

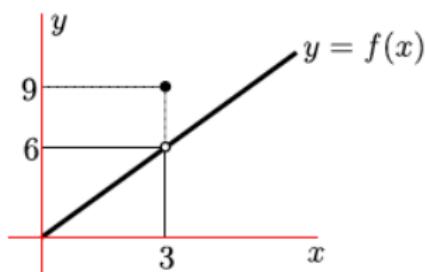
# MATH 100

Farid Aliniaiefard

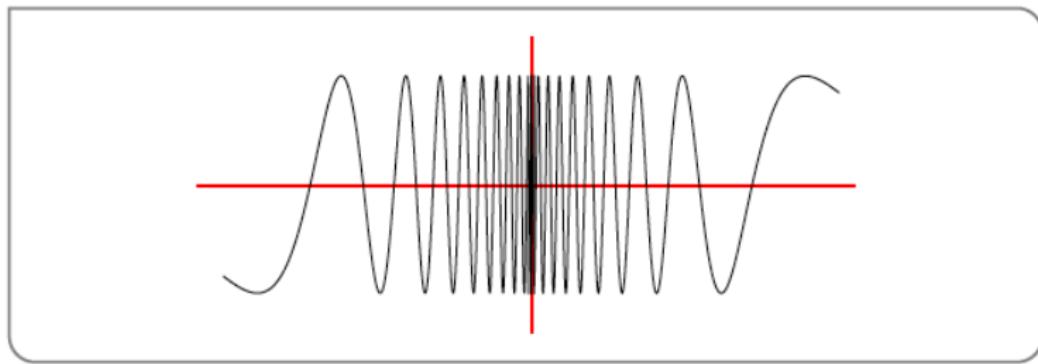
**University of British Columbia**

2019

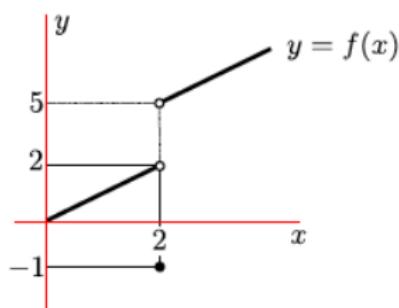
$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



$$f(x) = \sin\left(\frac{\pi}{x}\right)$$

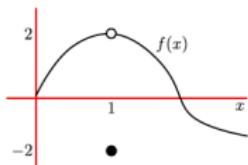


$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$



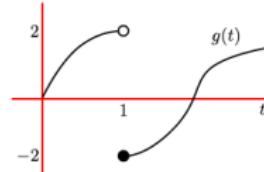
## Example

Consider the graph of the function  $f(x)$ .



## Example

Consider the graph of the function  $g(t)$ .



Then

$$\lim_{x \rightarrow 1^-} f(x) =$$

$$\lim_{x \rightarrow 1^+} f(x) =$$

$$\lim_{x \rightarrow 1} f(x) =$$

Then

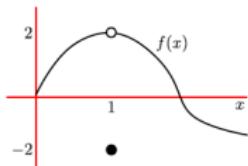
$$\lim_{t \rightarrow 1^-} g(t) =$$

$$\lim_{t \rightarrow 1^+} g(t) =$$

$$\lim_{t \rightarrow 1} g(t) =$$

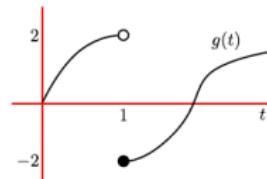
## Example

Consider the graph of the function  $f(x)$ .



## Example

Consider the graph of the function  $g(t)$ .



Then

$$\lim_{x \rightarrow 1^-} f(x) = 2$$

$$\lim_{x \rightarrow 1^+} f(x) = 2$$

$$\lim_{x \rightarrow 1} f(x) = 2$$

$$\lim_{t \rightarrow 1^-} g(t) = 2$$

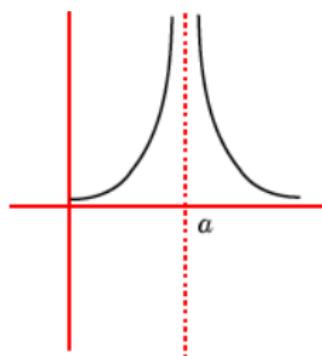
$$\lim_{t \rightarrow 1^+} g(t) = -2$$

$$\lim_{t \rightarrow 1} g(t) = DNE$$

# When the limit goes to infinity

## Example

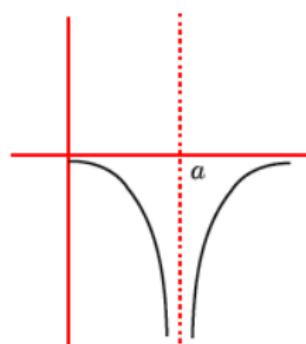
Consider the graph for the function  $f(x)$ .



$$\lim_{x \rightarrow a} f(x) = +\infty$$

## Example

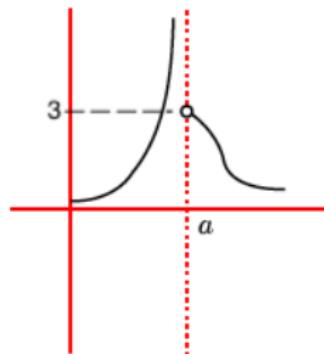
Consider the graph for the function  $g(x)$ .



$$\lim_{x \rightarrow a} g(x) = -\infty$$

## Example

Consider the graph for the function  $h(x)$ .

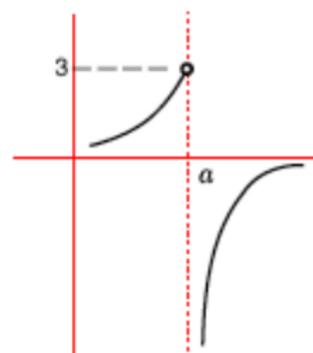


$$\lim_{x \rightarrow a^-} h(x) =$$

$$\lim_{x \rightarrow a^+} h(x) =$$

## Example

Consider the graph for the function  $s(x)$ .

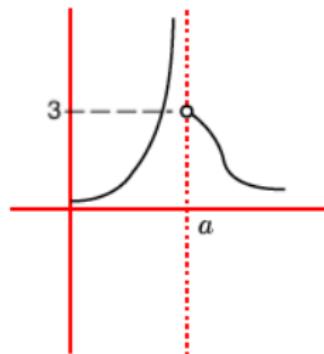


$$\lim_{x \rightarrow a^-} s(x) =$$

$$\lim_{x \rightarrow a^+} s(x) =$$

## Example

Consider the graph for the function  $h(x)$ .

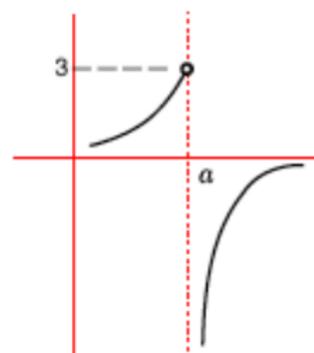


$$\lim_{x \rightarrow a^-} h(x) = +\infty$$

$$\lim_{x \rightarrow a^+} h(x) = 3$$

## Example

Consider the graph for the function  $s(x)$ .



$$\lim_{x \rightarrow a^-} s(x) = 3$$

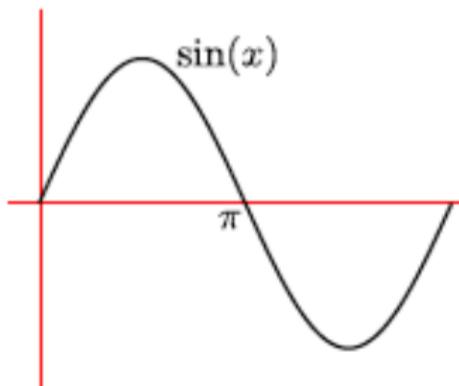
$$\lim_{x \rightarrow a^+} s(x) = -\infty$$

## Example

Consider the function

$$g(x) = \frac{1}{\sin(x)}.$$

Find the one-side limits of this function as  $x \rightarrow \pi$ .



$$\lim_{x \rightarrow \pi^-} \frac{1}{\sin(x)} = +\infty$$

$$\lim_{x \rightarrow \pi^+} \frac{1}{\sin(x)} = -\infty$$

## Second Session Outline

- ▶ Arithmetic of the Limits
- ▶ Limit of a ratio: what will happen if the limit of the denominator is zero. For example,

$$\lim_{x \rightarrow 0} \frac{1}{x^2} ? \quad \text{and} \quad \lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = ?$$

- ▶ Sandwich/ Squeeze/Pinch Theorem
- ▶ limit at infinity

## Arithmetic of the Limits

## Theorem

Let  $a, c \in \mathbb{R}$ . The following two limits hold

$$\lim_{x \rightarrow a} c = c \quad \lim_{x \rightarrow a} x = a$$

## Example

$$\lim_{x \rightarrow 3} -2 = -2 \quad \lim_{x \rightarrow -1} x = -1$$

## Theorem

**(Arithmetic of Limits)** Let  $a, c \in \mathbb{R}$ , let  $f(x)$  and  $g(x)$  be defined for all  $x$ 's that lie in some interval about  $a$  (but  $f$  and  $g$  need not be defined exactly at  $a$ ).

$$\lim_{x \rightarrow a} f(x) = F \quad \lim_{x \rightarrow a} g(x) = G$$

exists with  $F, G \in \mathbb{R}$ . Then the following limits hold

- ▶  $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$ —limit of the sum is the sum of the limits.

## Theorem

**(Arithmetic of Limits)** Let  $a, c \in \mathbb{R}$ , let  $f(x)$  and  $g(x)$  be defined for all  $x$ 's that lie in some interval about  $a$  (but  $f$  and  $g$  need not be defined exactly at  $a$ ).

$$\lim_{x \rightarrow a} f(x) = F \quad \lim_{x \rightarrow a} g(x) = G$$

exists with  $F, G \in \mathbb{R}$ . Then the following limits hold

- ▶  $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$  – limit of the sum is the sum of the limits.
- ▶  $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$  – limit of the difference is the difference of the limits.

## Theorem

**(Arithmetic of Limits)** Let  $a, c \in \mathbb{R}$ , let  $f(x)$  and  $g(x)$  be defined for all  $x$ 's that lie in some interval about  $a$  (but  $f$  and  $g$  need not be defined exactly at  $a$ ).

$$\lim_{x \rightarrow a} f(x) = F \quad \lim_{x \rightarrow a} g(x) = G$$

exists with  $F, G \in \mathbb{R}$ . Then the following limits hold

- ▶  $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$  – limit of the sum is the sum of the limits.
- ▶  $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$  – limit of the difference is the difference of the limits.
- ▶  $\lim_{x \rightarrow a} cf(x) = cF$ .

## Theorem

**(Arithmetic of Limits)** Let  $a, c \in \mathbb{R}$ , let  $f(x)$  and  $g(x)$  be defined for all  $x$ 's that lie in some interval about  $a$  (but  $f$  and  $g$  need not be defined exactly at  $a$ ).

$$\lim_{x \rightarrow a} f(x) = F \quad \lim_{x \rightarrow a} g(x) = G$$

exists with  $F, G \in \mathbb{R}$ . Then the following limits hold

- ▶  $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$  – limit of the sum is the sum of the limits.
- ▶  $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$  – limit of the difference is the difference of the limits.
- ▶  $\lim_{x \rightarrow a} cf(x) = cF$ .
- ▶  $\lim_{x \rightarrow a} (f(x).g(x)) = F.G$  – limit of the product is the product of the limits.

If  $G \neq 0$  then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}$$

## Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) =$$

## Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) =$$

## Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) =$$

## Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x \rightarrow 1} \frac{f(x)}{f(x) - g(x)} =$$

## Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x \rightarrow 1} \frac{f(x)}{f(x) - g(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x)} = \frac{3}{3 - 2} = 3.$$

## Example

$$\lim_{x \rightarrow 3} 4x^2 - 1 =$$

$$\lim_{x \rightarrow 2} \frac{x}{x - 1} =$$

## Example

$$\lim_{x \rightarrow 3} 4x^2 - 1 = 4 \times \lim_{x \rightarrow 3} x^2 - \lim_{x \rightarrow 3} 1 = 35.$$

$$\lim_{x \rightarrow 2} \frac{x}{x-1} = \frac{\lim_{x \rightarrow 2} x}{\lim_{x \rightarrow 2} x - \lim_{x \rightarrow 1} 1} = \frac{2}{2-1} = 2.$$

**Limit of a ratio: what will happen if the limit of the denominator is zero.**

## Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = DNE$$

## Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = DNE$$

- the **limit is**  $\pm\infty$ , eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x^4} = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{-x^2}{x^4} = \lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty.$$

## Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = DNE$$

- the **limit is**  $\pm\infty$ , eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x^4} = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{-x^2}{x^4} = \lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty.$$

- the **limit is** 0, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

## Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = DNE$$

- the **limit is  $\pm\infty$** , eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x^4} = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{-x^2}{x^4} = \lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty.$$

- the **limit is 0**, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

- the **limit exists and it nonzero**, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x} = 1.$$

## Theorem

Let  $n$  be a positive integer, let  $a \in R$  and let  $f$  be a function so that

$$\lim_{x \rightarrow a} f(x) = F$$

for some real number  $F$ . Then the following holds

$$\lim_{x \rightarrow a} (f(x))^n = \left( \lim_{x \rightarrow a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit.

## Theorem

Let  $n$  be a positive integer, let  $a \in R$  and let  $f$  be a function so that

$$\lim_{x \rightarrow a} f(x) = F$$

for some real number  $F$ . Then the following holds

$$\lim_{x \rightarrow a} (f(x))^n = \left( \lim_{x \rightarrow a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit.

Similarly, if

- ▶  $n$  is an even number and  $F > 0$ , or
- ▶  $n$  is an odd number and  $F$  is any real number

then

$$\lim_{x \rightarrow a} (f(x))^{1/n} = \left( \lim_{x \rightarrow a} f(x) \right)^{1/n} = F^{1/n}.$$

## Example

$$\lim_{x \rightarrow 4} x^{1/2} =$$

$$\lim_{x \rightarrow 4} (-x)^{1/2} =$$

$$\lim_{x \rightarrow 2} (4x^2 - 3)^{1/3} =$$

## Example

$$\lim_{x \rightarrow 4} x^{1/2} = 4^{1/2} = 2.$$

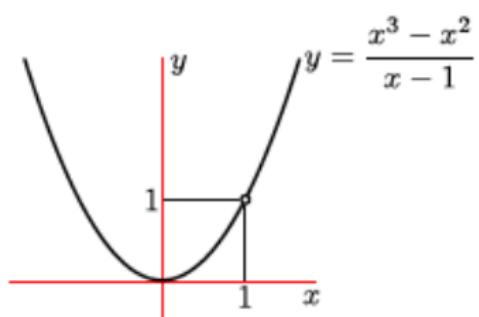
$$\lim_{x \rightarrow 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \rightarrow 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = (13)^{1/3}.$$

**Limit of a ratio: what will happen if the limit of the numerator and denominator are zero,  
for example,**

$$\lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = ?$$

$$\lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = ?$$



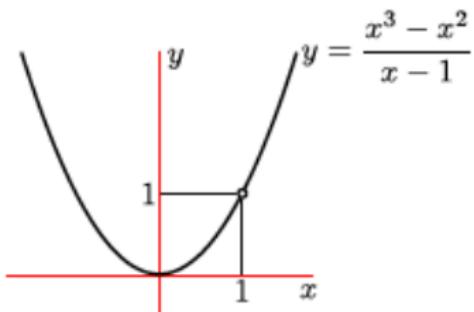
## Theorem

If  $f(x) = g(x)$  except when  $x = a$  then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

provided the limit of  $g$  exists.

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases} \Rightarrow \lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \rightarrow 1} x^2 = 1.$$

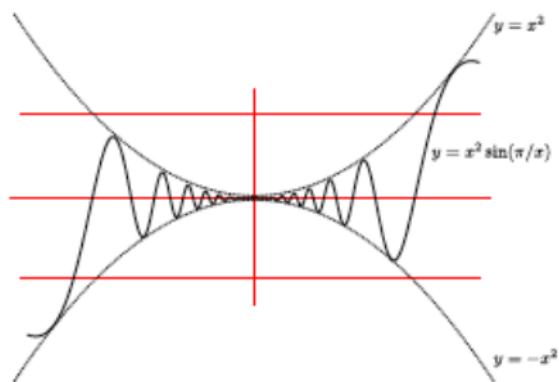
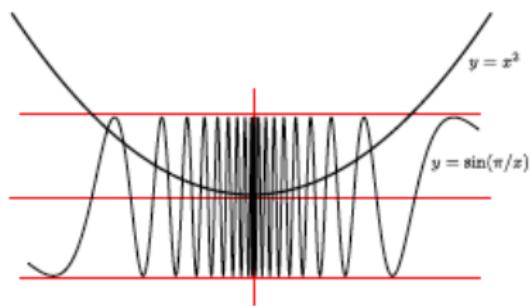


## Sandwich/ Squeeze/Pinch Theorem

## Example

Compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



## Example

Let  $f(x)$  be a function such that  $1 \leq f(x) \leq x^2 - 2x + 2$ . What is

$$\lim_{x \rightarrow 1} f(x)?$$

## Example

Let  $f(x)$  be a function such that  $1 \leq f(x) \leq x^2 - 2x + 2$ . What is

$$\lim_{x \rightarrow 1} f(x)?$$

## Solution

*Consider that*

$$\lim_{x \rightarrow 1} x = 1 \quad \text{and} \quad \lim_{x \rightarrow 1} x^2 - 2x + 2 = 1.$$

*Therefore, by the sandwich/pinch/squeeze theorem*

$$\lim_{x \rightarrow 1} f(x) = 1.$$

## Example

We want to compute

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{x}$$

By plug in some large numbers into  $\frac{1}{x}$  we have

-10000	-1000	-100	100	1000	10000
-0.0001	-0.001	-0.01	0.01	0.001	0.0001

We see that as  $x$  is getting bigger and positive the function  $\frac{1}{x}$  is getting closer to 0. Thus,

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0.$$

Moreover,

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

## **Limit at Infinity**

## Definition

**(Informal limit at infinity.)** We write

$$\lim_{x \rightarrow \infty} f(x) = L$$

when the value of the function  $f(x)$  gets closer and closer to  $L$  as we make  $x$  larger and larger and positive.

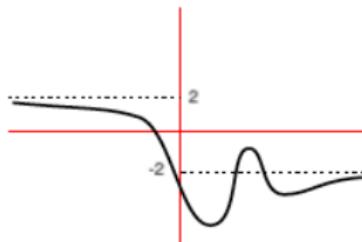
Similarly, we write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

when the value of the function  $f(x)$  gets closer and closer to  $L$  as we make  $x$  larger and larger and negative.

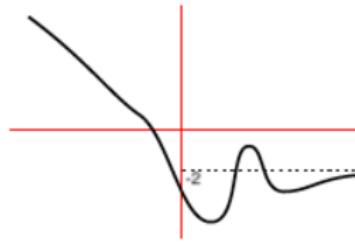
## Example

Consider the graph of the function  $f(x)$ .



## Example

Consider the graph of the function  $g(x)$ .



Then

$$\lim_{x \rightarrow \infty} f(x) =$$

$$\lim_{x \rightarrow -\infty} f(x) =$$

Then

$$\lim_{x \rightarrow \infty} g(x) =$$

$$\lim_{x \rightarrow -\infty} g(x) =$$

## Example

Consider the graph of the function  $f(x)$ .



## Example

Consider the graph of the function  $g(x)$ .



Then

$$\lim_{x \rightarrow \infty} f(x) = -2$$

$$\lim_{x \rightarrow -\infty} f(x) = 2$$

Then

$$\lim_{x \rightarrow \infty} g(x) = -2$$

$$\lim_{x \rightarrow -\infty} g(x) = +\infty$$

## **Review of the third session**

# Review

## Theorem

**sandwich (or squeeze or pinch)** Let  $a \in \mathbb{R}$  and let  $f, g, h$  be three functions so that

$$f(x) \leq g(x) \leq h(x)$$

for all  $x$  in an interval around  $a$ , except possibly at  $x = a$ . Then if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

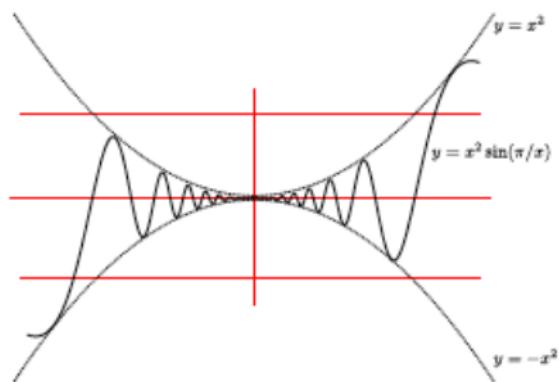
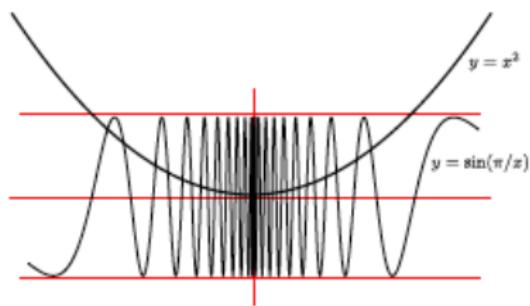
then it is also the case that

$$\lim_{x \rightarrow a} g(x) = L.$$

## Example

Compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



## Theorem

Let  $c \in \mathbb{R}$  then the following limits hold

$$\lim_{x \rightarrow +\infty} c = c \quad \lim_{x \rightarrow -\infty} c = c$$

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

# Outline For the Fourth Session

- ▶ Limit at Infinity

## Limit at Infinity

## Theorem

Let  $f(x)$  and  $g(x)$  be two functions for which the limits

$$\lim_{x \rightarrow \infty} f(x) = F \quad \lim_{x \rightarrow \infty} g(x) = G$$

exist. Then the following limits hold

$$\lim_{x \rightarrow \infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x \rightarrow \infty} f(x)g(x) = FG$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \text{provided } G \neq 0$$

and for rational numbers  $r$ ,

$$\lim_{x \rightarrow \infty} (f(x))^r = F^r$$

provided that  $f(x)^r$  is defined for all  $x$ .

The analogous results hold for limits to  $-\infty$ .



**Warning:** Consider that

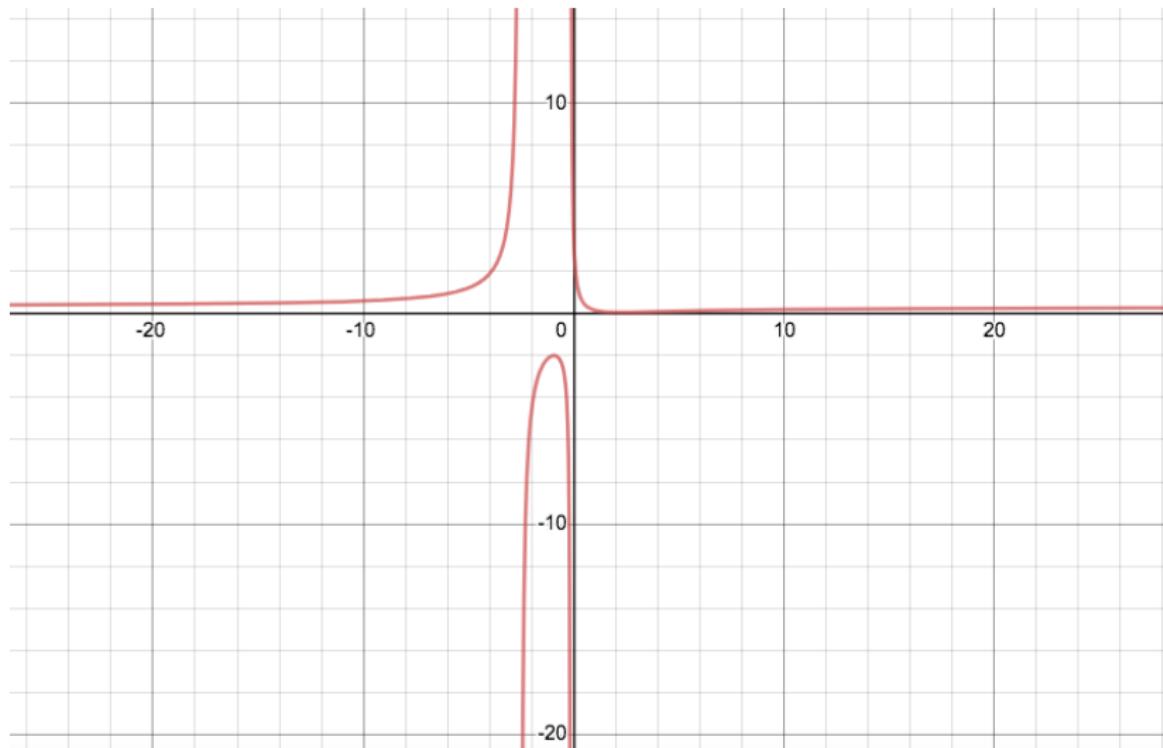
$$\lim_{x \rightarrow +\infty} \frac{1}{x^{1/2}} = 0$$

However,

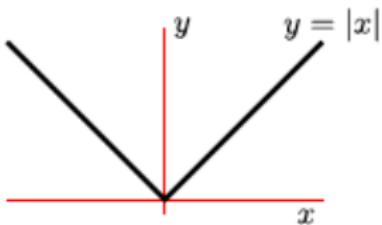
$$\lim_{x \rightarrow +\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because  $x^{1/2}$  is not defined for  $x < 0$ .

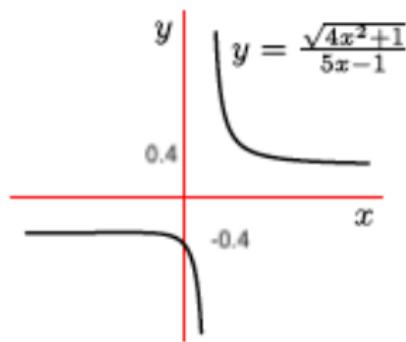
$$f(x) = \frac{x^2 - 3x + 4}{3x^2 + 8x + 1}$$



$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



$$y = \frac{\sqrt{4x^2 + 1}}{5x - 1}$$



## Theorem

Let  $a, c, H \in \mathbb{R}$  and let  $f, g, h$  be functions defined in an interval around  $a$  (but they need not be defined at  $x = a$ ), so that

$$\lim_{x \rightarrow a} f(x) = +\infty \quad \lim_{x \rightarrow a} g(x) = +\infty \quad \lim_{x \rightarrow a} h(x) = H$$

1.

$$\lim_{x \rightarrow a} (f(x) + g(x)) =$$

2.

$$\lim_{x \rightarrow a} (f(x) + h(x)) =$$

3.

$$\lim_{x \rightarrow a} (f(x) - g(x)) =$$

4.

$$\lim_{x \rightarrow a} (f(x) - h(x)) =$$

## Theorem

Let  $a, c, H \in \mathbb{R}$  and let  $f, g, h$  be functions defined in an interval around  $a$  (but they need not be defined at  $x = a$ ), so that

$$\lim_{x \rightarrow a} f(x) = +\infty \quad \lim_{x \rightarrow a} g(x) = +\infty \quad \lim_{x \rightarrow a} h(x) = H$$

1.

$$\lim_{x \rightarrow a} (f(x) + g(x)) = +\infty.$$

2.

$$\lim_{x \rightarrow a} (f(x) + h(x)) = +\infty.$$

3.

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \text{undetermined}.$$

4.

$$\lim_{x \rightarrow a} (f(x) - h(x)) = +\infty.$$

## Theorem

5.

$$\lim_{x \rightarrow a} cf(x) = \begin{cases} c > 0 \\ c = 0 \\ c < 0 \end{cases}$$

6.

$$\lim(f(x) \cdot g(x)) =$$

7.

$$\lim_{x \rightarrow a} (f(x) \cdot h(x)) = \begin{cases} H > 0 \\ H = 0 \\ H < 0 \end{cases}$$

8.

$$\lim_{x \rightarrow a} \frac{h(x)}{f(x)} =$$

## Theorem

5.

$$\lim_{x \rightarrow a} cf(x) = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$$

6.

$$\lim(f(x).g(x)) = +\infty.$$

7.

$$\lim_{x \rightarrow a} (f(x).h(x)) = \begin{cases} +\infty & H > 0 \\ undetermined & H = 0 \\ -\infty & H < 0 \end{cases}$$

8.

$$\lim_{x \rightarrow a} \frac{h(x)}{f(x)} = 0.$$

## Example

Consider the following three functions:

$$f(x) = x^{-2} \quad g(x) = 2x^{-2} \quad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \rightarrow 0} f(x) = +\infty \quad \lim_{x \rightarrow 0} g(x) = +\infty \quad \lim_{x \rightarrow 0} h(x) = +\infty.$$

Then

1.

$$\lim_{x \rightarrow 0} (f(x) - g(x)) =$$

2.

$$\lim_{x \rightarrow 0} (f(x) - h(x)) =$$

3.

$$\lim_{x \rightarrow 0} (g(x) - h(x)) =$$

## Example

Consider the following three functions:

$$f(x) = x^{-2} \quad g(x) = 2x^{-2} \quad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \rightarrow 0} f(x) = +\infty \quad \lim_{x \rightarrow 0} g(x) = +\infty \quad \lim_{x \rightarrow 0} h(x) = +\infty.$$

Then

1.

$$\lim_{x \rightarrow 0} (f(x) - g(x)) = \lim_{x \rightarrow 0} x^{-2} = \infty$$

2.

$$\lim_{x \rightarrow 0} (f(x) - h(x)) = \lim_{x \rightarrow 0} (1) = 1$$

3.

$$\lim_{x \rightarrow 0} (g(x) - h(x)) = \lim_{x \rightarrow 0} x^{-2} + 1 = \infty$$

# Outline For the Session Five

- ▶ Limit at Infinity
- ▶ Continuity
- ▶ Continuous from the left and from the right
- ▶ Arithmetic of continuity
- ▶ continuity of composites
- ▶ Intermediate Value Theorem

## Example

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

$x$	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{1}{x^2}$	100	10000	$10^6$		$10^6$	10000	100

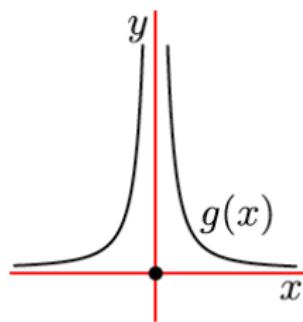
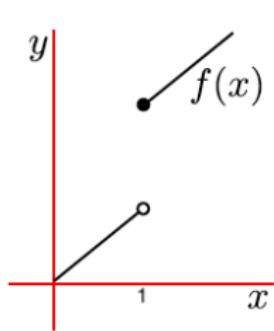
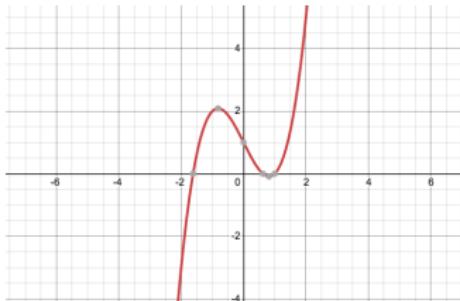
Consider that if

$$\lim_{x \rightarrow a} f(x) = \infty \quad \lim_{x \rightarrow a} g(x) = \infty$$

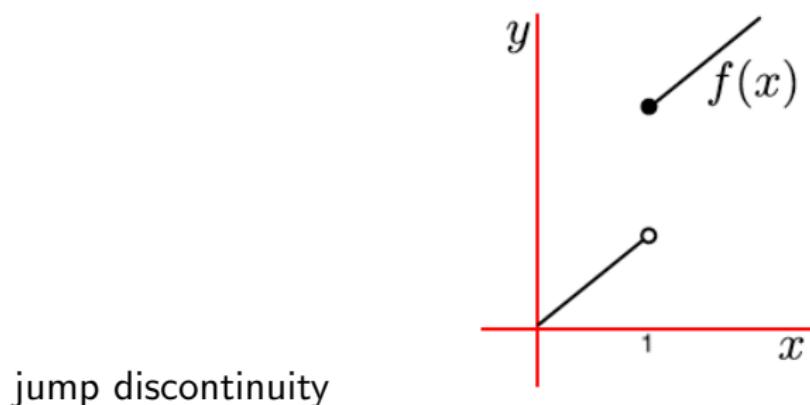
Then

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \text{undetermined}$$

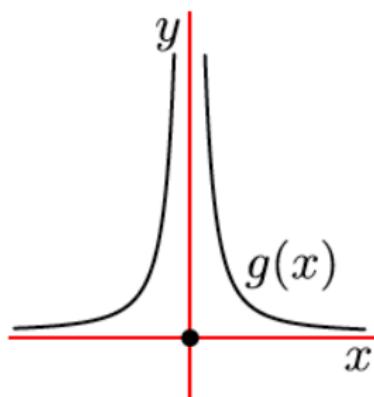
## Continuity



$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$



$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



infinite discontinuity

$$h(x) = \begin{cases} \frac{x^3 - x^2}{x - 1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$



removable discontinuity

# Outline - September 16, 2019

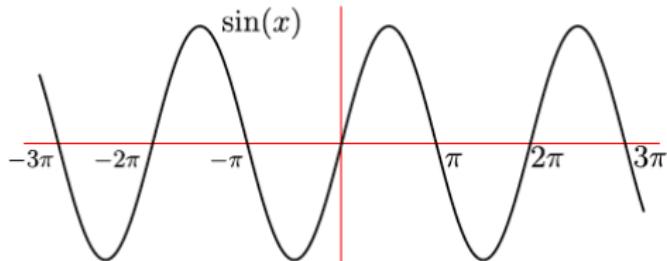
- ▶ **Section 1.6:**
  - ▶ Arithmetic of continuity
  - ▶ Continuity of composites
  - ▶ Intermediate Value Theorem
- ▶ **Section 2.1:**
  - ▶ Revisiting tangent lines

## Arithmetic of continuity

## Theorem

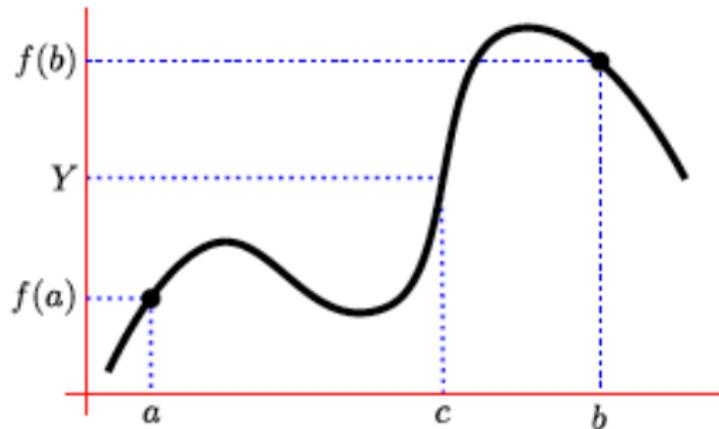
**(Arithmetic of continuity)** Let  $a, c \in \mathbb{R}$  and let  $f(x)$  and  $g(x)$  be functions that are continuous at  $a$ . Then the following functions are also continuous at  $x = a$ .

- ▶  $f(x) + g(x)$  and  $f(x) - g(x)$ ,
- ▶  $cf(x)$  and  $f(x)g(x)$ , and
- ▶  $\frac{f(x)}{g(x)}$  provided  $g(a) \neq 0$ .

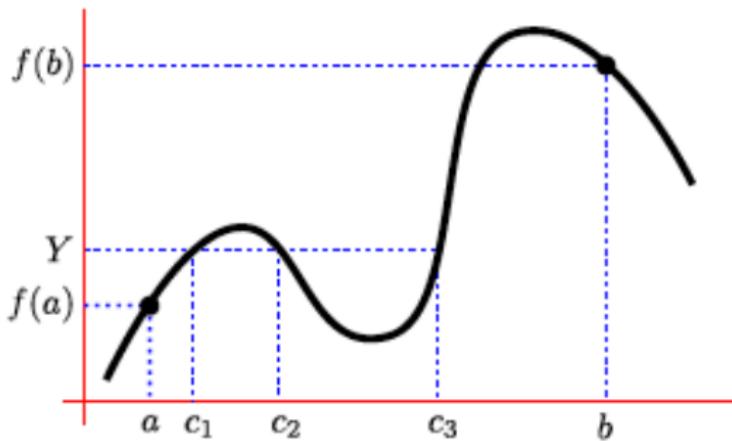


## Intermediate value theorem(IVT)

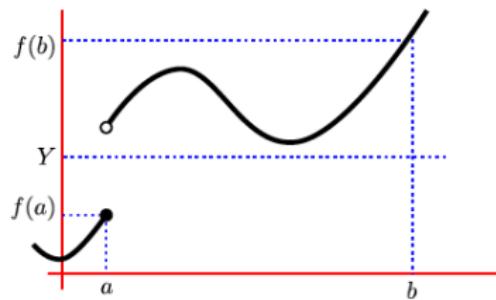
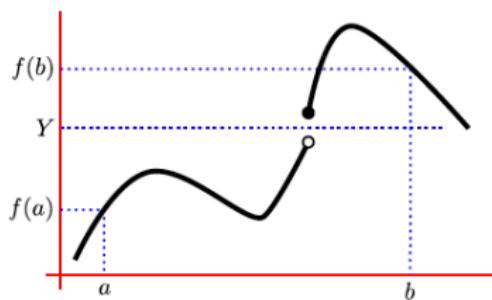
## Theorem (Intermediate value theorem(IVT))



## The existence not the uniqueness of $c$ in IVT



## Not continuous functions at $[a, b]$ do not satisfy IVT



## Revisiting tangent lines

## Revisiting tangent lines



$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \leftarrow \text{slope of the tangent line at } x = 1$$

## Definition of the derivative



$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

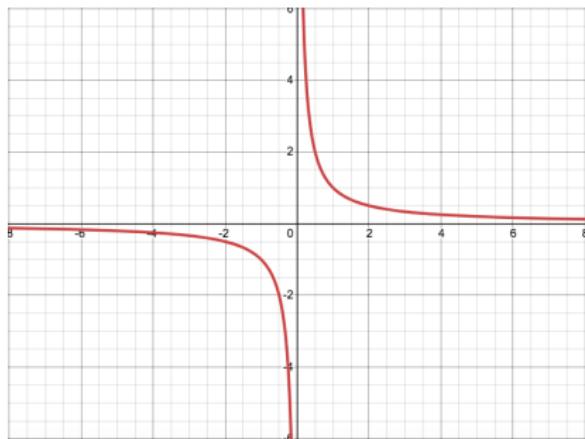


$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

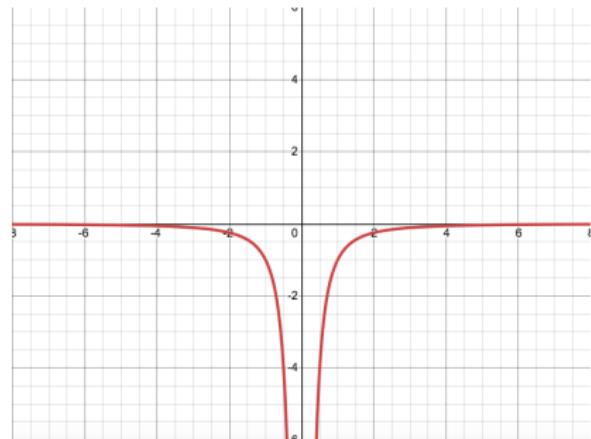
## Examples

- ▶  $f(x) = c$
- ▶  $f(x) = x$
- ▶  $f(x) = x^2$
- ▶  $f(x) = \frac{1}{x}$
- ▶  $f(x) = \sqrt{x}$
- ▶  $f(x) = |x|$

$y = \frac{1}{x}$  and its derivative  $-\frac{1}{x^2}$



$$y = \frac{1}{x}$$

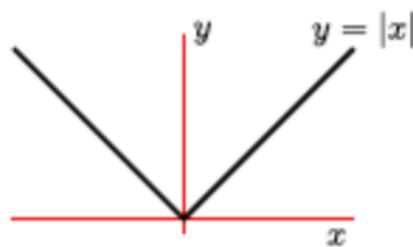


$$y = \frac{-1}{x^2}$$

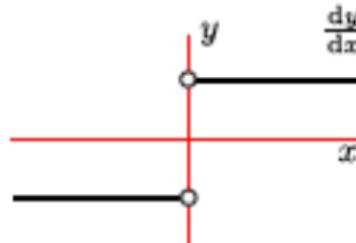
## Tangent lines to $y = \sqrt{x}$



The derivative of the function  $f(x) = |x|$ : not differentiable at  $x = 0$



The derivative of the function  $f(x) = |x|$

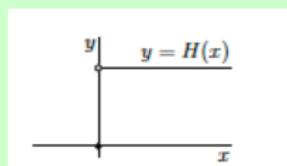


## Where a function is not differentiable at $x = a$ ?

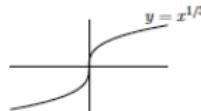
- ▶ Having a Sharp Corner at  $x = a$



- ▶ The function is not continuous at  $x = a$



- ▶ Having a tangent line, but the slope of the tangent line at  $x = a$  is infinity



# Outline - September 20, 2019

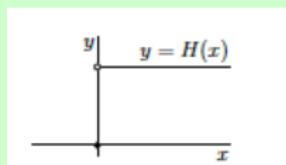
- ▶ **Section 2.2:**
  - ▶ Not differentiable examples
  - ▶ The relation between continuous and differentiable functions
- ▶ **Section 2.3:**
  - ▶ Interpretations of the derivative

## Where a function is not differentiable at $x = a$ ?

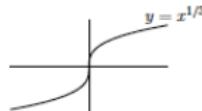
- ▶ Having a Sharp Corner at  $x = a$



- ▶ The function is not continuous at  $x = a$

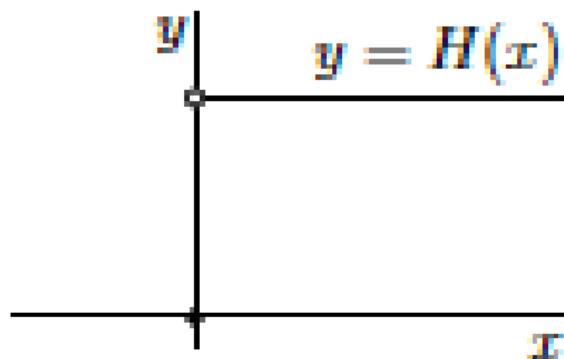


- ▶ Having a tangent line, but the slope of the tangent line at  $x = a$  is infinity



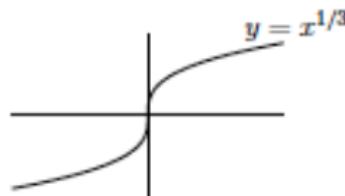
An example of a discontinuous and not differentiable function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$



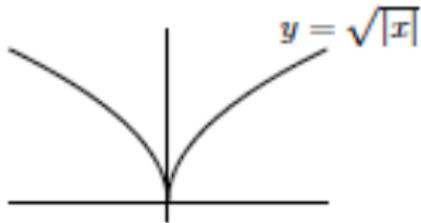
An example of a function with a tangent line with slope infinity at  $x = 0$

$$f(x) = x^{1/3}$$

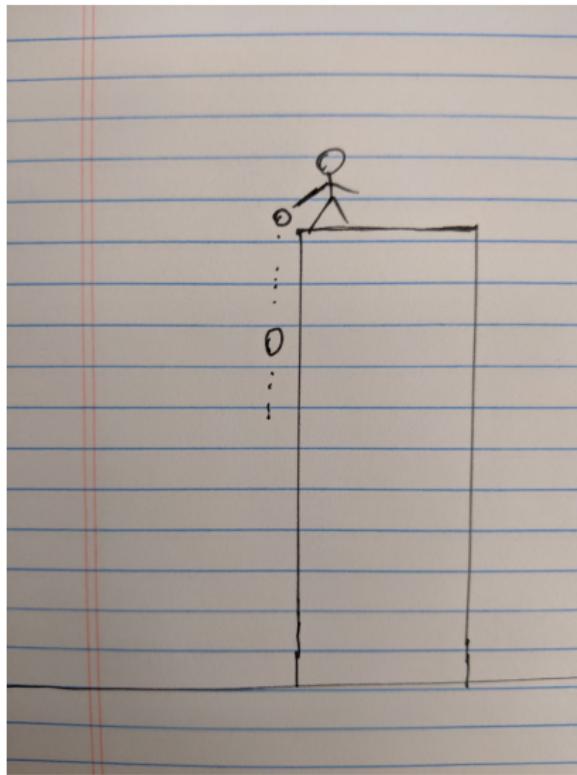


An example of a continuous and **not** differentiable function

$$y = \sqrt{|x|}$$



# Instantaneous rate of change



**average** rate of change of  $f(t)$  from  $t = a$  to  $t = a + h$  is

$$\frac{\text{change in } f(t) \text{ from } t = a \text{ to } t = a + h}{\text{length of time from } t = a \text{ to } t = a + h}$$

$$= \frac{f(a + h) - f(a)}{h}.$$

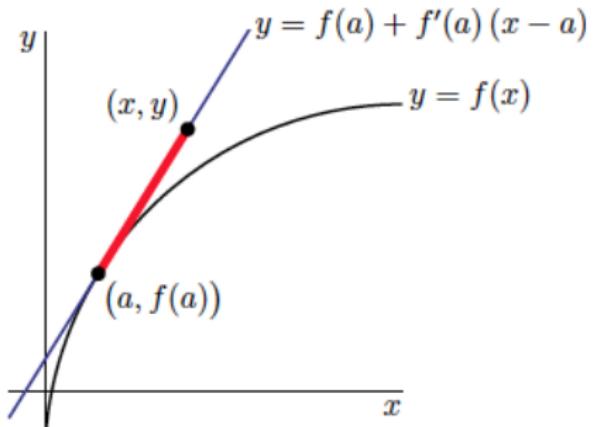
And so

**instantaneous** rate of change of  $f(t)$  at  $t = a$

$$= \lim_{h \rightarrow 0} [\text{average rate of change of } f(t) \text{ from } t = a \text{ to } t = a + h]$$

$$= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} = f'(a).$$

## Finding tangent line to a curve at $x = a$



$$y = f(a) + f'(a)(x - a)$$

# Outline - September 23, 2019

- ▶ **Section 2.4 and 2.5:**
  - ▶ Derivative of some simple functions
  - ▶ Tools
  - ▶ Examples

## A list of derivative of some simple functions:

$$\frac{d}{dx} 1 = 0$$

$$\frac{d}{dx} x = 1$$

$$\frac{d}{dx} x^2 = 2x$$

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}.$$

## A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \quad \frac{d}{dx}x = 1 \quad \frac{d}{dx}x^2 = 2x \quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

## Tools

Let  $f(x)$  and  $g(x)$  be differentiable functions and let  $c, d \in \mathbb{R}$ .

- ▶  $\frac{d}{dx}\{f(x) + g(x)\} = f'(x) + g'(x)$
- ▶  $\frac{d}{dx}\{f(x) - g(x)\} = f'(x) - g'(x)$
- ▶  $\frac{d}{dx}\{cf(x)\} = cf'(x)$

## Tools

Let  $f(x)$ ,  $g(x)$ , and  $h(x)$  be differentiable functions and let  $c, d \in \mathbb{R}$ .

- ▶  $\frac{d}{dx} \{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- ▶  $\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$

## Tools

Let  $f(x)$ ,  $g(x)$ , and  $h(x)$  be differentiable functions and let  $c, d \in \mathbb{R}$ .

- ▶  $\frac{d}{dx} \{ f(x)g(x) \} = f'(x)g(x) + g'(x)f(x)$
- ▶  $\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶  $\frac{d}{dx} \{ cf(x) + dg(x) \} = cf'(x) + dg'(x)$
- ▶  $\frac{d}{dx} \{ f(x)^2 \} = 2f(x)f'(x)$
- ▶  $\frac{d}{dx} \left\{ \frac{1}{g(x)} \right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$

## Tools

Let  $f(x)$ ,  $g(x)$ , and  $h(x)$  be differentiable functions and let  $c, d \in \mathbb{R}$ .

- ▶  $\frac{d}{dx} \{ f(x)g(x) \} = f'(x)g(x) + g'(x)f(x)$
- ▶  $\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶  $\frac{d}{dx} \{ cf(x) + dg(x) \} = cf'(x) + dg'(x)$
- ▶  $\frac{d}{dx} \{ f(x)^2 \} = 2f(x)f'(x)$
- ▶  $\frac{d}{dx} \left\{ \frac{1}{g(x)} \right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶  $\frac{d}{dx} \{ f(x)g(x)h(x) \} =$   
 $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- ▶  $\frac{d}{dx} \{ f(x)^n \} = nf^{n-1}(x)f'(x)$

## Tools

Let  $f(x)$ ,  $g(x)$ , and  $h(x)$  be differentiable functions and let  $c, d \in \mathbb{R}$ .

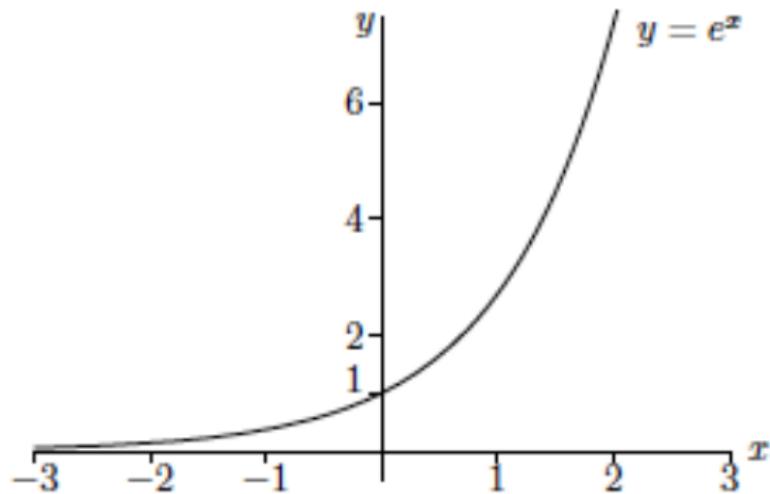
- ▶  $\frac{d}{dx} \{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- ▶  $\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶  $\frac{d}{dx} \{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- ▶  $\frac{d}{dx} \{f(x)^2\} = 2f(x)f'(x)$
- ▶  $\frac{d}{dx} \left\{ \frac{1}{g(x)} \right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶  $\frac{d}{dx} \{f(x)g(x)h(x)\} =$   
 $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- ▶  $\frac{d}{dx} \{f(x)^n\} = nf^{n-1}(x)f'(x)$
- ▶ Let  $a$  be a rational number, then

$$\frac{d}{dx} x^a = ax^{a-1}.$$

# Outline - September 25, 2019

- ▶ **Section 2.7 and 2.8:**
  - ▶ Derivative of exponential functions
  - ▶ Derivative of trigonometric functions

# The graph of $e^x$



# The graph of $q^x$ where $q > 1$



# YOUR TURN!

## Example

Find  $a$  such that the following function is continuous.

$$f(x) = \begin{cases} e^{x+a} & x < 0 \\ \sqrt{x+1} & x \geq 0 \end{cases}$$

## Example

We have

1.  $\log_q(xy) =$

- (a)  $\log_q(x) + \log_q(y)$
- (b)  $\log_q(x) \log_q(y)$

2.  $\log_q(x/y) =$

3.  $\log_q(x^r) =$

## Example

We have

$$1. \log_q(xy) = \log_q(x) + \log_q(y).$$

The reason for this is that

$$q^{\log_q(xy)} = xy = q^{\log_q(x)}q^{\log_q(y)} = q^{\log_q(x)+\log_q(y)}$$

Therefore,  $\log_q(xy) = \log(x) + \log(y)$ .

$$2. \log_q(x/y) = \log_q(x) - \log_q(y)$$

$$3. \log_q(x^r) = r \log_q(x)$$

## TOOLS:

$$\frac{d}{dx}(f \circ g)(x) = g'(x)f'(g(x))$$

## A list of derivative of some simple functions:

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = (\log_e a)a^x$$

## Example

Find the derivative of  $2^{\sqrt{x}}$ .

## Example

Find the derivative of  $2^{\sqrt{x}}$ .

## Example

Find  $a$  and  $b$  such that the following function is differentiable.

$$f(x) = \begin{cases} x^3 + a & x < 1 \\ e^{x-1} + bx & x \geq 1 \end{cases}$$

# Outline - September 30, 2019

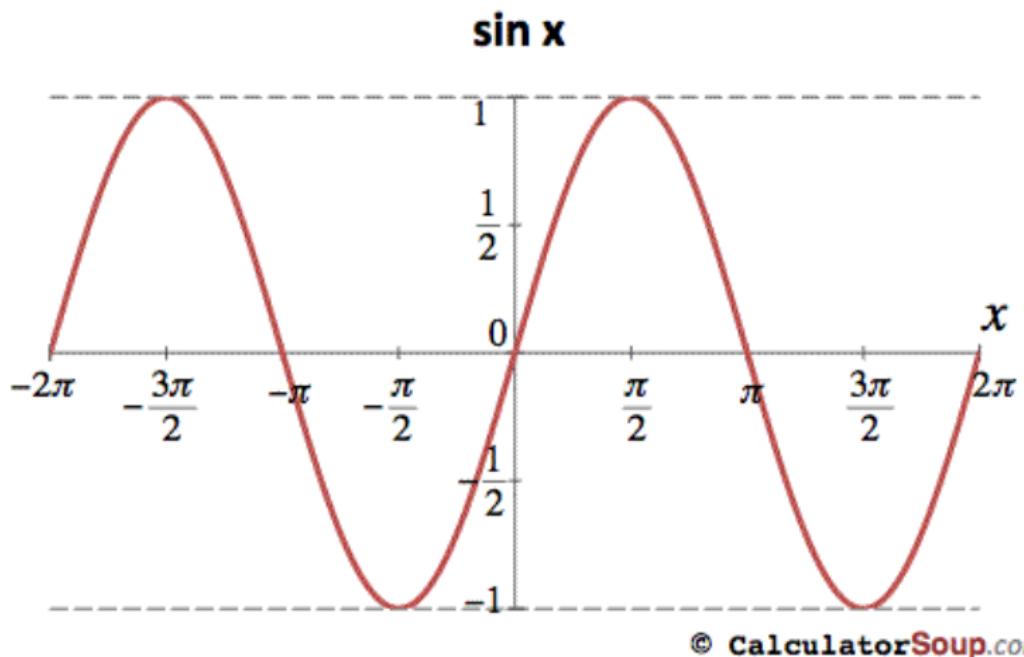
- ▶ **Section 2.8, 2.9, 0.6:**
  - ▶ Derivative of trigonometric functions
  - ▶ The chain rule
  - ▶ inverse of a function

## A list of derivative of some simple functions:

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} a^x = (\log_e a) a^x$$

$\sin(x)$  domain =  $\mathbb{R}$  range =  $[-1, 1]$

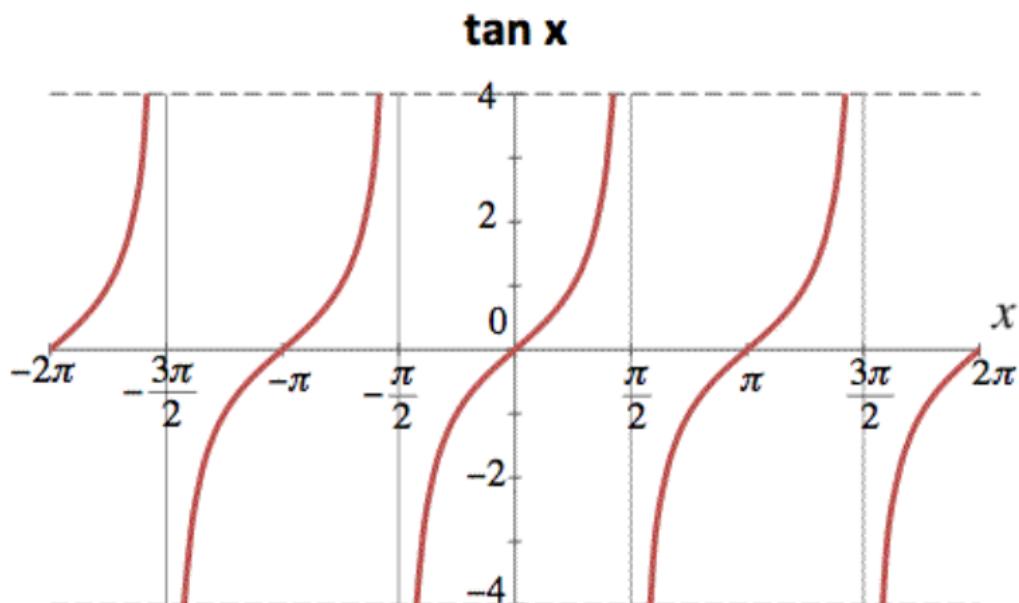


$\cos(x)$  domain =  $\mathbb{R}$  range =  $[-1, 1]$



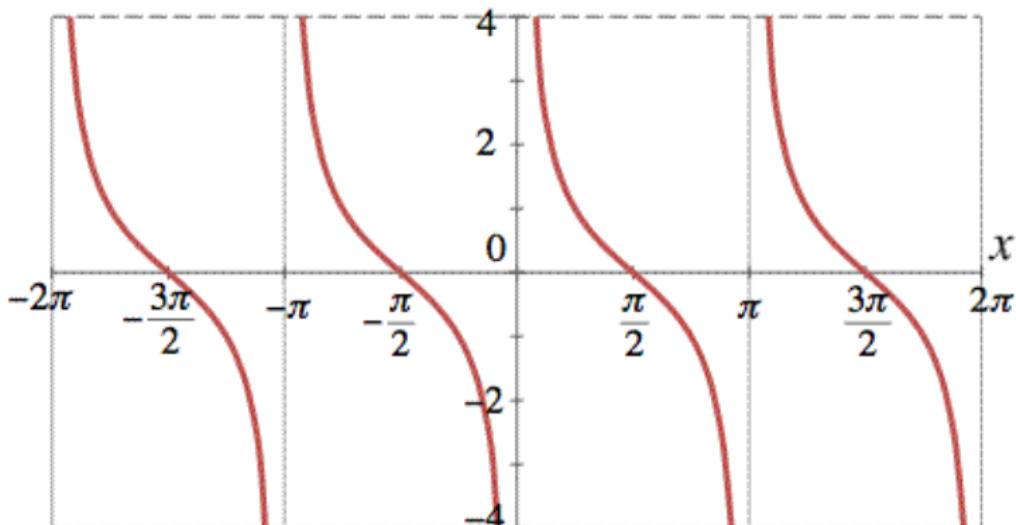
© CalculatorSoup.com

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \text{domain} = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R}$$



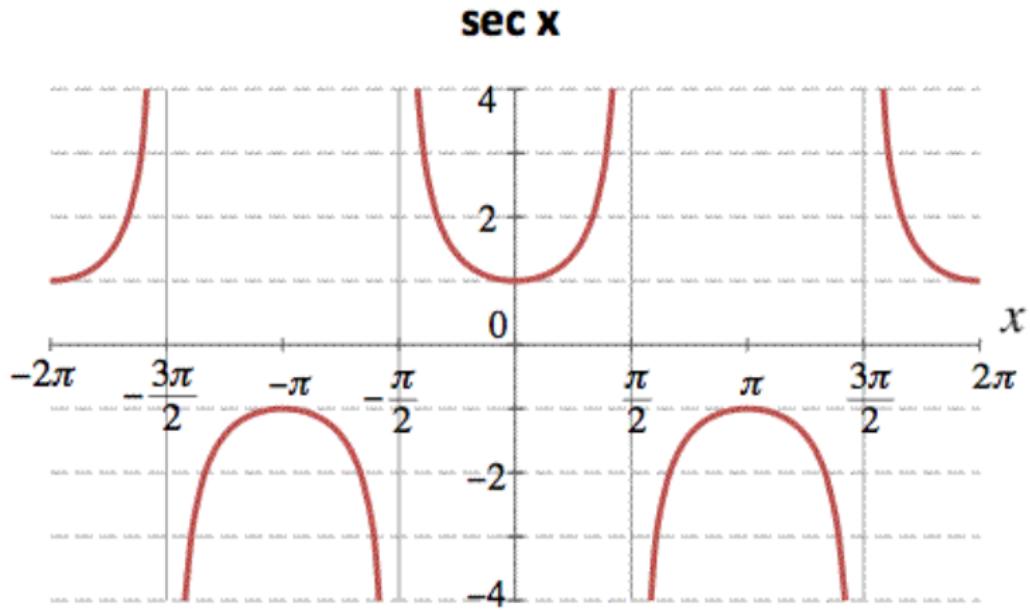
$$\cot(x) = \frac{\cos(x)}{\sin(x)} \text{ domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \text{ range} = \mathbb{R}$$

**cot x**



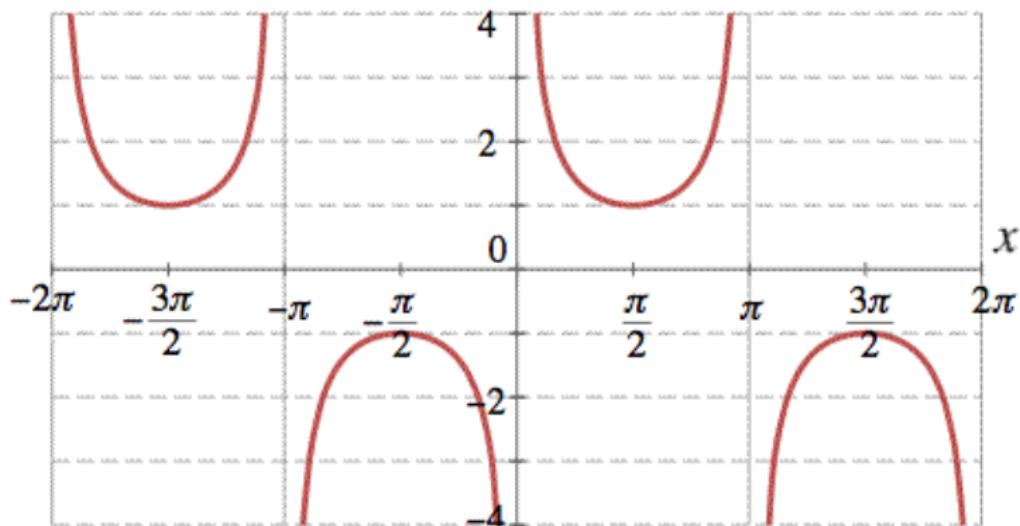
$$\sec(x) = \frac{1}{\cos(x)} \quad \text{domain} = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$$

$$\text{range} = \mathbb{R} - (-1, 1)$$



$$\csc(x) = \frac{1}{\sin(x)} \text{ domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \text{ range} = \mathbb{R} - (-1, 1)$$

### CSC X



## Derivative of $\sin(x)$

**Question:** Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of  $\sin(x)$  at  $x = 0$ .

## Derivative of $\sin(x)$

**Question:** Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of  $\sin(x)$  at  $x = 0$ .

**(sandwich (or squeeze or pinch) theorem )** Let  $a \in \mathbb{R}$  and let  $f, g, h$  be three functions so that  $f(x) \leq g(x) \leq h(x)$  for all  $x$  in an interval around  $a$ , except possibly at  $x = a$ . Then if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then it is also the case that

$$\lim_{x \rightarrow a} g(x) = L.$$

## An example of the application of the chain rule



- ▶ Your position at time  $t$  is  $x(t)$ .
- ▶ The temperature of the air at position  $x$  is  $f(x)$ .
- ▶ The temperature that you feel at time  $t$  is  $F(t) = f(x(t))$ .
- ▶ The instantaneous rate of change of temperature that you feel is  $F'(t)$ .

## The chain rule

### Theorem

Let  $f$  and  $g$  be differentiable functions then

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

## The chain rule

### Theorem

Let  $y = f(u)$  and  $u = g(x)$  be differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

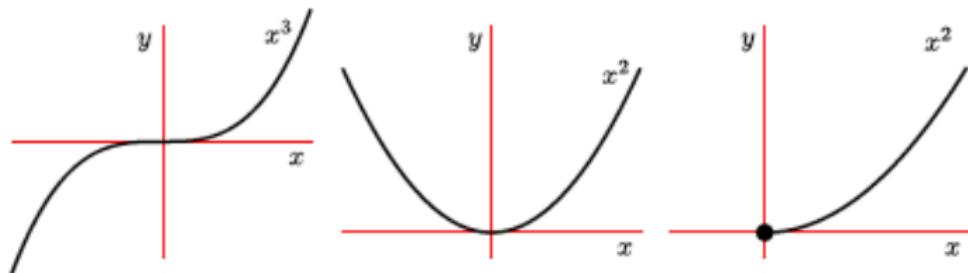
# Outline - October 2, 2019

- ▶ **Section 0.6, 2.10:**
  - ▶ Inverse of a function
  - ▶ Natural logarithm

input number  $x \mapsto f$  does “stuff” to  $x \mapsto$  return number  $y$

take output  $y \mapsto$  do “stuff” to  $y \mapsto$  return the original  
number  $x$

# One-to-one functions

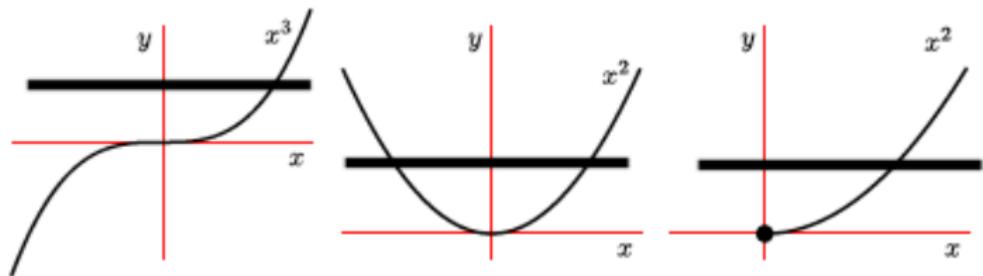


$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

$$\begin{array}{ccc} [0, \infty] & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

# One-to-one functions



$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

is one-to-one

$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is not one-to-one

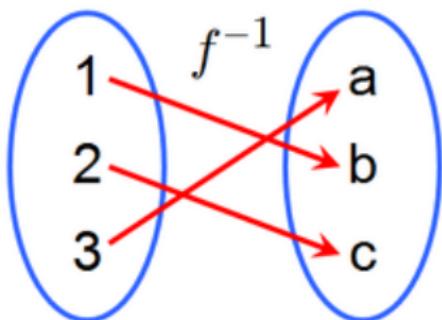
$$\begin{array}{ccc} [0, \infty) & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is one-to-one

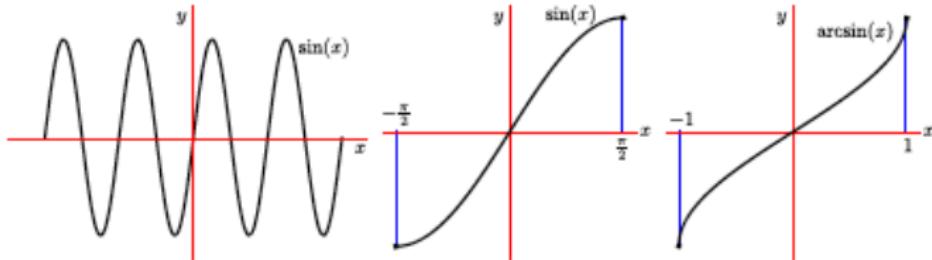
## Inverse of a functions



## Inverse of a functions



# Inverse of $\sin(x)$

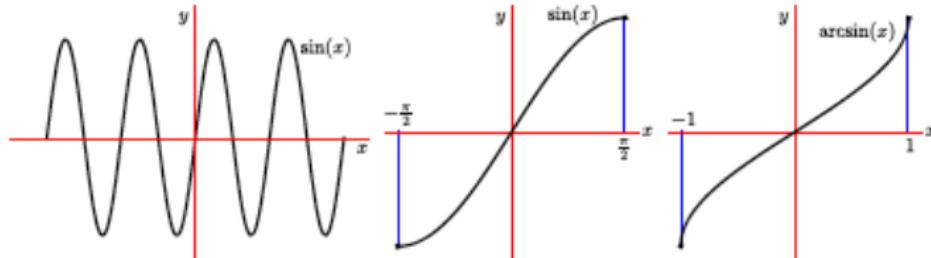


## Inverse of $\sin(x)$



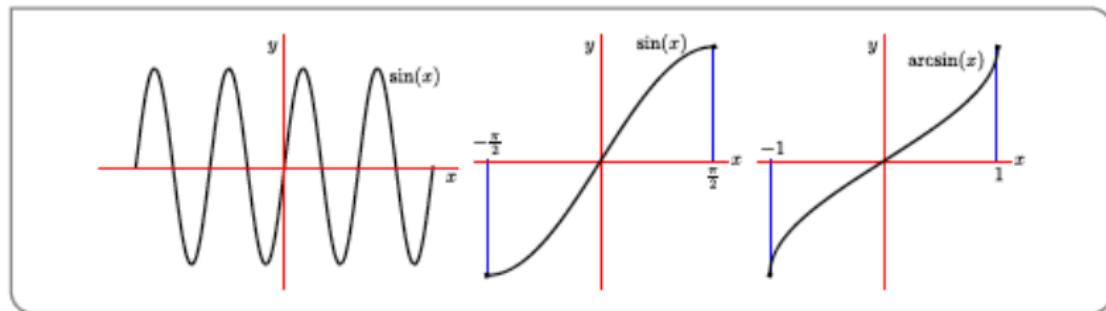
- ▶  $\sin(x)$  is not invertible on the domain  $\mathbb{R}$  because it is not one-to-one.

## Inverse of $\sin(x)$



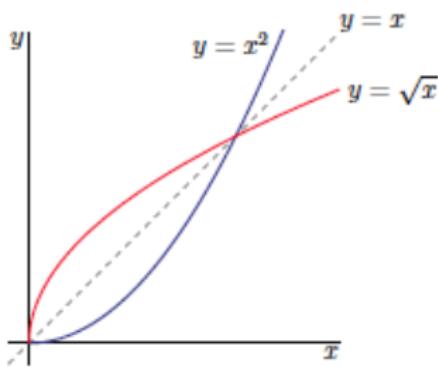
- ▶  $\sin(x)$  is not invertible on the domain  $\mathbb{R}$  because it is not one-to-one.
- ▶ If we look at  $\sin(x)$  on the domain  $[-\pi/2, \pi/2]$ , then it is one-to-one, and so it has an inverse.

## Inverse of $\sin(x)$



- ▶  $\sin(x)$  is not invertible on the domain  $\mathbb{R}$  because it is not one-to-one.
- ▶ If we look at  $\sin(x)$  on the domain  $[-\pi/2, \pi/2]$ , then it is one-to-one, and so it has an inverse.
- ▶ The inverse of  $\sin(x)$  is  $\arcsin(x)$  on the domain  $[-1, 1]$  and with the range  $[-\pi/2, \pi/2]$ .

## How to find the inverse of a function by its graph



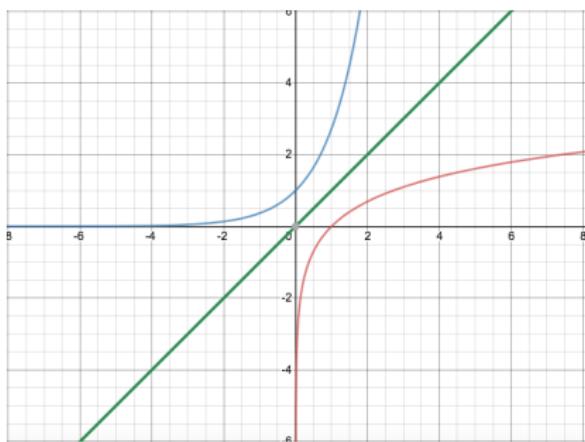
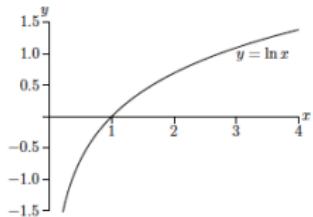
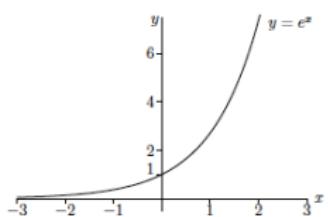
$$a^{\log_a x} = x$$

Remember that for  $a > 1$ ,

$$a^{\log_a x} = x,$$

$$\log_a x = \frac{\log_e x}{\log_e a}.$$

# The inverse of $e^x$



# Outline - October 4, 2019

- ▶ **Section 2.10 and 2.11:**
  - ▶ Natural logarithm
  - ▶ Implicit derivative

# Useful facts!

- ▶  $\frac{d}{dx} a^x = (\ln a) a^x.$
- ▶  $\log_a x = \frac{\ln x}{\ln a}$        $\ln x = \frac{\log_a x}{\log_a e}$        $a > 1.$
- ▶  $\ln(xy) = \ln x + \ln y.$
- ▶  $\ln(x/y) = \ln x - \ln y.$
- ▶  $\ln x^r = r \ln x.$

# Useful facts!

- ▶  $\frac{d}{dx} a^x = (\ln a) a^x.$
- ▶  $\log_a x = \frac{\ln x}{\ln a}$        $\ln x = \frac{\log_a x}{\log_a e}$        $a > 1.$
- ▶  $\ln(xy) = \ln x + \ln y.$
- ▶  $\ln(x/y) = \ln x - \ln y.$
- ▶  $\ln x^r = r \ln x.$
- ▶  $\frac{d}{dx} \ln x = \frac{1}{x}.$

# Useful facts!

- ▶  $\frac{d}{dx} a^x = (\ln a) a^x.$
- ▶  $\log_a x = \frac{\ln x}{\ln a}$        $\ln x = \frac{\log_a x}{\log_a e}$        $a > 1.$
- ▶  $\ln(xy) = \ln x + \ln y.$
- ▶  $\ln(x/y) = \ln x - \ln y.$
- ▶  $\ln x^r = r \ln x.$
- ▶  $\frac{d}{dx} \ln x = \frac{1}{x}.$
- ▶  $\frac{d}{dx} \ln |x| = \frac{1}{x}.$

# Useful facts!

- ▶  $\frac{d}{dx} a^x = (\ln a) a^x.$
- ▶  $\log_a x = \frac{\ln x}{\ln a}$        $\ln x = \frac{\log_a x}{\log_a e}$        $a > 1.$
- ▶  $\ln(xy) = \ln x + \ln y.$
- ▶  $\ln(x/y) = \ln x - \ln y.$
- ▶  $\ln x^r = r \ln x.$
- ▶  $\frac{d}{dx} \ln x = \frac{1}{x}.$
- ▶  $\frac{d}{dx} \ln |x| = \frac{1}{x}.$
- ▶  $\frac{d}{dx} \log_a x = \frac{1}{x \cdot \ln a}.$

# Useful facts!

- ▶  $\frac{d}{dx} a^x = (\ln a) a^x.$
- ▶  $\log_a x = \frac{\ln x}{\ln a}$        $\ln x = \frac{\log_a x}{\log_a e}$        $a > 1.$
- ▶  $\ln(xy) = \ln x + \ln y.$
- ▶  $\ln(x/y) = \ln x - \ln y.$
- ▶  $\ln x^r = r \ln x.$
- ▶  $\frac{d}{dx} \ln x = \frac{1}{x}.$
- ▶  $\frac{d}{dx} \ln |x| = \frac{1}{x}.$
- ▶  $\frac{d}{dx} \log_a x = \frac{1}{x \cdot \ln a}.$
- ▶  $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$

# Useful facts!

- ▶  $\frac{d}{dx} a^x = (\ln a)a^x.$
- ▶  $\log_a x = \frac{\ln x}{\ln a}$        $\ln x = \frac{\log_a x}{\log_a e}$        $a > 1.$
- ▶  $\ln(xy) = \ln x + \ln y.$
- ▶  $\ln(x/y) = \ln x - \ln y.$
- ▶  $\ln x^r = r \ln x.$
- ▶  $\frac{d}{dx} \ln x = \frac{1}{x}.$
- ▶  $\frac{d}{dx} \ln |x| = \frac{1}{x}.$
- ▶  $\frac{d}{dx} \log_a x = \frac{1}{x \cdot \ln a}.$
- ▶  $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$
- ▶  $\frac{d}{dx} |f(x)| = \frac{f'(x)}{|f(x)|}.$

# Outline - October 7, 2019

- ▶ **Section 2.11 and 2.12:**
  - ▶ Implicit derivative
  - ▶ Derivative of Trig functions

## Implicit derivative

$$\frac{d}{dx}x = \frac{d}{dx}e^{\ln x} \quad (\frac{d}{dx}x = \frac{d}{dx}e^y)$$

which is the same as

$$1 = \left(\frac{d}{dx}\ln x\right).e^{\ln x} \quad (1 = y'e^y).$$

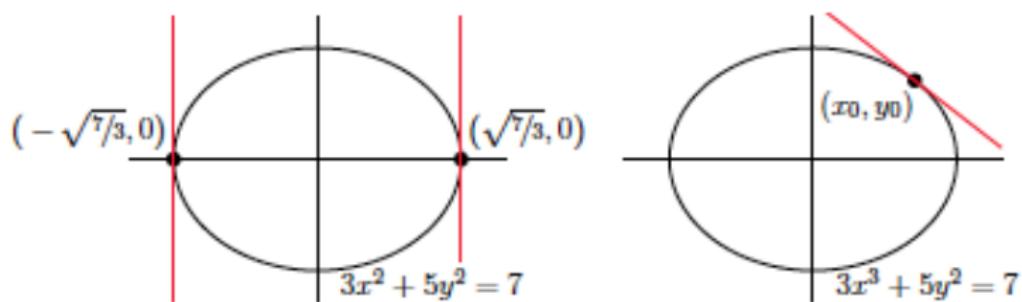
Note that  $e^{\ln x} = x$  ( $e^y = x$ ), thus

$$1 = \left(\frac{d}{dx}\ln x\right).x \quad (1 = y'x)$$

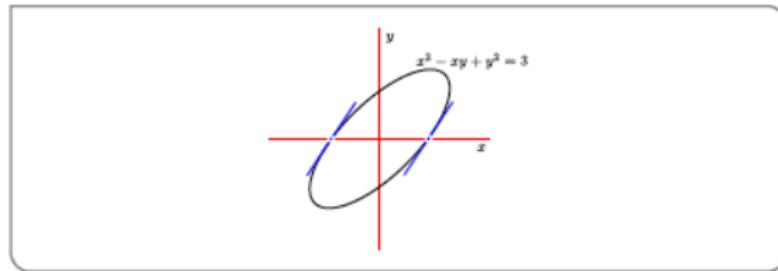
and so

$$\frac{d}{dx}\ln x = \frac{1}{x} \quad (y' = \frac{1}{x}).$$

$$3x^3 + 5y^2 = 7$$



$$x^2 - xy + y^2 = 3$$



$$x^{2/3} + y^{2/3} = 1$$



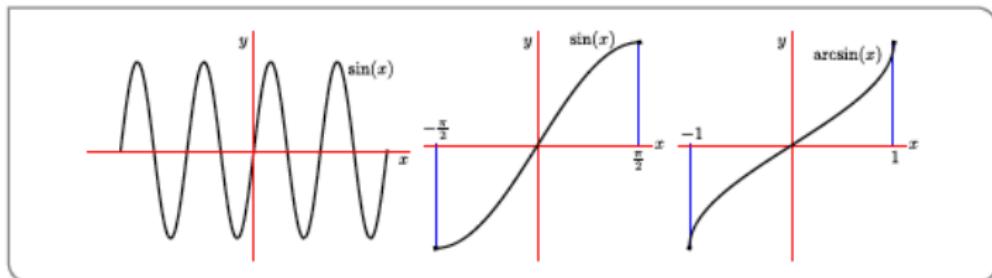
# Outline - October 9, 2019

- ▶ **Section 2.12:**
  - ▶ Derivative of Trig functions

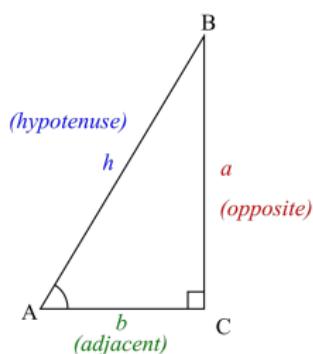
## Review of the inverse of a function

Remember that the inverse of a one-to-one function  $f(x)$  with domain  $A$  and range  $B$  is a function  $g(x)$  with domain  $B$  and range  $A$  such that

$$f(g(y)) = y \quad g(f(x)) = x \quad x \in A, y \in B.$$



# Trigonometry



- **sine:**  $\sin A = \frac{a}{h} = \frac{\text{opposite}}{\text{hypotenuse}}$
- **cosine:**  $\cos A = \frac{b}{h} = \frac{\text{adjacent}}{\text{hypotenuse}}$
- **tangent:**  $\tan A = \frac{a}{b} = \frac{\text{opposite}}{\text{adjacent}}$
- **cosecant:**  $\csc A = \frac{h}{a} = \frac{\text{hypotenuse}}{\text{opposite}}$
- **secant:**  $\sec A = \frac{h}{b} = \frac{\text{hypotenuse}}{\text{adjacent}}$
- **cotangent:**  $\cot A = \frac{b}{a} = \frac{\text{adjacent}}{\text{opposite}}$

$$\arcsin(\sin(x))$$

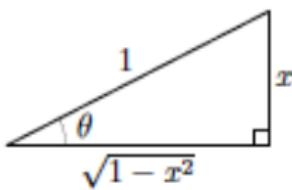
$\arcsin(\sin(x)) = \text{the unique angle } \theta \text{ between } -\pi/2 \text{ and } \pi/2$   
obeying that

$$\sin(x) = \sin(\theta).$$

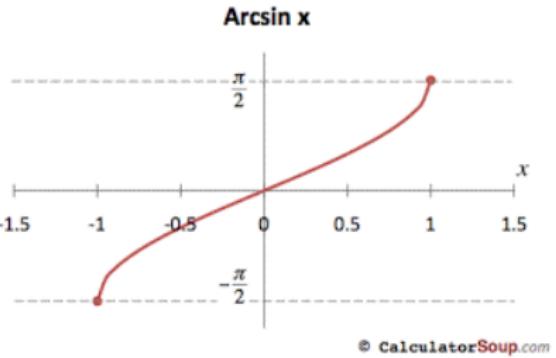
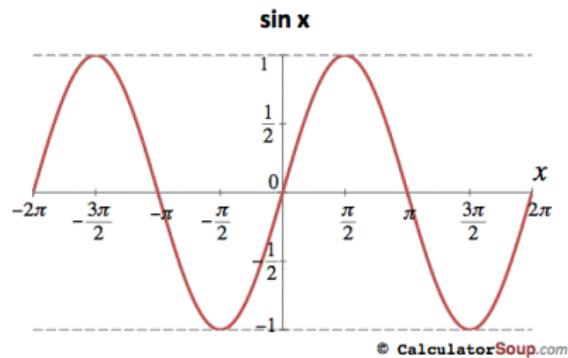
What is  $\arcsin(\sin(\frac{11\pi}{16}))$ ?



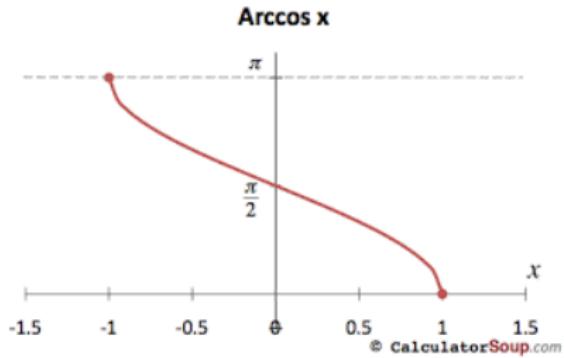
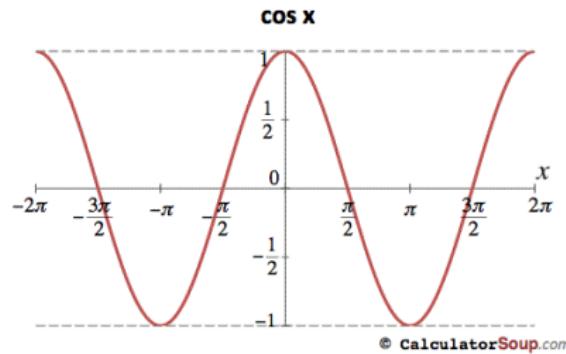
$$\cos(\arcsin(x)) = \sqrt{1 - x^2}$$



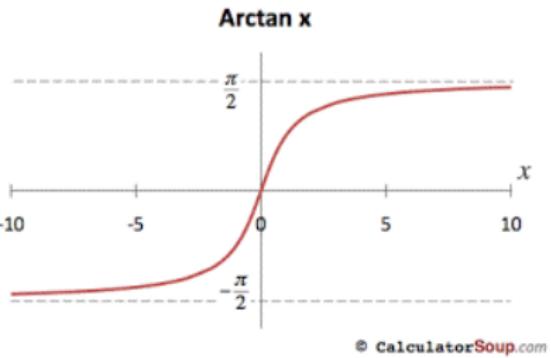
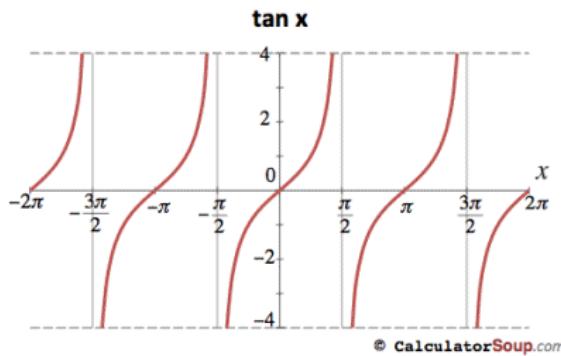
# Inverse of $\sin(x)$



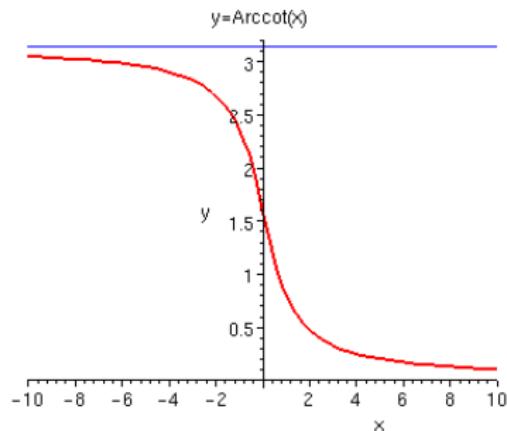
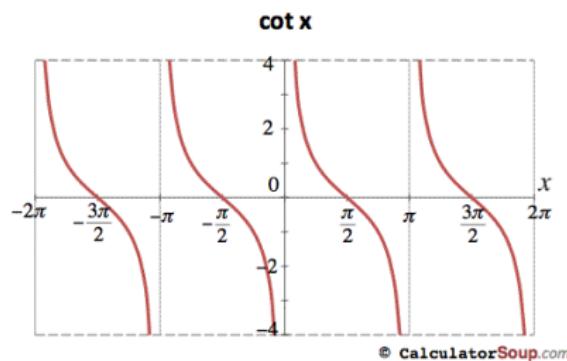
# Inverse of cos(x)



# Inverse of $\tan(x)$

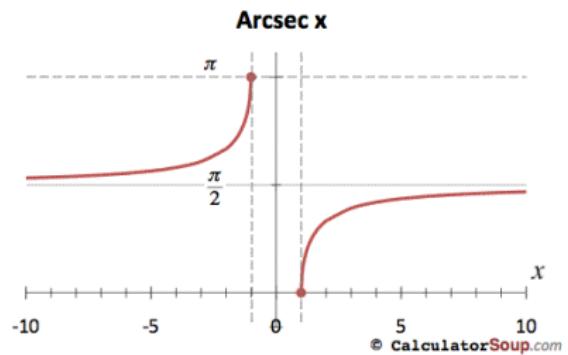
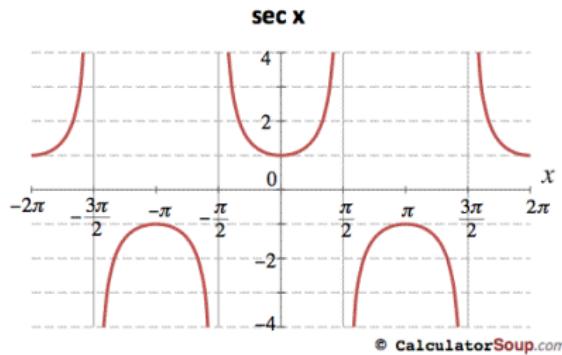


# Inverse of cotan(x)



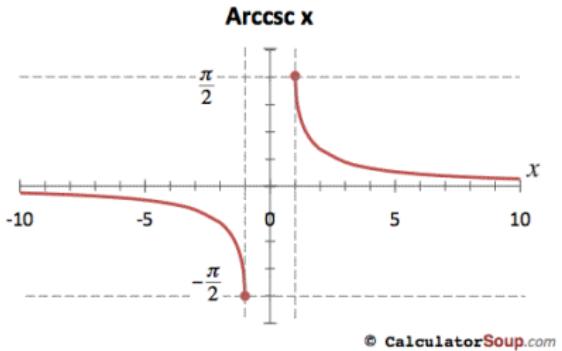
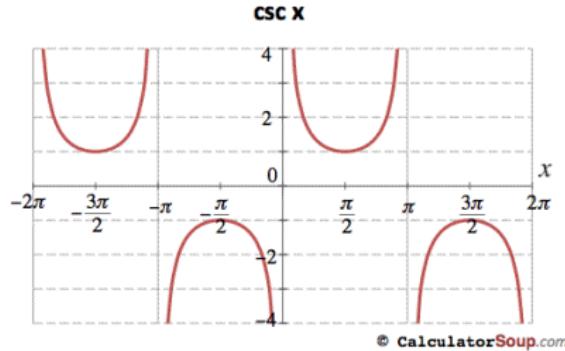
# Inverse of sec(x)

$$\text{arcsec}(x) = \arccos(1/x)$$

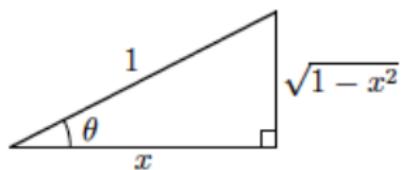


# Inverse of $\csc(x)$

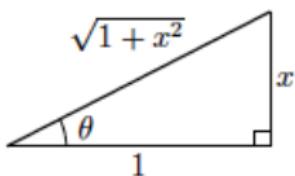
$$\text{arccsc}(x) = \arcsin(1/x)$$



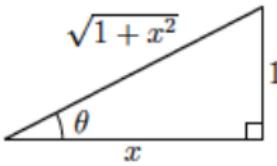
$$\sin(\theta) = \sin(\arccos(x)) = \sqrt{1 - x^2}$$



$$\cos^2(\arctan(x)) = \cos^2(\theta) = \frac{1}{1+x^2}.$$



$$\frac{1}{\csc^2(\theta)} = \sin^2(\theta) = 1 + x^2$$

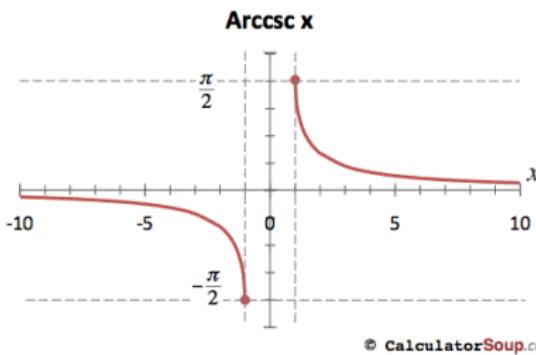
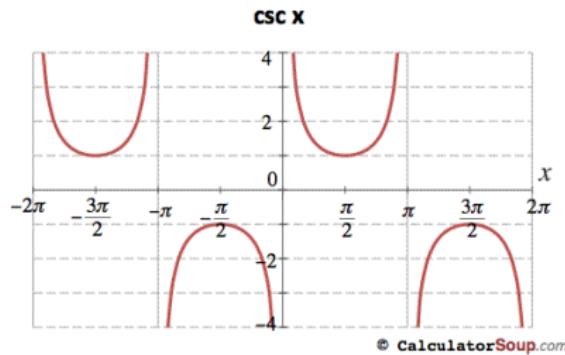


# Outline - October 11, 2019

- ▶ **Section 3.1:**
  - ▶ Derivative of Trig functions

# Inverse of $\csc(x)$

$$\text{arccsc}(x) = \arcsin(1/x)$$



## Derivative of the inverses of trigonometric functions in a nutshell

In a nutshell the derivatives of the inverse trigonometric functions are

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \text{arccsc}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \text{arcsec}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \text{arccot}(x) = -\frac{1}{1+x^2}$$

# The Application of Derivatives

## Velocity and Acceleration

If you are moving along the  $x$ -axis and your position at time  $t$  is  $x(t)$ , then

- ▶ your velocity at time  $t$  is  $v(t) = x'(t)$  and
- ▶ your acceleration at time  $t$  is  $a(t) = v'(t) = x''(t)$ .

Direction of your move with  $x(t) = t^3 - 3t + 2$

$t$	$(t - 1)(t + 1)$	$x'(t) = 3(t - 1)(t + 1)$	Direction
$t < -1$	<i>positive</i>	<i>positive</i>	<i>right</i>
$t = -1$	<i>zero</i>	<i>zero</i>	<i>halt</i>
$-1 < t < 1$	<i>negative</i>	<i>negative</i>	<i>left</i>
$t = 1$	<i>zero</i>	<i>zero</i>	<i>halt</i>
$t > 1$	<i>positive</i>	<i>positive</i>	<i>right</i>

And here is a schematic picture of the whole trajectory.



Direction of your move with  $x(t) = t^3 - 12t + 5$

$t$	$(t - 2)(t + 2)$	$x'(t) = 3(t - 2)(t + 2)$	Direction
$t < -2$	positive	positive	right
$t = -2$	zero	zero	halt
$-2 < t < 2$	negative	negative	left
$t = 2$	zero	zero	halt
$t > 2$	positive	positive	right

$t$	$your\ positionx(t)$	$x'(t)$	Direction
0	5	negative	left
$t = 2$	-11	zero	halt
$t = 10$	885	positive	right

# Outline - October 16, 2019

- ▶ **Section 3.2: Exponential Growth and Decay**
  - ▶ 3.1: Carbon Dating

**EXAM: Friday, October 18, Here in Class, at 2pm**

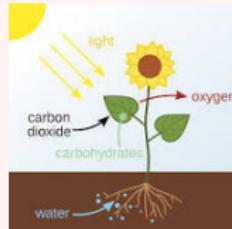
# Carbon Dating

*Cosmic ray hitting atmosphere*



$\text{Nitrogen (N)} \rightarrow \text{Carbon (C)}$

*Vegetation absorbs C through photosynth*



*Animals acquire C by eating plants*



*C decreases when animal dies*



More precisely, let  $Q(t)$  denote the amount of C (an element) in the plant or animal  $t$  years after it dies. The number of radioactive decays (rate of change) per unit time, at time  $t$ , is proportional to the amount of C present at time  $t$ , which is  $Q(t)$ . Thus

### Radioactive Decay

$$\frac{dQ}{dt}(t) = -kQ(t) \quad (1)$$

## Corollary

The function  $Q(t)$  satisfies the equation

$$\frac{dQ}{dt} = -kQ(t)$$

if and only if

$$Q(t) = Q(0).e^{-kt}$$

The half-life (the half-life of C is the length of time that it takes for half of the C to decay) is defined to be the time  $t_{1/2}$  which obeys

$$Q(t_{1/2}) = \frac{1}{2}.Q(0).$$

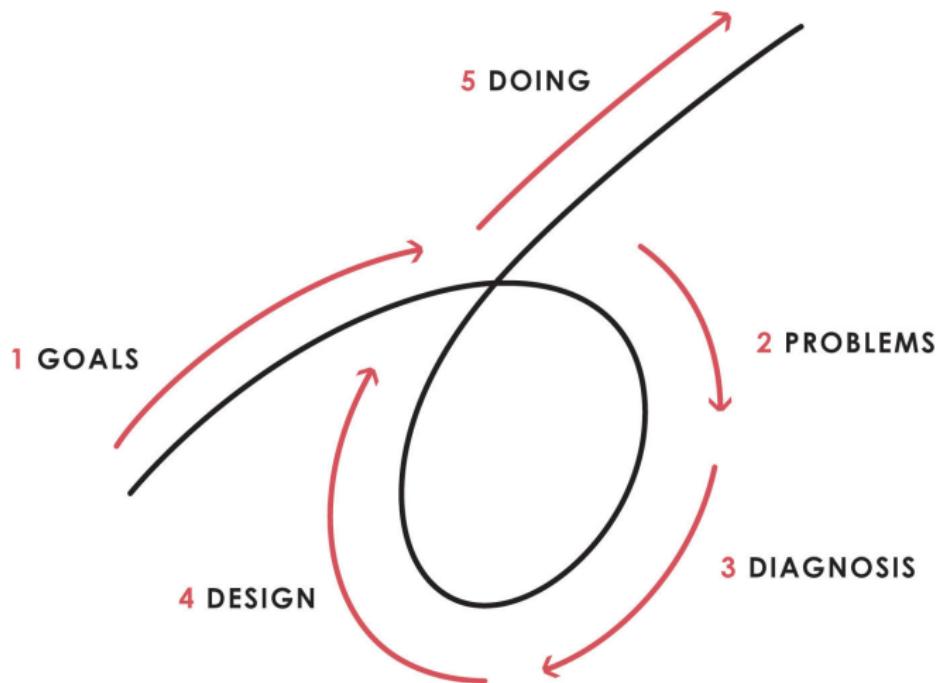
The half-life is related to the constant  $k$  by

$$t_{1/2} = \frac{\ln 2}{k}.$$

# Outline - October 21, 2019

- ▶ **Section 3.3.2: Newton's Law of Cooling**
  - ▶ 3.1: Newton's Law of Cooling

# No pain no gain

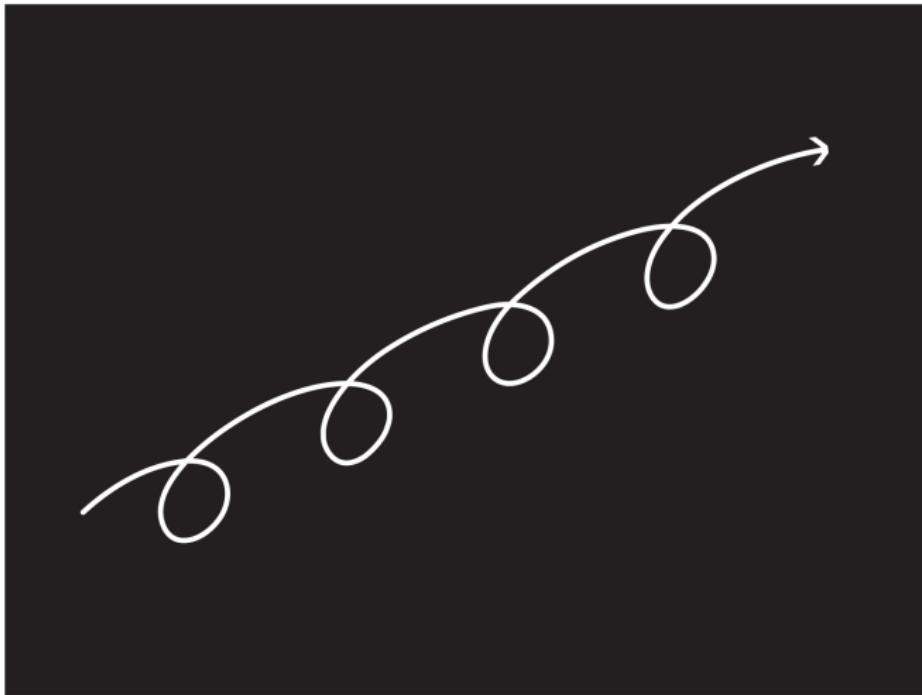


## Principles (Ray Dalio)

Most people



# Successful person



# Newton's Law of Cooling



$$\frac{dT}{dt}(t) = K [T(t) - A].$$

We have three possibilities:

- ▶  $T(t) > A \Rightarrow [T(t) - A] > 0$ , thus the temperature of the body is decreasing, so  $\frac{dT}{dt}$  must be negative, since  $\frac{dT}{dt}(t) = K [T(t) - A]$ , we must have  $K < 0$ .
- ▶  $T(t) < A \Rightarrow [T(t) - A] < 0$ , thus the temperature of the body is increasing, so  $\frac{dT}{dt}$  must be positive, since  $\frac{dT}{dt}(t) = K [T(t) - A]$ , we must have  $K < 0$ .
- ▶  $T(t) = A \Rightarrow [T(t) - A] = 0$ , thus the temperature of the body is no changing, so  $\frac{dT}{dt}$  must be zero, since  $\frac{dT}{dt}(t) = K [T(t) - A]$ . This does not impose any condition on  $K$ .

## Newton's Law of Cooling

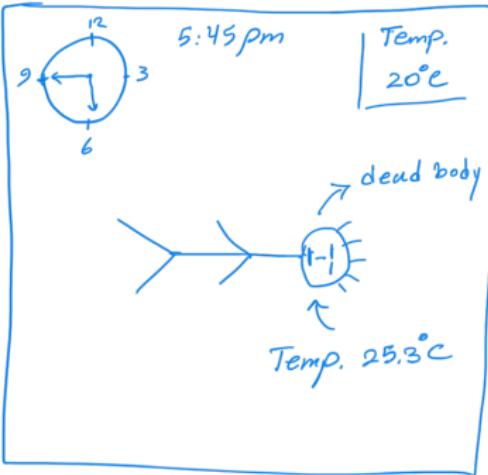
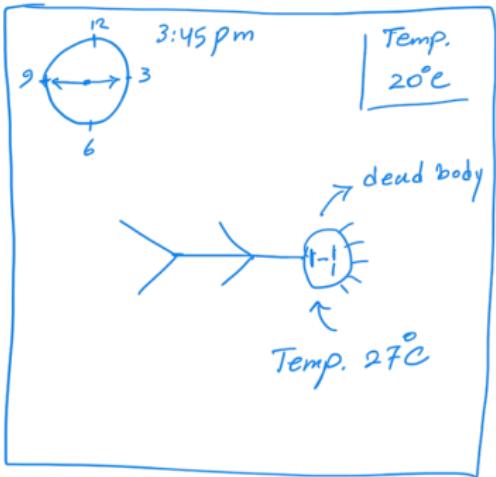
### Corollary

A differentiable function  $T(t)$  obeys the differential equation

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

if and only if

$$T(t) = [T(0) - A]e^{Kt} + A.$$



I entered this room at  
9:30 am and talk to him  
for 5 minutes.

# Outline - October 23, 2019

- ▶ **Section 3.3.3: Population Growth**
- ▶ **Section 3.2: Related Rates**

## Population Growth

Suppose that we wish to predict the size  $P(t)$  of a population as a function of the time  $t$ . So suppose that in average each couple produces  $\beta$  offspring (for some constant  $\beta$ ) and then dies. Then over the course of one generation since we have  $P(t)/2$  couples and each have produced  $\beta$  offspring, thus the population of the children of one generation is

$$\beta \frac{P(t)}{2}.$$

Let  $t_g$  be the life span of one generation, then

$$\begin{aligned} P(t + t_g) &= \beta \frac{P(t)}{2} \\ &= P(t) + \beta \frac{P(t)}{2} - P(t). \end{aligned}$$

Therefore,

$$P(t + t_g) - P(t) = \beta \frac{P(t)}{2} - P(t)$$

and so dividing both sides by  $t_g$ , we have

$$\begin{aligned}\frac{P(t + t_g) - P(t)}{t_g} &= \frac{1}{t_g} \left( \frac{\beta}{2} P(t) - P(t) \right) \\ &= \frac{1}{t_g} \left( \frac{\beta}{2} - 1 \right) P(t)\end{aligned}$$

Let  $\frac{1}{t_g} \left( \frac{\beta}{2} - 1 \right) = b$ , then

$$\frac{P(t + t_g) - P(t)}{t_g} = bP(t).$$

Approximately, we have

$$\frac{dP}{dt} = bP(t).$$

Moreover, same as the model for carbon dating we can write

$$P(t) = P(0)e^{bt}.$$

# Malthusian growth model

## Malthusian growth model

The model for the population growth is

$$\frac{dP}{dt} = bP(t)$$

and  $P(t)$  satisfies the above equation if and only if

$$P(t) = P(0)e^{bt}.$$

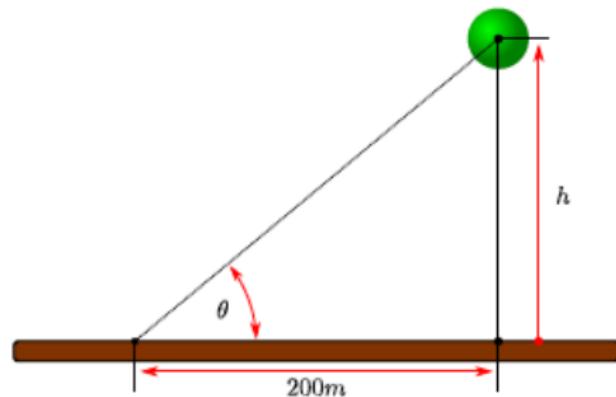
# Related Rates

## Volume of a sphere

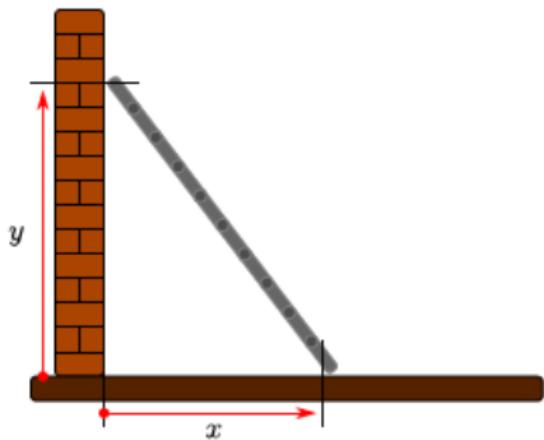
Remember that the volume of a sphere with radius  $r$  is

$$V = \frac{4}{3}\pi r^3.$$

# Helium Balloon



# Ladder



# Outline - October 25, 2019

- ▶ **Section 3.2: Related Rates: An Example**
- ▶ **Section 3.4.2 The Linear Approximation**
- ▶ **Section 3.4.3 The Quadratic Approximation**

# Shadow of the Ball

Similar triangles-ratio



# Approximation



This figure shows that the curve  $y = x$  and  $y = \sin(x)$  are almost the same when  $x$  is close to 0. Hence if we want the value of  $\sin(1/10)$  we just use this approximation  $y = x$  to get

$$\sin(1/10) \approx 1/10.$$

## The linear approximation

Given a function  $f(x)$  we want to have the approximating function to be a linear function that is  $F(x) = A + Bx$  for some constants  $A$  and  $B$ .



## The linear approximation

$$f(x) \approx F(x) = f(a) + f'(a)(x - a)$$

### Example

Estimate  $e^{0.01}$ ? So  $f(x) = e^x$  and  $a = 0$ .

# The quadratic approximation

In linear approximation we had

$$f(x) \approx F(x) = f(a) + f'(a)(x - a) \Rightarrow$$
$$f(a) = F(a) \quad \text{and} \quad f'(a) = F'(a).$$

We now want our approximation function to be a quadratic function of  $x$ , that is,  $F(x) = A + Bx + Cx^2$ . To have a good approximating function we choose  $A$ ,  $B$ , and  $C$  so that

- ▶  $f(a) = F(a)$
- ▶  $f'(a) = F'(a)$
- ▶  $f''(a) = F''(a)$

These conditions give us the following equations

$$F(x) = A + Bx + Cx^2 \Rightarrow F(a) = A + Ba + Ca^2 = f(a)$$

$$F'(x) = B + 2Cx \Rightarrow F'(a) = B + 2Ca = f'(a)$$

$$F''(x) = 2C \Rightarrow F''(a) = 2C = f''(a)$$

Solving these equations we can write  $A$ ,  $B$ , and  $C$  in terms of  $f(a)$ ,  $f'(a)$ , and  $f''(a)$ . So that

$$C = \frac{1}{2}f''(a)$$

$$B = f'(a) - af''(a)$$

$$A = f(a) - a[f'(a) - af''(a)] - \frac{1}{2}f''(a)a^2.$$

Consider that  $F(x) = A + Bx + CX^2$ , substituting  $A$ ,  $B$ , and  $C$ , we obtain

### Quadratic Approximation

$$F(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

Therefore,

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

# Outline - October 28, 2019

- ▶ **Section 3.4.3 The Quadratic Approximation**
- ▶ **Section 3.4.4 Taylor Polynomials**
- ▶ **Section 3.4.5 Some Examples**

# Linear Approximation

Approximate  $f(x)$  by  $F(x) = c_0 + c_1(x - a)$  such that

1.  $F(a) = f(a)$
2.  $F'(a) = f'(a)$

# Linear Approximation

Approximate  $f(x)$  by  $F(x) = c_0 + c_1(x - a)$  such that

1.  $F(a) = f(a)$
2.  $F'(a) = f'(a)$

Then

$$F(a) = c_0 = f(a) \quad F'(a) = c_1 = f'(a).$$

And so

$$F(x) = f(a) + f'(a)(x - a).$$

# Quadratic Approximation

Approximate  $f(x)$  by  $F(x) = c_0 + c_1(x - a) + c_2(x - a)^2$  such that

1.  $F(a) = f(a)$
2.  $F'(a) = f'(a)$
3.  $F''(a) = f''(a)$

## Quadratic Approximation

Approximate  $f(x)$  by  $F(x) = c_0 + c_1(x - a) + c_2(x - a)^2$  such that

1.  $F(a) = f(a)$
2.  $F'(a) = f'(a)$
3.  $F''(a) = f''(a)$

Then

$$F(a) = c_0 = f(a) \quad F'(a) = c_1 = f'(a) \quad F''(a) = 2c_2 = f''(a).$$

## Quadratic Approximation

Approximate  $f(x)$  by  $F(x) = c_0 + c_1(x - a) + c_2(x - a)^2$  such that

1.  $F(a) = f(a)$
2.  $F'(a) = f'(a)$
3.  $F''(a) = f''(a)$

Then

$$F(a) = c_0 = f(a) \quad F'(a) = c_1 = f'(a) \quad F''(a) = 2c_2 = f''(a).$$

And so

$$F(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

# Taylor Polynomial

We want to approximate  $f(x)$  with a polynomial  $T_n(x)$  of degree  $n$  of the form

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n$$

such that

1.  $T_n(a) = f(a),$

2.  $T'_n(a) = f'(a),$

⋮

n.  $T_n^{(n)}(a) = f^{(n)}(a).$

# Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) =$$

# Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

# Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) =$$

# Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) = c_1 = f'(a)$$

# Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) = c_1 = f'(a)$$

$$T''_n(x) = 2c_2 + 3 \times 2c_3(x - a) + \cdots + n(n - 1)c_n(x - a)^{n-2}$$

$$\Rightarrow T''_n(a) =$$

# Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) = c_1 = f'(a)$$

$$T''_n(x) = 2c_2 + 3 \times 2c_3(x - a) + \cdots + n(n - 1)c_n(x - a)^{n-2}$$

$$\Rightarrow T''_n(a) = 2c_2 = f''(a)$$

# Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) = c_1 = f'(a)$$

$$\begin{aligned} T''_n(x) &= 2c_2 + 3 \times 2c_3(x - a) + \cdots + n(n-1)c_n(x - a)^{n-2} \\ &\Rightarrow T''_n(a) = 2c_2 = f''(a) \end{aligned}$$

$$\begin{aligned} T_n^{(3)}(x) &= 3 \times 2c_3 + 4 \times 3 \times 2c_4(x - a) + \cdots + n(n-1)c_n(x - a)^{n-2} \\ &\Rightarrow T_n^{(3)}(a) = \end{aligned}$$

# Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) = c_1 = f'(a)$$

$$\begin{aligned} T''_n(x) &= 2c_2 + 3 \times 2c_3(x - a) + \cdots + n(n-1)c_n(x - a)^{n-2} \\ &\Rightarrow T''_n(a) = 2c_2 = f''(a) \end{aligned}$$

$$\begin{aligned} T_n^{(3)}(x) &= 3 \times 2c_3 + 4 \times 3 \times 2c_4(x - a) + \cdots + n(n-1)c_n(x - a)^{n-2} \\ &\Rightarrow T_n^{(3)}(a) = 6c_3 = f^{(3)}(a) \end{aligned}$$

# Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) = c_1 = f'(a)$$

$$\begin{aligned} T''_n(x) &= 2c_2 + 3 \times 2c_3(x - a) + \cdots + n(n-1)c_n(x - a)^{n-2} \\ &\Rightarrow T''_n(a) = 2c_2 = f''(a) \end{aligned}$$

$$\begin{aligned} T_n^{(3)}(x) &= 3 \times 2c_3 + 4 \times 3 \times 2c_4(x - a) + \cdots + n(n-1)c_n(x - a)^{n-2} \\ &\Rightarrow T_n^{(3)}(a) = 6c_3 = f^{(3)}(a) \end{aligned}$$

⋮

$$T_n^{(n)}(x) = n!c_n \Rightarrow T_n^{(n)}(a) = n!c_n$$

# Taylor Polynomial

We have

$$c_0 = f(a), c_1 = f'(a), c_2 = \frac{1}{2!}f''(a), c_3 = \frac{1}{3!}f^{(3)}(a), \dots, c_n = \frac{1}{n!}f^{(n)}(a)$$

and

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots + c_n(x - a)^n$$

we have that

$$\begin{aligned} f(x) &\approx T_n(x) = \\ &f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a) + \\ &\frac{1}{3!}f^{(3)}(a)(x - a)^3 + \dots + \frac{1}{n!}f^{(n)}(a)(x - a)^n \end{aligned}$$

# Taylor Polynomial

## Taylor Polynomial

Let  $a$  be a constant and let  $n$  be a non-negative integer. The  $n$ th degree Taylor polynomial for  $f(x)$  about  $x = a$  is

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2$$

$$+ \frac{1}{3!}f^{(3)}(a)(x - a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x - a)^n$$

or

$$T_n(x) = \sum_{k=0}^n \frac{1}{k!}f^{(k)}(a)(x - a)^k$$

The special case  $a = 0$  is called a Maclaurin polynomial.

# Outline - October 30, 2019

- ▶ **Section 3.4.5: Some Examples of Taylor Polynomial**
- ▶ **Section 3.4.8: The Error in the Taylor Polynomial Approximations**

# Taylor Polynomial

## Taylor Polynomial

Let  $a$  be a constant and let  $n$  be a non-negative integer. The  $n$ th degree Taylor polynomial for  $f(x)$  about  $x = a$  is

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2$$

$$+ \frac{1}{3!}f^{(3)}(a)(x - a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x - a)^n$$

or

$$T_n(x) = \sum_{k=0}^n \frac{1}{k!}f^{(k)}(a)(x - a)^k$$

The special case  $a = 0$  is called a Maclaurin polynomial.

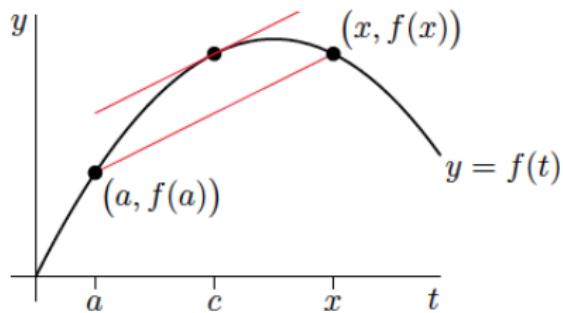
## Approximating $f(x)$ by the 0th Taylor polynomial about $x = a$

$$f(x) \approx T_0(x) = f(a).$$

Note that

$$\begin{aligned} f(x) &= f(x) + f(a) - f(a) \\ &= f(a) + (f(x) - f(a)) \frac{(x-a)}{(x-a)} \\ &= f(a) + \frac{f(x) - f(a)}{x-a} (x-a) \end{aligned} \tag{2}$$

$$f(x) = f(a) + \frac{f(x)-f(a)}{x-a}(x-a)$$



There is  $c$  strictly between  $x$  and  $a$  such that

$$f'(c) = \frac{f(x) - f(a)}{x - a}.$$

$$f(x) = f(a) + f'(c)(x-a) \text{ for some } c \text{ strictly between } a \text{ and } x.$$

$$f(x) = f(a) + f'(c)(x-a) \text{ for some } c \text{ strictly between } a \text{ and } x.$$

$$\Rightarrow f(x) - f(a) = f'(c)(x - a) \Rightarrow f(x) - T_0(x) = f'(c)(x - a)$$

### The error in constant approximation

$$R_0(x) = f(x) - T_0(x) = f'(c)(x - a)$$

for some  $c$  strictly between  $a$  and  $x$

## The error in linear approximation

$$R_1(x) = f(x) - T_1(x) = \frac{1}{2}f''(c)(x-a)^2$$

for some  $c$  strictly between  $a$  and  $x$

**Lagrange remainder theorem: The error when approximating function is  $T_n(x)$**

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{n+1}$$

for some  $c$  strictly between  $a$  and  $x$

## Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some  $c$  strictly between  $a$  and  $x$

### Remark

Consider that  $f(x) = R_n(x) + T_n(x)$  Therefore,

1. if  $0 \leq R_n(x) \leq E$ , then

$$T_n(x) \leq f(x) \leq T_n(x) + E.$$

2. if  $E \leq R_n(x) \leq 0$ , then

$$T_n(x) + E \leq f(x) \leq T_n(x).$$

# Outline - Nov. 1, 2019

- ▶ **Section 3.4.8: The Error in the Taylor Polynomial Approximations**
- ▶ **Section 3.5.1: Maxima and Minima**

## Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some  $c$  strictly between  $a$  and  $x$

## Accurate to $D$ decimal places

Generally we say that our estimate is “accurate to  $D$  decimal places” when

$$|\text{error}| < 0.5 \times 10^{-D}.$$

## Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some  $c$  strictly between  $a$  and  $x$

### Remark

Consider that  $f(x) = R_n(x) + T_n(x)$  Therefore,

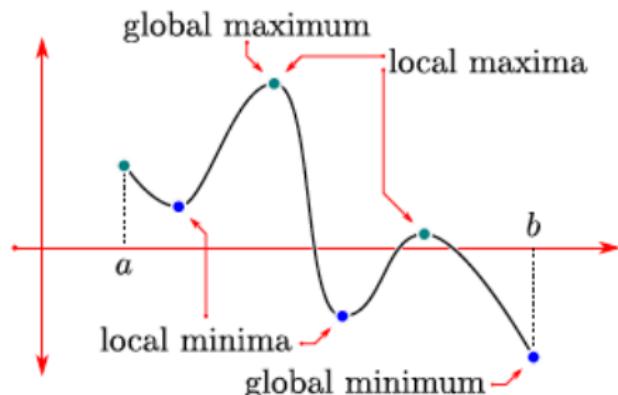
1. if  $0 \leq R_n(x) \leq E$ , then

$$T_n(x) \leq f(x) \leq T_n(x) + E.$$

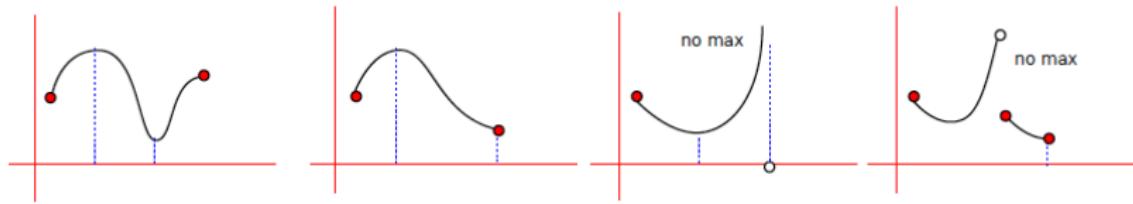
2. if  $E \leq R_n(x) \leq 0$ , then

$$T_n(x) + E \leq f(x) \leq T_n(x).$$

# Maximum and Minimum



# Continuity and global max/min



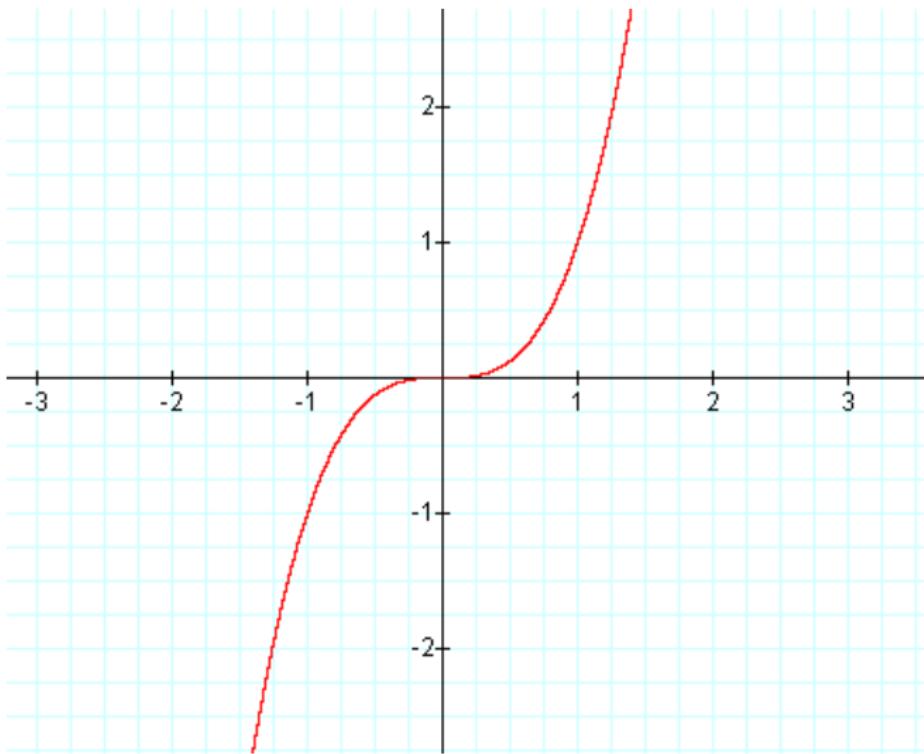
First one: Continuous/global min and max

Second one: Continuous/global min and max

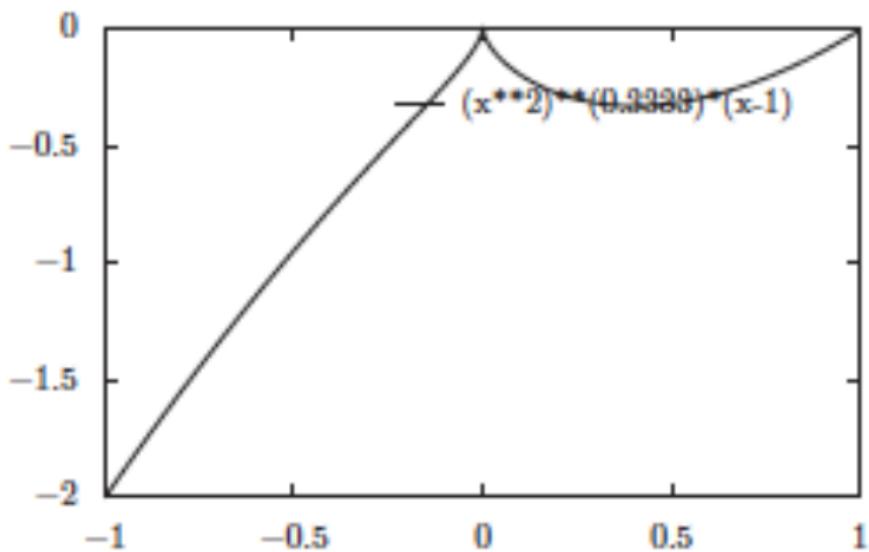
Third one: Not continuous/global min/no global max

Forth one: Not continuous/global min/no global max

If  $f'(c) = 0$ , then  $c$  is local max/min?!



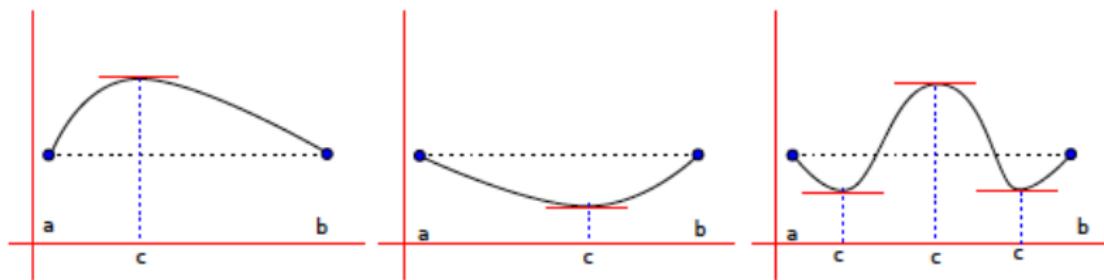
The graph of the function  $x^{5/3} - x^{2/3}$  for  $-1 \leq x \leq 1$



# Outline - Nov. 6, 2019

- ▶ **Section 2.13: MVT**
- ▶ **Section 3.6: Sketching Graphs**

# Rolle's Theorem



# Rolle's Theorem

## Rolle's Theorem

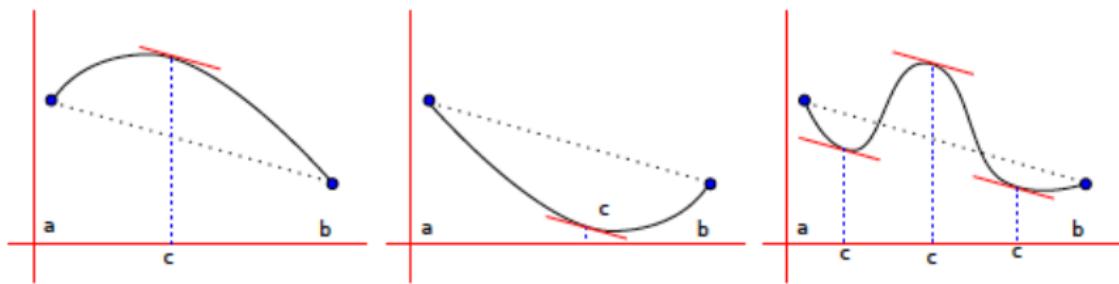
### Theorem

(CLP 2.13.1–Rolle's theorem) Let  $f$  be a function such that

- ▶  $f$  is continuous on  $[a, b]$ ,
- ▶  $f$  is differentiable on  $(a, b)$ ,
- ▶  $f(a) = f(b)$ .

Then there is a point  $c$  between  $a$  and  $b$  so that  $f'(c) = 0$ .

# The Mean Value Theorem (MVT)



## The Mean Value Theorem

## Theorem

(CLP 2.13.4–The mean value theorem) Let  $f$  be a function such that

- ▶  $f$  is continuous on  $[a, b]$ , and
- ▶  $f$  is differentiable on  $(a, b)$ .

Then there is a point  $c$  between  $a$  and  $b$  so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or equivalently,

$$f(b) - f(a) = (b - a)f'(c).$$

# Rolle's Theroem and IVT

## Rolle's Theorem

### Theorem

(CLP 2.13.1–Rolle's theorem) Let  $f$  be a function such that

- ▶  $f$  is continuous on  $[a, b]$ ,
- ▶  $f$  is differentiable on  $(a, b)$ ,
- ▶  $f(a) = f(b)$ .

Then there is a point  $c$  between  $a$  and  $b$  so that  $f'(c) = 0$ .

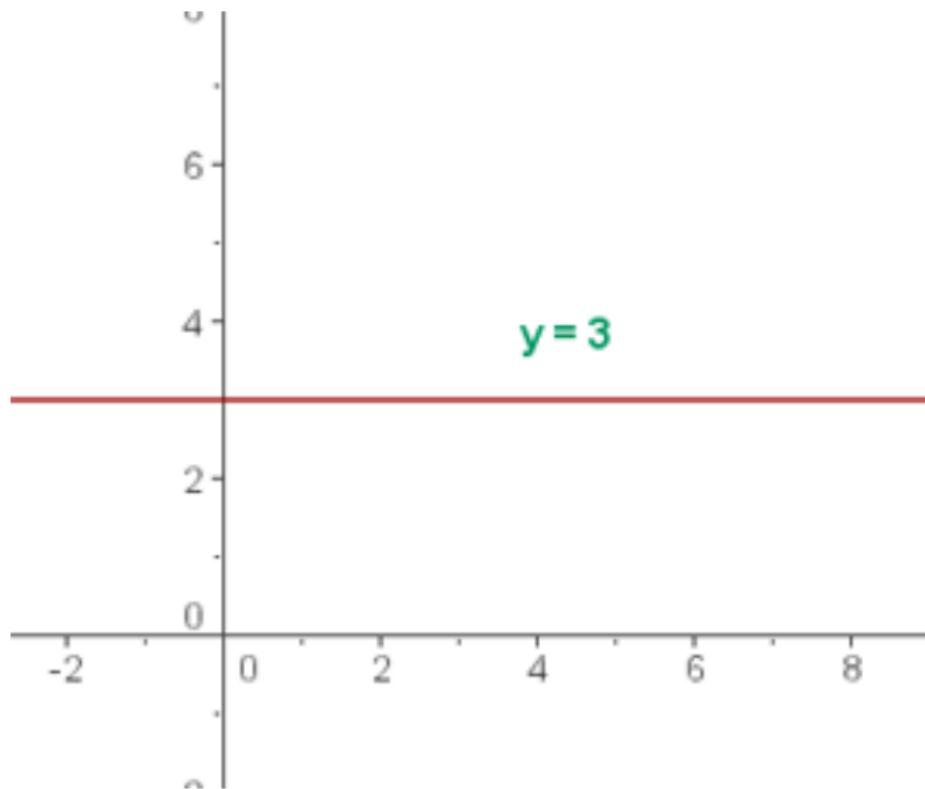
## Intermediate value theorem(IVT)

### Theorem

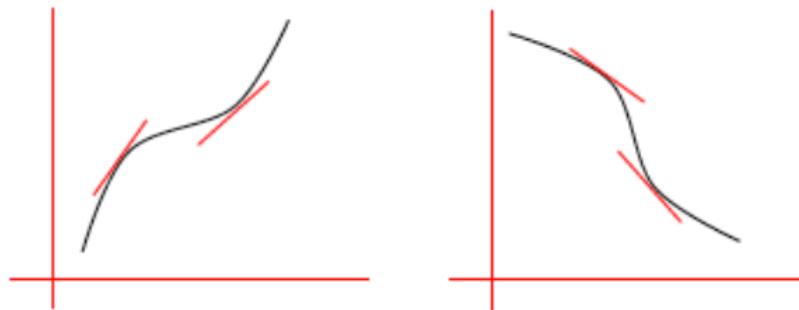
Let  $a < b$  and let  $f(x)$  be a function that is continuous at all points  $a \leq x \leq b$ . If  $Y$  is any number between  $f(a)$  and  $f(b)$  then there exists some number  $c \in [a, b]$  so that  $f(c) = Y$ .



If  $f'(x) = 0$  for all  $x \in (a, b)$ , then  $f(x)$  is constant on  $(a, b)$



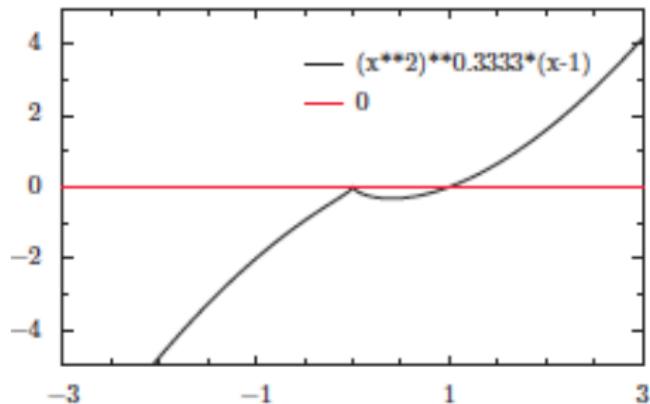
$f'(x) > 0$  then  $f$  is increasing;  $f'(x) < 0$  then  $f$  is decreasing



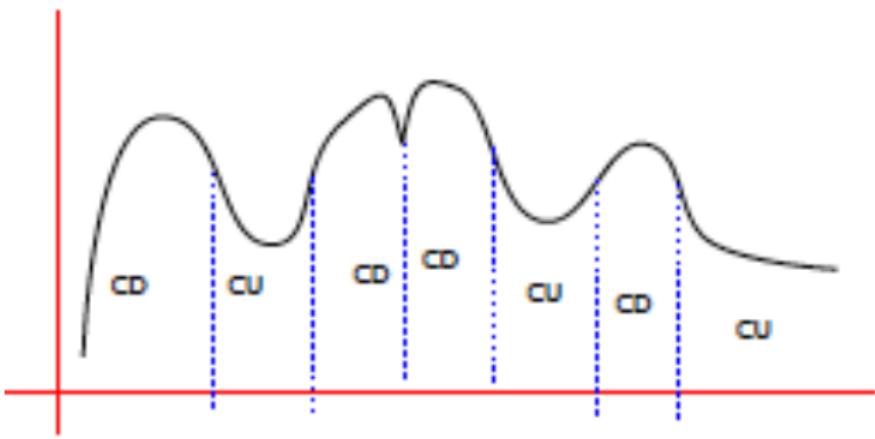
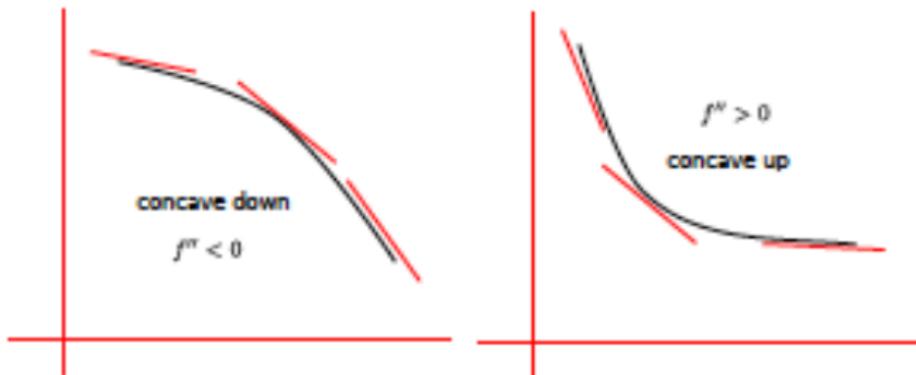
# When a critical or singular point of a continuous function is a local max/min



## Example



## Concave Up and Down



# Outline - Nov. 8, 2019

- ▶ **Section 3.6: Sketching Graphs**

# Different Level of Learning

- ▶ **Learning Objectives:** Be able to show that a differentiable function has exactly one or two (or more) zeros.

1. Recall and Memorize:

IVT, MVT, and Rolle's; and how we solve the question

2. Understand:

An Example by Farid

3. Apply:

Examples by You

3. Analyze:

Explain and Analyze Your Work

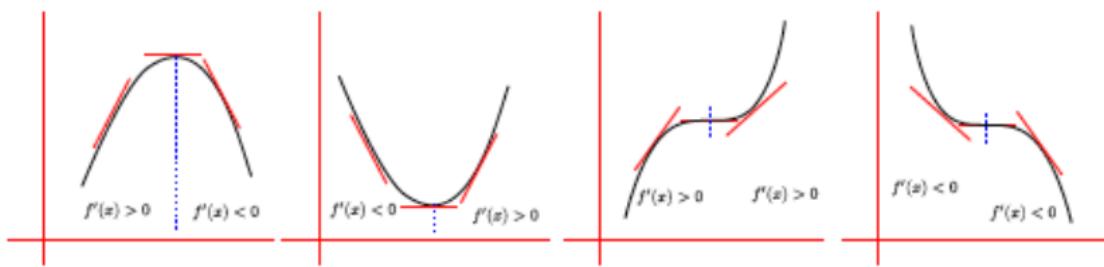
4. Evaluate:

Other's solution is Correct?

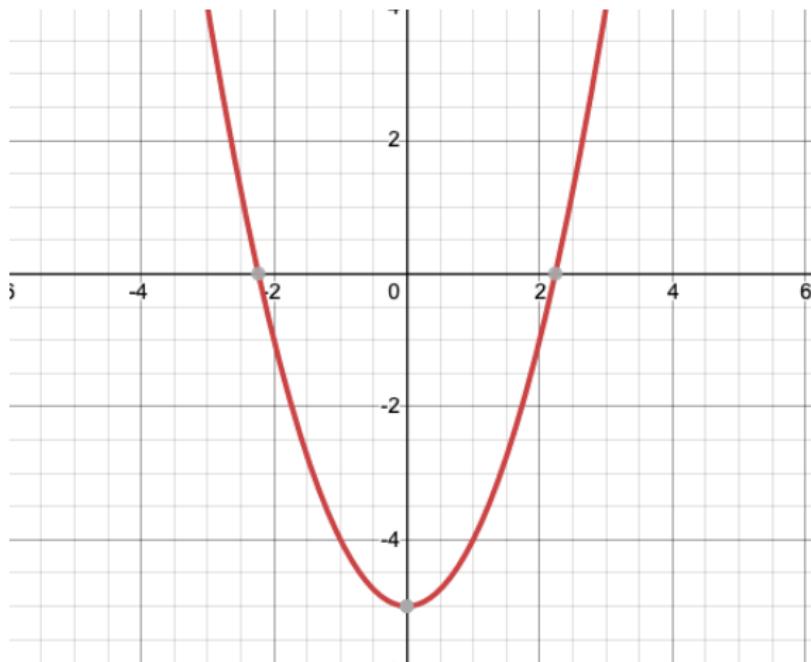
4. Create:

What questions do you give in the test?

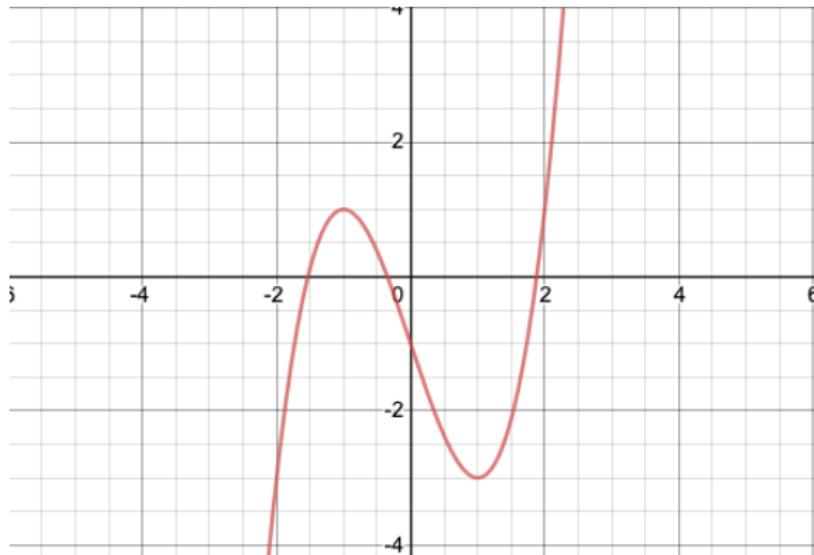
## Second Derivative Test



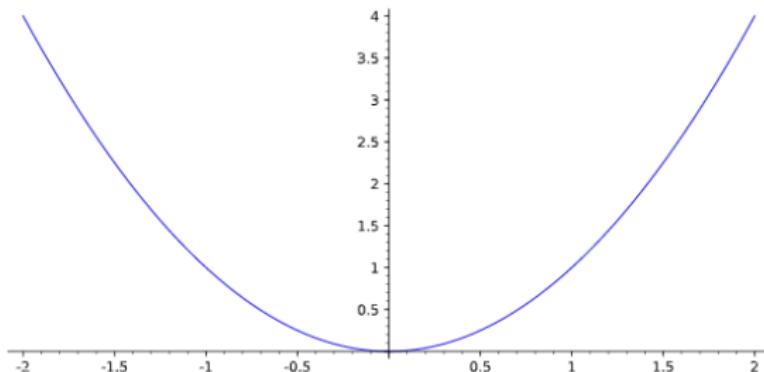
Consider the graph of  $x^2 - 5$



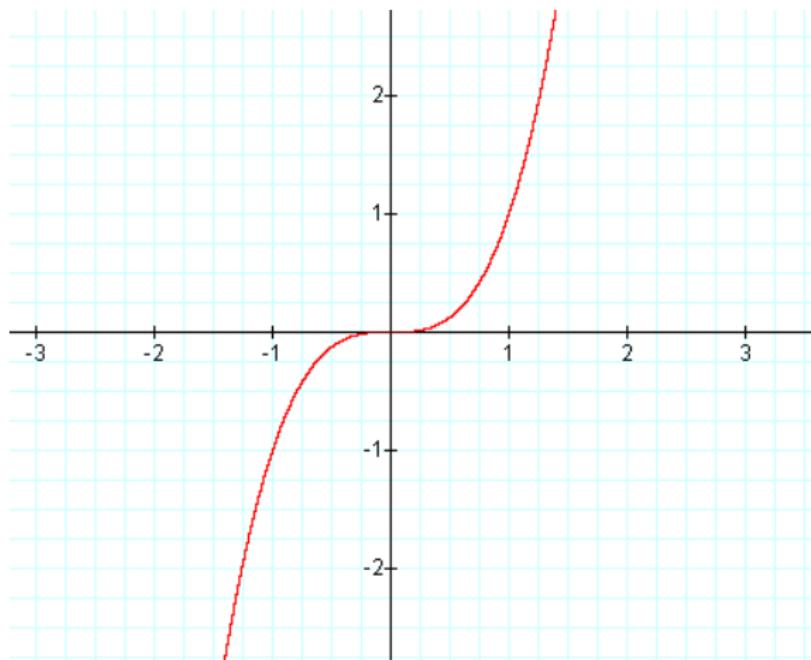
Consider the graph of  $x^3 - 3x - 1$



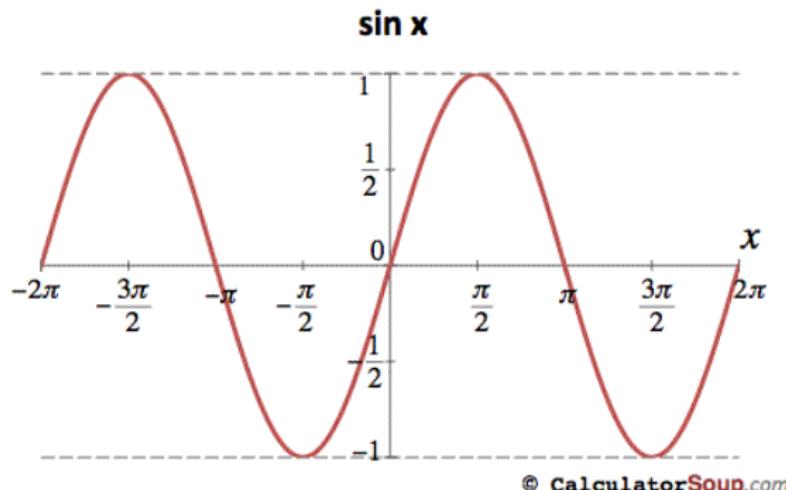
$f(x) = x^2$  is even



$f(x) = x^3$  is odd



$f(x) = \sin(x)$  is periodic



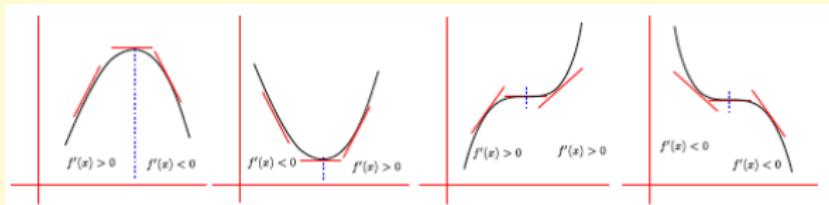
# Outline - Nov. 13, 2019

- ▶ A Quick Review
- ▶ Section 3.6: Sketching Graphs

## Theorem

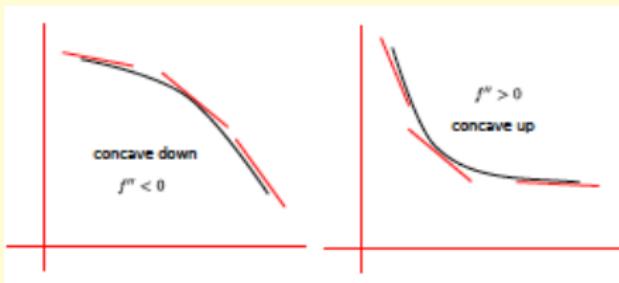
Let  $f$  be a continuous function and  $c$  be a singular or critical point. Then

- If  $f'$  changes from positive to negative at  $c$ , then  $f$  has a local max at  $c$ .
- If  $f'$  changes from negative to positive at  $c$ , then  $f$  has a local min at  $c$ .
- If  $f'$  does not change sign at  $c$ , then  $c$  is not a local max or min.

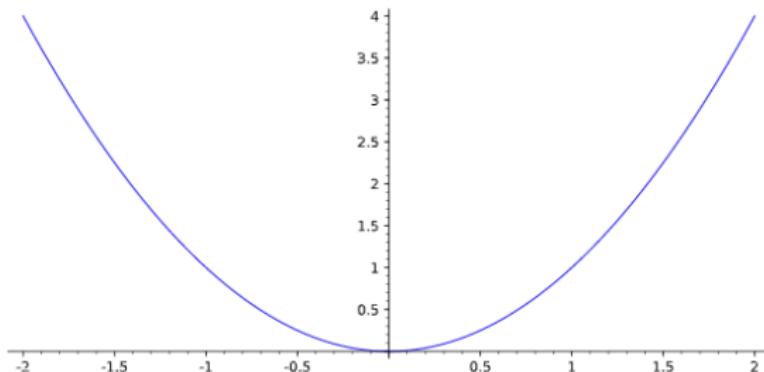


## Theorem

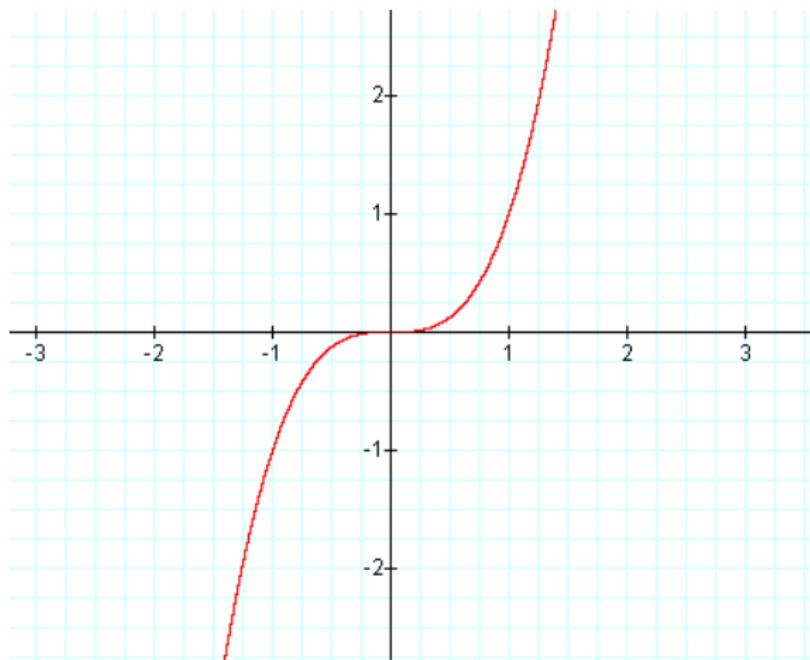
- If  $f''(x) > 0$  on  $I$ , then it is CU on  $I$ .
- If  $f''(x) < 0$ , then it is CD on  $I$ .



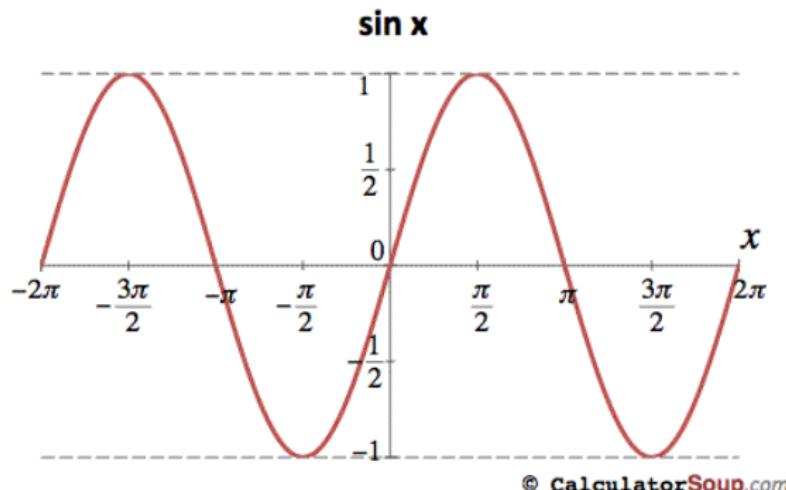
$f(x) = x^2$  is even



$f(x) = x^3$  is odd



$f(x) = \sin(x)$  is periodic



# Outline - Nov. 15, 2019

- ▶ **Section 3.6: Sketching Graphs**
- ▶ **Section 3.7: Indeterminate forms and L'Hopital's rule**

$$\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x} = \frac{0}{0} = ??? \quad \text{or} \quad \lim_{x \rightarrow +\infty} \frac{\ln(x)}{x} = \frac{\infty}{\infty} = ???$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^x = 1^\infty = ???$$

## Check-List

- **Sketching a graph.** A good check-list for sketching a graph.

- ▶ Domain
- ▶ Intercepts
- ▶ Symmetry
- ▶ Asymptotes
- ▶ Singular and critical points; Increasing/Decreasing
- ▶ Concavity and inflection points

# Different Level of Learning

- ▶ **Learning Objectives:** Be able to do all steps in the check list and sketch the graph

1. Recall and Memorize:

Check-List

2. Understand:

An Example by Farid

3. Apply:

Examples by You-Knowing what theorem or method you should use

4. Analyze:

Explain and Analyze Your Work

5. Evaluate:

Other's solution is Correct?

6. Create:

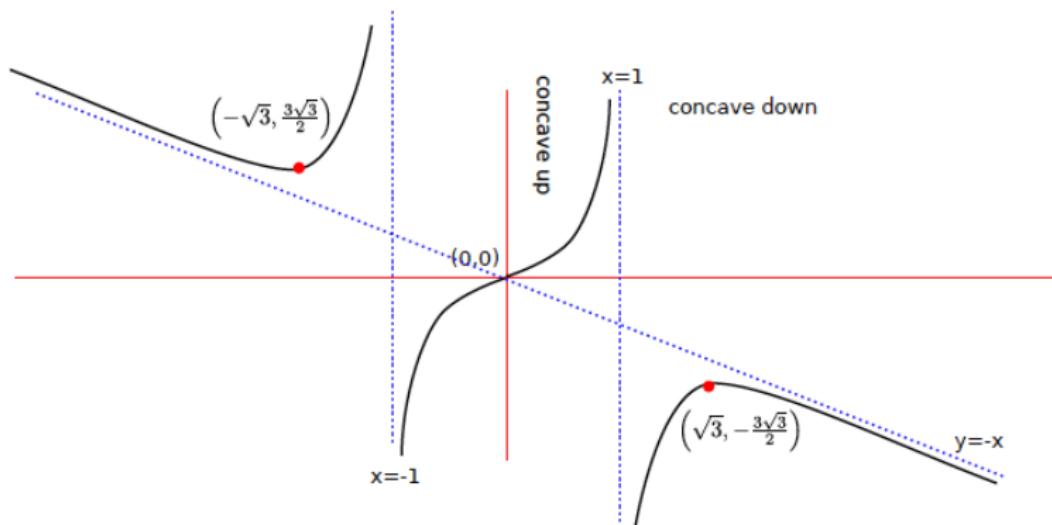
What questions do you give in the test?

$$f(x) = \frac{x^3}{1-x^2} \quad f'(x) = \frac{x^2(3-x^2)}{(1-x^2)^2}, \quad \text{and} \quad f''(x) = \frac{2x(3+x^2)}{(1-x^2)^3}.$$

$(-\infty, -\sqrt{3})$	$f'(x) < 0$	$D$	$f''(x) > 0$	$CU$
$x = -\sqrt{3}$	$f'(x) = 0$	$lmin$	$f''(x) > 0$	$CU$
$(-\sqrt{3}, -1)$	$f'(x) > 0$	$I$	$f''(x) > 0$	$CU$
$x = -1$	$NE$	$S$	$NE$	$NE$
$(-1, 0)$	$f'(x) > 0$	$I$	$f''(x) < 0$	$CD$
$x = 0$	$f'(x) = 0$	$C$	$f''(x) = 0$	$Inflection$
$(0, 1)$	$f'(x) > 0$	$I$	$f''(x) > 0$	$CU$
$x = 1$	$NE$	$S$	$NE$	$NE$
$(1, \sqrt{3})$	$f'(x) > 0$	$I$	$f''(x) < 0$	$CD$
$x = \sqrt{3}$	$f'(x) = 0$	$lmax$	$f''(x) < 0$	$CD$
$(\sqrt{3}, \infty)$	$f'(x) < 0$	$D$	$f''(x) < 0$	$CD$

Asymptotes:  $x = 1$  and  $x = -1$ , lmax:  $(\sqrt{3}, -\frac{3\sqrt{3}}{2})$  and lmin:  $(-\sqrt{3}, \frac{3\sqrt{3}}{2})$ . Also  $f(x)$  is **odd**.

$$f(x) = \frac{x^3}{1-x^2}$$



## Indeterminate forms and L'Hôpital's rule (CLP 3.7)

### Theorem

If  $\lim_{x \rightarrow a} f(x) = K$  and  $\lim_{x \rightarrow a} g(x) = L$ , then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{K}{L} \quad \text{provided } L \neq 0.$$

As an Example:

$$\lim_{x \rightarrow 2} \frac{x^2 - 1}{x + 1} = \frac{3}{3} = 1.$$

But what about this one

$$\lim_{x \rightarrow 1} \frac{x - 1}{x^2 - 1} = \frac{0}{0} = \text{?????}$$

$$\begin{aligned}&= \lim_{x \rightarrow 1} \frac{x - 1}{(x - 1)(x + 1)} \\&= \lim_{x \rightarrow 1} \frac{1}{x + 1} = \frac{1}{2}.\end{aligned}$$

## What you can do with

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \frac{0}{0} = ???$$

$$\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x} = \frac{0}{0} = ??? \quad \text{or} \quad \lim_{x \rightarrow +\infty} \frac{\ln(x)}{x} = \frac{\infty}{\infty} = ???$$

### Indeterminate,

Guillaume-Francois-Antoine Marquis de **L'Hôpital** (1661-1704)



- $\lim_{x \rightarrow 0^+} x \ln(x)$
- $\lim_{x \rightarrow \infty} x^{1/x}$
- $\lim_{x \rightarrow \infty} (1 + 3/x)^x$
- $\lim_{x \rightarrow \infty} \sqrt{4x^2 + 1} - \sqrt{x^2 - 3x}$



# Outline - Nov. 18, 2019

- ▶ A Quick Review
- ▶ Section 3.7: Indeterminate forms and L'Hôpital's rule

By the end of this section you will be able to compute limits by using L'Hôpital's rule when it's needed:

- (1) Change indeterminate forms of types

$$0 \times (\pm\infty) \quad 1^\infty \quad 0^0 \quad \infty^0 \quad \infty - \infty$$

to indeterminate forms of types

$$\pm\infty / \pm\infty \quad 0/0$$

and then use L'Hôpital's rule,

- (2) when it is better doing algebra than using L'Hôpital's rule.

## Indeterminate forms

$\pm\infty/\pm\infty$  and  $0/0$  are two indeterminate forms. Some other types are,

$$0 \times (\pm\infty) \quad 1^\infty \quad 0^0 \quad \infty^0 \quad \infty - \infty$$

If we have any of the above indeterminate forms, it is more likely that we can change it to a limit that in that limit we only need to take care of a limit of the form

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)},$$

and make it an indeterminate form of type  $\pm\infty/\pm\infty$  and  $0/0$ , and then we can use L'Hôpital's rule.

# L'Hôpital's rule

## Theorem

(CLP 3.7.2—L'Hôpital's Rule) Let  $f$  and  $g$  be differentiable functions and  $a$  either be a real number or  $\pm\infty$ .

Furthermore, suppose that either

- ▶  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ , or
- ▶  $\lim_{x \rightarrow a} f(x) = \pm\infty$  and  $\lim_{x \rightarrow a} g(x) = \pm\infty$

then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

provided that limit on the right-hand-side exists or is  $\pm\infty$ .