

Review of Consistent Quantum Theory, by
Robert B. Griffiths
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1 The quantum riddle

The book under review is devoted to a proposal for resolving the problem of the foundations of quantum mechanics. There are two parts: first the essentials of quantum theory, and then a variety of applications with emphasis on the paradoxical features of the theory and on measurement situations. The level of the book is comparable to that of other introductions to quantum theory, but it concentrates on promoting consistent family theory and thus omits a number of textbook topics that are not relevant to this. Something very similar to consistent family theory has also been considered by Omnès [O] and by Gell-Mann and Hartle [GH]. This theory is one instance of an important trend in foundations of quantum mechanics—the idea that measurement should not play a fundamental role.

While quantum mechanics is the fundamental theory of matter, its interpretation is controversial. The problem emerges from the mathematics of the theory. This mathematical apparatus is so simple and compelling that physicists are completely under its spell. But it leads directly to the difficulty.

The easiest way to understand the problem is to make the comparison to probability theory. In probability the structure is defined in terms of a set Ω . An element of the set Ω is called an *outcome*. In addition, there is a given family of subsets of Ω . Each such subset is called an *event*. The set Ω is itself an event. If E is an event, then its complement E' is also an event, representing the negation of E . Furthermore, if E and F are each events, then their intersection EF is an event, representing the conjunction of E and F . This family of events thus forms a Boolean algebra. A probability measure assigns to each event a number $P[E]$ with $0 \leq P[E] \leq 1$ in the closed interval from 0 to 1 in such a way that the usual probability axioms are satisfied.

The most common interpretation of this mathematical apparatus is the following. There is a situation that may be reproduced many times under independent, identical conditions. An outcome describes which events do or do not

happen. Thus the outcome ω for a particular replication may or may not belong to the set E . The probability of E is supposed to be close to the proportion of the replications for which E happens. The author of the book seems to endorse something close to this interpretation of probability, where he explains the notion of ensemble as “a large collection of systems” and says that “any particular system will be in some definite state, but this state is not known before the system is selected from the ensemble.” (pp. 73–74).

If the probability of an event is one, then one would expect to see the event happen in each replication. This leads to a kind of logic. Thus, for instance, if the probability of E and not F is zero, that is, if $P[EF'] = 0$, then in each replication in which E happens, F should also happen. That is, every (or almost every) outcome in E should also be in F .

In quantum mechanics the structure is defined in terms of a Hilbert space \mathcal{H} . A closed subspace of \mathcal{H} is called a *quantum event*. Since there is a one-to-one correspondence between closed subspaces and orthogonal projection operators, each closed subspace is identified with the corresponding projection operator. Each such projection operator is self-adjoint and satisfies $E^2 = E$. The identity operator I projects onto the entire Hilbert space \mathcal{H} , and it is regarded as a quantum event. If E is a projection onto a subspace representing a quantum event, then $E' = I - E$ projects onto the orthogonal complement and represents the negation of E . If E and F are each quantum events, then they are said to be compatible if the projections commute, $EF = FE$. If E and F are compatible quantum events, then their product EF projects onto the intersection and is a quantum event. A unit vector ψ in \mathcal{H} plays the role of a probability measure. The square of the length of the projected vector $E\psi$ is a number

$$P[E] = \|E\psi\|^2 \tag{1}$$

with $0 \leq P[E] \leq 1$. This is the probability of the quantum event E .

The analogy is close, but an extra feature of quantum mechanics makes all the difference. This is the existence of quantum events E, F that are not compatible. A standard example is when E is the event that the position of a particle is in a certain region, and F is the event that the momentum of the particle is in some other region. There is no definition of the conjunction of two such quantum events. Furthermore, there is no natural notion of outcome for all quantum events that would help us overcome this difficulty.

A *compatible family* of quantum events is a family of quantum events with the following properties. The family contains the sure event I , whenever E is in the family its negation E' is in the family, and whenever E, F are in the family, then the E, F are compatible, and the conjunction EF is in the family. Then this family forms a Boolean algebra, and the usual rules of probability hold. The interpretation of probabilities in terms of the proportion of outcomes is without problem. The puzzle is how one would be justified in restricting attention to a particular compatible family.

This raises the question: How is it possible that there is no general definition of conjunction of quantum events? Can someone forbid us to use the

word “and”? If so, who will assume this burden? There are several possible answers. One is to place the responsibility on the physicist performing an experiment. The traditional account is if E and F are incompatible quantum events, then there is no measurement that will simultaneously find out whether E happens and whether F happens. A measurement of one precludes the measurement of the other. Thus the question goes away. The measurement will single out a particular compatible family of quantum events. The probabilities given by quantum mechanics are relevant only to these events. They predict the frequency of outcomes in the usual way.

Other solutions are possible; again these may involve restricting attention to a special compatible family of quantum events. These events might correspond to questions about positions of particles (as in Bohmian mechanics [BZ]), or they might be certain macroscopic events (perhaps in some version of decoherence theory). In any case, such a restricted class of events would fall within the scope of ordinary probability theory. For better or worse, singling out such a class discards much of the Hilbert space symmetry of quantum mechanics.

The book under review takes yet another direction. There are two major themes. The first theme is relatively technical. The assertion is that it is useful to consider an extension of quantum mechanics to a more general consistent family (or framework) theory. In this theory one can do probability calculations with certain quantum events that are not compatible in the above sense, but only when certain consistency conditions are satisfied. The second theme has to do with the interpretation of quantum mechanics. The idea is that the difficulties mentioned above can be eliminated by imposing a new syntactic rule.

2 Consistent families

Say that there is a Boolean algebra of quantum events F associated with time instant 1 and another Boolean algebra of quantum events E associated with time instant 2. The quantum events E and F are not necessarily compatible. There is a construction that associates to the quantum event E a corresponding event E_2 and to the quantum event F a corresponding event F_1 in such a way that E_2 and F_1 are compatible. Unfortunately, the events E_2 and F_1 are associated with a different Hilbert space. Nevertheless one can still try to define probabilities of conjunctions using the original Hilbert space. Thus, given that the state is ψ at time 0, the probability of the conjunction E_2F_1 is

$$P[E_2F_1] = \|EF\psi\|^2. \quad (2)$$

The above definition ignores the unitary time evolution of quantum mechanics, but this is justified if we work in the Heisenberg picture. The real problem is that it will not satisfy the usual probability axioms. Thus, for example, it is not true in general that

$$P[E_2] = P[E_2F_1] + P[E_2F'_1]. \quad (3)$$

Since

$$\|E\psi\|^2 = \|EF\psi\|^2 + \|EF'\psi\|^2 + \langle EF'\psi, EF\psi \rangle + \langle EF\psi, EF'\psi \rangle, \quad (4)$$

the condition for this to happen is that the sum of the last two terms is zero. This is an example of a consistency condition. (In some treatments this is called a decoherence condition.)

Given the appropriate consistency conditions, the usual probability axioms will be satisfied. This is the idea behind the concept of *consistent family* or *framework* as expounded in the book. Each such framework corresponds to the usual setup of ordinary probability theory. The only remaining question is about the relation between different frameworks.

3 The single framework rule

The interpretation of quantum mechanics in the book is in the context of this extended notion of consistent family or framework. However the ordinary case of a compatible family may be viewed as special case of a framework. This could be an algebra of events defined only at time 1, or it could involve compatible events at time 1 and time 2 related (in the Heisenberg picture) by the time evolution. So the success of the interpretation should not hang on the more general notion; it would be impressive enough if it applied only to the special case of a compatible family.

The goal is clear: It is “constructing a rational approach to quantum theory that is not based on measurement as a fundamental principle” (p. xiv). The author states that, “paradoxes have sometimes been used to argue that the quantum world is not real, but is in some way created by human consciousness, or else that reality is a concept which only applies to the macroscopic domain immediately accessible to human experience.” (p. 10) He displays no sympathy for such views.

The difficulty to be overcome is displayed in several examples, explained in detail in the book. The Hardy paradox is typical. The system consists of two particles, each passing through one of two arms of an interferometer and then through one of two channels. The system has the following remarkable property. Whenever the first particle emerges in the E channel, then the second particle was earlier in the \bar{D} arm. Whenever the second particle emerges in the \bar{E} channel, then the first particle was earlier in the D arm. The probability that the first particle emerges in the E channel while the second particle emerges in the \bar{E} channel is $1/12$. However it is impossible for the first particle to have gone through the D arm while the second particle went through the \bar{D} arm. This is amazing, but it is also standard quantum mechanics.

In the language of quantum mechanics, the events E and \bar{E} are compatible and

$$P[E\bar{E}] = \frac{1}{12}. \quad (5)$$

The events E and \bar{D} are compatible, and

$$P[E\bar{D}'] = 0. \quad (6)$$

The events \bar{E} and D are compatible, and

$$P[\bar{E}D'] = 0. \quad (7)$$

Finally, the events D and \bar{D} are compatible, and

$$P[D\bar{D}] = 0. \quad (8)$$

The problem is this. If actual physical outcomes determine whether these events happen or not, then for some significant fraction of these outcomes E and \bar{E} both happen. If E happens, then \bar{D} must happen. If \bar{E} happens, then D must happen. So for these outcomes both \bar{D} and D happen. But this is impossible.

The usual way out is to say that only compatible events may be measured in any one experiment. So the probabilities only apply to these measured events. Pairs such as E, D are not compatible. However this exit is not available to the author, since he wants to develop a theory that is not based on measurement. His solution is breathtakingly simple: the single framework rule. It says that “when constructing a quantum description of a physical system it is necessary to restrict oneself to a single framework, or at least not mix results from incompatible frameworks.” (p. 217) In the example the events E, \bar{E}, D, \bar{D} do not fit into a single consistent family or framework, and so the laws of probability cannot be used to deduce that $P[D\bar{D}] \geq 1/12$.

How does this restriction work? He explains, “To be sure, there is never any harm in constructing as many alternative descriptions of a quantum system as one wants to, and writing them down on the same sheet of paper. The difficulty comes about when one wants to think of the results obtained by using incompatible frameworks as all referring simultaneously to the same physical system, or tries to combine the results of reasoning based on incompatible frameworks.” (p. 225) There is an element of choice: “In order to describe a physical system, a physicist must, of necessity, adopt some framework and this means choosing among many incompatible frameworks, no one of which is, from a fundamental point of view, more appropriate or more ‘real’ than any other.” (p. 364) The physicist can assert, “If E , then \bar{D} .” Another choice is, “If \bar{E} , then D .” But the conjunction “If E , then \bar{D} , and if \bar{E} , then D ” is meaningless.

The rule is that no more than one consistent family may be used at a time. This leads to a natural question: Is every possible consistent family available for use? For example, each quantum event E generates a consistent family consisting of $0, I, E, E'$. One event is that the position of a particle is in a certain region. Another event is that the momentum is in some other region. These two are not compatible, but they separately generate consistent families. There is also the event that some particular linear combination of position and momentum is in a given region. This has exactly the same mathematical status as the first two events, but its physical meaning is more obscure. Other events

may seem even more arbitrary. Does every quantum event, when regarded as a member of some consistent family, have a physical meaning? If so, then how does one find out this meaning? What do the probabilities predict? Or are some of the more complicated and arbitrary consistent families mere mathematical artifacts? If so, is there a systematic theory to select those that have physical meaning? The book treats many examples of consistent families, but it pays little attention to this more general issue.

4 Conclusion

There is a useful summary at the end of the book. The author makes three claims. (These might seem totally reasonable to a scientist not steeped in the orthodoxy of quantum mechanics.)

1. “Measurements play no fundamental role in quantum mechanics, just as they play no fundamental role in classical mechanics.”
2. “Quantum mechanics, like classical mechanics, is a local theory in the sense that the world can be understood without supposing that there are mysterious influences which propagate over long distances more rapidly than the speed of light.”
3. “Both quantum mechanics and classical mechanics are consistent with the notion of an *independent reality*, a real world whose properties and fundamental laws do not depend upon what human beings happen to believe, desire, or think. While this real world contains human beings, among other things, it existed long before the human race appeared on the surface of the earth, and our presence is not essential for it to continue.”

The theory that is to justify these claims relies on the single framework rule, which is a syntactic rule of a certain kind of logic. This is explicit in a letter by the Griffiths and Omnès in the May 2000 issue of Physics Today [GO]. They say that the principles of quantum mechanics “inevitably require some modification of the rules of propositional logic when dealing with quantum properties.” They are specific: “What we are recommending is a syntactical rule governing how logical expressions can be formed in a meaningful way, which prohibits combining propositions from distinct, incompatible consistent families. Each consistent family, on the other hand, constitutes a logic in which the usual rules of reasoning apply.”

The book does little to put this kind of quantum reasoning in a more general logical context. On the other hand, Isham [I] describes a logic in which mathematical objects are indexed by consistent families. He refers to this approach as “a type of neorealism.” For each quantum event E and each number p with $0 \leq p \leq 1$, the truth value of $P[E] = p$ is itself indexed by consistent families. This truth value belongs to a Heyting algebra that obeys the rules of intuitionistic logic. (A Heyting algebra is more general than a Boolean algebra; in particular the law of double negation need not hold.) Presumably a theoretical physicist could carry out all needed calculations in intuitionistic calculus, a subject that is already highly developed [B].

Near the end of the book there a passage that reads:

“The principle of unicity does not hold: there is not a unique exhaustive description of a physical system or a physical process. Instead, reality is such that it can be described in various alternative, incompatible ways, using descriptions that cannot be compared or combined.” (p. 369)

This concept of reality is defined negatively, and so its intent is elusive. Perhaps a clue could emerge from the fact that probabilities predict relative frequencies. Frequencies of what? The author is clear that there is no single overall objective outcome that describes whether each of the quantum events happens or not. In fact, this is ruled out by the non-existence of a “universal truth functional.” Perhaps the repetitions of the physical situation are partitioned according to the description that is being entertained—but then the partition would seem to depend on the choice of the physicist. The author is quite clear that there is no law of nature that specifies the framework that has to be employed in a particular circumstance. (p. 255) His advice is to say that “one has to learn how to choose the correct sample space for discussing a particular problem, and how to avoid carelessly combining results from incompatible sample spaces.” (p. 75) On the other hand, maybe the idea is that for each repetition there are outcomes associated with each of the individual consistent families, yet somehow these outcomes do not fit together to form an overall outcome, because they are making assertions about incomparable aspects of the physical system. Thus the notion of outcome in probability is also problematic. Or is there some other content to the probability predictions? This puzzle reflects the fundamental problem with the version of quantum theory in this book: A syntactical rule that merely restricts discourse does little to tell what the theory is about.

References

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