MATH100: Differential Calculus with Application to Physical Sciences and Engineering

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Contents

Chapter 1

Limits

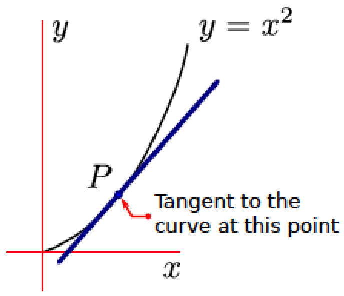
What does this mean

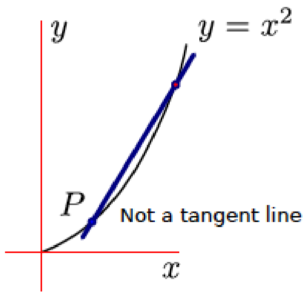
$$\lim_{x \to a} f(x) = L?$$

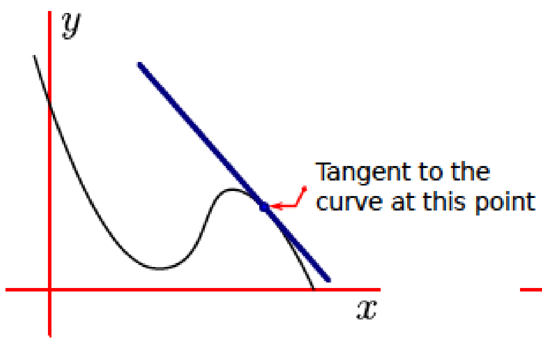
The "limit" appears when we want to

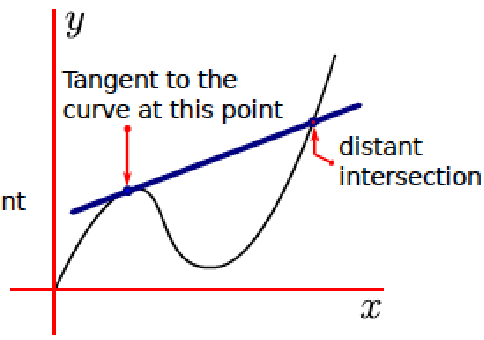
- find the tangent to a curve; or
- find the velocity of an object.

1.1 Tangent line









The **tangent line to a curve** $y = f(x)$ **at a point** P (if exists) is a line L that there is a neighborhood for P such that in that neighborhood the line L touches (does not cross) the curve $y = f(x)$ only at P (and not other points in that neighborhood).

The equation of a line

– The formula for a line that passes though (x_1, y_1) with slope m is

$$y = y_1 + m(x - x_1).$$

– Given two points (x_1, y_1) and (x_2, y_2) on a line, then the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

and the formula for the line then is

$$y = y_1 + m(x - x_1).$$

Example 1.1.1. *Find the equation of the line with slope* -3 *that passes through* $(1, 2)$.

Solution. *The equation of the line is*

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

Example 1.1.2. *Find the equation of the line that passes through* $(1, 2)$ *and* $(2, -1)$.

Solution. *First we find the slope which is*

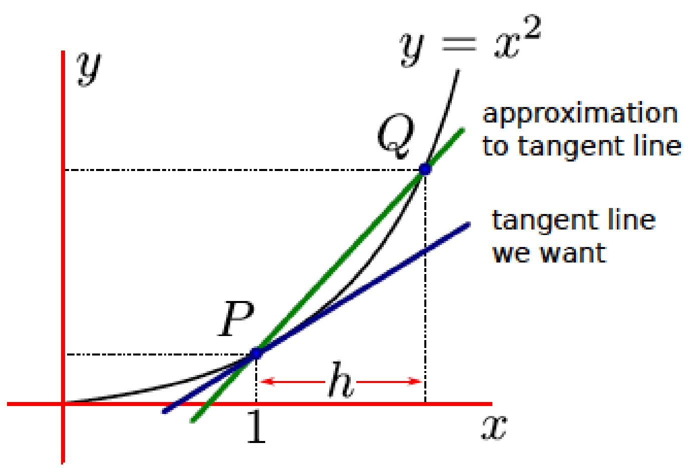
$$\frac{-1 - 2}{2 - 1} = -3.$$

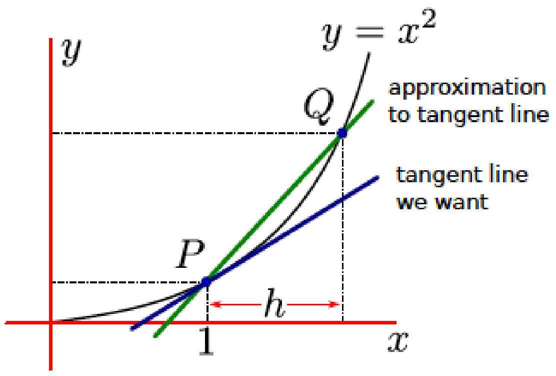
Then the equation of the line is

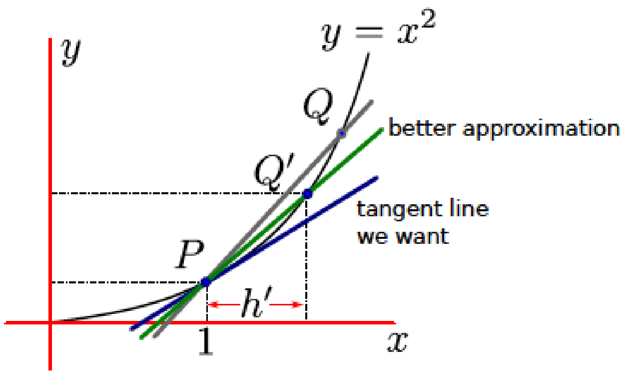
$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

The equation of a tangent line: Given a curve $y = f(x)$ and a point P on the curve, how to find the slope of the tangent to a curve at P: let do this through an example.

Example 1.1.3. *Find the tangent line to the curve* $y = x^2$ *that passes through* $P = (1, 1)$.







So we want to find the slope the line that passes through the points $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (1 + h, (1 + h)^2)$. The slope then is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1+h)^2 - 1^2}{(1+h) - 1} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(h+2)}{h} = 2 + h$$

h	$m = \frac{(1+h)^2 - 1^2}{(1+h)-1}$
0.1	2.1
0.01	2.01
0.001	2.001

When h gets smaller and smaller, the slope will be closer and closer to the slope of the tangent line to $y = x^2$ at $(1, 1)$, which the slope will be closer and closer to 2, we can write this more mathematically as

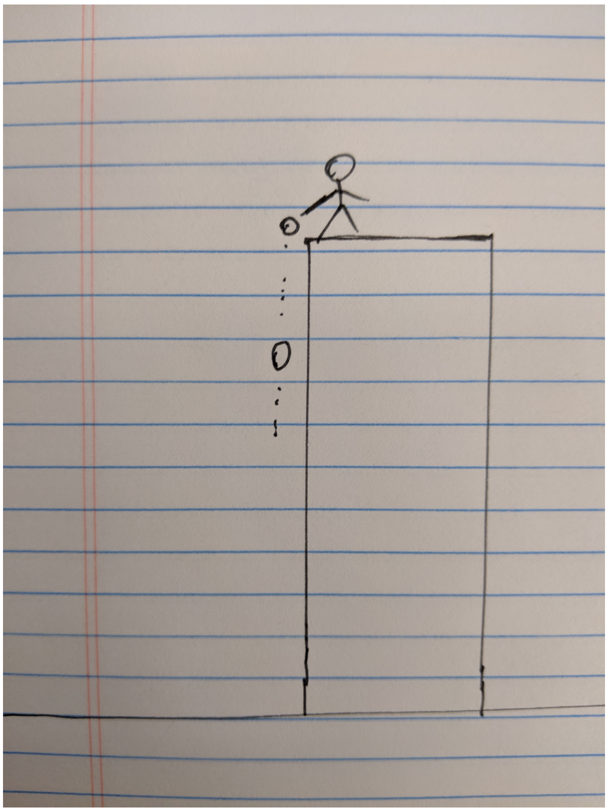
$$\lim_{h \to 0} \frac{(1+h)^2 - 1}{(1+h) - 1} = 2$$

Read: *the limit of* $\frac{(1+h)^2-1}{(1+h)-1}$ *as* h *approaches* 0 *is* 2.
Tangent line is

$$y = 1 + 2(x - 1) = 2x - 1.$$

1.2 Velocity

- Let t be elapsed time measured in second
- $S(t)$ be the distance the ball has fallen in meters
- What is $S(0)$? $S(0) = 0$.
- (**Galileo**) $S(t) = 4.9t^2$.



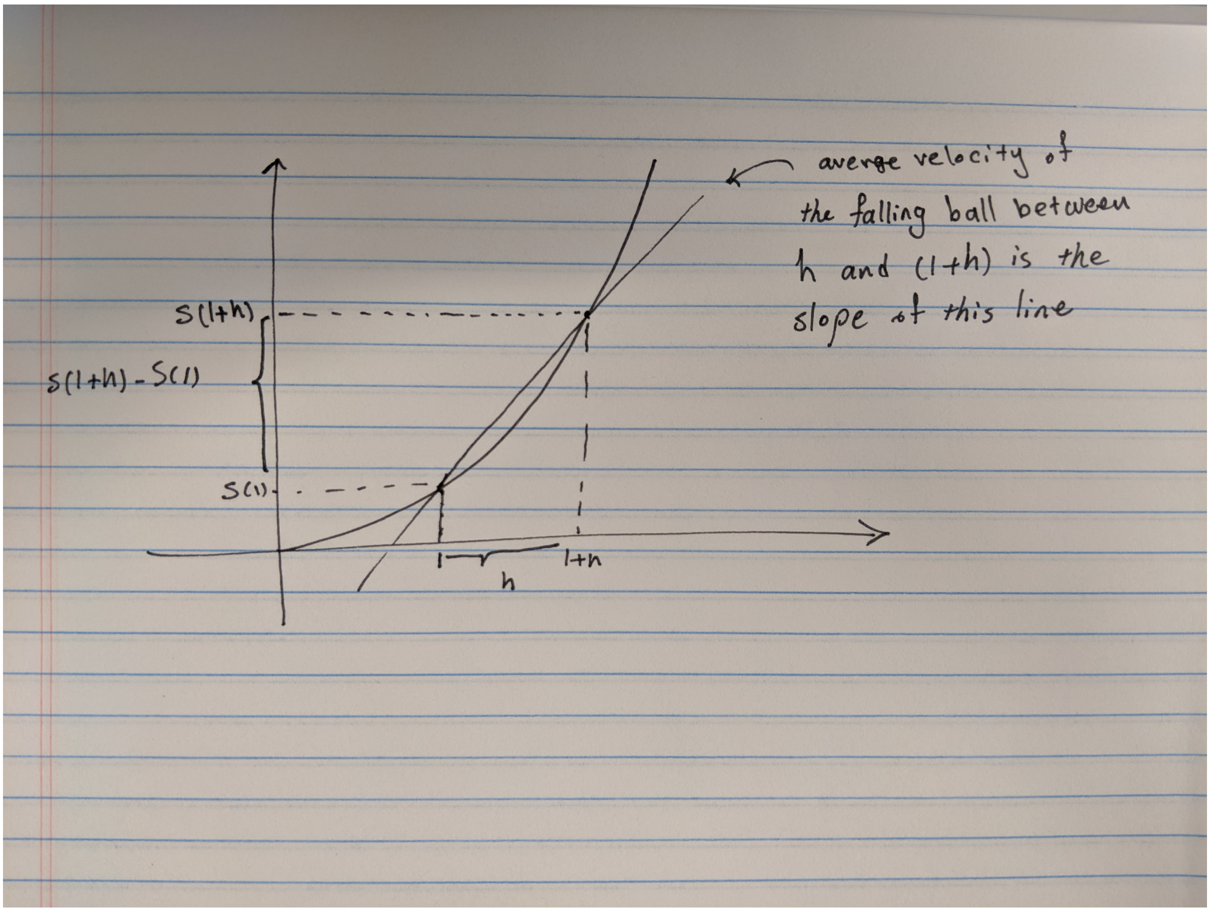
Question: How fast the ball is fallen after 1 second, that is, find $v(1)$, the velocity at $t = 1$?

$$\text{average velocity} = \frac{\text{difference in position}}{\text{difference in time}} = \frac{S(t_2) - S(t_1)}{t_2 - t_1}.$$

To answer the question we should find the average velocity of the falling ball between $(1 + h)$ and 1. So,

$$\text{average velocity when } (t_2 = 1 + h) \text{ and } (t_1 = 1)$$

$$= \frac{S(1+h) - S(1)}{h} = \frac{4.9(1+h)^2 - 4.9}{h} = 4.9(2 + h).$$



time window	average velocity
$1 \leq t \leq 1.1$	10.29
$1 \leq t \leq 1.01$	9.84
$1 \leq t \leq 1.01$	9.8049
$1 \leq t \leq 1.001$	9.80049

So we can write

$$v(1) = \lim_{h \to 0} \frac{S(1+h) - S(1)}{h} = 9.8.$$

More generally:

We define the instantaneous velocity at time $t = a$ to be the limit

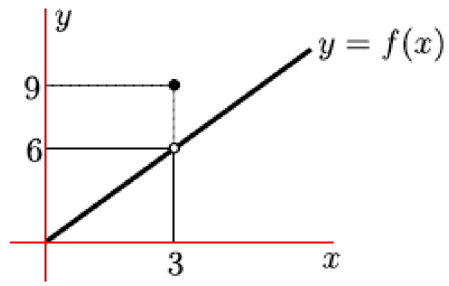
$$v(a) = \lim_{h \to 0} \frac{S(a+h) - S(a)}{h}$$

1.3 The limit of a function

To arrive at the definition of limit, we start with a very simple example.

Example 1.3.1. *Consider the following function*

$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



If we plug in some numbers very close to 3 (but not exactly 3) into the function we see

x	2.9	2.99	2.999	○	3.001	3.01	3.1
$f(x)$	5.8	5.98	5.998	○	6.002	6.02	6.2

So as x moves closer and closer to 3, without being exactly 3, we see that the function moves closer and closer to 6. We can then write this as

$$\lim_{x \to 3} f(x) = 6.$$

Definition. (Informal definition of limit) *We write*

$$\lim_{x \to a} f(x) = L.$$

if the value of the function $f(x)$ is sure to be arbitrary close to L whenever the value of x is close enough to a, without being exactly a.

Example 1.3.2. *Let $f(x) = \frac{x-2}{x^2+x-6}$ and find its limit as $x \to 2$.*

Solution. *We want to find*

$$\lim_{x \to 2} \frac{x-2}{x^2+x-6}.$$

Important point: *if we we compute $f(2)$, then we have $\frac{0}{0}$ which is undefined.*

Again we plug in numbers close to 2 and we have

x	1.9	1.99	1.999	○	2.001	2.01	2.1
$f(x)$	0.20408	0.20040	0.20004	○	0.19996	0.19960	0.19608

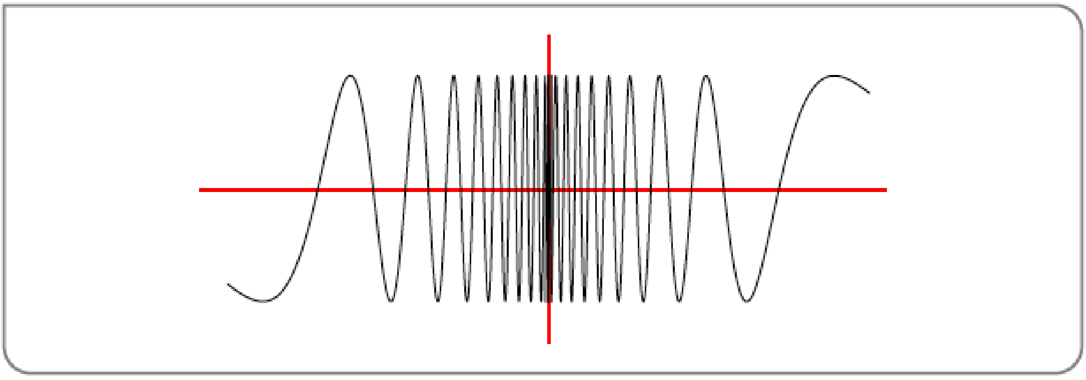
So

$$\lim_{x \to 2} \frac{x-2}{x^2+x-6} = 2.$$

Example 1.3.3. *Consider the following function $f(x) = sin(\pi/x)$. Find the limit as $x \to 0$ of $f(x)$.*

Solution. *When x is getting closer and closer to 0, it oscillates faster and faster. Since the function does not approach a single number as we bring x closer and closer to zero, the limit does not exist. Thus,*

$$\lim_{x\to 0} sin(\pi/x) = DNE$$



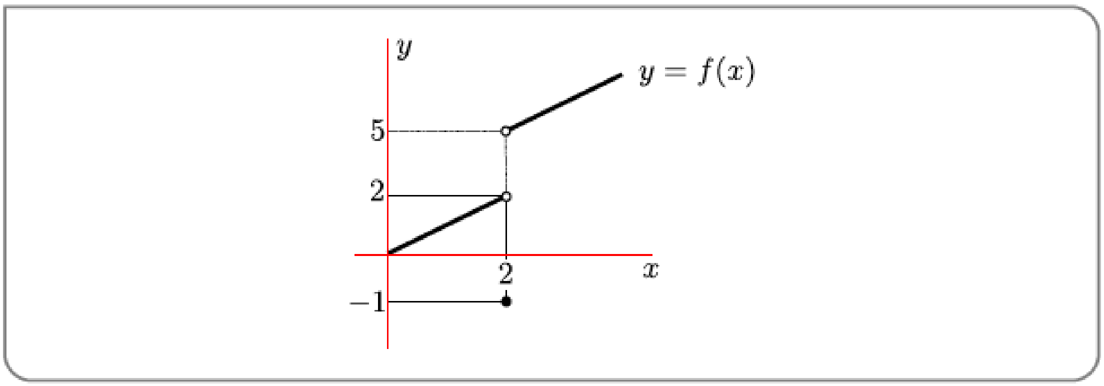
Example 1.3.4. *Consider the function*

$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x+3 & x > 2 \end{cases}$$

Find

$$\lim_{x\to 2} f(x).$$

Solution.



Let again plug in some numbers close to 2 (but not exactly 2)

x	1.9	1.99	1.999	∘	2.001	2.01	2.1
$f(x)$	1.9	1.99	1.999	∘	5.001	5.01	5.1

Now when we approach from below (or left), we seem to be getting closer to 2 $\left(\lim_{x\to 2^-} f(x) = 2\right)$, *but when we approach from above (or right) we seem to be getting closer to* 5 $\left(\lim_{x\to 2^+} f(x) = 5\right)$. *Since we are not approaching the same number the limit does not exists.*

$$\lim_{x\to 2} f(x) = DNE$$

Definition. *(***Informal definition of one-sided limits***) We write*

$$\lim_{x\to a^-} f(x) = K$$

when the value of $f(x)$ gets closer and closer to K when $x < a$ and x moves closer and closer to a. Since the x-values are always less than a, we say that x approaches a from below (or left). This is also often called the left-hand limit since the x-values lie to the left of a on a sketch of the graph.

We similarly write

$$\lim_{x\to a^+} f(x) = L$$

when the values of $f(x)$ gets closer and closer to L when $x > a$ and x moves closer and closer to a. For similar reason we say that x approaches a from above, and sometimes to this as the the right-hand limit.

Theorem 1.3.5.

$$\lim_{x\to a} f(x) = L \quad \textit{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \textit{and} \quad \lim_{x\to a^+} f(x) = L$$

- If the limit of $f(x)$ as x approaches a exists and is equal to L, then both the left-hand and right-hand limits exist and are equal to L.

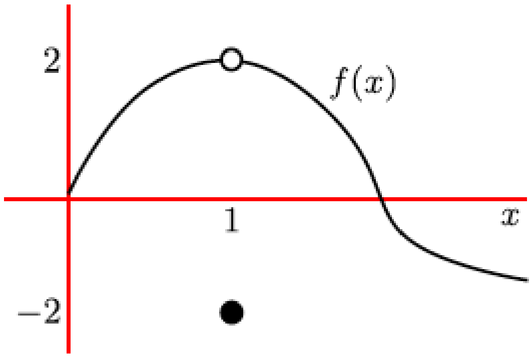
- If the left-hand and right-hand limits as x approaches a exist and are equal, then the limit as x approaches a exists and is equal to the one-sided limits.

Contrapositive of the above argument says

- If either of the left-hand and right-hand limits as x approaches a fail to exist, or if they both exist but are different, then the limit as x approaches a does not exist. AND,

- If the limit as x approaches a does not exist, then the left-hand and right-hand limits are either different or at least one of them does not exist.

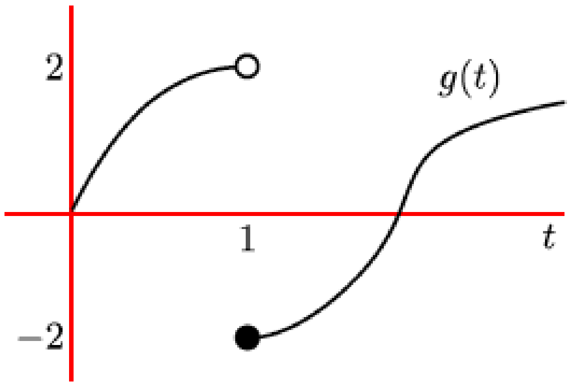
Example 1.3.6. *Consider the graph of the function $f(x)$.*



Then

$$\lim_{x\to 1^-} f(x) = 2 \qquad \lim_{x\to 1^+} f(x) = 2 \qquad \lim_{x\to 1} f(x) = 2$$

Example 1.3.7. *Consider the graph of the function $g(x)$.*

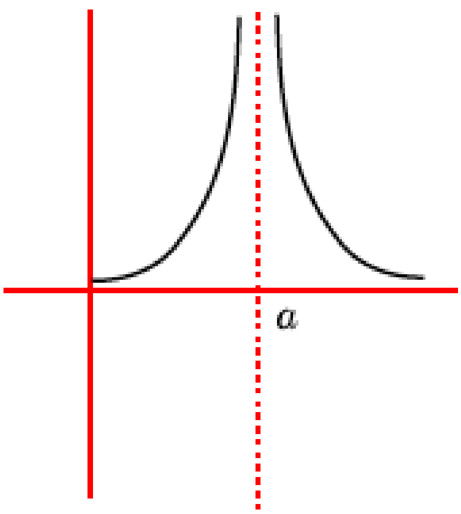


Then

$$\lim_{x\to 1^-} g(x) = 2 \qquad \lim_{x\to 1^+} g(x) = -2 \qquad \lim_{x\to 1} g(x) = DNE$$

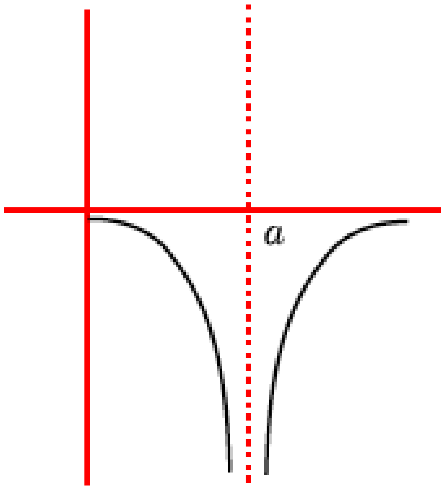
In the following example even though the limit doesn't exists when x approaches a, we can say more.

Example 1.3.8. *Consider the graph for the function $f(x)$.*



$$\lim_{x \to a} f(x) = +\infty$$

Example 1.3.9. *Consider the graph for the function $g(x)$.*



$$\lim_{x \to a} g(x) = -\infty$$

Definition. *We write*

$$\lim_{x \to a} f(x) = +\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as x gets closer and closer to a, without being exactly a.
Similarly, we write

$$\lim_{x \to a} f(x) = -\infty$$

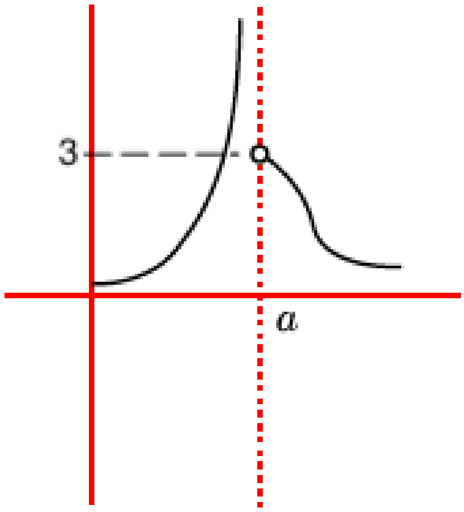
when the value of the function $f(x)$ becomes arbitrarily large and negative as x gets closer and closer to a, without being exactly a.

Example 1.3.10.

$$\lim_{x \to 0} \frac{1}{x^2} = +\infty \qquad \lim_{x \to 0} -\frac{1}{x^2} = -\infty$$

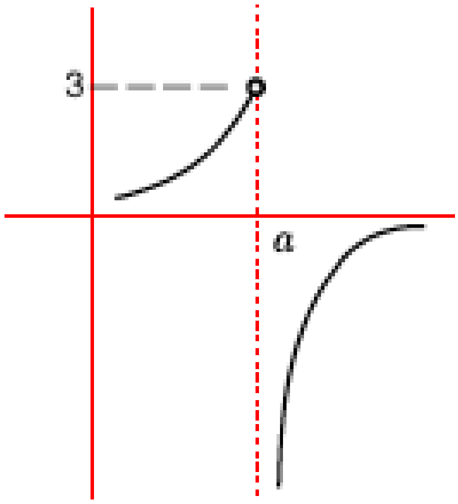
Important Point: Do not think of "$+\infty$" and "$-\infty$" in these statements as numbers. When we write $\lim\limits_{x \to a} f(x) = +\infty$, it says "the function $f(x)$ becomes arbitrary large as x approaches a".

Example 1.3.11. *Consider the graph for the function $h(x)$.*



$$\lim_{x\to a^-} h(x) = +\infty \qquad \lim_{x\to a^+} h(x) = 3 \qquad \lim_{x\to a} h(x) = DNE$$

Example 1.3.12. *Consider the graph for the function $s(x)$.*



$$\lim_{x\to a^-} s(x) = 3 \qquad \lim_{x\to a^+} s(x) = -\infty \qquad \lim_{x\to a} s(x) = DNE$$

Definition. *We write*

$$\lim_{x\to a^+} f(x) = +\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as x gets closer and closer to a from above (equivalently, from right), without being exactly a. Similarly, we write

$$\lim_{x\to a^+} f(x) = -\infty$$

when the values of the function $f(x)$ becomes arbitrarily large and negative as x gets closer and closer to a from above (equivalently, from right), without being exactly a.

The notation

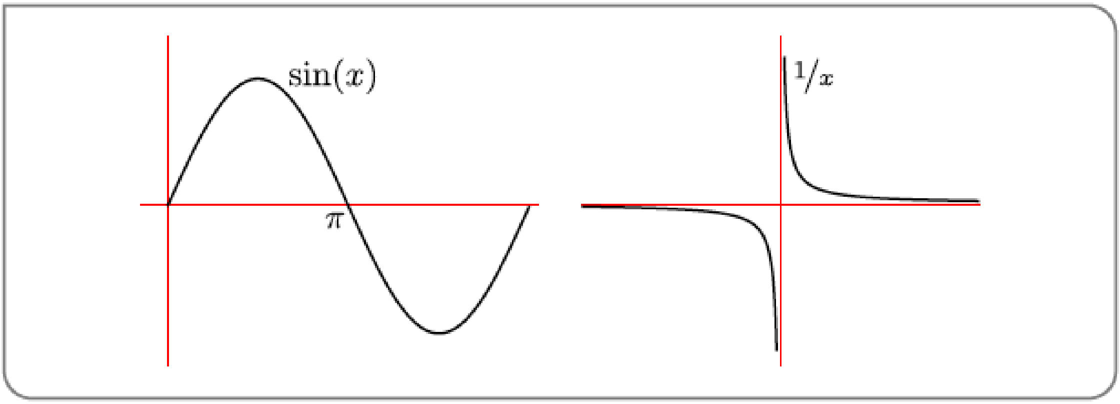
$$\lim_{x\to a^-} f(x) = +\infty \quad \textit{and} \quad \lim_{x\to a^-} f(x) = -\infty$$

has a similar meaning except that limits are approached form below (from left).

Example 1.3.13. *Consider the function*

$$g(x) = \frac{1}{sin(x)}.$$

Find the one-side limits of this function as $x \to \pi$.



- As $x \to \pi$ from the left, $\sin(x)$ is a small positive number that is getting closer and closer to zero. That is, as $x \to \pi^-$, we have that $\sin(x) \to 0$ through positive numbers (i.e. from above). Now look at the graph of $1/x$, and think what happens as we move $x \to 0^+$, the function is positive and becomes larger and larger.

  So as $x \to \pi$ from the left, $\sin(x) \to 0$ from above, and so $1/\sin(x) \to +\infty$.

- By very similar reasoning, as $x \to \pi$ from the right, $\sin(x)$ is a small negative number that gets closer and closer to zero. So as $x \to \pi$ from the right, $\sin(x) \to 0$ through negative numbers (i.e. from below) and so $1/\sin(x)$ to $-\infty$.

Thus

$$\lim_{x\to\pi^-} \frac{1}{\sin(x)} = +\infty \qquad \lim_{x\to\pi^+} \frac{1}{\sin(x)} = -\infty$$

1.4 Calculating Limits with Limit Laws

Let $a, c \in mathbbR$. The following two limits hold

$$\lim_{x \to a} c = c \qquad \lim_{x \to a} x = a$$

Theorem 1.4.1. *Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f and g need not to be defined exactly at a).*

$$\lim_{x \to a} f(x) = F \qquad \lim_{x \to a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. Then the following limits hold

- $\lim_{x \to a}(f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*
- $\lim_{x \to a}(f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*
- $\lim_{x \to a} cf(x) = cF$.
- $\lim_{x \to a}(f(x).g(x)) = F.G$*–limit of the product is the product of the limits.*
- *If $G \neq 0$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}$.*

Example 1.4.2.

Bibliography

[1] CLP1: Differential Calculus by J. Feldman, A. Rechnitzer, and E. Yeager.