\lim_{x\to 0}\frac{x}{x}=1.$$

Theorem

Let n be a positive integer, let $a \in R$ and let f be a function so that

$$\lim_{x \to a} f(x) = F$$

for some real number F. Then the following holds

$$\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = F^n$$

so that the limit of a power is the power of the limit.

Theorem

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so that the limit of a power is the power of the limit. Similarly, if

- *n is an even number and $F > 0$, or*
- *n is an odd number and F is any real number*

then

$$\lim_{x \to a} (f(x))^{1/n} = \left(\lim_{x \to a} f(x) \right)^{1/n} = F^{1/n}.$$

Example

$$\lim_{x\to 4} x^{1/2} =$$

$$\lim_{x\to 4} (-x)^{1/2} =$$

$$\lim_{x\to 2} (4x^2 - 3)^{1/3} =$$

Example

$$\lim_{x \to 4} x^{1/2} = 4^{1/2} = 2.$$

$$\lim_{x \to 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \to 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = (13)^{1/3}.$$

Limit of a ratio: what will happen if the limit of the numerator and denominator are zero, for example,

$$\lim_{x \to 1} \frac{x^3 - x^2}{x - 1} = ?$$

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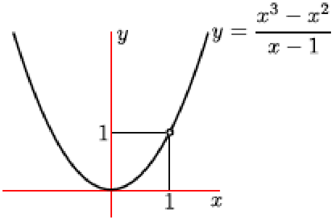


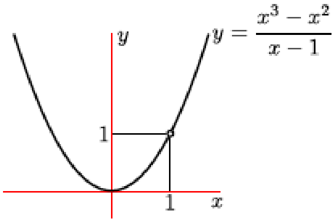
Theorem

If $f(x) = g(x)$ except when $x = a$ then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

provided the limit of g exists.

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases} \Rightarrow \lim_{x \to 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \to 1} x^2 = 1.$$

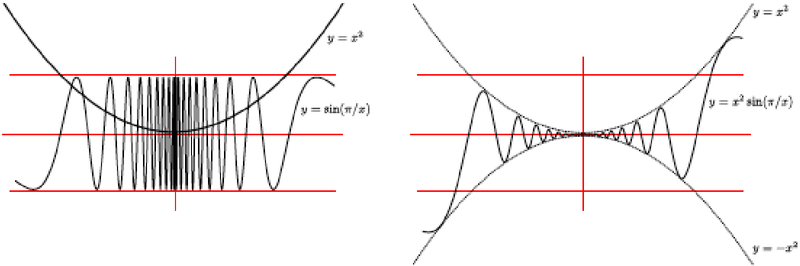


Sandwich/ Squeeze/Pinch Theorem

Example

Compute

$$\lim_{x \to 0} x^2 \sin(\frac{\pi}{x})$$



Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \to 1} f(x)?$$

Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \to 1} f(x)?$$

Solution

Consider that

$$\lim_{x \to 1} x = 1 \qquad \textit{and} \qquad \lim_{x \to 1} x^2 - 2x + 2 = 1.$$

Therefore, by the sandwich/pinch/squeeze theorem

$$\lim_{x \to 1} f(x) = 1.$$

Example

We want to compute

$$\lim_{x\to+\infty} \frac{1}{x} \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x}$$

By plug in some large numbers into $\frac{1}{x}$ we have

-10000	-1000	$-100\Vert \circ \Vert 100$	1000	10000
-0.0001	-0.001	$-0.01\Vert \circ \Vert 0.01$	0.001	0.0001

We see that as x is getting bigger and positive the function $\frac{1}{x}$ is getting closer to 0. Thus,

$$\lim_{x\to+\infty} \frac{1}{x} = 0.$$

Moreover,

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

Limit at Infinity

Definition

(**Informal limit at infinity.**) We write

$$\lim_{x \to \infty} f(x) = L$$

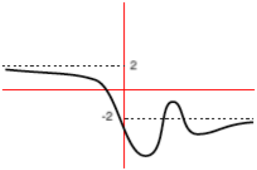
when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and positive.
Similarly, we write

$$\lim_{x \to -\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and negative.

Example

Consider the graph of the function $f(x)$.



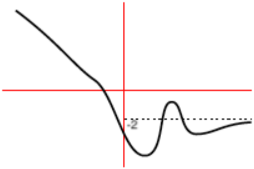
Then

$$\lim_{x \to \infty} f(x) =$$

$$\lim_{x \to -\infty} f(x) =$$

Example

Consider the graph of the function $g(x)$.



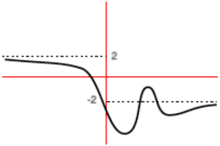
Then

$$\lim_{x \to \infty} g(x) =$$

$$\lim_{x \to -\infty} g(x) =$$

Example

Consider the graph of the function $f(x)$.



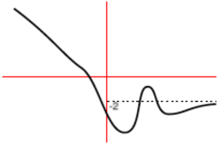
Then

$$\lim_{x \to \infty} f(x) = -2$$

$$\lim_{x \to -\infty} f(x) = 2$$

Example

Consider the graph of the function $g(x)$.



Then

$$\lim_{x \to \infty} g(x) = -2$$

$$\lim_{x \to -\infty} g(x) = +\infty$$

Review of the third session

Review

Theorem

sandwich (or squeeze or pinch) *Let* $a \in \mathbb{R}$ *and let* f, g, h *be three functions so that*

$$f(x) \leq g(x) \leq h(x)$$

for all x *in an interval around* a, *except possibly at* $x = a$. *Then if*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

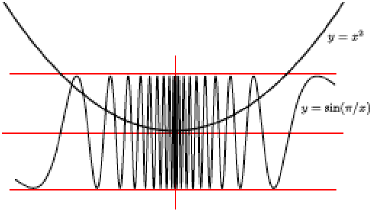
then it is also the case that

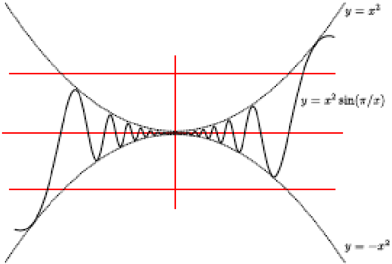
$$\lim_{x \to a} g(x) = L.$$

Example

Compute

$$\lim_{x \to 0} x^2 \sin(\frac{\pi}{x})$$





Theorem

Let $c \in \mathbb{R}$ then the following limits hold

$$\lim_{x \to +\infty} c = c \qquad \lim_{x \to -\infty} c = c$$

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

Outline For the Fourth Session

- Limit at Infinity

Limit at Infinity

Theorem

Let $f(x)$ and $g(x)$ be two functions for which the limits

$$\lim_{x\to\infty} f(x) = F \qquad \lim_{x\to\infty} = G$$

exist. Then the following limits hold

$$\lim_{x\to\infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x\to\infty} f(x)g(x) = FG$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \textit{provided } G \neq 0$$

and for rational numbers r,

$$\lim_{x\to\infty}(f(x))^r = F^r$$

provided that $f(x)^r$ is defined for all x.
The analogous results hold for limits to $-\infty$.

Warning: Consider that

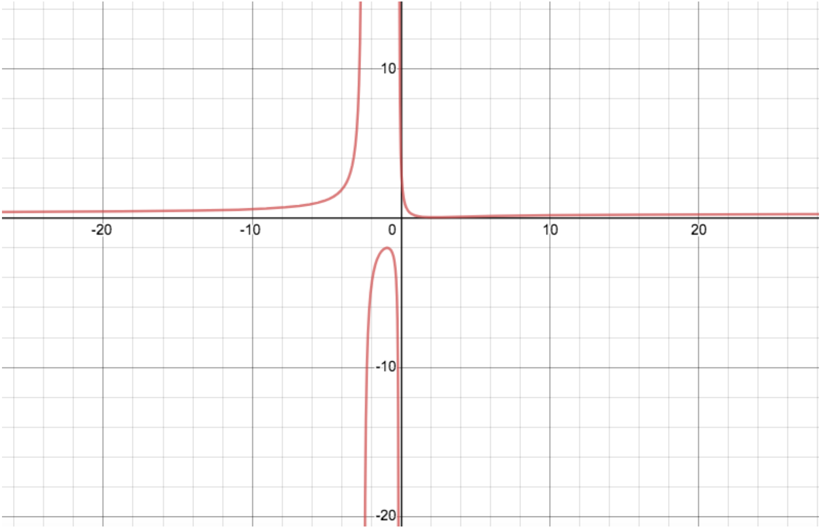
$$\lim_{x \to +\infty} \frac{1}{x^{1/2}} = 0$$

However,

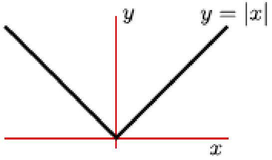
$$\lim_{x \to +\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because $x^{1/2}$ is not defined for $x < 0$.

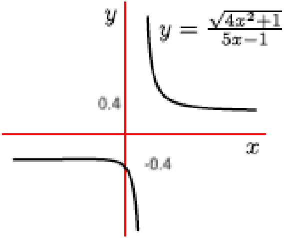
$$f(x) = \frac{x^2-3x+4}{3x^2+8x+1}$$



$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



$$y = \frac{\sqrt{4x^2+1}}{5x-1}$$



Theorem

Let $a, c, H \in \mathbb{R}$ *and let* f, g, h *be functions defined in an interval around* a *(but they need not be defined at* $x = a$*), so that*

$$\lim_{x \to a} f(x) = +\infty \qquad \lim_{x \to a} g(x) = +\infty \qquad \lim_{x \to a} h(x) = H$$

1. $$\lim_{x \to a} (f(x) + g(x)) =$$
2. $$\lim_{x \to a} (f(x) + h(x)) =$$
3. $$\lim_{x \to a} (f(x) - g(x)) =$$
4. $$\lim_{x \to a} (f(x) - h(x)) =$$

Theorem

Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that

$$\lim_{x \to a} f(x) = +\infty \qquad \lim_{x \to a} g(x) = +\infty \qquad \lim_{x \to a} h(x) = H$$

1. $$\lim_{x \to a} (f(x) + g(x)) = +\infty.$$
2. $$\lim_{x \to a} (f(x) + h(x)) = +\infty.$$
3. $$\lim_{x \to a} (f(x) - g(x)) = \textit{undetermined}.$$
4. $$\lim_{x \to a} (f(x) - h(x)) = +\infty.$$

Theorem

5.

$$\lim_{x \to a} cf(x) = \begin{cases} & c > 0 \\ & c = 0 \\ & c < 0 \end{cases}$$

6.

$$\lim(f(x).g(x)) =$$

7.

$$\lim_{x \to a}(f(x).h(x)) = \begin{cases} & H > 0 \\ & H = 0 \\ & H < 0 \end{cases}$$

8.

$$\lim_{x \to a} \frac{h(x)}{f(x)} =$$

Theorem

5.
$$\lim_{x\to a} cf(x) = \begin{cases} +\infty & c>0 \\ 0 & c=0 \\ -\infty & c<0 \end{cases}$$

6.
$$\lim(f(x).g(x)) = +\infty.$$

7.
$$\lim_{x\to a}(f(x).h(x)) = \begin{cases} +\infty & H>0 \\ \mathit{undetermined} & H=0 \\ -\infty & H<0 \end{cases}$$

8.
$$\lim_{x\to a}\frac{h(x)}{f(x)} = 0.$$

Example

Consider the following three functions:

$$f(x) = x^{-2} \qquad g(x) = 2x^{-2} \qquad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \to 0} f(x) = +\infty \qquad \lim_{x \to 0} g(x) = +\infty \qquad \lim_{x \to 0} h(x) = +\infty.$$

Then

1. $$\lim_{x \to 0} (f(x) - g(x)) =$$
2. $$\lim_{x \to 0} (f(x) - h(x)) =$$
3. $$\lim_{x \to 0} (g(x) - h(x)) =$$

Example

Consider the following three functions:

$$f(x) = x^{-2} \qquad g(x) = 2x^{-2} \qquad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x\to 0} f(x) = +\infty \qquad \lim_{x\to 0} g(x) = +\infty \qquad \lim_{x\to 0} h(x) = +\infty.$$

Then

1. $$\lim_{x\to 0}(f(x) - g(x)) = \lim_{x\to 0} x^{-2} = \infty$$
2. $$\lim_{x\to 0}(f(x) - h(x)) = \lim_{x\to 0}(1) = 1$$
3. $$\lim_{x\to 0}(g(x) - h(x)) = \lim_{x\to 0} x^{-2} + 1 = \infty$$

Outline For the Session Five

- Limit at Infinity
- Continuity
- Continuous from the left and from the right
- Arithmetic of continuity
- continuity of composites
- Intermediate Value Theorem

Example

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{1}{x^2}$	100	10000	10^6		10^6	10000	100

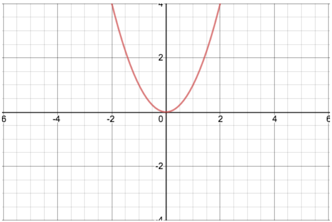
Consider that if

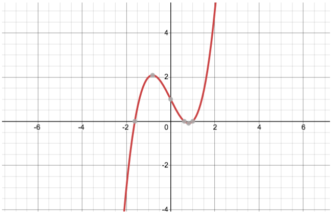
$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a} g(x) = \infty$$

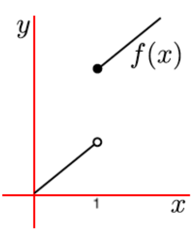
Then

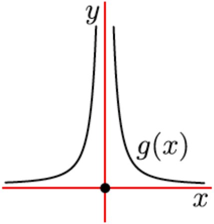
$$\lim_{x\to a} (f(x) - g(x)) = \text{undetermined}$$

Continuity

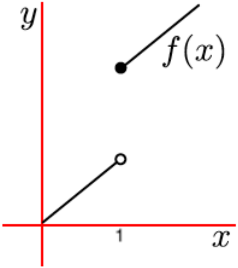

2
-2
-4
-2
0
2
4


4
2
-2
-4
-6
-4
-2
0
2
4
6


y
$f(x)$
1
x

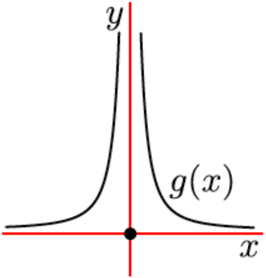

y
$g(x)$
x

$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$



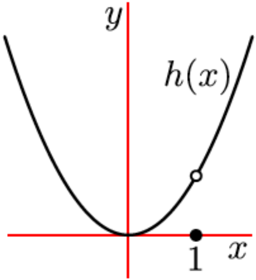
jump discontinuity

$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



infinite discontinuity

$$h(x) = \begin{cases} \frac{x^3 - x^2}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$



removable discontinuity

Outline - September 16, 2019

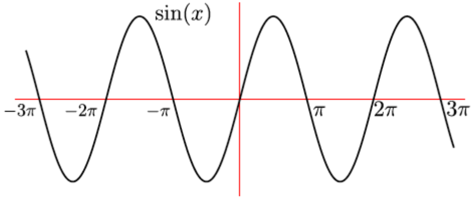
- **Section 1.6:**
  - Arithmetic of continuity
  - Continuity of composites
  - Intermediate Value Theorem
- **Section 2.1:**
  - Revisiting tangent lines

Arithmetic of continuity

Theorem

(**Arithmetic of continuity**) *Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at a. Then the following functions are also continuous at $x = a$.*

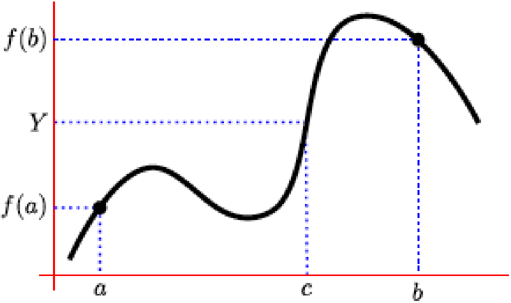
- $f(x) + g(x)$ *and* $f(x) - g(x)$,
- $cf(x)$ *and* $f(x)g(x)$, *and*
- $\frac{f(x)}{g(x)}$ *provided* $g(a) \neq 0$.


sin(x)
−3π
−2π
−π
π
2π
3π

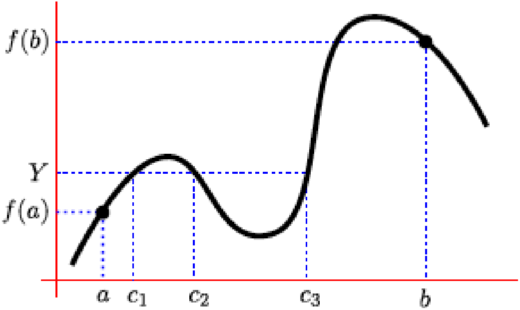
Intermediate value theorem(IVT)

Theorem

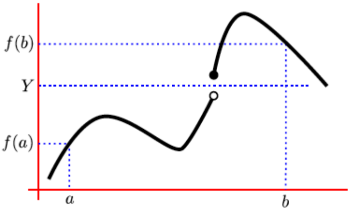
(Intermediate value theorem(IVT))

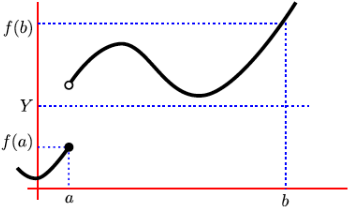


The existence not the uniqueness of c in IVT



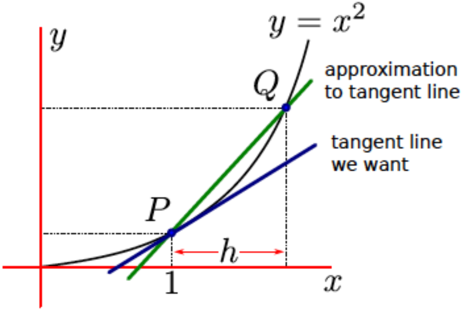
Not continuous functions at $[a, b]$ do not satisfy IVT





Revisiting tangent lines

Revisiting tangent lines



$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \leftarrow \quad \text{slope of the tangent line at } x = 1$$

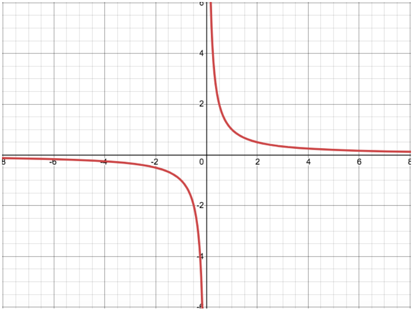
Definition of the derivative

- $$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
- $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

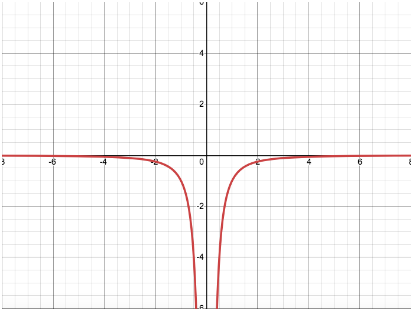
Examples

- $f(x) = c$
- $f(x) = x$
- $f(x) = x^2$
- $f(x) = \frac{1}{x}$
- $f(x) = \sqrt{x}$
- $f(x) = |x|$

$y = \frac{1}{x}$ and its derivative $-\frac{1}{x^2}$

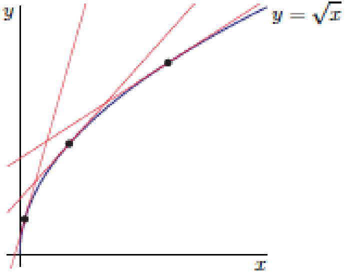


$$y = \frac{1}{x}$$

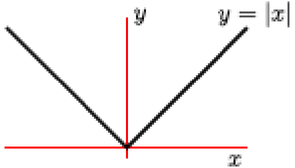


$$y = \frac{-1}{x^2}$$

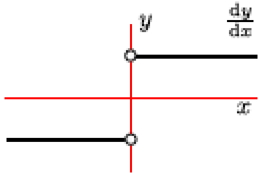
Tangent lines to $y = \sqrt{x}$



The derivative of the function $f(x) = |x|$: not differentiable at $x = 0$

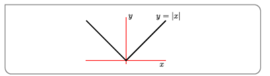


The derivative of the function $f(x) = |x|$

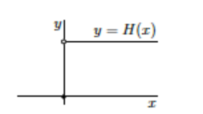


Where a function is not differentiable at $x = a$?

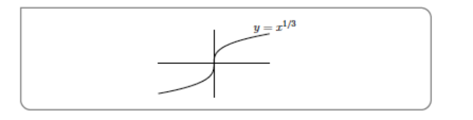
- Having a Sharp Corner at $x = a$



- The function is not continuous at $x = a$



- Having a tangent line, but the slope of the tangent line at $x = a$ is infinity

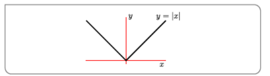


Outline - September 20, 2019

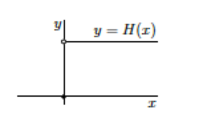
- **Section 2.2:**
  - Not differentiable examples
  - The relation between continuous and differentiable functions
- **Section 2.3:**
  - Interpretations of the derivative

Where a function is not differentiable at $x = a$?

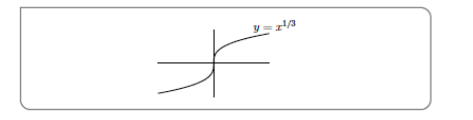
- Having a Sharp Corner at $x = a$



- The function is not continuous at $x = a$

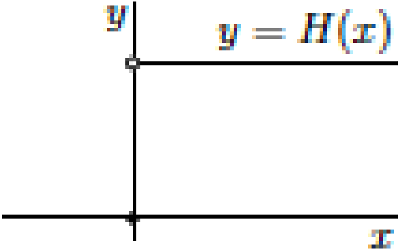


- Having a tangent line, but the slope of the tangent line at $x = a$ is infinity



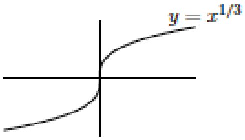
An example of a discontinuous and not differentiable function

$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$



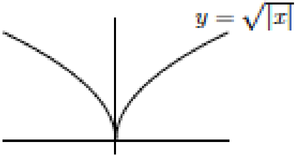
An example of a function with a tangent line with slope infinity at $x = 0$

$f(x) = x^{1/3}$

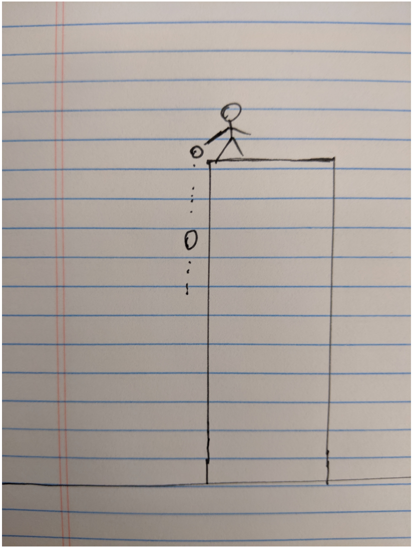


An example of a continuous and **not** differentiable function

$y = \sqrt{|x|}$



Instantaneous rate of change



average rate of change of $f(t)$ from $t = a$ to $t = a + h$ is

$$\frac{\text{change in } f(t) \text{ from } t = a \text{ to } t = a + h}{\text{length of time from } t = a \text{ to } t = a + h}$$

$$= \frac{f(a + h) - f(a)}{h}.$$

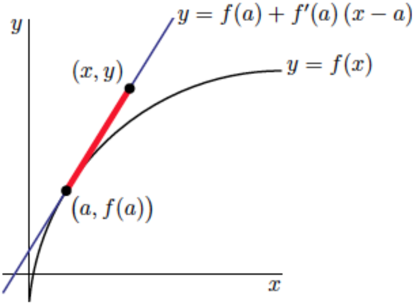
And so

instantaneous rate of change of $f(t)$ at $t = a$

$$= \lim_{h \to 0} [\text{average rate of change of } f(t) \text{ from } t = a \text{ to } t = a + h]$$

$$= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a).$$

Finding tangent line to a curve at $x = a$



A line segment of a tangent line

$$y = f(a) + f'(a)(x - a)$$

Outline - September 23, 2019

- **Section 2.4 and 2.5:**
  - Derivative of some simple functions
  - Tools
  - Examples

A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \qquad \frac{d}{dx}x = 1 \qquad \frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \qquad \frac{d}{dx}x = 1 \qquad \frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Tools

Let $f(x)$ and $g(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- $\frac{d}{dx}\{f(x) + g(x)\} = f'(x) + g'(x)$
- $\frac{d}{dx}\{f(x) - g(x)\} = f'(x) - g'(x)$
- $\frac{d}{dx}\{cf(x)\} = cf'(x)$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- $\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x)-g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- $\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$
- $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- $\frac{d}{dx}\{\frac{1}{g(x)}\} = \frac{-g'(x)}{g(x)^2} \qquad g(x) \neq 0$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- $\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$
- $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- $\frac{d}{dx}\{\frac{1}{g(x)}\} = \frac{-g'(x)}{g(x)^2} \qquad\qquad g(x) \neq 0$
- $\frac{d}{dx}\{f(x)g(x)h(x)\} =$ $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- $\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x)$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

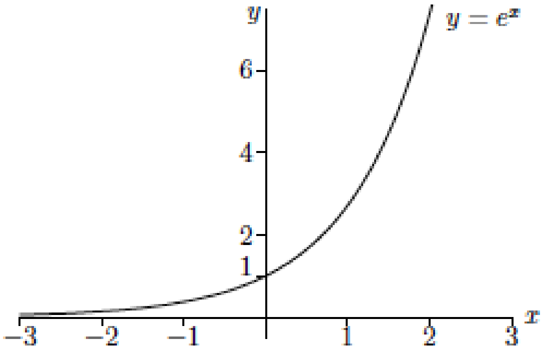
- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- $\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x)-g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$
- $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- $\frac{d}{dx}\{\frac{1}{g(x)}\} = \frac{-g'(x)}{g(x)^2} \qquad g(x) \neq 0$
- $\frac{d}{dx}\{f(x)g(x)h(x)\} =$
  $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- $\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x)$
- Let a be a rational number, then

$$\frac{d}{dx}x^a = ax^{a-1}.$$

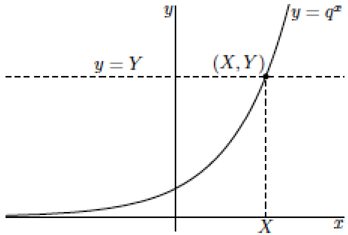
Outline - September 25, 2019

- **Section 2.7 and 2.8:**
  - Derivative of exponential functions
  - Derivative of trigonometric functions

The graph of e^x



The graph of q^x where $q > 1$



YOUR TURN!

Example

Find a such that the following function is continuous.

$$f(x) = \begin{cases} e^{x+a} & x < 0 \\ \sqrt{x+1} & x \geq 0 \end{cases}$$

Example

We have

1. $\log_q(xy) =$
   (a) $\log_q(x) + \log_q(y)$
   (b) $\log_q(x)\log_q(y)$
2. $\log_q(x/y) =$
3. $\log_q(x^r) =$

Example

We have

1. $\log_q(xy) = \log_q(x) + \log_q(y)$.
   The reason for this is that
   $$q^{\log_q(xy)} = xy = q^{\log_q(x)} q^{\log_q(y)} = q^{\log_q(x)+\log_q(y)}$$
   Therefore, $\log_q(xy) = \log(x) + \log(y)$.
2. $\log_q(x/y) = \log_q(x) - \log_q(y)$
3. $\log_q(x^r) = r \log_q(x)$

TOOLS:

$$\frac{d}{dx}(f \circ g)(x) = g'(x)f'(g(x))$$

A list of derivative of some simple functions:

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = (\log_e a)a^x$$

Example

Find the derivative of $2^{\sqrt{x}}$.

Example

Find the derivative of $2^{\sqrt{x}}$.

Example

Find a and b such that the following function is differentiable.

$$f(x) = \begin{cases} x^3 + a & x < 1 \\ e^{x-1} + bx & x \geq 1 \end{cases}$$

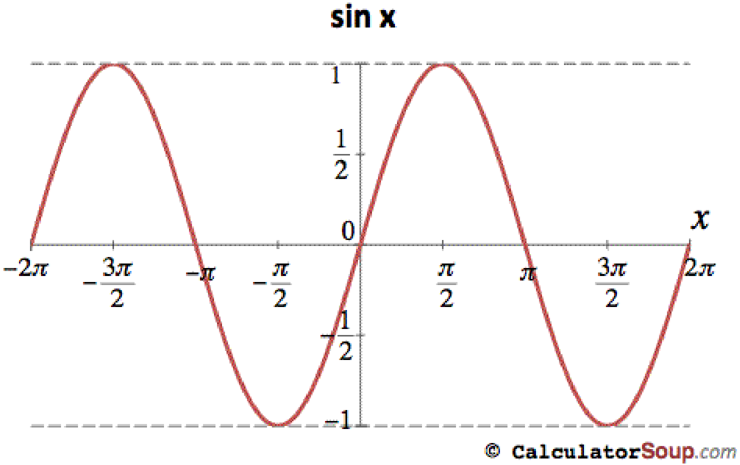
Outline - September 30, 2019

- **Section 2.8, 2.9, 0.6:**
  - Derivative of trigonometric functions
  - The chain rule
  - inverse of a function

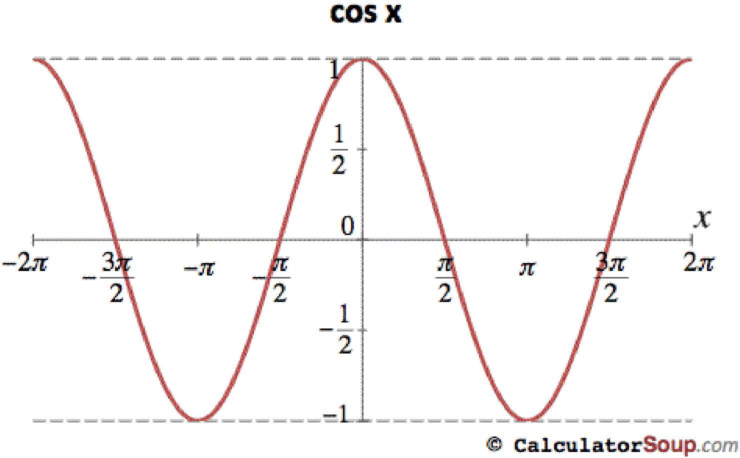
A list of derivative of some simple functions:

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = (\log_e a)a^x$$

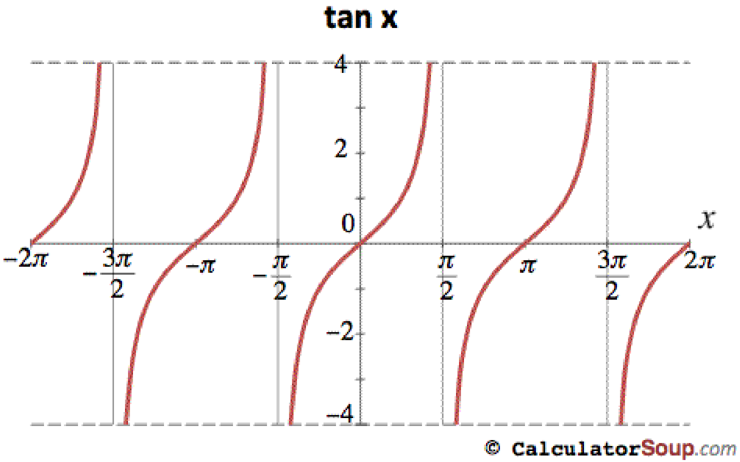
$\sin(x) \qquad \text{domain} = \mathbb{R} \qquad \text{range} = [-1, 1]$



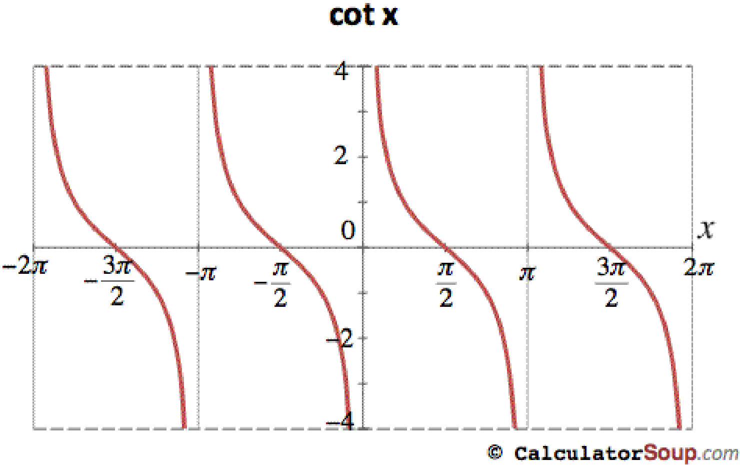
$\cos(x) \qquad \text{domain} = \mathbb{R} \qquad \text{range} = [-1, 1]$



$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \text{domain} = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R}$$

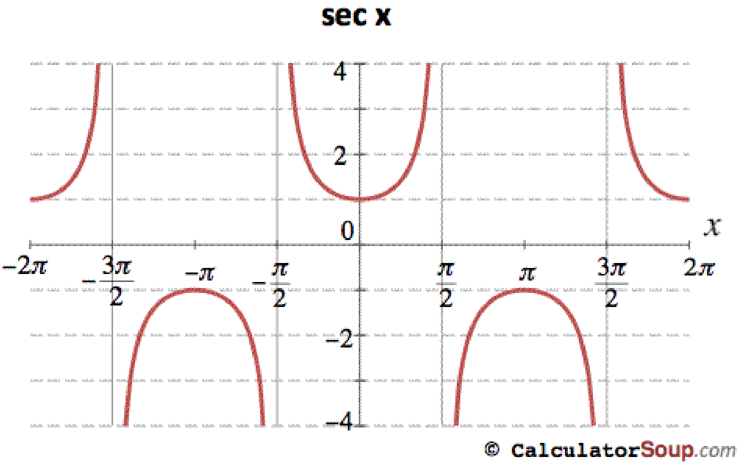


$$\cot(x) = \frac{\cos(x)}{\sin(x)} \quad \text{domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R}$$

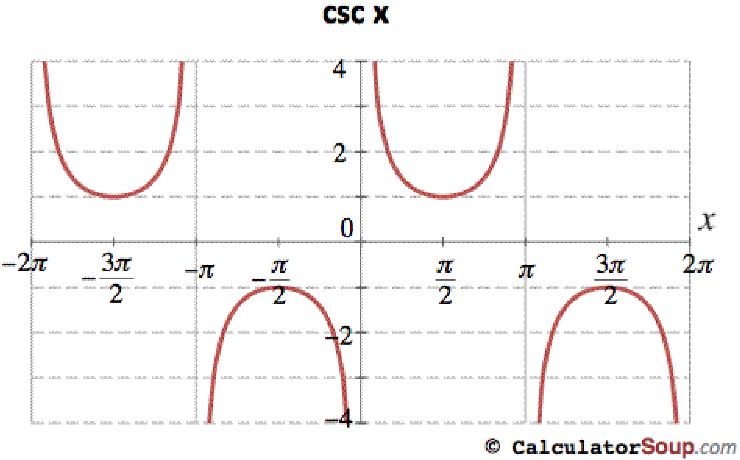


$$\sec(x) = \frac{1}{\cos(x)} \quad \text{domain} = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$$

$$\text{range} = \mathbb{R} - (-1, 1)$$



$$\csc(x) = \frac{1}{\sin(x)} \quad \text{domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R} - (-1, 1)$$



Derivative of $\sin(x)$

Question: Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of $\sin(x)$ at $x = 0$.

Derivative of $\sin(x)$

Question: Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of $\sin(x)$ at $x = 0$.

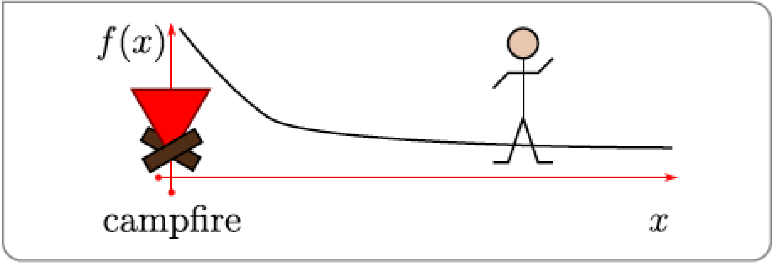
(**sandwich (or squeeze or pinch) theorem**) Let $a \in \mathbb{R}$ and let f, g, h be three functions so that $f(x) \leq g(x) \leq h(x)$ for all x in an interval around a, except possibly at $x = a$. Then if

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then it is also the case that

$$\lim_{x \to a} g(x) = L.$$

An example of the application of the chain rule



- Your position at time t is $x(t)$.
- The temperature of the air at position x is $f(x)$.
- The temperature that you feel at time t is $F(t) = f(x(t))$.
- The instantaneous rate of change of temperature that you feel is $F'(t)$.

The chain rule

Theorem

Let f and g be differentiable functions then

$$\frac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$$

The chain rule

Theorem

Let $y = f(u)$ and $u = g(x)$ be differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

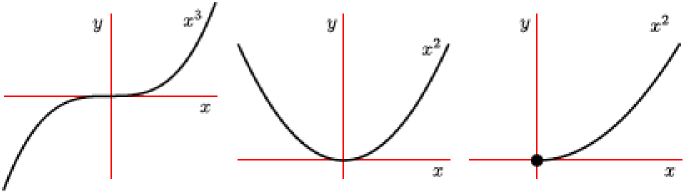
Outline - October 2, 2019

- **Section 0.6, 2.10:**
  - Inverse of a function
  - Natural logarithm

input number $x \mapsto f$ does "stuff" to $x \mapsto$ return number y

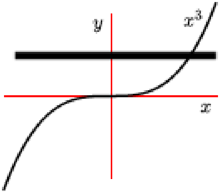
take output $y \mapsto$ do "stuff" to $y \mapsto$ return the original number x

One-to-one functions



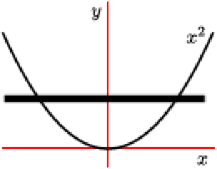
$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array} \qquad \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array} \qquad \begin{array}{ccc} [0, \infty] & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

One-to-one functions



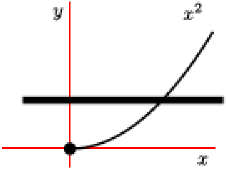
$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

is one-to-one



$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

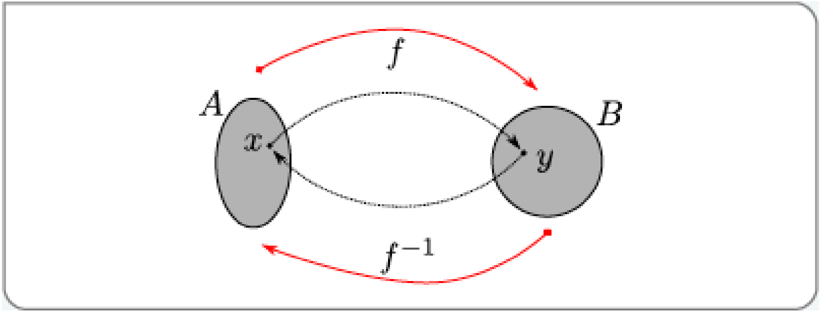
is not one-to-one



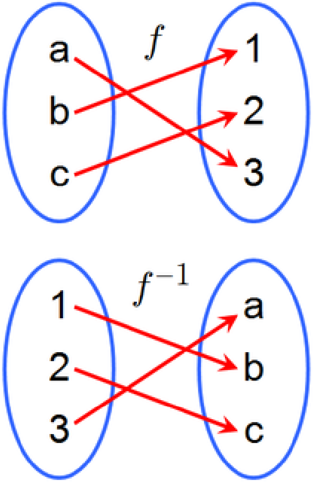
$$\begin{array}{ccc} [0, \infty) & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is one-to-one

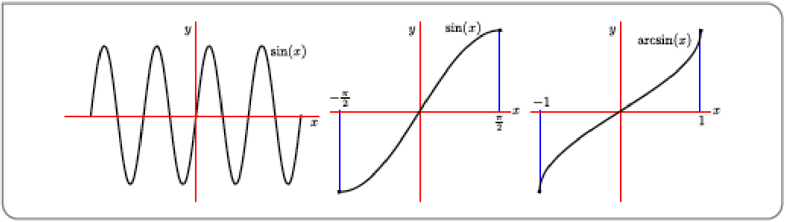
Inverse of a functions



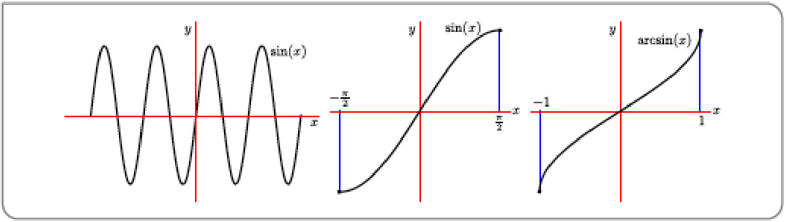
Inverse of a functions



Inverse of $\sin(x)$

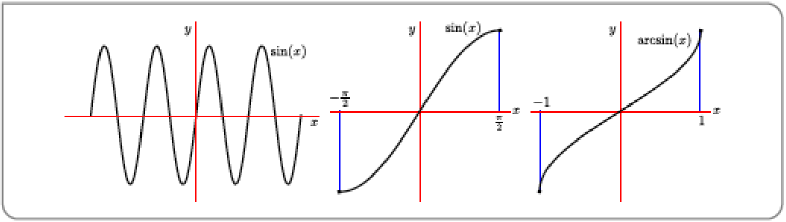


Inverse of $\sin(x)$



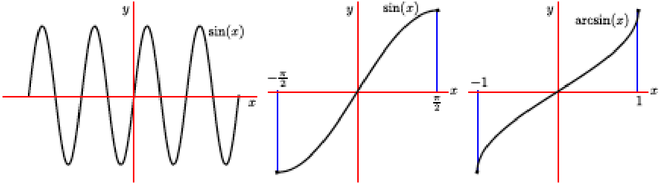
- $\sin(x)$ is not invertible on the domain $\mathbb{R}$ because it is not one-to-one.

Inverse of $\sin(x)$



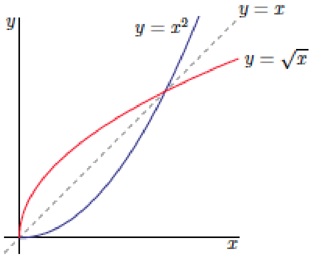
- $\sin(x)$ is not invertible on the domain $\mathbb{R}$ because it is not one-to-one.
- If we look at $\sin(x)$ on the domain $[-\pi/2, \pi/2]$, then it is one-to-one, and so it is has an inverse.

Inverse of sin(x)



- sin(x) is not invertible on the domain $\mathbb{R}$ because it is not one-to-one.
- If we look at sin(x) on the domain $[-\pi/2, \pi/2]$, then it is one-to-one, and so it is has an inverse.
- The inverse of sin(x) is arcsin(x) on the domain $[-1, 1]$ and with the range $[-\pi/2, \pi/2]$.

How to find the inverse of a function by its graph



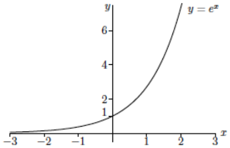
$a^{\log_a x} = x$

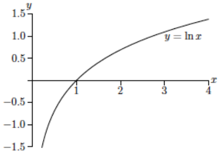
Remember that for $a > 1$,

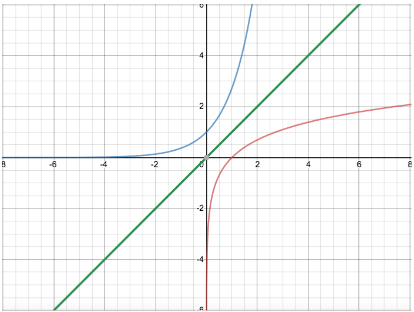
$$a^{\log_a x} = x,$$

$$\log_a x = \frac{\log_e x}{\log_e a}.$$

The inverse of e^x







Outline - October 4, 2019

- **Section 2.10 and 2.11:**
  - Natural logarithm
  - Implicit derivative

Useful facts!

- $\frac{d}{dx} a^x = (\ln a) a^x.$
- $\log_a x = \frac{\ln x}{\ln a} \qquad \ln x = \frac{\log_a x}{\log_a e} \qquad a > 1.$
- $\ln(xy) = \ln x + \ln y.$
- $\ln(x/y) = \ln x - \ln y.$
- $\ln x^r = r \ln x.$

Useful facts!

- $\frac{d}{dx} a^x = (\ln a) a^x.$
- $\log_a x = \frac{\ln x}{\ln a} \qquad \ln x = \frac{\log_a x}{\log_a e} \qquad a > 1.$
- $\ln(xy) = \ln x + \ln y.$
- $\ln(x/y) = \ln x - \ln y.$
- $\ln x^r = r \ln x.$
- $\frac{d}{dx} \ln x = \frac{1}{x}.$

Useful facts!

- $\frac{d}{dx} a^x = (\ln a) a^x$.
- $\log_a x = \frac{\ln x}{\ln a}$ $\qquad \ln x = \frac{\log_a x}{\log_a e}$ $\qquad a > 1$.
- $\ln(xy) = \ln x + \ln y$.
- $\ln(x/y) = \ln x - \ln y$.
- $\ln x^r = r \ln x$.
- $\frac{d}{dx} \ln x = \frac{1}{x}$.
- $\frac{d}{dx} \ln |x| = \frac{1}{x}$.

Useful facts!

- $\frac{d}{dx}a^x = (\ln a)a^x$.
- $\log_a x = \frac{\ln x}{\ln a}$ $\qquad \ln x = \frac{\log_a x}{\log_a e}$ $\qquad a > 1$.
- $\ln(xy) = \ln x + \ln y$.
- $\ln(x/y) = \ln x - \ln y$.
- $\ln x^r = r \ln x$.
- $\frac{d}{dx} \ln x = \frac{1}{x}$.
- $\frac{d}{dx} \ln |x| = \frac{1}{x}$.
- $\frac{d}{dx} \log_a x = \frac{1}{x. \ln a}$.

Useful facts!

- $\frac{d}{dx} a^x = (\ln a) a^x$.
- $\log_a x = \frac{\ln x}{\ln a} \qquad \ln x = \frac{\log_a x}{\log_a e} \qquad a > 1.$
- $\ln(xy) = \ln x + \ln y$.
- $\ln(x/y) = \ln x - \ln y$.
- $\ln x^r = r \ln x$.
- $\frac{d}{dx} \ln x = \frac{1}{x}$.
- $\frac{d}{dx} \ln |x| = \frac{1}{x}$.
- $\frac{d}{dx} \log_a x = \frac{1}{x . \ln a}$.
- $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$

Useful facts!

- $\frac{d}{dx} a^x = (\ln a) a^x$.
- $\log_a x = \frac{\ln x}{\ln a} \qquad \ln x = \frac{\log_a x}{\log_a e} \qquad a > 1.$
- $\ln(xy) = \ln x + \ln y$.
- $\ln(x/y) = \ln x - \ln y$.
- $\ln x^r = r \ln x$.
- $\frac{d}{dx} \ln x = \frac{1}{x}$.
- $\frac{d}{dx} \ln |x| = \frac{1}{x}$.
- $\frac{d}{dx} \log_a x = \frac{1}{x. \ln a}$.
- $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$
- $\frac{d}{dx} |f(x)| = \frac{f'(x)}{f(x)}$.

Outline - October 7, 2019

- **Section 2.11 and 2.12:**
  - Implicit derivative
  - Derivative of Trig functions

Implicit derivative

$$\frac{d}{dx}x = \frac{d}{dx}e^{\ln x} \qquad\qquad (\frac{d}{dx}x = \frac{d}{dx}e^{y})$$

which is the same as

$$1 = (\frac{d}{dx}\ln x).e^{\ln x} \qquad\qquad (1 = y'e^{y}).$$

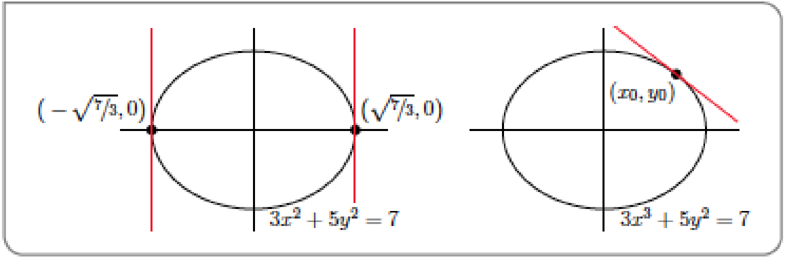
Note that $e^{\ln x} = x (e^{y} = x)$, thus

$$1 = (\frac{d}{dx}\ln x).x \qquad\qquad (1 = y'x)$$

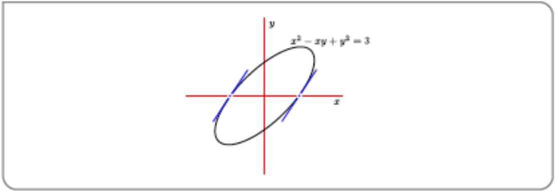
and so

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad\qquad (y' = \frac{1}{x}).$$

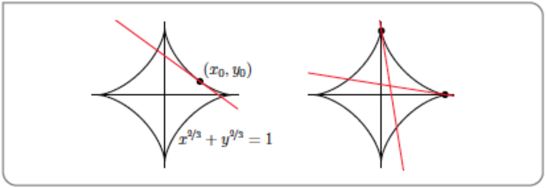
$3x^3 + 5y^2 = 7$



$x^2 - xy + y^2 = 3$



$x^{2/3} + y^{2/3} = 1$



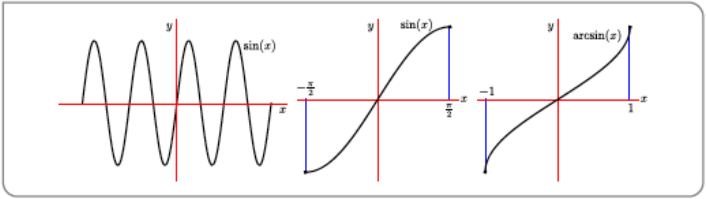
Outline - October 9, 2019

- **Section 2.12:**
  - Derivative of Trig functions

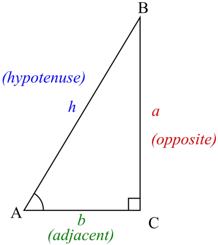
Review of the inverse of a function

Remember that the inverse of a one-to-one function $f(x)$ with domain A and range B is a function $g(x)$ with domain B and range A such that

$$f(g(y)) = y \qquad g(f(x)) = x \qquad x \in A, y \in B.$$



Trigonometry



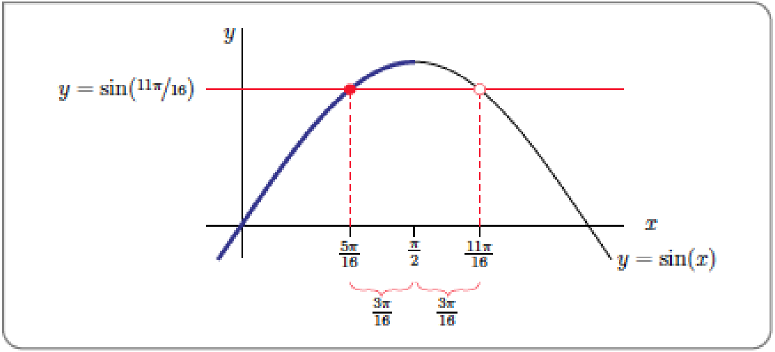
- **sine:** $\sin A = \dfrac{a}{h} = \dfrac{\text{opposite}}{\text{hypotenuse}}$
- **cosine:** $\cos A = \dfrac{b}{h} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
- **tangent:** $\tan A = \dfrac{a}{b} = \dfrac{\text{opposite}}{\text{adjacent}}$
- **cosecant:** $\csc A = \dfrac{h}{a} = \dfrac{\text{hypotenuse}}{\text{opposite}}$
- **secant:** $\sec A = \dfrac{h}{b} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$
- **cotangent:** $\cot A = \dfrac{b}{a} = \dfrac{\text{adjacent}}{\text{opposite}}$

arcsin(sin(x))

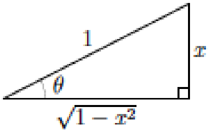
$$\arcsin(\sin(x)) = \text{the unique angle } \theta \text{ between } -\pi/2 \text{ and } \pi/2 \text{ obeying that}$$

$$\sin(x) = \sin(\theta).$$

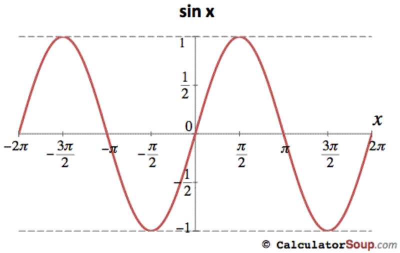
What is $\arcsin(\sin(\frac{11\pi}{16}))$?

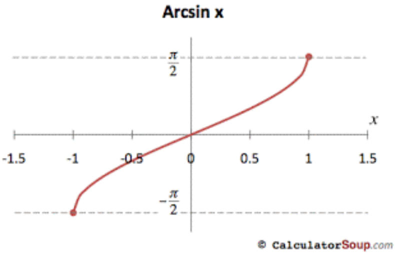


$\cos(\arcsin(x)) = \sqrt{1 - x^2}$

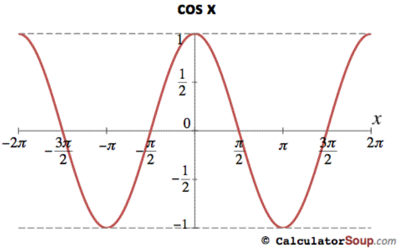


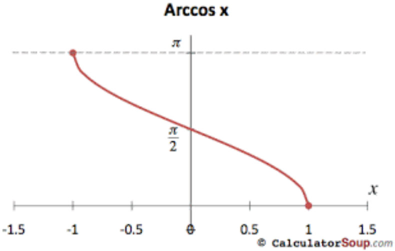
Inverse of sin(x)



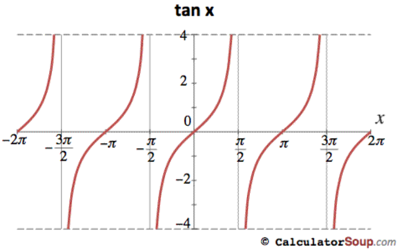


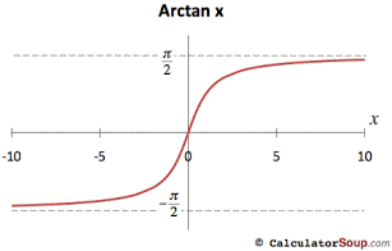
Inverse of $\cos(x)$



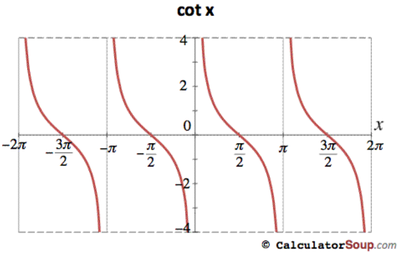


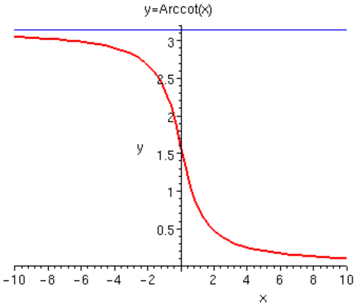
Inverse of $\tan(x)$





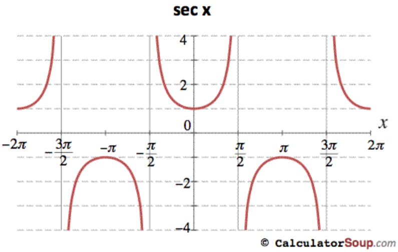
Inverse of $\cot\text{an}(x)$

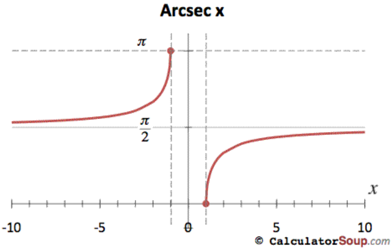




Inverse of sec(x)

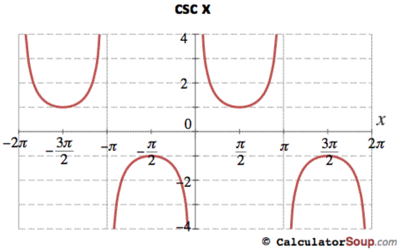
$$\text{arcsec}(x) = \arccos(1/x)$$

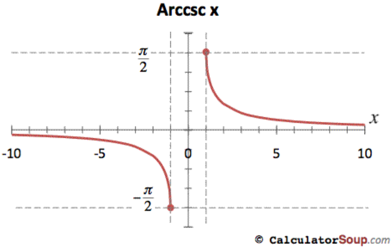




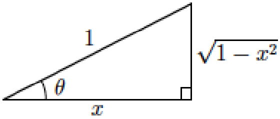
Inverse of $\csc(x)$

$$\operatorname{arccsc}(x) = \arcsin(1/x)$$

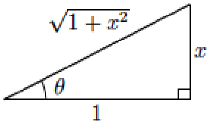




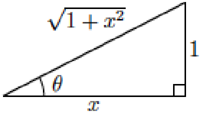
$\sin(\theta) = \sin(\arccos(x)) = \sqrt{1 - x^2}$



$$\cos^2(\arctan(x)) = \cos^2(\theta) = \frac{1}{1+x^2}.$$



$$\frac{1}{\csc^2(\theta)} = \sin^2(\theta) = 1 + x^2$$

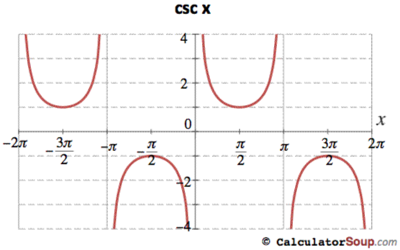


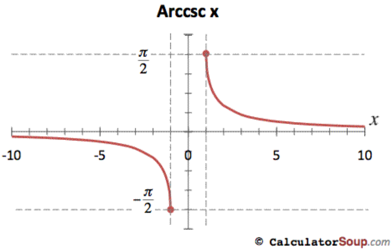
Outline - October 11, 2019

- **Section 3.1:**
  - Derivative of Trig functions

Inverse of $\csc(x)$

$$\operatorname{arccsc}(x) = \arcsin(1/x)$$





Derivative of the inverses of trigonometric functions in a nutshell

In a nutshell the derivatives of the inverse trigonometric functions are

$$\frac{d}{dx}\,\text{arcsin}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{arccsc}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\,\text{arccos}(x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\,\text{arctan}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arccot}(x) = -\frac{1}{1+x^2}$$

The Application of Derivatives

Velocity and Acceleration

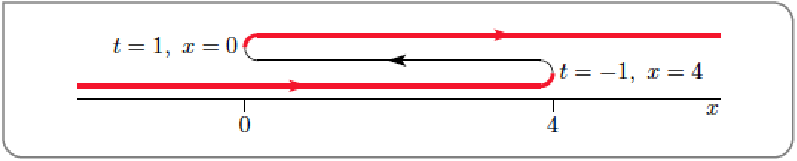
If you are moving along the x-axis and your position at time t is $x(t)$, then

- your velocity at time t is $v(t) = x'(t)$ and
- your acceleration at time t is $a(t) = v'(t) = x''(t)$.

Direction of your move with $x(t) = t^3 - 3t + 2$

t	$(t-1)(t+1)$	$x'(t) = 3(t-1)(t+1)$	Direction
$t < -1$	*positive*	*positive*	*right*
$t = -1$	*zero*	*zero*	*halt*
$-1 < t < 1$	*negative*	*negative*	*left*
$t = 1$	*zero*	*zero*	*halt*
$t > 1$	*positive*	*positive*	*right*

And here is a schematic picture of the whole trajectory.



Direction of your move with $x(t) = t^3 - 12t + 5$

t	$(t-2)(t+2)$	$x'(t) = 3(t-2)(t+2)$	Direction
$t < -2$	*positive*	*positive*	*right*
$t = -2$	*zero*	*zero*	*halt*
$-2 < t < 2$	*negative*	*negative*	*left*
$t = 2$	*zero*	*zero*	*halt*
$t > 2$	*positive*	*positive*	*right*

t	*your position* $x(t)$	$x'(t)$	Direction
0	5	*negative*	*left*
$t = 2$	-11	*zero*	*halt*
$t = 10$	885	*positive*	*right*

Outline - October 16, 2019

- **Section 3.2: Exponential Growth and Decay**
  - 3.1: Carbon Dating

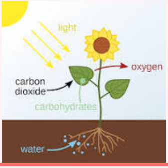
EXAM: Friday, October 18, Here in Class, at 2pm

Carbon Dating

Cosmic ray hitting atmosphere

$Nitrogen\ (N) \rightarrow Carbon(C)$

Vegetation absorbs C through photosynth



Animals acquire C by eating plants

C decreases when animal dies

More precisely, let $Q(t)$ denote the amount of C (an element) in the plant or animal t years after it dies. The number of radioactive decays (rate of change) per unit time, at time t, is proportional to the amount of C present at time t, which is $Q(t)$. Thus

Radioactive Decay

$$\frac{dQ}{dt}(t) = -kQ(t) \tag{1}$$

Corollary

The function $Q(t)$ satisfies the equation

$$\frac{dQ}{dt} = -kQ(t)$$

if and only if

$$Q(t) = Q(0).e^{-kt}$$

The half-life (the half-life of C is the length of time that it takes for half of the C to decay) is defined to be the time $t_{1/2}$ which obeys

$$Q(t_{1/2}) = \frac{1}{2}.Q(0).$$

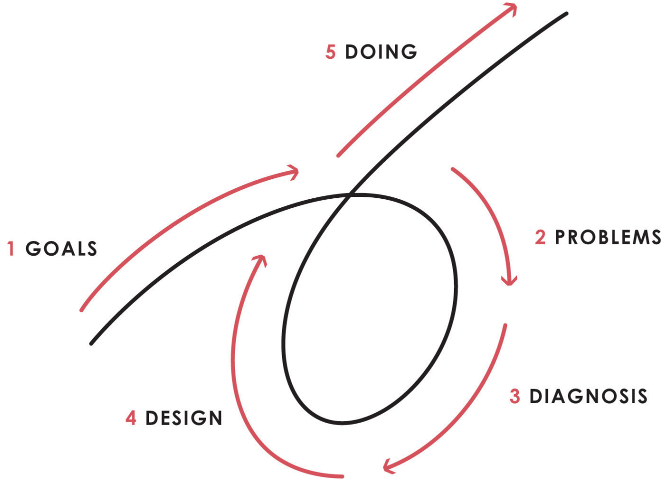
The half-life is related to the constant k by

$$t_{1/2} = \frac{\ln 2}{k}.$$

Outline - October 21, 2019

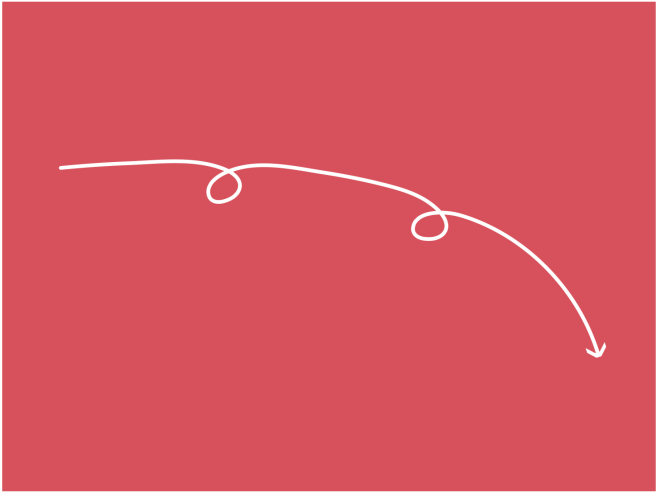
- **Section 3.3.2: Newton's Law of Cooling**
  - 3.1: Newton's Law of Cooling

No pain no gain

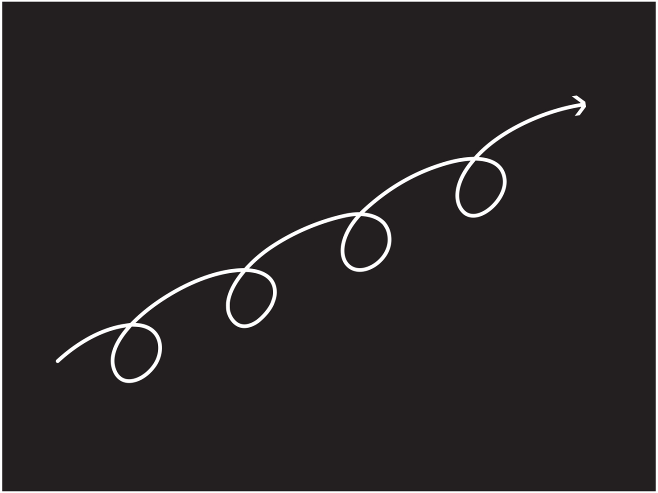


Principles (Ray Dalio)

Most people



Successful person



Newton's Law of Cooling

$$\frac{dT}{dt}(t) = K\left[T(t) - A\right].$$

We have three possibilities:

- $T(t) > A \Rightarrow [T(t) - A] > 0$, thus the temperature of the body is decreasing, so $\frac{dT}{dt}$ must be negative, since $\frac{dT}{dt}(t) = K\left[T(t) - A\right]$, we must have $K < 0$.

- $T(t) < A \Rightarrow [T(t) - A] < 0$, thus the temperature of the body is increasing, so $\frac{dT}{dt}$ must be positive, since $\frac{dT}{dt}(t) = K\left[T(t) - A\right]$, we must have $K < 0$.

- $T(t) = A \Rightarrow [T(t) - A] = 0$, thus the temperature of the body is no changing, so $\frac{dT}{dt}$ must be zero, since $\frac{dT}{dt}(t) = K\left[T(t) - A\right]$. This does not impose any condition on K.

Newton's Law of Cooling

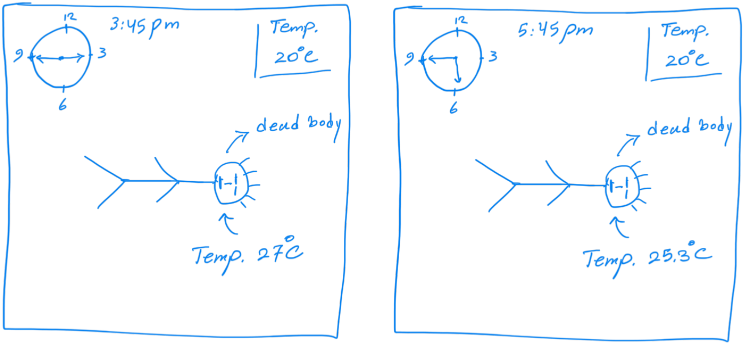
Corollary

A differentiable function $T(t)$ obeys the differential equation

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

if and only if

$$T(t) = [T(0) - A]e^{Kt} + A.$$



I entered this room at 9:30 am and talk to him for 5 minutes.

Outline - October 23, 2019

- **Section 3.3.3: Population Growth**
- **Section 3.2: Related Rates**

Population Growth

Suppose that we wish to predict the size $P(t)$ of a population as a function of the time t. So suppose that in average each couple produces β offspring (for some constant β) and then dies. Then over the course of one generation since we have $P(t)/2$ couples and each have produced β offspring, thus the population of the children of one generation is

$$\beta\frac{P(t)}{2}.$$

Let t_g be the life span of one generation, then

$$\begin{aligned} P(t+t_g) &= \beta\frac{P(t)}{2} \\ &= P(t) + \beta\frac{P(t)}{2} - P(t). \end{aligned}$$

Therefore,

$$P(t+t_g) - P(t) = \beta \frac{P(t)}{2} - P(t)$$

and so dividing both sides by t_g, we have

$$\begin{aligned} \frac{P(t+t_g) - P(t)}{t_g} &= \frac{1}{t_g}\left(\frac{\beta}{2}P(t) - P(t)\right) \\ &= \frac{1}{t_g}\left(\frac{\beta}{2} - 1\right)P(t) \end{aligned}$$

Let $\frac{1}{t_g}\left(\frac{\beta}{2} - 1\right) = b$, then

$$\frac{P(t+t_g) - P(t)}{t_g} = bP(t).$$

Approximately, we have

$$\frac{dP}{dt} = bP(t).$$

Moreover, same as the model for carbon dating we can write

$$P(t) = P(0)e^{bt}.$$

Malthusian growth model

Malthusian growth model

The model for the population growth is

$$\frac{dP}{dt} = bP(t)$$

and $P(t)$ satisfies the above equation if and only if

$$P(t) = P(0)e^{bt}.$$

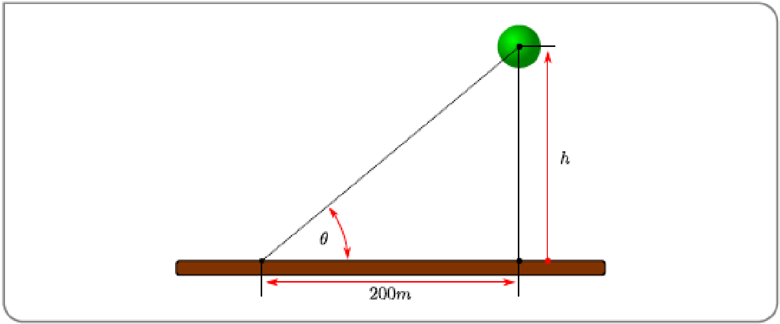
Related Rates

Volume of a sphere

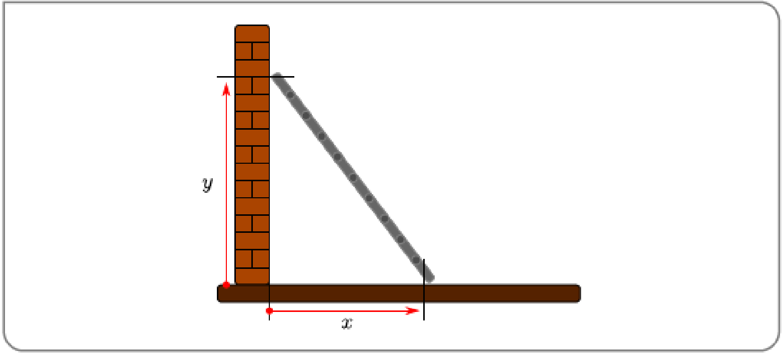
Remember that the volume of a sphere with radius r is

$$V = \frac{4}{3}\pi r^3.$$

Helium Balloon



Ladder

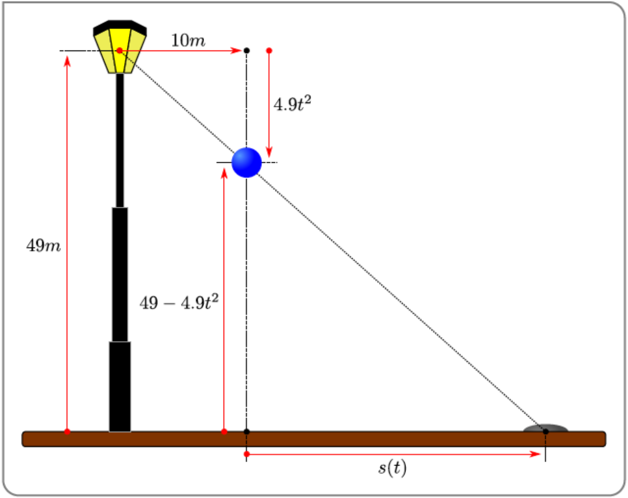


Outline - October 25, 2019

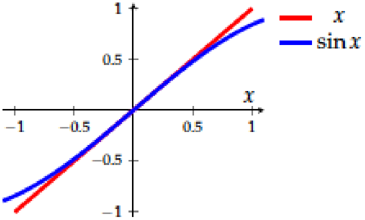
- **Section 3.2: Related Rates: An Example**
- **Section 3.4.2 The Linear Approximation**
- **Section 3.4.3 The Quadratic Approximation**

Shadow of the Ball

Similar triangles-ratio



Approximation

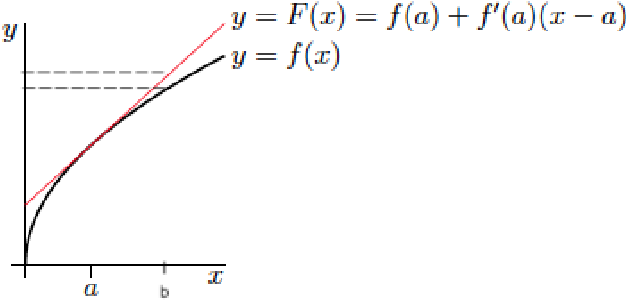


This figure shows that the curve $y = x$ and $y = \sin(x)$ are almost the same when x is close to 0. Hence if we want the value of $\sin(1/10)$ we just use this approximation $y = x$ to get

$$\sin(1/10) \approx 1/10.$$

The linear approximation

Given a function $f(x)$ we want to have the approximating function to be a linear function that is $F(x) = A + Bx$ for some constants A and B.



The linear approximation

$$f(x) \approx F(x) = f(a) + f'(a)(x - a)$$

Example

Estimate $e^{0.01}$? So $f(x) = e^x$ and $a = 0$.

The quadratic approximation

In linear approximation we had

$$f(x) \approx F(x) = f(a) + f'(a)(x-a) \Rightarrow$$

$$f(a) = F(a) \quad \text{and} \quad f'(a) = F'(a).$$

We now want our approximation function to be a quadratic function of x, that is, $F(x) = A + Bx + Cx^2$. To have a good approximating function we choose A, B, and C so that

- $f(a) = F(a)$
- $f'(a) = F'(a)$
- $f''(a) = F''(a)$

These conditions give us the following equations

$$F(x) = A + Bx + Cx^2 \quad \Rightarrow \quad F(a) = A + Ba + Ca^2 = f(a)$$

$$F'(x) = B + 2Cx \quad \Rightarrow \quad F'(a) = B + 2Ca = f'(a)$$

$$F''(x) = 2C \quad \Rightarrow \quad F'(a) = 2C = f''(a)$$

Solving these equation we can write A, B, and C in terms of $f(a)$, $f'(a)$, and $f''(a)$. So that

$$C = \frac{1}{2} f''(a)$$

$$B = f'(a) - af''(a)$$

$$A = f(a) - a[f'(a) - af''(a)] - \frac{1}{2} f''(a) a^2.$$

Consider that $F(x) = A + Bx + CX^2$, substituting A, B, and C, we obtain

Quadratic Approximation

$$F(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Therefore,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Outline - October 28, 2019

- **Section 3.4.3 The Quadratic Approximation**
- **Section 3.4.4 Taylor Polynomials**
- **Section 3.4.5 Some Examples**

Linear Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x - a)$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$

Linear Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x - a)$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$

Then

$$F(a) = c_0 = f(a) \qquad\qquad F'(a) = c_1 = f'(a).$$

And so

$$F(x) = f(a) + f'(a)(x - a).$$

Quadratic Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x-a) + c_2(x-a)^2$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$
3. $F''(a) = f''(a)$

Quadratic Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x - a) + c_2(x - a)^2$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$
3. $F''(a) = f''(a)$

Then

$$F(a) = c_0 = f(a) \qquad F'(a) = c_1 = f'(a) \qquad F''(a) = 2c_2 = f''(a).$$

Quadratic Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x-a) + c_2(x-a)^2$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$
3. $F''(a) = f''(a)$

Then

$$F(a) = c_0 = f(a) \qquad F'(a) = c_1 = f'(a) \qquad F''(a) = 2c_2 = f''(a).$$

And so

$$F(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Taylor Polynomial

We want to approximate $f(x)$ with a polynomial $T_n(x)$ of degree n of the form

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

such that

1. $T_n(a) = f(a)$,
2. $T_n'(a) = f'(a)$,

   $\vdots$

n. $T_n^{(n)}(a) = f^{(n)}(a)$.

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) =$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) =$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

$$T_n''(x) = 2c_2 + 3 \times 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n''(a) =$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

$$T_n''(x) = 2c_2 + 3 \times 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n''(a) = 2c_2 = f''(a)$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

$$T_n''(x) = 2c_2 + 3 \times 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n''(a) = 2c_2 = f''(a)$$

$$T_n^{(3)}(x) = 3 \times 2c_3 + 4 \times 3 \times 2c_4(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n^{(3)}(a) =$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

$$T_n''(x) = 2c_2 + 3 \times 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n''(a) = 2c_2 = f''(a)$$

$$T_n^{(3)}(x) = 3 \times 2c_3 + 4 \times 3 \times 2c_4(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n^{(3)}(a) = 6c_3 = f^{(3)}(a)$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

$$T_n''(x) = 2c_2 + 3 \times 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n''(a) = 2c_2 = f''(a)$$

$$T_n^{(3)}(x) = 3 \times 2c_3 + 4 \times 3 \times 2c_4(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n^{(3)}(a) = 6c_3 = f^{(3)}(a)$$

$$\vdots$$

$$T_n^{(n)}(x) = n!c_n \Rightarrow T_n^{(n)}(a) = n!c_n$$

Taylor Polynomial

We have

$$c_0 = f(a), c_1 = f'(a), c_2 = \frac{1}{2!}f''(a), c_3 = \frac{1}{3!}f^{(3)}(a), \ldots, c_n = \frac{1}{n!}f^{(n)}(a)$$

and

$$T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

we have that

$$f(x) \approx T_n(x) =$$

$$f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a) +$$

$$\frac{1}{3!}f^{(3)}(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Taylor Polynomial

Taylor Polynomial

Let a be a constant and let n be a non-negative integer. The nth degree Taylor polynomial for $f(x)$ about $x = a$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2$$

$$+\frac{1}{3!}f^{(3)}(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

or

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(a)(x-a)^k$$

The special case $a = 0$ is called a Maclaurin polynomial.

Outline - October 30, 2019

- **Section 3.4.5: Some Examples of Taylor Polynomial**
- **Section 3.4.8: The Error in the Taylor Polynomial Approximations**

Taylor Polynomial

Taylor Polynomial

Let a be a constant and let n be a non-negative integer. The nth degree Taylor polynomial for $f(x)$ about $x = a$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2$$

$$+\frac{1}{3!}f^{(3)}(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

or

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(a)(x-a)^k$$

The special case $a = 0$ is called a Maclaurin polynomial.

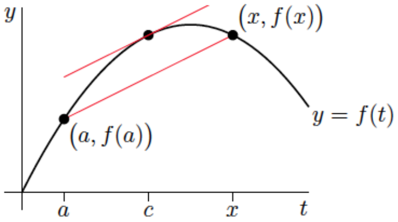
Approximating $f(x)$ by the $0th$ Taylor polynomial about $x = a$

$$f(x) \approx T_0(x) = f(a).$$

Note that

$$\begin{aligned}
f(x)& \\
&= f(x) + f(a) - f(a) \\
&= f(a) + (f(x) - f(a))\frac{(x-a)}{(x-a)} \qquad (2)\\
&= f(a) + \frac{f(x) - f(a)}{x - a}(x - a)
\end{aligned}$$

$f(x) = f(a) + \frac{f(x)-f(a)}{x-a}(x-a)$



There is c strictly between x and a such that

$$f'(c) = \frac{f(x) - f(a)}{x - a}.$$

$f(x) = f(a) + f'(c)(x-a)$ for some c strictly between a and x.

$f(x) = f(a) + f'(c)(x - a)$ for some c strictly between a and x.

$$\Rightarrow f(x) - f(a) = f'(c)(x-a) \Rightarrow f(x) - T_0(x) = f'(c)(x-a)$$

The error in constant approximation

$$R_0(x) = f(x) - T_0(x) = f'(c)(x-a)$$

for some c strictly between a and x

The error in linear approximation

$$R_1(x) = f(x) - T_1(x) = \frac{1}{2} f''(c)(x-a)^2$$

for some c strictly between a and x

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some c strictly between a and x

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some c strictly between a and x

Remark

Consider that $f(x) = R_n(x) + T_n(x)$ Therefore,

1. if $0 \leq R_n(x) \leq E$, then
$$T_n(x) \leq f(x) \leq T_n(x) + E.$$
2. if $E \leq R_n(x) \leq 0$, then
$$T_n(x) + E \leq f(x) \leq T_n(x).$$