Applications of Finite Math
The Mathematics of Power

1 Weighted Voting

A voting method in which the voters-referred to as players–do not all have an equal vote. That is to say some of the players have more power than others.

1.1 Motion

A vote in which players vote either in support (yes) or in opposition (no).

1.2 Weight

The number of votes a player controls.

1.3 Quota

The number of votes required to pass a motion.

1.4 Notation

We represent a weighted voting system with n players, $P_1, P_2, \ldots, P_n$, and corresponding weights $w_1, w_2, \ldots, w_n$, as

$$[q : w_1, w_2, \ldots, w_n]\,(w_1 \geq w_2 \geq \ldots \geq w_n)$$

where q represents the quota for the system.

1.5 Anarchy

$$[10 : 8, 7, 3, 2]$$

Here, since the quota is so small, both P_1 and P_2 can carray a motion, possibly having both the Yes and No votes meeting the quota. To avoid this issue we generally restrict

$$q > \frac{V}{2},$$

where V is the total number of votes for the system.

1.6 Gridlock

$$[21 : 8, 7, 3, 2]$$

Here, V = 20, but the quota is 21. This makes it impossible for any motion to ever carry. Thus, we restrict q such that

$$\frac{V}{2} < q \leq V$$

where unanimity (or 100% of the votes) is the most required for any system.

1.7 Dictators

$$[11 : 12, 5, 4]$$

Here, P_1 has enough votes to single handedly decide a motion and thus, we call P_1 a dictator. We say P_1 is a dictator when

$$w_1 \geq q$$

1.8 Dummies

Any voters who are not dictators when there is a dictator are considered dummies. Dummies' votes do not count. These types of players can also occur when there is not a dictator.

Example:

$$[30 : 10, 10, 10, 9]$$

P_4 is a dummy here because no matter how (s)he votes, P_1, P_2 and P_3 will always be required to meet the quota.

1.9 Veto Power

$$[12 : 9, 5, 4, 2]$$

Here, P_1 has the ability to prevent a motion from passing because $w_2 + w_3 + w_4 < q$, so P_1 is always required to pass any motion. However, P_1 can not single handedly pass a motion and therefore is not a dictator. Whenever

$$w < q$$

and

$$V - w < q$$

where ($V = \sum_{i=1}^{n} w_i$) we say that player has Veto Power.

1.10 Coalitions

Any set of players who might join forces to vote the same way. A coalition is represented using set notation

$$\{P_1, P_2, \ldots, P_n\}$$

If a coalition includes all the players in the voting system, then we call that coalition the Grand Coalition.

1.11 Critical Player

A player whose vote is necessary for a coalition to win. We say a player is a critical player when

$$W - w < q$$

where W is the weight of the coalition and w is the weight of the critical player(s).

1.11.1 How many?

If one takes a look at any set of players $P_1, P_2, \ldots, P_n$ then all the coalitions can be formed through a simple binary choice: Either the player is in the coalition, or the player is not in the coalition. This gives us 2^n total coalitions, where n is the number of players in the voting system. However, if we actually look at all the coalitions for some n–say $n = 3$, we notice we have the following sets

$$\begin{array}{cc} \{P_1, P_2, P_3\} & \{P_1, P_2\} \\ \{P_1, P_3\} & \{P_1\} \\ \{P_2, P_3\} & \{P_2\} \\ \{P_3\} & \{\} \end{array}$$

where we say the say $\{\}$ is the empty set, often denoted by ϕ. However, if there is an empty set, then there must be a Grand Coalition. Thus, we say that the case where there is a Grand Coalition and an empty coalition are the same. So, we can say that the actual number of coalitions is $2^n - 1$.

1.12 Banzhaf Power Index

Start by listing the possible winning coalitions. Then, determine the critical players for each winning coalition. The number of times the play P_n is a critical player is the critical count, B_n. The total critical counts, T, is

$$T = B_1 + B_2 + \ldots + B_n$$

The ration $\frac{B_n}{T}$ is the Banzhaf Power Index for P_n. We represent the Banzhaf Power Index for player n by

$$\beta_n = \frac{B_n}{T}$$

Example:

$$[101 : 99, 98, 3]$$

Coalition	Weight	Critical Players
$\{P_1, P_2\}$	197	P_1, P_2
$\{P_1, P_3\}$	102	P_1, P_3
$\{P_2, P_3\}$	101	P_2, P_3
$\{P_1, P_2, P_3\}$	200	none

$B_1 = 2, B_2 = 2, B_3 = 2$
$T = 6$
$\beta_1 = \frac{1}{3}, \beta_2 = \frac{1}{3}, \beta_3 = \frac{1}{3}$
We call the latter line the Banzhaf Power Distribution.

1.13 Shapley-Shubik Power Index

This power index is similar to the Banzhaf Power Index, but it uses Sequential Coalitions–a coalition in which the order that the players join is important. For example,

$$\{P_1, P_2, P_3\} \neq \{P_2, P_1, P_3\}$$

1.13.1 Pivotal Player

The player whose votes turn a losing coalition into a winning coalition. Note that for every sequential coalition, there is one (and only one) pivotal player. One finds the pivotal player by adding the votes sequentially for the members of the coalition, until the sum is larger than or equal to the quota.

1.13.2 Calculating the Shapley-Shubik Power Index

1. List all sequential coalitions.
2. Determine the pivotal player for each coalition
3. Count the number of times each player is a pivotal player. Call this value SS_n for the n^{th} player.
4. Find the ratios for each player $\sigma_n = \frac{SS_n}{T}$, where T is the number of coalitions.

Example:

$$[4 : 3, 2, 1]$$

Coalition	Weight	Pivotal Player
$\{P_1, P_2, P_3\}$	$3 + 2 = 5$	P_2
$\{P_1, P_3, P_2\}$	$3 + 1 = 4$	P_3
$\{P_2, P_1, P_3\}$	$1 + 3 = 4$	P_1
$\{P_2, P_3, P_1\}$	$1 + 2 = 3 + 3 = 6$	P_1
$\{P_3, P_1, P_2\}$	$2 + 1 = 3 + 3 = 6$	P_1
$\{P_3, P_2, P_1\}$	$2 + 3 = 5$	P_1

$\sigma_1 = \frac{2}{3}, \sigma_2 = \frac{1}{6}, \sigma_3 = \frac{1}{6}$

1.13.3 The Multiplication Rule

If there are m different ways to do X, and n different ways to do Y, then X and Y can be done together in $m \times n$ different ways.

Example:
An ice cream shop carries two different types of cones and 3 different flavors of ice cream. Then, by the multiplication rule, there are $3 \times 2 = 6$ different ways to make an ice cream cone.

1.13.4 The Factorial

The factorial is an operation on some number n, denoted

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 1$$

Example:

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

1.13.5 The Number of Sequential Coalitions

When creating a coalition, we pick the players as follows: Choose any one of the n players, then choose one of the n-1 remaining players, then one of the n-2 remaining players. Keep repeating this process until the very last player is chosen.

Using the multiplication rule, we can see that we have

$$n \times (n-1) \times (n-2) \times \ldots \times 1$$

ways to choose a coalition from n players. Thus, we can say that the number of sequential coalitions is simply $n!$ for a voting system with n players.