1. Adjoints

Lemma 1. Let $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ be a functor and let $\iota_0\colon\mathscr{D}_0\to\mathscr{D}$ be a full subcategory of \mathscr{D} . If for each object X of \mathscr{C} there exists an object Y of \mathscr{D}_0 and an isomorphism $\mathscr{F}X\stackrel{\sim}{\to} \iota_0(Y)$, then there exists a functor $\mathscr{F}_0\colon\mathscr{C}\to\mathscr{D}_0$ and a natural isomorphism $\theta_0 \colon \mathscr{F} \to \iota_0 \circ \mathscr{F}_0$. Moreover, \mathscr{F}_0 is unique up to unique isomorphism.

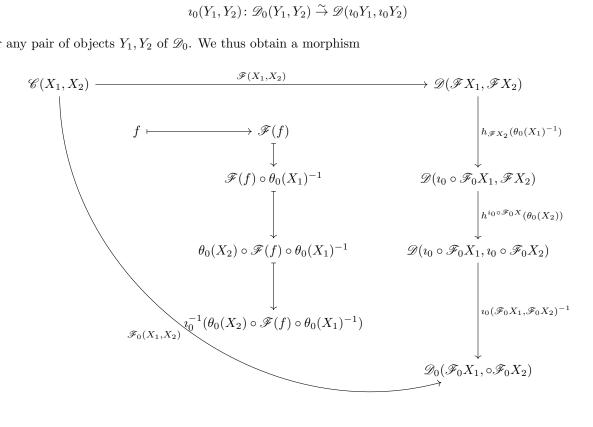
Proof. For each object X of \mathscr{C} , choose an object Y of \mathscr{D}_0 such that $\theta_0(X): \mathscr{F}X \cong \iota_0(Y)$ and define $\mathscr{F}_0(X) = Y$. Since \mathscr{F} is a functor we have a morphism

$$\mathscr{F}(X_1, X_2) \colon \mathscr{C}(X_1, X_2) \to \mathscr{D}(\mathscr{F}X_1, \mathscr{F}X_2).$$

for each pair of objects X_1 , X_2 of \mathscr{C} , and since \mathscr{D}_0 is a full subcategory, we also have an isomorphism

$$i_0(Y_1, Y_2) \colon \mathscr{D}_0(Y_1, Y_2) \xrightarrow{\sim} \mathscr{D}(i_0 Y_1, i_0 Y_2)$$

for any pair of objects Y_1, Y_2 of \mathcal{D}_0 . We thus obtain a morphism



Since all the morphisms involved respect composition, it's clear that this assignment is functorial and by construction makes the isomorphism $\theta_0 \colon \mathscr{F} \to \iota_0 \circ \mathscr{F}_0$ natural.

For uniqueness, suppose that $\mathscr{G}_0 \colon \mathscr{C} \to \mathscr{D}_0$ is another functor equipped with a natural isomorphism $\phi_0 \colon \mathscr{F} \to \iota_0 \circ \mathscr{G}_0$. We have by assumption an isomorphism of functors

$$i_0 \circ G_0 \xrightarrow{\phi_0^{-1}} \mathscr{F} \xrightarrow{\theta_0} i_0 \mathscr{F}_0.$$

Since i_0 is fully faithful, for each object X of $\mathscr C$ it reflects the isomorphism $\theta_0(X)\circ\varphi_0(X)^{-1}$ giving an isomorphism

$$i_0^{-1}(\theta_0(X)\circ\varphi_0(X)^{-1})=\eta(X)\colon\mathscr{G}_0(X)\longrightarrow\mathscr{F}_0(X).$$

To see that this is natural, we consider the diagram

$$\begin{array}{ccc} X & & \mathscr{G}_0(X) \xrightarrow{\eta(X)} \mathscr{F}_0(X) \\ \downarrow^f & & & \downarrow^{\mathscr{G}_0(f)} & \downarrow^{\mathscr{F}_0(f)} \\ X' & & & G_0(X') \xrightarrow{\eta(X')} \mathscr{F}_0(X') \end{array}$$

and note that from the two naturality squares

$$\begin{split} \mathscr{F}(X) &\xrightarrow{\theta_0(X)} \imath_0 \circ \mathscr{F}_0(X) & \mathscr{F}(X) \xrightarrow{\phi_0(X)} \imath_0 \circ \mathscr{G}_0(X) \\ \downarrow \mathscr{F}(f) & \downarrow \imath_0 \circ \mathscr{F}_0(f) \text{ and } & \downarrow \mathscr{F}(f) & \downarrow \imath_0 \circ \mathscr{G}_0(f) \\ \mathscr{F}(X') &\xrightarrow{\theta_0(X')} \imath_0 \circ \mathscr{F}_0(X') & \mathscr{F}(X') \xrightarrow{\phi_0(X')} \imath_0 \circ \mathscr{G}_0(X') \end{split}$$

we obtain

$$i_{0}(\mathscr{F}_{0}(f) \circ \eta(X)) = i_{0} \circ \mathscr{F}_{0}(f)) \circ \theta_{0}(X) \circ \varphi_{0}(X)^{-1}$$

$$= \theta_{0}(X') \circ \mathscr{F}(f) \circ \theta_{0}(X)^{-1} \circ \theta_{0}(X) \circ \varphi_{0}(X)^{-1}$$

$$= \theta_{0}(X') \circ \mathscr{F}(f) \circ \varphi_{0}(X)^{-1}$$

$$= \theta_{0}(X') \circ \varphi_{0}(X')^{-1} \circ i_{0} \circ \mathscr{G}_{0}(f)$$

$$= i_{0}(\eta(X')) \circ i_{0} \circ \mathscr{G}_{0}(f)$$

$$= i_{0}(\eta(X') \circ \mathscr{G}_{0}(f))$$

which implies $\mathscr{F}_0(f) \circ \eta(X) = \eta(X') \circ \mathscr{G}_0(f)$ because ι_0 is fully faithful.

Proposition 1. Consider the two functors $\mathcal{L}: \mathcal{C} \to \mathcal{D}$ and $\mathcal{R}: \mathcal{D} \to \mathcal{C}$. The following are equivalent:

(i) for each object X of \mathscr{C} and Y of \mathscr{D} , there is an isomorphism

$$\mathscr{D}(\mathscr{L}X,Y) \cong \mathscr{C}(X,\mathscr{R}Y)$$

natural in both X and Y,

(ii) there exist natural transformations

$$\epsilon \colon \operatorname{id}_{\mathscr{C}} \to \mathscr{R} \circ \mathscr{L} \text{ and } \eta \colon \mathscr{L} \circ \mathscr{R} \to \operatorname{id}_{\mathscr{D}}$$

called the unit and counit of adjunction, respectively, that make the diagrams

commute for all objects X of \mathscr{C} and Y of \mathscr{D} ,

(iii) for every object Y of \mathcal{D} , the functor

$$\mathscr{D}(\mathscr{L}(-),Y)\colon\mathscr{C}\to\mathfrak{Set}$$

is representable by $\mathcal{R}Y$.

If any of these conditions holds, we say that \mathcal{L} is left adjoint to \mathcal{R} (symmetrically, \mathcal{R} is right adjoint to \mathcal{L}), and write $\mathcal{L} \dashv \mathcal{R}$.

Note that condition (iii) implies that the right adjoint of \mathcal{L} is unique up to unique isomorphism and, symmetrically, the left adjoint of \mathcal{R} is as well.

Proof. $(i \Rightarrow ii)$

For each object X of \mathscr{C} and Y of \mathscr{D} define ϵ_X and η_Y to be the images of the identity morphism under the isomorphisms

$$\mathscr{D}(\mathscr{L}X,\mathscr{L}X) \stackrel{\sim}{\to} \mathscr{C}(X,\mathscr{R} \circ \mathscr{L}X)$$
 and $\mathscr{C}(\mathscr{R}Y,\mathscr{R}Y) \stackrel{\sim}{\to} \mathscr{D}(\mathscr{L} \circ \mathscr{R}Y,Y)$,

respectively. One sees that these are natural transformations by chasing the identities through the commutative diagrams

and

To establish the commutativity conditions, we chase the identities through the commutative diagrams

and

and use the definition of ϵ and η . This establishes (ii).

$$(ii \Rightarrow iii)$$

Define the morphisms

$$\mathscr{D}(\mathscr{L}X,Y) \overset{\mathscr{R}(\mathscr{L}X,Y)}{\longrightarrow} \mathscr{C}(\mathscr{R} \circ \mathscr{L}X,\mathscr{R}Y) \overset{h_{\mathscr{R}Y}(\epsilon_X)}{\longrightarrow} \mathscr{C}(X,\mathscr{R}Y)$$

and

$$\mathscr{C}(X,\mathscr{R}Y) \overset{\mathscr{L}(X,\mathscr{R}Y)}{\longrightarrow} \mathscr{D}(\mathscr{L}X,\mathscr{L} \circ \mathscr{R}Y) \overset{h^{\mathscr{L}X}(\eta_Y)}{\longrightarrow} \mathscr{D}(\mathscr{L}X,Y).$$

Using the naturality diagrams

$$\begin{array}{cccc} X & \xrightarrow{\epsilon_X} & \mathcal{R} \circ \mathcal{L}X & \mathcal{L} \circ \mathcal{R} \circ \mathcal{L}X & \xrightarrow{\eta_{\mathcal{L}X}} \mathcal{L}X \\ \downarrow^f & & \downarrow_{\mathcal{R} \circ \mathcal{L}(f)} \text{ and } & \downarrow_{\mathcal{R} \circ \mathcal{L}(g)} & \downarrow^g \\ \mathcal{R}Y & \xrightarrow{\epsilon_{\mathcal{R}Y}} & \mathcal{R} \circ \mathcal{L} \circ \mathcal{R}Y & \mathcal{L} \circ \mathcal{R}Y & \xrightarrow{\eta_Y} & Y \end{array}$$

we see

$$\begin{split} h_{\mathscr{R}Y}(\epsilon_X) \circ \mathscr{R}(\mathscr{L}X,Y) \circ h^{\mathscr{L}X}(\eta_Y) \circ \mathscr{L}(X,\mathscr{R}Y)(f) &=& \mathscr{R}(\eta_Y \circ \mathscr{L}(f)) \circ \epsilon_X \\ &=& \mathscr{R}(\eta_Y) \circ \mathscr{R} \circ \mathscr{L}(f) \circ \epsilon_X \\ &=& \mathscr{R}(\eta_Y) \circ \epsilon_{\mathscr{R}Y} \circ f \\ &=& f \end{split}$$

and

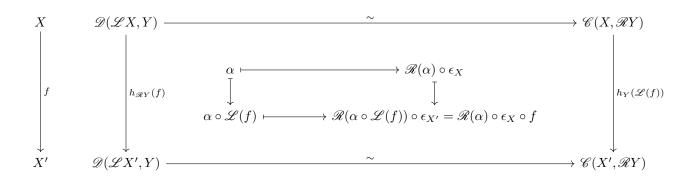
$$\begin{split} h^{\mathscr{L}X}(\eta_Y) \circ \mathscr{L}(X,\mathscr{R}Y) \circ h_{\mathscr{R}Y}(\epsilon_X) \circ \mathscr{R}(\mathscr{L}X,Y)(g) &= \eta_Y \circ \mathscr{L}(\mathscr{R}(g) \circ \epsilon_X) \\ &= \eta_Y \circ \mathscr{L} \circ \mathscr{R}(g) \circ \mathscr{L}(\epsilon_X) \\ &= g \circ \eta_{\mathscr{L}X} \circ \mathscr{L}(\epsilon_X) \\ &= g \end{split}$$

giving isomorphisms

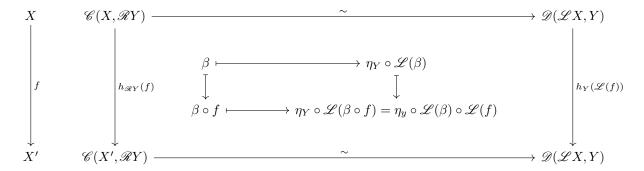
$$\mathscr{D}(\mathscr{L}X,Y)\cong\mathscr{C}(X,\mathscr{R}Y)$$

for each object X of \mathscr{C} .

Naturality in X follows directly from naturality of ϵ and η . Namely, given a morphism $f \in \mathcal{C}(X', X)$ we have the commutative diagram



and, similarly, the commutative diagram



This establishes the isomorphism of functors.

$$(iii \Rightarrow i)$$

First define the functor

$$\mathscr{L}_* \colon \operatorname{Fun}(\mathscr{D}^{\operatorname{op}}, \mathfrak{Set}) \to \operatorname{Fun}(\mathscr{C}^{\operatorname{op}}, \mathfrak{Set})$$

as follows. Given $\mathscr{F}: \mathscr{D} \to \mathfrak{Set}$ and an object X of \mathscr{C} define

$$\mathscr{L}_*(\mathscr{F})(X) = \mathscr{F} \circ \mathscr{L}(X).$$

Given a morphism $\eta \in \operatorname{Fun}(\mathscr{D}^{\operatorname{op}}, \mathfrak{Set})(\mathscr{F}_1, \mathscr{F}_2)$ define natural transformation $\mathscr{L}_*(\eta) \colon \mathscr{L}_*(\mathscr{F}_1) \to \mathscr{L}_*(\mathscr{F}_2)$ by

$$\mathscr{L}_*(\eta)_X = \eta_{\mathscr{L}X}.$$

Let $h_{-}^{\mathscr{D}}: \mathscr{D} \to \operatorname{Fun}(\mathscr{D}^{\operatorname{op}}, \mathfrak{Set})$ and $h_{-}^{\mathscr{C}}: \mathscr{C} \to \operatorname{Fun}(\mathscr{C}^{\operatorname{op}}, \mathfrak{Set})$ denote the Yoneda embeddings, which we note identify \mathscr{C} and \mathscr{D} as full subcategories of $\operatorname{Fun}(\mathscr{D}^{\operatorname{op}}, \mathfrak{Set})$ and $\operatorname{Fun}(\mathscr{C}^{\operatorname{op}}, \mathfrak{Set})$ by the Yoneda Lemma. By assumption, we have for each object Y of \mathscr{D} a representing object $\mathscr{R}Y$ of \mathscr{C} for the functor

$$\mathscr{L}_*(h_Y) = \mathscr{D}(\mathscr{L}(-), Y) \cong \mathscr{C}(-, \mathscr{R}Y) = h_-^{\mathscr{C}}(\mathscr{R}Y)$$

Hence by Lemma 1 we obtain a factorization

$$\mathscr{D} \xrightarrow{h_{-}^{\mathscr{D}}} \operatorname{Fun}\left(\mathscr{D}^{\operatorname{op}}, \mathfrak{Set}\right) \xrightarrow{\mathscr{L}_{*}} \operatorname{Fun}\left(\mathscr{C}^{\operatorname{op}}, \mathfrak{Set}\right)$$

by the same argument, giving an isomorphism of functors

$$\mathscr{C}(-,\mathscr{R}(-)) = h_{-}^{\mathscr{C}} \circ \mathscr{R} \cong \mathscr{L}_{*} \circ h_{-}^{\mathscr{D}} = \mathscr{D}(\mathscr{L}(-), -),$$

natural in Y.

Similarly, define the functor \mathscr{R}_* : Fun $(\mathscr{C}, \mathfrak{Set}) \to \operatorname{Fun}(\mathscr{D}, \mathfrak{Set})$ and note that using the co-Yoneda embeddings $h_{\mathscr{D}}^-$: $\mathscr{D} \to \operatorname{Fun}(\mathscr{C}, \mathfrak{Set})$ and $h_{\mathscr{D}}^-$: $\mathscr{C} \to \operatorname{Fun}(\mathscr{D}, \mathfrak{Set})$ we obtain a factorization

$$\mathscr{C} \xrightarrow{h_{\mathscr{C}}^{-}} \operatorname{Fun}\left(\mathscr{C}, \mathfrak{Set}\right) \xrightarrow{\mathscr{R}_{*}} \operatorname{Fun}\left(\mathscr{D}, \mathfrak{Set}\right)$$

giving an isomorphism of functors

$$\mathscr{D}(\mathscr{L}(-),-)=h_{\mathscr{D}}^{-}\circ\mathscr{L}\cong\mathscr{R}_{*}\circ h_{\mathscr{C}}^{-}=\mathscr{C}(-,\mathscr{R}(-)),$$

natural in X. This establishes (i).

Proposition 2. Let $\mathscr{C}, \mathscr{D}, \mathscr{D}'$ be categories equipped with adjunctions

$$\mathscr{C} \xleftarrow{\mathscr{L}} \mathscr{D} \quad and \quad \mathscr{D} \xleftarrow{\mathscr{L}'} \mathscr{D}'$$

Then

$$\mathscr{C} \stackrel{\mathscr{L}' \circ \mathscr{L}}{\underset{\mathscr{R} \circ \mathscr{R}'}{\longleftarrow}} \mathscr{D}'$$

is an adjunction.

Proof. The composition of functors is clearly well defined, so we obtain a natural isomorphism

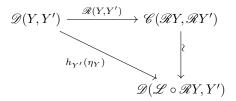
$$\mathcal{D}'(\mathcal{L}' \circ \mathcal{L}(X), Y') \cong \mathcal{D}(\mathcal{L}(X), \mathcal{R}'Y')$$
$$\cong \mathcal{C}(X, \mathcal{R} \circ \mathcal{R}'Y').$$

Definition 1. Given two categories \mathscr{C} and \mathscr{D} , an equivalence of categories is a pair of functors $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ and $\mathscr{G}:\mathscr{D}\to\mathscr{C}$ and natural isomorphisms $\epsilon\colon \mathrm{id}_\mathscr{C}\to\mathscr{G}\circ\mathscr{F},\,\eta\colon\mathscr{F}\circ\mathscr{G}\to\mathrm{id}_\mathscr{D}.$

Proposition 3. Let $\mathcal{L}: \mathcal{C} \to \mathcal{D}$ and $\mathcal{R}: \mathcal{C} \to \mathcal{D}$ be adjoint functors with unit $\epsilon: \mathrm{id}_{\mathcal{C}} \to \mathcal{R} \circ \mathcal{L}$ and counit $\eta: \mathcal{L} \circ \mathcal{R} \to \mathrm{id}_{\mathcal{D}}$.

- (i) The functor $\mathcal R$ is fully faithful if and only if η is an isomorphism,
- (ii) The functor \mathcal{L} is fully faithful if and only if ϵ is an isomorphism
- (iii) The following are equivalent:
 - (a) \mathcal{L} is an equivalence of categories,
 - (b) \mathcal{R} is an equivalence of categories,
 - (c) \mathcal{L} and \mathcal{R} are fully faithful. In this case, \mathcal{L} and \mathcal{R} are quasi-inverses of one another, and ϵ , η are both isomorphism.

Proof. For (i), we observe from the diagram



that $\mathscr{R}(Y,Y')$ is an isomorphism if and only if $h_{Y'}(\eta_Y)$ is an isomorphism. Hence \mathscr{R} is fully faithful if and only if $h_{Y'}(\eta_Y)$ is an isomorphism for all objects Y, Y' of \mathscr{D} , and we are reduced to showing this is equivalent to η_Y being an isomorphism.

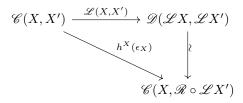
Clearly if η_Y is an isomorphism, then $h_{Y'}(\eta_Y)$ is an isomorphism for all objects Y, Y' of \mathscr{D} with inverse

$$h_{Y'}(\eta_Y^{-1}) \colon \mathscr{D}(\mathscr{L} \circ \mathscr{R}Y, Y')$$

Conversely, taking Y' = Y we obtain an inverse to η_Y by applying the the inverse morphism:

$$\eta_Y^{-1} = h_Y(\eta_Y)^{-1}(\eta_Y).$$

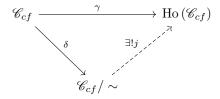
For (ii), we apply the same argument, mutatis mutandis, to the diagram



Part (iii) follows from the definition of an equivalence of categories and the uniqueness of the adjoints.

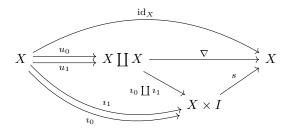
2. The Homotopy Category

Corollary 1. Let \mathscr{C} be a model category. Let $\gamma \colon \mathscr{C}_{cf} \to \operatorname{Ho}(\mathscr{C}_{cf})$ and $\delta \colon \mathscr{C}_{cf} \to \mathscr{C}_{cf} / \sim$ be the canonical functors. Then there is a unique isomorphism of categories making the diagram



commute. Furthermore, j is the identity on objects.

Proof. We show that \mathscr{C}_{cf}/\sim satisfies the same universal property as Ho (\mathscr{C}_{cf}) . We note that δ takes homotopy equivalences to isomorphisms and so by Proposition 1.2.8, δ also takes weak equivalences to isomorphisms. Let $\mathscr{F}:\mathscr{C}_{cf}\to\mathscr{D}$ be a functor that takes weak equivalences to isomorphisms. Given an object X, take the functorial cylinder object $X\times I$. This guarantees that $X\times I$ is cofibrant and fibrant, since $X\coprod X$ is also cofibrant and X is fibrant. Now note that we have a commutative diagram

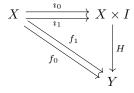


Since s and id_X are weak equivalences we have that i_0 and i_1 are both weak equivalences by 2-out-of-3. Hence

$$\mathscr{F}(s) \circ \mathscr{F}(\imath_0) = \mathscr{F}(s \circ \imath_0) = \mathrm{id}_{\mathscr{F}X} = \mathscr{F}(s \circ \imath_1) = \mathscr{F}(s) \circ \mathscr{F}(\imath_1)$$

implies $\mathscr{F}(i_0) = \mathscr{F}(i_1)$.

Given a homotopy



we have

$$\mathscr{F}(f_0) = \mathscr{F}(H \circ \iota_0) = \mathscr{F}(H) \circ \mathscr{F}(\iota_0) = \mathscr{F}(H) \circ \mathscr{F}(\iota_1) = \mathscr{F}(H \circ \iota_1) = \mathscr{F}(f_1)$$

so that \mathscr{F} identifies left homotopic maps. By duality, we see that \mathscr{F} also identifies right homotopic maps. This allows us to define a unique functor $\mathscr{G}:\mathscr{C}_{cf}/\sim\to\mathscr{D}$ by $\mathscr{G}\delta X=\mathscr{F}X$ and $\mathscr{G}\delta(f)=\mathscr{F}(f)$. The result now follows by uniqueness of universal objects.

Theorem 1. Suppose \mathscr{C} is a model category. Let $\gamma \colon \mathscr{C} \to \operatorname{Ho}(\mathscr{C})$ denote the canonical functor, Q the cofibrant replacement functor of \mathscr{C} , and R the fibrant replacement functor.

(i) The inclusion $\mathscr{C}_{cf} \to \mathscr{C}$ induces an equivalence of categories

$$\mathscr{C}_{cf}/\sim \stackrel{\sim}{\longrightarrow} \operatorname{Ho}\left(\mathscr{C}_{cf}\right) \longrightarrow \operatorname{Ho}\left(\mathscr{C}\right)$$

(ii) There are natural isomorphisms

$$\mathscr{C}(QRX, QRY)/\sim \xrightarrow{\sim} \operatorname{Ho}(C)(\gamma X, \gamma Y) \xrightarrow{\sim} \mathscr{C}(RQX, RQY)/\sim$$

In addition, there is a natural isomorphism $\operatorname{Ho}(C)(\gamma X, \gamma Y) \cong \mathscr{C}(QX, RY)/\sim$, and, if X is cofibrant and Y is fibrant, there is a natural isomorphism $\operatorname{Ho}(C)(\gamma X, \gamma Y) \cong \mathscr{C}(X, Y)/\sim$. In particular, $\operatorname{Ho}(C)$ is a category without moving to a higher universe.

- (iii) The functor $\gamma \colon \mathscr{C} \to \operatorname{Ho}(C)$ identifies left or right homotopic maps.
- (iv) If $f: A \longrightarrow B$ is a map in $\mathscr C$ such that γf is an isomorphism in $\operatorname{Ho}(C)$, then f is a weak equivalence.

Proof. Part (i) is just the composition of the isomorphism from the Corollary above and the equivalence $\operatorname{Ho}(\mathscr{C}_{cf}) \longrightarrow \operatorname{Ho}(C)$ of Proposition 1.2.3.

3. Quillen Adjunctions and Derived Functors

Definition 2. Let \mathscr{C} and \mathscr{D} be model categories.

- (1) A functor $\mathcal{L}: \mathcal{C} \to \mathcal{D}$ is left Quillen if \mathcal{L} is a left adjoint and preserves cofibrations and trivial cofibrations.
- (2) A functor $\mathscr{R}: \mathscr{D} \to \mathscr{C}$ is right Quillen if \mathscr{R} is a right adjoint and preserves fibrations and trivial fibrations.
- (3) An adjunction $(\mathcal{L}, \mathcal{R}, \phi)$, where ϕ is the natural isomorphism

$$\varphi_{XY} \colon \mathscr{D}(\mathscr{L}X,Y) \xrightarrow{\sim} \mathscr{C}(X,\mathscr{R}Y),$$

is called a Quillen adjunction if \mathcal{L} is left Quillen.

Remark 1. Hovey always uses η for the unit of adjunction and ϵ for the counit of adjunction; my notation is exactly the opposite.

Example 1. Let $\mathscr C$ be a model category and let I be a set. Equip the product category $\mathscr C^I$ with the product model structure. We can view an object of $\mathscr C^I$ as a discrete diagram of $\mathscr C$ and a morphism of this category as a morphisms of discrete diagrams. For example, in the case where I has two elements, a morphism between two objects, $f = (f_1, f_2) \colon X = (X_1, X_2) \to Y = (Y_1, Y_2)$ is just two morphisms of $\mathscr C$:

$$egin{array}{lll} X_1 & & X_2 \\ \downarrow f_1 & & \downarrow f_2 \\ Y_1 & & Y_2 \end{array}$$

with no commutativity relations.

As \mathscr{C} has all small limits by assumption, we can define a product functor $\lim_{I}:\mathscr{C}^{I}\to C$ which takes an object $X=(X_{i})_{I}$ of \mathscr{C}^{I} to the limit over the discrete diagram,

$$\lim_{I} X = \lim_{I} X_i = \prod_{i \in I} X_i.$$

For any morphism of $f \in \mathscr{C}^I(X,Y)$, $\lim_{I}(f)$ is the unique map induced by the universal property of limit, which is determined by the diagrams

$$\lim_{I} X \xrightarrow{\lim_{I} (f)} \lim_{I} Y$$

$$\downarrow \qquad \qquad \downarrow$$

$$X_{i} \xrightarrow{f_{i}} Y_{i}$$

indexed over I. For example, when I has two elements, the image of a morphism (f_1, f_2) : $(X_1, X_2) \to (Y_1, Y_2)$ is just the product map,

$$\begin{array}{cccc} X_1 \longleftarrow & X_1 \times X_2 \longrightarrow X_2 \\ \downarrow^{f_1} & & \downarrow^{\exists ! f_1 \times f_2} & \downarrow^{f_2} \\ Y_1 \longleftarrow & Y_1 \times Y_2 \longrightarrow Y_2 \end{array}$$

We can define a a diagonal functor $\Delta \colon \mathscr{C} \to \mathscr{C}^I$ which takes an object of X of \mathscr{C} to the discrete diagram with I copies of X. A morphism $f \in \mathscr{C}(X,X)$ defines a morphism of discrete diagrams by taking f in each component. We note that, by definition, a morphism $f \in \mathscr{C}^I(\Delta X,Y)$ is simply a collection of morphisms $f_i \in \mathscr{C}(X,Y_i)$, giving

$$\mathscr{C}^{I}(\Delta X, Y) = \prod_{i \in I} \mathscr{C}(X, Y_i)$$

With this observation, it's easy to see that this functor is left adjoint to \lim_I since we can view the universal property of products as a representability statement: The object $\lim_I Y_i = \prod_{i \in I} Y_i$ is the representing object of the functor $\prod_{i \in I} \mathscr{C}(-, Y_i)$, and hence

$$\mathscr{C}\left(-,\lim_{I}Y\right)=\mathscr{C}\left(-,\prod_{i\in I}Y_{i}\right)\cong\prod_{i\in I}\mathscr{C}\left(-,Y_{i}\right)=\mathscr{C}^{I}\left(-,Y\right)=\mathscr{C}^{I}\left(\Delta\left(-\right),Y\right).$$

By Proposition 1 we obtain the desired natural isomorphism

$$\mathscr{C}^{I}\left(\Delta X,Y\right)=\prod_{i\in I}\mathscr{C}\left(\Delta X,Y_{i}\right)\cong\mathscr{C}\left(\Delta X,\lim_{I}Y\right).$$

Note that by definition of the product model structure, Δ necessarily preserves cofibrations and weak equivalences, so this adjunction is Quillen.

Lemma 2. Let $\mathscr C$ and $\mathscr D$ be model categories and suppose $\mathscr L:\mathscr C\to\mathscr D$ is left adjoint to $\mathscr R:\mathscr D\to\mathscr C$. This adjunction is Quillen if and only if $\mathscr R$ is right Quillen.

Proof. By Lemma 1.1.10 of Hovey it suffices to show that given a fibration (resp. trivial fibration) $p \in \mathscr{D}(Y_1, Y_2)$, then $\mathscr{R}p \in \mathscr{C}(\mathscr{R}Y_1, \mathscr{R}Y_2)$ has the right lifting property with respect to all trivial cofibrations (resp. cofibrations). Let $f \in \mathscr{C}(X_1, X_2)$ be a trivial cofibration (resp. cofibration). To say that $\mathscr{R}p$ has the right lifting property with respect to f is to say that for every commutative diagram

$$X_1 \xrightarrow{\phi} \mathcal{R}Y_1$$

$$\downarrow^f \qquad \qquad \downarrow^{\mathcal{R}p}$$

$$X_2 \xrightarrow{\psi} \mathcal{R}Y_2$$

there is a lift $\ell: X_2 \to \mathscr{R}Y_1$ such that $\ell \circ f = \varphi$ and $\mathscr{R}p \circ \ell = \psi$. This is equivalent to the morphisms of sets

$$\mathscr{D}(\mathscr{L}X_2,Y_1) \cong \mathscr{C}(X_2,\mathscr{R}Y_1) \overset{h^{X_2}(\mathscr{R}p)}{\longrightarrow} \mathscr{C}(X_2,\mathscr{R}Y_2) \cong \mathscr{D}(\mathscr{L}X_2,Y_2)$$

and

$$\mathscr{D}(\mathscr{L}X_2,Y_1)\cong\mathscr{C}(X_2,\mathscr{R}Y_1)\overset{h_{\mathscr{R}Y_1}(f)}{\longrightarrow}\mathscr{C}(X_1,\mathscr{R}Y_1)\cong\mathscr{D}(\mathscr{L}X_1,Y_1)$$

being surjective. However, by the naturality squares

this is equivalent to $\mathcal{L}(f)$ having the left lifting property with respect to p. This is equivalent to $\mathcal{L}(f)$ being a cofibration (resp. trivial cofibration), which in turn is equivalent to the adjunction being Quillen.

Remark 2. (i) If $\mathcal{L}: \mathcal{C} \to \mathcal{D}$ and $\mathcal{R}: \mathcal{D} \to \mathcal{C}$ is a Quillen adjunction, then, by Ken Brown's Lemma, \mathcal{L} preserves weak equivalences between cofibrant objects and \mathcal{R} preserves weak equivalences between fibrant objects. By abuse of notation, we write $\mathcal{L}: \mathcal{C}_c \to \mathcal{D}$ and $\mathcal{R}: \mathcal{D}_f \to \mathcal{C}$ for the restrictions, then observe that we obtain

$$\begin{array}{ccc} \mathscr{C} & \stackrel{Q}{\longrightarrow} \mathscr{C}_{c} & \stackrel{\mathscr{L}}{\longrightarrow} \mathscr{D} \\ \downarrow^{\gamma_{\mathscr{C}}} & \downarrow^{\gamma_{\mathscr{C}_{c}}} & \stackrel{\exists!}{\longrightarrow} & \downarrow^{\gamma_{\mathscr{D}}} \\ \operatorname{Ho}(\mathscr{C}) & \stackrel{\operatorname{Ho}(Q)}{\longrightarrow} & \operatorname{Ho}(\mathscr{C}_{c}) & \stackrel{\operatorname{Ho}(\mathscr{L})}{\longrightarrow} & \operatorname{Ho}(\mathscr{D}) \end{array}$$

and

$$\mathcal{D} \xrightarrow{R} \mathcal{D}_{f} \xrightarrow{\mathscr{R}} \mathscr{C}$$

$$\downarrow^{\gamma_{\mathscr{D}}} \qquad \downarrow^{\gamma_{\mathscr{D}_{f}}} \xrightarrow{\exists !} \qquad \downarrow^{\gamma_{\mathscr{C}}}$$

$$\operatorname{Ho}(\mathscr{D}) \xrightarrow{\operatorname{Ho}(R)} \operatorname{Ho}(\mathscr{D}_{f}) \xrightarrow{\operatorname{Ho}(\mathscr{R})} \operatorname{Ho}(\mathscr{C})$$

by the universal property for the homotopy category.

We also note that given natural transformations $\eta: \mathcal{L} \to \mathcal{L}'$, $\nu: \mathcal{R} \to \mathcal{R}'$ between Quillen functors, we obtain natural transformations $\operatorname{Ho}(\eta): \operatorname{Ho}(\mathcal{L}) \to \operatorname{Ho}(\mathcal{L}')$ and $\operatorname{Ho}(\nu): \operatorname{Ho}(\mathcal{R}) \to \operatorname{Ho}(\mathcal{R}')$ defined by $\operatorname{Ho}(\eta)_X = \eta_X$ and $\operatorname{Ho}(\nu)_X = \nu_X$. Since all the functors involved preserve weak equivalences, these are indeed natural by Lemma 1.2.2.

Note that on objects,

$$\operatorname{Ho}(\mathscr{L})(X) = \mathscr{L}(QX) \text{ and } \operatorname{Ho}(\mathscr{R})(Y) = \mathscr{R}(QY).$$

(ii) This construction does not require Quillen functors. We only used the fact that these functors preserve weak equivalences between cofibrant (resp. fibrant) objects.

Definition 3. Let $\mathscr C$ and $\mathscr D$ be model categories.

(1) If $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ is a left Quillen functor, define the total left derived functor $L\mathscr{F}:\operatorname{Ho}(\mathscr{C})\to\operatorname{Ho}(D)$ to be the composition

$$\operatorname{Ho}(\mathscr{C}) \stackrel{\operatorname{Ho}(Q)}{\longrightarrow} \operatorname{Ho}(\mathscr{C}_c) \stackrel{\operatorname{Ho}(\mathscr{F})}{\longrightarrow} \operatorname{Ho}(\mathscr{D})$$

Given a natural transformation $\tau \colon \mathscr{F} \to \mathscr{F}'$ of left Quillen functors, define the total derived natural transformation $L\tau$ to be Ho $(\tau) \circ$ Ho (Q), so that $(L\tau)_X = \tau_{QX}$.

(2) If $\mathscr{G}: \mathscr{D} \to \mathscr{C}$ is a right Quillen functor, define the total right derived functor $R\mathscr{G}: \operatorname{Ho}(\mathscr{D}) \to \operatorname{Ho}(\mathscr{C})$ to be the composition

$$\operatorname{Ho}(\mathscr{D}) \stackrel{\operatorname{Ho}(R)}{\longrightarrow} \operatorname{Ho}(\mathscr{D}_f) \stackrel{\operatorname{Ho}(\mathscr{G})}{\longrightarrow} \operatorname{Ho}(\mathscr{C})$$

Given a natural transformation $\tau: \mathscr{G} \to \mathscr{G}'$ of right Quillen functors, define the total derived natural transformation $R\tau$ to be Ho $(\tau) \circ$ Ho (R), so that $(R\tau)_X = \tau_{RX}$.

Theorem 2. For every model category, \mathscr{C} , there is a natural isomorphism $\alpha \colon L(\mathrm{id}_{\mathscr{C}}) \to \mathrm{id}_{\mathrm{Ho}(\mathscr{C})}$. Also for every pair of left Quillen functors $F \colon \mathscr{C} \to \mathscr{D}$ and $F' \colon \mathscr{D} \to \mathscr{E}$, there is a natural isomorphism $m = m_{\mathscr{F}'\mathscr{F}} \colon L\mathscr{F}' \circ L\mathscr{F} \to L(\mathscr{F}' \circ \mathscr{F})$. These natural isomorphisms satisfy the following properties.

(1) An associativity coherence diagram is commutative. That is, if $\mathcal{F}: \mathcal{C} \to \mathcal{C}'$, $\mathcal{F}': \mathcal{C}' \to \mathcal{C}''$, and $\mathcal{F}'': \mathcal{C}'' \to \mathcal{C}'''$ are left Quillen functors, then the diagram

commutes.

(2) A left unit coherence diagram is commutative. That is, if $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ is a left Quillen functor, then the diagram

$$L(\mathrm{id}_{\mathscr{D}}) \circ L\mathscr{F} \xrightarrow{m} L(\mathrm{id}_{\mathscr{D}} \circ \mathscr{F})$$

$$\downarrow^{\alpha \circ L\mathscr{F}} \qquad \qquad \parallel$$

$$1_{\mathrm{Ho}(\mathscr{D})} \circ L\mathscr{F} = L\mathscr{F}$$

commutes.

(3) A right unit coherence diagram is commutative. That is, if $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ is a left Quillen functor, then the diagram

$$L\mathscr{F} \circ L(\mathrm{id}_{\mathscr{C}}) \xrightarrow{m} L(\mathscr{F} \circ \mathrm{id}_{\mathscr{C}})$$

$$\downarrow^{L\mathscr{F} \circ \alpha} \qquad \qquad \parallel$$

$$L\mathscr{F} \circ \mathrm{id}_{\mathrm{Ho}(\mathscr{C})} = L\mathscr{F}$$

commutes.

Proof. Let $i: \mathscr{C}_c \to \mathscr{C}$ be the natural inclusion. We observe from the functorial factorization

$$0 \xrightarrow{QX} X$$

that we obtain a natural transformation $q: i \circ Q \to \mathrm{id}_{\mathscr{C}}$. Taking the derived natural transformation $\mathrm{Ho}\,(q)$ gives the natural isomorphism

$$\operatorname{id}_{\operatorname{Ho}(\mathscr{C})} \stackrel{\sim}{\to} \operatorname{Ho}(i) \circ \operatorname{Ho}(Q) \stackrel{\operatorname{Ho}(q)}{\to} L(\operatorname{id}_{\mathscr{C}})$$

because q_X is a trivial fibration.

Define $m_{\mathscr{F}'\mathscr{F}}$ to be the collection of maps

$$L\mathscr{F}'\circ L\mathscr{F}X=\mathscr{F}'(Q(\mathscr{F}(QX)))\overset{F'(q_{\mathscr{F}(QX)})}{\longrightarrow}\mathscr{F}'(\mathscr{F}(QX))=L(\mathscr{F}'\circ\mathscr{F}(X))$$

which is natural in X as a functor on \mathscr{C} , for if $f' \in \mathscr{C}(X,X')$ then we have the commutative diagram

$$\begin{array}{c} QX \stackrel{q_X}{\longrightarrow} X \\ \downarrow_{Qf} & \downarrow_f \\ QX' \stackrel{q_{X'}}{\longrightarrow} X' \end{array}$$

which gives rise to the commutative diagram in \mathcal{D}

$$Q(\mathscr{F}(QX)) \xrightarrow{q_{\mathscr{F}(QX)}} \mathscr{F}(QX) \xrightarrow{\mathscr{F}(q_X)} \mathscr{F}(X)$$

$$\downarrow^{Q\mathscr{F}(Qf)} \qquad \downarrow^{\mathscr{F}(Qf)} \qquad \downarrow^{\mathscr{F}(f)}$$

$$Q(\mathscr{F}(QX')) \xrightarrow{q_{\mathscr{F}(QX')}} \mathscr{F}(QX') \xrightarrow{\mathscr{F}(q_{X'})} \mathscr{F}(X')$$

and hence a commutative diagram in $\mathscr E$

Since all functors involved preserve weak equivalences, $m_{\mathscr{F}'\mathscr{F}}$ is also natural in X as a functor on Ho (\mathscr{C}). Moreover, \mathscr{F} preserves cofibrant objects because it preserves cofibrations as a left Quillen functor, hence \mathscr{F}' preserves the weak equivalence $q_{\mathscr{F}(QX)}$ between cofibrant objects and thus it follows that $m_{\mathscr{F}'\mathscr{F}}$ is an isomorphism in Ho (\mathscr{E}).

For the associativity coherence diagram, we must show that

$$(\mathscr{F}'' \circ \mathscr{F}'(q_{\mathscr{F}QX})) \circ (F''(q_{\mathscr{F}'Q\mathscr{F}QX})) = (F''(q_{\mathscr{F}'\mathscr{F}QX})) \circ ((\mathscr{F}'' \circ Q \circ \mathscr{F}'(q_{\mathscr{F}QX})).$$

This follows from naturality of q and a construction similar to the one above.

The left unit coherence diagram commutes because by definition

$$m_{\mathrm{id}_{\mathscr{D}}\mathscr{F}}(X) = \mathrm{id}_{\mathscr{D}}(q_{\mathscr{F}QX}) = q_{\mathscr{F}QX}$$

and

$$\alpha \circ L\mathscr{F}(X) = \alpha_{\mathscr{F}OX} = q_{\mathscr{F}OX}$$

For the right unit coherence diagram, we have

$$m_{\mathscr{F}\mathrm{id}_{\mathscr{C}}}(X) = \mathscr{F}(q_{\mathrm{id}_{\mathscr{C}}(q_{QX})}) = \mathscr{F}(q_{QX}) \colon \mathscr{F}(QQX) \to \mathscr{F}(QX)$$

and

$$L\mathscr{F} \circ \alpha(X) = \mathscr{F}(Q(q_X)) \colon \mathscr{F}(QQX) \to \mathscr{F}(QX).$$

Given a cofibrant object X we have the commutative diagram

$$QQX \xrightarrow{q_{QX}} QX$$

$$\downarrow Q(q_{X}) \qquad \downarrow q_{X}$$

$$QX \xrightarrow{q_{X}} X$$

because q is natural. Since q_X is a weak equivalence between cofibrant objects it follow that $\mathscr{F}(q_X)$ is invertible in $\operatorname{Ho}(\mathscr{D})$ and hence

$$\mathscr{F}(q_{QX}) = \mathscr{F}Q(q_X).$$

Since every object of $\operatorname{Ho}(\mathscr{C})$ is weakly equivalent to a cofibrant object, this completes the proof.

Definition 4. Let \mathscr{C} , \mathscr{D} , and \mathscr{E} be categories and let $\mathscr{F},\mathscr{G}:\mathscr{C}\to\mathscr{D}$, $\mathscr{F}',\mathscr{G}':\mathscr{D}\to\mathscr{E}$ be functors. Given natural transformations $\eta:\mathscr{F}\to\mathscr{G}$ and $\nu:\mathscr{F}'\to\mathscr{G}'$, defin the horizontal composition $\eta*\nu:\mathscr{F}'\circ\mathscr{F}\to\mathscr{G}'\circ\mathscr{G}$ is the natural transformation defined by the collection of morphisms of \mathscr{E}

$$\mathcal{F}' \circ \mathcal{F}(X) \xrightarrow{\mathcal{F}'(\eta_X)} \mathcal{F}' \circ \mathcal{G}(X)$$

$$\downarrow^{\nu_{\mathcal{F}X}} \qquad \qquad \downarrow^{\nu_{\mathcal{G}X}}$$

$$\mathcal{G}' \circ \mathcal{F}(X) \xrightarrow{\mathcal{G}'(\eta_X)} \mathcal{G}' \circ \mathcal{G}(X)$$

Lemma 3. Let \mathscr{C} , \mathscr{D} , and \mathscr{E} be model categories. Let $\mathscr{F},\mathscr{G}:\mathscr{C}\to\mathscr{D}$ and $\mathscr{F}',\mathscr{G}':\mathscr{D}\to\mathscr{E}$ be left Quillen functors. Suppose $\eta\colon\mathscr{F}\to\mathscr{G}$ and $\nu\colon\mathscr{F}'\to\mathscr{G}'$ be natural transformations. If m is the composition isomorphism of the Theorem above, then the diagram

$$L\mathscr{F}' \circ L\mathscr{F} \xrightarrow{m} L(\mathscr{F}' \circ \mathscr{F})$$

$$\downarrow^{L\eta * L\nu} \qquad \qquad \downarrow^{L(\eta * \nu)}$$

$$L\mathscr{G}' \circ L\mathscr{G} \xrightarrow{m} L(\mathscr{G}' \circ \mathscr{G})$$

commutes.

Proof. We unravel the definitions. The morphism

$$L(\eta * \nu) \circ m_X : \mathscr{F}'Q\mathscr{F}QX \to \mathscr{G}'\mathscr{G}QX$$

is given by the composition across the top of the commutative diagram

$$\begin{split} \mathscr{F}'Q\mathscr{F}QX & \xrightarrow{\mathscr{F}'(q_{\mathscr{F}QX})} \mathscr{F}'\mathscr{F}QX & \xrightarrow{\nu_{\mathscr{F}QX}} \mathscr{G}'\mathscr{F}QX \\ \downarrow \mathscr{F}'Q(\eta_{QX}) & \downarrow \mathscr{F}'(\eta_{QX}) & \downarrow \mathscr{G}'(\nu_{QX}) \\ \\ \mathscr{F}'Q\mathscr{G}QX & \xrightarrow{\mathscr{F}'(q_{\mathscr{G}QX})} \mathscr{F}'\mathscr{G}QX & \xrightarrow{\nu_{\mathscr{G}QX}} \mathscr{G}'\mathscr{G}QX \\ \downarrow \nu_{Q\mathscr{G}QX} & \downarrow \nu_{\mathscr{G}QX} & \downarrow \\ \mathscr{G}'Q\mathscr{G}QX & \xrightarrow{\mathscr{G}'(q_{\mathscr{G}QX})} \mathscr{G}'\mathscr{G}QX & \Longrightarrow \mathscr{G}'\mathscr{G}QX \end{split}$$

and the morphism $m \circ L\eta * L\nu_X : \mathscr{F}'Q\mathscr{F}QX \to \mathscr{G}'\mathscr{G}QX$ is given by the composition across the bottom of the commutative diagram

$$\begin{split} \mathscr{F}'Q\mathscr{F}(QX) & \xrightarrow{\nu_{Q\mathscr{F}QX}} \mathscr{G}'Q\mathscr{F}QX & \xrightarrow{\mathscr{G}'(q_{\mathscr{F}QX})} \mathscr{G}'\mathscr{F}QX \\ \downarrow & & \downarrow \mathscr{G}'Q(\eta_{QX}) & \downarrow \mathscr{G}'(\eta_{QX}) & \downarrow \mathscr{G}'(\eta_{QX}) \\ \mathscr{F}'Q\mathscr{G}QX & \xrightarrow{\nu_{Q\mathscr{G}QX}} & \mathscr{G}'Q\mathscr{G}QX & \xrightarrow{\mathscr{G}'(q_{\mathscr{G}QX})} & \mathscr{G}'\mathscr{G}QX \end{split}$$

Chasing the bottom left side of each diagram gives us the desired equality.

Remark 3. Essentially, this says we have a 2-category (modulo some set theoretic issues...) with 0-cells model categories, 1-cells left Quillen functors, 2-cells the natural transformations and the homotopy category, total derived functor, and total derived natural transformation define a pseudo 2-functor to the 2-category of categories.

Lemma 4. Let \mathscr{C} be a model category and assume we have homotopic morphisms $f_0 \sim f_1 \in \mathscr{C}(X,Y)$. If Y is fibrant object, then we may always choose a fibrant path object. Dually, if X cofibrant we may always choose a cofibrant cylinder object.

Proof. Since Y is assumed to be fibrant, the pullback

$$Y \times Y \xrightarrow{\pi_0} Y$$

$$\downarrow^{\pi_1} \qquad \qquad \downarrow$$

$$Y \longrightarrow 0$$

is fibrant, as fibrations are stable under pullback. Choose a path object

$$Y \xrightarrow{\Delta} Y \times Y$$

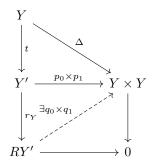
$$\downarrow t \qquad p_0 \times p_1 \qquad Y$$

$$Y'$$

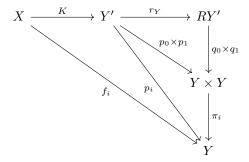
with $p_0 \times p_1$ a fibration, t a weak equivalence, and a homotopy

$$X \xrightarrow{K} Y' \downarrow_{p_i} \downarrow_{p_i}$$

Taking a fibrant replacement, $Y \stackrel{r_Y}{\to} RY'$, we obtain a lift



because $Y \times Y$ is fibrant and r_Y is a trivial cofibration, making RY' a path object. We also obtain a homotopy $r_Y \circ K$ since the diagrams



commute for i = 0, 1.

Lemma 5. Let \mathscr{C} and \mathscr{D} be model categories. Given a Quillen adjunction

$$\mathscr{C} \xrightarrow{\mathscr{F}} \mathscr{D}$$

we obtain a derived adjunction

$$\operatorname{Ho}\left(\mathscr{C}\right) \xrightarrow[R\mathscr{G}]{L\mathscr{F}} \operatorname{Ho}\left(\mathscr{D}\right)$$

Proof. Denote by ϵ and η the unit and counit of adjunction, respectively, and by $[-,-]_{\mathscr{C}}$, $[-,-]_{\mathscr{D}}$ the morphisms of the homotopy category. Also, recall that the isomorphism of adjunction is given by the morphisms

$$\mathscr{D}(\mathscr{F}X,Y) \xrightarrow{\mathscr{G}(\mathscr{F}X,Y)} \mathscr{C}(\mathscr{G}\mathscr{F}X,\mathscr{G}Y) \xrightarrow{h_{\mathscr{G}Y}(\epsilon_X)} \mathscr{C}(X,\mathscr{G}Y)$$

and

$$\mathscr{C}(X,\mathscr{G}Y) \xrightarrow{\mathscr{F}(X,\mathscr{G}Y)} \mathscr{D}(\mathscr{F}X,\mathscr{F}\mathscr{G}Y) \xrightarrow{h^{\mathscr{F}X}(\eta_Y)} \mathscr{D}(\mathscr{F}X,Y)$$

We first observe that we have natural isomorphisms

$$[L\mathscr{F}X,Y]_{\mathscr{D}}\cong\mathscr{D}(\mathscr{F}QX,RY)/\sim \text{ and } [X,R\mathscr{G}Y]_{\mathscr{C}}\cong\mathscr{C}(QX,\mathscr{G}RY)/\sim.$$

This reduces the problem to showing that the isomorphism of adjunction both preserves and reflects homotopies between cofibrant objects of \mathscr{C} and fibrant objects of \mathscr{D} , for then we can see that for any object X of \mathscr{C} and any object Y of \mathscr{D} , the isomorphism of adjunction descends to a well-defined isomorphism

Towards that end, assume that X is a cofibrant object of \mathscr{C} and Y is a fibrant object of \mathscr{D} . Given $f_0 \sim f_1 \in \mathscr{D}(\mathscr{F}X, Y)$, choose a homotopy from a fibrant path object.

$$Y \xrightarrow{\Delta} Y \times Y$$

$$\downarrow t \qquad p_0 \times p_1$$

$$\downarrow Y'$$

with $p_0 \times p_1$ a fibration, t a weak equivalence, and a homotopy

$$\mathcal{F}X \xrightarrow{K} Y' \downarrow_{p_i} \downarrow_{p_i} Y$$

Since \mathscr{G} is right Quillen it preserves products, fibrant objects, weak equivalences between fibrant objects, and (trivial) fibrations, hence we obtain a path object of \mathscr{C}

$$\mathscr{G}Y \xrightarrow{\mathscr{G}(\Delta)} \mathscr{G}(Y \times Y) \cong \mathscr{G}(Y) \times \mathscr{G}(Y)$$

$$\mathscr{G}(t) \xrightarrow{\mathscr{G}(p_0 \times p_1) = \mathscr{G}(p_0) \times \mathscr{G}(p_1)}$$

and a homotopy

$$X \xrightarrow{\mathscr{G}(K) \circ \epsilon_X} \mathscr{G}Y'$$

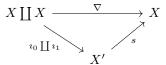
$$\mathscr{G}(f_i) \circ \epsilon_X \qquad \qquad \mathscr{G}Y$$

by the commutativity of the naturality square

$$\begin{array}{ccc} \mathscr{D}(\mathscr{F}X,Y') & \stackrel{\sim}{\longrightarrow} \mathscr{C}(X,\mathscr{G}Y') \\ & & & \downarrow^{h^{\mathscr{F}X}(p_i)} & & \downarrow^{h^X(\mathscr{G}(p_i))} \\ \mathscr{D}(\mathscr{F}X,Y) & \stackrel{\sim}{\longrightarrow} \mathscr{C}(X,\mathscr{G}Y) \end{array}$$

It's clear that this morphism is surjective and natural, so it remains to show that that it reflects homotopies.

Assume that there is a homotopy $\mathscr{G}(f_0) \circ \epsilon_X \sim \mathscr{G}(f_1) \circ \epsilon_X$ in \mathscr{C} . By duality, the argument above implies that we may choose a cofibrant cylinder object



and a homotopy

$$X \xrightarrow{\iota_i} X'$$

$$\mathscr{G}(f_i) \circ \epsilon_X \qquad \downarrow^H$$

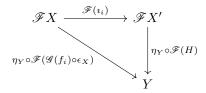
$$\mathscr{G}Y$$

As \mathscr{F} is left Quillen it preserves cofibrant objects, (trivial) cofibrations, and weak equivalences between cofibrant objects by which we obtain a cylinder object in \mathscr{D}

$$\mathscr{F}(X \coprod X) \cong \mathscr{F}X \coprod \mathscr{F}X \xrightarrow{\mathscr{F}(\nabla)} \mathscr{F}X$$

$$\mathscr{F}(\imath_0 \coprod \imath_1) = \mathscr{F}(\imath_0) \coprod \mathscr{F}(\imath_1) \qquad \mathscr{F}X'$$

and a homotopy



and we note that

$$\eta_Y \circ \mathscr{F} \circ \mathscr{G}(f_i) \circ \mathscr{F}(\epsilon_X) = f_i$$

implies that we have a homotopy $f_0 \sim f_1$, as desired. Therefore the induced map is injective, hence an isomorphism.

4. Quillen Equivalences

Definition 5. A Quillen adjunction $(\mathscr{F},\mathscr{G},\phi)\colon\mathscr{C}\to\mathscr{D}$ is a Quillen equivalence if and only if, for all cofibrant X in \mathscr{C} and fibrant Y in \mathscr{D} , a map $f\colon\mathscr{F}X\to Y$ is a weak equivalence in \mathscr{D} if and only if $\phi(f)\colon X\to\mathscr{G}Y$ is a weak equivalence in \mathscr{C} .

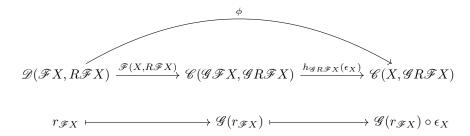
Remark 4. Note that a Quillen equivalence is *not* always an equivalence of categories. This could be thought of as a weak equivalence of model categories.

Proposition 4. Let $(\mathscr{F},\mathscr{G},\phi)\colon\mathscr{C}\to\mathscr{D}$ be a Quillen adjunction with unit $\epsilon\colon \mathrm{id}_\mathscr{C}\to\mathscr{G}\circ\mathscr{F}$ and counit $\eta\colon\mathscr{F}\circ\mathscr{G}\to\mathrm{id}_\mathscr{D}$. The following are equivalent:

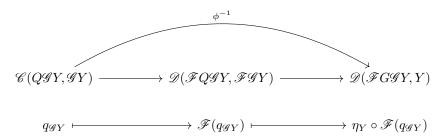
- (a) $(\mathcal{F}, \mathcal{G}, \phi)$ is a Quillen equivalence,
- (b) The composition $X \xrightarrow{\epsilon_X} \mathscr{GF}X \xrightarrow{\mathscr{G}(r_{\mathscr{F}}X)} \mathscr{G}R\mathscr{F}X$ is a weak equivalence for all cofibrant X, and the composition $\mathscr{F}Q\mathscr{G}Y \xrightarrow{\mathscr{F}(q_{\mathscr{G}}Y)} \mathscr{F}QY \xrightarrow{\eta_Y} Y$ is a weak equivalence for all fibrant Y,
- (c) The derived adjunction is an equivalence of categories.

Proof. (a)
$$\Rightarrow$$
 (b)

Assume X is cofibrant in \mathscr{C} and Y is fibrant in \mathscr{D} . Note that $r_{\mathscr{F}X}$ and $q_{\mathscr{G}Y}$ are both weak equivalences. From the diagrams



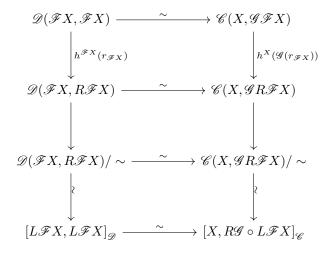
and



we see $\phi(r_{\mathscr{F}X}) = \mathscr{G}(r_{\mathscr{F}X} \circ \epsilon_X)$ and $\phi(\eta_Y \circ \mathscr{F}(q_{\mathscr{G}Y})) = q_{\mathscr{G}Y}$ imply $\mathscr{G}(r_{\mathscr{F}X}) \circ \epsilon_X$ and $\eta_Y \circ \mathscr{F}(q_{\mathscr{G}Y})$ are weak equivalences by the definition of a Quillen equivalence.

$$(b) \Rightarrow (c)$$

It suffices to show that the unit and counit of the adjunction are both isomorphisms. First assume that X is a cofibrant object of \mathscr{C} and note that $\mathscr{F}X$ is cofibrant in X. Chasing the identity through the diagram



we see that the image in the bottom right hand corner is

$$X \xrightarrow{\epsilon_X} \mathscr{GF}X \stackrel{\mathscr{G}(r_{\mathscr{F}X})}{\longrightarrow} \mathscr{G}R\mathscr{F}X$$

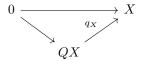
and thus for arbitrary X we obtain the unit of the derived adjunction

$$X \xrightarrow{q_X^{-1}} QX \xrightarrow{\epsilon_{QX}} \mathscr{GF}QX \xrightarrow{\mathscr{G}(r_{\mathscr{F}QX})} \mathscr{G}R\mathscr{F}QX$$

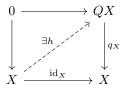
5. Appendix

Lemma 6. If X is a cofibrant object of \mathscr{C} , then $q_X \colon QX \to X$ is an isomorphism. Dually, if Y is a fibrant object of \mathscr{C} , then $r_Y \colon Y \to RY$ is an isomorphism.

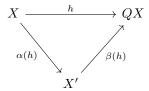
Proof. We have



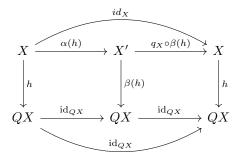
with q_X a trivial fibration, which gives a lift



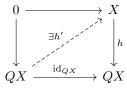
because X is cofibrant. We note that h is a weak equivalence by 2-out-of-3. By the functorial factorization we get



giving a retract



and hence h is a trivial fibration. This now gives a lift



and so we see

$$q_X = q_X \circ \mathrm{id}_{QX} = q_X \circ h \circ h' = \mathrm{id}_X \circ h' = h'.$$