Kernels for noncommutative projective schemes

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Abstract

We give a noncommutative geometric description of the internal Hom dg-category in the homotopy category of dg-categories between two noncommutative projective schemes in the style of Artin-Zhang. As an immediate application, we give a noncommutative projective derived Morita statement along the lines of Rickard and Orlov.

Keywords. noncommutative algebra, noncommutative projective schemes, derived categories, Fourier-Mukai transforms

1 Introduction

Derived categories in algebraic geometry have proven themselves to be an enormously useful tool in studying birational geometry [7, 10, 13], moduli theory [14, 28, 29], and have relations to other fields like representation theory of finite dimensional algebra [8, 18] and symplectic geometry [33, 34], through mirror symmetry [24].

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At the same time, as conceived by Artin and Zhang [4] under the name of noncommutative projective schemes, using tools of category theory and (commutative) algebraic geometry to understand the landscape of noncommutative graded rings has proven itself fruitful as well. Indeed, moduli spaces of point modules form the key technical tool in the classification of noncommutative $\mathbb{P}^2$'s [2, 3, 36, 37]. Furthermore, a natural conjecture of Artin classifies the noncommutative surfaces up to birational classification of noncommutative surfaces [1].

If one is seriously interested in using algebro-geometric techniques to study noncommutative graded rings, then focusing on and exploiting derived categories of noncommutative projective schemes is an obvious and seemingly fertile avenue. The structure of noncommutative projective schemes is shaped by moduli theory and birational geometry, almost the exact areas where derived categories realize their full power in (commutative) algebraic geometry. Additionally, Calabi-Yau noncommutative projective schemes can provide geometric interpretations for some non-geometric $\mathcal{N} = 2$ superconformal field theories compactifying IIB strings [5]. Some hints of this are already in the literature. Two glimpses of this are: Li and Zhao [25] show how Bridgeland stability for noncommutative $\mathbb{P}^2$'s provides access to the Minimal Model Program for commutative deformations of Hilbert schemes of points; Harder and Katzarkov [19] describe Homological Mirror Symmetry for four-dimensional quadratic Sklyanin algebras.

In a first course on derived categories in algebraic geometry, one learns that the power and influence of derived categories lies in the geometric notion of a kernel. Consider the derived categories of quasi-coherent sheaves, $\mathrm{D}(X)$ and $\mathrm{D}(Y)$, for two varieties X and Y. For a general exact functor $F : \mathrm{D}(X) \to \mathrm{D}(Y)$, one has almost no control; it is pure abstract smoke. However, in the vast majority of problems, we are lucky to not be interested in such a general functor, but one of a specific provenance. From an object $K \in \mathrm{D}(X \times Y)$ we can construct an exact functor

$$\begin{aligned}\Phi_K : \mathrm{D}(X) &\to \mathrm{D}(Y)\\ E &\mapsto q_*\left(K \otimes p^* E\right)\end{aligned}$$

called an integral transform, in obvious analogy with analysis, where $p : X \times Y \to X$ and $q : X \times Y \to Y$ are the projections. Continuing the analogy, we call K the kernel of the integral transform. Kernels and integral transforms categorify the notion of correspondences between varieties and they naturally arise in moduli theory as universal objects and in birational geometry as objects on resolutions of rational maps.

Moreover, if we work with dg-enhancements, then thanks to a theorem of Töen [38] we know that integral transforms are all we need to study. Precisely, in his seminal work [38], Töen showed that

1. (**existence of internal Hom**) the localization of the category of dg-categories at quasi-equivalences admits an internal Hom, $\mathbf{R}\,\underline{\mathrm{Hom}}$, and

2. (**geometric recognition**) the subcategory of the Hom between the dg-enhancements of $\mathrm{D}(X)$ and $\mathrm{D}(Y)$ consisting of quasi-functors commuting with coproducts is isomorphic in $\mathrm{Ho}\,(\mathrm{dgcat})$ to the enhancement of the derived category of the product $X \times Y$,

$$\mathbf{R}\underline{\mathrm{Hom}}_c\left(\mathcal{D}(X), \mathcal{D}(Y)\right) \cong \mathcal{D}(X \times Y).$$

It is important to not confuse the two issues; knowing (1) in no way helps with establishing (2). However, if we know that the functor in which we happen to be interested admits a lift to a dg quasi-functor, then by (2) it must be an integral transform and it must be geometric.

Predating Töen's result, Orlov had showed that any equivalence $F : \mathrm{D}^b(\mathrm{coh}\, X) \to \mathrm{D}^b(\mathrm{coh}\, Y)$ between smooth and projective varieties is actually isomorphic to Φ_K for some kernel K [31]. Being revisionist, we can say that, for varieties, equivalences lift to quasi-equivalences at the differential graded level. That is to say, if there is an exact equivalence of triangulated categories of homotopy categories, then there is a quasi-equivalence of the dg-categories. Post-Töen, Lunts and Orlov exhibited this lifting in vastly more generality for exact equivalences between derived categories of abelian categories [26].

If we are to study derived categories of noncommutative projective schemes, we are then at the intersection of Artin-Zhang and Kontsevich style noncommutative geometry. The most basic issue is to know what the kernels and integral transforms are, and whether the pleasant results in the context of (commutative) algebraic geometry persist in this noncommutative setting.

We can ask, for example, if we have an analogue of Orlov's result on the geometric nature of equivalences. Combining Töen's and Lunts-Orlov's results, one immediately concludes that if X and Y are noncommutative projective schemes and one has an equivalence $F : \mathrm{D}(X) \to \mathrm{D}(Y)$, then in fact one has a quasi-equivalence $\mathcal{F} : \mathcal{D}(X) \to \mathcal{D}(Y)$.

This is great, however, we are still stuck staring at the abstract smoke. To reap the benefits, we need to know that $\mathcal{F}$ is (noncommutative) geometric. That is, we need the noncommutative projective analogue of Töen's geometric recognition. Existence of the internal Hom and/or uniqueness of enhancements provides no guidance towards geometric recognition in any context. As such we encounter the following basic questions:

Question. *For noncommutative projective schemes X and Y, what noncommutative projective scheme is $X \times Y$? Does geometric recognition hold for X and Y (and $X \times Y$)?*

An intermediate issue is to provide a definition of integral transform in noncommutative projective geometry, which is entirely separate from the differential graded structure. No such creature has been observed in the literature. Encouragingly, one notes that geometric recognition holds in other settings beyond schemes:

- for higher derived stacks (using machinery of Lurie in place of Töen) in [9]
- and for various versions of matrix factorizations [6, 16, 32].

However, a look at the simpler question of graded Morita theory shows that answers for noncommutative projective schemes are already more complicated [43]. For commutative graded rings, Morita equivalence is the same thing as isomorphism.

Fortunately, there is really only one noncommutative projective scheme that deserves to be called $X \times Y$, the Segre product of X and Y. However, the answer to geometric recognition is clearly only positive with some homological restrictions on X and Y. It does not hold in general. Let us now state a version of the main result of this article.

Let A and B be connected graded k-algebras. We say that A and B form a *delightful couple* if they are both Ext-finite in the sense of [40], both are left and right Noetherian, and both satisfy $\chi^\circ(R)$ for $R = A, A^{\mathrm{op}}$ for A and $R = B, B^{\mathrm{op}}$ for B [4]. One can think of this requirement as Serre vanishing for the twisting sheaves plus some finite-dimensionality over k. Let X and Y be the associated noncommutative projective schemes, which we also say form a delightful couple.

Theorem 1.1. *Let X and Y be noncommutative projective schemes associated to a delightful couple, A and B, over a field k. Assume that both A and B are both generated in degree one. Then geometric recognition holds for X and Y. That is, there exists a quasi-equivalence*

$$\mathbf{R}\underline{\mathrm{Hom}}_c(\mathcal{D}(X), \mathcal{D}(Y)) \cong \mathcal{D}(X \times Y).$$

For a general delightful couple, geometric recognition holds, however one must step slightly outside the realm of noncommutative projective schemes, without losing the (noncommutative) geometry, to get the correct product. See Theorem 4.15 for the precise statement of the general result.

As an immediate corollary to Theorem 1.1, we get the following statement which is a geometricity of equivalences statement along lines of Orlov or Rickard.

Theorem 1.2. *Let X and Y be noncommutative projective schemes associated to a delightful couple over a field k, both of which are generated in* *degree one. If there is an exact equivalence $F : \mathrm{D}(X) \to \mathrm{D}(Y)$, then there exists an object K of $\mathrm{D}(X \times Y)$ whose associated integral transform Φ_K is an equivalence. That is, X and Y are Fourier-Mukai partners.*

Recall that Φ_K is introduced in this paper. For this, see the statement of Theorem 4.15. Note that this statement makes no reference to dg-categories and by restricting to commutative projective varieties recovers the analogous result there.

1.1 Conventions

We let k denote a field. Often, for ease of notation, $\mathcal{C}(X, Y)$ will be used to refer to the morphisms, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, between objects X and Y of a category $\mathcal{C}$. We shall also use an undecorated Hom again depending on the complexity of the notation. Whenever $\mathcal{C}$ has a natural enrichment over a category, $\mathcal{V}$, we will denote by $\underline{\mathcal{C}}(X, Y)$ the $\mathcal{V}$-object of morphisms. For example, the category of complexes of k-vector spaces, $C(k)$, can be endowed with the structure of a $C(k)$-enriched category using the Hom total complex, $\mathcal{C}(k)(C, D) := \underline{C}(k)(C, D)$ which has in degree n the k-vector space

$$\mathcal{C}(k)(C, D)^n = \prod_{m \in \mathbb{Z}} \mathrm{Mod}\, k\left(C^m, D^{m+n}\right)$$

and differential

$$d(f) = d_D \circ f + (-1)^{n+1} f \circ d_C.$$

It should be noted that $Z^0(\mathcal{C}(k)(C, D)) = C(k)(C, D)$.

2 Background on DG-Categories

Recall that a **dg-category**, $\mathcal{A}$, over k is a category enriched over the category of cochain complexes, $C(k)$, a **dg-functor**, $F\colon \mathcal{A} \to \mathcal{B}$ is a $C(k)$-enriched functor, a **morphism of dg-functors of degree** n, $\eta\colon F \to G$, is a $C(k)$-enriched natural transformation such that $\eta(A) \in \mathcal{B}(FA, GA)^n$ for all objects A of $\mathcal{A}$, and a **morphism of dg-functors** is a degree zero, closed morphism of dg-functors. We will denote by dgcat_k the 2-category of small $C(k)$-enriched categories, and by $\underline{\mathrm{dgcat}}_k(\mathcal{A},\mathcal{B})$ the dg-category of dg-functors from $\mathcal{A}$ to $\mathcal{B}$.

Recall also that for $\mathcal{A}$ and $\mathcal{B}$ small dg categories, we may define a dg-category $\mathcal{A}\otimes\mathcal{B}$ with objects $\mathrm{ob}(\mathcal{A})\times\mathrm{ob}(\mathcal{B})$ and morphisms

$$(\mathcal{A}\otimes\mathcal{B})\left((X,Y),(X',Y')\right)=\mathcal{A}(X,X')\otimes_k\mathcal{B}(Y,Y').$$

It is well known that there is an isomorphism

$$\mathrm{dgcat}_k(\mathcal{A}\otimes\mathcal{B},\mathcal{C})\cong\mathrm{dgcat}_k(\mathcal{A},\underline{\mathrm{dgcat}}_k(\mathcal{B},\mathcal{C})),$$

endowing dgcat_k with the structure of a symmetric monoidal closed category.

For any dg-category, $\mathcal{A}$, we denote by $Z^0(\mathcal{A})$ the category with objects those of $\mathcal{A}$ and morphisms

$$Z^0(\mathcal{A})(A_1,A_2):=Z^0(\mathcal{A}(A_1,A_2)).$$

By $H^0(\mathcal{A})$ we denote the category with objects those of $\mathcal{A}$ and morphisms

$$H^0(\mathcal{A})(A_1,A_2):=H^0(\mathcal{A}(A_1,A_2)).$$

Following [15], we say that two objects A_1, A_2 of a dg-category, $\mathcal{A}$, are **dg-isomorphic** (respectively, **homotopy equivalent**) if there is a morphism $f\in Z^0(\mathcal{A})(A_1,A_2)$ such that f (respectively, the image of f in $H^0(\mathcal{A})(A_1,A_2)$) is an isomorphism. In such a case, we say that f is a **dg-isomorphism** (respectively, **homotopy equivalence**).

2.1 The homotopy category of DG-Categories

We collect here some basic results on the model structure for dgcat_k. For any dg-functor $F\colon\mathcal{A}\to\mathcal{B}$, we say that F is

(i) **quasi-fully faithful** if for any two objects A_1, A_2 of $\mathcal{A}$ the morphism

$$F(A_1,A_2)\colon\mathcal{A}(A_1,\mathcal{A}_2)\to\mathcal{B}(FA_1,FA_2)$$

is a quasi-isomorphism of chain complexes,

(ii) **quasi-essentially surjective** if the induced functor

$$H^0(F)\colon H^0(\mathcal{A}) \to H^0(\mathcal{B})$$

is essentially surjective,

(iii) a **quasi-equivalence** if F is quasi-fully faithful and quasi-essentially surjective,

The localization of dgcat_k at the class of quasi-equivalences is the homotopy category, $\mathrm{Ho}\,(\mathrm{dgcat}_k)$. We will denote by $[\mathcal{A},\mathcal{B}]$ the morphisms of $\mathrm{Ho}\,(\mathrm{dgcat}_k)$.

2.2 dg Modules

For any small dg-category, $\mathcal{A}$, denote by $\mathrm{dgMod}\,(\mathcal{A})$ the dg-category of dg-functors $\underline{\mathrm{dgcat}}_k\,(\mathcal{A}^{\mathrm{op}}, \mathcal{C}\,(k))$, where $\mathcal{C}\,(k)$ denotes the dg-category of chain complexes equipped with the internal Hom from its symmetric monoidal closed structure. The objects of $\mathrm{dgMod}\,(\mathcal{A})$ will be called **dg $\mathcal{A}$ modules**. Since one may view the dg $\mathcal{A}^{\mathrm{op}}$ modules as what should reasonably be called left dg $\mathcal{A}$ modules, the terms right and left will be dropped in favor of dg $\mathcal{A}$ modules and dg $\mathcal{A}^{\mathrm{op}}$ modules, respectively. We note here that the somewhat vexing choice of terminology is such that we can view objects of $\mathcal{A}$ as dg $\mathcal{A}$ modules by way of the enriched Yoneda embedding

$$Y_{\mathcal{A}}\colon \mathcal{A} \to \mathrm{dgMod}\,(\mathcal{A})\,.$$

As a special case, we define for any two small dg-categories, $\mathcal{A}$ and $\mathcal{B}$, the category of dg $\mathcal{A}$-$\mathcal{B}$-bimodules to be $\mathrm{dgMod}\,(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$. We note here that the symmetric monoidal closed structure on dgcat_k allows us to view bimodules as morphisms of dg-categories by the isomorphism

$$\begin{aligned}
\mathrm{dgMod}\,(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}) &= \underline{\mathrm{dgcat}}_k\,(\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}, \mathcal{C}\,(k)) \\
&\cong \underline{\mathrm{dgcat}}_k(\mathcal{A}, \underline{\mathrm{dgcat}}_k\,(\mathcal{B}^{\mathrm{op}}, \mathcal{C}\,(k))) \\
&= \underline{\mathrm{dgcat}}_k\,(\mathcal{A}, \mathrm{dgMod}\,(\mathcal{B}))\,.
\end{aligned}$$

The image of a dg $\mathcal{A}$-$\mathcal{B}$-bimodule, E, is the dg-functor $\Phi_E(A) = E(A, -)$.

2.2.1 h-Projective dgModules

We say that a dg $\mathcal{A}$ module, N, is **acyclic** if $N(A)$ is an acyclic chain complex for all objects A of $\mathcal{A}$. A dg $\mathcal{A}$ module M is said to be **h-projective** if

$$H^0(\mathrm{dgMod}\,(\mathcal{A}))(M,N) := H^0(\mathrm{dgMod}\,(A)\,(M,N)) = 0$$

for every acyclic dg $\mathcal{A}$ module, N. The full dg-subcategory of $\mathrm{dgMod}\,(\mathcal{A})$ consisting of h-projectives will be called $\text{h-Proj}\,(\mathcal{A})$.

We always have a special class of h-projectives given by the representables, $h_A = \mathcal{A}(-,A)$ for if M is acyclic, then from the enriched Yoneda Lemma we have

$$\begin{aligned} H^0(\mathrm{dgMod}\,(A))(h_A, M) &:= H^0(\mathrm{dgMod}\,(\mathcal{A})\,(h_A, M)) \\ &\cong H^0(M(A)) = 0. \end{aligned}$$

Noting that the Yoneda Lemma applied to $H^0(\mathrm{dgMod}\,(A))$ immediately implies $\text{h-Proj}\,(\mathcal{A})$ is closed under homotopy equivalence, we denote the full dg-subcategory of $\text{h-Proj}\,(\mathcal{A})$ consisting of the dg $\mathcal{A}$ modules **homotopy equivalent to representables** by $\overline{\mathcal{A}}$.

An h-projective dg $\mathcal{A}$-$\mathcal{B}$-bimodule, E, is **right quasi-representable** if for every object A of $\mathcal{A}$ the dg $\mathcal{B}$ module $\Phi_E(A)$ is an object of $\overline{\mathcal{B}}$, and we will denote the full subcategory of $\text{h-Proj}\,(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ consisting of all right quasi-representables by $\text{h-Proj}\,(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})^{\mathrm{rqr}}$.

The dual notion, *h-injective*, is defined by reversing all the relevant arrows.

2.2.2 The Derived Category of a DG-Category

By definition, a degree zero closed morphism

$$\eta \in Z^0(\mathrm{dgMod}\,(\mathcal{A}))(M,N)$$

satisfies

$$\eta(A) \in Z^0(\mathcal{C}\,(k)\,(M(A), N(A))) = C\,(k)\,(M(A), N(A))$$

for all objects A of $\mathcal{A}$. Hence we are justified in the following definitions:

(i) η is a **quasi-isomorphism** if $\eta(A)$ is a quasi-isomorphism of chain complexes for all objects A of $\mathcal{A}$, and

(ii) η is a **fibration** if $\eta(A)$ is a degree-wise surjective morphism of complexes for all objects A of $\mathcal{A}$.

Equipping $C\,(k)$ with the standard projective model structure (see [21, Section 2.3]), these definitions endow $Z^0(\mathrm{dgMod}\,(\mathcal{A}))$ with the structure of a particularly nice cofibrantly generated model category (see [38, Section 3]). In analogy with the definition of the derived category of modules for a ring A, the **derived category of** $\mathcal{A}$ is defined to be the homotopy category

$$\mathrm{D}(\mathcal{A}) = \mathrm{Ho}\left(Z^0(\mathrm{dgMod}\,(\mathcal{A}))\right) = Z^0(\mathrm{dgMod}\,(\mathcal{A}))[\mathcal{W}^{-1}]$$

that is obtained by localizing $Z^0(\mathrm{dgMod}\,(\mathcal{A}))$ at the class, $\mathcal{W}$, of quasi-isomorphisms.

It can be shown (see [23, Section 3.5]) that for every dg $\mathcal{A}$ module, M, there exists an h-projective, N, and a quasi-isomorphism $N \to M$, which one calls an **h-projective resolution of** M. Moreover, it is not difficult to see that any quasi-isomorphism between h-projective objects is in fact a homotopy equivalence. It follows that there is an equivalence of categories between $H^0(\text{h-Proj}\,(\mathcal{A}))$ and $\mathrm{D}(\mathcal{A})$ for any small dg-category, $\mathcal{A}$.

It should be noted that this generalizes the notion of derived categories of modules over a commutative ring. Indeed, for a commutative ring, A, one associates to A the ringoid, $\mathcal{A}$, with one object, $*$, and morphisms, $\mathcal{A}(*,*)$, the complex with A in degree zero. One identifies the chain complexes of A modules enriched by the Hom total complex with $\mathrm{dgMod}\,(\mathcal{A})$, which is simply the full dg-subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{C}\,(k))$ comprised of all dg-functors. From this viewpoint it is easy to recognize the categories $Z^0(\mathrm{dgMod}\,(\mathcal{A}))$, $H^0(\mathrm{dgMod}\,(\mathcal{A}))$, and $\mathrm{D}(\mathcal{A})$, as the categories $C\,(A)$, $K(A)$, the usual category up to homotopy, and the derived category of $\mathrm{Mod}\,A$, respectively. In the language of [26], we say that $\text{h-Proj}\,(\mathcal{A})$ is a dg-enhancement of $\mathrm{D}(\mathrm{Mod}\,A)$.

2.3 Tensor Products of dg Modules

Let M be a dg $\mathcal{A}$ module, let N be a dg $\mathcal{A}^{\mathrm{op}}$ module, and let A, B be objects of $\mathcal{A}$. For ease of notation, we drop the functor notation $M(A)$ in favor of M_A and write $\mathcal{A}_{A,B}$ for the morphisms $\mathcal{A}(A, B)$. We have structure morphisms

$$M_{A,B} \in \mathcal{C}\left(k\right)\left(\mathcal{A}_{A,B}, \mathcal{C}\left(k\right)\left(M_B, M_A\right)\right) \cong \mathcal{C}\left(k\right)\left(M_B \otimes_k \mathcal{A}_{A,B}, M_A\right)$$

and

$$N_{A,B} \in \mathcal{C}\left(k\right)\left(\mathcal{A}_{A,B}, \mathcal{C}\left(k\right)\left(N_A, N_B\right)\right) \cong \mathcal{C}\left(k\right)\left(\mathcal{A}_{A,B} \otimes_k N_A, N_B\right),$$

which give rise to a unique morphism

$$M_B \otimes_k A_{A,B} \otimes_k N_A \to M_A \otimes_k N_A \oplus M_B \otimes_k N_B$$

induced by the universal properties of the biproduct. The two collections of morphisms given by projecting onto each factor induce morphisms

$$\Xi_1, \Xi_2 \colon \bigoplus_{A,B \in \mathrm{Ob}(\mathcal{A})} M_B \otimes_k \mathcal{A}_{A,B} \otimes_k N_A \to \bigoplus_{\mathcal{C} \in \mathrm{Ob}(\mathcal{A})} M_C \otimes_k N_C,$$

and we define the tensor product of M and N to be the coequalizer in $C\left(k\right)$

$$\bigoplus_{(i,j)\in\mathbb{Z}^2} M_j \otimes_k \mathcal{A}_{A,B} \otimes_k N_A \underset{\Xi_2}{\overset{\Xi_1}{\rightrightarrows}} \bigoplus_{\ell\in\mathbb{Z}} M_\ell \otimes_k N_\ell \longrightarrow M \otimes_{\mathcal{A}} N \ .$$

It is routine to check that a morphism $M \to M'$ of right dg $\mathcal{A}$ modules induces by the universal property for coequalizers a unique morphism

$$M \otimes_{\mathcal{A}} N \to M' \otimes_{\mathcal{A}} N$$

yielding a functor

$$- \otimes_{\mathcal{A}} N \colon \ \mathrm{dgMod}\left(\mathcal{A}\right) \to \mathcal{C}\left(k\right).$$

One extends this construction to bimodules as follows. Given objects E of $\mathrm{dgMod}\left(\mathcal{A} \otimes \mathcal{B}\right)$ and F of $\mathrm{dgMod}\left(\mathcal{B}^{\mathrm{op}} \otimes \mathcal{C}\right)$, we recall that we have associated to each a dg-functor

$$\Phi_E \colon \mathcal{A}^{\mathrm{op}} \to \mathrm{dgMod}\left(\mathcal{B}\right) \quad \text{and} \quad \Phi_F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{dgMod}\left(\mathcal{B}^{\mathrm{op}}\right)$$

by the symmetric monoidal closed structure on dgcat_k. For each pair of objects A of $\mathcal{A}$ and C of $\mathcal{C}$, we obtain dgModules

$$\Phi_E(A) = E(A, -) \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{C}\,(k) \text{ and } \Phi_F(C) = F(-, C) \colon \mathcal{B} \to \mathcal{C}\,(k)$$

and hence one may define the object $E \otimes_{\mathcal{B}} F$ of $\mathrm{dgMod}\,(\mathcal{A} \otimes \mathcal{C})$ by

$$(E \otimes_{\mathcal{B}} F)\,(A, C) = \Phi_E(A) \otimes_{\mathcal{B}} \Phi_F(C).$$

One can show that by a similar argument to the original that a morphism $E \to E'$ of $\mathrm{dgMod}\,(\mathcal{A} \otimes \mathcal{B})$ induces a morphism $E \otimes_{\mathcal{B}} F \to E' \otimes_{\mathcal{B}} F$ of $\mathrm{dgMod}\,(\mathcal{A} \otimes \mathcal{C})$, and a morphism $F \to F'$ of $\mathrm{dgMod}\,(\mathcal{B}^{\mathrm{op}} \otimes \mathcal{C})$ induces a morphism $E \otimes_{\mathcal{B}} F \to E \otimes_{\mathcal{B}} F'$ of $\mathrm{dgMod}\,(\mathcal{A} \otimes \mathcal{C})$.

Remark 2.1. Denote by $\mathcal{K}$ the dg-category with one object, $*$, and morphisms given by the chain complex

$$\mathcal{K}(*, *)^n = \begin{cases} k & n = 0 \\ 0 & n \neq 0 \end{cases}$$

with zero differential. This category serves as the unit of the symmetric monoidal structure on dgcat_k, so for small dg-categories, $\mathcal{A}$ and $\mathcal{C}$, we can always identify $\mathcal{A}$ with $\mathcal{A} \otimes \mathcal{K}$ and $\mathcal{C}$ with $\mathcal{K}^{\mathrm{op}} \otimes \mathcal{C}$. With this identification in hand, we obtain from taking $\mathcal{B} = \mathcal{K}$ in the latter construction a special case: Given a dg $\mathcal{A}^{\mathrm{op}}$ module, E, and a dg $\mathcal{C}$ module, F, we have a dg $\mathcal{A}$-$\mathcal{C}$-bimodule defined by the tensor product

$$(E \otimes F)\,(A, C) := (E \otimes_{\mathcal{K}} F)\,(A, C) = E(A) \otimes_k F(C).$$

2.4 Extensions of Morphisms Associated to Bimodules

Let E be a dg $\mathcal{A}$-$\mathcal{B}$-bimodule. Following [15, Section 3], we can extend the associated functor Φ_E to a dg-functor

$$\widehat{\Phi_E} \colon \mathrm{dgMod}\,(\mathcal{A}) \to \mathrm{dgMod}\,(\mathcal{B})$$

defined by $\widehat{\Phi_E}(M) = M \otimes_{\mathcal{A}} E$. Similarly, we have a dg-functor in the opposite direction

$$\widetilde{\Phi_E} \colon \mathrm{dgMod}\,(\mathcal{B}) \to \mathrm{dgMod}\,(\mathcal{A})$$

defined by $\widetilde{\Phi_E}(N) = \mathrm{dgMod}\,(\mathcal{B})\,(\Phi_E(-), N)$.

For any dg-functor $G\colon \mathcal{A} \to \mathcal{B}$ we denote by Ind_G the extension of the dg-functor

$$A \to \mathcal{B} \overset{Y_{\mathcal{B}}}{\to} \mathrm{dgMod}\,(\mathcal{B})$$

and its right adjoint by Res_G. By way of the enriched Yoneda Lemma we see that for any object A of $\mathcal{A}$ and any dg $\mathcal{B}$ module, N,

$$\mathrm{Res}_G(N)(A) = \mathrm{dgMod}\,(\mathcal{B})\,(h_{GA}, N) \cong N(GA).$$

We record here some useful propositions regarding extensions of dg-functors.

Proposition 2.2 ([15, Prop 3.2])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be small dg-categories. Let $F\colon \mathcal{A} \to \mathrm{dgMod}\,(\mathcal{B})$ and $G\colon \mathcal{A} \to \mathcal{B}$ be dg-functors.*

(i) *$\widehat{F}$ is left adjoint to $\widetilde{F}$,*

(ii) *$\widehat{F} \circ Y_{\mathcal{A}}$ is dg-isomorphic to F and $H^0(\widehat{F})$ is continuous,*

(iii) *$\widehat{F}(\text{h-Proj}\,(\mathcal{A})) \subseteq \text{h-Proj}\,(\mathcal{B})$ if and only if $F(A) \subseteq \text{h-Proj}\,(B)$,*

(iv) *$\mathrm{Res}_G(\text{h-Proj}\,(\mathcal{B})) \subseteq \text{h-Proj}\,(\mathcal{A})$ if and only if $\mathrm{Res}_G(\bar{\mathcal{B}}) \subseteq \text{h-Proj}\,(\mathcal{A})$; moreover, $H^0(\mathrm{Res}_G)$ is always continuous,*

(v) *$\mathrm{Ind}_G\colon\ \text{h-Proj}\,(\mathcal{A}) \to \text{h-Proj}\,(\mathcal{B})$ is a quasi-equivalence if G is a quasi-equivalence.*

Remark 2.3. 1. We note that for dg $\mathcal{A}$- and $\mathcal{A}^{\mathrm{op}}$ modules, M and N, part (i) implies that the dg-functors

$$- \otimes_{\mathcal{A}} N\colon\ \mathrm{dgMod}\,(\mathcal{A}) \to \mathcal{C}\,(k)$$

and

$$M \otimes_{\mathcal{A}} -\colon\ \mathrm{dgMod}\,(\mathcal{A}^{\mathrm{op}}) \to \mathcal{C}\,(k)$$

have right adjoints

$$\widetilde{N}(C) = \mathcal{C}\,(k)\,(N(-), C) \text{ and } \widetilde{M}(C) = \mathcal{C}\,(k)\,(M(-), C),$$

respectively. As an immediate consequence of the enriched Yoneda Lemma

$$h_A \otimes_{\mathcal{A}} N \cong N(A) \text{ and } M \otimes_{\mathcal{A}} h^A \cong M(A)$$

holds for any object A of $\mathcal{A}$.

2. Let $\Delta_{\mathcal{A}}$ denote the dg $\mathcal{A}$-$\mathcal{A}$-bimodule corresponding to the Yoneda embedding, $Y_{\mathcal{A}}$, under the isomorphism

$$\mathrm{dgMod}\left(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}\right) \cong \underline{\mathrm{dgcat}}_k \left(\mathcal{A}, \mathrm{dgMod}\left(\mathcal{A}\right)\right).$$

It's clear that we have a dg-functor

$$\Delta_{\mathcal{A}} \otimes_{\mathcal{A}} -\colon\ \mathrm{dgMod}\left(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}\right) \to \mathrm{dgMod}\left(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}\right)$$

and for any dg $\mathcal{A}$-$\mathcal{A}$-bimodule, E, we see that

$$(\Delta_{\mathcal{A}} \otimes_{\mathcal{A}} E)(A, A') = h_A \otimes_{\mathcal{A}} E(-, A') \cong E(A, A')$$

implies that $\Delta_{\mathcal{A}} \otimes_{\mathcal{A}} E \cong E$.

When starting with an h-projective we have a very nice extension of dg-functors:

Proposition 2.4 ([15, Lemma 3.4])**.** *If E is any h-projective dg $\mathcal{A}$-$\mathcal{B}$-bimodule, then the associated functor*

$$\Phi_E\colon \mathcal{A} \to \mathrm{dgMod}\left(\mathcal{B}\right)$$

factors through $\mathrm{h\text{-}Proj}\left(\mathcal{B}\right)$.

As a direct consequence of the penultimate proposition, this means that we can view the extension of Φ_E as a dg-functor

$$\widehat{\Phi_E} = - \otimes_{\mathcal{A}} E\colon\ \mathrm{h\text{-}Proj}\left(A\right) \to \mathrm{h\text{-}Proj}\left(B\right).$$

Put another way, tensoring with an h-projective $\mathcal{A}$-$\mathcal{B}$-bimodule preserves h-projectives.

One essential result about dgcat_k comes from Töen's result on the existence, and description of, the internal Hom in its homotopy category.

Theorem 2.5 ([38, Theorem 1.1], [15, 4.1])**.** *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be objects of* dgcat_k*. There exists a natural bijection*

$$[\mathcal{A},\mathcal{C}] \overset{1:1}{\longleftrightarrow} \mathrm{Iso}\left(H^0(\text{h-Proj}\,(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{C})^{\mathrm{rqr}})\right)$$

Moreover, the dg-category $\mathbf{R}\underline{\mathrm{Hom}}\,(\mathcal{B},\mathcal{C}) := \text{h-Proj}\,(\mathcal{B}^{\mathrm{op}}\otimes\mathcal{C})^{\mathrm{rqr}}$ *yields a natural bijection*

$$[\mathcal{A}\otimes\mathcal{B},\mathcal{C}] \overset{1:1}{\longleftrightarrow} [\mathcal{A},\mathbf{R}\underline{\mathrm{Hom}}\,(\mathcal{B},\mathcal{C})]$$

proving that the symmetric monoidal category $\mathrm{Ho}\,(\mathrm{dgcat}_k)$ *is closed.*

Corollary 2.6 ([38, 7.2], [15, Cor. 4.2])**.** *Given two dg categories $\mathcal{A}$ and $\mathcal{B}$,* $\mathbf{R}\underline{\mathrm{Hom}}\,(\mathcal{A},\text{h-Proj}\,(\mathcal{B}))$ *and* $\text{h-Proj}\,(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B})$ *are isomorphic in* $\mathrm{Ho}\,(\mathrm{dgcat}_k)$*. Moreover, there exists a quasi-equivalence*

$$\mathbf{R}\underline{\mathrm{Hom}}_c\,(\text{h-Proj}\,(\mathcal{A})\,,\text{h-Proj}\,(\mathcal{B})) \to \mathbf{R}\underline{\mathrm{Hom}}\,(\mathcal{A},\text{h-Proj}\,(\mathcal{B}))\,.$$

To get a sense of the value of this result, let us recall one application from [38, Section 8.3]. Let X and Y be quasi-compact and separated schemes over $\mathrm{Spec}\,k$. Recall the dg model for $\mathrm{D}(\mathrm{Qcoh}\,X)$, $\mathcal{L}_{\mathrm{qcoh}}(X)$, is the $\mathcal{C}\,(k)$-enriched subcategory of fibrant-cofibrant objects in the injective model structure on $C\,(\mathrm{Qcoh}\,X)$.

Theorem 2.7 ([38, Theorem 8.3])**.** *Let X and Y be quasi-compact, quasi-separated schemes over k. There exists an isomorphism in* $\mathrm{Ho}\,(\mathrm{dgcat}_k)$

$$\mathbf{R}\underline{\mathrm{Hom}}_c\,(\mathcal{L}_{\mathrm{qcoh}}X,\mathcal{L}_{\mathrm{qcoh}}Y) \cong \mathcal{L}_{\mathrm{qcoh}}(X\times_k Y)$$

which takes a complex $E\in\mathcal{L}_{\mathrm{qcoh}}(X\times_k Y)$ to the exact functor on the homotopy categories

$$\begin{aligned}\Phi_E : \mathrm{D}(\mathrm{Qcoh}\,X) &\to \mathrm{D}(\mathrm{Qcoh}\,Y)\\ M &\mapsto \mathbf{R}\pi_{2*}\left(E\overset{\mathbf{L}}{\otimes}\mathbf{L}\pi_1^*M\right)\end{aligned}$$

Proof. The first part of the statement is exactly as in [38]. The second part is implicit. □

3 Details on noncommutative projective schemes

3.1 Recollections and conditions

Noncommutative projective schemes were introduced by Artin and Zhang in [4]. We recall the definition.

Definition 3.1. Let N be a finitely-generated abelian group. We say that a k-algebra A is N-graded if there exists a decomposition as k modules

$$A = \bigoplus_{n \in N} A_n$$

with $A_n A_m \subset A_{n+m}$. One says that A is **connected graded** if it is $\mathbb{Z}$-graded with $A_0 = k$ and $A_n = 0$ for $n < 0$.

For algebraic geometers, the most common example is the homogeneous coordinate ring of a projective scheme. These are of course commutative. One has a plenitude of noncommutative examples.

Example 3.2. Let us take $k = \mathbf{C}$ and consider the following quotient of the free algebra

$$A_q := \mathbf{C}\langle x_0, \ldots, x_n \rangle / (x_i x_j - q_{ij} x_j x_i)$$

for $q_{ij} \in \mathbf{C}^\times$ with $q_{ij} = q_{ji}^{-1}$. These give noncommutative deformations of $\mathbf{P}^n$.

Example 3.3. Building off of Example 3.2, we recall the following class of noncommutative algebras of Kanazawa [22]. Pick $\phi \in \mathbf{C}$ and q_{ij} according to [22, Theorem 2.1] with $\prod_{i=1}^n q_{ij} = 1$ for all j. And set

$$A_q^\phi := A_q / \left(\sum_{i=0}^n x_i^{n+1} - \phi(n+1)(x_0 \cdots x_n) \right).$$

This is the noncommutative version of the homogeneous coordinate rings of the Hesse (or Dwork) pencil of Calabi-Yau hypersurfaces in $\mathbf{P}^n$.

Definition 3.4. Let M be a graded A module. We say that M has **right limited grading** if there exists a d such $M_{d'} = 0$ for $d' \geq d$. We define **left limited grading** analogously.

In general, good behavior requires some homological assumptions on the ring A. We recall two common such ones.

Definition 3.5. Let A be a connected graded k-algebra. Following Van den Bergh [40], we say that A is **Ext-finite** if for each $n \geq 0$ the ungraded Ext-groups are finite dimensional

$$\dim_k \operatorname{Ext}^n_A(k, k) < \infty.$$

Remark 3.6. The Ext's are taken in the category of left A modules, a priori.

Definition 3.7. Following Artin and Zhang [4], given a graded left module M, we say A satisfies $\chi^\circ(M)$ if $\underline{\operatorname{Ext}}^n_A(k, M)$ has right limited grading for each $n \geq 0$.

We recall some basic results on Ext-finiteness, essentially from [40, Section 4].

Proposition 3.8. *Assume that A and B are Ext-finite. Then*

1. *the ring $A \otimes_k B$ is Ext-finite.*
2. *the ring A^{op} is Ext-finite.*

Furthermore, if A is Ext-finite then A is finitely presented as a k-algebra.

Proof. See [40, Lemma 4.2] and the discussion preceding it. For the final statement, see the opening paragraphs of [12, Section 4.1]. □

For a connected graded k-algebra, A, one has the two-sided ideal

$$A_{\geq \ell} := \bigoplus_{n \geq \ell} A_n.$$

Definition 3.9. Let A be a finitely generated connected graded algebra. Recall that an element, m, of a module, M, is **torsion** if there is an n such that

$$A_{\geq n} m = 0.$$

We let τ denote the functor that takes a module, M, to its torsion submodule. The module M is torsion if $\tau M = M$.

The functor τ is right adjoint to the inclusion functor and we denote the counit $\eta \colon \tau \to 1_{\operatorname{Gr} A}$.

The following is likely well-known (under the assumption of a Noetherian ring the conclusion is contained in [4]). However the authors were unable to locate a convenient reference under the assumption of finitely generation, so we include the following

Proposition 3.10. *Let A be a connected graded k-algebra. If A is generated as a k-algebra by a finite set of elements of positive degree, then for each $\ell \in \mathbb{Z}$ the tails $A_{\geq \ell}$ are finitely generated A modules.*

Proof. Let $S = \{x_i\}_{i=1}^{g}$ be a set of generators for A as a k-algebra and let $d_i = \deg(x_i)$. By possibly relabeling, we may assume that $1 \leq d_1 \leq d_2 \leq \ldots \leq d_g$.

Fix ℓ. First we show that $A_{\geq \ell}$ is generated by the sum

$$\bigoplus_{r=\ell}^{\ell+d_g-1} A_r.$$

Take $a \in A_n$ with $\ell \leq n$. We induct on n. If $n \leq \ell + d_g - 1$, then a is generated by

$$\bigoplus_{r=\ell}^{\ell+d_g-1} A_r.$$

If $n > \ell + d_g - 1$, then we can write

$$a = \sum a_i b_i$$

with $n > \deg(a_i), \deg(b_i) > 0$. So a is generated by

$$\bigoplus_{r=n-d_g}^{n-1} A_r$$

By the induction hypothesis, we can generate any element of A_r for $n - d_g \leq r \leq n-1$ using

$$\bigoplus_{r=\ell}^{\ell+d_g-1} A_r.$$

It suffices to show that

$$\bigoplus_{r=\ell}^{\ell+d_g-1} A_r$$

is a finite dimensional k-vector space. Hence it is enough to show that A_r is a finite dimensional k-vector space for each r. Consider the free algebra $k\langle S\rangle$ as a graded algebra. By assumption there is a surjection

$$k\langle S\rangle \to A$$

so it suffices to show that there are only finitely many words of degree r in $k\langle S\rangle$.

Consider a word w of length N. We can write

$$w = \alpha X_1 \cdots X_n$$

where $X_i \in S$ and $\alpha \in k$, so

$$Nd_1 \leq \deg(w) \leq Nd_g.$$

If $r/d_1 < N$, then $w \notin k\langle S\rangle_r$. This implies

$$\{w \mid \deg(w) = r\} \subseteq \{w \mid 0 \leq \operatorname{len}(w) \leq r/d_1\}.$$

The latter has only finitely many elements since S has only finitely many elements. Hence $k\langle S\rangle_r$ is a finite dimensional vector space.

$\square$

Proposition 3.11. *Let A be a connected graded k-algebra. Denote by* $\operatorname{Tors} A$*, the full subcategory of* $\operatorname{Gr} A$ *consisting of all torsion modules. If A is finitely generated in positive degree, then* $\operatorname{Tors} A$ *is a Serre subcategory.*

Proof. Consider a short exact sequence

$$0 \to M' \to M \xrightarrow{p} M'' \to 0.$$

It's clear that if M is an object of $\operatorname{Tors} A$, then so are M' and M''. Hence it suffices to show that if M' and M'' are both objects of $\operatorname{Tors} A$, then so is M.

Fix an element $m \in M$. Since M'' is an object of $\operatorname{Tors} A$, there exists some n such that $A_{\geq n}p(m) = 0$ and hence $A_{\geq n}m \in M'$. By Proposition 3.10 the latter is finitely generated and so we can choose generators $m'_1, \dots, m'_t$ and integers $n_1, \dots, n_t$ such that $A_{\geq n_i} m'_i = 0$ for $i = 1, \dots, t$. Take $n_m = \max_{i=1,\dots,t}\{n + n_i\}$ so that $A_{\geq n_m} m = 0$, as desired. □

As such, we can form the quotient.

Definition 3.12. Let A be connected graded and finitely generated as a k-algebra. Then denote the quotient of the category of graded A modules by the subcategory of torsion modules as

$$\operatorname{QGr} A := \operatorname{Gr} A / \operatorname{Tors} A$$

Let

$$\pi : \operatorname{Gr} A \to \operatorname{QGr} A$$

denote the quotient functor. By Proposition 3.11 and [17, Cor. 1, III.3], π admits a fully faithful right adjoint which we denote by

$$\omega : \operatorname{QGr} A \to \operatorname{Gr} A.$$

Finally, we denote the composition $Q := \omega\pi$ and the unit of adjunction $\varepsilon \colon 1_{\operatorname{Gr} A} \to Q$.

The category $\operatorname{QGr} A$ is defined to be the quasi-coherent sheaves on the **noncommutative projective scheme** X.

Remark 3.13. Note that, traditionally speaking, X is not a space, in general. In the case A is commutative and finitely-generated by elements of degree 1, then a famous result of Serre says that there is an equivalence between $\mathrm{Qcoh}\, X$ and $\mathrm{QGr}\, A$, so that X is effectively $\mathrm{Proj}\, A$.

One can give more explicit descriptions of Q and τ.

Proposition 3.14. *Let A be a connected graded k-algebra and let M be a graded A module. If A is finitely generated, then*

$$\begin{aligned} \tau M &= \mathrm{colim}_n\, \underline{\mathrm{Gr}}A(A/A_{\geq n}, M) \\ QM &= \mathrm{colim}_n\, \underline{\mathrm{Gr}}A(A_{\geq n}, M). \end{aligned}$$

and $\varepsilon(M)(m) \in Q(M)$ is the class of the morphism

$$\begin{aligned} \varphi_m \colon A &\longrightarrow M \\ a &\longmapsto a \cdot m \end{aligned}$$

Proof. This is standard localization theory, see [35]. $\square$

3.2 Noncommutative Bi-projective Schemes

In studying questions of kernels and bimodules, we will have to move outside the realm of $\mathbb{Z}$-gradings. While one can generally treat N-graded k-algebras in our analysis, we limit the scope a bit and only consider $\mathbb{Z}^2$-gradings of the following form.

Definition 3.15. Let A and B be connected graded k-algebras. The tensor product $A \otimes_k B$ will be equipped with its natural bi-grading

$$(A \otimes_k B)_{n_1,n_2} = A_{n_1} \otimes_k B_{n_2}.$$

A **bi-bi module** for the pair (A, B) is a $\mathbb{Z}^2$-graded $A \otimes_k B$ module.

From bi-bi modules, we can produce A or B modules by taking slices of the gradings. For fixed $v \in \mathbb{Z}$ we have a functor

$$\begin{aligned} (-)_{*,v} \colon \mathrm{Gr}(A \otimes_k B) &\longrightarrow \mathrm{Gr}\, A \\ P &\longmapsto \textstyle\bigoplus_{m \in \mathbb{Z}} P_{m,v} \end{aligned}$$

and for fixed $u \in \mathbb{Z}$ a functor

$$(-)_{u,*} \colon \operatorname{Gr}(A \otimes_k B) \longrightarrow \operatorname{Gr} B$$

$$P \longmapsto \bigoplus_{n \in \mathbb{Z}} P_{u,n}$$

In the case that $A = B$, there is a particular bi-bi module of interest.

Definition 3.16. For A a finitely generated, connected graded k-algebra, we define Δ_A to be the A-A bi-bi module with

$$(\Delta_A)_{i,j} = A_{i+j}$$

and the natural left and right A actions. If the context is clear, we will often simply write Δ.

Notice that the forgetful functor $U_A \colon \operatorname{Gr}(A \otimes_k B) \to \operatorname{Gr} A$ sends a bi-bi module P to the $\mathbb{Z}$-graded sum

$$U_A(P) = \bigoplus_{u \in \mathbb{Z}} P_{u,*}$$

and similarly the forgetful functor $U_B \colon \operatorname{Gr}(A \otimes_k B) \to \operatorname{Gr} B$. Allowing for some repetition of notation we define for any bi-bi module

$$Q_A(P) := Q_A \circ U_A(P) \text{ and } Q_B(P) := Q_B \circ U_B(P)$$

and also

$$\tau_A(P) := \tau_A \circ U_A(P) \text{ and } \tau_B(P) := \tau_B \circ U_B(P).$$

In general, these will no longer be bi-bi modules, although under mild assumptions we can guarantee they will.

If A is finitely generated as a k-algebra, then we have a functor

$$Q'_A \colon \operatorname{Gr}(A \otimes_k B) \to \operatorname{Gr}(A \otimes_k B)$$

that takes a bi-bi module P to the bi-bi module $\bigoplus_{v \in \mathbb{Z}} Q_A(P_{*,v})$, where the $\mathbb{Z}^2$ grading is given by

$$Q'_A(P)_{u,v} := Q_A(P_{*,v})_u.$$

The two constructions are naturally isomorphic.

Lemma 3.17. *Assume A is finitely generated as a k-algebra. Let P be a bi-bi module. The natural map $Q'_A(P) \to Q_A(P)$ is an isomorphism and thus $Q_A(P)$ is also a bi-bi module. Similarly, $\tau_A(P)$ is also a bi-bi module.*

Symmetrically, if B is finitely generated as a k-algebra, then the natural map $Q'_B(P) \to Q_B(P)$ is an isomorphism and thus $Q_B(P)$ is also a bi-bi module. Similarly, $\tau_B(P)$ is also a bi-bi module.

Proof. By the universal property for coproducts we have a morphism

$$Q'_A(P) = \bigoplus_{v\in\mathbb{Z}} Q_A(P_{*,v}) \to Q_A\left(\bigoplus_{v\in\mathbb{Z}} P_{*,v}\right) = Q_A(P)$$

so it suffices to check that Q_A commutes with coproducts. Since coproducts commute with colimits it suffices to check that the natural morphism

$$\bigoplus_{v\in\mathbb{Z}} \underline{\mathrm{Gr}} A\left(A_{\geq n}, P_{*,v}\right) \to \underline{\mathrm{Gr}} A\left(A_{\geq n}, \bigoplus_{v\in\mathbb{Z}} P_{*,v}\right)$$

is an isomorphism. This holds provided $A_{\geq n}$ is finitely generated as a module, but this follows from the assumption that A is finitely generated as an algebra. This is Proposition 3.10. A similar argument works for τ_A. □

There are a couple notions of torsion for a bi-bi module that one can dream up. We use the following.

Definition 3.18. Let A and B be finitely generated, connected graded k-algebras, and let M be a bi-bi A-B module. We say that M is **torsion** if it lies in the smallest Serre subcategory containing A-torsion bi-bi modules and B-torsion bi-bi modules.

Lemma 3.19. *Let A and B be finitely generated, connected graded k-algebras. A bi-bi module P is torsion if and only if there exists a pair of integers, n_1, n_2, such that*

$$(A \otimes B)_{\geq n_1, \geq n_2} p = 0 \tag{1}$$

for all $p \in P$.

Furthermore

$$\tau_{A\otimes_k B}(P) = \mathrm{colim}_{n_1,n_2} \underline{\mathrm{Gr}}(A \otimes_k B)\left((A \otimes_k B) / (A_{\geq n_1} \otimes_k B_{\geq n_2}), P\right)$$

Proof. For ease of notation, denote by $\mathcal{T}$ the full subcategory of bi-bi modules satisfying Equation 1. Assume that the bi-bi modules P' and P'' both belong to $\mathcal{T}$ and that we have a short exact sequence of bi-bi modules

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0.$$

Since P'' belongs to $\mathcal{T}$, we may choose for any $p \in P$ integers ℓ_1 and ℓ_2 such that

$$(A \otimes_k B)_{\geq \ell_1, \geq \ell_2} p \subseteq P',$$

which is finitely generated by Proposition 3.10 by, say, $p_1, p_2, \ldots, p_k$. As elements of P', there exists for each i integers n_1^i and n_2^i such that

$$(A \otimes_k B)_{\geq n_1^i, \geq n_2^i} p_i = 0.$$

Taking $n_1 = \max_i \{n_1^i\}$ and $n_2 = \max_i \{n_2^i\}$ implies that P belongs to $\mathcal{T}$, and thus $\mathcal{T}$ is a Serre subcategory.

If P is either A-torsion, in which case $(A \otimes B)_{\geq n, \geq 0} p = 0$ for some n, or B-torsion, in which case $(A \otimes B)_{\geq 0, \geq n} p = 0$ for some n, then P belongs to $\mathcal{T}$. By minimality, $\mathcal{T}$ necessarily contains $\operatorname{Tors}(A \otimes_k B)$ and so it suffices to show that if

$$(A \otimes B)_{\geq n_1, \geq n_2} p = 0, \forall p \in P$$

then P lies in $\operatorname{Tors}(A \otimes_k B)$. Denote by $\tau_B P$ the B-torsion submodule of P and observe from the short exact sequence of bi-bi modules

$$0 \longrightarrow \tau_B P \longrightarrow P \longrightarrow P/\tau_B P \longrightarrow 0.$$

that it suffices to show $P/\tau_B P$ belongs to $\operatorname{Tors}(A \otimes_k B)$. For every $p \in P$, there is some n_1 such that $A_{\geq n_1} p$ is B-torsion. Hence, $A_{\geq n_1} \overline{p} = 0$ and $\overline{p}$ is A-torsion. Consequently, $P/\tau_B P$ is itself a torsion A module and therefore belongs to $\operatorname{Tors}(A \otimes_k B)$, as desired.

The final statement now follows immediately. □

We can compare our notion, $\tau_{A \otimes_k B}$, of torsion to the torsion of [40]

$$\tau^{\mathrm{VdB}}_{A \otimes_k B}(P) = \operatorname{colim}_{m,n} \underline{\operatorname{Gr}}(A \otimes_k B) \left(A/A_{\geq m} \otimes_k B/B_{\geq n}, P\right).$$

Lemma 3.20. *There exists a natural inclusion*

$$\tau^{\mathrm{VdB}}_{A\otimes_k B}(P) \stackrel{\nu_P}{\to} \tau_{A\otimes_k B}(P).$$

Proof. The surjections $A \to A/A_{\geq m}$ and $B \to B/B_{\geq n}$ induce morphisms

$$A \otimes_k B/A_{\geq m} \otimes_k B_{\geq n} \to A/A_{\geq m} \otimes_k B/B_{\geq n}$$

which in turn induce ν_P. □

Remark 3.21. In general, this inclusion is strict. For example, take $P = A/A_{\geq m} \otimes_k N$. One has

$$\tau_{A\otimes_k B}\left(A/A_{\geq m} \otimes_k N\right) = A/A_{\geq m} \otimes_k N$$

while

$$\tau^{\mathrm{VdB}}_{A\otimes_k B}\left(A/A_{\geq m} \otimes_k N\right) = A/A_{\geq m} \otimes_k \tau_B N.$$

One can form the quotient category

$$\mathrm{QGr}(A \otimes_k B) := \mathrm{Gr}(A \otimes_k B)/\operatorname{Tors}(A \otimes_k B).$$

Lemma 3.22. *The quotient functor*

$$\pi : \mathrm{Gr}(A \otimes_k B) \to \mathrm{QGr}(A \otimes_k B)$$

has a fully faithful right adjoint

$$\omega : \mathrm{QGr}(A \otimes_k B) \to \mathrm{Gr}(A \otimes_k B)$$

with

$$QM := \omega\pi M = \operatorname{colim}_{n_1,n_2} \underline{\mathrm{Gr}}(A \otimes_k B)(A_{\geq n_1} \otimes_k B_{\geq n_2}, M)$$

Proof. This is just an application of [17, Cor. 1, III.3]. □

For a given bi-bi module, a natural question one might ask is how Q_A, Q_B, and $Q_{A\otimes_k B}$ relate, as well as how τ_A, τ_B, and $\tau_{A\otimes_k B}$ relate.

Lemma 3.23. *Let A and B be finitely generated, connected graded k-algebras. For any complex of bi-bi modules, P, there exist natural morphisms of complexes*

$$\alpha_P^\ell : Q_A P \to Q_{A\otimes_k B} P$$
$$\alpha_P^r : Q_B P \to Q_{A\otimes_k B} P.$$

Moreover, the diagram

$$\begin{tikzcd}
P \arrow[r, "1_P"] \arrow[d, "\varepsilon_A(P)"] & P \arrow[d, "\varepsilon_{A\otimes_k B}(P)"] & P \arrow[l, "1_P"'] \arrow[d, "\varepsilon_B(P)"] \\
Q_A P \arrow[r, "\alpha_P^\ell"] & Q_{A\otimes_k B} P & Q_B P \arrow[l, "\alpha_P^r"']
\end{tikzcd}$$

commutes

Proof. We handle the case of α^ℓ and note that α^r follows from the same argument, mutatis mutandis.

First observe that it suffices to prove the result for P simply a bi-bi module and apply the result in each degree, for then all faces of the cube

$$\begin{tikzcd}
P^n \arrow[rr, "1_P^n"] \arrow[dr, "d^n"] \arrow[dd, "\varepsilon_A^n(P)"] & & P^n \arrow[dr, "d_P^n"] \arrow[dd, dashed, "\varepsilon_A^n(P)"' near start] & \\
& P^{n+1} \arrow[rr, "1_P^{n+1}" near start] \arrow[dd, "\varepsilon_A^{n+1}(P)" near start] & & P^{n+1} \arrow[dd, "\varepsilon_{A\otimes_k B}^{n+1}(P)"] \\
Q_A(P^n) \arrow[rr, dashed, "\alpha_{P^n}^\ell" near start] \arrow[dr, "d^n_{Q_A(P)}"] & & Q_{A\otimes_k B}(P^n) \arrow[dr, "d^n_{Q_{A\otimes_k B}}(P)"] & \\
& Q_A(P^{n+1}) \arrow[rr, "\alpha_{P^{n+1}}^\ell"] & & Q_{A\otimes_k B}(P^{n+1})
\end{tikzcd}$$

obviously commute for all n. Hence we assume that P is a bi-bi module.

Now fix an integer m and observe that $\underline{\mathrm{Gr}}A(A_{\geq m}, P)$ is a graded bi-bi module with $\mathbb{Z}^2$ grading given by

$$\underline{\mathrm{Gr}}A(A_{\geq m}, P)_{u,v} = \mathrm{Gr}\, A(A_{\geq m}, P_{*,v}(u)).$$

We have

$$\begin{aligned}\varphi^m_{u,v}\colon \underline{\mathrm{Gr}}A(A_{\geq m}, P)_{u,v} &\longrightarrow \underline{\mathrm{Gr}}(A\otimes_k B)(A_{\geq m}\otimes_k B, P)_{u,v}\\ f &\longmapsto \big\{a\otimes b\mapsto f(a)\cdot b := (1\otimes b)f(a)\big\}\end{aligned}$$

since $f(a)\in P_{*,v}(u)_r = P_{r+u,v}$ for any $a\in A_r$ and hence

$$f(a)\cdot b\in P_{r+u,v+s} = P(u,v)_{r,s}$$

for any $b\in B_s$. Since the choice of m was arbitrary, we have by the universal property for colimits the commutative diagram

$$\begin{tikzcd}\underline{\mathrm{Gr}}A(A_{\geq m}, P) \arrow[r, "\varphi_m"] \arrow[d] & \underline{\mathrm{Gr}}(A\otimes_k B)(A_{\geq m}\otimes_k B, P)_{u,v} \arrow[d]\\ Q_A(P) \arrow[r, dashed, "\exists!\alpha^\ell_P"] & Q_{A\otimes_k B}(P)\end{tikzcd}$$

For naturality, it suffices to show that

$$\begin{tikzcd}P_1 \arrow[d, "f"] & \underline{\mathrm{Gr}}A(A_{\geq m}, P_1) \arrow[r] & Q_A(P_1) \arrow[r, "\alpha^\ell_{P_1}"] \arrow[d, "Q_A(f)"] & Q_{A\otimes_k B}(P_1) \arrow[d, "Q_{A\otimes_k B}(f)"]\\ P_2 & & Q_A(P_2) \arrow[r, "\alpha^\ell_{P_2}"] & Q_{A\otimes_k B}(P_2)\end{tikzcd}$$

commutes for all m. It's clear from the colimit definition that $Q_A(f)$ applied to the class of a morphism $\varphi\colon A_{\geq m}\to P_1$ is the class of the composition, $[f\circ\varphi]$, which is mapped by $\alpha^\ell_{P_2}$ to the class of the morphism

$$a\otimes b\mapsto f\circ\varphi(a)\cdot b.$$

Along the other side of the square we see that $\alpha^\ell_{P_1}$ maps $[\varphi]$ to the class of

$$a\otimes b\mapsto 1\otimes\varphi(a)\cdot b$$

whose image under $Q_{A\otimes_k B}(f)$ is the class of the morphism

$$a\otimes b\mapsto f(\varphi(a)\cdot b) = f\circ\varphi(a)\cdot b$$

since f is a morphism of bi-bi modules.

By identifying P with $\underline{\mathrm{Gr}}A(A,P)$ we can see that image of an element p of P under the unit of adjunction $\varepsilon_A\colon 1_{\mathrm{Gr}\,A} \to Q_A$ is the class of the morphism $\varphi_p(a) = a \cdot p$ in $Q_A(P)$. Similarly, the unit of adjunction $\varepsilon_{A\otimes_k B}(P)$ takes an element p of P to the class of the morphism

$$\psi_p(a \otimes b) = a \otimes b \cdot p$$

in $Q_{A\otimes_k B}(P)$. From this observation it is clear that the result of applying α_P^ℓ to the image of p in $Q_A(P)$ is the class of the morphism

$$a \otimes b \mapsto \varphi_p(a) \cdot b = (a \cdot p) \cdot b = a \otimes b \cdot p = \psi_p(a \otimes b).$$

As a result, we see that α_P^ℓ factors the unit of adjunction

$$\varepsilon_{A\otimes_k B} = \alpha_P^\ell \circ \varepsilon_A.$$

□

Lemma 3.24. *Let A and B be finitely generated, connected graded k-algebras. There exist natural isomorphisms*

$$\begin{tikzcd} & Q_{A\otimes_k B}P \arrow[dl, "\gamma_P^\ell"'] \arrow[dr, "\gamma_P^r"] & \\ Q_B \circ Q_A(P) & & Q_A \circ Q_B(P) \end{tikzcd}$$

making the diagram

$$\begin{tikzcd} Q_AP \arrow[rr, "\alpha_P^\ell"] \arrow[dr, "\varepsilon_B(Q_AP)"'] & & Q_{A\otimes_k B}P \arrow[dl, "\gamma_P^\ell"] \arrow[dr, "\gamma_P^r"'] & & Q_BP \arrow[ll, "\alpha_P^r"'] \arrow[dl, "\varepsilon_A(Q_BP)"] \\ & Q_B \circ Q_A(P) & & Q_A \circ Q_B(P) & \end{tikzcd}$$

commute.

Proof. We handle the left case and note the other is symmetric. Since the tails, $B_{\geq n}$, of B are all finitely generated, the natural map

$$\begin{aligned} \gamma_P^l := \operatorname{colim}_{n,m} \underline{\mathrm{Gr}}(A \otimes_k B)\,(A_{\geq m} \otimes_k B_{\geq n}, P) \to \\ \operatorname{colim}_n \underline{\mathrm{Gr}}B\,(B_{\geq n}, \operatorname{colim}_m \underline{\mathrm{Gr}}A\,(A_{\geq m}, P)) \end{aligned}$$

induced by the isomorphisms

$$\begin{array}{ccc} \underline{\mathrm{Gr}}(A \otimes_k B)\,(A_{\geq m} \otimes_k B_{\geq n}, P) & \longrightarrow & \underline{\mathrm{Gr}}B\,(B_{\geq n}, \underline{\mathrm{Gr}}A\,(A_{\geq m}, P)) \\ \varphi & \longmapsto & (b \mapsto \varphi(- \otimes b)) \end{array}$$

is also an isomorphism.

For an element x of $Q_A(P)$ we can always choose a representative $\psi\colon A_{\geq n} \to P$. The image of x under α_P^ℓ is represented by the morphism

$$a \otimes b \mapsto \psi(a) \cdot b$$

and its image under γ_P^ℓ is represented by

$$b \mapsto \alpha_P^l \psi(- \otimes b) = \psi(-) \cdot b.$$

The image of x under $\varepsilon_A(Q_A P)$ is represented by the morphism

$$b \mapsto \psi(-) \cdot b$$

and the diagram commutes. □

Lemma 3.25. *Let A and B be finitely generated, connected graded k-algebras. There exist a commutative diagram*

$$\begin{tikzcd} \tau_A P & & \tau^{\mathrm{VdB}}_{A \otimes_k B} P \arrow[ll, "\xi_P^\ell"'] \arrow[rr, "\xi_P^r"] & & \tau_B P \\ & \tau_B \circ \tau_A(P) \arrow[ul, "\eta_B(\tau_A P)"] \arrow[ur, "\kappa_P^\ell"'] & & \tau_A \circ \tau_B(P) \arrow[ul, "\kappa_P^r"] \arrow[ur, "\eta_A(\tau_B P)"'] & \end{tikzcd}$$

with κ_P^ℓ and κ_P^r natural isomorphisms.

Proof. See [40, Lemma 4.5]. □

3.3 Derived functors

For a general Grothendieck category, we equip the category of chain complexes with the injective model structure. We compute the total right derived functors of a left exact functor, F, on an object, M, by applying F to the **fibrant replacement**, RM,

$$\mathbf{R}F(M) = F(RM).$$

When necessary to distinguish between multiple fibrant replacement functors, we will decorate with the relevant ring, e.g. $\mathbf{R}F(M) = F(R_A M)$.

Many of the relevant statements, e.g. checking that a morphism is a quasi-isomorphism, can be verified by passing to the level of the homotopy category, i.e. the derived category.

Definition 3.26. We say J is an F-acyclic if the natural morphism

$$F(J) \to \mathbf{R}F(J)$$

is a quasi-isomorphism.

A special class of F-acyclics are the h-injectives, which are all homotopy equivalent to a fibrant object. As such, one can use h-injective resolutions to compute derived functors when convenient.

We say that $\mathbf{R}F$ **commutes with coproducts** if the natural map

$$\bigoplus_{\gamma \in \Gamma} F(RX_\gamma) \to FR\left(\bigoplus_{\gamma \in \Gamma} X_\gamma\right)$$

is a quasi-isomorphism for any coproduct.

Lemma 3.27. *Let C and D be abelian categories and let $F \colon C \to D$ be left exact. If C is Grothendieck, then $\mathbf{R}F$ commutes with coproducts if and only if*

1. *F commutes with arbitrary coproducts of objects in C, and*
2. *Arbitrary coproducts of F-acyclics are F-acyclic.*

Proof. First assume that $\mathbf{R}F$ commutes with coproducts. Since F is left exact,

$$H^0\left(\mathbf{R}F(A)\right) \cong F(A)$$

for any object of C. So (1) is satisfied. Now assume that J_γ are F-acyclic. Then, we have the commutative diagram

$$\begin{array}{ccc} \bigoplus_\gamma F(J_\gamma) & \longrightarrow & \bigoplus_\gamma \mathbf{R}F\left(J_\gamma\right) \\ \downarrow & & \downarrow \\ F\left(\bigoplus_\gamma J_\gamma\right) & \longrightarrow & \mathbf{R}F\left(\bigoplus_\gamma J_\gamma\right) \end{array}$$

with vertical arrows quasi-isomorphisms. The upper horizontal arrow is a coproduct of quasi-isomorphisms so is also a quasi-isomorphism. Consequently, the lower horizontal arrow is a quasi-isomorphism and thus $\bigoplus J_\gamma$ is F-acyclic.

Conversely, assume that (1) and (2) hold. Given a collection $\{X_\gamma\}$ of objects of C, the map

$$\bigoplus_\gamma X_\gamma \to \bigoplus_\gamma RX_\gamma$$

is a quasi-isomorphism and $\bigoplus_\gamma RX_\gamma$ is F-acyclic. We can factor the natural map

$$\begin{tikzcd} \bigoplus_\gamma F(RX_\gamma) \arrow[rr] \arrow[dr] & & FR\left(\bigoplus_\gamma X_\gamma\right) \\ & F\left(\bigoplus_\gamma RX_\gamma\right) \arrow[ur] & \end{tikzcd}$$

Condition (1) says that the downward arrow is a quasi-isomorphism while condition (2) says that the upward arrow comes from applying F to a map of F-acyclic objects. As the $F \cong \mathbf{R}F$ on F-acyclic objects, F preserves quasi-isomorphisms between acyclic objects. Thus, the upward arrow is also a quasi-isomorphism. Consequently, the horizontal arrow is a quasi-isomorphism and $\mathbf{R}F$ commutes with coproducts. $\square$

For a given complex of bi-bi modules, P, the complex $\mathbf{R}Q_A P$ will not generally be a bi-bi module due to the fact that R_A does not commute with coproducts. However, the assumption that $\mathbf{R}Q_A$ commutes with coproducts says exactly that

$$\bigoplus_v Q_A R_A(P_{*,v}) \to \mathbf{R}Q_A P$$

is a quasi-isomorphism and the source is a complex of bi-bi modules. We therefore set

$$\mathbf{R}'Q_A P := \bigoplus_v Q_A R_A(P_{*,v}).$$

Lemma 3.28. *If I is an h-injective complex of bi-bi modules, then $I_{*,v}$ is an h-injective complex of A modules for any $v \in \mathbb{Z}$.*

Proof. We observe from the isomorphism of A modules

$$I_{*,v} \cong \operatorname{Gr} B(B(-v), I)$$

that one just needs to check that applying $\operatorname{Gr} B(B(-v), -)$ in each degree preserves h-injectivity. For the complex $I_{*,v}$ of A modules to be h-injective we need only show that

$$\mathrm{K}(\operatorname{Gr} A)(D, I_{*,v}) = 0$$

holds for all acyclic A modules, D. This follows from the tensor-hom adjunction

$$\begin{aligned} \mathrm{K}(\operatorname{Gr} A)(D, I_{*,v}) &\cong \mathrm{K}(\operatorname{Gr} A)(D, \operatorname{Gr} B(B(-v), I)) \\ &\cong \mathrm{K}(\operatorname{Gr}(A \otimes_k B))(D \otimes_k B(-v), I) = 0 \end{aligned}$$

because I is a h-injective bi-bi module and $D \otimes_k B(-v)$ is acyclic. □

Lemma 3.29. *The functor*

$$\omega : \text{h-Inj}\,(\operatorname{QGr} A) \to \text{h-Inj}\,(\operatorname{Gr} A)$$

is well-defined. Moreover, $H^0(\omega)$ is an equivalence with its essential image.

Proof. For the first statement, we just need to check that ω takes h-injective complexes to h-injective complexes. This is clear from the fact that ω is right adjoint to π, which is exact.

To see this is fully faithful, we recall that $\pi\omega \cong \mathrm{Id}$ so

$$\begin{aligned} \text{h-Inj}\,(\operatorname{Gr} A)\,(\omega M, \omega N) &\cong \text{h-Inj}\,(\operatorname{QGr} A)\,(\pi\omega M, N) \\ &\cong \text{h-Inj}\,(\operatorname{QGr} A)\,(M, N). \end{aligned}$$

□

Remark 3.30. Using Lemma 3.29, we can either use $\mathrm{h\text{-}Inj}\,(\mathrm{QGr}\,A)$ or its image under ω in $\mathrm{h\text{-}Inj}\,(\mathrm{Gr}\,A)$ as an enhancement of $\mathrm{D}(\mathrm{QGr}\,A)$.

Corollary 3.31. *Let A and B be finitely generated, connected graded k-algebras. If $\mathbf{R}Q_A$ and $\mathbf{R}Q_B$ both commute with coproducts, then, for a bi-bi module P, $\mathbf{R}Q_{A\otimes_k B}P$ is naturally quasi-isomorphic to $\mathbf{R}'Q_A(\mathbf{R}'Q_B P)$ and to $\mathbf{R}'Q_B(\mathbf{R}'Q_A P)$.*

Proof. We handle the quasi-isomorphism between the first two. The remaining part is analogous.

Let P be an object of $C\,(\mathrm{Gr}(A \otimes_k B))$. First observe that we have

$$\mathbf{R}Q_{A\otimes_k B}(P) = Q_{A\otimes_k B}(R_{A\otimes_k B}P) \cong Q_A \circ Q_B(R_{A\otimes_k B}P)$$

via $\gamma^r_{R_{A\otimes_k B}P}$. Note that we have two homotopy equivalences

$$Q_B R_B P_{v,*} \to Q_B R_B (R_{A\otimes_k B}P)_{v,*} \leftarrow Q_B(R_{A\otimes_k B}P)_{v,*}$$

which induce a quasi-isomorphism between $\mathbf{R}'Q_B P$ and $Q_B R_{A\otimes_k B}P$. Applying $\mathbf{R}Q_A$ preserves quasi-isomorphisms. Since $\mathbf{R}'Q_A$ is naturally quasi-isomorphic to $\mathbf{R}Q_A$, there is an induced natural quasi-isomorphism between $\mathbf{R}'Q_A(\mathbf{R}'Q_B P)$ and $\mathbf{R}'Q_A(Q_B R_{A\otimes_k B}P)$.

Therefore it suffices to show that

$$Q_A Q_B I \to \mathbf{R}'Q_A(Q_B I)$$

is a quasi-isomorphism for any h-injective, I. Since $\mathbf{R}'Q_A$ and $\mathbf{R}Q_A$ are naturally quasi-isomorphic, we can instead show that

$$Q_A Q_B I \to \mathbf{R}Q_A(Q_B I)$$

is a quasi-isomorphism, which is another way of saying that $Q_B I$ is Q_A-acyclic. Note that there is a short exact sequence of complexes

$$0 \to \bigoplus_n \underline{\mathrm{Gr}}(B)(B_{\geq n}, I) \to \bigoplus_n \underline{\mathrm{Gr}}(B)(B_{\geq n}, I) \to Q_B I \to 0$$

which induces an exact triangle in the derived category. Because the tails $A_{\geq m}$ are finitely generated, any map

$$A_{\geq m} \to Q_B I$$

lifts to a map

$$A_{\geq m} \to \underline{\mathrm{Gr}}(B)(B_{\geq n}, I).$$

This says that the sequence

$$0 \to \bigoplus_n Q_A \underline{\mathrm{Gr}}(B)(B_{\geq n}, I) \to \bigoplus_n Q_A \underline{\mathrm{Gr}}(B)(B_{\geq n}, I) \to Q_A Q_B I \to 0$$

remains exact. Using the fact that $\mathbf{R}Q_A$ commutes with coproducts, we get a map of triangles

$$\begin{array}{ccccc} \bigoplus_n Q_A \underline{\mathrm{Gr}}(B)(B_{\geq n}, I) & \longrightarrow & \bigoplus_n Q_A \underline{\mathrm{Gr}}(B)(B_{\geq n}, I) & \longrightarrow & Q_A Q_B I \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_n \mathbf{R}Q_A \underline{\mathrm{Gr}}(B)(B_{\geq n}, I) & \to & \bigoplus_n \mathbf{R}Q_A \underline{\mathrm{Gr}}(B)(B_{\geq n}, I) & \to & \mathbf{R}Q_A(Q_B I) \end{array}$$

where the first displayed vertical maps are quasi-isomorphisms since each $\underline{\mathrm{Gr}}(B)(B_{\geq n}, I)$ is h-injective for all n. Consequently, the map

$$Q_A Q_B I \to \mathbf{R}Q_A\,(Q_B I)$$

is also a quasi-isomorphism for any h-injective I. □

We can strengthen the previous statement if we assume some finite-dimensionality on $\mathbf{R}\tau$.

Proposition 3.32. *Assume that A is Ext-finite and there exists a fixed N such that $\mathbf{R}^n\tau_A = 0$ for all $n \geq N$. Then, the natural maps*

$$\mathbf{R}\tau_A(\mathbf{R}\tau_A M) \to \mathbf{R}\tau_A M$$
$$\mathbf{R}Q_A(M) \to \mathbf{R}Q_A(Q_A M)$$

are quasi-isomorphisms for any complex of graded A modules, M. Moreover, both $\mathbf{R}\tau_A(\mathbf{R}Q_A M)$ and $\mathbf{R}Q_A(\mathbf{R}\tau_A M)$ are acyclic.

Proof. It suffices to show that, for a cofibrant M, the object $\tau_A M$ is τ_A-acyclic. [12, Lemma 4.1.3] implies this for injective graded modules. An appeal to the spectral sequence whose E_1-page is $\mathbf{R}^p\tau_A(\tau_A M^q)$ plus the assumption that $\mathbf{R}\tau_A^p$ vanishes for sufficiently large p gives the acyclicity in general. A similar argument demonstrates the second quasi-isomorphism. $\square$

Remark 3.33. It seems like the assumption of finite dimensionality of $\mathbf{R}\tau_A$ is unnecessary.

Definition 3.34. We say an object M of $\mathcal{C}(\mathrm{Gr}\, A)$ is $\mathbf{R}\tau_A$-torsion free if $\mathbf{R}\tau_A M$ is acyclic.

We also have the following standard triangles of derived functors.

Lemma 3.35. *Let A and B be finitely generated connected graded algebras. Then, we have natural transformations*

$$\mathbf{R}\tau_A \longrightarrow R \longrightarrow \mathbf{R}Q_A$$

which when applied to any graded A module gives an exact triangle in the derived category. Analogous statements hold for graded B modules and bi-graded $A \otimes_k B$ modules.

Proof. The natural transformations are $\eta \circ R$ and $\varepsilon \circ R$. For the case of graded A modules (or graded B modules), this is well-known, see [12, Property 4.6]. For a bi-bi module, P, the sequence

$$0 \to \tau_{A\otimes_k B}P \to P \to Q_{A\otimes_k B}P$$

is exact. It suffices to prove that if $P = I$ is injective, then the whole sequence is actually exact. Here one can use the system of exact sequences

$$0 \to A_{\geq n_1} \otimes_k B_{\geq n_2} \to A \otimes_k B \to (A \otimes_k B)/A_{\geq n_1} \otimes_k B_{\geq n_2} \to 0$$

and exactness of $\underline{\mathrm{Gr}}(A \otimes_k B)(-, I)$ plus Lemmas 3.19 and 3.22 to get exactness. $\square$

Proposition 3.36. *Assume that A is Ext-finite. Then $\mathbf{R}\tau_A$ and $\mathbf{R}Q_A$ both commute with coproducts.*

Proof. See [40, Lemma 4.3] for $\mathbf{R}\tau_A$. Since coproducts are exact, using the triangle

$$\mathbf{R}\tau_A M \to RM \to \mathbf{R}Q_A M$$

we see that $\mathbf{R}\tau_A$ commutes with coproducts if and only if $\mathbf{R}Q_A$ commutes with coproducts. □

Corollary 3.37. *Assume that A and B are left Noetherian, and that $\mathbf{R}Q_A$ and $\mathbf{R}Q_B$ both commute with coproducts. There exist natural diagrams of complexes of bimodules*

$$\begin{tikzcd}
\mathbf{R}'Q_A P \arrow[dr, "\sim"] & & Q_A R_{A\otimes_k B} P \arrow[dl, "\sim"'] \arrow[dr, "\alpha^{\ell}_{R_{A\otimes_k B}P}"] & \\
& \mathbf{R}'Q_A(R_{A\otimes B}P) & & \mathbf{R}Q_{A\otimes_k B}P
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{R}'Q_B P \arrow[dr, "\sim"] & & Q_B R_{A\otimes_k B} P \arrow[dl, "\sim"'] \arrow[dr, "\alpha^{r}_{R_{A\otimes_k B}P}"] & \\
& \mathbf{R}'Q_B(R_{A\otimes B}P) & & \mathbf{R}Q_{A\otimes_k B}P
\end{tikzcd}$$

where the arrows labeled with $\sim$ are quasi-isomorphisms.

Proof. Because the canonical morphism $A \to A\otimes_k B$ is flat, the associated adjunction is Quillen by [20, Proposition 2.13], and hence the fibrant replacement $R_{A\otimes_k B}P$ is also fibrant when regarded as an object of $C\,(\mathrm{Gr}\,A)$. Since fibrancy is closed under retracts, $(R_{A\otimes_k B}P)_{*,v}$ is also fibrant for any v.

Therefore, the map

$$(R_{A\otimes_k B}P)_{*,v} \to R_A(R_{A\otimes_k B}P)_{*,v}$$

is a homotopy equivalence and remains such after application of Q_A. Consequently,

$$\begin{tikzcd} Q_A R_{A\otimes_k B} P \arrow[r, equal, "\sim"] & \bigoplus_v Q_A (R_{A\otimes_k B} P)_{*,v} \arrow[dl] \\ \bigoplus_v Q_A R_A (R_{A\otimes_k B} P)_{*,v} \arrow[r, equal] & \mathbf{R}' Q_A R_{A\otimes_k B} P \end{tikzcd}$$

is a quasi-isomorphism. Similarly, the map

$$R_A P_{*,v} \to R_A(R_{A\otimes_k B} P_{*,v})$$

is a homotopy equivalence and the map

$$\mathbf{R}' Q_A P \to \mathbf{R}' Q_A R_{A\otimes_k B} P$$

is a quasi-isomorphism. $\square$

Remark 3.38. We will denote the maps in the derived category resulting from the diagrams in Corollary 3.37 as

$$\beta_P^\ell : \mathbf{R}' Q_A P \to \mathbf{R} Q_{A\otimes_k B} P$$
$$\beta_P^r : \mathbf{R}' Q_B P \to \mathbf{R} Q_{A\otimes_k B} P.$$

We have an analogous definition for τ,

$$\mathbf{R}' \tau_A P := \bigoplus_v \tau_A R_A P_{*,v},$$

and an analogous result to Corollary 3.37.

Corollary 3.39. *Assume that A and B are left Noetherian, and that $\mathbf{R}\tau_A$ and $\mathbf{R}\tau_B$ both commute with coproducts. There exist natural diagrams of complexes of bimodules*

$$\begin{tikzcd} \mathbf{R}\tau^{\mathrm{VdB}}_{A\otimes_k B} P \arrow[dr] & & \mathbf{R}'\tau_A P \arrow[dl, "\sim"'] \\ & \mathbf{R}'\tau_A R_{A\otimes_k B} P & \end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{R}\tau^{\mathrm{VdB}}_{A\otimes_k B}P \arrow[dr] & & \mathbf{R}'\tau_B P \arrow[dl, "\sim"'] \\
& \mathbf{R}'\tau_B R_{A\otimes_k B}P &
\end{tikzcd}.$$

where the arrows labeled with $\sim$ are quasi-isomorphisms.

Proof. Establishing the quasi-isomorphism is completely analogous to the previous corollary. The only ambiguity is the map

$$\mathbf{R}\tau^{\mathrm{VdB}}_{A\otimes_k B}P \to \mathbf{R}'\tau_A R_{A\otimes_k B}P.$$

This comes from the composition of $\xi^r_{R_{A\otimes_k B}P}$ and the natural map

$$\begin{tikzcd}
\tau_A R_{A\otimes_k B}P \arrow[r, equal, "\sim"] & \bigoplus_v \tau_A(R_{A\otimes_k B}P)_{*,v} \arrow[dl] \\
\bigoplus_v \tau_A R_A(R_{A\otimes_k B}P)_{*,v} \arrow[r, equal] & \mathbf{R}'\tau_A R_{A\otimes_k B}P.
\end{tikzcd}$$

□

Remark 3.40. We denote the resulting maps in the derived category as

$$\mu^\ell_P : \mathbf{R}\tau^{\mathrm{VdB}}_{A\otimes B}P \to \mathbf{R}'\tau_A P$$
$$\mu^r_P : \mathbf{R}\tau^{\mathrm{VdB}}_{A\otimes B}P \to \mathbf{R}'\tau_B P.$$

Proposition 3.41. *Assume that A and B are left Noetherian and Ext-finite. Assume further that $\mathbf{R}\tau_A$ and $\mathbf{R}\tau_B$ are finite-dimensional. Then, we have natural isomorphisms in the derived category*

$$\delta^\ell_P : \mathbf{R}'Q_B(\mathbf{R}'Q_A P) \to \mathbf{R}Q_{A\otimes_k B}P$$
$$\delta^r_P : \mathbf{R}'Q_A(\mathbf{R}'Q_B P) \to \mathbf{R}Q_{A\otimes_k B}P.$$

Consequently, β^ℓ_P (respectively β^r_P) is an isomorphism if and only if $\mathbf{R}Q_A P$ (respectively $\mathbf{R}Q_B P$) is $\mathbf{R}\tau_B$ (respectively $\mathbf{R}\tau_A$) torsion-free.

Proof. Applying $\mathbf{R}'Q_A$ to the diagram from Corollary 3.37 gives

$$\begin{tikzcd}
\mathbf{R}'Q_A\mathbf{R}'Q_BP \arrow[dr, "\sim"] & \mathbf{R}'Q_AQ_BR_{A\otimes_k B}P \arrow[d, "\sim"'] \arrow[dr, "\mathbf{R}'Q_A(\alpha^r_{R_{A\otimes_k B}P})"] & \\
& \mathbf{R}'Q_A\mathbf{R}'Q_B(R_{A\otimes B}P) & \mathbf{R}'Q_A\mathbf{R}Q_{A\otimes_k B}P
\end{tikzcd}$$

Since $Q_{A\otimes_k B} = Q_A \circ Q_B$ an argument analogous to the proof of Proposition 3.32 shows that the natural map

$$\lambda^\ell_P : \mathbf{R}Q_{A\otimes_k B}(P) \to \mathbf{R}'Q_A\left(\mathbf{R}Q_{A\otimes_k B}(P)\right)$$

is a quasi-isomorphism. We set

$$\delta^\ell_P := \left(\lambda^\ell_P\right)^{-1}\mathbf{R}'Q_A(\beta^\ell_P).$$

Now we check that $\mathbf{R}'Q_A(\alpha^r_{R_{A\otimes_k B}P})$ is a quasi-isomorphism. By acyclicity, this reduces to checking that the natural map

$$Q_AQ_B(R_{A\otimes_k B}P) \to Q_A(Q_{A\otimes_k B}R_{A\otimes_k B}P) \cong Q_A(Q_A(Q_B(R_{A\otimes_k B}P)))$$

is a quasi-isomorphism. But the natural map

$$Q_A \to Q_A \circ Q_A$$

is always an isomorphism.

Assuming that

$$\mathbf{R}'Q_AP \to \mathbf{R}'Q_B(\mathbf{R}'Q_AP)$$

is a quasi-isomorphism, i.e. assuming that $\mathbf{R}'Q_AP$ is $\mathbf{R}\tau_B$ torsion free, we immediately get that the original map β^ℓ_P is a quasi-isomorphism. $\square$

By using the standard homological assumptions above, one has better statements for $P = \Delta$.

Proposition 3.42. *Let A be left and right Noetherian and assume that the conditions $\chi^\circ(A)$ holds as left and right A modules. Furthermore, assume that $\mathbf{R}\tau_A$ and $\mathbf{R}\tau_{A^{\mathrm{op}}}$ are finite dimensional. Then the maps*

$$\begin{tikzcd}
\mathbf{R}'Q_A\Delta \arrow[dr, "\beta^\ell_\Delta"] & & \mathbf{R}'Q_B\Delta \arrow[dl, "\beta^r_\Delta"'] \\
& \mathbf{R}Q_{A\otimes_k B}\Delta &
\end{tikzcd}$$

are quasi-isomorphisms.

Furthermore, the maps

$$\begin{tikzcd}
& \mathbf{R}\tau^{\mathrm{VdB}}_{A\otimes_k A^{\mathrm{op}}}\Delta \arrow[dl, "\mu^\ell_\Delta"'] \arrow[dr, "\mu^r_\Delta"] & \\
\mathbf{R}'\tau_A\Delta & & \mathbf{R}'\tau_{A^{\mathrm{op}}}\Delta
\end{tikzcd}$$

are also quasi-isomorphisms.

Proof. We have a triangle in $\mathrm{D}(\mathrm{Gr}(A \otimes_k A^{\mathrm{op}}))$

$$\mathbf{R}'\tau_{A^{\mathrm{op}}}(\mathbf{R}'Q_A\Delta) \to \mathbf{R}'Q_A\Delta \to \mathbf{R}'Q_{A^{\mathrm{op}}}(\mathbf{R}'Q_A\Delta) \to \mathbf{R}'\tau_{A^{\mathrm{op}}}(\mathbf{R}'Q_A\Delta)[1]$$

By Proposition 3.41, $\mathbf{R}'Q_{A^{\mathrm{op}}}(\mathbf{R}'Q_A\Delta) \cong \mathbf{R}'Q_{A\otimes_k A^{\mathrm{op}}}\Delta$, so it suffices to show that we have $\mathbf{R}'\tau_{A^{\mathrm{op}}}(\mathbf{R}'Q_A\Delta) = 0$.

First we note that for any bi-bi module, P, the natural morphism

$$\mathbf{R}'\tau_{A^{\mathrm{op}}}P \to P$$

is a quasi-isomorphism if and only if the natural morphism

$$\mathbf{R}\tau_{A^{\mathrm{op}}}P_{x,*} \to P_{x,*}$$

is a quasi-isomorphism for each $x \in \mathbb{Z}$. Moreover, for a right A module, M, if $H^j(M)$ is right limited for each j then $\mathbf{R}\tau_{A^{\mathrm{op}}}M \to M$ is a quasi-isomorphism. So it suffices to show that $(\mathbf{R}^j\tau_A\Delta)_{x,*}$ has right limited grading for each x and j. We have

$$\left(\mathbf{R}^j\tau_A\Delta\right)_{x,y} \cong \left(\mathbf{R}^j\tau_A\Delta_{*,y}\right)_x \cong \left(\mathbf{R}^j\tau_A A(y)\right)_x.$$

By [4, Cor. 3.6 (3)], we have $(\mathbf{R}^j\tau_A A(y))_x = 0$ for sufficiently large y and hence is right limited. This implies that

$$\mathbf{R}\tau_{A^{\mathrm{op}}}\left((\mathbf{R}\tau_A\Delta)_{x,*}\right) \to (\mathbf{R}\tau_A\Delta)_{x,*}$$

is a quasi-isomorphism, as desired. As this sits in a triangle,

$$\mathbf{R}'\tau_{A^{\mathrm{op}}}(\mathbf{R}'\tau_A\Delta) \to \mathbf{R}'\tau_A\Delta \to \mathbf{R}'Q_{A^{\mathrm{op}}}(\mathbf{R}'\tau_A\Delta) \to \mathbf{R}'\tau_{A^{\mathrm{op}}}(\mathbf{R}'\tau_A\Delta)[1]$$

we see that $\mathbf{R}'\tau_{A^{\mathrm{op}}}(\mathbf{R}'Q_A\Delta)$ is acyclic as desired. □

Hypotheses similar to those of Proposition 3.42 will appear often so we attach a name.

Definition 3.43. Let A and B be connected graded k-algebras. If A is Ext-finite, left and right Noetherian, satisfies $\chi^\circ(A)$ and $\chi^\circ(A^{\mathrm{op}})$, and has both $\mathbf{R}\tau_A$ and $\mathbf{R}\tau_{A^{\mathrm{op}}}$ finite-dimensional, then we say that A is **delightful**. If A and B are both delightful, then we say that A and B form a **delightful couple**.

3.4 Segre Products

Definition 3.44. Let A and B be connected graded k-algebras. The **Segre product** of A and B is the graded k-algebra

$$A \times_k B = \bigoplus_{i\in\mathbb{Z}} A_i \otimes_k B_i.$$

Proposition 3.45. *If A and B are connected graded k-algebras that are finitely generated in degree one, then $A\times_k B$ is finitely generated in degree one.*

Proof. If $\{x_i\}_{i=1}^r \subseteq A_1$ and $\{y_i\}_{i=1}^s \subseteq B_1$ are generators for A and B, respectively, then $A \otimes_k B$ is finitely generated by $\{x_i \otimes y_j\}_{i,j}$. □

As a nice corollary, we can relax the conditions on [42, Theorem 2.4] to avoid the Noetherian conditions on the Segre and tensor products.

Theorem 3.46 ([42, Theorem 2.4])**.** *Let A and B be finitely generated, connected graded k-algebras, and let $S = A \times_k B$, $T = A \otimes_k B$. If A and B are both generated in degree one, then there is an equivalence of categories*

$$\begin{aligned} \mathbb{V}\colon\ \mathrm{QGr}\, S &\longrightarrow \mathrm{QGr}\, T \\ E &\longmapsto \pi_T\left(T \otimes_S \omega_S E\right) \end{aligned}$$

Proof. As noted in Van Rompay's comments preceding the Theorem, the hypothesis is necessary only to ensure that $\mathrm{QGr}\, S$ and $\mathrm{QGr}\, T$ are well-defined. Thanks to Proposition 3.11 and Lemma 3.19, the equivalence follows by running the same argument. □

3.5 A Comparison with the Commutative Situation

To provide a touchstone for the reader, we interpret the definitions and results when A and B are commutative and finitely-generated by elements of weight 1. Then, $A = k[x_1, \ldots, x_n]/I_A$ and $B = k[y_1, \ldots, y_m]/I_B$ for some homogeneous ideals I_A, I_B. So $\mathrm{Spec}\, A$ is a closed $\mathbf{G}_m$-stable subscheme of affine space $\mathbf{A}^n$ and similarly for $\mathrm{Spec}\, B$. Let X and Y be the associated projective schemes. Then,

$$\mathrm{Spec}\, A \otimes_k B \subset \mathbf{A}^{n+m}$$

is $\mathbf{G}_m^2$-stable. The category $\mathrm{Gr}(A \otimes_k B)$ is equivalent to the $\mathbf{G}_m^2$-equivariant quasi-coherent sheaves on $\mathrm{Spec}\, A \otimes_k B$ with $\mathrm{Tors}(A \otimes_k B)$ being those modules supported on the subscheme corresponding to

$$(x_1, \ldots, x_n)(y_1, \ldots, y_m).$$

Descent then gives that

$$\mathrm{QGr}\,(A \otimes_k B) \cong \mathrm{Qcoh}(X \times Y).$$

The quotient $\mathrm{Gr}(A \otimes_k B)/\,\mathrm{Tors}(A \otimes_k B)$ is equivalent to $\mathrm{Qcoh}^{\mathbf{G}_m^2}(U \times V)$ for the quasi-affines $U = \mathrm{Spec}\, A \setminus 0$ and $V = \mathrm{Spec}\, B \setminus 0$. Since $U \times V$ is a $\mathbf{G}_m^2$ torsor over $X \times Y$ we have $\mathrm{Qcoh}^{\mathbf{G}_m^2}(U \times V) \cong \mathrm{Qcoh}(X \times Y)$.

3.6 Graded Morita Theory

This section demonstrates how the tools of dg-categories yield a nice perspective on derived graded Morita theory. One can compare with the well-known graded Morita statement in [43].

In order to utilize the machinery of dg-categories, we must first translate chain complexes of graded modules into dg-categories. While one can naïvely regard this category as a dg-category by way of an enriched Hom entirely analogous to the ungraded situation, the relevant statements of [38] are better suited to the perspective of functor categories. As such, we adapt the association of a ringoid with one object to a ring from Section 2.2 to the graded situation, considering instead a ringoid with multiple objects. This notion is standard, see eg [41].

Throughout this section, we will let $G = (G, +)$ be an abelian group, and let A and B be not necessarily commutative G-graded algebras over k. We will generally be concerned with the groups $\mathbb{Z}$ and $\mathbb{Z}^2$. In the sequel, there will be many instances where there are two simultaneous gradings on an object: homological degree and homogeneous degree. We avoid the latter term, preferring weight, and use degree solely when referring to homological degree.

For clarity, consider the example of a complex of G-graded left A modules, M. The degree n piece of M is the G-graded left A module M^n. The weight g piece of the graded module M^n is the A_0 module of homogeneous elements of (graded) degree g, M^n_g. Note that in this terminology, the usual morphisms of graded modules are the weight zero morphisms.

Definition 3.47. Denote by $\mathcal{C}\left(\operatorname{Gr} A\right)$ the dg-category with objects chain complexes of G-graded left A modules and morphisms defined as follows.

We say that a morphism $f\colon M \to N$ of degree p is a collection of morphisms $f^n\colon M^n \to N^{n+p}$ of weight zero. We denote by $\mathcal{C}\left(\operatorname{Gr} A\right)(M, N)^p$ the collection of all such morphisms, which we equip with the differential

$$d(f) = d_N \circ f + (-1)^{p+1} f \circ d_M$$

and define $\mathcal{C}\left(\operatorname{Gr} A\right)(M, N)$ to be the resulting chain complex. Composition is the usual composition of graded morphisms.

We denote by $\mathcal{C}\left(\operatorname{Gr}\left(A^{\mathrm{op}}\right)\right)$ the same construction with G-graded right A modules, which are equivalently left modules over the opposite ring, A^{op}.

Remark 3.48. One should note that the closed morphisms are precisely the morphisms of complexes $M \to N[p]$ and, in particular, the closed degree zero morphisms are precisely the usual morphisms of complexes.

Definition 3.49. To each G-graded k-algebra, A, associate the category $\mathcal{A}$ with objects the group G and morphisms given by

$$\mathcal{A}(g_1, g_2) = A_{g_2 - g_1}$$

and composition defined by the multiplication $A_{g_2-g_1} A_{g_3-g_2} \subseteq A_{g_3-g_1}$.

We regard $\mathcal{A}$ as a dg-category by considering the k module of morphisms, $\mathcal{A}(g_1, g_2)$, as the complex with $A_{g_2-g_1}$ in degree 0 and zero differential.

Lemma 3.50. *Let G be an abelian group. If A is a G-graded algebra over k and $\mathcal{A}$ the associated dg-category, then there is an isomorphism of dg-categories*

$$\mathcal{C}\,(\mathrm{Gr}\,A) \cong \mathrm{dgMod}\,(\mathcal{A})\,.$$

Proof. We first construct a dg-functor $F\colon \mathcal{C}\,(\mathrm{Gr}\,A) \to \mathrm{dgMod}\,(\mathcal{A})$. For each $g \in G$, denote by $A(g)[0]$ the complex with $A(g)$ in degree zero and consider the full subcategory of $\mathcal{C}\,(\mathrm{Gr}\,A)$ of all such complexes. We see that a morphism

$$f \in \mathcal{C}\,(\mathrm{Gr}\,A)\,(A(g)[0], M)^n$$

is just the data of a morphism $f^0\colon A(g) \to M^n$ which gives

$$\mathcal{C}\,(\mathrm{Gr}\,A)\,(A(g)[0], M)^n \cong \mathrm{Gr}\,A(A(g), M^n) \cong M^n_{-g}$$

and hence $M_{-g} := \mathcal{C}\,(\mathrm{Gr}\,A)\,(A(g)[0], M)$ is the complex with M^n_{-g} in degree n. In particular, when $M = A(h)[0]$, we have

$$\mathcal{C}\,(\mathrm{Gr}\,A)\,(A(g)[0], A(h)[0]) := A(h)[0]_{-g} = \mathcal{A}(g, h),$$

which allows us to identify this subcategory with $\mathcal{A}$ via the Yoneda embedding, $A(h)[0]$ corresponding to the representable functor $\mathcal{A}(-, h)$. Using this identification, we can define the image of M in $\mathrm{dgMod}\,(\mathcal{A})$ to be the dg-functor that takes an object $g \in G$ to

$$M_{-g} = \mathcal{C}\,(\mathrm{Gr}\,A)\,(A(g)[0], M)$$

with structure morphism

$$\mathcal{A}(g,h) \cong \mathcal{C}\,(\mathrm{Gr}\,A)\,(A(g)[0], A(h)[0]) \to \mathcal{C}\,(k)\,(M_{-h}, M_{-g})$$

induced by the representable functor $\mathcal{C}\,(\mathrm{Gr}\,A)\,(-, M)$. The image of a morphism $f \in \mathcal{C}\,(\mathrm{Gr}\,A)\,(M,N)$ is defined to be the natural transformation given by the collection of morphisms

$$h^{A(-g)[0]}(f)\colon \mathcal{C}\,(\mathrm{Gr}\,A)\,(A(-g)[0], M) \to \mathcal{C}\,(\mathrm{Gr}\,A)\,(A(-g)[0], N)$$

indexed by G.

Conversely, we note that the data of a functor $M\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}\,(k)$ is a collection of chain complexes, $M_g := M(g)$, indexed by G and morphisms of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & A_{g-h} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{C}\,(k)\,(M_g, M_h)^0 \arrow[r] & \mathcal{C}\,(k)\,(M_g, M_h)^1 \arrow[r] & \cdots
\end{tikzcd}$$

The non-zero arrow factors through $Z^0(\mathcal{C}\,(k)\,(M_g, M_h))$, so the structure morphism is equivalent to giving a morphism

$$A_{g-h} \to \mathcal{C}\,(k)\,(M_g, M_h)$$

and thus M determines a complex of graded A modules

$$\widetilde{M} = \bigoplus_{g \in G} M_{-g}.$$

A morphism $\eta\colon M \to N$ is simply a collection of natural transformations η^p such that for each $g \in G$ we have $\eta^p(g) \in \mathcal{C}\,(k)\,(M_g, N_g)^p$ and the naturality implies that $\eta^p(g)$ is A-linear. The natural transformation η^p thus determines a morphism

$$\bigoplus_{g \in G} \eta^p(-g) \in \mathcal{C}\left(\operatorname{Gr} A\right)\left(\widetilde{M}, \widetilde{N}\right)^p,$$

and hence η determines a morphism in $\mathcal{C}\left(\operatorname{Gr} A\right)\left(\widetilde{M}, \widetilde{N}\right)$, which is the collection of all such homogeneous components. This defines a dg-functor $\operatorname{dgMod}\left(\mathcal{A}\right) \to \mathcal{C}\left(\operatorname{Gr} A\right)$ which is clearly the inverse of F. □

Remark 3.51.

1. It is worth noting that it is natural from the ringoid perspective to reverse the weighting on the opposite ring in that, formally,
$$A^{\mathrm{op}}_g = \mathcal{A}^{\mathrm{op}}(0, g) = \mathcal{A}(g, 0) = A_{-g}$$
so that $\mathcal{A}^{\mathrm{op}}(-, h) = \mathcal{A}(h, -)$ is the representable functor corresponding to the left module $A^{\mathrm{op}}(h)$ by
$$\bigoplus_{g \in G} \mathcal{A}^{\mathrm{op}}(-g, h) = \bigoplus_{g \in G} \mathcal{A}(h, -g) = \bigoplus_{g \in G} A_{-(g+h)} = \bigoplus_{g \in G} A^{\mathrm{op}}_{g+h} = A^{\mathrm{op}}(h).$$
With this convention, when considering right modules, one can dispense with the formality of the opposite ring by constructing from a complex, M, the dg-functor $\mathcal{A} \to \mathcal{C}\left(k\right)$ mapping g to $M_g := \mathcal{C}\left(\operatorname{Gr}\left(A^{\mathrm{op}}\right)\right)\left(A(-g)[0], M\right)$.

2. For $\mathcal{A}$ the category associated to the k-algebra A, M an object of $\operatorname{Gr} A$, N an object of $\operatorname{Gr}(A^{\mathrm{op}})$, the k-vector space
$$N \otimes_{\mathcal{A}} M$$
is usually called the $\mathbb{Z}$-algebra tensor product. See [11, Section 4].

When $G = \mathbb{Z}^2$, and A, B are $\mathbb{Z}$-graded algebras over k, we denote by $\mathcal{C}\left(\operatorname{Gr} A^{\mathrm{op}} \otimes_k B\right)$ the dg-category of chain complexes of G-graded B-A-bimodules. We associate to the $\mathbb{Z}^2$-graded k-algebra $A^{\mathrm{op}} \otimes_k B$ the tensor product of the associated dg-categories, $\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}$. Note that in the identification

$$\mathcal{C}\left(\mathrm{Gr}\left(A^{\mathrm{op}} \otimes_k B\right)\right) \cong \mathrm{dgMod}\left(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}\right)$$

the weighting coming from the A module structure is reversed, as in the remark above.

From this construction, we have a dg-enhancement, $\mathrm{h\text{-}Proj}\left(\mathcal{A}\right)$, of the derived category of graded modules, $\mathrm{D}(\mathrm{Gr}\, A)$. Passing through the machinery of Corollary 2.6, we have an isomorphism in $\mathrm{Ho}\left(\mathrm{dgcat}_k\right)$

$$\mathbf{R}\underline{\mathrm{Hom}}_c\left(\mathrm{h\text{-}Proj}\left(\mathcal{A}\right), \mathrm{h\text{-}Proj}\left(\mathcal{B}\right)\right) \cong \mathrm{h\text{-}Proj}\left(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}\right).$$

This allows us to identify an object,

$$F \in \mathbf{R}\underline{\mathrm{Hom}}_c\left(\mathrm{h\text{-}Proj}\left(A\right), \mathrm{h\text{-}Proj}\left(B\right)\right)$$

with a dg $\mathcal{A}$-$\mathcal{B}$-bimodule, P, which in turn corresponds to a morphism $\Phi_P \colon \mathcal{A} \to \mathrm{h\text{-}Proj}\left(\mathcal{B}\right)$ by way of the symmetric monoidal closed structure on dgcat_k.

Following Section 3.3 of [15], we identify the homotopy equivalence class, $[P]_{\mathrm{Iso}}$, of P with $[\Phi_P] \in [\mathcal{A}, \mathrm{h\text{-}Proj}\left(\mathcal{B}\right)]$. The extension of Φ_P,

$$P \otimes_{\mathcal{A}} - = \widehat{\Phi_P} \colon\ \mathrm{h\text{-}Proj}\left(\mathcal{A}\right) \to \mathrm{h\text{-}Proj}\left(\mathcal{B}\right)$$

descends to a morphism $[\widehat{\Phi_P}] \in [\mathrm{h\text{-}Proj}\left(\mathcal{A}\right), \mathrm{h\text{-}Proj}\left(\mathcal{B}\right)]$ and induces a triangulated functor that commutes with coproducts

$$H^0(\widehat{\Phi_P}) \colon \mathrm{D}(\mathrm{Gr}\, A) \longrightarrow \mathrm{D}(\mathrm{Gr}\, B)$$

$$M \longmapsto P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M.$$

In particular, given an equivalence $f \colon \mathrm{D}(\mathrm{Gr}\, A) \to \mathrm{D}(\mathrm{Gr}\, B)$, we obtain from [26] a quasi-equivalence

$$F \colon\ \mathrm{h\text{-}Proj}\left(\mathcal{A}\right) \to \mathrm{h\text{-}Proj}\left(\mathcal{B}\right).$$

Tracing through the remarks above, we obtain an object

$$P \in \mathrm{h\text{-}Proj}\left(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}\right)$$

providing an equivalence

$$H^0(\widehat{\Phi_P})\colon \mathrm{D}(\mathrm{Gr}\, A) \to \mathrm{D}(\mathrm{Gr}\, B).$$

4 Derived Morita Theory for Noncommutative Projective Schemes

Let A and B be left Noetherian, connected graded k-algebras. We want to extend the ideas from Section 3.6 to cover dg-enhancements of $\mathrm{D}(\mathrm{QGr}\, A)$.

4.1 Repeated triangulated justifications

We recall a particularly nice type of property of objects in the setting of compactly generated triangulated categories. Many important properties are of this type, so we name it.

Definition 4.1. Let T be a compactly generated triangulated category. Let `P` be a property of objects of T. We say that `P` is **RTJ** if it satisfies the following three conditions.

- Whenever $A \to B \to C$ is a triangle in T and `P` holds for A and B, then `P` holds for C.
- If `P` holds for A, then `P` holds for the translate $A[1]$.
- Let I be a set and A_i be objects of T for each $i \in I$. If `P` holds for each A_i, then `P` holds for $\bigoplus_{i\in I} A_i$.

We say that `P` is **rtj** if it satisfies the following three conditions.

- Whenever $A \to B \to C$ is a triangle in T and `P` holds for A and B, then `P` holds for C.
- If `P` holds for A, then `P` holds for the translate $A[1]$.
- Let I be a finite set and A_i be objects of T for each $i \in I$. If `P` holds for each A_i, then `P` holds for $\bigoplus_{i\in I} A_i$.

Proposition 4.2. *Let* $\mathtt{P}$ *be an (rtj) RTJ property that holds for a set of compact generators of* T*. Then* $\mathtt{P}$ *holds for all objects of (*T^c*)* T*.*

Proof. Let P be the full triangulated subcategory of objects for which $\mathtt{P}$ holds. Then P

- contains a set of compact generators,
- is triangulated, and
- is closed under formation of (finite) coproducts.

Thus, P is all of (T^c) T. □

4.2 Vanishing of a tensor product

Recall from Section 2.3 that the tensor product of M, N over $\mathcal{A}$ is the truncation of the standard bar complex

$$(M \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N)_\ell = M \otimes_k \underbrace{\mathcal{A} \otimes_k \cdots \otimes_k \mathcal{A}}_{l} \otimes_k N.$$

As k is a field, everything is k-flat. As a consequence, $\overset{\mathbf{L}}{\otimes}_{\mathcal{A}}$ preserves quasi-isomorphisms in each entry. This justifies the notation and gives a particular model for the derived tensor product. In particular, for an h-projective E, the natural map

$$- \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} E \to - \otimes_{\mathcal{A}} E$$

is a quasi-isomorphism.

Definition 4.3. Let M be a complex of graded left A modules and let N be a complex of graded right A modules. We say that the pair satisfies $\bigstar(M, N)$ if the tensor product

$$\mathbf{R}\tau_{A^{\mathrm{op}}} N \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}Q_A M$$

is acyclic. If $\bigstar(M, N)$ holds for all M and N, then we say that A satisfies $\bigstar$.

Proposition 4.4. *Fix a finitely generated connected graded k-algebra, A. Assume that $\mathbf{R}\tau_A$ and $\mathbf{R}\tau_{A^{\mathrm{op}}}$ commute with coproducts. Then A satisfies $\bigstar$ if and only $\bigstar(A(u), A(v))$ holds for each $u, v \in \mathbb{Z}$.*

Proof. The necessity is clear, so assume that $\bigstar(A(u), A(v))$ holds for each $u, v \in \mathbb{Z}$. Note that $\bigstar(M, A(v))$ holds for all v is an RTJ property of M that holds for the set of compact generators $A(u), u \in \mathbb{Z}$. Thus, by Proposition 4.2, $\bigstar(M, A(v))$ holds for all v holds for all M in $\mathrm{D}(\mathrm{Gr}\, A)$. Similarly, we can consider the property of N in $\mathrm{D}(\mathrm{Gr}(A^{\mathrm{op}}))$: $\bigstar(M, N)$ holds for all objects M of $\mathrm{D}(\mathrm{Gr}\, A)$. This is also RTJ so $\bigstar(M, N)$ holds for all M and N. $\square$

There are also analogs of the projection formula in (commutative) algebraic geometry.

Proposition 4.5. *Fix a finitely generated connected graded k-algebra, A. Let P be a complex of bi-bi A modules and let M be a complex of graded left A modules. Assume $\mathbf{R}\tau_A$ commutes with coproducts. There is natural quasi-isomorphism*

$$(\mathbf{R}'\tau_A P) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \to \mathbf{R}\tau_A \left(P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \right).$$

Similarly, assume $\mathbf{R}Q_A$ commutes with coproducts. There is natural quasi-isomorphism

$$(\mathbf{R}'Q_A P) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \to \mathbf{R}Q_A \left(P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} M \right).$$

Proof. We treat the Q projection formula. The τ projection formula is analogous. Levelwise, we have the natural map

$$Q_A P \otimes_k \mathcal{A} \otimes_k \cdots \otimes_k M \to Q_P \left(P \otimes_k \mathcal{A} \otimes_k \cdots \otimes_k M \right)$$

which comes from the following. For any k module N, given

$$\psi \otimes_k N \in \underline{\mathrm{Gr}}A(A/A_{\geq m}, P) \otimes_k N$$

we naturally get

$$\begin{aligned} \widetilde{\psi} : A/A_{\geq m} &\to P \otimes_k N \\ a &\mapsto \psi(a) \otimes n. \end{aligned}$$

Taking the colimit gives the natural transformation. Note this also gives a natural map

$$Q_A P \otimes_{\mathcal{A}} M \to Q_A \left(P \otimes_{\mathcal{A}} M \right).$$

Since Q_A commutes with coproducts, if P is Q_A-acyclic, then so is $P \otimes_k N = P^{\oplus \dim_k N}$ where we interpret $\dim_k N$ as a set. Furthermore, the levelwise map is an isomorphism since, again, Q_A commutes with coproducts.

$\square$

For the hypothesis, recall Definition 3.43.

Proposition 4.6. *Assume A is delightful. Then $\bigstar$ holds for A.*

Proof. By Proposition 4.4, it suffices to check $\bigstar(M, A(v))$ for each v, which is equivalent to checking $\bigstar(M, \bigoplus_v A(v))$ because $\mathbf{R}\tau_{A^{\mathrm{op}}}$ and $- \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} -$ both commute with coproducts. While computing $\bigstar(M, \bigoplus_v A(v))$ only depends on the right A module structure of $\bigoplus_v A(v)$, if we recognize that Δ with its natural bi-bi structure

$$\bigoplus_v A(v) = \bigoplus_v \left(\bigoplus_u A_{u+v} \right) = \bigoplus_{u,v} A_{u+v} = \Delta,$$

restricts to $\bigoplus_v A(v)$ as a right A module, then it is equivalent to check $\bigstar(M, \Delta)$. This observation provides the advantage of being able to utilize the projection formulas as follows.

Apply $- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathbf{R}Q_A M$ to the diagram of Proposition 3.42 to obtain a diagram

$$\begin{tikzcd} \mathbf{R}\tau^{\mathrm{VdB}}_{A \otimes_k A^{\mathrm{op}}} \Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}Q_A M \arrow[d] \arrow[dr] & \\ \mathbf{R}'\tau_A \Delta \otimes^{\mathbf{L}}_{\mathcal{A}} \mathbf{R}Q_A(M) & \mathbf{R}'\tau_{A^{\mathrm{op}}} \Delta \otimes^{\mathbf{L}}_{\mathcal{A}} \mathbf{R}Q_A(M) \end{tikzcd}$$

with all arrows quasi-isomorphisms. We may now apply Proposition 4.5 to the left-hand side, because Δ is a bi-bi module, to obtain the quasi-isomorphism

$$\mathbf{R}'\tau_A \Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}Q_A M \cong \mathbf{R}\tau_A \left(\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}Q_A M \right) \cong \mathbf{R}\tau_A \left(\mathbf{R}Q_A M \right) = 0,$$

as desired. $\square$

4.3 Duality

One can regard the complex of bi-bi modules $\mathbf{R}Q_{A \otimes_k A^{\mathrm{op}}} \Delta$ as a sum of complexes of A modules

$$\mathbf{R}Q_{A \otimes_k A^{\mathrm{op}}} \Delta = \bigoplus_x (\mathbf{R}Q_{A \otimes_k A^{\mathrm{op}}} \Delta)_{*,x}$$

and define for any object, M, of $\mathcal{C}\left(\mathrm{Gr}\, A\right)$ the object

$$\mathbf{R}\underline{\mathrm{Hom}}_A(M, \mathbf{R}Q_{A \otimes_k A^{\mathrm{op}}} \Delta) = \bigoplus_x \mathbf{R} \,\mathrm{Hom}_A(M, \mathbf{R}Q_{A \otimes_k A^{\mathrm{op}}} \Delta_{*,x})$$

of $\mathcal{C}\left(\mathrm{Gr}\left(A^{\mathrm{op}}\right)\right)$. Consider the functor

$$\begin{aligned} (-)^{\vee} : \mathcal{C}\left(\mathrm{Gr}\, A\right)^{\mathrm{op}} &\to \mathcal{C}\left(\mathrm{Gr}\left(A^{\mathrm{op}}\right)\right) \\ M &\mapsto \mathbf{R}\underline{\mathrm{Hom}}_A\left(M, \mathbf{R}Q_{A \otimes_k A^{\mathrm{op}}} \Delta\right) \end{aligned}$$

Note that evaluation provides a natural transformation

$$\eta : \mathrm{Id} \to (-)^{\vee\vee}.$$

We also have the usual duality between left and right modules

$$\begin{aligned} (-)^{*} : \mathcal{C}\left(\mathrm{Gr}\, A\right)^{\mathrm{op}} &\to \mathcal{C}\left(\mathrm{Gr}\left(A^{\mathrm{op}}\right)\right) \\ M &\mapsto \mathbf{R}\underline{\mathrm{Hom}}_A\left(M, \Delta\right) \end{aligned}$$

with the associated natural transformation

$$\nu : \mathrm{Id} \to (-)^{**}.$$

It is easy to see that $(A(x))^*$ is $A(-x)$.

For any bi-bi module P, Hom-tensor adjunction gives a natural isomorphism

$$\mathrm{Hom}_{A^{\mathrm{op}}}(A_{\geq n}, \mathrm{Hom}_A(M, P)) \to \mathrm{Hom}_A(M, \mathrm{Hom}_{A^{\mathrm{op}}}(A_{\geq n}, P)).$$

Passing to the colimit, we get a natural map

$$Q_{A^{\mathrm{op}}}\left(\underline{\mathrm{Hom}}_A(M, P)\right) \to \underline{\mathrm{Hom}}_A(M, Q_{A^{\mathrm{op}}} P)$$

and therefore a natural map

$$\gamma_P : \mathbf{R}Q_{A^{\mathrm{op}}}\left(\mathbf{R}\underline{\mathrm{Hom}}_A(M, P)\right) \to \mathbf{R}\underline{\mathrm{Hom}}_A(M, \mathbf{R}'Q_{A^{\mathrm{op}}} P).$$

Note that the map γ_P is a quasi-isomorphism if we assume that M is compact.

Proposition 4.7. *We have a commutative diagram*

$$\begin{tikzcd}
{} & \mathbf{R}Q_A\mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathbf{R}\underline{\mathrm{Hom}}_A(M,\Delta),\Delta) \arrow[dr, "\gamma_\Delta"] & {} \\
\mathbf{R}Q_A M \arrow[ur, "\mathbf{R}Q_A(\nu_M)"] \arrow[dd, "\eta_{\mathbf{R}Q_A M}"'] & {} & \mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathbf{R}\underline{\mathrm{Hom}}_A(M,\Delta), \mathbf{R}'Q_A\Delta) \arrow[d, "\mathbf{R}Q_{A^{\mathrm{op}}}(-)"] \\
{} & {} & \mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathbf{R}Q_{A^{\mathrm{op}}}\mathbf{R}\underline{\mathrm{Hom}}_A(M,\Delta), \mathbf{R}'Q_{A^{\mathrm{op}}}\mathbf{R}'Q_A\Delta) \arrow[d, "\delta_\Delta \circ -"] \\
\mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{R}Q_A M, \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta), \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta) \arrow[ddr, "\mathbf{R}\underline{\mathrm{Hom}}(\mathbf{R}Q_A M, \delta_\Delta)"'] & {} & \mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathbf{R}Q_{A^{\mathrm{op}}}\mathbf{R}\underline{\mathrm{Hom}}_A(M,\Delta), \mathbf{R}Q_{A\otimes A^{\mathrm{op}}}\Delta) \\
{} & {} & \mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathbf{R}\underline{\mathrm{Hom}}_A(M, \mathbf{R}'Q_{A^{\mathrm{op}}}\Delta), \mathbf{R}Q_{A\otimes A^{\mathrm{op}}}\Delta) \arrow[u, "- \circ \gamma_\Delta"'] \\
{} & \mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{R}Q_A M, \mathbf{R}'Q_A\mathbf{R}'Q_{A^{\mathrm{op}}}\Delta), \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta) \arrow[ur, "- \circ \mathbf{R}Q_A(-)"'] & {}
\end{tikzcd}$$

Proof. The existence of this diagram follows from the existence of the underived version. The underived version is straightforward to verify. We suppress most the details but note that the image of a map $\phi : A_{\geq n_1} \to M$ from $Q_A M$ is the map

$$\begin{aligned}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(A_{\geq n_2}, M^*) &\to \underline{\mathrm{Hom}}_{A \otimes A^{\mathrm{op}}}(A_{\geq n_1} \otimes A_{\geq n_2}, \Delta) \\ \psi &\mapsto \big[a_1 \otimes a_2 \mapsto \psi(a_2)(\phi(a_1))\big].\end{aligned}$$

For clarity: the latter is the evaluation of $\psi(a_2)$ at $\phi(a_1)$. $\square$

Lemma 4.8. *Assume that A is delightful and M is compact object of $\mathrm{D}(\mathrm{Gr}(A))$. Then, $\eta_{\mathbf{R}Q_A M}$ is a quasi-isomorphism.*

Proof. We check that all the other maps in the diagram of Proposition 4.7 are quasi-isomorphisms.

First, since M is compact ν_M is a quasi-isomorphism and $\mathbf{R}Q_A$ preserves quasi-isomorphisms. Hence $\mathbf{R}Q_A \nu_M$ is also a quasi-isomorphism.

Next since M is compact so is M^*. Thus, γ_Δ is a quasi-isomorphism.

Since A is delightful, we know that $\mathbf{R}'Q_A\Delta$ is $\mathbf{R}\tau_{A^{\mathrm{op}}}$ torsion-free. Thus, applying $\mathbf{R}Q_{A^{\mathrm{op}}}$ to maps from any object to $\mathbf{R}'Q_A\Delta$ yields a quasi-isomorphism of morphism spaces.

The map δ_Δ is a quasi-isomorphism by Proposition 3.41. Derived morphism spaces preserve quasi-isomorphisms so it follows that $\delta_\Delta \circ -$ is a quasi-isomorphism.

Since M is compact, γ_Δ is a quasi-isomorphism and so is composition with it since derived morphism spaces preserve quasi-isomorphisms.

We again appeal to A being delightful to know that $\mathbf{R}'Q_{A^{\mathrm{op}}}\Delta$ is $\mathbf{R}\tau_A$ torsion free. Thus, $\mathbf{R}Q_A$ acting on morphism spaces from anything to $\mathbf{R}'Q_{A^{\mathrm{op}}}\Delta$ yields a quasi-isomorphism. Again composition with it remains a quasi-isomorphism.

Finally $(-)^\vee$ and $\mathbf{R}\underline{\mathrm{Hom}}(\mathbf{R}Q_A M, -)$ preserve quasi-isomorphisms so $\mathbf{R}\underline{\mathrm{Hom}}(\mathbf{R}Q_A M, \delta_\Delta)^\vee$ is also a quasi-isomorphism. $\square$

Lemma 4.9. *Assume that A is delightful. Then, there is a natural quasi-isomorphism between $\mathbf{R}Q_{A^{\mathrm{op}}}A(-x)$ and $(\mathbf{R}Q_A A(x))^\vee$.*

Proof. We have the following sequence of maps:

$$\begin{tikzcd}
\mathbf{R}Q_{A^{\mathrm{op}}}\mathbf{R}\underline{\mathrm{Hom}}_A(A(x),\Delta) \arrow[r, "\gamma_\Delta"] & \mathbf{R}\underline{\mathrm{Hom}}_A(A(x),\mathbf{R}'Q_{A^{\mathrm{op}}}\Delta) \arrow[dl, "\mathbf{R}Q_A(-)"] \\
\mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{R}Q_A A(x),\mathbf{R}'Q_A\mathbf{R}'Q_{A^{\mathrm{op}}}\Delta). &
\end{tikzcd}$$

The former is a quasi-isomorphism because $A(x)$ is compact. The latter is a quasi-isomorphism because A is delightful and so $\mathbf{R}'Q_{A^{\mathrm{op}}}\Delta$ is $\mathbf{R}\tau_A$ torsion free.

Using Proposition 3.41 gives a quasi-isomorphism

$$\mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{R}Q_A A(x),\mathbf{R}'Q_A\mathbf{R}'Q_{A^{\mathrm{op}}}\Delta) \cong$$
$$\mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{R}Q_A A(x),\mathbf{R}Q_{A\otimes A^{\mathrm{op}}}\Delta) = (\mathbf{R}Q_A A(x))^\vee.$$

On the other side, we have

$$A(-x) \cong \underline{\mathrm{Hom}}_A(A(x),\Delta) \cong \mathbf{R}\underline{\mathrm{Hom}}_A(A(x),\Delta).$$

Applying $\mathbf{R}Q_{A^{\mathrm{op}}}$ gives the final quasi-isomorphisms. □

Definition 4.10. Let $Q\mathcal{A}$ be the full dg-subcategory of $\mathcal{C}\,(\mathrm{Gr}\,A)$ with objects given by

$$\mathbf{R}\omega_A\pi A(x) := \omega_A R_{\mathrm{QGr}\,A}\pi_A A(x)$$

for all $x \in \mathbb{Z}$. Here $R_{\mathrm{QGr}\,A}$ is the fibrant replacement functor in $C(\mathrm{QGr}\,A)$.

Note that, since $\omega_A\pi_A \cong \mathrm{Id}$, all objects of $Q\mathcal{A}$ satisfy the condition that $\epsilon_M : M \to Q_A M$ is an isomorphism of complexes. Similarly, since ω_A has an exact left adjoint, ω_A preserves fibrations. Hence, each $\mathbf{R}\omega_A\pi A(x)$ is fibrant as a complex of graded A modules.

Lemma 4.11. *There exists a map of chain complexes*

$$\mathbf{R}Q_A A(x) \to \mathbf{R}\omega_A\pi_A A(x)$$

which is a quasi-isomorphism.

Proof. Consider the diagram

$$\begin{tikzcd}
\pi_A M \arrow[r] \arrow[d] & R_{\mathrm{QGr}\,A}\pi_A M \arrow[d] \\
\pi_A R_{\mathrm{Gr}\,A} M \arrow[r] & 0.
\end{tikzcd}$$

Since π_A is exact, the map $\pi_A M \to \pi_A R_{\mathrm{Gr}\,A} M$ is a trivial cofibration. Hence there exists a lift

$$\begin{tikzcd}
\pi_A M \arrow[r] \arrow[d] & R_{\mathrm{QGr}\,A}\pi_A M \arrow[d] \\
\pi_A R_{\mathrm{Gr}\,A} M \arrow[r] \arrow[ur] & 0.
\end{tikzcd}$$

which must also be a quasi-isomorphism. □

Corollary 4.12. *Assume that A is delightful. There is a quasi-equivalence between $\overline{(Q\mathcal{A})^{\mathrm{op}}}$ and $\overline{Q(\mathcal{A}^{\mathrm{op}})}$ which is isomorphic to $(-)^\vee$ at the level of derived categories.*

Proof. We can choose a h-injective complex bi-bi modules I such that there is a homotopy equivalence $\mathbf{R}Q_{A\otimes A^{\mathrm{op}}}\Delta \to I$ and the natural maps $I \to Q_{A^{\mathrm{op}}}I$ and $I \to Q_A I$ are isomorphisms. This comes from taking a cofibrant replacement of $\pi_{A\otimes A^{\mathrm{op}}}\mathbf{R}Q_{A\otimes A^{\mathrm{op}}}\Delta$ and applying $\omega_{A\otimes A^{\mathrm{op}}}$. Then, we have a homotopy equivalence

$$\mathbf{R}\underline{\mathrm{Hom}}_A(M, \mathbf{R}Q_{A\otimes A^{\mathrm{op}}}\Delta) \to \underline{\mathrm{Hom}}_A(M, I)$$

and we may replace $(-)^\vee$ by $\underline{\mathrm{Hom}}_A(M, I)$. We do so but we keep the same notation. Note the image of $(-)^\vee$ now consists of h-injective graded A^{op} modules.

From Lemma 4.8, we see $(-)^\vee$ is quasi-fully faithful on $Q\mathcal{A}$ and from Lemma 4.9 we see there is a quasi-isomorphism between $(\mathbf{R}Q_A A(x))^\vee$ and $\mathbf{R}Q_{A^{\mathrm{op}}}A(-x)$. It follows that $\mathbf{R}\omega_{A^{\mathrm{op}}}\pi_{A^{\mathrm{op}}}A(-x)$ and $(\mathbf{R}\omega_A\pi_A A(x))^\vee$ are quasi-isomorphic h-injective complexes. Hence, they are homotopy equivalent. Therefore, dg-functor

$$\begin{aligned}
\Xi : (Q\mathcal{A})^{\mathrm{op}} &\to \mathrm{dgMod}(Q\mathcal{A}^{\mathrm{op}}) \\
M &\mapsto \mathrm{Hom}(-, M^\vee)
\end{aligned}$$

has image homotopy equivalent to representable modules. Consequently, Ξ is quasi-fully-faithful overall. Its image consists of h-projective set of generators.

We get a induced functor

$$\overline{\Xi} : \overline{(Q\mathcal{A})^{\mathrm{op}}} \to \text{h-Proj}\, Q\mathcal{A}^{\mathrm{op}}.$$

The property that $\overline{\Xi}(M)$ is h-projective is rtj in M, as is the property that $\overline{\Xi}(M)$ is compact. Applying Proposition 4.2 shows that $\overline{\Xi}$ lands in $\overline{Q\mathcal{A}^{\mathrm{op}}}$.

The property that $\overline{\Xi}$ is quasi-fully faithful on $\mathrm{Hom}(M, N)$ for all N is rtj in M. Thus it suffices to prove that $\overline{\Xi}$ is quasi-fully faithful on $\mathrm{Hom}(C, N)$ for all N and some set of compact generators C. But since C is compact the condition, that $\overline{\Xi}$ is quasi-fully-faithful $\mathrm{Hom}(C, N)$ is rtj for N. Thus, we can reduce to checking quasi-fully-faithfulness on a set of compact generators. But we already saw that Ξ is quasi-fully-faithful on $Q\mathcal{A}$.

Finally, the condition that L is homotopy equivalent to $\overline{\Xi}(M)$ for some M is rtj in L and is true for a compact set of generators. Again Proposition 4.2 allows us to conclude that $\overline{\Xi}$ is quasi-essentially surjective and hence a quasi-equivalence. $\square$

We have a natural map

$$\begin{aligned} M^{\vee} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N &\to \mathbf{R}\underline{\mathrm{Hom}}_A(M, \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N) \\ \phi \otimes n &\mapsto (m \mapsto \phi(m) \otimes n). \end{aligned}$$

Lemma 4.13. *Assume that A is delightful, the map $N \to \mathbf{R}Q_A N$ is a quasi-isomorphism, and M is quasi-isomorphic to $\mathbf{R}Q_A M'$ for compact M'. Then both natural maps in the diagram*

$$\begin{tikzcd} M^{\vee} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N \arrow[r] & \mathbf{R}\underline{\mathrm{Hom}}_A(M, \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N) \\ & \mathbf{R}\underline{\mathrm{Hom}}_A(M, N) \arrow[u] \end{tikzcd}$$

are quasi-isomorphisms.

Proof. The vertical map comes from the map $\Delta \to \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta$. We have a triangle

$$\mathbf{R}\tau_{A\otimes_k A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N \to \Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N \to \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N.$$

Using Proposition 4.6 and the assumption that $N \to \mathbf{R}Q_A N$ is a quasi-isomorphisms, we have quasi-isomorphisms

$$\mathbf{R}\tau_{A\otimes_k A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N \cong \mathbf{R}\tau_{A\otimes_k A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}Q_A N \cong \mathbf{R}\tau_{A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}Q_A N \cong 0.$$

Thus, the map

$$N \cong \Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N \to \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N$$

is a quasi-isomorphism. So the map

$$\mathbf{R}\underline{\mathrm{Hom}}_A(M, N) \to \mathbf{R}\underline{\mathrm{Hom}}_A(M, \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N)$$

is also a quasi-isomorphism.

The property that the map

$$M^\vee \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N \to \mathbf{R}\underline{\mathrm{Hom}}_A(M, \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N)$$

is a quasi-isomorphism is rtj. Thus, we may reduce to checking the case $M = \mathbf{R}Q_A A(x)$ by Proposition 4.2. We have a commutative diagram

$$\begin{tikzcd}
\mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{R}Q_A A(x), \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N \arrow[r] \arrow[d] & \mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{R}Q_A A(x), \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N) \arrow[d] \\
\mathbf{R}\underline{\mathrm{Hom}}_A(A(x), \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N \arrow[r] & \mathbf{R}\underline{\mathrm{Hom}}_A(A(x), \mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N)
\end{tikzcd}$$

coming from the map $A(x) \to \mathbf{R}Q_A A(x)$. Since

$$\mathbf{R}Q_{A\otimes_k A^{\mathrm{op}}}\Delta \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N \quad \text{and} \quad \mathbf{R}Q_{A\otimes A^{\mathrm{op}}}\Delta$$

are $\mathbf{R}\tau_A$ torsion free, the vertical maps are quasi-isomorphisms. Since $A(x)$ is compact, $\mathbf{R}\underline{\mathrm{Hom}}_A(A(x), -) \cong \underline{\mathrm{Hom}}_A(A(x), -)$ commutes with coproducts. Consequently, $\underline{\mathrm{Hom}}_A(A(x), -)$ commutes with $\otimes_k$ and hence

$$\underline{\mathrm{Hom}}_A(A(x), P) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N \to \underline{\mathrm{Hom}}_A(A(x), P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} N)$$

is an isomorphism levelwise for any complex of bi-bi modules P. $\square$

4.4 Products

The following allows to re-express the tensor up to quasi-equivalence.

Proposition 4.14. *If A and B are both Ext-finite, left Noetherian, right Noetherian, and $\mathbf{R}\tau_A$ and $\mathbf{R}\tau_B$ have finite dimension, then the dg-functor*

$$\begin{aligned}\Upsilon : \text{h-Inj}(\mathrm{QGr}\, A \otimes_k B) &\to \text{h-Proj}(Q\mathcal{A} \otimes_k Q\mathcal{B}) \\ I &\mapsto \mathrm{Hom}_{\mathcal{C}(\mathrm{Gr}\, A\otimes_k B)}(R_{A\otimes_k B}(- \boxtimes -), \omega_{A\otimes_k B}(I))\end{aligned}$$

is well-defined and is a quasi-equivalence.

Proof. Note here that $M \boxtimes_k N$ is simply $M \otimes_k N$ as a complex of bi-bimodules.

We first reduce to the images of $\boxtimes$ and $\omega_{A\otimes_k B}$ inside $\mathcal{C}(\mathrm{Gr}(A \otimes_k B))$. Similar to Lemma 3.29, $\omega_{A\otimes_k B}$ is quasi-fully-faithful.

Next we check that $\boxtimes_k$ is quasi-fully-faithful. For general $M, M' \in \mathcal{C}(\mathrm{Gr}\, A)$ and $N, N' \in \mathcal{C}(\mathrm{Gr}\, B)$, we have a commutative diagram of natural maps

$$\begin{array}{ccc} \mathrm{Hom}(M, M') \otimes_k \mathrm{Hom}(N, N') & \xrightarrow{\boxtimes} & \mathrm{Hom}(M \otimes_k M', N \otimes_k N') \\ \downarrow & & \downarrow \\ \mathrm{Hom}(M, M' \otimes_k \mathrm{Hom}(N, N')) & \longrightarrow & \mathrm{Hom}(M, \mathrm{Hom}(N, M' \otimes_k N')) \end{array} \tag{2}$$

where the right vertical map is an isomorphism coming from Hom-$\otimes$ adjunction. If M and N are compact objects, then the other maps are quasi-isomorphisms. Since

$$\mathrm{Hom}(\mathbf{R}Q_A M, \mathbf{R}Q_A M'') \to \mathrm{Hom}(M, \mathbf{R}Q_A M'')$$
$$\mathrm{Hom}(\mathbf{R}Q_B N, \mathbf{R}Q_B N'') \to \mathrm{Hom}(N, \mathbf{R}Q_B N'')$$

are quasi-isomorphisms and all objects in the previous diagram are $\mathbf{R}\tau$-torsion free, it suffices to know that M is quasi-isomorphic to $\mathbf{R}Q_A\tilde{M}$ for some compact $\tilde{M}$ and similarly for N. So if we restrict attention to the objects $\mathbf{R}Q_A A(x)$ and $\mathbf{R}Q_B B(y)$, then all the maps in Diagram 2 are quasi-isomorphisms and $\boxtimes$ is quasi-fully-faithful.

Next we check that fibrant replacement

$$R := R_{\mathrm{Gr}\, A \otimes_k B} : Q\mathcal{A} \otimes_k Q\mathcal{B} \to R(Q\mathcal{A} \otimes_k Q\mathcal{B})$$

is a quasi-equivalence, where $R(Q\mathcal{A} \otimes_k Q\mathcal{B})$ denotes the essential image. For general complexes M, M' of graded A modules and N, N' of graded B modules, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(M \otimes_k N, M' \otimes_k N') \arrow[dd] \arrow[dr, "R"] & \\
& \mathrm{Hom}(R(M \otimes_k N), R(M' \otimes_k N')) \arrow[dl] \\
\mathrm{Hom}(M \otimes_k N, R(M' \otimes_k N')) &
\end{tikzcd}$$

The bottom diagonal map is a quasi-isomorphism since $R(M' \otimes_k N')$ is fibrant and the natural map $M \otimes_k N \to R(M \otimes_k N)$ is a quasi-isomorphism. The previous computation shows that, when M and N are compact, the tensor product of h-injective complexes is acyclic for $\mathrm{Hom}(M \otimes_k N, -)$ while $R(M' \otimes_k N')$ is fibrant and hence acyclic. Thus, taking M, M' from $Q\mathcal{A}$ and N, N' from $Q\mathcal{B}$, we see that the left vertical map is a quasi-isomorphism as it comes from applying a functor to a quasi-isomorphism between acyclic objects. Consequently, R is a quasi-fully-faithful. By definition of the image, it is essentially surjective and therefore is a quasi-equivalence.

Using morphisms in $\mathcal{C}(\mathrm{Gr}\, A \otimes_k B)$, for an object I of $\mathrm{h\text{-}Inj}(\mathrm{QGr}\, A \otimes_k B)$ we have a dg module

$$\begin{aligned} R(Q\mathcal{A} \otimes_k Q\mathcal{B})^{\mathrm{op}} &\to \mathcal{C}(k) \\ M &\mapsto \mathrm{Hom}(M, \omega(I)). \end{aligned}$$

which induces a dg-functor

$$\mathrm{h\text{-}Inj}(\mathrm{QGr}\, A \otimes_k B) \to \mathrm{dgMod}\, R(Q\mathcal{A} \otimes_k Q\mathcal{B})$$

We want to check that this induces an equivalence with $\text{h-Proj}\, R(Q\mathcal{A} \otimes_k Q\mathcal{B})$. If so, since Υ is a composition of this functor and R, which was already shown to be a quasi-equivalence, we get the desired conclusion.

Note first that Υ commutes with coproducts, up to quasi-isomorphism. Indeed, we have a sequence of quasi-isomorphisms: the map

$$\begin{array}{c} \mathrm{Hom}(R(\mathbf{R}\omega_A\pi_A A(x) \otimes_k \mathbf{R}\omega_B\pi_B B(y)), \omega_{A\otimes_k B} I) \\ \downarrow \\ \mathrm{Hom}(\mathbf{R}\omega_A\pi_A A(x) \otimes_k \mathbf{R}\omega_B\pi_B B(y), \omega_{A\otimes_k B} I) \end{array}$$

since $P \to RP$ is a quasi-isomorphism and $\omega_{A\otimes_k B} I$ is h-injective. Using adjunction, we have an isomorphism

$$\begin{gathered} \mathrm{Hom}_{\mathrm{Gr}\, A\otimes_k B}(\mathbf{R}\omega_A\pi_A A(x) \otimes_k \mathbf{R}\omega_B\pi_B B(y), \omega_{A\otimes_k B} I) \cong \\ \mathrm{Hom}_{\mathrm{QGr}\, A\otimes_k B}(R_{\mathrm{QGr}\, A}\pi_A A(x) \otimes_k R_{\mathrm{QGr}\, B}\pi_B B(y), I). \end{gathered}$$

Again since I is h-injective and $\otimes_k$ preserves quasi-isomorphisms since k is field, we get a quasi-isomorphism

$$\begin{array}{c} \mathrm{Hom}(R_{\mathrm{QGr}\, A}\pi_A A(x) \otimes_k R_{\mathrm{QGr}\, B}\pi_B B(y), I) \\ \downarrow \\ \mathrm{Hom}(\pi_A A(x) \otimes_k \pi_B B(y), I). \end{array}$$

Using adjunction again, we have an isomorphism

$$\begin{gathered} \mathrm{Hom}_{\mathrm{QGr\, A\otimes_k B}}(\pi_A A(x) \otimes_k \pi_B B(y), I) \cong \\ \mathrm{Hom}_{\mathrm{Gr}\, A\otimes_k B}(A(x) \otimes_k B(y), \omega_{A\otimes_k B} I). \end{gathered}$$

The end of these chains of quasi-isomorphisms commutes with coproducts since $\mathbf{R}\omega_{A\otimes_k B}$ commutes with coproducts with both A and B Ext-finite by Proposition 3.36.

Next, we note that

$$\Upsilon(R(\pi_A A(x) \otimes \pi_B B(y))) = \mathrm{Hom}(-, \mathbf{R}\omega_{A\otimes_k B}\pi_A A(x) \otimes_k \pi_B B(y))$$

and $\mathbf{R}\omega_{A\otimes_k B}\pi_A A(x) \otimes_k \pi_B B(y))$ and $R(\mathbf{R}\omega_A \pi_A A(x) \otimes \mathbf{R}\omega_B \pi_B B(y)$ are quasi-isomorphic. Since both objects are h-injective, they are homotopy equivalent. Thus, $\Upsilon(R(\pi_A A(x) \otimes \pi_B B(y)))$ is homotopy equivalent to a representable functor. Consequently, Υ is quasi-fully-faithful on the full subcategory consisting of the $R(\pi_A A(x) \otimes \pi_B B(y)$. Moreover, the images of these complexes are h-projective dg modules since h-projectivity is preserved under homotopy equivalence and representable modules are h-projective.

The property that $\Upsilon(I)$ is h-projective is RTJ in I since Υ commutes with coproducts. Thus, by Proposition 4.2, we see that Υ has image in $\text{h-Proj}\, R(Q\mathcal{A} \otimes Q\mathcal{B})$.

Similarly, the property that Υ is quasi-fully-faithful on $\operatorname{Hom}(M, N)$ for all N is RTJ. By Proposition 4.2 we just need to check that

$$R_{A\otimes_k B}(\pi_A A(x) \otimes \pi_B B(y))$$

have this property. But, the property that Υ is quasi-fully-faithful on

$$\operatorname{Hom}(R_{A\otimes_k B}(\pi_A A(x) \otimes \pi_B B(y)), M)$$

is RTJ in M. This is due to the quasi-isomorphism

$$\begin{array}{c} \operatorname{Hom}_{\operatorname{QGr} A\otimes_k B}(R\pi_A A(x) \otimes \pi_B B(y), I) \\ \big\downarrow \\ \operatorname{Hom}_{\operatorname{Gr} A\otimes_k B}(A(x) \otimes_k B(y), \omega_{A\otimes_k B} I) \end{array}$$

plus the facts that $\omega_{A\otimes_k B}$ commutes with coproducts and $A(x) \otimes_k B(y)$ are compact.

Thus, we just need to check that Υ is quasi-fully-faithful on the full subcategory consisting of the $R_{A\otimes_k B}(\pi_A A(x) \otimes \pi_B B(y))$ which we have seen. We can conclude that Υ is quasi-fully-faithful overall.

The property that $J \cong \Upsilon(I)$ in the homotopy category is RTJ in J and is satisfied by a compact set of generators. Thus, Υ is quasi-essentially surjective. Hence, Υ is a quasi-equivalence.

$\square$

4.5 The quasi-equivalence

Now we turn to the main result.

Theorem 4.15. *Let k be a field. Let A and B be connected graded k-algebras. If A and B form a delightful couple, then there is a natural quasi-equivalence*

$$F : \text{h-Inj}\left(\mathrm{QGr}(A^{\mathrm{op}} \otimes_k B)\right) \to \mathbf{R}\underline{\mathrm{Hom}}_c\left(\text{h-Inj}\left(\mathrm{QGr}\, A\right), \text{h-Inj}\left(\mathrm{QGr}\, B\right)\right)$$

such that for $P \in \mathrm{D}(\mathrm{QGr}(A^{\mathrm{op}} \otimes_k B))$, the exact functor $H^0(F(P))$ is isomorphic to

$$\Phi_P(M) := \pi_B\left(\mathbf{R}\omega_{A^{\mathrm{op}}\otimes_k B} P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}\omega_A M\right).$$

Proof. Appealing to Proposition 4.14 we have quasi-equivalences

$$\text{h-Inj}(\mathrm{QGr}\, A) \cong \text{h-Proj}(Q\mathcal{A})$$
$$\text{h-Inj}(\mathrm{QGr}\, B) \cong \text{h-Proj}(Q\mathcal{B}).$$

As $\mathbf{R}\underline{\mathrm{Hom}}_c(-,-)$ preserves quasi-equivalences, it suffices to construct a quasi-equivalence between $\mathbf{R}\underline{\mathrm{Hom}}_c(\text{h-Proj}\, Q\mathcal{A}, \text{h-Proj}\, Q\mathcal{B})$ and $\text{h-Inj}(\mathrm{QGr}\, A \otimes_k B)$.

Applying Corollary 2.6, it then suffices to provide a quasi-equivalence

$$\text{h-Inj}\left(\mathrm{QGr}(A^{\mathrm{op}} \otimes_k B)\right) \cong \text{h-Proj}\left((Q\mathcal{A})^{\mathrm{op}} \otimes Q\mathcal{B}\right)$$

In general, the inclusion $\mathcal{C} \to \overline{\mathcal{C}}$ induces a quasi-equivalence

$$\text{h-Proj}\left(\mathcal{C} \otimes \mathcal{D}\right) \cong \text{h-Proj}\left(\overline{\mathcal{C}} \otimes \mathcal{D}\right)$$

for two small dg-categories $\mathcal{C}$ and $\mathcal{D}$.

Using Corollary 4.12, we have a quasi-equivalence

$$\text{h-Proj}\left(\overline{(Q\mathcal{A})^{\mathrm{op}}} \otimes Q\mathcal{B}\right) \cong \text{h-Proj}\left(\overline{Q\mathcal{A}^{\mathrm{op}}} \otimes Q\mathcal{B}\right)$$

and thus a quasi-equivalence

$$\text{h-Proj}\left((Q\mathcal{A})^{\mathrm{op}} \otimes Q\mathcal{B}\right) \cong \text{h-Proj}\left(Q\mathcal{A}^{\mathrm{op}} \otimes Q\mathcal{B}\right).$$

From Proposition 4.14 we have the quasi-equivalence

$$\Upsilon : \text{h-Inj}(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B) \to \text{h-Proj}(Q\mathcal{A}^{\mathrm{op}} \otimes_k Q\mathcal{B}).$$

Combining these gives the desired quasi-equivalence.

Tracing out the quasi-equivalences, one just needs to note that

$$\begin{gathered}
\mathrm{Hom}(\mathbf{R}Q_A A(x)^\vee \otimes_k \mathbf{R}Q_B B(y), P) \cong \\
\mathrm{Hom}(\mathbf{R}Q_B B(y), \mathrm{Hom}(\mathbf{R}Q_A A(x)^\vee, \mathbf{R}\omega_{A^{\mathrm{op}}\otimes_k B} P)) \cong \\
\mathrm{Hom}(\mathbf{R}Q_B B(y), \mathbf{R}\omega_{A^{\mathrm{op}}\otimes_k B} P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}Q_A A(x))
\end{gathered}$$

using Proposition 4.6 and Lemma 4.13. This says that the induced continuous functor is

$$M \mapsto \pi_B \left(\mathbf{R}\omega_{A^{\mathrm{op}}\otimes_k B} P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathbf{R}\omega_A M \right).$$

□

The following statement is now a simple application of Theorem 4.15 and results of [26].

Corollary 4.16. *Let k be a field. Let A and B be a delightful couple of connected graded k-algebras. Assume that there exists an equivalence*

$$f : \mathrm{D}(\mathrm{QGr}\, A) \to \mathrm{D}(\mathrm{QGr}\, B).$$

Then there exists an object $P \in D(\mathrm{QGr}(A^{\mathrm{op}} \otimes_k B))$ such that

$$\Phi_P : \mathrm{D}(\mathrm{QGr}\, A) \to \mathrm{D}(\mathrm{QGr}\, B)$$

is an equivalence.

Proof. Applying [26, Theorem 1] we know there is a quasi-equivalence between the unique enhancements, i.e. there is an

$$F \in [\text{h-Inj}\,(\mathrm{QGr}\, A)\,, \text{h-Inj}\,(\mathrm{QGr}\, B)]$$

giving an equivalence

$$\begin{tikzcd} H^0(\text{h-Inj}\,(\mathrm{QGr}\,A)) \arrow[r, equal] \arrow[d, "H^0(F)"] & \mathrm{D}(\mathrm{QGr}\,A) \arrow[d] \\ H^0(\text{h-Inj}\,(\mathrm{QGr}\,B)) \arrow[r, equal] & \mathrm{D}(\mathrm{QGr}\,B) \end{tikzcd}$$

Then, by Theorem 4.15, there exists a $P \in \mathrm{D}(\mathrm{QGr}(A^{\mathrm{op}} \otimes_k B))$ such that $\Phi_P = H^0(F)$. $\square$

Remark 4.17. In particular, one can ask what the kernel associated to the identity functor on $\mathrm{D}(\mathrm{QGr}\,A)$. In this case, it is easy to see that

$$\Phi_{\pi_{A\otimes_k A^{\mathrm{op}}}\Delta} \cong \mathrm{Id}_{\mathrm{D}(\mathrm{QGr}\,A)}$$

justifying the notation.

One also has a statement for bounded and finitely generated category which is analogous to Orlov's theorem for equivalences between bounded derived categories of coherent sheaves on smooth and projective varieties.

Corollary 4.18. *Let A and B be a delightful couple of connected graded k-algebras with k a field. Assume that the enhancements of* $\mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\,A)$ *and* $\mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\,B)$ *are both smooth and proper as dg-categories. If there exists an equivalence*

$$f : \mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\,A) \to \mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\,B),$$

then there exists an object $P \in \mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\,A^{\mathrm{op}} \otimes_k B)$ *such that*

$$\Phi_P : \mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\,A) \to \mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\,B)$$

is an equivalence.

Proof. Since any generator of a smooth and proper dg-category must be a strong generator, both $\mathrm{QGr}\,A$ and $\mathrm{QGr}\,B$ necessarily have finite cohomological dimension. From [12, Lemma 3.4.2], we know that the compact objects in $\mathrm{D}(\mathrm{QGr}\, A)$ are exactly $\mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A)$. Using Corollary 4.12, we see that the enhancement of $\mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A^{\mathrm{op}})$ is smooth and proper and that

$$\mathrm{D}(\mathrm{QGr}\, A^{\mathrm{op}})^{c} \cong \mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A^{\mathrm{op}}).$$

Applying [26, Theorem 2.8] we know that there is a quasi-equivalence between the unique enhancements, i.e. there is an

$$F \in [\text{h-Inj}\,(\mathrm{QGr}\, A)^{c}\,, \text{h-Inj}\,(\mathrm{QGr}\, B)^{c}]$$

giving an equivalence

$$\begin{tikzcd}
H^0(\text{h-Inj}\,(\mathrm{QGr}\, A)^{c}) \arrow[r, equal] \arrow[d, "H^0(F)"] & \mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A) \arrow[d] \\
H^0(\text{h-Inj}\,(\mathrm{QGr}\, B)^{c}) \arrow[r, equal] & \mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, B)
\end{tikzcd}$$

There is then an induced quasi-equivalence between the two big categories, $\text{h-Inj}(\mathrm{QGr}\, A)$ and $\text{h-Inj}(\mathrm{QGr}\, B)$, and hence corresponds to an object in $\mathrm{D}(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)$ by Theorem 4.15. Since the induced functor takes compact objects to compact objects and $\text{h-Inj}\,(\mathrm{QGr}\, A)^{c}$ (an enhancement of $\mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A)$) is smooth as a dg-category, [39, Lemma 2.8] says the corresponding object of $\mathrm{D}(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)$ is a compact object. As $\mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A^{\mathrm{op}} \otimes_k B)$ is also smooth and proper, the same arguments of [12] give that

$$\mathrm{D}(\mathrm{QGr}\, A^{\mathrm{op}} \otimes_k B)^{c} \cong \mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A^{\mathrm{op}} \otimes_k B).$$

□

We wish to identify the kernels as objects of the derived category of an honest noncommutative projective scheme. In general, one can only hope that kernels obtained as above are objects of the derived category of a noncommutative (bi)projective scheme. However, we have the following special case in which we can collapse the $\mathbb{Z}^2$-grading to a $\mathbb{Z}$-grading.

Corollary 4.19. *Let A and B be a delightful couple of connected graded k-algebras with k a field that are both generated in degree one. Assume that there exists an equivalence*

$$f : \mathrm{D}(\mathrm{QGr}\, A) \to \mathrm{D}(\mathrm{QGr}\, B).$$

Then there exists an object $P \in D(\mathrm{QGr}(A^{\mathrm{op}} \times_k B))$ that induces an equivalence

$$\begin{aligned} \mathrm{D}(\mathrm{QGr}\, A) &\longrightarrow \mathrm{D}(\mathrm{QGr}\, B) \\ M &\longmapsto \pi_B \left(\mathbb{V}(P) \otimes^{\mathbf{L}} \mathbf{R}\omega_A M\right) \end{aligned}$$

Proof. The equivalence $\mathbb{V}$ of Theorem 3.46 extends naturally to a quasi-equivalence

$$\mathbb{V}\colon\ \text{h-Inj}\,(\mathrm{QGr}\, S) \to \text{h-Inj}\,(\mathrm{QGr}\, T)\,.$$

Now choose P such that $\mathbb{V}(P)$ is homotopy equivalent to the kernel obtained by an application of Corollary 4.16, so the desired equivalence is $\Phi_{\mathbb{V}(P)}$. □

Coming back to Example 3.3. We ask the following question.

Question 4.20. *Fix $q_{ij} \in \mathbf{C}$. Then two noncommutative projective schemes A_q^{ϕ} and $A_q^{\phi'}$ are isomorphic if and only if they are derived equivalent.*

In the commutative case, this is a derived Torelli statement which one can understand via matrix factorizations [30] and the Mather-Yau theorem [27].

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