MATH 142: EXAM 02
SOLUTIONS

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Answer the questions in the spaces provided on the question sheets and turn them in at the end of the class period.
Unless otherwise stated, all supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.
It is advised, although not required, that you check your answers.
By writing your name on the line below, you indicate that you have read and understand these directions.

Name: ______________________________

Tests	Points Earned	Points Possible	Problems	Points Earned	Points Possible
1		3	1		5
2		3	2		5
3		2	3		5
4		4	4		10
5		4	5		10
6		7	6		15
7		2	7		15
8		4	Subtotal		65
9		3	Bonus		5
10		3			
Subtotal		35	Total		100

Date: March 20, 2018.

1. Tests

Fill in the blanks.

1 (3 Points - n^{th} Term Test for Divergence)**.** The series $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$ or if $\lim_{n\to\infty} a_n$ does not exist. The test is inconclusive if $\lim_{n\to\infty} a_n = 0$.

2 (3 Points - Geometric Series)**.** The series $\sum_{n=0}^{\infty} ar^n$ converges if $|r| < 1$ and diverges if $1 \leq |r|$. If the series converges, then its sum is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

3 (2 Points - p-series)**.** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $1 < p$ and diverges if $p \leq 1$.

4 (4 Points - Integral Test)**.** Let $\{a_n\}$ be a sequence of positive terms. Suppose that f is a continuous, positive, decreasing function of the variable x for all $N \leq x$, for some positive integer N. If $f(n) = a_n$ for all integers $N \leq n$, then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,\mathrm{d}x$ either both converge or both diverge.

5 (4 Points - Comparison Test)**.** Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with non-negative terms. Suppose that for some integer N

$$d_n \leq a_n \leq c_n \text{ for all } N < n.$$

(a) If $\sum c_n$ converges, then $\sum a_n$ converges.
(b) If $\sum d_n$ diverges, then $\sum a_n$ diverges.

6 (7 Points - Limit Comparison Test)**.** Suppose that $0 < a_n$ and $0 < b_n$ for all $N \leq n$, for some positive integer N.

(1) If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$
then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(2) If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$
and $\sum b_n$ converges, then $\sum a_n$ converges.

(3) If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$
and $\sum b_n$ diverges, then $\sum a_n$ diverges.

7 (2 Points - Absolute Convergence Test)**.** If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.

8 (4 Points - Ratio Test)**.** Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

(a) The series converges absolutely if $\rho < 1$.
(b) The series diverges if $1 < \rho$ or $\rho = \infty$.
(c) The test is inconclusive if $\rho = 1$.

9 (3 Points - Root Test)**.** The $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \rho.$$

(a) The series converges absolutely if $\rho < 1$.
(b) The series diverges if $1 < \rho$ or $\rho = \infty$.
(c) The test is inconclusive if $\rho = 1$.

10 (3 Points - Alternating Series Test)**.** The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \dots$$

converges if all three of the following conditions are satisfied:

(1) $0 < u_n$ for all n,
(2) $u_{n+1} \leq u_n$ for all $N \leq n$, for some integer N, and
(3) $u_n \to 0$.

2. Problems

For each of the following problems, decide which of the tests listed above is appropriate and use it to show that the given series converges or diverges. **Clearly indicate the test you used and why the series fits the hypotheses of the test. Failure to do so will result in a zero on the problem.**

1 (5 Points)**.** Determine whether the series

$$\sum_{n=0}^{\infty}\frac{5(3^n)+2^{n+1}}{6^n}$$

converges or diverges. If the series converges, find the sum of the series.

Solution. We observe that

$$\frac{5(3^n)+2^{n+1}}{6^n} = \frac{5}{2^n} + \frac{2}{3^n}$$

which is the sum of terms of two convergent geometric series, so

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{5(3^n)+2^{n+1}}{6^n} &= \sum_{n=0}^{\infty}\left[\frac{5}{2^n}+\frac{2}{3^n}\right]\\ &= \sum_{n=0}^{\infty}\frac{5}{2^n}+\sum_{n=0}^{\infty}\frac{2}{3^n}\\ &= \frac{5}{1-\frac{1}{2}}+\frac{2}{1-\frac{1}{3}}\\ &= 5(2)+2\left(\frac{3}{2}\right)=10+3=13.\end{aligned}$$

2 (5 Points)**.** Determine whether the series

$$\sum_{n=0}^{\infty}\frac{e^n}{e^n+n}$$

converges or diverges.

Solution. We observe that

$$\lim_{n\to\infty}\frac{e^n}{e^n+n}\overset{\text{L'Hôpital}}{=}\lim_{n\to\infty}\frac{e^n}{e^n+1}\overset{\text{L'Hôpital}}{=}\lim_{n\to\infty}\frac{e^n}{e^n}=\lim_{n\to\infty}1=1$$

Therefore this series diverges by the n^{th} Term Test for Divergence.

3 (5 Points)**.** Determine whether the series

$$\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$$

converges or diverges.

Solution 1. First we observe that if $e<3\leq n$, then because $\ln(x)$ is an increasing function

$$1=\ln(e)<\ln(n)$$

and hence

$$\frac{1}{n}<\frac{\ln(n)}{n}.$$

Therefore the series

$$\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$$

diverges by the Comparison Test with the divergent Harmonic series.

Solution 2. We use the Limit Comparison Test with the divergent Harmonic Series. By computing

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n\ln(n)}{n} = \lim_{n\to\infty} \ln(n) = \infty$$

we see by Part (3) of the Limit Comparison Test that the series

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$

diverges.

Solution 3. We use the Integral Test. The function $f(x) = \ln(x)/x$ is continuous, positive, and decreasing for all $e \leq x$. To prove the latter claim, observe that

$$f'(x) = \frac{\frac{1}{x}\cdot x - \ln(x)}{x^2} = \frac{1-\ln(x)}{x^2} < 0$$

holds whenever $e < x$ because $1 = \ln(e)$ and $\ln(x)$ is an increasing function.

We observe that by using the substitution $u = \ln(x)$, $\mathrm{d}u = \mathrm{d}x/x$ then

$$\int_3^{\infty} \frac{\ln(x)}{x}\,\mathrm{d}x = \lim_{t\to\infty}\int_3^{t} \frac{\ln(x)}{x}\,\mathrm{d}x = \lim_{t\to\infty}\int_{\ln(3)}^{\ln(t)} u\,\mathrm{d}u = \lim_{t\to\infty} \frac{u^2}{2}\bigg|_{\ln(3)}^{\ln(t)} = \lim_{t\to\infty} \frac{\ln(t)^2 - \ln(3)^2}{2} = \infty.$$

Therefore the series

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n} = \frac{\ln(2)}{2} + \sum_{n=3}^{\infty} \frac{\ln(n)}{n}$$

diverges by the Integral Test.

4 (10 Points)**.** Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$$

converges or diverges.

Solution 1. We can rewrite

$$\frac{n+1}{n^2\sqrt{n}} = \frac{n}{n^2\sqrt{n}} + \frac{1}{n^2\sqrt{n}} = \frac{1}{n\sqrt{n}} + \frac{1}{n^2\sqrt{n}} = \frac{1}{n^{3/2}} + \frac{1}{n^{5/2}}.$$

Since both series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$$

are convergent p-series, we see that

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}} = \sum_{n=1}^{\infty} \left(\frac{1}{n^{3/2}} + \frac{1}{n^{5/2}} \right) = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$$

converges.

Solution 2. We use the Limit Comparison Test with the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

Because we are only concerned with $1 \leq n$, we have

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2\sqrt{n}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{3/2}(n+1)}{n^2\sqrt{n}} = \lim_{n\to\infty} \frac{n+1}{\sqrt{n}\sqrt{n}} = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

Therefore this series converges by Part (1) of the Limit Comparison Test.

Solution 3. We use the Integral Test. We can rewrite

$$\frac{n+1}{n^2\sqrt{n}} = \frac{n}{n^2\sqrt{n}} + \frac{1}{n^2\sqrt{n}} = \frac{1}{n\sqrt{n}} + \frac{1}{n^2\sqrt{n}} = n^{-3/2} + n^{-5/2}.$$

The functions $f_1(x) = x^{-3/2}$ and $f_2(x) = x^{-5/2}$ are both positive and continuous for $1 \leq x$. Computing the derivatives

$$f_1'(x) = -\frac{3}{2}x^{-5/2} \text{ and } f_2'(x) = -\frac{5}{2}x^{-7/2}$$

we see that both are decreasing for $1 \leq x$, and hence the function $f(x) = f_1(x) + f_2(x)$ is positive, continuous, and decreasing. Computing the Improper Integral of Type I we have

$$\begin{aligned}
\int_1^\infty \left(x^{-3/2} + x^{-5/2}\right) \mathrm{d}x &= \lim_{t\to\infty} \int_1^t \left(x^{-3/2} + x^{-5/2}\right) \mathrm{d}x \\
&= \lim_{t\to\infty} \left(\int_1^t x^{-3/2}\,\mathrm{d}x + \int_1^t x^{-5/2}\,\mathrm{d}x\right) \\
&= \lim_{t\to\infty} \left(-2x^{-1/2}\Big|_1^t + \frac{-2}{3}x^{-3/2}\Big|_1^t\right) \\
&= \lim_{t\to\infty} \left(-2(t^{-1/2} - 1) - \frac{2}{3}(t^{-3/2} - 1)\right) \\
&= \lim_{t\to\infty} \left(2 - \frac{1}{\sqrt{t}} + \frac{2}{3} - \frac{1}{\sqrt{t}^3}\right) \\
&= 2 - 0 + \frac{2}{3} - 0 \\
&= \frac{6}{3} + \frac{2}{3} \\
&= \frac{8}{3}.
\end{aligned}$$

Therefore the series

$$\sum_{n=1}^\infty \frac{n+1}{n^2\sqrt{n}}$$

converges by the Integral Test.

5 (10 Points)**.** Determine whether the series

$$\sum_{n=1}^\infty (-1)^n \frac{n^2(n+2)!}{n!3^{2n}}$$

converges conditionally, converges absolutely, or diverges.

Solution. We use the Ratio Test. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+1)^2(n+3)!}{(n+1)!3^{2n+2}} \frac{n!3^{2n}}{n^2(n+2)!} &= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{(n+3)!}{(n+2)!} \cdot \frac{3^{2n}}{3^{2n+2}} \cdot \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot (n+3) \cdot \frac{1}{3^2} \cdot \frac{1}{(n+1)} \\
&= \lim_{n\to\infty} \frac{(n+1)^2(n+3)}{9(n+1)n^2} \\
&= \lim_{n\to\infty} \frac{(n+1)(n+3)}{9n^2} \\
&= \lim_{n\to\infty} \frac{n^2+4n+3}{9n^2} \\
&= \frac{1}{9} < 1
\end{aligned}$$

Therefore this series converges absolutely by the Ratio Test.

6 (15 Points)**.** Determine whether the series

$$\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$$

converges or diverges.

Solution. We use the Ratio Test. We have

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{4n+3}{3n-5}\right)^n} = \lim_{n\to\infty} \frac{4n+3}{3n-5} = \frac{4}{3} > 1.$$

Therefore this series diverges by the Root Test.

7 (15 Points)**.** Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n^2+1}$$

converges conditionally, converges absolutely, or diverges.

Solution. First we observe that this series does not converge absolutely. By computing the limit

$$\lim_{n\to\infty} \frac{\frac{2n}{4n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} 2n^2 4n^2 + 1 = \frac{1}{2} > 1$$

we see that the series

$$\sum_{n=1}^{\infty} \frac{2n}{4n^2+1}$$

diverges by Part (1) of the Limit Comparison Test because the Harmonic Series diverges.

Next we try the Alternating Series Test. The first and third conditions are easy to verify: when $1 \leq n$ it's clear that

$$0 < u_n = \frac{2n}{4n^2+1}$$

holds and also

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2n}{4n^2+1} = \lim_{n\to\infty} \frac{n^2}{n^2} \cdot \frac{2/n}{4+1/n^2} = \lim_{n\to\infty} \frac{2/n}{4+1/n^2} = \frac{0}{4+0} = 0.$$

To conclude that this series converges by the Alternating Series Test, we need only verify that for some integer N, $u_{n+1} \leq u_n$ holds whenever $N \leq n$. Towards that end let $f(x) = \frac{2x}{4x^2+1}$ and observe that

$$f'(x) = \frac{2(4x^2+1) - (2x)(8x)}{(4x^2+1)^2} = \frac{8x^2+2-16x}{(4x^2+1)^2} = \frac{-8x^2+2}{(4x^2+1)^2} < 0$$

holds if and only if

$$-8x^2 + 2 < 0$$

which holds if and only if

$$\sqrt{\frac{2}{8}} = \sqrt{\frac{1}{4}} = \frac{1}{\sqrt{4}} = \frac{1}{2} < x.$$

Since f is a decreasing function if and only if $f'(x) < 0$, we see that

$$u_{n+1} = f(n+1) \leq f(n) = u_n$$

holds whenever $1 \leq n$. Therefore the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n^2+1}$$

converges conditionally.

8 (Bonus - 5 Points)**.** Determine whether the series

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

converges or diverges.

Solution. We use the Limit Comparison Test with the Harmonic Series. To evaluate the limit

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

we observe that if we let $h = \frac{1}{n}$, then

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

Therefore the series

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges by Part (1) of the Limit Comparison Test.