Math 122

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Calculus for Business Administration and Social Sciences

Math 122
Farman
5.1: Distance and Accumulated Change
Constant Functions
Linear Functions
Non-Linear Functions
Right Endpoint Estimates
Left Endpoint Estimates
Partitions
Left- and Right-Hand Sums
Applying Our Method

Outline

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Constant Functions

Suppose a car is traveling at 60 miles per hour for 2 hours. How far did the car go?

This is easy:

$$60\,\frac{\text{miles}}{\text{hour}} \cdot 2 \text{ hours} = 120 \text{ miles}.$$

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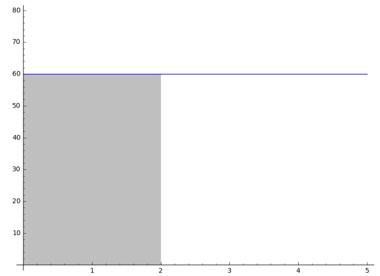
Constant Functions (cont.)

Geometrically, this is the area under the constant curve $y(t) = 60$ between $t = 0$ and $t = 2$:



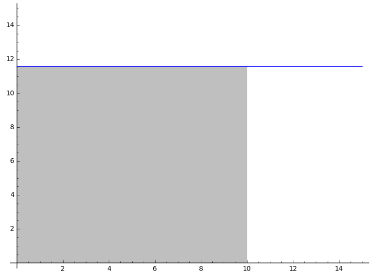
This says that under constant velocity, v, the position of the car, $s(t)$, relative to the starting point at time $0 \leq t$ is just

$$s(t) = v \cdot t.$$

Linear Functions

According to Car and Driver, a 2006 Bugatti Veyron is capable of an acceleration of $11.59\,\mathrm{m}/\mathrm{s}^2$. Assume the car starts at rest and accelerates at this constant rate.

By the observation in the last example, we can compute the velocity at time t as the area under the constant curve $y(t) = 11.59$:



Linear Functions (cont.)

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The velocity is linear: $v(t) = 11.59 \cdot t$. Hence the position, $s(t)$, is the area under the velocity curve:

Therefore the position at time t is:

$$\begin{aligned} s(t) &= \frac{1}{2}v(t)\cdot t = \frac{1}{2}(11.59\cdot t)\cdot t \\ &= \frac{11.59}{2}t^2. \end{aligned}$$

Non-Linear Functions

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What happens when the area is not a nice geometric object?

Can we tell how far a car traveled if we are given the following table of times and velocities?

time (sec)	0	2	4	6	8	10
speed (ft/sec)	20	30	38	44	48	50

This is clearly not linear:

$$\frac{30-20}{2-0} = 5 \text{ and } \frac{50-48}{10-8} = 1.$$

Non-Linear Functions (cont.)

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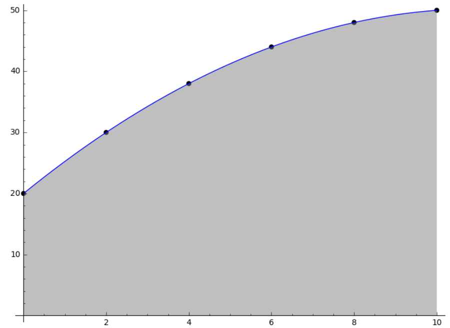
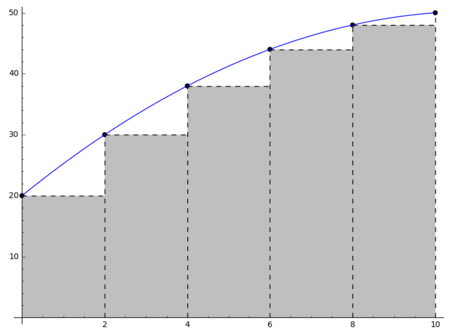
We can fit a curve to these points:



How do we compute the area of the shaded region?

Non-Linear Functions (cont.)

We could assume constant velocity between the two points and estimate. Say we assume the velocity is the velocity at the left endpoint:



Non-Linear Functions (cont.)

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This is an underestimate of the area.

- Each rectangle has width 2.
- The height of each rectangle is the height of the left endpoint.
- Our area estimate is:

$$\begin{aligned} 2(20 + 30 + 38 + 44 + 48) &= 2(180) \\ &= 360 \text{ feet.} \end{aligned}$$

Non-Linear Functions (cont.)

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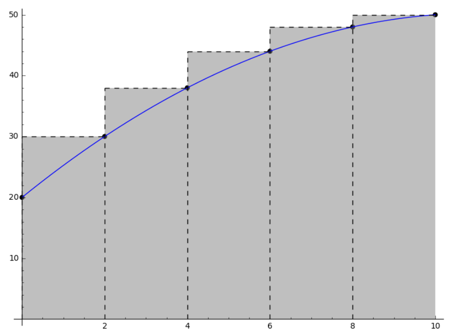
We could also assume the velocity is the velocity at the right endpoint:



Non-Linear Functions (cont.)

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This is an overestimate of the area.

- Each rectangle has width 2.
- The height of each rectangle is the height of the right endpoint.
- Our area estimate is:

$$\begin{aligned} 2(30 + 38 + 44 + 48 + 50) &= 2(210) \\ &= 420 \text{ feet.} \end{aligned}$$

Non-Linear Functions (Cont.)

This tells us:

- The distance traveled is **at least** 360 feet.
- The distance traveled is **at most** 420 feet.
- The distance traveled must be somewhere between these two.
- The average of these estimates is

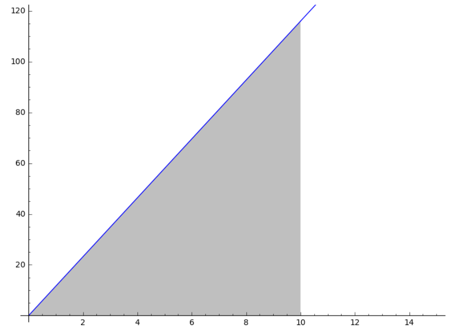
$$\frac{420+360}{2} = 390$$

  feet, which gives a better estimate.

Can we do better? If so, how?

Right Endpoint Estimates

We'll use the old linear velocity example, $v(t) = 11.59t$, to analyse these methods:

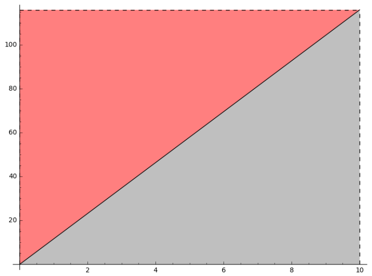


Two Equidistant Points

Say we use the two points $t = 0$ and $t = 10$. We know the area under the curve is given by:

$$\frac{1}{2}v(t)\cdot t.$$

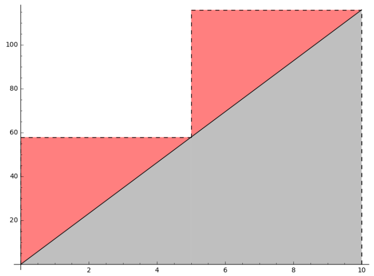
Our estimate is quite bad:



- Red is the error.
- Grey is the area.
- The estimate for the area is the sum of the red and grey areas.
- The error is equal to the actual area!

Three Equidistant Points

If we try three equidistant points, 0, $\frac{t}{2}$, and t, then we get:



- Visibly, this is a better estimate.
- The error is the area of the two red triangles.
- Both have base length $\frac{t}{2}$; here $t = 10$.
- The height of the left triangle is $v\left(\frac{t}{2}\right)$.
- The height of the right triangle is $v(t) - v\left(\frac{t}{2}\right)$.

Three Equidistant Points (cont.)

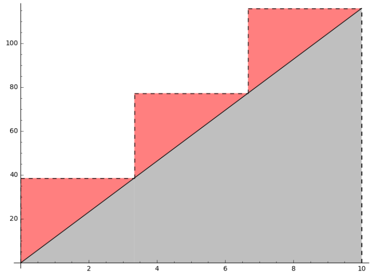
So, the total error is:

$$\begin{aligned}\frac{1}{2}\left[v(t) - v\left(\frac{t}{2}\right)\right]\frac{t}{2} + \frac{1}{2}v\left(\frac{t}{2}\right)\cdot\frac{t}{2} &= \frac{1}{2}\left[v(t) - v\left(\frac{t}{2}\right) + v\left(\frac{t}{2}\right)\right]\frac{t}{2}\\ &= \frac{1}{2}\left(\frac{1}{2}v(t)\cdot t\right).\end{aligned}$$

By adding one more point, we've reduced the error by a factor of two!

Four Equidistant Points

If we try four equidistant points, 0, $\frac{t}{3}$, $\frac{2t}{3}$, and t, then we get:



- Visibly, this is an even better estimate.
- All three red triangles have base length $\frac{t}{3}$.
- The height of the left triangle is $v\left(\frac{t}{3}\right)$.
- The height of the middle triangle is $v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right)$.
- The height of the right triangle is $v(t) - v\left(\frac{2t}{3}\right)$.

Four Equidistant Points (cont.)

So, the total error is:

$$\begin{aligned}\frac{1}{2}\left[v(t) - v\left(\frac{2t}{3}\right)\right]\frac{t}{3} + \frac{1}{2}\left[v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right)\right]\frac{t}{3} + \frac{1}{2}v\left(\frac{t}{3}\right)\frac{t}{3} &= \frac{1}{2}\left[v(t) - v\left(\frac{2t}{3}\right) + v\left(\frac{2t}{3}\right) - v\left(\frac{t}{3}\right) + v\left(\frac{t}{3}\right)\right]\frac{t}{3}\\ &= \frac{1}{3}\left(\frac{1}{2}v(t)\cdot t\right).\end{aligned}$$

By using four points, we've reduced the initial error by a factor of three!

$n+1$ Equidistant Points

If we use $n+1$ equidistant points,

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of n triangles. The k^{th} triangle, for $1 < k < n$, has:

- base length $\frac{t}{n}$,
- height $v(t_k) - v(t_{k-1})$,
- area
$$\frac{1}{2}\left[v\left(t_k\right) - v\left(t_{k-1}\right)\right]\frac{t}{n}$$

Remark 1

Note that $v(t_0) = v(0) = 0$.

$n+1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

$$\frac{1}{2}\left[v(t)-v(t_{k-1})+v(t_{k-1})-v(t_{k-2})+\ldots+v(t_2)-v(t_1)+v(t_1)-v(t_0)\right]\frac{t}{n} = \frac{1}{2}v(t)\cdot\frac{t}{n} = \frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right).$$

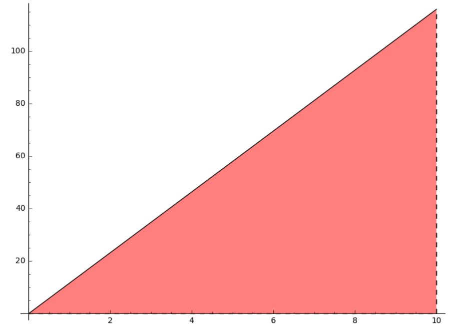
Therefore, if we use $n+1$ equidistant points, we have overestimated the area under $v(t)$ by

$$\frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right).$$

Left Estimate

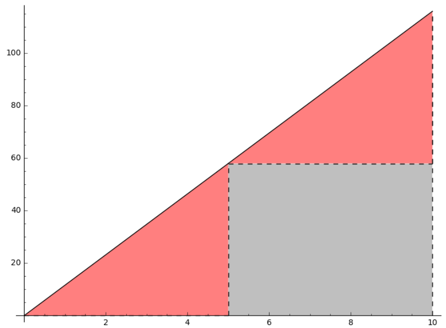
The situation for a left endpoint estimate is symmetric:

2 Equidistant Points:



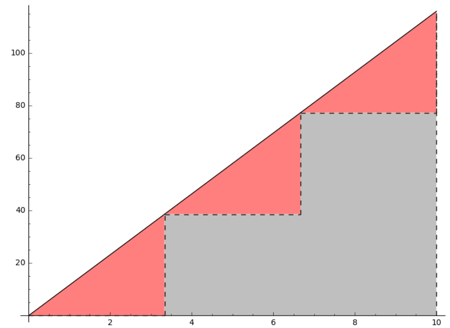
Our Estimate for the area here is **zero**. We have **underestimated** the area by $\frac{1}{2}v(t)\cdot t$.

3 Equidistant Points:



We have **underestimated** the area by $\frac{1}{2}\left(\frac{1}{2}v(t)\cdot t\right)$.

4 Equidistant Points:



We have **underestimated** the area by $\frac{1}{3}\left(\frac{1}{2}v(t)\cdot t\right)$.

Left Estimate

By the same analysis as with the right estimates, using $n+1$ equidistant points

$$t_0 = 0,\ t_1 = \frac{t}{n},\ t_2 = \frac{2t}{n},\ \ldots,\ t_{n-1} = \frac{(n-1)t}{n},\ t_n = t,$$

then we expect the error will be sum of the areas of n triangles. The k^{th} triangle, for $1 < k < n$, has:

- base length $\frac{t}{n}$,
- height $v(t_k) - v(t_{k-1})$,
- area

$$\frac{1}{2}\left[v\left(t_k\right) - v\left(t_{k-1}\right)\right]\frac{t}{n}$$

Remark 2

Note that $v(t_0) = v(0) = 0$.

$n+1$ Equidistant Points

Adding up the areas of each of the triangles, we get the total error:

$$\begin{aligned}\frac{1}{2}\left[v(t) - v(t_{k-1}) + v(t_{k-1}) - v(t_{k-2}) + \ldots + v(t_2) - v(t_1) + v(t_1) - v(t_0)\right]\frac{t}{n} &= \frac{1}{2}v(t)\cdot\frac{t}{n}\\ &= \frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right).\end{aligned}$$

Therefore, if we use $n+1$ equidistant points, we have **underestimated** the area under $v(t)$ by

$$\frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right).$$

More Is Better

- Using $n+1$ points for either a left or a right estimate, the absolute value of the error in estimating the area under the curve between 0 and $t = 10$ is given by

$$\frac{1}{n}\left(\frac{1}{2}v(t)\cdot t\right) = \frac{1}{n}\left(\frac{11.59}{2}100\right).$$

- This tells us that as n becomes large, the error decreases. That is, the more points, the better the estimate!
- As n grows larger, the right estimate **decreases** towards the actual area and the left estimate **increases** towards the actual area.

Right Error

Right Error Animation

LEFT ERROR

Left Error Animation

PARTITIONS OF AN INTERVAL

To generalize our methods to non-linear curves, we introduce some notation.

DEFINITION 1

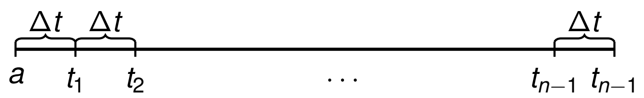
For a continuous function, f, on an interval $[a, b]$, a set of $n+1$ equidistant points,

$$t_0 = a < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$$

is called a *partition* of $[a, b]$.

PARTITIONS AND ESTIMATES

These $n+1$ points are called a partition because they partition $[a, b]$ into n smaller intervals of length Δt



where

$$\Delta t = \frac{b-a}{n}.$$

These n smaller intervals form the bases of the rectangles we use to estimate the area under a curve.

SUMS

DEFINITION 2

Let f be a continuous function on the interval $[a, b]$. Given a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

- The *Left-Hand Sum* is

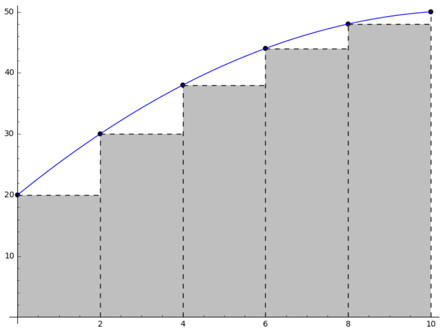
$$f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-2})\Delta t + f(t_{n-1})\Delta t.$$

- The *Right-Hand Sum* is

$$f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_{n-1})\Delta t + f(t_n)\Delta t.$$

Sums (cont.)

The Left-Hand Sum underestimates the area under our curve:



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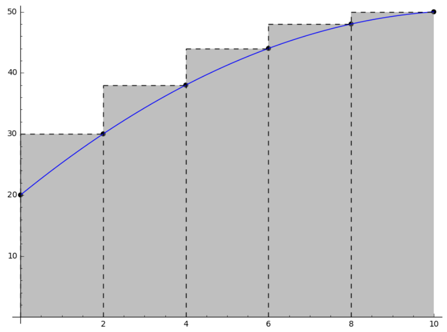
Sums (cont.)

The Right-Hand Sum overestimates the area under our curve:



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Sigma Notation

For ease of notation, we write the left-hand sum as

$$\sum_{i=0}^{n-1} f(t_i)\Delta t = f(t_0)\Delta t + \ldots + f(t_{n-1})\Delta t$$

and we write the right-hand sum as

$$\sum_{i=1}^{n} f(t_i)\Delta t = f(t_1)\Delta t + \ldots + f(t_n)\Delta t.$$

The letter i is the *index* of the summation and the letter n is the *upper bound* of the summation. The $i = 0$ underneath the sigma, Σ, indicates the sum starts at 0 and the upper bound indicates when to stop.

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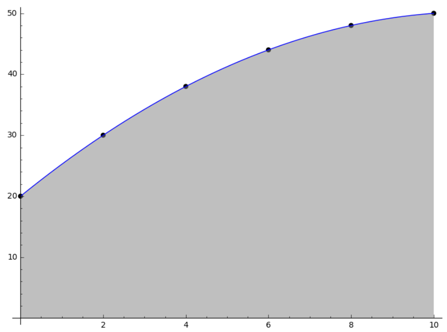
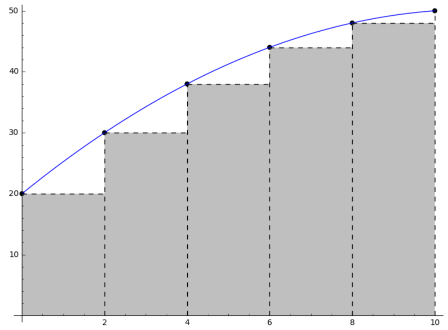
Generalizing Our Analysis

The entire point of our analysis of the linear velocity example was to improve our estimates for the non-linear curve



Generalizing our Analysis

When we use a Left-Hand Sum, we can't necessarily write down the error explicitly because the error isn't quite a triangle:



However, we can use differential calculus to get around this.

Linearization for Left-Hand Sums

Let f be a continuous function. Recall that if we take Δt sufficiently small, then we can use the Tangent Line Approximation,

$$f(t) \approx f'(a)(t-a) + f(a),$$

to ensure that f is basically a line whenever $a \leq t \leq a + \Delta t$.

Linearization for Left-Hand Sums (Cont.)

Say we want to find the area beneath a continuous curve, f, on the interval $[a, b]$.

- We can control the size of Δt by increasing the number of points in a partition

  $$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

  since

  $$\Delta t = \frac{b-a}{n}.$$

- This means that if we use enough points,

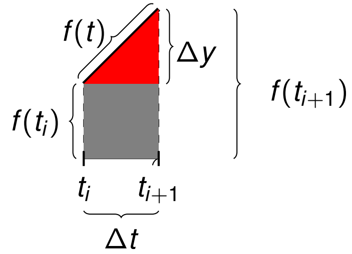
  $$f(t) \approx f'(t_i)(t - t_i) + f(t_i),$$

  whenever $t_i \leq t \leq t_{i+1}$, and in particular

  $$f(t_{i+1}) \approx f'(t_i)\Delta t + f(t_i).$$

Linearization for Left-Hand Sums (Cont.)

Using this linearization, we get the following picture on $[t_i, t_{i+1}]$:



By our previous analysis, the Left-Hand Sum underestimates the area under f on the interval $[t_i, t_{i+1}]$ by approximately

$$\frac{1}{2}\Delta y \Delta t = \frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t.$$

Linearization for Left-Hand Sums (cont.)

- By our work in Chapter 4, f attains a global maximum, M, and a global minimum, m, on $[a, b]$.
- This means we can bound the approximate error of the **underestimate** by

$$\frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t \leq \frac{1}{2}\left[M - m\right]\Delta t.$$

- Since $M - m$ is a fixed constant, this value goes to zero as n becomes large!
- This means we can compute the area under our curve to arbitrary precision by increasing the number of points in our partition.
- As we increase the number of points in our partition, the Left-Hand Sum **increases** towards the area under the curve.

Left Sum

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Left Estimate Animation

Linearization for Right-Hand Sums

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- Just as in the linear case, the analysis of the Right-Hand Sums is completely symmetric.
- After linearizing, the approximate error for the **overestimate** is

$$\frac{1}{2}\left[f(t_{i+1}) - f(t_i)\right]\Delta t \leq \frac{1}{2}\left[M - m\right]\Delta t.$$

- Again, as $M - m$ is a constant, this value goes to zero as n becomes large!
- This means we can compute the area under our curve to arbitrary precision by increasing the number of points in our partition.
- As we increase the number of points in our partition, the Right-Hand Sum **decreases** towards the area under the curve.

Right Sum

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Right Estimate Animation

Our Distance Traveled Example

Recall that we started this excursion with the following question:

Given the table of velocities and times

time (sec)	0	2	4	6	8	10
speed (ft/sec)	20	30	38	44	48	50

can we determine how far the car traveled?

Our Distance Traveled Example (cont.)

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It is possible to fit the data to the quadratic

$$v(t) = \frac{-1}{4}t^2 + \frac{11}{2}t + 20.$$

That is,

t	0	2	4	6	8	10
f(t)	20	30	38	44	48	50

This is the curve under which we've been attempting to estimate the area. Later, we'll be able to explicitly compute that the area under this curve–which represents the distance traveled over those ten seconds–is

$$\frac{1175}{3} = 391.\overline{6} \text{ feet}$$

Our Distance Traveled Example (cont.)

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With 5 equidistant points

- Our Left-Hand Sum estimated 360 feet,
- Our Right-Hand Sum estimated 420 feet,
- Our average estimated 390 feet, which was quite close.

Our Distance Traveled Example (cont.)

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Here is a table of Left-Hand Sums for $n+1$ points:

n	$\sum_{i=0}^{n-1} f(t_i)\Delta t$
10	376.25
100	390.1625
1,000	391.516625
10,000	391.65166625
100,000	391.6651666625

So we can see that as n increases, the Left-Hand Sums increase towards the actual area under the curve, as expected.

Our Distance Traveled Example (cont.)

Here is a table of Right-Hand Sums for $n+1$ points:

n	$\sum_{i=1}^{n} f(t_i)\Delta t$
10	406.25
100	393.1625
1,000	391.816625
10,000	391.68166625
100,000	391.6681666625

So we can see that as n increases, the Right-Hand Sums decrease towards the actual area under the curve, as expected.