

IMPROPER INTEGRALS

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Name: Solutions

Determine whether the following improper integrals converge or diverge.

TYPE I

1. $\int_1^{\infty} \frac{\ln(x)}{x} dx$

Let $u = \ln(x)$, $du = \frac{1}{x} dx$, so

$$\begin{aligned} \int_1^{\infty} \frac{\ln(x)}{x} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{\ln(x)}{x} dx = \lim_{t \rightarrow \infty} \int_0^{\ln(t)} u du \\ &= \lim_{t \rightarrow \infty} \left. \frac{1}{2} u^2 \right|_0^{\ln(t)} = \lim_{t \rightarrow \infty} \frac{1}{2} (\ln(t))^2 = \boxed{\infty} \end{aligned}$$

2. $\int_1^{\infty} \frac{1}{(2x+1)^3} dx$

Let $u = 2x+1$, $du = 2dx \Rightarrow \frac{1}{2} du = dx$ $u(1) = 2(1)+1 = 3$
 $u(t) = 2t+1$

$$\begin{aligned} \int_1^{\infty} \frac{1}{(2x+1)^3} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{(2x+1)^3} dx = \lim_{t \rightarrow \infty} \frac{1}{2} \int_3^{2t+1} u^{-3} du \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \left(\frac{-1}{2} \right) u^{-2} \Big|_3^{2t+1} = \lim_{t \rightarrow \infty} -\frac{1}{4} \left[\frac{1}{(2t+1)^2} - \frac{1}{9} \right] \\ &= -\frac{1}{4} \left[0 - \frac{1}{9} \right] = \boxed{\frac{1}{36}} \end{aligned}$$

3. $\int_{-\infty}^0 e^x dx$

$$\begin{aligned} \int_{-\infty}^0 e^x dx &= \lim_{t \rightarrow \infty} \int_{-t}^0 e^x dx = \lim_{t \rightarrow \infty} e^x \Big|_{-t}^0 = \lim_{t \rightarrow \infty} e^0 - e^{-t} \\ &= \lim_{t \rightarrow \infty} 1 - \frac{1}{e^t} = \boxed{1} \end{aligned}$$

4. $\int_{-\infty}^{\infty} x e^{-x^2} dx$

Let $u = -x^2$, so $du = -2x dx \Rightarrow \frac{-1}{2} du = x dx$

$$\int_0^{\infty} x e^{-x^2} dx = \lim_{t \rightarrow \infty} \int_0^t x e^{-x^2} dx = \lim_{t \rightarrow \infty} \frac{-1}{2} \int_0^{-t^2} e^u du$$

$$= \lim_{t \rightarrow \infty} \frac{-1}{2} e^u \Big|_0^{-t^2} = \lim_{t \rightarrow \infty} \frac{-1}{2} \left[\frac{1}{e^{t^2}} - 1 \right] = \frac{1}{2}$$

$$\int_{-\infty}^0 x e^{-x^2} dx = \lim_{t \rightarrow \infty} \int_{-t}^0 x e^{-x^2} dx = \lim_{t \rightarrow \infty} \frac{-1}{2} \int_{-t^2}^0 e^u du$$

$$= \lim_{t \rightarrow \infty} \frac{-1}{2} e^u \Big|_{-t^2}^0 = \lim_{t \rightarrow \infty} \frac{-1}{2} \left[1 - \frac{1}{e^{t^2}} \right] = -\frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} x e^{-x^2} dx = \frac{1}{2} + \left(-\frac{1}{2}\right) = \boxed{0}$$

TYPE II

5. $\int_0^1 \ln(x) dx$

$$\begin{aligned} \int_0^1 \ln(x) dx &= \lim_{t \rightarrow 0^+} \int_t^1 \ln(x) dx = \lim_{t \rightarrow 0^+} x \ln(x) \Big|_t^1 - x \Big|_t^1 \\ &= \lim_{t \rightarrow 0^+} [(0 - t \ln(t)) - (1 - t)] \\ &= \lim_{t \rightarrow 0^+} t - 1 - t \ln(t) = \boxed{-1} \end{aligned}$$

because

$$\lim_{t \rightarrow 0^+} t \ln(t) = \lim_{t \rightarrow 0^+} \frac{\ln(t)}{(\frac{1}{t})} \stackrel{L'H}{=} \lim_{t \rightarrow 0^+} \frac{-\frac{1}{t}}{(-\frac{1}{t^2})} = \lim_{t \rightarrow 0^+} \frac{t^2}{t} = \lim_{t \rightarrow 0^+} t = 0.$$

6. $\int_0^1 \frac{dx}{\sqrt{x}}$

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{x}} &= \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{\sqrt{x}} dx = \lim_{t \rightarrow 0^+} 2\sqrt{x} \Big|_t^1 \\ &= \lim_{t \rightarrow 0^+} (2 - 2\sqrt{t}) = 2 - 0 = \boxed{2} \end{aligned}$$