# A Formal Reduction of Sudoku Puzzle into SAT

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# Outline

1. Demonstration: Sudoku Puzzle Solver
2. Preliminaries: SAT
3. Formal Reduction: Encoding Sudoku in SAT
4. Applications of SAT
   - Search & Planning
   - Model Checking
   - Complexity Theory
5. Appendix

# Sudoku Puzzle

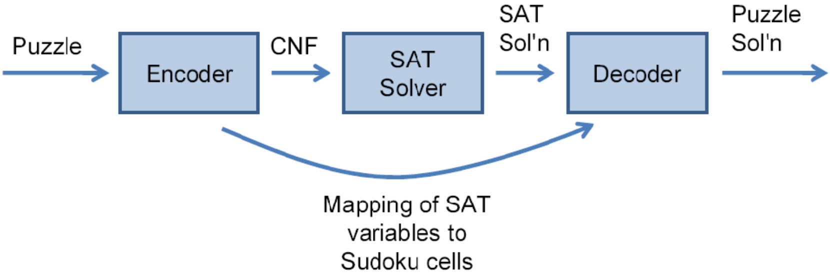
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 4 | | | | 5 | 3 | |
| 6 | | | 1 | 3 | | | | 9 |
| | 7 | | | | | 2 | | |
| | | | | 8 | | 4 | 7 | |
| | | | 3 | | 4 | | | |
| | 5 | 9 | | 6 | | | | |
| | | 6 | | | | | 9 | |
| 9 | | | | 7 | 1 | | | 5 |
| | 2 | 7 | | | | 1 | | |

# Sudoku Puzzle

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 4 | | | | 5 | 3 | |
| 6 | | | 1 | 3 | | | | 9 |
| | 7 | | | | | 2 | | |
| | | | | 8 | | 4 | 7 | |
| | | | 3 | | 4 | | | |
| | 5 | 9 | | 6 | | | | |
| | | 6 | | | | | 9 | |
| 9 | | | | 7 | 1 | | | 5 |
| | 2 | 7 | | | | 1 | | |

$6.10^{21}$ possible grids for a simple looking $9 * 9$ puzzle instance.

# Abstract Model



# Preliminaries: Propositional Logic

## Vocabulary

An alphabet of propositional logic consists of

- a (countable) infinite set $\mathcal{R} = \{p_0, p_1, p_2, \dots\}$ of propositional variables,
- the set logical connectives: $\circ \in \{\neg/1, \wedge/2, \vee/2, \rightarrow /2, \leftrightarrow /2$
- the special characters "(" and ")".

## Syntax

An atomic formula, atom, is a propositional variable.
The set of propositional formulas is the smaller set $\mathcal{L}(\mathcal{R})$ of strings over an alphabet of propositional logic with the following properties:

- if $F$ is an atomic formula, then $F \in \mathcal{L}(\mathcal{R})$.
- if $F \in \mathcal{L}(\mathcal{R})$, then $\neg F \in \mathcal{L}(\mathcal{R})$.
- if $\circ/2$ is a binary connective, $F, G \in \mathcal{L}(\mathcal{R})$, then $(F \circ G) \in \mathcal{L}(\mathcal{R})$.

# Truth Table Semantics

- The set of truth values $\mathcal{W}$ is the set $\{\top, \bot\}$.
- We consider the following functions on $\mathcal{W}$:
  - Negation $\neg^*/1$.
  - Conjunction $\wedge^*/2$.
  - Disjunction $\vee^*/2$.
  - Implication $\rightarrow^*/2$.
  - Equivalence $\leftrightarrow^*/2$.

| | | $\neg^*$ | $\wedge^*$ | $\vee^*$ | $\rightarrow^*$ | $\leftrightarrow^*$ |
|---|---|---|---|---|---|---|
| $\top$ | $\top$ | $\bot$ | $\top$ | $\top$ | $\top$ | $\top$ |
| $\top$ | $\bot$ | $\bot$ | $\bot$ | $\top$ | $\bot$ | $\bot$ |
| $\bot$ | $\top$ | $\top$ | $\bot$ | $\top$ | $\top$ | $\bot$ |
| $\bot$ | $\bot$ | $\top$ | $\bot$ | $\bot$ | $\top$ | $\top$ |

# Model based Semantics

An interpretation $\mathcal{I} = (\mathcal{W}, .^I)$ is a mapping $.^I : \mathcal{L}(\mathcal{R}) \rightarrow \mathcal{W}$ such that:

## SAT

A propositional satisfiability problem (SAT), consist of a formula $\phi \in \mathcal{L}(\mathcal{R})$, and is the problem to decide whether $\phi$ is satisfiable.

## Model

An interpretation $\mathcal{I} = (\mathcal{W}, .^I)$ is called a model for propositional formula $\phi$ , $\mathcal{I} \models \phi$ if $[\phi]^I = \top$ (i.e., $\mathcal{I}$ satisfies $\phi$). $\phi$ is unsatisfiable if it has no models.

# Propostional Satisfiability Problems

SAT is a combinatorial decision problem.

- Decision variant yes/no answer.
- Search variant find a model if $\phi$ is satisfiable.

## Example

- Let $\{p_1, p_2, p_3, p_4, p_5\} \subseteq \mathcal{R}$.

$$\begin{aligned}\phi = {} & (\neg p_1 \vee p_2) \wedge (\neg p_2 \vee p_1) \\ & \wedge (\neg p_1 \vee \neg p_2 \vee \neg p_3) \wedge (p_1 \vee p_2) \\ & \wedge (\neg p_4 \vee p_3) \wedge (\neg p_5 \vee p_3)\end{aligned}$$

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- $\{p_1, p_2\}$ is a model of $\phi$.

Hence, $\phi$ is satisfiable.

# Formal Reduction

## Idea:

Sudoku puzzle $\mathcal{S}$ is formulated as a CNF formula $\phi$ such that is

$$\phi \texttt{ is satisfiable iff } \mathcal{S} \texttt{ has a solution.}$$

## Sudoku Puzzle $\mathcal{S}$:

A Sudoku puzzle $\mathcal{S}$ is represented by a $\mathbb{N} * \mathbb{N}$ grid, which comprises of an $\sqrt{\mathbb{N}} * \sqrt{\mathbb{N}}$ sub-grids (also called boxes). Some of the entries in the grid are filled with numbers from 1 to $\mathbb{N}$, whereas other entries are left blank.

## Encoding Scheme ($\mathcal{S} \implies \phi$):

A SAT problem is represented as a propositional formula $\Phi$ where each variable $P_i$ is assigned 0 ($\mathbb{F}$) or 1 ($\mathbb{T}$) where i $\in (1, \cdots, n)$ In Sudoku each tuple $(r, c, v)$ denotes a variable which is true iff the cell in row $r$ and column $c$ is assigned a number $v$; $[r, c] = v$. The resulting set of formulas turn out to be $V = \{(r, c, v) \mid 1 \leq r, c, v \leq \text{n}\}$.

# Encoding Scheme ($\mathcal{S} \implies \phi$):

- There is at exactly one number in each cell
  $\phi_{cell.ex} := \phi_{cell.def} \wedge \phi_{cell.uniq}$
  There is at least one number for each cell
  $\phi_{cell.def} := \bigwedge_{r=1}^{n} \bigwedge_{c=1}^{n} \bigvee_{v=1}^{n} (r, c, v)$
  Each number appears at most one in each cell
  $\phi_{cell.uniq} := \bigwedge_{r=1}^{n} \bigwedge_{c=1}^{n} \bigwedge_{v_i=1}^{(n-1)} \bigwedge_{v_j=v_i+1}^{n} \neg(r, c, v_i) \vee \neg(r, c, v_j)$
- There is at exactly one number in each row
- There is at exactly one number in each column
- There is at exactly one number in each block

## Encodings ($\phi$):

$$\Phi_{extended} := \phi_{cell.ex} \wedge \phi_{row.ex} \wedge \phi_{col.ex} \wedge \phi_{blk.ex} \wedge \phi_{assign}.$$

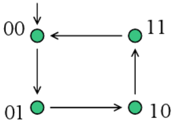
$$\Phi_{efficient} := \phi_{cell.ex} \wedge \phi_{row.uniq} \wedge \phi_{col.uniq} \wedge \phi_{blk.uniq} \wedge \phi_{assign}.$$

$$\Phi_{minimal} := \phi_{cell.def} \wedge \phi_{row.uniq} \wedge \phi_{col.uniq} \wedge \phi_{blk.uniq} \wedge \phi_{assign}.$$

# Applications

- Search & Planning
- Model Checking
- Complexity Theory ($P = NP$ or $P \neq NP$)

# Model Checking




Initial state: $I : \neg l \wedge \neg r$

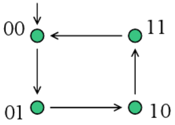
Transition: $R : \begin{pmatrix} l' = (l \neq r) \wedge \\ r' = \neg r \end{pmatrix}$

Safety property: $\mathbf{AG}\ (\neg l \vee \neg r)$

$$\Omega(2) : (\neg l_0 \wedge \neg r_0) \wedge \begin{pmatrix} l_1 = (l_0 \neq r_0) \wedge r_1 = \neg r_0 \wedge \\ l_2 = (l_1 \neq r_1) \wedge r_2 = \neg r_1 \end{pmatrix} \wedge \begin{pmatrix} (l_0 \wedge r_0) \vee \\ (l_1 \wedge r_1) \vee \\ (l_2 \wedge r_2) \end{pmatrix}$$

$\Omega(2)$ is unsatisfiable. $\Omega(3)$ is satisfiable.

# Model Checking

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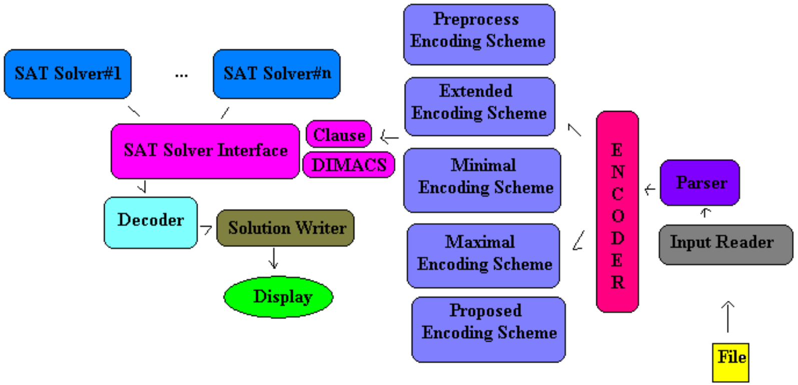
$\Omega(2)$ is unsatisfiable. $\Omega(3)$ is satisfiable.

Satisfying assignment gives the counter example to the safety property.

$$\mathcal{M} = \{r_1, l_2, l_3, r_3\}$$
$$\mathcal{M} = \{(\neg l_0, \neg r_0), (\neg l_1, r_1), (l_2, \neg r_2), (l_3, r_3)\}$$

# Sudoku Solver Architecture & Solver Demo



# For Further Reading


Stephen A. Cook. 1970
*The complexity of theorem-proving procedures.*.

Armin Biere, Marjin Heule, Hans van Marren and Toby Walsh. 2009
*Handbook of Satisfiability*.

**Q & A**