Proof

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September 2021

1 Introduction

Let an arbitrary star $\mathcal{S}$ have a length ℓ and N points, with $N, \ell \in \mathbb{Z}_{++}$. The angle a graphics turtle would need to rotate is:

$$\theta = \frac{360^\circ}{N} \times 2, \tag{1}$$

which has to repeat for N times, after drawing a line segment of ℓ each time.

Let α be the measure of the circumferential angles formed by the vertices of the star $\mathcal{S}$ at the circumference of the inscribing circle. It is easy to show that the radius of the circle r splits $\triangle AXY$ into two identical triangles and therefore $\angle XAO = \alpha/2$. Due to the same reason, the interior angle of the N-polygon formed by the star $\angle XOY = \frac{\theta}{4}$.

It should be easy to find out that $\alpha = 180^\circ - \theta$ since both angles fall on a straight line.

We inspect $\triangle AOB$ and $\triangle ADB$ knowing that $\angle ADB \coloneqq \alpha$. Therefore, the central angle $\angle AOB$ sharing the same arc with the circumferential angle has the measure of 2α.

$\triangle AOB$ is an isosceles triangle with the base angle measure of γ. Therefore, the measure of $\angle AOB$ is $\zeta \coloneqq 180^\circ - 2\gamma$. We can use the law of sines and write:

$$\frac{r}{\sin\gamma} = \frac{\overline{\mathrm{AB}}}{\sin\zeta}, \tag{2}$$

which makes it easy to write r as

$$\begin{aligned} r &= \overline{\mathrm{AB}}\frac{\sin\gamma}{\sin(180^\circ - 2\gamma)} \\ &= \overline{\mathrm{AB}}\frac{\sin\gamma}{\sin 2\gamma} \\ &= \overline{\mathrm{AB}}\frac{1}{\cos\gamma} \end{aligned} \tag{3}$$

We can write γ in terms of the circumferential angle α by simply stating

$$\gamma = \frac{1}{2}(180^\circ - 2\alpha) = 90^\circ - \alpha. \tag{4}$$

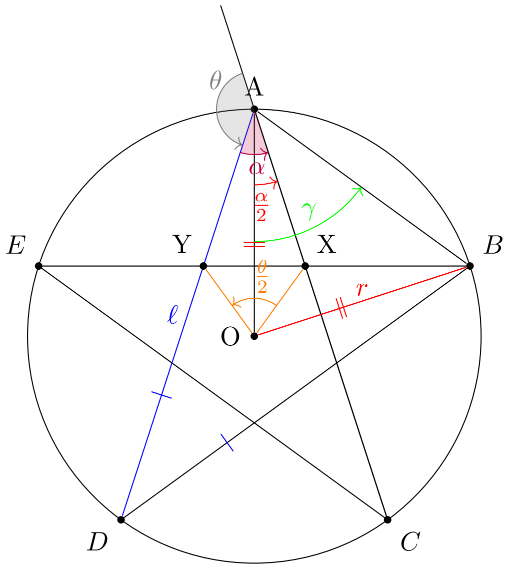

Figure 1: Solving the triangles

Now we can write r as follows

$$r = \overline{\mathrm{AB}}\frac{1}{\sin\alpha}. \tag{5}$$

We inspect $\triangle ABD$, which again is an isosceles triangle. The base angles have the measure of $90° - \alpha/2$ each. The law of sines enables us to write:

$$\frac{\ell}{\sin(90° - \alpha/2)} = \frac{\overline{\mathrm{AB}}}{\sin\alpha}. \tag{6}$$

Here we can compute:

$$\overline{\mathrm{AB}} = \ell\frac{\sin\alpha}{\cos\alpha/2}, \tag{7}$$

where the use of the sine of the double angle trigonometry identity allows us to write $\overline{\mathrm{AB}} = 2\ell\sin\alpha/2$. What is left now is finding substituting $\overline{\mathrm{AB}}$ in (5) in terms of θ and ℓ:

$$r = \frac{2\ell\sin\alpha/2}{\sin\alpha} = \ell\cos\alpha/2 \tag{8}$$

$$\therefore r = \frac{\ell}{2}\frac{1}{\sin\theta/2} = \frac{\ell}{2}\csc\left(\frac{360°}{N}\right) \qquad \blacksquare \tag{9}$$