

# MATH100: Differential Calculus with Application to Physical Sciences and Engineering

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# Chapter 1

## Limits

What does this mean

$$\lim_{x \rightarrow a} f(x) = L?$$

The "limit" appears when we want to

- find the tangent to a curve; or
- find the velocity of an object.

### 1.1 Tangent line



The **tangent line to a curve**  $y = f(x)$  at a point  $P$  (if exists) is a line  $L$  that there is a neighborhood for  $P$  such that in that neighborhood the line  $L$  touches (does not cross) the curve  $y = f(x)$  only at  $P$  (and not other points in that neighborhood).

## The equation of a line

- The formula for a line that passes through  $(x_1, y_1)$  with slope  $m$  is

$$y = y_1 + m(x - x_1).$$

- Given two points  $(x_1, y_1)$  and  $(x_2, y_2)$  on a line, then the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

and the formula for the line then is

$$y = y_1 + m(x - x_1).$$

**Example 1.1.1.** Find the equation of the line with slope  $-3$  that passes through  $(1, 2)$ .

**Solution.** The equation of the line is

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

**Example 1.1.2.** Find the equation of the line that passes through  $(1, 2)$  and  $(2, -1)$ .

**Solution.** First we find the slope which is

$$\frac{-1 - 2}{2 - 1} = -3.$$

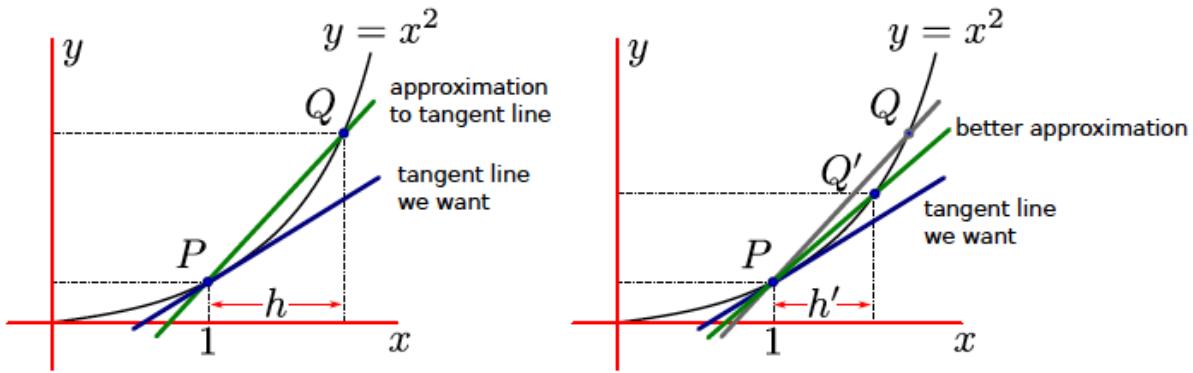
Then the equation of the line is

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

**The equation of a tangent line:** Given a curve  $y = f(x)$  and a point  $P$  on the curve, how to find the slope of the tangent to a curve at  $P$ : let do this through an example.

**Example 1.1.3.** Find the tangent line to the curve  $y = x^2$  that passes through  $P = (1, 1)$ .





So we want to find the slope the line that passes through the points  $(x_1, y_1) = (1, 1)$  and  $(x_2, y_2) = (1 + h, (1 + h)^2)$ . The slope then is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1 + h)^2 - 1^2}{(1 + h) - 1} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(h + 2)}{h} = 2 + h$$

$h$	$m = \frac{(1+h)^2 - 1^2}{(1+h) - 1}$
0.1	2.1
0.01	2.01
0.001	2.001

When  $h$  gets smaller and smaller, the slope will be closer and closer to the slope of the tangent line to  $y = x^2$  at  $(1, 1)$ , which the slope will be closer and closer to 2, we can write this more mathematically as

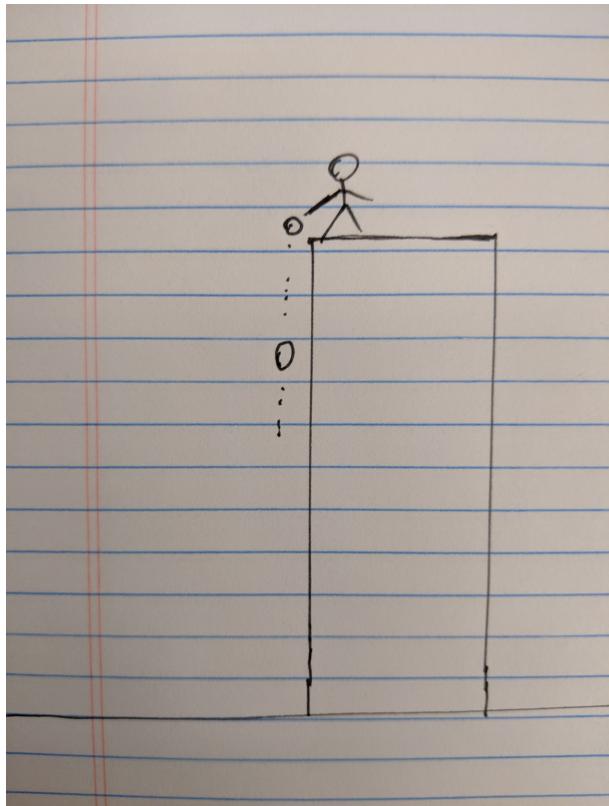
$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1^2}{(1 + h) - 1} = 2$$

**Read:** the limit of  $\frac{(1+h)^2 - 1^2}{(1+h) - 1}$  as  $h$  approaches 0 is 2.  
Tangent line is

$$y = 1 + 2(x - 1) = 2x - 1.$$

## 1.2 Velocity

- Let  $t$  be elapsed time measured in second
- $S(t)$  be the distance the ball has fallen in meters
- What is  $S(0)$ ?  $S(0) = 0$ .
- (**Galileo**)  $S(t) = 4.9t^2$ .



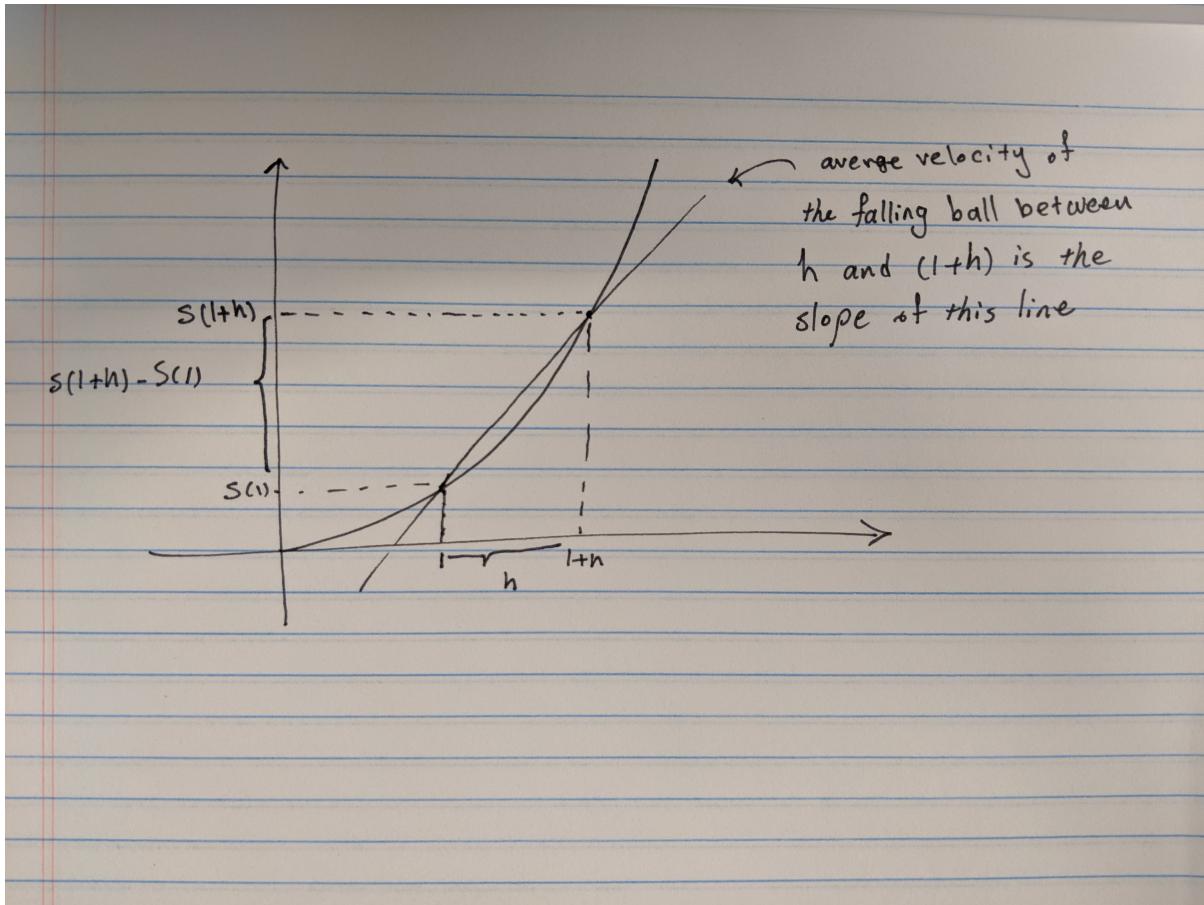
**Question:** How fast the ball is fallen after 1 second, that is, find  $v(1)$ , the velocity at  $t = 1$ ?

$$\text{average velocity} = \frac{\text{difference in position}}{\text{difference in time}} = \frac{S(t_2) - S(t_1)}{t_2 - t_1}.$$

To answer the question we should find the average velocity of the falling ball between  $(1 + h)$  and 1. So,

average velocity when  $(t_2 = 1 + h)$  and  $(t_1 = 1)$

$$= \frac{S(1 + h) - S(1)}{h} = \frac{4.9(1 + h)^2 - 4.9}{h} = 4.9(2 + h).$$



time window	average velocity
$1 \leq t \leq 1.1$	10.29
$1 \leq t \leq 1.01$	9.84
$1 \leq t \leq 1.01$	9.8049
$1 \leq t \leq 1.001$	9.80049

So we can write

$$v(1) = \lim_{h \rightarrow 0} \frac{S(1+h) - S(1)}{h} = 9.8.$$

More generally:

We define the instantaneous velocity at time  $t = a$  to be the limit

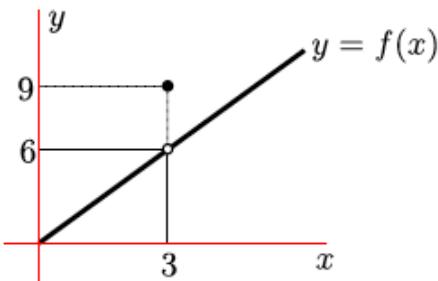
$$v(a) = \lim_{h \rightarrow 0} \frac{S(a+h) - S(a)}{h}$$

## 1.3 The limit of a function

To arrive at the definition of limit, we start with a very simple example.

**Example 1.3.1.** Consider the following function

$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



If we plug in some numbers very close to 3 (but not exactly 3) into the function we see

$x$	2.9	2.99	2.999	○	3.001	3.01	3.1
$f(x)$	5.8	5.98	5.998	○	6.002	6.02	6.2

So as  $x$  moves closer and closer to 3, without being exactly 3, we see that the function moves closer and closer to 6. We can then write this as

$$\lim_{x \rightarrow 3} f(x) = 6.$$

**Definition. (Informal definition of limit)** We write

$$\lim_{x \rightarrow a} f(x) = L.$$

if the value of the function  $f(x)$  is sure to be arbitrary close to  $L$  whenever the value of  $x$  is close enough to  $a$ , without being exactly  $a$ .

**Example 1.3.2.** Let  $f(x) = \frac{x-2}{x^2+x-6}$  and find its limit as  $x \rightarrow 2$ .

**Solution.** We want to find

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2+x-6}.$$

**Important point:** if we compute  $f(2)$ , then we have  $\frac{0}{0}$  which is undefined.

Again we plug in numbers close to 2 and we have

$x$	1.9	1.99	1.999	○	2.001	2.01	2.1
$f(x)$	0.20408	0.20040	0.20004	○	0.19996	0.19960	0.19608

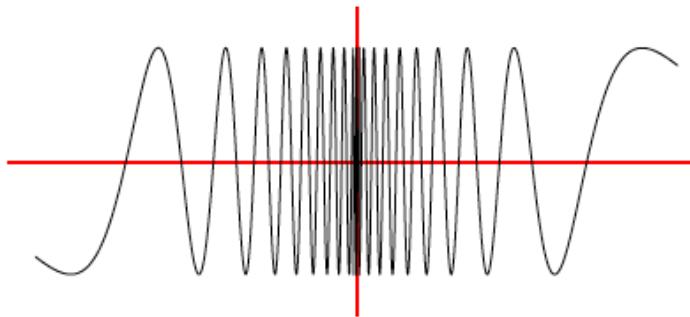
So

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2+x-6} = 2.$$

**Example 1.3.3.** Consider the following function  $f(x) = \sin(\pi/x)$ . Find the limit as  $x \rightarrow 0$  of  $f(x)$ .

**Solution.** When  $x$  is getting closer and closer to 0, it oscillates faster and faster. Since the function does not approach a single number as we bring  $x$  closer and closer to zero, the limit does not exist. Thus,

$$\lim_{x \rightarrow 0} \sin(\pi/x) = \text{DNE}$$



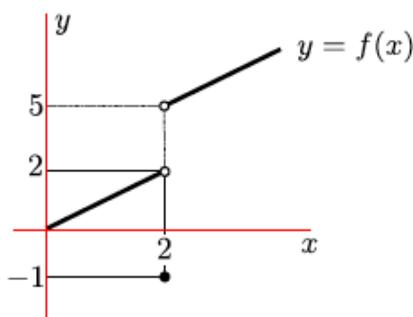
**Example 1.3.4.** Consider the function

$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$

Find

$$\lim_{x \rightarrow 2} f(x).$$

**Solution.**



Let again plug in some numbers close to 2 (but not exactly 2)

$x$	1.9	1.99	1.999	○	2.001	2.01	2.1
$f(x)$	1.9	1.99	1.999	○	5.001	5.01	5.1

Now when we approach from below (or left), we seem to be getting closer to 2 ( $\lim_{x \rightarrow 2^-} f(x) = 2$ ), but when we approach from above (or right) we seem to be getting closer to 5 ( $\lim_{x \rightarrow 2^+} f(x) = 5$ ). Since we are not approaching the same number the limit does not exist.

$$\lim_{x \rightarrow 2} f(x) = \text{DNE}$$

**Definition. (Informal definition of one-sided limits)** We write

$$\lim_{x \rightarrow a^-} f(x) = K$$

when the value of  $f(x)$  gets closer and closer to  $K$  when  $x < a$  and  $x$  moves closer and closer to  $a$ . Since the  $x$ -values are always less than  $a$ , we say that  $x$  approaches  $a$  from below (or left). This is also often called the left-hand limit since the  $x$ -values lie to the left of  $a$  on a sketch of the graph.

We similarly write

$$\lim_{x \rightarrow a^+} f(x) = L$$

when the values of  $f(x)$  gets closer and closer to  $L$  when  $x > a$  and  $x$  moves closer and closer to  $a$ . For similar reason we say that  $x$  approaches  $a$  from above, and sometimes to this as the the right-hand limit.

### Theorem 1.3.5.

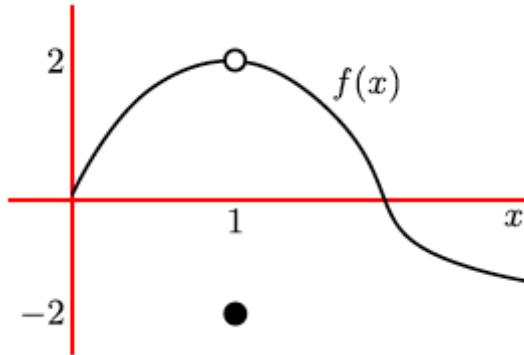
$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x) = L$$

- If the limit of  $f(x)$  as  $x$  approaches  $a$  exists and is equal to  $L$ , then both the left-hand and right-hand limits exist and are equal to  $L$ .
- If the left-hand and right-hand limits as  $x$  approaches  $a$  exist and are equal, then the limit as  $x$  approaches  $a$  exists and is equal to the one-sided limits.

Contrapositive of the above argument says

- If either of the left-hand and right-hand limits as  $x$  approaches  $a$  fail to exist, or if they both exist but are different, then the limit as  $x$  approaches  $a$  does not exist. AND,
- If the limit as  $x$  approaches  $a$  does not exist, then the left-hand and right-hand limits are either different or at least one of them does not exist.

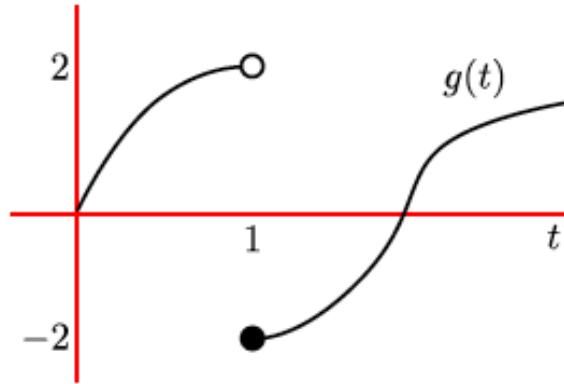
**Example 1.3.6.** Consider the graph of the function  $f(x)$ .



Then

$$\lim_{x \rightarrow 1^-} f(x) = 2 \quad \lim_{x \rightarrow 1^+} f(x) = 2 \quad \lim_{x \rightarrow 1} f(x) = 2$$

**Example 1.3.7.** Consider the graph of the function  $g(t)$ .



Then

$$\lim_{t \rightarrow 1^-} g(t) = 2 \quad \lim_{t \rightarrow 1^+} g(t) = -2 \quad \lim_{t \rightarrow 1} g(t) = DNE$$

In the following example even though the limit doesn't exist when  $x$  approaches  $a$ , we can say more.

**Example 1.3.8.** Consider the graph for the function  $f(x)$ .



$$\lim_{x \rightarrow a} f(x) = +\infty$$

**Example 1.3.9.** Consider the graph for the function  $g(x)$ .



$$\lim_{x \rightarrow a} g(x) = -\infty$$

**Definition.** We write

$$\lim_{x \rightarrow a} f(x) = +\infty$$

when the value of the function  $f(x)$  becomes arbitrarily large and positive as  $x$  gets closer and closer to  $a$ , without being exactly  $a$ .

Similarly, we write

$$\lim_{x \rightarrow a} f(x) = -\infty$$

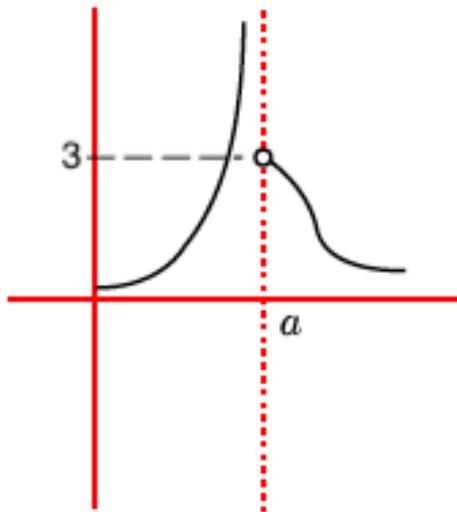
when the value of the function  $f(x)$  becomes arbitrarily large and negative as  $x$  gets closer and closer to  $a$ , without being exactly  $a$ .

**Example 1.3.10.**

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \lim_{x \rightarrow 0} -\frac{1}{x^2} = -\infty$$

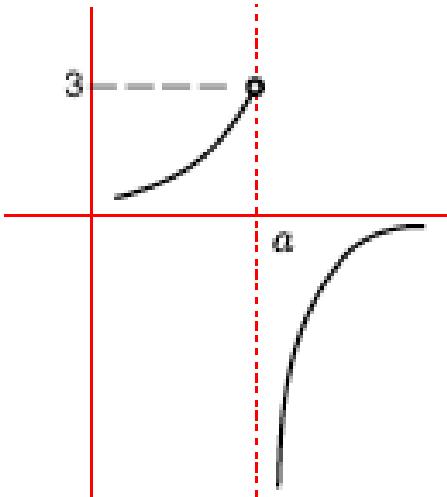
**Important Point:** Do not think of “ $+\infty$ ” and “ $-\infty$ ” in these statements as numbers. When we write  $\lim_{x \rightarrow a} f(x) = +\infty$ , it says “the function  $f(x)$  becomes arbitrary large as  $x$  approaches  $a$ ”.

**Example 1.3.11.** Consider the graph for the function  $h(x)$ .



$$\lim_{x \rightarrow a^-} h(x) = +\infty \quad \lim_{x \rightarrow a^+} h(x) = 3 \quad \lim_{x \rightarrow a} h(x) = \text{DNE}$$

**Example 1.3.12.** Consider the graph for the function  $s(x)$ .



$$\lim_{x \rightarrow a^-} s(x) = 3 \quad \lim_{x \rightarrow a^+} s(x) = -\infty \quad \lim_{x \rightarrow a} s(x) = \text{DNE}$$

**Definition.** We write

$$\lim_{x \rightarrow a^+} f(x) = +\infty$$

when the value of the function  $f(x)$  becomes arbitrarily large and positive as  $x$  gets closer and closer to  $a$  from above (equivalently, from right), without being exactly  $a$ . Similarly, we write

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

when the values of the function  $f(x)$  becomes arbitrarily large and negative as  $x$  gets closer and closer to  $a$  from above (equivalently, from right), without being exactly  $a$ .

The notation

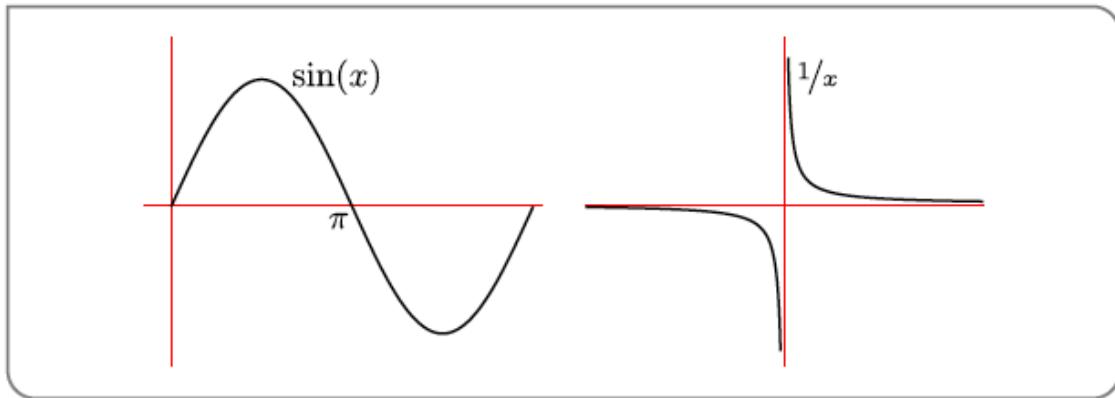
$$\lim_{x \rightarrow a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \rightarrow a^-} f(x) = -\infty$$

has a similar meaning except that limits are approached from below (from left).

**Example 1.3.13.** Consider the function

$$g(x) = \frac{1}{\sin(x)}.$$

Find the one-side limits of this function as  $x \rightarrow \pi$ .



- As  $x \rightarrow \pi$  from the left,  $\sin(x)$  is a small positive number that is getting closer and closer to zero. That is, as  $x \rightarrow \pi^-$ , we have that  $\sin(x) \rightarrow 0$  through positive numbers (i.e. from above). Now look at the graph of  $1/x$ , and think what happens as we move  $x \rightarrow 0^+$ , the function is positive and becomes larger and larger.

So as  $x \rightarrow \pi$  from the left,  $\sin(x) \rightarrow 0$  from above, and so  $1/\sin(x) \rightarrow +\infty$ .

- By very similar reasoning, as  $x \rightarrow \pi$  from the right,  $\sin(x)$  is a small negative number that gets closer and closer to zero. So as  $x \rightarrow \pi$  from the right,  $\sin(x) \rightarrow 0$  through negative numbers (i.e. from below) and so  $1/\sin(x)$  to  $-\infty$ .

Thus

$$\lim_{x \rightarrow \pi^-} \frac{1}{\sin(x)} = +\infty \qquad \lim_{x \rightarrow \pi^+} \frac{1}{\sin(x)} = -\infty$$

## 1.4 Calculating Limits with Limit Laws

**Theorem 1.4.1.** Let  $a, c \in \mathbb{R}$ . The following two limits hold

$$\lim_{x \rightarrow a} c = c \quad \lim_{x \rightarrow a} x = a$$

**Theorem 1.4.2. (Arithmetic of Limits)** Let  $a, c \in \mathbb{R}$ , let  $f(x)$  and  $g(x)$  be defined for all  $x$ 's that lie in some interval about  $a$  (but  $f$  and  $g$  need not to be defined exactly at  $a$ ).

$$\lim_{x \rightarrow a} f(x) = F \quad \lim_{x \rightarrow a} g(x) = G$$

exists with  $F, G \in \mathbb{R}$ . Then the following limits hold

- $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$  – limit of the sum is the sum of the limits.
- $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$  – limit of the difference is the difference of the limits.
- $\lim_{x \rightarrow a} cf(x) = cF$ .
- $\lim_{x \rightarrow a} (f(x).g(x)) = F.G$  – limit of the product is the product of the limits.
- If  $G \neq 0$  then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}$ .

**Example 1.4.3.** Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} g(x)} = \frac{3}{2} = 3.$$

**Example 1.4.4.**

$$\lim_{x \rightarrow 3} 4x^2 - 1 = 4 \times \lim_{x \rightarrow 3} x^2 - \lim_{x \rightarrow 3} 1 = 35.$$

$$\lim_{x \rightarrow 2} \frac{x}{x-1} = \frac{\lim_{x \rightarrow 2} x}{\lim_{x \rightarrow 2} x - \lim_{x \rightarrow 1} 1} = \frac{2}{2-1} = 2.$$

Consider that we apply the theorem Arithmetic of Limits to compute the limit of a ratio if the limit of denominator is not zero. **What will happen if the limit of denominator is zero:**

- the limit does not exist, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = DNE$$

– the limit is  $\pm\infty$ , eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x^4} = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{-x^2}{x^4} = \lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty.$$

– the limit is 0, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

– the limit exists and it nonzero, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x} = 1.$$

**Theorem 1.4.5.** Let  $n$  be a positive integer, let  $a \in R$  and let  $f$  be a function so that

$$\lim_{x \rightarrow a} f(x) = F$$

for some real number  $F$ . Then the following holds

$$\lim_{x \rightarrow a} (f(x))^n = \left( \lim_{x \rightarrow a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit. Similarly, if

- $n$  is an even number and  $F > 0$ , or
- $n$  is an odd number and  $F$  is any real number

then

$$\lim_{x \rightarrow a} (f(x))^{1/n} = \left( \lim_{x \rightarrow a} f(x) \right)^{1/n} = F^{1/n}.$$

### Example 1.4.6.

$$\lim_{x \rightarrow 4} x^{1/2} = 4^{1/2} = 2.$$

$$\lim_{x \rightarrow 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \rightarrow 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = 13^{1/3}$$

### Example 1.4.7. Compute the following limits.

$$1. \lim_{x \rightarrow 2} \frac{x^3 - x^2}{x - 1}$$

$$2. \lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1}$$

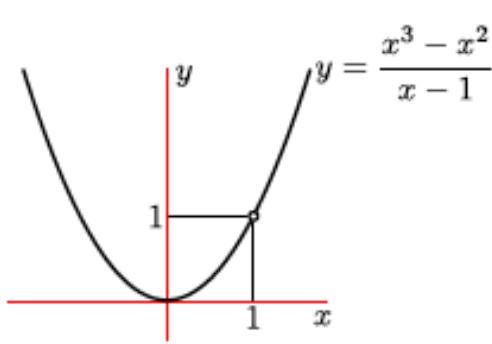
**Solution.** 1.  $\lim_{x \rightarrow 2} \frac{x^3 - x^2}{x - 1} = 4$ .

2. Consider that  $\lim_{x \rightarrow 1} x^3 - x^2 = 0$  and  $\lim_{x \rightarrow 1} x - 1 = 0$ . However,

$$\frac{x^3 - x^2}{x - 1} = \frac{x^2(x - 1)}{x - 1},$$

thus

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases}$$



And so

$$\lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \rightarrow 1} x^2 = 1.$$

The reasoning in the above example can be made more general:

**Theorem 1.4.8.** If  $f(x) = g(x)$  except when  $x = a$  then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

provided the limit of  $g$  exists.

We mostly use the above theorem when we end up with  $\frac{0}{0}$ .

**Example 1.4.9.** Compute

$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1}{h}.$$

**Solution.** Note that

$$\frac{(1 + h)^2 - 1}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(2 + h)}{h}.$$

Thus,

$$\frac{(1 + h)^2 - 1}{h} = \begin{cases} 2 + h & h \neq 0 \\ \text{undefined} & h = 0. \end{cases}$$

And so

$$\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1}{h} = \lim_{h \rightarrow 0} 2 + h = 2.$$

We now present a slightly harder example.

**Example 1.4.10.** Compute the limit

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1 + x} - 1}.$$

**Solution.** Both the limits of the numerator and denominator as  $x \rightarrow 0$  are 0, so we cannot use the Theorem Arithmetic of limits. We now can simply multiply the numerator and denominator by the conjugation of  $\sqrt{1+x} - 1$ , that is,  $\sqrt{1+x} + 1$ . We have

$$\begin{aligned}
 \frac{x}{\sqrt{1+x}-1} &= \frac{x}{\sqrt{1+x}-1} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} && \text{multiply by } \frac{\text{conjugate}}{\text{conjugate}} = 1 \\
 &= \frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)} && \text{bring things together} \\
 &= \frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x})^2 - 1 \cdot 1} && \text{since } (a-b)(a+b) = a^2 - b^2 \\
 &= \frac{x(\sqrt{1+x}+1)}{1+x-1} && \text{clean up a little} \\
 &= \frac{x(\sqrt{1+x}+1)}{x} && \\
 &= \sqrt{1+x}+1 && \text{cancel the } x
 \end{aligned}$$

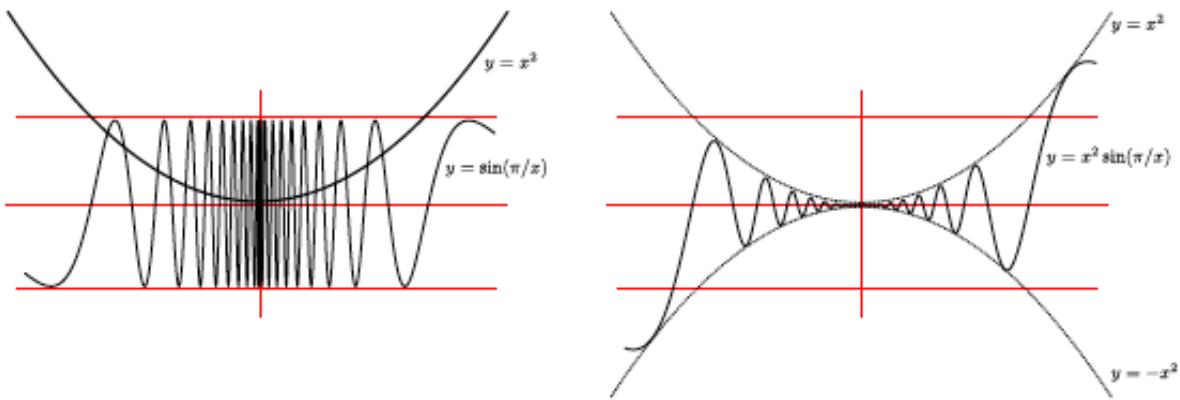
So now we have

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x}-1} &= \lim_{x \rightarrow 0} \sqrt{1+x}+1 \\
 &= \sqrt{1+0}+1 = 2
 \end{aligned}$$

Before we move to the next section and study the limits at infinity, we have one more theorem to state.

**Example 1.4.11.** Compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



**Solution.** It is not possible to simply use the theorem Arithmetic of Limits since the limit of  $\sin\left(\frac{\pi}{x}\right)$  as  $x \rightarrow 0$  does not exist. Since  $-1 \leq \sin(\theta) \leq 1$  for all real numbers  $\theta$ , we have

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1 \quad \text{for all } x \neq 0$$

Multiplying the above by  $x^2$  we see that

$$-x^2 \leq x^2 \sin\left(\frac{\pi}{x}\right) \leq x^2 \quad \text{for all } x \neq 0.$$

Since

$$\lim_{x \rightarrow 0} x^2 = \lim_{x \rightarrow 0} (-x^2) = 0$$

by the sandwich (or squeeze or pinch) theorem (look at below for the sandwich theorem) we have

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0.$$

**Theorem 1.4.12. (sandwich (or squeeze or pinch) theorem )** Let  $a \in \mathbb{R}$  and let  $f, g, h$  be three functions so that

$$f(x) \leq g(x) \leq h(x)$$

for all  $x$  in an interval around  $a$ , except possibly at  $x = a$ . Then if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then it is also the case that

$$\lim_{x \rightarrow a} g(x) = L.$$

**Example 1.4.13.** Let  $f(x)$  be a function such that  $1 \leq f(x) \leq x^2 - 2x + 2$ . What is

$$\lim_{x \rightarrow 1} f(x)?$$

**Solution.** Consider that

$$\lim_{x \rightarrow 1} x = 1 \quad \text{and} \quad \lim_{x \rightarrow 1} x^2 - 2x + 2 = 1.$$

Therefore, by the sandwich/pinch/squeeze theorem

$$\lim_{x \rightarrow 1} f(x) = 1.$$

## 1.5 Limits at Infinity

**Example 1.5.1.** We want to compute

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{x}$$

By plug in some large numbers into  $\frac{1}{x}$  we have

-10000	-1000	-100	o	100	1000	10000
-0.0001	0.001	-0.01	o	0.01	0.001	0.0001

We see that as  $x$  is getting bigger and positive the function  $\frac{1}{x}$  is getting closer to 0. Thus,

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0.$$

Moreover,

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

**Definition.** (Informal limit at infinity.) We write

$$\lim_{x \rightarrow \infty} f(x) = L$$

when the value of the function  $f(x)$  gets closer and closer to  $L$  as we make  $x$  larger and larger and positive.

Similarly, we write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

when the value of the function  $f(x)$  gets closer and closer to  $L$  as we make  $x$  larger and larger and negative.

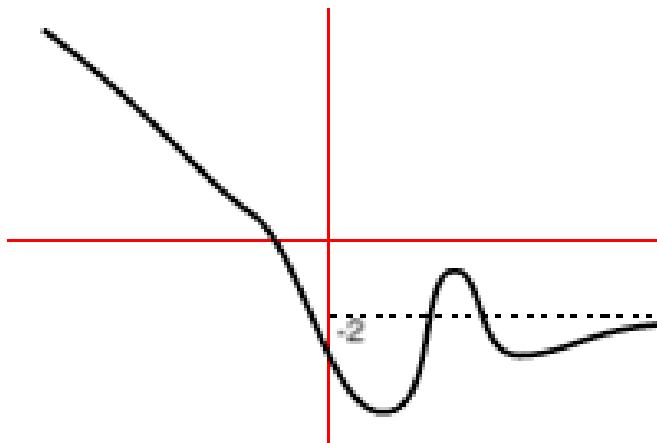
**Example 1.5.2.** Consider the graph of the function  $f(x)$ .



Then

$$\lim_{x \rightarrow \infty} f(x) = -2 \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = 2$$

**Example 1.5.3.** Consider the graph of the function  $g(x)$ .



Then

$$\lim_{x \rightarrow \infty} g(x) = -2 \quad \text{and} \quad \lim_{x \rightarrow -\infty} g(x) = +\infty$$

Same as usual we start with two very simple building blocks and build other limits from them.

**Theorem 1.5.4.** *Let  $c \in \mathbb{R}$  then the following limits hold*

$$\lim_{x \rightarrow +\infty} c = c \quad \lim_{x \rightarrow -\infty} c = c$$

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0 \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

**Theorem 1.5.5.** *Let  $f(x)$  and  $g(x)$  be two functions for which the limits*

$$\lim_{x \rightarrow \infty} f(x) = F \quad \lim_{x \rightarrow \infty} g(x) = G$$

*exist. Then the following limits hold*

$$\lim_{x \rightarrow \infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x \rightarrow \infty} f(x)g(x) = FG$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \text{provided } G \neq 0$$

*and for rational numbers  $r$ ,*

$$\lim_{x \rightarrow \infty} (f(x))^r = F^r$$

*provided that  $f(x)^r$  is defined for all  $x$ .*

*The analogous results hold for limits to  $-\infty$ .*

We need a little extra care with the powers of functions.

**Warning:** Consider that

$$\lim_{x \rightarrow +\infty} \frac{1}{x^{1/2}} = 0$$

However,

$$\lim_{x \rightarrow +\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because  $x^{1/2}$  is not defined for  $x < 0$ .

**Example 1.5.6.** *Compute the following limit:*

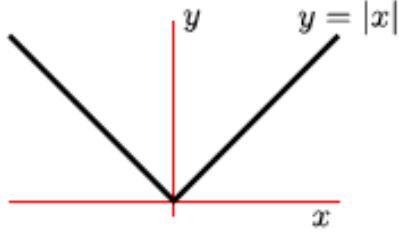
$$\lim_{x \rightarrow \infty} \frac{x^2 - 3x + 4}{3x^2 + 8x + 1}$$

**Solution.** By factoring  $x$  with largest exponent in the numerator and denominator we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2 - 3x + 4}{3x^2 + 8x + 1} &= \lim_{x \rightarrow \infty} \frac{x^2(1 + \frac{-3x}{x^2} + \frac{4}{x^2})}{x^2(3 + \frac{8x}{x^2} + \frac{1}{x^2})} = \lim_{x \rightarrow \infty} \frac{(1 + \frac{-3x}{x^2} + \frac{4}{x^2})}{(3 + \frac{8x}{x^2} + \frac{1}{x^2})} = \\ &\frac{\left(\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \frac{-3x}{x^2} + \lim_{x \rightarrow \infty} \frac{4}{x^2}\right)}{\left(\lim_{x \rightarrow \infty} 3 + \lim_{x \rightarrow \infty} \frac{8x}{x^2} + \lim_{x \rightarrow \infty} \frac{1}{x^2}\right)} = \frac{1}{3}. \end{aligned}$$

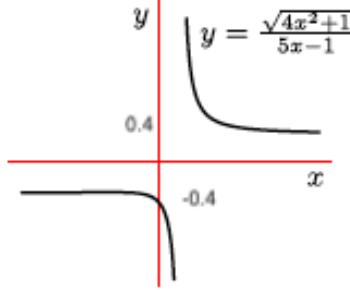
**Remark.** Note that

$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



**Example 1.5.7.** Compute the following limit:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{5x - 1}.$$



**Solution.** Factor the terms with the largest exponents in the numerator and denominator.

We have

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{5x - 1} = \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2(1 + \frac{1}{4x^2})}}{5x(1 - \frac{1}{5x})} = \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2} \sqrt{(1 + \frac{1}{4x^2})}}{5x(1 - \frac{1}{5x})} = \lim_{x \rightarrow \infty} \frac{2|x|}{5x} = \lim_{x \rightarrow \infty} \frac{2x}{5x} = \frac{2}{5}.$$

**Example 1.5.8.** Compute the following limit:

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{5x - 1}.$$

**Solution.** By the same kind of computation we have

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{5x - 1} = \lim_{x \rightarrow \infty} \frac{2|x|}{5x}.$$

Consider that since  $x$  is getting negative values, we have  $|x| = -x$ . Therefore,

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{5x - 1} = \lim_{x \rightarrow \infty} \frac{2|x|}{5x} = \lim_{x \rightarrow \infty} \frac{-2x}{5x} = \frac{-2}{5}.$$

**Example 1.5.9.** Compute the following limit:

$$\lim_{x \rightarrow \infty} (x^{7/5} - x).$$

**Solution.** We factor the term with the largest exponent, we have

$$\lim_{x \rightarrow \infty} (x^{7/5} - x) = \lim_{x \rightarrow \infty} x^{7/5} \left(1 - \frac{1}{x^{2/5}}\right) = \infty.$$

**Theorem 1.5.10.** Let  $a, c, H \in \mathbb{R}$  and let  $f, g, h$  be functions defined in an interval around  $a$  (but they need not be defined at  $x = a$ ), so that

$$\lim_{x \rightarrow a} f(x) = +\infty \quad \lim_{x \rightarrow a} g(x) = +\infty \quad \lim_{x \rightarrow a} h(x) = H$$

1.

$$\lim_{x \rightarrow a} (f(x) + g(x)) = +\infty.$$

2.

$$\lim_{x \rightarrow a} (f(x) + h(x)) = +\infty.$$

3.

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \text{undetermined}.$$

4.

$$\lim_{x \rightarrow a} (f(x) - h(x)) = +\infty.$$

5.

$$\lim_{x \rightarrow a} cf(x) = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$$

6.

$$\lim_{x \rightarrow a} (f(x).g(x)) = +\infty.$$

7.

$$\lim_{x \rightarrow a} (f(x).h(x)) = \begin{cases} +\infty & H > 0 \\ \text{undetermined} & H = 0 \\ -\infty & H < 0 \end{cases}$$

8.

$$\lim_{x \rightarrow a} \frac{h(x)}{f(x)} = 0.$$

**Example 1.5.11.** Consider the following three functions:

$$f(x) = x^{-2} \quad g(x) = 2x^{-2} \quad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \rightarrow 0} f(x) = +\infty \quad \lim_{x \rightarrow 0} g(x) = +\infty \quad \lim_{x \rightarrow 0} h(x) = +\infty.$$

Then

•

$$\lim_{x \rightarrow 0} (f(x) - g(x)) = \lim_{x \rightarrow 0} x^{-2} = \infty$$

•

$$\lim_{x \rightarrow 0} (f(x) - h(x)) = \lim_{x \rightarrow 0} (1) = 1$$

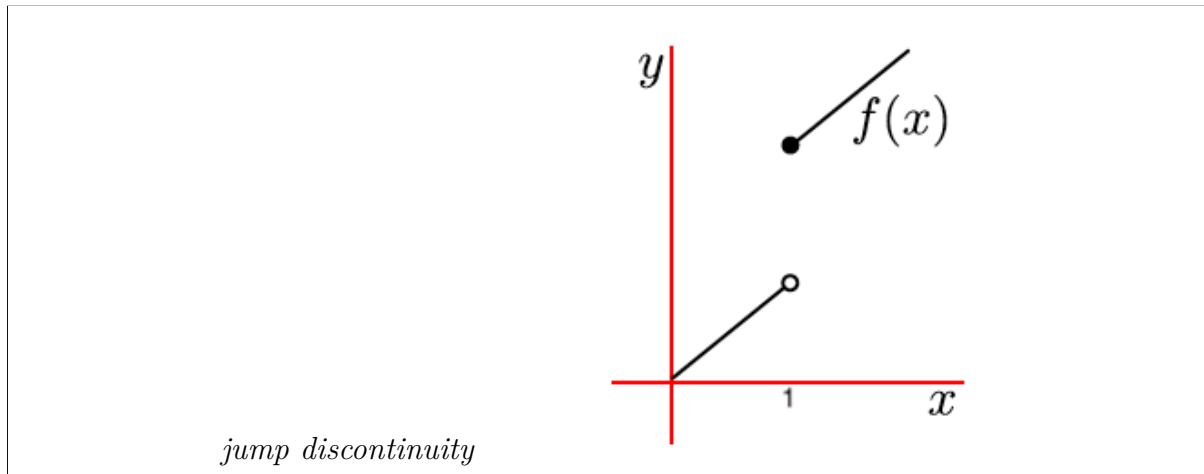
•

$$\lim_{x \rightarrow 0} (g(x) - h(x)) = \lim_{x \rightarrow 0} x^{-2} + 1 = \infty$$

## 1.6 Continuity

**Example 1.6.1.** Consider the function

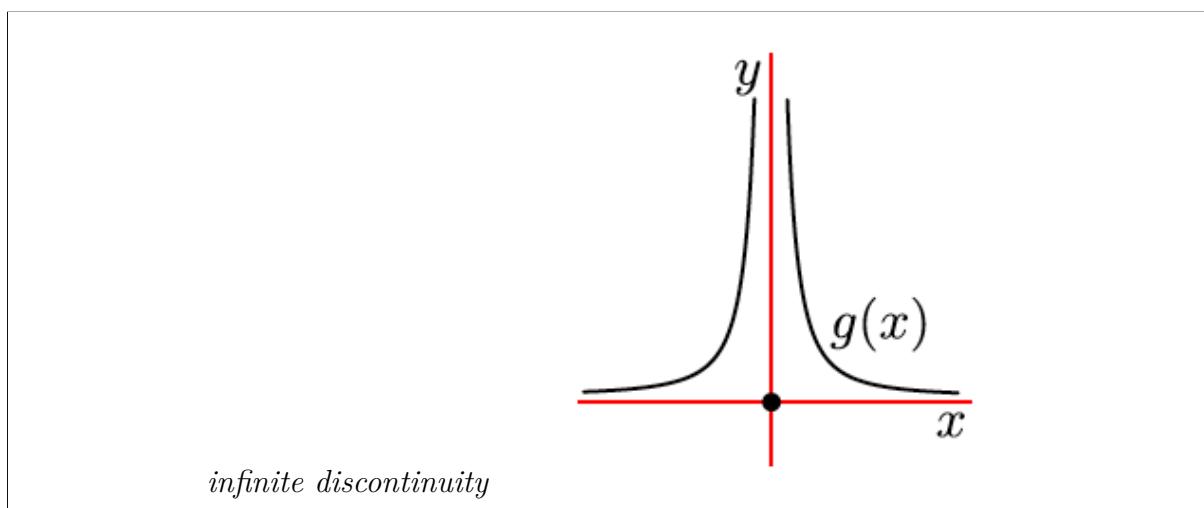
$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$



- The function  $f(x)$ , at  $x = 1$  is not continuous; however, it is continuous from the right at 1.
- The function  $f(x)$ , on  $[1, \infty)$  (for  $x \geq 1$ ) is continuous.
- The function  $f(x)$ , on  $(-\infty, -1)$  is continuous.

**Example 1.6.2.** Consider the function

$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

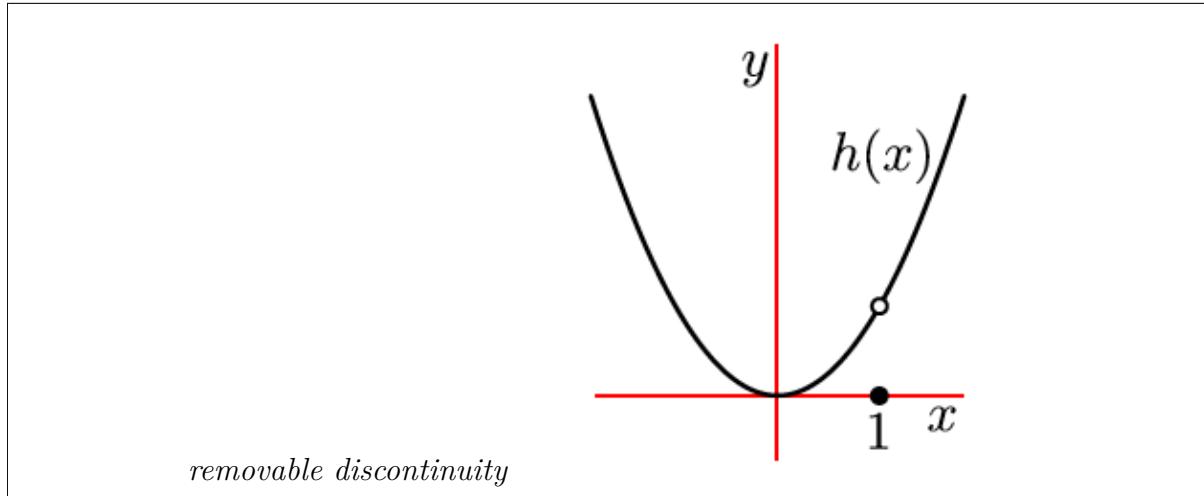


- the function  $g(x)$  is not continuous at 0 neither from the right nor the left.

- the function  $g(x)$  is continuous at all points in  $\mathbb{R}$  except 0.

**Example 1.6.3.** Consider the function

$$h(x) = \begin{cases} \frac{x^3-x^2}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$



- the function  $h(x)$  is not continuous at 1 neither from the right or the left.
- the function  $h(x)$  is continuous at all points in  $\mathbb{R}$  except 1.

**Definition.** A function  $f(x)$  is continuous from the right at  $a$  if

$$\lim_{x \rightarrow a^+} f(x) = f(a).$$

Similarly, a function  $f(x)$  is continuous from the left at  $a$  if

$$\lim_{x \rightarrow a^-} f(x) = f(a).$$

A function is continuous at  $a$  if it is continuous from the left and right at  $a$ , that is,

$$\lim_{x \rightarrow a^+} f(x) = f(a) = \lim_{x \rightarrow a^-} f(x).$$

**Definition.** A function  $f(x)$  is continuous on an interval  $[a, b]$  if

1.  $f(x)$  is continuous on  $(a, b)$ ,
2.  $f(x)$  is continuous from the right at  $a$ ,
3.  $f(x)$  is continuous from the left at  $b$ .

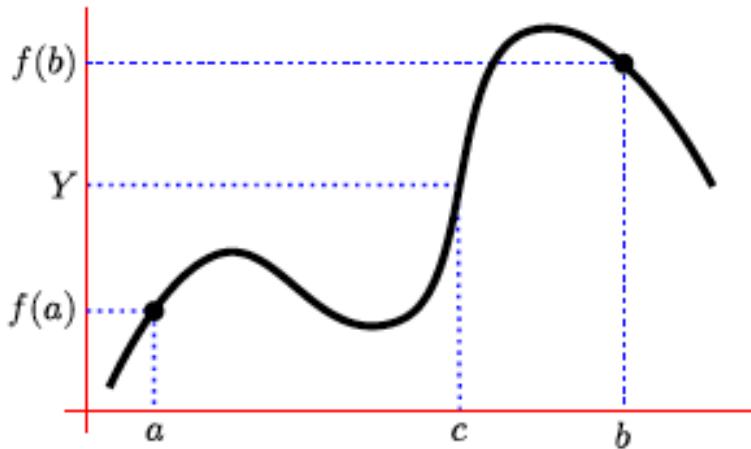
**Definition.** A function  $f(x)$  is continuous on an interval  $(a, b]$  if

1.  $f(x)$  is continuous on  $(a, b)$ ,
2.  $f(x)$  is continuous from the left at  $b$ .

**Definition.** A function  $f(x)$  is continuous on an interval  $[a, b]$  if

1.  $f(x)$  continuous on  $(a, b)$ ,
2.  $f(x)$  is continuous from the right at  $a$ .

**Theorem 1.6.4. (Intermediate value theorem(IVT))** Let  $a < b$  and let  $f(x)$  be a function that is continuous at all points  $a \leq x \leq b$ . If  $Y$  is any number between  $f(a)$  and  $f(b)$  then there exists some number  $c \in [a, b]$  so that  $f(c) = Y$ .



**Example 1.6.5.** Show that the function  $f(x) = x - 1 + \sin(\pi x/2)$  has a zero in  $0 \leq x \leq 1$ .

**Solution.** Consider that  $f(x)$  is a continuous function such that  $f(0) = -1$  and  $f(1) = 1$ . Therefore, by IVT, since  $f(0) = -1 \leq 0 \leq 1 = f(1)$ , we have  $f(c) = 0$  for some  $c \in [0, 1]$ .

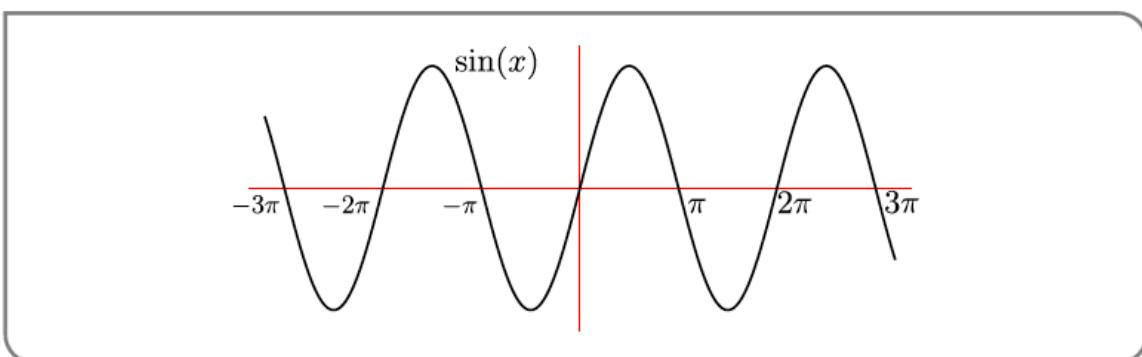
**Example 1.6.6.** Use the bisection method to find a zero of  $f(x) = x - 1 + \sin(\pi x/2)$  that lies between 0 and 1.

**Solution.** For learning about bisection method and how to solve this problem refer to Example 1.6.15 of the CLP book.

**Theorem 1.6.7.** If  $g$  is continuous at  $a$  and  $f(x)$  is continuous at  $g(a)$ , then  $(f \circ g)(x) = f(g(x))$  is continuous at  $x = a$ .

**Example 1.6.8.** Determine when the function  $h(x) = \sqrt{\sin(x)}$  is continuous.

**Solution.** Let  $f(x) = \sqrt{x}$  and  $g(x) = \sin(x)$ , then  $h(x) = (f \circ g)(x)$ . We only need to find out at what points  $\sin(x)$  is positive.



The  $\sin(x)$  is continuous if

$$x \in [2n\pi, (2n+1)\pi] \quad \text{for all natural numbers } n.$$

**Theorem 1.6.9.** Let  $a, c \in \mathbb{R}$  and let  $f(x)$  and  $g(x)$  be functions that are continuous at  $a$ . Then the following functions are also continuous at  $x = a$ .

- $f(x) + g(x)$  and  $f(x) - g(x)$ ,
- $cf(x)$  and  $f(x)g(x)$ , and
- $\frac{f(x)}{g(x)}$  provided  $g(a) \neq 0$ .

**Example 1.6.10.** Determine at what points in  $\mathbb{R}$  the function

$$f(x) = \frac{\sin(x)}{x^2 - 5x + 6}$$

is continuous.

**Solution.** Since both the functions  $\sin(x)$  and  $x^2 - 5x + 6$  are continuous at all points in  $\mathbb{R}$ , we only need to check when the denominator is zero. Note that

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

thus at  $x = 2, 3$  the denominator is zero, and so the function is continuous at all points of  $\mathbb{R}$  except  $x = 2, 3$ .

**Homework:**

Go to this link

<https://www.mooculus.osu.edu/textbook/mooculus.pdf> and download the book "MOOCULUS". Then do the following questions:

- all questions in page 35;
- in page 33 see why  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$ . Then do Questions 1-8 page 38;
- in page 42, do questions 1-10.

# Bibliography

- [1] CLP1: Differential Calculus by J. Feldman, A. Rechnitzer, and E. Yeager.