

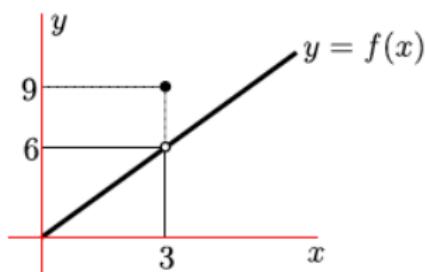
MATH 100

Farid Aliniaiefard

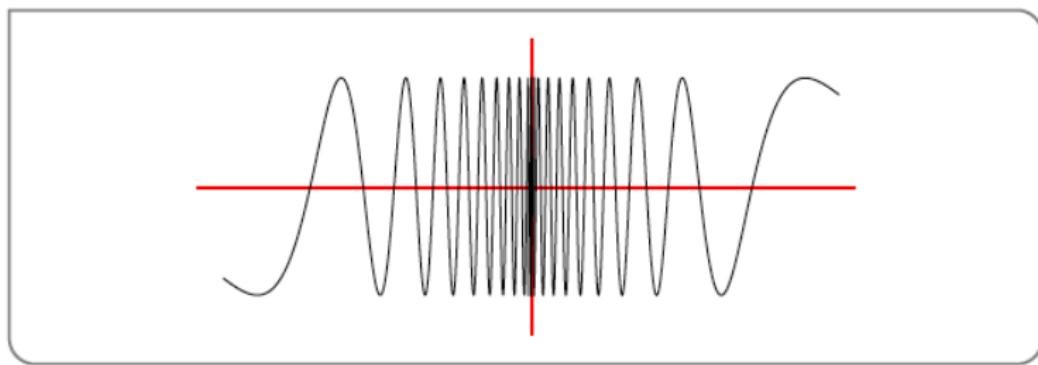
University of British Columbia

2019

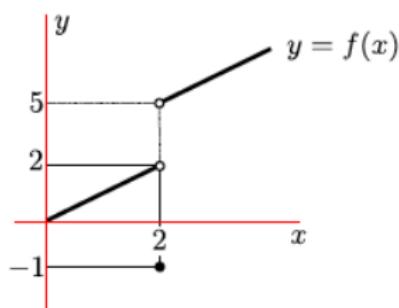
$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



$$f(x) = \sin\left(\frac{\pi}{x}\right)$$

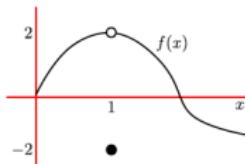


$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$



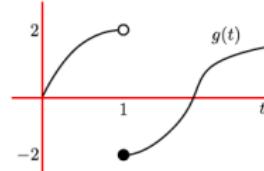
Example

Consider the graph of the function $f(x)$.



Example

Consider the graph of the function $g(t)$.



Then

$$\lim_{x \rightarrow 1^-} f(x) =$$

$$\lim_{x \rightarrow 1^+} f(x) =$$

$$\lim_{x \rightarrow 1} f(x) =$$

Then

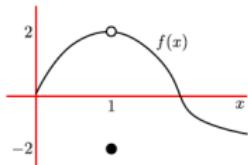
$$\lim_{t \rightarrow 1^-} g(t) =$$

$$\lim_{t \rightarrow 1^+} g(t) =$$

$$\lim_{t \rightarrow 1} g(t) =$$

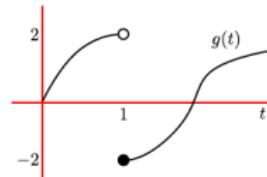
Example

Consider the graph of the function $f(x)$.



Example

Consider the graph of the function $g(t)$.



Then

$$\lim_{x \rightarrow 1^-} f(x) = 2$$

$$\lim_{x \rightarrow 1^+} f(x) = 2$$

$$\lim_{x \rightarrow 1} f(x) = 2$$

$$\lim_{t \rightarrow 1^-} g(t) = 2$$

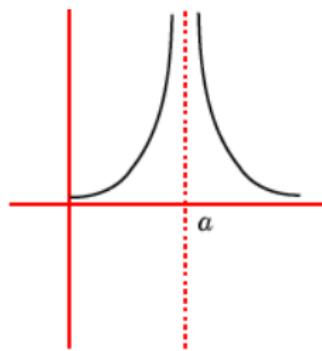
$$\lim_{t \rightarrow 1^+} g(t) = -2$$

$$\lim_{t \rightarrow 1} g(t) = DNE$$

When the limit goes to infinity

Example

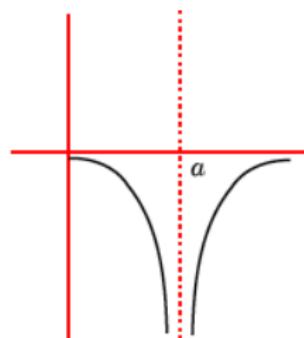
Consider the graph for the function $f(x)$.



$$\lim_{x \rightarrow a} f(x) = +\infty$$

Example

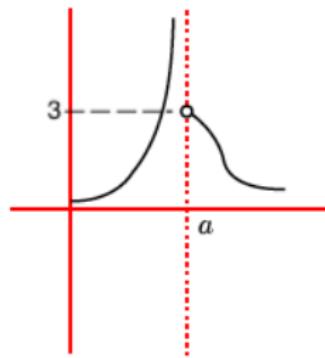
Consider the graph for the function $g(x)$.



$$\lim_{x \rightarrow a} g(x) = -\infty$$

Example

Consider the graph for the function $h(x)$.

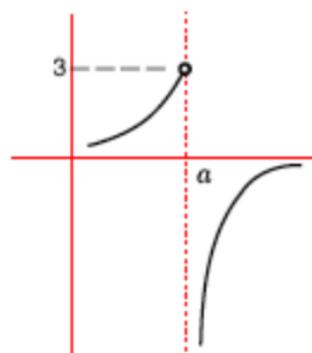


$$\lim_{x \rightarrow a^-} h(x) =$$

$$\lim_{x \rightarrow a^+} h(x) =$$

Example

Consider the graph for the function $s(x)$.

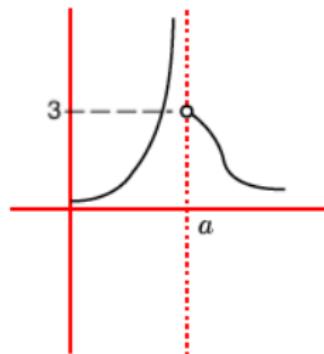


$$\lim_{x \rightarrow a^-} s(x) =$$

$$\lim_{x \rightarrow a^+} s(x) =$$

Example

Consider the graph for the function $h(x)$.

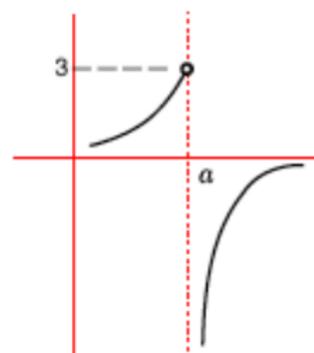


$$\lim_{x \rightarrow a^-} h(x) = +\infty$$

$$\lim_{x \rightarrow a^+} h(x) = 3$$

Example

Consider the graph for the function $s(x)$.



$$\lim_{x \rightarrow a^-} s(x) = 3$$

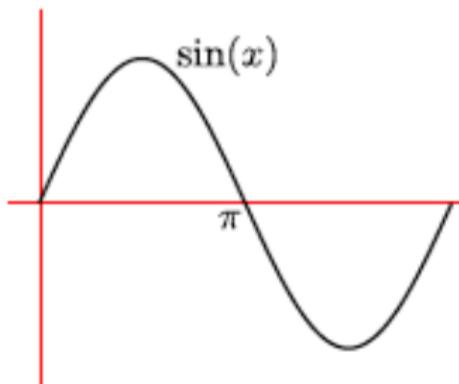
$$\lim_{x \rightarrow a^+} s(x) = -\infty$$

Example

Consider the function

$$g(x) = \frac{1}{\sin(x)}.$$

Find the one-side limits of this function as $x \rightarrow \pi$.



$$\lim_{x \rightarrow \pi^-} \frac{1}{\sin(x)} = +\infty$$

$$\lim_{x \rightarrow \pi^+} \frac{1}{\sin(x)} = -\infty$$

Second Session Outline

- ▶ Arithmetic of the Limits
- ▶ Limit of a ratio: what will happen if the limit of the denominator is zero. For example,

$$\lim_{x \rightarrow 0} \frac{1}{x^2} ? \quad \text{and} \quad \lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = ?$$

- ▶ Sandwich/ Squeeze/Pinch Theorem
- ▶ limit at infinity

Arithmetic of the Limits

Theorem

Let $a, c \in \mathbb{R}$. The following two limits hold

$$\lim_{x \rightarrow a} c = c \quad \lim_{x \rightarrow a} x = a$$

Example

$$\lim_{x \rightarrow 3} -2 = -2 \quad \lim_{x \rightarrow -1} x = -1$$

Theorem

(Arithmetic of Limits) Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x 's that lie in some interval about a (but f and g need not be defined exactly at a).

$$\lim_{x \rightarrow a} f(x) = F \quad \lim_{x \rightarrow a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. Then the following limits hold

- ▶ $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$ —limit of the sum is the sum of the limits.

Theorem

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- ▶ $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$ – limit of the sum is the sum of the limits.
- ▶ $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$ – limit of the difference is the difference of the limits.

Theorem

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- ▶ $\lim_{x \rightarrow a} cf(x) = cF$.

Theorem

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- ▶ $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$ – limit of the sum is the sum of the limits.
- ▶ $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$ – limit of the difference is the difference of the limits.
- ▶ $\lim_{x \rightarrow a} cf(x) = cF$.
- ▶ $\lim_{x \rightarrow a} (f(x).g(x)) = F.G$ – limit of the product is the product of the limits.

If $G \neq 0$ then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}$$

Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) =$$

Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) =$$

Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) =$$

Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

$$\lim_{x \rightarrow 1} 3f(x) = 3 \times \lim_{x \rightarrow 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x \rightarrow 1} 3f(x) - g(x) = 3 \times \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x \rightarrow 1} \frac{f(x)}{f(x) - g(x)} =$$

Example

Given

$$\lim_{x \rightarrow 1} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} g(x) = 2$$

We have

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$$\lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x \rightarrow 1} \frac{f(x)}{f(x) - g(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x)} = \frac{3}{3 - 2} = 3.$$

Example

$$\lim_{x \rightarrow 3} 4x^2 - 1 =$$

$$\lim_{x \rightarrow 2} \frac{x}{x - 1} =$$

Example

$$\lim_{x \rightarrow 3} 4x^2 - 1 = 4 \times \lim_{x \rightarrow 3} x^2 - \lim_{x \rightarrow 3} 1 = 35.$$

$$\lim_{x \rightarrow 2} \frac{x}{x-1} = \frac{\lim_{x \rightarrow 2} x}{\lim_{x \rightarrow 2} x - \lim_{x \rightarrow 1} 1} = \frac{2}{2-1} = 2.$$

Limit of a ratio: what will happen if the limit of the denominator is zero.

Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = DNE$$

Limit of a ratio: what will happen if the limit of denominator is zero:

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- the **limit is** $\pm\infty$, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x^4} = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{-x^2}{x^4} = \lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty.$$

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- the limit does **not exist**, eg.

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- the **limit is** 0, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = DNE$$

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- the **limit is** 0, eg.

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0.$$

- the **limit exists and it nonzero**, eg.

$$\lim_{x \rightarrow 0} \frac{x}{x} = 1.$$

Theorem

Let n be a positive integer, let $a \in R$ and let f be a function so that

$$\lim_{x \rightarrow a} f(x) = F$$

for some real number F . Then the following holds

$$\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit.

Theorem

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for some real number F . Then the following holds

$$\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit.

Similarly, if

- ▶ n is an even number and $F > 0$, or
- ▶ n is an odd number and F is any real number

then

$$\lim_{x \rightarrow a} (f(x))^{1/n} = \left(\lim_{x \rightarrow a} f(x) \right)^{1/n} = F^{1/n}.$$

Example

$$\lim_{x \rightarrow 4} x^{1/2} =$$

$$\lim_{x \rightarrow 4} (-x)^{1/2} =$$

$$\lim_{x \rightarrow 2} (4x^2 - 3)^{1/3} =$$

Example

$$\lim_{x \rightarrow 4} x^{1/2} = 4^{1/2} = 2.$$

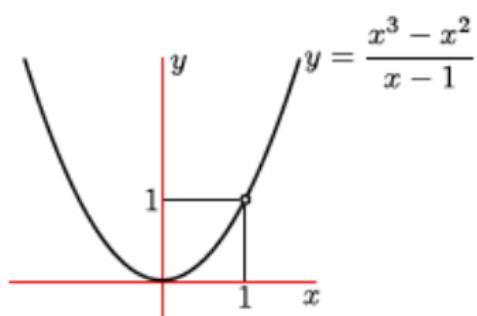
$$\lim_{x \rightarrow 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \rightarrow 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = (13)^{1/3}.$$

**Limit of a ratio: what will happen if the limit of the numerator and denominator are zero,
for example,**

$$\lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = ?$$

$$\lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = ?$$



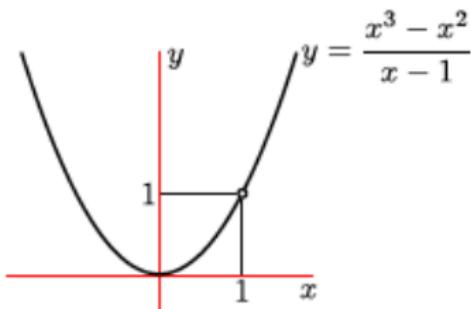
Theorem

If $f(x) = g(x)$ except when $x = a$ then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

provided the limit of g exists.

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases} \Rightarrow \lim_{x \rightarrow 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \rightarrow 1} x^2 = 1.$$

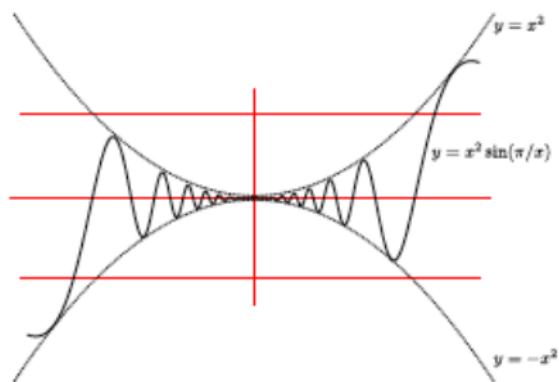
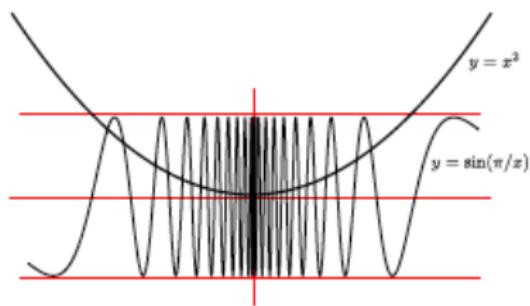


Sandwich/ Squeeze/Pinch Theorem

Example

Compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \rightarrow 1} f(x)?$$

Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \rightarrow 1} f(x)?$$

Solution

Consider that

$$\lim_{x \rightarrow 1} x = 1 \quad \text{and} \quad \lim_{x \rightarrow 1} x^2 - 2x + 2 = 1.$$

Therefore, by the sandwich/pinch/squeeze theorem

$$\lim_{x \rightarrow 1} f(x) = 1.$$

Example

We want to compute

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{x}$$

By plug in some large numbers into $\frac{1}{x}$ we have

-10000	-1000	-100	100	1000	10000
-0.0001	-0.001	-0.01	0.01	0.001	0.0001

We see that as x is getting bigger and positive the function $\frac{1}{x}$ is getting closer to 0. Thus,

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0.$$

Moreover,

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

Limit at Infinity

Definition

(Informal limit at infinity.) We write

$$\lim_{x \rightarrow \infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and positive.

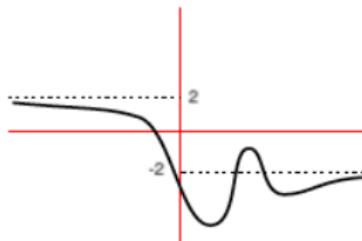
Similarly, we write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and negative.

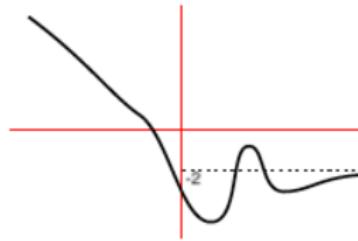
Example

Consider the graph of the function $f(x)$.



Example

Consider the graph of the function $g(x)$.



Then

$$\lim_{x \rightarrow \infty} f(x) =$$

$$\lim_{x \rightarrow -\infty} f(x) =$$

Then

$$\lim_{x \rightarrow \infty} g(x) =$$

$$\lim_{x \rightarrow -\infty} g(x) =$$

Example

Consider the graph of the function $f(x)$.



Example

Consider the graph of the function $g(x)$.



Then

$$\lim_{x \rightarrow \infty} f(x) = -2$$

$$\lim_{x \rightarrow -\infty} f(x) = 2$$

Then

$$\lim_{x \rightarrow \infty} g(x) = -2$$

$$\lim_{x \rightarrow -\infty} g(x) = +\infty$$

Review of the third session

Review

Theorem

sandwich (or squeeze or pinch) Let $a \in \mathbb{R}$ and let f, g, h be three functions so that

$$f(x) \leq g(x) \leq h(x)$$

for all x in an interval around a , except possibly at $x = a$. Then if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

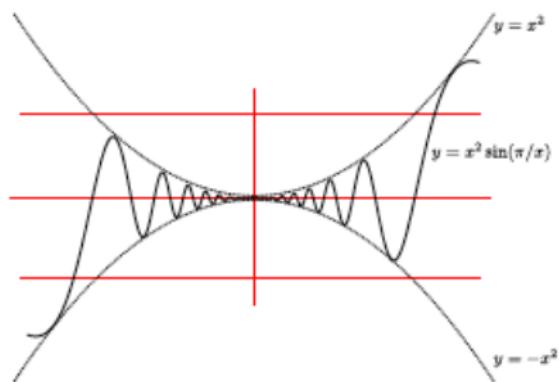
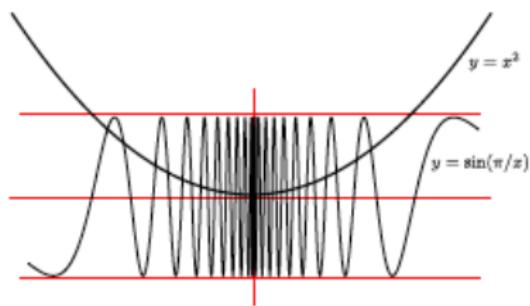
then it is also the case that

$$\lim_{x \rightarrow a} g(x) = L.$$

Example

Compute

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



Theorem

Let $c \in \mathbb{R}$ then the following limits hold

$$\lim_{x \rightarrow +\infty} c = c \quad \lim_{x \rightarrow -\infty} c = c$$

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

Outline For the Fourth Session

- ▶ Limit at Infinity

Limit at Infinity

Theorem

Let $f(x)$ and $g(x)$ be two functions for which the limits

$$\lim_{x \rightarrow \infty} f(x) = F \quad \lim_{x \rightarrow \infty} g(x) = G$$

exist. Then the following limits hold

$$\lim_{x \rightarrow \infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x \rightarrow \infty} f(x)g(x) = FG$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \text{provided } G \neq 0$$

and for rational numbers r ,

$$\lim_{x \rightarrow \infty} (f(x))^r = F^r$$

provided that $f(x)^r$ is defined for all x .

The analogous results hold for limits to $-\infty$.



Warning: Consider that

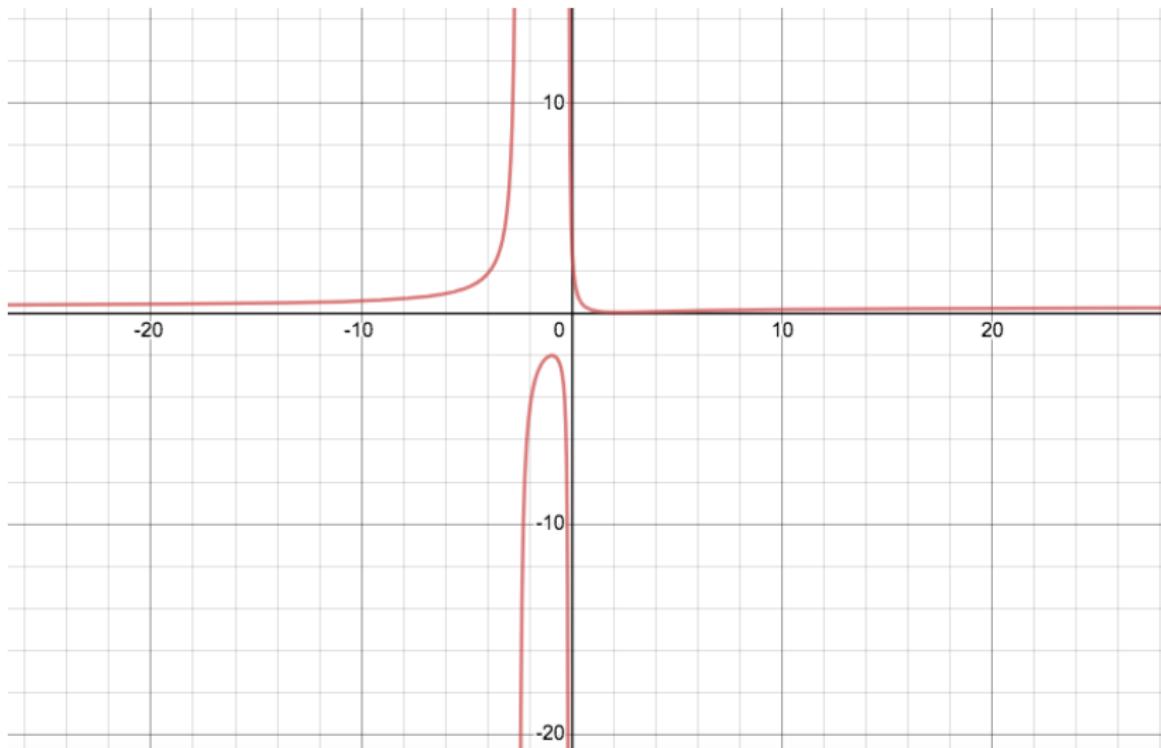
$$\lim_{x \rightarrow +\infty} \frac{1}{x^{1/2}} = 0$$

However,

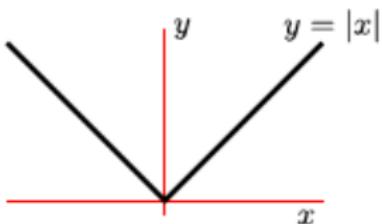
$$\lim_{x \rightarrow +\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because $x^{1/2}$ is not defined for $x < 0$.

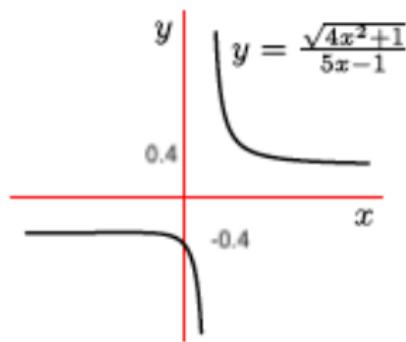
$$f(x) = \frac{x^2 - 3x + 4}{3x^2 + 8x + 1}$$



$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



$$y = \frac{\sqrt{4x^2 + 1}}{5x - 1}$$



Theorem

Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that

$$\lim_{x \rightarrow a} f(x) = +\infty \quad \lim_{x \rightarrow a} g(x) = +\infty \quad \lim_{x \rightarrow a} h(x) = H$$

1.

$$\lim_{x \rightarrow a} (f(x) + g(x)) =$$

2.

$$\lim_{x \rightarrow a} (f(x) + h(x)) =$$

3.

$$\lim_{x \rightarrow a} (f(x) - g(x)) =$$

4.

$$\lim_{x \rightarrow a} (f(x) - h(x)) =$$

Theorem

Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that

$$\lim_{x \rightarrow a} f(x) = +\infty \quad \lim_{x \rightarrow a} g(x) = +\infty \quad \lim_{x \rightarrow a} h(x) = H$$

1.

$$\lim_{x \rightarrow a} (f(x) + g(x)) = +\infty.$$

2.

$$\lim_{x \rightarrow a} (f(x) + h(x)) = +\infty.$$

3.

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \text{undetermined}.$$

4.

$$\lim_{x \rightarrow a} (f(x) - h(x)) = +\infty.$$

Theorem

5.

$$\lim_{x \rightarrow a} cf(x) = \begin{cases} c > 0 \\ c = 0 \\ c < 0 \end{cases}$$

6.

$$\lim(f(x) \cdot g(x)) =$$

7.

$$\lim_{x \rightarrow a} (f(x) \cdot h(x)) = \begin{cases} H > 0 \\ H = 0 \\ H < 0 \end{cases}$$

8.

$$\lim_{x \rightarrow a} \frac{h(x)}{f(x)} =$$

Theorem

5.

$$\lim_{x \rightarrow a} cf(x) = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$$

6.

$$\lim(f(x).g(x)) = +\infty.$$

7.

$$\lim_{x \rightarrow a} (f(x).h(x)) = \begin{cases} +\infty & H > 0 \\ undetermined & H = 0 \\ -\infty & H < 0 \end{cases}$$

8.

$$\lim_{x \rightarrow a} \frac{h(x)}{f(x)} = 0.$$

Example

Consider the following three functions:

$$f(x) = x^{-2} \quad g(x) = 2x^{-2} \quad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \rightarrow 0} f(x) = +\infty \quad \lim_{x \rightarrow 0} g(x) = +\infty \quad \lim_{x \rightarrow 0} h(x) = +\infty.$$

Then

1.

$$\lim_{x \rightarrow 0} (f(x) - g(x)) =$$

2.

$$\lim_{x \rightarrow 0} (f(x) - h(x)) =$$

3.

$$\lim_{x \rightarrow 0} (g(x) - h(x)) =$$

Example

Consider the following three functions:

$$f(x) = x^{-2} \quad g(x) = 2x^{-2} \quad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x \rightarrow 0} f(x) = +\infty \quad \lim_{x \rightarrow 0} g(x) = +\infty \quad \lim_{x \rightarrow 0} h(x) = +\infty.$$

Then

1.

$$\lim_{x \rightarrow 0} (f(x) - g(x)) = \lim_{x \rightarrow 0} x^{-2} = \infty$$

2.

$$\lim_{x \rightarrow 0} (f(x) - h(x)) = \lim_{x \rightarrow 0} (1) = 1$$

3.

$$\lim_{x \rightarrow 0} (g(x) - h(x)) = \lim_{x \rightarrow 0} x^{-2} + 1 = \infty$$

Outline For the Session Five

- ▶ Limit at Infinity
- ▶ Continuity
- ▶ Continuous from the left and from the right
- ▶ Arithmetic of continuity
- ▶ continuity of composites
- ▶ Intermediate Value Theorem

Example

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{1}{x^2}$	100	10000	10^6		10^6	10000	100

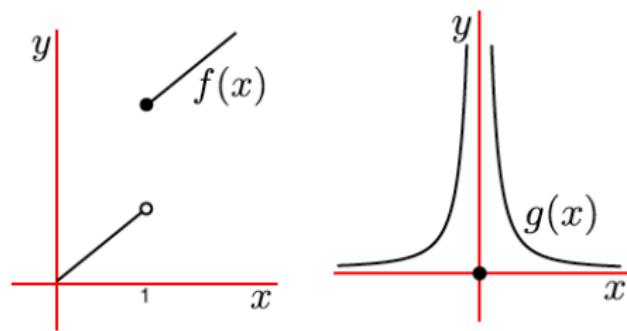
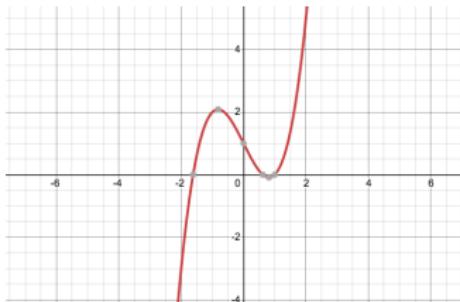
Consider that if

$$\lim_{x \rightarrow a} f(x) = \infty \quad \lim_{x \rightarrow a} g(x) = \infty$$

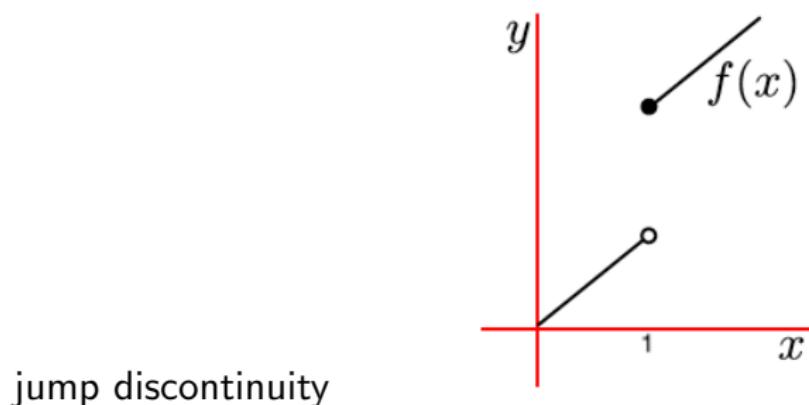
Then

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \text{undetermined}$$

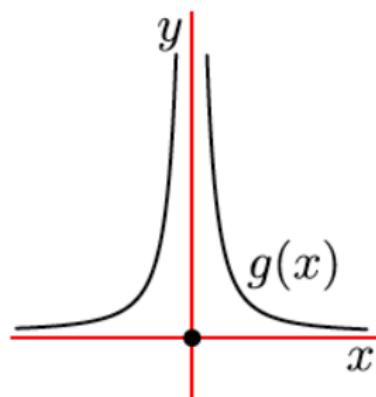
Continuity



$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$



$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



infinite discontinuity

$$h(x) = \begin{cases} \frac{x^3 - x^2}{x - 1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$



removable discontinuity

Outline - September 16, 2019

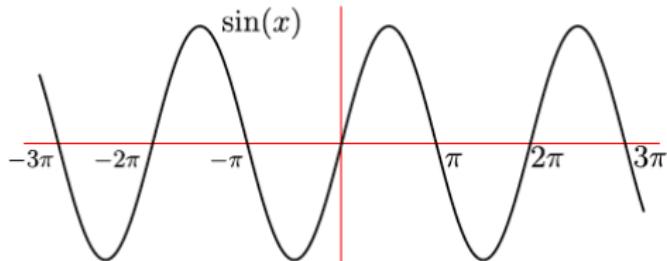
- ▶ **Section 1.6:**
 - ▶ Arithmetic of continuity
 - ▶ Continuity of composites
 - ▶ Intermediate Value Theorem
- ▶ **Section 2.1:**
 - ▶ Revisiting tangent lines

Arithmetic of continuity

Theorem

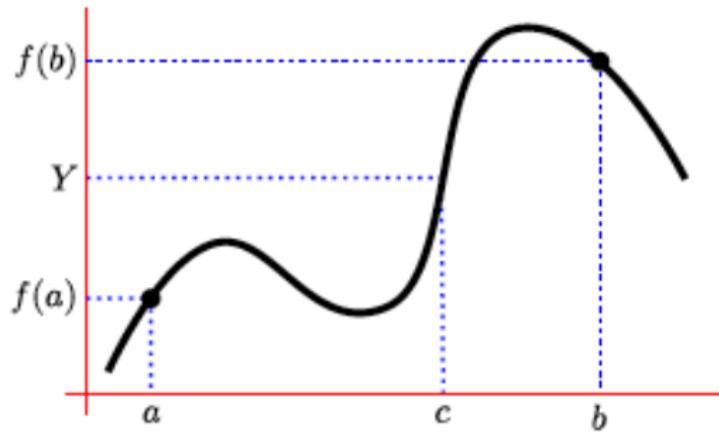
(Arithmetic of continuity) Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at a . Then the following functions are also continuous at $x = a$.

- ▶ $f(x) + g(x)$ and $f(x) - g(x)$,
- ▶ $cf(x)$ and $f(x)g(x)$, and
- ▶ $\frac{f(x)}{g(x)}$ provided $g(a) \neq 0$.

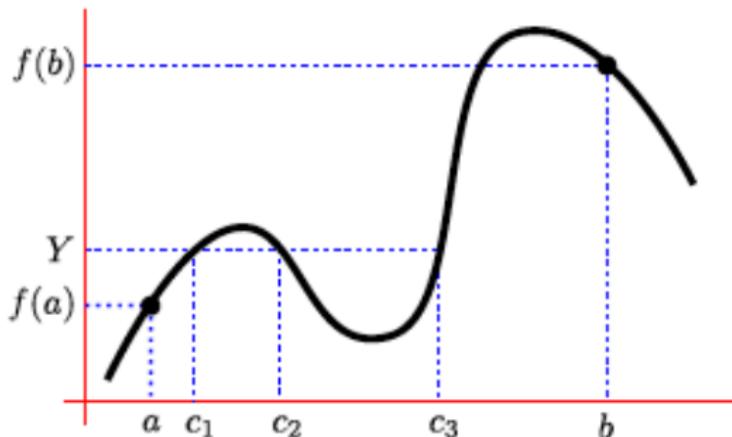


Intermediate value theorem(IVT)

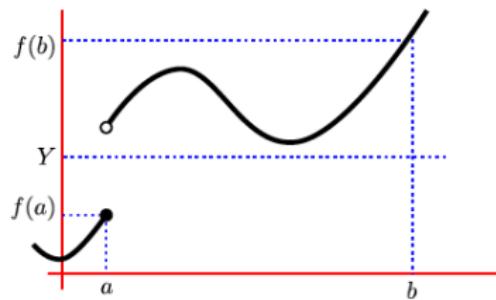
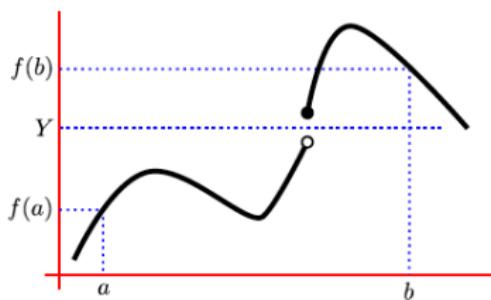
Theorem (Intermediate value theorem(IVT))



The existence not the uniqueness of c in IVT



Not continuous functions at $[a, b]$ do not satisfy IVT



Revisiting tangent lines

Revisiting tangent lines



$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \leftarrow \text{slope of the tangent line at } x = 1$$

Definition of the derivative



$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

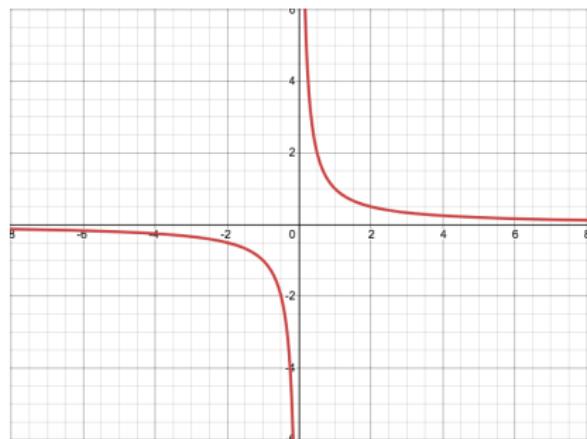


$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

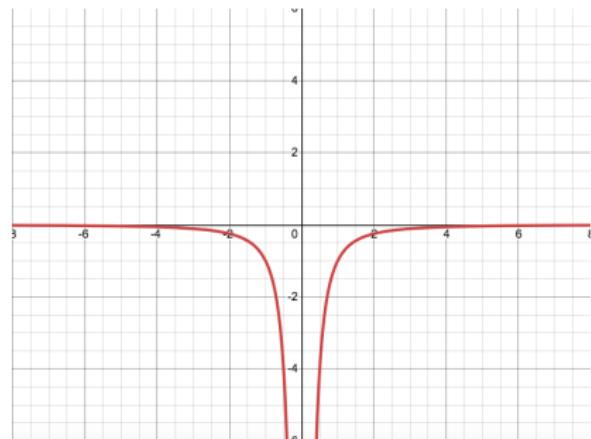
Examples

- ▶ $f(x) = c$
- ▶ $f(x) = x$
- ▶ $f(x) = x^2$
- ▶ $f(x) = \frac{1}{x}$
- ▶ $f(x) = \sqrt{x}$
- ▶ $f(x) = |x|$

$y = \frac{1}{x}$ and its derivative $-\frac{1}{x^2}$



$$y = \frac{1}{x}$$

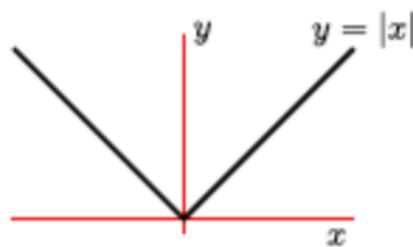


$$y = \frac{-1}{x^2}$$

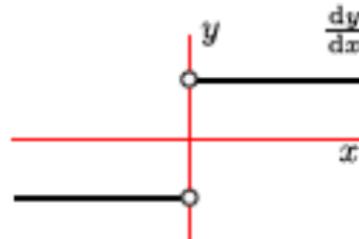
Tangent lines to $y = \sqrt{x}$



The derivative of the function $f(x) = |x|$: not differentiable at $x = 0$

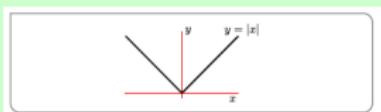


The derivative of the function $f(x) = |x|$

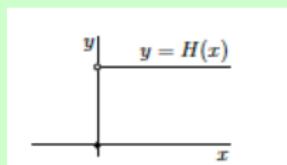


Where a function is not differentiable at $x = a$?

- ▶ Having a Sharp Corner at $x = a$



- ▶ The function is not continuous at $x = a$



- ▶ Having a tangent line, but the slope of the tangent line at $x = a$ is infinity



Outline - September 20, 2019

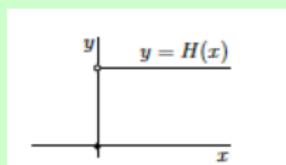
- ▶ **Section 2.2:**
 - ▶ Not differentiable examples
 - ▶ The relation between continuous and differentiable functions
- ▶ **Section 2.3:**
 - ▶ Interpretations of the derivative

Where a function is not differentiable at $x = a$?

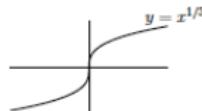
- ▶ Having a Sharp Corner at $x = a$



- ▶ The function is not continuous at $x = a$

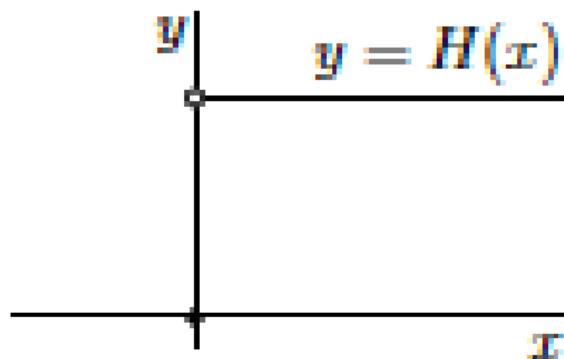


- ▶ Having a tangent line, but the slope of the tangent line at $x = a$ is infinity



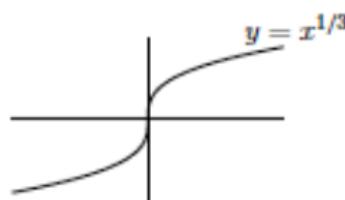
An example of a discontinuous and not differentiable function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$



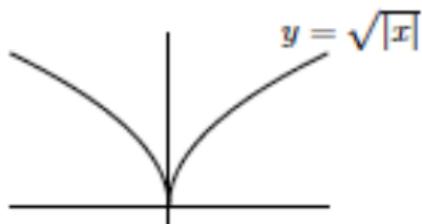
An example of a function with a tangent line with slope infinity at $x = 0$

$$f(x) = x^{1/3}$$

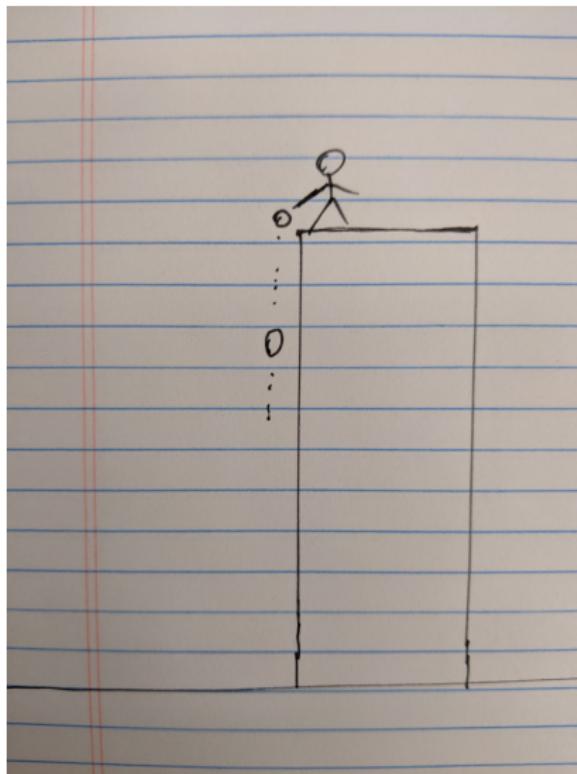


An example of a continuous and **not** differentiable function

$$y = \sqrt{|x|}$$



Instantaneous rate of change



average rate of change of $f(t)$ from $t = a$ to $t = a + h$ is

$$\frac{\text{change in } f(t) \text{ from } t = a \text{ to } t = a + h}{\text{length of time from } t = a \text{ to } t = a + h}$$

$$= \frac{f(a + h) - f(a)}{h}.$$

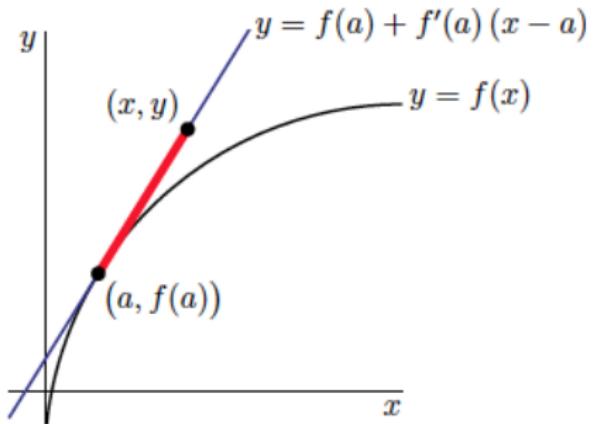
And so

instantaneous rate of change of $f(t)$ at $t = a$

$$= \lim_{h \rightarrow 0} [\text{average rate of change of } f(t) \text{ from } t = a \text{ to } t = a + h]$$

$$= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} = f'(a).$$

Finding tangent line to a curve at $x = a$



$$y = f(a) + f'(a)(x - a)$$

Outline - September 23, 2019

- ▶ **Section 2.4 and 2.5:**
 - ▶ Derivative of some simple functions
 - ▶ Tools
 - ▶ Examples

A list of derivative of some simple functions:

$$\frac{d}{dx} 1 = 0$$

$$\frac{d}{dx} x = 1$$

$$\frac{d}{dx} x^2 = 2x$$

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}.$$

A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \quad \frac{d}{dx}x = 1 \quad \frac{d}{dx}x^2 = 2x \quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Tools

Let $f(x)$ and $g(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- ▶ $\frac{d}{dx}\{f(x) + g(x)\} = f'(x) + g'(x)$
- ▶ $\frac{d}{dx}\{f(x) - g(x)\} = f'(x) - g'(x)$
- ▶ $\frac{d}{dx}\{cf(x)\} = cf'(x)$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- ▶ $\frac{d}{dx} \{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- ▶ $\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- ▶ $\frac{d}{dx} \{ f(x)g(x) \} = f'(x)g(x) + g'(x)f(x)$
- ▶ $\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶ $\frac{d}{dx} \{ cf(x) + dg(x) \} = cf'(x) + dg'(x)$
- ▶ $\frac{d}{dx} \{ f(x)^2 \} = 2f(x)f'(x)$
- ▶ $\frac{d}{dx} \left\{ \frac{1}{g(x)} \right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- ▶ $\frac{d}{dx} \{ f(x)g(x) \} = f'(x)g(x) + g'(x)f(x)$
- ▶ $\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶ $\frac{d}{dx} \{ cf(x) + dg(x) \} = cf'(x) + dg'(x)$
- ▶ $\frac{d}{dx} \{ f(x)^2 \} = 2f(x)f'(x)$
- ▶ $\frac{d}{dx} \left\{ \frac{1}{g(x)} \right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶ $\frac{d}{dx} \{ f(x)g(x)h(x) \} =$
 $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- ▶ $\frac{d}{dx} \{ f(x)^n \} = nf^{n-1}(x)f'(x)$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

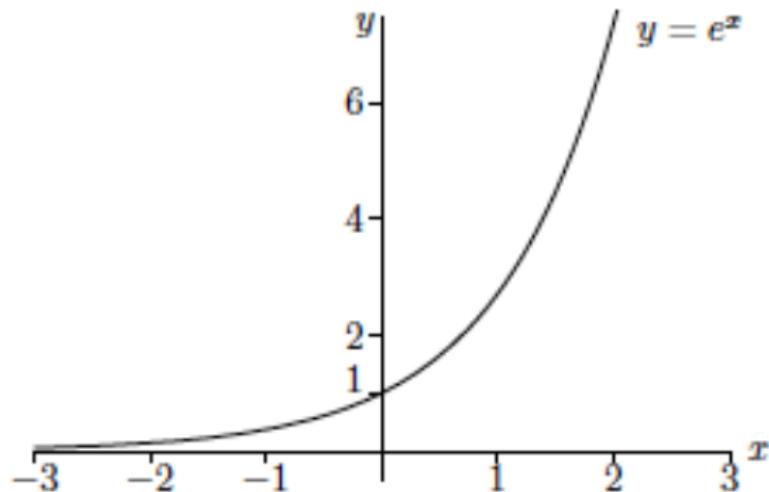
- ▶ $\frac{d}{dx} \{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- ▶ $\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶ $\frac{d}{dx} \{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- ▶ $\frac{d}{dx} \{f(x)^2\} = 2f(x)f'(x)$
- ▶ $\frac{d}{dx} \left\{ \frac{1}{g(x)} \right\} = \frac{-g'(x)}{g(x)^2} \quad g(x) \neq 0$
- ▶ $\frac{d}{dx} \{f(x)g(x)h(x)\} =$
 $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- ▶ $\frac{d}{dx} \{f(x)^n\} = nf^{n-1}(x)f'(x)$
- ▶ Let a be a rational number, then

$$\frac{d}{dx} x^a = ax^{a-1}.$$

Outline - September 25, 2019

- ▶ **Section 2.7 and 2.8:**
 - ▶ Derivative of exponential functions
 - ▶ Derivative of trigonometric functions

The graph of e^x



The graph of q^x where $q > 1$



YOUR TURN!

Example

Find a such that the following function is continuous.

$$f(x) = \begin{cases} e^{x+a} & x < 0 \\ \sqrt{x+1} & x \geq 0 \end{cases}$$

Example

We have

1. $\log_q(xy) =$

- (a) $\log_q(x) + \log_q(y)$
- (b) $\log_q(x) \log_q(y)$

2. $\log_q(x/y) =$

3. $\log_q(x^r) =$

Example

We have

$$1. \log_q(xy) = \log_q(x) + \log_q(y).$$

The reason for this is that

$$q^{\log_q(xy)} = xy = q^{\log_q(x)}q^{\log_q(y)} = q^{\log_q(x)+\log_q(y)}$$

Therefore, $\log_q(xy) = \log(x) + \log(y)$.

$$2. \log_q(x/y) = \log_q(x) - \log_q(y)$$

$$3. \log_q(x^r) = r \log_q(x)$$

TOOLS:

$$\frac{d}{dx}(f \circ g)(x) = g'(x)f'(g(x))$$

A list of derivative of some simple functions:

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = (\log_e a)a^x$$

Example

Find the derivative of $2^{\sqrt{x}}$.

Example

Find the derivative of $2^{\sqrt{x}}$.

Example

Find a and b such that the following function is differentiable.

$$f(x) = \begin{cases} x^3 + a & x < 1 \\ e^{x-1} + bx & x \geq 1 \end{cases}$$

Outline - September 30, 2019

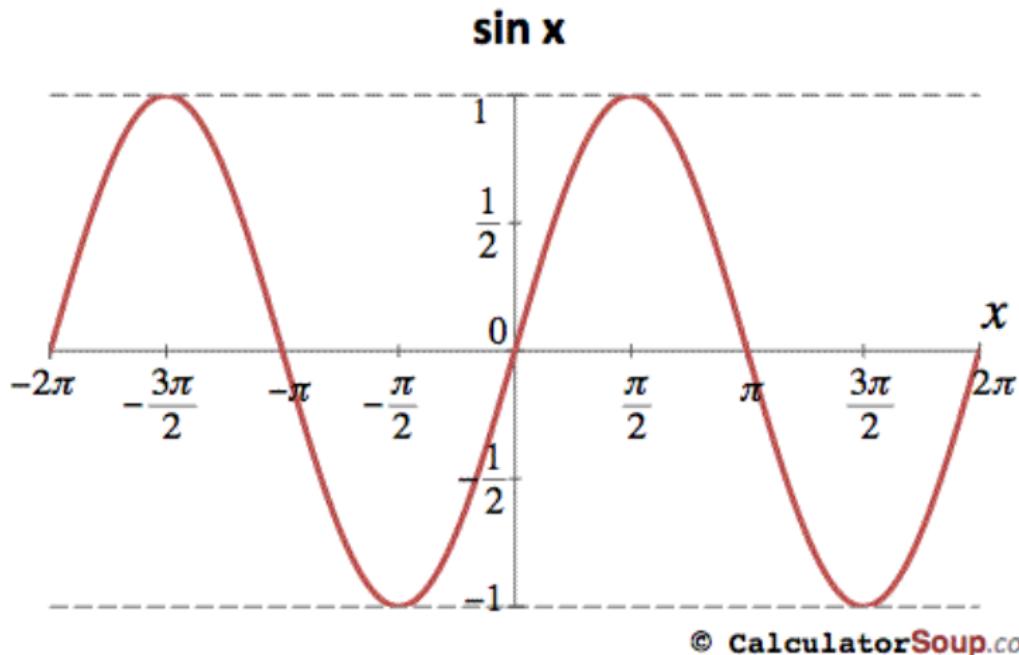
- ▶ **Section 2.8, 2.9, 0.6:**
 - ▶ Derivative of trigonometric functions
 - ▶ The chain rule
 - ▶ inverse of a function

A list of derivative of some simple functions:

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} a^x = (\log_e a) a^x$$

$\sin(x)$ domain = \mathbb{R} range = $[-1, 1]$

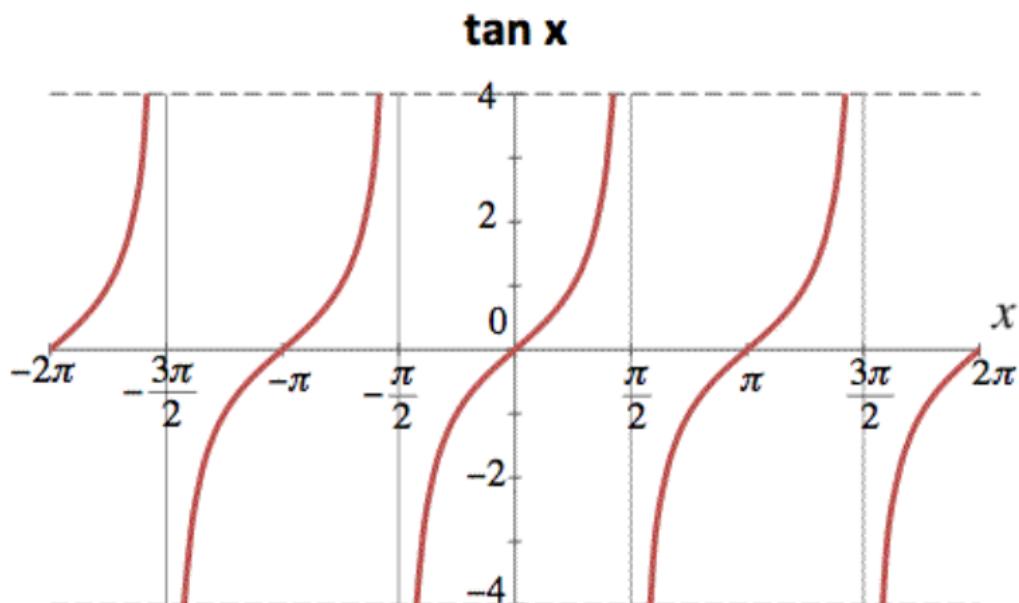


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$\cos(x)$ domain = \mathbb{R} range = $[-1, 1]$

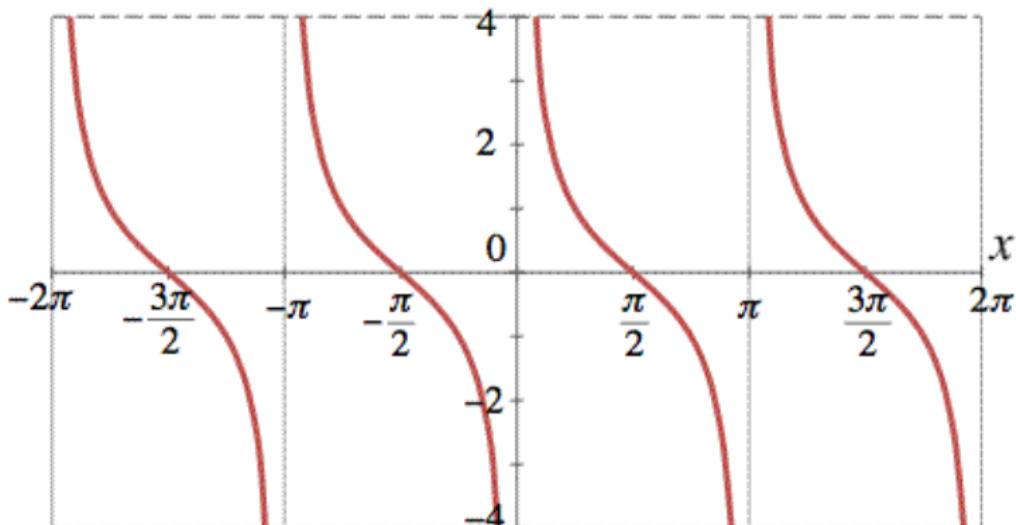


$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \text{domain} = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R}$$



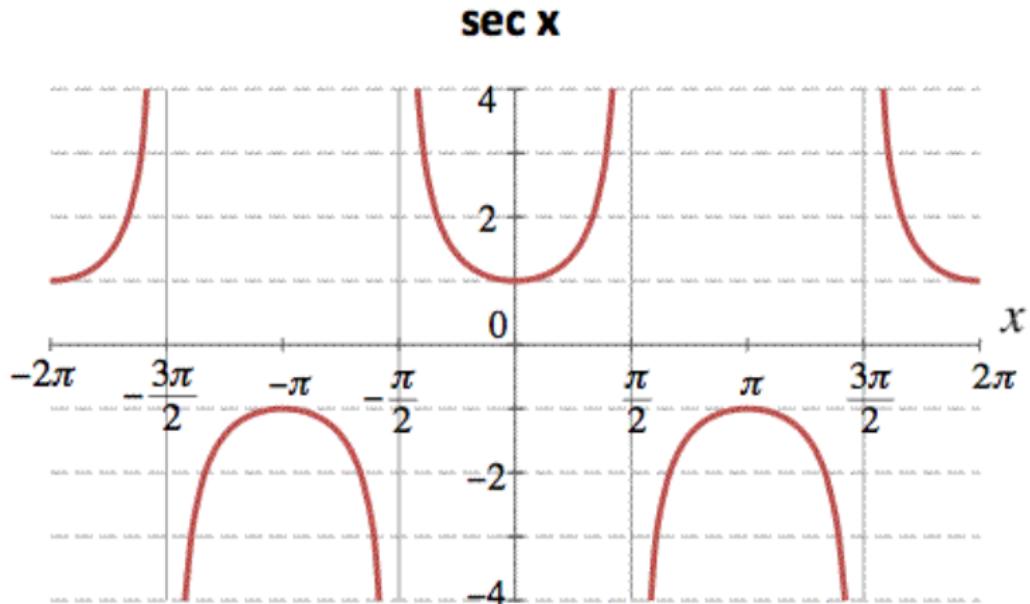
$$\cot(x) = \frac{\cos(x)}{\sin(x)} \text{ domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \text{ range} = \mathbb{R}$$

cot x



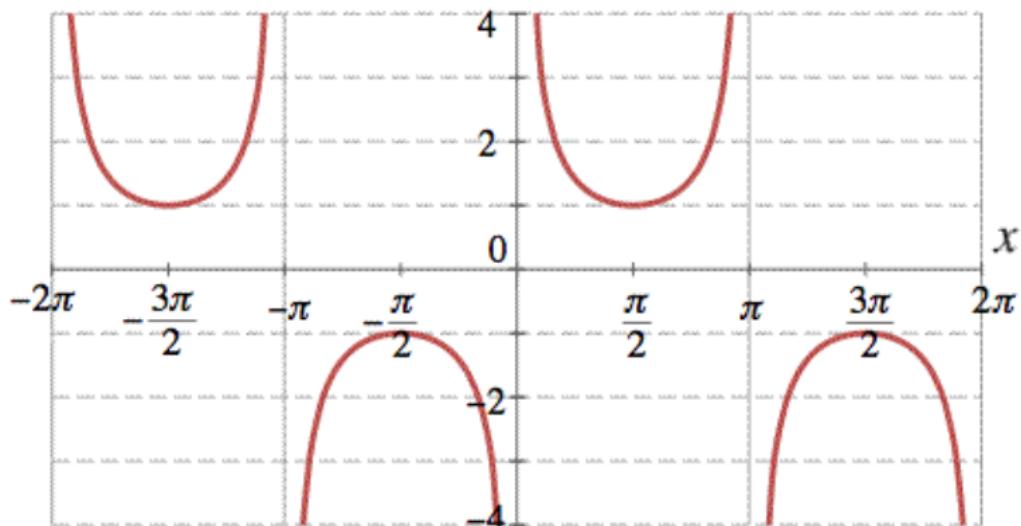
$$\sec(x) = \frac{1}{\cos(x)} \quad \text{domain} = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$$

$$\text{range} = \mathbb{R} - (-1, 1)$$



$$\csc(x) = \frac{1}{\sin(x)} \text{ domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \text{ range} = \mathbb{R} - (-1, 1)$$

CSC X



Derivative of $\sin(x)$

Question: Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of $\sin(x)$ at $x = 0$.

Derivative of $\sin(x)$

Question: Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of $\sin(x)$ at $x = 0$.

(sandwich (or squeeze or pinch) theorem) Let $a \in \mathbb{R}$ and let f, g, h be three functions so that $f(x) \leq g(x) \leq h(x)$ for all x in an interval around a , except possibly at $x = a$. Then if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then it is also the case that

$$\lim_{x \rightarrow a} g(x) = L.$$

An example of the application of the chain rule



- ▶ Your position at time t is $x(t)$.
- ▶ The temperature of the air at position x is $f(x)$.
- ▶ The temperature that you feel at time t is $F(t) = f(x(t))$.
- ▶ The instantaneous rate of change of temperature that you feel is $F'(t)$.

The chain rule

Theorem

Let f and g be differentiable functions then

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

The chain rule

Theorem

Let $y = f(u)$ and $u = g(x)$ be differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

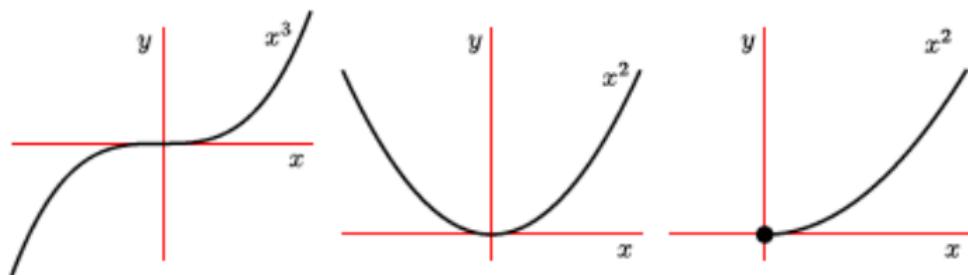
Outline - October 2, 2019

- ▶ **Section 0.6, 2.10:**
 - ▶ Inverse of a function
 - ▶ Natural logarithm

input number $x \mapsto f$ does “stuff” to $x \mapsto$ return number y

take output $y \mapsto$ do “stuff” to $y \mapsto$ return the original
number x

One-to-one functions

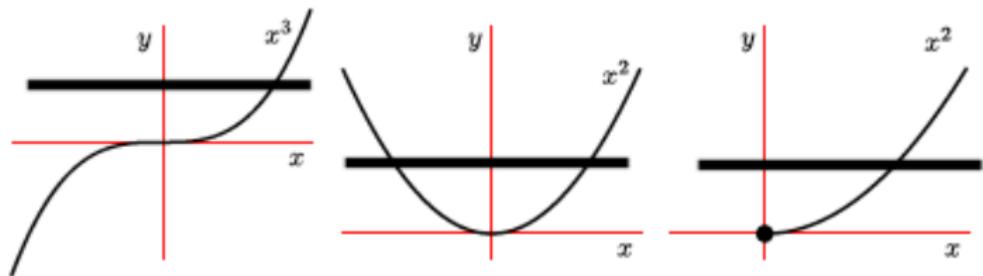


$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

$$\begin{array}{ccc} [0, \infty] & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

One-to-one functions



$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

is one-to-one

$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is not one-to-one

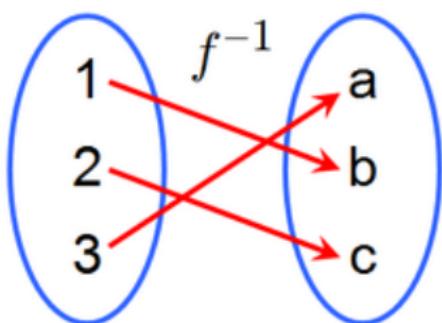
$$\begin{array}{ccc} [0, \infty) & \rightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is one-to-one

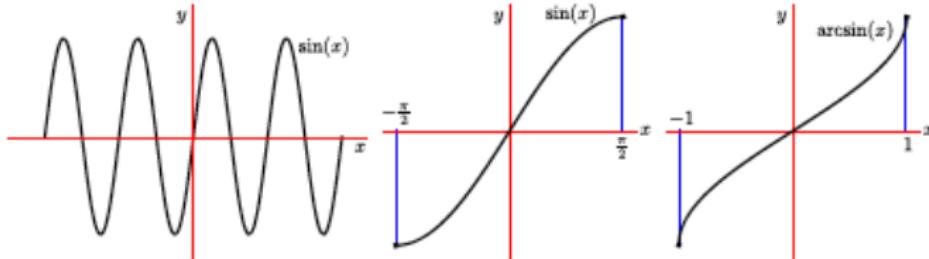
Inverse of a functions



Inverse of a functions



Inverse of $\sin(x)$

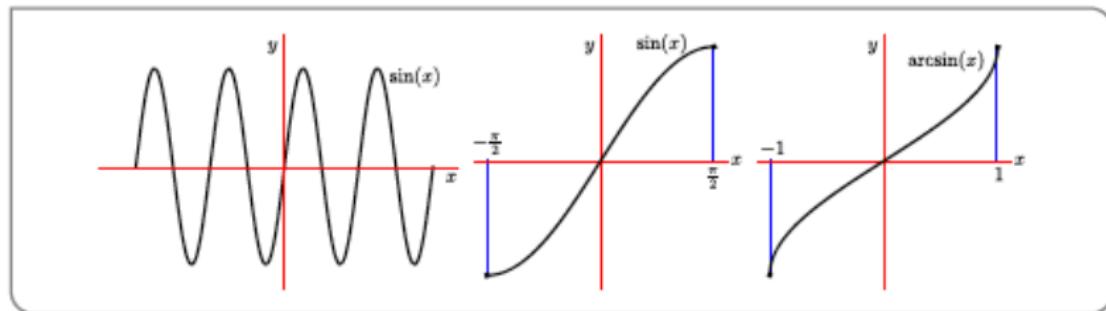


Inverse of $\sin(x)$



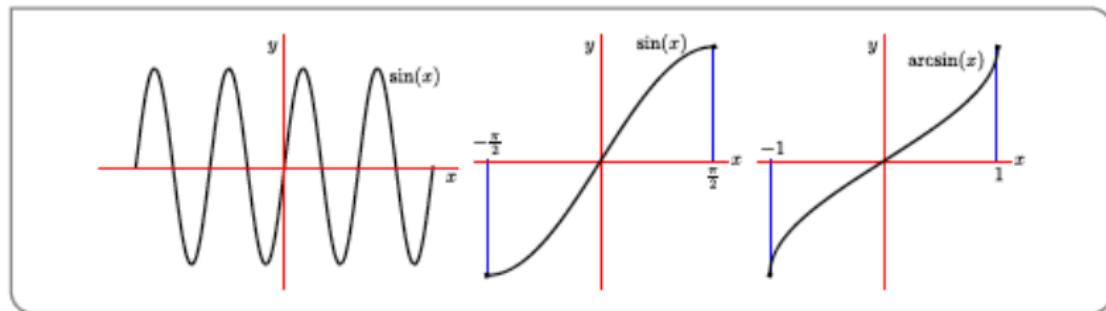
- ▶ $\sin(x)$ is not invertible on the domain \mathbb{R} because it is not one-to-one.

Inverse of $\sin(x)$



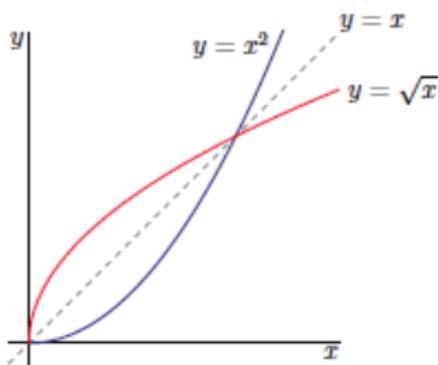
- ▶ $\sin(x)$ is not invertible on the domain \mathbb{R} because it is not one-to-one.
- ▶ If we look at $\sin(x)$ on the domain $[-\pi/2, \pi/2]$, then it is one-to-one, and so it has an inverse.

Inverse of $\sin(x)$



- ▶ $\sin(x)$ is not invertible on the domain \mathbb{R} because it is not one-to-one.
- ▶ If we look at $\sin(x)$ on the domain $[-\pi/2, \pi/2]$, then it is one-to-one, and so it has an inverse.
- ▶ The inverse of $\sin(x)$ is $\arcsin(x)$ on the domain $[-1, 1]$ and with the range $[-\pi/2, \pi/2]$.

How to find the inverse of a function by its graph



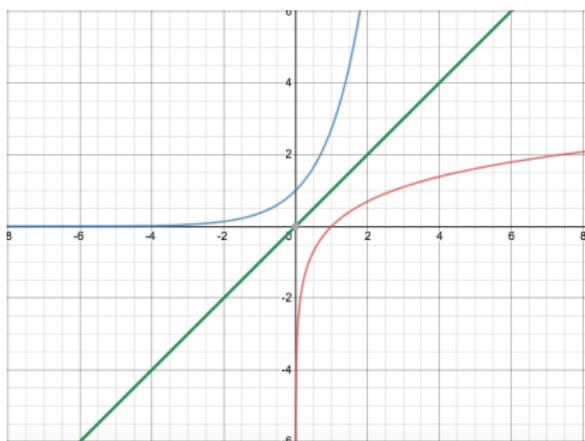
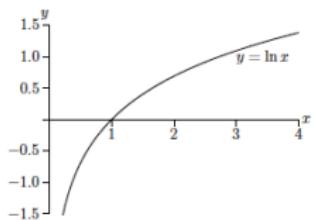
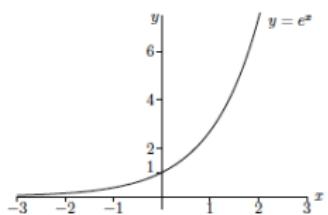
$$a^{\log_a x} = x$$

Remember that for $a > 1$,

$$a^{\log_a x} = x,$$

$$\log_a x = \frac{\log_e x}{\log_e a}.$$

The inverse of e^x



Outline - October 4, 2019

- ▶ **Section 2.10 and 2.11:**
 - ▶ Natural logarithm
 - ▶ Implicit derivative

Useful facts!

- ▶ $\frac{d}{dx} a^x = (\ln a) a^x.$
- ▶ $\log_a x = \frac{\ln x}{\ln a}$ $\ln x = \frac{\log_a x}{\log_a e}$ $a > 1.$
- ▶ $\ln(xy) = \ln x + \ln y.$
- ▶ $\ln(x/y) = \ln x - \ln y.$
- ▶ $\ln x^r = r \ln x.$

Useful facts!

- ▶ $\frac{d}{dx} a^x = (\ln a) a^x.$
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- ▶ $\ln x^r = r \ln x.$
- ▶ $\frac{d}{dx} \ln x = \frac{1}{x}.$

Useful facts!

- ▶ $\frac{d}{dx} a^x = (\ln a) a^x.$
- ▶ $\log_a x = \frac{\ln x}{\ln a}$ $\ln x = \frac{\log_a x}{\log_a e}$ $a > 1.$
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- ▶ $\frac{d}{dx} \ln x = \frac{1}{x}.$
- ▶ $\frac{d}{dx} \ln |x| = \frac{1}{x}.$

Useful facts!

- ▶ $\frac{d}{dx} a^x = (\ln a) a^x.$
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- ▶ $\ln x^r = r \ln x.$
- ▶ $\frac{d}{dx} \ln x = \frac{1}{x}.$
- ▶ $\frac{d}{dx} \ln |x| = \frac{1}{x}.$
- ▶ $\frac{d}{dx} \log_a x = \frac{1}{x \cdot \ln a}.$

Useful facts!

- ▶ $\frac{d}{dx} a^x = (\ln a) a^x.$
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- ▶ $\ln x^r = r \ln x.$
- ▶ $\frac{d}{dx} \ln x = \frac{1}{x}.$
- ▶ $\frac{d}{dx} \ln |x| = \frac{1}{x}.$
- ▶ $\frac{d}{dx} \log_a x = \frac{1}{x \cdot \ln a}.$
- ▶ $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$

Useful facts!

- ▶ $\frac{d}{dx} a^x = (\ln a)a^x.$
- ▶ $\log_a x = \frac{\ln x}{\ln a}$ $\ln x = \frac{\log_a x}{\log_a e}$ $a > 1.$
- ▶ $\ln(xy) = \ln x + \ln y.$
- ▶ $\ln(x/y) = \ln x - \ln y.$
- ▶ $\ln x^r = r \ln x.$
- ▶ $\frac{d}{dx} \ln x = \frac{1}{x}.$
- ▶ $\frac{d}{dx} \ln |x| = \frac{1}{x}.$
- ▶ $\frac{d}{dx} \log_a x = \frac{1}{x \cdot \ln a}.$
- ▶ $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$
- ▶ $\frac{d}{dx} |f(x)| = \frac{f'(x)}{|f(x)|}.$

Outline - October 7, 2019

- ▶ **Section 2.11 and 2.12:**
 - ▶ Implicit derivative
 - ▶ Derivative of Trig functions

Implicit derivative

$$\frac{d}{dx}x = \frac{d}{dx}e^{\ln x} \quad (\frac{d}{dx}x = \frac{d}{dx}e^y)$$

which is the same as

$$1 = \left(\frac{d}{dx}\ln x\right).e^{\ln x} \quad (1 = y'e^y).$$

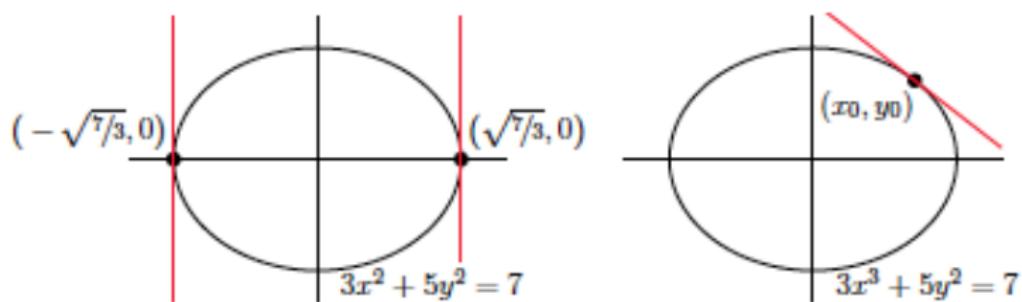
Note that $e^{\ln x} = x$ ($e^y = x$), thus

$$1 = \left(\frac{d}{dx}\ln x\right).x \quad (1 = y'x)$$

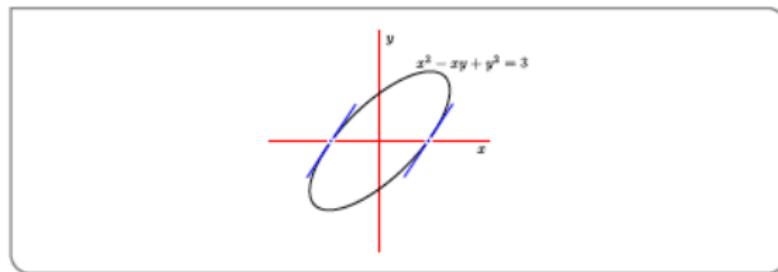
and so

$$\frac{d}{dx}\ln x = \frac{1}{x} \quad (y' = \frac{1}{x}).$$

$$3x^3 + 5y^2 = 7$$



$$x^2 - xy + y^2 = 3$$



$$x^{2/3} + y^{2/3} = 1$$



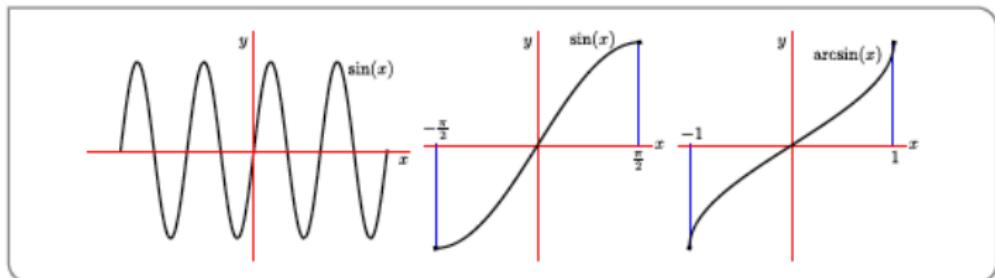
Outline - October 9, 2019

- ▶ **Section 2.12:**
 - ▶ Derivative of Trig functions

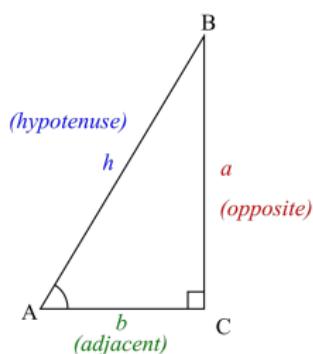
Review of the inverse of a function

Remember that the inverse of a one-to-one function $f(x)$ with domain A and range B is a function $g(x)$ with domain B and range A such that

$$f(g(y)) = y \quad g(f(x)) = x \quad x \in A, y \in B.$$



Trigonometry



- **sine:** $\sin A = \frac{a}{h} = \frac{\text{opposite}}{\text{hypotenuse}}$
- **cosine:** $\cos A = \frac{b}{h} = \frac{\text{adjacent}}{\text{hypotenuse}}$
- **tangent:** $\tan A = \frac{a}{b} = \frac{\text{opposite}}{\text{adjacent}}$
- **cosecant:** $\csc A = \frac{h}{a} = \frac{\text{hypotenuse}}{\text{opposite}}$
- **secant:** $\sec A = \frac{h}{b} = \frac{\text{hypotenuse}}{\text{adjacent}}$
- **cotangent:** $\cot A = \frac{b}{a} = \frac{\text{adjacent}}{\text{opposite}}$

$\arcsin(\sin(x))$

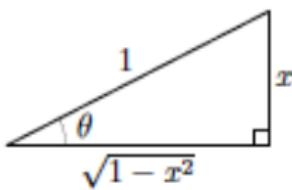
$\arcsin(\sin(x)) = \text{the unique angle } \theta \text{ between } -\pi/2 \text{ and } \pi/2$
obeying that

$$\sin(x) = \sin(\theta).$$

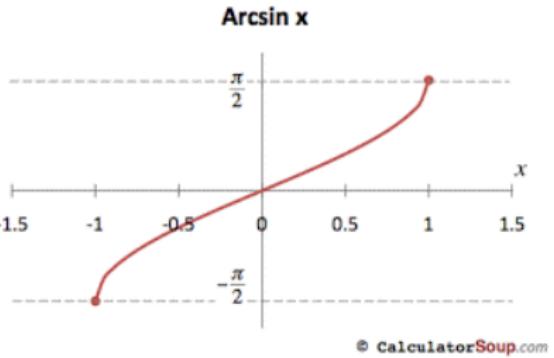
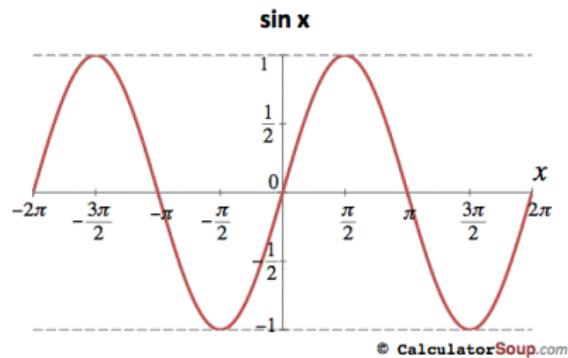
What is $\arcsin(\sin(\frac{11\pi}{16}))$?



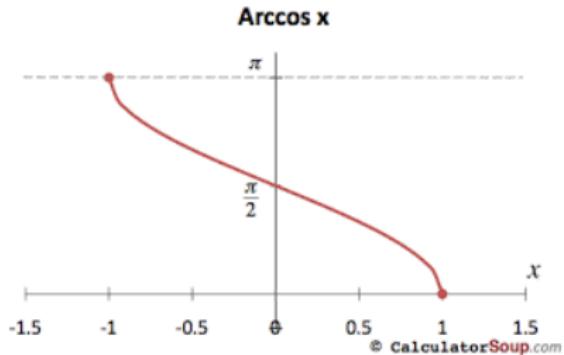
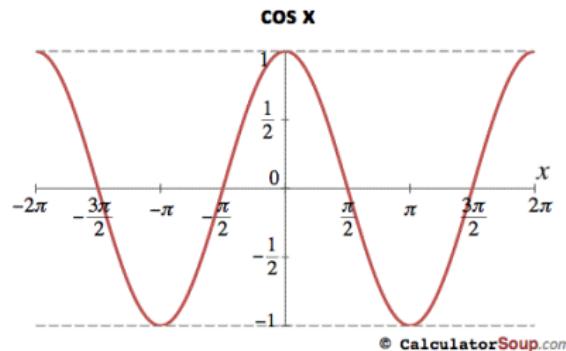
$$\cos(\arcsin(x)) = \sqrt{1 - x^2}$$



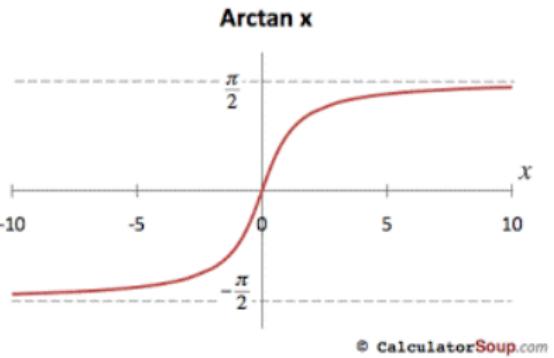
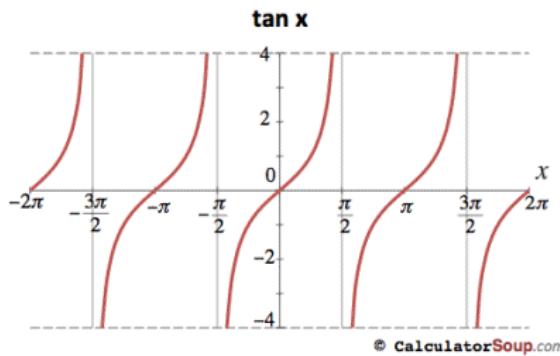
Inverse of $\sin(x)$



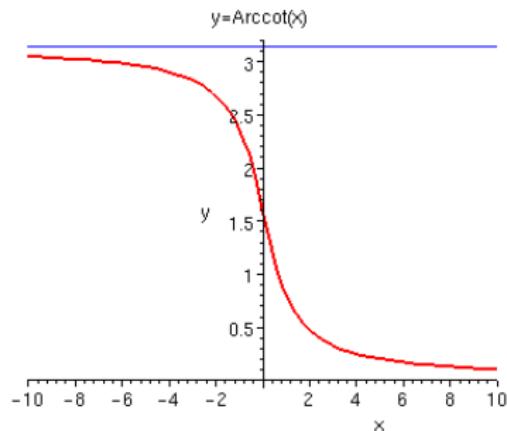
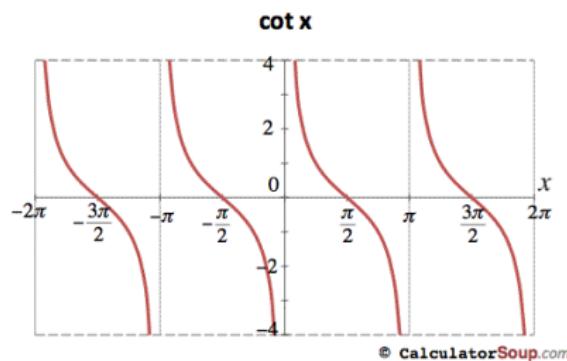
Inverse of $\cos(x)$



Inverse of $\tan(x)$

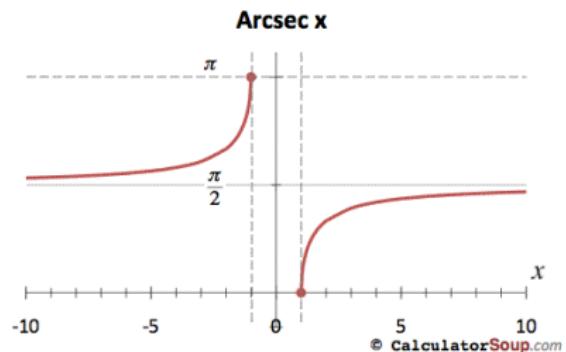
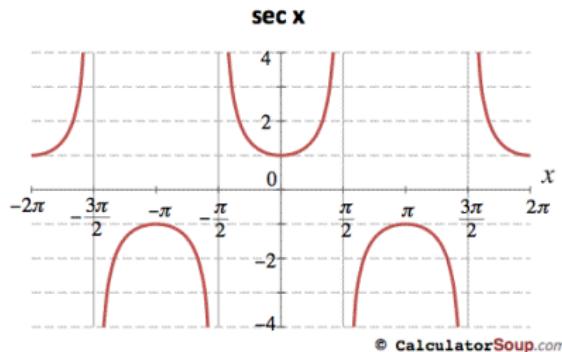


Inverse of cotan(x)



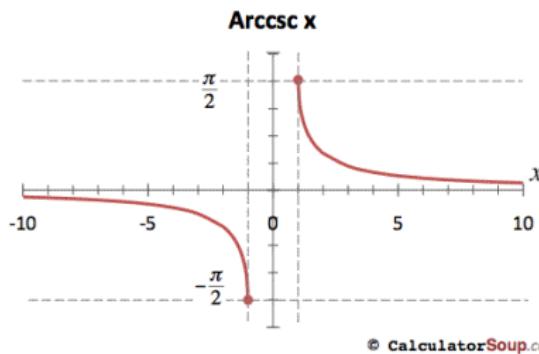
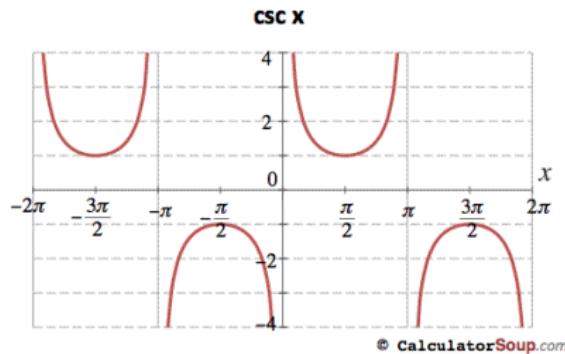
Inverse of sec(x)

$$\text{arcsec}(x) = \arccos(1/x)$$

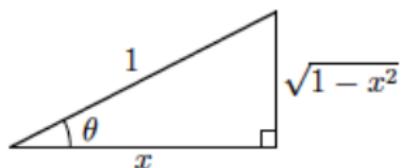


Inverse of csc(x)

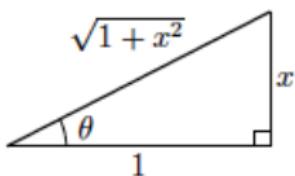
$$\text{arccsc}(x) = \arcsin(1/x)$$



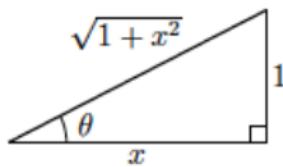
$$\sin(\theta) = \sin(\arccos(x)) = \sqrt{1 - x^2}$$



$$\cos^2(\arctan(x)) = \cos^2(\theta) = \frac{1}{1+x^2}.$$



$$\frac{1}{\csc^2(\theta)} = \sin^2(\theta) = 1 + x^2$$

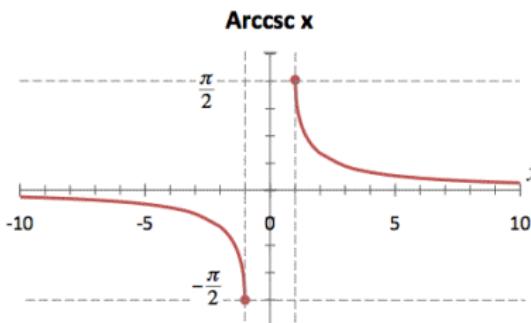
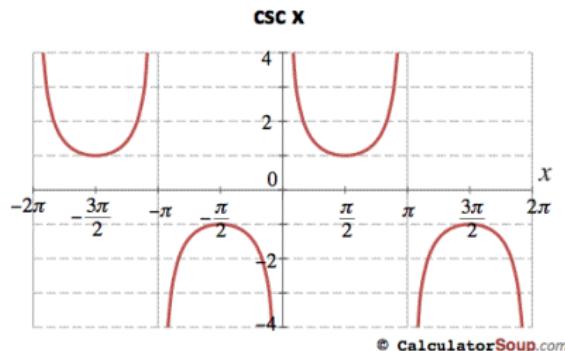


Outline - October 11, 2019

- ▶ **Section 3.1:**
 - ▶ Derivative of Trig functions

Inverse of $\csc(x)$

$$\text{arccsc}(x) = \arcsin(1/x)$$



Derivative of the inverses of trigonometric functions in a nutshell

In a nutshell the derivatives of the inverse trigonometric functions are

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \text{arccsc}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \text{arcsec}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \text{arccot}(x) = -\frac{1}{1+x^2}$$

The Application of Derivatives

Velocity and Acceleration

If you are moving along the x -axis and your position at time t is $x(t)$, then

- ▶ your velocity at time t is $v(t) = x'(t)$ and
- ▶ your acceleration at time t is $a(t) = v'(t) = x''(t)$.

Direction of your move with $x(t) = t^3 - 3t + 2$

t	$(t - 1)(t + 1)$	$x'(t) = 3(t - 1)(t + 1)$	Direction
$t < -1$	<i>positive</i>	<i>positive</i>	<i>right</i>
$t = -1$	<i>zero</i>	<i>zero</i>	<i>halt</i>
$-1 < t < 1$	<i>negative</i>	<i>negative</i>	<i>left</i>
$t = 1$	<i>zero</i>	<i>zero</i>	<i>halt</i>
$t > 1$	<i>positive</i>	<i>positive</i>	<i>right</i>

And here is a schematic picture of the whole trajectory.



Direction of your move with $x(t) = t^3 - 12t + 5$

t	$(t - 2)(t + 2)$	$x'(t) = 3(t - 2)(t + 2)$	Direction
$t < -2$	positive	positive	right
$t = -2$	zero	zero	halt
$-2 < t < 2$	negative	negative	left
$t = 2$	zero	zero	halt
$t > 2$	positive	positive	right

t	$your\ positionx(t)$	$x'(t)$	Direction
0	5	negative	left
$t = 2$	-11	zero	halt
$t = 10$	885	positive	right

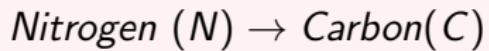
Outline - October 16, 2019

- ▶ **Section 3.2: Exponential Growth and Decay**
 - ▶ 3.1: Carbon Dating

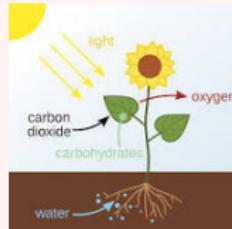
EXAM: Friday, October 18, Here in Class, at 2pm

Carbon Dating

Cosmic ray hitting atmosphere



Vegetation absorbs C through photosynth



Animals acquire C by eating plants



C decreases when animal dies



More precisely, let $Q(t)$ denote the amount of C (an element) in the plant or animal t years after it dies. The number of radioactive decays (rate of change) per unit time, at time t , is proportional to the amount of C present at time t , which is $Q(t)$. Thus

Radioactive Decay

$$\frac{dQ}{dt}(t) = -kQ(t) \quad (1)$$

Corollary

The function $Q(t)$ satisfies the equation

$$\frac{dQ}{dt} = -kQ(t)$$

if and only if

$$Q(t) = Q(0).e^{-kt}$$

The half-life (the half-life of C is the length of time that it takes for half of the C to decay) is defined to be the time $t_{1/2}$ which obeys

$$Q(t_{1/2}) = \frac{1}{2}.Q(0).$$

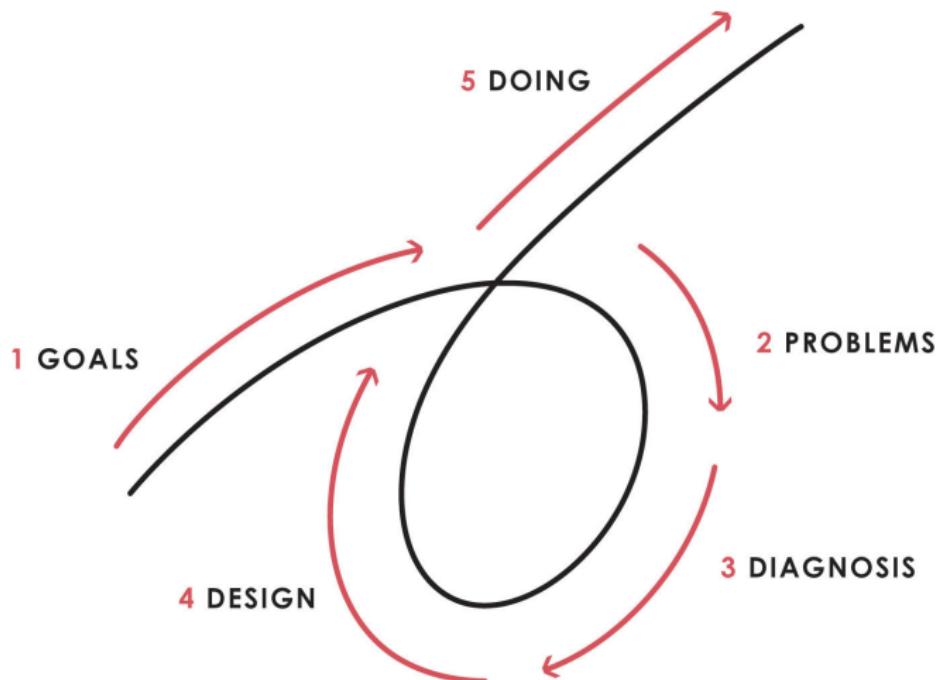
The half-life is related to the constant k by

$$t_{1/2} = \frac{\ln 2}{k}.$$

Outline - October 21, 2019

- ▶ **Section 3.3.2: Newton's Law of Cooling**
 - ▶ 3.1: Newton's Law of Cooling

No pain no gain

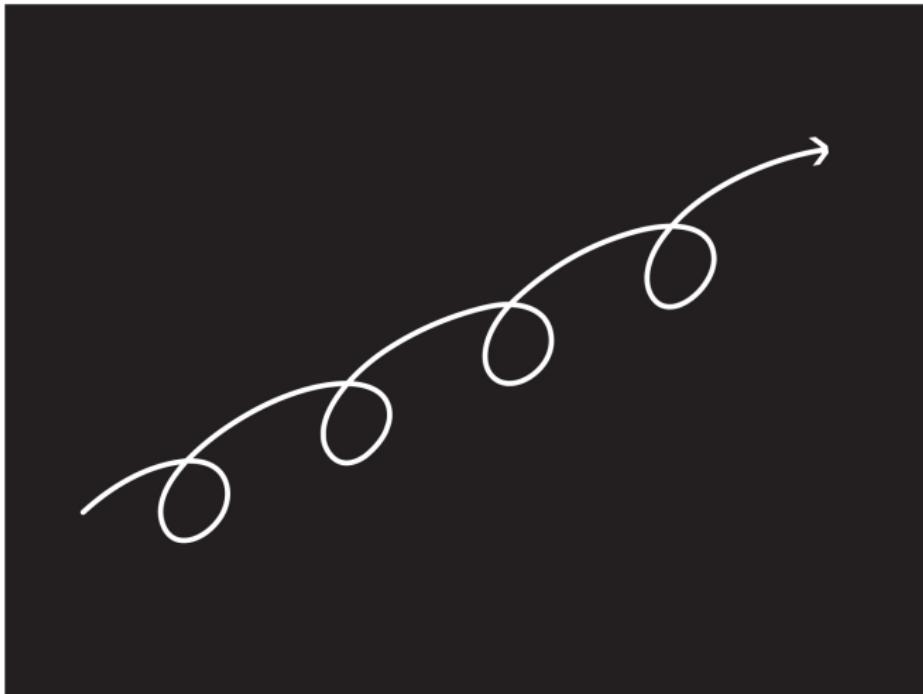


Principles (Ray Dalio)

Most people



Successful person



Newton's Law of Cooling



$$\frac{dT}{dt}(t) = K [T(t) - A].$$

We have three possibilities:

- ▶ $T(t) > A \Rightarrow [T(t) - A] > 0$, thus the temperature of the body is decreasing, so $\frac{dT}{dt}$ must be negative, since $\frac{dT}{dt}(t) = K [T(t) - A]$, we must have $K < 0$.
- ▶ $T(t) < A \Rightarrow [T(t) - A] < 0$, thus the temperature of the body is increasing, so $\frac{dT}{dt}$ must be positive, since $\frac{dT}{dt}(t) = K [T(t) - A]$, we must have $K < 0$.
- ▶ $T(t) = A \Rightarrow [T(t) - A] = 0$, thus the temperature of the body is no changing, so $\frac{dT}{dt}$ must be zero, since $\frac{dT}{dt}(t) = K [T(t) - A]$. This does not impose any condition on K .

Newton's Law of Cooling

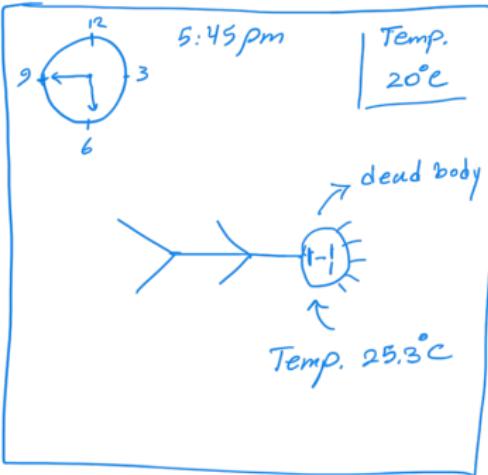
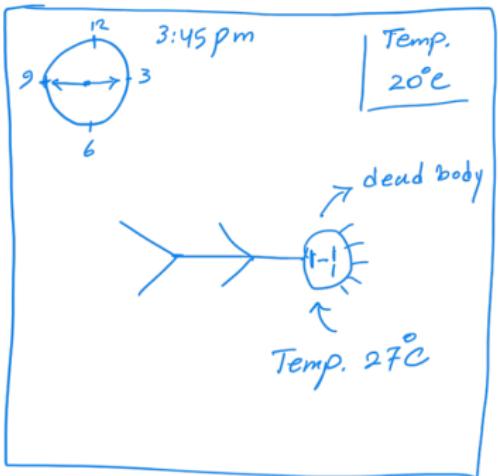
Corollary

A differentiable function $T(t)$ obeys the differential equation

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

if and only if

$$T(t) = [T(0) - A]e^{Kt} + A.$$



I entered this room at
9:30 am and talk to him
for 5 minutes.

Outline - October 23, 2019

- ▶ **Section 3.3.3: Population Growth**
- ▶ **Section 3.2: Related Rates**

Population Growth

Suppose that we wish to predict the size $P(t)$ of a population as a function of the time t . So suppose that in average each couple produces β offspring (for some constant β) and then dies. Then over the course of one generation since we have $P(t)/2$ couples and each have produced β offspring, thus the population of the children of one generation is

$$\beta \frac{P(t)}{2}.$$

Let t_g be the life span of one generation, then

$$\begin{aligned} P(t + t_g) &= \beta \frac{P(t)}{2} \\ &= P(t) + \beta \frac{P(t)}{2} - P(t). \end{aligned}$$

Therefore,

$$P(t + t_g) - P(t) = \beta \frac{P(t)}{2} - P(t)$$

and so dividing both sides by t_g , we have

$$\begin{aligned}\frac{P(t + t_g) - P(t)}{t_g} &= \frac{1}{t_g} \left(\frac{\beta}{2} P(t) - P(t) \right) \\ &= \frac{1}{t_g} \left(\frac{\beta}{2} - 1 \right) P(t)\end{aligned}$$

Let $\frac{1}{t_g} \left(\frac{\beta}{2} - 1 \right) = b$, then

$$\frac{P(t + t_g) - P(t)}{t_g} = bP(t).$$

Approximately, we have

$$\frac{dP}{dt} = bP(t).$$

Moreover, same as the model for carbon dating we can write

$$P(t) = P(0)e^{bt}.$$

Malthusian growth model

Malthusian growth model

The model for the population growth is

$$\frac{dP}{dt} = bP(t)$$

and $P(t)$ satisfies the above equation if and only if

$$P(t) = P(0)e^{bt}.$$

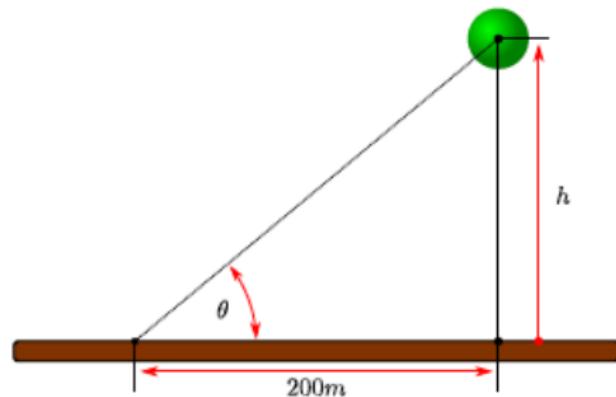
Related Rates

Volume of a sphere

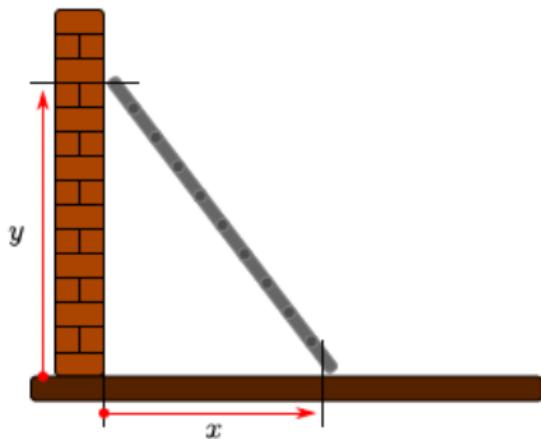
Remember that the volume of a sphere with radius r is

$$V = \frac{4}{3}\pi r^3.$$

Helium Balloon



Ladder

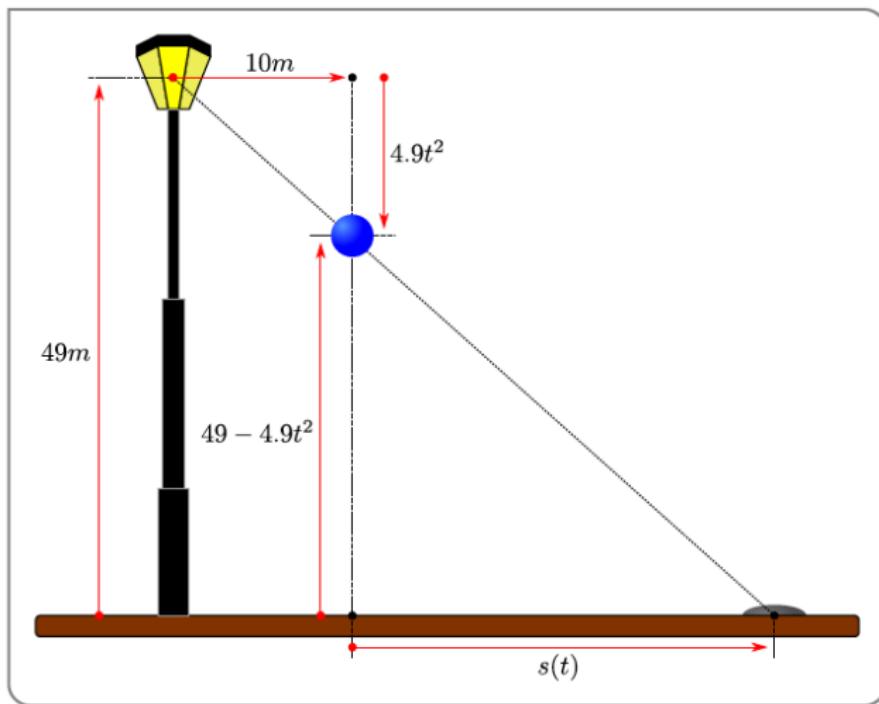


Outline - October 25, 2019

- ▶ **Section 3.2: Related Rates: An Example**
- ▶ **Section 3.4.2 The Linear Approximation**
- ▶ **Section 3.4.3 The Quadratic Approximation**

Shadow of the Ball

Similar triangles-ratio



Approximation



This figure shows that the curve $y = x$ and $y = \sin(x)$ are almost the same when x is close to 0. Hence if we want the value of $\sin(1/10)$ we just use this approximation $y = x$ to get

$$\sin(1/10) \approx 1/10.$$

The linear approximation

Given a function $f(x)$ we want to have the approximating function to be a linear function that is $F(x) = A + Bx$ for some constants A and B .



The linear approximation

$$f(x) \approx F(x) = f(a) + f'(a)(x - a)$$

Example

Estimate $e^{0.01}$? So $f(x) = e^x$ and $a = 0$.

The quadratic approximation

In linear approximation we had

$$f(x) \approx F(x) = f(a) + f'(a)(x - a) \Rightarrow$$
$$f(a) = F(a) \quad \text{and} \quad f'(a) = F'(a).$$

We now want our approximation function to be a quadratic function of x , that is, $F(x) = A + Bx + Cx^2$. To have a good approximating function we choose A , B , and C so that

- ▶ $f(a) = F(a)$
- ▶ $f'(a) = F'(a)$
- ▶ $f''(a) = F''(a)$

These conditions give us the following equations

$$F(x) = A + Bx + Cx^2 \Rightarrow F(a) = A + Ba + Ca^2 = f(a)$$

$$F'(x) = B + 2Cx \Rightarrow F'(a) = B + 2Ca = f'(a)$$

$$F''(x) = 2C \Rightarrow F''(a) = 2C = f''(a)$$

Solving these equations we can write A , B , and C in terms of $f(a)$, $f'(a)$, and $f''(a)$. So that

$$C = \frac{1}{2}f''(a)$$

$$B = f'(a) - af''(a)$$

$$A = f(a) - a[f'(a) - af''(a)] - \frac{1}{2}f''(a)a^2.$$

Consider that $F(x) = A + Bx + CX^2$, substituting A , B , and C , we obtain

Quadratic Approximation

$$F(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

Therefore,

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

Outline - October 28, 2019

- ▶ **Section 3.4.3 The Quadratic Approximation**
- ▶ **Section 3.4.4 Taylor Polynomials**
- ▶ **Section 3.4.5 Some Examples**

Linear Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x - a)$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$

Linear Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x - a)$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$

Then

$$F(a) = c_0 = f(a) \quad F'(a) = c_1 = f'(a).$$

And so

$$F(x) = f(a) + f'(a)(x - a).$$

Quadratic Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x - a) + c_2(x - a)^2$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$
3. $F''(a) = f''(a)$

Quadratic Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x - a) + c_2(x - a)^2$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$
3. $F''(a) = f''(a)$

Then

$$F(a) = c_0 = f(a) \quad F'(a) = c_1 = f'(a) \quad F''(a) = 2c_2 = f''(a).$$

Quadratic Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x - a) + c_2(x - a)^2$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$
3. $F''(a) = f''(a)$

Then

$$F(a) = c_0 = f(a) \quad F'(a) = c_1 = f'(a) \quad F''(a) = 2c_2 = f''(a).$$

And so

$$F(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

Taylor Polynomial

We want to approximate $f(x)$ with a polynomial $T_n(x)$ of degree n of the form

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n$$

such that

1. $T_n(a) = f(a),$

2. $T'_n(a) = f'(a),$

⋮

n. $T_n^{(n)}(a) = f^{(n)}(a).$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) =$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) =$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) = c_1 = f'(a)$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) = c_1 = f'(a)$$

$$T''_n(x) = 2c_2 + 3 \times 2c_3(x - a) + \cdots + n(n - 1)c_n(x - a)^{n-2}$$

$$\Rightarrow T''_n(a) =$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) = c_1 = f'(a)$$

$$T''_n(x) = 2c_2 + 3 \times 2c_3(x - a) + \cdots + n(n - 1)c_n(x - a)^{n-2}$$

$$\Rightarrow T''_n(a) = 2c_2 = f''(a)$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) = c_1 = f'(a)$$

$$\begin{aligned} T''_n(x) &= 2c_2 + 3 \times 2c_3(x - a) + \cdots + n(n-1)c_n(x - a)^{n-2} \\ &\Rightarrow T''_n(a) = 2c_2 = f''(a) \end{aligned}$$

$$\begin{aligned} T_n^{(3)}(x) &= 3 \times 2c_3 + 4 \times 3 \times 2c_4(x - a) + \cdots + n(n-1)c_n(x - a)^{n-2} \\ &\Rightarrow T_n^{(3)}(a) = \end{aligned}$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) = c_1 = f'(a)$$

$$\begin{aligned} T''_n(x) &= 2c_2 + 3 \times 2c_3(x - a) + \cdots + n(n-1)c_n(x - a)^{n-2} \\ &\Rightarrow T''_n(a) = 2c_2 = f''(a) \end{aligned}$$

$$\begin{aligned} T_n^{(3)}(x) &= 3 \times 2c_3 + 4 \times 3 \times 2c_4(x - a) + \cdots + n(n-1)c_n(x - a)^{n-2} \\ &\Rightarrow T_n^{(3)}(a) = 6c_3 = f^{(3)}(a) \end{aligned}$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T'_n(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1} \Rightarrow T'_n(a) = c_1 = f'(a)$$

$$\begin{aligned} T''_n(x) &= 2c_2 + 3 \times 2c_3(x - a) + \cdots + n(n-1)c_n(x - a)^{n-2} \\ &\Rightarrow T''_n(a) = 2c_2 = f''(a) \end{aligned}$$

$$\begin{aligned} T_n^{(3)}(x) &= 3 \times 2c_3 + 4 \times 3 \times 2c_4(x - a) + \cdots + n(n-1)c_n(x - a)^{n-2} \\ &\Rightarrow T_n^{(3)}(a) = 6c_3 = f^{(3)}(a) \end{aligned}$$

⋮

$$T_n^{(n)}(x) = n!c_n \Rightarrow T_n^{(n)}(a) = n!c_n$$

Taylor Polynomial

We have

$$c_0 = f(a), c_1 = f'(a), c_2 = \frac{1}{2!}f''(a), c_3 = \frac{1}{3!}f^{(3)}(a), \dots, c_n = \frac{1}{n!}f^{(n)}(a)$$

and

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots + c_n(x - a)^n$$

we have that

$$\begin{aligned} f(x) &\approx T_n(x) = \\ &f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a) + \\ &\frac{1}{3!}f^{(3)}(a)(x - a)^3 + \dots + \frac{1}{n!}f^{(n)}(a)(x - a)^n \end{aligned}$$

Taylor Polynomial

Taylor Polynomial

Let a be a constant and let n be a non-negative integer. The n th degree Taylor polynomial for $f(x)$ about $x = a$ is

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2$$

$$+ \frac{1}{3!}f^{(3)}(a)(x - a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x - a)^n$$

or

$$T_n(x) = \sum_{k=0}^n \frac{1}{k!}f^{(k)}(a)(x - a)^k$$

The special case $a = 0$ is called a Maclaurin polynomial.

Outline - October 30, 2019

- ▶ **Section 3.4.5: Some Examples of Taylor Polynomial**
- ▶ **Section 3.4.8: The Error in the Taylor Polynomial Approximations**

Taylor Polynomial

Taylor Polynomial

Let a be a constant and let n be a non-negative integer. The n th degree Taylor polynomial for $f(x)$ about $x = a$ is

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2$$

$$+ \frac{1}{3!}f^{(3)}(a)(x - a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x - a)^n$$

or

$$T_n(x) = \sum_{k=0}^n \frac{1}{k!}f^{(k)}(a)(x - a)^k$$

The special case $a = 0$ is called a Maclaurin polynomial.

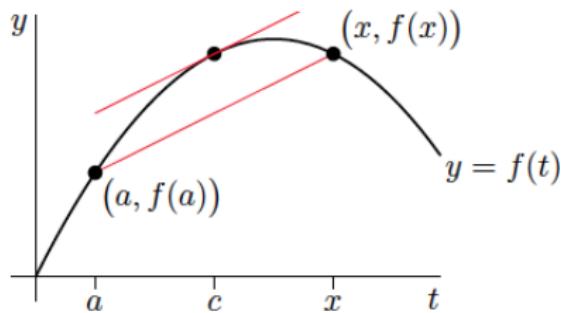
Approximating $f(x)$ by the 0th Taylor polynomial about $x = a$

$$f(x) \approx T_0(x) = f(a).$$

Note that

$$\begin{aligned} f(x) &= f(x) + f(a) - f(a) \\ &= f(a) + (f(x) - f(a)) \frac{(x - a)}{(x - a)} \\ &= f(a) + \frac{f(x) - f(a)}{x - a} (x - a) \end{aligned} \tag{2}$$

$$f(x) = f(a) + \frac{f(x)-f(a)}{x-a}(x-a)$$



There is c strictly between x and a such that

$$f'(c) = \frac{f(x) - f(a)}{x - a}.$$

$$f(x) = f(a) + f'(c)(x-a) \text{ for some } c \text{ strictly between } a \text{ and } x.$$

$$f(x) = f(a) + f'(c)(x-a) \text{ for some } c \text{ strictly between } a \text{ and } x.$$

$$\Rightarrow f(x) - f(a) = f'(c)(x - a) \Rightarrow f(x) - T_0(x) = f'(c)(x - a)$$

The error in constant approximation

$$R_0(x) = f(x) - T_0(x) = f'(c)(x - a)$$

for some c strictly between a and x

The error in linear approximation

$$R_1(x) = f(x) - T_1(x) = \frac{1}{2}f''(c)(x-a)^2$$

for some c strictly between a and x

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{n+1}$$

for some c strictly between a and x

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some c strictly between a and x

Remark

Consider that $f(x) = R_n(x) + T_n(x)$ Therefore,

1. if $0 \leq R_n(x) \leq E$, then

$$T_n(x) \leq f(x) \leq T_n(x) + E.$$

2. if $E \leq R_n(x) \leq 0$, then

$$T_n(x) + E \leq f(x) \leq T_n(x).$$

Outline - Nov. 1, 2019

- ▶ **Section 3.4.8: The Error in the Taylor Polynomial Approximations**
- ▶ **Section 3.5.1: Maxima and Minima**

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some c strictly between a and x

Accurate to D decimal places

Generally we say that our estimate is “accurate to D decimal places” when

$$|error| < 0.5 \times 10^{-D}.$$

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some c strictly between a and x

Remark

Consider that $f(x) = R_n(x) + T_n(x)$ Therefore,

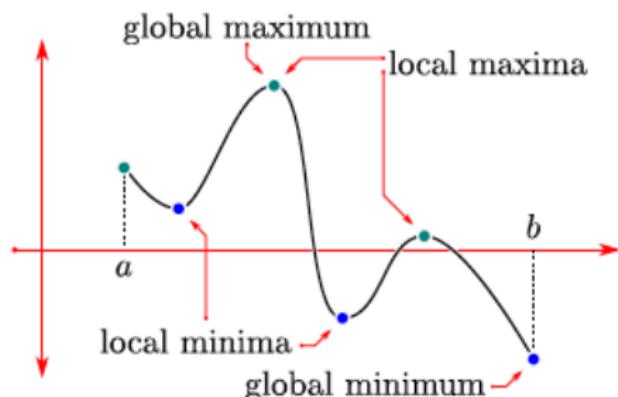
1. if $0 \leq R_n(x) \leq E$, then

$$T_n(x) \leq f(x) \leq T_n(x) + E.$$

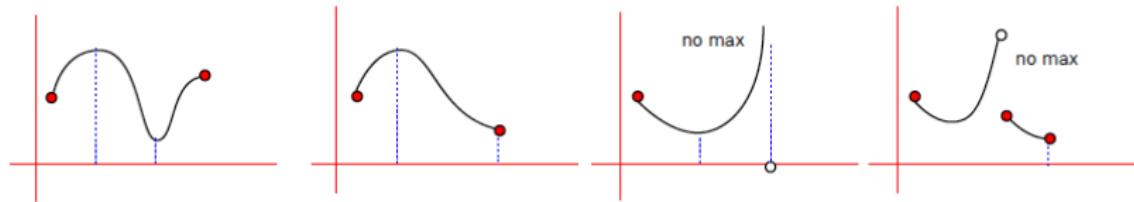
2. if $E \leq R_n(x) \leq 0$, then

$$T_n(x) + E \leq f(x) \leq T_n(x).$$

Maximum and Minimum



Continuity and global max/min



First one: Continuous/global min and max

Second one: Continuous/global min and max

Third one: Not continuous/global min/no global max

Forth one: Not continuous/global min/no global max

If $f'(c) = 0$, then c is local max/min?!

