MATH100: Differential Calculus with Application to Physical Sciences and Engineering

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Contents

Chapter 1

Limits

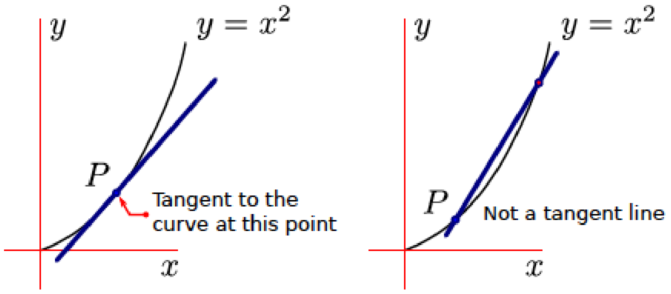
What does this mean

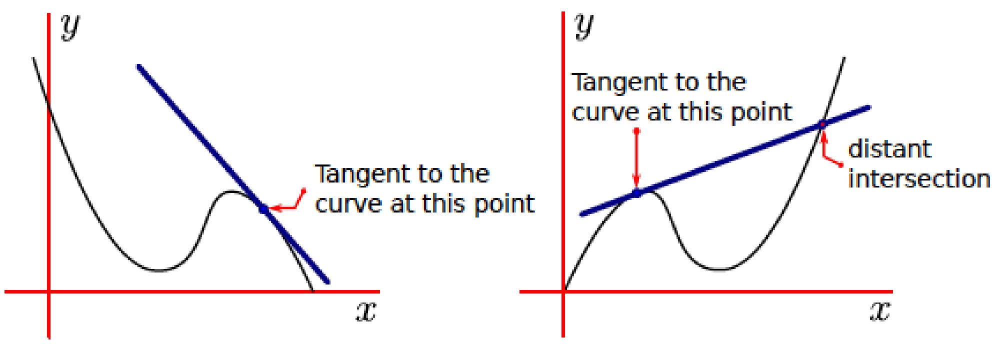
$$\lim_{x \to a} f(x) = L?$$

The "limit" appears when we want to

- find the tangent to a curve; or
- find the velocity of an object.

1.1 Tangent line





The **tangent line to a curve** $y = f(x)$ **at a point** P (if exists) is a line L that there is a neighborhood for P such that in that neighborhood the line L touches (does not cross) the curve $y = f(x)$ only at P (and not other points in that neighborhood).

The equation of a line

– The formula for a line that passes though (x_1, y_1) with slope m is

$$y = y_1 + m(x - x_1).$$

– Given two points (x_1, y_1) and (x_2, y_2) on a line, then the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

and the formula for the line then is

$$y = y_1 + m(x - x_1).$$

Example 1.1.1. *Find the equation of the line with slope* -3 *that passes through* $(1, 2)$.

Solution. *The equation of the line is*

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

Example 1.1.2. *Find the equation of the line that passes through* $(1, 2)$ *and* $(2, -1)$.

Solution. *First we find the slope which is*

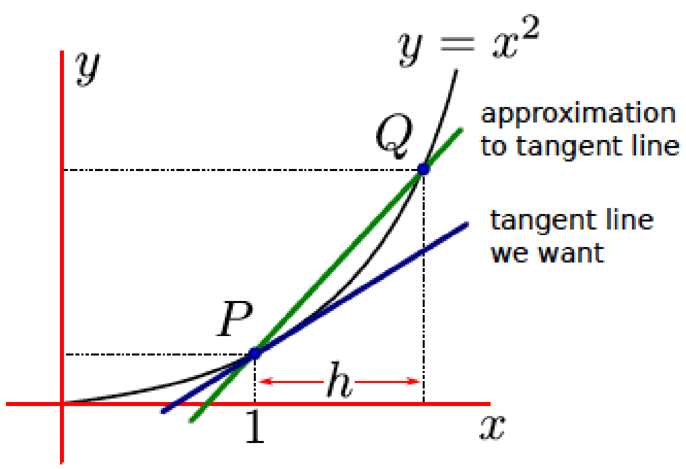
$$\frac{-1 - 2}{2 - 1} = -3.$$

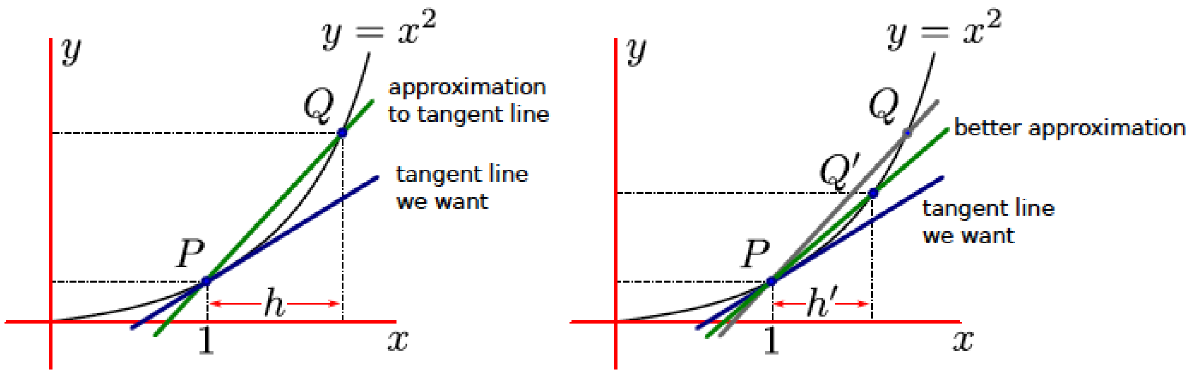
Then the equation of the line is

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

The equation of a tangent line: Given a curve $y = f(x)$ and a point P on the curve, how to find the slope of the tangent to a curve at P: let do this through an example.

Example 1.1.3. *Find the tangent line to the curve* $y = x^2$ *that passes through* $P = (1, 1)$.





So we want to find the slope the line that passes through the points $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (1 + h, (1 + h)^2)$. The slope then is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1+h)^2 - 1^2}{(1+h) - 1} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(h+2)}{h} = 2 + h$$

h	$m = \frac{(1+h)^2-1^2}{(1+h)-1}$
0.1	2.1
0.01	2.01
0.001	2.001

When h gets smaller and smaller, the slope will be closer and closer to the slope of the tangent line to $y = x^2$ at $(1, 1)$, which the slope will be closer and closer to 2, we can write this more mathematically as

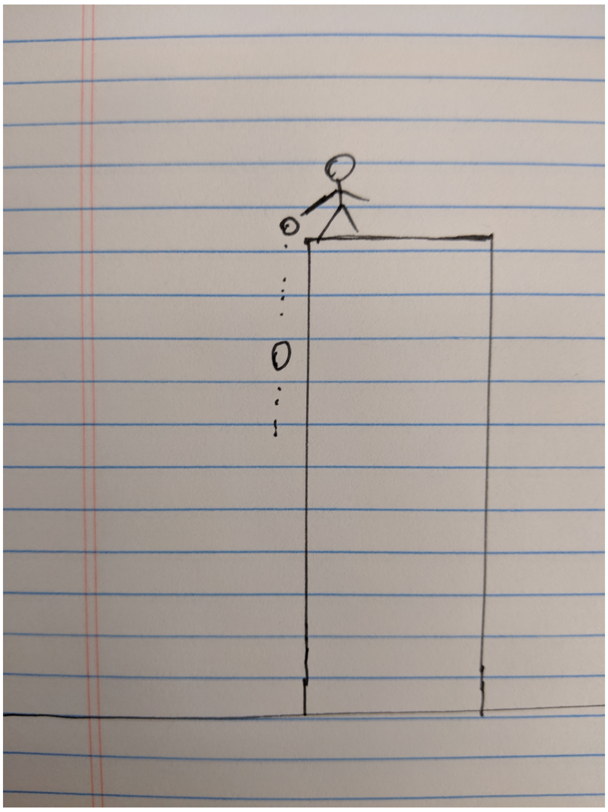
$$\lim_{h\to 0} \frac{(1+h)^2 - 1}{(1+h) - 1} = 2$$

Read: *the limit of* $\frac{(1+h)^2-1}{(1+h)-1}$ *as* h *approaches* 0 *is* 2.
Tangent line is

$$y = 1 + 2(x - 1) = 2x - 1.$$

1.2 Velocity

- Let t be elapsed time measured in second
- $S(t)$ be the distance the ball has fallen in meters
- What is $S(0)$? $S(0) = 0$.
- (**Galileo**) $S(t) = 4.9t^2$.



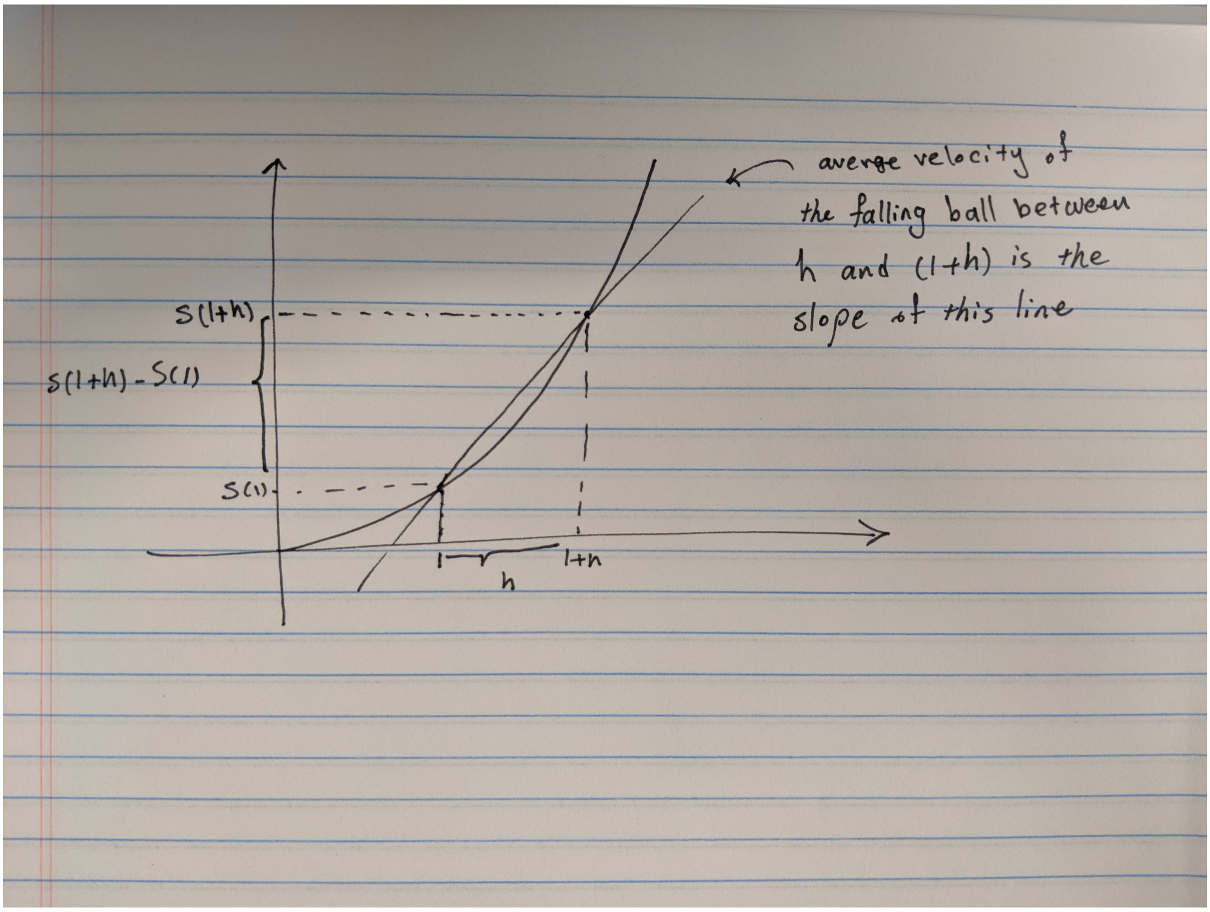
Question: How fast the ball is fallen after 1 second, that is, find $v(1)$, the velocity at $t = 1$?

$$\text{average velocity} = \frac{\text{difference in position}}{\text{difference in time}} = \frac{S(t_2) - S(t_1)}{t_2 - t_1}.$$

To answer the question we should find the average velocity of the falling ball between $(1 + h)$ and 1. So,

$$\text{average velocity when } (t_2 = 1 + h) \text{ and } (t_1 = 1)$$

$$= \frac{S(1 + h) - S(1)}{h} = \frac{4.9(1 + h)^2 - 4.9}{h} = 4.9(2 + h).$$



time window	average velocity
$1 \le t \le 1.1$	10.29
$1 \le t \le 1.01$	9.84
$1 \le t \le 1.01$	9.8049
$1 \le t \le 1.001$	9.80049

So we can write

$$v(1) = \lim_{h \to 0} \frac{S(1+h) - S(1)}{h} = 9.8.$$

More generally:

We define the instantaneous velocity at time $t = a$ to be the limit

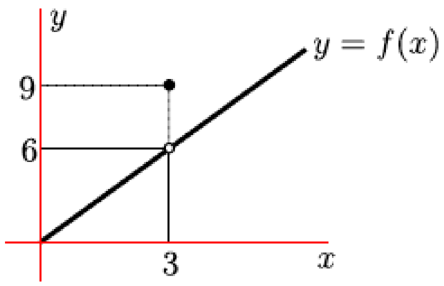
$$v(a) = \lim_{h \to 0} \frac{S(a+h) - S(a)}{h}$$

1.3 The limit of a function

To arrive at the definition of limit, we start with a very simple example.

Example 1.3.1. *Consider the following function*

$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



If we plug in some numbers very close to 3 *(but not exactly 3) into the function we see*

x	2.9	2.99	2.999	○	3.001	3.01	3.1
$f(x)$	5.8	5.98	5.998	○	6.002	6.02	6.2

So as x moves closer and closer to 3*, without being exactly* 3*, we see that the function moves closer and closer to* 6*. We can then write this as*

$$\lim_{x\to 3} f(x) = 6.$$

Definition. (Informal definition of limit) *We write*

$$\lim_{x\to a} f(x) = L.$$

if the value of the function $f(x)$ is sure to be arbitrary close to L whenever the value of x is close enough to a, without being exactly a.

Example 1.3.2. *Let $f(x) = \frac{x-2}{x^2+x-6}$ and find its limit as $x \to 2$.*

Solution. *We want to find*

$$\lim_{x\to 2} \frac{x-2}{x^2+x-6}.$$

Important point: *if we we compute $f(2)$, then we have $\frac{0}{0}$ which is undefined.*

Again we plug in numbers close to 2 *and we have*

x	1.9	1.99	1.999	○	2.001	2.01	2.1
$f(x)$	0.20408	0.20040	0.20004	○	0.19996	0.19960	0.19608

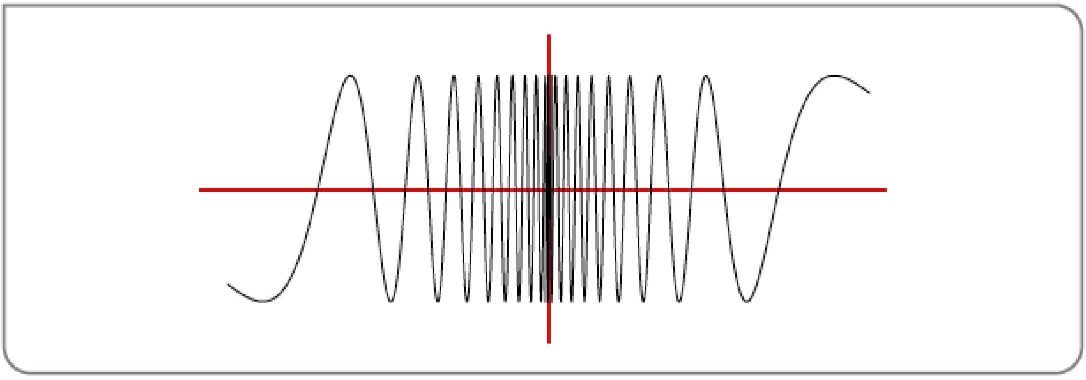
So

$$\lim_{x\to 2} \frac{x-2}{x^2+x-6} = 2.$$

Example 1.3.3. *Consider the following function $f(x) = sin(\pi/x)$. Find the limit as $x \to 0$ of $f(x)$.*

Solution. *When x is getting closer and closer to 0, it oscillates faster and faster. Since the function does not approach a single number as we bring x closer and closer to zero, the limit does not exist. Thus,*

$$\lim_{x\to 0} sin(\pi/x) = DNE$$



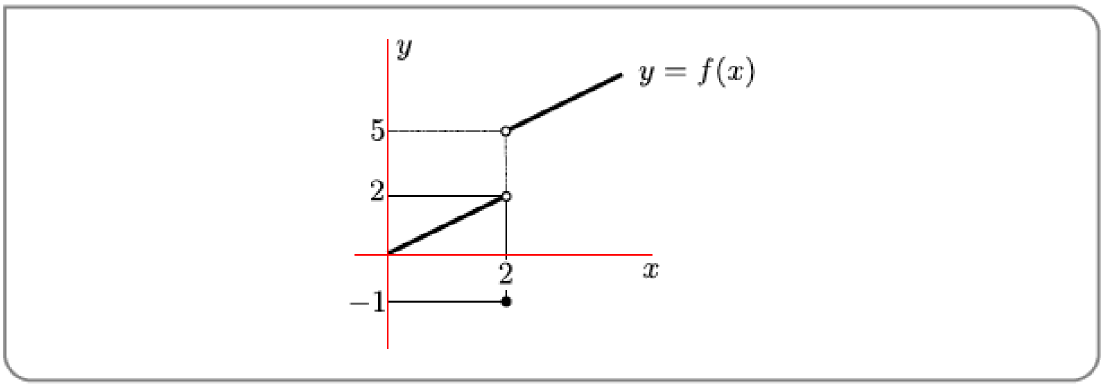
Example 1.3.4. *Consider the function*

$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x+3 & x > 2 \end{cases}$$

Find

$$\lim_{x\to 2} f(x).$$

Solution.



Let again plug in some numbers close to 2 (but not exactly 2)

x	1.9	1.99	1.999	○	2.001	2.01	2.1
$f(x)$	1.9	1.99	1.999	○	5.001	5.01	5.1

Now when we approach from below (or left), we seem to be getting closer to 2 $\left(\lim_{x\to 2^-} f(x) = 2\right)$*, but when we approach from above (or right) we seem to be getting closer to* 5 $\left(\lim_{x\to 2^+} f(x) = 5\right)$. *Since we are not approaching the same number the limit does not exists.*

$$\lim_{x\to 2} f(x) = DNE$$

Definition. *(***Informal definition of one-sided limits***) We write*

$$\lim_{x\to a^-} f(x) = K$$

when the value of $f(x)$ gets closer and closer to K when $x < a$ and x moves closer and closer to a. Since the x-values are always less than a, we say that x approaches a from below (or left). This is also often called the left-hand limit since the x-values lie to the left of a on a sketch of the graph.

We similarly write

$$\lim_{x\to a^+} f(x) = L$$

when the values of $f(x)$ gets closer and closer to L when $x > a$ and x moves closer and closer to a. For similar reason we say that x approaches a from above, and sometimes to this as the the right-hand limit.

Theorem 1.3.5.

$$\lim_{x\to a} f(x) = L \quad \textit{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \textit{and} \quad \lim_{x\to a^+} f(x) = L$$

- If the limit of $f(x)$ as x approaches a exists and is equal to L, then both the left-hand and right-hand limits exist and are equal to L.

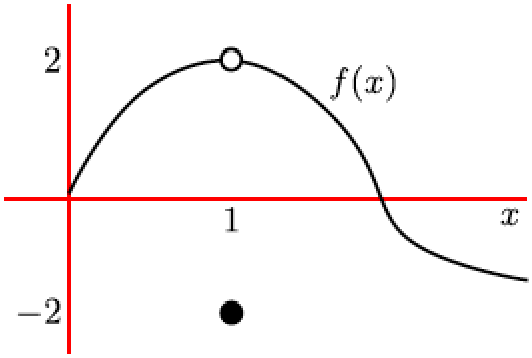
- If the left-hand and right-hand limits as x approaches a exist and are equal, then the limit as x approaches a exists and is equal to the one-sided limits.

Contrapositive of the above argument says

- If either of the left-hand and right-hand limits as x approaches a fail to exist, or if they both exist but are different, then the limit as x approaches a does not exist. AND,

- If the limit as x approaches a does not exist, then the left-hand and right-hand limits are either different or at least one of them does not exist.

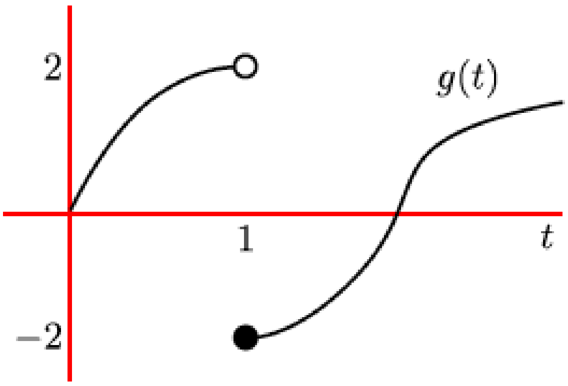
Example 1.3.6. *Consider the graph of the function $f(x)$.*



Then

$$\lim_{x\to 1^-} f(x) = 2 \qquad \lim_{x\to 1^+} f(x) = 2 \qquad \lim_{x\to 1} f(x) = 2$$

Example 1.3.7. *Consider the graph of the function $g(x)$.*

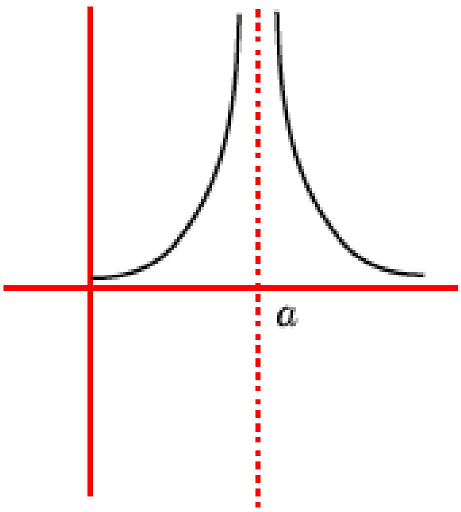


Then

$$\lim_{t\to 1^-} g(t) = 2 \qquad \lim_{t\to 1^+} g(t) = -2 \qquad \lim_{t\to 1} g(t) = DNE$$

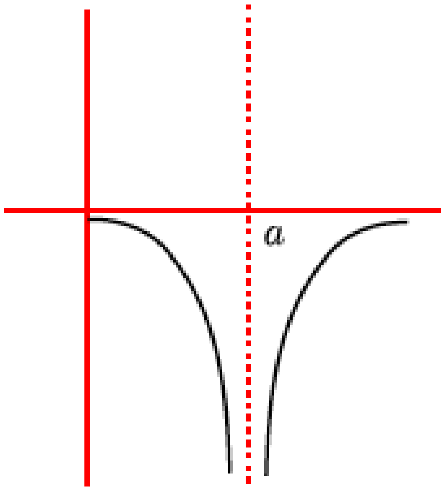
In the following example even though the limit doesn't exists when x approaches a, we can say more.

Example 1.3.8. *Consider the graph for the function $f(x)$.*



$$\lim_{x\to a} f(x) = +\infty$$

Example 1.3.9. *Consider the graph for the function $g(x)$.*



$$\lim_{x\to a} g(x) = -\infty$$

Definition. *We write*

$$\lim_{x\to a} f(x) = +\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as x gets closer and closer to a, without being exactly a.
Similarly, we write

$$\lim_{x\to a} f(x) = -\infty$$

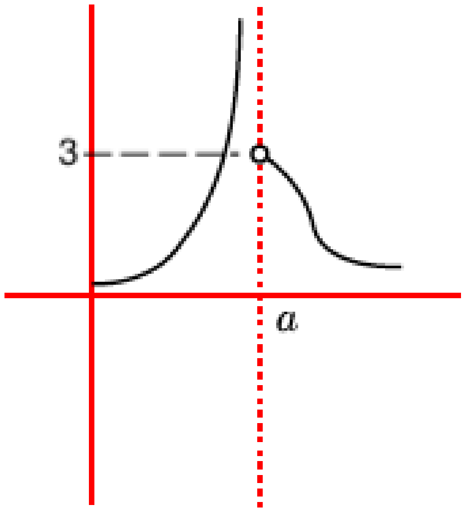
when the value of the function $f(x)$ becomes arbitrarily large and negative as x gets closer and closer to a, without being exactly a.

Example 1.3.10.

$$\lim_{x\to 0} \frac{1}{x^2} = +\infty \qquad \lim_{x\to 0} -\frac{1}{x^2} = -\infty$$

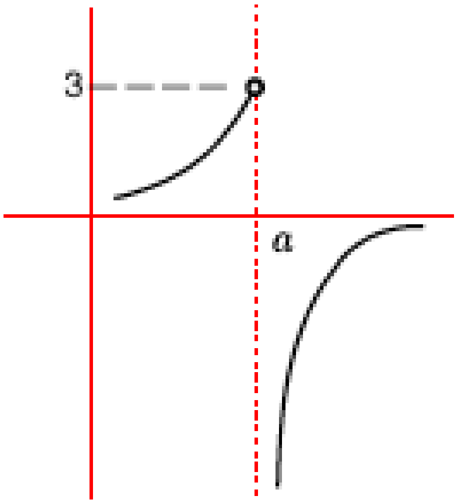
Important Point: Do not think of "$+\infty$" and "$-\infty$" in these statements as numbers. When we write $\lim_{x\to a} f(x) = +\infty$, it says "the function $f(x)$ becomes arbitrary large as x approaches a".

Example 1.3.11. *Consider the graph for the function $h(x)$.*



$$\lim_{x\to a^-} h(x) = +\infty \qquad \lim_{x\to a^+} h(x) = 3 \qquad \lim_{x\to a} h(x) = DNE$$

Example 1.3.12. *Consider the graph for the function $s(x)$.*



$$\lim_{x\to a^-} s(x) = 3 \qquad \lim_{x\to a^+} s(x) = -\infty \qquad \lim_{x\to a} s(x) = DNE$$

Definition. *We write*

$$\lim_{x\to a^+} f(x) = +\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as x gets closer and closer to a from above (equivalently, from right), without being exactly a. Similarly, we write

$$\lim_{x\to a^+} f(x) = -\infty$$

when the values of the function $f(x)$ becomes arbitrarily large and negative as x gets closer and closer to a from above (equivalently, from right), without being exactly a.

The notation

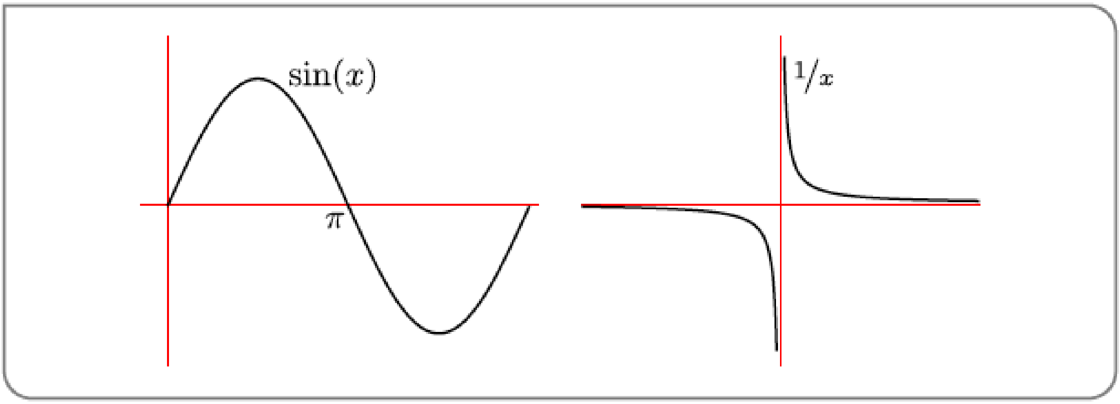
$$\lim_{x\to a^-} f(x) = +\infty \quad \textit{and} \quad \lim_{x\to a^-} f(x) = -\infty$$

has a similar meaning except that limits are approached form below (from left).

Example 1.3.13. *Consider the function*

$$g(x) = \frac{1}{sin(x)}.$$

Find the one-side limits of this function as $x \to \pi$.



- As $x \to \pi$ from the left, $\sin(x)$ is a small positive number that is getting closer and closer to zero. That is, as $x \to \pi^-$, we have that $\sin(x) \to 0$ through positive numbers (i.e. from above). Now look at the graph of $1/x$, and think what happens as we move $x \to 0^+$, the function is positive and becomes larger and larger.

  So as $x \to \pi$ from the left, $\sin(x) \to 0$ from above, and so $1/\sin(x) \to +\infty$.

- By very similar reasoning, as $x \to \pi$ from the right, $\sin(x)$ is a small negative number that gets closer and closer to zero. So as $x \to \pi$ from the right, $\sin(x) \to 0$ through negative numbers (i.e. from below) and so $1/\sin(x)$ to $-\infty$.

Thus

$$\lim_{x\to\pi^-} \frac{1}{\sin(x)} = +\infty \qquad\qquad \lim_{x\to\pi^+} \frac{1}{\sin(x)} = -\infty$$

1.4 Calculating Limits with Limit Laws

Theorem 1.4.1. *Let $a, c \in \mathbb{R}$. The following two limits hold*

$$\lim_{x\to a} c = c \qquad \lim_{x\to a} x = a$$

Theorem 1.4.2. *(Arithmetic of Limits) Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f and g need not to be defined exactly at a).*

$$\lim_{x\to a} f(x) = F \qquad \lim_{x\to a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. Then the following limits hold

- $\lim\limits_{x\to a}(f(x)+g(x)) = F+G$*–limit of the sum is the sum of the limits.*
- $\lim\limits_{x\to a}(f(x)-g(x)) = F-G$*–limit of the difference is the difference of the limits.*
- $\lim\limits_{x\to a} cf(x) = cF$.
- $\lim\limits_{x\to a}(f(x).g(x)) = F.G$*–limit of the product is the product of the limits.*
- *If* $G \neq 0$ *then* $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{F}{G}$.

Example 1.4.3. *Given*

$$\lim_{x\to 1} f(x) = 3 \quad \textit{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3\times \lim_{x\to 1} f(x) = 3\times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3\times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3\times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} f(x). \lim_{x\to 1} g(x) = 3\times 2 = 6.$$

$$\lim_{x\to 1}\frac{f(x)}{f(x)-g(x)} = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} f(x) - \lim_{x\to 1} g(x)} = \frac{3}{3-2} = 3.$$

Example 1.4.4.

$$\lim_{x\to 3} 4x^2 - 1 = 4\times \lim_{x\to 3} x^2 - \lim_{x\to 3} 1 = 35.$$

$$\lim_{x\to 2}\frac{x}{x-1} = \frac{\lim_{x\to 2} x}{\lim_{x\to 2} x - \lim_{x\to 1} 1} = \frac{2}{2-1} = 2.$$

Consider that we apply the theorem Arithmetic of Limits to compute the limit of a ratio if the limit of denominator is not zero. **What will happen if the limit of denominator is zero:**

– the limit does not exist, eg.

$$\lim_{x\to 0}\frac{x}{x^2} = \lim_{x\to 0}\frac{1}{x} = DNE$$

– the limit is $\pm\infty$, eg.

$$\lim_{x\to 0}\frac{x^2}{x^4}=\lim_{x\to 0}\frac{1}{x^2}=+\infty \qquad \text{or} \qquad \lim_{x\to 0}\frac{-x^2}{x^4}=\lim_{x\to 0}\frac{-1}{x^2}=-\infty.$$

– the limit is 0, eg.

$$\lim_{x\to 0}\frac{x^2}{x}=\lim_{x\to 0}x=0.$$

– the limit exists and it nonzero, eg.

$$\lim_{x\to 0}\frac{x}{x}=1.$$

Theorem 1.4.5. *Let n be a positive integer, let $a\in R$ and let f be a function so that*

$$\lim_{x\to a}f(x)=F$$

for some real number F. Then the following holds

$$\lim_{x\to a}(f(x))^n=\left(\lim_{x\to a}f(x)\right)^n=F^n$$

so that the limit of a power is the power of the limit. Similarly, if

- *n is an even number and $F>0$, or*
- *n is an odd number and F is any real number*

then

$$\lim_{x\to a}(f(x))^{1/n}=\left(\lim_{x\to a}f(x)\right)^{1/n}=F^{1/n}.$$

Example 1.4.6.

$$\lim_{x\to 4}x^{1/2}=4^{1/2}=2.$$

$$\lim_{x\to 4}(-x)^{1/2}=-4^{1/2}=\textit{not a real number}.$$

$$\lim_{x\to 2}(4x^2-3)^{1/3}=(4(2)^2-3)^{1/3}=13^{1/3}$$

Example 1.4.7. *Compute the following limits.*

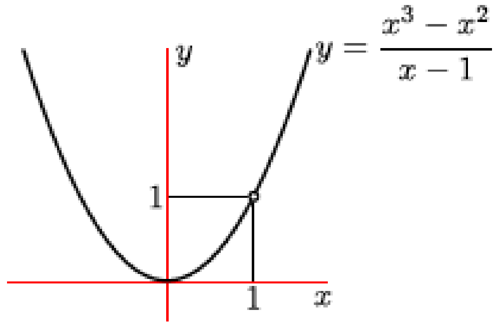
1. $\lim_{x\to 2}\frac{x^3-x^2}{x-1}$
2. $\lim_{x\to 1}\frac{x^3-x^2}{x-1}$

Solution. *1.* $\lim_{x\to 2}\frac{x^3-x^2}{x-1}=4.$
2. Consider that $\lim_{x\to 1}x^3-x^2=0$ *and* $\lim_{x\to 1}x-1=0$*. However,*

$$\frac{x^3-x^2}{x-1}=\frac{x^2(x-1)}{x-1},$$

thus

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \textit{undefined} & x = 1. \end{cases}$$



And so

$$\lim_{x \to 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \to 1} x^2 = 1.$$

The reasoning in the above example can be made more general:

Theorem 1.4.8. *If $f(x) = g(x)$ except when $x = a$ then*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

provided the limit of g exists.

We mostly use the above theorem when we end up with $\frac{0}{0}$.

Example 1.4.9. *Compute*

$$\lim_{h \to 0} \frac{(1 + h)^2 - 1}{h}.$$

Solution. *Note that*

$$\frac{(1 + h)^2 - 1}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(2 + h)}{h}.$$

Thus,

$$\frac{(1 + h)^2 - 1}{h} = \begin{cases} 2 + h & h \neq 0 \\ \textit{undefined} & h = 0. \end{cases}$$

And so

$$\lim_{h \to 0} \frac{(1 + h)^2 - 1}{h} = \lim_{h \to 0} 2 + h = 2.$$

We now present a slightly harder example.

Example 1.4.10. *Compute the limit*

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + x} - 1}.$$

Solution. *Both the limits of the numerator and denominator as $x \to 0$ are 0, so we cannot use the Theorem Arithmetic of limits. We now can simply multiply the numerator and denominator by the conjugation of $\sqrt{1+x}-1$, that is, $\sqrt{1+x}+1$. We have*

$$\begin{aligned}
\frac{x}{\sqrt{1+x}-1} &= \frac{x}{\sqrt{1+x}-1} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} && \text{multiply by } \frac{\text{conjugate}}{\text{conjugate}} = 1 \\
&= \frac{x\left(\sqrt{1+x}+1\right)}{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)} && \text{bring things together} \\
&= \frac{x\left(\sqrt{1+x}+1\right)}{\left(\sqrt{1+x}\right)^2 - 1\cdot 1} && \text{since } (a-b)(a+b) = a^2 - b^2 \\
&= \frac{x\left(\sqrt{1+x}+1\right)}{1+x-1} && \text{clean up a little} \\
&= \frac{x\left(\sqrt{1+x}+1\right)}{x} && \\
&= \sqrt{1+x}+1 && \text{cancel the } x
\end{aligned}$$

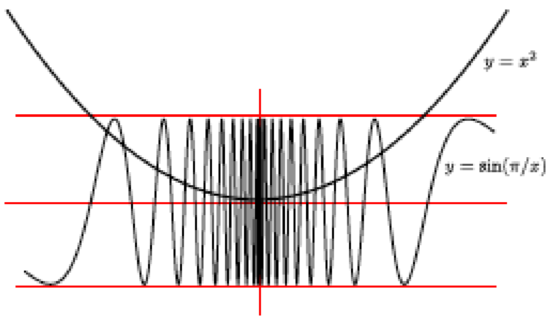
So now we have

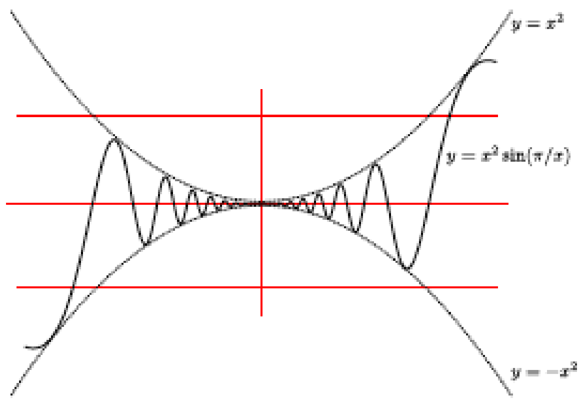
$$\begin{aligned}
\lim_{x\to 0} \frac{x}{\sqrt{1+x}-1} &= \lim_{x\to 0} \sqrt{1+x}+1 \\
&= \sqrt{1+0}+1 = 2
\end{aligned}$$

Before we move to the next section and study the limits at infinity, we have one more theorem to state.

Example 1.4.11. *Compute*

$$\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$$





Solution. *It is not possible to simply use the theorem Arithmetic of Limits since the limit of $\sin(\frac{\pi}{x})$ as $x \to 0$ does not exist. Since $-1 \le \sin(\theta) \le 1$ for all real numbers θ, we have*

$$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1 \qquad \textit{for all } x \neq 0$$

Multiplying the above by x^2 we see that

$$-x^2 \leq x^2 \sin(\frac{\pi}{x}) \leq x^2 \qquad \textit{for all } x \neq 0.$$

Since

$$\lim_{x \to 0} x^2 = \lim_{x \to 0}(-x^2) = 0$$

by the **sandwich (or squeeze or pinch) theorem** *(look at below for the sandwich theorem) we have*

$$\lim_{x \to 0} x^2 \sin(\frac{\pi}{x}) = 0.$$

Theorem 1.4.12. *(***sandwich (or squeeze or pinch) theorem** *) Let $a \in \mathbb{R}$ and let f, g, h be three functions so that*

$$f(x) \leq g(x) \leq h(x)$$

for all x in an interval around a, except possibly at $x = a$. Then if

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then it is also the case that

$$\lim_{x \to a} g(x) = L.$$

Example 1.4.13. *Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is*

$$\lim_{x \to 1} f(x)?$$

Solution. *Consider that*

$$\lim_{x \to 1} x = 1 \qquad \textit{and} \qquad \lim_{x \to 1} x^2 - 2x + 2 = 1.$$

Therefore, by the sandwich/pinch/squeeze theorem

$$\lim_{x \to 1} f(x) = 1.$$

1.5 Limits at Infinity

Example 1.5.1. *We want to compute*

$$\lim_{x \to +\infty} \frac{1}{x} \qquad \textit{and} \qquad \lim_{x \to -\infty} \frac{1}{x}$$

By plug in some large numbers into $\frac{1}{x}$ we have

-10000	-1000	$-100\|\circ\|100$	1000	10000
-0.0001	0.001	$-0.01\|\circ\|0.01$	0.001	0.0001

We see that as x is getting bigger and positive the function $\frac{1}{x}$ is getting closer to 0. Thus,

$$\lim_{x \to +\infty} \frac{1}{x} = 0.$$

Moreover,

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

Definition. *(***Informal limit at infinity.***) We write*

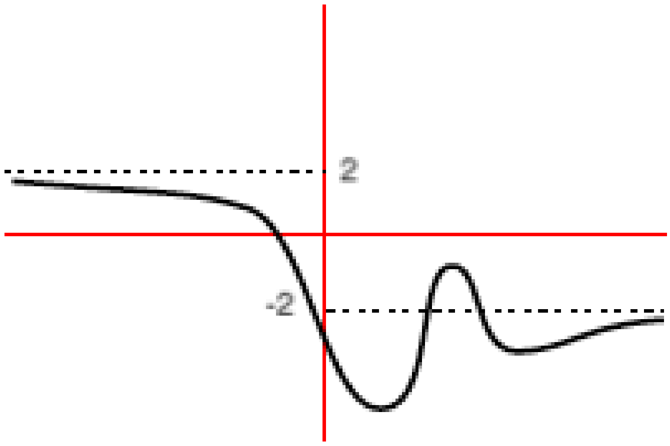
$$\lim_{x\to\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and positive.
Similarly, we write

$$\lim_{x\to-\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and negative.

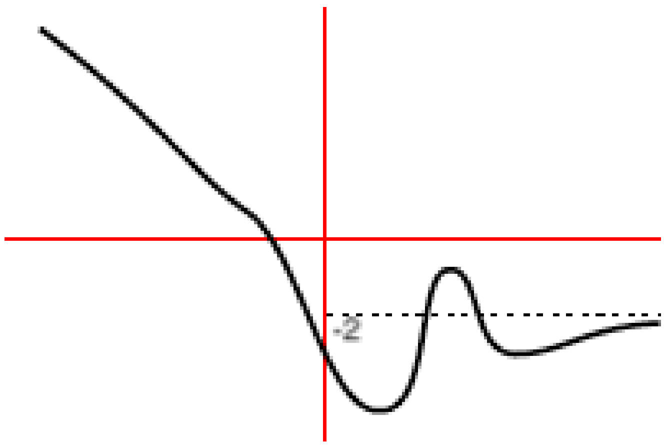
Example 1.5.2. *Consider the graph of the function $f(x)$.*



Then

$$\lim_{x\to\infty} f(x) = -2 \qquad \textit{and} \qquad \lim_{x\to-\infty} f(x) = 2$$

Example 1.5.3. *Consider the graph of the function $g(x)$.*



Then

$$\lim_{x\to\infty} g(x) = -2 \qquad \textit{and} \qquad \lim_{x\to-\infty} g(x) = +\infty$$

Same as usual we start with two very simple building blocks and build other limits from them.

Theorem 1.5.4. *Let $c \in \mathbb{R}$ then the following limits hold*

$$\lim_{x\to+\infty} c = c \qquad \lim_{x\to-\infty} c = c$$

$$\lim_{x\to+\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

Theorem 1.5.5. *Let $f(x)$ and $g(x)$ be two functions for which the limits*

$$\lim_{x\to\infty} f(x) = F \qquad \lim_{x\to\infty} = G$$

exist. Then the following limits hold

$$\lim_{x\to\infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x\to\infty} f(x)g(x) = FG$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \textit{provided } G \neq 0$$

and for rational numbers r,

$$\lim_{x\to\infty} (f(x))^r = F^r$$

provided that $f(x)^r$ is defined for all x.
The analogous results hold for limits to $-\infty$.

We need a little extra care with the posers of functions.
Warning: Consider that

$$\lim_{x\to+\infty} \frac{1}{x^{1/2}} = 0$$

However,

$$\lim_{x\to+\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because $x^{1/2}$ is not defined for $x < 0$.

Example 1.5.6. *Compute the following limit:*

$$\lim_{x\to\infty} \frac{x^2 - 3x + 4}{3x^2 + 8x + 1}$$

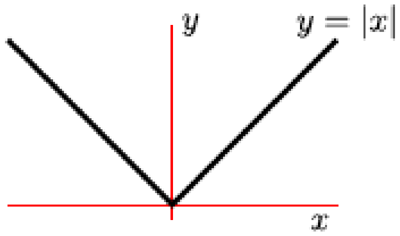
Solution. *By factoring x with largest exponent in the numerator and denominator we have*

$$\lim_{x\to\infty} \frac{x^2 - 3x + 4}{3x^2 + 8x + 1} = \lim_{x\to\infty} \frac{x^2(1 + \frac{-3x}{x^2} + \frac{4}{x^2})}{x^2(3 + \frac{8x}{3x^2} + \frac{1}{3x^2})} = \lim_{x\to\infty} \frac{(1 + \frac{-3x}{x^2} + \frac{4}{x^2})}{(3 + \frac{8x}{3x^2} + \frac{1}{3x^2})} =$$

$$\frac{(\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{-3x}{x^2} + \lim_{x\to\infty} \frac{4}{x^2})}{(\lim_{x\to\infty} 3 + \lim_{x\to\infty} \frac{8x}{3x^2} + \lim_{x\to\infty} \frac{1}{3x^2})} = \frac{1}{3}.$$

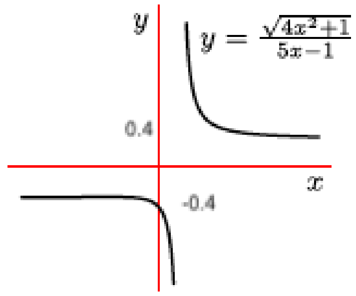
Remark. *Note that*

$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



Example 1.5.7. *Compute the following limit:*

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{5x-1}.$$



Solution. *Factor the terms with the largest exponents in the numerator and denominator. We have*

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{5x-1} = \lim_{x\to\infty} \frac{\sqrt{4x^2(1+\frac{1}{4x^2})}}{5x(1-\frac{1}{5x})} = \lim_{x\to\infty} \frac{\sqrt{4x^2}\sqrt{(1+\frac{1}{4x^2})}}{5x(1-\frac{1}{5x})} = \lim_{x\to\infty} \frac{2|x|}{5x} = \lim_{x\to\infty} \frac{2x}{5x} = \frac{2}{5}.$$

Example 1.5.8. *Compute the following limit:*

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{5x-1}.$$

Solution. *By the same kind of computation we have*

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{5x-1} = \lim_{x\to\infty} \frac{2|x|}{5x}.$$

Consider that since x is getting negative values, we have $|x| = -x$. Therefore,

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{5x-1} = \lim_{x\to\infty} \frac{2|x|}{5x} = \lim_{x\to\infty} \frac{-2x}{5x} = \frac{-2}{5}.$$

Example 1.5.9. *Compute the following limit:*

$$\lim_{x\to\infty} \left(x^{7/5} - x\right).$$

Solution. *We factor the term with the largest exponent, we have*

$$\lim_{x\to\infty}\left(x^{7/5}-x\right)=\lim_{x\to\infty}x^{7/5}\left(1-\frac{1}{x^{2/5}}\right)=\infty.$$

Theorem 1.5.10. *Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that*

$$\lim_{x\to a} f(x)=+\infty \qquad \lim_{x\to a} g(x)=+\infty \qquad \lim_{x\to a} h(x)=H$$

1. $$\lim_{x\to a}(f(x)+g(x))=+\infty.$$

2. $$\lim_{x\to a}(f(x)+h(x))=+\infty.$$

3. $$\lim_{x\to a}(f(x)-g(x))=\textit{undetermined.}$$

4. $$\lim_{x\to a}(f(x)-h(x))=+\infty.$$

5. $$\lim_{x\to a} cf(x)=\begin{cases}+\infty & c>0\\ 0 & c=0\\ -\infty & c<0\end{cases}$$

6. $$\lim(f(x).g(x))=+\infty.$$

7. $$\lim_{x\to a}(f(x).h(x))=\begin{cases}+\infty & H>0\\ \textit{undetermined} & H=0\\ -\infty & H<0\end{cases}$$

8. $$\lim_{x\to a}\frac{h(x)}{f(x)}=0.$$

Example 1.5.11. *Consider the following three functions:*

$$f(x) = x^{-2} \qquad g(x) = 2x^{-2} \qquad h(x) = x^{-2} - 1.$$

Then

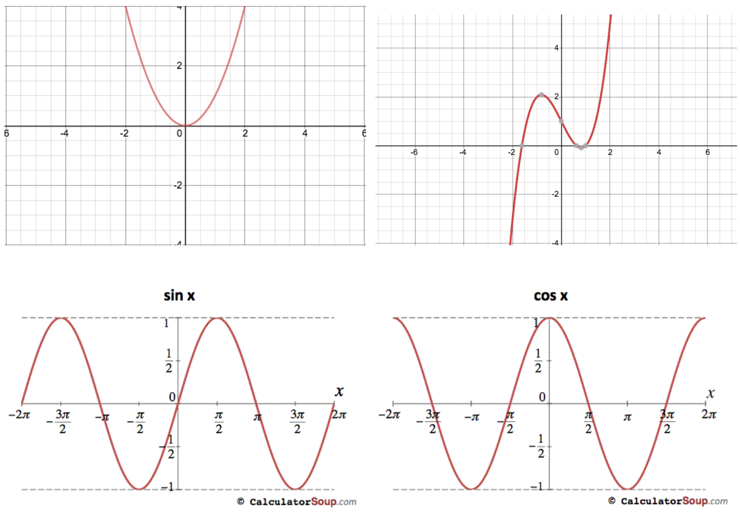
$$\lim_{x\to 0} f(x) = +\infty \qquad \lim_{x\to 0} g(x) = +\infty \qquad \lim_{x\to 0} h(x) = +\infty.$$

Then

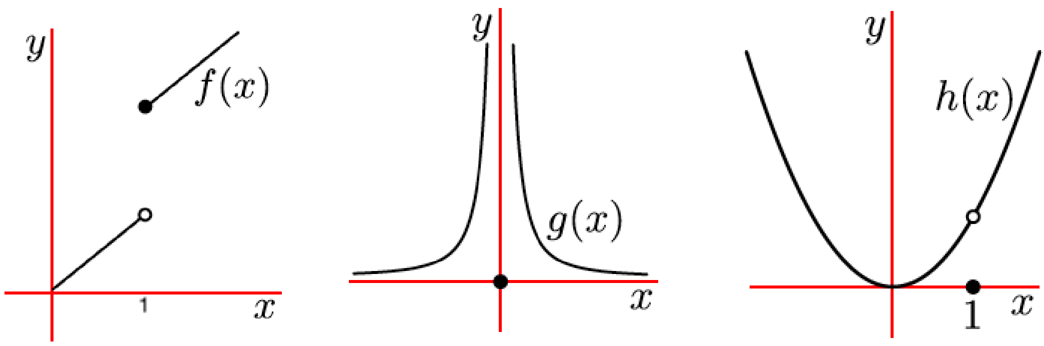
- $$\lim_{x\to 0}(f(x) - g(x)) = \lim_{x\to 0} x^{-2} = -\infty$$
- $$\lim_{x\to 0}(f(x) - h(x)) = \lim_{x\to 0}(1) = 1$$
- $$\lim_{x\to 0}(g(x) - h(x)) = \lim_{x\to 0} x^{-2} + 1 = \infty$$

1.6 Continuity

Look at all the following functions.



All of these functions are continuous. Roughly speaking, a function is continuous if it does not have any abrupt jumps. Now consider the following function.



These functions are not continuous. The function f, g, and h have abrupt jumps at $x = 2$, $x = 0$, and $x = 1$, respectively, so f is not continuous at a, g is not continuous at 0, and h is not continuous at 1.

Definition. *A function $f(x)$ is continuous at a if*

$$\lim_{x\to a} f(x) = f(a).$$

If a function is not continuous at a then it is said to be discontinuous at a. When we write that f is continuous without specifying a point, then typically this means that f is continuous at a for all $a \in \mathbb{R}$.. When we write that $f(x)$ is continuous on the open interval (a, b) then the function is continuous at every point c satisfying $a < c < b$.

From the above definition we immediately have that if f is continuous at a, then

1. $f(a)$ exists;
2. $\lim_{x\to a^-}$ exists and is equal to $f(a)$.

3. $\lim_{x\to a^+}$ exists and is equal to $f(a)$.

Definition. *A function is continuous from the left at a if*

$$\lim_{x\to a^-} = f(a).$$

And a function is continuous from the right at a if

$$\lim_{x\to a^-} = f(a).$$

Definition. *A function $f(x)$ is continuous on an interval $[a, b]$ if*

1. *$f(x)$ continuous on (a, b),*
2. *$f(x)$ is continuous form the right at a,*
3. *$f(x)$ is continuous form the left at b.*

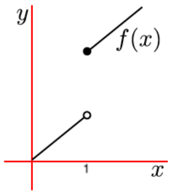
Definition. *A function $f(x)$ is continuous on an interval $(a, b]$ (on the interval $[a, b)$) if*

1. *$f(x)$ continuous on (a, b),*
2. *$f(x)$ is continuous form the left at b (from the right at a).*

Example 1.6.1. *Consider the function*

$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$

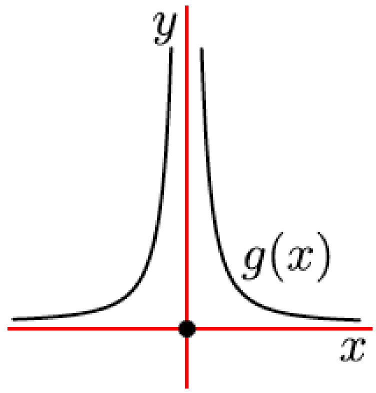
jump discontinuity



- $$\lim_{x \to 1^-} f(x) = 1 \qquad \lim_{x \to 1^+} f(x) = 3 \qquad f(1) = 3.$$
- *The function $f(x)$, at $x = 1$ is not continuous because the limit does not exist; however, it is continuous form the right at 1 since*
$$\lim_{x \to 1^+} f(x) = 3 = f(1).$$
- *The function $f(x)$, on $[1, \infty)$ (for $x \geq 1$) is continuous.*
- *The function $f(x)$, on $(-\infty, -1)$ is continuous.*

Example 1.6.2. *Consider the function*

$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



infinite discontinuity

- *Consider that*

$$\lim_{x\to 0^-} g(x) = \infty = \lim_{x\to 0^+} g(x) \qquad g(0) = 0.$$

Thus the function $g(x)$ is not continuous at 0 because

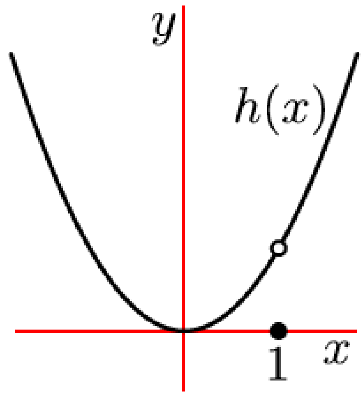
$$\lim_{x\to 0} g(x) = \infty \neq 0 = g(0).$$

It is not continuous at 0 from the left since $\lim_{x\to 0^-} g(x) = \infty \neq 0 = g(0)$ and not from the right since $\lim_{x\to 0^+} g(x) = \infty \neq 0 = g(0)$.

- *the function $g(x)$ is continuous at all points in $\mathbb{R}$ except 0.*

Example 1.6.3. *Consider the function*

$$h(x) = \begin{cases} \frac{x^3-x^2}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$



removable discontinuity

- $$\lim_{x \to 1^-} h(x) = 1 = \lim_{x \to 1^+} h(x) \qquad\qquad f(1) = 0.$$
- $$\lim_{x \to 1} = 1.$$
- *the function $h(x)$ is not continuous at 1 since*

  $$\lim_{x \to 1} h(x) = 1 \neq 0 = h(1).$$

  It is not continuous form the left since

  $$\lim_{x \to 1^-} h(x) = 1 \neq 0 = h(1)$$

  and not from the right since

  $$\lim_{x \to 1^+} h(x) = 1 \neq 0 = h(1).$$

- *the function $h(x)$ is continuous at all points in $\mathbb{R}$ except 1.*

Lemma 1.6.4. *Let $c \in \mathbb{R}$. The functions*

$$f(x) = x \qquad\qquad g(x) = c$$

are continuous everywhere on the real line.

Theorem 1.6.5. (**Arithmetic of continuity**) *Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at a. Then the following functions are also continuous at $x = a$.*

- $f(x) + g(x)$ *and* $f(x) - g(x)$,
- $cf(x)$ *and* $f(x)g(x)$, *and*
- $\frac{f(x)}{g(x)}$ *provided* $g(a) \neq 0$.

Theorem 1.6.6. *The following functions are continuous everywhere in their domains*

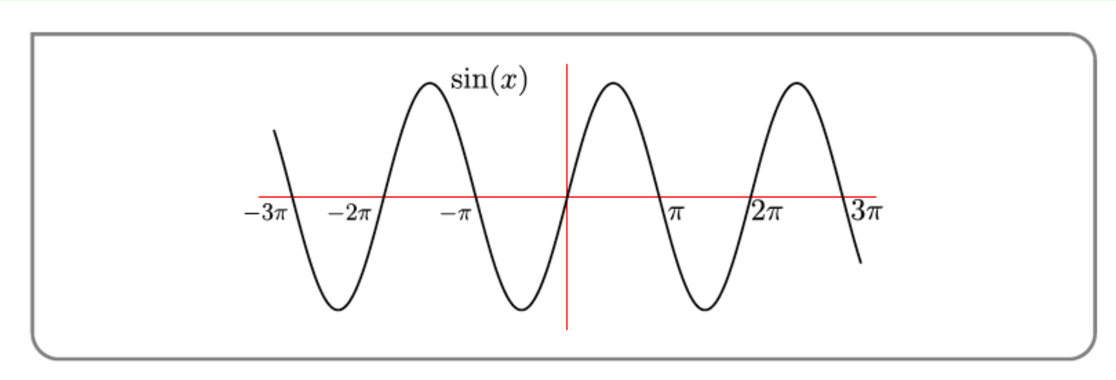
- *polynomials and rational functions (for example* $f(x) = x^5 + 4x^2 + 1$ *and* $g(x) = \frac{x^2+1}{x+1}$*)*
- *roots and powers (for example* $h(x) = \sqrt{x}$ *and* $r(x) = 2^x$*)*
- *trig functions and their inverses (for example* $k(x) = \sin(x)$ *and* $t(x) = \cos^{-1}(x)$*)*
- *exponentials and logarithms (for example* $s(x) = e^x$ *and* $q(x) = \ln x$*).*

Example 1.6.7. *Determine when the function* $f(x) = \frac{\sin(x)}{x^2-5x+6}$ *is continuous? Since both* $\sin(x)$ *and* $x^2 - 5x + 6$ *are continuous by the above theorem we only need to check when* $x^2 - 5x + 6 = 0$. *Note that* $x^2 - 5x + 6 = (x - 2)(x - 3)$, *thus this polynomial is only zero at* $x = 2$ *and* $x = 3$. *Therefore,* $f(x)$ *is continuous at all points in* $\mathbb{R}$ *except* 2 *and* 3.

Theorem 1.6.8. *If g is continuous at a and $f(x)$ is continuous at $g(a)$, then $(f \circ g)(x) = f(g(x))$ is continuous at $x = a$.*

Example 1.6.9. *Determine when the function $h(x) = \sqrt{\sin(x)}$ is continuous.*

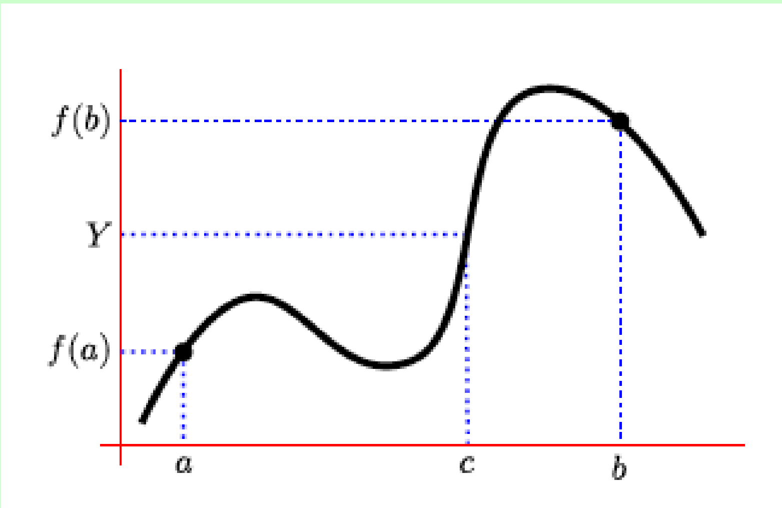
Solution. *Let $f(x) = \sqrt{x}$ and $g(x) = \sin(x)$, then $h(x) = (f \circ g)(x)$. We only need to find out at what points $\sin(x)$ is positive.*



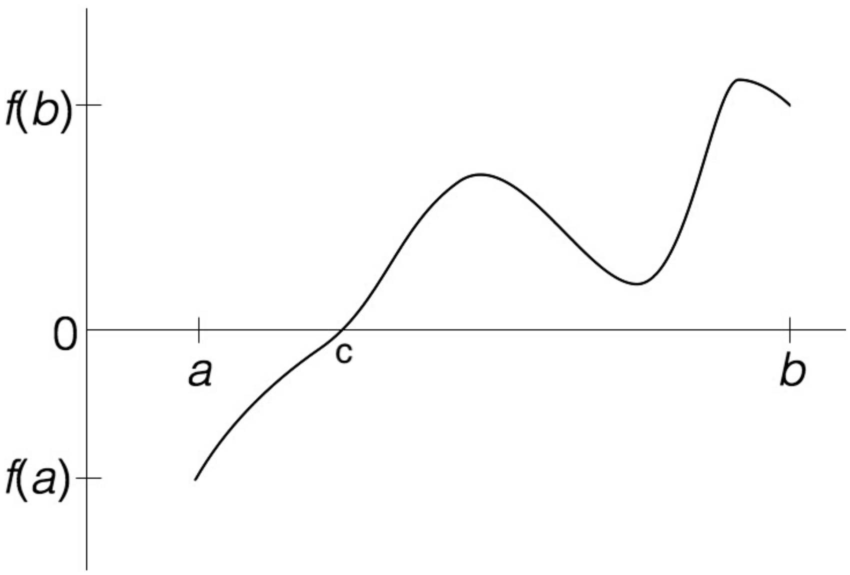
The function $\sqrt{\sin(x)}$ is continuous if

$$x \in [2n\pi, (2n+1)\pi] \qquad \text{for all natural numbers } n.$$

Theorem 1.6.10. (Intermediate value theorem(IVT)) *Let $a < b$ and let $f(x)$ be a function that is continuous at all points $a \leq x \leq b$. If Y is any number between $f(a)$ and $f(b)$ then there exists some number $c \in [a, b]$ so that $f(c) = Y$.*

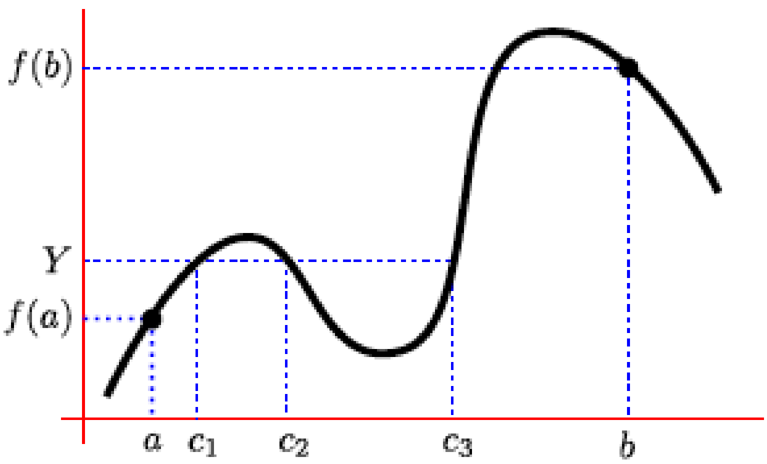


Remark. *One of the main application of the IVT theorem is showing a function f has* *a zero inside an interval. For example, in the following picture*



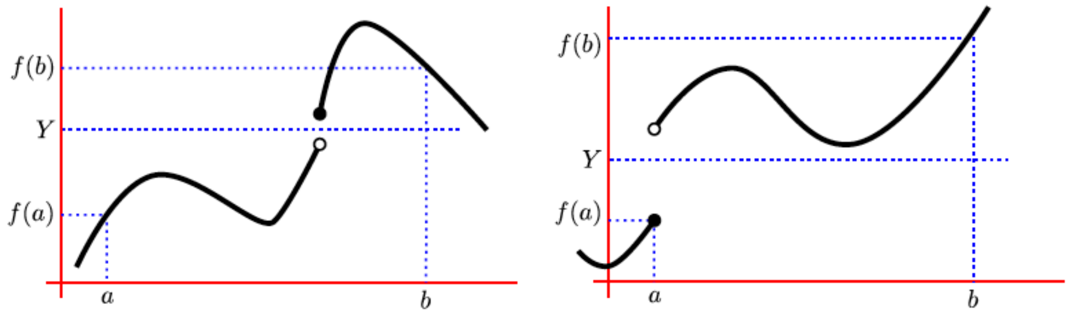
we can see that $f(a) < 0$ *and* $f(b) > 0$*, therefore by IVT, there is a number* c *between* a *and* b *such that* $f(c) = 0$*.*

Remark. *If* f *is continuous and* $f(a) \leq Y \leq f(b)$*, the IVT merely shows that there is a* $a \leq c \leq b$ *such that* $f(c) = Y$*, but it doesn't show how many of them exist. For example, in the following picture, we can see* $f(a) \leq Y \leq f(b)$*, and there are three numbers* c_1, c_2*, and* c_3 *such that* $f(c_1) = f(c_2) = f(c_3) = Y$*.*



Remark. *Consider that if the function* f *is not continuous at the interval* $[a, b]$ *then the IVT fails. In the following examples, even though* $f(a) \leq Y \leq f(b)$*, there is not a number*

$a \leq c \leq b$ such that $f(c) = Y$.



Example 1.6.11. *Show that the function $f(x) = x - 1 + sin(\pi x/2)$ has a zero in $0 \leq x \leq 1$.*

Solution. *Consider that $f(x)$ is a continuous function such that $f(0) = -1$ and $f(1) = 1$. Therefore, by IVT, since $f(0) = -1 \leq 0 \leq 1 = f(1)$, we have $f(c) = 0$ for some $c \in [0, 1]$.*

Example 1.6.12. *Use the bisection method to find a zero of $f(x) = x - 1 + \sin(\pi x/2)$ that lies between 0 and 1.*

Solution.

- *Let $a = 0$ and $b = 1$. Then*

$$f(0) = -1$$

$$f(1) = 1$$

- *Test the point in the middle $x = \frac{1+0}{2} = 0.5$,*

$$f(0.5) = 0.2071067813 > 0$$

- *Let $a = 0$ and $b = 0.5$. Then*

$$f(0) = -1$$

$$f(1) = 0.2071067813$$

  So by IVT, there is a zero in $[0, 0.5]$.

- *Test the point in the middle $x = \frac{0.5+0}{2} = 0.25$.*

$$f(0.25) = -0.3673165675 < 0.$$

- *Let $a = 0.25$, $b = 0.5$ where $f(0.25) < 0$ and $f(0.5) > 0$. By IVT there is a zero in the interval $[0.25, 0.5]$.*

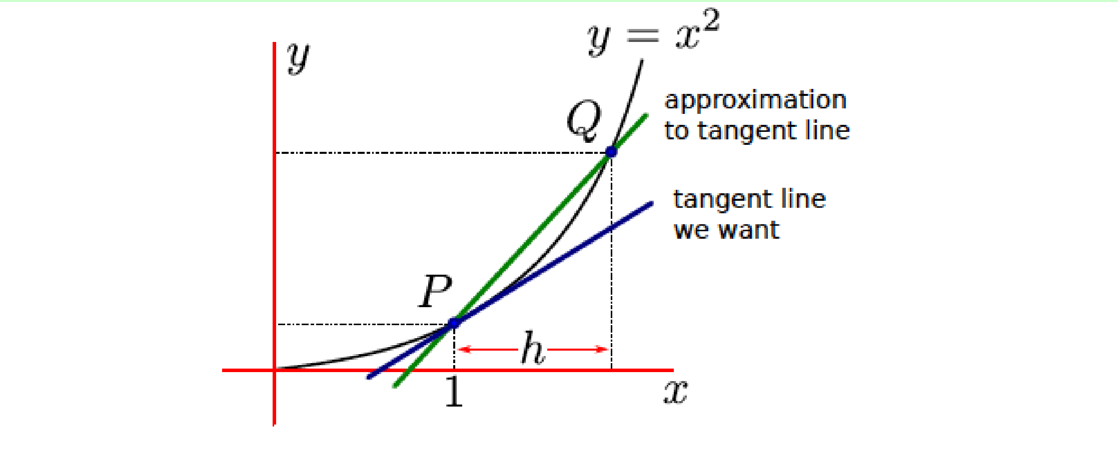
- *So without much work we know the location of a zero inside a range of length 1/4. Each iteration will halve the length of the range and we keep going until we reach the precision we need, though it is much easier to program a computer to do it.*

Chapter 2

Derivatives

2.1 Revisiting Tangent Lines

Example 2.1.1. *Find the slope of the tangent line to the curve $y = x^2$ that passes through $P = (1, 1)$.*



Solution. *Consider that the slope of the secant line is*

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

And the slope of the tangent line is the same as

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}.$$

Theorem 2.1.2. *Given a function $f(x)$ the slope of the tangent line at $x = a$ (if exists) is*

$$\lim_{x \to a} \frac{f(a+h) - f(a)}{h}.$$

2.2 Definition of the derivative

Definition. *(Derivative at a point) Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined on an open interval that contains a.*

- *The derivative of $f(x)$ at $x = a$ is denoted $f'(a)$ and is defined by*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{2.2.1}$$

  if the limit exists.

- *When the above limit exists, the function $f(x)$ is said to be differentiable at $x = a$. When the limit does not exist, the function $f(x)$ is said to be not differentiable at $x = a$.*

- *We can equivalently define the derivative $f'(a)$ by the limit*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

To see that these two definitions are the same, we set $x = a + h(x - a = h)$ and then when h approaches 0, we have x approaches a, and the limit in 2.2.1 becomes $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$.

Example 2.2.1. *Let $a, c \in \mathbb{R}$ be constants. Compute the derivative of the function $f(x) = c$ at $x = a$.*

Solution. *By the definition of the derivative, we have*

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} 0 \\ &= 0 \end{aligned}$$

Example 2.2.2. *Let $a \in \mathbb{R}$. Compute the limit of the function $g(x) = x$ at $x = a$.*

Solution. *By the definition of the derivative we have*

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{(a+h) - a}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1. \end{aligned}$$

We have so proved our first theorem which is the following.

Theorem 2.2.3. *(easiest derivative) Let $a, c \in \mathbb{R}$ and let $f(x) = c$ and $g(x) = x$. Then*

$$f'(a) = 0$$

and

$$g'(a) = 1.$$

Example 2.2.4. *Compute the derivative of $f(t) = t^2$ at $t = a$.*

Solution. *We have that*

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h\to 0} \frac{a^2 + h^2 + 2ah - a^2}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 2ah}{h} \\ &= \lim_{h\to 0} \frac{h(h+2a)}{h} \\ &= \lim_{h\to 0} h + 2a \\ &= 2a \end{aligned}$$

➤➤ We can tweak the derivative at a specific point a to obtain the derivative as a function x. We replace

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

with

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

This gives us the following definition.

Definition. *Let $f(x)$ be a function*

- *The derivative of $f(x)$ with respect to x is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  provided the limit exists.

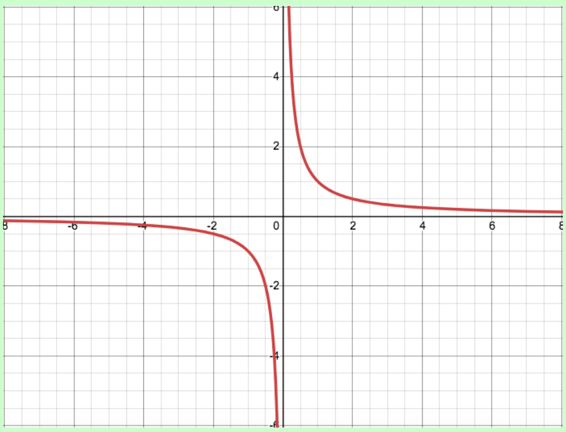
- *If the derivative $f'(x)$ exists for all $x \in (a,b)$ we say that f is differentiable on (a,b).*

- *Note that we will sometimes be a little sloppy with our discussion and simply write "f is differentiable" to mean "f is differentiable on an interval we are interested in" or "f is differentiable everywhere."*

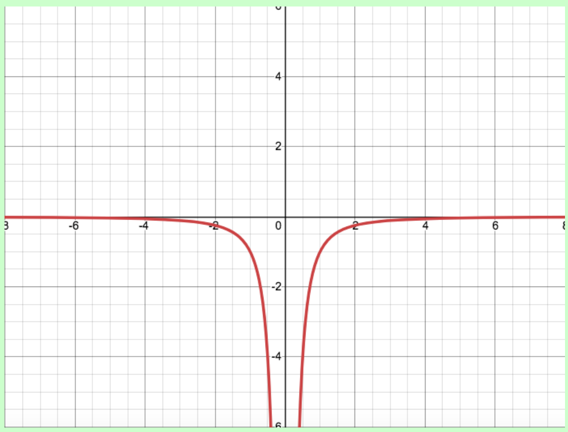
Example 2.2.5. *Let $f(x) = \frac{1}{x}$ and compute its derivative with respect to x.*

Solution. *We have that*

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{x+h} - \frac{1}{x} \right] \\
&= \lim_{h \to 0} \frac{1}{h} \frac{x - (x+h)}{x(x+h)} \\
&= \lim_{h \to 0} \frac{1}{h} \frac{-h}{x(x+h)} \\
&= -\frac{1}{x^2}.
\end{aligned}$$



$y = \dfrac{1}{x}$



$y = \dfrac{-1}{x^2}$

➤➤ Notice that the original function $f(x) = \frac{1}{x}$ was not defined at $x = 0$, and the derivative is also not defined at $x = 0$. This does happen more generally–if $f(x)$ is not defined at a particular point $x = a$, then the derivative will not exist at that point either.

Notation. There are several notation all used for "the derivative of $f(x)$ with respect to x"; however,

in this course we generally use the following notations

1. $f'(x)$. This notation is due to Lagrange, and we read it as "f-prime of x".
2. $\frac{df}{dx}$. This notation is duo to Leibniz, and we read it as "dee-f–dee–x".
3. $\frac{d}{dx}f$. We read this as dee–by–dee–x of f.

Example 2.2.6. *Compute the derivative, $f'(a)$, of the function $f(x) = \sqrt{x}$ at the point $x = a$ for any $a > 0$.*

Solution. *We have that*

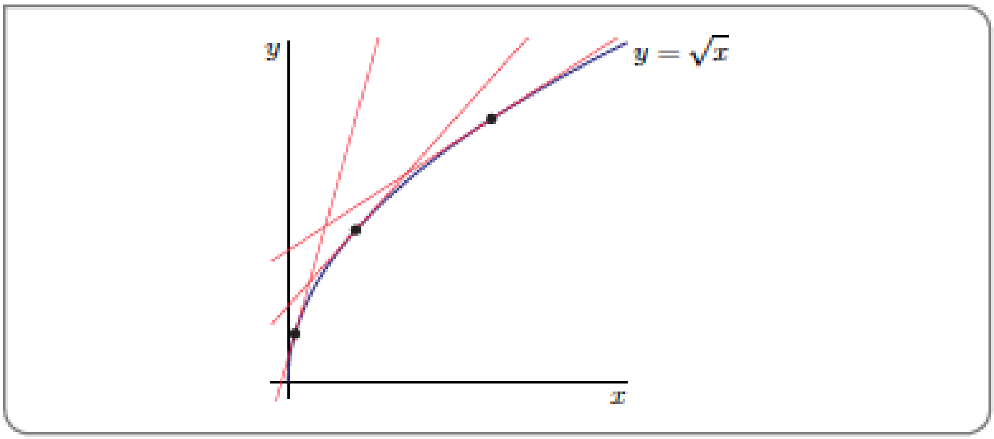
$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$

We now multiply the numerator and denominator by the conjugate of $\sqrt{x} - \sqrt{a}$, that is $\sqrt{x} + \sqrt{a}$. Then we have

$$\frac{\sqrt{x} - \sqrt{a}}{x - a} \times \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} = \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}}.$$

Therefore,

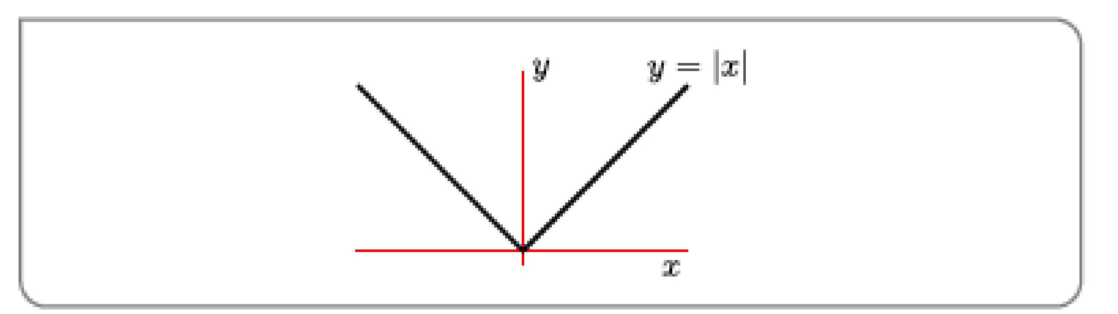
$$f'(a) = \lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \lim_{x\to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}.$$



Example 2.2.7. *Find the derivative, $f'(a)$, of the function $f(x) = |x|$ at the point $x = a$.*

Solution. Recall that

$$|x| = \begin{cases} -x & x < 0 \\ 0 & x = 0 \\ x & x > 0 \end{cases}$$



We should break our computation of the derivative into three cases depending on whether x is positive, negative, or zero.

- Assume $x > 0$. Then

$$\begin{aligned} \frac{df}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{|x+h| - |x|}{h} \end{aligned}$$

Since $x > 0$ and h is much more smaller than x, we have $x + h > 0$ and so $|x + h| = x + h$, moreover, since x is positive, $|x| = x$.

$$\begin{aligned} &\lim_{h \to 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h \to 0} \frac{x+h-x}{h} \\ &= \lim_{h \to 0} \frac{h}{h} = 1. \end{aligned}$$

- Assume $x < 0$. Then we have

$$\begin{aligned} \frac{df}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{|x+h| - |x|}{h} \end{aligned}$$

Since $x < 0$ and h is much more smaller than x, we have $x + h < 0$ and so $|x + h| = -(x + h)$, moreover, since $x < 0$ is positive, $|x| = -x$.

$$\begin{aligned} &\lim_{h \to 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h} \\ &= \lim_{h \to 0} \frac{-h}{h} = -1. \end{aligned}$$

- Assume $x = 0$. Then we have

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}\\ &= \lim_{h\to 0}\frac{|h|}{h}\end{aligned}$$

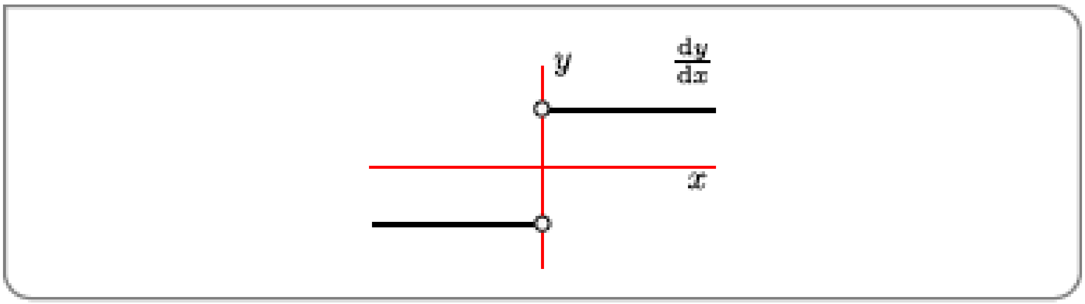
Consider that

$$\lim_{h\to 0^+}\frac{|h|}{h} = 1 \qquad \text{and} \qquad \lim_{h\to 0^-}\frac{|h|}{h} = -1$$

Therefore, this limit does not exist and so the function $|x|$ is not derivative at $x = 0$.

In summary:

$$\frac{d}{dx}|x| = \begin{cases} -1 & x < 0 \\ DNE & x = 0 \\ 1 & x > 0 \end{cases}$$

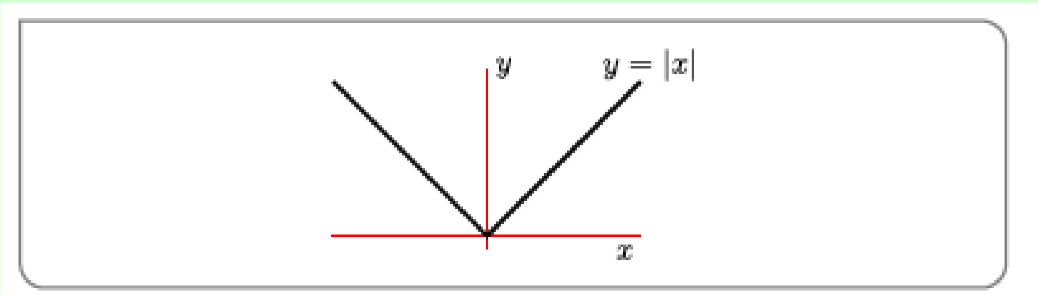


➤➤ **Where is the derivative undefined?** The derivative $f'(a)$ exists precisely when the limit

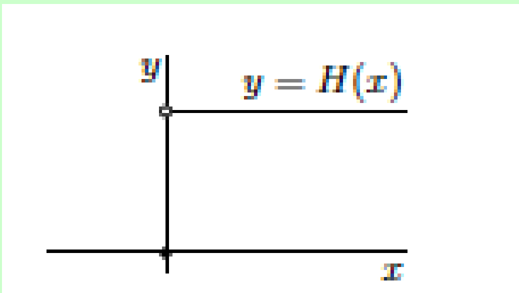
$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists. That limit is the slope of the tangent line to the curve $y=f(x)$ at $x=a$. Thus, that limit does not exist one of the following happens.

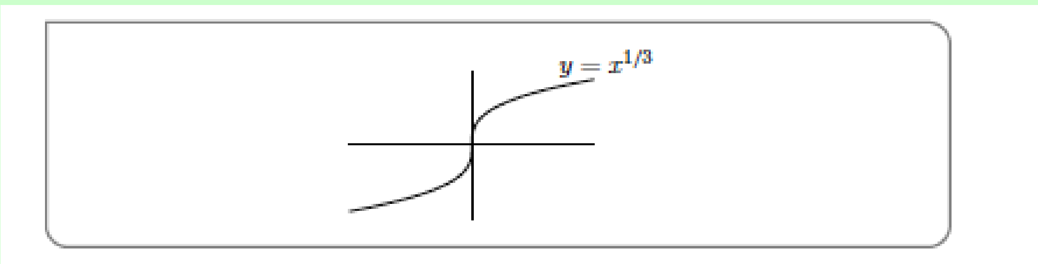
❶ The curve $y=f(x)$ does not have a tangent line at $x=a$ when it has a sharp corner at $x=a$, as an example $f(x)=|x|$ is not differentiable at $x=0$ since it has a sharp corner at $x=0$.



❷ When the curve does have a tangent line because it is not continuous at $x=a$. As an example, we have seen that $f(x)=H(x)=\begin{cases}0 & x\le 0\\ 1 & x>0\end{cases}$ does not have a tangent line at $x=0$ since it is not continuous at $x=0$.

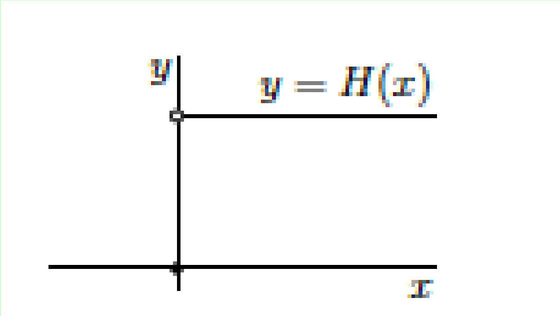


❸ When the curve has a tangent line at $x=a$ but the slope of the tangent line at $x=a$ is infinity. As an example, $f(x)=x^{1/3}$ is not differentiable at $x=0$ since it has a tangent line with slope infinity.



Example 2.2.8. *Verify that the function*

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$



does not have a tangent line at $x = 0$*.*

Solution. *Consider that if the tangent line exists then the following limit also must exists,*

$$\lim_{h\to 0} \frac{H(0+h) - H(0)}{h}.$$

Consider that

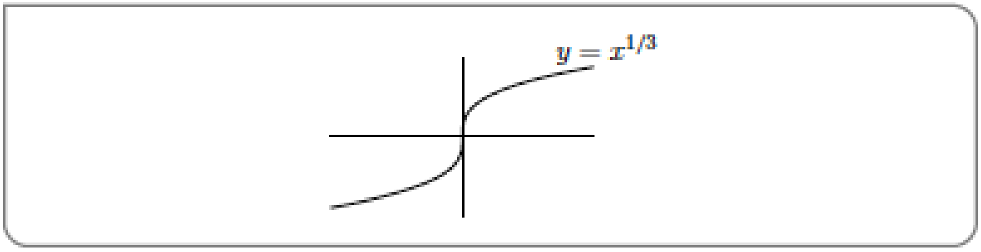
$$\lim_{h\to 0^+} \frac{H(0+h) - H(0)}{h} = \lim_{h\to 0^+} \frac{1}{h} = +\infty$$

and

$$\lim_{h\to 0^-} \frac{H(0+h) - H(0)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0.$$

Therefore, the limit does not exists.

Example 2.2.9. *Verify that the derivative of $f(x) = x^{1/3}$ at $x = 0$ does not exist.*



Solution. *You can already see in the graph that the derivative at $x = 0$ does not exist since the tangent line has infinite slope. However, we need a mathematical proof, and we should show that $f'(0)$ which is the same as the following limit*

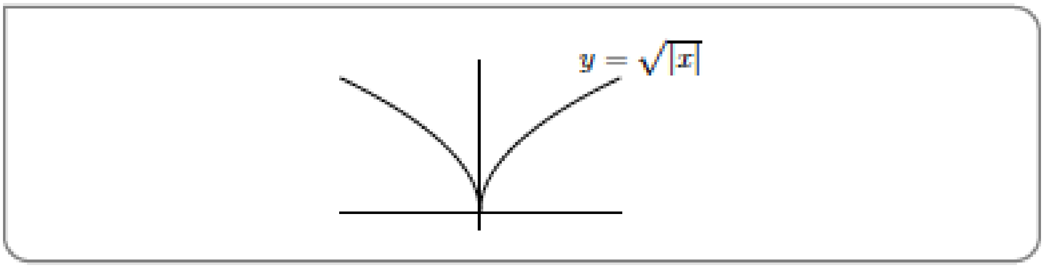
$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist. We have

$$\lim_{h\to 0} \frac{(0+h)^{1/3} - 0^{1/3}}{h} = \lim_{h\to 0} \frac{h^{1/3}}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = +\infty$$

(or we can say DNE).

Example 2.2.10. *Verify that the derivative of $f(x) = \sqrt{|x|}$ at $x = 0$ does not exist.*



Solution. *Even though you can see in the graph that at $x = 0$, the graph has a sharp corner, we also show that the following limit doesn't exist,*

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{|h|} - 0}{h}.$$

Consider that

$$\lim_{h\to 0^+} \frac{\sqrt{|h|}}{h} = \lim_{h\to 0^+} \frac{\sqrt{1}}{\sqrt{h}} = +\infty$$

(or DNE).

➤➤ What is the relation between continuity and differentiability?

Theorem 2.2.11. *If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.*

Theorem 2.2.12. *If $f(x)$ is not continuous at $x = a$, then it is not differentiable at $x = a$.*

2.3 Interpretations of the Derivative

Interpretation of the derivative:

- the instantaneous rate of change of a quality
- the slope of a curve.

➤➤ Instantaneous Rate of Change

Assume that we have the function $f(t)$ of the measuring of some quantity. Then

$$\text{average rate of change of } f(t) \text{ from } t = a \text{ to } t = a + h \text{ is}$$

$$\frac{\text{change in } f(t) \text{ from } t = a \text{ to } t = a + h}{\text{length of time from } t = a \text{ to } t = a + h}$$

$$= \frac{f(a+h) - f(a)}{h}.$$

And so

$$\text{instantaneous rate of change of } f(t) \text{ at } t = a$$

$$= \lim_{h \to 0} [\text{average rate of change of } f(t) \text{ from } t = a \text{ to } t = a + h]$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

Example 2.3.1. *You drop a ball from a tall building. After t seconds the ball has fallen a distance of $s(t) = 4.9t^2$ meters. What is the (instantaneous) velocity of the ball one second after it is dropped?*

Solution. *Since the instantaneous velocity is actually the instance rate of change if distance, we should compute $f'(1)$. Computing the limit of the derivative we have*

$$f'(t) = 9.8t \qquad \text{and} \qquad f'(1) = 9.8.$$

Example 2.3.2. *You are taking a walk and that as you walk, you are continuously measuring some quantity, like temperature, and that the measurement at time t is $f(t) = \sqrt{t}$. What is the average rate of $f(t) = \sqrt{t}$ from $t = 1$ to $t = 2$? What is the instantaneous rate of change of $f(t)$ at $t = 1$?*

Solution. *Consider that the average rate of change is*

$$\frac{f(a+h) - f(a)}{h}$$

The difference between times is h, so $h = 2 - 1 = 1$. Therefore, the average is

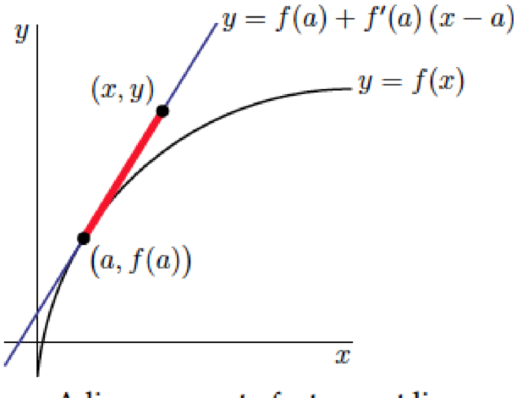
$$\frac{f(a+h) - f(a)}{h} = \frac{f(2) - f(1)}{1} = \sqrt{2} - 1.$$

The instantaneous rate of change is $f'(1)$. We already have seen that $f'(t) = \frac{1}{2\sqrt{t}}$, and so $f'(1) = \frac{1}{2}$.

➤➤ Slope and the formula for the tangent line to a curve at $x = a$

We know that the slope of a function at $x = a$ is $f'(a)$. To find the tangent line we need to find a point at the tangent line; however, for sure we know that $(a, f(a))$ is on the tangent line. So we should find the equation for a line that passes through $(a, f(a))$ with slope $f'(a)$ which is

$$y = f(a) + f'(a)(x - a).$$



A line segment of a tangent line

Theorem 2.3.3. *The tangent line to the curve $y = f(x)$ at $x = a$ is given by the equation*

$$y = f(a) + f'(a)(x - a)$$

provided the derivative $f'(a)$ exists.

Example 2.3.4. *Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.*

Solution. *By the above theorem we have the equation for the tangent line is*

$$y = f(a) + f'(a)(x - a).$$

Note that $a = 4$ and so $f(a) = f(4) = \sqrt{4} = 2$. We have already had $f'(x) = \frac{1}{2\sqrt{x}}$. Therefore, $f'(a) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. And so the equation for the tangent line is

$$y = 2 + \frac{1}{4}(x - 4).$$

2.4 Arithmetic of Derivatives - a Differentiation Toolbox

It is more efficient to have access to

- a list of derivatives of some simple functions and
- a collection of rules for breaking down complicated derivative computations into sequence of simple derivative computations.

➤➤ A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \qquad \frac{d}{dx}x = 1 \qquad \frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Tools

Theorem 2.4.1. *Let $f(x)$ and $g(x)$ be differentiable functions and let $c, d \in \mathbb{R}$. Then*

$$\frac{d}{dx}\{f(x)+g(x)\} = f'(x)+g'(x)$$
$$\frac{d}{dx}\{f(x)-g(x)\} = f'(x)-g'(x)$$
$$\frac{d}{dx}\{cf(x)\} = cf'(x)$$
$$\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x)+g'(x)f(x)$$
$$\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x)-g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$$

- ➤ $\frac{d}{dx}\{cf(x)+dg(x)\} = cf'(x)+dg'(x)$
- ➤ $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- ➤ $\frac{d}{dx}\{\frac{1}{g(x)}\} = \frac{-g'(x)}{g(x)^2} \qquad g(x) \neq 0$

2.5 Using the Arithmetic of Derivatives - Examples

Example 2.5.1.

$$\frac{d}{dx}4x+6 = 4.\frac{d}{dx}\{x\}+7.\frac{d}{dx}\{1\}$$
$$= 4.1+7.0 = 4.$$

Example 2.5.2.

$$\begin{aligned}\frac{d}{dx}\{x(4x+7)\} &= \frac{d}{dx}\{x\}(4x+7) + x\frac{d}{dx}\{4x+7\} \\ &= 4x+7+4x = 8x+7\end{aligned}$$

Example 2.5.3.

$$\begin{aligned}\frac{d}{dx}\{\frac{x}{(4x+7)}\} &= \frac{\frac{d}{dx}\{x\}(4x+7) - x\frac{d}{dx}\{4x+7\}}{(4x+7)^2} \\ &= \frac{4x+7-x(4)}{(4x+7)^2} = \frac{7}{(4x+7)^2}.\end{aligned}$$

Example 2.5.4. *Differentiate*

$$f(x) = \frac{x}{2x + \frac{1}{3x+1}}$$

Solution. *Let*

$$f_1(x) = x \quad \textit{and} \quad f_2(x) = 2x + \frac{1}{3x+1}$$

Let

$$f_3 = 2x \quad \textit{and} \quad f_4(x) = \frac{1}{3x+1}.$$

Therefore,

$$f(x) = \frac{f_1(x)}{f_2(x)}$$

and so

$$f'(x) = \frac{f_1'(x)f_2(x) - f_2'(x)f_1(x)}{f_2(x)^2}.$$

Consider that $f_1'(x) = 1$. *We have that* $f_2(x) = f_3(x) + f_4(x)$, *and thus*

$$f_2'(x) = f_3'(x) + f_4'(x) = 2 + \frac{-3}{(3x+1)^2}.$$

Therefore,

$$f'(x) = \frac{1 \times 2x + \frac{1}{3x+1} - 2 + \frac{-3}{(3x+1)^2}x}{(2x + \frac{1}{3x+1})^2}.$$

If we clean up this formula we have

$$f'(x) = \frac{6x+1}{(6x^2+2x+1)^2}.$$

Tools

Corollary 2.5.5. *Let n be an integer.*

$$\frac{d}{dx}\{f(x)g(x)h(x)\} = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x).$$

Example 2.5.6. *Let n be an integer, differentiate*

$$g(x) = x^n$$

Solution. *Let $f(x) = x$. Consider that $g(x) = f(x)^n$, by the above theorem we have*

$$g'(x) = nf(x)f(x)^{n-1} = n \times 1 \times x^{n-1} = nx^{n-1}.$$

Example 2.5.7. *Differentiate*

1. $f(x) = (3x+9)(x^2+4x^3)$
2. $\frac{4x^3-7x}{4x^2+1}$

Solution. *(1) Let $f_1(x) = 3x+9$ and $f_2(x) = x^2+4x^3$.*

$$f(x) = f_1(x)f_2(x) \qquad \text{and} \qquad f'(x) = f_1'(x)f_2(x) + f_2'(x)f_1(x).$$

Note that

$$f_1'(x) = 3 \qquad \text{and} \qquad f_2'(x) = 2x + 12x^2.$$

Therefore,

$$f'(x) = 3 \times (x^2+4x^3) + (2x+12x^2) \times (3x+9) = 18x + 117x^2 + 48x^3.$$

(2) Let $f_1(x) = 4x^3 - 7x$ and $f_2(x) = 4x^2+1$. Then

$$f(x) = \frac{f_1(x)}{f_2(x)} \qquad \text{and} \qquad f'(x) = \frac{f_1'(x)f_2(x) - f_2'(x)f_1(x)}{f_2(x)^2}.$$

Consider that

$$f_1'(x) = 12x^2 - 7 \qquad \text{and} \qquad f_2'(x) = 8x.$$

Therefore,

$$f'(x) = \frac{(12x^2-7)(4x^2+1) - (8x)(4x^3-7x)}{(4x^2+1)^2}.$$

After clean up,

$$f'(x) = \frac{16x^4 + 40x^2 - 7}{(4x^2+1)^2}.$$

Tools

Corollary 2.5.8. *Let a be a rational number, then*

$$\frac{d}{dx}x^a = ax^{a-1}.$$

Example 2.5.9. *The derivative of $f(x) = x^{1/3}$ is $f'(x) = (1/3)x^{1/3-1} = (1/3)x^{-2/3} = \frac{1}{x^{2/3}}$.*

Example 2.5.10. *Find the derivative of*

$$f(x) = \frac{(\sqrt{x}-1)(2-x)(1-x^2)}{\sqrt{x}(3+2x)}$$

Solution. *Let $f_1(x) = (\sqrt{x}-1)(2-x)(1-x^2)$ and $f_2(x) = \sqrt{x}(3+2x)$. Then*

$$f'(x) = \frac{f_1'(x)f_2(x) - f_2'(x)f_1(x)}{f_2(x)^2}. \tag{2.5.1}$$

Let $f_3(x) = (\sqrt{x}-1)$, $f_4(x) = (2-x)$ and $f_5(x) = (1-x^2)$. Then

$$f_1(x) = f_3(x)f_4(x)f_5(x)$$

and

$$f'(x) = f_3'(x)f_4(x)f_5(x) + f_3(x)f_4'(x)f_5(x) + f_3(x)f_4(x)f_5'(x).$$

Consider that

$$f_3'(x) = \frac{1}{2\sqrt{x}}, \quad f_4'(x) = -1, \text{ and } \quad f_5'(x) = -2x.$$

Therefore,

$$f_1'(x) = (\frac{1}{2\sqrt{x}})(2-x)(1-x^2) + (\sqrt{x}-1)(-1)(1-x^2) + (\sqrt{x}-1)(2-x)(-2x).$$

Also,

$$f_2'(x) = \frac{1}{2\sqrt{x}}(3+2x) + 2\sqrt{x}.$$

So we now substitute in 2.5.1 and we have the derivative of $f(x)$.

2.6 Derivatives of Exponential Functions

So far we have seen the derivative for some family of functions such as

- polynomials,

- rational functions, and powers and roots of rational functions.

➤➤ Looking for a function $f(x)$ such that $\frac{d}{dx}f(x) = f(x)$:

Now we want to find the derivative of exponential functions, that is, if $a > 0$ what is the derivative of $f(x) = a^x$. By the definition of the derivative, we have the derivative of $f(x) = a^x$ with respect to x is

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{(x+h)} - a^x}{h} = \lim_{h\to 0} a^x \frac{a^h - 1}{h} = a^x \lim_{h\to 0}\frac{a^h - 1}{h}.$$

For a moment let assume that

$$C(a) = \lim_{h\to 0}\frac{a^h - 1}{h}$$

exists. Therefore,

$$\frac{d}{dx}a^x = a^x C(a).$$

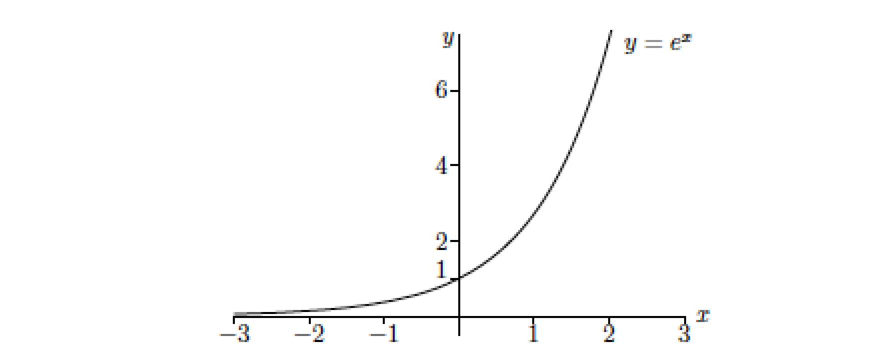
If we can find a such that $C(a) = 1$, it turns to be very useful.

Theorem 2.6.1. *$C(a) = 1$ when $a = e$, where*

$$\begin{aligned} e &= 2.71828182845904523 54\ldots \\ &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \end{aligned}$$

Therefore,

$$\frac{d}{dx}e^x = e^x \quad \textit{and} \quad \lim_{h\to 0}\frac{e^h - 1}{h}.$$



Therefore, we have

- $e^0 = 1$
- $e^{x+y} = e^x e^y$
- $e^{-x} = \frac{1}{e^x}$
- $(e^x)^y = e^{xy}$
- $$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to-\infty} e^x = 0.$$

Example 2.6.2. *Find a such that the following function is continuous.*

$$f(x) = \begin{cases} e^{x+a} & x < 0 \\ \sqrt{x+1} & x \geq 0 \end{cases}$$

Solution. *Consider that both functions e^{x+a} for any $a \in \mathbb{R}$ and $\sqrt{x+1}$ are continuous, so we only need to check that if this function is continuous at $x = 0$. Consider that if the function $f(x)$ is continuous at $x = 0$, then we must have*

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 1.$$

Note that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{x+a} = e^a$$

and

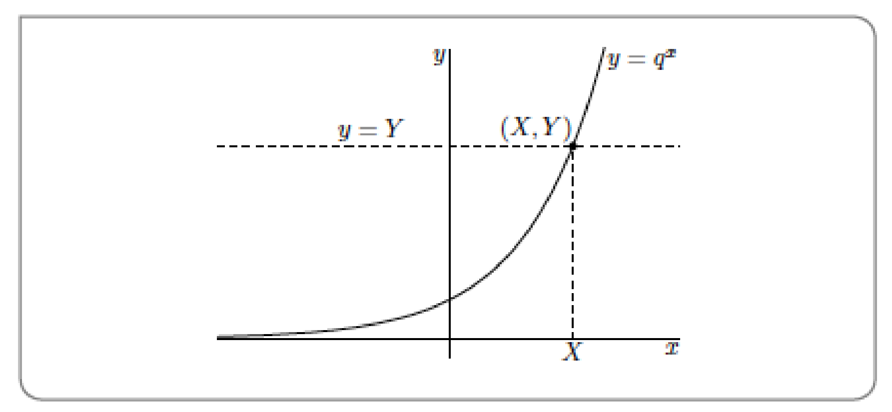
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x+1} = 1.$$

Therefore, we must have $e^a = 1$ and so $a = 0$.

➤➤ ➤Logarithmic Functions:

We are about to define "logarithm with base q". The number q can be any positive number, however, we restrict our attention to $q > 1$ since the only q's that are ever used are e (we will see what e is in the next pages). Let $q = 10$, then we have the following:

- $\lim_{x\to+\infty} 10^x = \infty$
- $\lim_{x\to-\infty} 10^x = 0$



Actually, the function $\log_q(x)$ is the inverse of the function q^x. For learning about inverse functions see section 0.6. So we have the following definition.

Definition. *Let $q > 1$. Then the logarithm with base q is defined by*

$$y = \log_q(x) \Leftrightarrow x = q^y$$

Obviously, we have the following.

$$\log_q(q^x) = x \qquad q^{\log_q(x)} = x$$

Example 2.6.3. *We have*

1. $\log_q(xy) = \log_q(x) + \log_q(y)$.
   The reason for this is that

   $$q^{\log_q(xy)} = xy = q^{\log_q(x)}q^{\log_q(y)} = q^{\log_q(x)+\log_q(y)}$$

   Therefore, $\log_q(xy) = \log(x) + \log(y)$.
2. $\log_q(x/y) = \log_q(x) - \log_q(y)$
3. $\log_q(x^r) = r\log_q(x)$

Example 2.6.4. *We now show that*

$$\log_q(x) = \frac{\log_{10}(x)}{\log_q(10)}.$$

Solution. *Let* $\log_q(x) = y$. *Then*

$$q^y = q^{\log_q(x)} = x.$$

We take log *base* 10 *of both sides*

$$\log_{10} q^y = \log_{10} x.$$

Then

$$y \log_{10} q = \log_{10}(x)$$

Therefore,

$$\log_q(x) = \frac{\log_{10}(x)}{\log_q(10)}.$$

Tools: Chain Rule

Theorem 2.6.5. *Let f and g be differentiable functions. Then*

$$\frac{d}{dx}(f \circ g)(x) = g'(x)f'(g(x)).$$

Let $a > 0$ and

$$a^x = e^{\log_e(a^x)} = e^{x \log_e a}$$

Let $f(x) = e^x$ and $g(x) = x \log_e a$. Then

$$(f \circ g)(x) = e^{x \log_e a} = a^x.$$

Therefore,

$$\frac{d}{dx}(f \circ g)(x) = g'(x)f'(g(x)) = (\log_e a)e^{x \log_e a} = (\log_e a)a^x.$$

The derivative of a^x

Theorem 2.6.6. *Let $a > 0$ and $f(x) = a^x$. Then*

$$\frac{d}{dx}a^x = (\log_e a)e^{x \log_e a} = (\log_e a)a^x.$$

Example 2.6.7. *Find the derivative of* $2^{\sqrt{x}}$.

Solution. *Let $f(x) = 2^x$ and $g(x) = \sqrt{x}$. Then*

$$f \circ g(x) = 2^{\sqrt{x}}.$$

Therefore,

$$\frac{d}{dx}2^x = g'(x)f'(g(x)).$$

We have $g'(x) = \frac{1}{2\sqrt{x}}$ *and* $f'(x) = (\log_e 2)2^x$. *Thus,*

$$\frac{d}{dx}2^x = g'(x)f'(g(x)) = \frac{1}{2\sqrt{x}}(\log_e 2)2^{\frac{1}{2\sqrt{x}}}.$$

Example 2.6.8. *Find a and b such that the following function is differentiable.*

$$f(x) = \begin{cases} x^3 + a & x < 1 \\ e^{x-1} + bx & x \geq 1 \end{cases}$$

Solution. *We first know that this function must be continuous, thus the left hand side limit at* 1 *must be equal to the right hand side limit at 1. We have*

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} e^{x-1} + bx = 1 + b$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^3 + a = 1 + a,$$

and so $1 + a = 1 + b \Rightarrow a = b$. *Moreover, by the definition of the derivative, if $f(x)$ is differentiable at* 1 *then we must have*

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

must exists. Therefore,

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h}$$

Note that

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{e^{x+h-1} + b(x+h) - (1+b)}{h} = \frac{d}{dx}|_{x=1}(e^{x-1} + bx) = b.$$

Also,

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = 3$$

Therefore, $b = 3$ *and* $a = 3$.

2.7 Derivatives of trigonometric functions

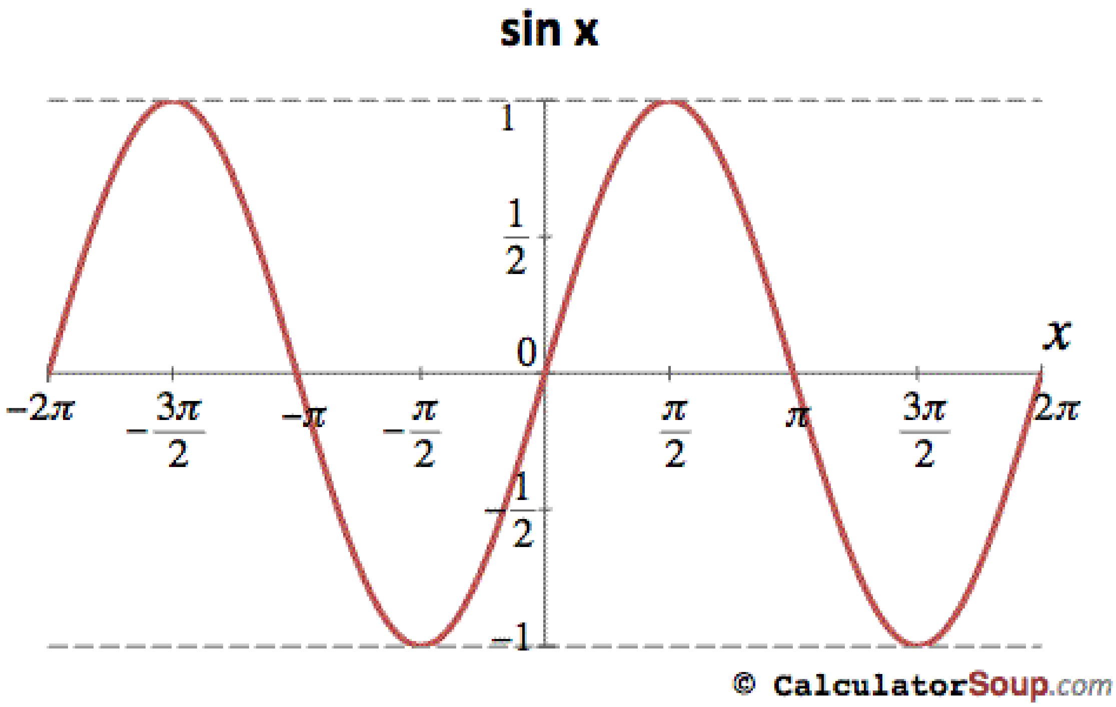
Remember that

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$$

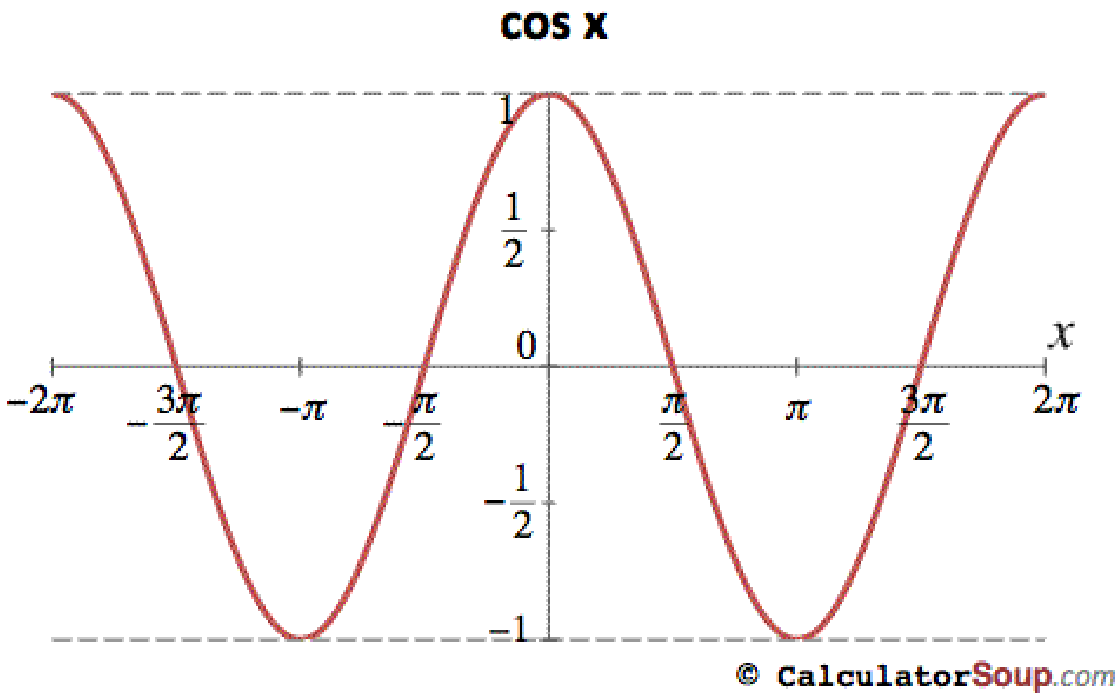
$$\csc(x) = \frac{1}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)}.$$

Let see the graph of each of this functions.

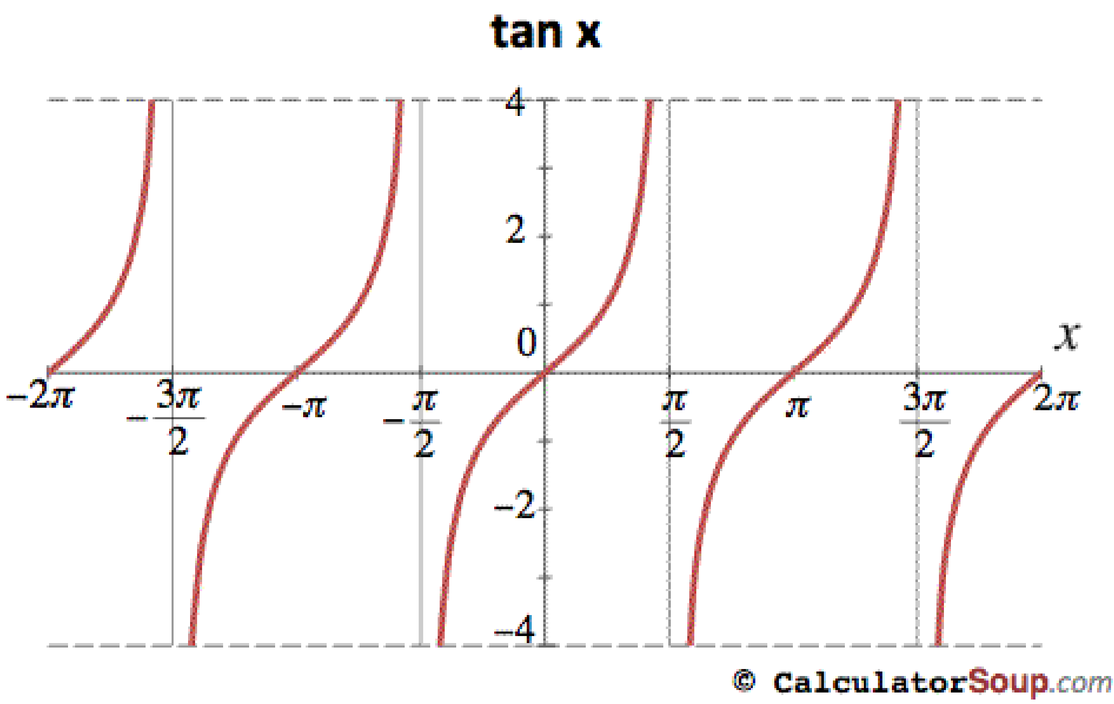
$\sin(x)$ domain $= \mathbb{R}$ range $= [-1, 1]$



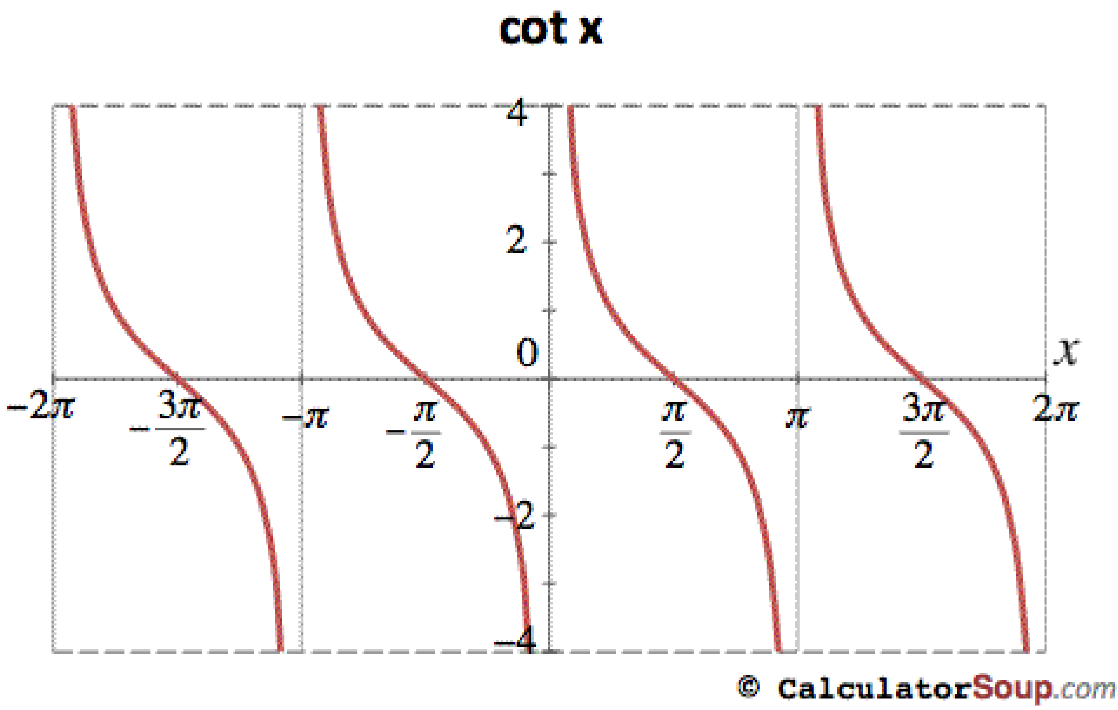
$\cos(x)$ domain $= \mathbb{R}$ range $= [-1, 1]$



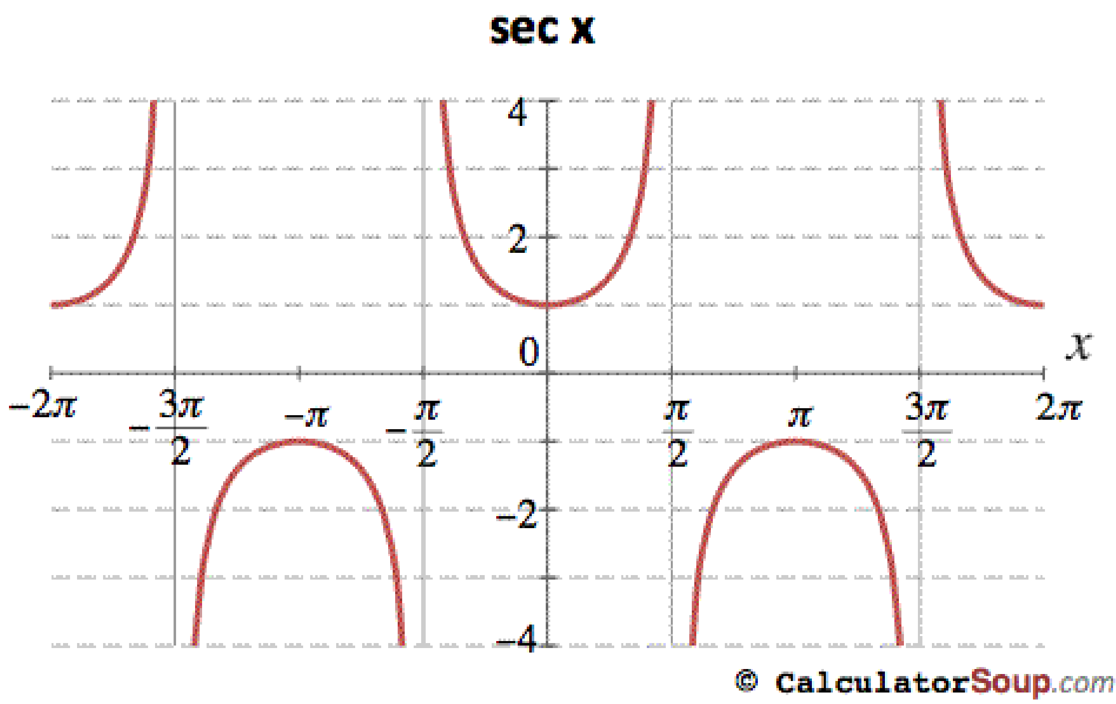
$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \text{domain} = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\} \qquad \text{range} = \mathbb{R}$$



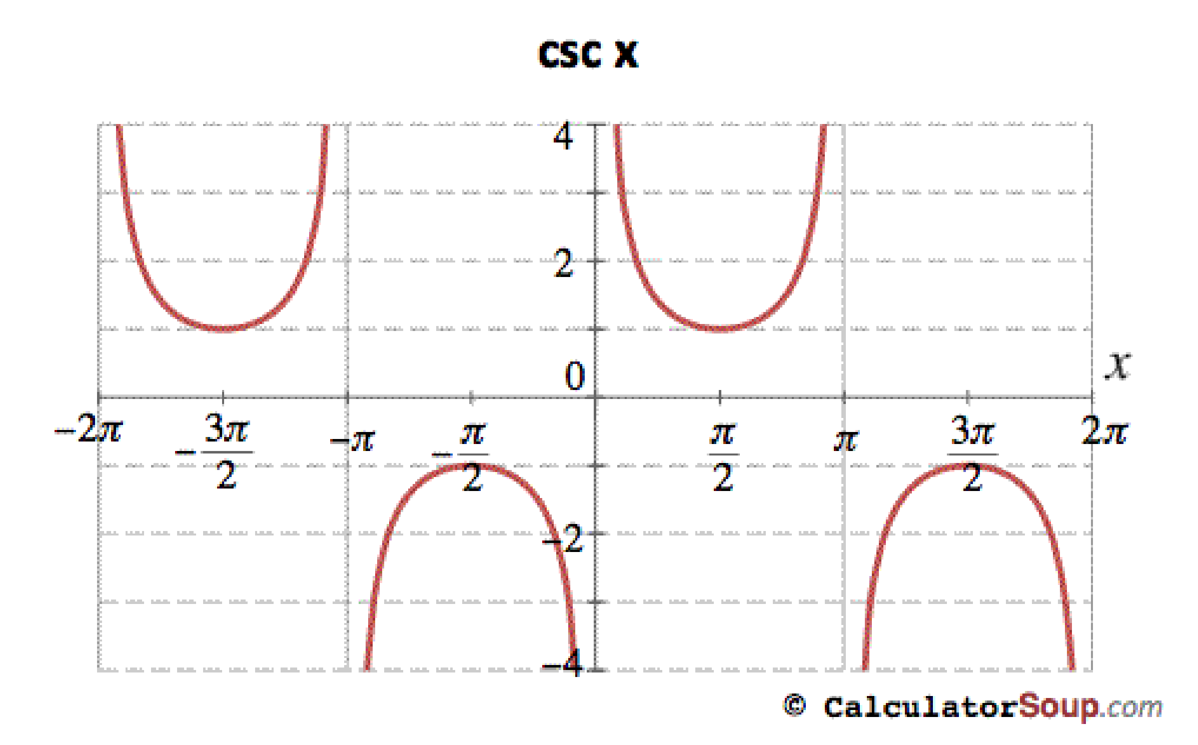
$$\cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \text{domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \qquad \text{range} = \mathbb{R}$$



$\sec(x) = \dfrac{1}{\cos(x)}$ domain $= \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$ range $= \mathbb{R} \setminus (-1, 1)$



$\csc(x) = \dfrac{1}{\sin(x)}$ domain $= \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$ range $= \mathbb{R} - (-1, 1)$



Question: Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of $\sin(x)$ at $x = 0$.

Solution. *Consider that by the definition of the derivative of a function, we have*

$$\frac{d}{dx}|_{x=0} \sin(x) = \lim_{h \to 0} \frac{\sin(0+h) - \sin(0)}{h} = \lim_{h \to 0} \frac{\sin(h)}{h}$$

Since $\lim_{h \to 0} \cos(h) = 1 = \lim_{h \to 0} 1$, *and* $\cos h \leq \frac{\sin h}{h} \leq 1$, *by the sandwich theorem we have*

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$$

The derivative of $\sin(x)$ and $\cos(x)$

Theorem 2.7.1. *If* $f(x) = \sin(x)$ *and* $g(x) = \cos(x)$*. Then*

$$f'(0) = \lim_{h \to 0} \frac{\sin h}{h} = 1 \qquad g'(0) = \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

And in general we have that

Theorem 2.7.2.

$$\frac{d}{dx} \sin(x) = \cos(x) \qquad \frac{d}{dx} \cos(x) = -\sin(x).$$

The derivative of trigonometric functions

Theorem 2.7.3.

$$\frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x) \qquad \frac{d}{dx}sec(x) = \sec(x)\tan(x).$$

2.8 One More Tool—the Chain Rule

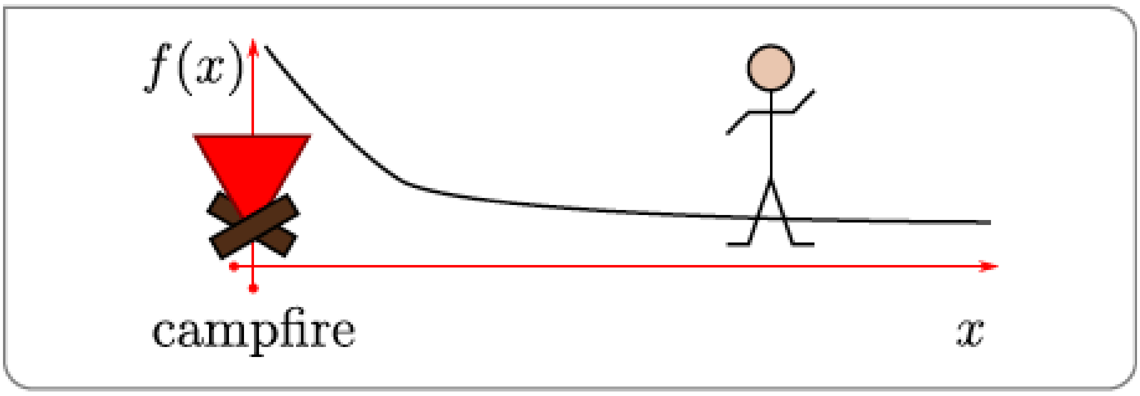
Write all of the tools that we have had so far.

You are out in the woods and are walking towards your camp fire. The heat form the fire means that the air temperature depends on your position. Let your position at time t be $x(t)$. The temperature of the air at position x is $f(x)$. What instantaneous rate of change of temperature do you feel at time t.

- Because your position at time t is $x(t)$, the temperature you feel at time t is $F(t) = f(x(t))$.

- The instantaneous rate of change of temperature that you feel is $F'(t)$. We have a complicated function, F(t), constructed by composing two simple functions, $x(t)$, and $f(x)$.

- We wish to compute the derivative, $F'(t) = \frac{d}{dt}f(x(t))$, of the complicated function $F(t)$ in terms of the derivatives, $x'(t)$ and $f'(x)$, of the two simple functions. This is exactly what the chain rule does.



The chain rule

Theorem 2.8.1. *Let $a \in \mathbb{R}$ and let $g(x)$ be a function that is differentiable at $x = a$. Now let $f(u)$ be a function that is differentiable at $u = g(a)$. Then the function $F(x) = f(g(x))$ is differentiable at $x = a$ and*

$$F'(a) = f'(g(a))g'(a).$$

Theorem 2.8.2. *Let f and g be differentiable functions then*

$$\frac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$$

Theorem 2.8.3. *Let $y = f(u)$ and $u = g(x)$ be differentiable functions, then*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example 2.8.4. *Find the derivative of $(\sin(x))^5$ with respect to x.*

Solution. *Let $g(x) = \sin(x)$ $f(x) = x^5$. Then*

$$f(g(x)) = (\sin(x))^5.$$

Therefore,

$$\frac{d}{dx}f(g(x)) = g'(x)f'(g(x)) = \cos(x)(5\sin(x)^4).$$

Example 2.8.5. *Find the derivative of $\cos(3x - 2)$ with respect to x.*

Solution. *Let $f(x) = \cos(x)$ and $g(x) = 3x - 2$. Then $f(g(x)) = \cos(3x - 2)$, and so*

$$\frac{d}{dx}\cos(3x - 2) = g'(x)f'(g(x)) = 3 \times -\sin(3x - 2).$$

Example 2.8.6. *Find the derivative of the function F with respect to t where*

$$F(x) = \sec x \qquad x(t) = e^{t+1}$$

Solution. *Consider that*

$$\frac{dF}{dt} = \frac{dF}{dx}\frac{dx}{dt}.$$

Note that

$$\frac{dF}{dx} = \sec(x)\tan(x) = \sec(e^{t+1})\tan(e^{t+1})$$

and also $\frac{dx}{dt} = e^{t+1}$. Therefore,

$$\frac{dF}{dt} = \frac{dF}{dx}\frac{dx}{dt} = e^{t+1}\sec(e^{t+1})\tan(e^{t+1}).$$

Example 2.8.7. *Find the derivative of* $2^{\sqrt{x}}$.

Solution. *Let* $f(x) = 2^x$ *and* $g(x) = \sqrt{x}$. *Then*

$$f \circ g(x) = 2^{\sqrt{x}}.$$

Therefore,

$$\frac{d}{dx} 2^x = g'(x) f'(g(x)).$$

We have $g'(x) = \frac{1}{2\sqrt{x}}$ *and* $f'(x) = (\log_e 2)2^x$. *Thus,*

$$\frac{d}{dx} 2^x = g'(x) f'(g(x)) = \frac{1}{2\sqrt{x}} (\log_e 2) 2^{\frac{1}{2\sqrt{x}}}.$$

Example 2.8.8. *Find the tangent line to the curve Find the derivative of* $y = \sin(2^{\sqrt{x}})$ *with respect to* x.

Solution. *Let* $f(x) = \sin(x)$ *and* $g(x) = 2^{\sqrt{x}}$. *Then* $f(g(x)) = \sin(2^{\sqrt{x}}) = y$. *Therefore,*

$$\frac{dy}{dx} = f'(g(x)).g'(x) = \cos(2^{\sqrt{x}}).\frac{1}{2\sqrt{x}} (\log_e 2) 2^{\frac{1}{2\sqrt{x}}}$$

2.9 Inverse Functions

Functions are really just rules for taking an input, processing it somehow and then returning an out put.

$$\text{input number } x \mapsto f \text{ does "stuff" to } x \mapsto \text{return number } y$$

In many situations it will turn out to be very useful if we can undo whatever it is that our functions has done. i.e,

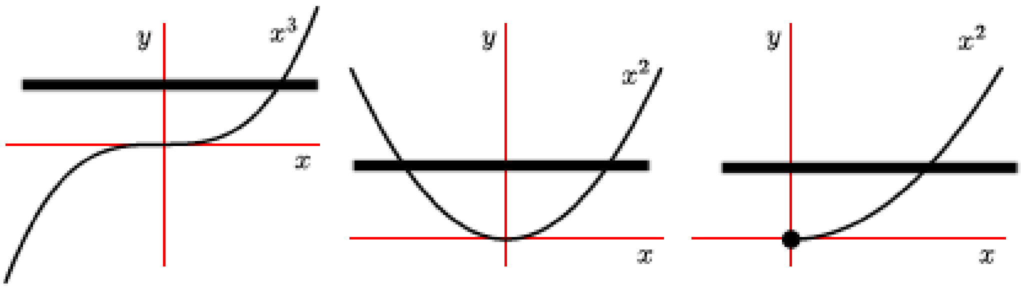
$$\text{take output } y \mapsto \text{do "stuff" to } y \mapsto \text{return the original number } x$$

Definition. *A function is one-to-one (injective) when it never takes the same value more than once. That is*

$$\text{if } x_1 \neq x_2 \text{ then } f(x_1) \neq f(x_2)$$

Definition. *A function is one-to-one if and only if no horizontal line* $y = c$ *intersects the graph* $y = f(x)$ *more than once.*

Example 2.9.1. *By the horizontal line test we will see that* $y = x^3$ *is one to one in its domain; however,* $y = x^2$ *is not one to one in its domain, but if we restrict* $y = x^2$ *to* *only zero or positive numbers is one to one.*



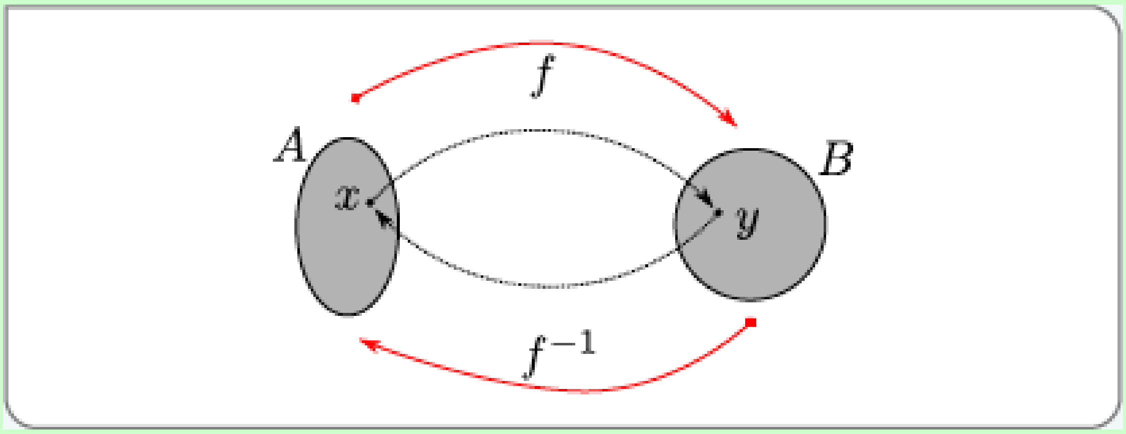
$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array} \qquad \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array} \qquad \begin{array}{ccc} [0,\infty) & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is one-to-oe *is not one-to-oe* *is one-to-oe*

Definition. *Let f be a one-to-one function with domain A and range B. Then its inverse function is denoted f^{-1} and has domain B and range A. It is defined by*

$$f^{-1}(y) = x \qquad \textit{whenever} \qquad f(x) = y$$

for any $y \in B$.



So

$$\begin{aligned} f^{-1}(f(x)) &= x \qquad \textit{for any} \quad x \in A \\ f(f^{-1}(y)) &= y \qquad \textit{for any} \quad y \in B \end{aligned}$$

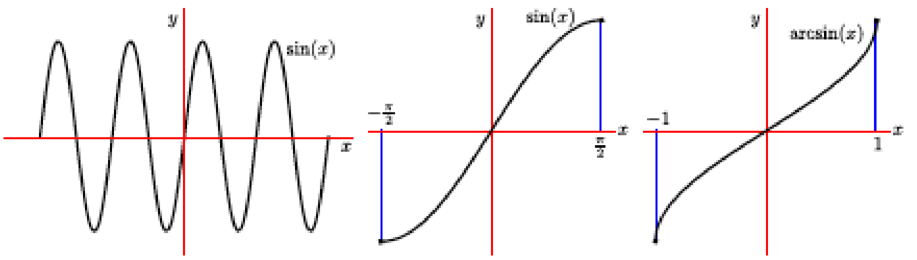
Caveat. We should be careful not to confuse $f^{-1}(x)$ with $\frac{1}{f(x)}$.

Example 2.9.2. *What is the inverse of the following one to one functions.*

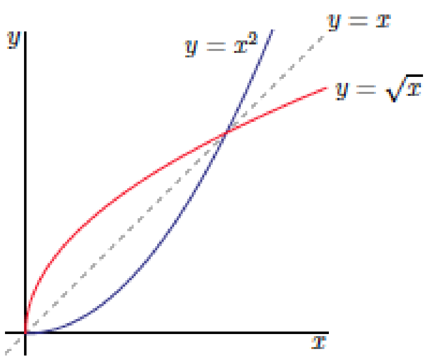
1. $f(x) = x^5 + 3$.
2. $g(x) = \sqrt{x-1}$ *on the domain* $x \geq 1$.

Solution. *(1) Consider that* $y = x^5 + 3$ *so* $y - 3 = x^5$*, and then* $x = \sqrt[5]{y-3}$*. Therefore,* $f^{-1}(x) = \sqrt[5]{x-3}$*.*
(2) Now consider that $y = \sqrt{x-1}$*, and so* $y^2 = x - 1$*. Thus* $x = y^2 + 1$*, and so* $y = x^2 + 1$ *is the inverse.*

Consider the graph of $\sin(x)$ on the domain $-\infty < x < +\infty$, it is easy to see that this function is not one to one, but when we consider this function on the domain $-\pi/2 \leq x \leq \pi/2$ this function is one to one and so it is invertible. The inverse of the function $\sin(x)$ is $\arcsin(x)$ and its domain is $[-1, 1]$ and its codomain is $[-\pi/2, \pi/2]$.



Finding the inverse of $f(x)$ **by its graph.** Assume that we have drawn the graph of a function, for example $y = x^2$ on the domain $x \geq 0$. To find its inverse which is $f^{-1}(x) = \sqrt{x}$, all you have to do is graph the function and then switch all x and y values in each point to graph the inverse. Just look at all those values switching places from the $f(x)$ function to its inverse $f^{-1}(x)$ (and back again), reflected over the line $y = x$.



2.10 The Natural Logarithm

Consider the logarithm base e–$\log_e x$ is the power that e must be raised to to give x. That is, $\log_e x$ is defined by

$$e^{\log_e x} = x,$$

consequently, it is the inverse of the exponential function with base e.

Moreover, note that

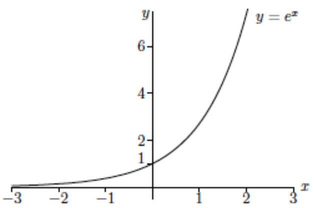
$$\log_q x = \frac{\log_e x}{\log_e q}$$

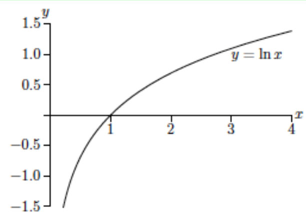
so we are free to chose a bases which is convenient for our purpose.

The logarithm with base e is called "natural logarithm". The "naturalness" of logarithms base e is exactly that this choice of base works very nicely in calculus.

There are several standard notation for the logarithm base e;

$$\log_e x = \log x = \ln x.$$





For any $x, y > 0$, the following hold:

- $e^{\log x} = x$,
- for any real number r, $\ln(e^r) = r$.
- for any $a > 1$, $\log_a x = \frac{\ln x}{\ln a}$ and $\ln x = \frac{\log_a x}{\log_a e}$.
- $\ln(xy) = \ln x + \ln(y)$
- $\ln(\frac{x}{y}) = \ln x - \ln y$, and $\ln(\frac{1}{y}) = -\ln y$
- $\ln(x^r) = r \ln x$
- $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to 0} \ln x = -\infty$.

➤ **What is the derivative of** $\ln x$
To find the derivative of $\ln x$ if you go with the definition of the derivative you don't arrive to something that looks good. However, we have

$$x = e^{\ln x}$$

and we know that if two functions are equal, then their derivative also are the same, therefore,

$$\frac{d}{dx}x = \frac{d}{dx}e^{\ln x}.$$

By the chain rule we have

$$1 = (\frac{d}{dx}\ln x)e^{\ln x}.$$

Thus,

$$1 = (\frac{d}{dx}\ln x)x$$

and so

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Derivative of $\ln x$

Theorem 2.10.1.

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Example 2.10.2. *Let* $f(x) = \ln 3x$. *Find* $f'(x)$.

Solution. *Consider that* $\ln 3x = \ln 3 + \ln x$ *so* $f'(x) = \ln 3 + \ln x$ *and therefore,*

$$f'(x) = \frac{1}{x}.$$

Example 2.10.3. *Let* $g(x) = \ln |x|$. *Find* $g'(x)$.

Since $|x|$ is not differentiable at $x = 0$, $g(x)$ is not differentiable at $x = 0$. Now we consider the following two cases:

- If $x > 0$, then $|x| = x$, and so

$$g'(x) = \frac{d}{dx}\ln x = \frac{1}{x}.$$

- if $x < 0$, then $|x| = -x$. Let $f_1(x) = \ln x$ and $f_2(x) = -x$, then

$$g(x) = f_1(f_2(x)) \Rightarrow g'(x) = f_1'(f_2(x))f_2'(x)$$

$$g'(x) = \frac{1}{-x} \times -1 = \frac{1}{x}.$$

What is $\frac{d}{dx}\log_a x$

So we want now find the derivative of $\log_a x$; consider that

$$\log_a x = \frac{\ln x}{\ln a}.$$

Therefore,

$$\frac{d}{dx}\log_a x = \frac{1}{\ln a}\frac{1}{x}.$$

➤ What is the derivative of $\ln f(x)$? Let $f_1(x) = \ln x$. Then

$$f_1(f(x)) = \ln f(x).$$

Therefore,

$$\frac{d}{dx}f_1(f(x)) = f_1'(f(x))f'(x) = \frac{1}{f(x)}f'(x).$$

Derivative of $\ln|f(x)|$

➤ Let $f_1(x) = \ln|x|$ and $f_2(x) = f(x)$. Then

$$f_1(f_2(x)) = \ln|f(x)|.$$

Therefore,

$$\frac{d}{dx}\ln|f(x)| = f_1'(f_2(x))f_2'(x) = \frac{1}{f(x)}f'(x)$$

➤ What is the derivative of a^x? We have

$$f(x) = a^x$$

$$\ln f(x) = x\ln a$$

$$\frac{d}{dx}\ln f(x) = \ln a.$$

Using the chain rule to process the left-hand side we have

$$\frac{f'(x)}{f(x)} = \ln a$$

Therefore,

$$f'(x) = f(x)\ln a.$$

Then

$$f'(x) = (\ln a)a^x.$$

Theorem 2.10.4.

$$\frac{d}{dx}a^x = \ln a.a^x \qquad \textit{for any } a > 0$$

$$\frac{d}{dx}\log_a x = \frac{1}{x.\ln a} \qquad \textit{for any } a > 0, a \neq 1$$

Logarithmic Differentiation

➤➤ **Logarithmic Differentiation** We want to compute the derivative of the multiplication of three functions

$$p(x) = f(x)g(x)h(x)$$

by using the logarithm properties. Consider that

$$|p(x)| = |f(x)||g(x)||h(x)|$$

$$\ln|p(x)| = \ln|f(x)||g(x)||h(x)| = \ln|f(x)| + \ln|g(x)| + \ln|h(x)|$$

Therefore,

$$\frac{d}{dx}\ln|p(x)| = \frac{d}{dx}\ln|f(x)| + \frac{d}{dx}\ln|g(x)| + \frac{d}{dx}\ln|h(x)|$$

Thus

$$\frac{p'(x)}{p(x)} = \frac{f'(x)}{f(x)}\frac{g'(x)}{g(x)}\frac{h'(x)}{h(x)} \Rightarrow p'(x) = p(x)(\frac{f'(x)}{f(x)}\frac{g'(x)}{g(x)}\frac{h'(x)}{h(x)}).$$

Example 2.10.5. *Compute the derivative of*

$$f(x) = \frac{(\sqrt{x}-1)(2-x)(1-x^2)}{\sqrt{x}(3+2x)}$$

Solution. *Check that* $\ln|f(x)| = \frac{f'(x)}{f(x)}$ *for any function* $f(x)$. *Since* $f(x)$ *can be negative and* $\ln$ *is not defined for negative numbers, we compute*

$$\ln|f(x)| = \ln\frac{|(\sqrt{x}-1)||(2-x)||(1-x^2)|}{|\sqrt{x}||(3+2x)|} =$$

$$\ln|\sqrt{x}-1| + \ln|(2-x)| + \ln|(1-x^2)| - \ln|\sqrt{x}| - \ln|3+2x|$$

Therefore,

$$\frac{f'(x)}{f(x)} = \frac{1/(2\sqrt{x})}{\sqrt{x}-1} + \frac{-1}{2-x} + \frac{-2x}{1-x^2} - \frac{1/(2\sqrt{x})}{\sqrt{x}} - \frac{2}{3+2x}.$$

So we have that

$$f'(x) = f(x)(\frac{1/(2\sqrt{x})}{\sqrt{x}-1} + \frac{-1}{2-x} + \frac{-2x}{1-x^2} - \frac{1/(2\sqrt{x})}{\sqrt{x}} - \frac{2}{3+2x})$$

2.11 Implicit Differentiation

Let's go back to see how we find the derivation of $y = \ln x$. We first have that $x = e^{\ln x}$ and then we showed that

$$\frac{d}{dx}x = \frac{d}{dx}e^{\ln x} \qquad (\frac{d}{dx}x = \frac{d}{dx}e^{y})$$

which is the same as

$$1 = (\frac{d}{dx}\ln x).e^{\ln x} \qquad (1 = y'e^{y}).$$

Note that $e^{\ln x} = x(e^{y} = x)$, thus

$$1 = (\frac{d}{dx}\ln x).x \qquad (1 = y'x)$$

and so

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad (y' = \frac{1}{x}).$$

Example 2.11.1. *Find the equation of the tangent line to $y = y^3 + xy + x^3$ at $x_0 = 1$.*

Solution. *Consider that for finding the tangent line we need a point on the curve and also the slope at the point. We already know that the point that we want to find the curve at has $x_0 = 1$, to find y_0 we plug in $x = 1$ in the equation and then solve it, so after plugging in $x = 1$ we have*

$$y = y^3 + y + 1$$

which we can see that $y^3 = -1$ and so $y = -1$. Therefore, we want to find the tangent line at $(x_0, y_0) = (1, -1)$. Rewrite the equation as

$$f(x) = f(x)^3 + xf(x) + x^3.$$

Now when we do the derivation with respect to x, we have

$$f'(x) = 3f'(x)f(x)^2 + f(x) + xf'(x) + 3x^2.$$

So we have that

$$f'(x_0) = 3f'(x_0)f(x_0)^2 + f'(x_0) + x_0f'(x_0) + 3x_0^2.$$

Consider that $x_0 = 1$ and $f(x_0) = y_0 = -1$. Therefore,

$$f'(1) = 3f'(1)f(1)^2 + f(1) + f'(1) + 3 \Rightarrow f'(1) = 3f'(1)(-1)^2 + (-1) + f'(1) + 3.$$

Consequently,

$$f'(1) = \frac{-2}{3}.$$

Therefore, the equation for the tangent line is

$$y = -1 + \frac{-2}{3}(x - 1) \Rightarrow 3y + 2y = -1.$$

In the next example, instead of substituting y by $f(x)$ we just keep in mind that y is a function of x and its derivative with respect to x is y'.

Example 2.11.2. *Let (x_0, y_0) be a point on the ellipse $3x^2 + 5y^2 = 7$. Find the equation for the tangent lines when $x = 1$ and y is positive. Then find an equation for the tangent line to the ellipse at a general point (x_0, y_0).*

Solution. *First we want to find the equation of the tangent line at $x = 1$ and y is positive. We need a point $(1, y_0)$ at the curve such that $y_0 > 0$. We again plug in $x = 1$ in the equation, then we have*

$$3(1)^2 + 5(y(1))^2 = 7 \Rightarrow (y(1))^2 = \frac{4}{5},$$

thus

$$y(1) = \pm\frac{2}{\sqrt{5}}.$$

Since we have that $y > 0$, thus we find the tangent line at $(1, \frac{2}{\sqrt{5}})$. Now we want to find y' at $x = 1$. Note that

$$6x + 10yy' = 0$$

so when $x = 1$

$$6 \times 1 + 10y(1)y'(1) = 0 \Rightarrow \qquad 6 + 10\frac{2}{\sqrt{5}}y'(1) = 0$$

thus

$$y'(1) = \frac{-6\sqrt{5}}{10 \times 2} = \frac{-3}{2\sqrt{5}}.$$

Therefore, the equation for the tangent line is

$$y = \frac{2}{\sqrt{5}} + \frac{-3}{2\sqrt{5}}(x - 1).$$

Solution. *(the tangent line at some arbitrarily point (x_0, y_0)) Now we want to find the tangent line at some arbitrarily point (x_0, y_0) where $y(x_0) = y_0$ on the curve. We need first to find the derivative at $y'(x_0)$ which is the slope of the tangent line. Consider that*

$$6x + 10yy' = 0$$

Thus,

$$y' = \frac{-6x}{10y} \Rightarrow \qquad y'(x_0) = \frac{-6x_0}{10y_0}$$

so the equation for the tangent line is

$$y = y_0 + \frac{-6x_0}{10y_0}(x - x_0),$$

thus

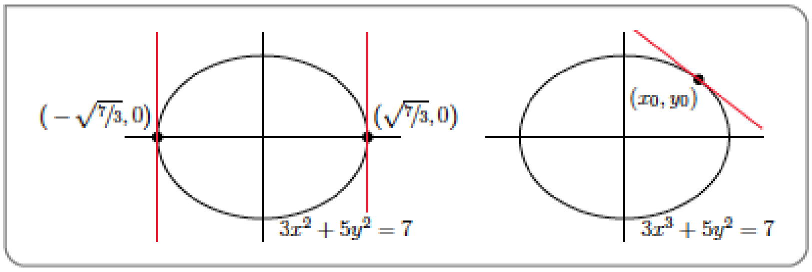
$$y = y_0 + \frac{-3x_0}{5y_0}x - \frac{-3x_0}{5y_0}x_0$$

and so

$$5yy_0 = 5y_0^2 - 3x_0x + 3x_0^2 \Rightarrow \qquad 5yy_0 + 3x_0x = 5y_0^2 + 3x_0^2$$

Note that $5y_0^2 + 3x_0^2 = 7$ *therefore, the equation for the tangent line is*

$$5yy_0 + 3x_0x = 7.$$



Example 2.11.3. *At which points does the curve* $x^2 - xy + y^2 = 3$ *cross the* x*-axis? Are the tangent lines to the curve at those points parallel?*

Solution. *Note that a curve cross the* x*-access when* $y = 0$*. Therefore, when we have* $y = 0$*, then* $x^2 = 3$*, and so* $x = \pm\sqrt{3}$*. Thus, both points* $(\sqrt{3}, 0)$ *and* $(-\sqrt{3}, 0)$ *are on the curve and we want to find the tangent lines at these two points. Consider that the implicit derivative of this curve is*

$$2x - y - xy' + 2yy' = 0$$

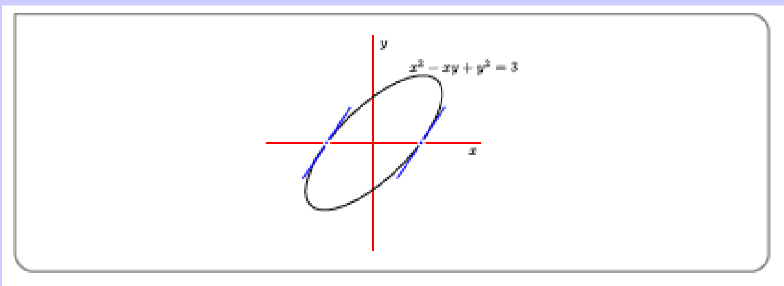
Now when $x = \sqrt{3}$ *and* $y = 0$*, then*

$$2\sqrt{3} - 0 - \sqrt{3}y'(\sqrt{3}) + 2 \times 0 \times y'(\sqrt{3}) = 0 \Rightarrow \qquad 2\sqrt{3} - \sqrt{3}y'(\sqrt{3}) = 0$$

and so $y'(\sqrt{3}) = 2$*.*
Now when $x = -\sqrt{3}$ *and* $y = 0$*, we have*

$$-2\sqrt{3} - 0 + \sqrt{3}y'(\sqrt{3}) + 2 \times 0 \times y'(\sqrt{3}) = 0 \Rightarrow \qquad -2\sqrt{3} + \sqrt{3}y'(\sqrt{3}) = 0.$$

Again we have $y'(-\sqrt{3}) = 2$*.*
Since the slope of the tangent lines at $(\sqrt{3}, 0)$ *and* $(-\sqrt{3}, 0)$ *are both* 2*, and since ant two lines with the same slope are parallel, we have that the two tangent lines are parallel.*



Example 2.11.4. *Let (x_0, y_0) be a point on the astroid*

$$x^{2/3} + y^{2/3} = 1.$$

Find the equation of the tangent line to the astroid at (x_0, y_0).

Solution. *Let assume that $x_0 \neq 0$ and $y_0 \neq 0$. Note that we have*

$$(2/3)x^{-1/3} + (2/3)y'y^{-1/3} = 0$$

Therefore,

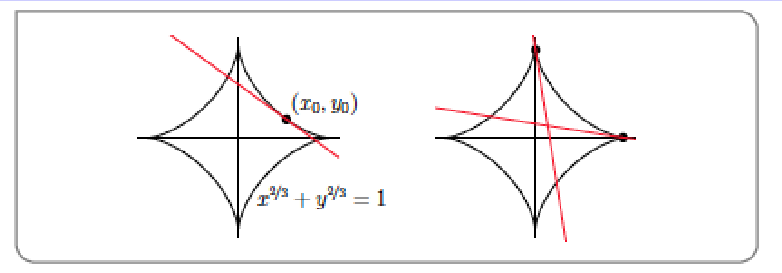
$$y' = -\sqrt[3]{\frac{y}{x}}$$

Therefore, the derivative at $x = x_0$ is

$$y'(x_0) = -\sqrt[3]{\frac{y_0}{x_0}}$$

and the equation for the tangent line is

$$y = y_0 - \sqrt[3]{\frac{y_0}{x_0}}(x - x_0).$$

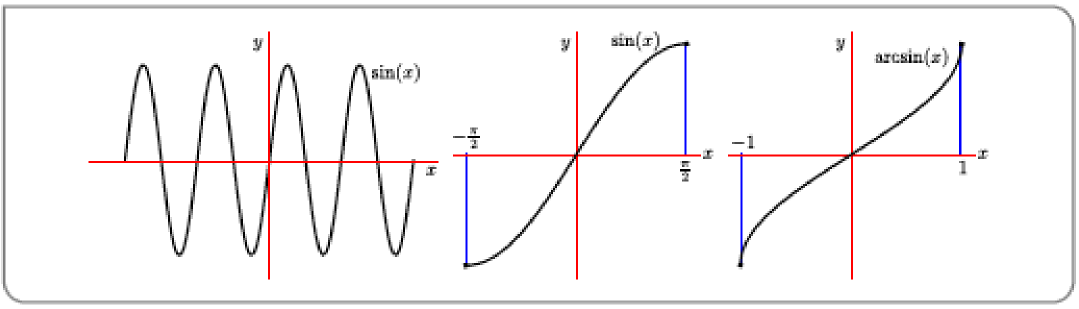


2.12 Inverse of trigonometric functions

Remember that the inverse of a one-to-one function $f(x)$ with domain A and range B is a function $g(x)$ with domain B and range A such that

$$f(g(y)) = y \qquad g(f(x)) = x \qquad x \in A, y \in B.$$

Consider the graph of $\sin(x)$ on the domain $-\infty < x < +\infty$, it is easy to see that this function is not one to one, but when we consider this function on the domain $-\pi/2 \le x \le \pi/2$ this function is one to one and so it is invertible. The inverse of the function $\sin(x)$ is $\arcsin(x)$ and its domain is $[-1, 1]$ and its codomain is $[-\pi/2, \pi/2]$.



arcsin(x)

Therefore we have that $\arcsin(x)$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ is the inverse of the function $\sin(x)$, thus we have

$$\sin(\arcsin(x)) = x \qquad \text{when} \quad -\pi/2 \le \arcsin(x) \le \pi/2$$

Now let talk about $\arcsin(\sin(x))$. Note that when $-\pi/2 \le x \le \pi/2$, then since $\sin(x)$ in this interval is the inverse of $\arcsin(x)$ we must have

$$\arcsin(\sin(x)) = x.$$

Therefore,

$$\arcsin(\sin(\pi/2)) = \pi/2 \qquad \text{and} \qquad \arcsin(\sin(\pi/6)) = \pi/6$$

Example 2.12.1. *1.* $\sin \pi/2 = 1$ *so* $\arcsin(1) = \pi/2$.

2. $\sin \pi/6 = 1/2$ *so* $\arcsin(1/2) = \pi/6$.

Example 2.12.2. *Consider that* $\sin(2\pi) = 0$*, and since* $\arcsin(x)$ *is defined on the interval* $[-1, 1]$*, so this makes sense to ask what is* $\arcsin(\sin(2\pi))$*? Can we say* $\arcsin(\sin(2\pi)) = 2\pi$*? For sure, no, because the range of* $\arcsin(x)$ *is* $[-\pi/2, \pi/2]$*. However,*

$\arcsin(\sin(x)) =$ *the unique angle* θ *between* $-\pi/2$ *and* $\pi/2$ *obeying* $\sin(\theta) = \sin(x)$*.*

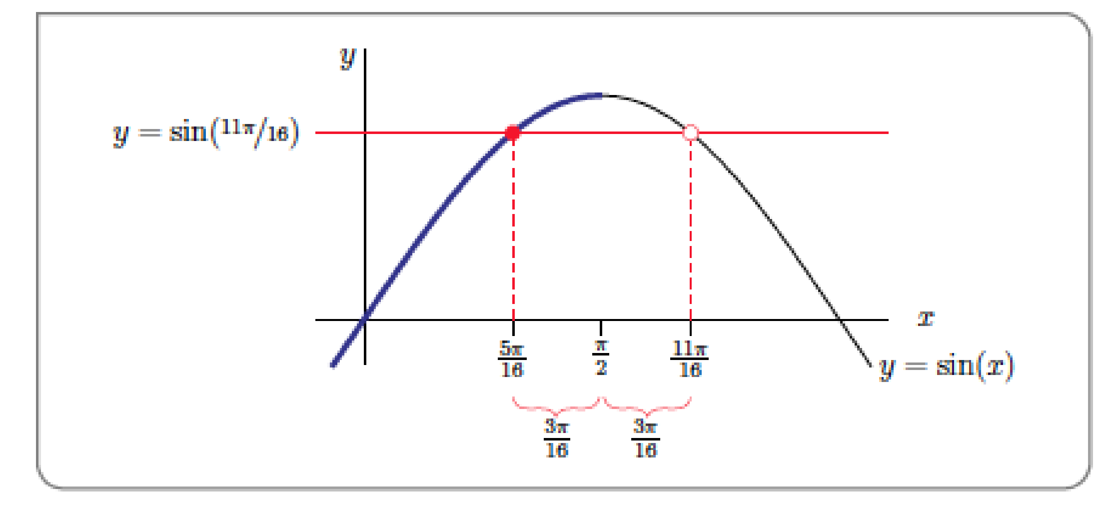
Therefore, $\arcsin(\sin(2\pi))$ *is the unique angle* θ *between* $-\pi/2$ *and* $\pi/2$ *obeying* $\sin(\theta) = \sin(2\pi) = 0$*, which is* 0*. Thus,* $\arcsin(\sin(2\pi)) = \arcsin(\sin(0)) = 0$*.*

Example 2.12.3. *What is* $\arcsin(\sin(\frac{11\pi}{16}))$*?*

Solution. $\arcsin(\sin(x)) =$ *the unique angle* θ *between* $-\pi/2$ *and* $\pi/2$ *obeying* $\sin(\theta) = \sin(x)$*.*

Therefore, $\arcsin(\sin(\frac{11\pi}{16}))$ *is the unique angle* θ *between* $-\pi/2$ *and* $\pi/2$ *obeying that*

$$\sin(\theta) = \sin(\frac{11\pi}{16}).$$



Note that for function $\sin(x)$ *we always have that*

$$\sin(\pi/2 + \alpha) = \sin(\pi/2 - \alpha)$$

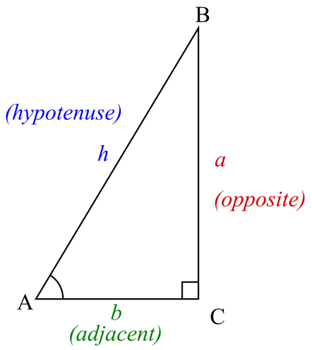
thus in the case of $\frac{11\pi}{16}$*, we have*

$$\sin(\frac{11\pi}{16}) = \sin(\pi/2 + \frac{3\pi}{16}) = \sin(\pi/2 - \frac{3\pi}{16}) = \sin(\frac{5\pi}{16}).$$

Since $\frac{5\pi}{16}$ *is between* $-\pi/2$ *and* $\pi/2$*, we must have*

$$\arcsin(\sin(\frac{11\pi}{16})) = \frac{5\pi}{16}.$$

➤➤ **Derivative of Inverse Trig Functions.**



- **sine:** $\sin A = \dfrac{a}{h} = \dfrac{\text{opposite}}{\text{hypotenuse}}$
- **cosine:** $\cos A = \dfrac{b}{h} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
- **tangent:** $\tan A = \dfrac{a}{b} = \dfrac{\text{opposite}}{\text{adjacent}}$
- **cosecant:** $\csc A = \dfrac{h}{a} = \dfrac{\text{hypotenuse}}{\text{opposite}}$
- **secant:** $\sec A = \dfrac{h}{b} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$
- **cotangent:** $\cot A = \dfrac{b}{a} = \dfrac{\text{adjacent}}{\text{opposite}}$

$\frac{d}{dx}\arcsin(x)$

We have already seen the $\arcsin(x)$, let just write

$$\theta(x) = \arcsin(x)$$

We are looking to find $\frac{d}{dx}\theta(x)$. We know that

$$\sin(\theta) = x$$

Using implicit derivation we have

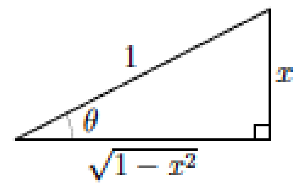
$$\cos(\theta).\frac{d\theta}{dx} = 1$$

and so

$$\frac{d\theta}{dx} = \frac{1}{\cos(\theta)}$$

Consider that $\theta = \arcsin(x)$, so by substitution in the above equation we have

$$\frac{d\theta}{dx} = \frac{1}{\cos(\arcsin(x))}.$$



We now try to find $\cos(\arcsin(x))$. Consider a right triangle where the length of hypotenuse is 1 and with vertical side x and the angle opposite to x is θ, then

$$\sin(\theta) = x.$$

Note that $\arcsin(x) = \theta$, and so

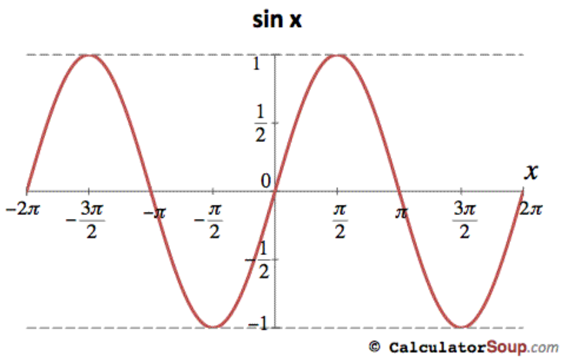
$$\cos(\arcsin(\theta)) = \cos(\theta) = \frac{1}{\sqrt{1-x^2}}.$$

Consequently,

$$\frac{d\theta}{dx} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}.$$

Therefore,

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$


sin x
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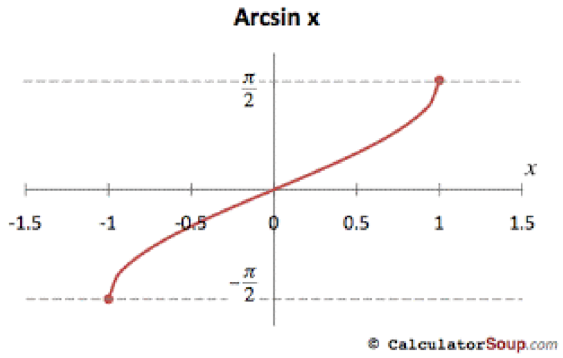
Arcsin x
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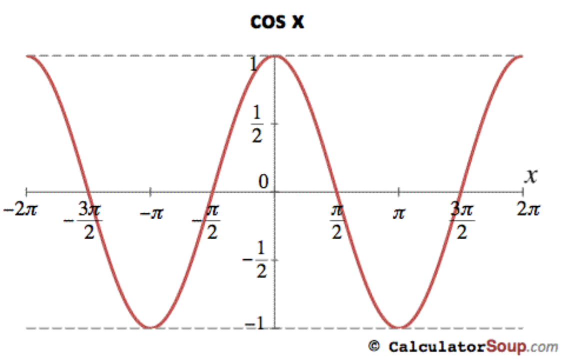
cos x
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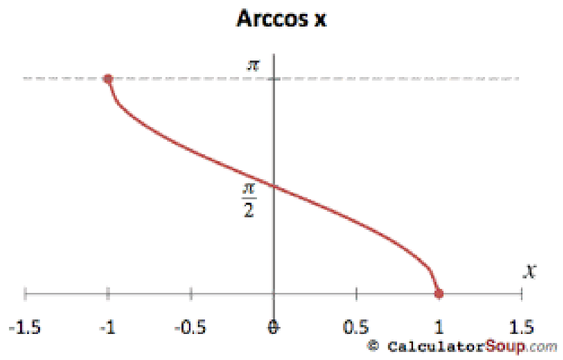
Arccos x
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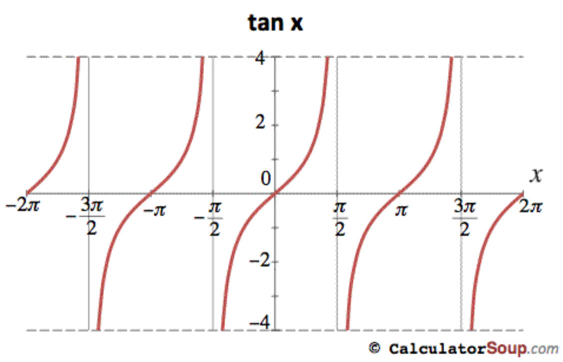
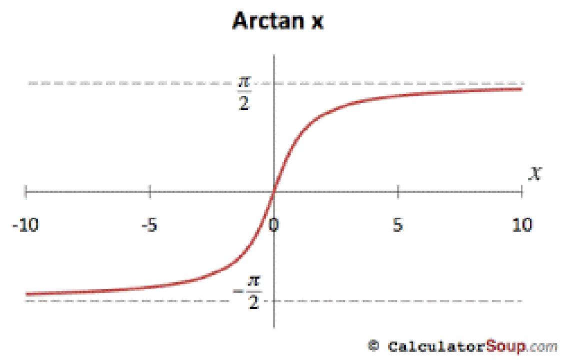
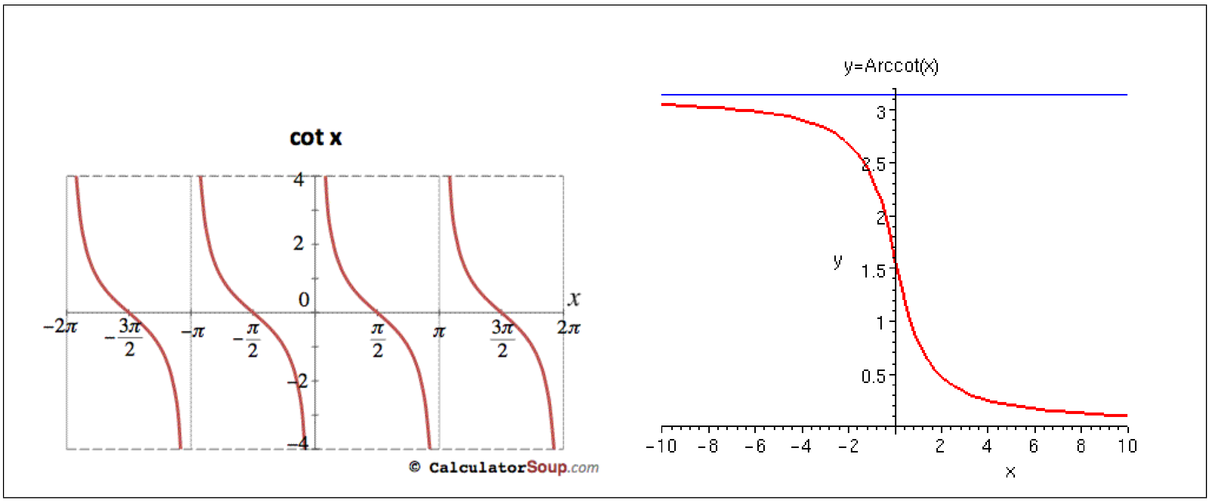

tan x
© CalculatorSoup.com


Arctan x
© CalculatorSoup.com


cot x
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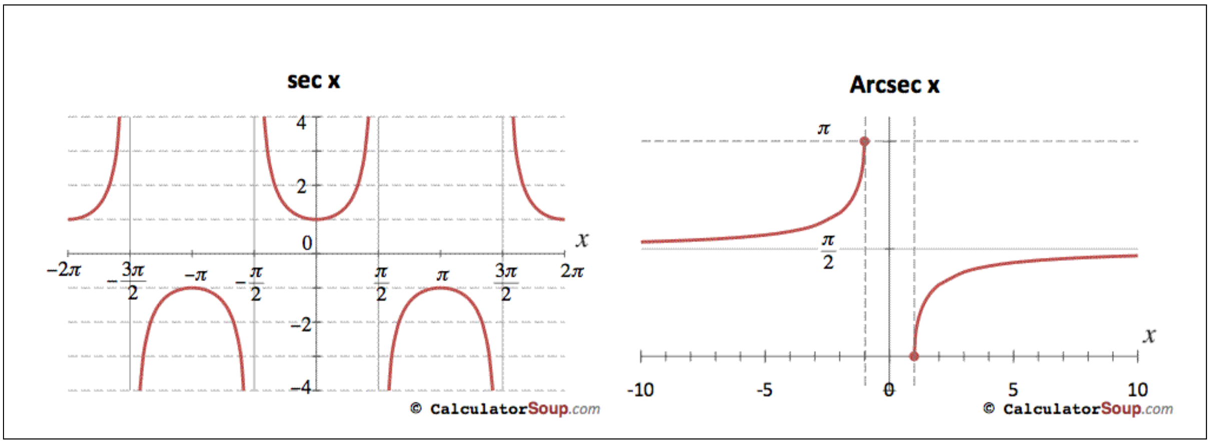
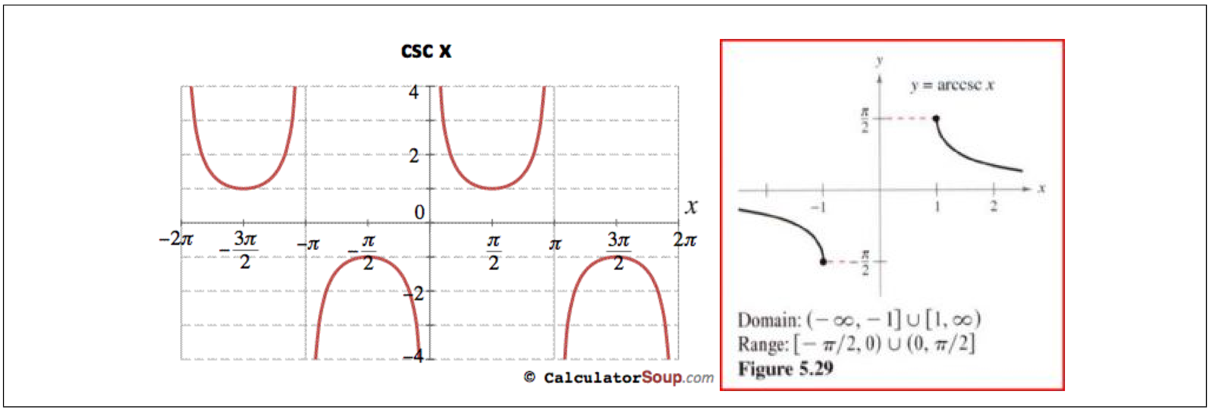
y=Arccot(x)
y
x


sec x
© CalculatorSoup.com
Arcsec x
© CalculatorSoup.com


csc x
© CalculatorSoup.com
y = arccsc x
Domain: $(-\infty,-1]\cup[1,\infty)$
Range: $[-\pi/2,0)\cup(0,\pi/2]$
Figure 5.29

Trigonometric functions

Definition. $\arcsin(x)$ *is defined for* $|x| \leq 1$*. It is the unique number obeying*

$$\sin(\arcsin(x)) = x \qquad \textit{and} \qquad -\pi/2 \leq \arcsin(x) \leq \pi/2$$

$\arccos(x)$ *is defined for* $|x| \leq 1$*. It is the unique number oneying*

$$\cos(\arccos(x)) = x \qquad \textit{and} \qquad 0 \leq \arccos(x) \leq \pi$$

$\arctan(x)$ *is defined for all* $x \in \mathbb{R}$*. It is the unique number obeying*

$$\tan(\arctan(x)) = x \qquad \textit{and} \qquad -\pi/2 < \arctan(x) < \pi/2$$

$\mathrm{arccot}(\mathrm{x})$ *is defined for all* $x \in \mathbb{R}$*. It is the unique number obeying*

$$\cot(\mathrm{arccot}(x)) = x \qquad \textit{and} \qquad 0 < \mathrm{arccot}(x) < \pi$$

$\mathrm{arcsec}(x) = \arccos(1/x)$ *is defined for* $|x| \geq 1$*. It is the unique number obeying*

$$\sec(\mathrm{arcsec}(x)) = x \qquad \textit{and} \qquad 0 \leq \mathrm{arcsec}(x) \leq \pi$$

$\mathrm{arccsc}(x) = \arcsin(1/x)$ *is defined for* $|x| \geq 1$*. It is unique number obeying*

$$\csc(\mathrm{arccsc}(x)) = x \qquad \textit{and} \qquad -\pi/2 \leq \mathrm{arccsc}(x) \leq \pi/2$$

$\frac{d}{dx}\arccos(x)$

Example 2.12.4. *We now wan to find the derivative of* $\theta = \arccos(x)$*. Consider that*

$$\cos(\arccos(x)) = x$$

thus we have

$$\cos(\theta) = x$$

Using implicit derivative we have

$$-\frac{d\theta}{dx}\sin(\theta) = 1.$$

Therefore,

$$\frac{d\theta}{dx} = \frac{-1}{\sin(\theta)} \Rightarrow \qquad \frac{d\theta}{dx} = \frac{1}{\sin(\arccos(x))}.$$

Again looking at the length of hypotenuse 1 and with horizontal side x, we have

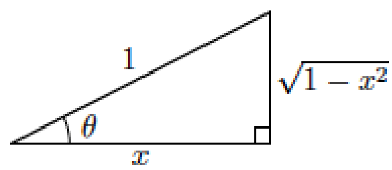
$$\cos(\theta) = x$$

and so

$$\sin(\theta) = \sqrt{1-x^2}.$$

Thus

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}.$$



$\frac{d}{dx}\arctan(x)$

Example 2.12.5. *Very similar steps give the derivative of* $\arctan(x)$.

Solution.

- *Start with* $\theta = \arctan(x)$, *so* $\tan(\theta) = x$.
- *Differentiate implicitly:*

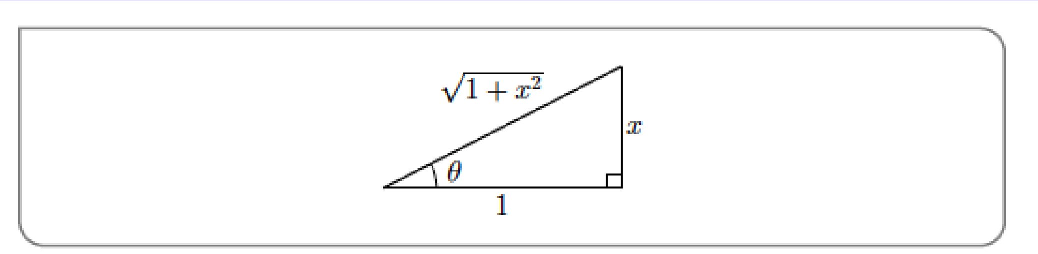
$$\sec^2(\theta)\frac{d\theta}{dx} = 1$$

$$\frac{d\theta}{dx} = \frac{1}{\sec^2(\theta)} = \cos^2\theta$$

  therefore,

$$\frac{d}{dx}\arctan(x) = \cos(\arctan(x)).$$

- *drawing the relevant triangle we have*



  from which we can see

$$\cos^2(\arctan(x)) = \cos^2(\theta) = \frac{1}{1+x^2}.$$

- *Thus* $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

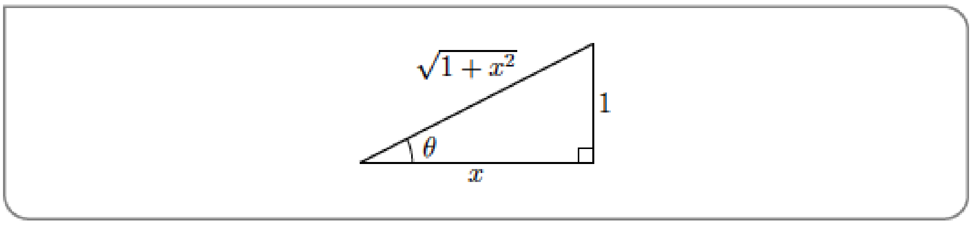
$\frac{d}{dx}\text{arccot}(x)$

Example 2.12.6. *An almost identical computation gives the derivative of* $\text{arccot}(x)$

- *Start with* $\theta = \text{arccot}(x)$, *so* $\cot(\theta) = x$.
- *Differentiate implicitly:*

$$-\csc^2(\theta)\frac{d\theta}{dx} = 1$$

$$\frac{d}{dx}\text{arccot}(x) = \frac{d\theta}{dx} = -\frac{1}{\csc(\theta)} = -\sin^2(\theta) = -\frac{1}{1+x^2}.$$



$\frac{d}{dx}\text{arccsc}(x)$

Example 2.12.7. *For the derivative of* arccsc *we can use the definition and the chain rule.*

$$\theta = \text{arccsc}(x)$$

then

$$\csc(\theta) = x \Rightarrow \sin(\theta) = \frac{1}{x}$$

therefore,

$$\theta = \arcsin(\frac{1}{x})$$

Now by using chain rule we can see

$$\frac{d\theta}{dx} = \frac{d}{dx}\arcsin(\frac{1}{x}) = \frac{1}{\sqrt{1-x^{-2}}}.\frac{-1}{x^2}.$$

To simplify we have

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{x^{-2}(x^2-1)}}.\frac{-1}{x^2} = \frac{1}{x^{-1}\sqrt{(x^2-1)}}.\frac{-1}{x^2}.$$

Note that $|x^{-1}|x^2 = |x|$. *Therefore, the above expression becomes*

$$\frac{1}{|x|\sqrt{(x^2-1)}}.$$

$\frac{d}{dx}\text{arcsec}(x)$

Example 2.12.8. *By the same method as the above, we have*

$$\frac{d}{dx}\text{arcsec}(x) = \frac{d}{dx}\text{arccos}(\frac{1}{x}) = -\frac{1}{\sqrt{1-1/x^2}}.(-\frac{1}{x^2}) = \frac{1}{|x|\sqrt{x^2-1}}.$$

Derivative of the inverses of trigonometric functions in a nutshell

In a nutshell the derivatives of the inverse trigonometric functions are

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\text{arccsc}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\text{arcsec}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\text{arccot}(x) = -\frac{1}{1+x^2}$$

Chapter 3

Application of Derivatives

3.1 Velocity and acceleration

Velocity and acceleration

If you are moving along the x-axis and your position at time t is $x(t)$, then

- your velocity at time t is $v(t) = x'(t)$ and
- your acceleration at time t is $a(t) = v'(t) = x''(t)$.

Example 3.1.1. *Consider the following situation.*

Suppose that you are moving along the x-axis and that at time t your position is given by

$$x(t) = t^3 - 3t + 2$$

- *If $x'(t) > 0$, then at that instant x is increasing, i.e., you are moving to the right.*
- *If $x'(t) = 0$, then at that instant you are not moving at all.*
- *If $x'(t) < 0$, then at that instant x is decreasing, i.e., you are moving to the left.*

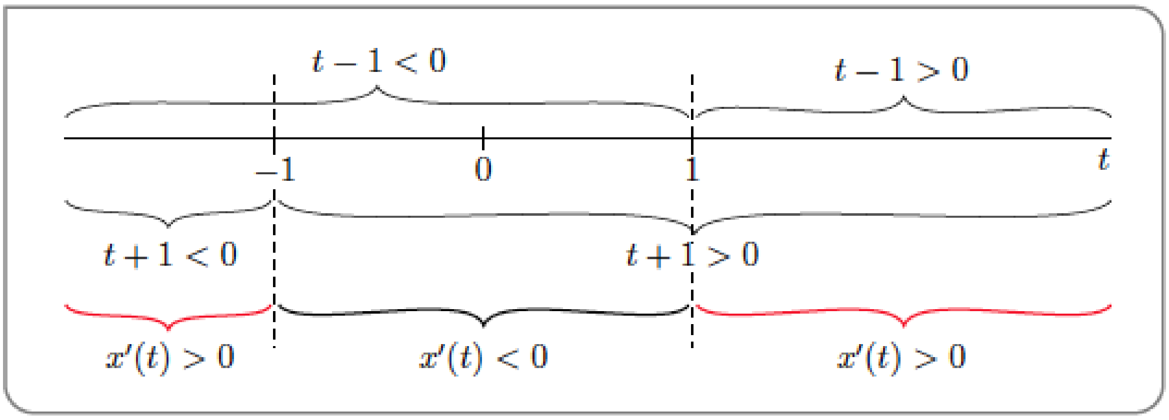
From the formula it is straight forward to see

$$v(t) = x'(t) = 3t^2 - 3 = 3(t^2 - 1) = 3(t+1)(t-1).$$

This is zero when $t = 1$ or $t = -1$.

t	$(t-1)(t+1)$	$x'(t) = 3(t-1)(t+1)$	*Direction(left or right)*
$t < -1$	*positive*	*positive*	*right*
$t = -1$	*zero*	*zero*	*halt*
$-1 < t < 1$	*negative*	*negative*	*left*
$t = 1$	*zero*	*zero*	*halt*
$t > 1$	*positive*	*positive*	*right*

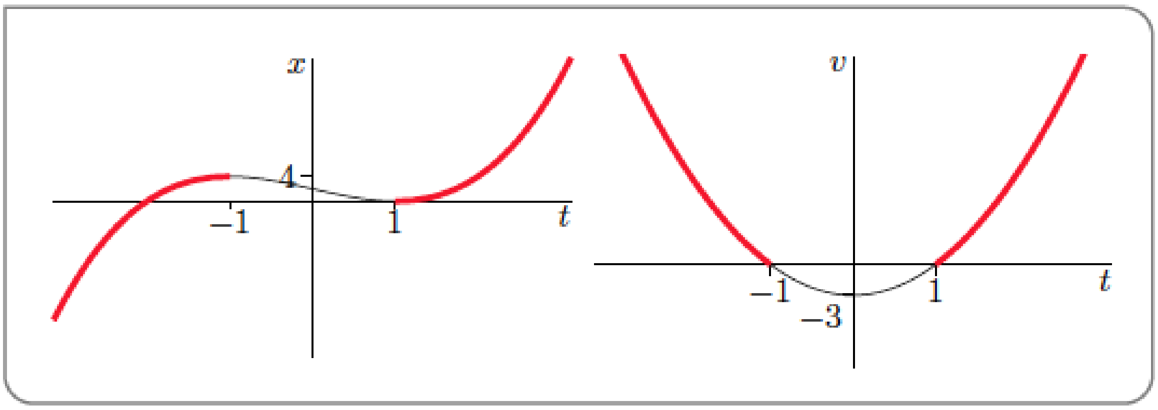
- *For all* $t < -1$*, both* $(t+1)$ *and* $(t-1)$ *are negative, so* $v(t) = x'(t) = 3(t+1)(t-1) > 0$.
- *For all* $-1 < t < 1$*, the factor* $(t+1) > 0$ *and the factor* $(t-1) < 0$ *so* $v(t) = x'(t) = 3(t+1)(t-1) < 0$.
- *For* $t > 1$*, both* $(t+1)$ *and* $(t-1)$ *are positive so* $v(t) = x'(t) = 3(t+1)(t-1) > 0$.



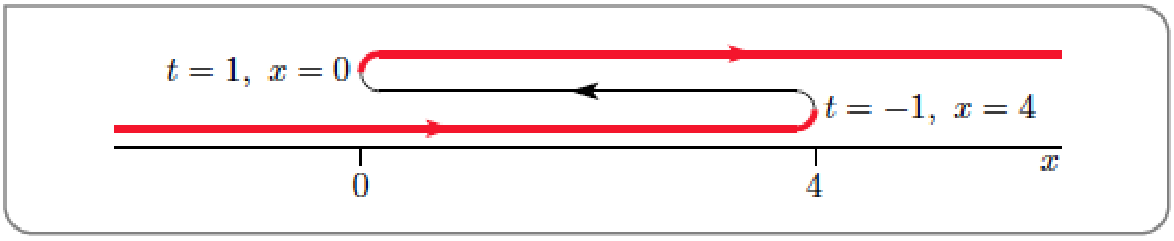
It is now easy to put together a mental image of your trajectory.

- *For* $t < -1$*,* $v(t) = x'(t) > 0$ *so* $x(t)$ *is increasing and you are moving to the right.*
- *At* $t = -1$*,* $v(-1) = 0$ *and you have come to a halt at position* $x(-1) = (-1)^3 - 3(-1) + 2 = 4$.
- *For* $-1 < t < 1$*,* $v(t) = x'(t) < 0$ *so* $x(t)$ *is decreasing and you are moving to the left.*
- *At* $t = +1$*,* $v(1) = 0$ *and you have again come to a halt, but now at position* $x(1) = 1^3 - 3 + 2 = 0$.
- *For* $t > 1$*,* $v(t) = x'(t) > 0$ *so that* $x(t)$ *is increasing and you are again moving to the right.*

Here is a sketch of the graphs of $x(t)$ *and* $v(t)$*.*



And here is a schematic picture of the whole trajectory.



Example 3.1.2. *In the above example, what distance in total you have traveled from $t = 0$ to $t = 2$?*

- *Consider that at time 0, you are at the position $x(0) = 0^3 - 3 \times 0 + 2 = 2$. From $t = 0$ to $t = 1$ since $x'(t) < 0$ you are moving to the left and at time 1 you will arrive to $x(1) = 1^3 + -3 \times 1 + 2 = -1$.*
- *And from $t = 1$ to $t = 2$ since $x'(t) > 0$ you are moving to the right and at $t = 2$ you are at $x(2) = 2^3 - 3 \times 2 + 2 = 4$.*
- *Therefore, from $t = 0$ to $t = 1$ you traveled form the position $x = 2$ to $x = -1$ and form $t = 1$ to $t = 2$, you moved from $x = -1$ to $x = 4$, so in total you traveled 8 meters.*

Example 3.1.3. *Suppose that you are moving along the x-axis and that at time t your position is given by*

$$x(t) = t^3 - 12t + 5$$

Find how many meters you have traveled from $t = 0$ to $t = 10$.

Solution.

Consider that $v(t) = x'(t) = 3t^2 - 12$. it follows that

$$v(t) = 3(t^2 - 4) = 3(t - 2)(t + 2).$$

We can see that $v(t) = 0$ when $t = 2$ or $t = -2$, we have $v(t) = 0$, so at this two times you are not moving.

However,

- *For $t < 2$, $v(t) = 3(t - 2)(t + 2) > 0$, and so you are moving to the right.*
- *For $-2 < t < 2$, $v(t) = 3(t - 2)(t + 2) < 0$, so you are moving to the left.*
- *For $t > 2$, $v(t) > 0$ and you are moving to the right.*

- *At* $t = 0$, *you are at* $x(0) = 0^3 - 12 \times 0 + 5 = 5$ *to the right of the origin. Then from* $t = 0$ *to* $t = 2$, *you move to the left since your velocity is negative, and your position is*

$$x(2) = 2^3 - 12 \times 2 + 5 = -11$$

So from $t = 0$ *to* $t = 2$ *you have moved from* 5 *meter to the right of the origin, to* -11 *meter to the left of origin, so until* $t = 2$ *you have traveled* 16 *meters.*

- *From* $t > 2$ *since the velocity is positive, you are moving to the right, and your position at* $t = 10$ *is* $x(10) = 10^3 - 12 \times 10 + 5 = 885$. *So from* $t = 2$ *to* $t = 10$, *you have moved form* -11 *meters to the left of the origin to* 885 *meter to the right of the origin, so you have moved* $11 + 885$ *meter.*

Therefore, the total distance that you have traveled is 16 (from 0 to 2)+896(from 2 to 10)=912 meter.

In a nutshell we have

t	$(t-2)(t+2)$	$x'(t) = 3(t-2)(t+2)$	*Direction*
$t < -2$	*positive*	*positive*	*right*
$t = -2$	*zero*	*zero*	*halt*
$-2 < t < 2$	*negative*	*negative*	*left*
$t = 2$	*zero*	*zero*	*halt*
$t > 2$	*positive*	*positive*	*right*

t	*your position*$x(t)$	$x'(t)$	*Direction*
0	5	*negative*	*left*
$t = 2$	-11	*zero*	*halt*
$t = 10$	885	*positive*	*right*

Example 3.1.4. *In this example, we are going to figure out how far a body falling from rest will fall in a given time period.*

- *We should start by defining some variables and their units. Denote*
  - *time in seconds by* t,
  - *mass* [1] *in kilograms by* m,
  - *distance fallen (in meters) at time* t *by* $s(t)$, *velocity (in m/sec) by* $v(t) = s'(t)$ *and acceleration (in* m/sec^2*) by* $a(t) = v'(t) = s''(t)$.
- *We have also the following information*

(1) Newton's second law: the force applied to the body at time t $=m.a(t)$.

(2) the force due to gravity acting on a body of mass m $= m.g$.

[1] The difference between mass and weight is that mass is the amount of matter in a material, while weight is a measure of how the force of gravity acts upon that mass.

(3) The constant g, called the acceleration of gravity, is about $9.8\mathrm{m/sec}^2$

- *What result we can get from the above information: We now that the body fallen from rest, so the initial velocity is* 0*, thus*

$$v(0) = 0.$$

Moreover, the force from the Newton's second law is the same as the forced applied to body, so

$$m.a(t) = \mathrm{forceduetogravity}$$

$$m.v'(t) = m.g$$

$$v'(t) = g$$

also $g = 9.8$ *thus*

$$v'(t) = 9.8$$

We can now guess that

$$v(t) = 9.8t + c$$

- *To find* c *we now that* $v(0) = 0$*, this*

$$v(0) = 9.8t + c = 0 \Rightarrow c = 0.$$

Now we want to find $s(t)$*, we have that*

$$s'(t) = v(t) = 9.8t$$

We can guess that

$$s(t) = \frac{9.8}{2}t^2 + c = 4.9t^2 + c,$$

to find c *we now that distance fallen at* $t = 0$ *is* 0*, thus*

$$s(0) = 4.9 \times 0^2 + c = 0$$

and so $c = 0$*. Therefore,*

$$s(t) = 4.9t^2.$$

Example 3.1.5. *A car's brakes can decelerate the car at* $64000km/hr^2$*. How fast can the car be driven if it must be able to stop within a distance of* $50m$*?*

Solution. *First be very careful about the units since in this problem we have the distance by meter but the acceleration is by* km/hr^2*. So in this question we transfer all distances to km.*
What information we have:

1. *We first have that* $a(t) = -64000km/hr^2$*.*
2. $$a(t) = v'(t) = x''(t)$$
3. *We can choose a coordinate system such that* $x(0) = 0$ *and the car stars braking at time* $t = 0$*.*
4. *If we let* t_{stop} *be the time that after the brake the car stopes, then* $x(t_{stop})$ *the stopping distance is* 0.05 *km.*
5. *We want to determine the maximum initial velocity* $v(0)$*.*

By (1) we know that $a(t) = -64000$*, and by (2)* $v'(t) = -64000$*. Thus,*

$$v(t) = -64000t + c$$

consider that $v(0) = c$*. Therefore, we want to find the maximum amount for* c*. Since* $x'(t) = v(t)$*, therefore,*

$$x'(t) = -64000t + c \qquad \Rightarrow \qquad x(t) = -\frac{64000t}{2}t^2 + ct + d$$

where $x(0) = d$*. We already have set up our system in a way that* $x(0) = 0$*. Thus,* $d = 0$*, and so*

$$x(t) = -\frac{64000t}{2}t^2 + ct$$

Note that at $t = t_{stop}$ *we have*

$$0.05 = x(t_{stop}) = -\frac{64000}{2}t_{stop}^2 + ct_{stop} \qquad \text{and} \qquad 0 = v(t_{stop}) = -64000t_{stop} + c$$

The latter gives us $t_{stop} = \frac{c}{64000}$*. The by plugging in the former equation we have*

$$0.05 = -\frac{64000}{2}(\frac{c}{64000})^2 + c\frac{c}{64000}$$

Therefore,

$$0.05 = -\frac{c^2}{2 \times 64000} + \frac{c^2}{64000} = \frac{2c^2 - c^2}{2 \times 64000}.$$

Consequently,

$$c^2 = 6400.$$

and so $c = 80$*. Therefore the velocity* $v(0)$ *must be at most* 80 *km.*

3.2 Exponential Growth and Decay

A First Look at Differential Equations This is Section 3.3 of the textbook

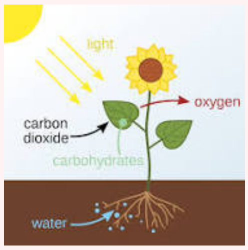
3.2.1 Carbon Dating

Cosmic rays hitting the atmosphere convert nitrogen into a radioactive isotope of carbon, C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide from the atmosphere through photosynthesis and animals acquire C by eating plants. When a plant or animal dies, it stops replacing its carbon and the amount of C begins to decrease through radioactive decay.

Cosmic ray hitting atmosphere

Nitrogen $(N) \to$ *Carbon* (C)

Vegetation absorbs C through photosynthesis



Animals acquire C by eating plants

C decreases when animal dies

More precisely, let $Q(t)$ denote the amount of C (an element) in the plant or animal t years after it dies. The number of radioactive decays (rate of change) per unit time, at time t, is proportional to the amount of C present at time t, which is $Q(t)$. Thus

Radioactive Decay

$$\frac{dQ}{dt}(t) = -kQ(t) \tag{3.2.1}$$

Now the question is which function $Q(t)$ satisfies the Radioactive Decay equation $\frac{dQ}{dt}(t) = -kQ(t)$? Consider that if

$$Q(t) = ce^{-kt}$$

where c is a constant, then we have

$$\frac{dQ}{dt}(t) = \frac{d}{dt}(ce^{-kt}) = -cke^{-kt} = -kQ(t).$$

Moreover, if $Q(t) = ce^{-kt}$, then $Q(0) = ce^{-k.0} = Q(0)$, so the equation becomes $Q(t) = Q(0)e^{-kt}$.

Therefore, $Q(t) = ce^{-kt}$ is one of the solutions to the equation above, but is it the only solution? we answer this question by claiming that $\frac{d}{dt}\left(\frac{Q(t)}{e^{-kt}}\right) = 0$ and thus we have $\frac{Q(t)}{e^{-kt}}$ is a constant c which means $Q(t) = ce^{-kt}$. Consider that

$$\frac{d}{dt}\left(\frac{Q(t)}{e^{-kt}}\right) = \frac{d}{dt}(e^{kt}Q(t)) = ke^{kt}Q(t) + e^{kt}Q'(t) = e^{kt}(kQ(t) + Q'(t)). \tag{3.2.2}$$

We already have $\frac{dQ}{dt}(t) = -kQ(t)$ which is equivalent to $Q'(t) + kQ(t) = 0$. Therefore, by looking at the equation 3.2.2, we have

$$\frac{d}{dt}\left(\frac{Q(t)}{e^{-kt}}\right) = 0$$

which proofs the claim.

Theorem 3.2.1. *A differentiable function $Q(t)$ obeys the differential equation*

$$\frac{dQ}{dt}(t) = -kQ(t)$$

if and only if there is a constant c such that

$$Q(t) = ce^{-kt}.$$

Corollary 3.2.2. *The function $Q(t)$ satisfies the equation*

$$\frac{dQ}{dt} = -kQ(t)$$

if and only if

$$Q(t) = Q(0).e^{-kt}$$

The half-life (the half-life of C is the length of time that it takes for half of the C to decay) is defined to be the time $t_{1/2}$ which obeys

$$Q(t_{1/2}) = \frac{1}{2}.Q(0).$$

The half-life is related to the constant k by

$$t_{1/2} = \frac{\ln 2}{k}.$$

Example 3.2.3. *The half-life of carbon is 5730 years. A particular piece of parchment contains about 64% as much C as plants to today. Estimate the age of the parchment.*

Solution. *Let $Q(t)$ denote the amount of C in the parchment t years after it was first created. We want to find a time t such that $Q(t) = 0.64Q(0)$. By the formula $t_{1/2} = \frac{\ln 2}{k}$ we have*

$$5730 = \frac{\ln 2}{k}.$$

Thus $k = \frac{\ln 2}{5730} = 0.000121$. Therefore, by the above corollary,

$$Q(t) = Q(0).e^{-kt}.$$

Since we want to find t such that $Q(t) = 0.64Q(0)$, thus the time t we are looking for is $0.64Q(0) = Q(0).e^{-kt}$. Therefore,

$$0.64 = e^{-kt},$$

so

$$-kt = \ln 0.64 \Rightarrow t = \frac{\ln 0.64}{-k} = \frac{\ln 0.64}{-0.000121} = 3700.$$

Example 3.2.4. *A scientist is studying a sample of the rare element implausium. With great effort he has produced a sample of pure implausium. The next day—17 hours later—he comes back to his lab and discovers that his sample is now only 37% pure. What is the half-life of the element.*

Solution. *Let $Q(t)$ denote the quantity of implausium at time t, measure in hours. Then we know that*

$$Q(t) = Q(0)e^{-kt} \Rightarrow Q(17) = Q(0)e^{-17k}$$

and since after 17 hours the quantity is 37% pure,

$$Q(17) = 0.37Q(0).$$

By the above two equation we have

$$Q(0)e^{-17k} = 0.37Q(0) \Rightarrow e^{-17k} = 0.37$$

and so

$$k = -\frac{\ln 0.37}{17} = 0.05849.$$

Therefore,

$$t_{1/2} = \frac{\ln 2}{k} = \frac{\ln 2}{0.05849} \approx 11.85.$$

3.2.2 Newtwon's Law of Cooling

Newton's Law of Cooling

The rate of change of temperature of an object is proportional to the difference in temperature between the object and its surroundings. The temperature of the surroundings is sometimes called the ambient temperature.

We translate this statement into the following differential equation.

Newton's Law of Cooling

$$\frac{dT}{dt}(t) = K\left[T(t) - A\right].$$

where $T(t)$ is the temperature of the object at time t, A is the temperature of its surroundings, and K is a constant of proportionality.

- This mathematical model of temperature change works well when studying a small object in a large, fixed temperature, environment. For example, a hot cup of coffee in a large room.

At any time there are three possibilities: base on these three situations we want to check that if K is positive, negative.

1. If $T(t) > A$, that is, if the body is warmer than its surroundings, we would expect heat to flow from the body into its surroundings and so we would expect the body to cool off so that $\frac{dT}{dt} < 0$. For this expectation since $T(A) - A > 0$ and $\frac{dT}{dt} < 0$ we need to have $K < 0$.

2. If $T(t) < A$, that is, the body is cooler than its surroundings, we would expect heat to flow from the surroundings into the body and so we would expect the body to warm up so that $\frac{dT}{dt}(t) > 0$. For this expectation since $T(A) - A < 0$ and $\frac{dT}{dt} > 0$, we need to have $K < 0$.

3. Finally if $T(t) = A$, that is, the body and its environment have the same temperature, we would not expect any heat to flow between the two and so we would expect that $\frac{dT}{dt}(t) = 0$. This does not impose any condition on K.

So how to solve the equation

$$\frac{dT}{dt}(t) = K\left[T(t) - A\right].$$

Let write $Q(t) = T(t) - A$. Then $T(t) = Q(t) + A$, and the equation becomes

$$\frac{d(Q(t) + A)}{dt} = KQ(t)$$

and so

$$\frac{Q(t)}{dt} = KQ(t).$$

This last equation is similar to the equation in 3.2.1, and so its solution is

$$Q(t) = ce^{Kt}$$

where c is a constant, and so by Corollary 3.2.2 we have

$$Q(t) = Q(0)e^{Kt}.$$

By substituting $Q(t) = T(t) - A$, we have

$$T(t) - A = (T(0) - A)e^{Kt}$$

thus

$$T(t) = (T(0) - A)e^{Kt} + A.$$

Newton's Law of Cooling

Corollary 3.2.5. *A differentiable function $T(t)$ obeys the differential equation*

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

if and only if

$$T(t) = [T(0) - A]e^{Kt} + A.$$

Example 3.2.6. *The temperature of a glass of iced tea is initially* 5°. *After* 5 *minutes, the tea has heated to* 10° *in a room where the air temperature is* 30°.

(a) *Determine the temperature as a function of time.*

(b) *What is the temperature after 10 minutes?*

(c) *Determine when tea will reach a temperature of* 20°.

Solution. *(a) We let* $T(t)$ *be the temperature of the tea* t *minutes after it was removed from the fridge, then the function of the temperature of the tea is*

$$T(t) = [T(0) - A]e^{Kt} + A$$

and since $A = 30$ *and* $T(0) = 5$, *the equation is*

$$T(t) = [5 - 30]e^{Kt} + 30 = -25e^{Kt} + 30.$$

However, still K *is unknown and we should find it. Consider that after* 5 *minutes the temperature of the tea is* 10°, *thus* $T(5) = 10$. *That is,*

$$10 = T(5) = -25e^{5K} + 30 \Rightarrow -20 = -25e^{5K} \Rightarrow 4/5 = e^{5K}$$

$$\Rightarrow 5K = \ln 4/5 \Rightarrow K = \frac{\ln 4/5}{5}.$$

Therefore the temperature at time t *is*

$$T(t) = -25e^{\frac{\ln 4/5}{5}t} + 30.$$

(b) The temperature after 10 *minutes is*

$$T(10) = -25e^{\frac{\ln 4/5}{5}10} + 30 =$$

$$-25e^{2\ln 4/5} + 30 = -25e^{\ln(4/5)^2} + 30 = -25 \times (4/5)^2 + 30 = 14$$

(c) The time that $T(t) = 20$, *should satisfies*

$$T(t) = -25e^{\frac{\ln 4/5}{5}t} + 30 = 20.$$

Thus

$$-10 = -25e^{\frac{\ln 4/5}{5}t} \Rightarrow 2/5 = e^{\frac{\ln 4/5}{5}t} \Rightarrow 2/5 = (4/5)^{(1/5)t} \Rightarrow t = 20.5$$

to 1 decimal place.

Example 3.2.7. *A dead body is discovered at 3:45pm in a room where the temperature is $20° C$. At that time the temperature of the body is $27° C$. Two hours later, at 5:45pm, the temperature of the body is $25.3° C$. What was the time of death? Note that the normal (adult human) body temperature is $37°$.*

Solution. *(a) We will assume that the body's temperature obeys Newton's law of cooling.*

- *Denote by $T(t)$ the temperature of the body at time t, with $t = 0$ corresponding to $3:45$pm. We want to find the time of death t_d. We can find the following data in the statement of the question*
  - *the ambient temperature is $20°$.*
  - *the temperature of the body when discovered: $T(0) = 27$*
  - *the temperature of the body two hours later: T(2)=25.3*
  - *the temperature at the time of death: $T(t_d) = 37$.*
- *by the Newton's law of cooling is*

$$T(t) = [T(0) - A]e^{Kt} + A = [27 - 20]e^{KT} + 20 = 20 + 7e^{Kt}.$$

Consider that $T(2) = 20 + 7e^{2K} = 25.3$, thus

$$5.3 = 7e^{2K} \Rightarrow 2K = \ln 5.3 \Rightarrow K = \frac{\ln(5.3/7)}{2} = -0.139.$$

We also have that $T(t_d) = 37$, therefore,

$$20 + 7e^{-0.139t_d} = 37.$$

Solving the equation, we get

$$7e^{-0.139t_d} = 17 \Rightarrow e^{-0.139t_d} = 17/7 \Rightarrow 0.139t_d$$

$$= \ln(17/7) \Rightarrow t_d = \frac{\ln(17/7)}{-0.139} = -6.38.$$

Since 6.38 hr is 6 hours and 22 minutes and 48 second, the time of death is 3:45 minus 6 hours and 22 minutes and 48 seconds which is 9:22:12.

Example 3.2.8. *A glass of room-temperature water is carried out onto a balcony from an apartment where the temperature is $22^\circ C$. After one minute the water has temperature $26^\circ C$ and after two minutes it has temperature $28^\circ C$. What is the outdoor temperature?*

Solution. *Let $T(t)$ be the temperature of the glass of water at time t. We have the following information from the statement of the problem*

- *The temperature of the water at the beginning is the same as the temperature of the room which is 22, thus $T(0) = 22$.*
- $T(1) = 26$
- $T(2) = 28$
- *let A be the ambient temperature, and we want to find A.*

Consider that the temperature of the glass of water satisfies the Newton's law of cooling, so

$$T(t) = [T(0) - A]e^{Kt} + A$$

where $T(0) = 22$. We have

$$T(1) = 26 = [22-A]e^{K}+A \qquad T(2) = 28 = [22-A]e^{2K}+A = [22-A](e^{K})^2+A$$

Therefore, from the former equation, we have

$$26 - A = (22 - A)e^{K} \Rightarrow e^{K} = \frac{26 - A}{22 - A}.$$

Therefore, from the latter equation and knowing that $e^{K} = \frac{26-A}{22-A}$, we have

$$28 = (22 - A)(\frac{26 - A}{22 - A})^2 + A.$$

Therefore,

$$\frac{28 - A}{22 - A} = \left(\frac{26 - A}{22 - A}\right)^2$$

Multiplying both sides by $(22 - A)^2$,

$$(28 - A)(22 - A) = (26 - A)^2 \Rightarrow 616 - 50A + A^2 = A^2 - 52A + 676$$

consequently, $60 = 2A$, and so $A = 30$.

3.2.3 Population Growth

Suppose that we wish to predict the size $P(t)$ of a population as a function of the time t. So suppose that in average each couple produces β offspring (for some constant β) and then dies. Then over the course of one generation since we have $P(t)/2$ couples and each have produced β offspring, thus the population of the children of one generation is

$$\beta \frac{P(t)}{2}.$$

Let t_g be the life span of one generation, then

$$P(t+t_g) = \beta \frac{P(t)}{2} = P(t) + \beta \frac{P(t)}{2} - P(t).$$

Therefore,

$$P(t+t_g) - P(t) = \beta \frac{P(t)}{2} - P(t)$$

and so dividing both sides by t_g, we have

$$\begin{aligned} \frac{P(t+t_g) - P(t)}{t_g} &= \frac{1}{t_g}\left(\frac{\beta}{2}P(t) - P(t)\right) \\ &= \frac{1}{t_g}\left(\frac{\beta}{2} - 1\right) P(t) \end{aligned}$$

Let $\frac{1}{t_g}\left(\frac{\beta}{2} - 1\right) = b$, then

$$\frac{P(t+t_g) - P(t)}{t_g} = bP(t).$$

Approximately, we have

$$\frac{dP}{dt} = bP(t).$$

Moreover, same as the model for carbon dating we can write

$$P(t) = P(0)e^{bt}.$$

Therefore we have the following model for population growth.

Malthusian growth model

The model for the population growth is

$$\frac{dP}{dt} = bP(t)$$

and $P(t)$ satisfies the above equation if and only if

$$P(t) = P(0)e^{bt}.$$

Example 3.2.9. *In 1927 the population of the world was about 2 billion. In 1974 it was about 4 billion. Estimate when it reached* 6 *bilion. What will the population of the world be in 2100, assuming the Malthusian growth model?*

Solution. *Let $P(t)$ be the world's population t years after 1927. Note that 1974 corresponded to* $t = 1974 - 27 = 47$.

- *the model that we have for the population is* $P(t) = P(0)e^{bt}$
- $P(0) = 2$ *and* $P(47) = 4$ *(*$1974 - 1927 = 74$*).*
- *we want to find* $P(173)$ *since* $2100 - 1927 = 173$.

The model becomes

$$P(t) = 2e^{bt}$$

and

$$P(47) = 4 = 2e^{47b}.$$

thus

$$2 = e^{47b} \Rightarrow \ln 2 = 47b \Rightarrow 47 = \frac{\ln 2}{b} \Rightarrow b = \frac{\ln 2}{47}.$$

Consequently the model is

$$P(t) = 2e^{\frac{\ln 2}{47}t}$$

• *To find out when the population reaches* 6 *billion, we should find t when* $P(t) = 2e^{\frac{\ln 2}{47}t} = 6$. *This gives*

$$e^{\frac{\ln 2}{47}t} = 3 \Rightarrow \frac{\ln 2}{47}t = \ln 3 \Rightarrow \frac{47 \times \ln 3}{\ln 2} = 74.5$$

So the time that the population is 6 *billion is* $1927 + 74.5 = 2010.5$.
• *Also*

$$P(173) = 2e^{\frac{\ln 2}{47}173} \text{ billion}$$

3.3 Related rates

This is section 3.2 of the textbook

Volume of a sphere

Remember that the volume of a sphere with radius r is

$$V = \frac{4}{3}\pi r^3.$$

Example 3.3.1. *A spherical balloon is being inflated at a rate of* $13cm^3/sec$. *How fast is the radius changing when the ballon has radius* $15cm$*?*

Solution. *The information that we can get from the statement of the problem are*

- *the balloon is spherical and so its volume is*

$$V = \frac{4}{3}\pi r(t)^3.$$

- $\frac{dV}{dt} = 13$
- *we want to find the rate of change of the radius* $\frac{dr}{dt}$ *when* $r = 15$.

We have that

$$\frac{dV}{dt} = 13 = 4\pi \frac{dr}{dt} r(t)^2$$

Since we want to compute $\frac{dr}{dt}$ *when* $r = 15$, *so by plugging in the above formula we have*

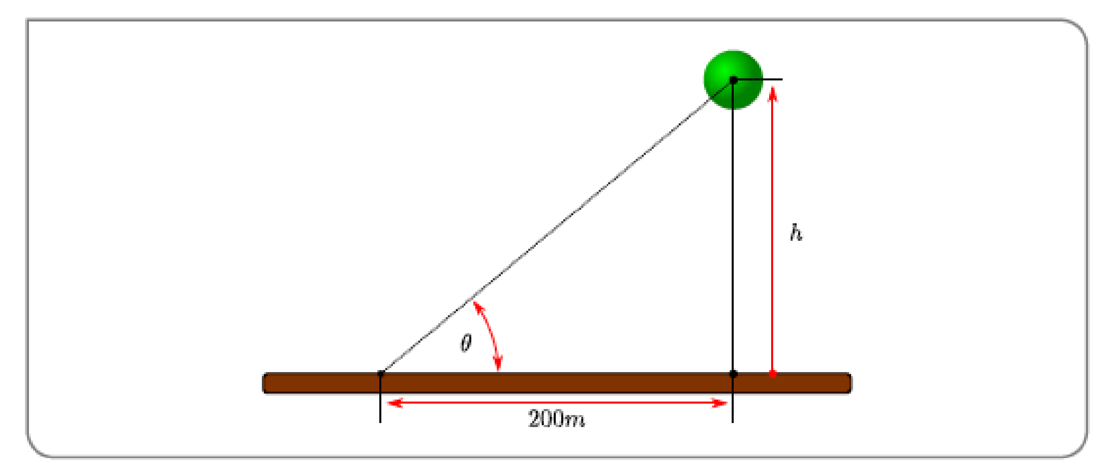
$$13 = 4\pi \frac{dr}{dt} 15^2.$$

Then

$$\frac{dr}{dt} = \frac{13}{4\pi \times 15^2}.$$

Example 3.3.2. *Consider a helium balloon rising vertically from a fixed point 200m away from you. You are trying to work out how fast it is rising. You observe that when it is at an angle of $\pi/4$ its angle is changing by 0.05 radians per second.*

Solution.



Denote the angle to be θ (in radians), the height of the balloon (in m) by h and time (in second) by t. The trigonometry tells us

$$h = 200.\tan\theta$$

and so

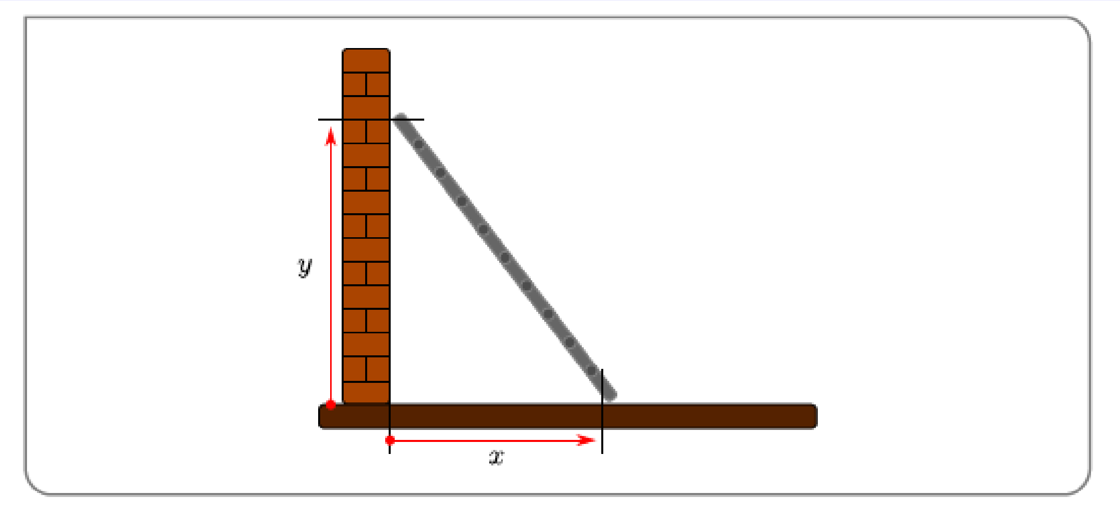
$$\frac{dh}{d\theta} = 200.\sec^2(\theta)\frac{d\theta}{dt}$$

We have also that $\frac{d\theta}{dt} = 0.05$ when $\theta = 0.05$. Therefore,

$$\frac{dh}{d\theta} = 200.\sec^2(\pi/4) \times 0.05 = 200 \times 0.05 \times \sqrt{2}^2 = 20m/s.$$

Example 3.3.3. *A 5 m ladder is leaning against a wall. The floor is quite slippery and the base of the latter slides out from the wall at a rate of $1m/s$. How fast is the top of the ladder sliding down the wall when the base of the ladder is $3m$ from the wall.*

Solution.



Consider that the length of the ladder is 5, and so

$$x^2 + y^2 = 25.$$

As in the statement the base of the latter sides out form the wall at a rate of $1m/s$, thus $\frac{dx}{dt} = 1m/s$. Now we want to see how fast ($\frac{dy}{dt}$) is the top of the latter sliding down the wall when the base of the ladder is $3m$ from the wall ($x = 3$). We need to find y when $x = 3$, since we have

$$x^2 + y^2 = 25$$

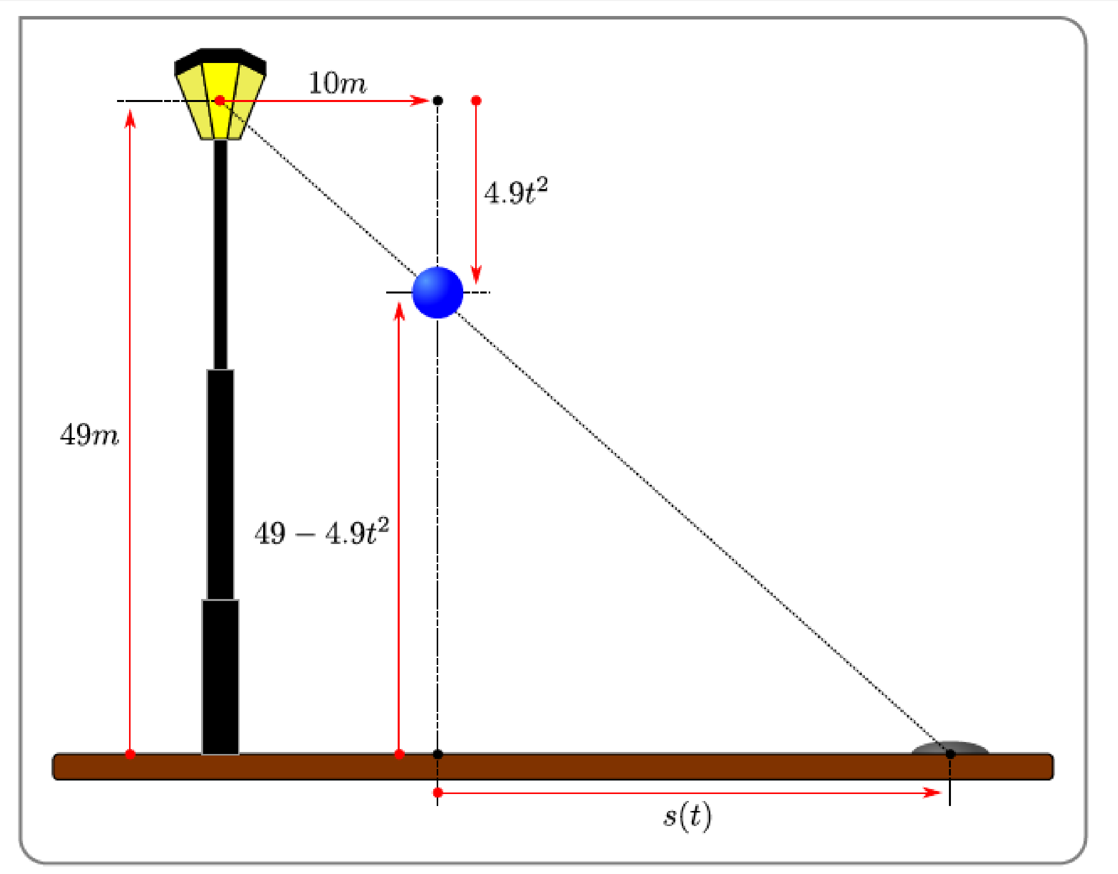
we have

$$3^2 + y^2 = 25 \Rightarrow y = 4.$$

Also,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow 2 \times 3 \times 1 = -2 \times 4 \times \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{4}{3}.$$

Example 3.3.4. *A ball is dropped from a height of* $49m$ *above level ground. The height of the ball at time* t *is* $h(t) = 49 - 4.9t^2 m$*. A light, which is also* $49m$ *above the ground, is* $10m$ *to the left of the ball's original position. As the ball descends, the shadow of the ball caused by the light moves across the ground. How fast is the shadow moving one second after the ball is dropped?*



Solution. *Let* $s(t)$ *be the distance from the shadow to the point on the ground directly underneath the ball. By similar triangles we have*

$$\frac{4.9t^2}{10} = \frac{49 - 4.9t^2}{s(t)}.$$

Therefore,

$$s(t) = \frac{10(49 - 4.9t^2)}{4.9t^2}$$

and so

$$s(t) = \frac{100}{t^2} - 10.$$

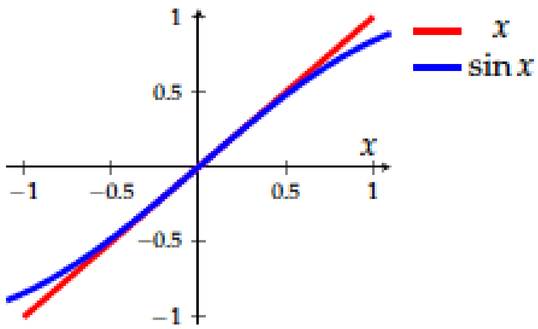
We have

$$s'(t) = -2\frac{100}{t^3}.$$

Consequently, $s'(1) = -200m/sec$.

3.4 Approximation functions near specific points-Tylor Polynomial

Consider the following figure.



This figure shows that the curve $y = x$ and $y = \sin(x)$ are almost the same when x is close to 0. Hence if we want the value of $\sin(1/10)$ we just use this approximation $y = x$ to get

$$\sin(1/10) \approx 1/10.$$

Approximating function

Given a function $f(x)$ that we wish to approximate close to some point $x = a$, and we need to find another function $F(x)$ (called approximating function) that

- is simple and easy to compute
- is a good approximation to $f(x)$ for x values close to a.
- Further, we need to be able to estimate the error $|f(x) - F(x)|$.

3.4.1 First Approximation-Linear Approximation

Linear approximation

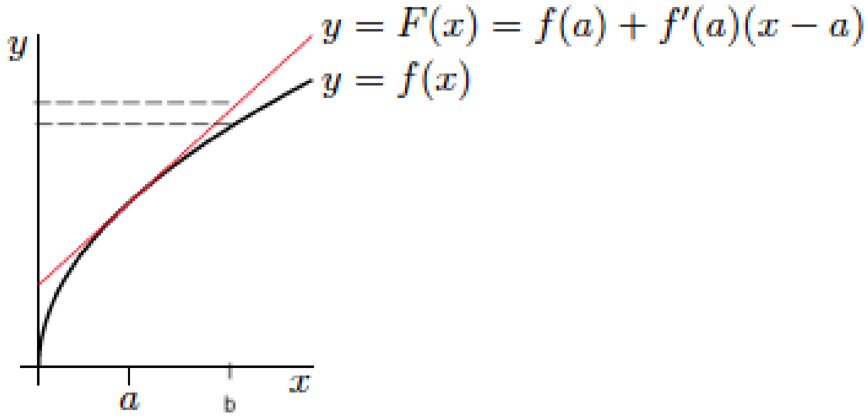
Given a function $f(x)$ we want to have the approximating function to be a linear function that is $F(x) = A + Bx$ for some constants A and B.

So to approximate $f(x)$ we find $F(x) = f(a) + f'(a)(x - a)$.

$$f(x) \approx f(a) + f'(a)(x - a)$$

If we want to approximate $f(b)$, **how do we find a?** the number a is a number such that

- it is close to b, and
- we can compute $f(a)$ and $f'(a)$.



Remark. Note that $F(a) = f(a)$ and $F'(a) = f'(a)$.

Example 3.4.1. *Use the linear approximation to estimate $e^{0.1}$.*

Solution. *Let $f(x) = e^x$. We want to have a linear approximation of $e^{0.1}$. Let $a = 0$. Then*

$$f(0) = 1 \qquad f'(0) = 1$$

So $F(x) = f(a) + f'(a)(x - a)$. Therefore,

$$e^{0.1} = f(0.1) \approx f(0) + f'(0)(0.1 - 0) = 1 + 0.1 = 1.01.$$

Remark. Consider that $e^{0.1} = 1.105170918...$ and so the linear approximation is almost correct to 3 digits.

Example 3.4.2. *Use a linear approximation to estimate $\sqrt{4.1}$.*

Solution. *Let $f(x) = \sqrt{x}$ and $a = 4$. Then*

$$f(4) = 2 \qquad\qquad f'(4) = \frac{1}{4}.$$

So

$$F(x) = f(a) + f'(a)(x-a) = 2 + \frac{1}{4}(x-4)$$

And so

$$\sqrt{4.1} = f(4.1) \approx F(4.1) = 2 + \frac{1}{4}(4.1-4) = 2.025$$

Remark. Consider that $\sqrt{4.1} = 2.024845673....$

3.4.2 Second approximation–the Quadratic Approximation

We now want our approximation function to be a quadratic function of x, that is, $F(x) = A + Bx + Cx^2$. To have a good approximating function we choose A, B, and C so that

- $f(a) = F(a)$
- $f'(a) = F'(a)$
- $f''(a) = F''(a)$

These conditions give us the following equations

$$F(x) = A + Bx + Cx^2 \quad \Rightarrow \quad F(a) = A + Ba + Ca^2 = f(a)$$

$$F'(x) = B + 2Cx \quad \Rightarrow \quad F'(a) = B + 2Ca = f'(a)$$

$$F''(x) = 2C \quad \Rightarrow \quad F'(a) = 2C = f''(a)$$

Solving these equation we can write A, B, and C in terms of $f(a)$, $f'(a)$, and $f''(a)$. So that

$$C = \frac{1}{2}f''(a)$$

$$B = f'(a) - af''(a)$$

$$A = f(a) - a[f'(a) - af''(a)] - \frac{1}{2}f''(a)a^2.$$

Consider that $F(x) = A + Bx + CX^2$, substituting A, B, and C, we obtain

Quadratic Approximation

$$F(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

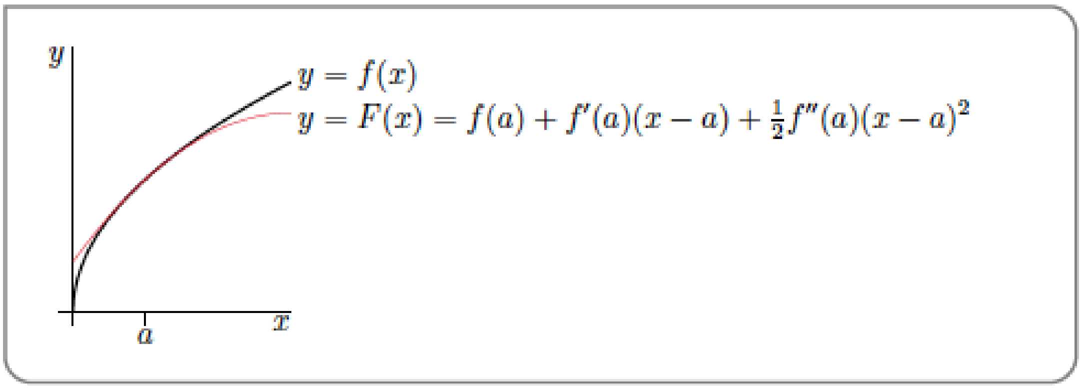
Therefore,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

How to estimate $f(b)$ using quadratic approximation:

We first find a such that

- it is close to b, and
- we can compute $f(a), f'(a)$, and $f''(a)$.



Example 3.4.3. *Use quadratic approximation to estimate $e^{0.1}$.*

Solution. *Set $f(x) = e^x$ and $a = 0$. Then*

$$\begin{aligned} f(x) &= e^x & f(0) &= 1 \\ f'(x) &= e^x & f'(0) &= 1 \\ f''(x) &= e^x & f''(0) &= 1 \end{aligned}$$

Therefore our quadratic approximating function is

$$F(x) = 1 + (x-0) + \frac{1}{2}(x-0)^2 = 1 + x + \frac{1}{2}x^2.$$

And

$$F(0.1) = 1.105$$

Recall that $e^{0.1} = 1.105170918...$, so the quadratic approximation is quite accurate with very little effort.

➤➤ Whirlwind Tour of Summation Notation

Assume that we need the sum of the first 11 squares:

$$1 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2$$

This becomes tedious. So we often skip the middle few terms and instead we write

$$1 + 2^2 + \cdots + 11^2.$$

More precisely we can write the above sum as

$$\sum_{k=1}^{11} k^2.$$

Let $m \le n$ be integers and let $f(x)$ be a function defined on the integers. Then we write

$$\sum_{k=m}^{n} f(k)$$

to mean the sum of $f(k)$ for k from m to n:

$$f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

Similarly we write

$$\sum_{i=m}^{n} a_i$$

to mean

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

for some set of coefficients $\{a_m, \ldots, a_n\}$.

Example 3.4.4.

- $$\sum_{k=3}^{7} \frac{1}{k^2} = \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}.$$

- $$\sum_{k=3}^{7} \frac{1}{k^2} = \sum_{i=3}^{7} \frac{1}{i^2} = \sum_{j=3}^{7} \frac{1}{j^2}$$

Factorial

Let n be a positive integer, then n-factorial, denoted $n!$, is the product

$$n! = n \times (n-1) \times \cdots \times 3 \times 2 \times 1$$

Furthermore, we use the convention that

$$0! = 1$$

The first few factorials are

$$1! = 1 \qquad 2! = 2 \qquad 3! = 6$$

$$4! = 24 \qquad 5! = 120 \qquad 6! = 720$$

3.5 Still Better Approximations–Taylor Polynomials

Let go back to linear and quadratic approximations.

• What we did in linear approximation. We start with a function $f(x)$ and we wanted to approximate this function by a function $F(x) = c_0 + c_1(x-a)$ such that

$$F(a) = f(a) \qquad F'(a) = f'(a).$$

Then

$$c_0 = f(a) \qquad c_1 = f'(a).$$

And so

$$F(x) = f(a) + f'(a)(x-a).$$

• What we did in quadratic approximation. We start with a function $f(x)$ and we wanted to approximate this function by a function $F(x) = c_0 + c_1(x-a) + c_2(x-a)^2$ such that

$$F(a) = f(a) \qquad F'(a) = f'(a) \qquad F''(a) = f''(a).$$

Then

$$c_0 = f(a) \qquad c_1 = f'(a) \qquad c_2 = \frac{1}{2}f''(a).$$

And so

$$F(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

•• **Taylor Polynomial.** We want to approximate $f(x)$ with a polynomial $T_n(x)$ of degree n of the form

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

such that

$$T_n(a) = f(a), \quad T_n'(a) = f'(a), \cdots, T_n^{(n)}(a) = f^{(n)}(a).$$

Note that

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

$$T_n''(x) = 2c_2 + 3\times 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2} \Rightarrow T_n''(a) = 2c_2 = f''(a)$$

$$T_n^{(3)}(x) = 3\times 2c_3 + 4\times 3\times 2c_4(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2} \Rightarrow T_n^{(3)}(a) = 6c_3 = f^{(3)}(a)$$

$$\vdots$$

$$T_n^{(n)}(x) = n!c_n \Rightarrow T_n^{(n)}(a) = n!c_n$$

Therefore,

$$c_0 = f(a) \qquad c_1 = f'(a) \qquad c_2 = \frac{1}{2!}f''(a) \qquad c_3 = \frac{1}{3!}f^{(3)}(a), \cdots, c_n = \frac{1}{n!}f^{(n)}(a).$$

Since

$$T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

we have that

$$f(x) \approx T_n(x) =$$

$$f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a) + \frac{1}{3!}f^{(3)}(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Taylor Polynomial

Let a be a constant and let n be a non-negative integer. The nth degree Taylor polynomial for $f(x)$ about $x = a$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f^{(3)}(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(a)(x-a)^k$$

The special case $a = 0$ is called a Maclaurin polynomial.

Example 3.5.1. *The first few Maclaurian polynomial of $f(x) = e^x$ are*

$$T_0(x) = 1 \qquad T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2}.$$

Moreover, since

$$f^{(n)}(x) = e^x \quad f^{(n)}(x) = 1 \qquad\qquad n = 0, 1, 2, \dots$$

we have

$$T_n(x) = \sum_{k=0}^{n} f^{(k)}(0)\frac{x^k}{k!} = \sum_{k=0}^{n} \frac{1}{k!}x^k.$$

Consider that

$$T_7(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}.$$

So

$$e^1 \approx T_7(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} = 2.718253968...$$

The true value of e is 2.718281828... so the approximation has an error of about 3×10^{-5}.

Example 3.5.2. *Find the first Taylor polynomial of $f(x) = x^{3/2}$ about $x = 9$.*

Solution. *To find the first Taylor polynomial we compute the following.*

$$f(x) = x^{3/2} \qquad\qquad f(9) = 27$$

$$f'(x) = \frac{3}{2}x^{1/2} \qquad\qquad f'(9) = \frac{9}{2}$$

Therefore,

$$T_1(x) = f(a) + f'(a)(x - a) = 3 + \frac{9}{2}(x - 9).$$

Example 3.5.3. *Compute the* $5th$ *Taylor polynomial for* $\ln x$ *about* $x = 1$. *(Note that* $\ln$ *is not defined at* $x = 0$ *so we can compute Maclaurin series.)*

Solution. *By the formula for the Taylor polynomial we should compute the following.*

$$
\begin{aligned}
f(x) &= \ln x & f(1) &= \ln 1 = 0 \\
f'(x) &= \frac{1}{x} & f'(1) &= \frac{1}{1} = 1 \\
f''(x) &= \frac{-1}{x^2} & f''(1) &= -1 \\
f^{(3)} &= \frac{2}{x^3} & f^{(3)}(1) &= 2 \\
f^{(4)}(x) &= \frac{-6}{x^4} & f^{(4)}(1) &= -6 \\
f^{(5)}(x) &= \frac{24}{x^5} & f^{(5)}(1) &= 24
\end{aligned}
$$

Therefore,

$$T_5(x) = 0+1.(x-1)+\frac{1}{2}.(-1).(x-1)^2+\frac{1}{6}.2.(x-1)^3+\frac{1}{24}.(-6).(x-1)^4+\frac{1}{120}.24.(x-1)^5 =$$

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5.$$

With a little work one can show that

$$T_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}(x-1)^k.$$

Example 3.5.4. *Find the 4th degree Maclaurin polynomial for* $\cos(x)$

Solution. *Since we want to find Maclaurin polynomial, we have* $a = 0$. *Also,*

$$f(x) = \cos(x) \qquad f(0) = 1$$

$$f'(x) = -\sin(x) \qquad f'(0) = 0$$

$$f''(x) = -\cos(x) \qquad f''(0) = -1$$

$$f'''(x) = \sin(x) \qquad f'''(0) = 0$$

$$f^{(4)}(x) = \cos(x) \qquad f^{(4)}(0) = 1$$

Substituting these in the equation for the Maclaurin polynomial gives

$$T_4(x) = 1 + 1.(0).x + \frac{1}{2}.(-1)x^2 + \frac{1}{6}.0.x^3 + \frac{1}{24}.(1).x^4$$

$$1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Moreover, we have

$$f^{(4)}(x) = \cos(x) \qquad f^{(4)}(0) = 1$$

$$f^{(5)}(x) = -\sin(x) \qquad f^{(5)}(0) = 0$$

$$f^{(6)}(x) = -\cos(x) \qquad f^{(6)}(0) = -1$$

$$f^{(7)}(x) = \sin(x) \qquad f^{(7)}(0) = 0$$

$$f^{(8)}(x) = \cos(x) \qquad f^{(8)}(0) = 1$$

Thus the 8th degree Maclaurin series is

$$T_8(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

Computing 2nth degree Maclaurin polynomial, we have

$$T_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k!)}.x^{2k}.$$

Example 3.5.5. *Find the 5th degree Maclaurin polynomial for* $\sin(x)$

Solution. *Let* $g(x) = \sin(x)$. *We have*

$$g(0) = 0, g'(0) = 1, g''(0) = 0, g'''(0) = -1, g^{(4)}(0) = 0, g^{(5)} = 1.$$

Hence

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

The $(2n+1)th$ *Maclaurin polynomial for* $\sin(x)$ *is*

$$T_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Remark. If $|x| \leq 1$ radians, then the magnitudes of the successive terms in the Taylor polynomial for $\sin(x)$ are bounded by

$$|x| \leq 1 \qquad \frac{1}{3!}|x|^3 \leq 1/6 \qquad \frac{1}{5!}|x|^5 \leq \frac{1}{120} \approx 0.0083$$

$$\frac{1}{7!}|x|^7 \leq \frac{1}{7!} \approx 0.0002 \qquad \frac{1}{9!}|x|^9 \leq \frac{1}{9!} \approx 0.000003 \qquad \frac{1}{11!}|x|^{11} \leq \frac{1}{11!} \approx 0.000000025$$

From these inequalities, it certainly looks like, for x not too large, even relatively low degree Taylor polynomials give very good approximations.

3.6 The Error in the Taylor Polynomial Approximations

When we approximate a function $f(x)$ by $F(x)$, the error is

$$error = R(x) = f(x) - F(x)$$

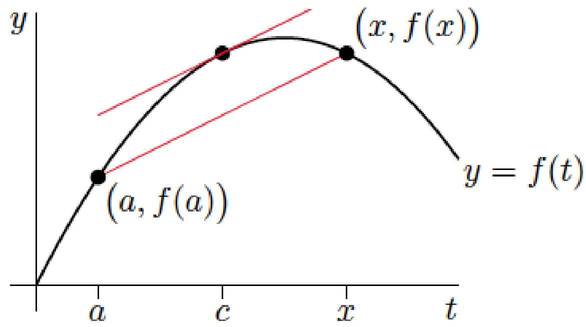
That is the difference between the function $f(x)$ and approximating function $F(x)$. It is not realistic to exactly find $R(x)$ since then $f(x) = F(x) + R(x)$ so we would like to find some relatively small M such that $|R(x)| = |f(x) - F(x)| \leq M$.

➤ We want to approximate the function $f(x)$ by the $0th$ Taylor polynomial about $x = a$ i.e., $f(a)$.

$$f(x) \approx T_0(x) = f(a).$$

Consider that

$$\begin{aligned} f(x) & \\ &= f(x) + f(a) - f(a) \\ &= f(a) + (f(x) - f(a))\frac{(x-a)}{(x-a)} \\ &= f(a) + \frac{f(x) - f(a)}{x-a}(x-a) \end{aligned} \tag{3.6.1}$$



This function is sometimes positive and sometimes negative, so by MVT there is c strictly between x and a such that

$$f'(c) = \frac{f(x) - f(a)}{x - a}.$$

Therefore,

$$f(x) = T_0(x) + f'(c)(x - a)$$

or equivalently,

The error in constant approximation

$$R_0(x) = f(x) - T_0(x) = f'(c)(x - a) \qquad \text{for some } c \text{ strictly between } a \text{ and } x$$

The error in linear approximation

$$R_1(x) = f(x) - T_1(x) = \frac{1}{2}f''(c)(x - a)^2 \qquad \text{for some } c \text{ strictly between } a \text{ and } x$$

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{n+1} \text{ for some } c \text{ strictly between } a \text{ and } x$$

Remark

Consider that

$$f(x) = R_n(x) + T_n(x)$$

Therefore,

1. if $0 \leq R_n(x) \leq E$, then

$$T_n(x) \leq f(x) \leq T_n(x) + E.$$

2. if $E \leq R_n(x) \leq 0$, then

$$T_n(x) + E \leq f(x) \leq T_n(x).$$

Accurate to D decimal places

Generally we say that our estimate is "accurate to D decimal places" when

$$|error| =< 0.5 \times 10^{-D}.$$

Example 3.6.1. *How well does the third Taylor polynomial for* $\ln(x)$ *about* $x = 1$ *estimate* $\ln(2)$. *(This asks find* M *such that* $|R_3(2)| < M$ *and then say the approximation is accurate to* D *decimal points).*

Solution. *We have*

$$f(x) = \ln(x) \qquad f(1) = 0$$

$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$

$$f''(x) = \frac{-1}{x^2} \qquad f''(1) = -1$$

$$f^{(3)}(x) = \frac{2}{x^3} \qquad f'''(1) = 2$$

$$f^{(4)}(x) = \frac{-6}{x^4}$$

$$T_3(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3.$$

So

$$T(2) = 1 - \frac{1}{2} + \frac{2}{3!} = \frac{5}{6}.$$

By Lagrange remainder theorem we have

$$R_3(2) = \frac{1}{4!}f^{(4)}(c)(x-1)^4 = \frac{1}{4!}f^{(4)}(c) = \frac{1}{4!}\frac{-6}{c^4}$$

for some $1 < c < 2$. *When* $1 < c < 2$, *we have that* $|\frac{-6}{c^4}| \leq 6$. *Therefore,*

$$|R_3(2)| = |\frac{1}{4!}\frac{-6}{c^4}| \leq \frac{1}{4!}6 = \frac{1}{4}.$$

So,

$$|R_3(2)| \leq \frac{1}{4} = 0.25 < 0.5 \times 10^{-0}$$

so it is accurate to 0 *decimal points.*
Moreover,

$$|f(2) - T_3(2)| = |R_3(2)| \leq \frac{1}{4} \Rightarrow |f(2) - \frac{5}{6}| \leq \frac{1}{4} \Rightarrow \frac{5}{6} - \frac{1}{4} \leq \ln(2) \leq \frac{5}{6} + \frac{1}{4}$$

Example 3.6.2. *Let* $f(x) = x^{3/2}$.

1. *Estimate* $f(9.1)$ *using the first Taylor polynomial about* $a = 9$ *as the approximating function.*
2. *Using Lagrange theorem find* E *(as small as possible) such that* $R_1(x) \le E$.
3. *Show that*
$$27.45 \le (9.1)^{3/2} \le 27.45 + 0.00125$$

Solution. *(1) To find the first Taylor polynomial we compute the following.*

$$f(x) = x^{3/2} \qquad\qquad f(9) = 27$$

$$f'(x) = \frac{3}{2}x^{1/2} \qquad\qquad f'(9) = \frac{9}{2}$$

$$f''(x) = \frac{3}{4\sqrt{x}}.$$

Therefore,

$$T_1(x) = f(a) + f'(a)(x-a) = 3 + \frac{9}{2}(x-9).$$

So

$$f(9.1) \approx T_1(9.1) = 3 + \frac{9}{2}(9.1-9) = 27 + \frac{9}{2} \times 0.1 = 27.45.$$

(2) Using Lagrange theorem we have

$$R_1(x) = \frac{1}{2}f''(c)(x-a)^2 \qquad \Rightarrow \qquad R_1(9.1) = \frac{1}{2}f''(c)(9.1-9)^2$$

for some c *strictly between* 9.1 *and* 9*. We have that*

$$f''(x) = \frac{3}{4}\frac{1}{\sqrt{x}} \qquad \Rightarrow \qquad f''(c) = \frac{3}{4}\frac{1}{\sqrt{c}}.$$

Since $9 < c < 9.1$ *we have*

$$f''(c) = \frac{3}{4}\frac{1}{\sqrt{c}} \le \frac{3}{4}\frac{1}{\sqrt{9}} = \frac{1}{4}.$$

Therefore,

$$R_1(9.1) = \frac{1}{2}f''(c)(9.1-9)^2 \le \frac{1}{2}\frac{1}{4}(0.1)^2 = \frac{1}{800} = 0.00125.$$

Thus,

$$E = 0.00125.$$

(3) Since $R_1(9.1) = \frac{1}{2}f''(c)(9.1-9)^2 > 0$ *is positive, by the above Remark, we have*

$$T_1(9.1) \le f(9.1) \le T_1(9.1) + E \Rightarrow 27.45 \le (9.1)^{3/2} \le 27.45 + 0.00125$$

(actually they differ by 0.001247695). Notice that the first 2 decimal places are correct.

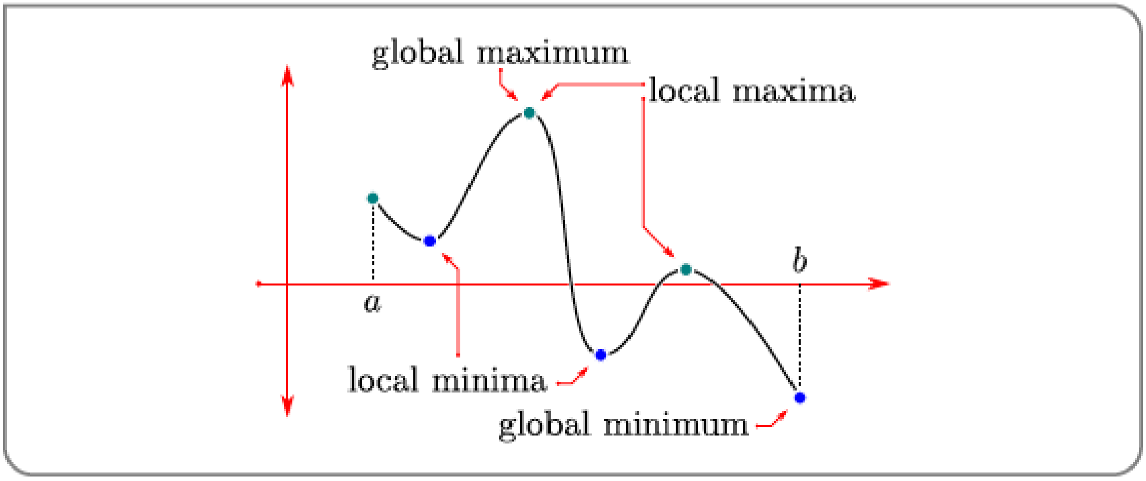
3.7 Optimization (section 3.5 in CLP)

3.7.1 Maximum and minimum values (section 3.5.1 in CLP)

Global maximum and global minimum (CLP 3.5.3)

Let f be a function with domain D.

- We say that f has a global maximum at a point c if $f(c) \geq f(x)$ for all $x \in D$. The value $f(c)$ is called the maximum value of f.
- Similarly we say that f has a global minimum at a point c if $f(c) \leq f(x)$ for all $x \in D$. And the value $f(c)$ is called the minimum value of f.
- The maximum and minimum values of f are called extreme values of f.
- Global max/min are sometimes called "absolute" max/min.

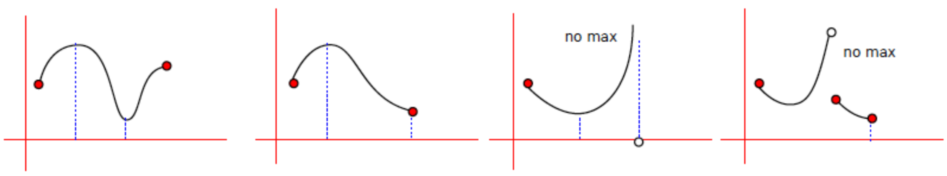


Local maximum and local minimum (CLP 3.5.3)

A function f has a local maximum at c if there are elements x in domain close to c form the left and right and $f(c) \geq f(x)$ for all x "close to" c. Similarly, f has a local minimum at c if $f(c) \leq f(x)$ for x near c.

Theorem: continuity and global max/min (CLP 3.5.10)

If $f(x)$ is continuous on the closed interval $[a, b]$ then f has an global maximum value $f(c)$ and a global minimum value $f(d)$ for some $c, d \in [a, b]$.



Definition: Critical and singular points (CLP 3.5.5)

Let $f(x)$ be a function and let c be a number in its domain.

- If $f'(c)$ exists and is equal to zero, then $x = c$ is a critical point of the function.
- If $f'(c)$ does not exist then $x = c$ is a singular point of the function.

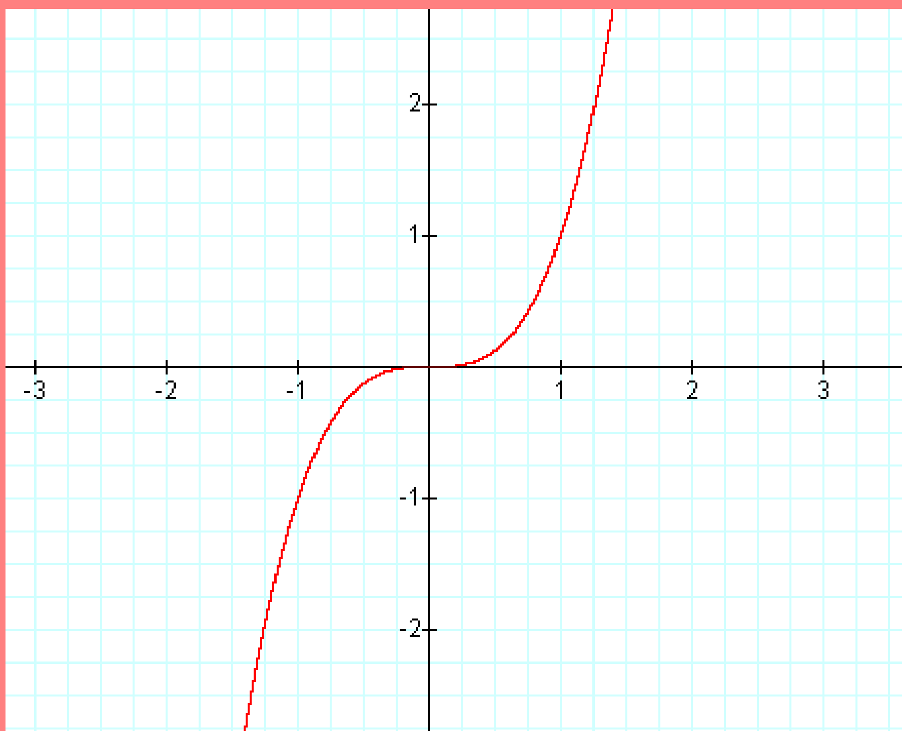
Caveat. Some books lump both these cases into “critical point”.

Fermat’s Theorem

Let f have a local maximum or minimum at $x = c$. If $f'(c)$ exists, the $f'(x) = 0$.

So Question: If $f'(c) = 0$ for some c, then c is a local maximum or minimum?! **No!** because

Example 3.7.1. *$f(x) = x^3$, then $f'(x) = 3x^2$ is zero if $x = 0$, but $x = 0$ is not a local maximum or minimum.*



Homework 1:

Go to this link https://www.mooculus.osu.edu/textbook/mooculus.pdf and download the book "MOOCULUS". Then do the following questions:

- all questions in page 35;
- in page 33 see why $\lim_{x\to 0} \frac{sin(x)}{x} = 1$. Then do Questions 1-8 page 38;
- in page 42, do questions 1-10.

Homework 2:

- First set of questions (CLICK) by Prof. DRAGOS GHIOCA.
- Second set of questions (CLICK) by Prof. DRAGOS GHIOCA.

Bibliography

[1] CLP1: Differential Calculus by J. Feldman, A. Rechnitzer, and E. Yeager.

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