MATH 142: EXAM 03

BLAKE FARMAN
UNIVERSITY OF SOUTH CAROLINA

Answer the questions in the spaces provided on the question sheets and turn them in at the end of the class period.
Unless otherwise stated, all supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc. By writing your name on the line below, you indicate that you have read and understand these directions.
It is advised, although not required, that you check your answers.

Name: ______________________________

Problem	Points Earned	Points Possible
1		25
2		25
3		25
4		25
Exam 1 Bonus		–
Exam 2 Bonus		–
Total		100

Date: February 13, 2018.

1. Problems

1 (25 Points)**.** Find the radius and interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}.$$

Solution. We use the Ratio Test. First we compute the limit

$$\rho = \lim_{n\to\infty}\left|\frac{3^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n x^n}\right| = \lim_{n\to\infty}\frac{3|x|}{n+1} = \lim_{n\to\infty} 3|x| \cdot \lim_{n\to\infty}\frac{1}{n+1} = 3|x|\cdot 0 = 0.$$

Since $\rho < 1$ holds for all x, the radius of convergence is $R = \infty$ and thus this series converges absolutely on $\mathbb{R} = (-\infty, \infty)$.

2 (25 Points)**.** Find the Maclaurin series for the function

$$f(x) = \frac{1}{(1-x)^3}.$$

Solution. First we recognize that on $(-1, 1)$ we have the power series

$$F(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

so differentiating on the left we obtain

$$\begin{aligned} F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(1-x)^{-1} \\ &= (-1)(1-x)^{-2}\frac{\mathrm{d}}{\mathrm{d}x}(1-x) \\ &= (-1)(1-x)^{-2}(-1) \\ &= (1-x)^{-2} \end{aligned}$$

and differentiating again we obtain

$$\begin{aligned} F''(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(1-x)^{-2} \\ &= (-2)(1-x)^{-3}\frac{\mathrm{d}}{\mathrm{d}x}(1-x) \\ &= (-2)(1-x)^{-3}(-1) \\ &= 2(1-x)^{-3} \\ &= 2\frac{1}{(1-x)^3} \end{aligned}$$

and hence

$$\frac{1}{2}F''(x) = \frac{1}{(1-x)^3}.$$

Applying Term-by-Term Differentiation to the series, we see that

$$F'(x) = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} x^n = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1}$$

and

$$F''(x) = \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} n x^{n-1} = \sum_{n=1}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)x^{n-2},$$

on $(-1,1)$. Therefore

$$\frac{1}{(1-x)^3} = \frac{1}{2}F''(x) = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2}$$

on $(-1,1)$.

3 (25 Points)**.** Find the Maclaurin series for $x\ln(1+2x)$.

Solution. First we recall that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ -1 \leq x \leq 1$$

so

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_{n=0}^{\infty} (-2x)^n = \sum_{n=0}^{\infty} (-1)^n 2^n x^n$$

holds so long as $|-2x| = 2|x| < 1$ or, equivalently, $|x| < 1/2$. Now we observe that using the change of variables $u = 1+2x$, $\mathrm{d}u/2 = \mathrm{d}x$, the integral of the left-hand side is

$$\int \frac{\mathrm{d}x}{1+2x} = \int \frac{\mathrm{d}u}{2u} = \frac{1}{2}\int \frac{\mathrm{d}u}{u} = \frac{1}{2}\ln(u) + c = \frac{1}{2}\ln(1+2x) + c$$

while applying Term-by-Term Integration to the series yields

$$\int \frac{\mathrm{d}x}{1+2x} = \sum_{n=0}^{\infty} \int \left[(-1)^n 2^n x^n\right] \mathrm{d}x = \sum_{n=0}^{\infty} (-1)^n 2^n \int x^n \,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n+1} x^{n+1}$$

and thus

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n+1} x^{n+1} = \frac{1}{2}\ln(1+2x) + c$$

on $(-1/2, 1/2)$. Evaluating the left-hand side of this equation at $x = 0$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n+1} 0^{n+1} = 0$$

while evaluating the right-hand side of this equation at $x = 0$ we have

$$\frac{1}{2}\ln(1+2(0)) + c = \frac{1}{2}\ln(1) + c = 0 + c = c,$$

from which it follows that $c = 0$ and hence

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n+1} x^{n+1} = \frac{1}{2}\ln(1+2x)$$

holds for $-1/2 < x < 1/2$. Multiplying both sides by $2x$ we obtain

$$x\ln(1+2x) = 2x\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n+1} x^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{n+1} x^{n+2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{n} x^{n+1}$$

for $-1/2 < x < 1/2$.

4 (25 Points)**.** Find the Maclaurin series for

$$\ln\left(\frac{1+x}{1-x}\right)$$

Solution. First, rewrite

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x).$$

We see that by making the change of variables $u = 1 + x$, $\mathrm{d}u = \mathrm{d}x$ we have

$$\int \frac{\mathrm{d}x}{1+x} = \int \frac{\mathrm{d}u}{u} = \ln(u) + c_1 = \ln(1+x) + c_1$$

and by making the change of variables $u = 1 - x$, $-\mathrm{d}u = \mathrm{d}x$

$$\int \frac{\mathrm{d}x}{1-x} = -\int \frac{\mathrm{d}u}{u} = -(\ln(u) + c_2) = -\ln(1-x) - c_2$$

so by letting $c = c_1 - c_2$, we have

$$\begin{aligned}
\int \left[\frac{1}{1+x} + \frac{1}{1-x}\right] \mathrm{d}x &= \int \frac{\mathrm{d}x}{1+x} + \int \frac{\mathrm{d}x}{1-x} \\
&= \ln(1+x) + c_1 - \ln(1-x) - c_2 \\
&= \ln(1+x) - \ln(1-x) + c.
\end{aligned}$$

Now for $-1 < x < 1$ we have

$$\begin{aligned}\frac{1}{1+x}+\frac{1}{1-x} &= \frac{1}{1-(-x)}+\frac{1}{1-x}\\ &= \sum_{n=0}^{\infty}(-x)^n+\sum_{n=0}^{\infty}x^n\\ &= \sum_{n=0}^{\infty}(-1)^n x^n + x^n\\ &= \sum_{n=0}^{\infty}((-1)^n+1)x^n\end{aligned}$$

When n is odd, we have $(-1)^n+1 = -1+1 = 0$, while when n is even, $(-1)^n+1 = 1+1 = 2$, so we can rewrite

$$\sum_{n=0}^{\infty}((-1)^n+1)x^n = \sum_{k=0}^{\infty}2x^{2k}$$

Applying Term-by-Term Integration we obtain

$$\ln(1+x)-\ln(1-x)+c = \sum_{k=0}^{\infty}\int\left[2x^{2k}\right]\mathrm{d}x = \sum_{k=0}^{\infty}2\int x^{2k}\,\mathrm{d}x = \sum_{k=0}^{\infty}\frac{2}{2k+1}x^{2k+1}$$

for $-1 < x < 1$. Evaluating the left-hand side at $x = 0$ we get

$$\ln(1+0)-\ln(1-0)+c = \ln(1)-\ln(1)+c = 0-0+c = c$$

while evaluating the right-hand side at $x = 0$ we get

$$c = \sum_{k=0}^{\infty}\frac{2}{2k+1}(0)^{2k+1} = 0.$$

Therefore

$$\ln(1+x)-\ln(1-x) = \sum_{k=0}^{\infty}\frac{2}{2k+1}x^{2k+1}$$

for $-1 < x < 1$.

5 (Bonus - Exam 1)**.** Decide whether

$$\int_2^{\infty}\frac{\mathrm{d}x}{x^2-1}$$

converges or diverges. If it converges, find the value of the integral.

Solution. By definition we have

$$\int_2^\infty \frac{\mathrm{d}x}{x^2-1} = \lim_{t\to\infty}\int_2^t \frac{\mathrm{d}x}{x^2-1}.$$

Factoring the denominator as $x^2 - 1 = (x+1)(x-1)$ we can do the definite integral by partial fraction decomposition as follows. Set

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

then clear denominators to get

$$1 = A(x+1) + B(x-1) = (A+B)x + (A-B)$$

and equate coefficients to obtain the system

$$\begin{aligned} 0 &= A + B \\ 1 &= A - B. \end{aligned}$$

Adding the two equations together gives $1 = 2A$, while subtracting the second equation from the first gives $-1 = 2B$. Thus $A = 1/2$, $B = -1/2$, and

$$\begin{aligned}
\lim_{t\to\infty}\int_2^t \frac{\mathrm{d}x}{x^2-1} &= \lim_{t\to\infty}\left(\frac{1}{2}\int_2^t \frac{\mathrm{d}x}{x-1} - \frac{1}{2}\int_2^t \frac{\mathrm{d}x}{x+1}\right) \\
&= \lim_{t\to\infty}\left(\frac{1}{2}\left[\ln|x-1| - \ln|x+1|\right]_2^t\right) \\
&= \lim_{t\to\infty}\left(\frac{\ln|t-1| - \ln|t+1| - \ln(2-1) + \ln(2+1)}{2}\right) \\
&= \lim_{t\to\infty}\frac{1}{2}\left(\ln\left|\frac{t-1}{t+1}\right| + \ln(3)\right).
\end{aligned}$$

Since both the natural logarithm and the absolute value functions are continuous we have

$$\lim_{t\to\infty}\ln\left|\frac{t-1}{t+1}\right| = \ln\left(\lim_{t\to\infty}\left|\frac{t-1}{t+1}\right|\right) = \ln\left|\lim_{t\to\infty}\frac{t-1}{t+1}\right| = \ln(1) = 0.$$

Therefore

$$\int_2^\infty \frac{\mathrm{d}x}{x^2-1} = \lim_{t\to\infty}\frac{1}{2}\left(\ln\left|\frac{t-1}{t+1}\right| + \ln(3)\right) = \frac{0+\ln(3)}{2} = \frac{\ln(3)}{2}.$$

Alternate Solution. This solution is much more difficult, in my opinion, but I include it becuase quite a few folks attempted this problem in this way.

We first make a couple of observations. The first is that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\csc(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\sin(\theta)^{-1} = -\sin(\theta)^{-2}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin(\theta) = -\frac{\cos(\theta)}{\sin^2(\theta)} = -\cot(\theta)\csc(\theta).$$

The second observation is that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\cot(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\tan(\theta)^{-1} = -\tan(\theta)^{-2}\frac{\mathrm{d}}{\mathrm{d}\theta}\tan(\theta) = \frac{\sec^2(\theta)}{\tan^2(\theta)} = \frac{1}{\cos^2(\theta)}\frac{\cos^2(\theta)}{\sin^2(\theta)} = -\csc^2(\theta).$$

Combining these two together we have the important observation, which will come in handy later:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\csc(\theta)+\cot(\theta)) = -\cot(\theta)\csc(\theta) - \csc^2(\theta) = -\csc(\theta)(\cot(\theta)+\csc(\theta))$$

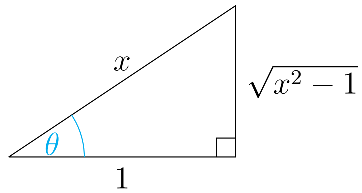
With these observations in hand, we can forge ahead with a trigonometric substition $x = \sec(\theta)$, so that $x^2 - 1 = \sec^2(\theta) - 1 = \tan^2(\theta)$, and $\mathrm{d}x = \sec(\theta)\tan(\theta)\,\mathrm{d}\theta$. The indefinite integral we will need to compute is

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^2-1} &= \int \frac{\sec(\theta)\tan(\theta)}{\tan^2(\theta)}\,\mathrm{d}\theta \\
&= \int \frac{\sec(\theta)}{\tan(\theta)}\,\mathrm{d}\theta \\
&= \int \frac{1}{\cos(\theta)}\frac{\cos(\theta)}{\sin(\theta)}\,\mathrm{d}\theta \\
&= \int \frac{1}{\sin(\theta)}\,\mathrm{d}theta \\
&= \int \csc(\theta)\,\mathrm{d}\theta
\end{aligned}$$

This is where we use our main observation above: take $u = \csc(\theta) + \cot(\theta)$ so that $-\,\mathrm{d}u = \csc(\theta)(\csc(\theta) + \cot(\theta))$ and then

$$\begin{aligned}
\int \csc(\theta)\,\mathrm{d}\theta &= \int \frac{\csc(\theta)(\csc(\theta)+\tan(\theta))}{\csc(\theta)+\cot(\theta)}\,\mathrm{d}\theta \\
&= -\int \frac{\mathrm{d}u}{u} \\
&= -\ln\left|\csc(\theta)+\cot(\theta)\right| + C
\end{aligned}$$

Now, we can look at the triangle



to rewrite our solution in terms of x as

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^2-1} &= -\ln\left(\frac{x}{\sqrt{x^2-1}}+\frac{1}{\sqrt{x^2-1}}\right)+C \\
&= -\ln\left(\frac{x+1}{\sqrt{x^2-1}}\right)+C \\
&= -\ln\left(\frac{\sqrt{x+1}\sqrt{x+1}}{\sqrt{x-1}\sqrt{x+1}}\right)+C \\
&= -\ln\left(\sqrt{\frac{x+1}{x-1}}\right)+C \\
&= -\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)+C
\end{aligned}$$

keeping in mind that we are only interested in $2 \leq x$.

Now we can evaluate the improper integral

$$\begin{aligned}
\int_2^\infty \frac{\mathrm{d}x}{x^2-1} &= \lim_{t\to\infty}\int_2^t \frac{\mathrm{d}x}{x^2-1} = \lim_{t\to\infty} -\frac{1}{2}\left[\ln\left(\frac{x+1}{x-1}\right)\right]_2^t \\
&= \lim_{t\to\infty} -\frac{1}{2}\left(\ln\left(\frac{t+1}{t-1}\right)-\ln\left(\frac{2+1}{2-1}\right)\right) \\
&= \lim_{t\to\infty} \frac{1}{2}\left(\ln(3)-\ln\left(\frac{t+1}{t-1}\right)\right) \\
&= \frac{1}{2}(\ln(3)-0) \\
&= \frac{\ln(3)}{2}.
\end{aligned}$$

6 (Bonus - Exam 2)**.** Determine whether the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{2n}{4n^2+1}$$

converges conditionally, converges absolutely, or diverges.

Solution. First we observe that this series does not converge absolutely. By computing the limit

$$\lim_{n\to\infty} \frac{\frac{2n}{4n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^2}{4n^2+1} = \frac{1}{2} > 1$$

we see that the series

$$\sum_{n=1}^{\infty} \frac{2n}{4n^2+1}$$

diverges by Part (1) of the Limit Comparison Test because the Harmonic Series diverges.

Next we try the Alternating Series Test. The first and third conditions are easy to verify: when $1 \le n$ it's clear that

$$0 < u_n = \frac{2n}{4n^2+1}$$

holds and also

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2n}{4n^2+1} = \lim_{n\to\infty} \frac{n^2}{n^2} \cdot \frac{2/n}{4+1/n^2} = \lim_{n\to\infty} \frac{2/n}{4+1/n^2} = \frac{0}{4+0} = 0.$$

To conclude that this series converges by the Alternating Series Test, we need only verify that for some integer N, $u_{n+1} \le u_n$ holds whenever $N \le n$. Towards that end let $f(x) = \frac{2x}{4x^2+1}$ and observe that

$$f'(x) = \frac{2(4x^2+1) - (2x)(8x)}{(4x^2+1)^2} = \frac{8x^2+2-16x}{(4x^2+1)^2} = \frac{-8x^2+2}{(4x^2+1)^2} < 0$$

holds if and only if

$$-8x^2 + 2 < 0$$

which holds if and only if

$$\sqrt{\frac{2}{8}} = \sqrt{\frac{1}{4}} = \frac{1}{\sqrt{4}} = \frac{1}{2} < x.$$

Since f is a decreasing function if and only if $f'(x) < 0$, we see that

$$u_{n+1} = f(n+1) \le f(n) = u_n$$

holds whenever $1 \le n$. Therefore the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n^2+1}$$

converges conditionally.