Noncommutative Projective Schemes

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Temple Graduate Student Conference in Algebra, Geometry, and Topology

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OUTLINE

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Theorem 1 (Serre)

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(ii) *there is an integer n_0, depending on $\mathscr{F}$ such that for each $0 < i$ and each $n_0 \leq n$, $H^i(X, \mathscr{F}(n)) = 0$.*

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Theorem 2 (Artin-Zhang)

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(H5) for every $1 \leq j$, $\underline{H}^j(\pi(M))$ is right bounded; i.e., there is an integer d_0 such that for all $d_0 \leq d$, $H^j(\pi(M)[d]) = 0$.

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(II) *Conversely, if A satisfies χ_1 and if (H4) and (H5) hold for every $\pi(M) \in \mathrm{qgr}$ -A, then A satisfies χ.*

Preliminaries

Let k be a Noetherian commutative ring, A a $\mathbb{Z}_{\geq 0}$-graded right Noetherian algebra over k. Denote by $\mathsf{Gr}\text{-}A$ (resp. $\mathsf{gr}\text{-}A$) the category of graded right A-modules (resp. finite) with morphisms

$$\mathsf{Hom}_{\mathsf{Gr}\text{-}A}(M, N) = \{f \in \mathsf{Hom}_A(M, N) \mid f(M_d) \subseteq N_d\}.$$

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- filtered colimits are exact,
- $\mathsf{Gr}\text{-}A$ has a generator: the functor $h^A\colon \mathsf{Gr}\text{-}A \to \mathfrak{Set}$ is faithful; for any morphism $M \to N$ the morphism

$$\mathrm{Hom}_{\mathsf{Gr}\text{-}A}(M, N) \longrightarrow \mathrm{Hom}_{\mathfrak{Set}}\left(h^A(M), h^A(N)\right)$$

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A full subcategory, $\mathscr{A}$, of an abelian category $\mathscr{C}$ is called a Serre (or épaisse/thick/dense) subcategory if for any short exact sequence

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The full subcategory Tors (resp. tors) of $\mathsf{Gr}\text{-}A$ (resp. $\mathsf{gr}\text{-}A$) with objects M of $\mathsf{Gr}\text{-}A$ (resp. $\mathsf{gr}\text{-}A$) satisfying

$$\tau(M) = \{m \in M \mid mA_{\geq s} = 0 \text{ for some } s\} = M$$

is a Serre subcategory.

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- $\operatorname{QGr}\text{-}A$ is abelian.
- There is a functor $\pi : \operatorname{Gr}\text{-}A \to \operatorname{QGr}\text{-}A$ that is the identity on objects and sends a morphism $f \in \operatorname{Hom}_{\operatorname{Gr}\text{-}A}(M, N)$ to its image, $\pi(f)$, in the colimit.

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- $\operatorname{qgr}\text{-}A$ is defined analogously.

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- Given two objects M, N of $\mathrm{Gr}\text{-}A$,

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- If in addition M is an object of $\mathsf{gr}\text{-}A$, then

$$\mathrm{Hom}_{\mathsf{QGr}\text{-}A}\left(\pi(M), \pi(N)\right) = \lim_{n \to \infty} \mathrm{Hom}_{\mathsf{Gr}\text{-}A}\left(M_{\geq n}, N\right)$$

where

$$M_{\geq n} = \bigoplus_{d \geq n} M_d.$$

Properties of π

- given an exact sequence

$$0 \longrightarrow K \xrightarrow{\ker f} M \xrightarrow{f} N \xrightarrow{\operatorname{coker} f} C \longrightarrow 0.$$

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- π preserves injectives.

Closed Objects

Definition 3

We say an object M of $\mathrm{Gr}\text{-}A$ is Tors-closed if M is torsion-free and any short exact sequence

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Remark 1

It's immediate that every torsion-free injective is Tors-closed.

Closed Objects

Proposition 1 (Gabriel)

For M an object of $\mathrm{Gr}\text{-}A$, the following are equivalent:

1. *Any exact sequence*

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2. *M is Tors-closed,*
3. *For any object N of $\mathrm{Gr}\text{-}A$*

$$\pi\colon \ \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(N, M) \cong \mathrm{Hom}_{\mathrm{QGr}\text{-}A}(\pi(N), \pi(M)).$$

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2. It can be shown (see Artin-Zhang, Prop 2.2) that if M is torsion, then so is $E(M)$. In the case that we have an injective object, Q, $\tau(Q)$ is injective and gives the decomposition $Q \cong \tau(Q) \oplus Q/\tau(Q) \cong \tau(Q) \oplus \omega\pi(Q)$. In fact, it follows that $Q/\tau(Q) \cong \omega\pi(Q)$ is injective.

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3. Every injective object of $\mathrm{QGr}\text{-}A$ is isomorphic to $\pi(Q/\tau(Q))$ for some injective object Q of $\mathrm{Gr}\text{-}A$.

Computing Ext

Since $\mathsf{QGr}\text{-}A$ has enough injectives, we can define Ext for $\mathsf{QGr}\text{-}A$. Let's compute $\mathrm{Ext}^i_{\mathsf{QGr}\text{-}A}(\pi(M), \pi(N))$:

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3. From the adjunction we get an isomorphism of complexes
$$\mathrm{Hom}_{\mathsf{QGr}\text{-}A}(\pi(M), \pi(Q^\cdot))) \cong \mathrm{Hom}_{\mathsf{Gr}\text{-}A}(M, \omega\pi(Q^\cdot))$$
and we see that
$$\mathrm{Ext}^i_{\mathsf{QGr}\text{-}A}(\pi(M), \pi(N)) \cong R^i\,\mathrm{Hom}_{\mathsf{Gr}\text{-}A}(M, \omega\pi(N))$$

Graded Hom

Definition 5

Define the graded modules

$$\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}(M, N) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M, N[d])$$

and

$$\underline{\mathrm{Hom}}_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(N)) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(N)[d]) .$$

Graded Ext

The right derived functors are

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Graded Ext (cont'd)

For $Q^{\cdot}$ an injective resolution of N,

$$\begin{aligned} \underline{\mathrm{Ext}}^i_{\mathrm{QGr}\text{-}A}(\pi(M),\pi(N)) &\cong h^i\left(\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}(M,\omega\pi(Q^{\cdot}))\right) \\ &\cong R^i\,\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}(M,\omega\pi(N)). \end{aligned}$$

Cohomology

Define the cohomology functors

$$H^i(\pi(M)) = \mathrm{Ext}^i_{\mathrm{QGr}\text{-}A}(\pi(A), \pi(M)) \cong h^i(\omega\pi(Q^\cdot))_0$$

and

$$\underline{H}^i(\pi(M)) = \bigoplus_{d\in\mathbb{Z}} H^i(\pi(M)[d]) \cong h^i(\omega\pi(Q^\cdot)).$$

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Definition 6

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(iii) We say M is bounded if it is left and right bounded.

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4. We say that $\chi_i(M)$ holds for an object of $\mathrm{Gr}\text{-}A$ if for all d and all $j \leq i$, there is an integer n_0 such that $\underline{\mathrm{Ext}}^j_{\mathrm{Gr}\text{-}A}(A/A_{\geq n}, M)_{\geq d}$ is an object of $\mathrm{gr}\text{-}A$ when $n_0 \leq n$.

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3. If $\chi_i^0(M)$ holds for A for every i, then we say that χ^0 holds for A.
4. We say that $\chi_i(M)$ holds for an object of $\mathrm{Gr}\text{-}A$ if for all d and all $j \le i$, there is an integer n_0 such that $\underline{\mathrm{Ext}}^j_{\mathrm{Gr}\text{-}A}(A/A_{\ge n}, M)_{\ge d}$ is an object of $\mathrm{gr}\text{-}A$ when $n_0 \le n$.
5. If χ_i holds for every object of $\mathrm{gr}\text{-}A$, then we say that χ_i holds for A.

The χ Condition

Definition 7

1. We say $\chi_i^0(M)$ holds if $\underline{\mathrm{Ext}}^j_{\mathrm{Gr}\text{-}A}(A_0, M)$ is bounded for all $j \leq i$.
2. If $\chi_i^0(M)$ holds for every object M of $\mathrm{gr}\text{-}A$, then we say that χ_i^0 holds for A.
3. If $\chi_i^0(M)$ holds for A for every i, then we say that χ^0 holds for A.
4. We say that $\chi_i(M)$ holds for an object of $\mathrm{Gr}\text{-}A$ if for all d and all $j \leq i$, there is an integer n_0 such that $\underline{\mathrm{Ext}}^j_{\mathrm{Gr}\text{-}A}(A/A_{\geq n}, M)_{\geq d}$ is an object of $\mathrm{gr}\text{-}A$ when $n_0 \leq n$.
5. If χ_i holds for every object of $\mathrm{gr}\text{-}A$, then we say that χ_i holds for A.
6. If χ_i holds for every i, then we say that χ holds for A.

Thank you!