

Applications of Finite Math

1 Voting Theory

1.1 Fairness Criteria

1.1.1 The Majority Criterion:

A candidate with the majority of the first-place votes should win the election.

1.1.2 The Condorcet Criterion:

The Condorcet Candidate should always win the election. That is to say, when the candidates are compared two at a time, the candidate who beats each of the other candidates (the Condorcet Candidate) should always win.

1.1.3 The Monotonicity Criterion:

Suppose candidate X wins an election, but there is (for some reason) another [subsequent] election. If the only voting changes are those that *positively* affect candidate X, then candidate X should win the subsequent election.

1.1.4 The Independence-of-Irrelevant-Alternatives (IIA) Criterion:

Suppose candidate X wins an election, but there is (for some reason) another [subsequent] election. If the only change is another candidate drops out, or is disqualified, then candidate X should still win. The converse, which states the winner should not be penalized by the introduction of a candidate who has no chance of winning, is also true.

1.2 The Building Blocks

1.2.1 The Preference Ballot

A preference ballot is a ballot, by which, the voter casts their vote giving an ordered list of each candidate in the election, in descending order, starting with their favorite candidate.

Example:

$$\begin{bmatrix} 1st & A \\ 2nd & B \\ 3rd & C \\ 4th & D \end{bmatrix}$$

1.2.2 The Preference Schedule

A preference schedule is a condensed representation of all the ballots cast in an election.

Example:

Suppose there is an election between four candidates, A, B, C and D. The results of this election are shown below:

$$\begin{bmatrix} 1st & A \\ 2nd & B \\ 3rd & C \\ 4th & D \end{bmatrix} \begin{bmatrix} 1st & C \\ 2nd & B \\ 3rd & D \\ 4th & A \end{bmatrix} \begin{bmatrix} 1st & D \\ 2nd & C \\ 3rd & B \\ 4th & A \end{bmatrix} \begin{bmatrix} 1st & B \\ 2nd & D \\ 3rd & C \\ 4th & A \end{bmatrix}$$
$$\begin{bmatrix} 1st & C \\ 2nd & D \\ 3rd & B \\ 4th & A \end{bmatrix} \begin{bmatrix} 1st & A \\ 2nd & B \\ 3rd & C \\ 4th & D \end{bmatrix} \begin{bmatrix} 1st & A \\ 2nd & B \\ 3rd & C \\ 4th & D \end{bmatrix} \begin{bmatrix} 1st & A \\ 2nd & B \\ 3rd & C \\ 4th & D \end{bmatrix}$$
$$\begin{bmatrix} 1st & C \\ 2nd & B \\ 3rd & D \\ 4th & A \end{bmatrix} \begin{bmatrix} 1st & C \\ 2nd & B \\ 3rd & D \\ 4th & A \end{bmatrix} \begin{bmatrix} 1st & B \\ 2nd & D \\ 3rd & C \\ 4th & A \end{bmatrix} \begin{bmatrix} 1st & A \\ 2nd & B \\ 3rd & C \\ 4th & D \end{bmatrix}$$

Then, the preference schedule would look like this:

$$\begin{bmatrix} \#Voters & 5 & 3 & 2 & 1 & 1 \\ 1st & A & C & B & D & C \\ 2nd & B & B & D & C & D \\ 3rd & C & D & C & B & B \\ 4th & D & A & A & A & A \end{bmatrix}$$

1.2.3 Transitivity

Assume a voter is casting their vote in an election which has four candidates, A, B, C and D. Assume then, that the voter's preference ballot is as follows:

$$\begin{bmatrix} 1st & A \\ 2nd & B \\ 3rd & C \\ 4th & D \end{bmatrix}$$

The Law of Transitivity says that when a voter prefers a candidate A to candidate B, and a candidate B to candidate C, then the statement that said voter prefers candidate A to candidate C is true.

Consequences: A useful consequence of transitivity, which we will use later, is in a head-to-head comparison between candidates, say X and Y, the only factor we need consider is how high up the list the candidates are: the higher up candidate, in a head-to-head comparison, will get the vote.

Transitivity also allows us to decide what to do in the case of a withdrawal or disqualification. For example,

Assume that there has been an election between four candidates, A, B, C and D. After the election, a candidate, C, is disqualified for foul play. Given the following ballot:

$$\begin{bmatrix} 1st & B \\ 2nd & C \\ 3rd & A \\ 4th & D \end{bmatrix}$$

Then, during the subsequent election, we can transform the voter's ballot, without having to hold an actual election, like so:

$$\begin{bmatrix} 1st & A \\ 2nd & B \\ 3rd & C \\ 4th & D \end{bmatrix} \sim \sim \begin{bmatrix} 1st & A \\ 2nd & B \\ 3rd & D \end{bmatrix}$$

Because of the Law of Transitivity, we can do this without worry of misrepresenting the voter's preferences.

1.3 Voting Methods

1.3.1 The Plurality Method

The Plurality Method states that the candidate with the most first place votes is the candidate who wins the election.

Example:

$$\begin{bmatrix} \#Voters & 49 & 48 & 3 \\ 1st & A & B & C \\ 2nd & B & E & B \\ 3rd & C & D & E \\ 4th & D & C & D \\ 5th & E & A & A \end{bmatrix}$$

By Plurality, A is the winner with 49 first-place votes.

While A does hold more first-place votes than any other candidate, one must also take into consideration that there are in fact 100 total voters and A does *not* have a majority vote. In fact, there are actually 51 5th place votes for A—more than the 49 first place votes!

This is, in fact, a violation of the Majority Criterion. Furthermore, compared head-to-head with another candidate, say B for example, we notice that using the Law of Transitivity, B has 51 first place votes and A has only 49. Referring back to the Condorcet Criterion, we find that this method violates that criterion as well.

Example:

$$\begin{bmatrix} \#Voters & 4 & 2 & 6 & 3 \\ 1st & A & B & C & B \\ 2nd & B & C & A & A \\ 3rd & C & A & B & C \end{bmatrix}$$

By the Plurality Method, we can see that C wins the election with 6 first-place votes.

However, assume A either withdraws or is disqualified from the election. We can then use the Law of Transitivity to transform the above preference schedule as follows:

$$\begin{bmatrix} \#Voters & 4 & 2 & 6 & 3 \\ 1st & A & B & C & B \\ 2nd & B & C & A & A \\ 3rd & C & A & B & C \end{bmatrix} \longrightarrow \begin{bmatrix} \#Voters & 4 & 2 & 6 & 3 \\ 1st & B & B & C & B \\ 2nd & C & C & B & C \end{bmatrix}$$

Using this new preference schedule, we can see that the new election winner using the Plurality Method is B, with a total of 9 first-place votes. This is a violation of the Independence-of-Irrelevant-Alternatives Criterion!

Insincere Voting Delving even further in the viability of the Plurality Method, we come across another startling flaw:

Insincere Voting: Knowing that the candidate one wishes to win can not, one instead votes for a lesser choice with a higher chance of winning.

Take for example the following two preference schedules: real preferences (left) and actual votes (right).

$$\begin{bmatrix} \#Voters & 49 & 48 & 3 \\ 1st & A & B & C \\ 2nd & B & E & B \\ 3rd & C & D & E \\ 4th & D & C & D \\ 5th & E & A & A \end{bmatrix} \begin{bmatrix} \#Voters & 49 & 48 & 3 \\ 1st & A & B & B \\ 2nd & B & E & C \\ 3rd & C & D & E \\ 4th & D & C & D \\ 5th & E & A & A \end{bmatrix}$$

One can see that the only real difference is that three people who preferred candidate C changed their votes so that B was their first choice. These three votes alone were enough to change the entire set of election results such that B would win the election, instead of A.

1.3.2 The Borda Count Method

In an election with n candidates, each place (1st, 2nd, 3rd, ..., nth) carries a weight varying from n down to 1, where each vote cast adds that many points to the candidate's total. A vote for 1st place carries n points, and a vote for nth place carries 1 point. The candidate who accumulates the most points wins the election.

Example:

<i>#Voters</i>	3	2	1
<i>1st</i>	A	C	A
<i>2nd</i>	B	B	C
<i>3rd</i>	C	A	B

$$\text{A: } 3(3) + 2(1) + 1(3) = 14$$

$$\text{B: } 3(2) + 2(2) + 1(1) = 11$$

$$\text{C: } 3(1) + 2(3) + 1(2) = 11$$

By the Borda Count Method, A is the winner with 14 points.

Example:

<i>#Voters</i>	6	2	3
<i>1st</i>	A	B	C
<i>2nd</i>	B	C	D
<i>3rd</i>	C	D	B
<i>4th</i>	D	A	A

$$\text{A: } 6(4) + 2(1) + 3(1) = 29$$

$$\text{B: } 6(3) + 2(4) + 3(2) = 32$$

$$\text{C: } 6(2) + 2(3) + 3(4) = 30$$

$$\text{D: } 6(1) + 2(2) + 3(3) = 19$$

By the Borda Count Method, B would win the election with 32 points.

However, we can see that A has both a majority of the votes (6 first-place votes of 9 total votes) and is also the Condorcet Candidate. However, the winner here is B, not A, with 32 points. This shows that The Borda Count Method violates both the Majority Criterion and the Condorcet Criterion.

Example:

$$\begin{bmatrix} \#Voters & 4 & 2 & 6 & 3 \\ 1st & A & B & C & B \\ 2nd & B & C & A & A \\ 3rd & C & A & B & C \end{bmatrix}$$

$$\text{A: } 4(3) + 2(1) + 6(2) + 3(2) = 32$$

$$\text{B: } 4(2) + 2(3) + 6(1) + 3(3) = 29$$

$$\text{C: } 4(1) + 2(2) + 6(3) + 3(1) = 29$$

By the Borda Count Method, the winner here is A.

Suppose, then, that B is disqualified. The new preference schedule would be as follows:

$$\begin{bmatrix} \#Voters & 4 & 2 & 6 & 3 \\ 1st & A & C & C & A \\ 2nd & C & A & A & C \end{bmatrix}$$

$$\text{A: } 4(2) + 2(1) + 6(1) + 3(2) = 22$$

$$\text{C: } 4(1) + 2(2) + 6(2) + 3(1) = 23$$

Now, by the Borda Count Method, C is the winner! This is a violation of the IIA Criterion!

1.3.3 The Plurality-with-Elimination Method

The Plurality-with-Elimination Method—otherwise known as the Instant Runoff Voting—counts the number of first place votes, eliminating the lowest candidate (or candidates in the case of a tie) until someone has a majority in first-place votes.

Example:

$$\begin{bmatrix} \#Voters & 14 & 10 & 8 & 4 & 1 \\ 1st & A & C & D & B & C \\ 2nd & B & B & C & D & D \\ 3rd & C & D & B & C & B \\ 4th & D & A & A & A & A \end{bmatrix}$$

Round 1:

Highest: A with 14 first-place votes.

Lowest: B with 4 first-place votes.

The new preference schedule is now:

$$\begin{bmatrix} \#Voters & 14 & 10 & 8 & 4 & 1 \\ 1st & A & C & D & D & C \\ 2nd & C & D & C & C & D \\ 3rd & D & A & A & A & A \end{bmatrix}$$

Round 2:

Highest: A with 14 first-place votes.

Lowest: C with 11 first-place votes.

The new preference schedule is now:

$$\begin{bmatrix} \#Voters & 14 & 10 & 8 & 4 & 1 \\ 1st & A & D & D & D & D \\ 2nd & D & A & A & A & A \end{bmatrix}$$

Round 3:

Highest: D with 23 first-place votes. Lowest: A with 14 first-place votes.
Thus, D is the winner.

Comparing D and C, head-to-head, we have the following preference schedule:

$$\begin{bmatrix} \#Voters & 14 & 10 & 8 & 4 & 1 \\ 1st & C & C & D & D & C \\ 2nd & D & D & C & C & D \end{bmatrix}$$

C has 25 first-place votes, while D only has 12. This is a violation of the Condorcet Criterion!

Example:

$$\begin{bmatrix} \#Voters & 7 & 8 & 10 & 4 \\ 1st & A & B & C & A \\ 2nd & B & C & A & C \\ 3rd & C & A & B & B \end{bmatrix}$$
Round 1:

Highest: A with 11 first-place votes.
Lowest: B with 8 first-place votes.

The new preference schedule is now:

$$\begin{bmatrix} \#Voters & 7 & 8 & 10 & 4 \\ 1st & A & C & C & A \\ 2nd & C & A & A & C \end{bmatrix}$$
Round 2:

Highest: C with 18 first-place votes.

Lowest: A with 11 first-place votes.

Thus, C is the winner.

Suppose a second vote occurs between the same candidates. The results are depicted in the following preference schedule:

<i>#Voters</i>	7	8	14
<i>1st</i>	<i>A</i>	<i>B</i>	<i>C</i>
<i>2nd</i>	<i>B</i>	<i>C</i>	<i>A</i>
<i>3rd</i>	<i>C</i>	<i>A</i>	<i>B</i>

Round 1:

Highest: C with 14 first-place votes.

Lowest: A with 7 first-place votes.

The new preference schedule is now:

<i>#Voters</i>	7	8	14
<i>1st</i>	<i>B</i>	<i>B</i>	<i>C</i>
<i>2nd</i>	<i>C</i>	<i>C</i>	<i>B</i>

Round 2:

Highest: B with 15 first-place votes.

Lowest: C with 14 first-place votes.

Thus, B is the winner.

We can see here, the only change made in the voting was a positive increase in C's count. Thus, this voting method violates the Monotonicity Criterion.

Example:

$$\begin{bmatrix} \#Voters & 4 & 2 & 6 & 3 \\ 1st & A & B & C & B \\ 2nd & B & C & A & A \\ 3rd & C & A & B & C \end{bmatrix}$$

Using the Plurality-with-Elimination method, we can show that B is the winner with 9 first-place votes.

Round 1:

Highest: C with 6 first-place votes.

Lowest: A with 4 first-place votes.

The new preference schedule is now:

$$\begin{bmatrix} \#Voters & 4 & 2 & 6 & 3 \\ 1st & B & B & C & B \\ 2nd & C & C & B & C \end{bmatrix}$$

Round 2:

Highest: B with 9 first-place votes.

Lowest: C with 6 first-place votes.

Thus, B is the winner.

Now, if we were to remove C from the election, we would find that A would win with 10 first-place votes. This is a violation of the IIA Criterion.

We start with the following preference schedule:

$$\begin{bmatrix} \#Voters & 4 & 2 & 6 & 3 \\ 1st & A & B & A & B \\ 2nd & B & A & B & A \end{bmatrix}$$

Round 1:

Highest: A with 10 first-place votes.

Lowest: B with 5 first-place votes.

Thus, A is the winner.

1.3.4 The Pairwise Comparisons Method

Each candidate is compared, head-to-head, with each other candidate. Whoever wins gets a point, and the candidate with the most points wins the election.

Example:

<i>#Voters</i>	14	10	8	4	1
<i>1st</i>	A	C	D	B	C
<i>2nd</i>	B	B	C	D	D
<i>3rd</i>	C	D	B	C	B
<i>4th</i>	D	A	A	A	A

*A*vs.*B* : 14vs.23 *B*vs.*C* : 18vs.19 *C*vs.*D* : 15vs.12

*A*vs.*C* : 14vs.23 *B*vs.*D* : 28vs.9

*A*vs.*D* : 14vs.23

Thus, C wins the election with 3 points, B with 2, D with 1 and A with 0.

Example:

<i>#Voters</i>	2	6	4	1	1	4	4
<i>1st</i>	A	B	B	C	C	D	E
<i>2nd</i>	D	A	A	B	D	A	C
<i>3rd</i>	C	C	D	A	A	E	D
<i>4th</i>	B	D	E	D	B	C	B
<i>5th</i>	E	E	C	E	E	B	A

$A \text{ vs. } B : 7 \text{ vs. } 15$ $B \text{ vs. } C : 10 \text{ vs. } 12$ $D \text{ vs. } E : 18 \text{ vs. } 4$
 $A \text{ vs. } C : 16 \text{ vs. } 6$ $B \text{ vs. } D : 11 \text{ vs. } 11$
 $A \text{ vs. } D : 13 \text{ vs. } 9$ $B \text{ vs. } E : 14 \text{ vs. } 8$
 $A \text{ vs. } E : 18 \text{ vs. } 4$

The results are A with 3 points, B with 2.5, C with 2, D with 1.5 and E with 1 point. Thus, A is the winner.

When we drop C, we get the following preference schedule:

$$\left[\begin{array}{c|ccccccc} \#Voters & 2 & 6 & 4 & 1 & 1 & 4 & 4 \\ \hline 1st & A & B & B & B & D & D & E \\ 2nd & D & A & A & A & A & A & D \\ 3rd & B & D & D & D & B & E & B \\ 4th & E & E & E & E & E & B & A \end{array} \right]$$

Which is equivalent to:

$$\left[\begin{array}{c|ccccc} \#Voters & 2 & 11 & 1 & 4 & 4 \\ \hline 1st & A & B & D & D & E \\ 2nd & D & A & A & A & D \\ 3rd & B & D & B & E & B \\ 4th & E & E & E & B & A \end{array} \right]$$

When we do the pairwise comparisons, we get:

$A \text{ vs. } B : 7 \text{ vs. } 15$ $B \text{ vs. } D : 11 \text{ vs. } 11$ $D \text{ vs. } E : 18 \text{ vs. } 4$
 $A \text{ vs. } D : 13 \text{ vs. } 9$ $B \text{ vs. } E : 14 \text{ vs. } 8$
 $A \text{ vs. } E : 18 \text{ vs. } 4$

This ends up with A getting 2 points, B 2.5, D 1.5 and E 0, meaning B is the winner. Therefore, the Pairwise Comparisons Method violates the IIA Criterion.

Another issue with Pairwise Comparisons: It's long! But just how long is it to calculate a winner with Pairwise Comparisons?

Let's take a look at a race with n candidates, $C_1, C_2, C_3, \dots, C_n$. In order to calculate a winner with the Pairwise Comparisons method, we must compare the first candidate to each and every other candidate. In order to make sure we don't make extra comparisons, we will start with the first candidate, C_1 , and compare it to all other $n - 1$ candidates. We will then take the second candidate, C_2 , and compare it to all other candidates, excluding the previous candidate, C_1 , which accounts for a total of $n - 2$ more comparisons.

If we keep going on down the line, all the way to the $(n - 1)th$ candidate, we will see that for the i th candidate, we will make $n - i$ comparisons. So, the total number of comparisons made in an election will be the sum of all comparisons made for each candidate, or

$$\sum_{i=1}^{n-1} (n - i)$$

If we only have a few candidates running, say C_1, C_2, C_3 so that $n = 3$, this isn't too big. We must compare C_1 to C_2 and C_3 , then C_2 to C_3 . Thus, for the first candidate, we have $n - 1 = 3 - 1 = 2$ comparisons and for the second candidate, we have $n - 2 = 3 - 2 = 1$ comparison. So, the total comparisons when $n = 3$, we have:

$$\sum_{i=1}^{n-1} (n - i) = [3 - 1] + [3 - 2] = 3$$

As you might imagine, and we've seen before, if you have many more than two or three candidates, the number of comparisons become very large, very fast. In fact, we can show a much simpler way to calculate exactly how many comparisons are made, for any number of candidates, without complicated sums. If you think of the way we make comparisons, we start with the first candidate and move all the way to the second to last, or $(n-1)th$, candidate. We never compare the last candidate to any other candidates, because we have already compared previous candidates to the last candidate.

So, we can see that the total number of comparisons looks like

$$S = (n - 1) + (n - 2) + \dots + (n - (n - 2)) + (n - (n - 1))$$

total comparisons, which we can reduce to

$$S = (n - 1) + (n - 2) + \dots + 2 + 1$$

total comparisons. Noting that addition is commutative, we can rearrange this equation, without consequence, as follows:

$$S = 1 + 2 + \dots + (n - 2) + (n - 1)$$

.

Now, if one is to look at these two equations on top of one another,

$$S = (n - 1) + (n - 2) + \dots + 2 + 1$$

$$S = 1 + 2 + \dots + (n - 2) + (n - 1)$$

we can notice an interesting property. When terms that are on top of each other are added together, i.e. $(n - 1)$ and 1 , we come up with n every time. So, if we were to take the series and add them together, we would start with

$$[(n - 1) + (n - 2) + \dots + 2 + 1] + [1 + 2 + \dots + (n - 2) + (n - 1)]$$

Keeping in mind that the two sums inside the brackets are both equal to S , we can say

$$[(n - 1) + (n - 2) + \dots + 2 + 1] + [1 + 2 + \dots + (n - 1) + (n - 1)] = [S] + [S] = 2S$$

Then, using our previous observation, we can combine the terms of the series and we end up with

$$2S = n + n + n + \dots + n$$

Now, we know that we only made comparisons between the first $(n - 1)$ candidates, so we know that there are exactly $(n - 1)$ terms in the previous equation. Thus, we can write

$$2S = (n - 1)(n)$$

Then, solving for S , we get

$$S = (n - 1)(n)/2$$

So, we can safely assume that for any number of candidates, n , using the Pairwise Comparisons method we have the total number of comparisons as

$$\sum_{i=1}^{n-1} (n-i) = (n-1)(n)/2 \quad (1)$$

In fact, we can actually generalize this form to something even more powerful. Noting that

$$\sum_{i=1}^{n-1} (n-i) = (n-1) + (n-2) + \dots + 2 + 1$$

is the same in both directions, we can go as far as to say

$$\sum_{i=1}^{n-1} (n-i) = 1 + 2 + \dots + (n-2) + (n-1) = \sum_{i=1}^{n-1} i$$

We can then generalize this formula to the following form:

$$\sum_{i=1}^{n-1} i = (n-1)(n)/2$$

Replacing $(n-1)$ with a sum of a full n terms, we can even go as far as to say that the sum of any n consecutive integers,

$$\sum_{i=1}^n i = (n)(n+1)/2 \quad (2)$$

Now, with this we can really take a good look at just how long the Pairwise Comparisons Method truly is. As we saw before, with three candidates, we have a total of three comparisons. Using our new method for calculating the total number of comparisons, we can see that when $n = 3$, the total number of comparisons is, as expected,

$$(n-1)(n)/2 = (3-1)(3)/2 = (2)(3)/2 = 3$$

Now, let's take a look at something a little larger. Suppose there were six candidates—double the number of candidates from the first example. We would then have

$$(n-1)(n)/2 = (6-1)(6)/2 = 15$$

For twice as many candidates, we already have five times the number of comparisons. You can imagine just how tedious it would be to have to do 15 different head-to-head comparisons.

1.4 Rankings

The idea behind ranking is to come up with a way to not only determine the winner of the election, but to also determine who came in second, third, etc.

We'll use the example from the text to create rankings for each of the methods:

$$\begin{bmatrix} \#Voters & 14 & 10 & 8 & 4 & 1 \\ 1st & A & C & D & B & C \\ 2nd & B & B & C & D & D \\ 3rd & C & D & B & C & B \\ 4th & D & A & A & A & A \end{bmatrix}$$

1.4.1 Extended Plurality

In order to rank with the Plurality method, you simply arrange the candidates by the number of first place votes received.

We'll also use something which looks very much like a preference schedule to depict the rankings.

$$\begin{bmatrix} Place & 1 & 2 & 3 & 4 \\ Candidate & A & C & D & B \\ Votes & 14 & 11 & 8 & 4 \end{bmatrix}$$

1.4.2 Extended Borda Count

Assume the results of a Borda Count Election are as follows:

A: 79 points
 B: 106 points
 C: 104 points
 D: 81 points

$$\begin{bmatrix} Place & 1 & 2 & 3 & 4 \\ Candidate & B & C & D & A \\ BordaPoints & 106 & 104 & 81 & 79 \end{bmatrix}$$

1.4.3 Extended Plurality-with-Elimination

Rank in reverse order of elimination. The first person eliminated is the last ranked candidate.

$$\begin{bmatrix} \textit{Place} & & 1 & 2 & 3 & 4 \\ \textit{Candidate} & & D & A & C & B \\ \textit{Round} & \textit{Eliminated} & & 3 & 2 & 1 \end{bmatrix}$$

1.4.4 Extended Pairwise Comparisons

$$\begin{bmatrix} \textit{Place} & 1 & 2 & 3 & 4 \\ \textit{Candidate} & C & B & D & A \\ \textit{Points} & 3 & 2 & 1 & 0 \end{bmatrix}$$

1.5 Recursion

A process which is recursive is a process in which the current step of the process depends on the previous steps of the process.

1.5.1 Recursive Rankings

To calculate rankings recursively, simple run the voting method of choice to determine the winner. Then, remove that candidate from the preference schedule and run the method again. The winner from the first round takes first place, the winner from the second round takes second place and so on.

Example: Recursive Plurality

$$\begin{bmatrix} \#Voters & 14 & 10 & 8 & 4 & 1 \\ 1st & A & C & D & B & C \\ 2nd & B & B & C & D & D \\ 3rd & C & D & B & C & B \\ 4th & D & A & A & A & A \end{bmatrix}$$

A is the winner with 14 votes. The plurality schedule for round 2 becomes:

$$\begin{bmatrix} \#Voters & 14 & 10 & 8 & 4 & 1 \\ 1st & B & C & D & B & C \\ 2nd & C & B & C & D & D \\ 3rd & D & D & B & C & B \end{bmatrix}$$

B wins this round with 18 votes, meaning B takes second place and the plurality schedule becomes:

$$\begin{bmatrix} \#Voters & 14 & 10 & 8 & 4 & 1 \\ 1st & C & C & D & D & C \\ 2nd & D & D & C & C & D \end{bmatrix}$$

C wins this round with 25 votes, meaning C takes third place and D takes fourth.

Thus, we have the following rankings:

$$\begin{bmatrix} Rank & 1 & 2 & 3 & 4 \\ Candidate & A & B & C & D \end{bmatrix}$$