Noncommutative Projective Schemes

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Thesis Defense

Outline

1. Packing Chromatic Number

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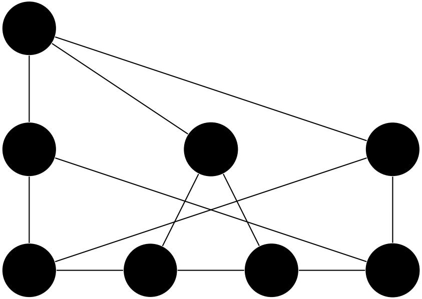
OUTLINE

- Packing colorings were inspired by a frequency assignment problem in broadcasting.
- This coloring was first introduced by Goddard, Harris, Hedetniemi, Hedetniemi, and Rall (2008) where it was called *broadcast coloring*.
- Brešar, Klavžar, and Rall (2007) were the first to use the term packing coloring.

Definition

Let G be a simple connected graph of order n and let i be a positive integer. $X_i \subseteq V(G)$ is called an *i-packing* if vertices in X_i are pairwise distance more than i apart. A *packing coloring* of G is a partition $X = \{X_1, X_2, X_3, \ldots, X_k\}$ of $V(G)$ such that each color class X_i is an i-packing. The minimum order k of a packing coloring is called the *packing chromatic number* of G, denoted by $\chi_\rho(G)$.

Example



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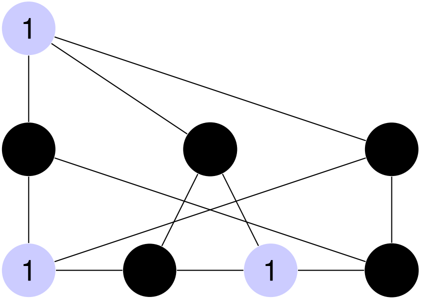

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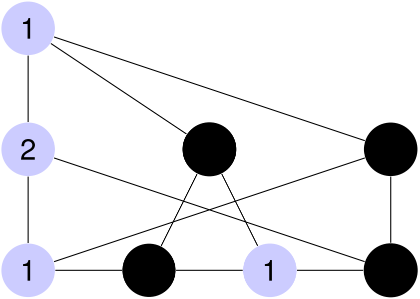

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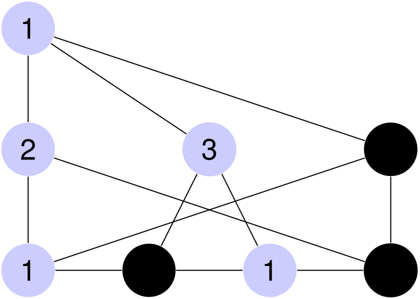

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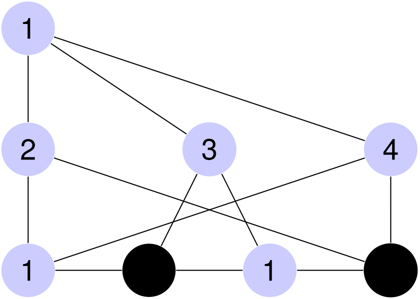

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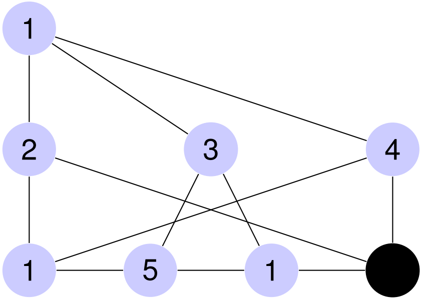

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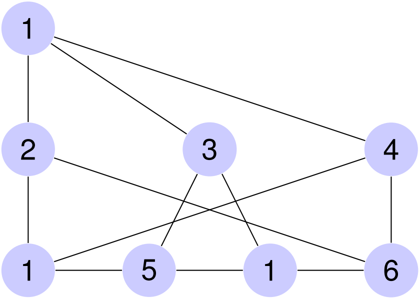

The Packing Chromatic Number

- Goddard et al. (2008) investigated, amongst others, the packing chromatic number of paths, trees, and the infinite square lattice, $\mathbb{Z}^2$. They found that for the square lattice, $9 \leq \chi_\rho(\mathbb{Z}^2) \leq 23$.
- Most recently the bounds have been improved to $12 \leq \chi_\rho(\mathbb{Z}^2) \leq 17$.
- $\chi_\rho(G)$ for lattices, trees, and Cartesian products in general was also considered by Brešar et al. (2007) and Finbow and Rall in 2010.
- C., Hattingh, Jonck examined the lower bound for 3-regular graphs and provide a class of graphs which achieve the bound $\chi_\rho(G) \geq 4$.

- Goddard et al. (2008) showed that finding $\chi_\rho(G)$ for general graphs is NP-complete and deciding whether $\chi_\rho(G) \leq 4$ is also NP-complete.
- Fiala and Golovach (2010) showed that the decision whether a tree allows a packing coloring with at most k classes is NP-complete.

Surprise!

Theorem (Sloper, 2004):

The infinite 3-regular tree has packing chromatic number 7.

Eccentric Coloring

Definition:

An eccentric coloring of a graph $G = (V, E)$ is a function $color : V \to \mathbb{N}$ such that

1. For all $u, v \in V$, $(color(u) = color(v)) \Rightarrow d(u, v) > color(u)$
2. For all $v \in V$, $color(v) \leq e(v)$ where $e(v) = \max_{u \in V}\{d(v, u)\}$

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We inductively define a complete binary tree B_i:

1. $B_1 :=$ 1 vertex, the root. This vertex is Level 1.
2. $B_h :=$ Start with B_{h-1} and append 2 new leaves to each leaf of B_{h-1}. The new leaves are Level h.

The height of a complete binary tree is $h = d(root, leaf) + 1$.

SLOPER'S THEOREM

THEOREM (SLOPER, 2004):

Any complete binary tree of height of three or more is eccentrically colorable with 7 colors or less.

Defintion: An *expandable eccentric coloring* of a complete binary tree $T = (V, E)$ is a coloring such that

- For all $u, v \in V$, (color(u)=color(v))$\Rightarrow d(u, v) > color(u)$

Sloper's Construction

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- color(v)=6 and color(u)=7 $\Rightarrow$ $d(u, v) \geq 5$

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- For all $u \in V$, color(u)≤ 7

Example

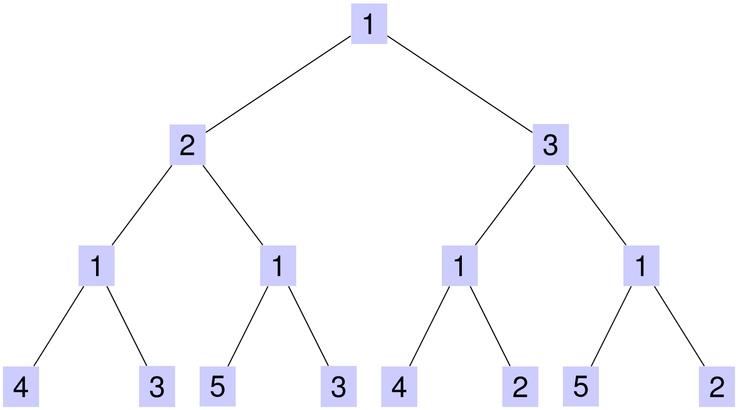

Figure: Expandable eccentric coloring

The Proof

Lemma:

An expandable eccentric coloring of a complete binary tree of height n can be extended to an expandable eccentric coloring of height $(n+1)$.

Sketch of Proof: Note that we may assume n is odd. Then all vertices at level n are colored 1. Consider a leaf u at level $n+1$ and its grandparent p. If $color(p) \in \{4, 5, 6, 7\}$ then u and its sibling, say v, are assigned the colors 2 and 3 (order does not matter).

$color(p) \in \{4, 5, 6, 7\}$

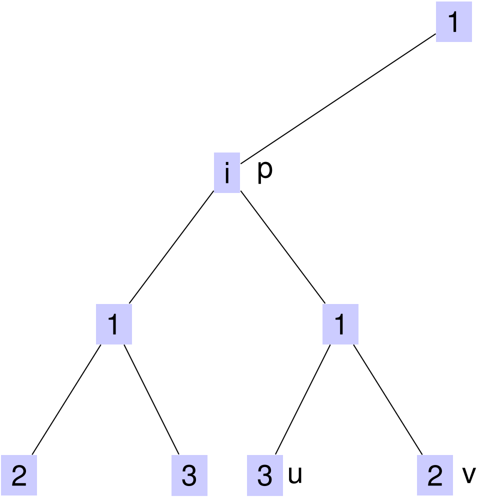

FIGURE: A coloring of level $n + 1$ if $color(p) \in \{4, 5, 6, 7\}$

The Proof (cont.)

We now consider the case when $\mathit{color}(p) = 2$ (the case when $\mathit{color}(p) = 3$ is handled similarly).

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The Proof (cont.)

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Since for any grandchild j of p, $d(j, p) = 2$, we have $color(j) \neq 2$.
Let u, v, w, z be p's grandchildren with pairs of siblings $\{u, v\}$ and $\{w, z\}$. We consider all vertices at distance at most 6 from u, v, w, and z (any vertex at distance 7 from u, v, w, or z must be on an odd level and is already colored 1). By rule 5, two of p's grandchildren (which are not siblings) must receive the color 3. Without loss of generality suppose $color(v) = color(z) = 3$.

$color(p) = 2$

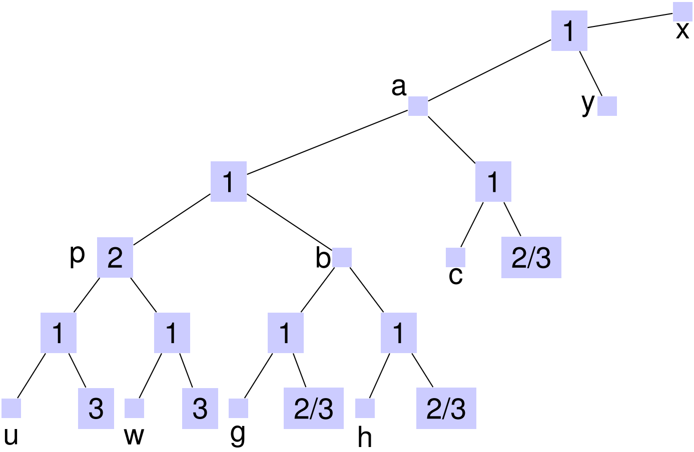
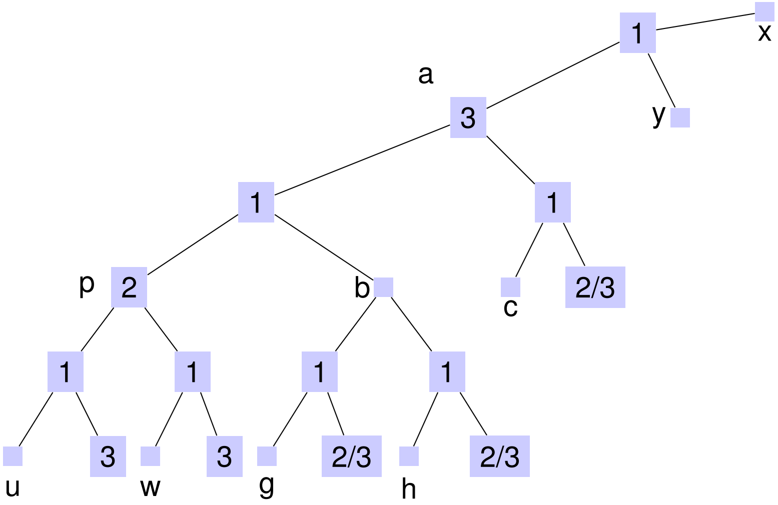


FIGURE: The subtree examined when $color(p) = 2$

$color(p) = 2$ AND $color(a) = 3$



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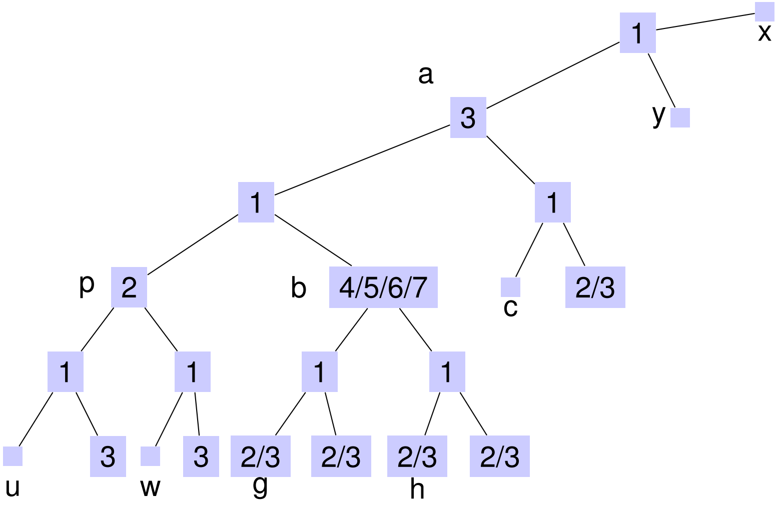
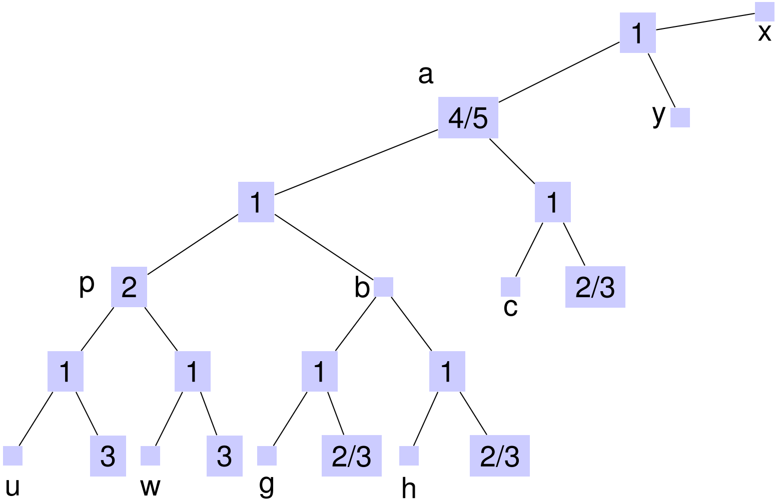


$color(p) = 2$ and $color(a) \in \{4, 5\}$



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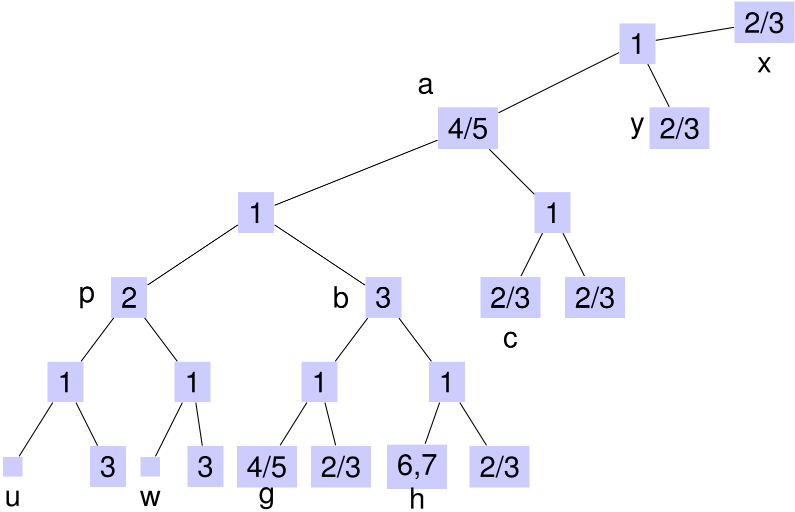
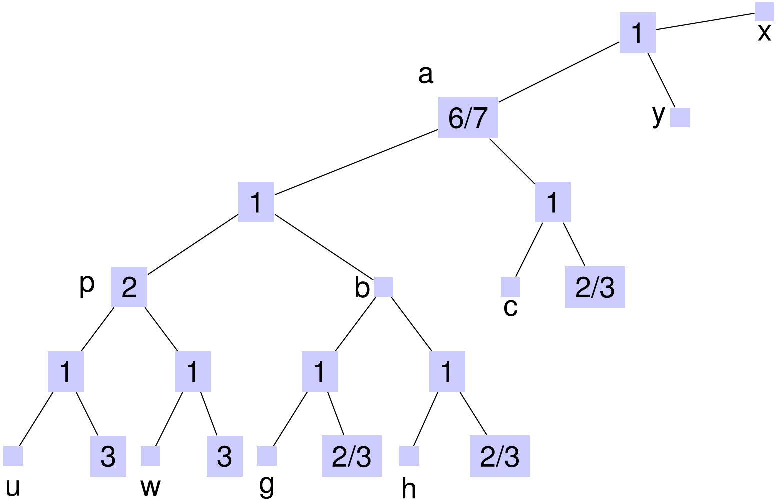


$color(p) = 2$ AND $color(a) \in \{6, 7\}$



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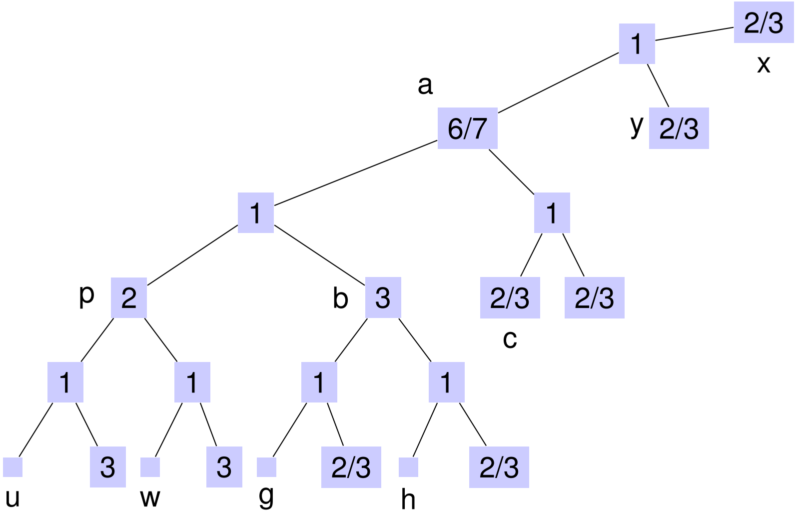

SURPRISE!

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Definition:

A k-ary tree is a tree T such that for all $v \in V(T)$, $d_T(v) \leq k+1$. We can inductively define the complete k-ary tree, T_i:

1. $T_1 :=$ 1 vertex, the root
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The height of a complete k-ary tree is $h = d(root, leaf) + 1$.

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The height of a complete k-ary tree is $h = d(root, leaf) + 1$.

Theorem (Sloper, 2004):

No complete k-ary tree, $k \geq 3$, of height h, $h \geq 4$ is eccentrically broadcast-colorable.

d-regular Graphs

Theorem (C., Griggs, Lu 2015$^+$)

Let G be a d-regular graph with girth g. If $d \geq 4$, then $\chi_\rho(G) \geq g - 1$.

d-regular Graphs

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Let G be a d-regular graph with girth g. If $d \geq 4$, then $\chi_\rho(G) \geq g - 1$.

Proof: Let $k = g - 2$ and assume $\chi_\rho(G) \leq k$. Then there is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ such that for any $1 \leq i \leq k$, and two distinct vertices $u, v \in V_i$, $d(u, v) \geq i$. For any vertex u, let $N_i(u)$ be the set of vertices of distance at most i from u. Similarly, for any edge uv, let $N_i(uv)$ be the set of vertices of distance at most i from u or v.
Note that the induced graph on $N_i(u)$ (for $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$) is a tree depending only on i. Similarly, the induced graph of G on $N_i(uv)$ (for $i = 1, \ldots, \lfloor \frac{k}{2} \rfloor - 1$) is a tree depending on i.

Thus,

$$|N_i(u)| = 1 + d + d(d-1) + \cdots + d(d-1)^{i-1} = \frac{d(d-1)^i - 2}{d-2}$$

and

$$|N_i(uv)| = 2(1 + (d-1) + \cdots + (d-1)^{i-1}) = \frac{2d(d-1)^i - 2}{d-2}.$$

Now, observe that $|V_{2i} \cap N_i(u)| \leq 1$ for $i = 1, 2, \ldots, \lfloor \frac{k}{2} \rfloor$ and $|V_{2i-1} \cap N_{i-1}(uv)| \leq 1$ for $i = 1, 2, \ldots, \lceil \frac{k}{2} \rceil$.
Thus,

$$|V_{2i}| \leq \frac{n}{|N_i(u)|} = \frac{n(d-2)}{d(d-1)^i - 2} \quad \text{for} \quad 1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$$

and

$$|V_{2i-1}| \leq \frac{n}{|N_i(uv)|} = \frac{n(d-2)}{2d(d-1)^i - 2} \quad \text{for} \quad 1 \leq i \leq \left\lceil \frac{k}{2} \right\rceil.$$

d-regular Graphs

As the series $\phi_i(d) := \sum_{i=1}^{\infty} \frac{d-2}{d(d-1)^i-2}$ and $\nu_i(d) := \sum_{i=1}^{\infty} \frac{d-2}{2d(d-1)^i-2}$ converge and are decreasing functions of d, we have

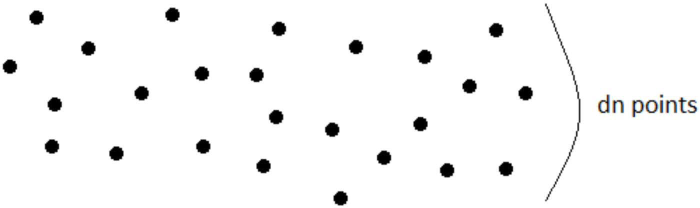
$$\begin{aligned}
n &= \sum_{i=1}^{k} |V_i| \\
&= \sum_{i=1}^{\lceil \frac{k}{2} \rceil} |V_{2i-1}| + \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} |V_{2i}| \\
&\leq \sum_{i=1}^{\lceil \frac{k}{2} \rceil} \frac{n(d-2)}{2d(d-1)^i-2} + \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \frac{n(d-2)}{d(d-1)^i-2} \\
&< n(\phi_i(d) + \nu_i(d)) \\
&< n.
\end{aligned}$$

Random Graphs

- Erdős and Rényi are given credit for first implementing the use of random graphs in probabilistic proofs of the existence of graphs with special properties such as arbitrarily large girth and chromatic number.
- The first combinatorial structures to be studied probabilistically were tournaments.
- Applications of random graphs are found in all areas in which complex networks need to be modeled.

Let $\mathcal{G}_{n,d}$ denote the uniform probability space of d-regular graphs on the n vertices $\{1, 2, \ldots, n\}$ where dn is even. Consider a set of dn points partitioned into n subsets $v_1, v_2, \ldots, v_n$ of d points each. Apply a perfect matching of the points into $\frac{1}{2}dn$ pairs. A pairing corresponds to a multigraph in which the subsets are the vertices and the pairs are edges.

Configuration Model



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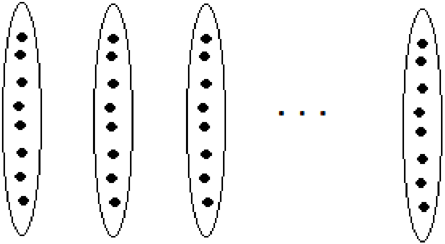

Configuration Model

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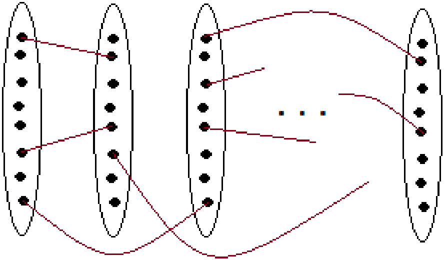
Random
Graphs

Main Result



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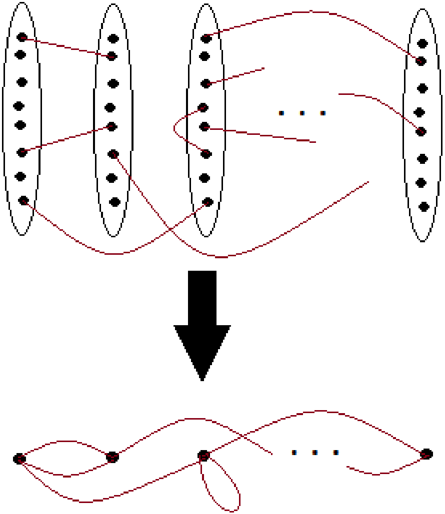

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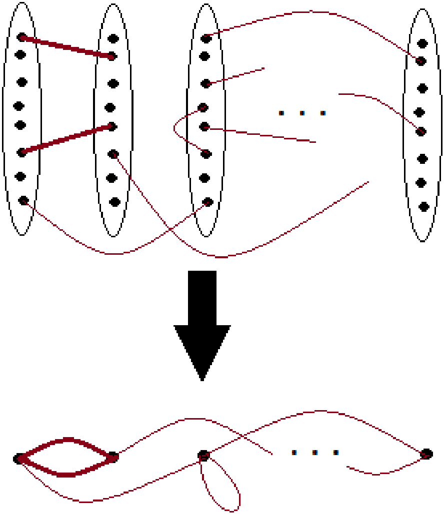
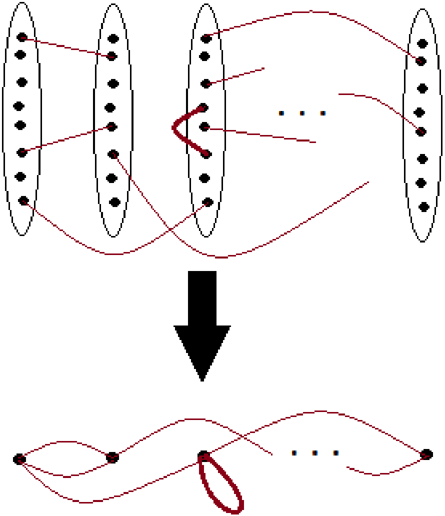


CONFIGURATION MODEL



Configuration Model

- The configuration model was first given by Bollobás in 1979.
- Bender and Canfield (1978) show that the probability that $G_{n,d}$ is simple is $(1 + o(1))\exp(\frac{1-d^2}{4})$.
- Let D be the diameter of $G_{n,d}$. Bollobás and de la Vega (1982) showed that with high probability $D = (1 + o(1))\log_{d-1}(n)$.

MAIN RESULT

THEOREM (C., GRIGGS, LU 2015$^+$):

For any integer $d \geq 4$, there exists a positive constant c_d such that

$$\mathbf{P}(\chi_\rho(G_{n,d}) \geq c_d n) = 1 - o_n(1).$$

Proof

Lemma 1

Let u be a vertex of $G_{n,d}$ and let $N_i(u)$ denote the set of vertices of distance at most i from u in $G_{n,d}$ where $1 \leq i \leq (1+o(1))D/2$, where D is the diameter of $G_{n,d}$. With probability 1-o(1), for all u, $|N_i(u)| \in \{f_i(d), f_i(d)-1\}$.

Lemma 2

Let uv be an edge of $G_{n,d}$ and let $N_i(uv)$ denote the set of vertices of distance at most i from u or v in $G_{n,d}$, where $1 \leq i \leq (1+o(1))D/2$. With probability $1-o(1)$, for all u, v, $|N_i(uv)| \in \{g_i(d), g_i(d)-1\}$.

Proof

Lemma 3

Let u be a vertex of $G_{n,d}$ and let $N_i(u)$ denote the set of vertices of distance at most i from u in $G_{n,d}$ where $(1+o(1))D/2 \leq i \leq 3D/4$. With probability 1-o(1), for all u, $\frac{f_i(d)}{2} \leq |N_i(u)| \leq f_i(d)$.

Lemma 4

Let uv be an edge of $G_{n,d}$ and let $N_i(uv)$ denote the set of vertices of distance at most i from u or v in $G_{n,d}$, where $(1+o(1))D/2 \leq i \leq 3D/4$. With probability $1-o(1)$, for all u, v, $\frac{g_i(d)}{2} \leq |N_i(uv)| \leq g_i(d)$.

Proof

Finally, we must consider the case when $i \in [3D/4+1, D]$. Define $G^i = (V, E^i)$ where $E^i = \{(u,v) : d(u,v) \le i \text{ in } G\}$. Note that as $\alpha(G^i_{n,d}) \le \alpha(G^j_{n,d})$ for $i \ge j$, where α is the independence number, we have that $|V_i| \le \alpha(G^i_{n,d}) \le \alpha(G^j_{n,d}) \le \frac{2}{f_i(d)} n$ for $i \ge j$. Thus, $\sum_{i=3/4D}^{D} |V_i| \le \sum_{j=(1+o(1))D/2}^{3/4D} \frac{2}{f_j(d)} n < \frac{\epsilon}{6} n$.

Thank you!