Probability Sheet 1

1) $$P(\text{both } W) = \frac{\binom{n-2}{2}}{\binom{n}{4}} = \frac{\frac{\cancel{(n-2)}\cancel{(n-3)}}{2}}{\frac{n(n-1)\cancel{(n-2)}\cancel{(n-3)}}{4!}}$$

$$= 12/n(n-1)$$

$$P(\text{neither } W) = \binom{n-2}{4} \Big/ \binom{n}{4} = \frac{\frac{(n-2)!}{4!(n-6)!}}{\frac{(n!)}{4!(n-4)!}}$$

$$= \frac{(n-2)!(n-4)!}{n!(n-6)!} = \frac{(n-4)(n-5)}{n(n-1)}$$

Therefore: $$12 = 2(n-4)(n-5) \Rightarrow \overbrace{(n-4)}^{3}\overbrace{(n-5)}^{2} = 6$$

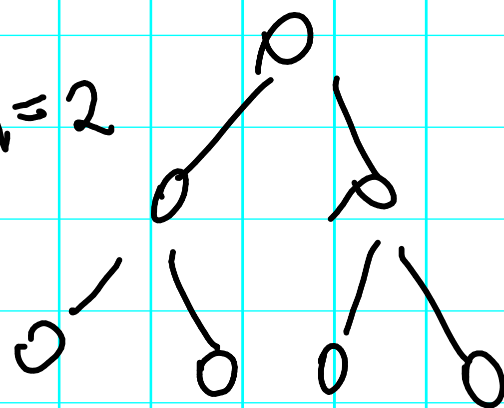
$$\Rightarrow n-4 = 3 \Rightarrow \boxed{n=7}$$

2) (i) 1 place out of 2^n-1 places. $\therefore$

$$P = 1/(2^n-1)$$

(ii)

$n=2$



- Round 1: diff. sides
- Both must win all until finals.

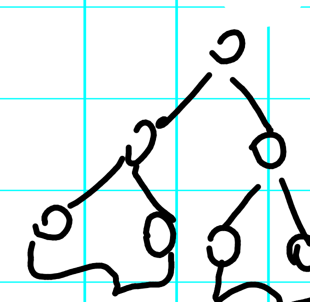
$$P(\text{diff. sides}) = \frac{2^{n-1}}{2^n-1} =$$

$$P(\text{win until final}) = \frac{1}{2^{n-1}}$$

$$\therefore P(\text{meet final}) = \frac{1}{2^{2n-2}} \cdot \frac{2^{n-1}}{2^n-1} = \frac{1}{2^{n-1}\cdot(2^n-1)}$$

(iii)

$$P(\text{meet 1st}) = 1/(N-1)$$



$$P(\text{meet 2nd}) = (2/(N-1))\cdot\frac{1}{4}$$

$$P(\text{meet 3rd}) = (4/(N-1))\cdot\frac{1}{4^2}$$

$$P(\text{meet n-th}) = (2^{n-1}/N-1)\cdot\frac{1}{2^{2n-2}} =$$

$$P(\text{meet some}) \simeq \sum_{k=1}^{n} \frac{2^{k-1}}{N-1} \cdot \frac{1}{2^{2k-2}} = \frac{1}{N-1} \sum_{1}^{n} \frac{1}{2^{k-1}}$$

$$= \frac{1}{N-1} \cdot \frac{2^{-n}-1}{\frac{1}{2}-1} = \frac{1}{N-1} \cdot \frac{1-N}{\frac{N}{2}-N} = + \frac{1}{N/2} =$$

$$= \frac{2}{N} = 2^{1-n}$$

3) # of half-decks: $C(52,26)$

# of good ": $C(26,13) \times C(26,13)$

$$\frac{1}{P} = \frac{52!}{26!\,26!} \cdot \frac{13!\,13!}{26!} \cdot \frac{13! \cdot 13!}{26!} = \frac{52!\,(13!)^4}{(26!)^4}$$

$$n! \cong \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

$$\therefore \frac{1}{P} = \frac{\sqrt{104\pi} \cdot \cancel{4\pi^2}\,\cancel{13^2} \cdot \left(\frac{52}{\cancel{e}}\right)^{52} \cdot \left(\frac{13}{\cancel{e}}\right)^{52}}{4 \cdot \cancel{13^2}\,\cancel{4\pi^2} \left(\frac{26}{\cancel{e}}\right)^{104}}$$

$$\frac{1}{P} = \sqrt{104\pi} \cdot \cancel{676}^{52} / 4 \cdot \cancel{676}^{2}$$

$$= \sqrt{6.5\pi} \Rightarrow p \hat{=} 0.22$$

1

4) $6n$ dice at random.

$$\# \text{good conf.} = \# \text{perm de } \{\overbrace{1,1,\dots,1}^{n},\overbrace{2,\dots,2}^{n},\dots,6\}$$

$$= (6n)!/(n!)^6$$

order of $1_n, 2_n, \dots$ doesn't matter.

$$\# \text{all config.} = 6^{6n}$$

Therefore, the prob we're looking for is

$$p = \frac{(6n)!}{(n!)^6} \cdot \left(\frac{1}{6}\right)^{6n}.$$

Using Stirling's:

$$p \cong \frac{\sqrt{12\pi n} \cdot \cancel{(6n/e)^{6n}}}{(2\pi n)^3 \cdot \cancel{\left(\frac{n}{e}\right)^{6n}}} \cdot \cancel{6^{-6n}} = \frac{1}{8 \cdot 2 \cdot 3^{-\frac{1}{2}} \cdot \pi^{3-\frac{1}{2}} \cdot n^{3-\frac{1}{2}}}$$

$$= c \cdot n^{-5/2} \quad \text{where} \quad c = 1/16 \cdot 3^{-1/2} \cdot \pi^{5/2}. \quad \blacksquare$$

5) We know that $P(\Omega)=1$ and that $A\subseteq\Omega$. Thus $P(A)\leq 1$. ∎

$\because A\cup A^c=\Omega \Rightarrow P(A)+\underbrace{P(A^c)}_{\geq 0}=1$

6) (i) $P(A)+P(A^c\cap(B\cup C)) \overset{\because \text{disjoint}}{=}$

$= P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(AB)-P(AC)-P(BC)$
$+P(ABC)$

from which the result follows.

(ii) B : divisible by 3
C : divisible by 5
A : divisible by 7

$n(A^c\cap(B\cup C)) = n(B)+n(C)-n(BC)-n(AC)-n(AB)+n(ABC)$

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