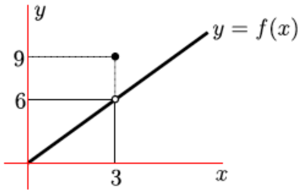
MATH 100

Farid Aliniaeifard

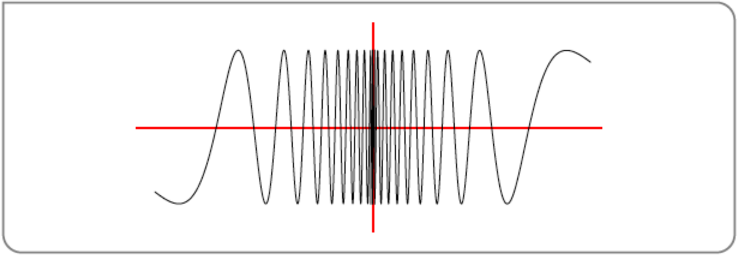
University of British Columbia

2019

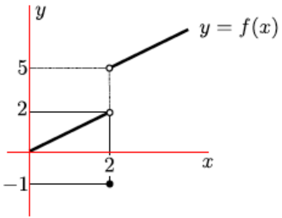
$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



$f(x) = \sin(\frac{\pi}{x})$

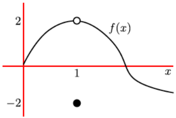


$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$



Example

Consider the graph of the function $f(x)$.



Then

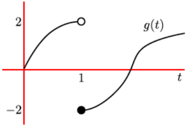
$$\lim_{x \to 1^-} f(x) =$$

$$\lim_{x \to 1^+} f(x) =$$

$$\lim_{x \to 1} f(x) =$$

Example

Consider the graph of the function $g(t)$.



Then

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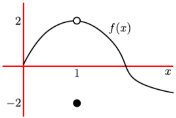

Then

$$\lim_{x \to 1^-} f(x) = 2$$

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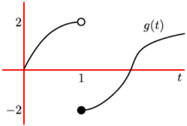
Example

Consider the graph of the function $g(t)$.



Then

$$\lim_{t \to 1^-} g(t) = 2$$

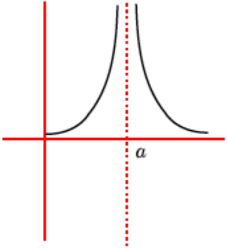
$$\lim_{t \to 1^+} g(t) = -2$$

$$\lim_{t \to 1} g(t) = DNE$$

When the limit goes to infinity

Example

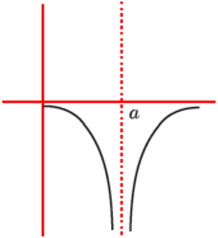
Consider the graph for the function $f(x)$.



$$\lim_{x \to a} f(x) = +\infty$$

Example

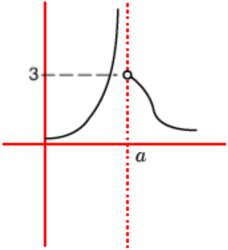
Consider the graph for the function $g(x)$.



$$\lim_{x \to a} g(x) = -\infty$$

Example

Consider the graph for the function $h(x)$.

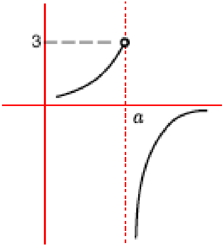


$$\lim_{x \to a^-} h(x) =$$

$$\lim_{x \to a^+} h(x) =$$

Example

Consider the graph for the function $s(x)$.

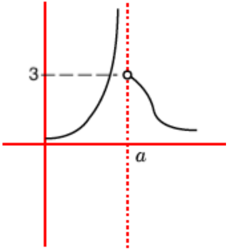


$$\lim_{x \to a^-} s(x) =$$

$$\lim_{x \to a^+} s(x) =$$

Example

Consider the graph for the function $h(x)$.

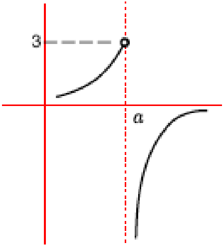


$$\lim_{x \to a^-} h(x) = +\infty$$

$$\lim_{x \to a^+} h(x) = 3$$

Example

Consider the graph for the function $s(x)$.



$$\lim_{x \to a^-} s(x) = 3$$

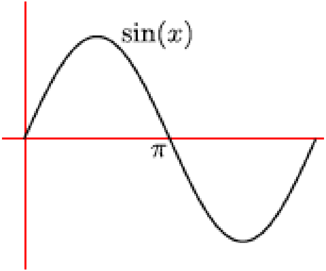
$$\lim_{x \to a^+} s(x) = -\infty$$

Example

Consider the function

$$g(x) = \frac{1}{sin(x)}.$$

Find the one-side limits of this function as $x \to \pi$.



$$\lim_{x \to \pi^-} \frac{1}{sin(x)} = +\infty$$

$$\lim_{x \to \pi^+} \frac{1}{sin(x)} = -\infty$$

Second Session Outline

- Arithmetic of the Limits
- Limit of a ratio: what will happen if the limit of the denominator is zero. For example,

$$\lim_{x\to 0} \frac{1}{x^2}? \quad \text{and} \quad \lim_{x\to 1} \frac{x^3 - x^2}{x-1} =?$$

- Sandwich/ Squeeze/Pinch Theorem
- limit at infinity

Arithmetic of the Limits

Theorem

Let $a, c \in \mathbb{R}$. The following two limits hold

$$\lim_{x \to a} c = c \qquad \lim_{x \to a} x = a$$

Example

$$\lim_{x \to 3} -2 = -2 \qquad \lim_{x \to -1} x = -1$$

Theorem

*(**Arithmetic of Limits**) Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f and g need not to be defined exactly at a).*

$$\lim_{x \to a} f(x) = F \qquad \lim_{x \to a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. Then the following limits hold

- $\lim\limits_{x \to a}(f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*

Theorem

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- $\lim\limits_{x \to a}(f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*

Theorem

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- $\lim_{x \to a} cf(x) = cF$.

Theorem

(**Arithmetic of Limits**) *Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f and g need not to be defined exactly at a).*

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- $\lim_{x\to a}(f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*
- $\lim_{x\to a}(f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*
- $\lim_{x\to a} cf(x) = cF$.
- $\lim_{x\to a}(f(x).g(x)) = F.G$*–limit of the product is the product of the limits.*

If $G \neq 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}$$

Example

Given

$$\lim_{x \to 1} f(x) = 3 \quad \text{and} \quad \lim_{x \to 1} g(x) = 2$$

We have

$$\lim_{x \to 1} 3f(x) =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$\lim_{x\to 1} 3f(x) - g(x) =$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

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$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} f(x). \lim_{x\to 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x\to 1} \frac{f(x)}{f(x) - g(x)} =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

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$$\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} f(x). \lim_{x\to 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x\to 1} \frac{f(x)}{f(x) - g(x)} = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} f(x) - \lim_{x\to 1} g(x)} = \frac{3}{3-2} = 3.$$

Example

$$\lim_{x \to 3} 4x^2 - 1 =$$

$$\lim_{x \to 2} \frac{x}{x-1} =$$

Example

$$\lim_{x \to 3} 4x^2 - 1 = 4 \times \lim_{x \to 3} x^2 - \lim_{x \to 3} 1 = 35.$$

$$\lim_{x \to 2} \frac{x}{x-1} = \frac{\lim_{x \to 2} x}{\lim_{x \to 2} x - \lim_{x \to 1} 1} = \frac{2}{2-1} = 2.$$

Limit of a ratio: what will happen if the limit of the denominator is zero.

Limit of a ratio: what will happen if the limit of denominator is zero:

– the limit does **not exist**, eg.

$$\lim_{x \to 0} \frac{x}{x^2} = \lim_{x \to 0} \frac{1}{x} = DNE$$

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- the limit does **not exist**, eg.

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- the **limit is** $\pm\infty$, eg.

$$\lim_{x\to 0} \frac{x^2}{x^4} = \lim_{x\to 0} \frac{1}{x^2} = +\infty \qquad \text{or} \qquad \lim_{x\to 0} \frac{-x^2}{x^4} = \lim_{x\to 0} \frac{-1}{x^2} = -\infty.$$

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– the **limit is** 0, eg.

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0.$$

Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x\to 0}\frac{x}{x^2}=\lim_{x\to 0}\frac{1}{x}=DNE$$

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- the **limit is** 0, eg.

$$\lim_{x\to 0}\frac{x^2}{x}=\lim_{x\to 0}x=0.$$

- the **limit exists and it nonzero**, eg.

$$\lim_{x\to 0}\frac{x}{x}=1.$$

Theorem

Let n be a positive integer, let $a \in R$ and let f be a function so that

$$\lim_{x \to a} f(x) = F$$

for some real number F. Then the following holds

$$\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit.

Theorem

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so that the limit of a power is the power of the limit. Similarly, if

- *n is an even number and $F > 0$, or*
- *n is an odd number and F is any real number*

then

$$\lim_{x \to a} (f(x))^{1/n} = \left(\lim_{x \to a} f(x) \right)^{1/n} = F^{1/n}.$$

Example

$$\lim_{x \to 4} x^{1/2} =$$

$$\lim_{x \to 4} (-x)^{1/2} =$$

$$\lim_{x \to 2} (4x^2 - 3)^{1/3} =$$

Example

$$\lim_{x \to 4} x^{1/2} = 4^{1/2} = 2.$$

$$\lim_{x \to 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \to 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = (13)^{1/3}.$$

Limit of a ratio: what will happen if the limit of the numerator and denominator are zero, for example,

$$\lim_{x \to 1} \frac{x^3 - x^2}{x - 1} = ?$$

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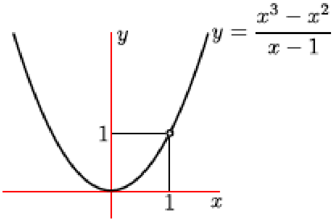


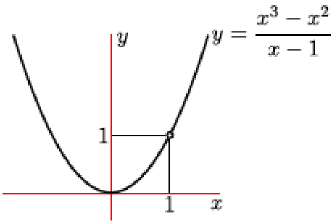
Theorem

If $f(x) = g(x)$ except when $x = a$ then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

provided the limit of g exists.

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases} \Rightarrow \lim_{x \to 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \to 1} x^2 = 1.$$

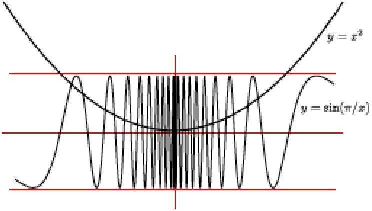


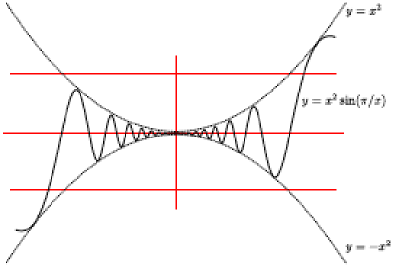
Sandwich/ Squeeze/Pinch Theorem

Example

Compute

$$\lim_{x \to 0} x^2 \sin(\frac{\pi}{x})$$





Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \to 1} f(x)?$$

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Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \to 1} f(x)?$$

Solution

Consider that

$$\lim_{x \to 1} x = 1 \qquad \textit{and} \qquad \lim_{x \to 1} x^2 - 2x + 2 = 1.$$

Therefore, by the sandwich/pinch/squeeze theorem

$$\lim_{x \to 1} f(x) = 1.$$

Example

We want to compute

$$\lim_{x\to+\infty}\frac{1}{x} \qquad \text{and} \qquad \lim_{x\to-\infty}\frac{1}{x}$$

By plug in some large numbers into $\frac{1}{x}$ we have

-10000	-1000	-100	$\circ$	100	1000	10000
-0.0001	-0.001	-0.01	$\circ$	0.01	0.001	0.0001

We see that as x is getting bigger and positive the function $\frac{1}{x}$ is getting closer to 0. Thus,

$$\lim_{x\to+\infty}\frac{1}{x}=0.$$

Moreover,

$$\lim_{x\to-\infty}\frac{1}{x}=0.$$

Definition

(**Informal limit at infinity.**) We write

$$\lim_{x \to \infty} f(x) = L$$

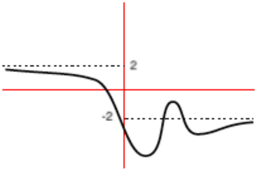
when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and positive.
Similarly, we write

$$\lim_{x \to -\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and negative.

Example

Consider the graph of the function $f(x)$.



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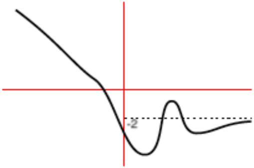

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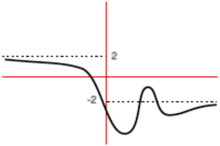


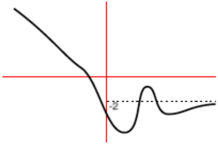
Then

$$\lim_{x\to\infty} f(x) = -2$$

$$\lim_{x\to-\infty} f(x) = 2$$

Example

Consider the graph of the function $g(x)$.



Then

$$\lim_{x\to\infty} g(x) = -2$$

$$\lim_{x\to-\infty} g(x) = +\infty$$

Theorem

Let $c \in \mathbb{R}$ then the following limits hold

$$\lim_{x \to +\infty} c = c \qquad \lim_{x \to -\infty} c = c$$

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

Theorem

Let $f(x)$ and $g(x)$ be two functions for which the limits

$$\lim_{x\to\infty} f(x) = F \qquad\qquad \lim_{x\to\infty} = G$$

exist. Then the following limits hold

$$\lim_{x\to\infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x\to\infty} f(x)g(x) = FG$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \textit{provided } G \neq 0$$

and for rational numbers r,

$$\lim_{x \to \infty} (f(x))^r = F^r$$

provided that $f(x)^r$ is defined for all x.
The analogous results hold for limits to $-\infty$.

Warning: Consider that

$$\lim_{x \to +\infty} \frac{1}{x^{1/2}} = 0$$

However,

$$\lim_{x \to +\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because $x^{1/2}$ is not defined for $x < 0$.