A Formal Reduction of Sudoku Puzzle into SAT

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Outline

1. Demonstration: Sudoku Puzzle Solver
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3. Formal Reduction: Encoding Sudoku in SAT
4. Applications of SAT
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    - Model Checking
    - Zero Knowledge Proof
    - Complexity Theory
5. Appendix

Sudoku Puzzle

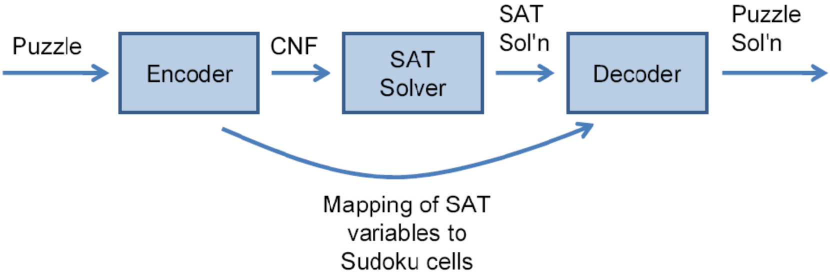
		4				5	3	
6			1	3				9
	7					2		
				8		4	7	
			3		4			
	5	9		6				
		6					9	
9				7	1			5
	2	7				1		

Sudoku Puzzle

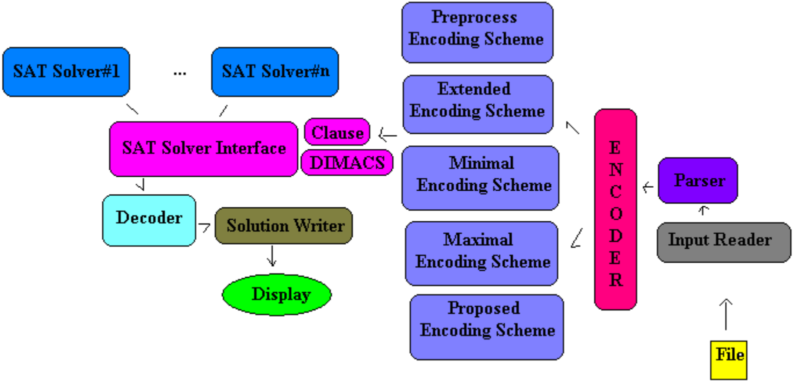
		4				5	3	
6			1	3				9
	7					2		
				8		4	7	
			3		4			
	5	9		6				
		6					9	
9				7	1			5
	2	7				1		

6.10^{21} possible grids for a simple looking $9 * 9$ puzzle instance.

Abstract Model



Sudoku Solver Architecture



Preliminaries: Propositional Logic

Vocabulary

An alphabet of propositional logic consists of

- a (countable) infinite set $\mathcal{R} = \{p_0, p_1, p_2, \ldots\}$ of propositional variables,
- the set logical connectives: $\circ \in \{\neg/1, \wedge/2, \vee/2, \rightarrow /2, \leftrightarrow /2$
- the special characters "(" and ")".

Syntax

An atomic formula, atom, is a propositional variable.
The set of propositional formulas is the smaller set $\mathcal{L}(\mathcal{R})$ of strings over an alphabet of propositional logic with the following properties:

- if F is an atomic formula, then $F \in \mathcal{L}(\mathcal{R})$.
- if $F \in \mathcal{L}(\mathcal{R})$, then $\neg F \in \mathcal{L}(\mathcal{R})$.
- if $\circ/2$ is a binary connective, $F, G\mathcal{L}(\mathcal{R})$, then $(F \circ G) \in \mathcal{L}(\mathcal{R})$.

Truth Table Semantics

- The set of truth values $\mathcal{W}$ is the set $\{\top, \bot\}$.
- We consider the following functions on $\mathcal{W}$:
  - Negation $\neg^*/1$.
  - Conjunction $\wedge^*/2$.
  - Disjunction $\vee^*/2$.
  - Implication $\rightarrow^*/2$.
  - Equivalence $\leftrightarrow^*/2$.

		$\neg^*$	$\wedge^*$	$\vee^*$	$\rightarrow^*$	$\leftrightarrow^*$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$

Model based Semantics

An interpretation $\mathcal{I} = (\mathcal{W}, .^I)$ is a mapping $.^I : \mathcal{L}(\mathcal{R}) \rightarrow \mathcal{W}$ such that:

SAT

A propositional satisfiability problem (SAT), consist of a formula $\phi \in \mathcal{L}(\mathcal{R})$, and is the problem to decide whether ϕ is satisfiable.

Model

An interpretation $\mathcal{I} = (\mathcal{W}, .^I)$ is called a model for propositional formula ϕ , $\mathcal{I} \models \phi$ if $[\phi]^I = \top$ (i.e., $\mathcal{I}$ satisfies ϕ). ϕ is unsatisfiable if it has no models.

Propostional Satisfiability Problems

SAT is a combinatorial decision problem.

- Decision variant yes/no answer.
- Search variant find a model if ϕ is satisfiable.

Example

- Let $\{p_1, p_2, p_3, p_4, p_5\} \subseteq \mathcal{R}$.

$$\begin{aligned}\phi = & (\neg p_1 \vee p_2) \wedge (\neg p_2 \vee p_1) \\ & \wedge (\neg p_1 \vee \neg p_2 \vee \neg p_3) \wedge (p_1 \vee p_2) \\ & \wedge (\neg p_4 \vee p_3) \wedge (\neg p_5 \vee p_3)\end{aligned}$$

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- $\{p_1, p_2\}$ is a model of ϕ.

Hence, ϕ is satisfiable.

Formal Reduction

Idea:

Sudoku puzzle $\mathcal{S}$ is formulated as a CNF formula ϕ such that is

$$\phi \texttt{ is satisfiable iff } \mathcal{S} \texttt{ has a solution.}$$

Sudoku Puzzle $\mathcal{S}$:

A Sudoku puzzle $\mathcal{S}$ is represented by a $\mathbb{N} * \mathbb{N}$ grid, which comprises of an $\sqrt{\mathbb{N}} * \sqrt{\mathbb{N}}$ sub-grids (also called boxes). Some of the entries in the grid are filled with numbers from 1 to $\mathbb{N}$, whereas other entries are left blank.

Encoding Scheme ($\mathcal{S} \implies \phi$):

A SAT problem is represented as a propositional formula Φ where each variable P_i is assigned 0 ($\mathbb{F}$) or 1 ($\mathbb{T}$) where i $\in (1, \cdots, n)$ In Sudoku each tuple (r, c, v) denotes a variable which is true iff the cell in row r and column c is assigned a number v; $[r, c] = v$. The resulting set of formulas turn out to be $V = \{(r, c, v) \mid 1 \leq r, c, v \leq \mathrm{n}\}$.

Encoding Scheme ($\mathcal{S} \implies \phi$):

- There is at exactly one number in each cell
  $\phi_{cell.ex} := \phi_{cell.def} \wedge \phi_{cell.uniq}$
  There is at least one number for each cell
  $\phi_{cell.def} := \bigwedge_{r=1}^{n} \bigwedge_{c=1}^{n} \bigvee_{v=1}^{n} (r, c, v)$
  Each number appears at most one in each cell
  $\phi_{cell.uniq} := \bigwedge_{r=1}^{n} \bigwedge_{c=1}^{n} \bigwedge_{v_i=1}^{(n-1)} \bigwedge_{v_j=v_i+1}^{n} \neg(r, c, v_i) \vee \neg(r, c, v_j)$
- There is at exactly one number in each row
- There is at exactly one number in each column
- There is at exactly one number in each block

Encodings (ϕ):

$$\Phi_{extended} := \phi_{cell.ex} \wedge \phi_{row.ex} \wedge \phi_{col.ex} \wedge \phi_{blk.ex} \wedge \phi_{assign}.$$

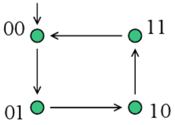
$$\Phi_{efficient} := \phi_{cell.ex} \wedge \phi_{row.uniq} \wedge \phi_{col.uniq} \wedge \phi_{blk.uniq} \wedge \phi_{assign}.$$

$$\Phi_{minimal} := \phi_{cell.def} \wedge \phi_{row.uniq} \wedge \phi_{col.uniq} \wedge \phi_{blk.uniq} \wedge \phi_{assign}.$$

Applications

- Search & Planning
- Model Checking
- Zero Knowledge Proof $\mathcal{S} \overset{\Pi}{\leftarrow} V$
- Complexity Theory ($\mathrm{P} = \mathrm{NP}$ or $\mathrm{P} \neq \mathrm{NP}$)

Model Checking



Initial state: $I : \neg l \wedge \neg r$

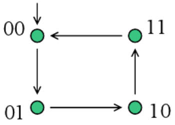
Transition: $R: \begin{pmatrix} l' = (l \neq r) \wedge \\ r' = \neg r \end{pmatrix}$

Safety property: $\mathbf{AG}\,(\neg l \vee \neg r)$

$$\Omega(2) : (\neg l_0 \wedge \neg r_0) \wedge \begin{pmatrix} l_1 = (l_0 \neq r_0) \wedge r_1 = \neg r_0 \wedge \\ l_2 = (l_1 \neq r_1) \wedge r_2 = \neg r_1 \end{pmatrix} \wedge \begin{pmatrix} (l_0 \wedge r_0) \vee \\ (l_1 \wedge r_1) \vee \\ (l_2 \wedge r_2) \end{pmatrix}$$

$\Omega(2)$ is unsatisfiable. $\Omega(3)$ is satisfiable.

Model Checking



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Satisfying assignment gives the counter example to the safety property.

$$\mathcal{M} = \{r_1, l_2, l_3, r_3\}$$
$$\mathcal{M} = \{(\neg l_0, \neg r_0), (\neg l_1, r_1), (l_2, \neg r_2), (l_3, r_3)\}$$

For Further Reading

Stephen A. Cook. 1970
The complexity of theorem-proving procedures..

Armin Biere, Marjin Heule, Hans van Marren and Toby Walsh. 2009
Handbook of Satisfiability.

Q & A