Efficient MUS Enumeration of Horn Formulae
with Applications to Axiom Pinpointing

M. Fareed Arif[1], Carlos Mencía[1], and Joao Marques-Silva[1,2]

[1] CASL, University College Dublin, Ireland
`muhammad.arif.1@ucdconnect.ie,carlos.mencia@ucd.ie,jpms@ucd.ie`
[2] INESC-ID, IST, ULisboa, Portugal

Abstract. The enumeration of minimal unsatisfiable subsets (MUSes) finds a growing number of practical applications, that includes a wide range of diagnosis problems. As a concrete example, the problem of axiom pinpointing in the $\mathcal{EL}$ family of description logics (DLs) can be modeled as the enumeration of the group-MUSes of Horn formulae. In turn, axiom pinpointing for the $\mathcal{EL}$ family of DLs finds important applications, such as debugging medical ontologies, of which SNOMED CT is the best known example. The main contribution of this paper is to develop an efficient group-MUS enumerator for Horn formulae, HG-MUS, that finds immediate application in axiom pinpointing for the $\mathcal{EL}$ family of DLs. In the process of developing HGMUS, the paper also identifies performance bottlenecks of existing solutions. The new algorithm is shown to outperform all alternative approaches when the problem domain targeted by group-MUS enumeration of Horn formulae is axiom pinpointing for the $\mathcal{EL}$ family of DLs, with a representative suite of examples taken from different medical ontologies.

1 Introduction

Description Logics (DLs) are well-known knowledge representation formalisms [4]. DLs find a wide range of applications in computer science, including the semantic web and representation of ontologies, but also in medical bioinformatics. Axiom pinpointing represents the problem of computing one minimal axiom set (denoted *MinA*), which explains a subsumption relation in an ontology [49]. Example applications of axiom pinpointing include context-based reasoning, error-tolerant reasoning [33], and ontology debugging and revision [50,27]. Axiom pinpointing for different description logics (DLs) has been studied extensively for more than a decade, with related work in the mid 90s [3,49,43,38,5,7,26,54,50,8,51,6,40,41,32,52,1,34].

The $\mathcal{EL}$ family of DLs is well-known for being tractable. Despite being inexpressive, the $\mathcal{EL}$ family of DLs, concretely by using the more expressive $\mathcal{EL}^+$, has been used for representing ontologies in the medical sciences, including the well-known SNOMED CT ontology [56]. Work on axiom pinpointing for the $\mathcal{EL}$ family of DLs can be traced to 2006, namely the CEL tool [5]. Later, in 2009, the use of SAT was proposed for axiom pinpointing in the $\mathcal{EL}$ family of DLs [51,57,52], concretely for the more expressive description logic $\mathcal{EL}^+$. This seminal work proposed a propositional Horn encoding that can be exponentially smaller than earlier work [5,7,8]. Moreover, the use of SAT for axiom pinpointing for the $\mathcal{EL}$ family of DLs, named EL⁺SAT [51,57,52], was

shown to consistently outperform earlier work, concretely CEL [5]. Recent work [1] proposes the EL2MCS tool that builds on these propositional encodings, but exploits the relationship between axiom pinpointing and MUS enumeration; concretely, it relies on explicit hitting set dualization [31]. This tool is evaluated in [1], where it is shown to achieve conclusive performance gains over earlier work. The relationship between axiom pinpointing and MUS enumeration was also studied elsewhere [34]. Instead of exploiting hitting set dualization, this alternative approach exploits the enumeration of prime implicants [34].

The main contribution of this paper is to develop an efficient group-MUS enumerator for Horn formulae, referred to as HGMUS, that finds immediate application in axiom pinpointing for the $\mathcal{EL}$ family of DLs. In the process of developing HGMUS, the paper also identifies performance bottlenecks of existing solutions, in particular EL$^+$SAT [51,52]. The new group-MUS enumerator for Horn formulae builds on the large body of recent work on problem solving with SAT oracles. This includes, among others, MUS extraction [12], MCS extraction and enumeration [35], and partial MUS enumeration [45,29,30]. HGMUS also exploits earlier work on solving Horn propositional formulae [18,39], and develops novel algorithms for MUS extraction in propositional Horn formulae. The experimental results, using well-known problem instances, demonstrate conclusive performance improvements over all other existing approaches, in most cases by several orders of magnitude.

The paper is organized as follows. Section 2 introduces the notation and definitions used throughout the paper. Section 3 reviews recent work on MUS enumeration, which serves as the basis for HGMUS. Afterwards, the new group-MUS enumerator HGMUS is described in Section 4. Section 5 compares HGMUS with existing alternatives. Experimental results on well-known problem instances from axiom pinpointing for the $\mathcal{EL}$ family of DLs are analyzed in Section 6. The paper concludes in Section 7.

2 Preliminaries

We assume familiarity with propositional logic [13] and consider propositional Boolean formulae in Conjunctive Normal Form (CNF). A CNF formula $\mathcal{F}$ is defined over a set of Boolean variables $V(\mathcal{F}) = \{x_1, ..., x_n\}$ as a conjunction of clauses $(c_1 \wedge ... \wedge c_m)$. A clause c is a disjunction of literals $(l_1 \vee ... \vee l_k)$ and a literal l is either a variable x or its negation $\neg x$. We refer to the set of literals appearing in $\mathcal{F}$ as $L(\mathcal{F})$. Formulae can be alternatively represented as sets of clauses, and clauses as sets of literals.

A truth assignment, or interpretation, is a mapping $\mu : X \rightarrow \{0, 1\}$, with $X = V(\mathcal{F})$ also used to represent the variables of $\mathcal{F}$. If all the variables in X are assigned a truth value, μ is referred to as a *complete* assignment. Interpretations can be also seen as conjunctions or sets of literals. Truth valuations are lifted to clauses and formulae as follows: μ satisfies a clause c if it contains at least one of its literals. Given a formula $\mathcal{F}$, μ satisfies $\mathcal{F}$ (written $\mu \models \mathcal{F}$) if it satisfies all its clauses, being μ referred to as a *model* of $\mathcal{F}$.

Given two formulae $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F}$ entails $\mathcal{G}$ (written $\mathcal{F} \models \mathcal{G}$) iff all the models of $\mathcal{F}$ are also models of $\mathcal{G}$. $\mathcal{F}$ and $\mathcal{G}$ are equivalent (written $\mathcal{F} \equiv \mathcal{G}$) iff $\mathcal{F} \models \mathcal{G}$ and $\mathcal{G} \models \mathcal{F}$.

A formula $\mathcal{F}$ is satisfiable ($\mathcal{F} \nvDash \bot$) if there exists a model for it. Otherwise it is unsatisfiable ($\mathcal{F} \vDash \bot$). SAT is the decision problem of determining the satisfiability of a propositional formula. This problem is in general NP-*complete* [16].

Some applications require computing certain types of models. In this paper, we will make use of maximal models, i.e. models such that a set-wise maximal subset of the variables are assigned value 1:

Definition 1. *(MxM).* *Let $\mathcal{F}$ be a satisfiable propositional formula, $\mu \vDash \mathcal{F}$ a model of $\mathcal{F}$ and $P \subseteq X$ the set of variables appearing in μ with positive polarity. μ is a maximal model (MxM) of $\mathcal{F}$ iff $\mathcal{F} \cup P \nvDash \bot$ and for all $v \in X \setminus P$, $\mathcal{F} \cup P \cup \{v\} \vDash \bot$.*

Herein, we will denote a maximal model by P, i.e. the set of its positive literals.

Horn formulae constitute an important subclass of propositional logic. These are composed of Horn clauses, which have at most one positive literal. Satisfiability of Horn formulae is decidable in polynomial time [18,24,39].

Given an unsatisfiable formula $\mathcal{F}$, the following subsets represent different notions regarding (set-wise) minimal unsatisfiability and maximal satisfiability [31,35]:

Definition 2. *(MUS).* *$\mathcal{M} \subseteq \mathcal{F}$ is a* Minimally Unsatisfiable Subset *(MUS) of $\mathcal{F}$ iff $\mathcal{M}$ is unsatisfiable and $\forall c \in \mathcal{M}, \mathcal{M} \setminus \{c\}$ is satisfiable.*

Definition 3. *(MCS).* *$\mathcal{C} \subseteq \mathcal{F}$ is a* Minimal Correction Subset *(MCS) iff $\mathcal{F} \setminus \mathcal{C}$ is satisfiable and $\forall c \in \mathcal{C}, \mathcal{F} \setminus (\mathcal{C} \setminus \{c\})$ is unsatisfiable.*

Definition 4. *(MSS).* *$\mathcal{S} \subseteq \mathcal{F}$ is a* Maximal Satisfiable Subset *(MSS) iff $\mathcal{S}$ is satisfiable and $\forall c \in \mathcal{F} \setminus \mathcal{S}, \mathcal{S} \cup \{c\}$ is unsatisfiable.*

An MSS is the complement of an MCS. MUSes and MCSes are closely related by the well-known hitting set duality [47,10,14,55]: Every MCS (MUS) is an irreducible hitting set of all MUSes (MCSes) of $\mathcal{F}$. In the worst case, there can be an exponential number of MUSes and MCSes [31,42]. Besides, MCSes are related to the MaxSAT problem, which consists in finding an assignment satisfying as many clauses as possible. The smallest MCS (largest MSS) represents an optimal solution to MaxSAT.

Motivated by several applications, MUSes and related concepts have been extended to CNF formulae where clauses are partitioned into disjoint sets called groups [31].

Definition 5. *(Group-Oriented MUS).* *Given an explicitly partitioned unsatisfiable CNF formula $\mathcal{F} = \mathcal{G}_0 \cup ... \cup \mathcal{G}_k$, a group-oriented MUS (or group-MUS) of $\mathcal{F}$ is a set of groups $\mathcal{G} \subseteq \{\mathcal{G}_1, ..., \mathcal{G}_k\}$, such that $\mathcal{G}_0 \cup \mathcal{G}$ is unsatisfiable, and for every $\mathcal{G}_i \in \mathcal{G}, \mathcal{G}_0 \cup (\mathcal{G} \setminus \mathcal{G}_i)$ is satisfiable.*

Note the special role $\mathcal{G}_0$ (*group-0*); this group consists of *background* clauses that are included in every group-MUS. Because of $\mathcal{G}_0$ a group-MUS, as opposed to MUS, can be empty. Nevertheless, in this paper we assume that $\mathcal{G}_0$ is satisfiable.

Equivalently, the related concepts of group-MCS and group-MSS can be defined in the same way. We omit these definitions here due to lack of space. In the case of MaxSAT, the use of groups is investigated in detail in [23].

Algorithm 1: EMUS [45] / MARCO [30]

Input: $\mathcal{F}$ a CNF formula

1 $I \leftarrow \{p_i \mid c_i \in \mathcal{F}\}$ // Variable p_i picks clause c_i

2 $\mathcal{Q} \leftarrow \emptyset$

3 while true **do**

4 $(st, P) \leftarrow \texttt{MaximalModel}(\mathcal{Q})$

5   **if not** st **then return**

6 $\mathcal{F}' \leftarrow \{c_i \mid p_i \in P\}$ // Pick selected clauses

7   **if not** $\text{SAT}(\mathcal{F}')$ **then**

8 $\mathcal{M} \leftarrow \texttt{ComputeMUS}(\mathcal{F}')$

9 $\texttt{ReportMUS}(\mathcal{M})$

10 $b \leftarrow \{\neg p_i \mid c_i \in \mathcal{M}\}$ // Negative clause blocking the MUS

11   **else**

12 $b \leftarrow \{p_i \mid p_i \in I \setminus P\}$ // Positive clause blocking the MCS

13 $\mathcal{Q} \leftarrow \mathcal{Q} \cup \{b\}$

---

3   MUS Enumeration in Horn Formulae

Enumeration of MUSes has been the subject of research that can be traced to the seminal work of Reiter [47]. A well-known family of algorithms uses (explicit) minimal hitting set dualization [14,10,31]. The organization of these algorithms can be summarized as follows. First compute all the MCSes of a CNF formula. Second, MUSes are obtained by computing the minimal hitting sets of the set of MCSes. The main drawback of explicit minimal hitting set dualization is that, if the number of MCSes is exponentially large, these approaches will be unable to compute MUSes, even if the total number of MUSes is small. As a result, recent work considered what can be described as implicit minimal hitting set dualization [29,45,30]. In these approaches, either an MUS or an MCS is computed at each step of the algorithm, with the guarantee that one or more MUSes will be computed at the outset. In some settings, implicit minimal hitting set dualization is the only solution for finding some MUSes of a CNF formula. As pointed out in this recent work, implicit minimal hitting set dualization aims to complement, but not replace, the explicit dualization alternative, and in some settings where enumeration of MCSes is feasible, explicit minimal hitting set dualization may be the preferred option [45,30].

Algorithm 1 shows the EMUS enumeration algorithm [45], also used in the most recent version of MARCO [30]. It relies on a two-solver approach aimed at enumerating the MUSes/MCSes of an unsatisfiable formula $\mathcal{F}$. On the one hand, a formula $\mathcal{Q}$ is used to enumerate subsets of $\mathcal{F}$. This formula is defined over a set of variables $I = \{p_i \mid c_i \in \mathcal{F}\}$, each one of them associated with one clause $c_i \in \mathcal{F}$. Iteratively until $\mathcal{Q}$ becomes unsatisfiable, EMUS computes a maximal model P of $\mathcal{Q}$ and tests the satisfiability of the corresponding subformula $\mathcal{F}' \subseteq \mathcal{F}$. If it is satisfiable, $\mathcal{F}'$ represents an MSS of $\mathcal{F}$, and the clause $I \setminus P$ is added to $\mathcal{Q}$, preventing the algorithm from generating any subset of the MSS (superset of the MCS) again. Otherwise, if $\mathcal{F}'$ is unsatisfiable, it is reduced to an MUS $\mathcal{M}$, which is blocked adding to $\mathcal{Q}$ a clause made of the variables

in I associated with $\mathcal{M}$ with negative polarity. This way, no superset of $\mathcal{M}$ will be generated. Algorithm 1 is guaranteed to find all MUSes and MCSes of $\mathcal{F}$, in a number of iterations that corresponds to the sum of the number of MUSes and MCSes.

This paper considers the concrete problem of enumerating the group-MUSes of an unsatisfiable Horn formula. As highlighted earlier, and as discussed later in the paper, enumeration of the group-MUSes of unsatisfiable Horn formulae finds important applications in axiom pinpointing for the $\mathcal{EL}$ family of DLs, including $\mathcal{EL}^+$. It should be observed that the difference between the enumeration of plain MUSes of Horn formulae and the enumeration of group-MUSes is significant. First, enumeration of group-MUSes of Horn formulae cannot be achieved in total polynomial time, unless P = NP. This is an immediate consequence from the fact that axiom pinpointing for the $\mathcal{EL}$ family of DLs cannot be achieved in total polynomial time, unless P = NP [7], and that axiom pinpointing for the $\mathcal{EL}$ family of DLs can be reduced in polynomial time to group-MUS enumeration of Horn formulae [1]. Second, enumeration of MUSes of Horn formulae can be achieved in total polynomial time (actually with polynomial delay) [44].

Given the above, a possible approach for enumerating group-MUSes of Horn formulae is to use an existing solution, either based on explicit or implicit minimal hitting set dualization. For example, the use of explicit minimal hitting dualization was recently proposed in [1]. Alternatively, either EMUS [45] or the different versions of MARCO [29,30] could be used, as also pointed out in [34].

This paper opts instead to exploit the implicit minimal hitting set dualization approach [29,45,30], but develops a solution that is specific to the problem formulation. This solution is described in the next section.

4 Algorithm for Group-MUS Enumeration in Horn Formulae

This section describes HGMUS, a novel and efficient group-MUS enumerator for Horn formulae based on implicit minimal hitting set dualization. In this section, $\mathcal{H}$ denotes the group of clauses $\mathcal{G}_0$, i.e. the background clauses. Moreover, $\mathcal{I}$ denotes the set of (individual) groups of clauses, with $\mathcal{I} = \{\mathcal{G}_1, \ldots, \mathcal{G}_k\}$. So, the unsatisfiable group-Horn formula corresponds to $\mathcal{F} = \mathcal{H} \cup \mathcal{I}$. Also, in this section, the formula $\mathcal{Q}$ shown in Algorithm 1 is defined on a set of variables associated to the groups in $\mathcal{I}$. For the problem instances considered later in the paper (obtained from axiom pinpointing for the $\mathcal{EL}$ family of DLs), each group of clauses contains a single unit clause. However, the algorithm would work for arbitrary groups of clauses.

4.1 Organization

The high-level organization of HGMUS mimics that of EMUS/MARCO (see Algorithm 1), with a few essential differences. First, the satisfiability testing step (because it operates on Horn formulae) uses the dedicated linear time algorithm LTUR [39]. LTUR can be viewed as one-sided unit propagation, since only variables assigned value 1 are propagated. Moreover, the simplicity of LTUR enables very efficient implementations, that use adjacency lists for representing clauses instead of the now more commonly

---

Algorithm 2: Computation of Maximal Models

Input: $\mathcal{Q}$ a CNF formula
Output: (st, P): with st a Boolean and P an MxM (if it exists)

1 $(P, U, B) \leftarrow (\{\{x\} \mid \neg x \notin L(\mathcal{Q})\}, \{\{x\} \mid \neg x \in L(\mathcal{Q})\}, \emptyset)$
2 $(st, P, U) \leftarrow \texttt{InitialAssignment}(\mathcal{Q} \cup P)$
3 **if not** st **then return** $(\textsf{false}, \emptyset)$
4 **while** $U \neq \emptyset$ **do**
5 $\quad\mid\quad l \leftarrow \texttt{SelectLiteral}(U)$
6 $\quad\mid\quad (st, \mu) \leftarrow \texttt{SAT}(\mathcal{Q} \cup P \cup B \cup \{l\})$
7 $\quad\mid\quad$ **if** st **then** $(P, U) \leftarrow \texttt{UpdateSatClauses}(\mu, P, U)$
8 $\quad\mid\quad$ **else** $(U, B) \leftarrow (U \setminus \{l\}, B \cup \{\neg l\})$
9 **return** $(\textsf{true}, P)$ $\qquad\qquad\qquad\qquad$ // P is an MxM of $\mathcal{Q}$

---

used watched literals. Second, the problem formulation motivates using a dedicated MUS extraction algorithm, which is shown to be more effective in this concrete case than other well-known approaches [12]. Third, we also highlight important aspects of the EMUS/MARCO implicit minimal hitting set dualization approach, which we claim have been overlooked in earlier work [57,52].

4.2 Computing Maximal Models

The use of maximal models for computing either MCSes of a formula or a set of clauses that contain an MUS was proposed in earlier work [45], which exploited SAT with preferences for computing maximal models [21,48]. The use of SAT with preferences for computing maximal models is also exploited in related work [51,52].

Computing maximal models of a formula $\mathcal{Q}$ can be reduced to the problem of extracting an MSS of a formula $\mathcal{Q}'$ [35], where the clauses of $\mathcal{Q}$ are hard and, for each variable $x_i \in V(\mathcal{Q})$, it includes a unit soft clause $c_i \equiv \{x_i\}$. Also, recent work [35,22,9,37] has shown that state-of-the-art MCS/MSS computation approaches outperform SAT with preferences. HGMUS uses a dedicated algorithm based on the LinearSearch MCS extraction algorithm [35], due to its good performance in MCS enumeration. Since all soft clauses are unit, it can also be related with the novel Literal-Based eXtractor algorithm [37]. Shown in Algorithm 2, it relies on making successive calls to a SAT solver. It maintains three sets of literals: P, an under-approximation of an MxM (i.e. positive literals s.t. $\mathcal{Q} \cup P \nvDash \bot$), B, with negative literals $\neg l$ such that $\mathcal{Q} \cup P \cup \{l\} \vDash \bot$ (i.e. *backbone literals*), and U, with the remaining set of positive literals to be tested. Initially, P and U are initialized from a model $\mu \vDash \mathcal{Q}$, P (U) including the literals appearing with positive (negative) polarity in μ. Then, iteratively, it tries to extend P with a new literal $l \in U$, by testing the satisfiability of $\mathcal{Q} \cup P \cup B \cup \{l\}$. If it is satisfiable, all the literals in U satisfied by the model (including l) are moved to P. Otherwise, l is removed from U and $\neg l$ is added to B. This algorithm has a query complexity of $\mathcal{O}(|V(\mathcal{Q})|)$.

Algorithm 2 integrates a new technique, which consists in pre-initializing P with the pure positive literals appearing in $\mathcal{Q}$ and U with the remaining ones (line 1), and then requiring the literals of P to be satisfied by the initial assignment (line 2). It can

be easily proved that these pure literals are included in all MxMs of $\mathcal{Q}$, so a number of calls to the SAT solver could be avoided. Moreover, the SAT solver will never branch on these variables, easing the decision problems. This technique is expected to be effective in HGMUS. Note that, in this context, $\mathcal{Q}$ is made of two types of clauses: positive clauses blocking MCSes of the Horn formula, and negative clauses blocking MUSes. So, with this technique, the computation of MxMs is restricted to the variables representing groups appearing in some MUS of the Horn formula.[1]

4.3 Adding Blocking Clauses

One important aspect of HGMUS are the blocking clauses created and added to the formula $\mathcal{Q}$ (see Algorithm 1). These follow what was first proposed in EMUS [45] and MARCO [29,30]. For each MUS, the blocking clause consists of a set of negative literals, requiring at least one of the clauses in the MUS *not* to be included in future selected sets of clauses. For each MCS, the blocking clause consists of a set of positive literals, requiring at least one of the clauses in the MCS to be included in future selected sets of clauses. The way MCSes are handled is essential to prevent that MCS and sets containing the same MCS to be selected again. Although conceptually simple, it can be shown that existing approaches may not guarantee that supersets of MCSes (or subsets of the MSSes) are not selected. As argued later, this is the case with EL$^+$SAT [57,52].

4.4 Deciding Satisfiability of Horn Formulae

It is well-known that Horn formulae can be decided in linear time [18,24,39]. HGMUS implements the LTUR algorithm [39]. There are important reasons for this choice. First, LTUR is expected to be more efficient than plain unit propagation, since only variables assigned value 1 need to be propagated. Second, most implementations of unit propagation in CDCL SAT solvers (i.e. that use watched literals) are not guaranteed to run in linear time [20]; this is for example the case will *all* implementations of Minisat [19] and its variants, for which unit propagation runs in worst-case quadratic time. As a result, using an off-the-shelf SAT solver and exploiting only unit propagation (as is done for example in earlier work [51,52,34]) is unlikely to be the most efficient solution. Besides the advantages listed above, the use of a linear time algorithm for deciding the satisfiability of Horn formulae turns out to be instrumental for MUS extraction, as shown in the next section. In order to use LTUR for MUS extraction, an incremental version has been implemented, which allows for the incremental addition of clauses to the formula and incremental identification of variables assigned value 1. Clearly, the amortized run time of LTUR, after adding $m = |\mathcal{F}|$ clauses, is $\mathcal{O}(||\mathcal{F}||)$, with $||\mathcal{F}||$ the number of literals appearing in $\mathcal{F}$.

4.5 MUS Extraction in Horn Formulae

For arbitrary CNF formulae, a number of approaches exist for MUS extraction, with the most commonly used one being the deletion-based approach [11,12], but other alterna-

---

[1] SATPin [34] also exploits this insight of *relevant* variables, but not in the computation of MxMs, as SATPin does not compute MxMs.

Algorithm 3: Insertion-based [17] MUS extraction using LTUR [39]

Input: $\mathcal{H}$, denotes the $\mathcal{G}_0$ clauses; $\mathcal{I}$, denotes the set of (individual) group clauses
Output: $\mathcal{M}$, denotes the computed MUS

```
1  (M, cr) ← (H, 0)
2  LTUR_prop(M, M)                    // Start by propagating G0 clauses
3  while true do
4  |   if cr > 0 then
5  |   |   M ← M ∪ {cr}               // Add transition clause cr to M
6  |   |   if not LTUR_prop(M, {cr}) then
7  |   |   |   LTUR_undo(M, M)
8  |   |   |   return M \ H    // Remove G0 clauses from computed MUS
9  |   S ← ∅
10 |   while true do
11 |   |   cr ← SelectRemoveClause(I)   // Target transition clause
12 |   |   S ← S ∪ {cr}
13 |   |   if not LTUR_prop(M ∪ S, {cr}) then
14 |   |   |   I ← S \ {cr}              // Update working set of groups
15 |   |   |   LTUR_undo(M, S)
16 |   |   |   break              // cr represents a transition clause
```

tives include the QuickXplain algorithm [25] and the more recent Progression algorithm [36]. It is also well-known and generally accepted that, due to its query complexity, the insertion-based algorithm [17] for MUS extraction is in practice not competitive with existing alternatives [12].

Somewhat surprisingly, this is not the case with Horn formulae when (an incremental implementation of) the LTUR algorithm is used. A modified insertion-based MUS extraction algorithm that exploits LTUR is shown in Algorithm 3. LTUR_prop propagates the consequences of adding some new set of clauses, given some existing incremental context. LTUR_undo unpropagates the consequences of adding some set of clauses (in order), given some existing incremental context. The organization of the algorithm mimics the standard insertion-based MUS extraction algorithm [17], but the use of the incremental LTUR yields run time complexity that improves over other approaches. Consider the operation of the standard insertion-based algorithm [17], in which clauses are iteratively added to the working formula. When the formula becomes unsatisfiable, a *transition clause* [12] has been identified, which is then added to the MUS being constructed. The well-known query complexity of the insertion-based algorithm is $\mathcal{O}(m \times k)$ where m is the number of clauses and k is the size of a largest MUS. Now consider that the incremental LTUR algorithm is used. To find the first transition clause, the amortized run time is $\mathcal{O}(||\mathcal{F}||)$. Clearly, this holds true for *any* transition clause, and so the run time of MUS extraction with the LTUR algorithm becomes $\mathcal{O}(|\mathcal{M}| \times ||\mathcal{F}||)$, where $\mathcal{M} \subseteq \mathcal{I}$ is a largest MUS. Algorithm 3 highlights the main differences with respect to a standard insertion-based MUS extraction algorithm. In contrast, observe that for a deletion-based algorithm the run time complexity will

be $\mathcal{O}(|\mathcal{I}| \times ||\mathcal{F}||)$. In situations where the sizes of MUSes are much smaller than the number of groups in $\mathcal{I}$, this difference can be significant. As a result, when extracting MUSes from Horn formulae, and when using a polynomial time incremental decision procedure, an insertion-based algorithm should be used instead of other more commonly used alternatives.

5 Comparison with Existing Alternatives

This section compares HGMUS with the group-MUS enumerators used in EL$^+$SAT [51,52], and SATPin [34]. The experimental comparison with these enumerators as well as other approaches for axiom pinpointing for the $\mathcal{EL}$ family of DLs is provided in Section 6.

5.1 EL$^+$SAT

The best known SAT-based approach for axiom pinpointing is EL$^+$SAT [51,57,52]. EL$^+$SAT is composed of two main phases. The first phase compiles the axiom pinpointing problem to a Horn formula. The second phase enumerates the so-called MinAs, and corresponds to group-MUS enumeration for this Horn formula [1].. Although existing references emphasize the enumeration of MinAs (MUSes) using an AllSAT approach (itself inspired by an AllSMT approach [28]), the connection with MUS enumeration is immediate [1]. More importantly, EL$^+$SAT shares a number of similarities with implicit minimal hitting set dualization, but also crucial differences, which we now analyze.

Similar to EMUS, EL$^+$SAT selects subformulae of an unsatisfiable Horn formula. This is achieved with a SAT solver that always assigns variables value 1 when branching [52]. This corresponds to solving SAT with preferences [21,48], and so it corresponds to computing a maximal model, inasmuch the same way as EMUS operates.

In EL$^+$SAT, the approach for deciding the satisfiability of Horn subformulae is based on running the unit propagation engine of a CDCL SAT solver. As explained earlier, this can be inefficient when compared with the dedicated LTUR algorithm for Horn formulae [39]. Moreover, in EL$^+$SAT, MUSes are extracted with what can be viewed as a deletion-based algorithm [11,12]. Although more efficient alternatives are suggested, none is as asymptotically as efficient as the dedicated algorithm proposed in Section 4.5.

Finally, the most important drawback is the blocking of sets of clauses that do not contain an MUS/MinA. In our setting of implicit minimal hitting set dualization, this represents one MCS. The approach used in EL$^+$SAT consists of creating a blocking clause solely based on the decision variables (which are *always* assigned value 1) [57,52][2]. Thus, the learned clauses, although blocking one MCS (and corresponding MSS), do *not* block supersets of MCSes (and the corresponding subsets of the MSSes). This can result in exponentially more iterations than necessary, and explains in part the poor performance of EL$^+$SAT in practice. It should be further observed that this drawback becomes easier to spot once the problem is described as MUS enumeration by implicit minimal hitting set dualization.

---

[2] The clause learning mechanism used in EL$^+$SAT is detailed in [52, page 17, first paragraph].

5.2 SATPin

SATPin [34] represents a recent SAT-based alternative for axiom pinpointing for the $\mathcal{EL}$ family of DLs, that focuses on optimizing the low-level implementation details of the CDCL SAT solver, including the use of incremental SAT solving. As indicated above, HGMUS opts to revisit instead the LTUR [39] algorithm from the late 80s, since it is guaranteed to run in linear time for Horn formulae, and can be implemented with small overhead. The SATPin approach is presented in terms of iteratively computing prime implicants. However, computing a prime implicant is tightly related with extracting an MUS [15]. As a result, some aspects of the organization of SATPin can be related with those of EL$^+$SAT, namely the procedure for extracting MUSes/MinAs. Although the actual enumeration of candidate sets is not detailed in [34], the description of SATPin suggests the use of model enumeration with some essential pruning techniques.

6 Experimental Results

This section evaluates group MUS enumerators for Horn formulae obtained from axiom pinpointing problems for the $\mathcal{EL}$ family of DLs, particularly applied to medical ontologies. A set of standard benchmarks is considered. These have been used in earlier work, e.g. [5,51,32,1,34].

Since all experiments consist of converting axiom pinpointing problems into group MUS enumeration problems, the tool that uses HGMUS as its back-end is named EL2MUS. Thus, in this section, the results for EL2MUS illustrate the performance of the group-MUS enumerator described in this paper.

6.1 Experimental Setup

Each considered instance represents the problem of explaining a particular subsumption relation (query) entailed in a medical ontology. Four medical ontologies[3] are considered: GALEN [46], GENE [2], NCI [53] and SNOMED CT [56]. For GALEN, we consider two variants: FULL-GALEN and NOT-GALEN. The most important ontology is SNOMED CT and, due to its huge size, it also produces the hardest axiom pinpointing instances. For each ontology (including the GALEN variants) 100 queries are considered; 50 random (which are expected to be easier) and 50 sorted (expected to have a large number of minimal explanations) queries. So, there are 500 queries in total. Given an ontology and a subsumption query, the encoding proposed in [51,52] produces a Horn formula and a set of axioms (variables) which may be responsible for the subsumption relation. This can be transformed into a group-MUS enumeration problem where the original Horn formula forms group-0 and each axiom constitutes a group containing only a unit clause.

Two different experiments were considered by applying two different simplification techniques to the problem instances, both of which were proposed in [52]. The first one

---

[3] GENE, GALEN and NCI ontologies are freely available at http://lat.inf.tu-dresden.de/~meng/toyont.html. The SNOMED CT ontology was requested from IHTSDO under a nondisclosure license agreement.
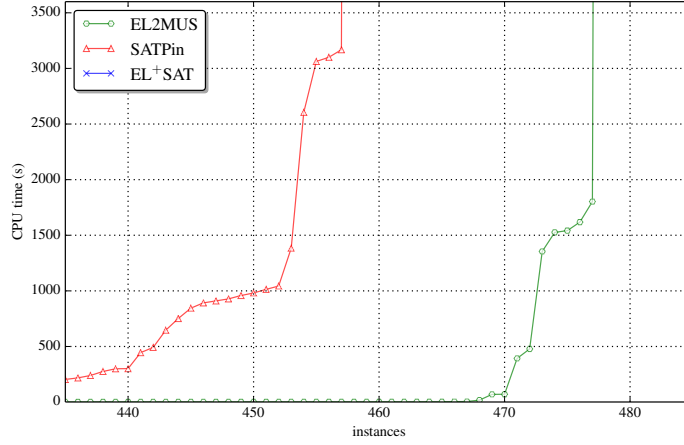
Fig. 1: Cactus plot comparing EL$^+$SAT, SATPin and EL2MUS on the COI instances

uses the Cone-Of-Influence (COI) reduction. These are reduced instances in both the size of the Horn formula and the number of axioms, but are still quite large. Similar techniques are exploited in related work [5,32,34]. The second one considers the more effective reduction technique (which we refer to as x2), consisting in applying the COI technique, re-encoding the Horn formula into a reduced ontology, and encoding this ontology again into a Horn formula. This results in small Horn formulae, which will be useful to evaluate the algorithms when there are a large number of MUSes/MCSes.

The experiments compare EL2MUS to different algorithms, namely EL$^+$SAT [51,52], CEL [5], JUST [32] and SATPin [34]. EL$^+$SAT [52] has been shown to outperform CEL [5], whereas SATPin [34] has been shown to outperform the MUS enumerator MARCO [30].

The comparison with CEL and JUST imposes a number of constraints. First, CEL only computes 10 MinAs, so all comparisons with CEL only consider reporting the first 10 MinAs/MUSes. Also, CEL uses a simplification technique similar to COI, so CEL is considered in the first experiments. Second, JUST operates on selected subsets of $\mathcal{EL}^+$, i.e. the description logic used in most medical ontologies. As a result, all comparisons with JUST consider solely the problem instances for which JUST can compute correct results. JUST accepts the simplified x2 ontologies, so it is considered in the second experiments. The comparison with these tools is presented at the end of the section.

EL2MUS interfaces the SAT solver Minisat 2.2 [19] for computing maximal models. All the experiments were performed on a Linux cluster (2 GHz) and the algorithms were given a time limit of 3600s and a memory limit of 4 GB.

6.2 COI Instances

Figure 1 summarizes the results for EL$^+$SAT, SATPin and EL2MUS. EL$^+$SAT does not show in the plot due to its poor performance. As can be observed, EL2MUS terminates for more instances than any of the other tools. Figure 2 shows scatter plots comparing the different tools. As can be concluded, and with a few outliers, the performance of
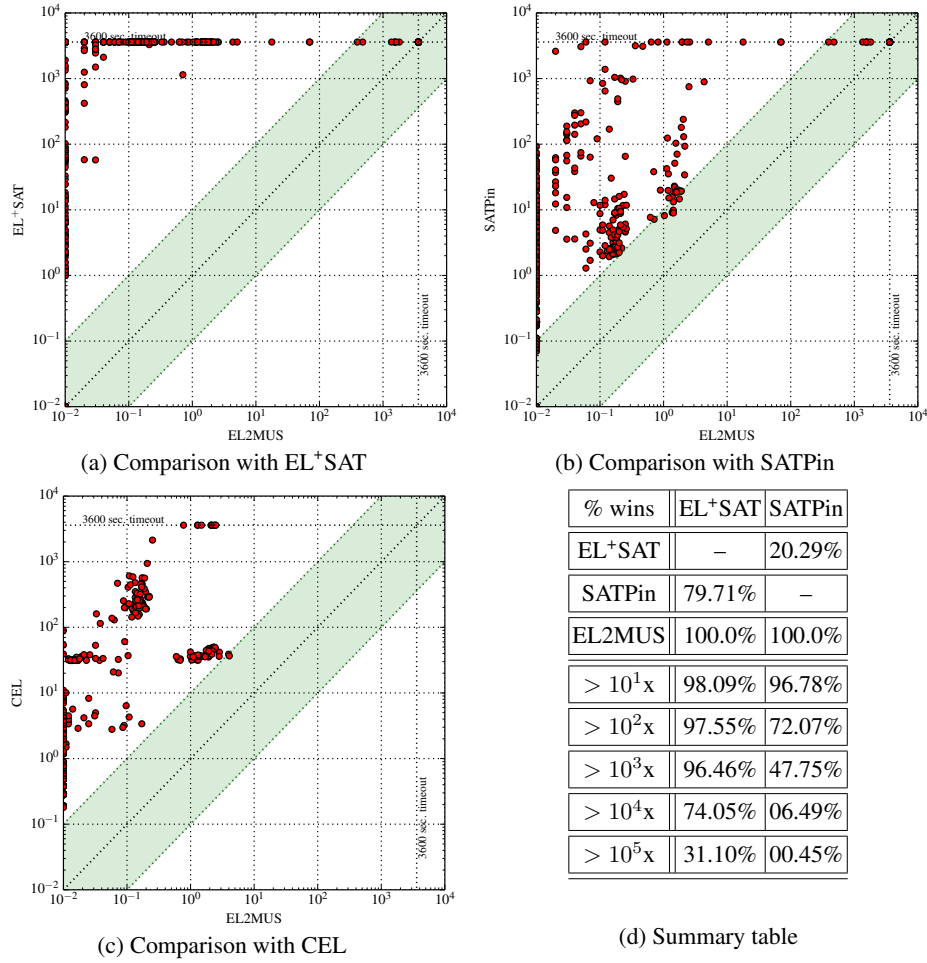
(a) Comparison with EL$^+$SAT

(b) Comparison with SATPin

(c) Comparison with CEL

% wins	EL$^+$SAT	SATPin
EL$^+$SAT	–	20.29%
SATPin	79.71%	–
EL2MUS	100.0%	100.0%
$> 10^1$x	98.09%	96.78%
$> 10^2$x	97.55%	72.07%
$> 10^3$x	96.46%	47.75%
$> 10^4$x	74.05%	06.49%
$> 10^5$x	31.10%	00.45%

(d) Summary table

Fig. 2: Scatter plots for COI instances

EL2MUS exceeds the performance of any of the other tools by at least one order of magnitude (and often by more). Figure 2d summarizes the results in the scatter plots, where the percentages shown are computed for problem instances for which at least one of the tools takes more than 0.001s. CEL is not shown in the table due to the special constraints mentioned above. As can be observed, EL2MUS outperforms any of the other tools in all of the problem instances and, for many cases, with two or more orders of magnitude improvement.

6.3 x2 Instances

The x2 instances are significantly simpler than the COI instances. Thus, whereas the COI instances can serve to assess the scalability of each approach, the x2 instances highlight the expected performance in representative settings. Figure 3a summarizes the
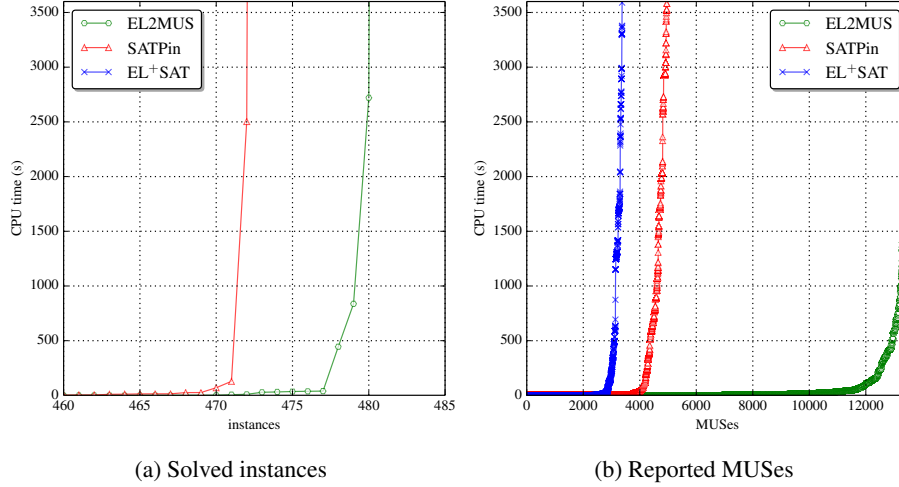
(a) Solved instances   (b) Reported MUSes

Fig. 3: Cactus plots comparing EL$^+$SAT, SATPin and EL2MUS on the x2 instances

performance of the tools EL$^+$SAT, SATPin and EL2MUS. As with the COI instances, EL$^+$SAT does not show in the plot due to its poor performance. Moreover, and as before in terms of terminated instances, EL2MUS exhibits an observable performance edge.

A pairwise comparison between the different tools is summarized in Figure 4. Although not as impressive as for the COI instances, EL2MUS still consistently outperforms all other tools. Figure 4d summarizes the results, where as before the percentages shown are computed for problem instances for which at least one of the tools takes more than 0.001s. Observe that, for these easier instances, SATPin becomes competitive with EL2MUS. Nevertheless, for instances taking more than 0.1s, EL2MUS outperforms SATPin on 100% of the instances. Thus, the 67.69% shown in the table result from instances for which both SATPin and EL2MUS take at most 0.04s. The summary table also lists the number of computed MUSes for the 19 instances for which EL2MUS does *not* terminate (all of the other tools also do not terminate for these 19 instances). EL2MUS computes 9948 MUSes in total. As can be observed from the table, the other tools lag behind, and compute significantly fewer MUSes. The comparison with EL$^+$SAT and SATPin, reveals that EL2MUS computes respectively in excess of a factor of 10 and of 5 more MUSes.

EL2MUS not only terminates on more instances than any other approach and computes more MUSes for the unsolved instances; it also reports the sequences of MUSes much faster. Figure 3b shows, for each computed MUS over the whole set of instances, the time each MUS was reported. This figure compares EL$^+$SAT, SATPin and EL2MUS, as these are the only methods able to report MUSes from the beginning. The results confirm that EL2MUS is able to find many more MUSes in less time than the alternatives.

These experimental results suggest that, not only is EL2MUS the best performing axiom pinpointing tool, on both the COI and x2 problem instances, but it is also the one that is expected to scale better for more challenging problem instances, given the results on the COI instances.
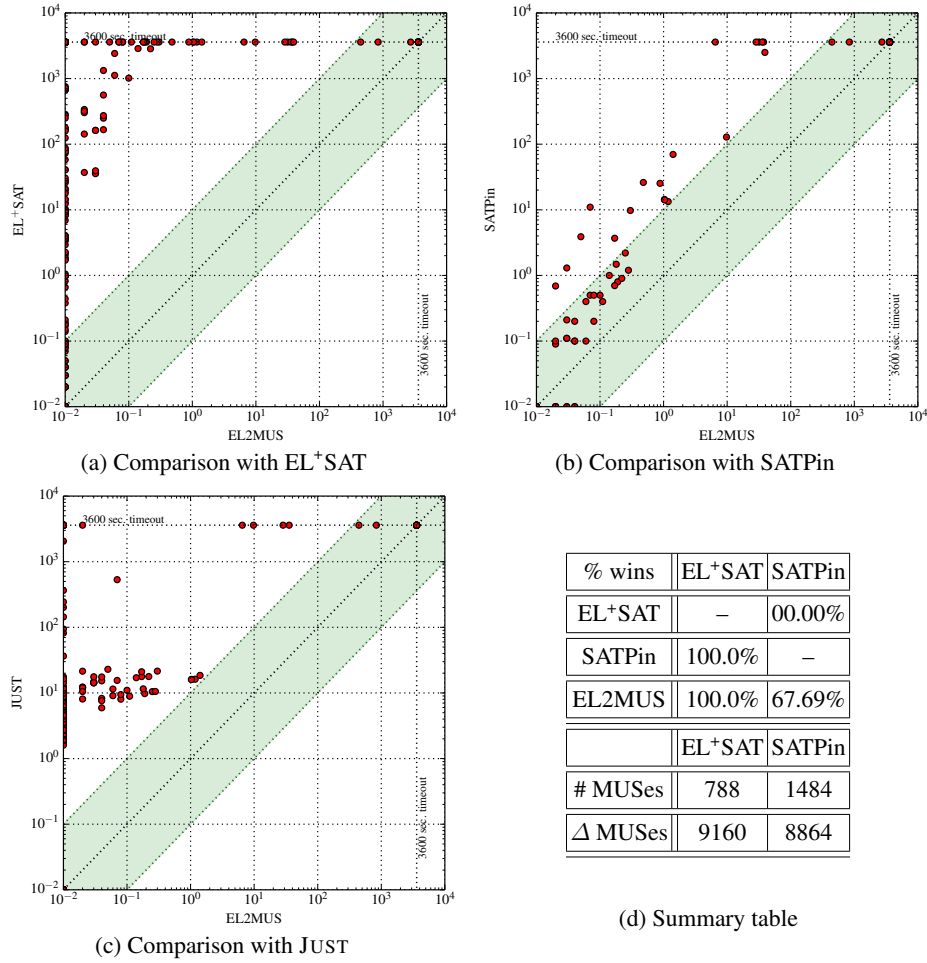
(a) Comparison with EL⁺SAT

(b) Comparison with SATPin

(c) Comparison with JUST

% wins	EL$^+$SAT	SATPin
EL$^+$SAT	–	00.00%
SATPin	100.0%	–
EL2MUS	100.0%	67.69%

	EL$^+$SAT	SATPin
# MUSes	788	1484
Δ MUSes	9160	8864

(d) Summary table

Fig. 4: Scatter plots for x2 instances

6.4 Assessment of Non SAT-Based Axiom Pinpointing Tools

Figure 2c and Figure 4c show scatter plots comparing EL2MUS with CEL [5] and JUST [32], respectively for the COI and x2 instances[4]. As indicated earlier, CEL only computes 10 MinAs, and so the run times shown are for computing the first 10 MinAs. As can be observed, the performance edge of EL2MUS is clear, with the performance gap exceeding 1 order of magnitude almost without exception. Moreover, JUST [32] is a recent state of the art axiom pinpointing tool for the less expressive $\mathcal{ELH}$ DL. Thus, not all subsumption relations can be represented and analyzed. The results shown are for the subsumption relations for which JUST gives the correct results. In total, 382 instances could be considered and are shown in the plot. As before, the performance

---

[4] The other scatter plots are not shown, but the conclusions are the same.

edge of EL2MUS is clear, with the performance gap exceeding 1 order of magnitude without exception. In this case, since the x2 instances are in general much simpler, the performance gap is even more significant.

7 Conclusions & Future Work

Enumeration of group MUS for Horn formulae finds important applications, including axiom pinpointing for the $\mathcal{EL}$ family of DLs. Since the $\mathcal{EL}$ family of DLs is widely used for representing medical ontologies, namely with $\mathcal{EL}^+$, enumeration of group MUSes for Horn formulae represents a promising and strategic application of SAT technology. This includes, among others, SAT solvers, MCS extractors and enumerators, and MUS extractors and enumerators. This paper develops a highly optimized group MUS enumerator for Horn formulae, which is shown to extensively outperform any other existing approach. Performance gains are almost without exception at least one order of magnitude, and most often significantly more than that. More importantly, the experimental results demonstrate that SAT-based approaches are by far the most effective approaches for axiom pinpointing for the $\mathcal{EL}$ family of DLs. When compared with other non SAT-based approaches, the performance gains are also conclusive.

Future work will exploit integration of additional recent work on SAT-based problem solving, e.g. in MCS enumeration and MUS enumeration, to further improve performance of axiom pinpointing.

References

1. M. F. Arif and J. Marques-Silva. Towards efficient axiom pinpointing of EL+ ontologies. *CoRR*, abs/1503.08454, March 2015. Available from http://arxiv.org/abs/1503.08454.

2. M. Ashburner, C. A. Ball, J. A. Blake, D. Botstein, H. Butler, J. M. Cherry, A. P. Davis, K. Dolinski, S. S. Dwight, J. T. Eppig, and et al. Gene ontology: tool for the unification of biology. *Nature genetics*, 25(1):25–29, 2000.

3. F. Baader and B. Hollunder. Embedding defaults into terminological knowledge representation formalisms. *J. Autom. Reasoning*, 14(1):149–180, 1995.

4. F. Baader, I. Horrocks, and U. Sattler. Description logics. In V. L. Frank van Harmelen and B. Porter, editors, *Handbook of Knowledge Representation*, Foundations of Artificial Intelligence, chapter 3, pages 135 – 179. Elsevier, 2008.

5. F. Baader, C. Lutz, and B. Suntisrivaraporn. CEL - A polynomial-time reasoner for life science ontologies. In *IJCAR*, pages 287–291, 2006.

6. F. Baader and R. Peñaloza. Axiom pinpointing in general tableaux. *J. Log. Comput.*, 20(1):5–34, 2010.

7. F. Baader, R. Peñaloza, and B. Suntisrivaraporn. Pinpointing in the description logic $\mathcal{EL}^+$. In *KI*, pages 52–67, 2007.

8. F. Baader and B. Suntisrivaraporn. Debugging SNOMED CT using axiom pinpointing in the description logic $\mathcal{EL}^+$. In *KR-MED*, 2008.

9. F. Bacchus, J. Davies, M. Tsimpoukelli, and G. Katsirelos. Relaxation search: A simple way of managing optional clauses. In *AAAI*, pages 835–841, 2014.

10. J. Bailey and P. J. Stuckey. Discovery of minimal unsatisfiable subsets of constraints using hitting set dualization. In *PADL*, pages 174–186, 2005.

11. R. R. Bakker, F. Dikker, F. Tempelman, and P. M. Wognum. Diagnosing and solving over-determined constraint satisfaction problems. In *IJCAI*, pages 276–281, 1993.

12. A. Belov, I. Lynce, and J. Marques-Silva. Towards efficient MUS extraction. *AI Commun.*, 25(2):97–116, 2012.

13. A. Biere, M. Heule, H. van Maaren, and T. Walsh, editors. *Handbook of Satisfiability*, volume 185 of *Frontiers in Artificial Intelligence and Applications*. IOS Press, 2009.

14. E. Birnbaum and E. L. Lozinskii. Consistent subsets of inconsistent systems: structure and behaviour. *J. Exp. Theor. Artif. Intell.*, 15(1):25–46, 2003.

15. A. R. Bradley and Z. Manna. Checking safety by inductive generalization of counterexamples to induction. In *FMCAD*, pages 173–180, 2007.

16. S. A. Cook. The complexity of theorem-proving procedures. In *STOC*, pages 151–158, 1971.

17. J. L. de Siqueira N. and J.-F. Puget. Explanation-based generalisation of failures. In *ECAI*, pages 339–344, 1988.

18. W. F. Dowling and J. H. Gallier. Linear-time algorithms for testing the satisfiability of propositional Horn formulae. *J. Log. Program.*, 1(3):267–284, 1984.

19. N. Eén and N. Sörensson. An extensible SAT-solver. In *SAT*, pages 502–518, 2003. MiniSat 2.2 is available from: https://github.com/niklasso/minisat.git.

20. I. Gent. Optimal implementation of watched literals and more general techniques. *Journal of Artificial Intelligence Research*, 48:231–252, 2013.

21. E. Giunchiglia and M. Maratea. Solving optimization problems with DLL. In *ECAI*, pages 377–381, 2006.

22. É. Grégoire, J. Lagniez, and B. Mazure. An experimentally efficient method for (MSS, CoMSS) partitioning. In *AAAI*, pages 2666–2673, 2014.

23. F. Heras, A. Morgado, and J. Marques-Silva. MaxSAT-based encodings for group MaxSAT. *AI Commun.*, 28(2):195–214, 2015.

24. A. Itai and J. A. Makowsky. Unification as a complexity measure for logic programming. *J. Log. Program.*, 4(2):105–117, 1987.

25. U. Junker. QuickXplain: Preferred explanations and relaxations for over-constrained problems. In *AAAI*, pages 167–172, 2004.

26. A. Kalyanpur, B. Parsia, M. Horridge, and E. Sirin. Finding all justifications of OWL DL entailments. In *ISWC*, pages 267–280, 2007.

27. A. Kalyanpur, B. Parsia, E. Sirin, and B. C. Grau. Repairing unsatisfiable concepts in owl ontologies. pages 170–184, 2006.

28. S. K. Lahiri, R. Nieuwenhuis, and A. Oliveras. SMT techniques for fast predicate abstraction. In *CAV*, pages 424–437, 2006.

29. M. H. Liffiton and A. Malik. Enumerating infeasibility: Finding multiple MUSes quickly. In *CPAIOR*, pages 160–175, 2013.

30. M. H. Liffiton, A. Previti, A. Malik, and J. Marques-Silva. Fast, flexible MUs enumeration. *Constraints*, 2015. Online version: http://link.springer.com/article/10.1007/s10601-015-9183-0.

31. M. H. Liffiton and K. A. Sakallah. Algorithms for computing minimal unsatisfiable subsets of constraints. *J. Autom. Reasoning*, 40(1):1–33, 2008.

32. M. Ludwig. Just: a tool for computing justifications w.r.t. ELH ontologies. In *ORE*, 2014.

33. M. Ludwig and R. Peñaloza. Error-tolerant reasoning in the description logic $\mathcal{EL}$. In *JELIA*, pages 107–121, 2014.

34. N. Manthey and R. Peñaloza. Exploiting SAT technology for axiom pinpointing. Technical Report LTCS 15-05, Chair of Automata Theory, Institute of Theoretical Computer Science, Technische Universität Dresden, April 2015. Available from https://ddll.inf.tu-dresden.de/web/Techreport3010.

35. J. Marques-Silva, F. Heras, M. Janota, A. Previti, and A. Belov. On computing minimal correction subsets. In *IJCAI*, pages 615–622, 2013.

36. J. Marques-Silva, M. Janota, and A. Belov. Minimal sets over monotone predicates in Boolean formulae. In *CAV*, pages 592–607, 2013.

37. C. Mencía, A. Previti, and J. Marques-Silva. Literal-based MCS extraction. In *IJCAI*, 2015. To Appear.

38. T. A. Meyer, K. Lee, R. Booth, and J. Z. Pan. Finding maximally satisfiable terminologies for the description logic $\mathcal{ALC}$. In *AAAI*, pages 269–274, 2006.

39. M. Minoux. LTUR: A simplified linear-time unit resolution algorithm for Horn formulae and computer implementation. *Inf. Process. Lett.*, 29(1):1–12, 1988.

40. K. Moodley, T. Meyer, and I. J. Varzinczak. Root justifications for ontology repair. In *RR*, pages 275–280, 2011.

41. H. H. Nguyen, N. Alechina, and B. Logan. Axiom pinpointing using an assumption-based truth maintenance system. In *DL*, 2012.

42. B. O'Sullivan, A. Papadopoulos, B. Faltings, and P. Pu. Representative explanations for over-constrained problems. In *AAAI*, pages 323–328, 2007.

43. B. Parsia, E. Sirin, and A. Kalyanpur. Debugging OWL ontologies. In *WWW*, pages 633–640, 2005.

44. R. Peñaloza and B. Sertkaya. On the complexity of axiom pinpointing in the EL family of description logics. In *KR*, 2010.

45. A. Previti and J. Marques-Silva. Partial MUS enumeration. In *AAAI*, pages 818–825, 2013.

46. A. L. Rector and I. R. Horrocks. Experience building a large, re-usable medical ontology using a description logic with transitivity and concept inclusions. In *Workshop on Ontological Engineering*, pages 414–418, 1997.

47. R. Reiter. A theory of diagnosis from first principles. *Artif. Intell.*, 32(1):57–95, 1987.

48. E. D. Rosa and E. Giunchiglia. Combining approaches for solving satisfiability problems with qualitative preferences. *AI Commun.*, 26(4):395–408, 2013.

49. S. Schlobach and R. Cornet. Non-standard reasoning services for the debugging of description logic terminologies. In *IJCAI*, pages 355–362, 2003.
50. S. Schlobach, Z. Huang, R. Cornet, and F. van Harmelen. Debugging incoherent terminologies. *J. Autom. Reasoning*, 39(3):317–349, 2007.
51. R. Sebastiani and M. Vescovi. Axiom pinpointing in lightweight description logics via Horn-SAT encoding and conflict analysis. In *CADE*, pages 84–99, 2009.
52. R. Sebastiani and M. Vescovi. Axiom pinpointing in large $\mathcal{EL}^+$ ontologies via SAT and SMT techniques. Technical Report DISI-15-010, DISI, University of Trento, Italy, April 2015. Under Journal Submission. Available as http://disi.unitn.it/~rseba/elsat/elsat_techrep.pdf.
53. N. Sioutos, S. de Coronado, M. W. Haber, F. W. Hartel, W. Shaiu, and L. W. Wright. NCI thesaurus: A semantic model integrating cancer-related clinical and molecular information. *Journal of Biomedical Informatics*, 40(1):30–43, 2007.
54. E. Sirin, B. Parsia, B. C. Grau, A. Kalyanpur, and Y. Katz. Pellet: A practical OWL-DL reasoner. *J. Web Sem.*, 5(2):51–53, 2007.
55. J. Slaney. Set-theoretic duality: A fundamental feature of combinatorial optimisation. In *ECAI*, pages 843–848, 2014.
56. K. A. Spackman, K. E. Campbell, and R. A. Côté. SNOMED RT: a reference terminology for health care. In *AMIA*, 1997.
57. M. Vescovi. *Exploiting SAT and SMT Techniques for Automated Reasoning and Ontology Manipulation in Description Logics*. PhD thesis, University of Trento, 2011.