

MATH100: Differential Calculus with Application to Physical Sciences and Engineering

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Chapter 1

Limits

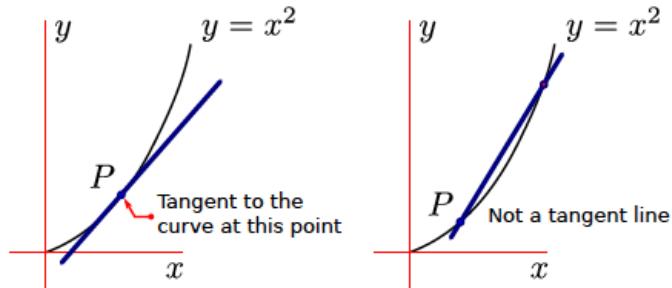
What does this mean

$$\lim_{x \rightarrow a} f(x) = L?$$

The "limit" appears when we want to

- find the tangent to a curve; or
- find the velocity of an object.

1.1 Tangent line



The **tangent line to a curve** $y = f(x)$ at a point P (if exists) is a line L that there is a neighborhood for P such that in that neighborhood the line L touches (does not cross) the curve $y = f(x)$ only at P (and not other points in that neighborhood).

The equation of a line

- The formula for a line that passes through (x_1, y_1) with slope m is

$$y = y_1 + m(x - x_1).$$

- Given two points (x_1, y_1) and (x_2, y_2) on a line, then the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

and the formula for the line then is

$$y = y_1 + m(x - x_1).$$

Example 1.1.1. Find the equation of the line with slope -3 that passes through $(1, 2)$.

Solution. The equation of the line is

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

Example 1.1.2. Find the equation of the line that passes through $(1, 2)$ and $(2, -1)$.

Solution. First we find the slope which is

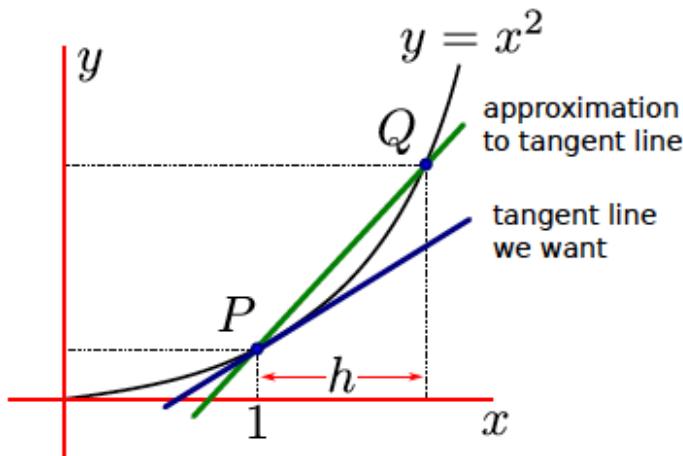
$$\frac{-1 - 2}{2 - 1} = -3.$$

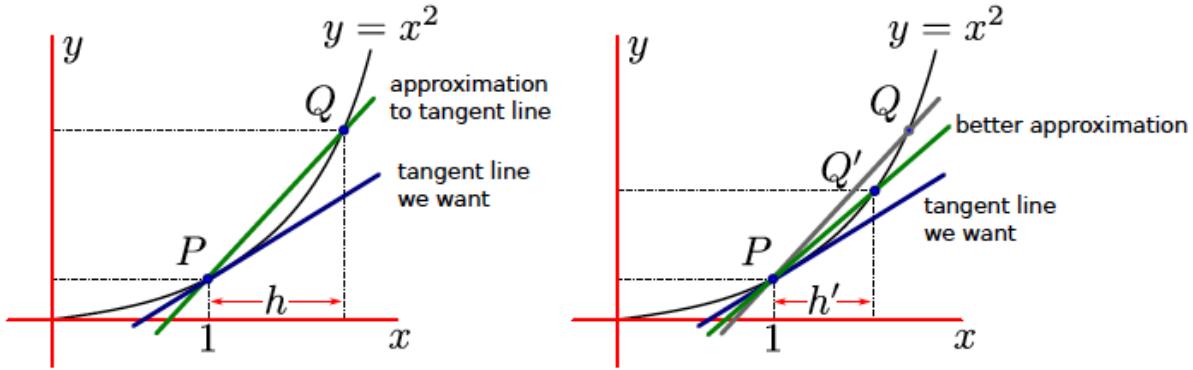
Then the equation of the line is

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

The equation of a tangent line: Given a curve $y = f(x)$ and a point P on the curve, how to find the slope of the tangent to a curve at P : let do this through an example.

Example 1.1.3. Find the tangent line to the curve $y = x^2$ that passes through $P = (1, 1)$.





So we want to find the slope the line that passes through the points $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (1 + h, (1 + h)^2)$. The slope then is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1 + h)^2 - 1^2}{(1 + h) - 1} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(h + 2)}{h} = 2 + h$$

h	$m = \frac{(1+h)^2 - 1^2}{(1+h) - 1}$
0.1	2.1
0.01	2.01
0.001	2.001

When h gets smaller and smaller, the slope will be closer and closer to the slope of the tangent line to $y = x^2$ at $(1, 1)$, which the slope will be closer and closer to 2, we can write this more mathematically as

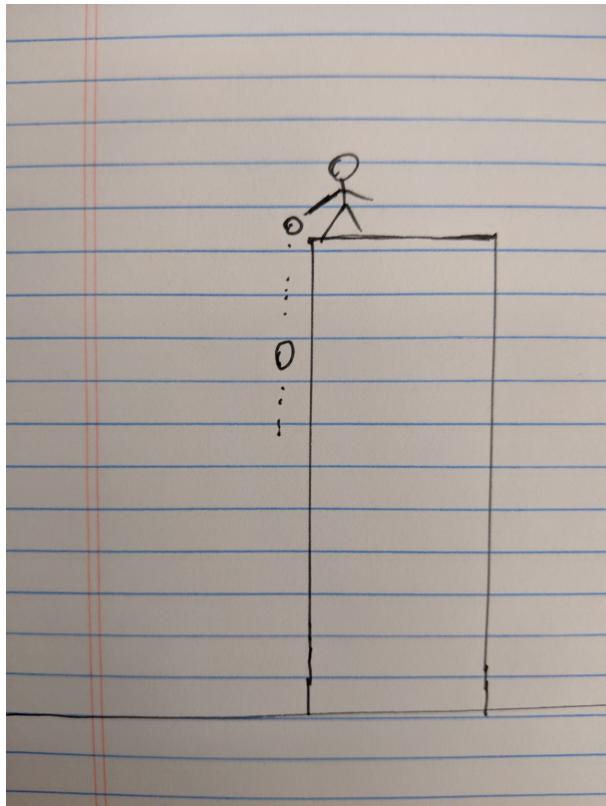
$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1^2}{(1 + h) - 1} = 2$$

Read: the limit of $\frac{(1+h)^2 - 1^2}{(1+h) - 1}$ as h approaches 0 is 2.
Tangent line is

$$y = 1 + 2(x - 1) = 2x - 1.$$

1.2 Velocity

- Let t be elapsed time measured in second
- $S(t)$ be the distance the ball has fallen in meters
- What is $S(0)$? $S(0) = 0$.
- (**Galileo**) $S(t) = 4.9t^2$.



Question: How fast the ball is fallen after 1 second, that is, find $v(1)$, the velocity at $t = 1$?

$$\text{average velocity} = \frac{\text{difference in position}}{\text{difference in time}} = \frac{S(t_2) - S(t_1)}{t_2 - t_1}.$$

To answer the question we should find the average velocity of the falling ball between $(1 + h)$ and 1. So,

average velocity when $(t_2 = 1 + h)$ and $(t_1 = 1)$

$$= \frac{S(1 + h) - S(1)}{h} = \frac{4.9(1 + h)^2 - 4.9}{h} = 4.9(2 + h).$$



time window	average velocity
$1 \leq t \leq 1.1$	10.29
$1 \leq t \leq 1.01$	9.84
$1 \leq t \leq 1.01$	9.8049
$1 \leq t \leq 1.001$	9.80049

So we can write

$$v(1) = \lim_{h \rightarrow 0} \frac{S(1+h) - S(1)}{h} = 9.8.$$

More generally:

We define the instantaneous velocity at time $t = a$ to be the limit

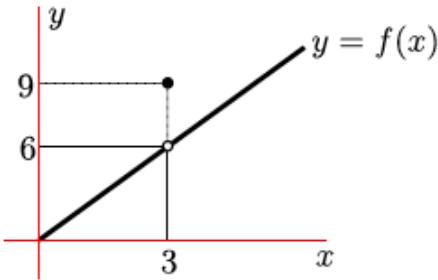
$$v(a) = \lim_{h \rightarrow 0} \frac{S(a+h) - S(a)}{h}$$

1.3 The limit of a function

To arrive at the definition of limit, we start with a very simple example.

Example 1.3.1. Consider the following function

$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



If we plug in some numbers very close to 3 (but not exactly 3) into the function we see

x	2.9	2.99	2.999	○	3.001	3.01	3.1
$f(x)$	5.8	5.98	5.998	○	6.002	6.02	6.2

So as x moves closer and closer to 3, without being exactly 3, we see that the function moves closer and closer to 6. We can then write this as

$$\lim_{x \rightarrow 3} f(x) = 6.$$

Definition. (Informal definition of limit) We write

$$\lim_{x \rightarrow a} f(x) = L.$$

if the value of the function $f(x)$ is sure to be arbitrary close to L whenever the value of x is close enough to a , without being exactly a .

Example 1.3.2. Let $f(x) = \frac{x-2}{x^2+x-6}$ and find its limit as $x \rightarrow 2$.

Solution. We want to find

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2+x-6}.$$

Important point: if we compute $f(2)$, then we have $\frac{0}{0}$ which is undefined.

Again we plug in numbers close to 2 and we have

x	1.9	1.99	1.999	○	2.001	2.01	2.1
$f(x)$	0.20408	0.20040	0.20004	○	0.19996	0.19960	0.19608

So

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2+x-6} = 2.$$

Example 1.3.3. Consider the following function $f(x) = \sin(\pi/x)$. Find the limit as $x \rightarrow 0$ of $f(x)$.

Solution. When x is getting closer and closer to 0, it oscillates faster and faster. Since the function does not approach a single number as we bring x closer and closer to zero, the limit does not exist. Thus,

$$\lim_{x \rightarrow 0} \sin(\pi/x) = \text{DNE}$$



Example 1.3.4. Consider the function

$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$

Find

$$\lim_{x \rightarrow 2} f(x).$$

Solution.



Let again plug in some numbers close to 2 (but not exactly 2)

x	1.9	1.99	1.999	○	2.001	2.01	2.1
$f(x)$	1.9	1.99	1.999	○	5.001	5.01	5.1

Now when we approach from below (or left), we seem to be getting closer to 2 ($\lim_{x \rightarrow 2^-} f(x) = 2$), but when we approach from above (or right) we seem to be getting closer to 5 ($\lim_{x \rightarrow 2^+} f(x) = 5$). Since we are not approaching the same number the limit does not exist.

$$\lim_{x \rightarrow 2} f(x) = \text{DNE}$$

Definition. (Informal definition of one-sided limits) We write

$$\lim_{x \rightarrow a^-} f(x) = K$$

when the value of $f(x)$ gets closer and closer to K when $x < a$ and x moves closer and closer to a . Since the x -values are always less than a , we say that x approaches a from below (or left). This is also often called the left-hand limit since the x -values lie to the left of a on a sketch of the graph.

We similarly write

$$\lim_{x \rightarrow a^+} f(x) = L$$

when the values of $f(x)$ gets closer and closer to L when $x > a$ and x moves closer and closer to a . For similar reason we say that x approaches a from above, and sometimes to this as the the right-hand limit.

Theorem 1.3.5.

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x) = L$$

- If the limit of $f(x)$ as x approaches a exists and is equal to L , then both the left-hand and right-hand limits exist and are equal to L .
- If the left-hand and right-hand limits as x approaches a exist and are equal, then the limit as x approaches a exists and is equal to the one-sided limits.

Contrapositive of the above argument says

- If either of the left-hand and right-hand limits as x approaches a fail to exist, or if they both exist but are different, then the limit as x approaches a does not exist. AND,
- If the limit as x approaches a does not exist, then the left-hand and right-hand limits are either different or at least one of them does not exist.

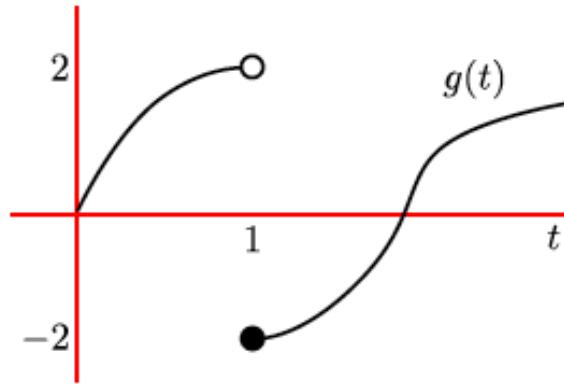
Example 1.3.6. Consider the graph of the function $f(x)$.



Then

$$\lim_{x \rightarrow 1^-} f(x) = 2 \quad \lim_{x \rightarrow 1^+} f(x) = 2 \quad \lim_{x \rightarrow 1} f(x) = 2$$

Example 1.3.7. Consider the graph of the function $g(t)$.



Then

$$\lim_{t \rightarrow 1^-} g(t) = 2 \quad \lim_{t \rightarrow 1^+} g(t) = -2 \quad \lim_{t \rightarrow 1} g(t) = DNE$$

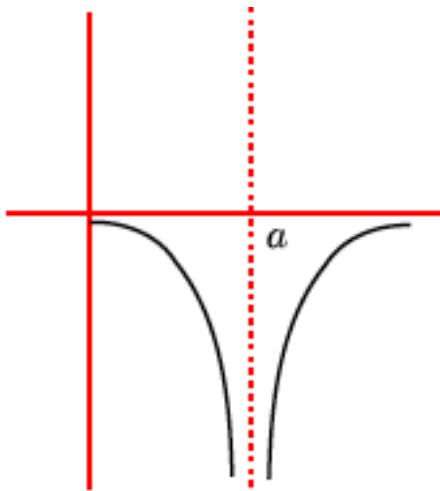
In the following example even though the limit doesn't exists when x approaches a , we can say more.

Example 1.3.8. Consider the graph for the function $f(x)$.



$$\lim_{x \rightarrow a} f(x) = +\infty$$

Example 1.3.9. Consider the graph for the function $g(x)$.



$$\lim_{x \rightarrow a} g(x) = -\infty$$

Definition. We write

$$\lim_{x \rightarrow a} f(x) = +\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as x gets closer and closer to a , without being exactly a .

Similarly, we write

$$\lim_{x \rightarrow a} f(x) = -\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and negative as x gets closer and closer to a , without being exactly a .

Example 1.3.10.

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \lim_{x \rightarrow 0} -\frac{1}{x^2} = -\infty$$

Important Point: Do not think of “ $+\infty$ ” and “ $-\infty$ ” in these statements as numbers. When we write $\lim_{x \rightarrow a} f(x) = +\infty$, it says “the function $f(x)$ becomes arbitrary large as x approaches a ”.

Example 1.3.11. Consider the graph for the function $h(x)$.



$$\lim_{x \rightarrow a^-} h(x) = +\infty \quad \lim_{x \rightarrow a^+} h(x) = 3 \quad \lim_{x \rightarrow a} h(x) = \text{DNE}$$

Example 1.3.12. Consider the graph for the function $s(x)$.



$$\lim_{x \rightarrow a^-} s(x) = 3 \quad \lim_{x \rightarrow a^+} s(x) = -\infty \quad \lim_{x \rightarrow a} s(x) = \text{DNE}$$

Definition. We write

$$\lim_{x \rightarrow a^+} f(x) = +\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as x gets closer and closer to a from above (equivalently, from right), without being exactly a . Similarly, we write

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

when the values of the function $f(x)$ becomes arbitrarily large and negative as x gets closer and closer to a from above (equivalently, from right), without being exactly a .

The notation

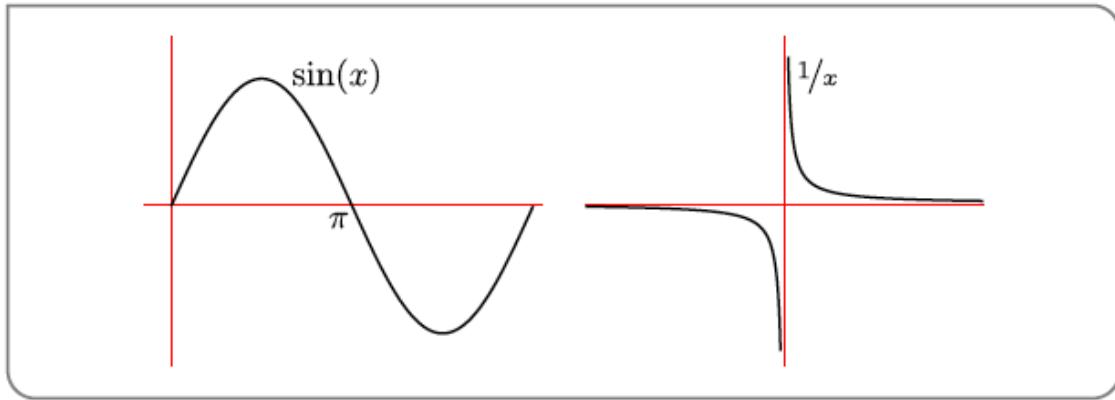
$$\lim_{x \rightarrow a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \rightarrow a^-} f(x) = -\infty$$

has a similar meaning except that limits are approached from below (from left).

Example 1.3.13. Consider the function

$$g(x) = \frac{1}{\sin(x)}.$$

Find the one-side limits of this function as $x \rightarrow \pi$.



- As $x \rightarrow \pi$ from the left, $\sin(x)$ is a small positive number that is getting closer and closer to zero. That is, as $x \rightarrow \pi^-$, we have that $\sin(x) \rightarrow 0$ through positive numbers (i.e. from above). Now look at the graph of $1/x$, and think what happens as we move $x \rightarrow 0^+$, the function is positive and becomes larger and larger.
So as $x \rightarrow \pi$ from the left, $\sin(x) \rightarrow 0$ from above, and so $1/\sin(x) \rightarrow +\infty$.
- By very similar reasoning, as $x \rightarrow \pi$ from the right, $\sin(x)$ is a small negative number that gets closer and closer to zero. So as $x \rightarrow \pi$ from the right, $\sin(x) \rightarrow 0$ through negative numbers (i.e. from below) and so $1/\sin(x)$ to $-\infty$.

Thus

$$\lim_{x \rightarrow \pi^-} \frac{1}{\sin(x)} = +\infty \qquad \lim_{x \rightarrow \pi^+} \frac{1}{\sin(x)} = -\infty$$

1.4 Calculating Limits with Limit Laws

Theorem 1.4.1. Let $a, c \in \mathbb{R}$. The following two limits hold

$$\lim_{x \rightarrow a} c = c \quad \lim_{x \rightarrow a} x = a$$

Theorem 1.4.2. (Arithmetic of Limits) Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x 's that lie in some interval about a (but f and g need not to be defined exactly at a).

$$\lim_{x \rightarrow a} f(x) = F \quad \lim_{x \rightarrow a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. Then the following limits hold

- $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$ – limit of the sum is the sum of the limits.
- $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$ – limit of the difference is the difference of the limits.
- $\lim_{x \rightarrow a} cf(x) = cF$.
- $\lim_{x \rightarrow a} (f(x).g(x)) = F.G$ – limit of the product is the product of the limits.
- If $G \neq 0$ then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}$.

Example 1.4.3. Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} g(x)} = \frac{3}{2} = 3.$$

Example 1.4.4.

$$\lim_{x \rightarrow 3} 4x^2 - 1 = 4 \times \lim_{x \rightarrow 3} x^2 - \lim_{x \rightarrow 3} 1 = 35.$$

$$\lim_{x \rightarrow 2} \frac{x}{x-1} = \frac{\lim_{x \rightarrow 2} x}{\lim_{x \rightarrow 2} x - \lim_{x \rightarrow 1} 1} = \frac{2}{2-1} = 2.$$

Consider that we apply the theorem Arithmetic of Limits to compute the limit of a ratio if the limit of denominator is not zero. **What will happen if the limit of denominator is zero:**

- the limit does not exist, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = DNE$$

– the limit is $\pm\infty$, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x^4} = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{-x^2}{x^4} = \lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty.$$

– the limit is 0, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

– the limit exists and it nonzero, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x} = 1.$$

Theorem 1.4.5. Let n be a positive integer, let $a \in R$ and let f be a function so that

$$\lim_{x \rightarrow a} f(x) = F$$

for some real number F . Then the following holds

$$\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit. Similarly, if

- n is an even number and $F > 0$, or
- n is an odd number and F is any real number

then

$$\lim_{x \rightarrow a} (f(x))^{1/n} = \left(\lim_{x \rightarrow a} f(x) \right)^{1/n} = F^{1/n}.$$

Example 1.4.6.

$$\lim_{x \rightarrow 4} x^{1/2} = 4^{1/2} = 2.$$

$$\lim_{x \rightarrow 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \rightarrow 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = 13^{1/3}$$

Example 1.4.7. Compute the following limits.

$$1. \lim_{x \rightarrow 2} \frac{x^3 - x^2}{x - 1}$$

$$2. \lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1}$$

Solution. 1. $\lim_{x \rightarrow 2} \frac{x^3 - x^2}{x - 1} = 4$.

2. Consider that $\lim_{x \rightarrow 1} x^3 - x^2 = 0$ and $\lim_{x \rightarrow 1} x - 1 = 0$. However,

$$\frac{x^3 - x^2}{x - 1} = \frac{x^2(x - 1)}{x - 1},$$

thus

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases}$$



And so

$$\lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \rightarrow 1} x^2 = 1.$$

The reasoning in the above example can be made more general:

Theorem 1.4.8. If $f(x) = g(x)$ except when $x = a$ then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

provided the limit of g exists.

We mostly use the above theorem when we end up with $\frac{0}{0}$.

Example 1.4.9. Compute

$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1}{h}.$$

Solution. Note that

$$\frac{(1 + h)^2 - 1}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(2 + h)}{h}.$$

Thus,

$$\frac{(1 + h)^2 - 1}{h} = \begin{cases} 2 + h & h \neq 0 \\ \text{undefined} & h = 0. \end{cases}$$

And so

$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1}{h} = \lim_{h \rightarrow 0} 2 + h = 2.$$

We now present a slightly harder example.

Example 1.4.10. Compute the limit

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1 + x} - 1}.$$

Solution. Both the limits of the numerator and denominator as $x \rightarrow 0$ are 0, so we cannot use the Theorem Arithmetic of limits. We now can simply multiply the numerator and denominator by the conjugation of $\sqrt{1+x} - 1$, that is, $\sqrt{1+x} + 1$. We have

$$\begin{aligned}
 \frac{x}{\sqrt{1+x}-1} &= \frac{x}{\sqrt{1+x}-1} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} && \text{multiply by } \frac{\text{conjugate}}{\text{conjugate}} = 1 \\
 &= \frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)} && \text{bring things together} \\
 &= \frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x})^2 - 1 \cdot 1} && \text{since } (a-b)(a+b) = a^2 - b^2 \\
 &= \frac{x(\sqrt{1+x}+1)}{1+x-1} && \text{clean up a little} \\
 &= \frac{x(\sqrt{1+x}+1)}{x} && \\
 &= \sqrt{1+x}+1 && \text{cancel the } x
 \end{aligned}$$

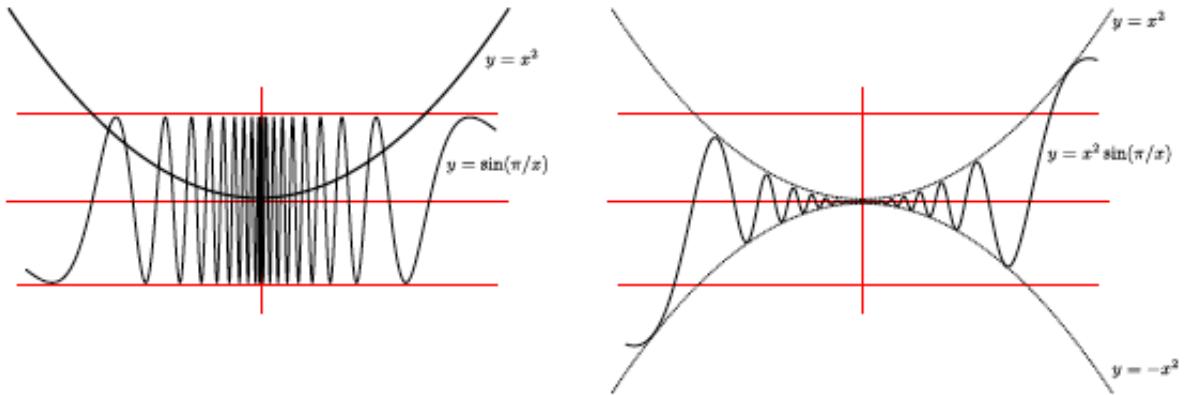
So now we have

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x}-1} &= \lim_{x \rightarrow 0} \sqrt{1+x}+1 \\
 &= \sqrt{1+0}+1 = 2
 \end{aligned}$$

Before we move to the next section and study the limits at infinity, we have one more theorem to state.

Example 1.4.11. Compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



Solution. It is not possible to simply use the theorem Arithmetic of Limits since the limit of $\sin\left(\frac{\pi}{x}\right)$ as $x \rightarrow 0$ does not exist. Since $-1 \leq \sin(\theta) \leq 1$ for all real numbers θ , we have

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1 \quad \text{for all } x \neq 0$$

Multiplying the above by x^2 we see that

$$-x^2 \leq x^2 \sin\left(\frac{\pi}{x}\right) \leq x^2 \quad \text{for all } x \neq 0.$$

Since

$$\lim_{x \rightarrow 0} x^2 = \lim_{x \rightarrow 0} (-x^2) = 0$$

by the sandwich (or squeeze or pinch) theorem (look at below for the sandwich theorem) we have

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0.$$

Theorem 1.4.12. (sandwich (or squeeze or pinch) theorem) Let $a \in \mathbb{R}$ and let f, g, h be three functions so that

$$f(x) \leq g(x) \leq h(x)$$

for all x in an interval around a , except possibly at $x = a$. Then if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then it is also the case that

$$\lim_{x \rightarrow a} g(x) = L.$$

Example 1.4.13. Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \rightarrow 1} f(x)?$$

Solution. Consider that

$$\lim_{x \rightarrow 1} x = 1 \quad \text{and} \quad \lim_{x \rightarrow 1} x^2 - 2x + 2 = 1.$$

Therefore, by the sandwich/pinch/squeeze theorem

$$\lim_{x \rightarrow 1} f(x) = 1.$$

1.5 Limits at Infinity

Example 1.5.1. We want to compute

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{x}$$

By plug in some large numbers into $\frac{1}{x}$ we have

-10000	-1000	-100	100	1000	10000
-0.0001	0.001	-0.01	0.01	0.001	0.0001

We see that as x is getting bigger and positive the function $\frac{1}{x}$ is getting closer to 0. Thus,

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0.$$

Moreover,

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

Definition. (Informal limit at infinity.) We write

$$\lim_{x \rightarrow \infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and positive.

Similarly, we write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and negative.

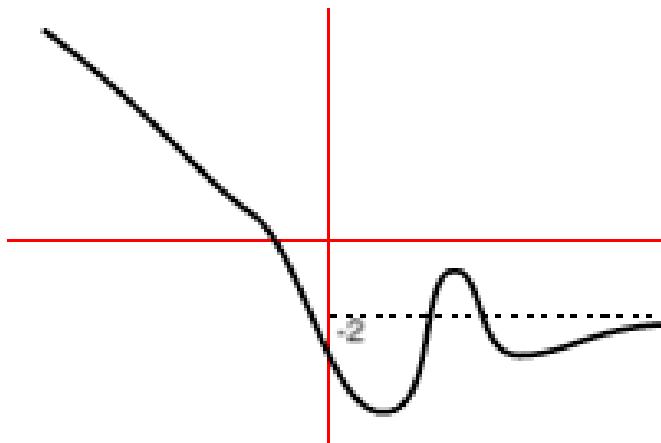
Example 1.5.2. Consider the graph of the function $f(x)$.



Then

$$\lim_{x \rightarrow \infty} f(x) = -2 \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = 2$$

Example 1.5.3. Consider the graph of the function $g(x)$.



Then

$$\lim_{x \rightarrow \infty} g(x) = -2 \quad \text{and} \quad \lim_{x \rightarrow -\infty} g(x) = +\infty$$

Same as usual we start with two very simple building blocks and build other limits from them.

Theorem 1.5.4. *Let $c \in \mathbb{R}$ then the following limits hold*

$$\lim_{x \rightarrow +\infty} c = c \quad \lim_{x \rightarrow -\infty} c = c$$

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0 \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

Theorem 1.5.5. *Let $f(x)$ and $g(x)$ be two functions for which the limits*

$$\lim_{x \rightarrow \infty} f(x) = F \quad \lim_{x \rightarrow \infty} g(x) = G$$

exist. Then the following limits hold

$$\lim_{x \rightarrow \infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x \rightarrow \infty} f(x)g(x) = FG$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \text{provided } G \neq 0$$

and for rational numbers r ,

$$\lim_{x \rightarrow \infty} (f(x))^r = F^r$$

provided that $f(x)^r$ is defined for all x .

The analogous results hold for limits to $-\infty$.

We need a little extra care with the posers of functions.

Warning: Consider that

$$\lim_{x \rightarrow +\infty} \frac{1}{x^{1/2}} = 0$$

However,

$$\lim_{x \rightarrow +\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because $x^{1/2}$ is not defined for $x < 0$.

Example 1.5.6. *Compute the following limit:*

$$\lim_{x \rightarrow \infty} \frac{x^2 - 3x + 4}{3x^2 + 8x + 1}$$

Solution. By factoring x with largest exponent in the numerator and denominator we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2 - 3x + 4}{3x^2 + 8x + 1} &= \lim_{x \rightarrow \infty} \frac{x^2(1 + \frac{-3x}{x^2} + \frac{4}{x^2})}{x^2(3 + \frac{8x}{x^2} + \frac{1}{x^2})} = \lim_{x \rightarrow \infty} \frac{(1 + \frac{-3x}{x^2} + \frac{4}{x^2})}{(3 + \frac{8x}{x^2} + \frac{1}{x^2})} = \\ &\frac{\left(\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \frac{-3x}{x^2} + \lim_{x \rightarrow \infty} \frac{4}{x^2}\right)}{\left(\lim_{x \rightarrow \infty} 3 + \lim_{x \rightarrow \infty} \frac{8x}{x^2} + \lim_{x \rightarrow \infty} \frac{1}{x^2}\right)} = \frac{1}{3}. \end{aligned}$$

Remark. Note that

$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



Example 1.5.7. Compute the following limit:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{5x - 1}.$$



Solution. Factor the terms with the largest exponents in the numerator and denominator. We have

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{5x - 1} = \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2(1 + \frac{1}{4x^2})}}{5x(1 - \frac{1}{5x})} = \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2} \sqrt{(1 + \frac{1}{4x^2})}}{5x(1 - \frac{1}{5x})} = \lim_{x \rightarrow \infty} \frac{2|x|}{5x} = \lim_{x \rightarrow \infty} \frac{2x}{5x} = \frac{2}{5}.$$

Example 1.5.8. Compute the following limit:

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{5x - 1}.$$

Solution. By the same kind of computation we have

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{5x - 1} = \lim_{x \rightarrow \infty} \frac{2|x|}{5x}.$$

Consider that since x is getting negative values, we have $|x| = -x$. Therefore,

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{5x - 1} = \lim_{x \rightarrow \infty} \frac{2|x|}{5x} = \lim_{x \rightarrow \infty} \frac{-2x}{5x} = \frac{-2}{5}.$$

Example 1.5.9. Compute the following limit:

$$\lim_{x \rightarrow \infty} (x^{7/5} - x).$$

Solution. We factor the term with the largest exponent, we have

$$\lim_{x \rightarrow \infty} (x^{7/5} - x) = \lim_{x \rightarrow \infty} x^{7/5} \left(1 - \frac{1}{x^{2/5}}\right) = \infty.$$

Theorem 1.5.10. Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that

$$\lim_{x \rightarrow a} f(x) = +\infty \quad \lim_{x \rightarrow a} g(x) = +\infty \quad \lim_{x \rightarrow a} h(x) = H$$

1.

$$\lim_{x \rightarrow a} (f(x) + g(x)) = +\infty.$$

2.

$$\lim_{x \rightarrow a} (f(x) + h(x)) = +\infty.$$

3.

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \text{undetermined}.$$

4.

$$\lim_{x \rightarrow a} (f(x) - h(x)) = +\infty.$$

5.

$$\lim_{x \rightarrow a} cf(x) = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$$

6.

$$\lim_{x \rightarrow a} (f(x).g(x)) = +\infty.$$

7.

$$\lim_{x \rightarrow a} (f(x).h(x)) = \begin{cases} +\infty & H > 0 \\ \text{undetermined} & H = 0 \\ -\infty & H < 0 \end{cases}$$

8.

$$\lim_{x \rightarrow a} \frac{h(x)}{f(x)} = 0.$$

Example 1.5.11. Consider the following three functions:

$$f(x) = x^{-2} \quad g(x) = 2x^{-2} \quad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \rightarrow 0} f(x) = +\infty \quad \lim_{x \rightarrow 0} g(x) = +\infty \quad \lim_{x \rightarrow 0} h(x) = +\infty.$$

Then

•

$$\lim_{x \rightarrow 0} (f(x) - g(x)) = \lim_{x \rightarrow 0} x^{-2} = -\infty$$

•

$$\lim_{x \rightarrow 0} (f(x) - h(x)) = \lim_{x \rightarrow 0} (1) = 1$$

•

$$\lim_{x \rightarrow 0} (g(x) - h(x)) = \lim_{x \rightarrow 0} x^{-2} + 1 = \infty$$

1.6 Continuity

Look at all the following functions.



All of these functions are continuous. Roughly speaking, a function is continuous if it does not have any abrupt jumps. Now consider the following function.



These functions are not continuous. The function f , g , and h have abrupt jumps at $x = 2$, $x = 0$, and $x = 1$, respectively, so f is not continuous at 2, g is not continuous at 0, and h is not continuous at 1.

Definition. A function $f(x)$ is continuous at a if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

If a function is not continuous at a then it is said to be discontinuous at a . When we write that f is continuous without specifying a point, then typically this means that f is continuous at a for all $a \in \mathbb{R}$. When we write that $f(x)$ is continuous on the open interval (a, b) then the function is continuous at every point c satisfying $a < c < b$.

From the above definition we immediately have that if f is continuous at a , then

1. $f(a)$ exists;
2. $\lim_{x \rightarrow a^-} f(x)$ exists and is equal to $f(a)$.

3. $\lim_{x \rightarrow a^+}$ exists and is equal to $f(a)$.

Definition. A function is continuous from the left at a if

$$\lim_{x \rightarrow a^-} = f(a).$$

And a function is continuous from the right at a if

$$\lim_{x \rightarrow a^+} = f(a).$$

Definition. A function $f(x)$ is continuous on an interval $[a, b]$ if

1. $f(x)$ continuous on (a, b) ,
2. $f(x)$ is continuous form the right at a ,
3. $f(x)$ is continuous form the left at b .

Definition. A function $f(x)$ is continuous on an interval $(a, b]$ (*on the interval $[a, b)$*) if

1. $f(x)$ continuous on (a, b) ,
2. $f(x)$ is continuous form the left at b (*from the right at a*).

Example 1.6.1. Consider the function

$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$



- $\lim_{x \rightarrow 1^-} f(x) = 1 \quad \lim_{x \rightarrow 1^+} f(x) = 3 \quad f(1) = 3.$
- The function $f(x)$, at $x = 1$ is not continuous because the limit does not exist; however, it is continuous from the right at 1 since

$$\lim_{x \rightarrow 1^+} f(x) = 3 = f(1).$$

- The function $f(x)$, on $[1, \infty)$ ($\text{for } x \geq 1$) is continuous.
- The function $f(x)$, on $(-\infty, -1)$ is continuous.

Example 1.6.2. Consider the function

$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



- Consider that

$$\lim_{x \rightarrow 0^-} g(x) = \infty = \lim_{x \rightarrow 0^+} g(x) \quad g(0) = 0.$$

Thus the function $g(x)$ is not continuous at 0 because

$$\lim_{x \rightarrow 0} g(x) = \infty \neq 0 = g(0).$$

It is not continuous at 0 from the left since $\lim_{x \rightarrow 0^-} g(x) = \infty \neq 0 = g(0)$ and not from the right since $\lim_{x \rightarrow 0^+} g(x) = \infty \neq 0 = g(0)$.

- the function $g(x)$ is continuous at all points in \mathbb{R} except 0.

Example 1.6.3. Consider the function

$$h(x) = \begin{cases} \frac{x^3 - x^2}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$



- $\lim_{x \rightarrow 1^-} h(x) = 1 = \lim_{x \rightarrow 1^+} h(x) \quad f(1) = 0.$
- $\lim_{x \rightarrow 1} h(x) = 1.$
- the function $h(x)$ is not continuous at 1 since

$$\lim_{x \rightarrow 1} h(x) = 1 \neq 0 = h(1).$$

It is not continuous from the left since

$$\lim_{x \rightarrow 1^-} h(x) = 1 \neq 0 = h(1)$$

and not from the right since

$$\lim_{x \rightarrow 1^+} h(x) = 1 \neq 0 = h(1).$$

- the function $h(x)$ is continuous at all points in \mathbb{R} except 1.

Lemma 1.6.4. Let $c \in \mathbb{R}$. The functions

$$f(x) = x \quad g(x) = c$$

are continuous everywhere on the real line.

Theorem 1.6.5. (Arithmetic of continuity) Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at a . Then the following functions are also continuous at $x = a$.

- $f(x) + g(x)$ and $f(x) - g(x)$,
- $cf(x)$ and $f(x)g(x)$, and
- $\frac{f(x)}{g(x)}$ provided $g(a) \neq 0$.

Theorem 1.6.6. The following functions are continuous everywhere in their domains

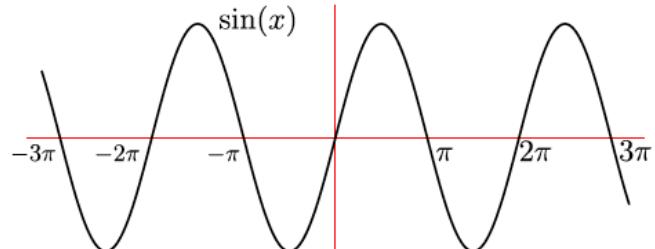
- polynomials and rational functions (for example $f(x) = x^5 + 4x^2 + 1$ and $g(x) = \frac{x^2+1}{x+1}$)
- roots and powers (for example $h(x) = \sqrt{x}$ and $r(x) = 2^x$)
- trig functions and their inverses (for example $k(x) = \sin(x)$ and $t(x) = \cos^{-1}(x)$)
- exponentials and logarithms (for example $s(x) = e^x$ and $q(x) = \ln x$).

Example 1.6.7. Determine when the function $f(x) = \frac{\sin(x)}{x^2 - 5x + 6}$ is continuous? Since both $\sin(x)$ and $x^2 - 5x + 6$ are continuous by the above theorem we only need to check when $x^2 - 5x + 6 = 0$. Note that $x^2 - 5x + 6 = (x - 2)(x - 3)$, thus this polynomial is only zero at $x = 2$ and $x = 3$. Therefore, $f(x)$ is continuous at all points in \mathbb{R} except 2 and 3.

Theorem 1.6.8. If g is continuous at a and $f(x)$ is continuous at $g(a)$, then $(f \circ g)(x) = f(g(x))$ is continuous at $x = a$.

Example 1.6.9. Determine when the function $h(x) = \sqrt{\sin(x)}$ is continuous.

Solution. Let $f(x) = \sqrt{x}$ and $g(x) = \sin(x)$, then $h(x) = (f \circ g)(x)$. We only need to find out at what points $\sin(x)$ is positive.



The function $\sqrt{\sin(x)}$ is continuous if

$$x \in [2n\pi, (2n+1)\pi] \quad \text{for all natural numbers } n.$$

Theorem 1.6.10. (Intermediate value theorem(IVT)) Let $a < b$ and let $f(x)$ be a function that is continuous at all points $a \leq x \leq b$. If Y is any number between $f(a)$ and $f(b)$ then there exists some number $c \in [a, b]$ so that $f(c) = Y$.



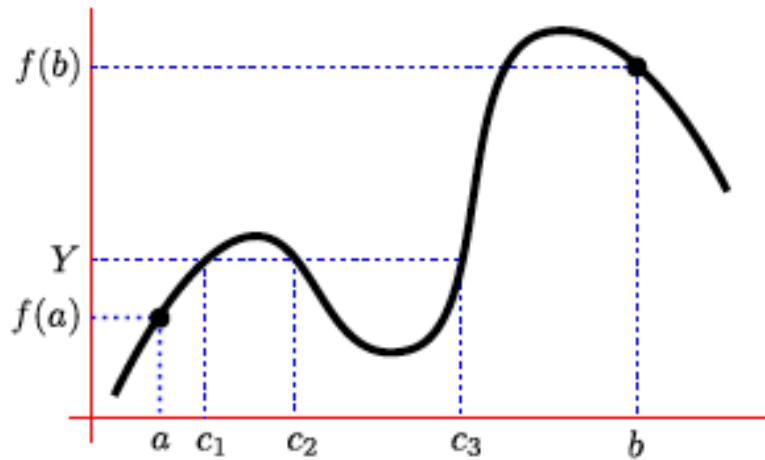
Remark. One of the main application of the IVT theorem is showing a function f has

a zero inside an interval. For example, in the following picture



we can see that $f(a) < 0$ and $f(b) > 0$, therefore by IVT, there is a number c between a and b such that $f(c) = 0$.

Remark. If f is continuous and $f(a) \leq Y \leq f(b)$, the IVT merely shows that there is a $a \leq c \leq b$ such that $f(c) = Y$, but it doesn't show how many of them exist. For example, in the following picture, we can see $f(a) \leq Y \leq f(b)$, and there are three numbers c_1, c_2 , and c_3 such that $f(c_1) = f(c_2) = f(c_3) = Y$.



Remark. Consider that if the function f is not continuous at the interval $[a, b]$ then the IVT fails. In the following examples, even though $f(a) \leq Y \leq f(b)$, there is not a number

$a \leq c \leq b$ such that $f(c) = Y$.



Example 1.6.11. Show that the function $f(x) = x - 1 + \sin(\pi x/2)$ has a zero in $0 \leq x \leq 1$.

Solution. Consider that $f(x)$ is a continuous function such that $f(0) = -1$ and $f(1) = 1$. Therefore, by IVT, since $f(0) = -1 \leq 0 \leq 1 = f(1)$, we have $f(c) = 0$ for some $c \in [0, 1]$.

Example 1.6.12. Use the bisection method to find a zero of $f(x) = x - 1 + \sin(\pi x/2)$ that lies between 0 and 1.

Solution.

- Let $a = 0$ and $b = 1$. Then

$$f(0) = -1$$

$$f(1) = 1$$

- Test the point in the middle $x = \frac{1+0}{2} = 0.5$,

$$f(0.5) = 0.2071067813 > 0$$

- Let $a = 0$ and $b = 0.5$. Then

$$f(0) = -1$$

$$f(1) = 0.2071067813$$

So by IVT, there is a zero in $[0, 0.5]$.

- Test the point in the middle $x = \frac{0.5+0}{2} = 0.25$.

$$f(0.25) = -0.3673165675 < 0.$$

- Let $a = 0.25$, $b = 0.5$ where $f(0.25) < 0$ and $f(0.5) > 0$. By IVT there is a zero in the interval $[0.25, 0.5]$.

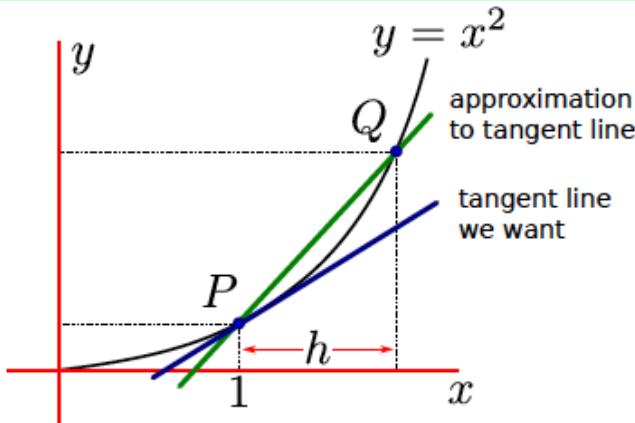
- So without much work we know the location of a zero inside a range of length $1/4$. Each iteration will halve the length of the range and we keep going until we reach the precision we need, though it is much easier to program a computer to do it.

Chapter 2

Derivatives

2.1 Revisiting Tangent Lines

Example 2.1.1. Find the slope of the tangent line to the curve $y = x^2$ that passes through $P = (1, 1)$.



Solution. Consider that the slope of the secant line is

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

And the slope of the tangent line is the same as

$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}.$$

Theorem 2.1.2. Given a function $f(x)$ the slope of the tangent line at $x = a$ (if exists) is

$$\lim_{x \rightarrow a} \frac{f(a+h) - f(a)}{h}.$$

2.2 Definition of the derivative

Definition. (*Derivative at a point*) Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined on an open interval that contains a .

- The derivative of $f(x)$ at $x = a$ is denoted $f'(a)$ and is defined by

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \quad (2.2.1)$$

if the limit exists.

- When the above limit exists, the function $f(x)$ is said to be differentiable at $x = a$. When the limit does not exist, the function $f(x)$ is said to be not differentiable at $x = a$.
- We can equivalently define the derivative $f'(a)$ by the limit

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

To see that these two definitions are the same, we set $x = a + h$ ($x - a = h$) and then when h approaches 0, we have x approaches a , and the limit in 2.2.1 becomes $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$.

Example 2.2.1. Let $a, c \in \mathbb{R}$ be constants. Compute the derivative of the function $f(x) = c$ at $x = a$.

Solution. By the definition of the derivative, we have

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c - c}{h} \\ &= \lim_{h \rightarrow 0} 0 \\ &= 0 \end{aligned}$$

Example 2.2.2. Let $a \in \mathbb{R}$. Compute the limit of the function $g(x) = x$ at $x = a$.

Solution. By the definition of the derivative we have

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a+h) - a}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} \\ &= \lim_{h \rightarrow 0} 1 \\ &= 1. \end{aligned}$$

We have so proved our first theorem which is the following.

Theorem 2.2.3. (easiest derivative) Let $a, c \in \mathbb{R}$ and let $f(x) = c$ and $g(x) = x$. Then

$$f'(a) = 0$$

and

$$g'(a) = 1.$$

Example 2.2.4. Compute the derivative of $f(t) = t^2$ at $t = a$.

Solution. We have that

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 + h^2 + 2ah - a^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 + 2ah}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(h+2a)}{h} \\ &= \lim_{h \rightarrow 0} h + 2a \\ &= 2a \end{aligned}$$

►► We can tweak the derivative at a specific point a to obtain the derivative as a function x . We replace

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

with

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

This gives us the following definition.

Definition. Let $f(x)$ be a function

- The derivative of $f(x)$ with respect to x is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

- If the derivative $f'(x)$ exists for all $x \in (a, b)$ we say that f is differentiable on (a, b) .
- Note that we will sometimes be a little sloppy with our discussion and simply write “ f is differentiable” to mean “ f is differentiable on an interval we are interested in” or “ f is differentiable everywhere.”

Example 2.2.5. Let $f(x) = \frac{1}{x}$ and compute its derivative with respect to x .

Solution. We have that

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{x+h} - \frac{1}{x} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{x - (x+h)}{x(x+h)} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{-h}{x(x+h)} \\ &= -\frac{1}{x^2}. \end{aligned}$$



$$y = \frac{1}{x}$$

$$y = \frac{-1}{x^2}$$

►► Notice that the original function $f(x) = \frac{1}{x}$ was not defined at $x = 0$, and the deriva-

tive is also not defined at $x = 0$. This does happen more generally—if $f(x)$ is not defined at a particular point $x = a$, then the derivative will not exist at that point either.

Notation. There are several notation all used for “the derivative of $f(x)$ with respect to x ”; however,

in this course we generally use the following notations

1. $f'(x)$. This notation is due to Lagrange, and we read it as “ f -prime of x ”.
2. $\frac{df}{dx}$. This notation is due to Leibniz, and we read it as “dee- f -dee- x ”.
3. $\frac{d}{dx}f$. We read this as dee-by-dee- x of f .

Example 2.2.6. Compute the derivative, $f'(a)$, of the function $f(x) = \sqrt{x}$ at the point $x = a$ for any $a > 0$.

Solution. We have that

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$

We now multiply the numerator and denominator by the conjugate of $\sqrt{x} - \sqrt{a}$, that is $\sqrt{x} + \sqrt{a}$. Then we have

$$\frac{\sqrt{x} - \sqrt{a}}{x - a} \times \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} = \frac{(x - a)}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}}.$$

Therefore,

$$f'(a) = \lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \lim_{x \rightarrow a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}.$$



Example 2.2.7. Find the derivative, $f'(a)$, of the function $f(x) = |x|$ at the point $x = a$.

Solution. Recall that

$$|x| = \begin{cases} -x & x < 0 \\ 0 & x = 0 \\ x & x > 0 \end{cases}$$



We should break our computation of the derivative into three cases depending on whether x is positive, negative, or zero.

- Assume $x > 0$. Then

$$\begin{aligned} \frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} \end{aligned}$$

Since $x > 0$ and h is much more smaller than x , we have $x+h > 0$ and so $|x+h| = x+h$, moreover, since x is positive, $|x| = x$.

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} = 1. \end{aligned}$$

- Assume $x < 0$. Then we have

$$\begin{aligned} \frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} \end{aligned}$$

Since $x < 0$ and h is much more smaller than x , we have $x+h < 0$ and so $|x+h| = -(x+h)$, moreover, since $x < 0$ is positive, $|x| = -x$.

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h \rightarrow 0} \frac{-(x+h) - (-x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h} = -1. \end{aligned}$$

- Assume $x = 0$. Then we have

$$\begin{aligned}\frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|h|}{h}\end{aligned}$$

Consider that

$$\lim_{h \rightarrow 0^+} \frac{|h|}{h} = 1 \quad \text{and} \quad \lim_{h \rightarrow 0^-} \frac{|h|}{h} = -1$$

Therefore, this limit does not exist and so the function $|x|$ is not derivative at $x = 0$.

In summary:

$$\frac{d}{dx}|x| = \begin{cases} -1 & x < 0 \\ DNE & x = 0 \\ 1 & x > 0 \end{cases}$$

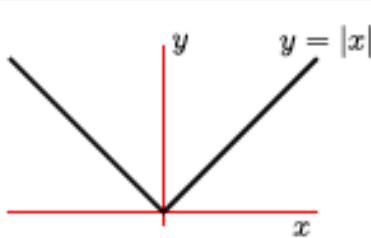


►► Where is the derivative undefined? The derivative $f'(a)$ exists precisely when the limit

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

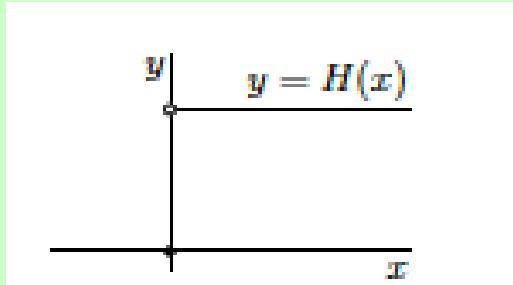
exists. That limit is the slope of the tangent line to the curve $y = f(x)$ at $x = a$. Thus, that limit does not exist one of the following happens.

- ❶ The curve $y = f(x)$ does not have a tangent line at $x = a$ when it has a sharp corner at $x = a$, as an example $f(x) = |x|$ is not differentiable at $x = 0$ since it has a sharp corner at $x = 0$.



- ❷ When the curve does have a tangent line because it is not continuous at $x = a$.

As an example, we have seen that $f(x) = H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$ does not have a tangent line at $x = 0$ since it is not continuous at $x = 0$.



- ❸ When the curve has a tangent line at $x = a$ but the slope of the tangent line at $x = a$ is infinity. As an example, $f(x) = x^{1/3}$ is not differentiable at $x = 0$ since it has a tangent line with slope infinity.



Example 2.2.8. Verify that the function

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$



does not have a tangent line at $x = 0$.

Solution. Consider that if the tangent line exists then the following limit also must exists,

$$\lim_{h \rightarrow 0} \frac{H(0 + h) - H(0)}{h}.$$

Consider that

$$\lim_{h \rightarrow 0^+} \frac{H(0 + h) - H(0)}{h} = \lim_{h \rightarrow 0^+} \frac{1}{h} = +\infty$$

and

$$\lim_{h \rightarrow 0^-} \frac{H(0 + h) - H(0)}{h} = \lim_{h \rightarrow 0^-} \frac{0}{h} = 0.$$

Therefore, the limit does not exists.

Example 2.2.9. Verify that the derivative of $f(x) = x^{1/3}$ at $x = 0$ does not exist.



Solution. You can already see in the graph that the derivative at $x = 0$ does not exist since the tangent line has infinite slope. However, we need a mathematical proof, and we should show that $f'(0)$ which is the same as the following limit

$$\lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h}$$

does not exist. We have

$$\lim_{h \rightarrow 0} \frac{(0 + h)^{1/3} - 0^{1/3}}{h} = \lim_{h \rightarrow 0} \frac{h^{1/3}}{h} = \lim_{h \rightarrow 0} \frac{1}{h^{2/3}} = +\infty$$

(or we can say DNE).

Example 2.2.10. Verify that the derivative of $f(x) = \sqrt{|x|}$ at $x = 0$ does not exist.



Solution. Even though you can see in the graph that at $x = 0$, the graph has a sharp corner, we also show that the following limit doesn't exist,

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{|h|} - 0}{h}.$$

Consider that

$$\lim_{h \rightarrow 0^+} \frac{\sqrt{|h|}}{h} = \lim_{h \rightarrow 0^+} \frac{\sqrt{1}}{\sqrt{h}} = +\infty$$

(or DNE).

»» What is the relation between continuity and differentiability?

Theorem 2.2.11. *If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.*

Theorem 2.2.12. *If $f(x)$ is not continuous at $x = a$, then it is not differentiable at $x = a$.*

2.3 Interpretations of the Derivative

Interpretation of the derivative:

- the instantaneous rate of change of a quality
- the slope of a curve.

»» Instantaneous Rate of Change

Assume that we have the function $f(t)$ of the measuring of some quantity. Then

average rate of change of $f(t)$ from $t = a$ to $t = a + h$ is

$$\begin{aligned} & \frac{\text{change in } f(t) \text{ from } t = a \text{ to } t = a + h}{\text{length of time from } t = a \text{ to } t = a + h} \\ &= \frac{f(a + h) - f(a)}{h}. \end{aligned}$$

And so

$$\begin{aligned} & \text{instantaneous rate of change of } f(t) \text{ at } t = a \\ &= \lim_{h \rightarrow 0} [\text{average rate of change of } f(t) \text{ from } t = a \text{ to } t = a + h] \\ &= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} = f'(a). \end{aligned}$$

Example 2.3.1. You drop a ball from a tall building. After t seconds the ball has fallen a distance of $s(t) = 4.9t^2$ meters. What is the (instantaneous) velocity of the ball one second after it is dropped?

Solution. Since the instantaneous velocity is actually the instance rate of change if distance, we should compute $f'(1)$. Computing the limit of the derivative we have

$$f'(t) = 9.8t \quad \text{and} \quad f'(1) = 9.8.$$

Example 2.3.2. You are taking a walk and that as you walk, you are continuously measuring some quantity, like temperature, and that the measurement at time t is $f(t) = \sqrt{t}$. What is the average rate of $f(t) = \sqrt{t}$ from $t = 1$ to $t = 2$? What is the instantaneous rate of change of $f(t)$ at $t = 1$?

Solution. Consider that the average rate of change is

$$\frac{f(a+h) - f(a)}{h}$$

The difference between times is h , so $h = 2 - 1 = 1$. Therefore, the average is

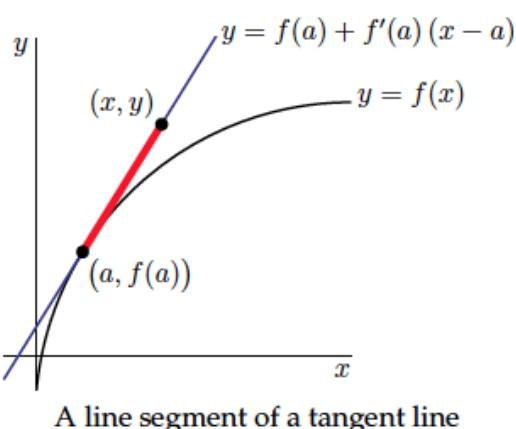
$$\frac{f(a+h) - f(a)}{h} = \frac{f(2) - f(1)}{1} = \sqrt{2} - 1.$$

The instantaneous rate of change is $f'(1)$. We already have seen that $f'(t) = \frac{1}{2\sqrt{t}}$, and so $f'(1) = \frac{1}{2}$.

►► Slope and the formula for the tangent line to a curve at $x = a$

We know that the slope of a function at $x = a$ is $f'(a)$. To find the tangent line we need to find a point at the tangent line; however, for sure we know that $(a, f(a))$ is on the tangent line. So we should find the equation for a line that passes through $(a, f(a))$ with slope $f'(a)$ which is

$$y = f(a) + f'(a)(x - a).$$



Theorem 2.3.3. The tangent line to the curve $y = f(x)$ at $x = a$ is given by the equation

$$y = f(a) + f'(a)(x - a)$$

provided the derivative $f'(a)$ exists.

Example 2.3.4. Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution. By the above theorem we have the equation for the tangent line is

$$y = f(a) + f'(a)(x - a).$$

Note that $a = 4$ and so $f(a) = f(4) = \sqrt{4} = 2$. We have already had $f'(x) = \frac{1}{2\sqrt{x}}$. Therefore, $f'(a) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. And so the equation for the tangent line is

$$y = 2 + \frac{1}{4}(x - 4).$$

2.4 Arithmetic of Derivatives - a Differentiation Toolbox

It is more efficient to have access to

- a list of derivatives of some simple functions and
- a collection of rules for breaking down complicated derivative computations into sequence of simple derivative computations.

►► A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \quad \frac{d}{dx}x = 1 \quad \frac{d}{dx}x^2 = 2x \quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Tools

Theorem 2.4.1. Let $f(x)$ and $g(x)$ be differentiable functions and let $c, d \in \mathbb{R}$. Then

$$\frac{d}{dx}\{f(x) + g(x)\} = f'(x) + g'(x)$$

$$\frac{d}{dx}\{f(x) - g(x)\} = f'(x) - g'(x)$$

$$\frac{d}{dx}\{cf(x)\} = cf'(x)$$

$$\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$$

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$$

► $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$

► $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$

► $\frac{d}{dx}\left\{\frac{1}{g(x)}\right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$

2.5 Using the Arithmetic of Derivatives - Examples

Example 2.5.1.

$$\begin{aligned} \frac{d}{dx}4x + 6 &= 4 \cdot \frac{d}{dx}\{x\} + 7 \cdot \frac{d}{dx}\{1\} \\ &= 4 \cdot 1 + 7 \cdot 0 = 4. \end{aligned}$$

Example 2.5.2.

$$\begin{aligned}\frac{d}{dx}\{x(4x+7)\} &= \frac{d}{dx}\{x\}(4x+7) + x\frac{d}{dx}\{4x+7\} \\ &= 4x+7+4x = 8x+7\end{aligned}$$

Example 2.5.3.

$$\begin{aligned}\frac{d}{dx}\left\{\frac{x}{(4x+7)}\right\} &= \frac{\frac{d}{dx}\{x\}(4x+7) - x\frac{d}{dx}\{4x+7\}}{(4x+7)^2} \\ &= \frac{4x+7-x(4)}{(4x+7)^2} = \frac{7}{(4x+7)^2}.\end{aligned}$$

Example 2.5.4. Differentiate

$$f(x) = \frac{x}{2x + \frac{1}{3x+1}}$$

Solution. Let

$$f_1(x) = x \quad \text{and} \quad f_2(x) = 2x + \frac{1}{3x+1}$$

Let

$$f_3 = 2x \quad \text{and} \quad f_4(x) = \frac{1}{3x+1}.$$

Therefore,

$$f(x) = \frac{f_1(x)}{f_2(x)}$$

and so

$$f'(x) = \frac{f'_1(x)f_2(x) - f'_2(x)f_1(x)}{f_2(x)^2}.$$

Consider that $f'_1(x) = 1$. We have that $f_2(x) = f_3(x) + f_4(x)$, and thus

$$f'_2(x) = f'_3(x) + f'_4(x) = 2 + \frac{-3}{(3x+1)^2}.$$

Therefore,

$$f'(x) = \frac{1 \times 2x + \frac{1}{3x+1} - 2 + \frac{-3}{(3x+1)^2}x}{(2x + \frac{1}{3x+1})^2}.$$

If we clean up this formula we have

$$f'(x) = \frac{6x+1}{(6x^2+2x+1)^2}.$$

Tools

Corollary 2.5.5. Let n be an integer.

$$\frac{d}{dx}\{f(x)g(x)h(x)\} = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x).$$

Example 2.5.6. Let n be an integer, differentiate

$$g(x) = x^n$$

Solution. Let $f(x) = x$. Consider that $g(x) = f(x)^n$, by the above theorem we have

$$g'(x) = nf(x)f(x)^{n-1} = n \times 1 \times x^{n-1} = nx^{n-1}.$$

Example 2.5.7. Differentiate

$$1. \quad f(x) = (3x + 9)(x^2 + 4x^3)$$

$$2. \quad \frac{4x^3 - 7x}{4x^2 + 1}$$

Solution. (1) Let $f_1(x) = 3x + 9$ and $f_2(x) = x^2 + 4x^3$.

$$f(x) = f_1(x)f_2(x) \quad \text{and} \quad f'(x) = f'_1(x)f_2(x) + f'_2(x)f_1(x).$$

Note that

$$f'_1(x) = 3 \quad \text{and} \quad f'_2(x) = 2x + 12x^2.$$

Therefore,

$$f'(x) = 3 \times (x^2 + 4x^3) + (2x + 12x^2) \times (3x + 9) = 18x + 117x^2 + 48x^3.$$

(2) Let $f_1(x) = 4x^3 - 7x$ and $f_2(x) = 4x^2 + 1$. Then

$$f(x) = \frac{f_1(x)}{f_2(x)} \quad \text{and} \quad f'(x) = \frac{f'_1(x)f_2(x) - f'_2(x)f_1(x)}{f_2(x)^2}.$$

Consider that

$$f'_1(x) = 12x^2 - 7 \quad \text{and} \quad f'_2(x) = 8x.$$

Therefore,

$$f'(x) = \frac{(12x^2 - 7)(4x^2 + 1) - (8x)(4x^3 - 7x)}{(4x^2 + 1)^2}.$$

After clean up,

$$f'(x) = \frac{16x^4 + 40x^2 - 7}{(4x^2 + 1)^2}.$$

Tools

Corollary 2.5.8. Let a be a rational number, then

$$\frac{d}{dx}x^a = ax^{a-1}.$$

Example 2.5.9. The derivative of $f(x) = x^{1/3}$ is $f'(x) = (1/3)x^{1/3-1} = (1/3)x^{-2/3} = \frac{1}{x^{2/3}}$.

Example 2.5.10. Find the derivative of

$$f(x) = \frac{(\sqrt{x}-1)(2-x)(1-x^2)}{\sqrt{x}(3+2x)}$$

Solution. Let $f_1(x) = (\sqrt{x}-1)(2-x)(1-x^2)$ and $f_2(x) = \sqrt{x}(3+2x)$. Then

$$f'(x) = \frac{f'_1(x)f_2(x) - f'_2(x)f_1(x)}{f_2(x)^2}. \quad (2.5.1)$$

Let $f_3(x) = (\sqrt{x}-1)$, $f_4(x) = (2-x)$ and $f_5(x) = (1-x^2)$. Then

$$f_1(x) = f_3(x)f_4(x)f_5(x)$$

and

$$f'(x) = f'_3(x)f_4(x)f_5(x) + f_3(x)f'_4(x)f_5(x) + f_3(x)f_4(x)f'_5(x).$$

Consider that

$$f'_3(x) = \frac{1}{2\sqrt{x}}, \quad f'_4(x) = -1, \quad \text{and} \quad f'_5(x) = -2x.$$

Therefore,

$$f'_1(x) = \left(\frac{1}{2\sqrt{x}}\right)(2-x)(1-x^2) + (\sqrt{x}-1)(-1)(1-x^2) + (\sqrt{x}-1)(2-x)(-2x).$$

Also,

$$f'_2(x) = \frac{1}{2\sqrt{x}}(3+2x) + 2\sqrt{x}.$$

So we now substitute in 2.5.1 and we have the derivative of $f(x)$.

2.6 Derivatives of Exponential Functions

So far we have seen the derivative for some family of functions such as

- polynomials,
- rational functions, and powers and roots of rational functions.

►► Looking for a function $f(x)$ such that $\frac{d}{dx}f(x) = f(x)$:

Now we want to find the derivative of exponential functions, that is, if $a > 0$ what is the derivative of $f(x) = a^x$. By the definition of the derivative, we have the derivative of $f(x) = a^x$ with respect to x is

$$\frac{d}{dx}a^x = \lim_{h \rightarrow 0} \frac{a^{(x+h)} - a^x}{h} = \lim_{h \rightarrow 0} a^x \frac{a^h - 1}{h} = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}.$$

For a moment let assume that

$$C(a) = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$$

exists. Therefore,

$$\frac{d}{dx}a^x = a^x C(a).$$

If we can find a such that $C(a) = 1$, it turns to be very useful.

Theorem 2.6.1. $C(a) = 1$ when $a = e$, where

$$e = 2.7182818284590452354\dots$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

Therefore,

$$\frac{d}{dx}e^x = e^x \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{e^h - 1}{h}.$$



Therefore, we have

- $e^0 = 1$
- $e^{x+y} = e^x e^y$
- $e^{-x} = \frac{1}{e^x}$
- $(e^x)^y = e^{xy}$
- $\lim_{x \rightarrow \infty} e^x = \infty \quad \lim_{x \rightarrow -\infty} e^x = 0.$

Example 2.6.2. Find a such that the following function is continuous.

$$f(x) = \begin{cases} e^{x+a} & x < 0 \\ \sqrt{x+1} & x \geq 0 \end{cases}$$

Solution. Consider that both functions e^{x+a} for any $a \in \mathbb{R}$ and $\sqrt{x+1}$ are continuous, so we only need to check that if this function is continuous at $x = 0$. Consider that if the function $f(x)$ is continuous at $x = 0$, then we must have

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0) = 1.$$

Note that

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} e^{x+a} = e^a$$

and

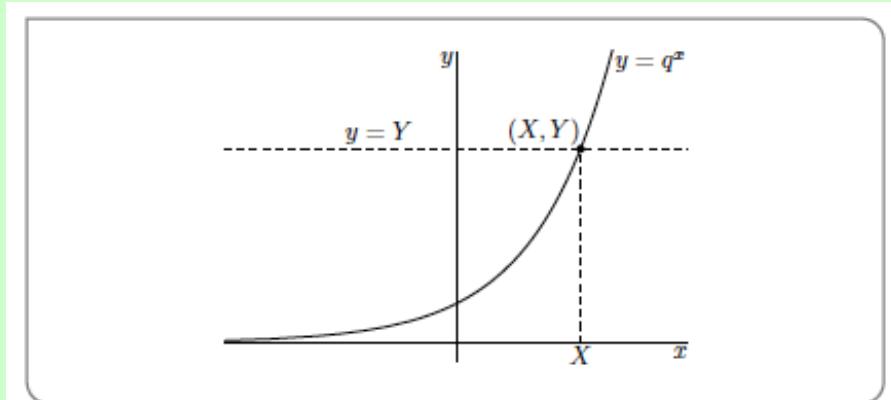
$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sqrt{x+1} = 1.$$

Therefore, we must have $e^a = 1$ and so $a = 1$.

►►► Logarithmic Functions:

We are about to define “logarithm with base q ”. The number q can be any positive number, however, we restrict our attention to $q > 1$ since the only q ’s that are ever used are e (we will see what e is in the next pages). Let $q = 10$, then we have the following:

- $\lim_{x \rightarrow +\infty} 10^x = \infty$
- $\lim_{x \rightarrow -\infty} 10^x = 0$



Actually, the function $\log_q(x)$ is the inverse of the function q^x . For learning about inverse functions see section 0.6. So we have the following definition.

Definition. Let $q > 1$. Then the logarithm with base q is defined by

$$y = \log_q(x) \Leftrightarrow x = q^y$$

Obviously, we have the following.

$$\log_q(q^x) = x \quad q^{\log_q(x)} = x$$

Example 2.6.3. We have

1. $\log_q(xy) = \log_q(x) + \log_q(y)$.
The reason for this is that

$$q^{\log_q(xy)} = xy = q^{\log_q(x)}q^{\log_q(y)} = q^{\log_q(x)+\log_q(y)}$$

Therefore, $\log_q(xy) = \log(x) + \log(y)$.

2. $\log_q(x/y) = \log_q(x) - \log_q(y)$
3. $\log_q(x^r) = r \log_q(x)$

Example 2.6.4. We now show that

$$\log_q(x) = \frac{\log_{10}(x)}{\log_q(10)}.$$

Solution. Let $\log_q(x) = y$. Then

$$q^y = q^{\log_q(x)} = x.$$

We take log base 10 of both sides

$$\log_{10} q^y = \log_{10} x.$$

Then

$$y \log_{10} q = \log_{10}(x)$$

Therefore,

$$\log_q(x) = \frac{\log_{10}(x)}{\log_q(10)}.$$

Tools: Chain Rule

Theorem 2.6.5. Let f and g be differentiable functions. Then

$$\frac{d}{dx}(f \circ g)(x) = g'(x)f'(g(x)).$$

Let $a > 0$ and

$$a^x = e^{\log_e(a^x)} = e^{x \log_e a}$$

Let $f(x) = e^x$ and $g(x) = x \log_e a$. Then

$$(f \circ g)(x) = e^{x \log_e a} = a^x.$$

Therefore,

$$\frac{d}{dx}(f \circ g)(x) = g'(x)f'(g(x)) = (\log_e a)e^{x \log_e a} = (\log_e a)a^x.$$

The derivative of a^x

Theorem 2.6.6. Let $a > 0$ and $f(x) = a^x$. Then

$$\frac{d}{dx}a^x = (\log_e a)e^{x \log_e a} = (\log_e a)a^x.$$

Example 2.6.7. Find the derivative of $2^{\sqrt{x}}$.

Solution. Let $f(x) = 2^x$ and $g(x) = \sqrt{x}$. Then

$$f \circ g(x) = 2^{\sqrt{x}}.$$

Therefore,

$$\frac{d}{dx} 2^x = g'(x) f'(g(x)).$$

We have $g'(x) = \frac{1}{2\sqrt{x}}$ and $f'(x) = (\log_e 2)2^x$. Thus,

$$\frac{d}{dx} 2^x = g'(x) f'(g(x)) = \frac{1}{2\sqrt{x}} (\log_e 2) 2^{\frac{1}{2\sqrt{x}}}.$$

Example 2.6.8. Find a and b such that the following function is differentiable.

$$f(x) = \begin{cases} x^3 + a & x < 1 \\ e^{x-1} + bx & x \geq 1 \end{cases}$$

Solution. We first know that this function must be continuous, thus the left hand side limit at 1 must be equal to the right hand side limit at 1. We have

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} e^{x-1} + bx = 1 + b$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^3 + a = 1 + a,$$

and so $1 + a = 1 + b \Rightarrow a = b$. Moreover, by the definition of the derivative, if $f(x)$ is differentiable at 1 then we must have

$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

must exists. Therefore,

$$\lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h}$$

Note that

$$\lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^+} \frac{e^{x+h-1} + b(x+h) - (1+b)}{h} = \frac{d}{dx}|_{x=1} (e^{x-1} + bx) = b.$$

Also,

$$\lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} = 3$$

Therefore, $b = 3$ and $a = 3$.

2.7 Derivatives of trigonometric functions

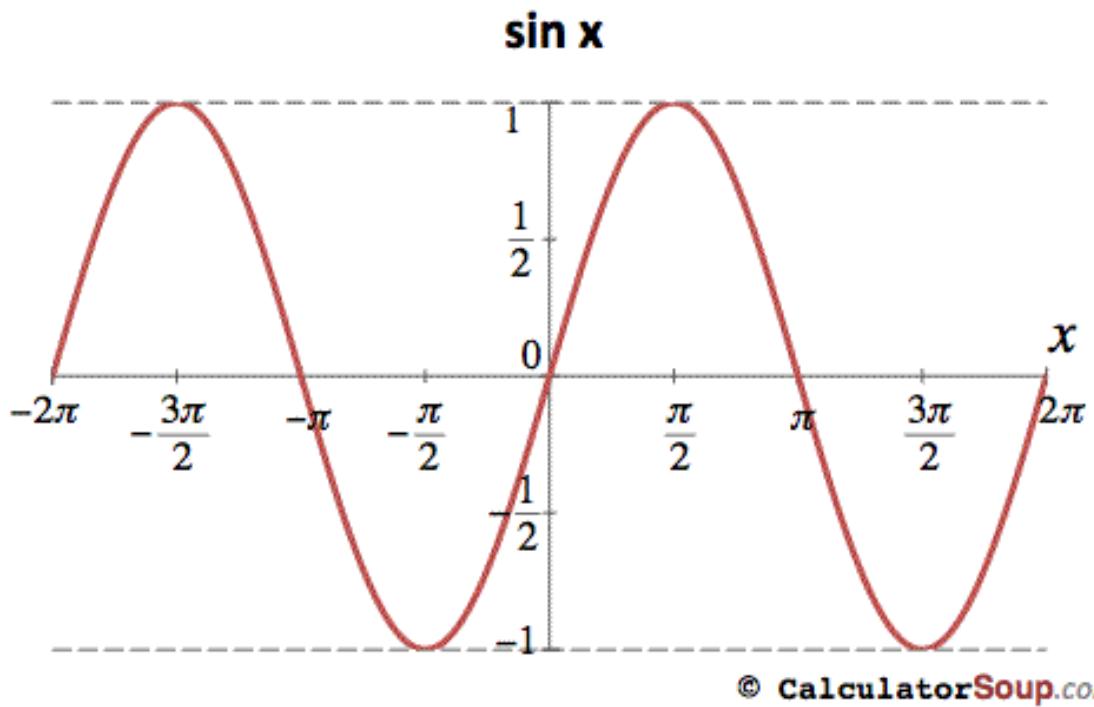
Remember that

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$$

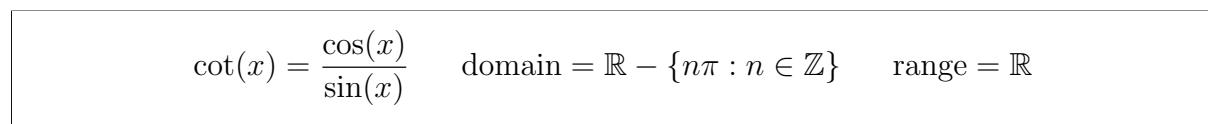
$$\csc(x) = \frac{1}{\sin(x)} \quad \sec(x) = \frac{1}{\cos(x)}.$$

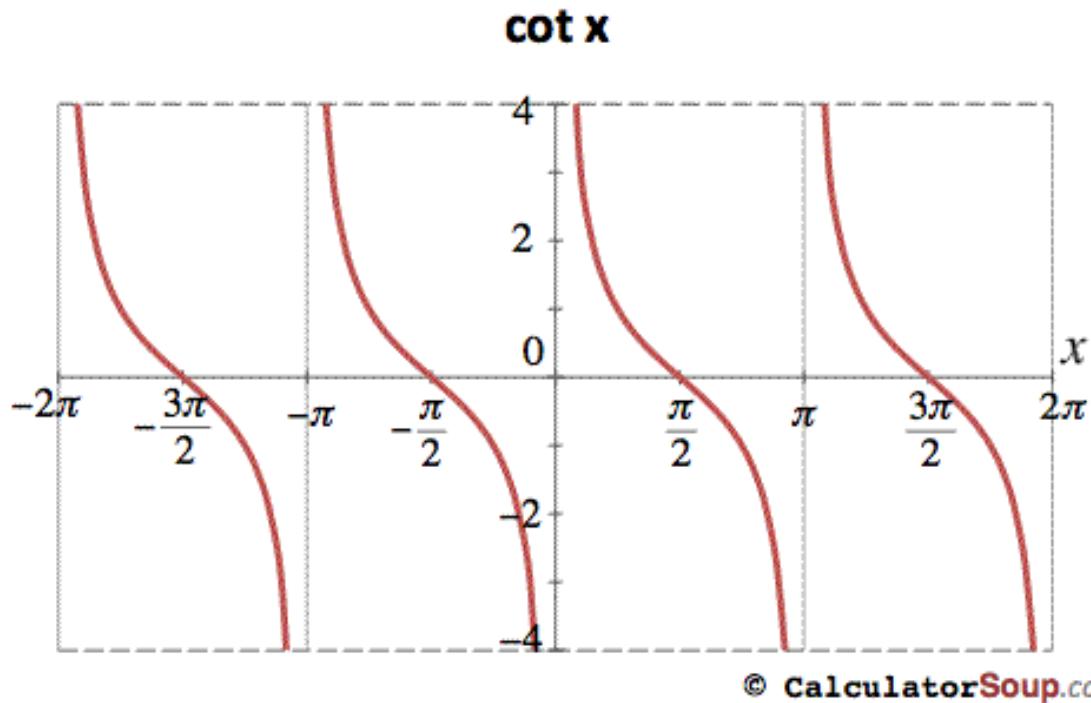
Let see the graph of each of this functions.

$$\sin(x) \quad \text{domain} = \mathbb{R} \quad \text{range} = [-1, 1]$$

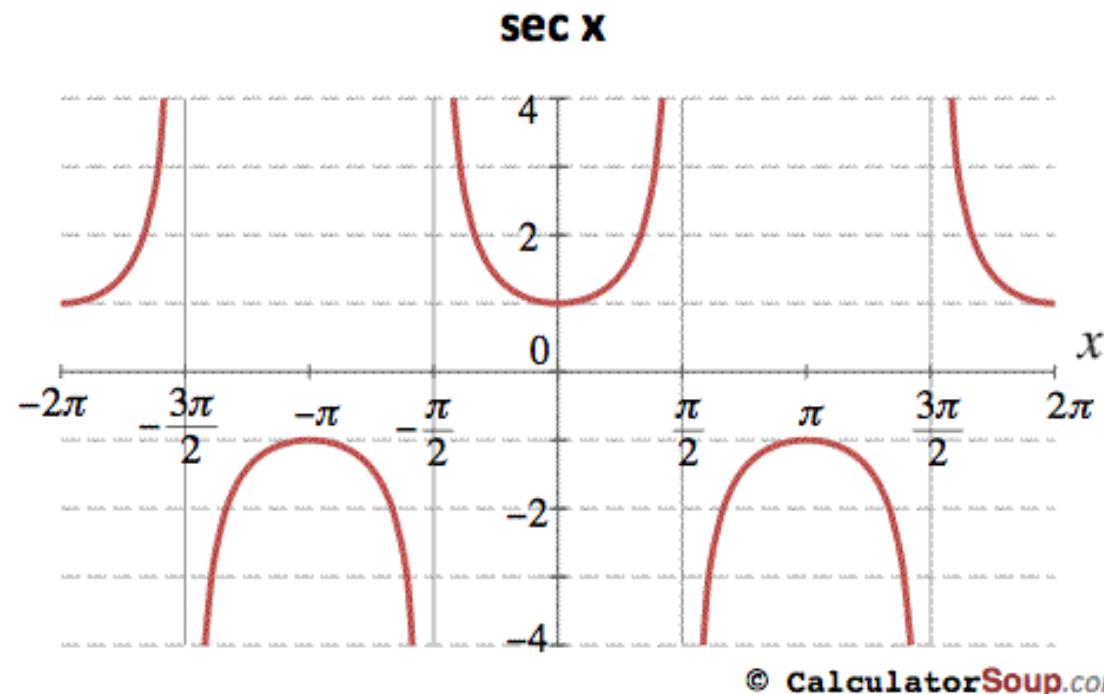


$$\cos(x) \quad \text{domain} = \mathbb{R} \quad \text{range} = [-1, 1]$$





$$\sec(x) = \frac{1}{\cos(x)} \quad \text{domain} = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R} \setminus (-1, 1)$$



$$\csc(x) = \frac{1}{\sin(x)} \quad \text{domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R} - (-1, 1)$$



Question: Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of $\sin(x)$ at $x = 0$.

Solution. Consider that by the definition of the derivative of a function, we have

$$\frac{d}{dx}|_{x=0} \sin(x) = \lim_{h \rightarrow 0} \frac{\sin(0 + h) - \sin(0)}{h} = \lim_{h \rightarrow 0} \frac{\sin(h)}{h}$$

Since $\lim_{h \rightarrow 0} \cos(h) = 1 = \lim_{h \rightarrow 0} 1$, and $\cos h \leq \frac{\sin h}{h} \leq 1$, by the sandwich theorem we have

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$$

The derivative of $\sin(x)$ and $\cos(x)$

Theorem 2.7.1. If $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Then

$$f'(0) = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \quad g'(0) = \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0.$$

And in general we have that

Theorem 2.7.2.

$$\frac{d}{dx} \sin(x) = \cos(x) \quad \frac{d}{dx} \cos(x) = -\sin(x).$$

The derivative of trigonometric functions

Theorem 2.7.3.

$$\frac{d}{dx} \tan(x) = \sec^2(x) \quad \frac{d}{dx} \cot(x) = -\csc^2(x)$$

$$\frac{d}{dx} \csc(x) = -\csc(x) \cot(x) \quad \frac{d}{dx} \sec(x) = \sec(x) \tan(x).$$

2.8 One More Tool—the Chain Rule

Write all of the tools that we have had so far.

You are out in the woods and are walking towards your camp fire. The heat from the fire means that the air temperature depends on your position. Let your position at time t be $x(t)$. The temperature of the air at position x is $f(x)$. What instantaneous rate of change of temperature do you feel at time t .

- Because your position at time t is $x(t)$, the temperature you feel at time t is $F(t) = f(x(t))$.
- The instantaneous rate of change of temperature that you feel is $F'(t)$. We have a complicated function, $F(t)$, constructed by composing two simple functions, $x(t)$, and $f(x)$.
- We wish to compute the derivative, $F'(t) = \frac{d}{dt}f(x(t))$, of the complicated function $F(t)$ in terms of the derivatives, $x'(t)$ and $f'(x)$, of the two simple functions. This is exactly what the chain rule does.



The chain rule

Theorem 2.8.1. Let $a \in \mathbb{R}$ and let $g(x)$ be a function that is differentiable at $x = a$. Now let $f(u)$ be a function that is differentiable at $u = g(a)$. Then the function $F(x) = f(g(x))$ is differentiable at $x = a$ and

$$F'(a) = f'(g(a))g'(a).$$

Theorem 2.8.2. Let f and g be differentiable functions then

$$\frac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$$

Theorem 2.8.3. Let $y = f(u)$ and $u = g(x)$ be differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

Example 2.8.4. Find the derivative of $(\sin(x))^5$ with respect to x .

Solution. Let $g(x) = \sin(x)$ $f(x) = x^5$. Then

$$f(g(x)) = (\sin(x))^5.$$

Therefore,

$$\frac{d}{dx}f(g(x)) = g'(x)f'(g(x)) = \cos(x)(5\sin(x)^4).$$

Example 2.8.5. Find the derivative of $\cos(3x - 2)$ with respect to x .

Solution. Let $f(x) = \cos(x)$ and $g(x) = 3x - 2$. Then $f(g(x)) = \cos(3x - 2)$, and so

$$\frac{d}{dx}\cos(3x - 2) = g'(x)f'(g(x)) = 3 \times -\sin(3x - 2).$$

Example 2.8.6. Find the derivative of the function F with respect to t where

$$F(x) = \sec x \quad x(t) = e^{t+1}$$

Solution. Consider that

$$\frac{dF}{dt} = \frac{dF}{dx} \frac{dx}{dt}.$$

Note that

$$\frac{dF}{dx} = \sec(x)\tan(x) = \sec(e^{t+1})\tan(e^{t+1})$$

and also $\frac{dx}{dt} = e^{t+1}$. Therefore,

$$\frac{dF}{dt} = \frac{dF}{dx} \frac{dx}{dt} = e^{t+1} \sec(e^{t+1})\tan(e^{t+1}).$$

Example 2.8.7. Find the derivative of $2^{\sqrt{x}}$.

Solution. Let $f(x) = 2^x$ and $g(x) = \sqrt{x}$. Then

$$f \circ g(x) = 2^{\sqrt{x}}.$$

Therefore,

$$\frac{d}{dx} 2^x = g'(x)f'(g(x)).$$

We have $g'(x) = \frac{1}{2\sqrt{x}}$ and $f'(x) = (\log_e 2)2^x$. Thus,

$$\frac{d}{dx} 2^x = g'(x)f'(g(x)) = \frac{1}{2\sqrt{x}}(\log_e 2)2^{\frac{1}{2\sqrt{x}}}.$$

Example 2.8.8. Find the tangent line to the curve $y = \sin(2^{\sqrt{x}})$ with respect to x .

Solution. Let $f(x) = \sin(x)$ and $g(x) = 2^{\sqrt{x}}$. Then $f(g(x)) = \sin(2^{\sqrt{x}}) = y$. Therefore,

$$\frac{dy}{dx} = f'(g(x)).g'(x) = \cos(2^{\sqrt{x}}).\frac{1}{2\sqrt{x}}(\log_e 2)2^{\frac{1}{2\sqrt{x}}}$$

2.9 Inverse Functions

Functions are really just rules for taking an input, processing it somehow and then returning an output.

input number $x \mapsto f$ does “stuff” to $x \mapsto$ return number y

In many situations it will turn out to be very useful if we can undo whatever it is that our functions has done. i.e,

take output $y \mapsto$ do “stuff” to $y \mapsto$ return the original number x

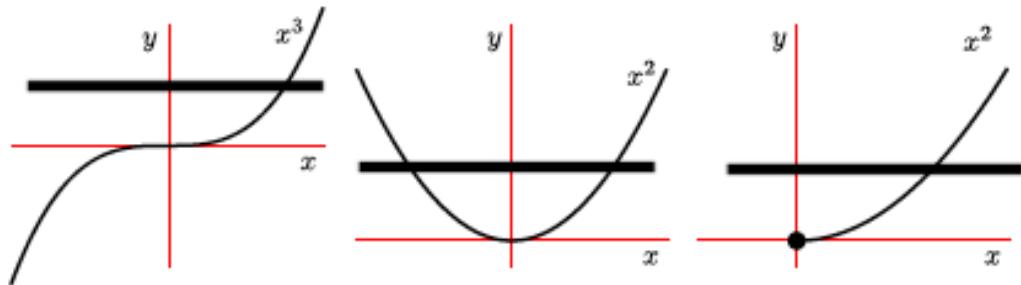
Definition. A function is one-to-one (injective) when it never takes the same value more than once. That is

$$\text{if } x_1 \neq x_2 \text{ then } f(x_1) \neq f(x_2)$$

Definition. A function is one-to-one if and only if no horizontal line $y = c$ intersects the graph $y = f(x)$ more than once.

Example 2.9.1. By the horizontal line test we will see that $y = x^3$ is one to one in its domain; however, $y = x^2$ is not one to one in its domain, but if we restrict $y = x^2$ to

only zero or positive numbers is one to one.



$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

is one-to-one

$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is not one-to-one

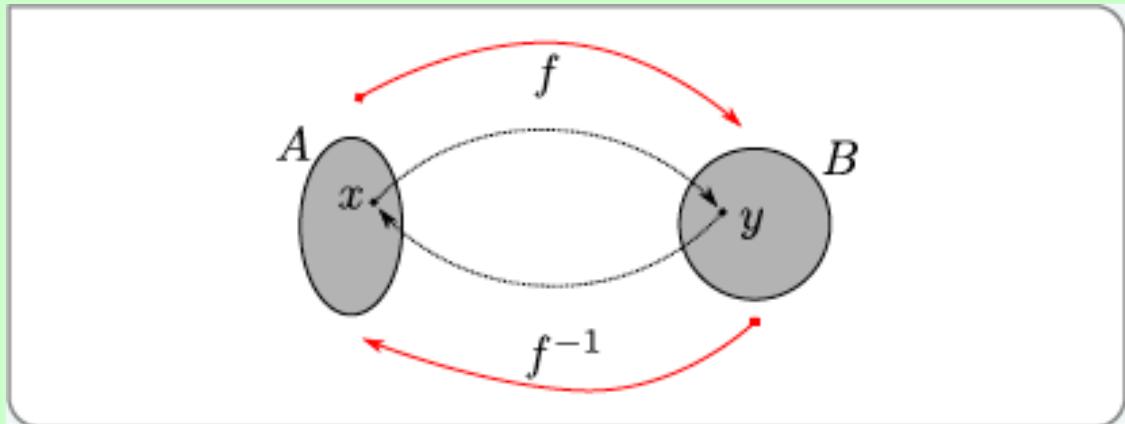
$$\begin{array}{ccc} [0, \infty] & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is one-to-one

Definition. Let f be a one-to-one function with domain A and range B . Then its inverse function is denoted f^{-1} and has domain B and range A . It is defined by

$$f^{-1}(y) = x \quad \text{whenever} \quad f(x) = y$$

for any $y \in B$.



So

$$f^{-1}(f(x)) = x \quad \text{for any } x \in A$$

$$f(f^{-1}(y)) = y \quad \text{for any } y \in B$$

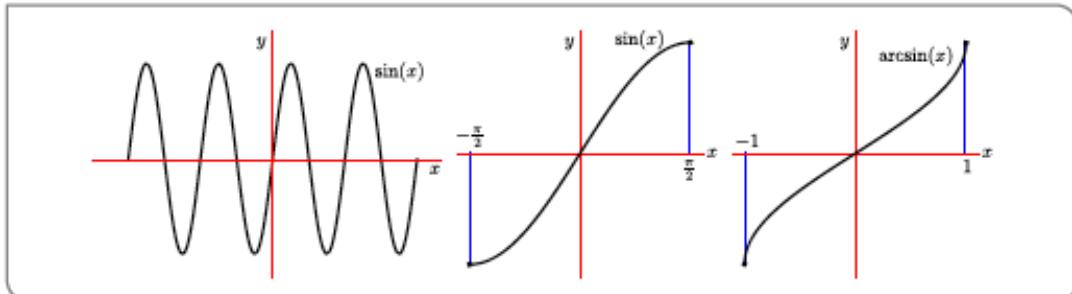
Caveat. We should be careful not to confuse $f^{-1}(x)$ with $\frac{1}{f(x)}$.

Example 2.9.2. What is the inverse of the following one to one functions.

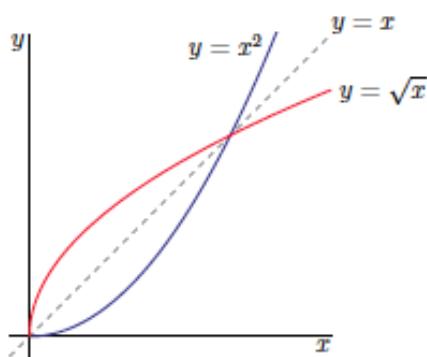
1. $f(x) = x^5 + 3$.
2. $g(x) = \sqrt{x-1}$ on the domain $x \geq 1$.

Solution. (1) Consider that $y = x^5 + 3$ so $y - 3 = x^5$, and then $x = \sqrt[5]{y-3}$. Therefore, $f^{-1}(x) = \sqrt[5]{x-3}$.
(2) Now consider that $y = \sqrt{x-1}$, and so $y^2 = x - 1$. Therefore, $y = \sqrt{x-1}$. Thus, $f^{-1}(x) = \sqrt{x-1}$.

Consider the graph of $\sin(x)$ on the domain $-\infty < x < +\infty$, it is easy to see that this function is not one to one, but when we consider this function on the domain $-\pi/2 \leq x \leq \pi/2$ this function is one to one and so it is invertible. The inverse of the function $\sin(x)$ is $\arcsin(x)$ and its domain is $[-1, 1]$ and its codomain is $[-\pi/2, \pi/2]$.



Finding the inverse of $f(x)$ by its graph. Assume that we have drawn the graph of a function, for example $y = x^2$ on the domain $x \geq 0$. To find its inverse which is $f^{-1}(x) = \sqrt{x}$, all you have to do is graph the function and then switch all x and y values in each point to graph the inverse. Just look at all those values switching places from the $f(x)$ function to its inverse $f^{-1}(x)$ (and back again), reflected over the line $y = x$.



2.10 The Natural Logarithm

Consider the logarithm base e — $\log_e x$ is the power that e must be raised to to give x . That is, $\log_e x$ is defined by

$$e^{\log_e x} = x,$$

consequently, it is the inverse of the exponential function with base e .

Moreover, note that

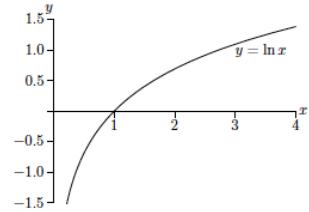
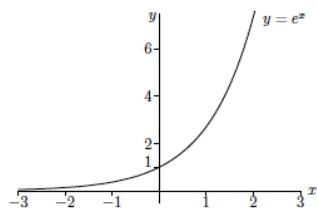
$$\log_q x = \frac{\log_e x}{\log_e q}$$

so we are free to chose a bases which is convenient for our purpose.

The logarithm with base e is called “natural logarithm”. The “naturalness” of logarithms base e is exactly that this choice of base works very nicely in calculus.

There are several standard notation for the logarithm base e ;

$$\log_e x = \log x = \ln x.$$



For any $x, y > 0$, the following hold:

- $e^{\log x} = x$,
- for any real number r , $\ln(e^r) = r$.
- for any $a > 1$, $\log_a x = \frac{\ln x}{\ln a}$ and $\ln x = \frac{\log_a x}{\log_a e}$.
- $\ln(xy) = \ln x + \ln(y)$
- $\ln(\frac{x}{y}) = \ln x - \ln y$, and $\ln(\frac{1}{y}) = -\ln y$
- $\ln(x^r) = r \ln x$
- $\lim_{x \rightarrow \infty} \ln x = \infty$ and $\lim_{x \rightarrow 0} \ln x = -\infty$.

► **What is the derivative of $\ln x$**

To find the derivative of $\ln x$ if you go with the definition of the derivative you don't arrive to something that looks good. However, we have

$$x = e^{\ln x}$$

and we know that if two functions are equal, then their derivative also are the same, therefore,

$$\frac{d}{dx}x = \frac{d}{dx}e^{\ln x}.$$

By the chain rule we have

$$1 = \left(\frac{d}{dx}\ln x\right)e^{\ln x}.$$

Thus,

$$1 = \left(\frac{d}{dx}\ln x\right)x$$

and so

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Derivative of $\ln x$

Theorem 2.10.1.

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Example 2.10.2. Let $f(x) = \ln 3x$. Find $f'(x)$.

Solution. Consider that $\ln 3x = \ln 3 + \ln x$ so $f'(x) = \ln 3 + \ln x$ and therefore,

$$f'(x) = \frac{1}{x}.$$

Example 2.10.3. Let $g(x) = \ln |x|$. Find $g'(x)$.

Since $|x|$ is not differentiable at $x = 0$, $g(x)$ is not differentiable at $x = 0$. Now we consider the following two cases:

- If $x > 0$, then $|x| = x$, and so

$$g'(x) = \frac{d}{dx}\ln x = \frac{1}{x}.$$

- if $x < 0$, then $|x| = -x$. Let $f_1(x) = \ln x$ and $f_2(x) = -x$, then

$$g(x) = f_1(f_2(x)) \Rightarrow g'(x) = f'_1(f_2(x))f'_2(x)$$

$$g'(x) = \frac{1}{-x} \times -1 = \frac{1}{x}.$$

What is $\frac{d}{dx} \log_a x$

So we want now find the derivative of $\log_a x$; consider that

$$\log_a x = \frac{\ln x}{\ln a}.$$

Therefore,

$$\frac{d}{dx} \log_a x = \frac{1}{\ln a} \frac{1}{x}.$$

► What is the derivative of $\ln f(x)$? Let $f_1(x) = \ln x$. Then

$$f_1(f(x)) = \ln f(x).$$

Therefore,

$$\frac{d}{dx} f_1(f(x)) = f'_1(f(x)) f'(x) = \frac{1}{f(x)} f'(x).$$

Derivative of $\ln |f(x)|$

► Let $f_1(x) = \ln |x|$ and $f_2(x) = f(x)$. Then

$$f_1(f_2(x)) = \ln |f(x)|.$$

Therefore,

$$\frac{d}{dx} \ln |f(x)| = f'_1(f_2(x)) f'_2(x) = \frac{1}{f(x)} f'(x)$$

► What is the derivative of a^x ? We have

$$f(x) = a^x$$

$$\ln f(x) = x \ln a$$

$$\frac{d}{dx} \ln f(x) = \ln a.$$

Using the chain rule to process the left-hand side we have

$$\frac{f'(x)}{f(x)} = \ln a$$

Therefore,

$$f'(x) = f(x) \ln a.$$

Then

$$f'(x) = (\ln a) a^x.$$

Theorem 2.10.4.

$$\frac{d}{dx} a^x = \ln a \cdot a^x \quad \text{for any } a > 0$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \cdot \ln a} \quad \text{for any } a > 0, a \neq 1$$

Logarithmic Differentiation

►► Logarithmic Differentiation We want to compute the derivative of the multiplication of three functions

$$p(x) = f(x)g(x)h(x)$$

by using the logarithm properties. Consider that

$$\ln p(x) = \ln f(x)g(x)h(x) = \ln f(x) + \ln g(x) + \ln h(x)$$

Therefore,

$$\frac{d}{dx} \ln p(x) = \frac{d}{dx} \ln f(x) + \frac{d}{dx} \ln g(x) + \frac{d}{dx} \ln h(x)$$

Thus

$$\frac{p'(x)}{p(x)} = \frac{f'(x)}{f(x)} \frac{g'(x)}{g(x)} \frac{h'(x)}{h(x)} \Rightarrow p'(x) = p(x) \left(\frac{f'(x)}{f(x)} \frac{g'(x)}{g(x)} \frac{h'(x)}{h(x)} \right).$$

Example 2.10.5. Compute the derivative of

$$f(x) = \frac{(\sqrt{x} - 1)(2 - x)(1 - x^2)}{\sqrt{x}(3 + 2x)}$$

Solution. Check that $\ln |f(x)| = \frac{f'(x)}{f(x)}$ for any function $f(x)$. Since $f(x)$ can be negative and \ln is not defined for negative numbers, we compute

$$\ln |f(x)| = \ln \frac{|(\sqrt{x} - 1)|(2 - x)|(1 - x^2)|}{|\sqrt{x}||(3 + 2x)|} =$$

$$\ln |\sqrt{x} - 1| + \ln |(2 - x)| + \ln |(1 - x^2)| - \ln |\sqrt{x}| - \ln |3 + 2x|$$

Therefore,

$$\frac{f'(x)}{f(x)} = \frac{1/(2\sqrt{x})}{\sqrt{x} - 1} + \frac{-1}{2 - x} + \frac{-2x}{1 - x^2} - \frac{1/(2\sqrt{x})}{\sqrt{x}} - \frac{2}{3 + 2x}.$$

So we have that

$$f'(x) = f(x) \left(\frac{1/(2\sqrt{x})}{\sqrt{x} - 1} + \frac{-1}{2 - x} + \frac{-2x}{1 - x^2} - \frac{1/(2\sqrt{x})}{\sqrt{x}} - \frac{2}{3 + 2x} \right)$$

Homework 1:

Go to this link
<https://www.mooculus.osu.edu/textbook/mooculus.pdf> and download the book "MOOCULUS". Then do the following questions:

- all questions in page 35;
- in page 33 see why $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$. Then do Questions 1-8 page 38;
- in page 42, do questions 1-10.

Homework 2:

- First set of questions (CLICK) by Prof. DRAGOS GHIOCA.
- Second set of questions (CLICK) by Prof. DRAGOS GHIOCA.

Bibliography

- [1] CLP1: Differential Calculus by J. Feldman, A. Rechnitzer, and E. Yeager.

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