Proof

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1 Introduction

Let an arbitrary star $\mathcal{S}$ have a length ℓ and N points, with $N, \ell \in \mathbb{Z}_{++}$. To draw this star, as shown in g. 1, the angle that the graphics turtle would need to rotate in each of the N line segments of length ℓ is:

$$\theta = \frac{360^\circ}{N} \times 2. \tag{1}$$

Let α be the measure of the circumferential angles formed by the vertices of the star $\mathcal{S}$ at the circumference of the inscribing circle. It is easy to show that the radius of the circle r splits $\triangle AXY$ into two identical triangles and therefore $\angle XAO = \alpha/2$. Due to the same reason, the interior angle of the N-polygon formed by the star $\angle XOY = \frac{\theta}{4}$.

It should be easy to find out that $\alpha = 180^\circ - \theta$ since both angles fall on a straight line. These two angles are *supplementary*.

We inspect $\triangle AOB$ and $\triangle ADB$ knowing that $\angle ADB \coloneqq \alpha$ due to symmetry. Therefore, the central angle $\angle BOA$ sharing the same arc with the circumferential angle has the measure of 2α.

$\triangle AOB$ is an isosceles triangle with the base angle measure of γ. Therefore, the measure of $\angle BOA$ is $\zeta \coloneqq 180^\circ - 2\gamma$. We can use the law of sines and write:

$$\frac{r}{\sin\gamma} = \frac{\overline{\mathrm{AB}}}{\sin\zeta}, \tag{2}$$

which makes it easy to write r as

$$\begin{aligned} r &= \overline{\mathrm{AB}}\frac{\sin\gamma}{\sin(180^\circ - 2\gamma)} \\ &= \overline{\mathrm{AB}}\frac{\sin\gamma}{\sin 2\gamma} \\ &= \overline{\mathrm{AB}}\frac{1}{2\cos\gamma}, \end{aligned} \tag{3}$$

where the last step comes from the trigonometric identity of the sine of a double-angle. We can write γ in terms of the circumferential angle α since $\zeta = 2\alpha =$

$180° - 2\gamma$ (all from $\triangle AOB$). Thus:

$$\gamma = \frac{1}{2}(180° - 2\alpha) = 90° - \alpha. \tag{4}$$

Now we can write r using the cosine of the complementary angle as follows

$$r = \frac{\overline{\mathrm{AB}}}{2\sin\alpha}. \tag{5}$$

We inspect $\triangle ABD$, which again is an isosceles triangle. The base angles have the measure of $\gamma + \alpha/2 = 90° - \alpha/2$ each. The law of sines enables us to write:

$$\frac{\ell}{\sin(90° - \alpha/2)} = \frac{\overline{\mathrm{AB}}}{\sin\alpha}. \tag{6}$$

Here we can compute:

$$\overline{\mathrm{AB}} = \ell\frac{\sin\alpha}{\cos\alpha/2} \tag{7}$$

using the trigonometric identity of the sine of a double-angle again. What is left now is finding r in terms of N and ℓ, which is obtained by substituting $\overline{\mathrm{AB}}$ of (7) into (5):

$$r = \frac{\ell}{2\cos\alpha/2} = \frac{\ell}{2\sin\frac{360°}{N}} \tag{8}$$

$$\therefore r = \frac{\ell}{2}\csc\left(\frac{360°}{N}\right) \qquad \blacksquare \tag{9}$$

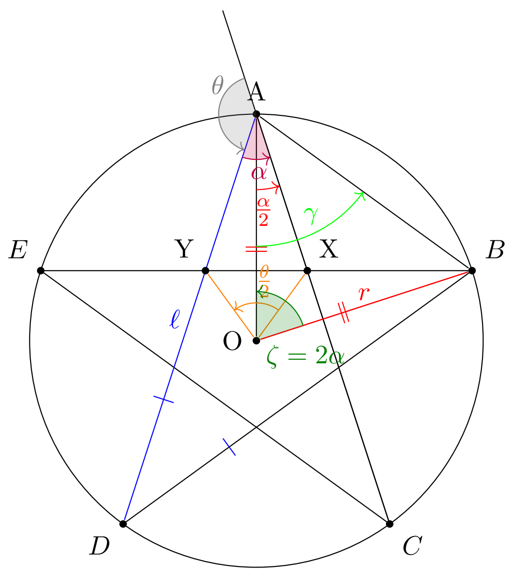

Figure 1: Solving the triangles