MATH100: Differential Calculus with Application to Physical Sciences and Engineering

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Contents

0.1 Indeterminate forms and L'Hôpital's rule (CLP 3.7)

Let us go back to limits again. We recall that

Theorem 0.1.1. *If* $\lim_{x\to a} f(x) = K$ *and* $\lim_{x\to a} g(x) = L$, *then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{K}{L} \qquad \text{provided } L \neq 0.$$

So,

$$\lim_{x\to 2} \frac{x^2-1}{x+1} = \frac{3}{3} = 1.$$

But what about this one

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{0}{0} = ?????$$

we realized that we have to simplify first before taking the limit

$$\begin{aligned} &= \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)} \\ &= \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}. \end{aligned}$$

Also,

$$\lim_{x\to 0^+} \frac{x}{x^2} = \lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

But what happens when we get something more substantial like

$$\lim_{x\to 0} \frac{\sin(x)}{x} = ????$$

we can solve this one by doing quite a bit of geometry and then we have the limit is

$$= 1$$

What can we do with

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = \frac{0}{0} = ??? \qquad \text{or} \qquad \lim_{x\to +\infty} \frac{\log(x)}{x} = \frac{\infty}{\infty} = ???$$

We will soon see the L'Hopital's rule and easily we can find the above limits.

Definition (CLP 3.71 and 3.7.2)

Consider the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}.$$

- If
$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$
then we refer to the limit as an indeterminate form of type $\frac{0}{0}$.
- If
$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$$
then we refer to the limit as an indeterminate form of type $\frac{\pm\infty}{\pm\infty}$.

There are other types of indeterminate forms as well. Some other types are,

$$0 \times (\pm\infty) \qquad 1^{\infty} \qquad 0^0 \qquad \infty^0 \qquad \infty - \infty$$

(**warning:** $\frac{\pm\infty}{0}$ and $\frac{0}{\pm\infty}$ are not in the above list.) If we have any of the above indeterminate forms, it is more likely that we can change it to a limit that in that limit we only need to take care of a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)},$$

and that's where we can use L'Hôpital's rule.

Theorem 0.1.2. *(CLP 3.7.2—L'Hôpital's Rule) Let f and g be differentiable functions and a either be a real number or $\pm\infty$. Furthermore, suppose that either*

- $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, *or*
- $\lim_{x\to a} f(x) = \pm \lim_{x\to a} g(x) = \pm\infty$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that limit on the right-hand-side exists or is $\pm\infty$.

Indeterminate forms $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$ can all be dealt with the following way

Example 0.1.3. *Compute*

$$\lim_{x\to 0} \frac{\cos(x)-1}{x}.$$

Solution. *Consider that $\lim_{x\to 0} \cos(x) - 1 = 0$ and $\lim_{x\to 0} = 0$, thus this is a $0/0$ indeterminate form, and since both functions are differentiable, we use l'Hôpital's rule. So by l'Hôpital's rule, and so*

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = \lim_{x\to 0} \frac{-\sin(x)}{1} = 0.$$

Example 0.1.4. Indeterminate form $0 \times (\pm\infty)$ can all be dealt with the following way

Compute

$$\lim_{x\to 0^+} x\ln(x).$$

Solution. *Note that $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+} \ln(x) = -\infty$. So we have an indeterminate of the form $0.(-\infty)$. Consider that*

$$\begin{aligned}
\lim_{x\to 0^+} x\ln(x) &= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} && \textit{indeterminate form of type } -\infty/\infty \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} && \textit{by L'Hôpital's rule} \\
&= \lim_{x\to 0^+} -x = 0.
\end{aligned}$$

Indeterminate forms 1^∞, 0^0, and ∞^0 can all be dealt with the following way

Example 0.1.5.

$$\lim_{x\to\infty} x^{\frac{1}{x}}$$

Solution. *Note that* $\lim_{x\to\infty} x = \infty$ *and* $\lim_{x\to\infty} \frac{1}{x} = 0$*. So we have an indeterminate form of type* ∞^0*. Let*

$$y = \lim_{x\to\infty} x^{\frac{1}{x}}.$$

Then

$$\begin{aligned}
\ln y &= \ln(\lim_{x\to\infty} x^{\frac{1}{x}}) \\
&= \lim_{x\to\infty} \ln(x^{\frac{1}{x}}) && x^{\frac{1}{x}} \textit{ is continuous} \\
&= \lim_{x\to\infty} \frac{1}{x}\ln(x) = \lim_{x\to\infty} \frac{\ln(x)}{x} && \textit{indeterminate form of type } \infty/\infty \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{1}{x} && \textit{by l'Hôpital's rule} \\
&= 0
\end{aligned}$$

Therefore, $\ln y = 0$ *and so* $1 = y = \lim_{x\to\infty} x^{\frac{1}{x}}$.

Indeterminate form $\infty - \infty$: you should change it to an indeterminate of the form $0/0$ or $\pm\infty / \pm\infty$.

Example 0.1.6. *Compute*

$$\lim_{x\to\infty} \sqrt{x^2+4x} - \sqrt{x^2-3x}.$$

Solution. *Consider that this limit gives an indeterminate form of type $\infty - \infty$. We can write*

$$\begin{aligned}
\sqrt{x^2+4x} - \sqrt{x^2-3x} &= x\sqrt{1+\frac{4}{x}} - x\sqrt{1-\frac{3}{x}} && x \textit{ is positive} \\
&= x\left(\sqrt{1+\frac{4}{x}} - \sqrt{1-\frac{3}{x}}\right) \\
&= \frac{\sqrt{1+\frac{4}{x}} - \sqrt{1-\frac{3}{x}}}{\frac{1}{x}}
\end{aligned}$$

Therefore,

$$\lim_{x\to\infty} \sqrt{x^2+4x} - \sqrt{x^2-3x} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x}} - \sqrt{1-\frac{3}{x}}}{\frac{1}{x}}$$

So we change the limit to a limit that is an indeterminate of type $0/0$, and now we can use L'Hôpital's rule; and using the rule we have the limit is equal to $7/2$.

Example 0.1.7. *Compute*

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Solution. *Consider that this limit gives an indeterminate form of type ∞/∞. Let see what will happen if we use L'Hôpital's rule,*

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} &= \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} && \text{indeterminate form of type } \infty/\infty \\
& && \text{using L'Hôpital's rule again} \\
&= \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} && \text{indeterminate form of type } \infty/\infty
\end{aligned}$$

Seems that we are in a loop if we continue using L'Hôpital's rule; what if we do some algebra here.

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} &= \lim_{x\to\infty} \left(\frac{e^x}{e^x}\right) \frac{1 - e^{-2x}}{1 + e^{-2x}} \\
&= \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} \\
&= 1
\end{aligned}$$

Example 0.1.8. *Sometimes you need to use L'Hôpital's rule more than once. Consider*

$$\lim_{x\to-\infty} \frac{x^2}{e^{1-x}}.$$

Note that $\lim_{x\to-\infty} x^2 = \infty$ and $\lim_{x\to-\infty} e^{1-x} = \infty$, so we have an indeterminate form of type ∞/∞. Thus using L'Hôpital's rule

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x^2}{e^{1-x}} &= \lim_{x\to-\infty} \frac{2x}{-e^{1-x}} && \text{indeterminate of type } -\infty/-\infty \\
&= \lim_{x\to-\infty} \frac{2}{e^{1-x}} && \text{by L'Hôpital's rule} \\
&= 0.
\end{aligned}$$

Example 0.1.9. *Compute the limit*

$$\lim_{x\to\infty} \frac{\ln(\ln(x))}{\sqrt{x}}.$$

Solution. *Consider that*

$$\lim_{x\to\infty} \ln(\ln(x)) = \infty \qquad \textit{and} \qquad \lim_{x\to\infty} \sqrt{x} = \infty.$$

So we have a ∞/∞ indeterminate form. We can use l'Hôpital's rule. Note that

$$\frac{d}{dx}\ln(\ln(x)) = \frac{1}{x\ln(x)} \qquad \textit{and} \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Therefore,

$$\begin{aligned}
\lim_{x\to\infty} \ln(\ln(x)) &= \lim_{x\to\infty} \frac{\frac{1}{x\ln(x)}}{\frac{1}{2\sqrt{x}}} && \textit{by L'Hôpital} \\
&= \lim_{x\to\infty} \frac{2\sqrt{x}}{x\ln(x)} = \lim_{x\to\infty} \frac{2}{\sqrt{x}\ln(x)} = 0 && \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}
\end{aligned}$$

Example 0.1.10. *Compute*

$$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x.$$

Solution. *Consider that this limit gives an indeterminate form of type 1^∞. We can write*

$$\begin{aligned}
y &= \lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x \\
\ln y &= \ln\left(\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x\right) \\
&= \lim_{x\to\infty} \ln\left(1+\frac{3}{x}\right)^x \\
&= \lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right) \\
&= \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} && \textit{indeterminate form of type } 0/0\textit{, using L'Hôpital's rule} \\
&= \lim_{x\to\infty} \frac{\frac{-3}{x^2}.\frac{1}{(1+\frac{3}{x})}}{\frac{-1}{x^2}} && \frac{d}{dx}\ln\left(1+\frac{3}{x}\right) = \frac{-3}{x^2}.\frac{1}{(1+\frac{3}{x})},\ \frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2} \\
&= \lim_{x\to\infty} \frac{3}{1+\frac{3}{x}} = 3
\end{aligned}$$

0.2 Optimization (CLP 3.5)

Many application of mathematics to "real world" problems consists of finding the maximum and minimum of some function subject to various constraint. For example, minimize the cost, minimize travel time, maximize efficiency, etc. We will solve some of these questions by using calculus. Many of this questions are very easy in terms of mathematics; however, the main obstacle here is translating the problem to calculus. The only way to be expert in this translation is by **practice**.

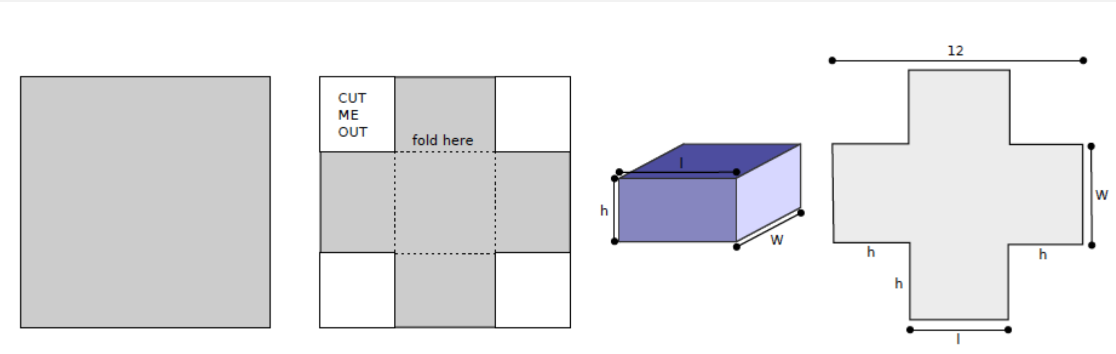
In general to answer this kind of questions, you need to

- Draw a diagram.
- Variables—assign variables to the quantities in the problem.
- Find some relation between the variables.
- Reduce to a function of 1 variable.
- Find the domain, the possible values that can be assigned to the variable.
- Max/Min: find the absolute max/min by using methods that we have studied, for example "closed interval method."

Example 0.2.1. *You are given a square of cardboard (12cm by 12cm) and you need to cut out squares from the corners of your sheet so that you may fold it into a box. How large should these cut-out squares be so as to maximize the volume of the box.*

Solution.

First we draw a picture



(Assign variables.) *So we need to maximize the volume of the box with*

- *width w,*
- *length l, and*
- *height h.*

All should be in cm. And we should make volume V in cm^3.

(Relation between variables.) *We have*

- *the volume is $V = lwh$,*
- *the height is given by the size that we cut out i.e., h,*
- *consider that the length and width are equal and given by*

$$l = w = 12 - 2h.$$

(Reduce to function of 1 variable.) *So the volume is $V = h(12 - 2h)^2$. Therefore, we want to find the maximum of the function $V = h(12 - 2h)^2$.*

(Find the domain.) *Consider that h must not be negative and also*

$$l = w = 12 - 2h \geq 0 \Rightarrow 12 \geq 2h \Rightarrow 6 \geq h.$$

Thus, $0 \leq h \leq 6$.

(Max/Min.) *Now since V is continuous we can use closed interval method, that is finding the critical and singular point, and then the maximum in the interval $0 \leq h \leq 6$ is the largest value of the function at points 0, 6, and singular and critical points. Note that*

$$V = h(144 - 48h + 4h^2) = 144h - 48h^2 + 4h^3,$$

and so

$$\frac{dV}{dh} = 144 - 96h + 12h^2 = 12(h - 6)(h - 2).$$

Thus the critical points are $h = 6$ and $h = 2$. We have

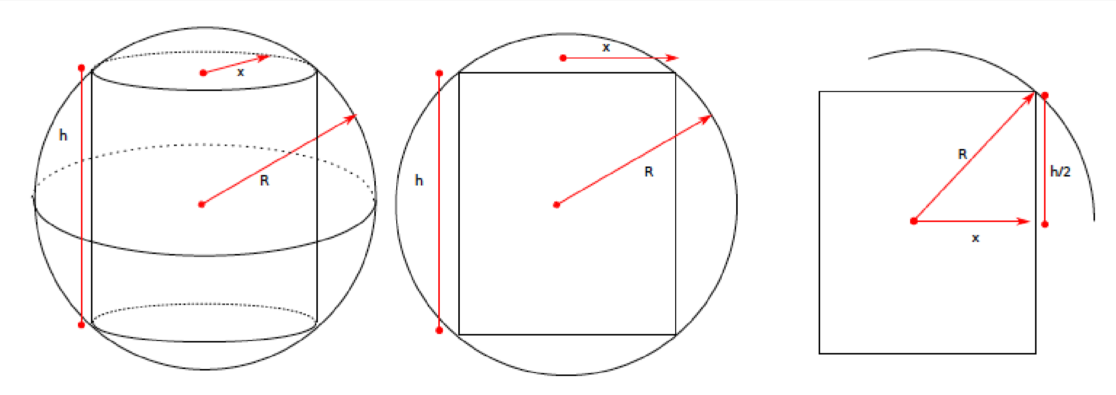
$$V(6) = 0 \quad V(0) = 0 \qquad V(2) = 128cm^3.$$

Therefore, the maximum is $128xm^3$. Actually the question asks for how big we cut the squares, so answer is $h = 2cm$; we cut 2cm squares from the corners.

Example 0.2.2. *Find the dimensions of the largest (in terms of volume) circular cylinder (the usual sort of cylinder we are used to playing with) that can be inscribed (put inside) a sphere of radius 5cm. What fraction of the sphere does the cylinder occupy?*

Solution.

We draw the following diagrams.



(Assign variables.) *Let V be the volume of the cylinder. A cylinder is described by its height and radius. We denote the hight of the cylinder by h and its radius by x.*

(Relation between variables.) *Consider that the volume of sphere is $\frac{4}{3}\pi R^3 = \frac{4}{3}\pi 5^3 = \frac{500}{3}\pi$, and the volume of cylinder is $V = \pi x^2 h$.*
We know want to find the relation between variables. Look at the cross section. We have

$$5^2 = x^2 + (\frac{h}{2})^2.$$

Therefore,

$$x = \sqrt{5^2 - \frac{h^2}{4}}.$$

(Reduce to a function of one variable.) *The volume of the cylinder is*

$$V = \pi x^2 h = \pi h \left(5^2 - \frac{h^2}{4}\right).$$

(Domain.) *Now it's time to find all possible values for h. Consider that the volume and height can't be negative so we must have $h \geq 0$ and*

$$V = \pi h \left(25 - \frac{h^2}{4}\right) \geq 0.$$

Thus

$$0 \leq 25 - \frac{h^2}{4} \Rightarrow h \leq 10.$$

Therefore,

$$0 \leq h \leq 10.$$

(Max/Min.) *We now want to maximize the volume, that is, we should find the global maximum. Consider that*

$$V = 25\pi h - \pi\frac{h^3}{4} \qquad \Rightarrow \qquad \frac{dV}{dh} = 25\pi - 3\pi\frac{h^2}{4}.$$

To find the critical points, set

$$25\pi - 3\pi\frac{h^2}{4} = 0 \qquad \Rightarrow h = \pm\frac{10}{\sqrt{3}}.$$

Remark that h most be positive so we have

$$h = \frac{10}{\sqrt{3}}.$$

Note that at $h = 10$ and $h = 0$, the volume is zero, so we have the maximal volume when $h = \frac{10}{\sqrt{3}}$.
The question asks about the dimension of the cylinder (height and radius), so we need to find the radius too. The radius is

$$x = \sqrt{5^2 - \frac{(\frac{10}{\sqrt{3}})^2}{4}} = 5\sqrt{\frac{2}{3}}.$$

And so the maximum volume is

$$\pi x^2 h = \pi\left(5\sqrt{\frac{2}{3}}\right)^2 \frac{10}{\sqrt{3}} = \frac{500\pi}{3\sqrt{3}}.$$

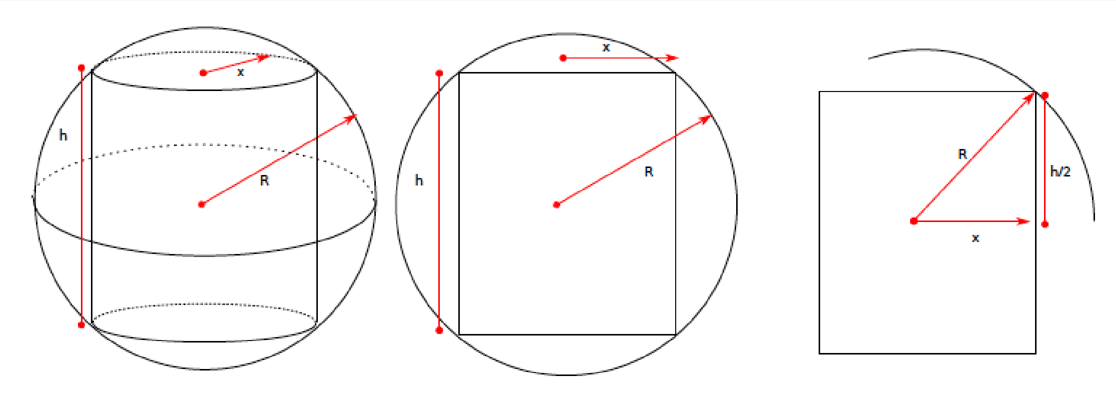
The fraction that the cylinder occupies is

$$\frac{\textit{the volume of the cylinder}}{\textit{the volume of the sphere}} = \frac{\frac{500\pi}{3\sqrt{3}}}{\frac{500\pi}{3}} = \frac{1}{\sqrt{3}}.$$

Example 0.2.3. *Find the dimensions of the largest (in terms of volume) circular cylinder (the usual sort of cylinder we are used to playing with) that can be inscribed (put inside) a sphere. What fraction of the sphere does the cylinder occupy?*

Solution.

We draw the following diagrams.



(Assign variables.) *Let R be the radius of the sphere. Let V be the volume of the cylinder. A cylinder is described by its height and radius. We denote the hight of the cylinder by h and its radius by x.*

(Relation between variables.) *Consider that the volume of sphere is $\frac{4}{3}\pi R^3$, and the volume of cylinder is $V = \pi x^2 h$.*
We know want to find the relation between variables. Look at the cross section. We have

$$R^2 = x^2 + (\frac{h}{2})^2.$$

Therefore,

$$x = \sqrt{R^2 - \frac{h^2}{4}}.$$

(Reduce to a function of one variable.) *The volume of the cylinder is*

$$V = \pi x^2 h = \pi h \left(R^2 - \frac{h^2}{4} \right).$$

(Domain.) *Now it's time to find all possible values for h. Consider that the volume and height can't be negative so we must have $h \geq 0$ and*

$$V = \pi h \left(R^2 - \frac{h^2}{4} \right) \geq 0.$$

Thus

$$0 \leq R^2 - \frac{h^2}{4} \Rightarrow h \leq 2R.$$

Therefore,

$$0 \leq h \leq 2R.$$

(Max/Min.) *We now want to maximize the volume, that is, we should find the global maximum. Consider that*

$$V = R^2\pi h - \pi\frac{h^3}{4} \qquad \Rightarrow \qquad \frac{dV}{dh} = R^2\pi - 3\pi\frac{h^2}{4}.$$

To find the critical points, set

$$R^2\pi - 3\pi\frac{h^2}{4} = 0 \qquad \Rightarrow h = \pm\frac{2R}{\sqrt{3}}.$$

Remark that h most be positive so we have

$$h = \frac{2R}{\sqrt{3}}.$$

Note that at $h = 2R$ and $h = 0$, the volume is zero, so we have the maximal volume when $h = \frac{2R}{\sqrt{3}}$.
The question asks about the dimension of the cylinder (height and radius), so we need to find the radius too. The radius is

$$x = \sqrt{R^2 - \frac{(\frac{2R}{\sqrt{3}})^2}{4}} = R\sqrt{\frac{2}{3}}.$$

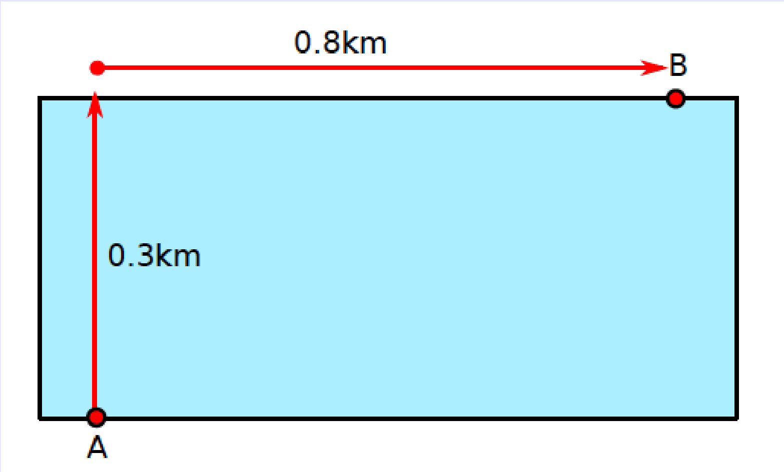
And so the maximum volume is

$$\pi x^2 h = \pi\left(R\sqrt{\frac{2}{3}}\right)^2\frac{2R}{\sqrt{3}} = \frac{4R^3\pi}{3\sqrt{3}}.$$

The fraction that the cylinder occupies is

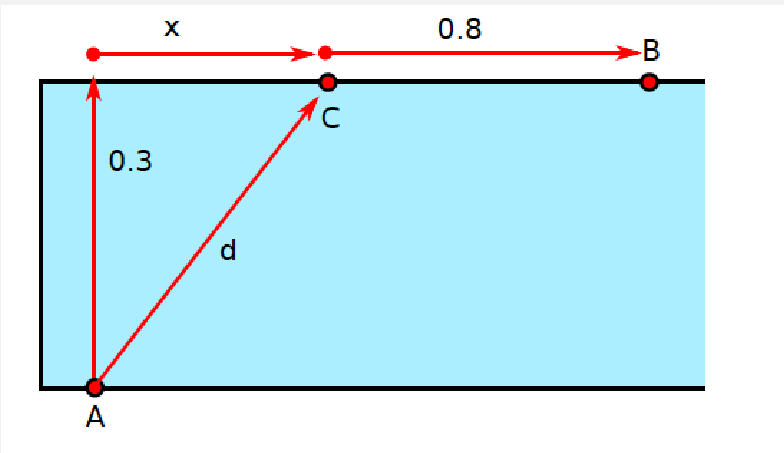
$$\frac{\textit{the volume of the cylinder}}{\textit{the volume of the sphere}} = \frac{\frac{4R^3\pi}{3\sqrt{3}}}{\frac{4R^3\pi}{3}} = \frac{1}{\sqrt{3}}.$$

Example 0.2.4. *You need to cross a small canal to get from point A to point B. The canal is 300m wide and point B is 800m from the closet point on the other side. You can row at 6km/h and run at 10km/h. To which point on the opposite side of the canal should you row to in order to minimize your travel time from A to B?*



Solution. *Remember that if you travel with the speed of v km/h in t hours, you travel $x = vt$ km. So your travel time is $t = \frac{x}{v}$.*

(Diagram and assigning variabels.) *In this question, we need to minimize the total travel time which is made up of the rowing time across the canal and the running time along the canal. Let us say we row to a point C which is x km along the bank.*



(Relations and reducing to a function of one variable.)

- *The row time* T_1 *from* A *to* C *is* $T_1 = \frac{1}{6}\sqrt{0.3^2 + x^2}$.
- *The running time* T_2 *form* C *to* B *is* $T_2 = \frac{1}{10}(0.8 - x)$.
- *Total time is*

$$T = T_1 + T_2 = \frac{1}{6}\sqrt{0.3^2 + x^2} + \frac{1}{10}(0.8 - x)$$

(Domain.) *Now we want to minimize* T. *The domain for* T *is* $0 \leq x \leq 0.8$.

(Max/Min.) *Again we use the closed interval method to find the global minimum. Consider that*

$$\begin{aligned}\frac{dT}{dx} &= \frac{x}{6\sqrt{0.3^2 + x^2}} - \frac{1}{10} \\ &= \frac{5x - 3\sqrt{0.3^2 + x^2}}{30\sqrt{0.3^2 + x^2}}.\end{aligned}$$

When $\frac{dT}{dx} = 0$, *we have*

$$5x - 3\sqrt{0.3^2 + x^2} = 0 \quad \Rightarrow \quad 3\sqrt{0.3^2 + x^2} = 5x \Rightarrow 9(0.3^2 + x^2) = 25x^2$$

$$\Rightarrow 9(0.3)^2 + 9x^2 = 25x^2 \Rightarrow 9(0.3)^2 = 16x^2 \Rightarrow x = \pm\frac{9}{40}.$$

But x *must be positive, so* $x = \frac{9}{40}$. *We have*

$$T(0) = \frac{13}{100} \qquad T(0.8) = \frac{73}{60} \qquad T(9/40) = \frac{3}{25}.$$

Therefore, the global minimum is at $x = \frac{9}{40}$.
Now we can answer the question, travel time is minimized when we row to a point $\frac{9}{40}$ *km along the opposite bank.*

0.2.1 How to compute the derivative of $f(x)^{g(x)}$

Given differentiable functions $f(x)$ and $g(x)$, compute the derivative of

$$f(x)^{g(x)}.$$

Let

$$y = f(x)^{g(x)}.$$

So we want to compute y'. We have

$$\begin{aligned}\ln y &= \ln f(x)^{g(x)} \\ \ln y &= g(x)\ln f(x).\end{aligned}$$

Therefore,

$$\frac{y'}{y} = g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}.$$

Thus,

$$y' = \left(g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}\right) y$$

which means

$$y' = \left(g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}\right) f(x)^{g(x)}.$$

Example 0.2.5. *Find*

$$\frac{d}{dx}\sin(x)^{\cos(x)}.$$

Solution. *Let*

$$y = \sin(x)^{\cos(x)} \Rightarrow \ln y = \ln(\sin(x)^{\cos(x)}) \Rightarrow \ln y = \cos(x)\ln(\sin(x)).$$

So we have

$$\frac{y'}{y} = -\sin(x)\ln(\sin(x)) + \cos(x)\frac{-\cos(x)}{\sin(x)}.$$

Therefore,

$$\begin{aligned} y' &= (-\sin(x)\ln(\sin(x)) + \cos(x)\frac{-\cos(x)}{\sin(x)})y \\ &= (-\sin(x)\ln(\sin(x)) + \cos(x)\frac{-\cos(x)}{\sin(x)})\sin(x)^{\cos(x)}.\end{aligned}$$

Chapter 1

Antiderivatives

Learning Objectives

By the end of this section,

- given a derivative $\frac{dy}{dx}$, you will be able to find what is the original function $y = f(x)$;
- you will be able to find a function $F(x)$ such that $F'(x) = f(x)$ and $F(b) = B$.

Bridge to the next calculus

Finally, we come to the last topic that will be covered in this course—Antiderivative. This topic is actually will be the focus of the next calculus subject. In this course, a large portion of the concepts either were directly using derivative or it was an application of the derivative.

Pre-assessment

We have $F'(x) = 4x^3 + 1$ and $F(1) = 10$. Then

1. $F(x) = x^4 + x + 10$
2. $F(x) = 4x^4 + x + 5$
3. $F(x) = x^4 + x + 8$
4. None of the above.

Pre-example. Note that

$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$$

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + 1\right) = x^2$$

$$\frac{d}{dx}\left(\frac{1}{3}x^3 - \pi\right) = x^2$$

So what is the antiderivative of x^2 (a function that its derivative is x^2)?

Definition

Definition. *A function F is called an antiderivative of f on an interval I when*

$$F'(x) = f(x) \qquad \text{for all } x \in I$$

Why would we want to do this? It turns out that idea is crucial to solving "differential equations" like the one we saw for population models. Also it helps us solve problems like "A particle travels with velocity blah. What is its position?"

Theorem

Theorem 1.0.1. *If F is an antiderivative of f on an interval I, then the most general antiderivative is $F(x) + C$ where C is an arbitrary constant.*

Examples

Now we find the antiderivative of

- $f(x) = \sin(x)$—$F(x) = -\cos(x) + C$
- $f(x) = \cos(x)$ $F(x) = \sin(x) + C$
- $f(x) = x^n$, $n \neq -1$ $F(x) = \frac{1}{n+1}x^{n+1} + C$
- $f(x) = \frac{1}{x}$ $F(x) = \ln|x| + C$ (be careful with this one)

Since we know the derivative of trig functions and inverse trig functions we also know that

$$\begin{array}{cc} f(x) = \sec^2(x) & F(x) = \tan(x) + C \\ f(x) = \frac{1}{\sqrt{1-x^2}} & F(x) = \arcsin(x) + C \\ f(x) = \frac{1}{1+x^2} & F(x) = \arctan(x) + C \end{array}$$

Example

Example 1.0.2. • *Find the antiderivative of*

$$f(x) = 3e^x + \frac{2}{1+x^2}.$$

The antiderivative is

$$F(x) = 3e^x + 2\arctan(x) + C$$

• *Find the antiderivative for*

$$g(t) = 2\cos(t) + \frac{t^3 - 7\sqrt{t}}{t}.$$

Note that

$$g(t) = 2\cos(t) + t^2 + \frac{7\sqrt{t}}{t} = 2\cos(x) + t^2 + \frac{7}{\sqrt{t}}.$$

Therefore, its antiderivative is

$$G(t) = 2\sin(t) + \frac{t^3}{3} - 14t^{1/2} + C.$$

Example

Example 1.0.3. *Find $F(x)$ if $F'(x) = 6x^2 - 18x + 14$ and $F(0) = 1$.*

Solution. *We have that*

$$F(x) = 2x^3 - 9x^2 + 14x + C.$$

Since $F(0) = 1$, then

$$F(0) = 2(0)^3 - 9(0)^2 + 14(0) + C = 1 \Rightarrow C = 1.$$

Therefore,

$$F(x) = 2x^3 - 9x^2 + 14x + 1.$$

Post-assessment

We have $F'(x) = 4x^3 + 1$ and $F(1) = 10$. Then

1. $F(x) = x^4 + x + 10$
2. $F(x) = 4x^4 + x + 5$
3. $F(x) = x^4 + x + 8$
4. None of the above.

Solution. *Consider that $F'(x) = 4x^3+1$, so $F(x) = \frac{1}{4}4x^4+x+C$; we have that $F(1) = 10$, so $F(1) = 1 + 1 + C = 10$. Therefore, $C = 8$. Consequently, $F(x) = x^4 + x + 8$.*

Summary

- The antiderivative of a function $f(x)$ is a function $F(x)$ that $F'(x) = f(x)$; and
- the most general antiderivative is $F(x)+C$ where C is an arbitrary constant.

Next:

Find $F(x)$ if $F''(x) = 6x^2 - 18x + 14$ and $F(0) = -8$, $F(10) = -\frac{5}{2}$.

Pre-assessment

Find $F(x)$ if $F''(x) = 6x^2 - 18x + 14$ and $F(0) = -8$, $F(10) = -\frac{5}{2}$.

1. $F(x) = \frac{1}{2}x^4 - 3x^3 + 7x^2 - 8$
2. $F(x) = \frac{1}{2}x^4 - 3x^3 + 7x^2 + x - 8$
3. $F(x) = 2x^4 - 3x^3 + 7x^2 - 8$
4. $F(x) = 2x^4 - 3x^3 + 7x^2 + x - 8$

Example

Find $F(x)$ if $F''(x) = \sin(x) + 6x$ and $F(0) = \pi$ and $F(\pi) = \pi^2$.

Solution. *Consider that*

$$F'(x) = -\cos(x) + 3x^2 + C$$

and so

$$F(x) = -\sin(x) + x^2 + Cx + D.$$

We have that

$$F(0) = D = \pi$$

$$F(\pi) = \pi^2 + C\pi + D = \pi^2 + C\pi + \pi = \pi^2 \Rightarrow C\pi + \pi = 0 \Rightarrow C = -1.$$

Therefore,

$$F(x) = -\sin(x) + x^2 - x + \pi.$$

Post-assessment

Find $F(x)$ if $F''(x) = 6x^2 - 18x + 14$ and $F(0) = -8,\ F(1) = -\frac{5}{2}$.

1. $F(x) = \frac{1}{2}x^4 - 3x^3 + 7x^2 - 8$
2. $F(x) = \frac{1}{2}x^4 - 3x^3 + 7x^2 + x - 8$
3. $F(x) = 2x^4 - 3x^3 + 7x^2 - 8$
4. $F(x) = 2x^4 - 3x^3 + 7x^2 + x - 8$

Solution. *Consider that*

$$F'(x) = 2x^3 - 9x^2 + 14x + C$$

and

$$F(x) = \frac{x^4}{2} - 3x^3 + 7x^2 + Cx + D.$$

We have

$$F(0) = -8 \Rightarrow D = -8, \quad \textit{and so} \quad F(x) = \frac{x^4}{2} - 3x^3 + 7x^2 + Cx - 8.$$

Moreover,

$$F(10) = -\frac{5}{2} \Rightarrow \frac{1^4}{2} - 3 + 7(1)^2 + 1C - 8 = -\frac{5}{2} \Rightarrow C = 1.$$

Consequently,

$$F(x) = \frac{x^4}{2} - 3x^3 + 7x^2 + x - 8.$$

Chapter 2

Review

2.1 How to compute the derivative of $f(x)^{g(x)}$

$\frac{d}{dx} f(x)^{g(x)}$

Given differentiable functions $f(x)$ and $g(x)$, compute the derivative of

$$f(x)^{g(x)}.$$

Let

$$y = f(x)^{g(x)}.$$

So we want to compute y'. We have

$$\begin{aligned}\ln y &= \ln f(x)^{g(x)} \\ \ln y &= g(x) \ln f(x).\end{aligned}$$

Therefore,

$$\frac{y'}{y} = g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)}.$$

Thus,

$$y' = \left(g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)} \right) y$$

which means

$$y' = \left(g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)} \right) f(x)^{g(x)}.$$

Example

Find

$$\frac{d}{dx}x^{\sin(x)}.$$

1. $\frac{d}{dx}x^{\sin(x)} = (\ln x^{\cos(x)} + \frac{\sin(x)}{x})x^{\sin(x)}$.
2. $\frac{d}{dx}x^{\cos(x)} = (\ln x^{\sin(x)} - \frac{\cos(x)}{x})x^{\sin(x)}$.
3. $\frac{d}{dx}x^{\sin(x)} = (\ln x^{\sin(x)} + \frac{\cos(x)}{x})x^{\sin(x)}$.
4. $\frac{d}{dx}x^{\sin(x)} = (\ln x^{\sin(x)} - \frac{\cos(x)}{x})x^{\sin(x)}$.

Solution

Let $y = x^{\sin(x)}$. Then

$$\begin{aligned}
\ln y = \ln x^{\sin(x)} &= \sin(x)\ln x && \Rightarrow \\
\frac{d}{dx}\ln y &= \frac{d}{dx}\sin(x)\ln x && \Rightarrow \\
\frac{y'}{y} &= (\cos(x)\ln x + \frac{\sin(x)}{x})y && \Rightarrow \\
y' &= (\ln x^{\cos(x)} + \frac{\sin(x)}{x})x^{\sin(x)}.
\end{aligned}$$

Example

Example 2.1.1. *Find*

$$\frac{d}{dx}\sin(x)^{\cos(x)}.$$

Solution. *Let*

$$y = \sin(x)^{\cos(x)} \Rightarrow \ln y = \ln(\sin(x)^{\cos(x)}) \Rightarrow \ln y = \cos(x)\ln(\sin(x)).$$

So we have

$$\frac{y'}{y} = -\sin(x)\ln(\sin(x)) + \cos(x)\frac{-\cos(x)}{\sin(x)}.$$

Therefore,

$$\begin{aligned}
y' &= (-\sin(x)\ln(\sin(x)) + \cos(x)\frac{-\cos(x)}{\sin(x)})y \\
&= (-\sin(x)\ln(\sin(x)) + \cos(x)\frac{-\cos(x)}{\sin(x)})\sin(x)^{\cos(x)}.
\end{aligned}$$

2.2 Newton's law of cooling

Newton's Law of Cooling

$$\frac{dT}{dt}(t) = K\left[T(t) - A\right].$$

where $T(t)$ is the temperature of the object at time t, A is the temperature of its surroundings, and K is a constant of proportionality.

Newton's Law of Cooling

Corollary 2.2.1. *A differentiable function $T(t)$ obeys the differential equation*

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

if and only if

$$T(t) = [T(0) - A]e^{Kt} + A.$$

Newton's Law of Cooling

The temperature of a glass of iced tea is initially 5°. After 5 minutes, the tea has heated to 10° in a room where the air temperature is 30°.
What is the temperature after 10 minutes?

1. 11
2. 12
3. 13
4. 14

Example 2.2.2. *The temperature of a glass of iced tea is initially 5°. After 5 minutes, the tea has heated to 10° in a room where the air temperature is 30°.*

(a) Determine the temperature as a function of time.

(b) What is the temperature after 10 minutes?

(c) Determine when tea will reach a temperature of 20°.

Solution

(a) We let $T(t)$ be the temperature of the tea t minutes after it was removed from the fridge, then the function of the temperature of the tea is

$$T(t) = [T(0) - A]e^{Kt} + A$$

and since $A = 30$ and $T(0) = 5$, the equation is

$$T(t) = [5 - 30]e^{Kt} + 30 = -25e^{Kt} + 30.$$

However, still K is unknown and we should find it. Consider that after 5 minutes the temperature of the tea is $10°$, thus $T(5) = 10$. That is,

$$10 = T(5) = -25e^{5K} + 30 \Rightarrow -20 = -25e^{5K} \Rightarrow 4/5 = e^{5K}$$

$$\Rightarrow 5K = \ln 4/5 \Rightarrow K = \frac{\ln 4/5}{5}.$$

Therefore the temperature at time t is

$$T(t) = -25e^{\frac{\ln 4/5}{5}t} + 30.$$

(b) The temperature after 10 minutes is

$$T(10) = -25e^{\frac{\ln 4/5}{5}10} + 30 =$$

$$-25e^{2\ln 4/5} + 30 = -25e^{\ln(4/5)^2} + 30 = -25 \times (4/5)^2 + 30 = 14$$

(c) The time that $T(t) = 20$, should satisfies

$$T(t) = -25e^{\frac{\ln 4/5}{5}t} + 30 = 20.$$

Thus

$$-10 = -25e^{\frac{\ln 4/5}{5}t} \Rightarrow 2/5 = e^{\frac{\ln 4/5}{5}t} \Rightarrow 2/5 = (4/5)^{(1/5)t} \Rightarrow t = 20.5$$

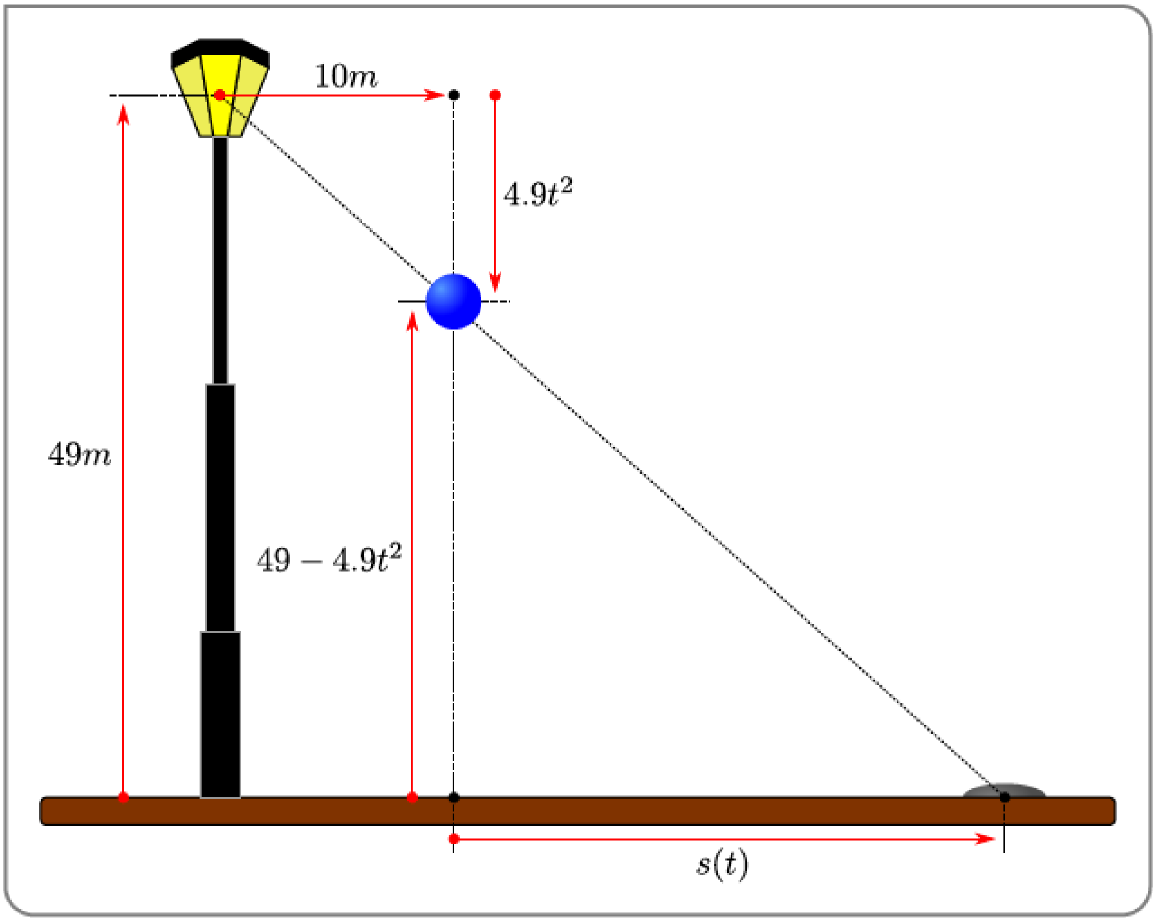
to 1 decimal place.

2.3 Related rates

Related Rates

A ball is dropped from a height of 49m above level ground. The height of the ball at time t is $h(t) = 49 - 4.9t^2$m. A light, which is also 49m above the ground, is 10m to the left of the ball's original position. As the ball descends, the shadow of the ball caused by the light moves across the ground. How fast is the shadow moving one second after the ball is dropped?

1. -100 2. -200 3. 100 4. 200



Solution. *Let $s(t)$ be the distance from the shadow to the point on the ground directly underneath the ball. By similar triangles we have*

$$\frac{4.9t^2}{10} = \frac{49 - 4.9t^2}{s(t)}.$$

Therefore,

$$s(t) = \frac{10(49 - 4.9t^2)}{4.9t^2}$$

and so

$$s(t) = \frac{100}{t^2} - 10.$$

We have

$$s'(t) = -2\frac{100}{t^3}.$$

Consequently, $s'(1) = -200m/sec.$

2.4 Taylor Polynomial

Taylor Polynomial

Let a be a constant and let n be a non-negative integer. The nth degree Taylor polynomial for $f(x)$ about $x = a$ is

$$T_n(x) = f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f^{(3)}(a)(x-a)^3+\cdots+\frac{1}{n!}f^{(n)}(a)(x-a)^n$$

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k$$

The special case $a = 0$ is called a Maclaurin polynomial.

Maclaurin polynomial for $\sin(x)$

Example. Find the 5th degree Maclaurin polynomial for $\sin(x)$.

1. $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$
2. $T_5(x) = x + \frac{x^3}{3!} - \frac{x^5}{5!}$
3. $T_5(x) = x + \frac{x^3}{3} - \frac{x^5}{5}$
4. $T_5(x) = 1 + \frac{x^2}{2!} - \frac{x^4}{4!}$

Solution. *Let* $g(x) = \sin(x)$. *We have*

$$g(0) = 0, g'(0) = 1, g''(0) = 0, g'''(0) = -1, g^{(4)}(0) = 0, g^{(5)} = 1.$$

Hence

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

The $(2n+1)$*th Maclaurin polynomial for* $\sin(x)$ *is*

$$T_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Third Taylor polynomial of $\ln(x)$

Which of the following is the third Taylor polynomial of $\ln x$ about $x = 1$.

1. $1 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3$
2. $1 + (x-1) - \frac{1}{2}(x-1)^2 - \frac{2}{3!}(x-1)^3$
3. $(x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3$
4. $(x-1) - \frac{1}{2}(x-1)^2 - \frac{2}{3!}(x-1)^3$

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1} \text{ for some } c \text{ strictly between } a \text{ and } x$$

Estimate $\ln(2)$

We use the third Taylor polynomial for $\ln(x)$ about $x = 1$ to estimate $\ln(2)$. Then which of the following is more accurate.

1. $|R_3(2)| \leq 1$.
2. $|R_3(2)| \leq \frac{1}{2}$.
3. $|R_3(2)| \leq \frac{1}{4}$.
4. $|R_3(2)| = 0$.

This contains the solution for previous two questions

We have

$$f(x) = \ln(x) \qquad f(1) = 0$$

$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$

$$f''(x) = \frac{-1}{x^2} \qquad f''(1) = -1$$

$$f^{(3)}(x) = \frac{2}{x^3} \qquad f'''(1) = 2$$

$$f^{(4)}(x) = \frac{-6}{x^4}$$

$$T_3(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3.$$

So

$$T(2) = 1 - \frac{1}{2} + \frac{2}{3!} = \frac{5}{6}.$$

By Lagrange remainder theorem we have

$$R_3(2) = \frac{1}{4!}f^{(4)}(c)(x-1)^4 = \frac{1}{4!}f^{(4)}(c) = \frac{1}{4!}\frac{-6}{c^4}$$

for some $1 < c < 2$. When $1 < c < 2$, we have that $|\frac{-6}{c^4}| \leq 6$. Therefore,

$$|R_3(2)| = |\frac{1}{4!}\frac{-6}{c^4}| \leq \frac{1}{4!}6 = \frac{1}{4}.$$

So,

$$|R_3(2)| \leq \frac{1}{4} = 0.25 < 0.5 \times 10^{-0}$$

so it is accurate to 0 decimal points.

Domain

The domain of $f(x) = x(3-x)^{1/3}$ is

1. $x \leq 3$ 2. $x \geq 3$ 3. $0 \leq x \leq 3$ 4. $\mathbb{R}$.

Solution. *The domain of the function $f(x)$ is $\mathbb{R}$.*

limits

Let $f(x) = x(3-x)^{1/3}$. Then $\lim_{x\to\infty} f(x) =$ ..... and $\lim_{x\to\infty} f(x) =$ ......

1. $-\infty, -\infty$ 2. $\infty, -\infty$ 3. $-\infty, \infty$ 4. ∞, ∞

Solution. *The answer is the first choice.*

Derivative of $f(x)$

Let $f(x) = x(3-x)^{1/3}$. Then

1. $\frac{d}{dx}f(x) = -\frac{4x-9}{3(3-x)^{2/3}}$.
2. $\frac{d}{dx}f(x) = \frac{4x-9}{3(3-x)^{2/3}}$.
3. $\frac{d}{dx}f(x) = (x-3)^{1/3} - \frac{1}{3(3-x)^{2/3}}$.
4. $\frac{d}{dx}f(x) = (x-3)^{1/3} + \frac{1}{3(3-x)^{2/3}}$.

Solution. *We have that*

$$\frac{d}{dx}f(x) = (3-x)^{1/3} - \frac{x}{3(3-x)^{2/3}} = \frac{3(3-x)-x}{3(3-x)^{2/3}} = -\frac{4x-9}{3(3-x)^{2/3}}$$

Singular/Critical

Let $f(x) = x(3-x)^{1/3}$. Then

1. $f(x)$ has a singular point at $x = 2.25$ and a critical point at $x = 3$.
2. $f(x)$ has singular points at $x = 2.25$ and $x = 3$.
3. $f(x)$ has a singular point at $x = 3$ and a critical point at $x = 2.25$.
4. $f(x)$ has critical points at $x = 2.25$ and $x = 3$.

Solution. *Note that*

$$f'(x) = -\frac{4x-9}{3(3-x)^{2/3}}.$$

So $f'(x) = 0$ when $x = 9/4 = 2.25$; so $x = 2.25$ is the only critical point, and also the regular point is $x = 3$ because the derivative is not defined at that point.

Global max/min

Let $f(x) = x(3-x)^{1/3}$. Find the global max/min (if any) of $f(x)$ on the interval $[0, 4]$

1. $f(x)$ has a global max at $x = 2.25$ and has a global min at $x = 4$.
2. $f(x)$ has a global max at $x = 4$ and has a global min at $x = 2.25$.
3. $f(x)$ has a global max at $x = 2.25$ and has no global min.
4. $f(x)$ has no global max and has a global min at $x = 4$.

Solution. *Note that $f(x)$ is continuous and we want to find the global max/min, so the most useful theorem here is the closed interval method. Since*

$$f(0) = 0, f(4) = -4, f(2.25) = 2.25(0.75)^{1/3}, f(3) = 0,$$

by the closed interval method we have that the global max is at $x = 2.25$ and the global minimum is at $x = 4$.

Increasing/Decreasing

Let $f(x) = x(3-x)^{1/3}$. Find where the function $f(x)$ is increasing and where it is decreasing.

1. $f(x)$ is increasing on $(-\infty, 2.25) \cup (3, \infty)$, and it is decreasing on $(2.25, 3)$.
2. $f(x)$ is decreasing on $(-\infty, 2.25) \cup (3, \infty)$, and it is increasing on $(2.25, 3)$.
3. $f(x)$ is decreasing on $(-\infty, 2.25)$, and it is increasing on $(2.25, \infty)$.
4. $f(x)$ is increasing on $(-\infty, 2.25)$, and it is decreasing on $(2.25, \infty)$.

Solution. *We have that*

	$(-\infty, 2.25)$	$x = 2.25$	$(2.25, 3)$	$x = 3$	$(3, \infty)$
$f'(x)$	$+$	0	$-$	DNE	$-$

So the right answer is 4.

Local max/min

Let $f(x) = x(3-x)^{1/3}$. Find the local max/min (if any) of $f(x)$.

1. $f(x)$ has a local min at $x = 2.25$ and has a local max at $x = 3$.
2. $f(x)$ has a local max at $x = 2.25$ and has a local min at $x = 3$.
3. $f(x)$ has a local max at $x = 2.25$ and has no local min.
4. $f(x)$ has a local min at $x = 3$ and has no local max.

Solution. *Looking at the previous question we have that there is a local max at $x = 2.25$ since the function is first increasing and then decreasing around $x = 2.25$. It does not have a local min.*

Second derivative

Let $f(x) = x(3-x)^{1/3}$. Find the second derivative of $f(x)$.

1. $f''(x) = \frac{-4x-18}{9(3-x)^{5/3}}$
2. $f''(x) = \frac{4x+18}{9(3-x)^{5/3}}$
3. $f''(x) = \frac{-4x+18}{9(3-x)^{5/3}}$
4. $f''(x) = \frac{4x-18}{9(3-x)^{5/3}}$

Solution. *The right answer is 4.*

Concavity

Let $f(x) = x(3-x)^{1/3}$. Where $f(x)$ is concave up and where it is concave down.

1. Concave down on $(-\infty, 4.5)$ and concave up $(4.5, \infty)$.
2. Concave down on $(-\infty, 3)$ and concave up $(3, \infty)$.
3. Concave down on $(-\infty, 3) \cup (4.5, \infty)$ and concave up $(3, 4.5)$.
4. Concave up on $(-\infty, 3) \cup (4.5, \infty)$ and concave down on $(3, 4.5)$.

Solution. *Note that*

$$f''(x) = \frac{4x - 18}{9(3-x)^{5/3}}.$$

Thus $f''(x) = 0$ at $x = 4.5$ and it doesn't exist at $x = 3$. We have that

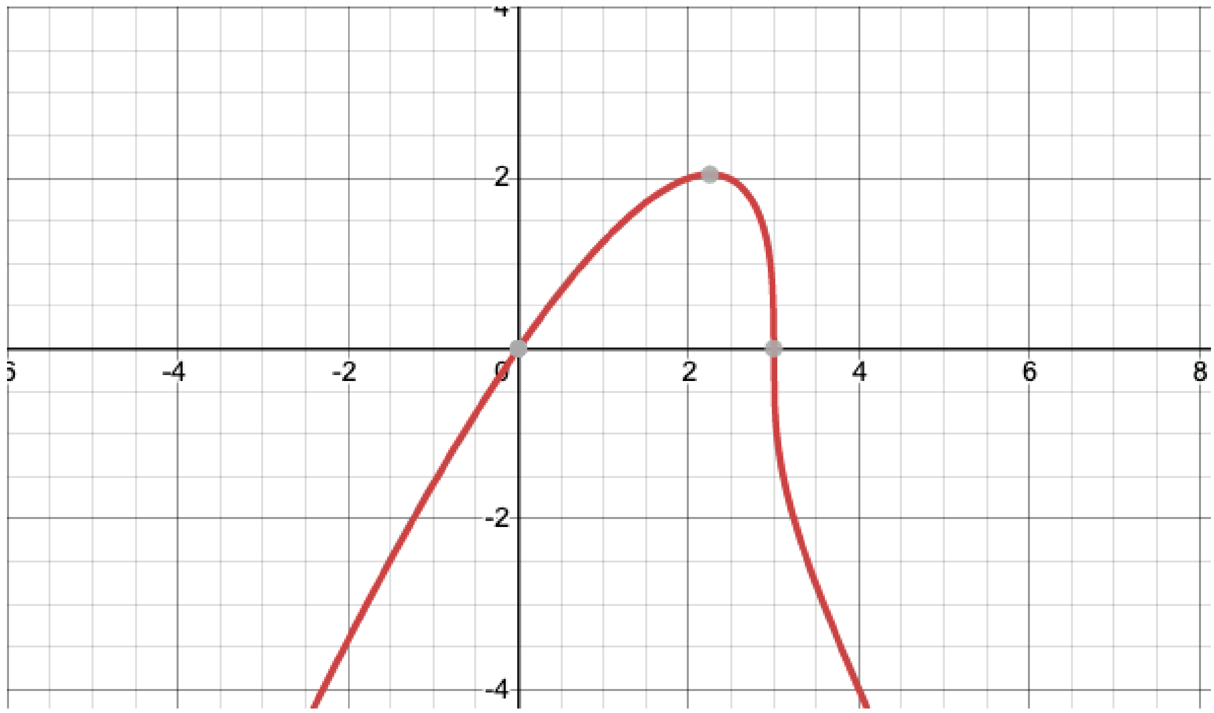
	$(-\infty, 3)$	$x = 3$	$(3, 4.5)$	$x = 4.5$	$(4.5, \infty)$
$f''(x)$	$-$	DNE	$+$	0	$-$
concavity	D		U		D

Inflection points

Let $f(x) = x(3-x)^{1/3}$. Find the inflection point(s) of $f(x)$.

1. The function has only one inflection point at $x = 3$.
2. The function has only one inflection point at $x = 4.5$.
3. The function has inflection points at $x = 3$ and $x = 4.5$
4. The function has no inflection points.

Solution. *Therefore, by the solution of the previous question, the inflection points are at $x = 3$ and $x = 4.5$.*


4
2
-2
-4
-6
-4
-2
0
2
4
6
8

Homework 1:

Go to this link https://www.mooculus.osu.edu/textbook/mooculus.pdf and download the book "MOOCULUS". Then do the following questions:

- all questions in page 35;
- in page 33 see why $\lim_{x\to 0} \frac{sin(x)}{x} = 1$. Then do Questions 1-8 page 38;
- in page 42, do questions 1-10.

Homework 2:

- Do Worksheets! by Lior Silberman
- Course website, Jamie Juul

Bibliography

[1] CLP1: Differential Calculus by J. Feldman, A. Rechnitzer, and E. Yeager.

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