

# Ornstein-Uhlenbeck and renormalization semigroups

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## Abstract

The Ornstein-Uhlenbeck semigroup combines Gaussian diffusion with the flow of a linear vector field. In infinite dimensional settings there can be non-Gaussian invariant measures. This gives a context for one version of the renormalization group. The adjoint of the Ornstein-Uhlenbeck semigroup with respect to an invariant measure need not be an Ornstein-Uhlenbeck semigroup. This adjoint is the appropriate semigroup to analyze the local stability of the invariant measure under the renormalization group.

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## 1 Introduction

The Ornstein-Uhlenbeck semigroup is a semigroup of linear operators defined by explicit formulas. The action of the semigroup on measures consists of convolution by a Gaussian with a certain covariance  $C_t$  followed by the flow of a linear vector field  $\exp(-tA)$ . One expects that if such a semigroup has an invariant measure, then it will be unique, and it will be Gaussian. Indeed this happens in all non-degenerate finite dimensional situations. However in infinite dimensional situations it is considerably easier to have several invariant measures. The structure of such invariant measures has been clarified in work of Bogachev, Röckner, and Schmuland [2]. The first part of this paper reviews this theory.

The renormalization group gives a profound analysis of certain phenomena in statistical mechanics and quantum field theory. In certain versions it may be realized as a combination of a scaling with a Gaussian convolution. In this context it is a special case of the Ornstein-Uhlenbeck semigroup. In this paper one goal is to characterize this special case abstractly.

The version of the renormalization group that is considered here arises in considering the long distance behavior of a random field. Physically, one wants to integrate the fluctuations over a distance scale from 1 to  $e^t$ , with  $t > 0$ , and then scale distance by a factor  $e^{-t}$ . What remains is the effective fluctuations at distances greater than  $e^t$ , scaled so as to appear as taking place at distances greater than 1. In the Ornstein-Uhlenbeck setting this idea is implemented instead by first scaling distance by a factor  $e^{-t}$  and then integrating the fluctuations over a distance scale from  $e^{-t}$  to 1. The scaling acts on a function  $\phi$  by  $(\exp(-tA)\phi)(x) = \phi(e^t x)e^{(\nu/2-1)t}$ . Here  $\nu$  is the dimension of space. This scaling is chosen so as to leave the Gaussian measure with covariance

$(-\Delta)^{-1}$  invariant. The integration convolves the measure on the space of functions  $\phi$  by a Gaussian measure with covariance  $(-\Delta)^{-1}(\exp(e^{-2t}\Delta) - \exp(\Delta))$ . In the wave number representation the second factor  $\exp(e^{-2t}\Delta) - \exp(\Delta)$  becomes multiplication by  $\exp(-e^{-2t}k^2) - \exp(-k^2)$ . Thus it is mainly concentrated on wave numbers in the shell from 1 to  $e^t$ . This corresponds to distances in the shell from  $e^{-t}$  to 1.

This view of the renormalization group illuminates some of its properties. Fixed points of the renormalization group are invariant measures for the Ornstein-Uhlenbeck process. Unfortunately, this approach does not immediately provide a parameterization of translation invariant measures, and most of the analysis is done in practice with related nonlinear equations. However it is shown that the adjoint of the generator of the Ornstein-Uhlenbeck semigroup with respect to the invariant measure is the linearization of this nonlinear flow at this fixed point. Finally, the last part of the paper presents other infinite dimensional examples for which it is easy to exhibit non-Gaussian invariant measures.

## 2 Measures on Hilbert space

In this section we establish notation and basic facts about measures on Hilbert spaces [8, 1]. Let  $E$  be a separable real Hilbert space. We consider probability measures  $\mu$  defined on the Borel  $\sigma$ -algebra of subsets of  $E$ . Each such probability measure is automatically a Radon measure. We may consider it as an element of the dual space of the space of bounded continuous functions on  $E$ . In particular, we may speak of the weak topology on the space of such measures. Then a sequence  $\mu_n$  of probability measures converges weakly to a probability measure  $\mu$  if for every bounded continuous function  $F$  on  $E$  the integrals  $\int F(\phi) d\mu_n(\phi) \rightarrow \int F(\phi) d\mu(\phi)$  as  $n \rightarrow \infty$ .

It is convenient to describe a probability measure by its Fourier transform (characteristic function). Let  $E^*$  be the dual space of  $E$ . The Fourier transform is the function on  $E^*$  defined by

$$\hat{\mu}(u) = \int \exp(i\langle u, \phi \rangle) d\mu(\phi). \quad (2.1)$$

The Fourier transform determines the measure.

When the appropriate integrals converge the mean  $m$  in  $E$  and the covariance  $C : E^* \rightarrow E$  are defined by

$$\langle u, m \rangle = \int \langle u, \phi \rangle d\mu(\phi) \quad (2.2)$$

and

$$\langle u, Cv \rangle = \int \langle u, \phi - m \rangle \langle v, \phi - m \rangle d\mu(\phi). \quad (2.3)$$

Let  $u_j$  be an orthonormal basis for  $E^*$ . The square of the radial distance from the mean is

$$\|\phi - m\|_E^2 = \sum_j \langle u_j, \phi - m \rangle^2. \quad (2.4)$$

This integral of this is

$$\int \|\phi - m\|^2 d\mu(\phi) = \sum_j \langle u_j, Cu_j \rangle = \text{tr}(C). \quad (2.5)$$

So when this is finite, the covariance operator  $C$  from  $E^*$  to  $E$  is a positive trace class operator. This calculation shows that a trace condition on covariance operators is natural.

The Sazonov topology on  $E^*$  is the coarsest topology on  $E^*$  such that  $\langle u, Cu \rangle$  is continuous in  $u$  for every  $C : E^* \rightarrow E$  that is a positive trace class operator. Sazonov's theorem says that a function  $\chi$  on  $E^*$  is a Fourier transform of a probability measure if and only if it is positive definite, continuous in the Sazonov topology, and satisfies  $\chi(0) = 1$ .

The special case of Gaussian probability measures is particularly important. Let  $E$  be a separable Hilbert space, and let  $E^*$  be its dual space. Let  $m$  be in  $E$  and let  $C$  be a positive and bounded linear map from  $E^*$  to  $E$ . Then a Gaussian probability measure  $\mu$  with mean  $m$  and covariance  $C$  is a measure such that for each  $u$  in  $E^*$  the Fourier transform is

$$\hat{\mu}(u) = \int \exp(i\langle u, \phi \rangle) d\mu(\phi) = \exp(i\langle u, m \rangle) \exp(-\frac{1}{2}\langle u, Cu \rangle). \quad (2.6)$$

It follows automatically that  $C : E^* \rightarrow E$  is a positive trace class operator. This is the necessary and sufficient condition for  $C$  to be the covariance of a Gaussian measure. For Gaussian measures it is particularly convenient to characterize weak convergence of measures. The condition is that the means  $m_n \rightarrow m$  in  $E$  and the covariances  $C_n \rightarrow C$  in the trace norm as operators from  $E^*$  to  $E$ .

Consider a Gaussian measure  $\mu$  with mean zero and with covariance  $C$  with trivial null space. Let  $H(C)^*$  be the completion of the Hilbert space with the norm  $\|u\|_{H(C)^*}^2 = \langle u, Cu \rangle$ . This is a natural space of Gaussian random variables with finite covariance. Let  $H(C)$  be the dual space of  $H(C)^*$ . Then  $H(C)$  has the norm  $\|\phi\|_{H(C)}^2 = \langle \phi, C^{-1}\phi \rangle$ . Furthermore,  $C : H(C)^* \rightarrow H(C)$  is an isomorphism. Since  $E^* \subset H(C)^*$  we have  $H(C) \subset E$ . In general  $H(C)$  will be a subset of  $E$  of measure zero.

Now let  $T : H(C) \rightarrow H(C)$  be a bounded operator with bounded adjoint  $T^* : H(C)^* \rightarrow H(C)^*$ . It is known [1] that  $T$  extends to a measurable map  $T : E \rightarrow E$  and that the measure  $T[\mu]$  is Gaussian with mean zero and covariance  $TCT^*$ . It is worth noting that these operators also have Hilbert space adjoints. In fact  $CT^*C^{-1} : H(C) \rightarrow H(C)$  is a bounded operator with bounded adjoint  $C^{-1}TC : H(C)^* \rightarrow H(C)^*$ . Thus the measure  $CT^*C^{-1}[\mu]$  is Gaussian with mean zero and covariance  $CT^*C^{-1}TC$ .

### 3 The Ornstein-Uhlenbeck semigroup

This section reviews information about Ornstein-Uhlenbeck semigroups in a framework close to that of Bogachev, Röckner, Schmuland [2]. An Ornstein-Uhlenbeck semigroup (or Mehler semigroup) is a semigroup given by convolution by a Gaussian and rescaling. The setting is a separable real Hilbert space  $E$  with dual  $E^*$ . The first datum is a bounded operator  $Q$  from  $E^*$  to  $E$ . We shall assume that it is a positive trace class operator. Thus it is a covariance operator.

The second datum is a closed operator  $A$  acting in  $E$  such that  $-A$  is the generator of a strongly continuous semigroup  $\exp(-tA)$ , for  $t \geq 0$  acting in  $E$ . It is also assumed that the  $-A^*$  is the generator of the adjoint semigroup  $\exp(-tA^*) = \exp(-tA)^*$  acting in  $E^*$ .

Define the family of covariance operators

$$C_t = \int_0^t \exp(-sA)Q \exp(-sA^*) ds \quad (3.1)$$

These are also to be positive trace class operators. The  $C_t$  are covariance operators of mean zero Gaussian probability measures  $\mu_t$  supported on  $E$ .

The *Ornstein-Uhlenbeck semigroup*  $\exp(-tL)$  for  $t \geq 0$  is defined for bounded continuous functions  $F$  on  $E$  by

$$(\exp(-tL)F)(\phi) = \int F(\exp(-At)\phi + \chi) d\mu_t(\chi). \quad (3.2)$$

Thus it is equivalent to convolution followed by rescaling. The special form of the covariances ensures that this is indeed a one-parameter semigroup of operators. Sometimes it is convenient to describe the action of the semigroup in terms of exponential functions  $e_u(\phi) = \exp(i\langle u, \phi \rangle)$ . Thus

$$\exp(-tL)e_u = e_{\exp(-tA^*)u} \exp\left(-\frac{1}{2}\langle u, C_t u \rangle\right). \quad (3.3)$$

We can examine the action of the *dual semigroup*  $\exp(-tL^\dagger)$  on measures. It is easy to see that

$$\int \int F(\exp(-At)\phi + \chi) d\mu_t(\chi) d\nu(\phi) = \int \int F(\psi + \chi) d\exp(-At)[\nu](\psi) d\mu_t(\chi). \quad (3.4)$$

Here  $\exp(-At)[\nu]$  denotes the image of  $\nu$  under the map  $\exp(-At)$ . This says that  $\exp(-tL^\dagger)$  is a rescaling followed by a convolution by  $\mu_t$ :

$$\exp(-tL^\dagger)\nu = \exp(-At)[\nu] * \mu_t. \quad (3.5)$$

The evolution  $\nu_t = \exp(-tL^\dagger)\nu$  may also be described in terms of Fourier transforms. Thus

$$\hat{\nu}_t(u) = \hat{\nu}(\exp(-A^*t)u) \hat{\mu}_t(u). \quad (3.6)$$

Make the supplementary assumption that

$$C = \int_0^\infty \exp(-sA)Q \exp(-sA^*) ds \quad (3.7)$$

converges in trace norm. Then it is easy to see that the Gaussian measure  $\mu$  with covariance  $C$  is an invariant measure for the dual semigroup. In this case the covariance operators  $C_t$  have the simple explicit representation

$$C_t = C - \exp(-tA)C \exp(-tA^*). \quad (3.8)$$

Furthermore, by differentiation

$$AC + CA^* = Q. \quad (3.9)$$

One very special situation is *detailed balance*, which is a formulation of time reversibility. The criterion for this is that  $AQ = QA^*$ . Then  $C$  has a much more explicit form. In fact,  $AC = CA^* = \frac{1}{2}Q$ . However we shall not confine ourselves to this case.

Let  $\mu$  be the mean zero Gaussian measure with covariance  $C$ . Then  $\mu$  is a fixed point for the dual Ornstein-Uhlenbeck semigroup. Suppose that  $A$  is such that there is a measure  $\sigma$  invariant under the flow, so that  $\exp(-tA)[\sigma] = \sigma$ . Then  $\nu = \mu * \sigma$  is another fixed point for this semigroup. It is possible that  $\sigma$  and  $\nu$  are not Gaussian.

**Theorem 3.1** (Bogachev, Röckner, Schmuland) *Consider a dual Ornstein-Uhlenbeck semigroup with Gaussian invariant measure  $\mu$  obtained as the limit of the  $\mu_t$ . The measure  $\nu$  is a fixed point of the dual Ornstein-Uhlenbeck semigroup if and only if  $\nu$  is a convolution  $\nu = \mu * \sigma$ , where  $\sigma$  is invariant under the flow  $\exp(-tA)$ .*

The condition that  $\nu$  be a fixed point may be written in terms of Fourier transforms. It is

$$\hat{\nu}(u) = \hat{\nu}(\exp(-A^*t)u)\hat{\mu}_t(u). \quad (3.10)$$

Since the limit of  $\hat{\mu}_t(u)$  is  $\hat{\mu}(u)$ , the limit  $\chi(u)$  of  $\hat{\nu}(\exp(-A^*t)u)$  must also exist. In [2] it is shown that from the Sazonov theorem it follows that  $\chi(u) = \hat{\sigma}(u)$  is the Fourier transform of a measure  $\sigma$ . Thus  $\hat{\nu} = \hat{\sigma}\hat{\mu}$  and so  $\nu = \sigma * \mu$ .

**Theorem 3.2** Consider a dual Ornstein-Uhlenbeck semigroup with Gaussian invariant measure  $\mu$  obtained as the limit of the  $\mu_t$ . Let  $\bar{\sigma}$  be a probability measure such that under the flow  $\exp(-tA)[\bar{\sigma}] \rightarrow \sigma$  as  $t \rightarrow \infty$ . Let  $\bar{\mu}$  be a probability measure such that under the dual Ornstein-Uhlenbeck semigroup  $\exp(-tA)[\bar{\mu}] * \mu_t \rightarrow \mu$  as  $t \rightarrow \infty$ . Let  $\bar{\nu} = \bar{\sigma} * \bar{\mu}$ . Then under the dual Ornstein-Uhlenbeck semigroup  $\nu_t = \exp(-tA)[\bar{\nu}] * \mu_t \rightarrow \nu = \sigma * \mu$  as  $t \rightarrow \infty$ .

This is because  $\nu_t = \exp(-tA)[\bar{\sigma} * \bar{\mu}] * \mu_t = \exp(-tA)[\bar{\sigma}] * \exp(-tA)[\bar{\mu}] * \mu_t$ . The result follows from the continuity of convolution in the weak topology [8].

The action of the process on Gaussian measures is to send the measure with mean  $m$  and covariance  $G$  to the measure with mean  $\exp(-tA)m$  and covariance  $\exp(-tA)G\exp(-tA^*) + C_t$ . A fixed point is given by a mean  $m$  with  $\exp(-tA)m = m$  and a covariance  $G$  such that  $\exp(tA)G\exp(-tA^*) + C_t = G$ . Such a covariance is obtained by taking a Gaussian measure with covariance  $S$  such that  $\exp(-tA)S\exp(-tA^*) = S$ . Then  $G = C + S$  is a solution. The infinitesimal forms of these equations are  $AS + SA^* = 0$  and  $AG + GA^* = Q$ .

There are interesting Hilbert spaces associated with these covariances. Let  $G$  be the non-degenerate covariance of an invariant measure. Then  $\langle u, Gv \rangle$  is an inner product on the Hilbert space  $H(G)^*$ . Since  $G - \exp(-tA)G\exp(-tA^*) \geq 0$ , the semigroup  $\exp(-tA^*)$  is a contraction semigroup on this Hilbert space. If  $S$  is the non-degenerate covariance of a Gaussian measure that is invariant under the flow, then there is another Hilbert space  $H(S)^*$  associated with this covariance. Since  $\exp(-tA)S\exp(-tA^*) = S$ , the semigroup  $\exp(-tA^*)$  is a semigroup of isometries acting on this Hilbert space. Suppose that  $\exp(-tA^*)$  is actually a group of unitary operators acting in the Hilbert space  $H_S^*$ . Then  $A^*$  is a skew-adjoint operator acting in  $H_S^*$ . Its Hilbert space adjoint is  $S^{-1}AS = -A^*$ .

**Theorem 3.3** Let  $S$  be a non-degenerate positive quadratic form such that  $\exp(-tA)S\exp(-tA^*) = S$ . Suppose that the  $\exp(-tA^*)$  are unitary in  $H_S^*$ , so that  $A^*$  is a skew-adjoint operator acting in  $H_S^*$ . Let  $Q$  be a non-degenerate positive quadratic form that defines a bounded operator from  $H(S)^*$  to  $H(S)$  such that

$$C = \int_0^\infty \exp(-sA)Q\exp(-sA^*) ds \quad (3.11)$$

exists and is a bounded operator for  $H(S)^*$  to  $H(S)$ . Then  $A^*$  has absolutely continuous spectrum as an operator in  $H_S^*$ .

Proof: The operator  $S^{-1}Q : H(S)^* \rightarrow H(S)^*$  is bounded. Write the equation as

$$S^{-1}C = \int_0^\infty S^{-1}\exp(-sA)Q\exp(-sA^*) ds = \int_0^\infty \exp(sA^*)S^{-1}Q\exp(-sA^*) ds. \quad (3.12)$$

Then  $S^{-1}Q$  is a positive operator in  $H_S^*$  and  $S^{-1}C$  is bounded in  $H_S^*$ . The result is then a well known fact in quantum scattering theory [7].

The Ornstein-Uhlenbeck semigroup  $\exp(-tL)$  sends positive functions to positive functions and sends 1 to 1. Hence if  $\nu$  is an invariant probability measure, the semigroup is a contraction on  $L^1(E, \mu)$  and on  $L^\infty(E, \mu)$ . By interpolation, it is a contraction on  $L^2(E, \mu)$ .

In the case of a Gaussian fixed point  $\nu$ , there is a spectral theory of the operator  $\exp(-tL)$  acting on  $L^2(E, \nu)$ , even in the non-selfadjoint case [3]. Let  $e_u(\phi) = \exp(i\langle u, \phi \rangle)$  and let  $h_u = e_u \exp(\frac{1}{2}\langle u, Gu \rangle)$ . Since  $C_t = G - \exp(-tA)G \exp(-tA^*)$ , the action of the semigroup is

$$\exp(-tL)h_u = h_{\exp(-tA^*)u}. \quad (3.13)$$

Define the Wick map by

$$:e_u := h_u. \quad (3.14)$$

As is well known, this sends monomials  $u_1 \cdots u_m$  of degree  $m$  into Wick polynomials :  $u_1 \cdots u_m$  : of degree  $m$  that are orthogonal to all polynomials of lower degree. (These are closely related to Hermite polynomials.) The action of the Ornstein-Uhlenbeck semigroup on such a polynomial is

$$\exp(-tL) : u_1 \cdots u_m : = : \exp(-tA^*)u_1 \cdots \exp(-tA^*)u_m :. \quad (3.15)$$

The closed linear span of the Wick polynomials of degree  $m$  is a subspace  $M_m$  of  $L^2(E, \nu)$ . The entire space  $L^2(E, \nu)$  is the direct sum of the subspaces of this form as  $m = 0, 1, 2, 3, \dots$ . This is the Fock space decomposition. The action of the Ornstein-Uhlenbeck semigroup  $\exp(-tL)$  is determined on each invariant subspace  $M_n$  by the natural action of  $\exp(-tA^*)$  on symmetric tensors of rank  $m$  in this subspace.

## 4 The generator

The discussion of generators is delicate, because of domain questions. We consider generators defined on smooth cylindrical functions  $F$  on  $E$ . These are functions of the form  $F = f(u_1, \dots, u_k)$  where  $f$  is a smooth function on  $\mathbf{R}^k$  with bounded derivatives of all orders and  $u_1, \dots, u_k$  each belong to  $E^*$ . Then the differential  $\nabla F = \sum_j \partial f / \partial x_j(u_1, \dots, u_k)u_j$  is a function on  $E$  with values in  $E^*$ .

The Ornstein-Uhlenbeck semigroup has a generator given by the partial differential operator  $-L$  described by

$$-LF = \frac{1}{2} \nabla \cdot Q \nabla F - A\phi \cdot \nabla F. \quad (4.1)$$

The dot is a reminder of the pairing between  $E$  and  $E^*$ . The first term involves the divergence of the vector field obtained by applying  $Q$  to the differential. The second term requires further comment. The semigroup  $\exp(-tA)$  acting in  $E$  has a generator  $-A$ , and the semigroup  $\exp(-tA)^*$  acting in  $E^*$  has generator  $-A^*$ . The function  $F$  must be a smooth cylindrical function based on elements of the domain of  $A^*$ . Then  $A^* \nabla F$  is a function on  $E$  with values in  $E^*$ . We consider  $\phi$  as the identity from  $E$  to  $E$ . So the second term is defined to be  $-\phi \cdot A^* \nabla F$ , a scalar function on  $E$ .

In infinite dimensions there is no Lebesgue measure, and the density of a measure with respect to Lebesgue measure is not defined. Nevertheless, it is possible to write formal equations for the density in analogy with the finite dimensional situation. The adjoint action of the semigroup on densities is given by

$$-L^\dagger \rho = \frac{1}{2} \nabla \cdot Q \nabla \rho + \nabla \cdot (A\phi\rho). \quad (4.2)$$

Set  $\rho_t = \exp(-tL^\dagger)\rho_0$ . The partial differential equation is

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \nabla \cdot Q \nabla \rho + \nabla \cdot (A\phi\rho). \quad (4.3)$$

Set  $\rho_t = 1/Z_t \exp(-V_t)$ . Then

$$\frac{\partial V}{\partial t} = \frac{1}{2} \nabla \cdot Q \nabla V + A\phi \cdot V - \frac{1}{2} \nabla V \cdot Q \nabla V. \quad (4.4)$$

This is a non-linear partial differential equation, and indeed the renormalization group is often viewed in non-linear form [4]. It is not a rigorous equation, as the density does not exist. However the logarithmic derivative of the density  $-\nabla V$  has a better hope of existence, at least in some cases. There is a corresponding equation for this quantity that deserves investigation.

If  $V^*$  is a fixed point, then the linearization about this fixed point is

$$\frac{\partial V}{\partial t} = \frac{1}{2} \nabla \cdot Q \nabla V + A\phi \cdot \nabla V - \nabla V^* \cdot Q \nabla V. \quad (4.5)$$

This is the linearized renormalization group. Unless the fixed point is Gaussian, this will not be an Ornstein-Uhlenbeck semigroup. However, a rigorous relation to the Ornstein-Uhlenbeck semigroup will appear in the next section.

## 5 The adjoint semigroup

This section treats the adjoint semigroup of an Ornstein-Uhlenbeck semigroup with respect to an invariant probability measure. If the invariant measure is not Gaussian, then the adjoint semigroup will be a diffusion semigroup that is not an Ornstein-Uhlenbeck semigroup.

Let  $\nu$  be an invariant measure for the semigroup  $\exp(tL^\dagger)$ . Then  $\exp(-tL)$  acts in the Hilbert space  $L^2(E, \nu)$ . The *adjoint semigroup*  $\exp(-tL^*)$  acts in the same Hilbert space. It is defined by

$$\int F \exp(-tL)G d\nu = \int \exp(-tL^*)F G d\nu. \quad (5.1)$$

The significance of the adjoint semigroup is that it describes the stability of the invariant measure to multiplicative perturbations. A multiplicative perturbation is given by multiplying the invariant measure  $\nu$  by a function  $F$ . The equation may also be written

$$\exp(-tL^\dagger)(F\nu) = (\exp(-tL^*)F)\nu. \quad (5.2)$$

**Proposition 5.1** *Let  $\exp(-tL)$  be an Ornstein-Uhlenbeck semigroup give by the covariance  $Q$  and flow  $\exp(-tA)$ . Let  $\nu$  be a Gaussian invariant measure with covariance  $G$ . Then the adjoint semigroup  $\exp(-tL^*)$  in  $L^2(E, \nu)$  has covariance  $Q$  and flow  $G \exp(-tA^*)G^{-1}$ .*

Proof: From  $C_t = G - \exp(-tA)G \exp(-tA^*) \geq 0$  we know that  $\exp(-tA^*)$  is a contraction semigroup on  $H(G)^*$  and  $\exp(-tA)$  is a contraction semigroup on  $H(G)$ . Since  $\exp(-tA^*)$  is a contraction in  $H(G)^*$  and  $G$  is an isomorphism from  $H(G)^*$  to  $H(G)$ , it follows that  $G \exp(tA^*)G^{-1}$  is a contraction on  $H(G)$ . Alternatively, since  $\exp(-tA)$  is a contraction in  $H(G)$ , its Hilbert space adjoint  $G \exp(tA^*)G^{-1}$  is a contraction on  $H(G)$ . In any case, it follows that  $G \exp(tA^*)G^{-1}$  extends to a measurable map on  $E$ .

To show that this is the flow associated with  $\exp(-tL^*)$  it is sufficient to compute with exponential functions. It is easy to check that

$$\int \bar{e}_v \exp(-tL)e_u d\nu = \exp(\langle \exp(-tA^*)u, Gv \rangle) \exp(-\frac{1}{2}\langle u, Gu \rangle) \exp(-\frac{1}{2}\langle v, Gv \rangle), \quad (5.3)$$

which is real. The corresponding matrix element of the adjoint  $\exp(-tL^*)$  is obtained by reversing the role of  $u$  and  $v$ . However this is equivalent to replacing  $\exp(-tA^*)$  by  $G^{-1} \exp(-tA)G$ .

The description of the adjoint associated with a non-Gaussian measure is more challenging. For this, it is useful to have the concept of logarithmic gradient. Consider a probability measure  $\nu$ , and let  $Q$  be bounded from  $E^*$  to  $E$ . In some circumstances  $\nu$  has a *logarithmic gradient*  $\mathbf{B}$  associated with  $Q$ . This is obtained formally by taking the differential of the logarithm of the (non-existent) density and then applying  $Q$  to get a vector field. Thus  $\mathbf{B}$  is a Borel map from  $E$  to  $E$  and satisfies

$$\int \langle u, Q\nabla F \rangle d\nu = - \int F \langle u, \mathbf{B} \rangle d\nu \quad (5.4)$$

for all smooth cylindrical functions  $F$  and all  $u$  in  $E^*$ .

**Proposition 5.2** *Let  $\nu$  be a Gaussian probability measure with covariance  $G$ . Suppose that  $Q$  is bounded from  $H(G)^*$  to  $H(G)$ . Then  $QG^{-1}$  is bounded from  $H(G)$  to  $H(G)$ , and the logarithmic gradient is the measurable extension  $\mathbf{B} = -QG^{-1}$  to a function from  $E$  to  $E$ .*

The formula for the logarithmic gradient has a variant that involves a kind of divergence.

**Proposition 5.3** *Let  $W$  be a function from  $E$  to  $E^*$  of the form  $W = \sum_k F_k u_k$ , where each  $F_k$  is a smooth cylindrical scalar function, each  $u_k$  is in  $E^*$ , and the sum is finite. Then*

$$\int \nabla \cdot QW d\nu = - \int \mathbf{B} \cdot W d\nu. \quad (5.5)$$

**Theorem 5.4** *Consider the Ornstein-Uhlenbeck semigroup with generator  $-L$  and invariant measure  $\nu$ . Suppose that  $\nu$  has a logarithmic gradient  $\mathbf{B}$  with respect to  $Q$ . Then the Hilbert space adjoint in  $L^2(E, \nu)$  is given by*

$$-L^*F = \frac{1}{2}\nabla \cdot Q\nabla F + A\phi \cdot \nabla F + \mathbf{B} \cdot \nabla F. \quad (5.6)$$

In the theorem  $F$  is a smooth cylindrical function such that  $\nabla F$  is a function from  $E$  to  $E^*$ . Then  $\mathbf{B}$  is a function from  $E$  to  $E$ , so the pairing in the last term gives a scalar function on  $E$ .

Proof: From the proposition

$$-\int \mathbf{B} \cdot F \nabla G d\nu = \int \nabla \cdot QF \nabla G d\nu = \int \nabla F \cdot Q \nabla G d\nu + \int F \nabla \cdot Q \nabla G d\nu. \quad (5.7)$$

Similarly,

$$-\int \mathbf{B} \cdot \nabla F d\nu = \int \nabla \cdot Q \nabla FG d\nu = \int \nabla F \cdot Q \nabla G d\nu + \int \nabla \cdot Q \nabla FG d\nu. \quad (5.8)$$

Finally, from  $0 = -\int L(FG) d\nu$  and the proposition we obtain

$$0 = \frac{1}{2} \int \nabla \cdot Q \nabla (FG) d\nu + \int A\phi \cdot \nabla (FG) d\nu = \frac{1}{2} \int \mathbf{B} \cdot \nabla (FG) d\nu + \int A\phi \cdot \nabla (FG) d\nu. \quad (5.9)$$

This may be rewritten as

$$0 = \frac{1}{2} \int F \mathbf{B} \cdot \nabla G d\nu + \frac{1}{2} \int \mathbf{B} \cdot \nabla FG d\nu + \int FA\phi \cdot \nabla G d\nu + \int A\phi \cdot \nabla FG d\nu. \quad (5.10)$$

These three equations together give

$$\int F \, LG \, d\nu = \int L^* F \, G \, d\nu \quad (5.11)$$

with  $L^*$  as given above.

In the Gaussian case, when the original vector field that defines  $-L$  is  $-A$ , the vector field that defines the adjoint  $-L^*$  is  $A - QG^{-1} = -GA^*G^{-1}$ . The condition that  $L$  is self-adjoint is that  $AG = GA^* = \frac{1}{2}Q$ . This is equivalent to the detailed balance condition  $AQ = QA^*$ .

Say that  $G = RS$  where  $R$  is invertible. Then since  $AS + SA^* = 0$  we obtain  $-GA^*G^{-1} = -RSA^*S^{-1}R^{-1} = RAR^{-1}$ . The corresponding flow is  $R\exp(tA)R^{-1}$ . So in this case the flow is conjugate to the flow  $\exp(tA)$ .

## 6 The abstract renormalization group

This section is intended to show how the renormalization group that is conventional in the study of random fields fits into the general picture of an Ornstein-Uhlenbeck semigroup. As before we consider a separable Hilbert space  $E$  with dual space  $E^*$ . There is a semigroup  $\exp(-tA)$  acting in  $E$  with dual semigroup  $\exp(-tA^*)$  acting in  $E^*$ . We are interested in the situation when there is a covariance operator  $S : E^* \rightarrow E$  that is invariant under the semigroup, that is,  $\exp(-tA)S\exp(-tA^*) = S$ . Then there are dual Hilbert spaces  $H(S)$  and  $H(S)^*$  with  $H(S) \subset E$  and  $E^* \subset H(S)^*$ . The semigroup  $\exp(-tA)$  is unitary on  $H(S)$ , while  $\exp(-tA^*)$  is unitary on  $H(S)^*$ .

We introduce one more Hilbert space  $H$ , and we identify the Hilbert space  $H$  with its dual space  $H^*$ . We assume that

$$E^* \subset H^* = H \subset E \quad (6.1)$$

where each inclusion is a continuous dense injection.

Let  $D$  be a skew-adjoint operator acting in  $H$ , so  $D^* = -D$ . Then  $\exp(tD)$  is unitary on  $H$ . The first hypothesis on the generator  $-A$  is that

$$-A = D - 1. \quad (6.2)$$

**Proposition 6.1** Suppose that  $\exp(-tA)S\exp(-tA^*) = S$  and that  $-A = D - 1$ , where  $D$  is skew-adjoint. Then for every Borel function  $f$  it follows that  $\exp(-tA)f(S)\exp(-tA^*) = e^{-2t}f(e^{2t}S)$ .

Proof: The hypotheses imply that  $\exp(-tD)S\exp(tD) = e^{2t}S$ . But this is a unitary equivalence, so  $\exp(-tD)f(S)\exp(tD) = f(e^{2t}S)$ . However this translates back to  $\exp(-tA)f(S)\exp(-tA^*) = e^{-2t}f(e^{2t}S)$ .

The second hypothesis is that the covariance in the Ornstein-Uhlenbeck process is of the form

$$Q = w(S^{-1}), \quad (6.3)$$

where  $w$  is a strictly positive function on the positive real axis. We also want to assume that  $w$  has finite integral over the entire positive real axis. Then the integral  $W$  of  $w$  with  $W(0) = 0$  is an increasing bounded function. The example to keep in mind is  $w(u) = e^{-u}$  with integral  $W(u) = 1 - e^{-u}$ .

**Theorem 6.2** Suppose the two hypotheses are satisfied. Let  $W$  be the integral of  $w$  with  $W(0) = 0$ . Then

$$C_t = \int_0^t \exp(-t'A)Q\exp(-t'A^*) dt' = \frac{1}{2}S[W(S^{-1}) - W(e^{-2t}S^{-1})]. \quad (6.4)$$

Proof: From the previous result

$$C_t = \int_0^t e^{-2t'} g(e^{-2t} S^{-1}) dt' = \frac{1}{2} S \int_{e^{-2t} S^{-1}}^{S^{-1}} w(u) du. \quad (6.5)$$

This proposition indicates that

$$C = \frac{1}{2} S W(S^{-1}) \quad (6.6)$$

is the covariance of the limiting Gaussian measure that is invariant under the Ornstein-Uhlenbeck semigroup. Clearly the covariance  $G = C + aS$  is also invariant for each  $a > 0$ .

We need to verify when such a covariance operator  $C : E^* \rightarrow E$  is actually the covariance of a Gaussian measure on  $E$ . Let  $J : E \rightarrow H$  be an isomorphism. Then  $J^* : H \rightarrow E^*$  is also an isomorphism. It is easy to see that  $J$  restricted to  $H$  is a bounded operator from  $H$  to  $H$ . Then  $J^*$  is also a bounded operator from  $H$  to  $H$ , and  $E^*$  is the range of  $J^*$ . The condition is that  $J C J^*$  is a positive trace class operator from  $H$  to  $H$ . If  $J$  is Hilbert-Schmidt from  $H$  to  $H$  and  $C$  is bounded on  $H$ , then this condition is satisfied. However this is a somewhat special situation.

We shall be interested in the following in covariance operators that are translation invariant. The natural setting for this is the Hilbert space  $H = L^2(\mathbf{R}^\nu, d^\nu x)$ , regarded as self-dual. Let  $E^*$  be a Hilbert space of moderately smooth and decaying test functions, and let  $E$  be the dual Hilbert space of moderately rough and increasing tempered distributions. Then  $E^* \subset H^* = H \subset E$ . Suppose that  $C$  is a translation invariant operator that is trace class from  $E^*$  to  $E$ . Thus  $C$  is a translation invariant operator such that  $J C J^*$  is trace class from  $H$  to  $H$ . Then there is a translation invariant Gaussian measure on  $E$  with covariance  $C$ . If  $C$  is translation invariant, then it is an integral operator with a positive definite kernel of the form  $c(x - y)$ . Then the Fourier transform  $\hat{c}(k)$  is a positive function.

**Proposition 6.3** *If the covariance has Fourier transform  $\hat{c}$  in  $L^1 + L^\infty$ , then the Gaussian measure is defined on a Hilbert space  $E$  of tempered distribution.*

Proof: Take  $J$  of the form

$$(Jf)(x) = \int j_1(x) j_2(x - x') f(k) d^\nu x. \quad (6.7)$$

In the Fourier transform representation it acts by

$$(\hat{J}\hat{f})(x) = \int \hat{j}_1(k - k') \hat{j}_2(k) \hat{f}(k) d^\nu k / (2\pi)^k. \quad (6.8)$$

Take  $j_1$  and  $j_2$  each in  $L^2$ . Then  $\hat{j}_1$  and  $\hat{j}_2$  are each in  $L^2$ . Then  $J$  is a Hilbert-Schmidt operator. Similarly  $J^*$  is Hilbert-Schmidt. In the case when  $\hat{c}$  is in  $L^\infty$ , it follows that  $C$  is a bounded operator and  $J C J^*$  is trace class. In the other case, when  $\hat{c}$  is in  $L^1$ , require also that  $\hat{j}_2$  be in  $L^\infty$ . Then  $\sqrt{\hat{c}}$  is in  $L^2$ , and  $J C^{\frac{1}{2}}$  is Hilbert-Schmidt. So again  $J C J^*$  is trace class.

The classic example of a covariance with Fourier transform in  $L^\infty$  is  $(-\Delta + m^2)^{-1}$  with  $m > 0$ . An example where it is only in  $L^1 + L^\infty$  is  $(-\Delta)^{-1}$  in dimensions  $\nu > 2$ .

The trouble with distributions is that it is difficult to perform non-linear operations. Therefore it is useful to find circumstances when the measure is concentrated on a space of functions.

**Proposition 6.4** *If the covariance has Fourier transform  $\hat{c}$  in  $L^1$ , then the Gaussian measure is defined on a Hilbert space  $E$  of functions.*

Proof: Take  $J$  so that

$$(Jf)(x) = j(x)f(x). \quad (6.9)$$

In the Fourier transform representation it acts by

$$(\hat{J}\hat{f})(x) = \int \hat{j}(k - k')\hat{f}(k) d^{\nu}k / (2\pi)^k. \quad (6.10)$$

Take  $j$  in  $L^2$ , so that  $\hat{j}$  is also in  $L^2$ . If  $\hat{c}$  is in  $L^1$ , then  $\sqrt{\hat{c}}$  is in  $L^2$ , and  $J\hat{C}^{\frac{1}{2}}$  is Hilbert-Schmidt. So  $J\hat{C}J^*$  is trace class.

An example of a covariance to which this result applies is  $(-\frac{d^2}{dx^2} + m^2)^{-1}$  with  $m > 0$ , in dimension one.

## 7 The renormalization group

The renormalization group (RG) is of course a realization of this abstract scheme. The Hilbert space is  $H = L^2(\mathbf{R}^{\nu}, d^{\nu}x)$ . The translation group acts in this space by unitary operators. However another important unitary group is given by  $\exp(tD)$ . Here  $D$  is the skew-adjoint dilation generator

$$D = x \cdot \nabla + \frac{\nu}{2}. \quad (7.1)$$

So  $\exp(tD)u(x) = u(e^t x)e^{t\nu/2}$  is indeed a unitary group of *scaling* transformations.

Take the dimension  $\nu$  of space to satisfy  $\nu > 2$ . Then the covariance of greatest primary interest is defined and is given by  $S = (-\Delta)^{-1}$ . There are weighted scaling transformations that are appropriate to this covariance. The vector field  $-A$  is given by

$$-A = D - 1 \quad (7.2)$$

with adjoint

$$-A^* = -D - 1. \quad (7.3)$$

Thus on the Hilbert space  $H(S)^*$  defined by  $(-\Delta)^{-1}$  the operator  $A^*$  is skew-adjoint and  $\exp(-tA^*)$  is unitary. Similarly, on the Hilbert space  $H(S)$  defined by  $-\Delta$  the operator  $A$  is skew-adjoint and  $\exp(-tA)$  is unitary. It is not hard to see that the space  $E$  and the dual space  $E^*$  may be chosen so that  $\exp(-tA)$  and  $\exp(-tA^*)$  are strongly continuous semigroups. Since the semigroups are determined by scaling, this is accomplished by taking the operator  $J$  defining the space  $E$  to be defined by functions with appropriate growth conditions under scaling.

The covariance  $Q$  is taken to be a function of  $S^{-1} = -\Delta$ . One convenient choice for  $Q$  is  $Q = 2\alpha \exp(\alpha\Delta)$ . In wave number space this is a cutoff that only allows wave numbers up to the order of  $1/\sqrt{\alpha}$ . This corresponds to smearing so that only distances greater than  $\sqrt{\alpha}$  are considered. In other words, the noise is generated more or less uniformly over distances greater than this cutoff value.

The relation between  $D$  and  $\Delta$  is expressed by the commutation relation  $[D, \Delta] = -2\Delta$ . This integrates to  $\exp(tD)q(\Delta)\exp(-tD) = q(e^{-2t}\Delta)$ . It follows that

$$\exp(-tA)q(\Delta)\exp(-tA^*) = e^{-2t}q(e^{-2t}\Delta). \quad (7.4)$$

From this it is easy to compute  $C_t$ . The result is

$$C_t = (-\Delta)^{-1} (\exp(\alpha e^{-2t}\Delta) - \exp(\alpha\Delta)). \quad (7.5)$$

In wave number space this is a cutoff that only allows wave numbers in the range from  $1/\sqrt{\alpha}$  to  $e^t/\sqrt{\alpha}$ . In position space this corresponds to a restriction to the region from  $e^{-t}\sqrt{\alpha}$  to  $\sqrt{\alpha}$ .

The action of the renormalization group is to first map the measure by  $\exp(-tA)$  and then convolve with the Gaussian measure with covariance  $C_t$ . The mapping with  $\exp(-tA)$  takes the spatial range from  $\sqrt{\alpha}$  to  $e^t\sqrt{\alpha}$  to the range from  $e^{-t}\sqrt{\alpha}$  to  $\sqrt{\alpha}$ . Then the convolution works on that range. The result is to integrate out distances out to  $e^t\sqrt{\alpha}$  and then rescale to a fixed maximum distance  $\sqrt{\alpha}$ .

The limiting covariance is

$$C = (-\Delta)^{-1} (1 - \exp(\alpha\Delta)). \quad (7.6)$$

Let us look at the action of the RG on a translation invariant Gaussian measure  $\nu$  with covariance  $G = C + F$ . The result is a Gaussian measure  $\nu_t$  with covariance  $C + \exp(-tA)F\exp(-tA^*)$ . If  $F = f(\Delta)$ , then  $\exp(-tA)F\exp(-tA^*) = \exp(-2t)f(e^{-2t}\Delta)$ . Write  $f(\Delta) = -\Delta^{-1}h(\Delta)$ . Then  $\exp(-tA)F\exp(-tA^*) = -\Delta^{-1}h(e^{-2t}\Delta) \rightarrow -\Delta^{-1}h(0)$  as  $t \rightarrow \infty$ . The higher order terms are irrelevant. All that survives is  $C + a(-\Delta)^{-1}$ , where  $a = h(0)$ .

**Theorem 7.1** *Let  $\exp(-tL)$  be the Ornstein-Uhlenbeck semigroup corresponding to the renormalization group, with flow  $\exp(-tA)$  generated by  $-A = D - 1$ . Let  $\nu$  be the invariant Gaussian measure with covariance  $G = C + aS$ , where  $C$  is given above and  $S = (-\Delta)^{-1}$  and  $a > 0$ . Then  $G = RS$ , where  $R, S$  commute and  $R$  is self-adjoint and invertible. Furthermore, the Hilbert space adjoint semigroup  $\exp(-tL^*)$  is the Ornstein-Uhlenbeck semigroup with flow  $G\exp(-tA^*)G^{-1} = R\exp(tA)R^{-1}$  generated by  $-GA^*G^{-1} = RAR^{-1}$ .*

The Fock space analysis shows that the action of the adjoint semigroup  $\exp(-tL^*)$  on  $L^2(E, \nu)$  is given on Wick products by the action of  $R^{-1}\exp(tA^*)R$ . Thus

$$\exp(-tL^*) : u_1 \cdots u_m := R^{-1} \exp(tA^*) R u_1 \cdots R^{-1} \exp(tA^*) R u_m : . \quad (7.7)$$

So the solution is determined by the flow

$$(\exp(tA^*)f)(x) = f(e^t x) e^{(1+\nu/2)t}. \quad (7.8)$$

We can write this as

$$(\exp(tA^*)f) = [f(e^t x) e^{\nu t}] e^{(1-\nu/2)t}. \quad (7.9)$$

The factor  $f(e^t) e^{\nu t}$  approaches a constant times a delta function, while the remaining factor decreases exponentially. So it would appear that the fixed point is stable under local perturbations.

To see the phenomena of interest, one must leave the Hilbert space and look at translation invariant perturbations [9]. This is more difficult to make rigorous, but the calculation is illuminating. Consider functions of the form

$$V_m(\phi) = \int v(x_1, \dots, x_m) : \phi(x_1) \dots \phi(x_m) : dx_1 \dots dx_m, \quad (7.10)$$

where the function  $v(x_1, \dots, x_m)$  is symmetric. The stability is calculated by looking at the action of the adjoint scaling by  $\exp(tA^*)$  on the kernels  $v(x_1, \dots, x_n)$ . The result is the same expression with  $v(x_1, \dots, x_n)$  replaced by

$$v_t(x_1, \dots, x_m) = v(e^t x_1, \dots, e^t x_n) e^{m(\nu/2+1)t}. \quad (7.11)$$

In field theory one looks for translation invariant solutions. Thus this is taken as a function of  $m-1$  difference variables. In order to conserve the  $m-1$  dimensional integral, one needs a factor  $e^{(m-1)\nu t}$ . So we write

$$v_t(x_1, \dots, x_m) = [v(e^t x_1, \dots, e^t x_n) e^{(m-1)\nu t}] e^{(m+\nu-m\nu/2)t}. \quad (7.12)$$

The first factor

$$v(e^t x_1, \dots, e^t x_n) e^{(m-1)\nu t} \rightarrow c \delta(x_1 - x_2) \cdots \delta(x_{m-1} - x_m) \quad (7.13)$$

in the sense of distributions as  $t \rightarrow \infty$ . Here  $c$  is the total  $m-1$  dimensional integral. (Assume that  $c \neq 0$ .) What happens to the remaining coefficient  $e^{(m+\nu-m\nu/2)t}$  depends on the sign of  $m+\nu-m\nu/2$ . If this is negative, then the coefficient will go to zero and there will be a simplification. This happens when  $1/m + 1/\nu < 1/2$ . So when  $m = 4$  this happens for  $\nu > 4$ , and when  $m = 6$  this happens for  $\nu > 3$ , and so on. On the other hand, if sign is positive, then there will be a linear instability. When  $m = 4$  this happens when  $\nu < 4$ .

According to renormalization group lore, it is interesting to consider both integer and non-integer dimensions. There should be non-Gaussian fixed points for any dimension  $\nu$  greater than 2 but less than 4. One way to approach non-integer dimension is to replace the generator of the RG flow by

$$-A = D - 1 - \epsilon. \quad (7.14)$$

One can think of the  $\epsilon$  in the formula as replacing dimension  $\nu$  by dimension  $\nu - \epsilon$ . The flow  $\exp(-tA)$  leaves invariant the Hilbert space with norm  $\langle w, (-\Delta)^{\frac{1+\epsilon}{2}} w \rangle$ . The adjoint flow  $\exp(-tA^*)$  leaves invariant the Hilbert space with norm  $\langle u, (-\Delta)^{-\frac{1+\epsilon}{2}} u \rangle$ . The RG calculations above may be repeated. If again  $Q = 2\alpha \exp(\alpha\Delta)$ , the corresponding covariances are

$$C_t = \alpha \int_{e^{-2t}}^1 u^{\frac{\epsilon}{2}} \exp(\alpha u \Delta) du. \quad (7.15)$$

Recently Brydges, Dimock, and Hurd [5] considered this framework for dimension  $4 - \epsilon$ . For  $\epsilon > 0$  small they proved the existence of a non-Gaussian fixed point bifurcating off the Gaussian fixed point at dimension 4.

## 8 A dissipative example

The RG example suggests the program of looking for other pairs consisting of a diffusion given by  $Q$  and a linear vector field  $-A$  from which one can get more than one invariant measure. In the case when  $E$  is finite dimensional and the covariances  $C_t$  determined by  $Q$  are non-degenerate, there is at most one invariant measure for the Ornstein-Uhlenbeck semigroup, and that is the Gaussian measure with mean zero and covariance  $C$ . However in infinite dimensions there are other examples, including a Gaussian measure with non-zero mean. It was shown by Da Prato and Zabczyk [6] that this is possible by taking  $A$  to have a zero eigenvalue and making a suitable choice of diffusion  $Q$ .

Consider the Hilbert space  $E = L^2(\mathbf{R}_+, dx)$  of square integrable functions on the half line  $\mathbf{R}_+ = [0, \infty)$ . For this example we identify  $E$  with its dual space  $E^*$ . Let  $D = d/dx$ , the generator of left translations on the space of functions. This generates a semigroup  $\exp(tD)$  for  $t \geq 0$ . The spectrum of  $D$  consists of the entire closed left half plane. Each point in the open left half plane is an eigenvalue of  $D$ , and the corresponding eigenfunctions are decaying exponentials. Thus there is a sense in which  $D$  is quite dissipative. The adjoint  $D^* = -d/dx$  with a vanishing boundary condition at 0. It generates a semigroup  $\exp(tD^*)$  for  $t \geq 0$  of right translations with zero filling in the missing gap. These are isometries, so there are no Hilbert space eigenvalues. In this example the vector field is given by  $-A = D + \epsilon$  for  $\epsilon > 0$ . The  $\epsilon > 0$  compensates for the dissipation of  $D$ .

The covariance  $Q$  is given by an integral operator with kernel  $q(x, y)$ . For instance, one could take  $q(x, y) = \exp(-x^2/2) \exp(-y^2/2)$ . Then  $C_t$  is an integral operator with kernel

$$C_t(x, y) = \int_0^t e^{2\epsilon s} q(x+s, y+s) ds. \quad (8.1)$$

If the function  $q(x, y)$  has suitable decay, then the limiting covariance exists and has the kernel

$$C(x, y) = \int_0^\infty e^{2\epsilon s} q(x + s, y + s) ds. \quad (8.2)$$

This gives the mean zero Gaussian fixed point  $\mu$ . If  $\epsilon > 0$  there is another fixed point. In fact, the group  $\exp(-tA)$  leaves invariant the function  $m(x) = c \exp(-\epsilon x)$ . So the point mass concentrated on  $m$  is a measure  $\sigma$  that flow invariant. The convolution of  $\mu$  and  $\sigma$  is a Gaussian fixed point  $\nu$  that has non-zero mean  $m$ . By taking a mixture of point masses with different values of  $c$  one can also obtain a non-Gaussian measures  $\sigma$  whose convolution with  $\mu$  is a non-Gaussian fixed point  $\nu$ .

The stationary measure with mean  $m$  is concentrated on functions close to  $m(x)$ , but with some extra noise, especially just to the right of the origin. The reason this non-selfadjoint example works is that while the semigroup  $\exp(-tA)$  has a fixed point  $m$ , it dissipates the noise faster than it is generated. Most of the noise is generated just to the right of the origin, but the semigroup moves everything to the left, and left of the origin is oblivion.

There can be no example of this sort in the case when  $A$  is skew-adjoint. Suppose there were a vector  $m$  with  $\exp(-tA)m = m$ . Then we would have  $\exp(-tA^*)m = m$ . It would follow that  $\langle w, C_t w \rangle = t \langle w, Qw \rangle$  does not converge as  $t \rightarrow \infty$ .

## 9 A conservative example

This section treats a variant of the Da Prato and Zabczyk example. The Hilbert space is  $H = L^2(\mathbf{R}, dx)$ , regarded as self-dual. The group  $\exp(-tA)$  is unitary. The obvious choice is to take  $-A = D$ , where  $D = d/dx$  is the generator of left translations. Everything moves to the left at constant velocity. Notice that the generator  $-A$  has only absolutely continuous spectrum.

The operator  $Q$  is an integral operator with kernel  $q(x, y)$ . For instance, we could take  $q(x, y) = \exp(-x^2/2) \exp(-y^2/2)$ . The operator  $C_t$  has kernel

$$c_t(x, y) = \int_0^t q(x + s, y + s) ds. \quad (9.1)$$

The limiting operator  $C$  has kernel

$$c(x, y) = \int_0^\infty q(x + s, y + s) ds. \quad (9.2)$$

However now the kernel  $c(x, y)$  will not decay for  $x$  and  $y$  near  $-\infty$ . It will, however, remain bounded.

The Hilbert space is supporting the measure is  $E = L^2(\mathbf{R}, j(x)^2 dx)$ , where  $j(x)$  is a bounded function with appropriate decay properties at infinity. The dual space  $E^*$  is  $L^2(\mathbf{R}, j(x)^{-2} dx)$ . The pairing between  $E^*$  and  $E$  is given by the inner product  $\langle u, w \rangle = \int u(x)w(x) dx$ . Thus we have  $E^* \subset H^* = H \subset E$ . The space  $E$  is taken large enough so that the Gaussian measures  $\mu_t$  and  $\mu$  with covariances  $C_t$  and  $C$  are concentrated on  $E$ .

The invariant measure  $\mu$  is concentrated on functions that are close to zero for large positive  $x$ , but that experience some external noise near the origin. For large negative  $x$  each such function is close to some random constant value. Since this example is conservative, the noise has a permanent effect, and this is reflected in the strong correlations for negative values of  $x$ .

There are many other invariant measures. Each translation invariant (stationary) random process on the line defines an flow invariant measure  $\sigma$  for the semigroup  $\exp(-tA)$ . There are

many examples, both Gaussian and non-Gaussian. Then the convolution of  $\sigma$  with  $\mu$  is a measure that is invariant under the Ornstein-Uhlenbeck process.

Perhaps the simplest example is when  $\sigma$  is the measure defined by the stationary Gaussian Markov process with covariance  $S$  given by

$$s(x, y) = \frac{1}{2m} \exp(-m|x - y|). \quad (9.3)$$

The inverse covariance is  $S^{-1} = -d^2/dx^2 + m^2$ . Even for this Gaussian example it is not trivial to compute the logarithmic gradient of  $\nu$ . This quantity involves the inverse  $G^{-1}$  of the covariance  $G$  of  $\nu$ . This is determined by the sum  $G = C + S$ . The inverse may be written in the form

$$G^{-1} = (C + S)^{-1} = (1 + S^{-1}C)^{-1}S^{-1}. \quad (9.4)$$

The computation of the first factor on the right is equivalent to solving an integral equation.

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## References

- [1] V. I. Bogachev, *Gaussian Measures*, (Mathematical Surveys and Monographs, vol. 62), American Mathematical Society, Providence, 1998.
- [2] V. I. Bogachev, M. Röckner, and B. Schmuland, *Generalized Mehler semigroups and applications*, Probability Theory and Related Fields **105** (1996), 193–225.
- [3] A. Chojnowska-Michalik and B. Goldys, *Nonsymmetric Ornstein-Uhlenbeck semigroup as second quantized operator*, J. Math. Kyoto Univ. **36** (1996), 481–498.
- [4] D. Brydges and T. Kennedy, ‘*Mayer expansions and the Hamilton-Jacobi equation*, J. Statistical Physics **48** (1987), 19–49.
- [5] D. Brydges, J. Dimock, and T. R. Hurd, *A non-Gaussian fixed point for  $\phi^4$  in  $4-\epsilon$  dimensions*, Commun. Math. Phys. **198** (1998), 111–156.
- [6] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, (Encyclopedia of Mathematics and Its Applications, vol. 45), Cambridge University Press, Cambridge, 1992.
- [7] T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann. **162** (1966), 258–279.
- [8] N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan, *Probability Distributions on Banach Spaces*, D. Reidel (Kluwer), Dordrecht, 1987.
- [9] C. Wieczorkowski, *Running coupling expansion for the renormalized  $\phi_4^4$  trajectory from renormalization invariance*, J. Statistical Phys. **89** (1997), 929–945.