Problem : 2

A) False

S_3 has order $6 = 2 \cdot 3$ and is non-abelian.

There is a standard theorem: if $|G| = pq$ with primes $p < q$ and $p \nmid (q-1)$, then G is cyclic (hence abelian).

But when $p \mid (q-1)$ nonabelian semidirect product may exist.

B) True.

Every group of order p^2 is abelian.

let $|G| = p^2$. The center $Z(G)$ is nontrivial (class equation). If $|Z(G)| = p^2$ then $G = Z(G)$ is abelian. If $|Z(G)| = p$ then $G/(Z(G))$ has order p hence is cyclic, implying G is abelian.

Thus in all cases G is abelian.

If $G \cong C_{p^2}$ (cyclic) then there is exactly one subgroup of order p.

If $G \cong C_p \times C_p$ then there are exactly $p+1$ subgroups of order p.

So, G has $p+1$ subgroups of order p if and only if $G \cong C_p \times C_p$.

then G is abelian.

C) True

Let the distinct conjugates be $H_1 \ldots\ldots H_k$ where $H_1 = H$. Each conjugate has $|H|$ elements. Two distinct conjugates intersect in a subgroup of size at most $|H|-1$. also $k = [G : N_G(H)]$, where $N_G(H)$ is the normalizer of H. Since $H \subseteq N_G(H)$ we have,

$$k = [G : N_G(H)] \leq [G : H].$$

D) False

$G = S_3$. take $N = \langle (123) \rangle$ (order 3), which is normal and cyclic. The quotient S_3/N has order 2 and is cyclic. But S_3 is non-abelian.

So both N and G/N cyclic does not force G abelian.

G) False

work in the infinite cyclic group $\langle x \rangle$. Let $a = x$ and $b = x^2$. Then $a^4 = b^2$, and $ab = ba$. But $(ab)^6 = (x \cdot x^2)^6 = x^{18} \neq e$. So the conclusion $(ab)^6 = e$ is not generally true without extra hypotheses.

I) True

If there is exactly one subgroup of order p^n then that subgroup is unique and therefore normal.