MATH100: Differential Calculus with Application to Physical Sciences and Engineering

University of British Columbia

Farid Aliniaeifard

September 20, 2019

Contents

Chapter 1

Limits

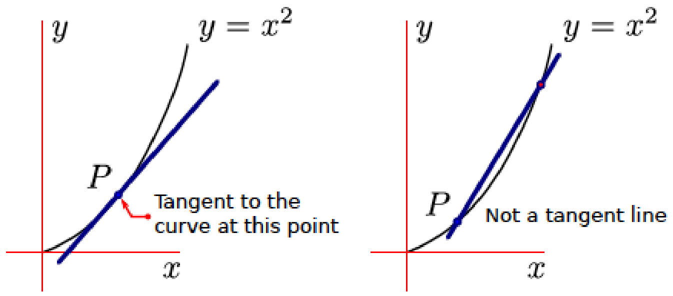
What does this mean

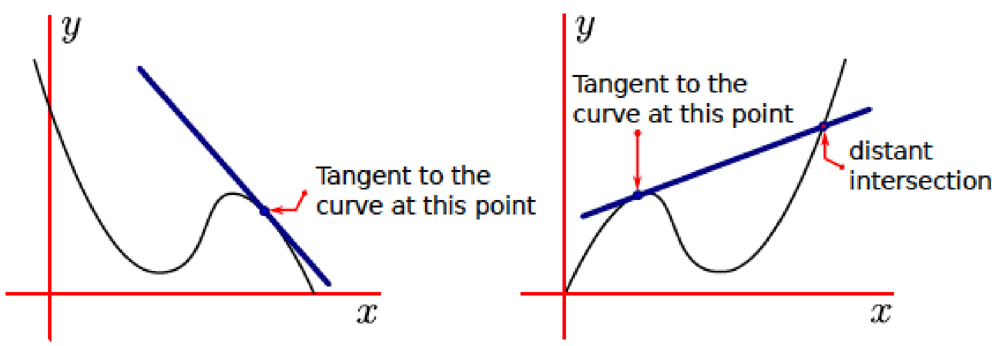
$$\lim_{x \to a} f(x) = L?$$

The "limit" appears when we want to

- find the tangent to a curve; or
- find the velocity of an object.

1.1 Tangent line





The **tangent line to a curve** $y = f(x)$ **at a point** P (if exists) is a line L that there is a neighborhood for P such that in that neighborhood the line L touches (does not cross) the curve $y = f(x)$ only at P (and not other points in that neighborhood).

The equation of a line

– The formula for a line that passes though (x_1, y_1) with slope m is

$$y = y_1 + m(x - x_1).$$

– Given two points (x_1, y_1) and (x_2, y_2) on a line, then the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

and the formula for the line then is

$$y = y_1 + m(x - x_1).$$

Example 1.1.1. *Find the equation of the line with slope* -3 *that passes through* $(1, 2)$.

Solution. *The equation of the line is*

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

Example 1.1.2. *Find the equation of the line that passes through* $(1, 2)$ *and* $(2, -1)$.

Solution. *First we find the slope which is*

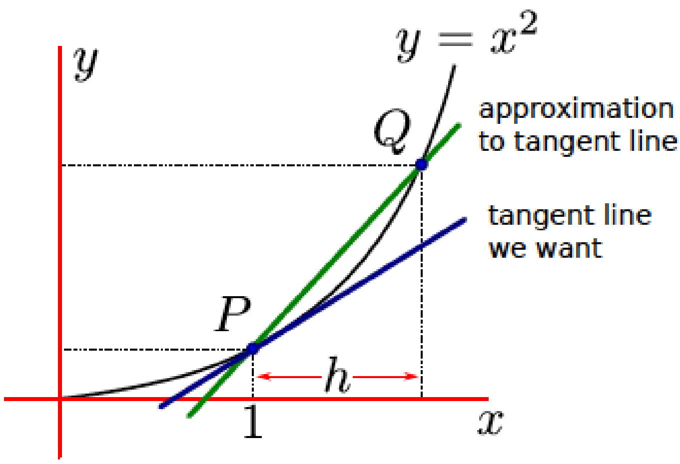
$$\frac{-1 - 2}{2 - 1} = -3.$$

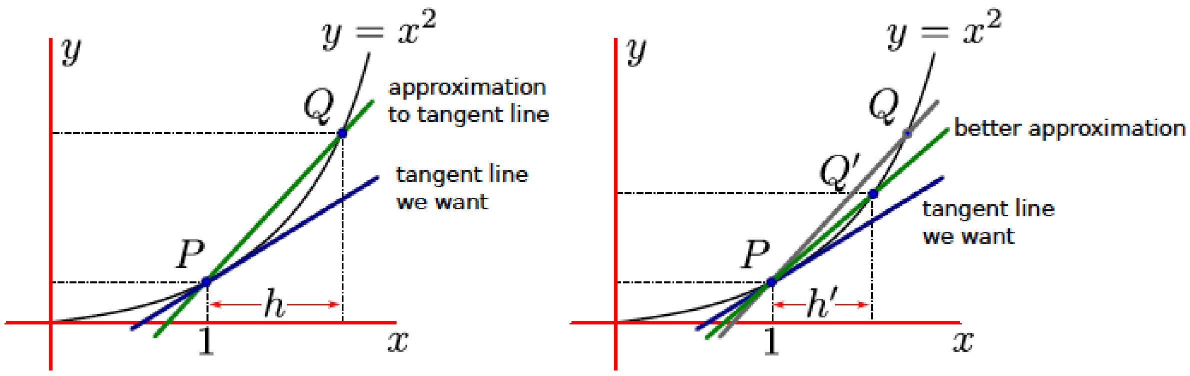
Then the equation of the line is

$$y = 2 + (-3)(x - 1), \text{ so } y = 5 - 3x.$$

The equation of a tangent line: Given a curve $y = f(x)$ and a point P on the curve, how to find the slope of the tangent to a curve at P: let do this through an example.

Example 1.1.3. *Find the tangent line to the curve* $y = x^2$ *that passes through* $P = (1, 1)$.





So we want to find the slope the line that passes through the points $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (1 + h, (1 + h)^2)$. The slope then is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1+h)^2 - 1^2}{(1+h) - 1} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(h+2)}{h} = 2 + h$$

h	$m = \frac{(1+h)^2-1^2}{(1+h)-1}$
0.1	2.1
0.01	2.01
0.001	2.001

When h gets smaller and smaller, the slope will be closer and closer to the slope of the tangent line to $y = x^2$ at $(1, 1)$, which the slope will be closer and closer to 2, we can write this more mathematically as

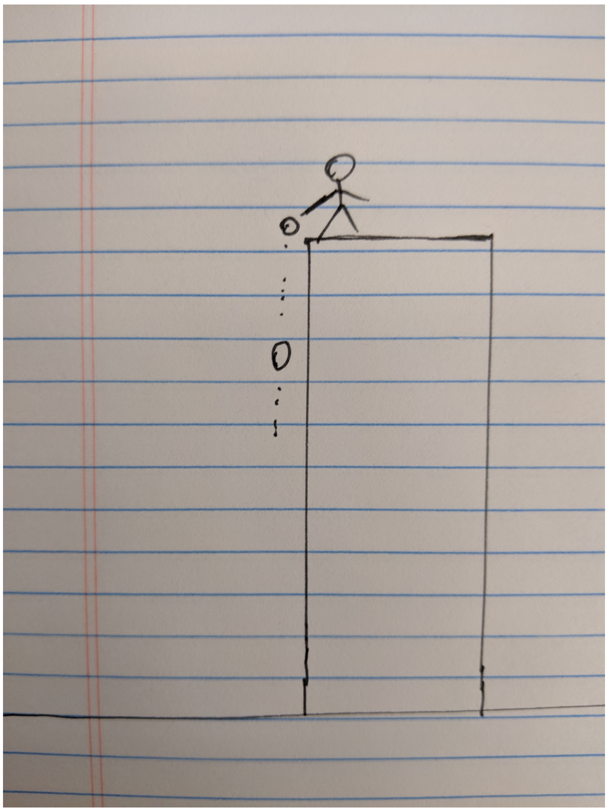
$$\lim_{h \to 0} \frac{(1+h)^2 - 1}{(1+h) - 1} = 2$$

Read: *the limit of* $\frac{(1+h)^2-1}{(1+h)-1}$ *as* h *approaches* 0 *is* 2.
Tangent line is

$$y = 1 + 2(x - 1) = 2x - 1.$$

1.2 Velocity

- Let t be elapsed time measured in second
- $S(t)$ be the distance the ball has fallen in meters
- What is $S(0)$? $S(0) = 0$.
- (**Galileo**) $S(t) = 4.9t^2$.



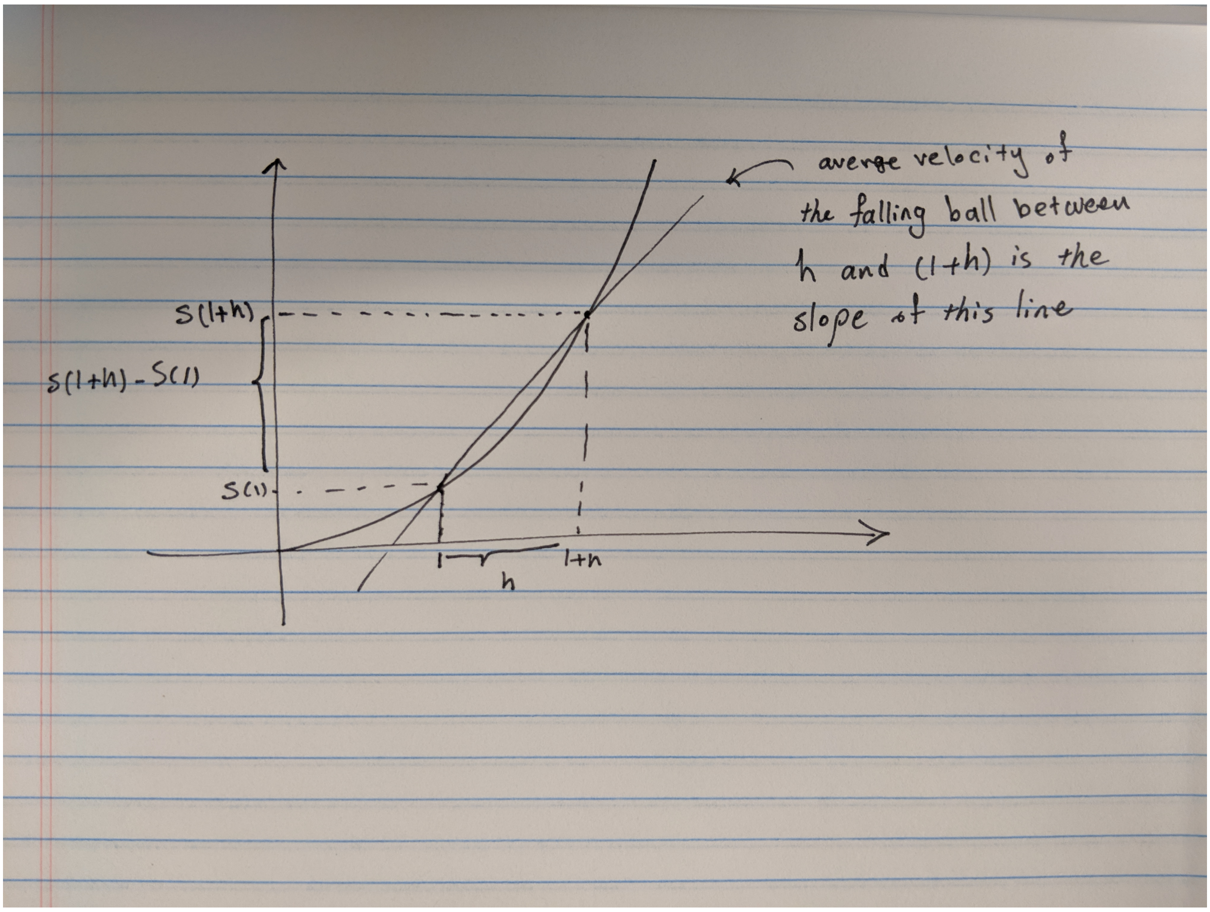
Question: How fast the ball is fallen after 1 second, that is, find $v(1)$, the velocity at $t = 1$?

$$\text{average velocity} = \frac{\text{difference in position}}{\text{difference in time}} = \frac{S(t_2) - S(t_1)}{t_2 - t_1}.$$

To answer the question we should find the average velocity of the falling ball between $(1 + h)$ and 1. So,

$$\text{average velocity when } (t_2 = 1 + h) \text{ and } (t_1 = 1)$$

$$= \frac{S(1+h) - S(1)}{h} = \frac{4.9(1+h)^2 - 4.9}{h} = 4.9(2 + h).$$



time window	average velocity
$1 \leq t \leq 1.1$	10.29
$1 \leq t \leq 1.01$	9.84
$1 \leq t \leq 1.01$	9.8049
$1 \leq t \leq 1.001$	9.80049

So we can write

$$v(1) = \lim_{h \to 0} \frac{S(1+h) - S(1)}{h} = 9.8.$$

More generally:

We define the instantaneous velocity at time $t = a$ to be the limit

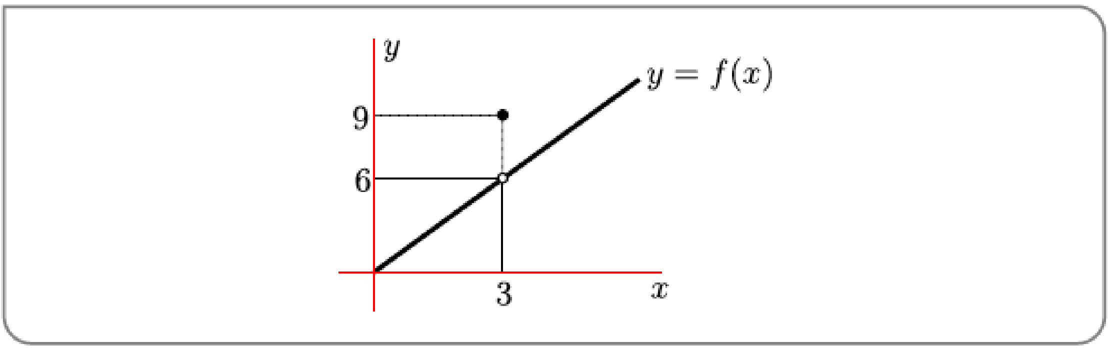
$$v(a) = \lim_{h \to 0} \frac{S(a+h) - S(a)}{h}$$

1.3 The limit of a function

To arrive at the definition of limit, we start with a very simple example.

Example 1.3.1. *Consider the following function*

$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



If we plug in some numbers very close to 3 *(but not exactly 3) into the function we see*

x	2.9	2.99	2.999	$\circ$	3.001	3.01	3.1
$f(x)$	5.8	5.98	5.998	$\circ$	6.002	6.02	6.2

So as x moves closer and closer to 3*, without being exactly* 3*, we see that the function moves closer and closer to* 6*. We can then write this as*

$$\lim_{x\to 3} f(x) = 6.$$

Definition. (Informal definition of limit) *We write*

$$\lim_{x\to a} f(x) = L.$$

if the value of the function $f(x)$ is sure to be arbitrary close to L whenever the value of x is close enough to a, without being exactly a.

Example 1.3.2. *Let $f(x) = \frac{x-2}{x^2+x-6}$ and find its limit as $x \to 2$.*

Solution. *We want to find*

$$\lim_{x\to 2} \frac{x-2}{x^2+x-6}.$$

Important point: *if we we compute $f(2)$, then we have $\frac{0}{0}$ which is undefined.*

Again we plug in numbers close to 2 *and we have*

x	1.9	1.99	1.999	$\circ$	2.001	2.01	2.1
$f(x)$	0.20408	0.20040	0.20004	$\circ$	0.19996	0.19960	0.19608

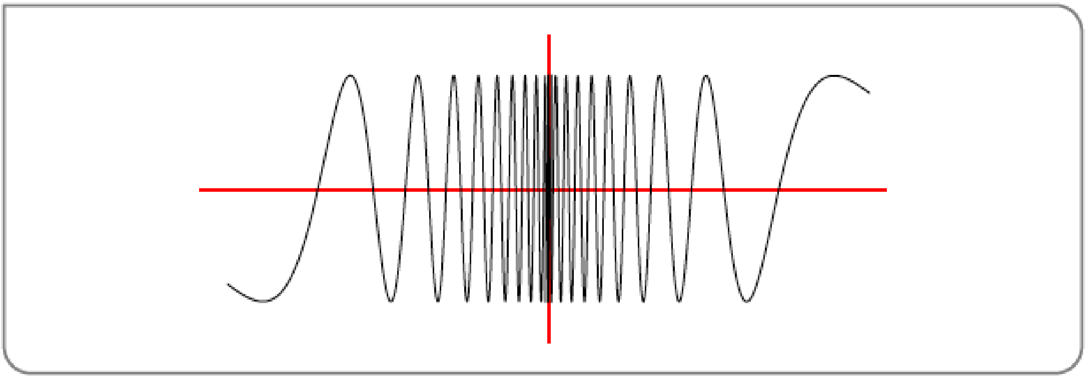
So

$$\lim_{x\to 2} \frac{x-2}{x^2+x-6} = 2.$$

Example 1.3.3. *Consider the following function $f(x) = sin(\pi/x)$. Find the limit as $x \to 0$ of $f(x)$.*

Solution. *When x is getting closer and closer to 0, it oscillates faster and faster. Since the function does not approach a single number as we bring x closer and closer to zero, the limit does not exist. Thus,*

$$\lim_{x\to 0} sin(\pi/x) = DNE$$



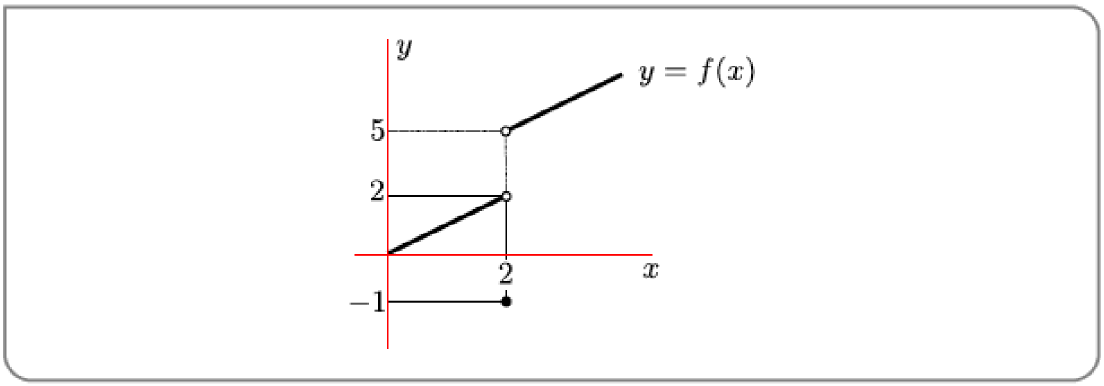
Example 1.3.4. *Consider the function*

$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x+3 & x > 2 \end{cases}$$

Find

$$\lim_{x\to 2} f(x).$$

Solution.



Let again plug in some numbers close to 2 (but not exactly 2)

x	1.9	1.99	1.999	∘	2.001	2.01	2.1
$f(x)$	1.9	1.99	1.999	∘	5.001	5.01	5.1

Now when we approach from below (or left), we seem to be getting closer to 2 $\left(\lim_{x\to 2^-} f(x) = 2\right)$, *but when we approach from above (or right) we seem to be getting closer to* 5 $\left(\lim_{x\to 2^+} f(x) = 5\right)$. *Since we are not approaching the same number the limit does not exists.*

$$\lim_{x\to 2} f(x) = DNE$$

Definition. *(***Informal definition of one-sided limits***) We write*

$$\lim_{x\to a^-} f(x) = K$$

when the value of $f(x)$ gets closer and closer to K when $x < a$ and x moves closer and closer to a. Since the x-values are always less than a, we say that x approaches a from below (or left). This is also often called the left-hand limit since the x-values lie to the left of a on a sketch of the graph.

We similarly write

$$\lim_{x\to a^+} f(x) = L$$

when the values of $f(x)$ gets closer and closer to L when $x > a$ and x moves closer and closer to a. For similar reason we say that x approaches a from above, and sometimes to this as the the right-hand limit.

Theorem 1.3.5.

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

- If the limit of $f(x)$ as x approaches a exists and is equal to L, then both the left-hand and right-hand limits exist and are equal to L.

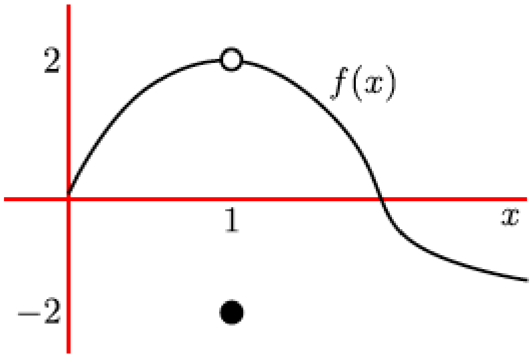
- If the left-hand and right-hand limits as x approaches a exist and are equal, then the limit as x approaches a exists and is equal to the one-sided limits.

Contrapositive of the above argument says

- If either of the left-hand and right-hand limits as x approaches a fail to exist, or if they both exist but are different, then the limit as x approaches a does not exist. AND,

- If the limit as x approaches a does not exist, then the left-hand and right-hand limits are either different or at least one of them does not exist.

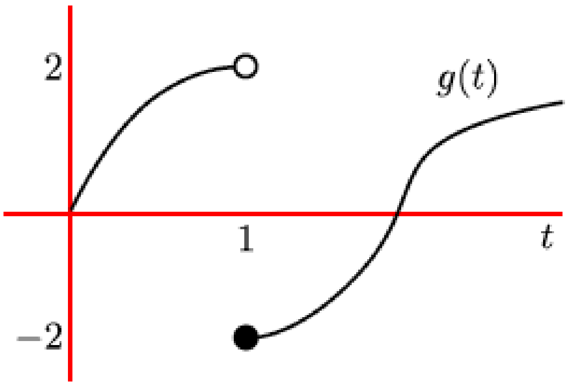
Example 1.3.6. *Consider the graph of the function $f(x)$.*



Then

$$\lim_{x\to 1^-} f(x) = 2 \qquad \lim_{x\to 1^+} f(x) = 2 \qquad \lim_{x\to 1} f(x) = 2$$

Example 1.3.7. *Consider the graph of the function $g(x)$.*

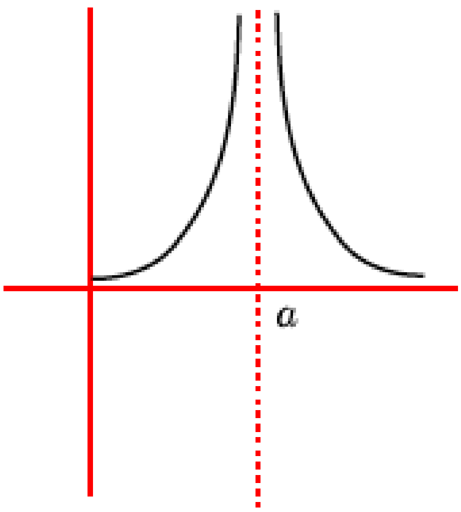


Then

$$\lim_{t\to 1^-} g(t) = 2 \qquad \lim_{t\to 1^+} g(t) = -2 \qquad \lim_{t\to 1} g(t) = DNE$$

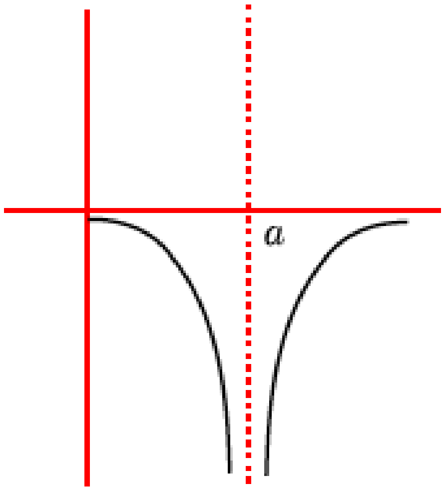
In the following example even though the limit doesn't exists when x approaches a, we can say more.

Example 1.3.8. *Consider the graph for the function $f(x)$.*



$$\lim_{x \to a} f(x) = +\infty$$

Example 1.3.9. *Consider the graph for the function $g(x)$.*



$$\lim_{x \to a} g(x) = -\infty$$

Definition. *We write*

$$\lim_{x \to a} f(x) = +\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as x gets closer and closer to a, without being exactly a.
Similarly, we write

$$\lim_{x \to a} f(x) = -\infty$$

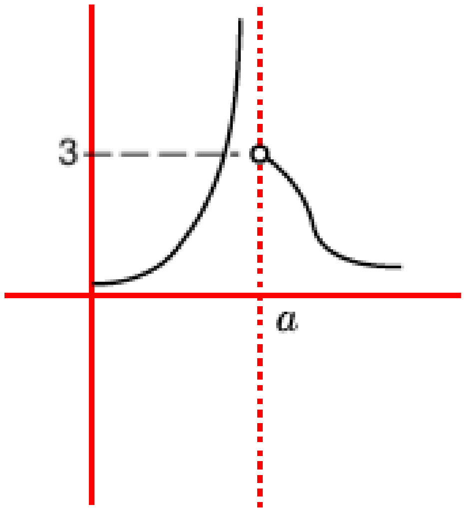
when the value of the function $f(x)$ becomes arbitrarily large and negative as x gets closer and closer to a, without being exactly a.

Example 1.3.10.

$$\lim_{x \to 0} \frac{1}{x^2} = +\infty \qquad \lim_{x \to 0} -\frac{1}{x^2} = -\infty$$

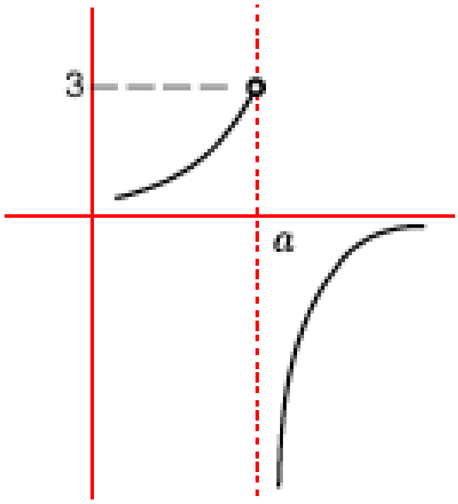
Important Point: Do not think of "$+\infty$" and "$-\infty$" in these statements as numbers. When we write $\lim_{x \to a} f(x) = +\infty$, it says "the function $f(x)$ becomes arbitrary large as x approaches a".

Example 1.3.11. *Consider the graph for the function $h(x)$.*



$$\lim_{x\to a^-} h(x) = +\infty \qquad \lim_{x\to a^+} h(x) = 3 \qquad \lim_{x\to a} h(x) = DNE$$

Example 1.3.12. *Consider the graph for the function $s(x)$.*



$$\lim_{x\to a^-} s(x) = 3 \qquad \lim_{x\to a^+} s(x) = -\infty \qquad \lim_{x\to a} s(x) = DNE$$

Definition. *We write*

$$\lim_{x\to a^+} f(x) = +\infty$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as x gets closer and closer to a from above (equivalently, from right), without being exactly a. Similarly, we write

$$\lim_{x\to a^+} f(x) = -\infty$$

when the values of the function $f(x)$ becomes arbitrarily large and negative as x gets closer and closer to a from above (equivalently, from right), without being exactly a.

The notation

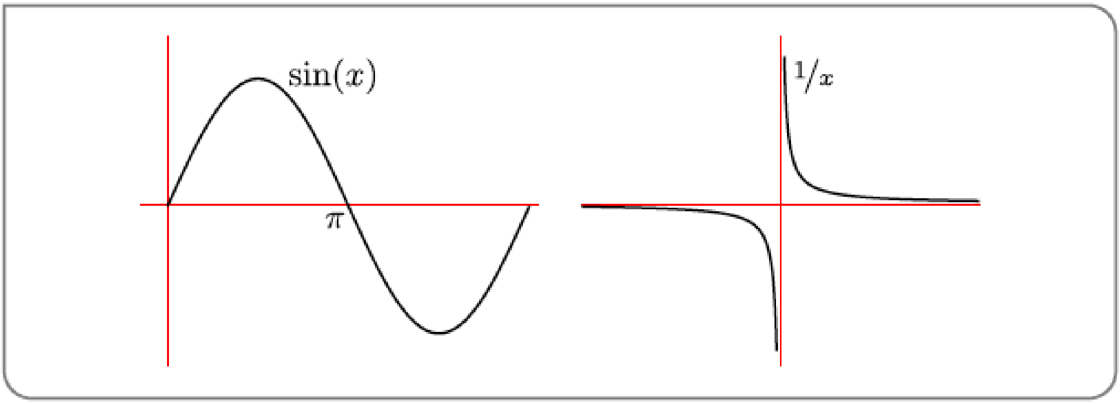
$$\lim_{x\to a^-} f(x) = +\infty \quad \textit{and} \quad \lim_{x\to a^-} f(x) = -\infty$$

has a similar meaning except that limits are approached form below (from left).

Example 1.3.13. *Consider the function*

$$g(x) = \frac{1}{sin(x)}.$$

Find the one-side limits of this function as $x \to \pi$.



- As $x \to \pi$ from the left, $\sin(x)$ is a small positive number that is getting closer and closer to zero. That is, as $x \to \pi^-$, we have that $\sin(x) \to 0$ through positive numbers (i.e. from above). Now look at the graph of $1/x$, and think what happens as we move $x \to 0^+$, the function is positive and becomes larger and larger.

  So as $x \to \pi$ from the left, $\sin(x) \to 0$ from above, and so $1/\sin(x) \to +\infty$.

- By very similar reasoning, as $x \to \pi$ from the right, $\sin(x)$ is a small negative number that gets closer and closer to zero. So as $x \to \pi$ from the right, $\sin(x) \to 0$ through negative numbers (i.e. from below) and so $1/\sin(x)$ to $-\infty$.

Thus

$$\lim_{x\to\pi^-} \frac{1}{\sin(x)} = +\infty \qquad\qquad \lim_{x\to\pi^+} \frac{1}{\sin(x)} = -\infty$$

1.4 Calculating Limits with Limit Laws

Theorem 1.4.1. *Let $a, c \in \mathbb{R}$. The following two limits hold*

$$\lim_{x\to a} c = c \qquad \lim_{x\to a} x = a$$

Theorem 1.4.2. *(Arithmetic of Limits) Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f and g need not to be defined exactly at a).*

$$\lim_{x\to a} f(x) = F \qquad \lim_{x\to a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. Then the following limits hold

- $\lim\limits_{x\to a}(f(x)+g(x)) = F+G$*–limit of the sum is the sum of the limits.*
- $\lim\limits_{x\to a}(f(x)-g(x)) = F-G$*–limit of the difference is the difference of the limits.*
- $\lim\limits_{x\to a} cf(x) = cF$.
- $\lim\limits_{x\to a}(f(x).g(x)) = F.G$*–limit of the product is the product of the limits.*
- *If $G \neq 0$ then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{F}{G}$.*

Example 1.4.3. *Given*

$$\lim_{x\to 1} f(x) = 3 \quad \textit{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} f(x). \lim_{x\to 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x\to 1}\frac{f(x)}{f(x)-g(x)} = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} f(x) - \lim_{x\to 1} g(x)} = \frac{3}{3-2} = 3.$$

Example 1.4.4.

$$\lim_{x\to 3} 4x^2 - 1 = 4 \times \lim_{x\to 3} x^2 - \lim_{x\to 3} 1 = 35.$$

$$\lim_{x\to 2}\frac{x}{x-1} = \frac{\lim_{x\to 2} x}{\lim_{x\to 2} x - \lim_{x\to 1} 1} = \frac{2}{2-1} = 2.$$

Consider that we apply the theorem Arithmetic of Limits to compute the limit of a ratio if the limit of denominator is not zero. **What will happen if the limit of denominator is zero:**

– the limit does not exist, eg.

$$\lim_{x\to 0}\frac{x}{x^2} = \lim_{x\to 0}\frac{1}{x} = DNE$$

– the limit is $\pm\infty$, eg.

$$\lim_{x\to 0}\frac{x^2}{x^4}=\lim_{x\to 0}\frac{1}{x^2}=+\infty \qquad \text{or} \qquad \lim_{x\to 0}\frac{-x^2}{x^4}=\lim_{x\to 0}\frac{-1}{x^2}=-\infty.$$

– the limit is 0, eg.

$$\lim_{x\to 0}\frac{x^2}{x}=\lim_{x\to 0}x=0.$$

– the limit exists and it nonzero, eg.

$$\lim_{x\to 0}\frac{x}{x}=1.$$

Theorem 1.4.5. *Let n be a positive integer, let $a\in R$ and let f be a function so that*

$$\lim_{x\to a}f(x)=F$$

for some real number F. Then the following holds

$$\lim_{x\to a}(f(x))^n=\left(\lim_{x\to a}f(x)\right)^n=F^n$$

so that the limit of a power is the power of the limit. Similarly, if

- *n is an even number and $F>0$, or*
- *n is an odd number and F is any real number*

then

$$\lim_{x\to a}(f(x))^{1/n}=\left(\lim_{x\to a}f(x)\right)^{1/n}=F^{1/n}.$$

Example 1.4.6.

$$\lim_{x\to 4}x^{1/2}=4^{1/2}=2.$$

$$\lim_{x\to 4}(-x)^{1/2}=-4^{1/2}=\textit{not a real number.}$$

$$\lim_{x\to 2}(4x^2-3)^{1/3}=(4(2)^2-3)^{1/3}=13^{1/3}$$

Example 1.4.7. *Compute the following limits.*

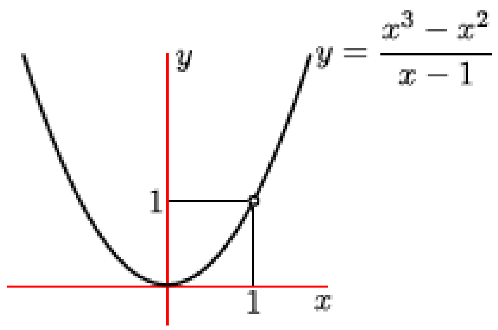
1. $\lim_{x\to 2}\frac{x^3-x^2}{x-1}$
2. $\lim_{x\to 1}\frac{x^3-x^2}{x-1}$

Solution. *1.* $\lim_{x\to 2}\frac{x^3-x^2}{x-1}=4$.
2. Consider that $\lim_{x\to 1}x^3-x^2=0$ *and* $\lim_{x\to 1}x-1=0$. *However,*

$$\frac{x^3-x^2}{x-1}=\frac{x^2(x-1)}{x-1},$$

thus

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \textit{undefined} & x = 1. \end{cases}$$



And so

$$\lim_{x \to 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \to 1} x^2 = 1.$$

The reasoning in the above example can be made more general:

Theorem 1.4.8. *If $f(x) = g(x)$ except when $x = a$ then*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

provided the limit of g exists.

We mostly use the above theorem when we end up with $\frac{0}{0}$.

Example 1.4.9. *Compute*

$$\lim_{h \to 0} \frac{(1 + h)^2 - 1}{h}.$$

Solution. *Note that*

$$\frac{(1 + h)^2 - 1}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{h(2 + h)}{h}.$$

Thus,

$$\frac{(1 + h)^2 - 1}{h} = \begin{cases} 2 + h & h \neq 0 \\ \textit{undefined} & h = 0. \end{cases}$$

And so

$$\lim_{h \to 0} \frac{(1 + h)^2 - 1}{h} = \lim_{h \to 0} 2 + h = 2.$$

We now present a slightly harder example.

Example 1.4.10. *Compute the limit*

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + x} - 1}.$$

Solution. *Both the limits of the numerator and denominator as $x \to 0$ are 0, so we cannot use the Theorem Arithmetic of limits. We now can simply multiply the numerator and denominator by the conjugation of $\sqrt{1+x}-1$, that is, $\sqrt{1+x}+1$. We have*

$$\begin{aligned}
\frac{x}{\sqrt{1+x}-1} &= \frac{x}{\sqrt{1+x}-1} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} && \text{multiply by } \frac{\text{conjugate}}{\text{conjugate}} = 1 \\
&= \frac{x\left(\sqrt{1+x}+1\right)}{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)} && \text{bring things together} \\
&= \frac{x\left(\sqrt{1+x}+1\right)}{\left(\sqrt{1+x}\right)^2 - 1 \cdot 1} && \text{since } (a-b)(a+b) = a^2 - b^2 \\
&= \frac{x\left(\sqrt{1+x}+1\right)}{1+x-1} && \text{clean up a little} \\
&= \frac{x\left(\sqrt{1+x}+1\right)}{x} \\
&= \sqrt{1+x}+1 && \text{cancel the } x
\end{aligned}$$

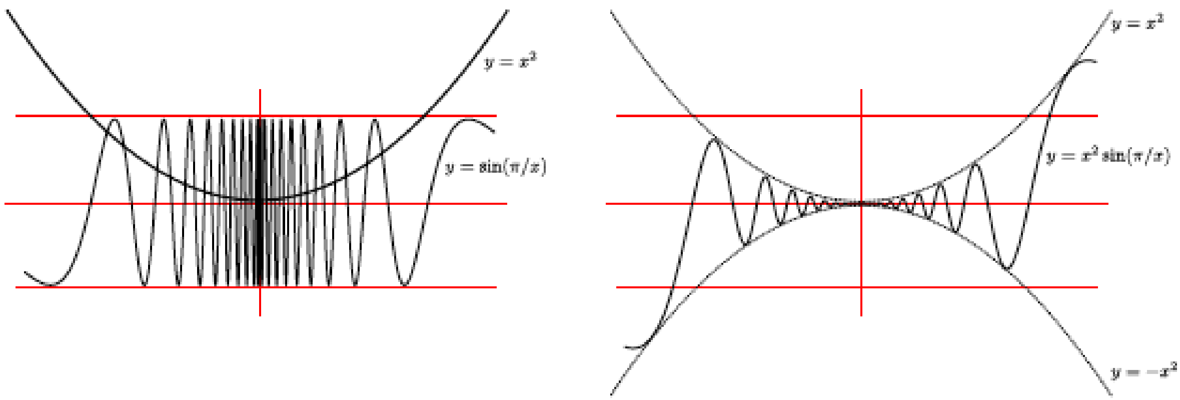
So now we have

$$\begin{aligned}
\lim_{x\to 0} \frac{x}{\sqrt{1+x}-1} &= \lim_{x\to 0} \sqrt{1+x}+1 \\
&= \sqrt{1+0}+1 = 2
\end{aligned}$$

Before we move to the next section and study the limits at infinity, we have one more theorem to state.

Example 1.4.11. *Compute*

$$\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$$



Solution. *It is not possible to simply use the theorem Arithmetic of Limits since the limit of $\sin(\frac{\pi}{x})$ as $x \to 0$ does not exist. Since $-1 \le \sin(\theta) \le 1$ for all real numbers θ, we have*

$$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1 \qquad \textit{for all } x \neq 0$$

Multiplying the above by x^2 we see that

$$-x^2 \leq x^2 \sin(\frac{\pi}{x}) \leq x^2 \qquad \text{for all } x \neq 0.$$

Since

$$\lim_{x \to 0} x^2 = \lim_{x \to 0}(-x^2) = 0$$

by the **sandwich (or squeeze or pinch) theorem** *(look at below for the sandwich theorem) we have*

$$\lim_{x \to 0} x^2 \sin(\frac{\pi}{x}) = 0.$$

Theorem 1.4.12. *(***sandwich (or squeeze or pinch) theorem** *) Let $a \in \mathbb{R}$ and let f, g, h be three functions so that*

$$f(x) \leq g(x) \leq h(x)$$

for all x in an interval around a, except possibly at $x = a$. Then if

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then it is also the case that

$$\lim_{x \to a} g(x) = L.$$

Example 1.4.13. *Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is*

$$\lim_{x \to 1} f(x)?$$

Solution. *Consider that*

$$\lim_{x \to 1} x = 1 \qquad \text{and} \qquad \lim_{x \to 1} x^2 - 2x + 2 = 1.$$

Therefore, by the sandwich/pinch/squeeze theorem

$$\lim_{x \to 1} f(x) = 1.$$

1.5 Limits at Infinity

Example 1.5.1. *We want to compute*

$$\lim_{x \to +\infty} \frac{1}{x} \qquad \text{and} \qquad \lim_{x \to -\infty} \frac{1}{x}$$

By plug in some large numbers into $\frac{1}{x}$ we have

-10000	-1000	$-100 \| \circ \| 100$	1000	10000
-0.0001	0.001	$-0.01 \| \circ \| 0.01$	0.001	0.0001

We see that as x is getting bigger and positive the function $\frac{1}{x}$ is getting closer to 0. Thus,

$$\lim_{x \to +\infty} \frac{1}{x} = 0.$$

Moreover,

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

Definition. *(***Informal limit at infinity.***) We write*

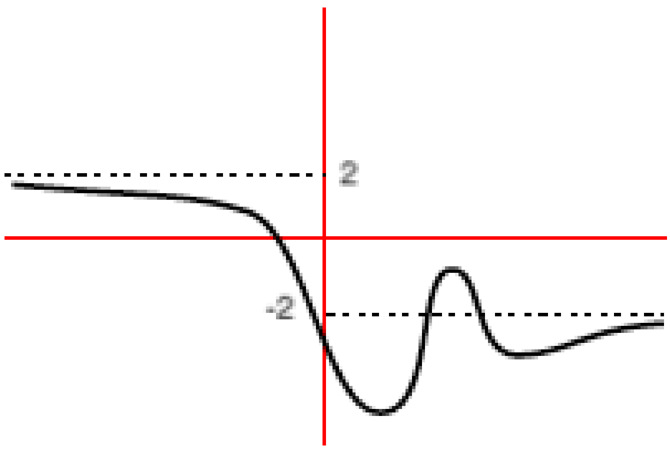
$$\lim_{x\to\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and positive.
Similarly, we write

$$\lim_{x\to-\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and negative.

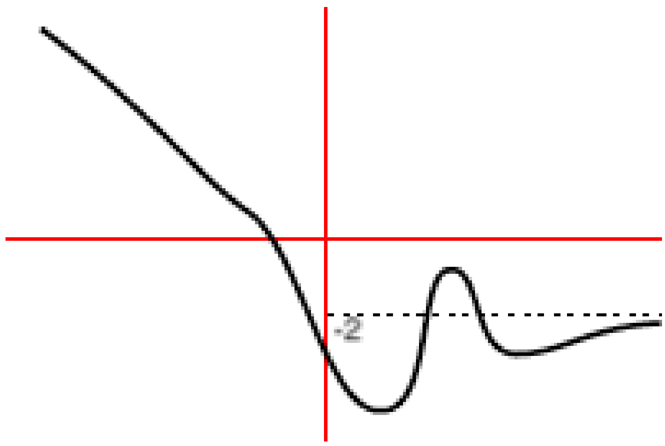
Example 1.5.2. *Consider the graph of the function $f(x)$.*



Then

$$\lim_{x\to\infty} f(x) = -2 \qquad \textit{and} \qquad \lim_{x\to-\infty} f(x) = 2$$

Example 1.5.3. *Consider the graph of the function $g(x)$.*



Then

$$\lim_{x\to\infty} g(x) = -2 \qquad \textit{and} \qquad \lim_{x\to-\infty} g(x) = +\infty$$

Same as usual we start with two very simple building blocks and build other limits from them.

Theorem 1.5.4. *Let $c \in \mathbb{R}$ then the following limits hold*

$$\lim_{x\to+\infty} c = c \qquad \lim_{x\to-\infty} c = c$$

$$\lim_{x\to+\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

Theorem 1.5.5. *Let $f(x)$ and $g(x)$ be two functions for which the limits*

$$\lim_{x\to\infty} f(x) = F \qquad \lim_{x\to\infty} = G$$

exist. Then the following limits hold

$$\lim_{x\to\infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x\to\infty} f(x)g(x) = FG$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \textit{provided } G \neq 0$$

and for rational numbers r,

$$\lim_{x\to\infty} (f(x))^r = F^r$$

provided that $f(x)^r$ is defined for all x.
The analogous results hold for limits to $-\infty$.

We need a little extra care with the posers of functions.
Warning: Consider that

$$\lim_{x\to+\infty} \frac{1}{x^{1/2}} = 0$$

However,

$$\lim_{x\to+\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because $x^{1/2}$ is not defined for $x < 0$.

Example 1.5.6. *Compute the following limit:*

$$\lim_{x\to\infty} \frac{x^2 - 3x + 4}{3x^2 + 8x + 1}$$

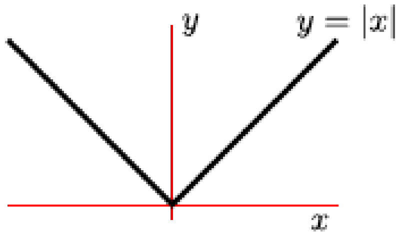
Solution. *By factoring x with largest exponent in the numerator and denominator we have*

$$\lim_{x\to\infty} \frac{x^2 - 3x + 4}{3x^2 + 8x + 1} = \lim_{x\to\infty} \frac{x^2(1 + \frac{-3x}{x^2} + \frac{4}{x^2})}{x^2(3 + \frac{8x}{3x^2} + \frac{1}{3x^2})} = \lim_{x\to\infty} \frac{(1 + \frac{-3x}{x^2} + \frac{4}{x^2})}{(3 + \frac{8x}{3x^2} + \frac{1}{3x^2})} =$$

$$\frac{(\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{-3x}{x^2} + \lim_{x\to\infty} \frac{4}{x^2})}{(\lim_{x\to\infty} 3 + \lim_{x\to\infty} \frac{8x}{3x^2} + \lim_{x\to\infty} \frac{1}{3x^2})} = \frac{1}{3}.$$

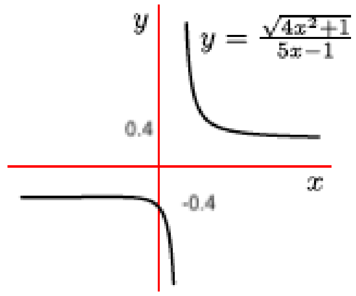
Remark. *Note that*

$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



Example 1.5.7. *Compute the following limit:*

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{5x-1}.$$



Solution. *Factor the terms with the largest exponents in the numerator and denominator. We have*

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{5x-1} = \lim_{x\to\infty} \frac{\sqrt{4x^2(1+\frac{1}{4x^2})}}{5x(1-\frac{1}{5x})} = \lim_{x\to\infty} \frac{\sqrt{4x^2}\sqrt{(1+\frac{1}{4x^2})}}{5x(1-\frac{1}{5x})} = \lim_{x\to\infty} \frac{2|x|}{5x} = \lim_{x\to\infty} \frac{2x}{5x} = \frac{2}{5}.$$

Example 1.5.8. *Compute the following limit:*

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{5x-1}.$$

Solution. *By the same kind of computation we have*

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{5x-1} = \lim_{x\to\infty} \frac{2|x|}{5x}.$$

Consider that since x is getting negative values, we have $|x| = -x$. Therefore,

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{5x-1} = \lim_{x\to\infty} \frac{2|x|}{5x} = \lim_{x\to\infty} \frac{-2x}{5x} = \frac{-2}{5}.$$

Example 1.5.9. *Compute the following limit:*

$$\lim_{x\to\infty} \left(x^{7/5} - x\right).$$

Solution. *We factor the term with the largest exponent, we have*

$$\lim_{x\to\infty}\left(x^{7/5}-x\right)=\lim_{x\to\infty}x^{7/5}\left(1-\frac{1}{x^{2/5}}\right)=\infty.$$

Theorem 1.5.10. *Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that*

$$\lim_{x\to a} f(x)=+\infty \qquad \lim_{x\to a} g(x)=+\infty \qquad \lim_{x\to a} h(x)=H$$

1. $$\lim_{x\to a}(f(x)+g(x))=+\infty.$$

2. $$\lim_{x\to a}(f(x)+h(x))=+\infty.$$

3. $$\lim_{x\to a}(f(x)-g(x))=\textit{undetermined}.$$

4. $$\lim_{x\to a}(f(x)-h(x))=+\infty.$$

5. $$\lim_{x\to a} cf(x)=\begin{cases}+\infty & c>0\\ 0 & c=0\\ -\infty & c<0\end{cases}$$

6. $$\lim(f(x).g(x))=+\infty.$$

7. $$\lim_{x\to a}(f(x).h(x))=\begin{cases}+\infty & H>0\\ \textit{undetermined} & H=0\\ -\infty & H<0\end{cases}$$

8. $$\lim_{x\to a}\frac{h(x)}{f(x)}=0.$$

Example 1.5.11. *Consider the following three functions:*

$$f(x) = x^{-2} \qquad g(x) = 2x^{-2} \qquad h(x) = x^{-2} - 1.$$

Then

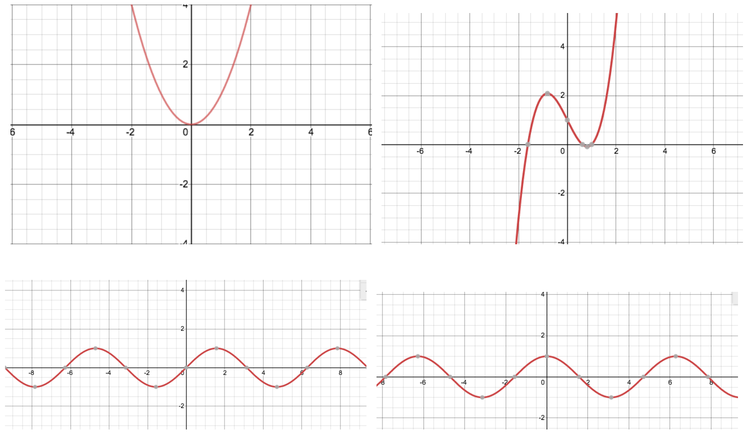
$$\lim_{x\to 0} f(x) = +\infty \qquad \lim_{x\to 0} g(x) = +\infty \qquad \lim_{x\to 0} h(x) = +\infty.$$

Then

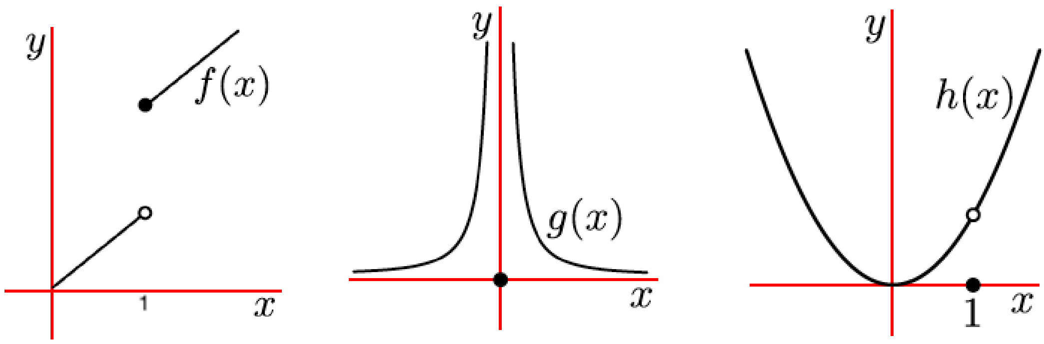
- $$\lim_{x\to 0}(f(x) - g(x)) = \lim_{x\to 0} x^{-2} = -\infty$$
- $$\lim_{x\to 0}(f(x) - h(x)) = \lim_{x\to 0}(1) = 1$$
- $$\lim_{x\to 0}(g(x) - h(x)) = \lim_{x\to 0} x^{-2} + 1 = \infty$$

1.6 Continuity

Look at all the following functions.



All of these functions are continuous. Roughly speaking, a function is continuous if it does not have any abrupt jumps. Now consider the following function.



These functions are not continuous. The function f, g, and h have abrupt jumps at $x = 2$, $x = 0$, and $x = 1$, respectively, so f is not continuous at a, g is not continuous at 0, and h is not continuous at 1.

Definition. *A function $f(x)$ is continuous at a if*

$$\lim_{x \to a} f(x) = f(a).$$

If a function is not continuous at a then it is said to be discontinuous at a. When we write that f is continuous without specifying a point, then typically this means that f is continuous at a for all $a \in \mathbb{R}$.. When we write that $f(x)$ is continuous on the open interval (a, b) then the function is continuous at every point c satisfying $a < c < b$.

From the above definition we immediately have that if f is continuous at a, then

1. $f(a)$ exists;
2. $\lim_{x \to a^-}$ exists and is equal to $f(a)$.
3. $\lim_{x \to a^+}$ exists and is equal to $f(a)$.

Definition. *A function is continuous from the left at a if*

$$\lim_{x \to a^-} = f(a).$$

And a function is continuous from the right at a if

$$\lim_{x \to a^-} = f(a).$$

Definition. *A function $f(x)$ is continuous on an interval $[a, b]$ if*

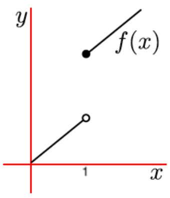
1. *$f(x)$ continuous on (a, b),*
2. *$f(x)$ is continuous form the right at a,*
3. *$f(x)$ is continuous form the left at b.*

Definition. *A function $f(x)$ is continuous on an interval $(a, b]$ (on the interval $[a, b)$) if*

1. *$f(x)$ continuous on (a, b),*
2. *$f(x)$ is continuous form the left at b (from the right at a).*

Example 1.6.1. *Consider the function*

$$f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$$



jump discontinuity

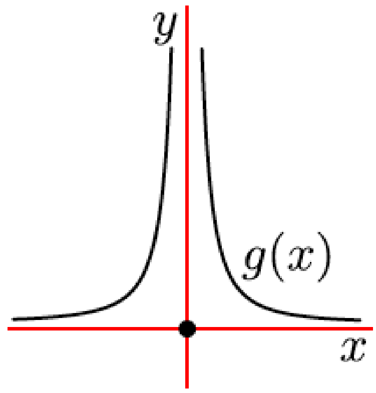
- $$\lim_{x \to 1^-} f(x) = 1 \qquad \lim_{x \to 1^+} f(x) = 3 \qquad f(1) = 3.$$
- *The function $f(x)$, at $x = 1$ is not continuous because the limit does not exist; however, it is continuous form the right at 1 since*

$$\lim_{x \to 1^+} f(x) = 3 = f(1).$$

- *The function $f(x)$, on $[1, \infty)$ (for $x \geq 1$) is continuous.*
- *The function $f(x)$, on $(-\infty, -1)$ is continuous.*

Example 1.6.2. *Consider the function*

$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



infinite discontinuity

- *Consider that*

$$\lim_{x \to 0^-} g(x) = \infty = \lim_{x \to 0^+} g(x) \qquad g(0) = 0.$$

Thus the function $g(x)$ is not continuous at 0 because

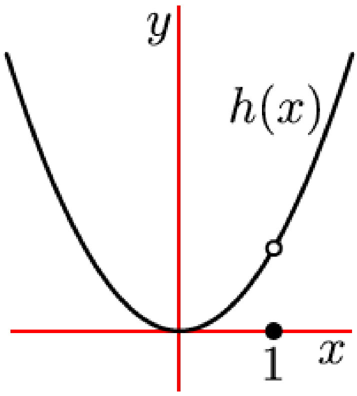
$$\lim_{x \to 0} g(x) = \infty \neq 0 = g(0).$$

It is not continuous at 0 from the left since $\lim_{x \to 0^-} g(x) = \infty \neq 0 = g(0)$ and not from the right since $\lim_{x \to 0^+} g(x) = \infty \neq 0 = g(0)$.

- *the function $g(x)$ is continuous at all points in $\mathbb{R}$ except 0.*

Example 1.6.3. *Consider the function*

$$h(x) = \begin{cases} \frac{x^3-x^2}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$



removable discontinuity

- $$\lim_{x\to 1^-} h(x) = 1 = \lim_{x\to 1^+} h(x) \qquad\qquad f(1) = 0.$$
- $$\lim_{x\to 1} = 1.$$
- *the function* $h(x)$ *is not continuous at* 1 *since*
$$\lim_{x\to 1} h(x) = 1 \neq 0 = h(1).$$
It is not continuous form the left since
$$\lim_{x\to 1^-} h(x) = 1 \neq 0 = h(1)$$
and not from the right since
$$\lim_{x\to 1^+} h(x) = 1 \neq 0 = h(1).$$
- *the function* $h(x)$ *is continuous at all points in* $\mathbb{R}$ *except* 1.

Lemma 1.6.4. *Let* $c \in \mathbb{R}$. *The functions*

$$f(x) = x \qquad\qquad g(x) = c$$

are continuous everywhere on the real line.

Theorem 1.6.5. (**Arithmetic of continuity**) *Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at a. Then the following functions are also continuous at $x = a$.*

- $f(x) + g(x)$ *and* $f(x) - g(x)$,
- $cf(x)$ *and* $f(x)g(x)$, *and*
- $\frac{f(x)}{g(x)}$ *provided* $g(a) \neq 0$.

Theorem 1.6.6. *The following functions are continuous everywhere in their domains*

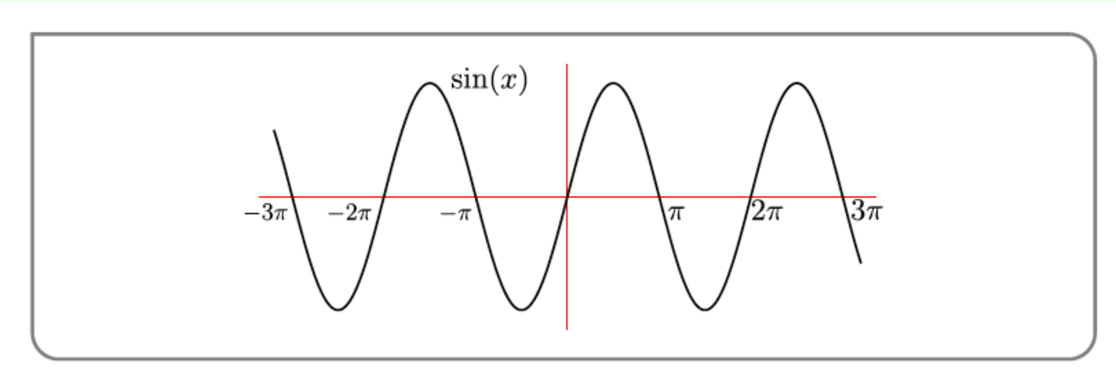
- *polynomials and rational functions (for example* $f(x) = x^5 + 4x^2 + 1$ *and* $g(x) = \frac{x^2+1}{x+1}$*)*
- *roots and powers (for example* $h(x) = \sqrt{x}$ *and* $r(x) = 2^x$*)*
- *trig functions and their inverses (for example* $k(x) = \sin(x)$ *and* $t(x) = \cos^{-1}(x)$*)*
- *exponentials and logarithms (for example* $s(x) = e^x$ *and* $q(x) = \ln x$*).*

Example 1.6.7. *Determine when the function* $f(x) = \frac{\sin(x)}{x^2-5x+6}$ *is continuous? Since both* $\sin(x)$ *and* $x^2 - 5x + 6$ *are continuous by the above theorem we only need to check when* $x^2 - 5x + 6 = 0$. *Note that* $x^2 - 5x + 6 = (x-2)(x-3)$, *thus this polynomial is only zero at* $x = 2$ *and* $x = 3$. *Therefore,* $f(x)$ *is continuous at all points in* $\mathbb{R}$ *except* 2 *and* 3.

Theorem 1.6.8. *If* g *is continuous at* a *and* $f(x)$ *is continuous at* $g(a)$, *then* $(f \circ g)(x) = f(g(x))$ *is continuous at* $x = a$.

Example 1.6.9. *Determine when the function* $h(x) = \sqrt{\sin(x)}$ *is continuous.*

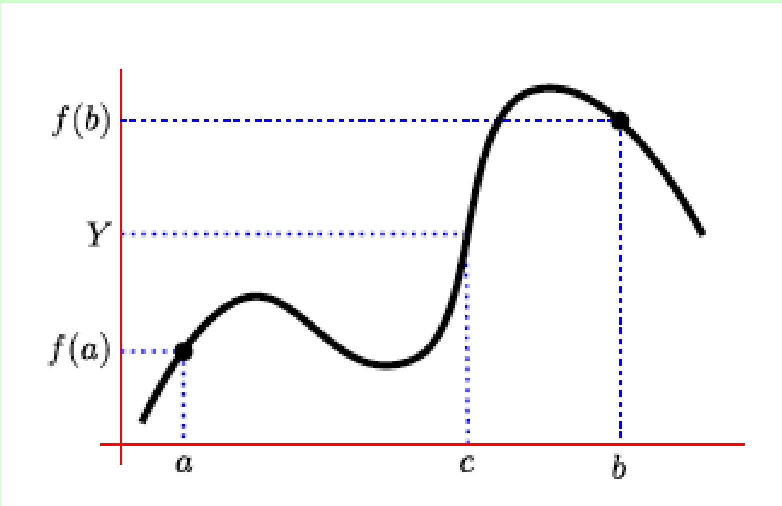
Solution. *Let* $f(x) = \sqrt{x}$ *and* $g(x) = \sin(x)$*, then* $h(x) = (f \circ g)(x)$*. We only need to find out at what points* $\sin(x)$ *is positive.*



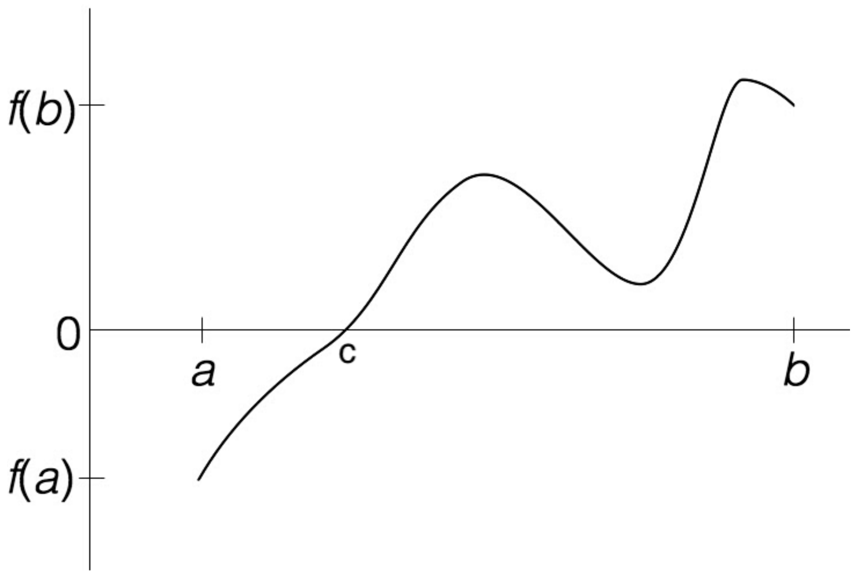
The function $\sqrt{\sin(x)}$ *is continuous if*

$$x \in [2n\pi, (2n+1)\pi] \qquad \text{for all natural numbers } n.$$

Theorem 1.6.10. (Intermediate value theorem(IVT)) *Let* $a < b$ *and let* $f(x)$ *be a function that is continuous at all points* $a \leq x \leq b$*. If* Y *is any number between* $f(a)$ *and* $f(b)$ *then there exists some number* $c \in [a, b]$ *so that* $f(c) = Y$*.*

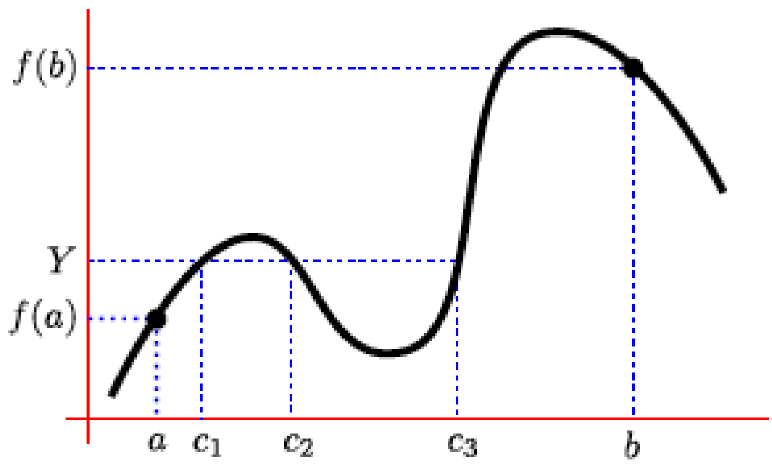


Remark. *One of the main application of the IVT theorem is showing a function* f *has* *a zero inside an interval. For example, in the following picture*



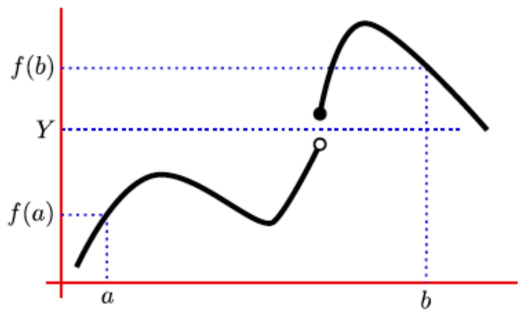
we can see that $f(a) < 0$ and $f(b) > 0$, therefore by IVT, there is a number c between a and b such that $f(c) = 0$.

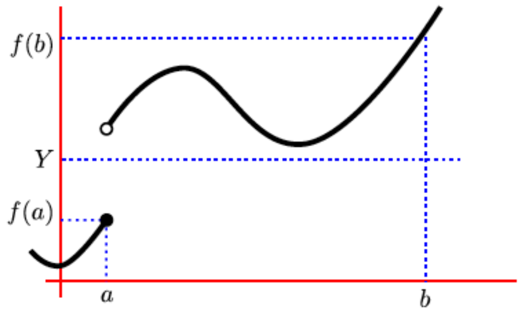
Remark. *If f is continuous and $f(a) \leq Y \leq f(b)$, the IVT merely shows that there is a $a \leq c \leq b$ such that $f(c) = Y$, but it doesn't show how many of them exist. For example, in the following picture, we can see $f(a) \leq Y \leq f(b)$, and there are three numbers c_1, c_2, and c_3 such that $f(c_1) = f(c_2) = f(c_3) = Y$.*



Remark. *Consider that if the function f is not continuous at the interval $[a, b]$ then the IVT fails. In the following examples, even though $f(a) \leq Y \leq f(b)$, there is not a number*

$a \le c \le b$ such that $f(c) = Y$.





Example 1.6.11. *Show that the function $f(x) = x - 1 + sin(\pi x/2)$ has a zero in $0 \le x \le 1$.*

Solution. *Consider that $f(x)$ is a continuous function such that $f(0) = -1$ and $f(1) = 1$. Therefore, by IVT, since $f(0) = -1 \le 0 \le 1 = f(1)$, we have $f(c) = 0$ for some $c \in [0, 1]$.*

Example 1.6.12. *Use the bisection method to find a zero of $f(x) = x - 1 + \sin(\pi x/2)$ that lies between 0 and 1.*

Solution.

- *Let $a = 0$ and $b = 1$. Then*

$$f(0) = -1$$

$$f(1) = 1$$

- *Test the point in the middle $x = \frac{1-0}{2} = 0.5$,*

$$f(0.5) = 0.2071067813 > 0$$

- *Let $a = 0$ and $b = 0.5$. Then*

$$f(0) = -1$$

$$f(1) = 0.2071067813$$

  So by IVT, there is a zero in $[0, 0.5]$.

- *Test the point in the middle $x = \frac{0.5-0}{2} = 0.25$.*

$$f(0.25) = -0.3673165675 < 0.$$

- *Let $a = 0.25$, $b = 0.5$ where $f(0.25) < 0$ and $f(0.5) > 0$. By IVT there is a zero in the interval $[0.25, 0.5]$.*

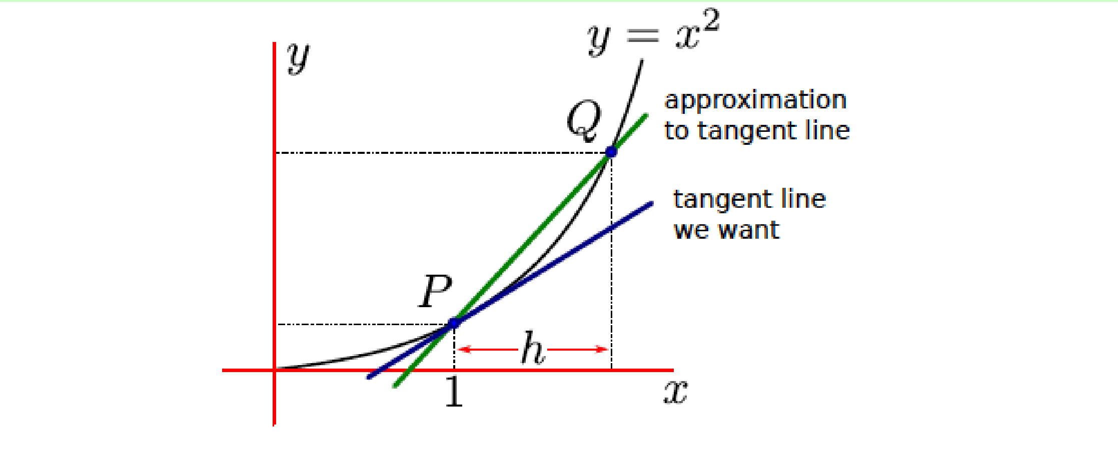
- *So without much work we know the location of a zero inside a range of length 1/4. Each iteration will halve the length of the range and we keep going until we reach the precision we need, though it is much easier to program a computer to do it.*

Chapter 2

Derivatives

2.1 Revisiting Tangent Lines

Example 2.1.1. *Find the slope of the tangent line to the curve $y = x^2$ that passes through $P = (1, 1)$.*



Solution. *Consider that the slope of the secant line is*

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

And the slope of the tangent line is the same as

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}.$$

Theorem 2.1.2. *Given a function $f(x)$ the slope of the tangent line at $x = a$ (if exists) is*

$$\lim_{x \to a} \frac{f(a+h) - f(a)}{h}.$$

2.2 Definition of the derivative

Definition. *(Derivative at a point) Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined on an open interval that contains a.*

- *The derivative of $f(x)$ at $x = a$ is denoted $f'(a)$ and is defined by*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{2.2.1}$$

  if the limit exists.

- *When the above limit exists, the function $f(x)$ is said to be differentiable at $x = a$. When the limit does not exist, the function $f(x)$ is said to be not differentiable at $x = a$.*

- *We can equivalently define the derivative $f'(a)$ by the limit*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

To see that these two definitions are the same, we set $x = a + h(x - a = h)$ and then when h approaches 0, we have x approaches a, and the limit in 2.2.1 becomes $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$.

Example 2.2.1. *Let $a, c \in \mathbb{R}$ be constants. Compute the derivative of the function $f(x) = c$ at $x = a$.*

Solution. *By the definition of the derivative, we have*

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} 0 \\ &= 0 \end{aligned}$$

Example 2.2.2. *Let $a \in \mathbb{R}$. Compute the limit of the function $g(x) = x$ at $x = a$.*

Solution. *By the definition of the derivative we have*

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{(a+h) - a}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1. \end{aligned}$$

We have so proved our first theorem which is the following.

Theorem 2.2.3. *(easiest derivative) Let $a, c \in \mathbb{R}$ and let $f(x) = c$ and $g(x) = x$. Then*

$$f'(a) = 0$$

and

$$g'(a) = 1.$$

Example 2.2.4. *Compute the derivative of $f(t) = t^2$ at $t = a$.*

Solution. *We have that*

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h\to 0} \frac{a^2 + h^2 + 2ah - a^2}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 2ah}{h} \\ &= \lim_{h\to 0} \frac{h(h+2a)}{h} \\ &= \lim_{h\to 0} h + 2a \\ &= 2a \end{aligned}$$

➤➤ We can tweak the derivative at a specific point a to obtain the derivative as a function x. We replace

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

with

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

This gives us the following definition.

Definition. *Let $f(x)$ be a function*

- *The derivative of $f(x)$ with respect to x is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  provided the limit exists.

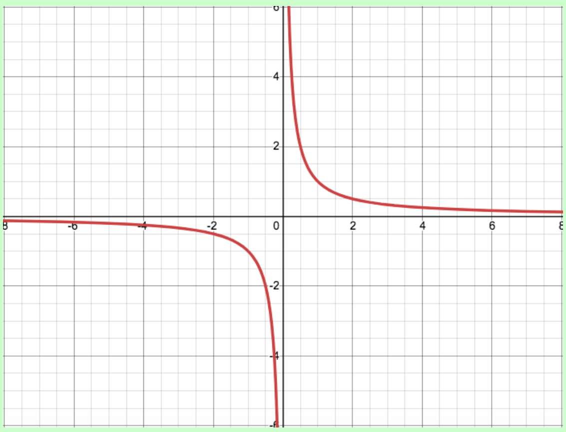
- *If the derivative $f'(x)$ exists for all $x \in (a, b)$ we say that f is differentiable on (a, b).*

- *Note that we will sometimes be a little sloppy with our discussion and simply write "f is differentiable" to mean "f is differentiable on an interval we are interested in" or "f is differentiable everywhere."*

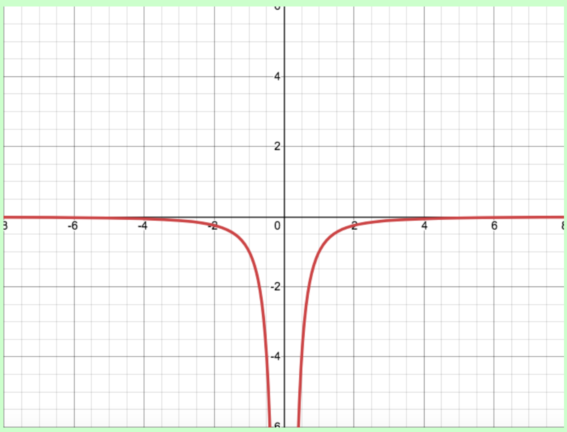
Example 2.2.5. *Let $f(x) = \frac{1}{x}$ and compute its derivative with respect to x.*

Solution. *We have that*

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{x+h} - \frac{1}{x} \right] \\ &= \lim_{h \to 0} \frac{1}{h} \frac{x - (x+h)}{x(x+h)} \\ &= \lim_{h \to 0} \frac{1}{h} \frac{-h}{x(x+h)} \\ &= -\frac{1}{x^2}. \end{aligned}$$



$$y = \frac{1}{x}$$



$$y = \frac{-1}{x^2}$$

➤➤ Notice that the original function $f(x) = \frac{1}{x}$ was not defined at $x = 0$, and the derivative is also not defined at $x = 0$. This does happen more generally–if $f(x)$ is not defined at a particular point $x = a$, then the derivative will not exist at that point either.

Notation. There are several notation all used for "the derivative of $f(x)$ with respect to x"; however,

in this course we generally use the following notations

1. $f'(x)$. This notation is due to Lagrange, and we read it as "f-prime of x".
2. $\frac{df}{dx}$. This notation is duo to Leibniz, and we read it as "dee-f–dee–x".
3. $\frac{d}{dx}f$. We read this as dee–by–dee–x of f.

Example 2.2.6. *Compute the derivative, $f'(a)$, of the function $f(x) = \sqrt{x}$ at the point $x = a$ for any $a > 0$.*

Solution. *We have that*

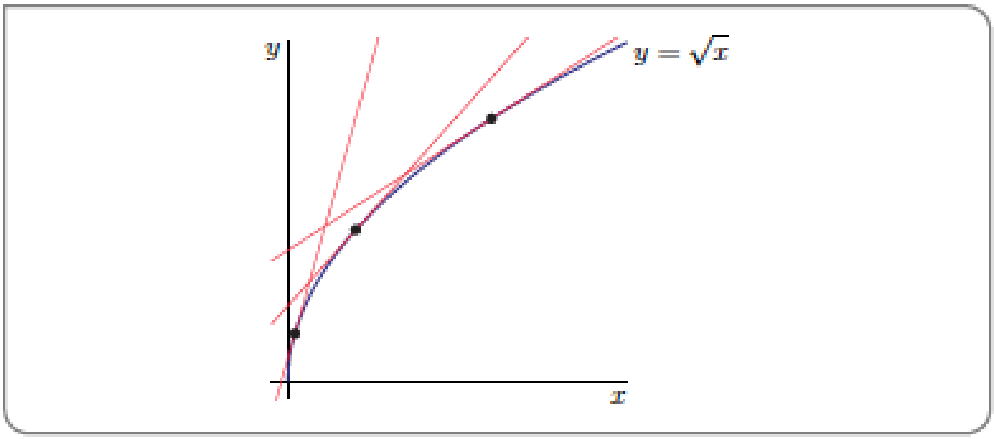
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$

We now multiply the numerator and denominator by the conjugate of $\sqrt{x} - \sqrt{a}$, that is $\sqrt{x} + \sqrt{a}$. Then we have

$$\frac{\sqrt{x} - \sqrt{a}}{x - a} \times \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} = \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}}.$$

Therefore,

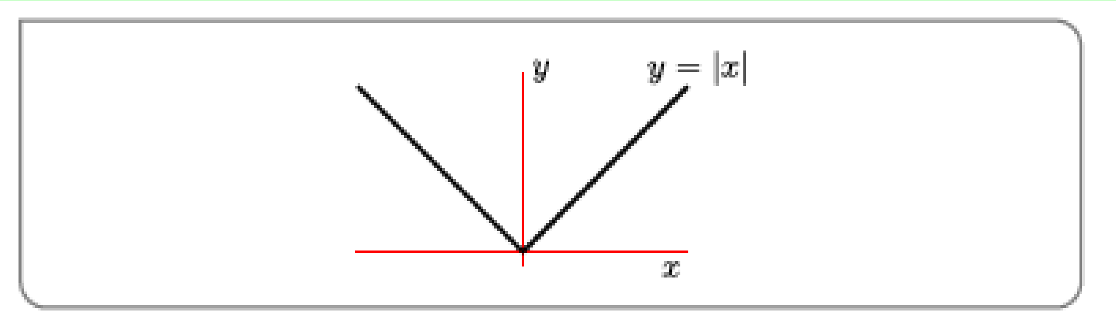
$$f'(a) = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}.$$



Example 2.2.7. *Find the derivative, $f'(a)$, of the function $f(x) = |x|$ at the point $x = a$.*

Solution. Recall that

$$|x| = \begin{cases} -x & x < 0 \\ 0 & x = 0 \\ x & x > 0 \end{cases}$$



We should break our computation of the derivative into three cases depending on whether x is positive, negative, or zero.

- Assume $x > 0$. Then

$$\begin{aligned} \frac{df}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{|x+h| - |x|}{h} \end{aligned}$$

Since $x > 0$ and h is much more smaller than x, we have $x + h > 0$ and so $|x + h| = x + h$, moreover, since x is positive, $|x| = x$.

$$\begin{aligned} &\lim_{h\to 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h\to 0} \frac{x+h-x}{h} \\ &= \lim_{h\to 0} \frac{h}{h} = 1. \end{aligned}$$

- Assume $x < 0$. Then we have

$$\begin{aligned} \frac{df}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{|x+h| - |x|}{h} \end{aligned}$$

Since $x < 0$ and h is much more smaller than x, we have $x + h < 0$ and so $|x + h| = -(x + h)$, moreover, since $x < 0$ is positive, $|x| = -x$.

$$\begin{aligned} &\lim_{h\to 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h\to 0} \frac{-(x+h) - (-x)}{h} \\ &= \lim_{h\to 0} \frac{-h}{h} = -1. \end{aligned}$$

- Assume $x = 0$. Then we have

$$\begin{aligned} \frac{df}{dx} &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h} \end{aligned}$$

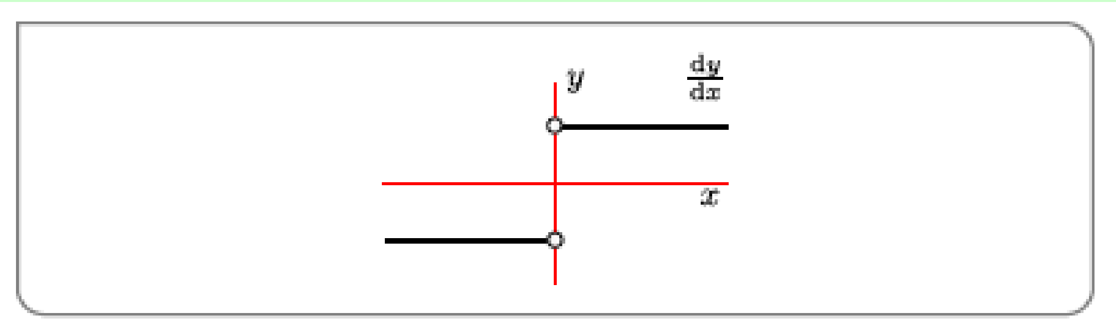
Consider that

$$\lim_{h \to 0^+} \frac{|h|}{h} = 1 \qquad \text{and} \qquad \lim_{h \to 0^-} \frac{|h|}{h} = -1$$

Therefore, this limit does not exist and so the function $|x|$ is not derivative at $x = 0$.

In summary:

$$\frac{d}{dx}|x| = \begin{cases} -1 & x < 0 \\ DNE & x = 0 \\ 1 & x > 0 \end{cases}$$

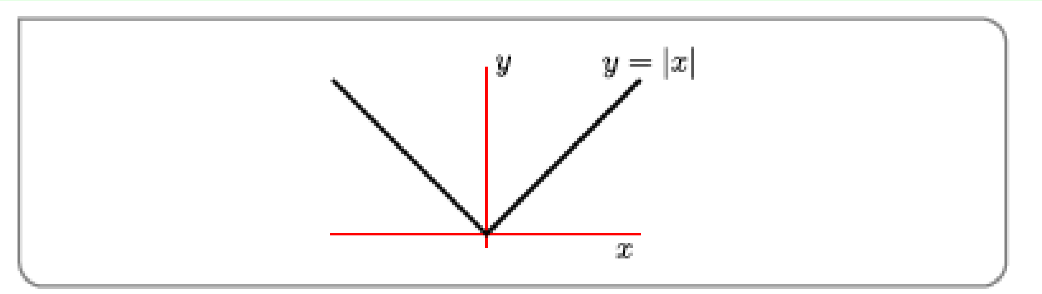


➤➤ **Where is the derivative undefined?** The derivative $f'(a)$ exists precisely when the limit

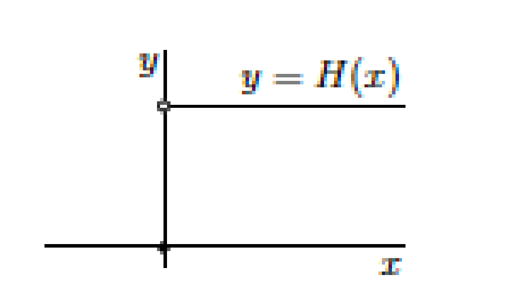
$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists. That limit is the slope of the tangent line to the curve $y=f(x)$ at $x=a$. Thus, that limit does not exist one of the following happens.

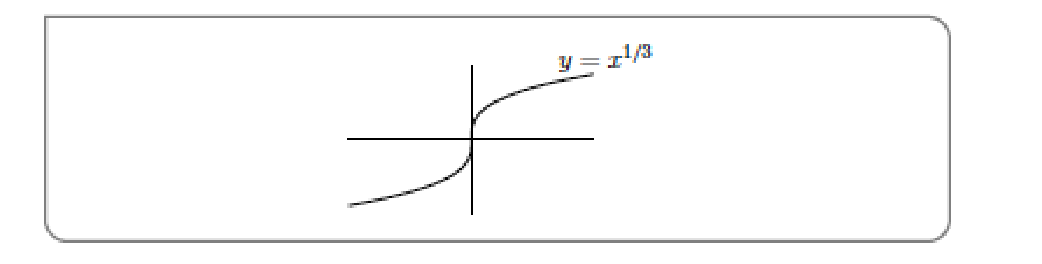
❶ The curve $y=f(x)$ does not have a tangent line at $x=a$ when it has a sharp corner at $x=a$, as an example $f(x)=|x|$ is not differentiable at $x=0$ since it has a sharp corner at $x=0$.



❷ When the curve does have a tangent line because it is not continuous at $x=a$. As an example, we have seen that $f(x)=H(x)=\begin{cases}0 & x\le 0\\ 1 & x>0\end{cases}$ does not have a tangent line at $x=0$ since it is not continuous at $x=0$.

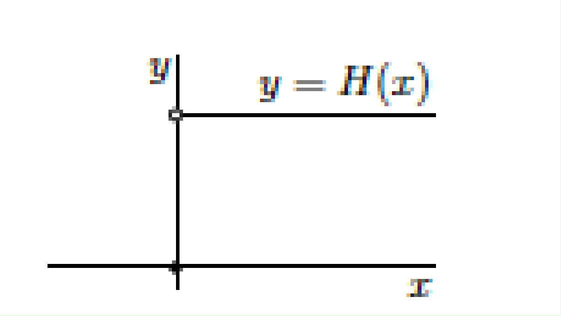


❸ When the curve has a tangent line at $x=a$ but the slope of the tangent line at $x=a$ is infinity. As an example, $f(x)=x^{1/3}$ is not differentiable at $x=0$ since it has a tangent line with slope infinity.



Example 2.2.8. *Verify that the function*

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$



does not have a tangent line at $x = 0$.

Solution. *Consider that if the tangent line exists then the following limit also must exists,*

$$\lim_{h \to 0} \frac{H(0+h) - H(0)}{h}.$$

Consider that

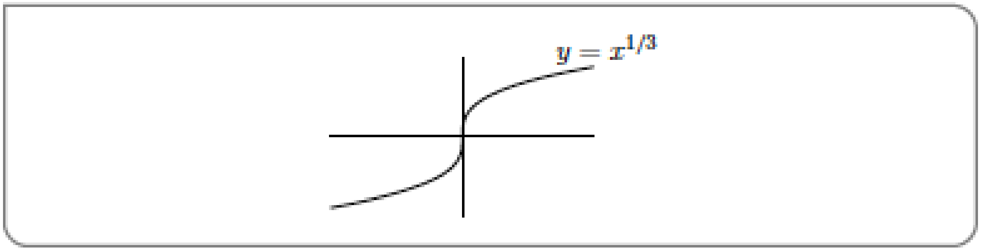
$$\lim_{h \to 0^+} \frac{H(0+h) - H(0)}{h} = \lim_{h \to 0^+} \frac{1}{h} = +\infty$$

and

$$\lim_{h \to 0^-} \frac{H(0+h) - H(0)}{h} = \lim_{h \to 0^-} \frac{0}{h} = 0.$$

Therefore, the limit does not exists.

Example 2.2.9. *Verify that the derivative of $f(x) = x^{1/3}$ at $x = 0$ does not exist.*



Solution. *You can already see in the graph that the derivative at $x = 0$ does not exist since the tangent line has infinite slope. However, we need a mathematical proof, and we should show that $f'(0)$ which is the same as the following limit*

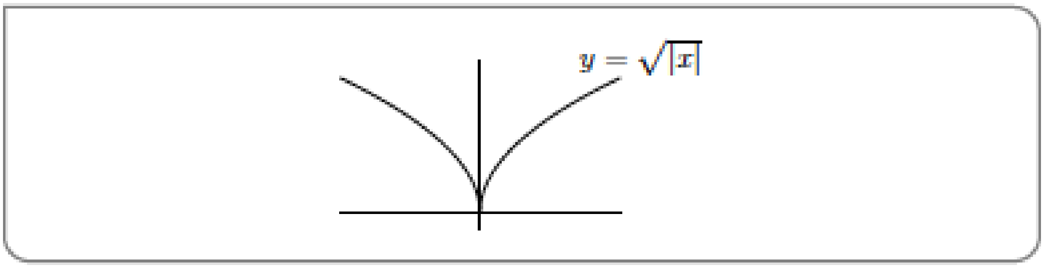
$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

does not exist. We have

$$\lim_{h\to 0}\frac{(0+h)^{1/3}-0^{1/3}}{h}=\lim_{h\to 0}\frac{h^{1/3}}{h}=\lim_{h\to 0}\frac{1}{h^{2/3}}=+\infty$$

(or we can say DNE).

Example 2.2.10. *Verify that the derivative of $f(x) = \sqrt{|x|}$ at $x = 0$ does not exist.*



Solution. *Even though you can see in the graph that at $x = 0$, the graph has a sharp corner, we also show that the limit*

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{\sqrt{|h|}-0}{h}.$$

Consider that

$$\lim_{h\to 0^+}\frac{\sqrt{|h|}}{h}=\lim_{h\to 0^+}\frac{\sqrt{1}}{\sqrt{h}}=+\infty$$

(or DNE).

➤➤ What is the relation between continuity and differentiability?

Theorem 2.2.11. *If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.*

Theorem 2.2.12. *If $f(x)$ is not continuous at $x = a$, then it is not differentiable at $x = a$.*

2.3 Interpretations of the Derivative

Interpretation of the derivative:

- the instantaneous rate of change of a quality
- the slope of a curve.

➤➤ Instantaneous Rate of Change

Assume that we have the function $f(t)$ of the measuring of some quantity. Then

$$\text{average rate of change of } f(t) \text{ from } t = a \text{ to } t = a + h \text{ is}$$

$$\frac{\text{change in } f(t) \text{ from } t = a \text{ to } t = a + h}{\text{length of time from } t = a \text{ to } t = a + h}$$

$$= \frac{f(a + h) - f(a)}{h}.$$

And so

$$\text{instantaneous rate of change of } f(t) \text{ at } t = a$$

$$= \lim_{h \to 0} \text{average rate of change of } f(t) \text{ from } t = a \text{ to } t = a + h$$

$$= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a).$$

Example 2.3.1. *You drop a ball from a tall building. After t seconds the ball has fallen a distance of $s(t) = 4.9t^2$ meters. What is the (instantaneous) velocity of the ball one second after it is dropped?*

Solution. *Since the instantaneous velocity is actually the instance rate of change if distance, we should compute $f'(1)$. Computing the limit of the derivative we have*

$$f'(t) = 9.8t \qquad \text{and} \qquad f'(1) = 9.8.$$

Example 2.3.2. *You are taking a walk and that as you walk, you are continuously measuring some quantity, like temperature, and that the measurement at time t is $f(t) = \sqrt{t}$. What is the average rate of $f(t) = \sqrt{t}$ from $t = 1$ to $t = 2$? What is the instantaneous rate of change of $f(t)$ at $t = 1$?*

Solution. *Consider that the average rate of change is*

$$\frac{f(a+h) - f(a)}{h}$$

The difference between times is h, so $h = 2 - 1 = 1$. Therefore, the average is

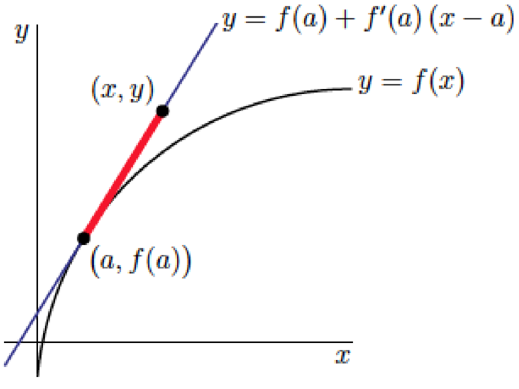
$$\frac{f(a+h) - f(a)}{h} = \frac{f(2) - f(1)}{1} = \sqrt{2} - 1.$$

The instantaneous rate of change is $f'(1)$. We already have seen that $f'(t) = \frac{1}{2\sqrt{t}}$, and so $f'(1) = \frac{1}{2}$.

➤➤ Slope and the formula for the tangent line to a curve at $x = a$

We know that the slope of a function at $x = a$ is $f'(a)$. To find the tangent line we need to find a point at the tangent line; however, for sure we know that $(a, f(a))$ is on the tangent line. So we should find the equation for a line that passes through $(a, f(a))$ with slope $f'(a)$ which is

$$y = f(a) + f'(a)(x - a).$$



A line segment of a tangent line

Theorem 2.3.3. *The tangent line to the curve $y = f(x)$ at $x = a$ is given by the equation*

$$y = f(a) + f'(a)(x - a)$$

provided the derivative $f'(a)$ exists.

Example 2.3.4. *Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.*

Solution. *By the above theorem we have the equation for the tangent line is*

$$y = f(a) + f'(a)(x - a).$$

Note that $a = 4$ and so $f(a) = f(4) = \sqrt{4} = 2$. We also already have had $f'(x) = \frac{1}{2\sqrt{x}}$. Therefore, $f'(a) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. And so the equation for the tangent line is

$$y = 2 + \frac{1}{4}(x - 4).$$

Homework:

Go to this link
https://www.mooculus.osu.edu/textbook/mooculus.pdf and download the book "MOOCULUS". Then do the following questions:

- all questions in page 35;
- in page 33 see why $\lim_{x\to 0} \frac{sin(x)}{x} = 1$. Then do Questions 1-8 page 38;
- in page 42, do questions 1-10.

Bibliography

[1] CLP1: Differential Calculus by J. Feldman, A. Rechnitzer, and E. Yeager.

➤➤