# Noncommutative Projective Schemes

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Temple Graduate Student Conference in Algebra, Geometry, and Topology

# Outline

1. Serre Finiteness

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# Commutative Case

**Theorem 1 (Serre)**

*Let $X$ be a projective scheme over a noetherian ring $A$, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on $X$ over $\operatorname{Spec} A$. Let $\mathscr{F}$ be a coherent sheaf on $X$. Then:*

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(i) *for each $0 \leq i$, $H^i(X, \mathscr{F})$ is a finitely generated $A$-module;*

(ii) *there is an integer $n_0$, depending on $\mathscr{F}$ such that for each $0 < i$ and each $n_0 \leq n$, $H^i(X, \mathscr{F}(n)) = 0$.*

# NONCOMMUTATIVE CASE



**THEOREM 2 (ARTIN-ZHANG)**

*Let $A$ be a right noetherian $\mathbb{Z}_{\geq 0}$-graded algebra over a commutative noetherian ring $k$ satisfying $\chi$ and let $\pi(M)$ be an object of $\operatorname{qgr}$-$A$.*

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*(H5) for every $1 \leq j$, $\underline{H}^j(\pi(M))$ is right bounded; i.e., there is an integer $d_0$ such that for all $d_0 \leq d$, $H^j(\pi(M)[d]) = 0$.*

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(II) *Conversely, if $A$ satisfies $\chi_1$ and if (H4) and (H5) hold for every $\pi(M) \in \mathrm{qgr}\text{-}A$, then $A$ satisfies $\chi$.*

# Preliminaries

Let $k$ be a Noetherian commutative ring, $A$ a $\mathbb{Z}_{\geq 0}$-graded right Noetherian ring. Denote by $\operatorname{Gr}\text{-}A$ (resp. $\operatorname{gr}\text{-}A$) the category of graded right $A$-modules (resp. finite) with morphisms

$$\operatorname{Hom}_{\operatorname{Gr}\text{-}A}(M, N) = \{f \in \operatorname{Hom}_A(M, N) \mid f(M_d) \subseteq N_d\}.$$

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- every family of objects has a coproduct,
- filtered colimits are exact,
- Gr-$A$ has a generator: the functor $h^A\colon \operatorname{Gr}\text{-}A \to \mathfrak{Set}$ is faithful; for any morphism $M \to N$ the morphism

$$\operatorname{Hom}_{\operatorname{Gr}\text{-}A}(M, N) \longrightarrow \operatorname{Hom}_{\mathfrak{Set}}\left(h^A(M), h^A(N)\right)$$

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**Definition 1**

A full subcategory, $\mathscr{A}$, of an abelian category $\mathscr{C}$ is called a Serre (or épaisse/thick/dense) subcategory if for any short exact sequence

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The full subcategory Tors (resp. tors) of Gr-$A$ (resp. gr-$A$) with objects $M$ of Gr-$A$ (resp. gr-$A$) satisfying

$$\tau(M) = \{m \in M \mid mA_{\geq s} = 0 \text{ for some } s\} = M$$

is a Serre subcategory.

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The category $\mathscr{I}$ is filtered.

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- $\operatorname{QGr}\text{-}A$ is abelian.
- There is a functor $\pi : \operatorname{Gr}\text{-}A \to \operatorname{QGr}\text{-}A$ that is the identity on objects and sends a morphism $f \in \operatorname{Hom}_{\operatorname{Gr}\text{-}A}(M, N)$ to its image, $\pi(f)$, in the colimit.

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- $\operatorname{qgr}\text{-}A$ is defined analogously.

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- If in addition $M$ is an object of gr-$A$, then

$$\operatorname{Hom}_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(N)) = \lim_{n\to\infty} \operatorname{Hom}_{\mathrm{Gr}\text{-}A}(M_{\geq n}, N)$$

where

$$M_{\geq n} = \bigoplus_{d \geq n} M_d.$$

# Properties of $\pi$

- given an exact sequence

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- $\pi$ preserves injectives.

# Closed Objects



**Definition 3**

We say an object $M$ of $\mathrm{Gr}\text{-}A$ is Tors-closed if $M$ is torsion-free and any short exact sequence

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## Remark 1

It's immediate that every torsion-free injective is Tors-closed.

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**Proposition 1 (Gabriel)**

*For $M$ an object of $\mathrm{Gr}\text{-}A$, the following are equivalent:*

1. *Any exact sequence*

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2. *$M$ is $\mathrm{Tors}$-closed,*
3. *For any object $N$ of $\mathrm{Gr}\text{-}A$*
$$\pi \colon \operatorname{Hom}_{\mathrm{Gr}\text{-}A}(N, M) \cong \operatorname{Hom}_{\mathrm{QGr}\text{-}A}(\pi(N), \pi(M)).$$

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3. Every injective object of QGr -$A$ is isomorphic to $\pi(Q/\tau(Q))$ for some injective object $Q$ of Gr -$A$.

# Computing Ext

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2. $\pi(Q^{\cdot})$ is an injective resolution of $\pi(N)$ by the comments above, so

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3. From the adjunction we get an isomorphism of complexes
$$\mathrm{Hom}_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(Q^\cdot))) \cong \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M, \omega\pi(Q^\cdot))$$
and we see that
$$\mathrm{Ext}^i_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(N)) \cong R^i\,\mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M, \omega\pi(N))$$

# Graded Hom

**Definition 5**

Define the graded modules

$$\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}(M, N) = \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M, N[d])$$

and

$$\underline{\mathrm{Hom}}_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(N)) = \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(N)[d]).$$

# GRADED Ext

The right derived functors are

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# Graded Ext (cont'd)

For $Q^{\cdot}$ an injective resolution of $N$,

$$\begin{aligned}\underline{\mathrm{Ext}}^i_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(N)) &\cong h^i\left(\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}(M, \omega\pi(Q^{\cdot}))\right) \\ &\cong R^i\,\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}(M, \omega\pi(N)).\end{aligned}$$

Define the cohomology functors

$$H^i(\pi(M)) = \operatorname{Ext}^i_{\mathrm{QGr}\text{-}A}(\pi(A), \pi(M)) \cong h^i(\omega\pi(Q^\cdot))_0$$

and

$$\underline{H}^i(\pi(M)) = \bigoplus_{d \in \mathbb{Z}} H^i(\pi(M)[d]) \cong h^i(\omega\pi(Q^\cdot)).$$

# Bounded Modules

**Definition 6**

Let $M$ be an object of $\mathrm{Gr}\text{-}A$.

(i) We say $M$ is left bounded if there exists some $\ell$ such that $M_d = 0$ for all $d \leq \ell$.

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(iii) We say $M$ is bounded if it is left and right bounded.

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2. If $\chi_i^0(M)$ holds for every object $M$ of gr-$A$, then we say that $\chi_i^0$ holds for $A$.
3. If $\chi_i^0(M)$ holds for $A$ for every $i$, then we say that $\chi^0$ holds for $A$.
4. We say that $\chi_i(M)$ holds for an object of Gr-$A$ if for all $d$ and all $j \leq i$, there is an integer $n_0$ such that $\underline{\mathrm{Ext}}^j_{\mathrm{Gr}\text{-}A}(A/A_{\geq n}, M)_{\geq d}$ is an object of gr-$A$ when $n_0 \leq n$.

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## Definition 7

1. We say $\chi_i^0(M)$ holds if $\underline{\mathrm{Ext}}^j_{\mathrm{Gr}\text{-}A}(A_0, M)$ is bounded for all $j \leq i$.
2. If $\chi_i^0(M)$ holds for every object $M$ of $\mathrm{gr}\text{-}A$, then we say that $\chi_i^0$ holds for $A$.
3. If $\chi_i^0(M)$ holds for $A$ for every $i$, then we say that $\chi^0$ holds for $A$.
4. We say that $\chi_i(M)$ holds for an object of $\mathrm{Gr}\text{-}A$ if for all $d$ and all $j \leq i$, there is an integer $n_0$ such that $\underline{\mathrm{Ext}}^j_{\mathrm{Gr}\text{-}A}(A/A_{\geq n}, M)_{\geq d}$ is an object of $\mathrm{gr}\text{-}A$ when $n_0 \leq n$.
5. If $\chi_i$ holds for every object of $\mathrm{gr}\text{-}A$, then we say that $\chi_i$ holds for $A$.

# THE $\chi$ CONDITION

## DEFINITION 7

1. We say $\chi_i^0(M)$ holds if $\underline{\mathrm{Ext}}^j_{\mathrm{Gr}\text{-}A}(A_0, M)$ is bounded for all $j \leq i$.
2. If $\chi_i^0(M)$ holds for every object $M$ of $\mathrm{gr}\text{-}A$, then we say that $\chi_i^0$ holds for $A$.
3. If $\chi_i^0(M)$ holds for $A$ for every $i$, then we say that $\chi^0$ holds for $A$.
4. We say that $\chi_i(M)$ holds for an object of $\mathrm{Gr}\text{-}A$ if for all $d$ and all $j \leq i$, there is an integer $n_0$ such that $\underline{\mathrm{Ext}}^j_{\mathrm{Gr}\text{-}A}(A/A_{\geq n}, M)_{\geq d}$ is an object of $\mathrm{gr}\text{-}A$ when $n_0 \leq n$.
5. If $\chi_i$ holds for every object of $\mathrm{gr}\text{-}A$, then we say that $\chi_i$ holds for $A$.
6. If $\chi_i$ holds for every $i$, then we say that $\chi$ holds for $A$.

Thank you!