1. Preliminaries

Let k be a Noetherian commutative ring, A a $\mathbb{Z}_{\geq 0}$-graded right Noetherian ring. Denote by $\mathrm{Gr}\text{-}A$ (resp. $\mathrm{gr}\text{-}A$) the category of graded right A-modules (resp. finite) with morphisms

$$\mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M,N) = \{f \in \mathrm{Hom}_A(M,N) \mid f(M_d) \subseteq N_d\}.$$

This is a Grothendieck category with injective envelopes. That is,

- $\mathrm{Gr}\text{-}A$ is abelian (zero object, finite biproducts, all kernels and cokernels, monics and epics are normal–every monic is a kernel and every epic is a cokernel),
- every family of objects has a direct limit/filtered colimit,
- the presheaf $h^A\colon \mathrm{Gr}\text{-}A \to \mathfrak{Set}$ is faithful; for any morphism $M \to N$ the morphism

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M,N) &\longrightarrow \mathrm{Hom}_{\mathfrak{Set}}\left(h^A(M), h^A(N)\right) \\ f &\longmapsto h^A(f) \end{aligned}$$

  is injective.

Definition 1. A full subcategory, $\mathscr{A}$, of $\mathscr{C}$ is called a Serre (or épaisse/thick/dense) subcategory if for any short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

of $\mathscr{C}$, X is an object of $\mathscr{A}$ if and only if both X' and X'' are.

Denote by Tors (resp. tors) the Serre subcategory of torsion modules (resp. finite), where a module M is called torsion if

$$\tau(M) = \{m \in M \mid xA_{\geq s} = 0 \text{ for some } s\} = M$$

2. Quotient Categories

Throughout, let $\mathscr{C}$ be an abelian category.

Definition 2. Let X be an object of $\mathscr{C}$. For two subobjects $i_1\colon X_1 \to X$ and $i_2\colon X_2 \to X$ denote by $X_1 \cap X_2$ the fibered product

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r, "x_2'"] \arrow[d, "x_1'"] & X_1 \arrow[d, "i_1"] \\ X_2 \arrow[r, "i_2"] & X \end{tikzcd}$$

and denote by $X_1 + X_2$ the fibered coproduct

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r, "i_2'"] \arrow[d, "i_1'"] & X_1 \arrow[d, "u_1"] \\ X_2 \arrow[r, "u_2"] & X_1 + X_2. \end{tikzcd}$$

These are both subobjects of X and endow the subobjects of X with lattice structure under the relation

$$X_1 \leq X_2$$

if there exists a monomorphism making the diagram

$$\begin{tikzcd} X_1 \arrow[rr, dashed, "\exists!"] \arrow[dr, "i_1"'] & & X_2 \arrow[dl, "i_2"] \\ & X & \end{tikzcd}$$

commute.

Remark 1. Alternatively, one can construct $X_1 + X_2$ as the image of the morphism s below

$$\begin{tikzcd} X_1 \arrow[dr] \arrow[ddr, bend right, "i_1"'] & & X_2 \arrow[dl] \arrow[ddl, bend left, "i_2"] \\ & X_1 \coprod X_2 \arrow[d, dashed, "\exists! s"] & \\ & X. & \end{tikzcd}$$

Definition 3. Given an object X of $\mathscr{C}$, an essential extension is a monomorphism $i\colon X \to E$ such that for any non-zero subobject $E' \to E$, $E \cap X$ is non-zero.

If E is an injective object, then we say that i is an injective envelope/hull.

Proposition 1. *Let $f : X \to Y$ be a morphism of $\mathscr{C}$. The following are equivalent*

(1) For any subobject $Y' \to Y$, in the pullback diagram

$$\begin{tikzcd} f^{-1}(Y') \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

$f^{-1}(Y') = 0$ implies $Y' = 0$, and

(2) if $\zeta : Z \to X$ is a morphism such that $\ker(f \circ \zeta) = \ker f$, then f is a monomorphism.

In particular, f is an essential extension if and only if whenever $f \circ \zeta$ is a monomorphism, ζ is a morphism.

Proposition 2. *Let Q be an object of $\mathscr{C}$. The following are equivalent.*

(a) Q is injective,

(b) every morphism $X \to Q$ lifts over monics,

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[dl, dashed, "\exists"] \\ & Q & \end{tikzcd}$$

(c) the presheaf of abelain groups h_Q is exact, and

(d) every short exact sequence

$$0 \longrightarrow Q \longrightarrow X \longrightarrow X/Q \longrightarrow 0$$

splits.

Definition 4. We say that a subobject, X', of an object, X, is an $\mathscr{A}$-subobject of X if X' is an object of $\mathscr{A}$. We say that an $\mathscr{A}$-subobject, X', is maximal if for every $\mathscr{A}$-subobject X'' we have a commutative diagram

$$\begin{tikzcd} X'' \arrow[rr, dashed, "\exists! h"] \arrow[dr] & & X' \arrow[dl] \\ & X & \end{tikzcd}$$

If X has no non-zero $\mathscr{A}$ subobjects, then we say that X is $\mathscr{A}$-torsionfree.

Proposition 3. *Let X and Y be objects of $\mathscr{C}$. The collection of pairs of subobjects (X', Y') such that X/X' and Y' are objects of $\mathscr{A}$ is directed by the relation*

$$(X', Y') \leq (X'', Y'')$$

if $X'' \leq X'$ and $Y' \leq Y''$.

Moreover, the system of Abelian groups

$$\mathrm{Hom}_{\mathscr{C}}(X', Y/Y')$$

induced by pairs (X', Y') above is a directed system with morphisms

$$\mathrm{Hom}_{\mathscr{C}}(X', Y/Y') \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X'', Y'')$$

$$(X' \to Y/Y') \longmapsto (X'' \to X' \to Y/Y' \to Y/Y'')$$

whenever $(X', Y') \leq (X'', Y'')$.

Definition 5. Define the quotient category, $\mathscr{C}/\mathscr{A}$, to be the category with objects the objects of $\mathscr{C}$ and morphisms

$$\mathrm{Hom}_{\mathscr{C}/\mathscr{A}}(X, Y) = \mathrm{colim}_{(X',Y')} \mathrm{Hom}_{\mathscr{C}}(X', Y/Y').$$

Let $\pi\colon \mathscr{C} \to \mathscr{C}/\mathscr{A}$ be the canonical projection functor, defined by $\pi(X) = X$ and sending a morphism $f\colon X \to Y$ to its image, $\pi(f)$, in the colimit.

Lemma 1. *The quotient category, $\mathscr{C}/\mathscr{A}$, is an additive category and π is an additive functor.*

Lemma 2. *Let $f\colon X \to Y$ be a morphism of $\mathscr{C}$. We have a factorization of f*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{coim}\, f"'] & & Y \\ & f(X) \arrow[ur, "\mathrm{im}\, f"'] & \end{tikzcd}$$

and an exact sequence

$$0 \longrightarrow K \xrightarrow{\ker f} X \xrightarrow{f} Y \xrightarrow{\mathrm{coker}\, f} C \longrightarrow 0.$$

Then

(i) $\pi(f) = 0$ if and only if $f(X)$ is an object of $\mathscr{A}$,

(ii) $\pi(f)$ is a monomorphism if and only if K is an object of $\mathscr{A}$, and

(iii) $\pi(f)$ is an epimorphism if and only if C is an object of $\mathscr{A}$.

Lemma 3. *For any morphism $f\colon X \to Y$ of $\mathscr{C}$, we have an exact sequence*

$$0 \longrightarrow K \xrightarrow{\ker f} X \xrightarrow{f} Y \xrightarrow{\mathrm{coker}\, f} C \longrightarrow 0.$$

The morphism $\pi(f)$ has a kernel and a cokernel,

$$0 \longrightarrow \mathcal{K} \xrightarrow{\ker \pi(f)} \pi(X) \xrightarrow{\pi(f)} \pi(Y) \xrightarrow{\operatorname{coker} \pi(f)} \mathcal{C} \longrightarrow 0.$$

Moreover, $\pi(\ker f)$ induces an isomorphism $\pi(K) \cong \mathcal{K}$ and $\pi(\operatorname{coker} f)$ induces an isomorphism $\pi(C) \cong \mathcal{C}$.

Lemma 4. *Given an exact sequence*

$$0 \longrightarrow K \xrightarrow{\ker f} X \xrightarrow{f} Y \xrightarrow{\operatorname{coker} f} C \longrightarrow 0.$$

of $\mathscr{C}$, f is an isomorphism if and only if K and C are both objects of $\mathscr{A}$.

Proposition 4. *The quotient category $\mathscr{C}/\mathscr{A}$ is an abelian category and π is an exact functor.*

3. Proj

Denote by QGr-A (resp. qgr-A) the quotient category Gr-A/ Tors (resp. gr-A/ tors). It can be shown that QGr-A is an Ab 5 category; see III.4 of Des Catègories Abélienne. We view QGr-A as the analogue of quasi-coherent sheaves and qgr-A as the analogue of coherent sheaves.

Definition 6. (i) Let $\mathscr{C}$, and $\mathscr{C}'$ be k-linear abelian categories; that is categories enriched over Mod-k. For X and X' objects of $\mathscr{C}$ and $\mathscr{C}'$, a morphism of pairs

$$(\mathscr{C}, X) \to (\mathscr{C}', X')$$

is a pair (f, θ) consisting of an isomorphism $\theta\colon f(X) \to X'$ and a k-linear functor $f\colon \mathscr{C} \to \mathscr{C}'$; that is, the canonical morphism

$$\operatorname{Hom}_{\mathscr{C}}(A, B) \longrightarrow \operatorname{Hom}_{\mathscr{C}'}(f(A), f(B)).$$

is k-linear.

(a) A morphism of pairs is said to be an isomorphism if f is an equivalence of categories.
(b) A morphism of pairs is said to be right exact if f preserves direct limits.
(c) Two morphisms of pairs (f, θ) and (f', θ') are said to be equivalent if there is a natural isomorphism $\eta\colon f \to f'$ compatible with θ and θ'.

(ii) Given two k-linear abelian categories $\mathscr{C}$ and $\mathscr{C}'$ equipped with autoequivalences $s : \mathscr{C} \to \mathscr{C}$ and $s' : \mathscr{C}' \to \mathscr{C}'$, and objects X of $\mathscr{C}$ and X' of $\mathscr{C}$, a morphism of triples

$$(\mathscr{C}, X, s) \to (\mathscr{C}', X', s')$$

is a triple (f, θ, μ) with $f : \mathscr{C} \to \mathscr{C}'$ a k-linear functor, $\theta : f(X) \to X'$ an isomorphism, and $\mu : f \circ s \to s' \circ f$ a natural isomorphism.

(a) A morphism of triples is said to be right exact if f preserves direct limits.
(b) Two morphisms of triples (f_1, θ_1, μ_1) and (f_2, θ_2, μ_2) are said to be equivalent if there exists a natural isomorphism $\eta : f_1 \to f_2$ such that

$$\theta_1 = \theta_2 \circ \eta(A)$$

and for all objects A of $\mathscr{C}$

$$(s' \circ \eta(A)) \circ \mu_1 = \mu_2 \circ \eta(s(A)).$$

(c) A morphism of triples is said to be an isomorphism if f is an equivalence of categories.

(iii) Let s be the twist functor, $s(M) = M[1]$, $s^d(M) = M[d]$, which is an automorphism of $\mathrm{Gr}\text{-}A$. Since $\mathrm{QGr}\text{-}A$ is a quotient category, it inherits this autoequivalence in an obvious way. The general (resp. Noetherian) projective scheme of A, $\mathrm{Proj}\, A$ (resp $\mathrm{proj}\, A$), is the pair $(\mathrm{QGr}\text{-}A, \pi(A))$ (resp. $(\mathrm{qgr}\text{-}A, \pi(A))$).

(iv) A morphism $F : \mathrm{Proj}\, B \to \mathrm{Proj}\, A$ is an equivalence class of right exact morphisms of pairs $(\mathrm{QGr}\text{-}A, \pi(A)) \to (\mathrm{QGr}\text{-}B, \pi(B))$.

(v) A morphism of general schemes is an equivalence class of morphisms of triples $(\mathrm{QGr}\text{-}A, \pi(A), s_A) \to (\mathrm{QGr}\text{-}B, \pi(B), s_B)$.

Analogous definitions are made for $\mathrm{proj}\, A$ by substituting $\mathrm{QGr}\text{-}A$ with $\mathrm{qgr}\text{-}A$ as necessary.

The next two propositions describe the morphisms of $\mathrm{QGr}\text{-}A$ explicitly.

Proposition 5. *Given two objects M, N of* $\mathrm{Gr}\text{-}A$*,*

$$\mathrm{Hom}_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(N)) = \mathrm{colim}_{M'} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M', N/\tau(N)).$$

Proof. Consider the indexing category $\mathscr{I}$ with objects pairs of subobjects (M', N') such that M/M' and N' are objects of Tors and morphisms induced by the relation $\leq$ defined above. We note that because N' is a subobject of $\tau(N)$ for all N', the full subcategory, $\mathscr{J}$, with objects $(M', \tau(N))$ is cofinal. Therefore

$$\mathrm{Hom}_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(N)) = \mathrm{colim}_{\mathscr{I}} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M', N/N') = \mathrm{colim}_{\mathscr{J}} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M', N/\tau(N)).$$

■

Proposition 6. *If M is an object of* $\mathrm{qgr}\text{-}A$*, then*

$$\mathrm{Hom}_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(N)) = \lim_{n\to\infty} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M_{\geq n}, N)$$

where

$$M_{\geq n} = \bigoplus_{d \geq n} M_d.$$

Proof. Let $M' \to M$ be a subobject with torsion quotient. By definition, for each $m \in M$ there is an n_m such that $mA_{\geq n_m} \subseteq M'$. Let $m_1, \ldots, m_s$ be a set of generators for M and let $n = \max\{\deg(m_i) + n_{m_i} \mid i = 1, \ldots, s\}$. The subobject $M_{\geq n}$ has torsion quotient, $M/M_{\geq n}$, and we get the kernel diagram

$$\begin{tikzcd} M_{\geq n} \arrow[r] \arrow[rr, bend left, "0"] \arrow[dr, dashed, "\exists!"] & M \arrow[r] & M/M' \\ & M' \arrow[u] & \end{tikzcd}$$

because for each $m \in M_d$ with $n \leq d$ we can write

$$m = a_1 m_1 + \ldots a_s m_s$$

and by construction we have

$$n_{m_i} = (\deg(m_i) + n_{m_i}) - deg(m_i) \leq n - \deg(m_1) \leq d - \deg(m_1) = \deg(a_i).$$

With $\mathscr{I}$ as above, we see that for any object (M', N') of $\mathscr{I}$ there exists some n such that

$$(M', N') \leq (M_{\geq n}, N)$$

and hence the full subcategory $\mathscr{J}$ with objects $(M_{\geq n}, N)$ is cofinal. Therefore

$$\begin{aligned} \operatorname{Hom}_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(N)) &= \operatorname{colim}_{\mathscr{I}} \operatorname{Hom}_{\mathrm{Gr}\text{-}A}(M', N') \\ &= \operatorname{colim}_{\mathscr{J}} \operatorname{Hom}_{\mathrm{Gr}\text{-}A}(M_{\geq n}, N) \\ &= \lim_{n\to\infty} \operatorname{Hom}_{\mathrm{Gr}\text{-}A}(M_{\geq n}, N). \end{aligned}$$

■

4. The Section Functor

Lemma 5. *If X is an object of $\mathscr{C}$, then the following are equivalent.*

(1) Given a short exact sequence

$$0 \longrightarrow K \xrightarrow{\ker f} Z \xrightarrow{f} Y \xrightarrow{\operatorname{coker} f} C \longrightarrow 0$$

with K and C objects of $\mathscr{A}$, then the canonical morphism

$$h_X(f)\colon h_X(Y) \to h_X(Z)$$

is an isomorphism,

(2) X is $\mathscr{A}$-torsionfree and any short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{\operatorname{coker} f} C \longrightarrow 0$$

with C an object of $\mathscr{A}$ splits, and

(3) For any object Y of $\mathscr{C}$, $\pi\colon \mathscr{C} \to \mathscr{C}/\mathscr{A}$ induces an isomorphism

$$\operatorname{Hom}_{\mathscr{C}}(Y, X) \cong \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(Y), \pi(X)).$$

Proof. (1) $\implies$ (2). Given an $\mathscr{A}$-subobject $i\colon X' \to X$, then we have the short exact sequence

$$0 \longrightarrow X' \xrightarrow{i} X \xrightarrow{\operatorname{coker} i} X/X' \longrightarrow 0$$

both X' and 0 are objects of $\mathscr{A}$, hence an isomorphism

$$h_X(\operatorname{coker} i)\colon \operatorname{Hom}_{\mathscr{C}}(X/X', X) \to \operatorname{Hom}_{\mathscr{C}}(X, X)$$

which implies that $\operatorname{coker} i$ is monic. Therefore $\operatorname{coker} i \circ i = 0$ implies $i = 0$.

Now, if we let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{p} C \longrightarrow 0$$

be a short exact sequence with C an object of $\mathscr{A}$, then the isomorphism

$$h_X(f)\colon \operatorname{Hom}_{\mathscr{C}}(Y, X) \to \operatorname{Hom}_{\mathscr{C}}(X, X)$$

yields a section $s\colon Y \to X$ of f, so the sequence splits.

(2) $\implies$ (3). Let Y be an object of $\mathscr{C}$. Given a morphism $f : \pi(Y) \to \pi(X)$, we lift to a morphism $f'\colon Y' \to X/X'$ with Y/Y' and X' objects of $\mathscr{A}$. Since we have assumed that X has no non-trivial $\mathscr{A}$-subobjects, it follows that $X/X' = X$. By dualizing the relevant theorems on fiber products, this gives the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Y' \arrow[r, "j"] \arrow[d, "f'"] & Y \arrow[r, "\operatorname{coker} j"] \arrow[d, "f''"] & Y/Y' \arrow[r] \arrow[d, dashed, "\exists! h"] & 0 \\
0 \arrow[r] & X \arrow[r, "i"] & Y \coprod_{Y'} X \arrow[r, "\operatorname{coker} i"] & \left(Y \coprod_{Y'} X\right)/X \arrow[r] & 0
\end{tikzcd}$$

and with h an isomorphism. Since Y/Y' was assumed to be an object of $\mathscr{A}$, so too is $\left(Y \coprod_{Y'} X\right)/X$ and thus there exists a section $s : Y \coprod_{Y'} X \to X$ of i so that

$$f' = id_X \circ f' = s \circ i \circ f' = s \circ f'' \circ j.$$

By commutativity of the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathscr{C}}(Y, X) \arrow[rr, "_ \circ j"] \arrow[dr] & & \operatorname{Hom}_{\mathscr{C}}(Y', X) \arrow[dl] \\
& \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(Y), \pi(X)) &
\end{tikzcd}$$

we see that $\pi(s \circ f'') = f$ and thus

$$\operatorname{Hom}_{\mathscr{C}}(Y, X) \to \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(Y), \pi(X))$$

is surjective. For injectivity, suppose that $f\colon Y \to X$ satisfies $\pi(f) = 0$. Then $f(Y)$ is an object of $\mathscr{A}$ and from the short exact sequence

$$0 \longrightarrow f(Y) \xrightarrow{\operatorname{im} f} X \xrightarrow{\operatorname{coker} f} C \longrightarrow 0$$

we see that $\operatorname{im} f = 0$. Therefore $f = \operatorname{im} f \circ \operatorname{coim} f = 0$, as desired.

(3) $\implies$ (1). Let

$$0 \longrightarrow K \xrightarrow{i} Z \xrightarrow{f} Y \xrightarrow{p} C \longrightarrow 0$$

be an exact sequence with K and C objects of $\mathscr{A}$. We have the commutative diagram

$$\begin{tikzcd}
Z \arrow[d, "f"] & \operatorname{Hom}_{\mathscr{C}}(Y, X) \arrow[r] \arrow[d, "h_X(f)"] & \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(Y), \pi(X)) \arrow[d, "h_{\pi(X)}(\pi(f))"] \\
Y & \operatorname{Hom}_{\mathscr{C}}(Z, X) \arrow[r, "\sim"] & \operatorname{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(Z), \pi(X))
\end{tikzcd}$$

with $h_{\pi(X)}(\pi(f))$ an isomorphism because $\pi(f)$ is. Therefore $h_X(f)$ is an isomorphism, as desired. ∎

Definition 7. (i) If X is an object of $\mathscr{C}$ satisfying any of the conditions in Lemma 5, then we say that X is $\mathscr{A}$-closed.

(ii) A morphism $X \to Y$ is an $\mathscr{A}$-envelope if in the exact sequence

$$0 \longrightarrow K \longrightarrow X \longrightarrow Y \longrightarrow C \longrightarrow 0$$

Y is $\mathscr{A}$-closed, and both K and C are objects of $\mathscr{A}$.

Lemma 6. *If X has a maximal $\mathscr{A}$-subobject, $X_{\mathscr{A}}$, then $X/X_{\mathscr{A}}$ is $\mathscr{A}$-torsionfree.*

Proof. Let $j\colon Y \to X/X_{\mathscr{A}}$ be a monic with Y an object of $\mathscr{A}$. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\ker p'"] \arrow[d, dashed, "\exists! h"] & X \times_{X/X_{\mathscr{A}}} Y \arrow[r, "p'"] \arrow[d, "i'"] & Y \arrow[r, "\operatorname{coker} p'"] \arrow[d, "j"] & C \arrow[d, dashed, "\exists! h'"] \\
0 \arrow[r] & X_{\mathscr{A}} \arrow[r, "i"] & X \arrow[r, "p"] & X/X_{\mathscr{A}} \arrow[r, "\operatorname{coker} p"] & 0
\end{tikzcd}$$

with h an isomorphism, and h' monic, hence an isomorphism. The top row gives us the short exact sequence

$$0 \longrightarrow K \xrightarrow{\ker p'} X \times_{X/X_{\mathscr{A}}} Y \xrightarrow{p'} Y \xrightarrow{\operatorname{coker} p'} 0$$

with K and Y objects of $\mathscr{A}$, hence $X \times_{X/X_{\mathscr{A}}} Y$ is also an object of $\mathscr{A}$. By maximality of $X_{\mathscr{A}}$, i' factors through i uniquely,

$$\begin{tikzcd}
X \times_{X/X_{\mathscr{A}}} Y \arrow[rr, dashed, "i''"] \arrow[dr, "i'"'] & & X_{\mathscr{A}} \arrow[dl, "i"] \\
& X &
\end{tikzcd}$$

and so we see

$$j \circ p' = p \circ i = p \circ (i \circ i'') = (p \circ i) \circ i'' = 0$$

implies, because p' is epic, that $j = 0$. Therefore $X/X_{\mathscr{A}}$ is $\mathscr{A}$-torsionfree, as desired. ■

Lemma 7. *If $\mathscr{C}$ is such that every object of $\mathscr{C}$ has a maximal $\mathscr{A}$-subobject and every $\mathscr{A}$-torsionfree object has a monomorphism to an $\mathscr{A}$-closed object, then every object of $\mathscr{C}$ has an $\mathscr{A}$-envelope.*

Proof. Let X be an object of $\mathscr{C}$ and let $X_{\mathscr{A}}$ be its maximal $\mathscr{A}$-subobject, so we have the short exact sequence

$$0 \longrightarrow X_{\mathscr{A}} \xrightarrow{i} X \xrightarrow{p} X/X_{\mathscr{A}} \longrightarrow 0.$$

By assumption, there exists an $\mathscr{A}$-closed object Y and a short exact sequence

$$0 \longrightarrow X/X_{\mathscr{A}} \xrightarrow{j} Y \xrightarrow{q} C \longrightarrow 0$$

from which we construct the pullback

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\ker q'"] \arrow[d, dashed, "\exists! h"] & q^{-1}(C_{\mathscr{A}}) \arrow[r, "q'"] \arrow[d, "k'"] & C_{\mathscr{A}} \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & X/X_{\mathscr{A}} \arrow[r, "j"] & Y \arrow[r, "q"] & C \arrow[r] & 0,
\end{tikzcd}$$

with h an isomorphism. Then from the short exact sequence

$$0 \longrightarrow X/X_{\mathscr{A}} \cong K \xrightarrow{\ker q'} q^{-1}(C_{\mathscr{A}}) \xrightarrow{q'} C_{\mathscr{A}} \longrightarrow 0$$

it suffices to show that $q^{-1}(C_{\mathscr{A}})$ is $\mathscr{A}$-closed.

It's clear that $q^{-1}(C_{\mathscr{A}})$ is $\mathscr{A}$-torsionfree because it is a subobject of the $\mathscr{A}$-closed object Y. If we have any short exact sequence

$$0 \longrightarrow q^{-1}(C_{\mathscr{A}}) \xrightarrow{s} A \xrightarrow{\operatorname{coker} s} B \longrightarrow 0$$

with B an object of $\mathscr{A}$, then by Lemma 1 there is a unique morphism $\varphi : A \to Y$ such that

$$k' = \varphi \circ s = h_Y(s)(\varphi).$$

Now we have the commutative diagram

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists ! r"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "\varphi"'] & & \\
& q^{-1}(C_{\mathscr{A}}) \arrow[r, "q'"] \arrow[d, "k'"] & C_{\mathscr{A}} \arrow[d, "k"] \\
& Y \arrow[r, "q"] & C
\end{tikzcd}$$

from which we see that

$$k' \circ id_{q^{-1}(C_{\mathscr{A}})} = k' = \varphi \circ s = (k' \circ r) \circ s = k'(\circ r \circ s)$$

and thus $r \circ s = id_{q^{-1}(C_{\mathscr{A}})}$. Therefore $q^{-1}(C_{\mathscr{A}})$ is $\mathscr{A}$-closed by Lemma 5.2, as desired. ■

Lemma 8. *If $\pi \colon \mathscr{C} \to \mathscr{C}/\mathscr{A}$ has a right adjoint, $\omega \colon \mathscr{C}/\mathscr{A} \to \mathscr{C}$, then*

(1) for each object Y of $\mathscr{C}$, $\omega\pi(Y)$ is $\mathscr{A}$-closed,

(2) for Y an object of $\mathscr{C}$, the morphism $\eta_{\pi(Y)} : \pi\omega\pi(Y) \to \pi(Y)$ is an isomorphism, and

(3) ω is fully faithful.

Proof. (1) Given an exact sequence

$$0 \longrightarrow K \longrightarrow Z \xrightarrow{f} Y \longrightarrow C \longrightarrow 0$$

with K and C objects of $\mathscr{A}$, we have that $\pi(f)$ is an isomorphism and hence $h_{\pi(Y)}(\pi(f))$ is also an isomorphism. From the adjunction we get the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(X, \omega\pi(Y)) \arrow[r, "\sim"] \arrow[d, "h_{\omega\pi(Y)}(f)"] & \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(X), \pi(Y)) \arrow[d, "h_{\pi(Y)}(\pi(f))"] \\
\mathrm{Hom}_{\mathscr{C}}(Z, \omega\pi(Y)) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}(\pi(Z), \pi(Y))
\end{tikzcd}$$

which shows that $h_{\pi(Y)}\pi(F)$ is an isomorphism. Therefore $\omega\pi(Y)$ is $\mathscr{A}$-closed by part 1 of Lemma 5.

(2) We have the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(\omega\pi(Y), \omega\pi(Y)) \arrow[rr, "\sim"] \arrow[dr, "\sim"] & & \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}(\pi\omega\pi(Y), \pi\omega\pi(Y)) \arrow[dl, "h_{\pi\omega\pi(Y)}(\eta_{\pi(Y)})"] \\
& \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}(\pi\omega\pi(Y), \pi(Y)) &
\end{tikzcd}$$

since for any morphism $f \colon \omega\pi(Y) \to \omega\pi(Y)$, the image under the adjunction isomorphism is just $\eta_{\pi(Y)} \circ \pi(f)$. This immediately implies that $h_{\pi\omega\pi(Y)}(\eta_{\pi(Y)})$ is an isomorphism, and hence so is $\eta_{\pi(Y)}$.

(3) Since ω being fully faithful is equivalent to η being a natural isomorpism, this is a consequence of the definition of $\mathscr{C}/\mathscr{A}$. Indeed, every object of $\mathscr{C}/\mathscr{A}$ is $\pi(X)$ for some object X of $\mathscr{C}$, and the result follows.

■

Theorem 1. *The following are equivalent.*

(1) $\pi \colon \mathscr{C} \to \mathscr{C}/\mathscr{A}$ has a right adjoint, and

(2) Every object of $\mathscr{A}$ has a maximal $\mathscr{A}$-subobject and every $\mathscr{A}$-torsionfree object has a monomorphism into an $\mathscr{A}$-closed object.

Proof. First assume that π has a right adjoint, $\omega\colon \mathscr{C}/\mathscr{A} \to \mathscr{C}$, and let Y be an object of $\mathscr{C}$. There are then two natural transformations of adjunction, $\varepsilon : \mathrm{id}_{\mathscr{C}} \to \omega\pi$ (unit) and $\eta : \pi\omega \to \mathrm{id}_{\mathscr{C}/\mathscr{A}}$ (counit), the latter being an isomorphism by Lemma 8. It follows from the commutative diagram

$$\begin{tikzcd} \pi(Y) \arrow[r, "\pi(\varepsilon_Y)"] \arrow[dr, "\mathrm{id}_{\pi(Y)}"'] & \pi\omega\pi(Y) \arrow[d, "\eta_{\pi(Y)}"] \\ & \pi(Y) \end{tikzcd}$$

that $\pi(\varepsilon_Y) = \eta^{-1}_{\pi(Y)}$ is an isomorphism, whence in the short exact sequence

$$0 \longrightarrow K \longrightarrow Y \xrightarrow{\varepsilon_Y} \omega\pi(Y) \longrightarrow C \longrightarrow 0$$

both K and C are objects of $\mathscr{A}$. We show that K is the desired subobject. Indeed, let $j\colon Y' \to Y$ be a subobject of Y with Y' and object of $\mathscr{A}$. We have the commutative diagram

$$\begin{tikzcd} Y' \arrow[rr, "\varepsilon_Y \circ j"] \arrow[dr, "\mathrm{coim}\,(\varepsilon_Y \circ j)"'] & & \omega\pi(Y) \\ & \varepsilon_Y(Y') \arrow[ur, "\mathrm{im}\,(\varepsilon_Y \circ j)"'] & \end{tikzcd}$$

and we note that because $\omega\pi(Y)$ is $\mathscr{A}$-closed and $\varepsilon_Y(Y') \cong Y'/(Y' \cap K)$ is an object of $\mathscr{A}$, the monic $\mathrm{im}\,(\varepsilon_Y \circ j)$ is zero. Therefore by the kernel diagram

$$\begin{tikzcd} Y' \arrow[r, "j"] \arrow[rr, bend left, "0"] \arrow[dr, dashed, "\exists! j'"'] & Y \arrow[r, "\varepsilon_Y"] & \omega\pi(Y) \\ & K \arrow[u] & \end{tikzcd}$$

we see that j' is monic, and K is maximal, as desired.

Conversely, assume that every object of $\mathscr{C}$ has a maximal $\mathscr{A}$-subobject and every $\mathscr{A}$-torsionfree object has a monomorphism into an $\mathscr{A}$-closed object. Let Y be an object of $\mathscr{C}$. By Lemma 7, Y has an $\mathscr{A}$-envelope $Y \to E$. Hence $\pi(Y) \cong \pi(E)$ and by the natural isomorphisms

$$\mathrm{Hom}_{\mathscr{C}}\left(\_, E\right) \cong \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(\_), \pi(E)\right) \cong \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(\_), \pi(Y)\right),$$

the presheaf $\mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(\_), \pi(Y)\right)$ on $\mathscr{C}$ is representable. Therefore π admits a right adjoint. ∎

Definition 8. If π has a right adjoint, then we say that $\mathscr{A}$ is a localizing subcategory.

Corollary 1. *Assume that π has a right adjoint, $\omega\colon \mathscr{C}/\mathscr{A} \to \mathscr{C}$. Then*

(1) $\mathscr{A}$-envelopes are unique up to unique isomorphism,
(2) for every object X of $\mathscr{C}$, $\omega\pi(X) \cong E$, where $X \to E$ is an $\mathscr{A}$-envelope of X,

Proof. (1) By the proof of Theorem 1, an $\mathscr{A}$-envelope of an object Y of $\mathscr{C}$ represents the presheaf $\mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(\_), Y\right)$ and thus is unique up to unique isomorphism.
(2) This is immediate from Yoneda's Lemma.

∎

Lemma 9. *Assume that $\mathscr{A}$ is a localizing subcategory, X, Y, objects of $\mathscr{C}$, $X_{\mathscr{A}}$, $Y_{\mathscr{A}}$, their maximal $\mathscr{A}$-subobjects. A morphism $f\colon X \to Y$ induces a morphism*

$$\begin{tikzcd} X_{\mathscr{A}} \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[d, "q"] \\ & X/X_{\mathscr{A}} \arrow[r, dashed, "\exists! h"] & Y/Y_{\mathscr{A}} \end{tikzcd}$$

and the morphism $\pi(f)$ is an essential extension if and only if h is.

Proof. We first note that $\pi(p)$ and $\pi(h)$ are isomorphisms, hence essential extensions, so

$$\pi(f) = \pi(q)^{-1} \circ \pi(h) \circ \pi(p)$$

is an essential extension if and only if $\pi(h)$ is. Hence it suffices to assume that $X_{\mathscr{A}} = Y_{\mathscr{A}} = 0$ and $h = f$.

Assume first that $\pi(f)$ is an essential extension. Given a subobject $k : Y' \to Y$ we get the pullback

$$\begin{tikzcd} \pi(Y' \times_Y X) \arrow[r, "\pi(k')"] \arrow[d, "\pi(f')"] & \pi(X) \arrow[d, "\pi(f)"] \\ \pi(Y') \arrow[r, "\pi(k)"] & \pi(Y) \end{tikzcd}$$

because π is exact. We note that so long as Y' is not an object of $\mathscr{A}$, $\pi(Y')$ is not zero. Since Y was assumed to be $\mathscr{A}$-torsionfree, this is equivalent to Y' being non-zero. Therefore $\pi(Y' \times_Y X)$ is non-zero whenever Y' is non-zero because $\pi(f)$ is essential and hence so is $Y' \times_Y X$.

Conversely, assume that f is an essential extension. Given $i : \pi(Z) \to \pi(Y)$ a non-zero subobject, we may lift to a morphism

$$0 \longrightarrow K \xrightarrow{\ker j} Z' \xrightarrow{j} Y$$

with Z/Z' and K objects of $\mathscr{A}$ since k is monic and Y has no non-zero $\mathscr{A}$-subobjects. Since f is an essential extension we have the non-zero pullback

$$\begin{tikzcd} Z'/K \times_Y X \arrow[r, "k"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Z/K \arrow[r, "\operatorname{im} j"] & Y \end{tikzcd}$$

As K is an object of $\mathscr{A}$, the short exact sequence

$$0 \longrightarrow K \xrightarrow{\ker j} Z' \xrightarrow{\operatorname{coim} j} Z'/K \longrightarrow 0$$

gives the isomorphism

$$\pi(Z'/K) \cong \pi(Z') \cong \pi(Z).$$

Therefore

$$\pi(Z'/K \times_Y X) \cong \pi(Z'/K) \times_{\pi(Y)} \pi(X) \cong \pi(Z) \times_{\pi(Y)} \pi(X)$$

is non-zero, as desired. ∎

Lemma 10. *If Q is an $\mathscr{A}$-closed injective, then $\pi(Q)$ is injective.*

Proof. Given a short exact sequence

$$0 \longrightarrow \pi(Q) \xrightarrow{\ s\ } \pi(X) \xrightarrow{\operatorname{coker} s} \pi(X/Q) \longrightarrow 0$$

it is enough to show that s is a section; that is, there exists a morphism $r\colon \pi(X) \to \pi(Q)$ such that $r \circ s = \mathrm{id}_{\pi(Q)}$. We can lift s to a morphism

$$0 \longrightarrow K \xrightarrow{\ker t} Q' \xrightarrow{\ t\ } X/X'$$

with K, Q/Q', and X' objects of $\mathscr{A}$. Since we have assumed that Q is $\mathscr{A}$-closed, the diagram

$$\begin{tikzcd} K \arrow[r, "\ker t"] \arrow[rr, bend left, "0"] & Q' \arrow[r, "i"] & Q \end{tikzcd}$$

commutes and thus we see that $\ker t = 0$ because i is a monomorphism. Since Q was assumed to be injective, we have the lift

$$\begin{tikzcd} 0 \arrow[r] & Q' \arrow[r, "t"] \arrow[d, "i"] & X/X' \arrow[dl, dashed, "\exists r"] \\ & Q. & \end{tikzcd}$$

If we let $q\colon X \to X/X'$ be the canonical projection, then we have the diagram

$$\begin{tikzcd} \pi(Q') \arrow[r, "\pi(t)"] \arrow[d, "\pi(i)"'] & \pi(X/X') \arrow[dl, "\pi(r)"'] \\ \pi(Q) \arrow[r, "s"] & \pi(X) \arrow[u, "\pi(q)"'] \end{tikzcd}$$

with $\pi(i)$ and $\pi(q)$ isormoprhisms, the top left triangle commutative, and the outer square commutative. Therefore

$$\mathrm{id}_{\pi(Q)} \circ \pi(i) = \pi(i) = \pi(r) \circ \pi(t) = \pi(r) \circ \pi(q) \circ s \circ \pi(i)$$

implies, because $\pi(i)$ is an isomorphism, that

$$(\pi(r) \circ \pi(q)) \circ s = \mathrm{id}_{\pi(Q)},$$

as desired. ∎

Lemma 11. *If $i\colon X \to E$ is an injective envelope and X is $\mathscr{A}$-torsionfree, then E is $\mathscr{A}$-closed and the morphism $\pi(i)\colon \pi(X) \to \pi(E)$ is an injective envelope.*

Proof. Since E is injective, every short exact sequence

$$0 \longrightarrow E \longrightarrow A \longrightarrow B \longrightarrow 0$$

splits. To see that E is $\mathscr{A}$-closed, it then suffices by Lemma 5.2 to show that E is $\mathscr{A}$-torsionfree. Given an $\mathscr{A}$-subobject $j : E' \to E$, we have the pullback

$$\begin{tikzcd} E' \times_E X \arrow[r, "j'"] \arrow[d, "i'"] & X \arrow[d, "i"] \\ E' \arrow[r, "j"] & E \end{tikzcd}$$

and the morphism i' gives $E' \times_E X$ E-subobject structure, hence is an object of $\mathscr{A}$. Since X is $\mathscr{A}$-torsionfree by assumption, $E' \times_E X = 0$ and thus E' is also zero because i is essential.

By Lemma 10 we see that $\pi(E)$ is injective, so it remains to show that $\pi(i)$ is essential. To see this, we note that the assumption $\mathscr{A}$ is a localizing subcategory in Lemma 9 was only used to produce maximal $\mathscr{A}$-subobjects, and hence the same argument shows that $\pi(i)$ is essential. Therefore $\pi(i)$ is an injective envelope. ■

Proposition 7. *Assume that $\mathscr{A}$ is a localizing subcategory. If $\mathscr{C}$ has injective envelopes, then*

(i) $\mathscr{C}/\mathscr{A}$ has injective envelopes,

(ii) Every injective object of $\mathscr{C}/\mathscr{A}$ is isomorphic to $\pi(Q)$ for some $\mathscr{A}$-closed injective, Q, and

(iii) Every injective object Q of $\mathscr{C}$ is isomorphic to $E \oplus \omega(Q_2)$, where $Q_{\mathscr{A}} \to E$ is an injective envelope of the maximal $\mathscr{A}$-subobject of Q and Q_2 is an injective object of $\mathscr{C}/\mathscr{A}$.

Proof. (i) Given an object $\pi(X)$ of $\mathscr{C}/\mathscr{A}$, let $X_{\mathscr{A}}$ be the maximal $\mathscr{A}$-subobject of X. Since $\mathscr{C}$ has injective envelopes and $X/X_{\mathscr{A}}$ is $\mathscr{A}$-torsionfree, an injective envelope $X/X_{\mathscr{A}} \to E$ gives the injective envelope

$$\pi(X) \cong \pi(X/X_{\mathscr{A}}) \to \pi(E)$$

by Lemma 11.

(ii) Given an injective object $\pi(Q)$ of $\mathscr{C}/\mathscr{A}$, $\omega\pi(Q)$ is $\mathscr{A}$-closed by Lemma 8.1 and is injective because π is exact. Therefore by Lemma 8.3, $\pi(Q) \cong \pi(\omega\pi(Q))$.

(iii) Let Q be an injective object of $\mathscr{C}$, let $i\colon Q_{\mathscr{A}} \to Q$ be its maximal $\mathscr{A}$-subobject, and let $j\colon Q_{\mathscr{A}} \to E$ be an injective envelope. Since Q is injective we have the lift

$$\begin{tikzcd} 0 \arrow[r] & Q_{\mathscr{A}} \arrow[r, "j"] \arrow[d, "i"] & E \arrow[dl, dashed, "\exists k"] \\ & Q & \end{tikzcd}$$

with k a monomorphism because j is essential and $\ker(k \circ j) = \ker i = 0$. Because E is injective we get the split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, shift left, "k"] & Q \arrow[l, shift left, "r"] \arrow[r, shift left, "p"] & Q/E \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$

so we need only show that Q/E is an $\mathscr{A}$-closed injective, for then $\pi(Q/E)$ is injective by Lemma 10, and $Q/E \cong \omega\pi(Q/E)$.

The fact that Q/E is injective follows from the fact that both Q and E are injective. Thus every monomorphism out of Q/E splits, so by Lemma 5.2 it is enough to show that Q/E is $\mathscr{A}$-torsionfree. Given an $\mathscr{A}$-subobject $\varphi\colon X \to E$, the fact that $Q/Q_{\mathscr{A}}$ is $\mathscr{A}$-torsionfree gives the kernel diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[rrr, bend left, "0"] \arrow[drr, dashed, "\exists! h"'] & Q/E \arrow[r, "s"] & Q \arrow[r] & Q/Q_{\mathscr{A}} \\ & & Q_{\mathscr{A}} \arrow[u, "i"] & \end{tikzcd}$$

Therefore

$$\varphi = \mathrm{id}_{Q/E} \circ \varphi = p \circ s \circ \varphi = p \circ i \circ h = p \circ k \circ j \circ h = 0,$$

as desired.

∎

Corollary 2. *Assume that $\mathscr{A}$ is a localizing subcategory and that $\mathscr{C}$ has injective envelopes. If the injective envelope of an object of $\mathscr{A}$ is a morphism of $\mathscr{A}$, then*

(i) the maximal subobject of an injective is injective and thus the $\mathscr{A}$-envelope of an injective, Q, is $Q \to Q/Q_{\mathscr{A}}$, where $Q_{\mathscr{A}}$ is the maximal $\mathscr{A}$-subobject,

(ii) π preserves injectives, and

Proof. (i) Let Q be an injective object of $\mathscr{C}$ and $i\colon Q_{\mathscr{A}} \to Q$ its maximal $\mathscr{A}$-subobject. Given an injective envelope $j\colon Q_{\mathscr{A}} \to E$, we have the lift

$$\begin{tikzcd} 0 \arrow[r] & Q_{\mathscr{A}} \arrow[r, "j"] \arrow[d, "i"] & E \arrow[dl, dashed, "\exists k"] \\ & Q & \end{tikzcd}$$

with k a monomorphism because j is essential. Since E is assumed to be an object of $\mathscr{A}$, k factors through i uniquely,

$$\begin{tikzcd} E \arrow[r, dashed, "\exists!\varphi"] \arrow[dr, "k"'] & Q_{\mathscr{A}} \arrow[d, "i"] \\ & Q. \end{tikzcd}$$

So we see that

$$k \circ j \circ \varphi = i \circ \varphi = k = k \circ id_E$$

implies that $j \circ \varphi = id_E$ and

$$i \circ \varphi \circ j = k \circ j = i = i \circ id_{Q_{\mathscr{A}}}$$

implies $\varphi \circ j = id_{Q_{\mathscr{A}}}$. Hence φ is an isomorphism. Therefore by Proposition 7 we have the short exact sequence

$$0 \longrightarrow Q_{\mathscr{A}} \longrightarrow Q_{\mathscr{A}} \oplus Q/Q_{\mathscr{A}} \longrightarrow Q/Q_{\mathscr{A}} \longrightarrow 0$$

and $Q/Q_{\mathscr{A}}$ is $\mathscr{A}$-closed, as desired.

(ii) If Q is an injective object of $\mathscr{C}$, then by the above $Q/Q_{\mathscr{A}}$ is $\mathscr{A}$-closed and hence $\pi(Q) \cong \pi(Q/Q_{\mathscr{A}})$ is injective by Lemma 10.

∎

Proposition 8. *Let A be a right noetherian $\mathbb{Z}_{\geq 0}$ graded algebra over a commutative noetherian ring, k. If $i \in \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M, N)$ is an essential extension, then*

(a) the right bounds of N and M are equal; that is, if there exists some $0 \ll n$ such that $M_d = 0$ for all $n \leq d$, then $N_d = 0$ for all $n \leq d$, and

(b) if M is torsion, then so is N.

Proof. See Proposition 2.2 on page 234 of *Noncommutative Projective Schemes.* ∎

Remark 2. In this case, every object M of Gr-A has a maximal Tors-subobject, $\tau(M)$, and hence Tors is a localizing subcategory. In particular, every injective object, Q, of Gr-A decomposes as $\tau(Q) \oplus Q/\tau(Q)$

with $Q/\tau(Q)$ a Tors-closed object, and $\omega\pi(Q) \cong Q/\tau(Q)$ and the injective objects of $\mathrm{QGr}\text{-}A$ are precisely $\pi(Q/Q(\tau))$ for Q an injective object of $\mathrm{Gr}\text{-}A$.

5. Cohomology

Corollary 3. *Assume that $\mathscr{A}$ is a localizing subcategory and that $\mathscr{C}$ has injective envelopes. If the injective envelope of an object of $\mathscr{A}$ is a morphism of $\mathscr{A}$, then for objects X and Y of $\mathscr{C}$*

$$\mathrm{Ext}^i_{\mathscr{C}/\mathscr{A}}\left(\pi(X),\pi(Y)\right) \cong R^i\,\mathrm{Hom}_{\mathscr{C}}\left(X,\omega\pi(Y)\right).$$

Proof. Take an injective resolution

$$Q^{\cdot} : 0 \longrightarrow Y = Q^0 \xrightarrow{d^0} Q^1 \xrightarrow{d^1} \cdots$$

of Y. By Corollary 2.ii, $\pi(Q^{\cdot})$ is an injective resolution of $\pi(Y)$. Using the natural transformation $\varepsilon\colon id_{\mathscr{C}} \to \omega\pi$ we have for each n an isomorphism of adjunction

$$\Phi^n\colon\ \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(X),\pi(Q^n)\right) \longrightarrow \mathrm{Hom}_{\mathscr{C}}\left(X,\omega\pi(Q^n)\right)$$

$$\varphi \longmapsto \omega(\varphi)\circ\varepsilon_X$$

and so we get an isomorphism of chain complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(X),\pi(Q^{\cdot})\right) : 0 \arrow[r] & \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(X),\pi(Y)\right) \arrow[r, "h^{\pi(X)}(\pi(d^0))"] \arrow[d, "\Phi^0"] & \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(X),\pi(Q^1)\right) \arrow[r, "h^{\pi(X)}(\pi(d^1))"] \arrow[d, "\Phi^1"] & \cdots \\
\mathrm{Hom}_{\mathscr{C}}\left(X,\omega\pi(Q^{\cdot})\right) : 0 \arrow[r] & \mathrm{Hom}_{\mathscr{C}}\left(X,\omega\pi(Y)\right) \arrow[r, "h^X(\omega\pi(d^0))"] & \mathrm{Hom}_{\mathscr{C}}\left(X,\omega\pi(Q^1)\right) \arrow[r, "h^X(\omega\pi(d^1))"] & \cdots
\end{tikzcd}$$

since for $\varphi \in \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(X),Q^n\right)$ we have

$$h^X(\omega\pi(d^n))\circ\Phi^n(\varphi) = \omega\pi(d^n)\circ\omega(\varphi)\circ\varepsilon_X = \omega(\pi(d^n)\circ\varphi)\circ\varepsilon_X = \Phi^{n+1}\circ h^X(\pi(d^n))(\varphi).$$

Therefore

$$\mathrm{Ext}^i_{\mathscr{C}/\mathscr{A}}\left(\pi(X),\pi(Y)\right) \cong h^i\left(\mathrm{Hom}_{\mathscr{C}/\mathscr{A}}\left(\pi(X),\pi(Q^{\cdot})\right)\right) \cong h^i\left(\mathrm{Hom}_{\mathscr{C}}\left(X,\omega\pi(Q^{\cdot})\right)\right) \cong R^i\,\mathrm{Hom}_{\mathscr{C}}\left(X,\omega\pi(Y)\right)$$

∎

From here on, let A be a right Noetherian $\mathbb{Z}_{\geq 0}$-graded algebra over a commutative Noetherian ring k, $\mathscr{C} = \mathrm{Gr}\text{-}A$, $\mathscr{A} = \mathrm{Tors}$, $\mathscr{C}/\mathscr{A} = \mathrm{QGr}\text{-}A$.

Definition 9. Define the graded modules

$$\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}\left(M,N\right) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}\left(M,N[d]\right)$$

and

$$\underline{\mathrm{Hom}}_{\mathrm{QGr}\text{-}A}\left(\pi(M),\pi(N)\right) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}\text{-}A/A}\left(\pi(M),\pi(N)[d]\right).$$

Proposition 9. *The right derived functors of* $\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}(M,N)$ *and* $\underline{\mathrm{Hom}}_{\mathrm{QGr}\text{-}A}(\pi(M),\pi(N))$ *are*

$$\underline{\mathrm{Ext}}^i_{\mathrm{Gr}\text{-}A}(M,N) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Ext}^i_{\mathrm{Gr}\text{-}A}(M, N[d])$$

and

$$\underline{\mathrm{Ext}}^i_{\mathrm{QGr}\text{-}A}(\pi(M),\pi(N)) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Ext}^i_{\mathrm{QGr}\text{-}A}(\pi(M), \pi(N)[d]) .$$

Moreover, for $Q^\cdot$ *an injective resolution of* N*,*

$$\underline{\mathrm{Ext}}^i_{\mathrm{QGr}\text{-}A}(\pi(M),\pi(N)) \cong h^i\left(\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}(M, \omega\pi(Q^\cdot))\right) \cong R^i\,\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}(M, \omega\pi(N)) .$$

Proof. The first is a consequence of the shift operator, s, being an automorphism of $\mathrm{Gr}\text{-}A$ and cohomology being an additive functor. In particular, if $Q^\cdot$ is an injective resolution of N, then we have

$$\begin{aligned} h^i(\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}(M, Q^\cdot) &= h^i\left(\bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M, (Q^\cdot)[d])\right) \\ &\cong \bigoplus_{d\in\mathbb{Z}} h^i\left(\mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M, (Q^\cdot)[d])\right) \\ &= \bigoplus_{d\in\mathbb{Z}} \mathrm{Ext}^i_{\mathrm{Gr}\text{-}A}(M, N[d]) . \end{aligned}$$

Similarly, for $\mathrm{QGr}\text{-}A$ we have

$$\begin{aligned} h^i(\underline{\mathrm{Hom}}_{\mathrm{QGr}\text{-}A}(M, (Q^\cdot)[d]) &= h^i\left(\bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{QGr}\text{-}A}(M, (Q^\cdot[d]))\right) \\ &\cong \bigoplus_{d\in\mathbb{Z}} h^i\left(\mathrm{Hom}_{\mathrm{QGr}\text{-}A}(M, (Q^\cdot[d]))\right) \\ &= \bigoplus_{d\in\mathbb{Z}} \mathrm{Ext}^i_{\mathrm{QGr}\text{-}A}(M, N[d]) . \\ &\cong \bigoplus_{d\in\mathbb{Z}} R^i\,\mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M, \omega\pi(N)[d]) . \\ &\cong \bigoplus_{d\in\mathbb{Z}} h^i(\mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M, \omega\pi(Q^\cdot)[d]) \\ &\cong h^i\left(\bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(M, \omega\pi(Q^\cdot)[d])\right) \\ &= h^i\left(\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}(M, \omega\pi(Q^\cdot))\right) \end{aligned}$$

■

Proposition 10. *For any object* M *of* $\mathrm{Gr}\text{-}A$*, the canonical morphism*

$$\mathrm{Hom}_{\mathrm{Gr}\text{-}A}(A, M) \longrightarrow M_0$$

$$\varphi \longmapsto \varphi(1)$$

is an isomorhism. In particular, taking degree zero is an exact because A is a projective object in $\mathrm{Gr}\text{-}A$ *and, moreover,* $\mathrm{Hom}_{\mathrm{Gr}\text{-}A}(A, M[d]) \cong M_d$.

Definition 10. Let M be an object of $\mathrm{Gr}\text{-}A$ and $Q^\cdot$ an injective resolution. Define the cohomology functors

$$H^i(\pi(M)) = \mathrm{Ext}^i_{\mathrm{QGr}\text{-}A}(\pi(A), \pi(M)) \cong h^i(\mathrm{Hom}_{\mathrm{Gr}\text{-}A}(A, \omega\pi(Q^\cdot))) \cong h^i(\omega\pi(Q^\cdot)_0) \cong h^i(\omega\pi(Q^\cdot))_0,$$

and note that the last isomorphism follows from the fact that taking degree 0 is an exact functor. Define the graded cohomology functors

$$\underline{H}^i(\pi(M)) = \bigoplus_{d\in\mathbb{Z}} H^i(\pi(M)[d]) \cong \bigoplus_{d\in\mathbb{Z}} h^i(\omega\pi(Q^\cdot))_d \cong h^i(\omega\pi(Q^\cdot)).$$

Remark 3. Note that for M an object of $\mathrm{Gr}\text{-}A$ we have

$$H^0(\pi(M)) = \mathrm{Ext}^0_{\mathrm{QGr}\text{-}A}(\pi(A), \pi(M)) \cong R^0\,\mathrm{Hom}_{\mathrm{Gr}\text{-}A}(A, \omega\pi(M)) = \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(A, \omega\pi(M)) \cong \omega\pi(M)_0$$

and so it follows that $\underline{H}^0(\pi(M)) \cong \omega\pi(M)$.

Proposition 11. *Let M be an object of* $\mathrm{Gr}\text{-}A$ *and let N be an object of* $\mathrm{QGr}\text{-}A$. *Then*

(a) For $i \geq 0$

$$\underline{\mathrm{Ext}}^i_{\mathrm{QGr}\text{-}A}(\pi(N), \pi(M)) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}^i_{\mathrm{Gr}\text{-}A}(N_{\geq n}, M)$$

and

$$\underline{H}^i(\pi(M)) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}^i_{\mathrm{Gr}\text{-}A}(A_{\geq n}, M).$$

(b) There is an exact sequence

$$0 \longrightarrow \tau(M) \longrightarrow M \longrightarrow \underline{H}^0(\pi(M)) \longrightarrow \lim\nolimits_{n\to\infty} \underline{\mathrm{Ext}}^1_{\mathrm{Gr}\text{-}A}(A/A_{\geq n}, M) \longrightarrow 0$$

and for $i \geq 1$,

$$\underline{H}^i(\pi(M)) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}^{i+1}_{\mathrm{Gr}\text{-}A}(A/A_{\geq n}, M) \cong h^{i+1}(\tau(Q^\cdot))$$

for Q an injective resolution of M.

(c) $\underline{H}^i(\pi(M))$ is an object of Tors *if $i \geq 1$.*

(d) $\underline{\mathrm{Ext}}^i_{\mathrm{QGr}\text{-}A}(\pi(N), \pi(M))$ and $\underline{H}^i(\pi(M))$ are compatible with direct limits of objects $\pi(M)$.

Proof. (a) Let $Q^\cdot$ be an injective resolution of M. For each $0 \leq i$ and for each d we have

$$\lim_{n\to\infty} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}\left(N_{\geq n}, Q^i[d]\right) \quad\cong\quad \mathrm{Hom}_{\mathrm{QGr}\text{-}A}\left(\pi(N), \pi(Q^i)[d]\right).$$

Since $\mathrm{Gr}\text{-}A$ is an Ab 5 category we have

$$\begin{aligned}
\lim_{n\to\infty} \underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}\left(N_{\geq n}, Q^i\right) &= \lim_{n\to\infty} \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}\left(N_{\geq n}, Q^i[d]\right) \\
&\cong \bigoplus_{d\in\mathbb{Z}} \lim_{n\to\infty} \mathrm{Hom}_{\mathrm{Gr}\text{-}A}\left(N_{\geq n}, Q^i[d]\right) \\
&\cong \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{QGr}\text{-}A}\left(\pi(N), \pi(Q^i)[d]\right) \\
&= \underline{\mathrm{Hom}}_{\mathrm{QGr}\text{-}A}\left(\pi(N), \pi(Q^i)\right).
\end{aligned}$$

Next we note that

$$\left(\lim_{n\to\infty}\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}\left(N_{\geq n},Q^{\cdot}\right)\right)_i \cong \lim_{n\to\infty}\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}\left(N_{\geq n},Q^{i}\right) \cong \underline{\mathrm{Hom}}_{\mathrm{QGr}\text{-}A}\left(\pi(N),\pi(Q^{i})\right)$$

and hence

$$\begin{aligned}
\lim_{n\to\infty}\underline{\mathrm{Ext}}^i_{\mathrm{Gr}\text{-}A}\left(N_{\geq n},M\right) &= \lim_{n\to\infty} h^i(\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}\left(N_{\geq n},Q^{\cdot}\right))\\
&\cong h^i\left(\lim_{n\to\infty}\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}\left(N_{\geq n},Q^{\cdot}\right)\right)\\
&\cong h^i\left(\underline{\mathrm{Hom}}_{\mathrm{QGr}\text{-}A}\left(\pi(N),\pi(Q^{\cdot})\right)\right)\\
&\cong \underline{\mathrm{Ext}}^i_{\mathrm{QGr}\text{-}A}\left(\pi(N),\pi(M)\right)
\end{aligned}$$

That $\underline{H}^i(\pi(M)) \cong \lim_{n\to\infty}\underline{\mathrm{Ext}}^i_{\mathrm{Gr}\text{-}A}\left(A_{\geq n},M\right)$ follows by taking $N=A$.
The short exact sequence

$$0 \longrightarrow A_{\geq n} \longrightarrow A \longrightarrow A/A_{\geq n} \longrightarrow 0$$

gives rise to a long exact sequence

$$\begin{aligned}
0 \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}\left(A/A_{\geq n},M\right) \longrightarrow M \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}\left(A_{\geq n},M\right) \\
\longrightarrow \underline{\mathrm{Ext}}^1_{\mathrm{Gr}\text{-}A}\left(A/A_{\geq n},M\right) \longrightarrow \mathrm{Ext}^1_{\mathrm{Gr}\text{-}A}\left(A,M\right)=0 \longrightarrow \dots
\end{aligned}$$

We see that

$$\begin{aligned}
\lim_{n\to\infty}\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}\left(A_{\geq n},M\right) &\cong \underline{\mathrm{Hom}}_{\mathrm{QGr}\text{-}A}\left(\pi(A),\pi(M)\right)\\
&\cong \underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}\left(A,\omega\pi(M)\right)\\
&\cong \omega\pi(M)\\
&\cong \underline{H}^0(\pi(M))
\end{aligned}$$

and so by taking limits we get the exact sequence

$$0 \longrightarrow \lim_{n\to\infty}\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}\left(A/A_{\geq n},M\right) \longrightarrow M \longrightarrow \underline{H}^0(\pi(M)) \longrightarrow \lim_{n\to\infty}\underline{\mathrm{Ext}}^1_{\mathrm{Gr}\text{-}A}\left(A/A_{\geq n},M\right) \longrightarrow 0$$

It remains only to show that $\lim_{n\to\infty}\underline{\mathrm{Hom}}_{\mathrm{Gr}\text{-}A}\left(A/A_{\geq n},M\right) \cong \tau(M)$. ■