Throughout, let $\mathscr{C}$ be an abelian category.

1. Preliminaries

Definition 1. A full subcategory, $\mathscr{A}$, of $\mathscr{C}$ is called a Serre (or épaisse/thick/dense) subcategory if for any short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

of $\mathscr{C}$, X is an object of $\mathscr{A}$ if and only if both X' and X'' are.

Definition 2. Let X be an object of $\mathscr{C}$. For two subobjects $i_1\colon X_1 \to X$ and $i_2\colon X_2 \to X$ denote by $X_1 \cap X_2$ the fibered product

$$\begin{tikzcd}
X_1 \cap X_2 \arrow[r, "x_2'"] \arrow[d, "x_1'"] & X_1 \arrow[d, "i_1"] \\
X_2 \arrow[r, "i_2"] & X
\end{tikzcd}$$

and denote by $X_1 + X_2$ the fibered coproduct

$$\begin{tikzcd}
X_1 \cap X_2 \arrow[r, "i_2'"] \arrow[d, "i_1'"] & X_1 \arrow[d, "u_1"] \\
X_2 \arrow[r, "u_2"] & X_1 + X_2.
\end{tikzcd}$$

These are both subobjects of X and endow the subobjects of X with lattice structure under the relation

$$X_1 \leq X_2$$

if there exists a monomorphism making the diagram

$$\begin{tikzcd}
X_1 \arrow[rr, dashed, "\exists!"] \arrow[dr, "i_1"'] & & X_2 \arrow[dl, "i_2"] \\
& X &
\end{tikzcd}$$

commute.

Remark 1. Alternatively, one can construct $X_1 + X_2$ as the image of the morphism s below

$$\begin{tikzcd}
X_1 \arrow[dr] \arrow[ddr, bend right, "i_1"'] & & X_2 \arrow[dl] \arrow[ddl, bend left, "i_2"] \\
& X_1 \coprod X_2 \arrow[d, dashed, "\exists! s"] & \\
& X. &
\end{tikzcd}$$

Definition 3. We say that a subobject, X', of an object, X, is an $\mathscr{A}$-subobject of X if X' is an object of $\mathscr{A}$. We say that an $\mathscr{A}$-subobject, X', is maximal if for every $\mathscr{A}$-subobject X'' we have a commutative diagram

$$\begin{tikzcd}
X'' \arrow[rr, dashed, "\exists! h"] \arrow[dr] & & X' \arrow[dl] \\
& X &
\end{tikzcd}$$

If X has no non-zero $\mathscr{A}$ subobjects, then we say that X is $\mathscr{A}$-torsionfree.

Proposition 1. *Let X and Y be objects of $\mathscr{C}$. The collection of pairs of subobjects (X', Y') such that X/X' and Y' are objects of $\mathscr{A}$ is directed by the relation*

$$(X', Y') \leq (X'', Y'')$$

if $X'' \leq X'$ and $Y' \leq Y''x$.

Moreover, the system of Abelian groups

$$\operatorname{Hom}_{\mathscr{C}}(X', Y/Y')$$

induced by pairs (X', Y') above is a directed system with morphisms

$$\operatorname{Hom}_{\mathscr{C}}(X', Y/Y') \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X'', Y'')$$

$$(X' \to Y/Y') \longmapsto (X'' \to X' \to Y/Y' \to Y/Y'')$$

whenever $(X', Y') \leq (X'', Y'')$.

Definition 4. Define the quotient category, $\mathscr{C}/\mathscr{A}$, to be the category with objects the objects of $\mathscr{C}$ and morphisms

$$\operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(X, Y) = \operatorname{colim}_{(X', Y')} \operatorname{Hom}_{\mathscr{C}}(X', Y/Y').$$

Let $\pi\colon \mathscr{C} \to \mathscr{C}/\mathscr{A}$ be the canonical projection functor, defined by $\pi(X) = X$ and sending a morphism $f\colon X \to Y$ to its image, $\pi(f)$, in the colimit.

Lemma 1. *The quotient category, $\mathscr{C}/\mathscr{A}$, is an additive category and π is an additive functor.*

Lemma 2. *Let $f\colon X \to Y$ be a morphism of $\mathscr{C}$. We have a factorization of f*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\operatorname{coim} f"'] & & Y \\ & f(X) \arrow[ur, "\operatorname{im} f"'] & \end{tikzcd}$$

and an exact sequence

$$0 \longrightarrow K \xrightarrow{\ker f} X \xrightarrow{f} Y \xrightarrow{\operatorname{coker} f} C \longrightarrow 0.$$

Then

(i) $\pi(f) = 0$ if and only if $f(X)$ is an object of $\mathscr{A}$,
(ii) $\pi(f)$ is a monomorphism if and only if K is an object of $\mathscr{A}$, and
(iii) $\pi(f)$ is an epimorphism if and only if C is an object of $\mathscr{A}$.

Lemma 3. *For any morphism $f\colon X \to Y$ of $\mathscr{C}$, we have an exact sequence*

$$0 \longrightarrow K \xrightarrow{\ker f} X \xrightarrow{f} Y \xrightarrow{\operatorname{coker} f} C \longrightarrow 0.$$

The morphism $\pi(f)$ has a kernel and a cokernel,

$$0 \longrightarrow \mathcal{K} \xrightarrow{\ker \pi(f)} \pi(X) \xrightarrow{\pi(f)} \pi(Y) \xrightarrow{\operatorname{coker} \pi(f)} \mathcal{C} \longrightarrow 0.$$

Moreover, $\pi(\ker f)$ induces an isomorphism $\pi(K) \cong \mathcal{K}$ and $\pi(\operatorname{coker} f)$ induces an isomorphism $\pi(C) \cong \mathcal{C}$.

Lemma 4. *Given an exact sequence*

$$0 \longrightarrow K \xrightarrow{\ker f} X \xrightarrow{f} Y \xrightarrow{\operatorname{coker} f} C \longrightarrow 0.$$

of $\mathscr{C}$, f is an isomorphism if and only if K and C are both objects of $\mathscr{A}$.

Proposition 2. *The quotient category $\mathscr{C}/\mathscr{A}$ is an abelian category and π is an exact functor.*

2. The Section Functor

Lemma 5. *If X is an object of $\mathscr{C}$, then the following are equivalent.*

(1) Given a short exact sequence

$$0 \longrightarrow K \xrightarrow{\ker f} Z \xrightarrow{f} Y \xrightarrow{\operatorname{coker} f} C \longrightarrow 0$$

with K and C objects of $\mathscr{A}$, then the canonical morphism

$$h_X(f)\colon h_X(Y) \to h_X(Z)$$

is an isomorphism,

(2) X is $\mathscr{A}$-torsionfree and any short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{\operatorname{coker} f} C \longrightarrow 0$$

with C an object of $\mathscr{A}$ splits, and

(3) For any object Y of $\mathscr{C}$, $\pi\colon \mathscr{C} \to \mathscr{C}/\mathscr{A}$ induces an isomorphism

$$\operatorname{Hom}_{\mathscr{C}}(Y, X) \cong \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(Y), \pi(X)).$$

Proof. (1) $\implies$ (2). Given an $\mathscr{A}$-subobject $i\colon X' \to X$, then we have the short exact sequence

$$0 \longrightarrow X' \xrightarrow{i} X \xrightarrow{\operatorname{coker} i} X/X' \longrightarrow 0$$

both X' and 0 are objects of $\mathscr{A}$, hence an isomorphism

$$h_X(\operatorname{coker} i)\colon \operatorname{Hom}_{\mathscr{C}}(X/X', X) \to \operatorname{Hom}_{\mathscr{C}}(X, X)$$

which implies that $\operatorname{coker} i$ is monic. Therefore $\operatorname{coker} i \circ i = 0$ implies $i = 0$.

Now, if we let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{p} C \longrightarrow 0$$

be a short exact sequence with C an object of $\mathscr{A}$, then the isomorphism

$$h_X(f)\colon \operatorname{Hom}_{\mathscr{C}}(Y, X) \to \operatorname{Hom}_{\mathscr{C}}(X, X)$$

yields a section $s\colon Y \to X$ of f, so the sequence splits.

(2) $\implies$ (3). Let Y be an object of $\mathscr{C}$. Given a morphism $f : \pi(Y) \to \pi(X)$, we lift to a morphism $f'\colon Y' \to X/X'$ with Y/Y' and X' objects of $\mathscr{A}$. Since we have assumed that X has no non-trivial $\mathscr{A}$-subobjects, it follows that $X/X' = X$. By dualizing the relevant theorems on fiber products, this gives the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Y' \arrow[r, "j"] \arrow[d, "f'"] & Y \arrow[r, "\operatorname{coker} j"] \arrow[d, "f''"] & Y/Y' \arrow[r] \arrow[d, dashed, "\exists! h"] & 0 \\
0 \arrow[r] & X \arrow[r, "i"] & Y \coprod_{Y'} X \arrow[r, "\operatorname{coker} i"] & \left(Y \coprod_{Y'} X\right)/X \arrow[r] & 0
\end{tikzcd}$$

and with h an isomorphism. Since Y/Y' was assumed to be an object of $\mathscr{A}$, so too is $\left(Y \coprod_{Y'} X\right)/X$ and thus there exists a section $s : Y \coprod_{Y'} X \to X$ of i so that

$$f' = id_X \circ f' = s \circ i \circ f' = s \circ f'' \circ j.$$

By commutativity of the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathscr{C}}(Y, X) \arrow[rr, "_ \circ j"] \arrow[dr] & & \operatorname{Hom}_{\mathscr{C}}(Y', X) \arrow[dl] \\
& \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(Y), \pi(X)) &
\end{tikzcd}$$

we see that $\pi(s \circ f'') = f$ and thus

$$\operatorname{Hom}_{\mathscr{C}}(Y, X) \to \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(Y), \pi(X))$$

is surjective. For injectivity, suppose that $f \colon Y \to X$ satisfies $\pi(f) = 0$. Then $f(Y)$ is an object of $\mathscr{A}$ and from the short exact sequence

$$0 \longrightarrow f(Y) \xrightarrow{\operatorname{im} f} X \xrightarrow{\operatorname{coker} f} C \longrightarrow 0$$

we see that $\operatorname{im} f = 0$. Therefore $f = \operatorname{im} f \circ \operatorname{coim} f = 0$, as desired.

(3) $\implies$ (1). Let

$$0 \longrightarrow K \xrightarrow{i} Z \xrightarrow{f} Y \xrightarrow{p} C \longrightarrow 0$$

be an exact sequence with K and C objects of $\mathscr{A}$. We have the commutative diagram

$$\begin{tikzcd}
Z \arrow[d, "f"] & \operatorname{Hom}_{\mathscr{C}}(Y, X) \arrow[r] \arrow[d, "h_X(f)"] & \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(Y), \pi(X)) \arrow[d, "h_{\pi(X)}(\pi(f))"] \\
Y & \operatorname{Hom}_{\mathscr{C}}(Z, X) \arrow[r, "\sim"] & \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(Z), \pi(X))
\end{tikzcd}$$

with $h_{\pi(X)}(\pi(f))$ an isomorphism because $\pi(f)$ is. Therefore $h_X(f)$ is an isomorphism, as desired. ■

Definition 5. (i) If X is an object of $\mathscr{C}$ satisfying any of the conditions in Lemma 5, then we say that X is $\mathscr{A}$-closed.

(ii) A morphism $X \to Y$ is an $\mathscr{A}$-envelope if in the exact sequence

$$0 \longrightarrow K \longrightarrow X \longrightarrow Y \longrightarrow C \longrightarrow 0$$

Y is $\mathscr{A}$-closed, and both K and C are objects of $\mathscr{A}$.

Lemma 6. *If X has a maximal $\mathscr{A}$-subobject, $X_{\mathscr{A}}$, then $X/X_{\mathscr{A}}$ is $\mathscr{A}$-torsionfree.*

Proof. Let $j \colon Y \to X/X_{\mathscr{A}}$ be a monic with Y an object of $\mathscr{A}$. We have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\ker p'"] \arrow[d, dashed, "\exists! h"] & X \times_{X/X_{\mathscr{A}}} Y \arrow[r, "p'"] \arrow[d, "i'"] & Y \arrow[r, "\operatorname{coker} p'"] \arrow[d, "j"] & C \arrow[d, dashed, "\exists! h'"] \\ 0 \arrow[r] & X_{\mathscr{A}} \arrow[r, "i"] & X \arrow[r, "p"] & X/X_{\mathscr{A}} \arrow[r, "\operatorname{coker} p"] & 0 \end{tikzcd}$$

with h an isomorphism, and h' monic, hence an isomorphism. The top row gives us the short exact sequence

$$0 \longrightarrow K \xrightarrow{\ker p'} X \times_{X/X_{\mathscr{A}}} Y \xrightarrow{p'} Y \xrightarrow{\operatorname{coker} p'} 0$$

with K and Y objects of $\mathscr{A}$, hence $X \times_{X/X_{\mathscr{A}}} Y$ is also an object of $\mathscr{A}$. By maximality of $X_{\mathscr{A}}$, i' factors through i uniquely,

$$\begin{tikzcd} X \times_{X/X_{\mathscr{A}}} Y \arrow[rr, dashed, "i''"] \arrow[dr, "i'"'] & & X_{\mathscr{A}} \arrow[dl, "i"] \\ & X & \end{tikzcd}$$

and so we see

$$j \circ p' = p \circ i = p \circ (i \circ i'') = (p \circ i) \circ i'' = 0$$

implies, because p' is epic, that $j = 0$. Therefore $X/X_{\mathscr{A}}$ is $\mathscr{A}$-torsionfree, as desired. ■

Lemma 7. *If $\mathscr{C}$ is such that every object of $\mathscr{C}$ has a maximal $\mathscr{A}$-subobject and every $\mathscr{A}$-torsionfree object has a monomorphism to an $\mathscr{A}$-closed object, then every object of $\mathscr{C}$ has an $\mathscr{A}$-envelope.*

Proof. Let X be an object of $\mathscr{C}$ and let $X_{\mathscr{A}}$ be its maximal $\mathscr{A}$-subobject, so we have the short exact sequence

$$0 \longrightarrow X_{\mathscr{A}} \xrightarrow{i} X \xrightarrow{p} X/X_{\mathscr{A}} \longrightarrow 0.$$

By assumption, there exists an $\mathscr{A}$-closed object Y and a short exact sequence

$$0 \longrightarrow X/X_{\mathscr{A}} \xrightarrow{j} Y \xrightarrow{q} C \longrightarrow 0$$

from which we construct the pullback

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\ker q'"] \arrow[d, dashed, "\exists! h"] & q^{-1}(C_{\mathscr{A}}) \arrow[r, "q'"] \arrow[d, "k'"] & C_{\mathscr{A}} \arrow[r] \arrow[d, "k"] & 0 \\ 0 \arrow[r] & X/X_{\mathscr{A}} \arrow[r, "j"] & Y \arrow[r, "q"] & C \arrow[r] & 0, \end{tikzcd}$$

with h an isomorphism. Then from the short exact sequence

$$0 \longrightarrow X/X_{\mathscr{A}} \cong K \xrightarrow{\ker q'} q^{-1}(C_{\mathscr{A}}) \xrightarrow{q'} C_{\mathscr{A}} \longrightarrow 0$$

it suffices to show that $q^{-1}(C_{\mathscr{A}})$ is $\mathscr{A}$-closed.

It's clear that $q^{-1}(C_{\mathscr{A}})$ is $\mathscr{A}$-torsionfree because it is a subobject of the $\mathscr{A}$-closed object Y. If we have any short exact sequence

$$0 \longrightarrow q^{-1}(C_{\mathscr{A}}) \xrightarrow{s} A \xrightarrow{\operatorname{coker} s} B \longrightarrow 0$$

with B an object of $\mathscr{A}$, then by Lemma 1 there is a unique morphism $\varphi : A \to Y$ such that

$$k' = \varphi \circ s = h_Y(s)(\varphi).$$

Now we have the commutative diagram

$$\begin{tikzcd} A \arrow[rrd, bend left, "0"] \arrow[rd, dashed, "\exists! r"] \arrow[rdd, bend right, "\varphi"'] & & \\ & q^{-1}(C_{\mathscr{A}}) \arrow[r, "q'"] \arrow[d, "k'"] & C_{\mathscr{A}} \arrow[d, "k"] \\ & Y \arrow[r, "q"] & C \end{tikzcd}$$

from which we see that

$$k' \circ id_{q^{-1}(C_{\mathscr{A}})} = k' = \varphi \circ s = (k' \circ r) \circ s = k'(\circ r \circ s)$$

and thus $r \circ s = id_{q^{-1}(C_{\mathscr{A}})}$. Therefore $q^{-1}(C_{\mathscr{A}})$ is $\mathscr{A}$-closed by Lemma 5.2, as desired. ∎

Lemma 8. *If $\pi \colon \mathscr{C} \to \mathscr{C}/\mathscr{A}$ has a right adjoint, $\omega \colon \mathscr{C}/\mathscr{A} \to \mathscr{C}$, then*

(1) for each object Y of $\mathscr{C}$, $\omega\pi(Y)$ is $\mathscr{A}$-closed,
(2) for Y an object of $\mathscr{C}$, the morphism $\eta_{\pi(Y)} : \pi\omega\pi(Y) \to \pi(Y)$ is an isomorphism, and
(3) ω is fully faithful.

Proof. (1) Given an exact sequence

$$0 \longrightarrow K \longrightarrow Z \xrightarrow{f} Y \longrightarrow C \longrightarrow 0$$

with K and C objects of $\mathscr{A}$, we have that $\pi(f)$ is an isomorphism and hence $h_{\pi(Y)}(\pi(f))$ is also an isomorphism. From the adjunction we get the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(X, \omega\pi(Y)) \arrow[r, "\sim"] \arrow[d, "h_{\omega\pi(Y)}(f)"] & \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(X), \pi(Y)) \arrow[d, "h_{\pi(Y)}(\pi(f))"] \\ \mathrm{Hom}_{\mathscr{C}}(Z, \omega\pi(Y)) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(Z), \pi(Y)) \end{tikzcd}$$

which shows that $h_{\pi(Y)}\pi(F)$ is an isomorphism. Therefore $\omega\pi(Y)$ is $\mathscr{A}$-closed by part 1 of Lemma 5.

(2) We have the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(\omega\pi(Y), \omega\pi(Y)) \arrow[rr, "\sim"] \arrow[dr, "\sim"] & & \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}(\pi\omega\pi(Y), \pi\omega\pi(Y)) \arrow[dl, "h_{\pi\omega\pi(Y)}(\eta_{\pi(Y)})"] \\ & \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}(\pi\omega\pi(Y), \pi(Y)) & \end{tikzcd}$$

since for any morphism $f \colon \omega\pi(Y) \to \omega\pi(Y)$, the image under the adjunction isomorphism is just $\eta_{\pi(Y)} \circ \pi(f)$. This immediately implies that $h_{\pi\omega\pi(Y)}(\eta_{\pi(Y)})$ is an isomorphism, and hence so is $\eta_{\pi(Y)}$.

(3) Since ω being fully faithful is equivalent to η being a natural isomorpism, this is a consequence of the definition of $\mathscr{C}/\mathscr{A}$. Indeed, every object of $\mathscr{C}/\mathscr{A}$ is $\pi(X)$ for some object X of $\mathscr{C}$, and the result follows.

∎

Theorem 1. *The following are equivalent.*

(1) $\pi \colon \mathscr{C} \to \mathscr{C}/\mathscr{A}$ has a right adjoint, and
(2) Every object of $\mathscr{A}$ has a maximal $\mathscr{A}$-subobject and every $\mathscr{A}$-torsionfree object has a monomorphism into an $\mathscr{A}$-closed object.

Proof. First assume that π has a right adjoint, $\omega\colon \mathscr{C}/\mathscr{A} \to \mathscr{C}$, and let Y be an object of $\mathscr{C}$. There are then two natural transformations of adjunction, $\varepsilon : \mathrm{id}_{\mathscr{C}} \to \omega\pi$ and $\eta : \pi\omega \to \mathrm{id}_{\mathscr{C}/\mathscr{A}}$, the latter being an isomorphism by Lemma 8. It follows from the commutative diagram

$$\begin{tikzcd} \pi(Y) \arrow[r, "\pi(\varepsilon_Y)"] \arrow[dr, "\mathrm{id}_{\pi(Y)}"'] & \pi\omega\pi(Y) \arrow[d, "\eta_{\pi(Y)}"] \\ & \pi(Y) \end{tikzcd}$$

that $\pi(\varepsilon_Y) = \eta^{-1}_{\pi(Y)}$ is an isomorphism, whence in the short exact sequence

$$0 \longrightarrow K \longrightarrow Y \xrightarrow{\varepsilon_Y} \omega\pi(Y) \longrightarrow C \longrightarrow 0$$

both K and C are objects of $\mathscr{A}$. We show that K is the desired subobject. Indeed, let $j\colon Y' \to Y$ be a subobject of Y with Y' and object of $\mathscr{A}$. We have the commutative diagram

$$\begin{tikzcd} Y' \arrow[rr, "\varepsilon_Y \circ j"] \arrow[dr, "\mathrm{coim}\,(\varepsilon_Y \circ j)"'] & & \omega\pi(Y) \\ & \varepsilon_Y(Y') \arrow[ur, "\mathrm{im}\,(\varepsilon_Y \circ j)"'] & \end{tikzcd}$$

and we note that because $\omega\pi(Y)$ is $\mathscr{A}$-closed and $\varepsilon_Y(Y') \cong Y'/(Y' \cap K)$ is an object of $\mathscr{A}$, the monic $\mathrm{im}\,(\varepsilon_Y \circ j)$ is zero. Therefore by the kernel diagram

$$\begin{tikzcd} Y' \arrow[r, "j"] \arrow[rr, bend left, "0"] \arrow[dr, dashed, "\exists! j'"'] & Y \arrow[r, "\varepsilon_Y"] & \omega\pi(Y) \\ & K \arrow[u] & \end{tikzcd}$$

we see that j' is monic, and K is maximal, as desired.

Conversely, assume that every object of $\mathscr{C}$ has a maximal $\mathscr{A}$-subobject and every $\mathscr{A}$-torsionfree object has a monomorphism into an $\mathscr{A}$-closed object. Let Y be an object of $\mathscr{C}$. By Lemma 7, Y has an $\mathscr{A}$-envelope $Y \to E$. Hence $\pi(Y) \cong \pi(E)$ and by the natural isomorphisms

$$\mathrm{Hom}_{\mathscr{C}}\left(\_, E\right) \cong \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(\_), \pi(E)\right) \cong \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(\_), \pi(Y)\right),$$

the presheaf $\mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(\_), \pi(Y)\right)$ on $\mathscr{C}$ is representable. Therefore π admits a right adjoint. ■

Definition 6. If π has a right adjoint, then we say that $\mathscr{A}$ is a localizing subcategory.

Corollary 1. *Assume that π has a right adjoint, $\omega\colon \mathscr{C}/\mathscr{A} \to \mathscr{C}$. Then*

(1) $\mathscr{A}$-envelopes are unique up to unique isomorphism,

(2) for every object X of $\mathscr{C}$, $\omega\pi(X) \cong E$, where $X \to E$ is an $\mathscr{A}$-envelope of X,

Proof. (1) By the proof of Theorem 1, an $\mathscr{A}$-envelope of an object Y of $\mathscr{C}$ represents the presheaf $\mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(\_), Y\right)$ and thus is unique up to unique isomorphism.

(2) This is immediate from Yoneda's Lemma.

■

Lemma 9. *Assume that $\mathscr{A}$ is a localizing subcategory, X, Y, objects of $\mathscr{C}$, $X_{\mathscr{A}}$, $Y_{\mathscr{A}}$, their maximal $\mathscr{A}$-subobjects. A morphism $f\colon X \to Y$ induces a morphism*

$$\begin{tikzcd} X_{\mathscr{A}} \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[d, "q"] \\ & X/X_{\mathscr{A}} \arrow[r, dashed, "\exists! h"] & Y/Y_{\mathscr{A}} \end{tikzcd}$$

and the morphism $\pi(f)$ is an essential extension if and only if h is.

Proof. We first note that $\pi(p)$ and $\pi(h)$ are isomorphisms, hence essential extensions, so

$$\pi(f) = \pi(q)^{-1} \circ \pi(h) \circ \pi(p)$$

is an essential extension if and only if $\pi(h)$ is. Hence it suffices to assume that $X_{\mathscr{A}} = Y_{\mathscr{A}} = 0$ and $h = f$.

Assume first that $\pi(f)$ is an essential extension. Given a subobject $k : Y' \to Y$ we get the pullback

$$\begin{tikzcd} \pi(Y' \times_Y X) \arrow[r, "\pi(k')"] \arrow[d, "\pi(f')"] & \pi(X) \arrow[d, "\pi(f)"] \\ \pi(Y') \arrow[r, "\pi(k)"] & \pi(Y) \end{tikzcd}$$

because π is exact. We note that so long as Y' is not an object of $\mathscr{A}$, $\pi(Y')$ is not zero. Since Y was assumed to be $\mathscr{A}$-torsionfree, this is equivalent to Y' being non-zero. Therefore $\pi(Y' \times_Y X)$ is non-zero whenever Y' is non-zero because $\pi(f)$ is essential and hence so is $Y' \times_Y X$.

Conversely, assume that f is an essential extension. Given $i : \pi(Z) \to \pi(Y)$ a non-zero subobject, we may lift to a morphism

$$0 \longrightarrow K \xrightarrow{\ker j} Z' \xrightarrow{j} Y$$

with Z/Z' and K objects of $\mathscr{A}$ since k is monic and Y has no non-zero $\mathscr{A}$-subobjects. Since f is an essential extension we have the non-zero pullback

$$\begin{tikzcd} Z'/K \times_Y X \arrow[r, "k"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Z/K \arrow[r, "\operatorname{im} j"] & Y \end{tikzcd}$$

As K is an object of $\mathscr{A}$, the short exact sequence

$$0 \longrightarrow K \xrightarrow{\ker j} Z' \xrightarrow{\operatorname{coim} j} Z'/K \longrightarrow 0$$

gives the isomorphism

$$\pi(Z'/K) \cong \pi(Z') \cong \pi(Z).$$

Therefore

$$\pi(Z'/K \times_Y X) \cong \pi(Z'/K) \times_{\pi(Y)} \pi(X) \cong \pi(Z) \times_{\pi(Y)} \pi(X)$$

is non-zero, as desired. ∎

Lemma 10. *If Q is an $\mathscr{A}$-closed injective, then $\pi(Q)$ is injective.*

Proof. Given a short exact sequence

$$0 \longrightarrow \pi(Q) \xrightarrow{s} \pi(X) \xrightarrow{\operatorname{coker} s} \pi(X/Q) \longrightarrow 0$$

it is enough to show that s is a section; that is there exists a morphism $r\colon \pi(X) \to \pi(Q)$ such that $r \circ s = \mathrm{id}_{\pi(Q)}$. We can lift s to a morphism

$$0 \longrightarrow K \xrightarrow{\ker t} Q' \xrightarrow{t} X/X'$$

with K, Q/Q', and X' objects of $\mathscr{A}$. Since we have assumed that Q is $\mathscr{A}$-closed, the diagram

$$\begin{tikzcd} K \arrow[r, "\ker t"] \arrow[rr, bend left, "0"] & Q' \arrow[r, "i"] & Q \end{tikzcd}$$

commutes and thus we see that $\ker t = 0$ because i is a monomorphism. Since Q was assumed to be injective, we have the lift

$$\begin{tikzcd} 0 \arrow[r] & Q' \arrow[r, "t"] \arrow[d, "i"] & X/X' \arrow[dl, dashed, "\exists r"] \\ & Q. & \end{tikzcd}$$

If we let $q\colon X \to X/X'$ be the canonical projection, then we have the diagram

$$\begin{tikzcd} \pi(Q') \arrow[r, "\pi(t)"] \arrow[d, "\pi(i)"'] & \pi(X/X') \arrow[dl, "\pi(r)"'] \\ \pi(Q) \arrow[r, "s"] & \pi(X) \arrow[u, "\pi(q)"'] \end{tikzcd}$$

with $\pi(i)$ and $\pi(q)$ isormoprhisms, the top left triangle commutative, and the outer square commutative. Therefore

$$\mathrm{id}_{\pi(Q)} \circ \pi(i) = \pi(i) = \pi(r) \circ \pi(t) = \pi(r) \circ \pi(q) \circ s \circ \pi(i)$$

implies, because $\pi(i)$ is an isomorphism, that

$$(\pi(r) \circ \pi(q)) \circ s = \mathrm{id}_{\pi(Q)},$$

as desired. ■

Lemma 11. *If $i\colon X \to E$ is an injective envelope and X is $\mathscr{A}$-torsionfree, then E is $\mathscr{A}$-closed and the morphism $\pi(i)\colon \pi(X) \to \pi(E)$ is an injective envelope.*

Proof. Since E is injective, every short exact sequence

$$0 \longrightarrow E \longrightarrow A \longrightarrow B \longrightarrow 0$$

splits. To see that E is $\mathscr{A}$-closed, it then suffices by Lemma 5.2 to show that E is $\mathscr{A}$-torsionfree. Given an $\mathscr{A}$-subobject $j : E' \to E$, we have the pullback

$$\begin{tikzcd} E' \times_E X \arrow[r, "j'"] \arrow[d, "i'"] & X \arrow[d, "i"] \\ E' \arrow[r, "j"] & E \end{tikzcd}$$

and the morphism i' gives $E' \times_E X$ E-subobject structure, hence is an object of $\mathscr{A}$. Since X is $\mathscr{A}$-torsionfree by assumption, $E' \times_E X = 0$ and thus E' is also zero because i is essential.

By Lemma 10 we see that $\pi(E)$ is injective, so it remains to show that $\pi(i)$ is essential. To see this, we note that the assumption $\mathscr{A}$ is a localizing subcategory in Lemma 9 was only used to produce maximal $\mathscr{A}$-subobjects, and hence the same argument shows that $\pi(i)$ is essential. Therefore $\pi(i)$ is an injective envelope. ∎

Proposition 3. *Assume that $\mathscr{A}$ is a localizing subcategory. If $\mathscr{C}$ has injective envelopes, then*

(i) $\mathscr{C}/\mathscr{A}$ has injective envelopes,

(ii) Every injective object of $\mathscr{C}/\mathscr{A}$ is isomorphic to $\pi(Q)$ for some $\mathscr{A}$-closed injective, Q, and

(iii) Every injective object Q of $\mathscr{C}$ is isomorphic to $E \oplus \omega(Q_2)$, where $Q_{\mathscr{A}} \to E$ is an injective envelope of the maximal $\mathscr{A}$-subobject of Q and Q_2 is an injective object of $\mathscr{C}/\mathscr{A}$.

Proof. (i) Given an object $\pi(X)$ of $\mathscr{C}/\mathscr{A}$, let $X_{\mathscr{A}}$ be the maximal $\mathscr{A}$-subobject of X. Since $\mathscr{C}$ has injective envelopes and $X/X_{\mathscr{A}}$ is $\mathscr{A}$-torsionfree, an injective envelope $X/X_{\mathscr{A}} \to E$ gives the injective envelope

$$\pi(X) \cong \pi(X/X_{\mathscr{A}}) \to \pi(E)$$

by Lemma 11.

(ii) Given an injective object $\pi(Q)$ of $\mathscr{C}/\mathscr{A}$, $\omega\pi(Q)$ is $\mathscr{A}$-closed by Lemma 8.1 and is injective because π is exact. Therefore by Lemma 8.3, $\pi(Q) \cong \pi(\omega\pi(Q))$.

(iii) Let Q be an injective object of $\mathscr{C}$, let $i\colon Q_{\mathscr{A}} \to Q$ be its maximal $\mathscr{A}$-subobject, and let $j\colon Q_{\mathscr{A}} \to E$ be an injective envelope. Since Q is injective we have the lift

$$\begin{tikzcd}
0 \arrow[r] & Q_{\mathscr{A}} \arrow[r, "j"] \arrow[d, "i"] & E \arrow[dl, dashed, "\exists k"] \\
& Q &
\end{tikzcd}$$

with k a monomorphism because j is essential and $\ker(k \circ j) = \ker i = 0$. Because E is injective we get the split exact sequence

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, shift left, "k"] & Q \arrow[l, shift left, "r"] \arrow[r, shift left, "p"] & Q/E \arrow[l, shift left, "s"] \arrow[r] & 0
\end{tikzcd}$$

so we need only show that Q/E is an $\mathscr{A}$-closed injective, for then $\pi(Q/E)$ is injective by Lemma 10, and $Q/E \cong \omega\pi(Q/E)$.

The fact that Q/E is injective follows from the fact that both Q and E are injective. Thus every monomorphism out of Q/E splits, so by Lemma 5.2 it is enough to show that Q/E is $\mathscr{A}$-torsionfree. Given an $\mathscr{A}$-subobject $\varphi\colon X \to E$, the fact that $Q/Q_{\mathscr{A}}$ is $\mathscr{A}$-torsionfree gives the kernel diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[rrr, bend left, "0"] \arrow[drr, dashed, "\exists! h"'] & Q/E \arrow[r, "s"] & Q \arrow[r] & Q/Q_{\mathscr{A}} \\
& & Q_{\mathscr{A}} \arrow[u, "i"] &
\end{tikzcd}$$

Therefore

$$\varphi = \mathrm{id}_{Q/E} \circ \varphi = p \circ s \circ \varphi = p \circ i \circ h = p \circ k \circ j \circ h = 0,$$

as desired. ∎

Corollary 2. *Assume that $\mathscr{A}$ is a localizing subcategory and that $\mathscr{C}$ has injective envelopes. If the injective envelope of an object of $\mathscr{A}$ is a morphism of $\mathscr{A}$, then*

(i) the maximal subobject of an injective is injective and thus the $\mathscr{A}$-envelope of an injective, Q, is $Q \to Q/Q_{\mathscr{A}}$, where $Q_{\mathscr{A}}$ is the maximal $\mathscr{A}$-subobject,

(ii) π preserves injectives, and

Proof. (i) Let Q be an injective object of $\mathscr{C}$ and $i\colon Q_{\mathscr{A}} \to Q$ its maximal $\mathscr{A}$-subobject. Given an injective envelope $j\colon Q_{\mathscr{A}} \to E$, we have the lift

$$\begin{tikzcd} 0 \arrow[r] & Q_{\mathscr{A}} \arrow[r, "j"] \arrow[d, "i"] & E \arrow[dl, dashed, "\exists k"] \\ & Q & \end{tikzcd}$$

with k a monomorphism because j is essential. Since E is assumed to be an object of $\mathscr{A}$, k factors through i uniquely,

$$\begin{tikzcd} E \arrow[r, dashed, "\exists! \varphi"] \arrow[dr, "k"'] & Q_{\mathscr{A}} \arrow[d, "i"] \\ & Q. \end{tikzcd}$$

So we see that

$$k \circ j \circ \varphi = i \circ \varphi = k = k \circ id_E$$

implies that $j \circ \varphi = id_E$ and

$$i \circ \varphi \circ j = k \circ j = i = i \circ id_{Q_{\mathscr{A}}}$$

implies $\varphi \circ j = id_{Q_{\mathscr{A}}}$. Hence φ is an isomorphism. Therefore by Proposition 3 we have the short exact sequence

$$0 \longrightarrow Q_{\mathscr{A}} \longrightarrow Q_{\mathscr{A}} \oplus Q/Q_{\mathscr{A}} \longrightarrow Q/Q_{\mathscr{A}} \longrightarrow 0$$

and $Q/Q_{\mathscr{A}}$ is $\mathscr{A}$-closed, as desired.

(ii) If Q is an injective object of $\mathscr{C}$, then by the above $Q/Q_{\mathscr{A}}$ is $\mathscr{A}$-closed and hence $\pi(Q) \cong \pi(Q/Q_{\mathscr{A}})$ is injective by Lemma 10. ∎

Remark 2. When A is a right Noetherian $\mathbb{N}$-graded ring, $\mathscr{C} = \mathrm{Gr}-A$, and $\mathscr{A} = \mathrm{Tors}$, then $\mathscr{A}$ is closed under essential extensions and hence under injective envelopes. In particular, every injective module, Q, decomposes as $\tau(Q) \oplus Q/\tau(Q)$ with $Q/\tau(Q)$ an $\mathscr{A}$-closed object, and $\omega\pi(Q) \cong Q/\tau(Q)$.

3. Cohomology

Corollary 3. *Assume that $\mathscr{A}$ is a localizing subcategory and that $\mathscr{C}$ has injective envelopes. If the injective envelope of an object of $\mathscr{A}$ is a morphism of $\mathscr{A}$, then for objects X and Y of $\mathscr{C}$*

$$\mathrm{Ext}^i_{\mathscr{C}/\mathscr{A}}(\pi(X), \pi(Y)) \cong R^i \operatorname{Hom}_{\mathscr{C}}(X, \omega\pi(Y)).$$

Proof. Take an injective resolution

$$Q^{\cdot} : 0 \longrightarrow Y = Q^0 \xrightarrow{d^0} Q^1 \xrightarrow{d^1} \cdots$$

of Y. By Corollary 2.ii, $\pi(Q^{\cdot})$ is an injective resolution of $\pi(Y)$. Using the natural transformation $\varepsilon\colon id_{\mathscr{C}} \to \omega\pi$ we have for each n an isomorphism of adjunction

$$\Phi^n\colon \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(X), \pi(Q^n)) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X, \omega\pi(Q^n))$$

$$\varphi \longmapsto \omega(\varphi) \circ \varepsilon_X$$

and so we get an isomorphism of chain complexes

$$\begin{array}{ccccccc} \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(X), \pi(Q^{\cdot})) : 0 \longrightarrow & \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(X), \pi(Y)) & \xrightarrow{h^{\pi(X)}(\pi(d^0))} & \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(X), \pi(Q^1)\right) & \xrightarrow{h^{\pi(X)}(\pi(d^1))} \cdots \\ & \downarrow{\Phi^0} & & \downarrow{\Phi^1} & \\ \operatorname{Hom}_{\mathscr{C}}(X, \omega\pi(Q^{\cdot})) : 0 \longrightarrow & \operatorname{Hom}_{\mathscr{C}}(X, \omega\pi(Y)) & \xrightarrow{h^X(\omega\pi(d^0))} & \operatorname{Hom}_{\mathscr{C}}\left(X, \omega\pi(Q^1)\right) & \xrightarrow{h^X(\omega\pi(d^1))} \cdots \end{array}$$

since for $\varphi \in \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(X), Q^n)$ we have

$$h^X(\omega\pi(d^n)) \circ \Phi^n(\varphi) = \omega\pi(d^n) \circ \omega(\varphi) \circ \varepsilon_X = \omega(\pi(d^n) \circ \varphi) \circ \varepsilon_X = \Phi^{n+1} \circ h^X(\pi(d^n))(\varphi).$$

Therefore

$$\operatorname{Ext}^i_{\mathscr{C}/\mathscr{A}}(\pi(X), \pi(Y)) \cong H^i\left(\operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(X), \pi(Q^{\cdot}))\right) \cong H^i\left(\operatorname{Hom}_{\mathscr{C}}(X, \omega\pi(Q^{\cdot}))\right) \cong R^i \operatorname{Hom}_{\mathscr{C}}(X, \omega\pi(Y))$$

∎

From here on, let A be a right Noetherian $\mathbb{N}$-graded algebra over a commutative Noetherian ring k, $\mathscr{C} = \operatorname{Gr} - A$, $\mathscr{A} = \operatorname{Tors}$, $\mathscr{C}/\mathscr{A} = \operatorname{QGr} - A$.

Definition 7. Define the graded modules

$$\underline{\operatorname{Hom}}_{\mathscr{C}}(M, N) = \bigoplus_{d \in \mathbb{Z}} \operatorname{Hom}_{\mathscr{C}}(M, N[d])$$

and

$$\underline{\operatorname{Hom}}_{\mathscr{C}/\mathscr{A}}(\pi(M), \pi(N)) = \bigoplus_{d \in \mathbb{Z}} \operatorname{Hom}_{\mathscr{C}/A}(\pi(M), \pi(N)[d]).$$

Proposition 4. *The right derived functors of* $\underline{\operatorname{Hom}}_{\mathscr{C}}(M, N)$ *and* $\underline{\operatorname{Hom}}_{\mathscr{C}/\mathscr{A}}(\pi(M), \pi(N))$ *are*

$$\underline{\operatorname{Ext}}^i_{\mathscr{C}}(M, N) = \bigoplus_{d \in \mathbb{Z}} \operatorname{Ext}^i_{\mathscr{C}}(M, N[d])$$

and

$$\underline{\operatorname{Ext}}^i_{\mathscr{C}/\mathscr{A}}(\pi(M), \pi(N)) = \bigoplus_{d \in \mathbb{Z}} \operatorname{Ext}^i_{\mathscr{C}/\mathscr{A}}(\pi(M), \pi(N)[d]).$$

Moreover, for $Q^{\cdot}$ *an injective resolution of* N,

$$\underline{\operatorname{Ext}}^i_{\mathscr{C}/\mathscr{A}}(\pi(M), \pi(N)) \cong H^i\left(\underline{\operatorname{Hom}}_{\mathscr{C}}(M, \omega\pi(Q^{\cdot}))\right) \cong R^i\, \underline{\operatorname{Hom}}_{\mathscr{C}}(M, \omega\pi(N)).$$

Proof. The first is a consequence of the shift operator s being an autoequivalence of $\mathscr{C}$ and cohomology being an additive functor. In particular, if $Q^{\cdot}$ is an injective resolution of N, then we have

$$H^i(\underline{\mathrm{Hom}}_{\mathscr{C}}\left(M, s^d(Q^{\cdot})\right) = H^i\left(\bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}\left(M, s^d(Q^{\cdot})\right)\right) \cong \bigoplus_{d\in\mathbb{Z}} H^i\left(\mathrm{Hom}_{\mathscr{C}}\left(M, s^d(Q^{\cdot})\right)\right) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Ext}^i_{\mathscr{C}}\left(M, N[d]\right).$$

Similarly, for $\mathscr{C}/\mathscr{A}$ we have

$$\begin{aligned}
H^i(\underline{\mathrm{Hom}}_{\mathscr{C}/\mathscr{A}}\left(M, s^d(Q^{\cdot})\right) &= H^i\left(\bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(M, s^d(Q^{\cdot})\right)\right) \\
&\cong \bigoplus_{d\in\mathbb{Z}} H^i\left(\mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(M, s^d(Q^{\cdot})\right)\right) \\
&= \bigoplus_{d\in\mathbb{Z}} \mathrm{Ext}^i_{\mathscr{C}/\mathscr{A}}\left(M, N[d]\right). \\
&\cong \bigoplus_{d\in\mathbb{Z}} R^i\,\mathrm{Hom}_{\mathscr{C}}\left(M, \omega\pi(N)[d]\right). \\
&\cong \bigoplus_{d\in\mathbb{Z}} H^i(\mathrm{Hom}_{\mathscr{C}}\left(M, \omega\pi(Q^{\cdot})[d]\right) \\
&\cong H^i\left(\bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}\left(M, \omega\pi(Q^{\cdot})[d]\right)\right) \\
&= H^i\left(\underline{\mathrm{Hom}}_{\mathscr{C}}\left(M, \omega\pi(Q^{\cdot})\right)\right)
\end{aligned}$$

■