Problem 2

Proof that $(\mathbb{Q}, +, \cdot)$ is a field

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0 \right\}$$

where $\frac{p}{q}$ denotes the equivalence class of the pair (p, q) under the relation $(p,q) \sim (p', q') \leftrightarrow pq' = p'q$. Define for represent the usual formulas

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}, \quad \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}.$$

well-definedness.

If $\frac{p}{q} = \frac{p'}{q'}$ and $\frac{r}{s} = \frac{r'}{s'}$ (i.e $pq' = p'q$ and $rs' = r's$), then

$$\frac{ps + rq}{qs} = \frac{p's' + r'q'}{q's'}$$

Multiplying both sides by $qsq's'$ and using the equalities $pq' = p'q$ and $rs' = r's$ shows the numerators become equal; similarly for multiplication. Hence the formulas give the same result on equivalence classes, so addition and multiplication are well-defined on $\mathbb{Q}$.

closure:-

If $p/q,\ r/s \in \mathbb{Q}$ then $ps + rq$ and pr are integers and $qs \neq 0$, so $\frac{ps+rq}{qs}, \frac{pr}{qs} \in \mathbb{Q}$. Thus $\mathbb{Q}$ is closed under $+$ and $\cdot$.

Associativity:

Associativity of addition and multiplication follows from associativity in $\mathbb{Z}$ after computing by the fraction formulas. For example,

$$(p/q + r/s) + t/u = \frac{ps+rq}{qs} + \frac{t}{u} = \frac{(ps+rq)u+tqs}{qsu}$$

$$= \frac{psu+rqu+tqs}{qsu}$$

and,

$$p/q + (r/s + t/u) = \frac{p}{q} + \frac{ru+ts}{su} = \frac{p(su)+q(ru+ts)}{qsu}$$

$$= \frac{psu+rqu+tqs}{qsu}$$

So, they are equal. The same of computation proves associativity of multiplication.

Commutativity:

From commutativity in $\mathbb{Z}$,

$$p/q + r/s = \frac{ps+rq}{qs} = \frac{rq+ps}{sq} = r/s + p/q,$$

and, $$p/q \cdot r/s = \frac{pr}{qs} = \frac{rp}{sq} = r/s \cdot p/q$$

Identities :

Additive identity : $0 = \frac{0}{1}$, since $p/q + \frac{0}{1} = p/q$

multiplicative identity : $1 = 1/1$ since $p/q \cdot 1/1 = p/q$

Additive inverses :

for p/q take $-p/q = -p/q$ then

$$p/q + -p/q = \frac{pq + (-p)q}{q^2} = \frac{0}{q^2} = 0$$

multiplicative inverses (non zero elements) :

If $p/q \neq 0$ then $p \neq 0$. Define

$$\left(\frac{p}{q}\right)^{-1} = \frac{p}{q}$$

Then $p/q \cdot q/p = \frac{pq}{qp} = 1$. well-definedness holds because if $p/q = \frac{p'}{q'}$ with $p, q' \neq 0$ then $pq' = p'q$ implies $\frac{q}{p} = \frac{q'}{p'}$

Distributivity :

For $p/q, \frac{r}{s}, t/u \in \mathbb{Q}$

$$\frac{p}{q}(r/s + t/u) = p/q \cdot \frac{ru + ts}{su}$$

$$= \frac{p(ru + ts)}{qsu} = \frac{pru + pts}{qsu}$$

$$= \frac{pr}{qs} + \frac{pt}{qu}$$

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So multiplication distributes over addition.

All field axioms (closure, associativity, commutativity, additive and multiplicative identities, additive inverses for every element, multiplicative inverse for every nonzero element and distributivity) are satisfied.

Therefore $(\mathbb{Q}, +, \cdot)$ is a field.