# Gluing Problems

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## Problem 1

In this problem $k$ is your favorite field and $p(x) \in k[x]$ is your favorite monicdegree $2g+2$ polynomial

$$p(x) = x^{2g+2} + a_1 x^{2g+1} + a_2 x^{2g} + \cdots + a_{2g+1} x + a_{2g+2}$$

with distinct roots in $\overline{k}$. Set

$$q(u) = a_{2g+2} x^{2g+2} + a_{2g+1} x^{2g+1} + \cdots + a_1 x + 1.$$

1. Prove that the affine scheme $U := \operatorname{Spec}(k[x,y]/(y^2 - p(x))$ is normal. (One approach is to prove directly that the localization of $R := k[x,y]/(y^2 - p(x))$ at a prime is either a field or a discrete valuation ring. Another approach is to use Serre's Criteria for normality, a criteria you should google if you haven't seen it before.)

2. Construct an open immersion $U \to X$ for $X$ the projectivization of $k[X,Y,Z]/(Y^2 Z^{2g} - p(X))$. (Geometrically this is the closure of $U \subset \mathbb{A}^2_k$ in $\mathbb{P}^2_k$.)

3. Give an alternative description of $X$ as the scheme obtained by glueing together a finite collection of affine schemes. (You can do this by looking at the three different dehomogenizations of $Y^2 Z^{2g} - p(X)$.)

4. Define $Y$ to be the scheme obtained by glueing $U$ to $V = \operatorname{Spec}(k[u,v]/(v^2 - q(u))$ by the isomorphism

$$k[x,y]/(y^2 - p(x))[x^{-1}] \to k[u,v]/(v^2 - q(u))[u^{-1}],$$
$$x \mapsto u^{-1},$$
$$y \mapsto v/u^{g+1}.$$

   Prove that $Y$ is well-defined (i.e. that the morphism actually is an isomorphism), that $Y$ contains $U$ as a dense open subset, and that $Y$ is NOT isomorphism to $X$. (For the last part, one approach is to show that $X$ is normal but $Y$ is not.)

5. Construct a morphism $Y \to X$ that restricts to the identity $U \subset Y \to U \subset X$. Can you construct a morphism $X \to Y$ that restricts to the identity $U \subset X \to U \subset Y$?