

NP-complete Problems: Reductions

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Outline

- 1 Reductions
- 2 Showing **NP**-completeness
- 3 Independent Set \rightarrow Vertex Cover
- 4 3-SAT \rightarrow Independent Set
- 5 SAT \rightarrow 3-SAT
- 6 All of **NP** \rightarrow SAT
- 7 Using SAT-solvers

Informally

We say that a search problem A is reduced to a search problem B and write $A \rightarrow B$, if a polynomial time algorithm for B can be used (as a black box) to solve A in polynomial time.

Reduction: $A \rightarrow B$

instance I of A

Reduction: $A \rightarrow B$

instance I of A

Algorithm for A

Algorithm for B

Reduction: $A \rightarrow B$

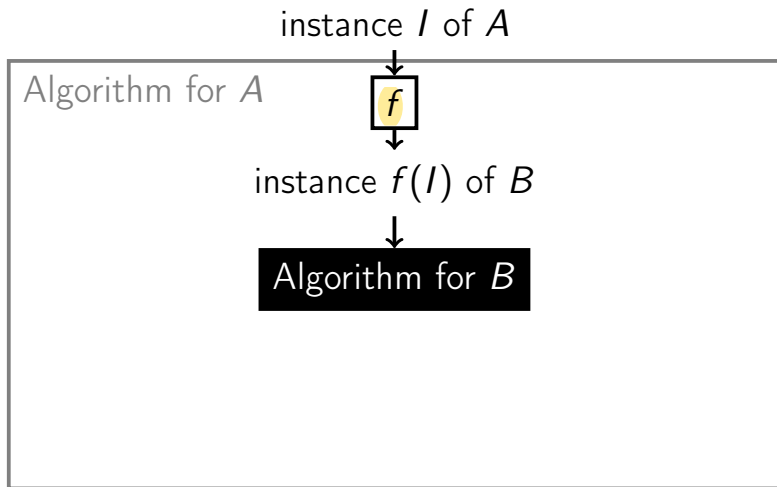
instance I of A



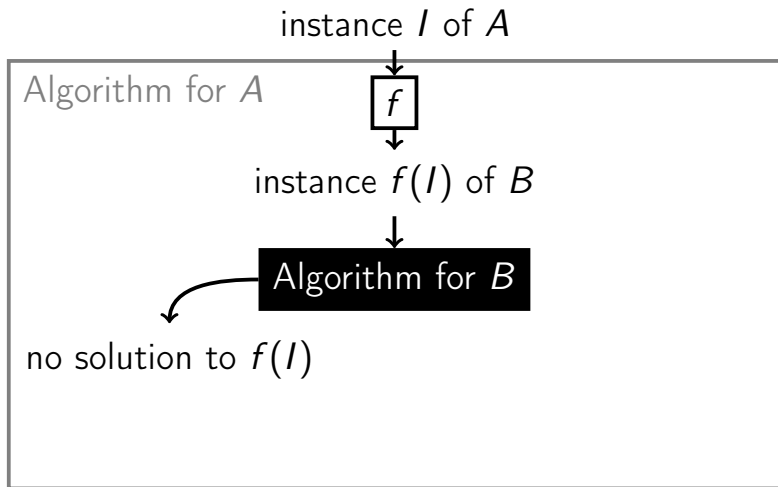
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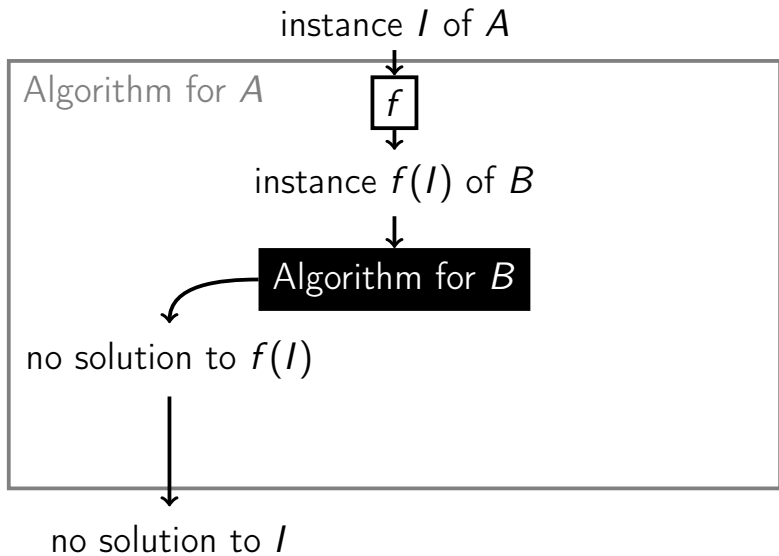
Reduction: $A \rightarrow B$



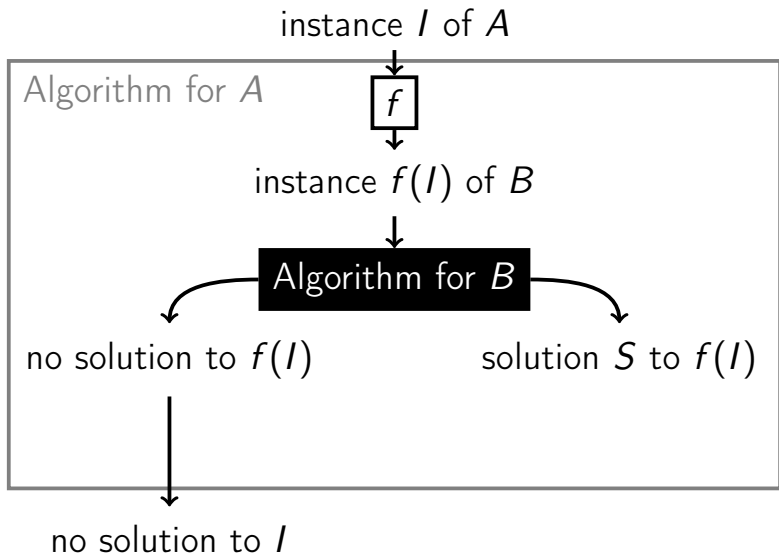
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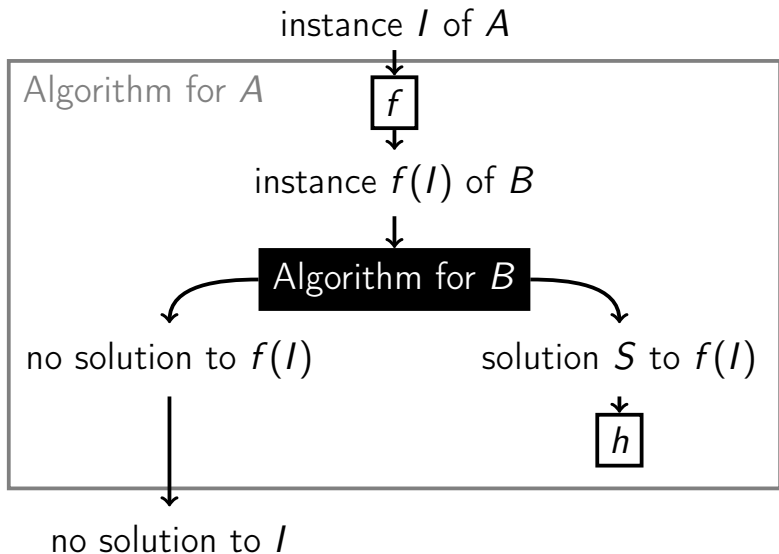
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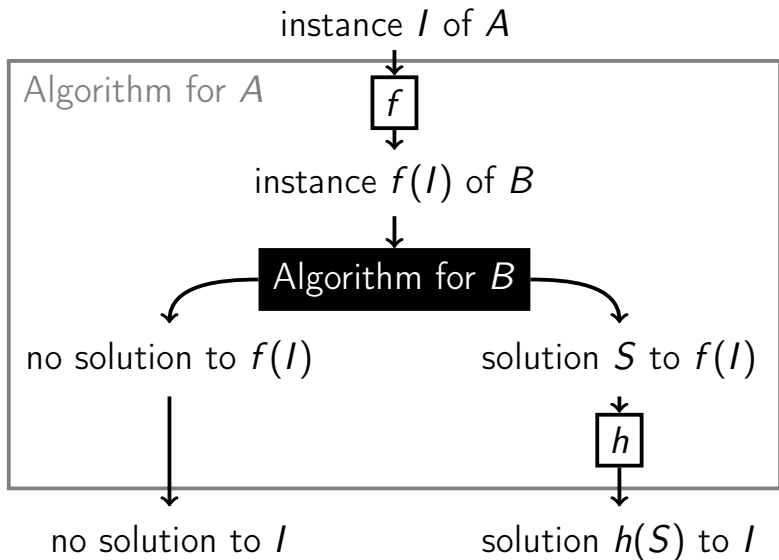
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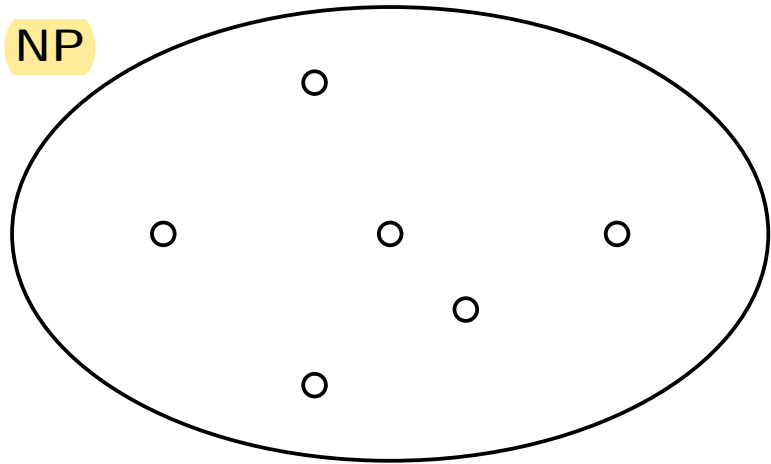


Formally

Definition

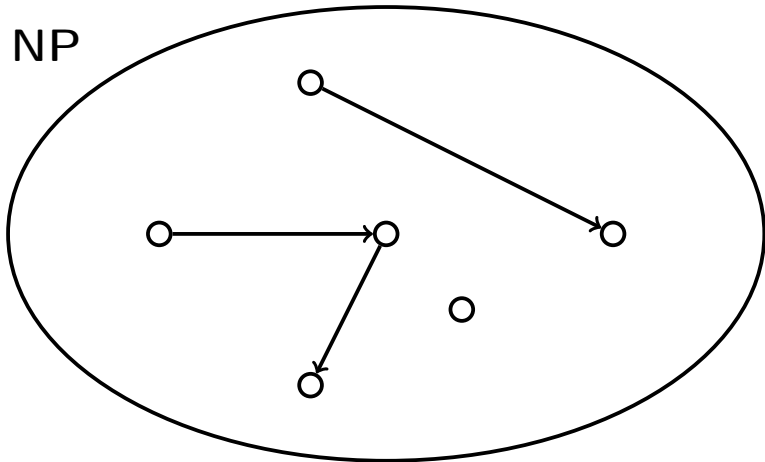
We say that a search problem A is reduced to a search problem B and write $A \rightarrow B$, if there exists a polynomial time algorithm f that converts any instance I of A into an instance $f(I)$ of B , together with a polynomial time algorithm h that converts any solution S to $f(I)$ back to a solution $h(S)$ to A . If there is no solution to $f(I)$, then there is no solution to I .

Graph of Search Problems



Graph of Search Problems

NP



NP-complete Problems

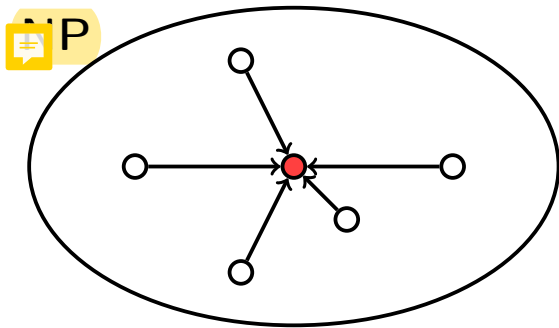
Definition

A search problem is called **NP-complete** if all other search problems reduce to it.

NP-complete Problems

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Do they exist?

It's not at all immediate that **NP**-complete problems even exist. We'll see later that all hard problems that we've seen in the previous part are in fact **NP**-complete!

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Two ways of using $A \rightarrow B$:

- 1 if B is easy (can be solved in polynomial time), then so is A
- 2 if A is hard (cannot be solved in polynomial time), then so is B

Reductions Compose

Lemma

If $A \rightarrow B$ and $B \rightarrow C$, then $A \rightarrow C$.

Proof

- The reductions $A \rightarrow B$ and $B \rightarrow C$ are given by pairs of polytime algorithms (f_{AB}, h_{AB}) and (f_{BC}, h_{BC}) .

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- To transform an instance I_A of A to an instance I_C of C we apply a polytime algorithm $f_{BC} \circ f_{AB}$: $I_C = f_{BC}(f_{AB}(I_A))$.

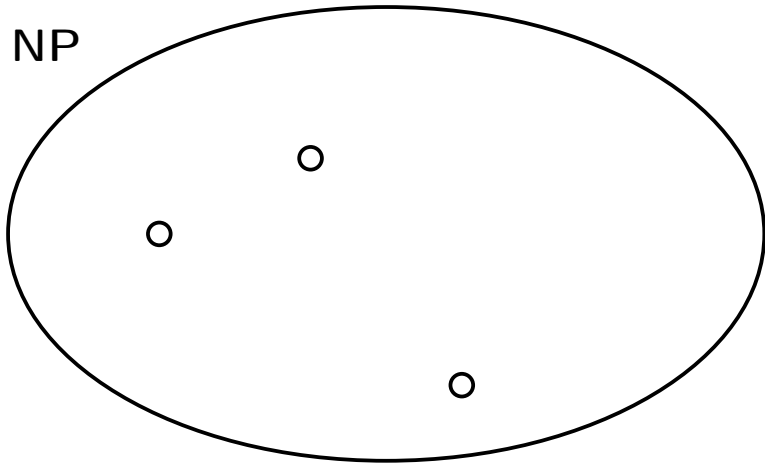
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- To transform a solution S_C to I_C to a solution S_A to I_A we apply a polytime algorithm $h_{AB} \circ h_{BC}$:
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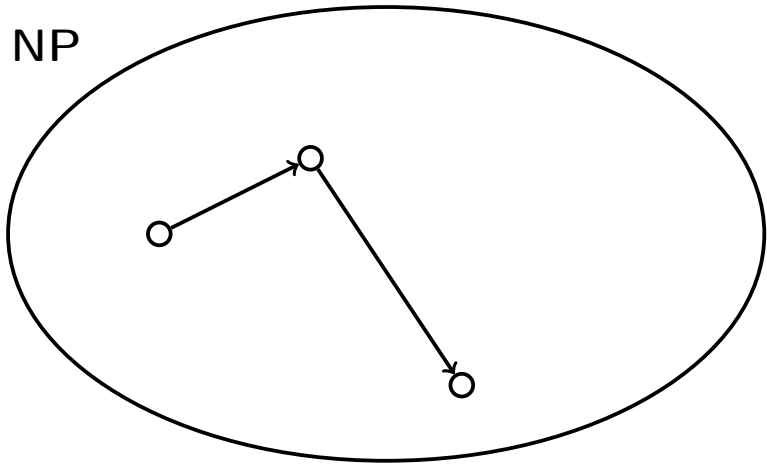
Pictorially

NP



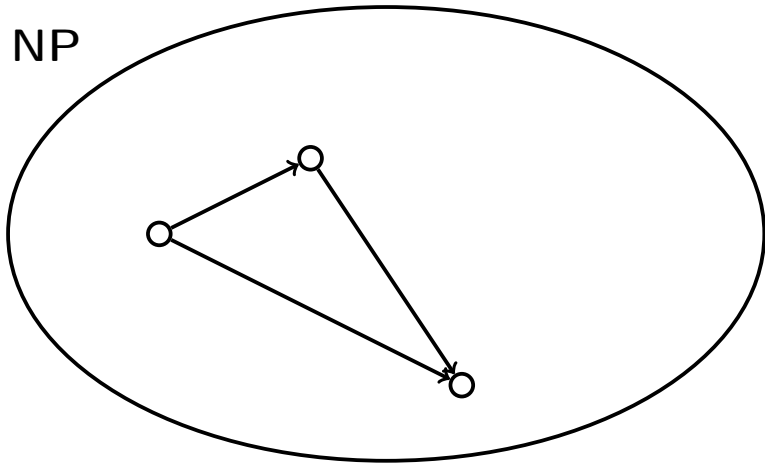
Pictorially

NP



Pictorially

NP



Showing **NP**-completeness

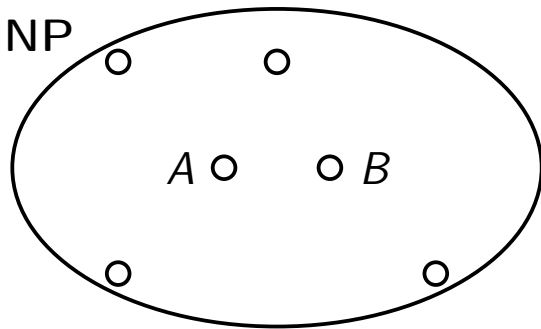
Corollary

If $A \rightarrow B$ and A is **NP**-complete, then so is B .

Showing **NP**-completeness

Corollary

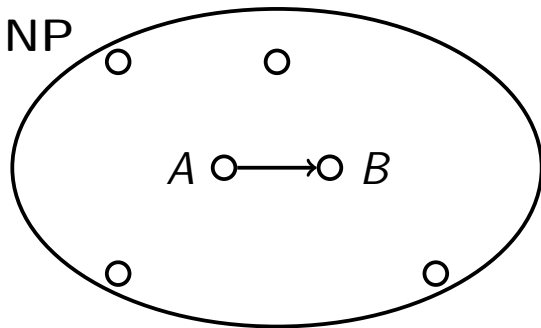
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Corollary

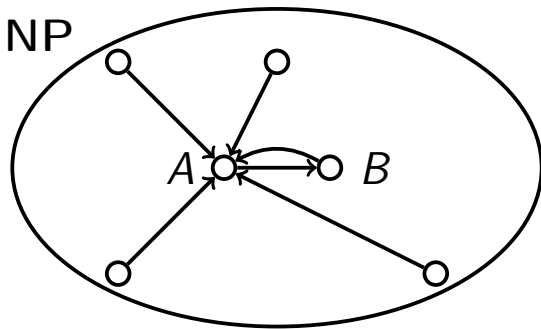
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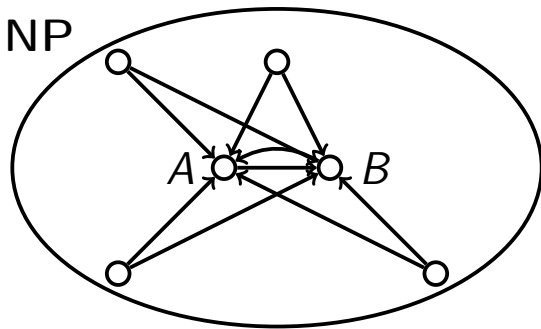
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Showing **NP**-completeness

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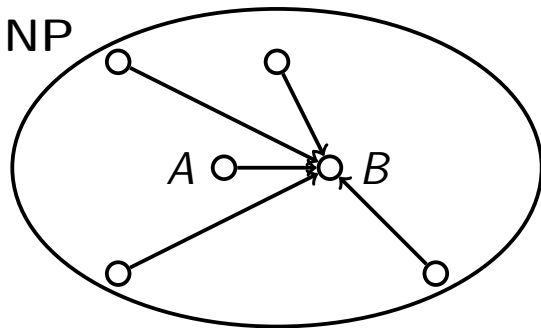
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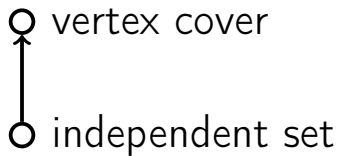
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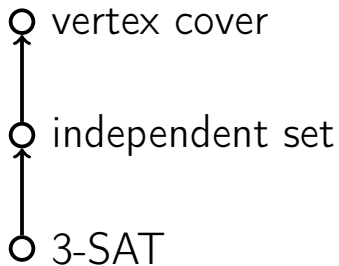
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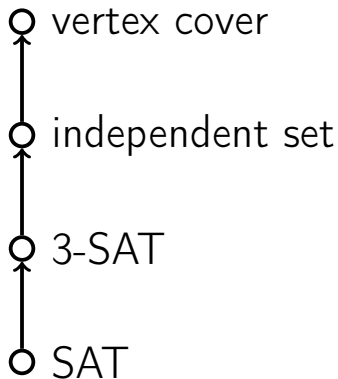
Plan



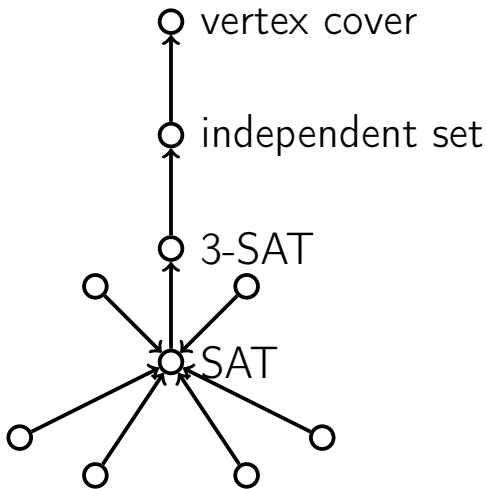
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Independent set

Input: A graph and a budget b .

Output: A subset of at least b vertices such that no two of them are adjacent.

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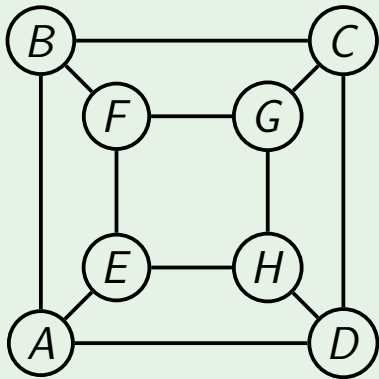
Output: A subset of at least b vertices such that no two of them are adjacent.

Vertex cover

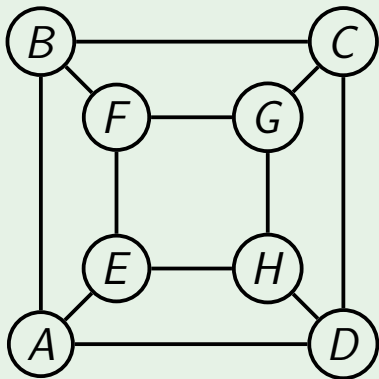
Input: A graph and a budget b .

Output: A subset of at most b vertices that touches every edge.

Example

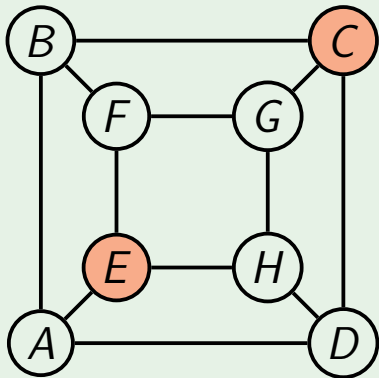


Example



Independent sets:

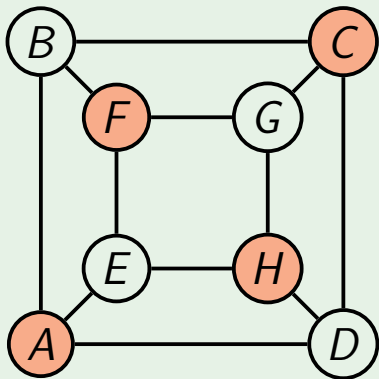
Example



Independent sets:

$\{E, C\}$

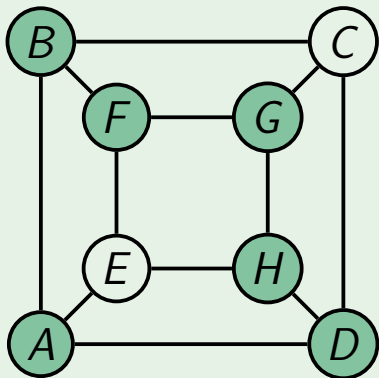
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Independent sets:

$\{E, C\}$ $\{A, C, F, H\}$

Example



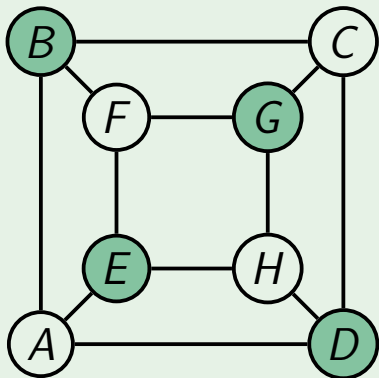
Independent sets:

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Vertex covers:

$\{A, B, D, F, G, H\}$

Example



Independent sets:

$\{E, C\}$ $\{A, C, F, H\}$

Vertex covers:

$\{A, B, D, F, G, H\}$

$\{B, D, E, G\}$

Lemma

I is an independent set of $G(V, E)$, if and only if $V - I$ is a vertex cover of G .

Proof

- \Rightarrow If I is an independent set, then there is no edge with both endpoints in I .
Hence $V - I$ touches every edge.
- \Leftarrow If $V - I$ touches every edge, then each edge has at least one endpoint in $V - I$.
Hence I is an independent set. \square

Reduction

Independent set \rightarrow vertex cover: to check whether $G(V, E)$ has an independent set of size at least b , check whether G has a vertex cover of size at most $|V| - b$:

- $f(G(V, E), b) = (G(V, E), |V| - b)$
- $h(S) = V - S$

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3-SAT

Input: Formula F in 3-CNF (a collection of clauses each having at most three literals).

Output: An assignment of Boolean values to the variables of F satisfying all clauses, if exists.

Goal

Design a polynomial time algorithm that, given a 3-CNF formula F , outputs a graph G and an integer b , such that:

F is satisfiable, if and only if G has an independent set of size at least b .

We need to find an assignment of Boolean values to variables, such that each clause contains at least one satisfied literal.

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Example

- Setting $x = 1, y = 1, z = 1$ satisfies a formula $(x \vee y \vee z)(x \vee \bar{y})(y \vee \bar{z})$.

We need to find an assignment of Boolean values to variables, such that each clause contains at least one satisfied literal.

Example

- Setting $x = 1, y = 1, z = 1$ satisfies a formula $(x \vee y \vee z)(x \vee \bar{y})(y \vee \bar{z})$.
- Setting $x = 1, y = 0, z = 0$ also satisfies it: $(x \vee y \vee z)(x \vee \bar{y})(y \vee \bar{z})$.

Alternatively, we need to select at least one literal from each clause, such that the set of selected literals is consistent: it does not contain a literal ℓ together with its negation $\bar{\ell}$.

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- Consistent: $\{x, x, \bar{z}\}$, $\{x, x, y\}$, $\{x, \bar{y}, \bar{z}\}$.

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- Consistent: $\{x, x, \bar{z}\}$, $\{x, x, y\}$, $\{x, \bar{y}, \bar{z}\}$.
- Inconsistent: $\{y, \bar{y}, \bar{z}\}$, $\{z, x, \bar{z}\}$.

Using Alternative Statement

$$(x \vee y \vee z)(x \vee \bar{y})(y \vee \bar{z})(z \vee \bar{x})(\bar{x} \vee \bar{y} \vee \bar{z})$$

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(y)

(\bar{y})

(\bar{z})

(\bar{x})

(\bar{y})

(x)

(z)

(x)

(y)

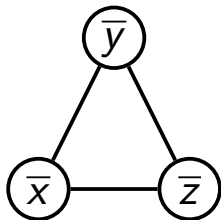
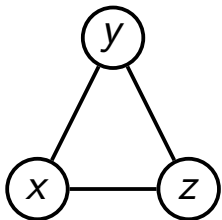
(z)

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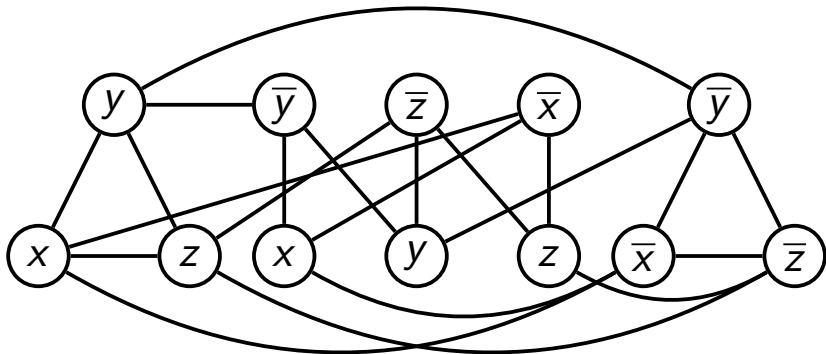
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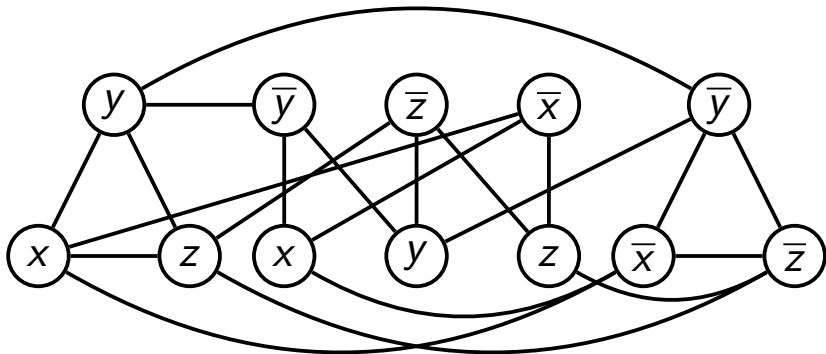
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the formula is satisfiable iff the resulting graph has independent set of size 5

Transforming an Instance

- For each clause of the input formula F , introduce three (or two, or one) vertices in G labeled with the literals of this clause. Join every two of them.

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- F is satisfiable if and only if G has independent set of size equal to the number of clauses in F .
- Transformation takes polynomial time.

Transforming a Solution

- Given a solution S for G , just read the labels of the vertices from S to get a satisfying assignment of F (takes polynomial time).

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- Given a solution S for G , just read the labels of the vertices from S to get a satisfying assignment of F (takes polynomial time).
- If there is no solution for G , then F is unsatisfiable: indeed, a satisfying assignment for F would give a required independent set in G .

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Goal

Transform a CNF formula into an equisatisfiable 3-CNF formula. That is, reduce a problem to its special case.

Transforming an Instance

- We need to get rid of clauses of length more than 3 in an input formula

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- Introduce a fresh variable y and replace C with the following two clauses:
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- The second clause is shorter than C
- Repeat while there is a long clause

Running time

The running time of the transformation is polynomial: at each iteration we replace a clause with a shorter clause and a 3-clause. Hence the total number of iterations is at most the total number of literals of the initial formula.

Correctness

Lemma

The formulas $F = (\ell_1 \vee \ell_2 \vee A) \dots$ and $F' = (\ell_1 \vee \ell_2 \vee y)(\bar{y} \vee A) \dots$ are equisatisfiable.

Proof

$$F = (\ell_1 \vee \ell_2 \vee A) \dots$$

$$F' = (\ell_1 \vee \ell_2 \vee y)(\bar{y} \vee A) \dots$$

\Rightarrow If either ℓ_1 or ℓ_2 is satisfied, set $y = 0$.
Otherwise A must be satisfied. Then set $y = 1$.

\Leftarrow If $(\ell_1 \vee \ell_2 \vee y)(\bar{y} \vee A)$ are satisfied, then
so is $(\ell_1 \vee \ell_2 \vee A)$. □

Transforming a Solution

Given a satisfying assignment for F' , just throw away the values of all new variables (y 's) to get a satisfying assignment of the initial formula.

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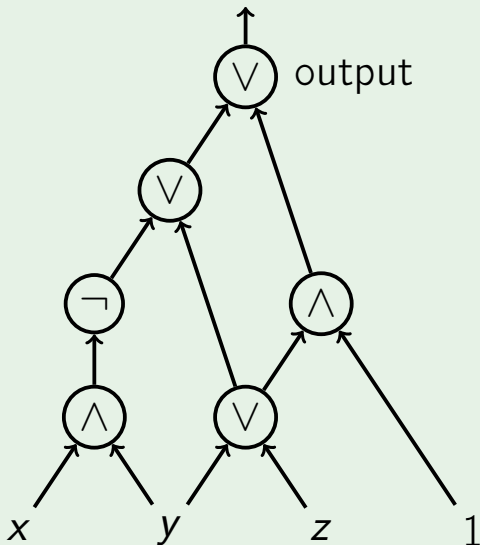
Show that **every** search problem reduces to SAT.

Goal

Show that **every** search problem reduces to SAT.

Instead, we show that any problem reduces to Circuit SAT problem, which, in turn, reduces to SAT.

Circuit



Definition

A **circuit** is a directed acyclic graph of in-degree at most 2. Nodes of in-degree 0 are called **inputs** and are marked by Boolean variables and constants. Nodes of in-degree 1 and 2 are called **gates**: gates of in-degree 1 are labeled with NOT, gates of in-degree 2 are labeled with AND or OR. One of the sinks is marked as **output**.

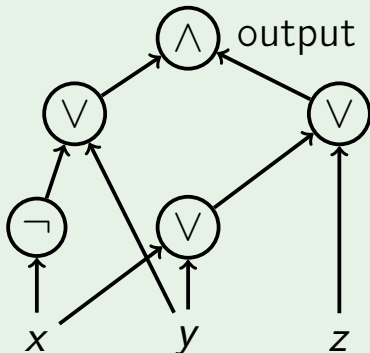
Circuit-SAT

Input: A circuit.

Output: An assignment of Boolean values to the input variables of the circuit that makes the output true.

SAT is a special case of Circuit-SAT as a CNF formula can be represented as a circuit:

Example: $(x \vee y \vee z)(y \vee \bar{x})$



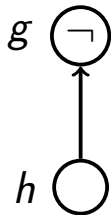
Circuit-SAT \rightarrow SAT

To reduce Circuit-SAT to SAT, we need to design a polynomial time algorithm that for a given circuit outputs a CNF formula which is satisfiable, if and only if the circuit is satisfiable

Idea

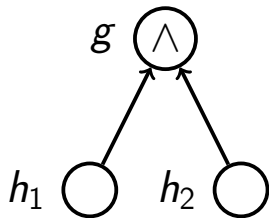
- Introduce a Boolean variable for each gate
- For each gate, write down a few clauses that describe the relationship between this gate and its direct predecessors

NOT Gates



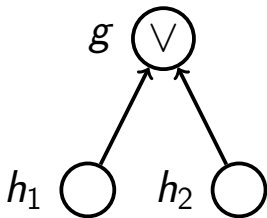
$$(h \vee g)(\bar{h} \vee \bar{g})$$

AND Gates



$$(h_1 \vee \bar{g})(h_2 \vee \bar{g})(\bar{h}_1 \vee \bar{h}_2 \vee g)$$

OR Gates



$$(\overline{h_1} \vee g)(\overline{h_2} \vee g)(h_1 \vee h_2 \vee \overline{g})$$

Output Gate

$$g \bigcirc \text{output} \quad (g)$$

- The resulting CNF formula is consistent with the initial circuit: in any satisfying assignment of the formula, the value of g is equal to the value of the gate labeled with g in the circuit

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- Therefore, the CNF formula is equisatisfiable to the circuit
- The reduction takes polynomial time

Goal

Reduce every search problem to Circuit-SAT.

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- We know that there exists an algorithm \mathcal{C} that takes an instance I of A and a candidate solution S and checks whether S is a solution for I in time polynomial in $|I|$

Goal

Reduce every search problem to Circuit-SAT.

- Let A be a search problem
- We know that there exists an algorithm \mathcal{C} that takes an instance I of A and a candidate solution S and checks whether S is a solution for I in time polynomial in $|I|$
- In particular, $|S|$ is polynomial in $|I|$

Turn an Algorithm into a Circuit

- Note that a computer is in fact a circuit (of constant size!) implemented on a chip

Turn an Algorithm into a Circuit

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- Each step of the algorithm $\mathcal{C}(I, S)$ is performed by this computer's circuit
- This gives a circuit of size polynomial in $|I|$ that has input bits for I and S and outputs whether S is a solution for I (a separate circuit for each input length)

Reduction

To solve an instance I of the problem A :

- take a circuit corresponding to $\mathcal{C}(I, \cdot)$

Reduction

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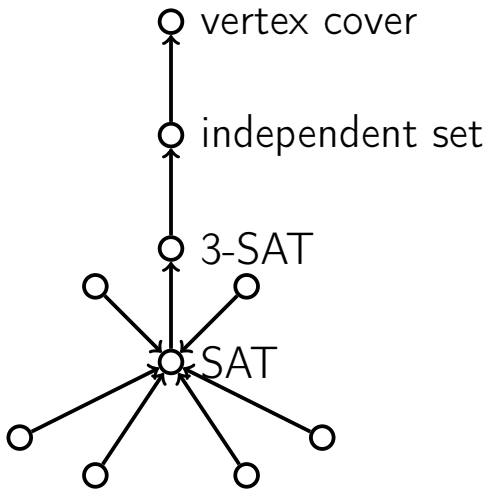
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Reduction

To solve an instance I of the problem A :

- take a circuit corresponding to $\mathcal{C}(I, \cdot)$
- the inputs to this circuit encode candidate solutions
- use a Circuit-SAT algorithm for this circuit to find a solution (if exists)

Summary



Outline

- 1 Reductions
- 2 Showing **NP**-completeness
- 3 Independent Set \rightarrow Vertex Cover
- 4 3-SAT \rightarrow Independent Set
- 5 SAT \rightarrow 3-SAT
- 6 All of **NP** \rightarrow SAT
- 7 Using SAT-solvers

Sudoku Puzzle

This part

A simple and efficient Sudoku solver

SAT: Theory and Practice

Theory: we have no algorithm checking the satisfiability of a CNF formula F with n variables in time $\text{poly}(|F|) \cdot 1.99^n$

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Practice: SAT-solvers routinely solve instances with thousands of variables

Solving Hard Problems in Practice

An easy way to solve a hard combinatorial problem in practice:

- Reduce the problem to SAT (many problems are reduced to SAT in a natural way)

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An easy way to solve a hard combinatorial problem in practice:

- Reduce the problem to SAT (many problems are reduced to SAT in a natural way)
- Use a SAT solver

Sudoku Puzzle

Goal: fill in with digits the partially completed 9×9 grid so that each row, each column, and each of the nine 3×3 subgrids contains all the digits from 1 to 9.

Example

Variables

There will be $9 \times 9 \times 9 = 729$ Boolean variables: for $1 \leq i, j, k \leq 9$, $x_{ijk} = 1$, if and only if the cell $[i, j]$ contains the digit k

Exactly One Is True

Clauses expressing the fact that exactly one of the literals ℓ_1, ℓ_2, ℓ_3 is equal to 1:

$$(\ell_1 \vee \ell_2 \vee \ell_3)(\bar{\ell}_1 \vee \bar{\ell}_2)(\bar{\ell}_1 \vee \bar{\ell}_3)(\bar{\ell}_2 \vee \bar{\ell}_3)$$

Constraints

- Cell $[i, j]$ contains exactly one digit:
 $\text{ExactlyOneOf}(x_{ij1}, x_{ij2}, \dots, x_{ij9})$

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- k appears exactly once in row i :
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- k appears exactly once in column j :
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- k appears exactly once in a 3×3 block:
 $\text{ExactlyOneOf}(x_{11k}, x_{12k}, \dots, x_{33k})$

Constraints

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- k appears exactly once in a 3×3 block:
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- $[i, j]$ already contains k : (x_{ijk})

Resulting Formula

State-of-the-art SAT-solvers find a satisfying assignment for the resulting formula in blink of an eye, though the corresponding search space has size about $2^{729} \approx 10^{220}$