APPROXIMATE INTEGRATION

BLAKE FARMAN

Introduction

The idea of this section is to be able to estimate integrals of functions for which there are no known elementary antiderivatives with methods that are generally 'better' than simply using rectangles. The methods in this section have the advantage of being extremely straightforward to implement in your favorite mathematics software (Sage, Mathematica, Matlab, Maple, etc.).

Notation. For both the Trapezoidal Rule and Simpson's Rule, we provide geometric interpretations of the derivation. For ease of presentation, we will make some assumptions that simplify this geometric presentation. We will assume throughout that f is a non-negative, continuous function on $[a, b]$, however it need only be continuous on $[a, b]$ for this to work, as can be seen from the alternate formulations.

Let n be an integer and define

$$\Delta x = \frac{b-a}{n},$$

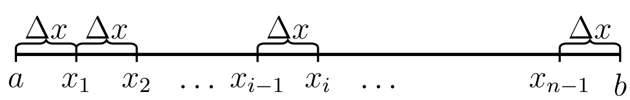
which provides a partition of $[a, b]$ into n subintervals

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{i-1}, x_i],\ \ldots,\ [x_{n-1}, x_n]$$

using the $n + 1$ points

$$x_0 = a, x_1 = a + \Delta x, \ldots , x_i = a + i\Delta x, \ldots , x_n = b$$

as pictured below:



Recollection. First, we recall the definition of the definite integral:

Definition 1 (Definite Integral)**.** If f is a funciton defined on $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $x_i^* \in [x_{i-1}, x_i]$ is any point.

The sum in the definition is often referred to as a *Riemann Sum*. This sum is a function defined for each integer that approximates the area under the curve using rectangles of width Δx, along the x-axis, and height $f(x_i^*)$, along the y-axis. Since the choice of the point used in the approximation of the area is aribtrary, one often introduces and names a few of these Riemann sums that are especially handy for computation, which we recall here.

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Left Endpoint:

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

Right Endpoint:

$$R_n = \sum_{i=i}^{n} f(x_i)\Delta x$$

Midpoint:

$$M_n = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x, \text{ where } \bar{x}_i = \frac{x_{i-1}+x_i}{2}$$

For the latter, we make the following useful observation. Given a partition $x_i = a + i\Delta x$, it is sometimes useful to refine this partition to

$$t_i = a + i\frac{(b-a)}{2n} = a + i\Delta t$$

so that

$$x_i = t_{2i},\ 1 \leq i \leq n$$

and

$$\bar{x}_i = t_{2i+1},\ 0 \leq i \leq n-1.$$

By observing that

$$\Delta x = \frac{b-a}{n} = \frac{2(b-a)}{2n} = 2\Delta t$$

we can then express the midpoint sum using the refinement as

$$M_n = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \sum_{i=0}^{n-1} 2f(t_{2i+1})\Delta t.$$

This observation will be so important in the discussion surrouding Simpson's Rule that we include the following concrete example.

Example 2. Consider the partition of $[0,6]$ into 3 subintervals of length

$$\Delta x = \frac{6-0}{3} = 2.$$

The points that make up the partition are

$$x_0 = 0,\ x_1 = 2,\ x_2 = 4,\ x_3 = 6$$

and the midpoints for this partition are

$$\begin{aligned}
\bar{x}_1 &= \frac{x_0+x_1}{2} = \frac{0+2}{2} = 1 \\
\bar{x}_2 &= \frac{x_1+x_2}{2} = \frac{2+4}{2} = 3 \\
\bar{x}_3 &= \frac{x_2+x_3}{2} = \frac{6+4}{2} = 5.
\end{aligned}$$

If we refine this partition using twice as many points, we have

$$\Delta t = \frac{6-0}{6} = 1.$$

The points that make up the partition are

$$t_0 = 0, t_1 = 1, t_2 = 2, t_3 = 3, t_4 = 4, t_5 = 5, t_6 = 6$$

and we can view the midpoints as those t_i with odd index:

$$\bar{x}_1 = t_1 \ \bar{x}_2 = t_3 \ \bar{x}_3 = t_5$$

For a funciton f defined on $[0, 6]$ we have

$$\Delta x = 2 = 2(1) = 2\Delta t$$

and

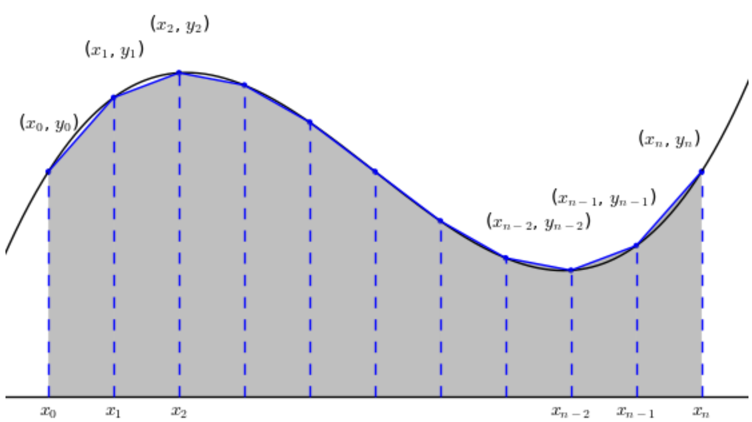
$$\begin{aligned} M_3 &= \big(f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3)\big)\,\Delta x \\ &= \big(f(t_1) + f(t_3) + f(t_5)\big)\,2\Delta t \\ &= \big(2f(t_1) + 2f(t_3) + 2f(t_5)\big)\,\Delta t \end{aligned}$$

Trapezoidal Approximations

We name the function values associated with the partition of $[a, b]$

$$y_0 = f(a),\ y_1 = f(x_1),\ \ldots,\ y_n = f(x_n)$$

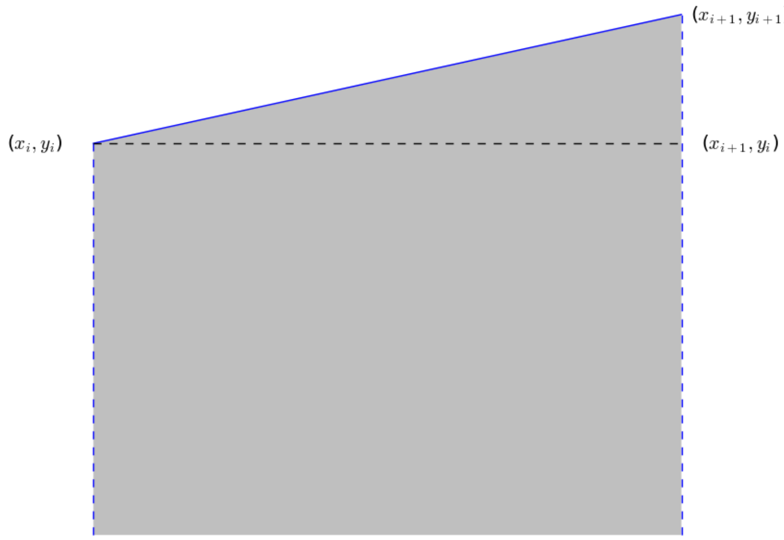
and, on each subinterval $[x_i, x_{i+1}]$, we construct a trapezoid connecting the points $[x_i, y_i]$ and $[x_{i+1}, y_{i+1}]$ to get something that looks like the following



To estimate the integral

$$\int_a^b f(x)\,\mathrm{d}x$$

we need only sum the areas of the trapezoids. The i^{th} trapezoid looks like



so its area can be computed as the sum of the area of the triangle on top and the area of the rectangle on the bottom

$$\frac{1}{2}(y_{i+1} - y_i)\Delta x + y_i \Delta x = \frac{1}{2}(y_{i+1} - y_i + 2y_i)\Delta x = \frac{y_i + y_{i+1}}{2}\Delta x.$$

To get an approximation to the area underneath the curve f we now add up the areas of the n trapezoids

$$\begin{aligned} &\frac{y_0 + y_1}{2}\Delta x + \frac{y_1 + y_2}{2}\Delta x + \ldots + \frac{y_{n-2} + y_{n-1}}{2}\Delta x + \frac{y_{n-1} + y_n}{2}\Delta x \\ &= (y_0 + y_1 + y_1 + y_2 + \ldots y_{n-2} + y_{n-1} + y_{n-1} + y_n)\frac{\Delta x}{2}. \end{aligned}$$

We observe that y_0 and y_n appear in the sum only once, but each other y_i appears twice: once on the subinterval $[x_i, x_{i+1}]$ and once on the subinterval $[x_{i+1}, x_{i+2}]$. Therefore we have

Theorem 3 (The Trapezoidal Rule)**.** *To approximate the definite integral*

$$\int_a^b f(x)\,\mathrm{d}x$$

by n trapezoids of base width Δx use

$$T_n = \frac{y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-2} + 2y_{n-1} + y_n}{2}\Delta x.$$

Remark 4. Alternatively, one can construct T_n as the average of the left and right endpoint sums

$$\begin{aligned} T_n &= \frac{L_n + R_n}{2} \\ &= \frac{(y_0 + y_1 + \ldots + y_{n-1})\,\Delta x + (y_1 + y_2 + \ldots + y_n)\,\Delta x}{2} \\ &= \frac{y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n}{2}\Delta x \end{aligned}$$

From this point of view, it is easy to see that

$$\lim_{n\to\infty} T_n = \frac{\lim_{n\to\infty} L_n + \lim_{n\to\infty} R_n}{2} = \frac{\int_a^b f(x)\,\mathrm{d}x + \int_a^b f(x)\,\mathrm{d}x}{2} = \int_a^b f(x)\,\mathrm{d}x$$

Example 5. Use the Trapezoidal Rule to approximate

$$\int_1^2 x^2\,\mathrm{d}x$$

using $n = 4$ trapezoids.

Solution. First we compute

$$\Delta x = \frac{2-1}{4} = \frac{1}{4}$$

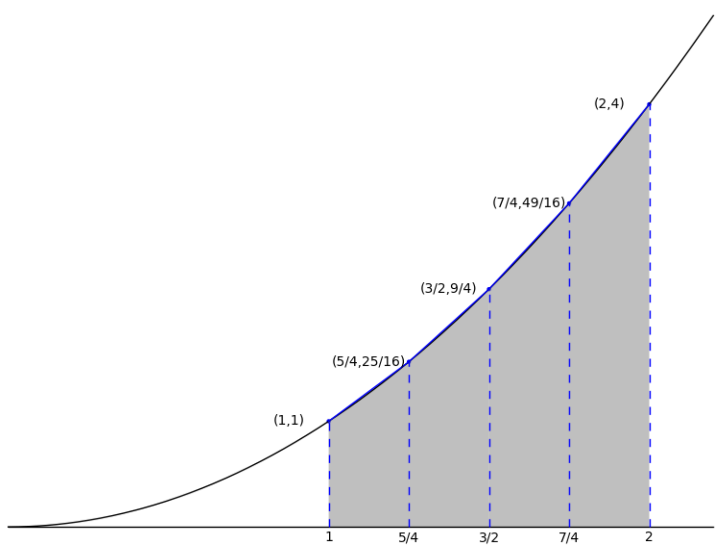
so our partition is

$$x_0 = 1,\ x_1 = 5/4,\ x_2 = 3/2,\ x_3 = 7/4,\ x_4 = 2$$

and

$$y_0 = 1,\ y_1 = \frac{25}{16},\ y_2 = \frac{9}{4},\ y_3 = \frac{49}{16},\ y_4 = 4.$$

We can sketch the trapezoids we're using for our approximation



The estimate is then

$$\begin{aligned} T &= \left(1 + 2\cdot\frac{25}{16} + 2\cdot\frac{9}{4} + 2\cdot\frac{49}{16} + 4\right)\frac{1/4}{2} \\ &= \left(\frac{8}{8} + \frac{25}{8} + \frac{36}{8} + \frac{49}{8} + \frac{32}{8}\right)\frac{1}{8} \\ &= \frac{150}{64} \\ &= \frac{75}{32} \end{aligned}$$

Since we know how to compute the actual area, we see that the estimate has an error of

$$\begin{aligned}\int_1^2 x^2\,\mathrm{d}x - \frac{75}{32} &= \frac{2^3-1^3}{3} - \frac{75}{32}\\ &= \frac{7}{3} - \frac{75}{32}\\ &= \frac{7(32)-3(75)}{96}\\ &= \frac{210+14-210-15}{96}\\ &= -\frac{1}{96}\\ &= -0.01041\overline{6}.\end{aligned}$$

Thus the trapezoidal rule gives a slight over-estimate here. □

Simpson's Rule: Approximations Using Parabolas

Once again, we let

$$\Delta x = \frac{b-a}{2},$$

take an evenly spaced partition

$$x_0 = a,\ x_1 = a+\Delta x,\ x_2 = a + 2\Delta x,\ \ldots,\ x_{n-1} = a + (n-1)\Delta x,\ x_n = b,$$

and denote $y_i = f(x_i)$ for $0 \le i \le n$. However, this time we require that n **is an even integer**.

We make this stipulation because we will fit a parabola to three points

$$(x_{i-1}, y_{i-1}), (x_i, y_i), (x_{i+1}, y_{i+1})$$

and use the area under the parabola on the interval $[x_{i-1}, x_{i+1}]$ as an estimate for

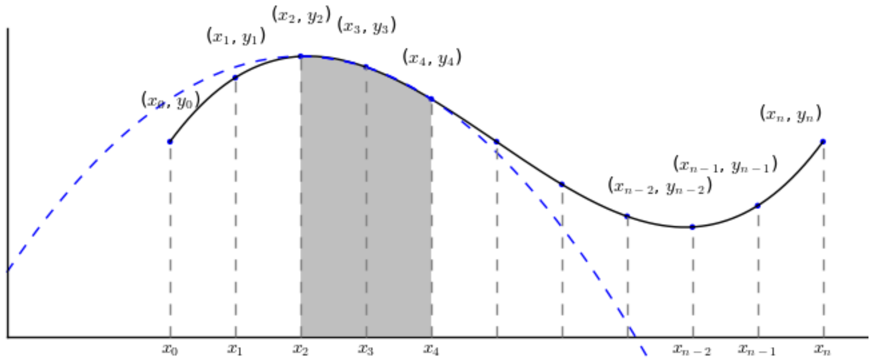
$$\int_{x_i-1}^{x_{i+1}} f(x)\,\mathrm{d}x = \int_{x_{i-1}}^{x_i} f(x)\,\mathrm{d}x + \int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x.$$

Since the partition subdivides $[a, b]$ into n subintervals and each parabola estimates the area on two of these subintervals, we must use $n/2$ parabolas, and hence n **must** be even for the method to work.

To get a feel for what is happening here, depicted below is a curve with a parabola fitted to the three points

$$(x_2, y_2), (x_3, y_3), (x_4, y_4).$$

The shaded region represents the area under the parabola, $p(x)$, which approximates the area under the curve f on the interval $[x_2, x_4]$.

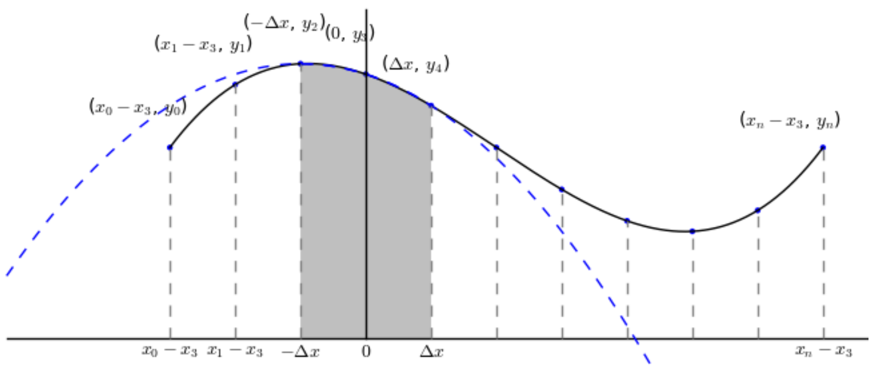


For each of the $m = n/2$ subintervals,

$$[x_1, x_3], [x_2, x_4], \ldots, [x_{n-4}, x_{n-2}], [x_{n-2}, x_n]$$

we must compute the area under the parabolas and then sum them up. To make our task a little simpler, we make the following observation.

We note that if we shift our graph above to the left by x_3 units, this rigid transformation does not change the area of the shaded region, as seen in the graph of $f(x+x_3)$ and $p(x+x_3)$ below



So we see that instead of trying to fit a parabola to the points

$$(x_2, y_2), (x_3, y_3), (x_4, y_4)$$

we can fit a parabola, $q(x)$, to the points

$$(-\Delta x, y_{i-1}), (0, y_i), (\Delta x, y_{i+1})$$

so that the shifted parabola

$$p(x) = q(x - x_3)$$

satisfies

$$\begin{aligned} p(x_2) &= q(x_2 - x_3) = q(x_2 - (x_2 + \Delta x)) = q(-\Delta x) = y_2 \\ p(x_3) &= q(x_3 - x_3) = q(0) = y_3 \\ p(x_4) &= q(x_4 - x_3) = q(x_3 + \Delta x - x_3) = q(\Delta x) = y_4 \end{aligned}$$

and

$$\int_{x_2}^{x_4} p(x)\,\mathrm{d}x = \int_{-\Delta x}^{\Delta x} q(x)\,\mathrm{d}x.$$

This small observation will simplify our task greatly.

Next, we set about computing all the areas we need. The first thing we need is a parabola for each of $i = 1, 3, 5, \ldots, n-1$. For such an i we take the three points $(-\Delta x, y_{i-1})$, $(0, y_i)$, and $(\Delta x, y_{i+1})$. The parabola we want will have the form

$$q(x) = Ax^2 + Bx + C$$

and it must satisfy the three equations

$$\begin{aligned} y_i &= q(0) = A(0)^2 + B(0) + C = C \\ y_{i-1} &= q(-\Delta x) = A(\Delta x)^2 - B(\Delta x) + C \\ y_{i+1} &= q(-\Delta x) = A(\Delta x)^2 - B(\Delta x) + C \end{aligned}$$

The first equation gives us $C = y_i$. If we add together the other two equations we see that

$$y_{i-1} + y_{i+1} = 2A(\Delta x)^2 + 2y_i$$

so we have

$$2A(\Delta x)^2 = y_{i-1} + y_{i+1} - 2y_i.$$

This is enough information to solve for A and B in terms of y_{i-1}, y_i, y_{i+1}, and Δx, but we can be a little more clever here.

Since x^2 and the constant function are even functions, and x is an odd function, the area under $q(x)$ on $[-\Delta x, \Delta x]$ is given by

$$\begin{aligned} \int_{-\Delta x}^{\Delta x} q(x)\,\mathrm{d}x &= \int_{-\Delta x}^{\Delta x} (Ax^2 + Bx + C)\,\mathrm{d}x \\ &= A\int_{-\Delta x}^{\Delta x} x^2\,\mathrm{d}x + B\int_{-\Delta x}^{\Delta x} x\,\mathrm{d}x + C\int_{-\Delta x}^{\Delta x} \mathrm{d}x \\ &= 2A\int_0^{\Delta x} x^2\,\mathrm{d}x + 0 + 2C\int_0^{\Delta x} \mathrm{d}x \\ &= \frac{2}{3}A(\Delta x)^3 + 2C\Delta x \\ &= \frac{\Delta x}{3}\left(2A(\Delta x)^2 + 6C\right) \\ &= \frac{\Delta x}{3}\left(y_{i-1} + y_{i+1} - 2y_i + 6y_i\right) \\ &= \frac{\Delta x}{3}\left(y_{i-1} + 4y_i + y_{i+1}\right). \end{aligned}$$

Now, as we observed above, the parabola

$$p(x) = q(x - x_i)$$

satisfies

$$\begin{aligned} p(x_{i-1}) &= q(x_{i-1} - (x_{i-1} + \Delta x)) = q(-\Delta x) = y_{i-1} \\ p(x_i) &= q(x_i - x_i) = q(0) = y_i \\ p(x_{i+1}) &= q(x_i + \Delta x - x_i) = q(\Delta x) = y_{i+1} \end{aligned}$$

and

$$\int_{x_{i-1}}^{x_{i+1}} p(x)\,\mathrm{d}x = \int_{-\Delta x}^{\Delta x} q(x)\,\mathrm{d}x = \frac{\Delta x}{3}(y_{i-1} + 4y_i + y_{i+1})$$

Hence we have the approximation

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x \quad &\approx \quad \frac{\Delta x}{3}(y_0 + 4y_1 + y_2) + \frac{\Delta x}{3}(y_2 + 4y_3 + y_4) + \dots \\ &+ \quad \frac{\Delta x}{3}(y_{n-4} + 4y_{n-3} + y_{n-2}) + \frac{\Delta x}{3}(y_{n-2} + 4y_{n-1} + y_n)\end{aligned}$$

We observe that the terms y_0 and y_n will only appear once, and the terms $y_1, y_3, \dots, y_{n-3}, y_{n-1}$ with odd subscript all appear with coefficient 4 exactly once. All other terms with even subscript, y_{2j} for $j = 1, 2, \dots, n/2$, all appear twice: once in the approximation over the interval $[x_{2j-2}, x_{2j}]$ and once in the approximation over the interval $[x_{2j}, x_{2j+2}]$. Therefore we have

Theorem 6 (Simpson's Rule)**.** *The integral*

$$\int_a^b f(x)\,\mathrm{d}x$$

can be approximated using $n/2$ parabolas by

$$S_n = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 2y_{n-2} + 4y_{n-1} + y_n\right),$$

where

$$\Delta x = \frac{b-a}{n},$$

the interval $[a, b]$ is partitioned by

$$x_0 = a,\ x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ \dots,\ x_{n-1} = a + (n-1)\Delta x,\ x_n = b,$$

and $y_i = f(x_i)$.

Example 7. Use Simpson's rule to estimate

$$\int_0^2 5x^4\,\mathrm{d}x$$

using two parabolas.

Solution. First we compute

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

and so we have a partition

$$x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2.$$

The y-coordinates of the points we need are

$$y_0 = 0, y_1 = \frac{5}{16}, y_2 = 5, y_3 = \frac{405}{16}, y_4 = 80.$$

Thus the estimate is given by

$$\begin{aligned} S &= \frac{1}{6}(0 + \frac{5}{4} + 10 + \frac{405}{4} + 80) \\ &= \frac{1}{24}(5 + 40 + 405 + 320) \\ &= \frac{770}{24} \\ &= \frac{2 \cdot 5 \cdot 7 \cdot 11}{2^3 \cdot 3} \\ &= \frac{385}{12} \\ &= 32 + \frac{1}{12}. \end{aligned}$$

□

Alternate Construction of Simpson's Rule. To see clearly that Simpson's Rule actually gives an approximation to the definite integral, it is easiest to reformulate the sum in terms of the Midpoint and Trapezoidal rules.

Assume that $n = 2k$ is an even integer. Let $\Delta x = (b-a)/n$, $x_i = a + i\Delta x$, and $y_i = f(x_i)$. Following the discussion at the end of Section , observe that we can express the midpoint sum for k intervals as

$$M_k = \sum_{i=1}^{k-1} 2y_{2i+1}\Delta x = (2y_1 + 2y_3 + \ldots + 2y_{n-1})\,\Delta x$$

Similarly, using k intervals the right endpoint sum can be expresed as

$$R_k = \sum_{i=1}^{k} y_{2i}\left(\frac{b-a}{k}\right) = (2y_0 + 2y_2 + \ldots + 2y_{n-2})\,\Delta x$$

and the left endpoint sum can be expressed as

$$L_k = \sum_{i=0}^{k-1} y_{2i}\left(\frac{b-a}{k}\right) = (2y_2 + 2y_4 + \ldots + 2y_n)\,\Delta x$$

so that the approximation of the area by k trapezoids is

$$T_k = \frac{L_k + R_k}{2} = \frac{2y_0 + 4y_2 + \ldots + 4y_{n-2} + 2y_n}{2}\Delta x = (y_0 + 2y_2 + \ldots + 2y_{n-2} + y_n)\,\Delta x$$

Now we observe that

$$\begin{aligned} \frac{2M_k + T_k}{3} &= \frac{(4y_1 + 4y_3 + \ldots + 4y_{n-1})\,\Delta x + (y_0 + 2y_2 + \ldots + 2y_{n-2} + y_n)\,\Delta x}{3} \\ &= \frac{y_0 + 4y_1 + 2y_2 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n}{3}\Delta x \\ &= S_n = S_{2k} \end{aligned}$$

From this observation, it is now clear that

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{k\to\infty} S_{2k} \\
&= \lim_{k\to\infty} \frac{2M_k + T_k}{3} \\
&= \frac{2\lim_{k\to\infty} M_k + \lim_{k\to\infty} T_k}{3} \\
&= \frac{2\int_a^b f(x)\,\mathrm{d}x + \int_a^b f(x)\,\mathrm{d}x}{3} \\
&= \frac{3\int_a^b f(x)\,\mathrm{d}x}{3} \\
&= \int_a^b f(x)\,\mathrm{d}x.
\end{aligned}$$

Error Analysis. Now that we have these two methods for approximation, the natural question to ask is, how good are these approximations? To answer this question, we need to first define how we measure the error.

Definition 8. For a partition of $[a, b]$ into n subintervals, we define the following values

$$\begin{aligned}
E_M &= \int_a^b f(x)\,\mathrm{d}x - M_n \\
E_T &= \int_a^b f(x)\,\mathrm{d}x - T_n \\
E_S &= \int_a^b f(x)\,\mathrm{d}x - S_n
\end{aligned}$$

to be the error incurred by approximating the definite integral by the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule with n subintervals, respectively.

It should be noted that because

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} M_n = \lim_{n\to\infty} T_n = \int_a^b f(x)\,\mathrm{d}x$$

we have

$$\lim_{n\to\infty} E_M = \lim_{n\to\infty} E_T = \lim_{n\to\infty} E_S = 0.$$

This means, as we increase the number of intervals, the error from the estimation tends towards zero. That is to say, more intervals gives a better estimate. However, this doesn't explicitly tell us how the error varies with the number of intervals. For this, we have the following theorems.

Theorem 9 (Midpoint and Trapezoidal Error Bounds)**.** *If f'' is continuous and $\left|f''(x)\right| \leq K$ on $[a, b]$, then the error, E_M, in the midpoint approximation and the error, E_T, in the trapezoidal approximation of*

$$\int_a^b f(x)\,\mathrm{d}x$$

satisfy

$$|E_M| \leq \frac{K(b-a)^3}{12n^2} \text{ and } |E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

Theorem 10 (Simpson's Error Bounds)**.** *If the fourth derivative, $f^{(4)}$, is continuous and $\left|f^{(4)}(x)\right| \leq K$ on $[a,b]$, then the error, E_S, in the Simpson's rule approximation of*

$$\int_a^b f(x)\,\mathrm{d}x$$

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

Proof. Omitted. □

Remark 11. Since both of these bounds rely on derivatives, it follows that any polynomial of degree at most 1, the Trapezoidal Approximation has no error, and for any polynomial of degree at most 3, the Simpson's Rule Approximation has no error, regardless of the choice of n.

For example, if we consider a polynomial

$$\ell(x) = Mx + B$$

then $\ell''(x) = 0$, so for any interval $[a,b]$ we have

$$|E_T| \leq \frac{0(b-a)^3}{12n^2} = 0$$

implies that $E_T = 0$ and thus

$$\int_a^b \ell(x)\,\mathrm{d}x = T_n.$$

Similarly, for any polynomial

$$p(x) = Ax^3 + Bx^2 + Cx + D$$

we have $p^{(4)}(x) = 0$, so

$$|E_S| \leq \frac{0(b-a)^5}{180n^4} = 0$$

implies that $E_S = 0$ and thus

$$\int_a^b p(x)\,\mathrm{d}x = S_n.$$

Example 12. As an interesting special case of the last statement in the remark, consider the following cubic polynomial. Let $a < c < b$ be real numbers and define

$$p(x) = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc.$$

Partition the interval $[a,b]$ using the points

$$x_0 = a,\ x_1 = \frac{a+b}{2}, x_2 = b$$

so

$$\begin{aligned} S_2 &= \frac{p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)}{3}\left(\frac{b-a}{2}\right) \\ &= 4p\left(\frac{a+b}{2}\right)\frac{b-a}{6} \\ &= p\left(\frac{a+b}{2}\right)\frac{2(b-a)}{3} \end{aligned}$$

By Theorem 10 we see that

$$|E_S| \le \frac{0(b-a)^5}{180n^4} = 0$$

Hence

$$\int_a^b p(x)\,\mathrm{d}x = p\left(\frac{a+b}{2}\right)\frac{2(b-a)}{3}$$

In particular, if $c = (a+b)/2$, then we see that

$$\int_a^b p(x)\,\mathrm{d}x = 0.$$

Equivalently, one could express this as observing that the shifted cubic

$$\begin{aligned} p(x+c) &= (x+c-a)(x+c-c)(x+c-b) \\ &= x(x-(a-c))(x-(b-c)) \\ &= x\left(x - \frac{a-b}{2}\right)\left(x - \frac{b-a}{2}\right) \\ &= x\left(x + \frac{b-a}{2}\right)\left(x - \frac{b-a}{2}\right) \\ &= x\left(x^2 - \left(\frac{b-a}{2}\right)^2\right) \\ &= x^3 - \left(\frac{b-a}{2}\right)^2 x \end{aligned}$$

is an (obviously) odd function.

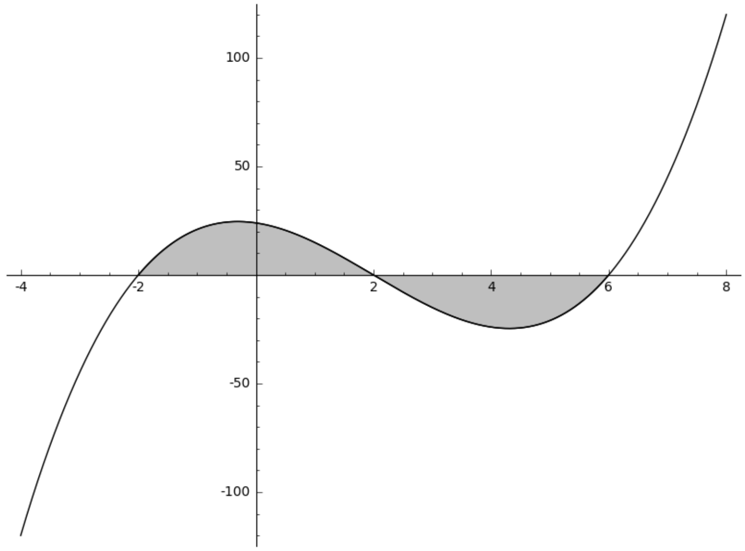
Concretely, we could take the polynomial

$$p(x) = (x+2)(x-2)(x-6) = x^3 - 6x^2 - 4x + 24$$

and

$$\int_{-2}^{6} p(x)\,\mathrm{d}x = p\left(\frac{6+(-2)}{2}\right)\frac{2(6-(-2))}{3}x = \frac{8}{3}p\left(\frac{4}{2}\right) = \frac{8}{3}p(2) = 0$$

which tells us that there is a certain symmetry about a degree three polynomial with evenly spaced roots! That is to say, in the graph



the two shaded regions are the same size.

Example 13. Find an upper bound for the error in estimating

$$\int_0^2 5x^4 \, \mathrm{d}x$$

using Simpson's Rule with $n = 4$.

Solution. Since we're using $f(x) = 5x^4$, we note that

$$f^{(4)}(x) = 210 = M$$

serves as an upper bound on $[0, 2]$, so

$$|E_S| \leq \frac{120(2)^5}{180(4)^4} = \frac{120}{180 \cdot 2^3} = \frac{2}{3 \cdot 8} = \frac{1}{12}$$

□

Example 14. Determine the number of subintervals, n, needed to estimate

$$\int_0^2 5x^4 \, \mathrm{d}x$$

to an error of magnitude less than 10^{-4} using Simpson's Rule.

Solution. We want to have

$$|E_S| \leq \frac{M(b-a)^5}{180n^4} = \frac{120(2^5)}{180n^4} = \frac{2^6}{3n^4} < 10^{-4}.$$

So, all we need to do is require that

$$\frac{2^6 \cdot 10^4}{3} < n^4$$

or, equivalently, that

$$\frac{\sqrt[4]{2^6}\sqrt[4]{10^4}}{\sqrt[4]{3}} = \frac{2^{3/2} \cdot 10}{\sqrt{4}3} < n.$$

Since

$$\frac{2^{3/2} \cdot 10}{\sqrt{43}} \approx 21.5$$

it follows that we can ensure the error is less than 10^{-4} by taking $22 \leq n$. $\square$