College Algebra

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Chapter 1

The Cartesian Plane

1.1 Sets

In the sciences, knowledge is often derived from experimentation in a process known as *inductive reasoning*. While these methods may provide insight into a problem, inductive reasoning is ***not*** a valid method of producing mathematical truth. Instead, mathematicians produce new knowledge from known truths using logical inference in a process known as *deductive reasoning*. In fact, all mathematical knowledge is derived from only *nine* statements that are assumed to be true — called **axioms**. These axioms and the truths derived from them are known as Zermelo-Fraenkel Set Theory[1].

For this reason, sets and set operations are unavoidable in any careful treatment of mathematics. While our treatment of algebra is careful, this is ***not*** meant to be a course on the foundations of mathematics. We present here the rudiments of a set theory that includes only the pieces absolutely necessary for the development of the material relevant to basic algebra. We begin with the intuitive definition of a set.

1.1.1 Representations of Sets

Definition 1.1.1 A **set** is a collection of objects called **elements** or **members** of the set. We specify that an object x is an element of the set S by writing $x \in S$ and say "x is an element (or member) of S" $\Diamond$

Sets are effectively mathematical containers for *things*. While there is no restriction on what kinds of *things* we can place in a set, we will focus on sets that contain numbers. For small collections of *things*, we can write down the elements explicitly.

Definition 1.1.2 Roster Notation. Roster notation specifies the elements of a set as a comma separated list surrounded by curly braces. $\Diamond$

Most interesting sets either have more elements than we *want* to write down or more elements than we *can* write down. For large sets, we use a sort of description of the elements.

Definition 1.1.3 Set-Builder Notation. Set-builder notation specifies the members of a set using a variable and condition for membership in the set. $\Diamond$

[1] `en.wikipedia.org/wiki/Zermelo-Fraenkel_set_theory`

The condition for membership must always be a statement that is either true or false. It allows us to test whether a *thing* belongs in the set.

Example 1.1.4 Roster and Set-Builder Notation.

(a) Express the set with elements 1, 2, 3, and 4 in Roster Notation.

(b) Express the set that contains all positive, even integers in Set-Builder Notation.

□

1.1.2 Subsets and Equality

In order to work with sets effectively, we need a way to compare sets analogous to how we compare numbers. The simplest comparison we can make is *sameness*. For numbers, this is important because the same number can have two different representations, such as

$$4 = \frac{8}{2} \quad \text{or} \quad 7 = 3 \times 2 + 1.$$

The idea for sets is the same —a set might have more than one representation.

Definition 1.1.5 Let A and B be sets. We say the sets A and B are **equal** and write $A = B$ if A and B have the exact same elements. $\Diamond$

Example 1.1.6 The sets $\{1, 2, 3\}$ and $\{3, 1, 2\}$ are equal. $\square$

Example 1.1.7 The sets $\{1, 2, 3\}$ and $\{1, 2, 4\}$ are ***not*** equal. $\square$

Sets have a particularly odd quirk: they do ***not*** keep track of repetition.

Example 1.1.8 The sets $\{1, 1, 1, 1\}$ and $\{1\}$ are equal. $\square$

The other comparison we can make is an *order* relationship. For any two numbers, we indicate which one is the smaller of the two using the symbol $\leq$, such as

$$2 \leq 2 \quad \text{or} \quad \pi \leq 4.$$

For sets, we have a similar sort of ordering.

Definition 1.1.9 Subset. Let A and B be sets. We say that A is a **subset** of B and write $A \subseteq B$ if every element of A is also an element of B. $\Diamond$

Remark 1.1.10 Every set A is a subset of itself, $A \subseteq A$.

Example 1.1.11 The set $\{1, 2\}$ is a subset of the set $\{1, 2, 3\}$. The set $\{1, 2, 3\}$ is ***not*** a subset of the set $\{1, 2\}$. $\square$

A very nice property of numbers is that any two numbers, x and y, ***must*** satisfy

$$x \leq y \quad \text{or} \quad y \leq x.$$

Unfortunately, this is not the case for sets. It may be the case that two sets A and B satisfy $A \not\subseteq B$ and $B \not\subseteq A$. We call such sets **incomparable**

Example 1.1.12 The set $\{1, 2, 3\}$ is ***not*** a subset of $\{4, 5\}$ and the set $\{4, 5\}$ is ***not*** a subset of the set $\{1, 2, 3\}$. $\square$

Finally, we note that in some instances we may wish to exclude the possibility that two objects are the same in an order relationship. With numbers, we use the $<$ symbol to indicate that one number is *strictly* smaller than the other, such as

$$1 < 2 \quad \text{or} \quad 7 < 1125.$$

There is a strongly analogous notational change if we want to emphasize that one set is a subset of another, but the two sets are not the same.

Definition 1.1.13 Proper Subset. Let A and B be sets. We say that A is a **proper subset** of B and write $A \subset B$ if every element of A is also an element of B and $A \neq B$. $\Diamond$

Example 1.1.14 The set $\{1, 2\}$ is a proper subset of the set $\{1, 2, 3\}$. We could write either $\{1, 2\} \subseteq \{1, 2, 3\}$ or $\{1, 2\} \subset \{1, 2, 3\}$. We use the latter when it is important to emphasize that the two sets are not the same. $\square$

1.1.3 Special Sets

Certain sets appear so frequently that mathematicians use an agreed upon symbol to refer to the set. We collect a list of these sets below.

Definition 1.1.15 The Empty Set. The **empty set** is the set with no elements, $\emptyset$. $\Diamond$

Remark 1.1.16 For convenience, the The Empty Set is a subset of *every* set.

Definition 1.1.17 The Natural Numbers. The set of **natural numbers** is the set with elements all positive integers,

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

◊

Definition 1.1.18 The Integers. The set of **integers** is the set with elements all integers,

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

◊

Definition 1.1.19 The Rational Numbers. The set of **rational numbers** is the set with elements all rational numbers,

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z},\, b \in \mathbb{Z},\, b \neq 0 \right\}.$$

◊

Definition 1.1.20 The Real Numbers. The set of **real numbers** is the set with elements all real numbers, $\mathbb{R}$. ◊

These sets of real numbers are related in the following way

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Each of the containments is proper because, for example,

- Every natural number is an integer, but $0 \in \mathbb{Z}$ and $0 \notin \mathbb{N}$.
- Every integer $n = n/1$ is a rational number, but $1/2 \in \mathbb{Q}$ and $1/2 \notin \mathbb{Z}$.
- Every rational number is a real number, but $\sqrt{2} \in \mathbb{R}$ and $\sqrt{2} \notin \mathbb{Q}$.

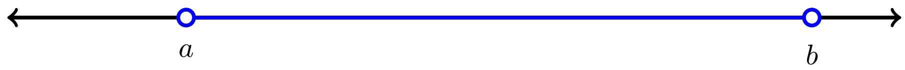
1.1.4 Intervals

A particularly special class of subsets of $\mathbb{R}$ are the *intervals*. These sets arise frequently, so it is important to specify the notation and a graphical representation for each one.

Definition 1.1.21 Open Interval. Let a and b be real numbers and assume that $a < b$. An **open interval** is a set that takes one of the following forms.

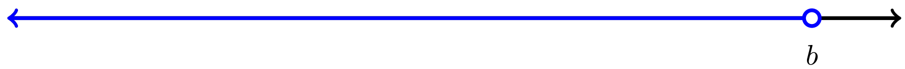
$$\begin{aligned}
(a, b) &= \{x \in \mathbb{R} \mid a < x < b\}, \\
(a, \infty) &= \{x \in \mathbb{R} \mid a < x\}, \\
(-\infty, b) &= \{x \in \mathbb{R} \mid x < b\}, \text{ or} \\
(-\infty, \infty) &= \mathbb{R}.
\end{aligned}$$

◊



(a) (a, b)

(b) (a, ∞)



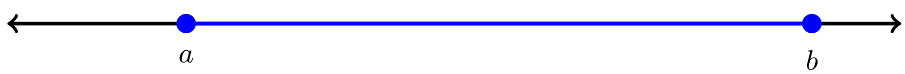
(c) $(-\infty, b)$

Figure 1.1.22 The open intervals

Definition 1.1.23 Closed Interval. Let a and b be real numbers and assume that $a < b$. A **closed interval** is a set that takes one of the following forms

$$\begin{aligned}
[a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\}, \\
[a, \infty) &= \{x \in \mathbb{R}; \mid a \leq x\}, \text{ or} \\
(-\infty, b] &= \{x \in \mathbb{R} \mid x \leq b\}. \\
(-\infty, \infty) &= \mathbb{R}.
\end{aligned}$$

◊



(a) $[a, b]$

(b) $[a, \infty)$

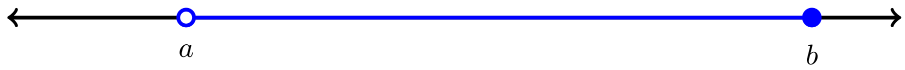
(c) $(-\infty, b]$

Figure 1.1.24 The closed intervals

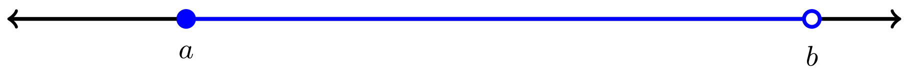
Definition 1.1.25 Half-Open Interval. Let a and b be real numbers and assume that $a < b$. A **half-open interval** from a to b takes one of the following forms.

$$\begin{aligned}
(a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\}, \\
[a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\},
\end{aligned}$$

◊



(a) $(a, b]$



(b) $[a, b)$

Figure 1.1.26 The half-open intervals

1.1.5 Basic Set Operations

There are two common operations that we will frequently need to perform on sets.

Definition 1.1.27 Intersection. Let A and B be sets. The **intersection** of A and B is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

$\Diamond$

Remark 1.1.28 The intersection is the largest set that is a subset of A and a subset of B.

Definition 1.1.29 Union. Let A and B be sets. The **union** of A and B is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

$\Diamond$

Remark 1.1.30 In mathematics, the word *or* is **inclusive**. This means the condition $x \in A$ or $x \in B$ allows for the possibility that $x \in A$ *and* $x \in B$.

Remark 1.1.31 The union is the smallest set that contains both A and B as subsets.

Example 1.1.32 Intersections and Unions of Intervals.

(a) Find the intersection of the intervals $[0, 2)$ and $(1, 3]$. Sketch the intersection on a number line.

(b) Find the union of the two intervals $[-1, 2)$ and $(1, 3]$. Sketch the union on a number line.

□

1.1.6 Worksheet: Sets

Objectives

- Use roster and set-builder notation to express a set,
- Find the intersection and union of two sets.

1. Write the real numbers that are at least as large as 2 and less than 5 in set notation.

2. Write the integers that are at least as large as 2 and less than 5 in set notation.

3. Write $[0, \infty) \cap (-\infty, 5)$ in set notation. Graph the result on a number line.

4. Write $[0, 5) \cap (3, 5)$ in set notation. Graph the result on a number line.

5. Write $[0, 5) \cup (3, 5)$ in set notation. Graph the result on a number line.

6. Write $[0, \infty) \cup (-\infty, 5)$ in set notation. Graph the result on a number line.

1.2 Cartesian Coordinates

In mathematics, we are interested in studying how quantities change together. We will restrict our attention to two quantities at a time and will often find it useful to visualize these relationships. The Cartesian plane is the natural setting for these visualizations.

1.2.1 The Cartesian Plane

Definition 1.2.1 The **Cartesian plane** is a grid system used to describe locations using two perpendicular directions. The horizontal direction is usually called the x-**axis** and the vertical direction is usually called the y-**axis**. The plane is partitioned into four pieces called **quadrants**, depicted in Figure 1.2.2

◊

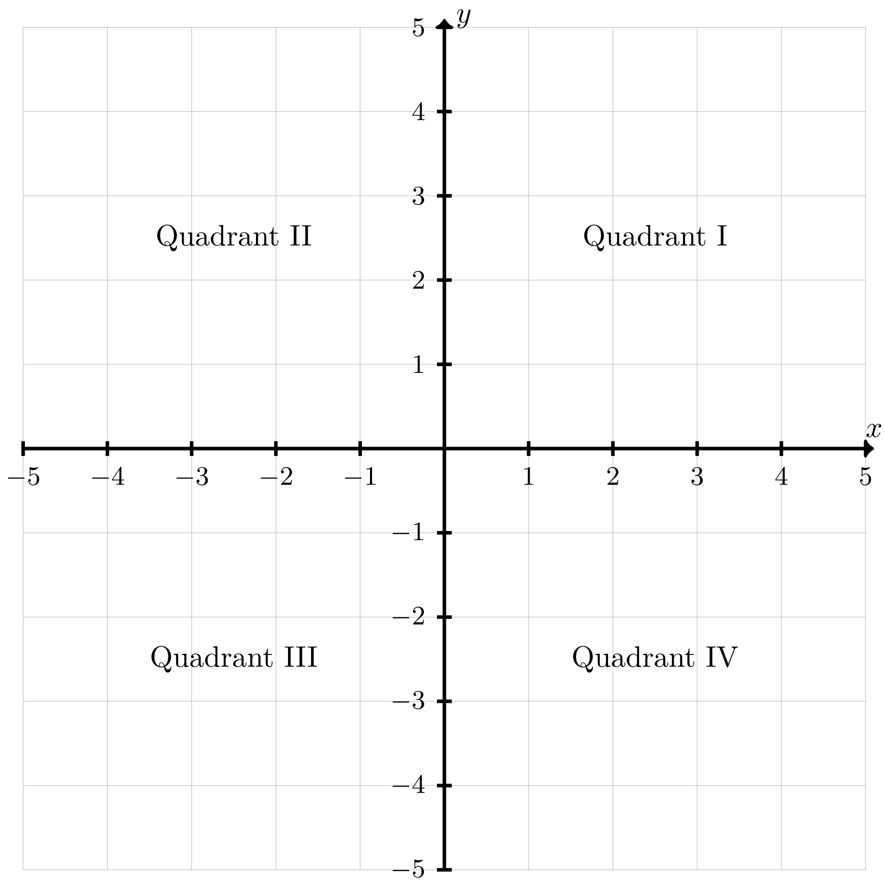


Figure 1.2.2 The Cartesian Plane

The type of *things* we are interested in locating will vary by application. In general, we will refer to these simply as **points**. A point is located in the plane by its position relative to the x-axis, called the x-**coordinate**, and relative to the y-axis, called the y-**coordinate**. By convention, we locate a point by first giving the x-coordinate, then the y-coordinate.

Definition 1.2.3 An **ordered pair**, (x, y), is the data of two real numbers. The collection of all ordered pairs of real numbers is

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

◊

We encode the position of a point using its x-coordinate and y-coordinate using an ordered pair with the x-coordinate first and the y-coordinate second.

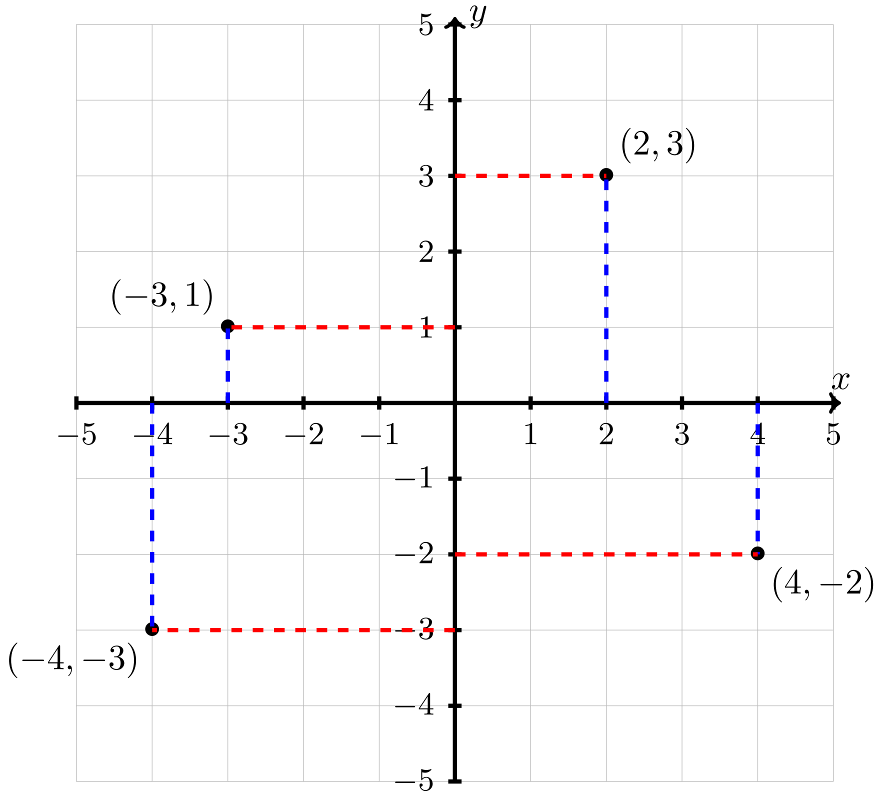


Figure 1.2.4 The x- and y-Coordinates of several points in the plane.

Worksheet: Cartesian Coordinates

Objectives

- Plot points in the plane using Cartesian coordinates,
- Identify the quadrant in which a point lies.

In this activity, we will practice plotting some points in the plane.

1. Plot the following points in the Cartesian plane.

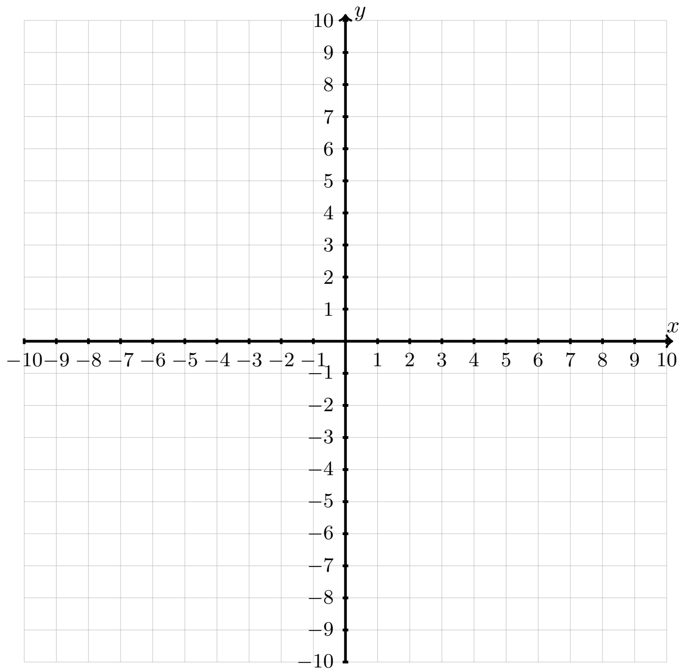


Figure 1.2.5

$A = (-1, 3)$ $B = (-2, -1)$ $C = (3, 5.5)$

$D = (3, -4)$ $E = (-8, -2)$ $F = (5.5, 3)$

2. For each of the points that you plotted above, indicate the quadrant that contains the point.

A lies in quadrant __________ C lies in quadrant __________ E lies in quadrant __________

B lies in quadrant __________ D lies in quadrant __________ F lies in quadrant __________

1.3 Graphing Equations

When two quantities vary together, the relationship is modeled by an equation in two variables. By convention, mathematicians use the variables x and y. Often, it is convenient to visualize this relationship using a graph.

1.3.1 Solutions to an Equation

Definition 1.3.1 Solution to an Equation. A **solution** to an equation in the variables x and y is an ordered pair (a, b) that make the equation true when

1. the first coordinate, a, is substituted into the equation for x, and

2. the second coordinate, b, is substituted into the equation for y.

$\Diamond$

Example 1.3.2 Consider the equation $y = 2x - 2$. The ordered pair $(1, 0)$ is a solution to this equation because

$$0 = 2(1) - 2 = 2 - 2$$

is a true statement. The ordered pair $(3, 5)$ is ***not*** a solution to this equation because

$$5 = 2(3) - 2 = 6 - 2 = 4$$

is a false statement. $\square$

Definition 1.3.3 Graph of an Equation. The **graph** of an equation in the variables x and y consists of all the points in the plane that are solutions to the equation. $\Diamond$

Most equations that we will encounter will have infinitely many solutions. Since we cannot find all of the solutions, we will mostly rely on studying the shape of certain graphs and some specific solutions.

Definition 1.3.4 y-intercept. A Solution to an Equation of the form $(0, y)$ is called a **y-intercept**. $\Diamond$

Definition 1.3.5 x-intercept. A Solution to an Equation of the form $(x, 0)$ is called an **x-intercept**. $\Diamond$

Use the graph of the equation $y = 2x - 2$ to answer the questions below.

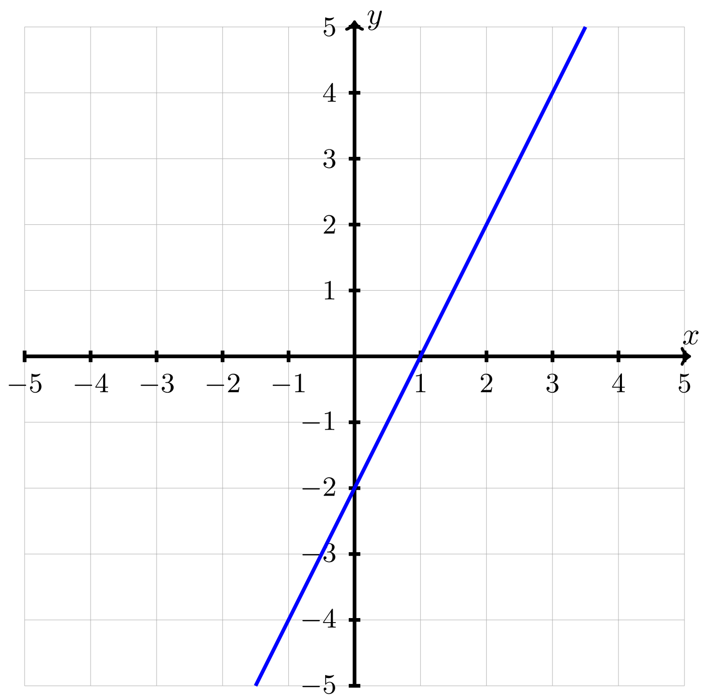


Figure 1.3.6 The graph of the equation $y = 2x - 2$.

Example 1.3.7

(a) Find the coordinates of the y-intercept and plot it on the graph.

(b) Find the coordinates of the x-intercept and plot it on the graph.

□

1.3.2 Horizontal Lines

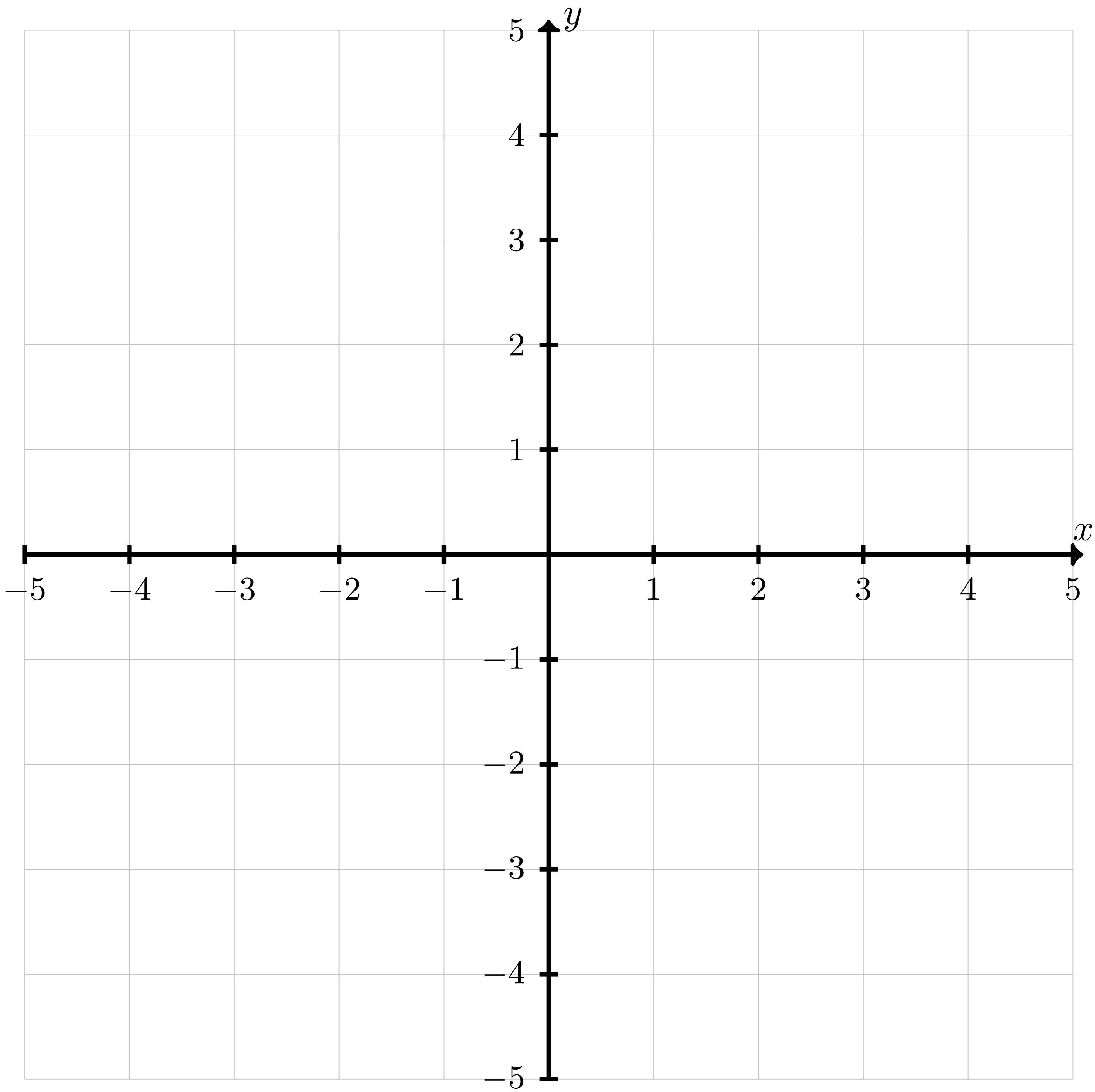
The equation

$$y = a$$

where a is a constant has solutions of the form (x, a), where x is allowed to be *any* real number. The graph of this equation is a horizontal line.

Examples

Example 1.3.8 Graph the equation $y = 0$.



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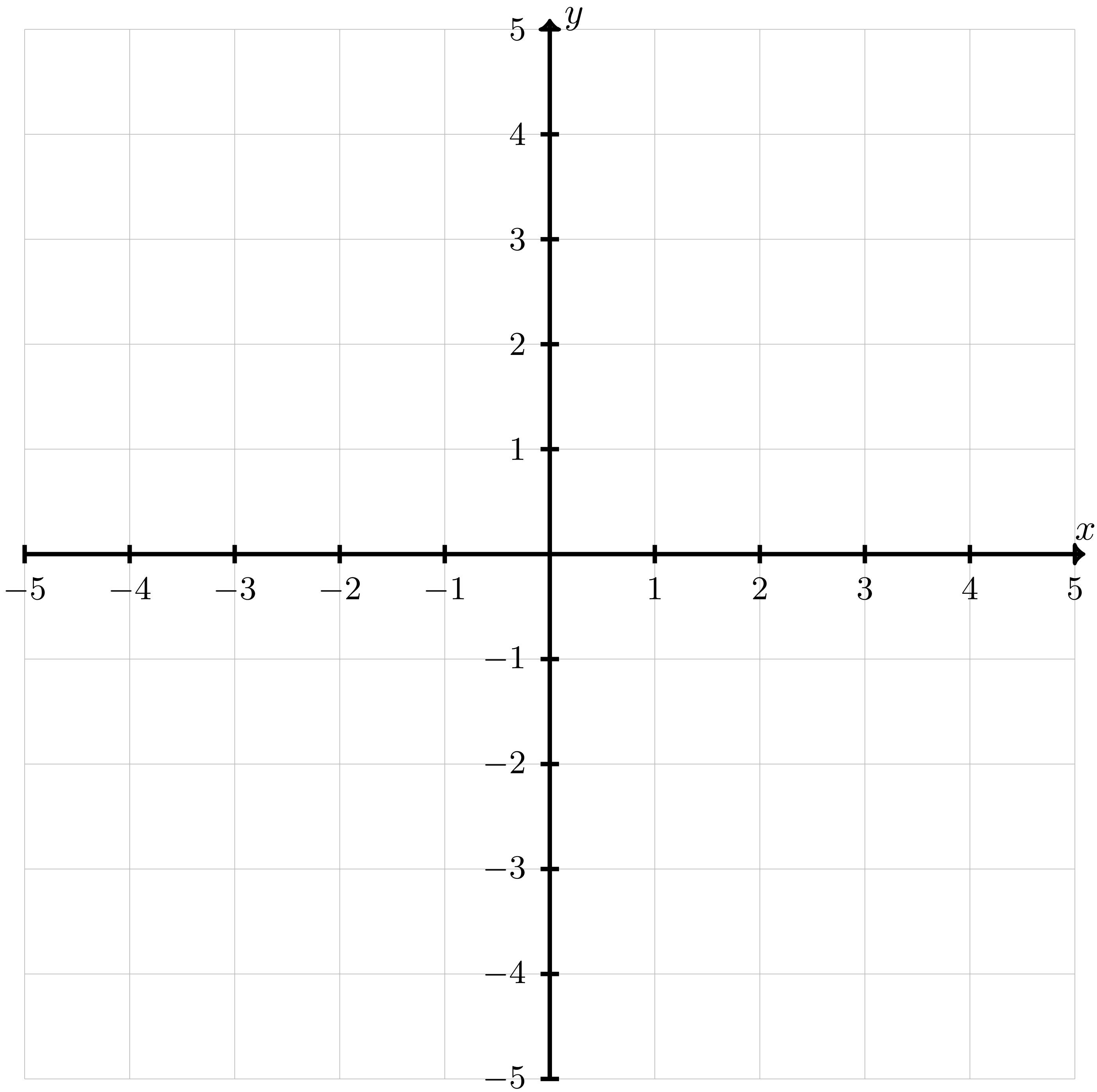
Example 1.3.9 Graph the equation $3y = 7$.



□

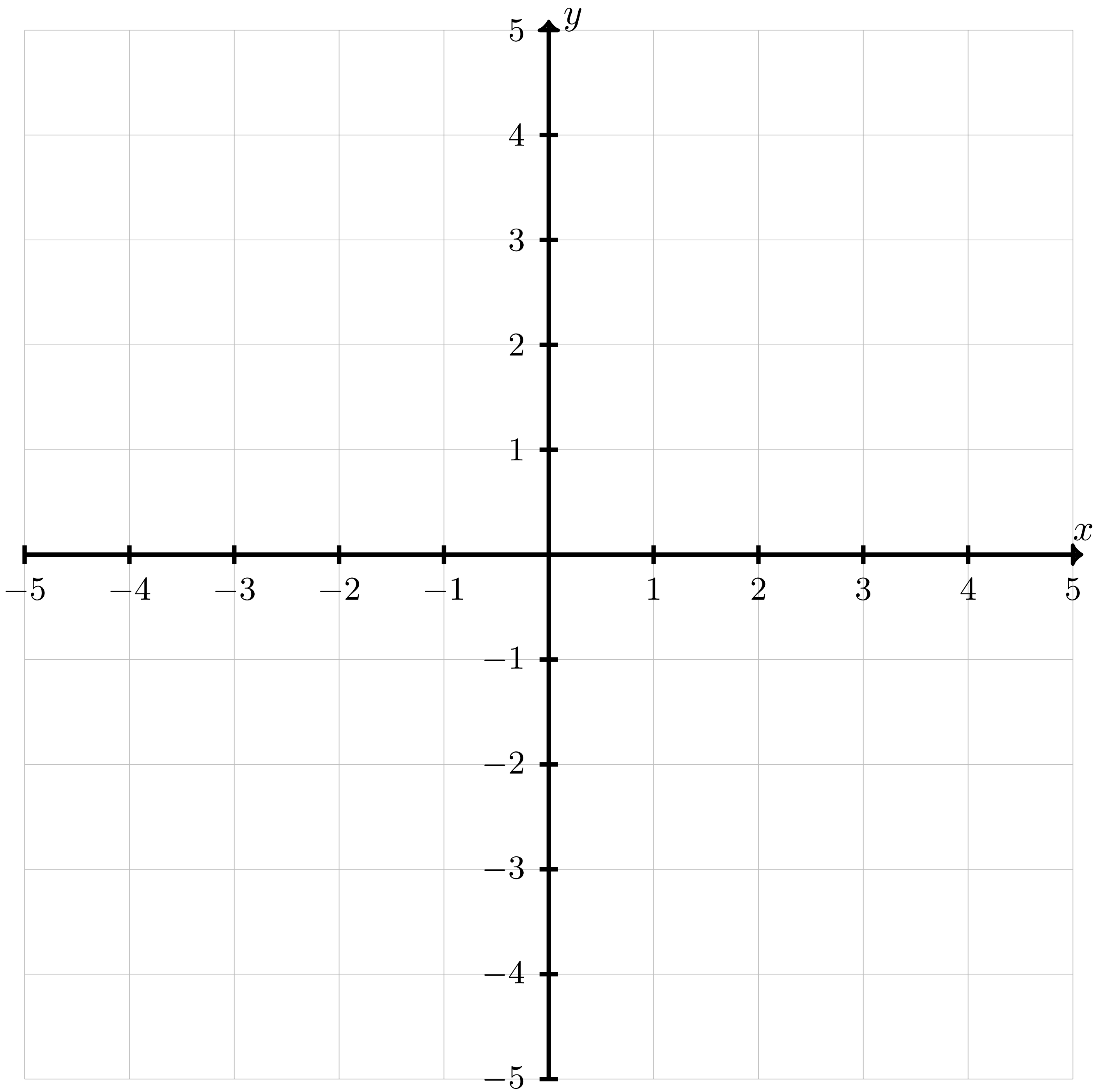
1.3.3 Vertical Lines

The equation

$$x = a$$

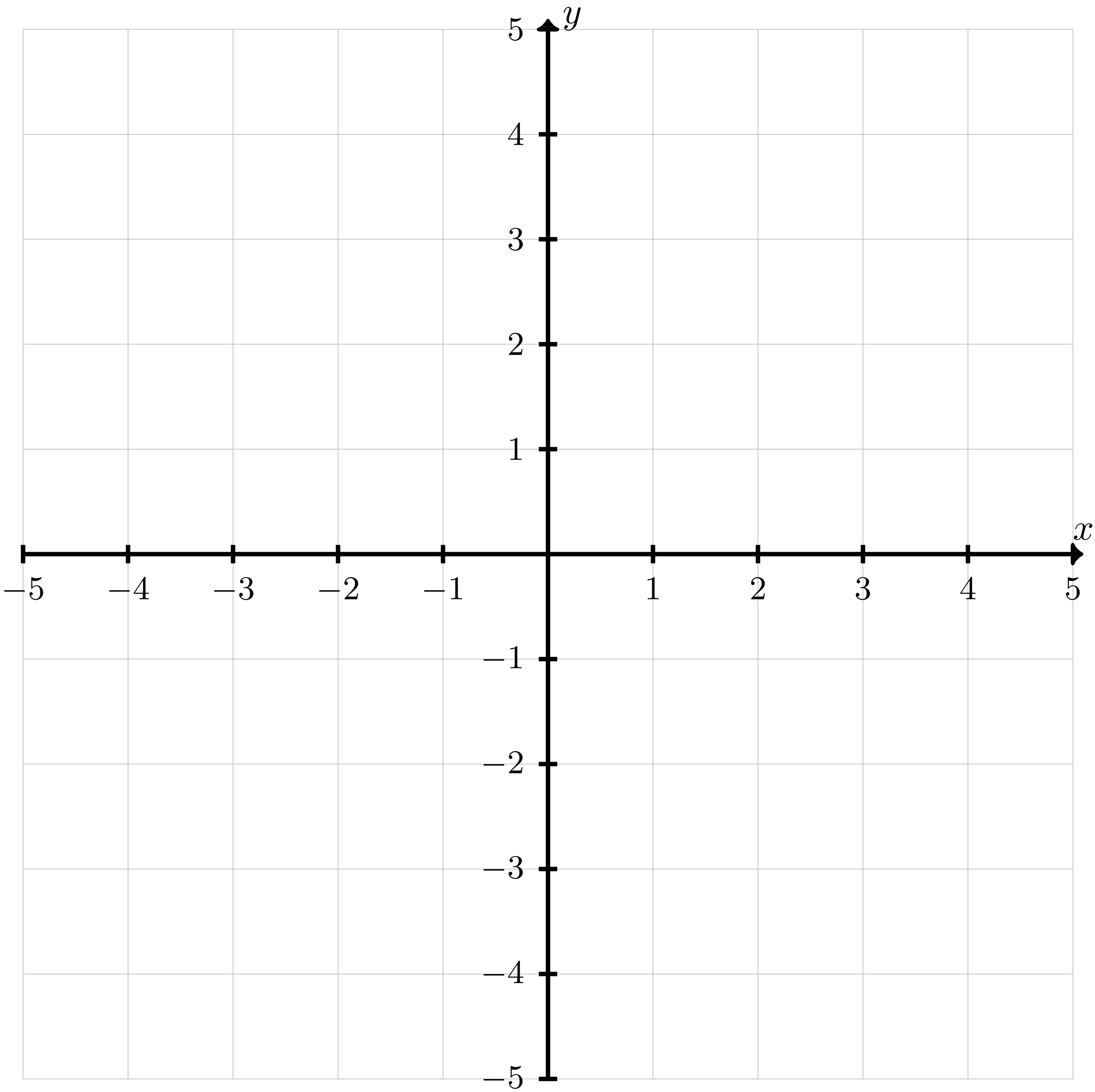
where a is a constant has solutions of the form (a, y), where y is allowed to be *any* real number. The graph of this equation is a vertical line.

Example 1.3.10 Graph the equation $x = 0$.



$\square$

Example 1.3.11 Graph the equation $3x = 2$.



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1.3.4 Worksheet: Graphing Equations

Objectives

- Identify points in the plane that are solutions to equations in two variables.

Consider the equation $2x - 3y = 6$. The graph of this equation is given below. Use the equation and the graph to answer the following questions.

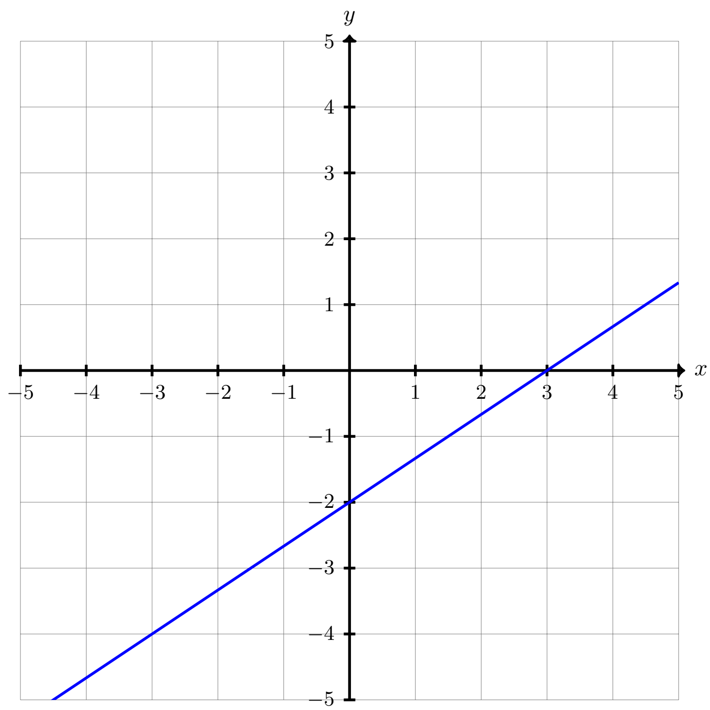


Figure 1.3.12 Graph of $2x - 3y = 6$.

1. Find the value of x such that the point $(x, 0)$ lies on the graph of the equation. Plot this point on the graph above and label it.

2. Find the value of y such that the point $(0, y)$ lies on the graph of the equation. Plot this point on the graph above and label it.

3. Is the point $(-3, -4)$ on the graph of the equation? Justify your answer using the equation of the graph.

4. Is the point $\left(4, \frac{1}{2}\right)$ on the graph of the equation? Justify your answer using the equation of the graph.

1.4 The Distance Formula

Within the Cartesian plane, we will often want to determine the distance between two points. In this section, we extend the notion of distance between two points on the number line to the Cartesian plane.

1.4.1 Distance in One Dimension

Definition 1.4.1 Assume x and y are any two numbers. The **distance** from x to y on the number line is given by the formula

$$|x-y| = \sqrt{(x-y)^2}$$

◊

The number $|x-y|$ measures the length of the segment of the number line starting at x and ending at y.

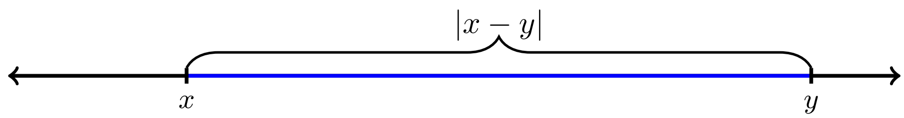


Figure 1.4.2 The distance from x to y on the number line.

Of course, we could also ask about the length of the segment of the number line starting at y and ending at x. Figure 1.4.2 suggests that these two numbers should be the same. We can verify this algebraically using Definition 1.4.1: the distance from y to x on the number line is

$$\begin{aligned}
|y-x| &= \sqrt{(y-x)^2} && \text{Factor } -1 \text{ from } y-x\\
&= \sqrt{\left((-1)(x-y)\right)^2} && \text{Expand}\\
&= \sqrt{(-1)(x-y)(-1)(x-y)} && \text{Reorder terms}\\
&= \sqrt{(-1)(-1)(x-y)(x-y)} && \text{Simplify}\\
&= \sqrt{(x-y)^2}\\
&= |x-y|
\end{aligned}$$

We say the distance is **symmetric** to signify that the distance between x and y whether you start measuring at x and end at y, or if you start measuring at y and end at x.

1.4.2 Distance in Two Dimensions

Our goal for this section is to measure the distance between two points in the plane using Definition 1.4.1 First, we must agree on what that statement should mean. In mathematics, whenever a term might be ambiguous, we remove the ambiguity by defining the term explicitly.

Definition 1.4.3 The **distance** between the points (x_1, y_1) and (x_2, y_2) in the Cartesian plane is the length of the line segment connecting the two points.

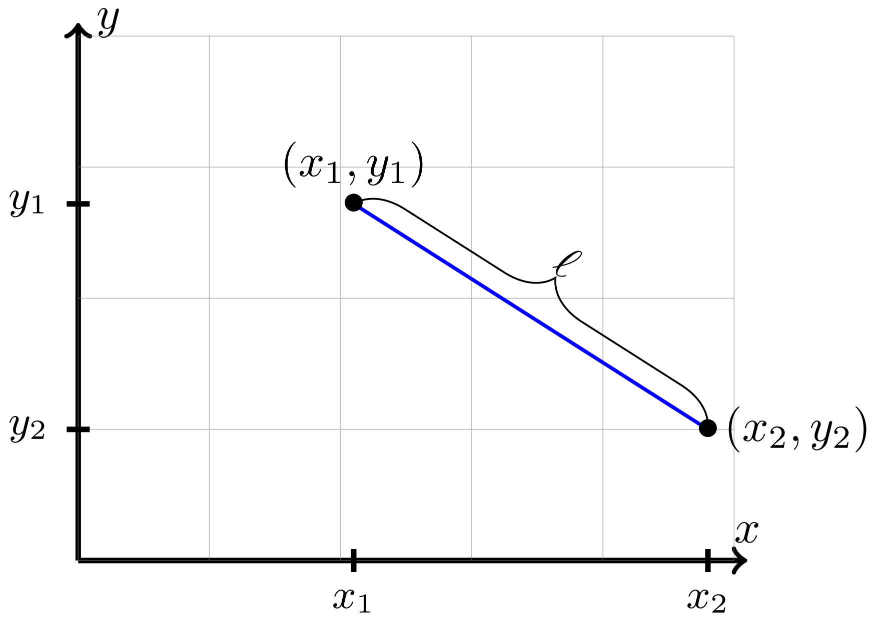


Figure 1.4.4 The line segment of length ℓ connecting (x_1, y_1) and (x_2, y_2)

◊

To measure the length of this line, we need a little help from geometry. If our line segment were either horizontal or vertical, then we could use Definition 1.4.1 to measure the parallel segment of the x-axis or y-axis, respectively. To take advantage of these observations, we draw one horizontal line and one vertical line to form a right triangle.

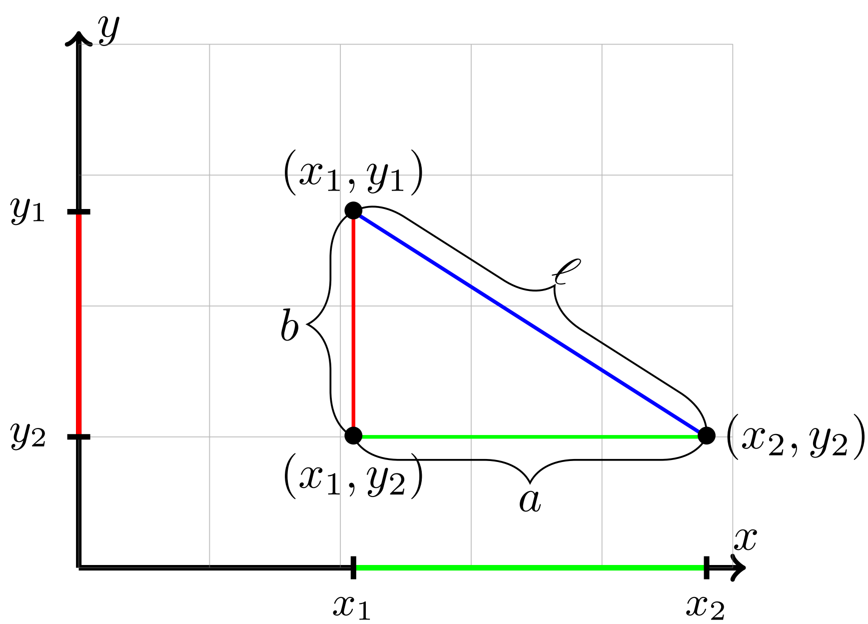


Figure 1.4.5 The triangle formed by adding in the point (x_1, y_2)

From Figure 1.4.5, we can see the length of the horizontal side is the distance between the points x_1 and x_2 along the x-axis. Similarly, the length of the vertical side is the distance between the points y_1 and y_2 along the y-axis.

Using Definition 1.4.1, we have

$$a = |x_1 - x_2| = \sqrt{(x_1 - x_2)^2} \quad \text{and} \quad b = |y_1 - y_2| = \sqrt{(y_1 - y_2)^2}.$$

Since we know two of the three sides, we can use the Pythagorean Theorem to find the third side.

Theorem 1.4.6 The Pythagorean Theorem. *The right triangle in Figure 1.4.7 satisfies the equation* $a^2 + b^2 = c^2$

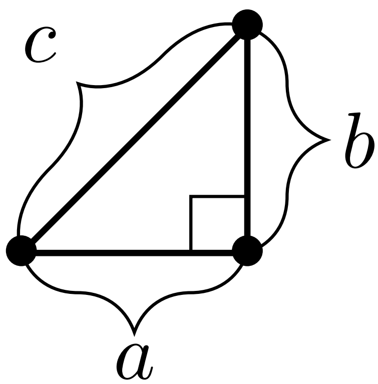


Figure 1.4.7 *Right triangle with sides of length a, b, and c.*

Square the two side lengths

$$a^2 = \sqrt{(x_1 - x_2)^2}^2 = (x_1 - x_2)^2 \quad \text{and} \quad b^2 = \sqrt{(y_1 - y_2)^2} = (y_1 - y_2)^2.$$

By the Pythagorean Theorem, the square of the distance between the points (x_1, y_1) and (x_2, y_2) satisfies

$$\ell^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

Taking the square root of both sides yields the distance formula in the plane.

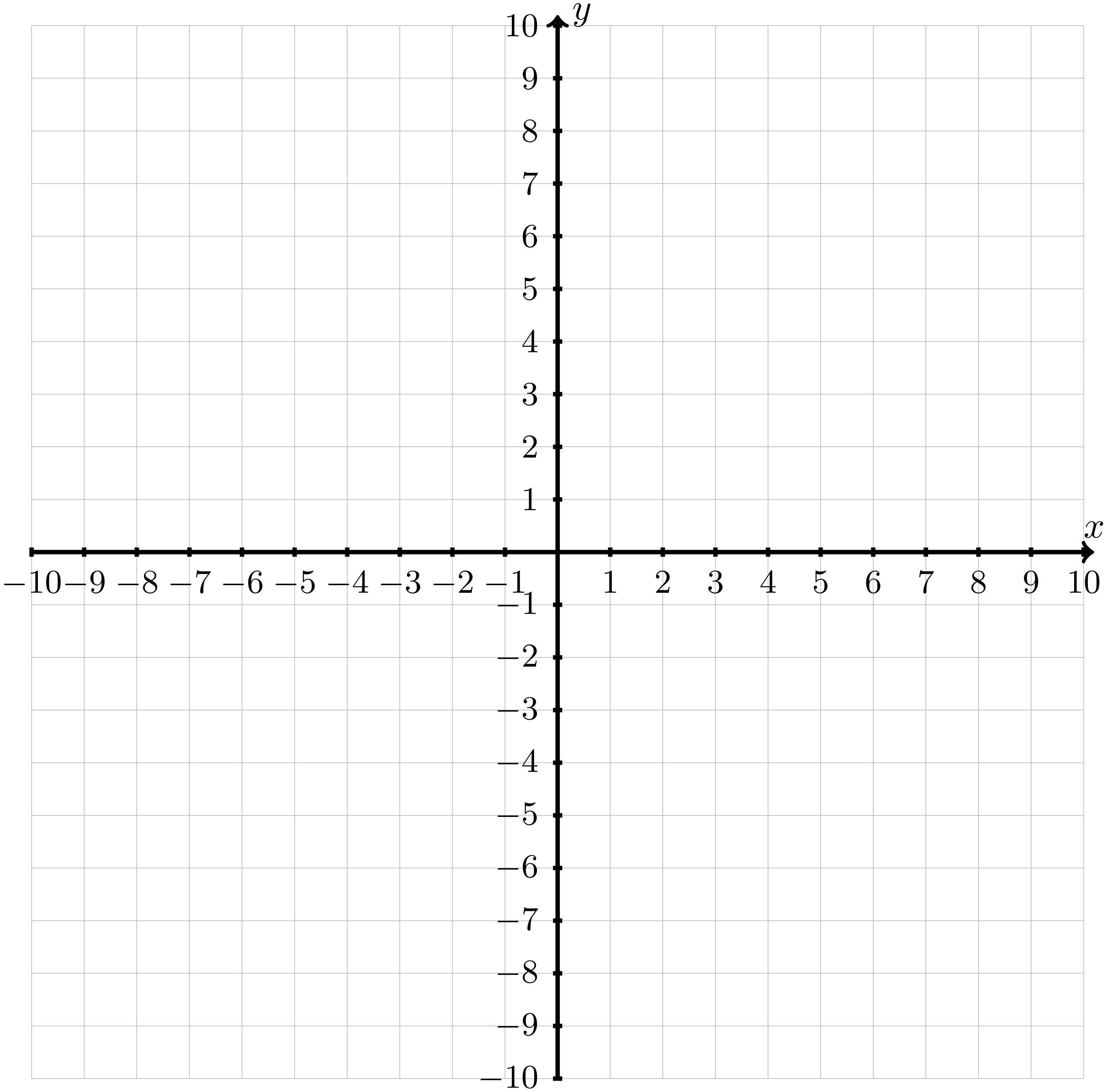
Formula 1.4.8 The Distance Formula. *The distance from the point* (x_1, y_1) *to the point* (x_2, y_2) *in the Cartesian plane is given by*

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Remark 1.4.9 Just like the distance between numbers, the distance between two points in the plane is symmetric. We can verify this using Formula 1.4.8:

$$\begin{aligned}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} &= \sqrt{\left((-1)(x_1 - x_2)\right)^2 + \left((-1)(y_1 - y_2)\right)^2}\\ &= \sqrt{(-1)^2(x_1 - x_2)^2 + (-1)^2(y_1 - y_2)^2}\\ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\end{aligned}$$

Example 1.4.10 Plot the points $(3, -2)$ and $(4, 7)$ in the plane. Find the length of the line segment that connects these two points.



□

1.4.3 Worksheet: The Distance Formula

Objectives

- Find the distance between two points in the Cartesian plane.

Use to answer the following questions.

1. Plot each pair of points in the plane. Find the length of the line segment that connects them.

(a) $(3, 2)$ and $(-2, -10)$.

(b) $(1, 3)$ and $(4, 7)$

(c) $(0, 0)$ and $(3, 4)$

2. Find the perimeter of the triangle below.

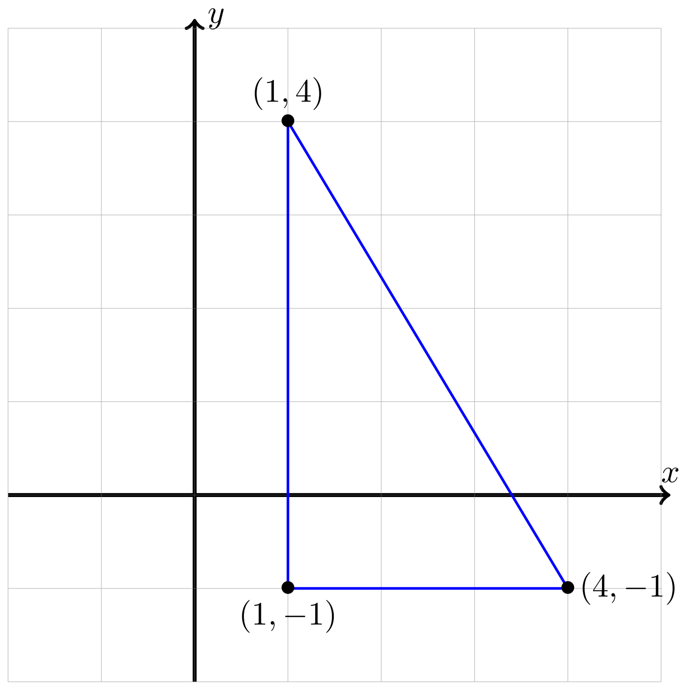


Figure 1.4.11

1.5 The Midpoint Formula

Within the Cartesian plane, we will often want to determine the midpoint between two points. In this section, we extend the notion of midpoint between two points on the number line to the Cartesian plane.

1.5.1 Midpoint in One Dimension

Definition 1.5.1 The **midpoint** between two numbers x and y on the number line is given by the formula

$$\frac{x+y}{2},$$

as depicted in Figure 1.5.2. ◊

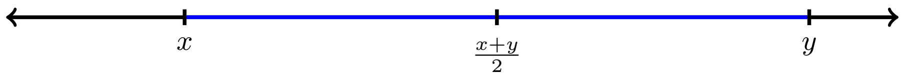


Figure 1.5.2 Midpoint between two numbers

Remark 1.5.3 The midpoint is so named because it is halfway between the numbers x and y. We can check this using Definition 1.4.1: the distance from x to the midpoint is

$$\begin{aligned}\left|x-\frac{x+y}{2}\right| &= \left|\frac{2x}{2}-\frac{x+y}{2}\right|\\ &= \left|\frac{2x-x-y}{2}\right|\\ &= \frac{|x-y|}{2}\end{aligned}$$

and the distance from the midpoint to y is

$$\begin{aligned}\left|\frac{x+y}{2}-y\right| &= \left|\frac{x+y}{2}-\frac{2y}{2}\right|\\ &= \left|\frac{x+y-2y}{2}\right|\\ &= \frac{|x-y|}{2}.\end{aligned}$$

1.5.2 Midpoint in Two Dimensions

Similar to the distance formula, we can find the midpoint of a line segment between the points (x_1, y_1) and (x_2, y_2) in the Cartesian plane using a triangle. We use Definition 1.5.1 to find the midpoint of the vertical and horizontal sides of the triangle, and use those as the coordinates for the midpoint as depicted in Figure 1.5.4.

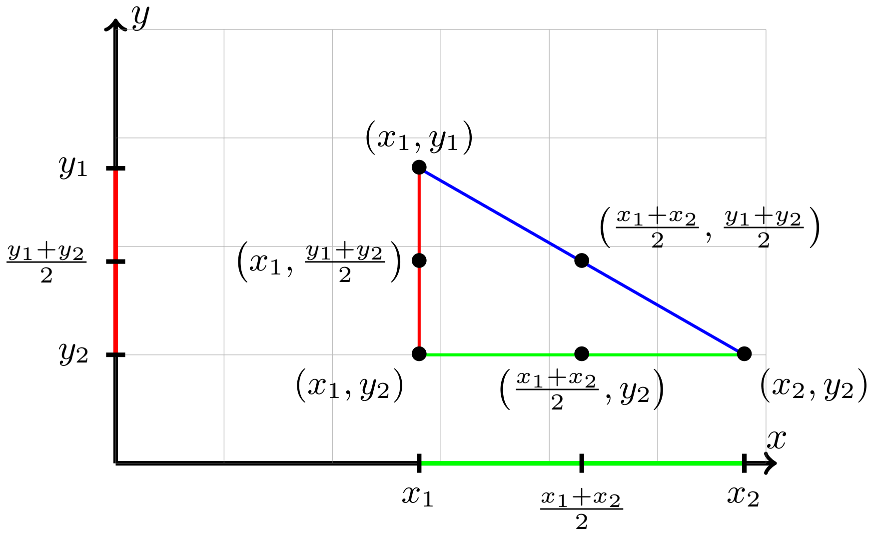


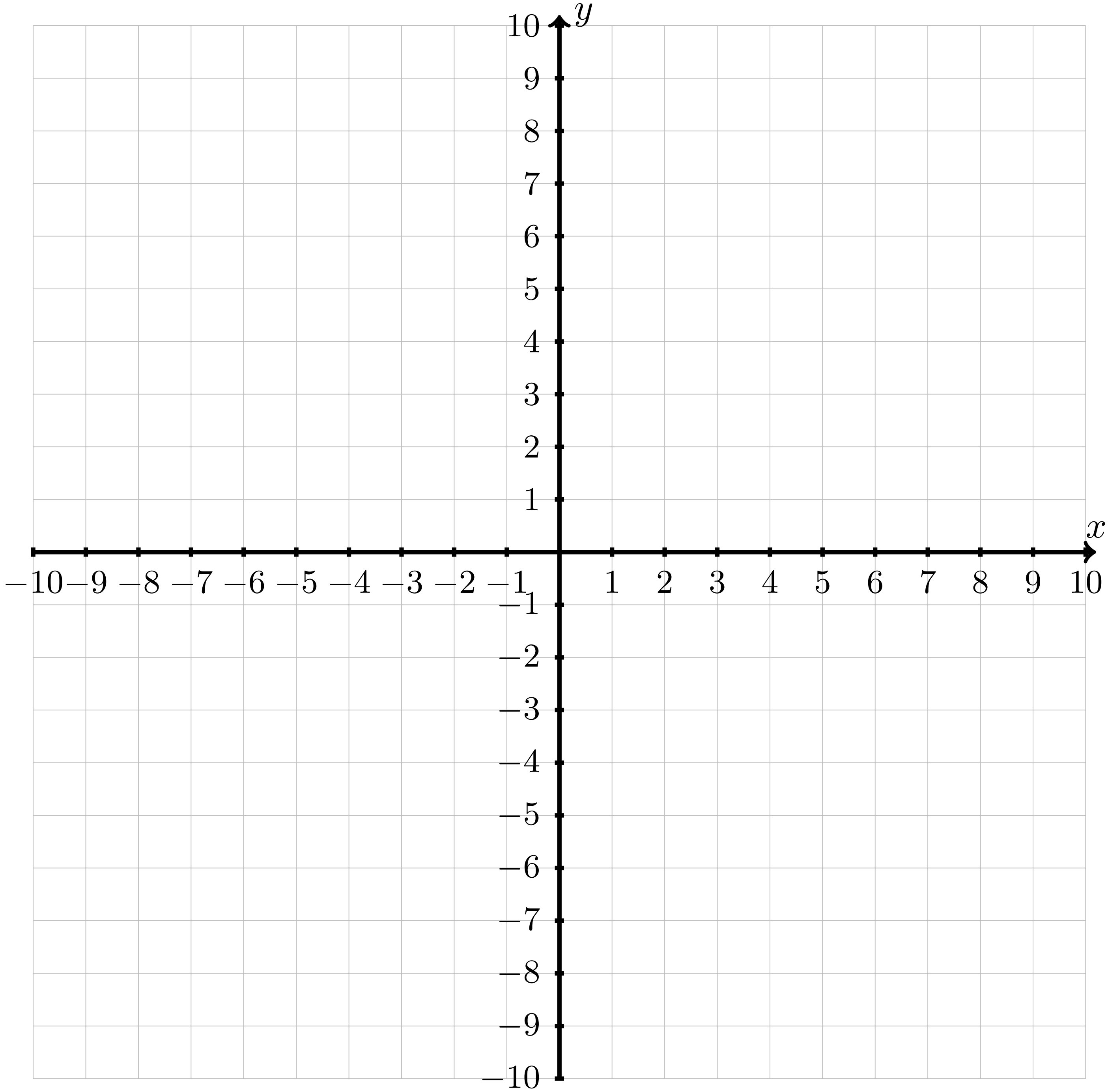
Figure 1.5.4 The triangle formed by adding in the point (x_1, y_2).

Definition 1.5.5 The Midpoint Formula. The **midpoint** between (x_1, y_1) and (x_2, y_2) in the Cartesian plane is the point

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

◊

Example 1.5.6 Plot the points $(1, -3)$ and $(-3, 5)$ in the Cartesian plane. Use The Midpoint Formula to find the midpoint between to find the midpoint of the line segment connecting these two points, then plot it.



□

Remark 1.5.7 Just as with the midpoint on the real line, the midpoint is halfway between the points (x_1, y_1) and (x_2, y_2). The easiest way to see this is true is to make two smaller triangles inside the bigger triangle, as depicted in Figure 1.5.8.

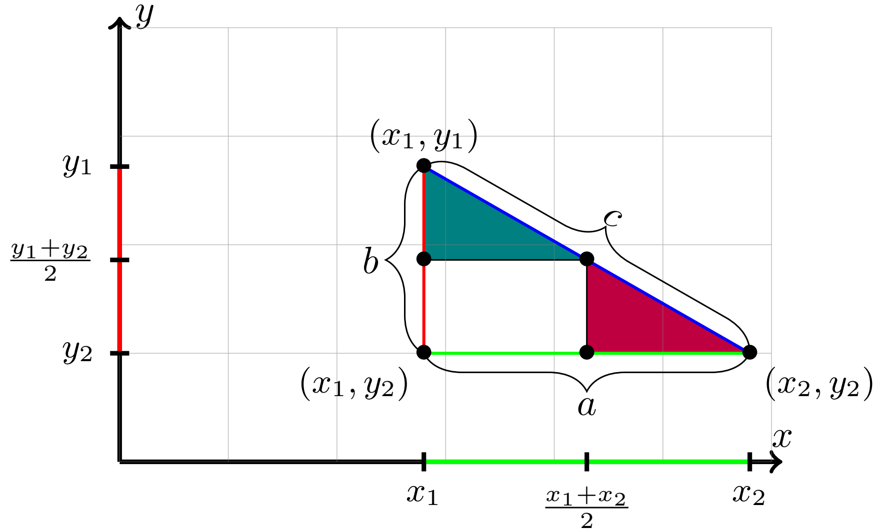


Figure 1.5.8 The triangle formed by adding in the point (x_1, y_2).

The two shaded triangles have the same side lengths. The horizontal side has length

$$\frac{a}{2} = \frac{|x_1 - x_2|}{2}$$

and the vertical side has length

$$\frac{b}{2} = \frac{|y_1 - y_2|}{2}.$$

Using The Pythagorean Theorem, the third side of each triangle has length

$$\begin{aligned}
\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} &= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} \\
&= \sqrt{\frac{a^2 + b^2}{4}} \\
&= \frac{\sqrt{a^2 + b^2}}{\sqrt{4}} \\
&= \frac{\sqrt{c^2}}{2} \\
&= \frac{c}{2},
\end{aligned}$$

which is exactly half the distance between the points (x_1, y_1) and (x_2, y_2).

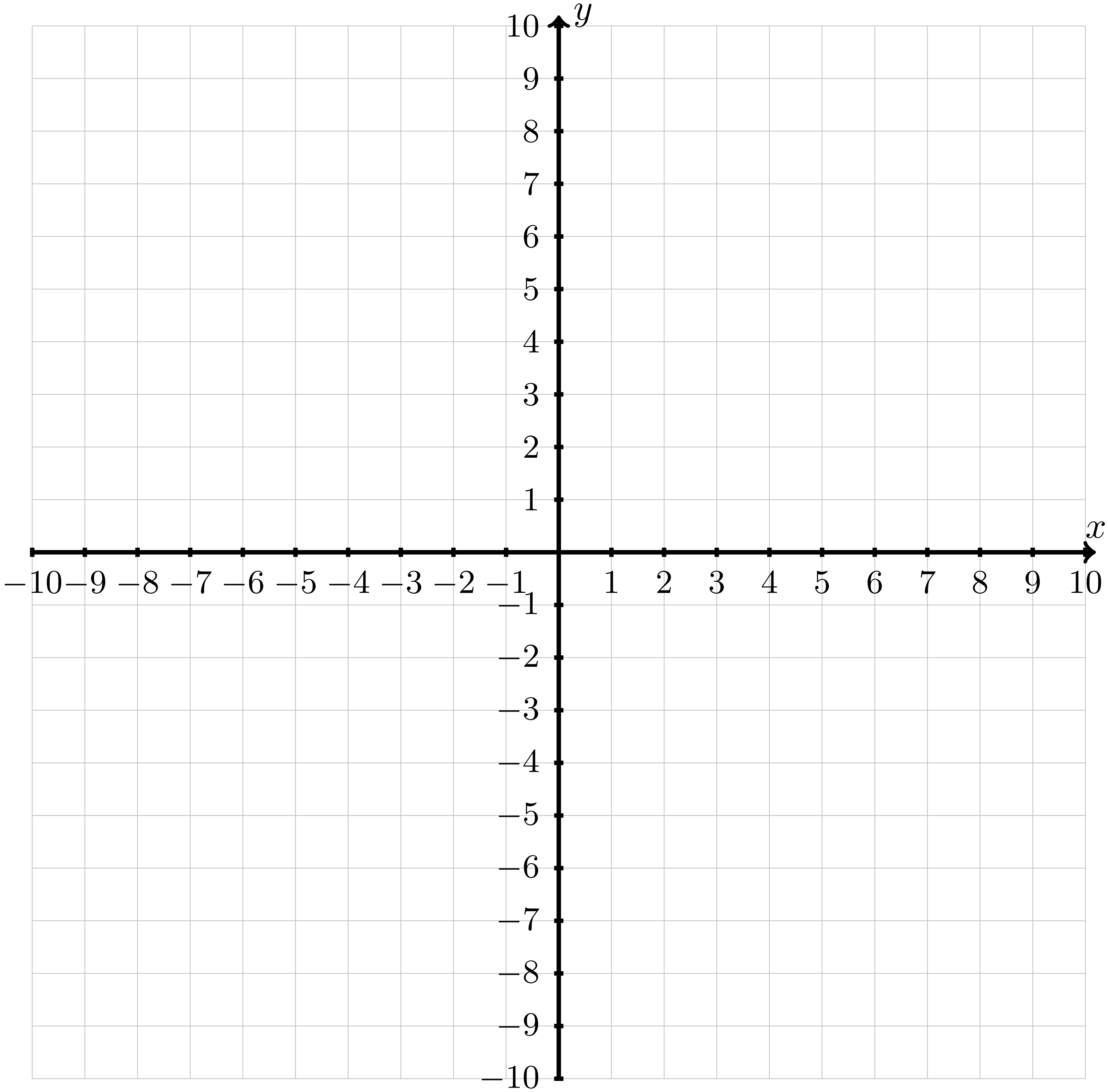
1.5.3 Worksheet: The Midpoint Formula

Objectives

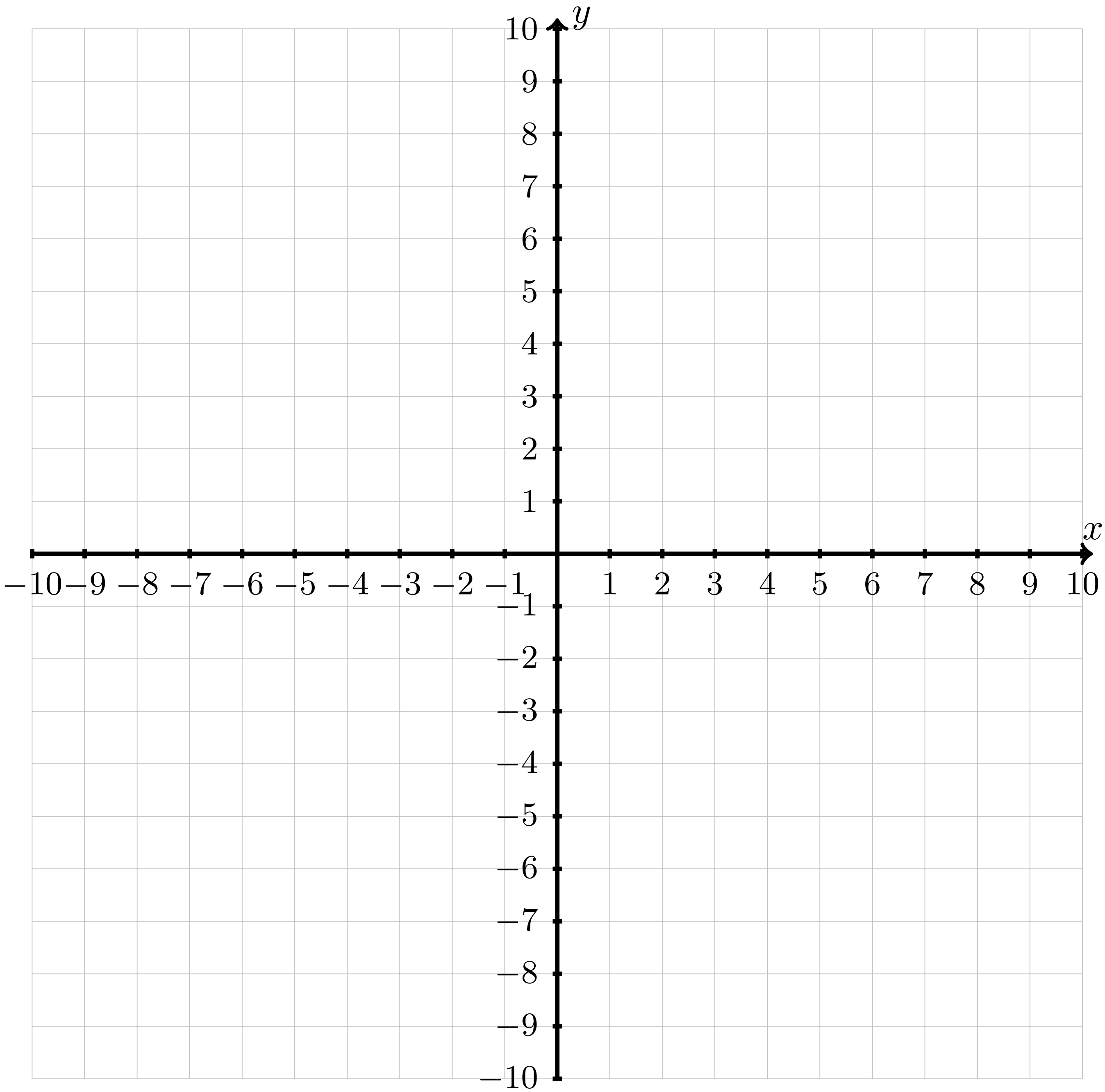
- Find the midpoint between two points in the Cartesian plane.

Use the The Midpoint Formula to answer the following questions.

1. Plot the points $(1, 3)$ and $(3, -1)$ in the plane. Find the midpoint of the line segment that connects these two points, then plot it.



2. Plot the points $(3, 2)$ and $(-5, -10)$ in the plane. Find the midpoint of the line segment that connects these two points, then plot it.



1.6 Circles

In this section, we study circles in the Cartesian plane.

Definition 1.6.1 The **circle** with radius r centered at (h, k) consists of all points in the Cartesian plane r units away from (h, k) as depicted in Figure 1.6.2. ◊

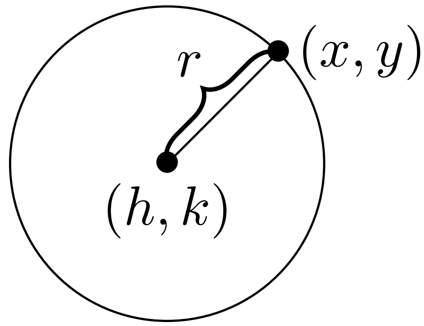


Figure 1.6.2 The circle of radius r centered at (h, k).

To find the equation of the circle with radius r and center (h, k) we use The Distance Formula. A point (x, y) is on this circle if the distance between (x, y) and (h, k) is exactly r:

$$r = \sqrt{(x-h)^2 + (y-k)^2}.$$

To make this simpler, we square each side of the equation.

Definition 1.6.3 The Standard Form of a Circle. The **standard form of a circle** with radius r and center (h, k) is

$$(x-h)^2 + (y-k)^2 = r^2.$$

◊

Definition 1.6.4 A **diameter** of a circle is a line that connects two points on the circle through the center. The length of a diameter of a circle of radius r is $2r$, and the midpoint of the diameter is the center of the circle. ◊

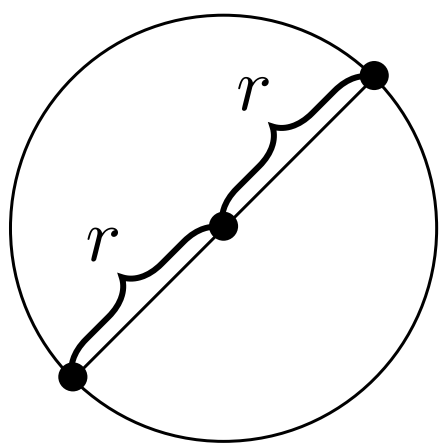


Figure 1.6.5 A diameter of the circle of radius r centered at (h, k).

Worksheet: Circles

Objectives

- Use the standard form of a circle to identify the center and radius.
- Use the standard form of a circle to produce a graph.

1. Write down the equation of the circle of radius $\sqrt{2}$ centered at $(-3, 1)$.

2. Find the center and radius of the circle

$$(x-3)^2 + (y-17)^2 = 1225.$$

Sketch a graph of the equation.

3. Find the radius of the circle with center $(2,3)$ that passes through the point $(-1,7)$. Write the equation of this circle in standard form. Sketch a graph of the circle.

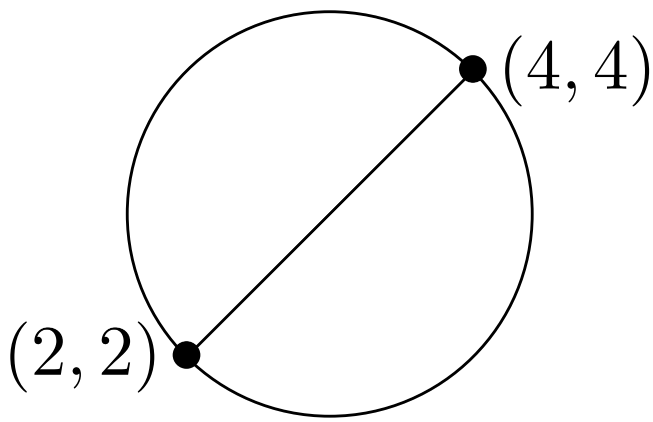


Figure 1.6.6 A circle with $(2,2)$ and $(4,4)$ the endpoints of a diameter.

4. The endpoints of a diameter of a circle are $(2,2)$ and $(4,4)$ as pictured above. Write the equation for this circle in standard form.

Chapter 2

Functions and Graphs

2.1 Functions

In nearly every branch of mathematics, functions are the primary objects of study. Functions provide a robust language for describing relationships between sets, and serve as the basic building blocks for modern development of algebra, geometry, trigonometry, calculus, and beyond. We develop techniques for analyzing and visualizing functions that are foundational in subsequent mathematics courses and also applicable outside of mathematics — in science, technology, and engineering— to model the behavior of systems.

2.1.1 Algebraic Functions

Definition 2.1.1 Function. Assume A and B are sets. A **function from** A **to** B, written $f\colon A \to B$, is a rule that assigns to each element $x \in A$ a unique element $f(x) \in B$.

- The set A is the **domain** of f. The domain of f is the set of all allowable input values to the function f.
- The set B is the **codomain** of f. The codomain of f is the set that contains all possible output values of f.
- The **range** of f is the set of all possible output values

$$f(A) = \{f(x) \mid x \in A\}.$$

◊

While functions are defined for arbitrary sets, we will restrict our attention to functions of the form $f\colon A \to \mathbb{R}$, where A is some subset of $\mathbb{R}$. By convention, we will use the variable x to stand for *some* input value from the domain of f.

It is common to refer to $f(x)$ —read "f of x" — as the **value of f at x** or the **image of x under f**. We will normally use the variable y to stand for *some* output value in the range of f. This relationship is made explicit by writing $y = f(x)$ or saying that y is a function of x. Generally, the value of y is determined by a choice of a specific $x \in A$. We call x the **independent variable** and y the **dependent variable**.

When we start from two sets A and B and write down a rule that assigns to each $x \in A$ an element $f(x) \in B$, we are defining a function **explicitly**. Most interesting subsets of the real numbers are simply too large to provide an explicit definition. To get around this problem, we often rely on algebraic expression involving a variable, x, to define our functions. These expressions are built using the operations of addition, subtraction, multiplication, division, and roots, and are called **algebraic functions**.

When we write down an equation like

$$y = \frac{1}{x}$$

we are defining a function **implicitly**. We have not explicitly stated the domain of the function, so this function has an **implied domain**, that consists of all real numbers for which the expression makes sense.

Example 2.1.2

(a) Find the implied domain and the range of the implicit function $y = 1/x$.

(b) Find the domain and range of the function $f(x) = 3x + 7$.

(c) Find the domain and range of the function $f(x) = x^2$.

$\square$

2.1.2 Comparing Functions

Whenever we define a new mathematical object, it is important to understand when two objects are *the same*. For functions with the same domain and codomain, we measure equality based on the output.

Definition 2.1.3 Equality of Functions. Two functions $f\colon A \to B$ and $g\colon A \to B$ are **equal** if for every $x \in A$, $f(x) = g(x)$.

We write $f = g$ to indicate that f and g represent the same function. ◊

Example 2.1.4

(a) The functions $f, g \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x^2}$ and $g(x) = |x|$ are equal.

(b) The functions $f, g \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x^2}$ and $g(x) = x$ are ***not*** equal.

$\square$

2.1.3 Graphing Functions

The ability to visualize a function often provides an immense amount of insight into how the function behaves. The graph of a function is defined in a manner similar to the graph of an equation.

Definition 2.1.5 Graph of a Function. Assume $A \subseteq \mathbb{R}$. The **graph** of a function $f\colon A \to \mathbb{R}$ is the set of points

$$\{(x, f(x)) \mid x \in A\} \subseteq \mathbb{R}^2$$

◊

When the function is implicitly defined by an equation in the variables x and y, the Graph of an Equation will coincide with the Graph of a Function. While most of the functions we consider will be defined by an equation in the variables x and y, it is not the case that all equations in x and y define y as a function of x.

Theorem 2.1.6 The Vertical Line Test. *An equation in the variables x and y is defines y as a function of x if and only if every vertical line intersects the graph of the relation in* ***at most*** *one point.*

Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by the rule $f(x) = x + 2$.

Example 2.1.7

(a) Verify that f is a function algebraically.

(b) Verify that f is a function visually.

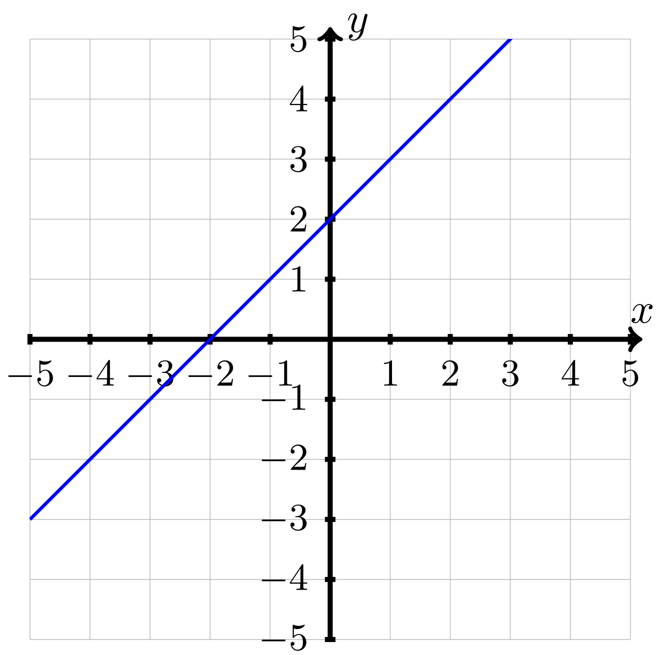


Figure 2.1.8 The graph of $y = x + 2$

□

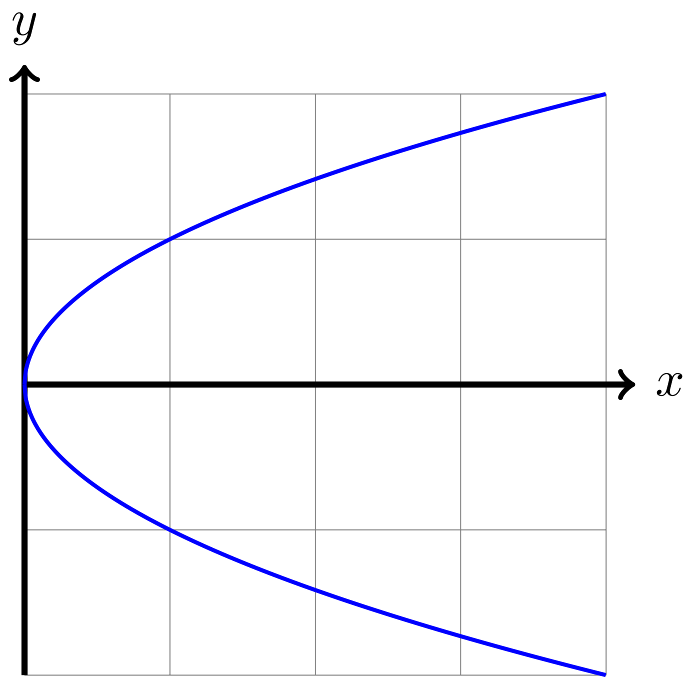


Figure 2.1.9 The graph of the equation $x = y^2$.

Example 2.1.10 Explain why the equation $x = y^2$ does ***not*** define y as a function of x. □

2.1.4 Functions

Objectives

- Find the implied domain and codomain of an algebraic function.
- Find the range of a function.
- Determine whether an equation defines y as a function of x.

1. For each of the following functions, find the implied domain, codomain, and range.

(a) $f(x) = \sqrt{x}$

(b) $g(x) = \frac{1}{x}$

(c) $h(x) = |x|$

2. Evaluate the function at the given value.

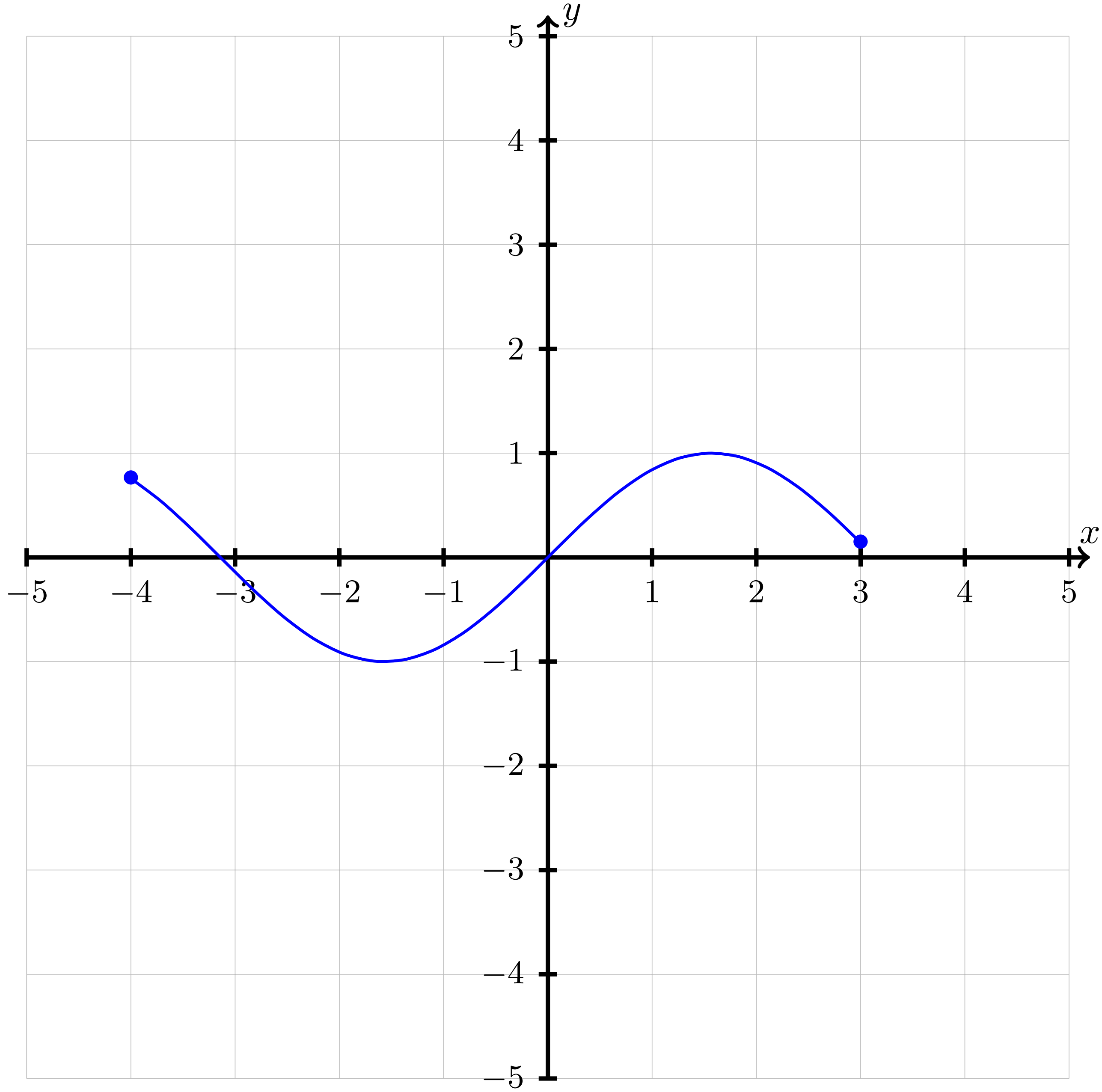
(a) $f(x) = 3x^2 + 2x - 7$; $x = 2$.

(b) $g(x) = x - \sqrt{x} + \frac{7}{x}$; $x = 49$.

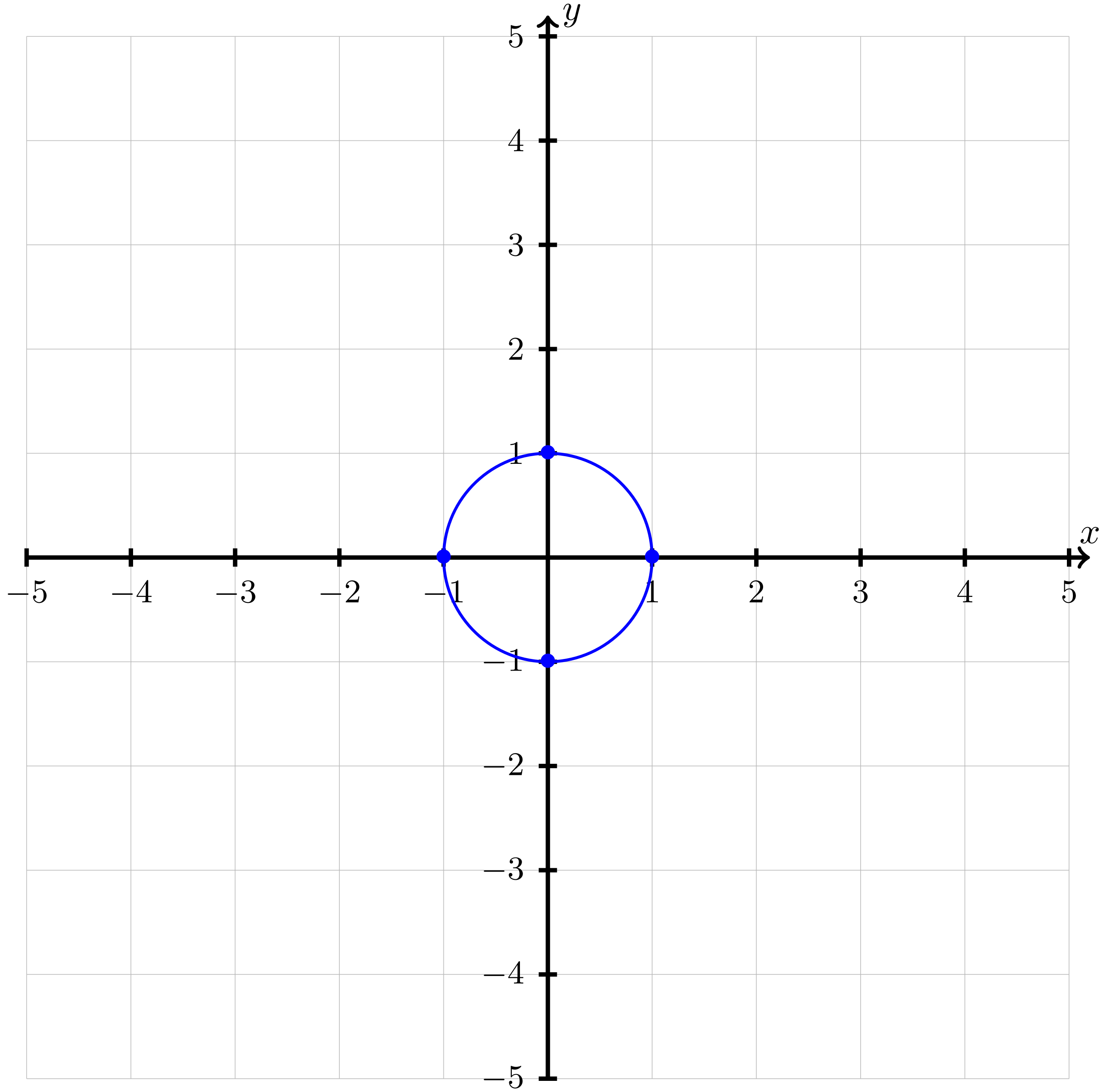
(c) $h(x) = |x| + 3x - 5$; $x = -11$.

3. For each of the graphs, find the domain and range. Use the The Vertical Line Test to determine whether y is a function of x.

(a)



(b)



2.2 Graph Transformations

Graphing arbitrary functions can be a difficult task and the general methods are beyond the scope of basic algebra. However, if we know the graph of a function, f, and we can write another function, g, in terms of f in a specific way, then we can use the graph of $f(x)$ to graph $g(x)$ using *transformations*. In this section, we discuss three types of transformations: translation, scaling, and reflection.

2.2.1 Vertical Translation

Definition 2.2.1 Vertical Translation. Let c be a positive number. We say the function $g(x)$ is a **vertical translation** of the function $f(x)$ if $g(x) = f(x) \pm c$.

- The graph of $g(x) = f(x)+c$ is obtained by shifting the graph of $y = f(x)$ up the y-axis by c units.
- The graph of $g(x) = f(x)-c$ is obtained by shifting the graph of $y = f(x)$ down the y-axis by c-units.

$\Diamond$

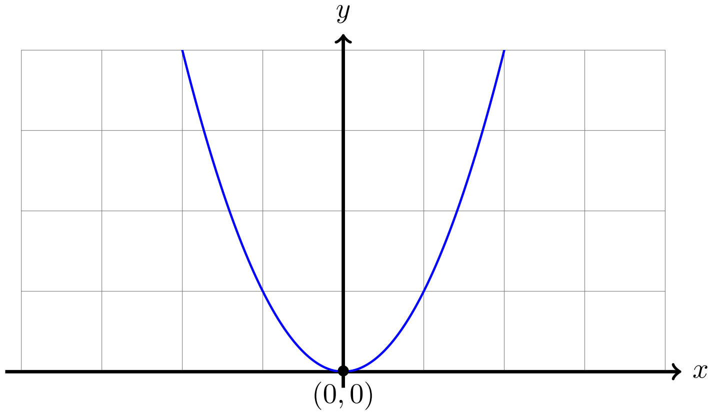
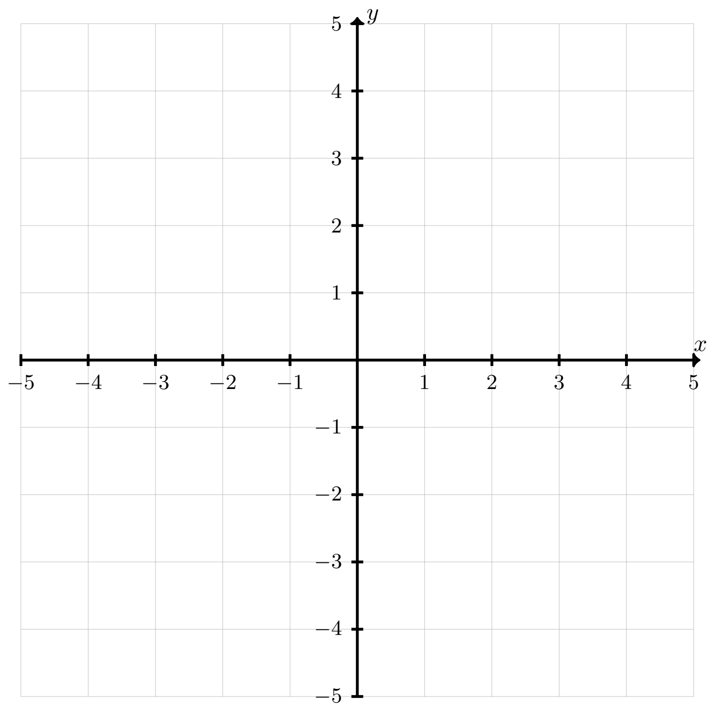


Figure 2.2.2 The graph of $y = x^2$

Example 2.2.3 Use the graph of $y = x^2$ to graph the functions $g(x) = x^2 + 2$ and $h(x) = x^2 - 3$.



□

2.2.2 Horizontal Translation

Definition 2.2.4 Horizontal Translation. Let c be a positive number. We say the function $g(x)$ is a **horizontal translation** of the function $f(x)$ if $g(x) = f(x \pm c)$.

- The graph of $g(x) = f(x-c)$ is obtained by shifting the graph of $y = f(x)$ to the right along the x-axis by c units.
- The graph of $g(x) = f(x+c)$ is obtained by shifting the graph of $y = f(x)$ to the left along the x-axis by c-units.

$\Diamond$

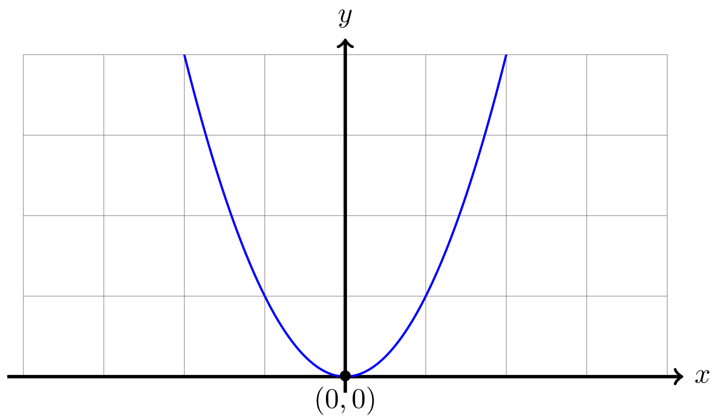
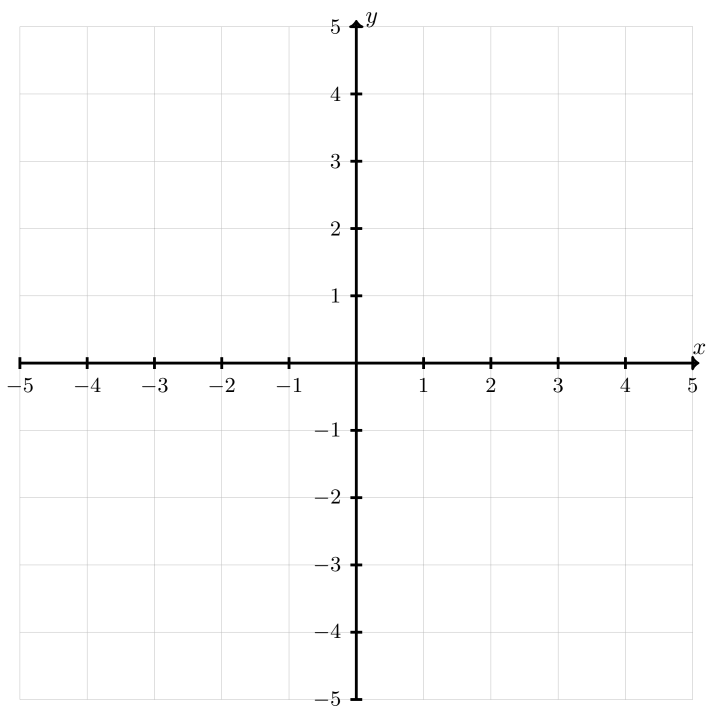


Figure 2.2.5 The graph of $y = x^2$

Example 2.2.6 Use the graph of $y = x^2$ to graph the functions $g(x) = (x + 2)^2$ and $h(x) = (x - 3)^2$.



$\square$

2.2.3 Vertical Scaling

Definition 2.2.7 Vertical Scaling. Let $c > 1$ be a number. We say the function $g(x)$ is a **vertical scaling** of the function $f(x)$ if $g(x) = cf(x)$ or $g = \frac{1}{c}f(x)$.

- The graph of $g(x) = cf(x)$ is obtained by stretching the graph of $y = f(x)$ vertically by a factor of c.
- The graph of $g(x) = \frac{1}{c}f(x)$ is obtained by compressing the graph of $y = f(x)$ vertically by a factor of c.

$\Diamond$

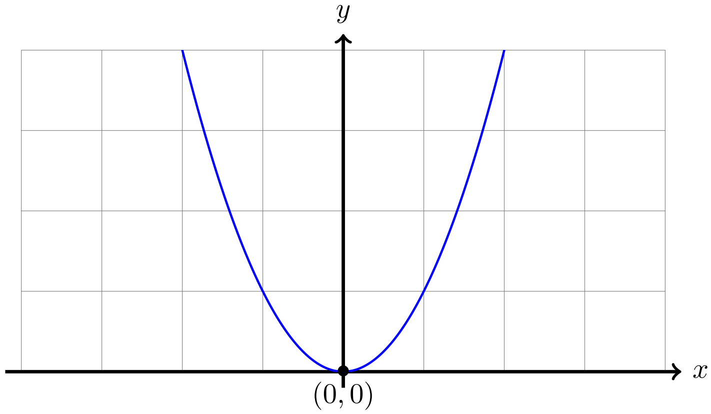
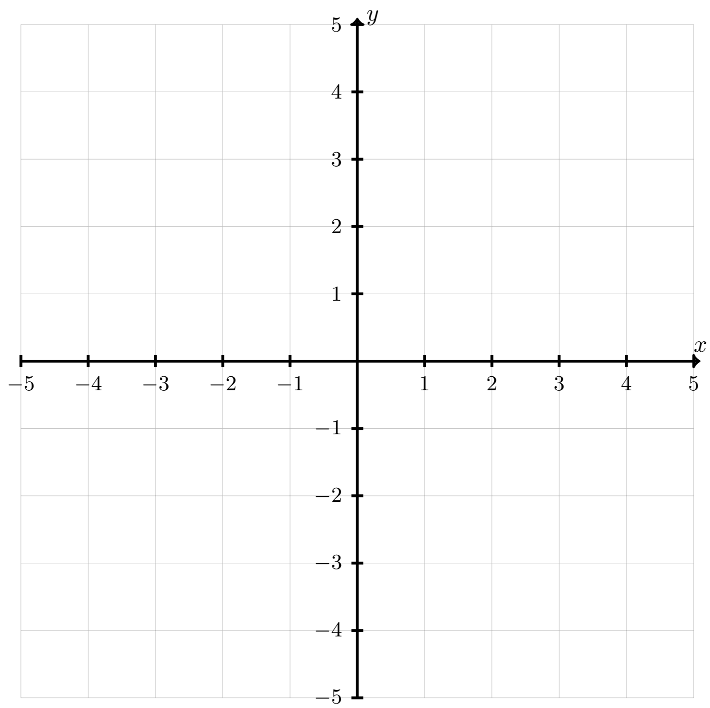


Figure 2.2.8 The graph of $y = x^2$

Example 2.2.9 Use the graph of $y = x^2$ to graph the functions $g(x) = 2x^2$ and $h(x) = \frac{2}{3}x^2$.



□

2.2.4 Reflection

Definition 2.2.10 Reflection. We say the function $g(x)$ is a **reflection** of $f(x)$ if either $g(x) = -f(x)$ or $g(x) = f(-x)$.

- The graph of $g(x) = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ over the x-axis.
- The graph of $g(x) = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ over the y-axis.

$\Diamond$

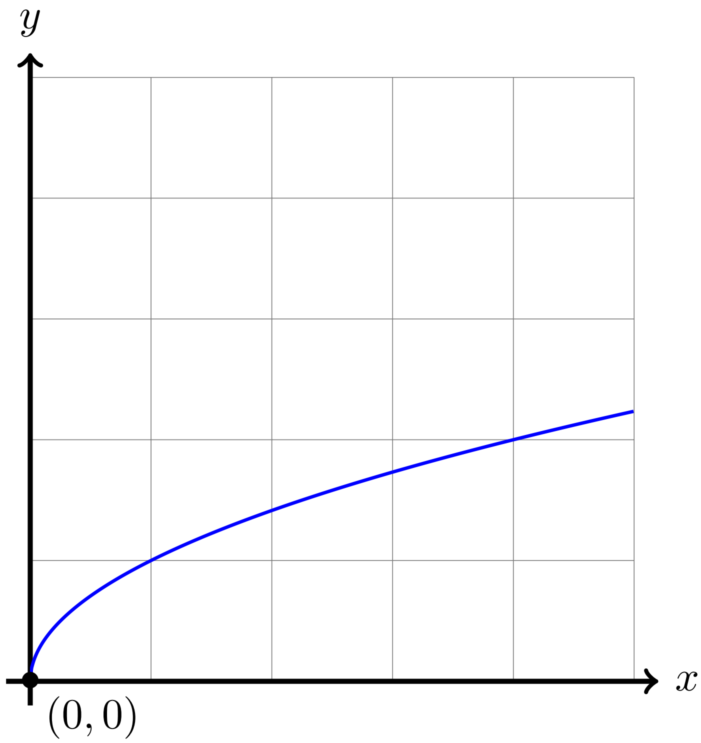
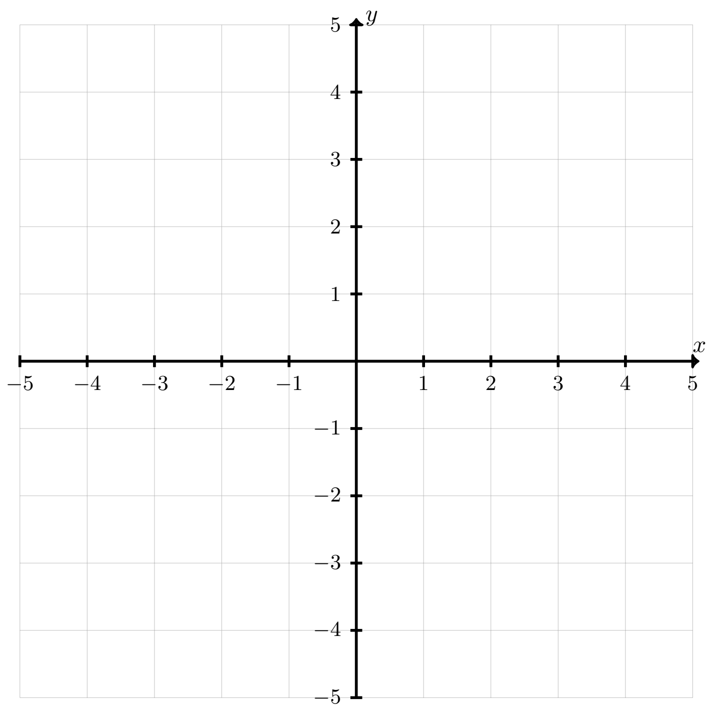


Figure 2.2.11 The graph of $y = \sqrt{x}$

Example 2.2.12 Use the graph of $y = \sqrt{x}$ to graph the functions $g(x) = -\sqrt{x}$ and $h(x) = \sqrt{-x}$.



□

2.2.5 Horizontal Scaling

Definition 2.2.13 Horizontal Scaling. Let $c > 1$ be a number. We say the function $g(x)$ is a **horizontal scaling** of the function $f(x)$ if $g(x) = f(cx)$ or $g(x) = f\left(\frac{1}{c}x\right)$.

- The graph of $g(x) = f(cx)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of c.
- The graph of $g(x) = f\left(\frac{1}{c}x\right)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of c.

$\Diamond$

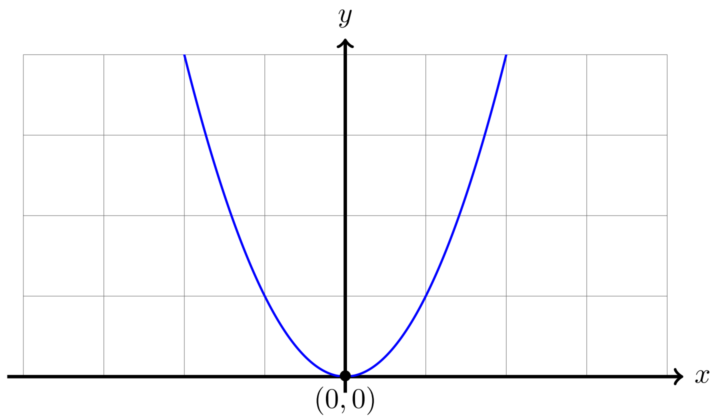


Figure 2.2.14 The graph of $y = x^2$

Example 2.2.15 Use the graph of $y = x^2$ to graph the functions $g(x) = (2x)^2$ and $h(x) = \left(\frac{2}{3}x\right)^2$. $\square$

2.2.6 Combining Graph Transformations

Most functions are obtained from a basic function using multiple transformations. When graphing a function, the order in which transformations are applied is important.

Algorithm 2.2.16 Graphing Functions using Transformations.

1. *Horizontal Translation*
2. *Scaling*
3. *Reflection*
4. *Vertical Translation*

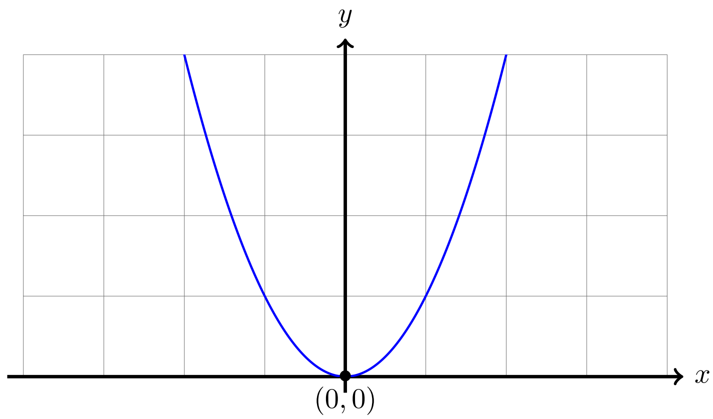
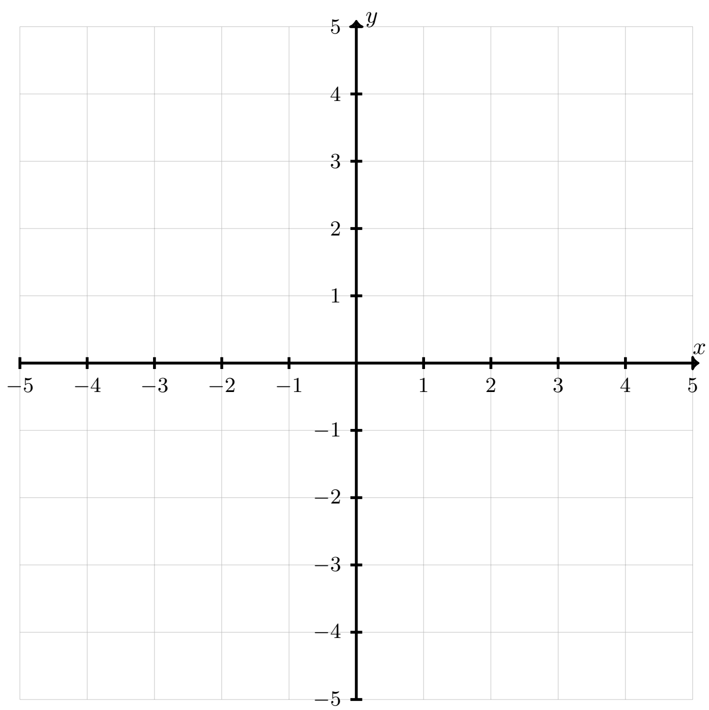


Figure 2.2.17 The graph of $y = x^2$

Example 2.2.18 Use the graph of $y = x^2$ to graph the function $g(x) = 2(x+3)^2 - 2$.



□

2.2.7 Worksheet: Graph Transformations

Objectives

Use graphs of basic functions and graph transformations to graph general functions:

- Vertical Translation
- Horizontal Translation
- Vertical Scaling,
- Horizontal Scaling, and
- Reflection.

1. Write down the graph transformations, in order, needed to graph the function $y = -3(x+2)^3 - 1$ using the graph of $y = x^3$. For each transformation, indicate the intermediate function.

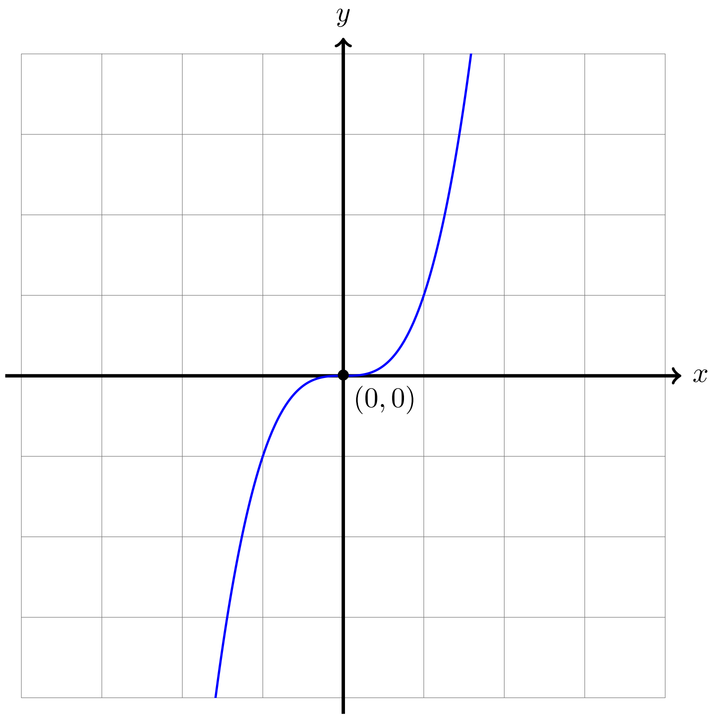
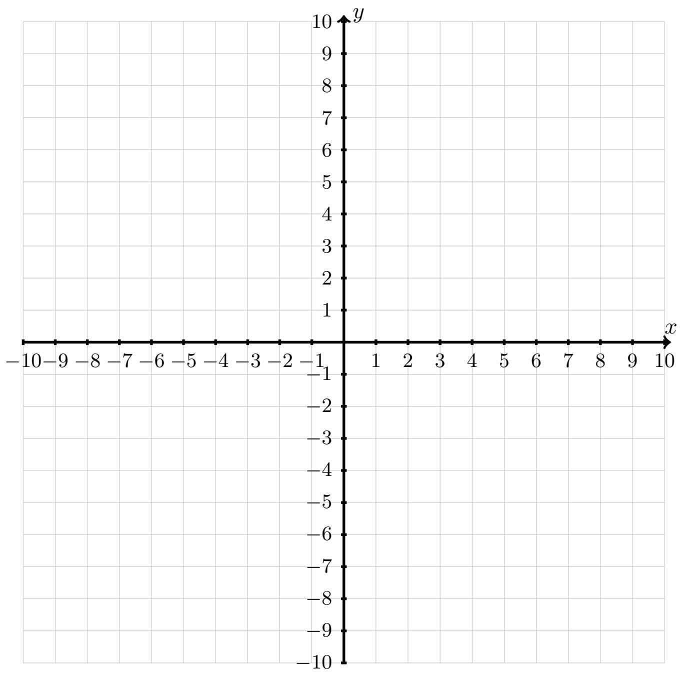
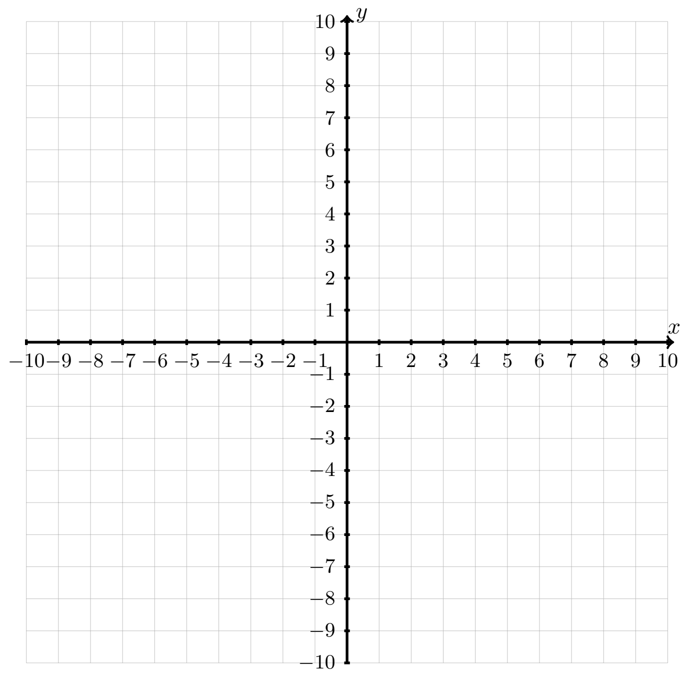
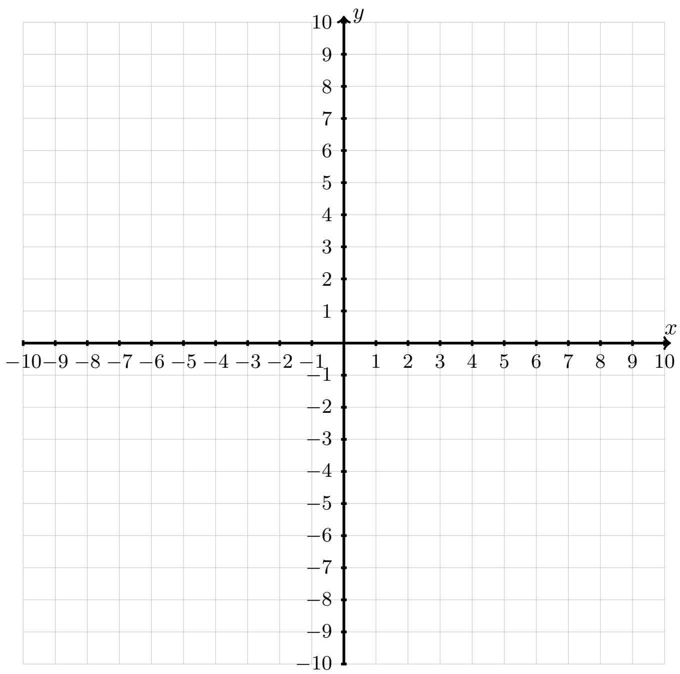


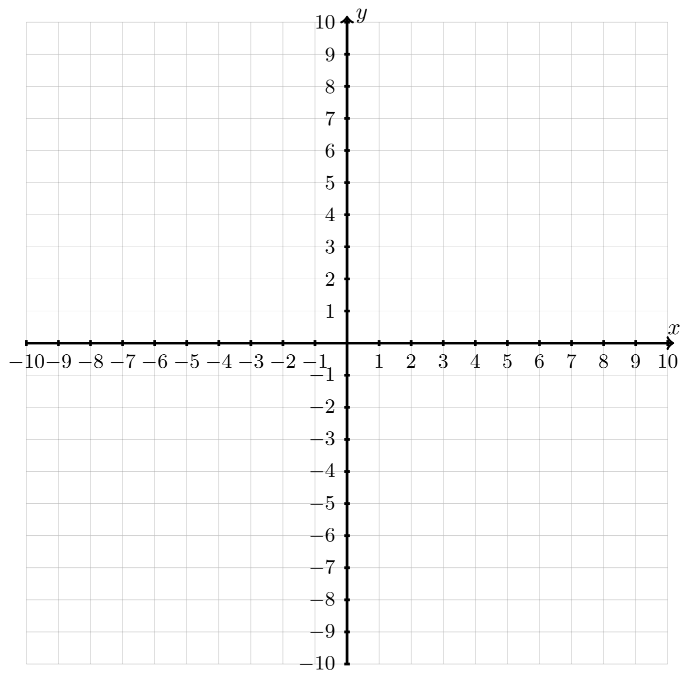
Figure 2.2.19 The graph of $y = x^3$

2. Use the graph of $y = x^3$ and your list above to graph $y = -3(x+2)^3 - 1$









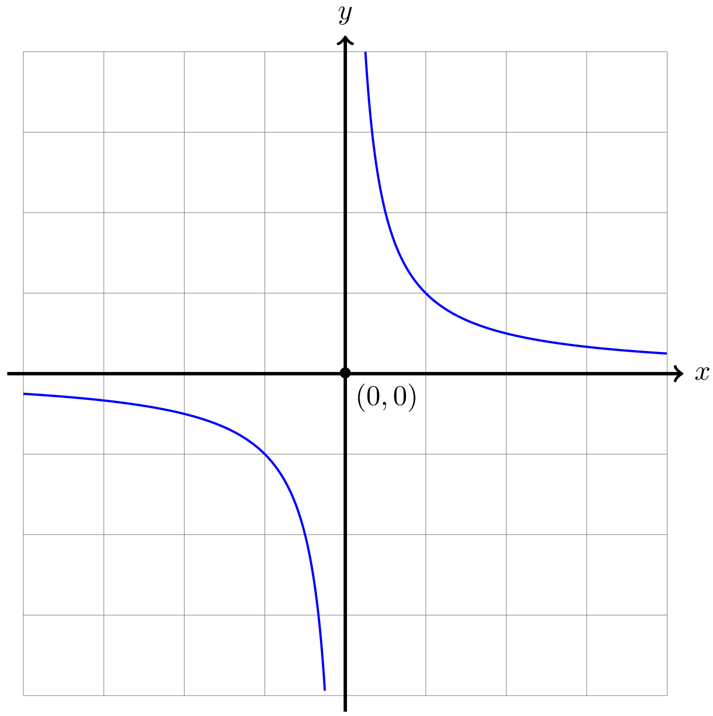


Figure 2.2.20 The graph of $y = 1/x$

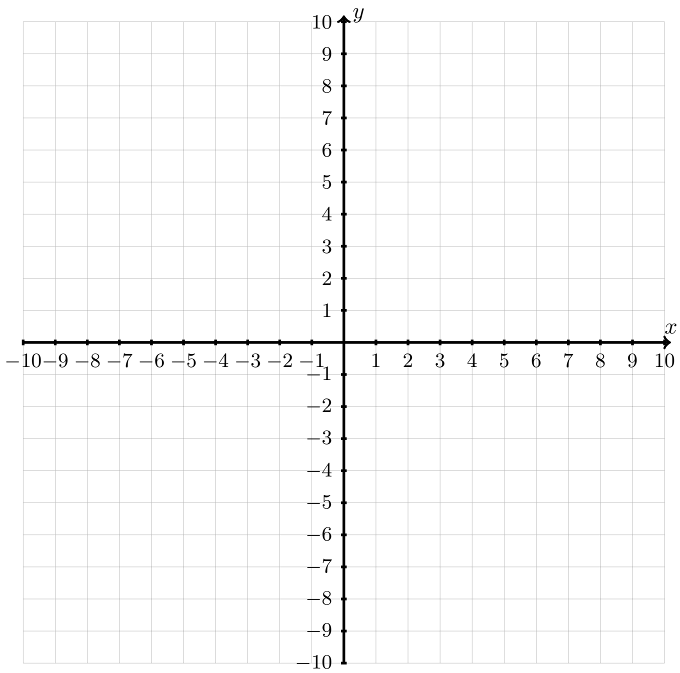
3. Write down the graph transformations, in order, needed to graph the function

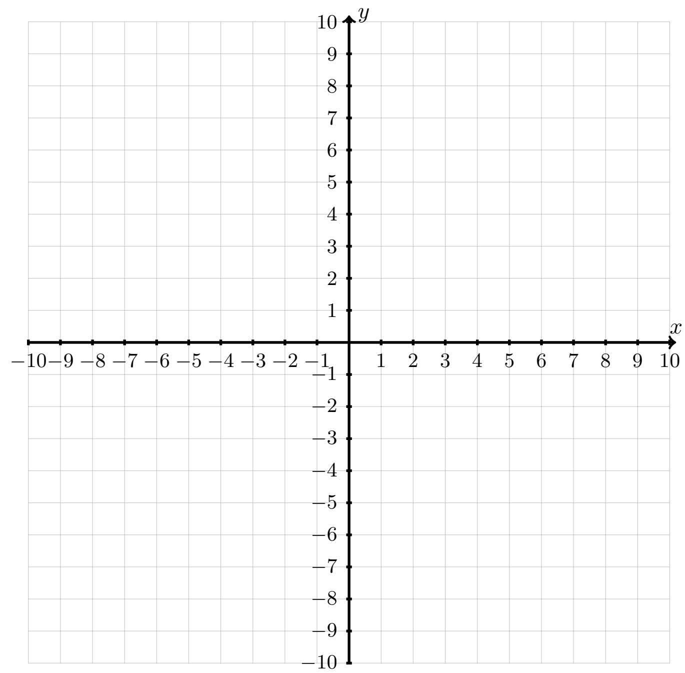
$$y = \frac{-2}{x-3} + 1$$

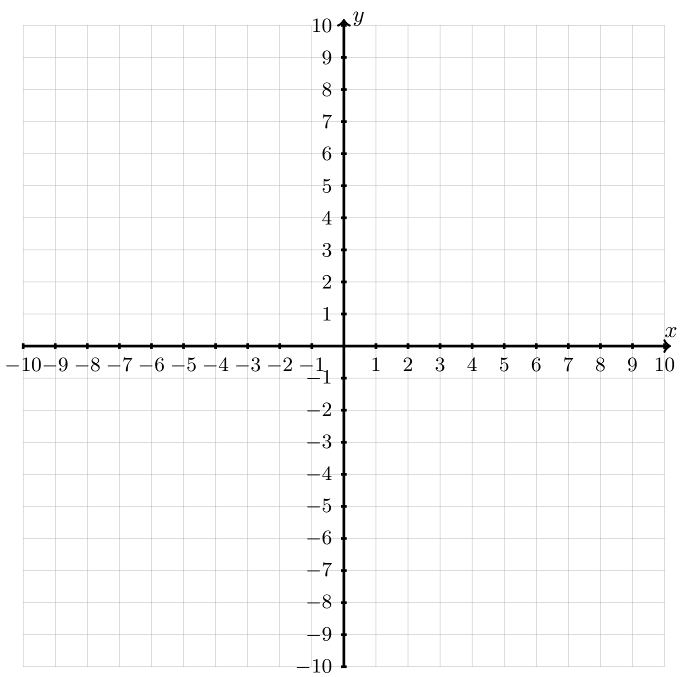
using the graph of $y = 1/x$. For each transformation, indicate the intermediate function.

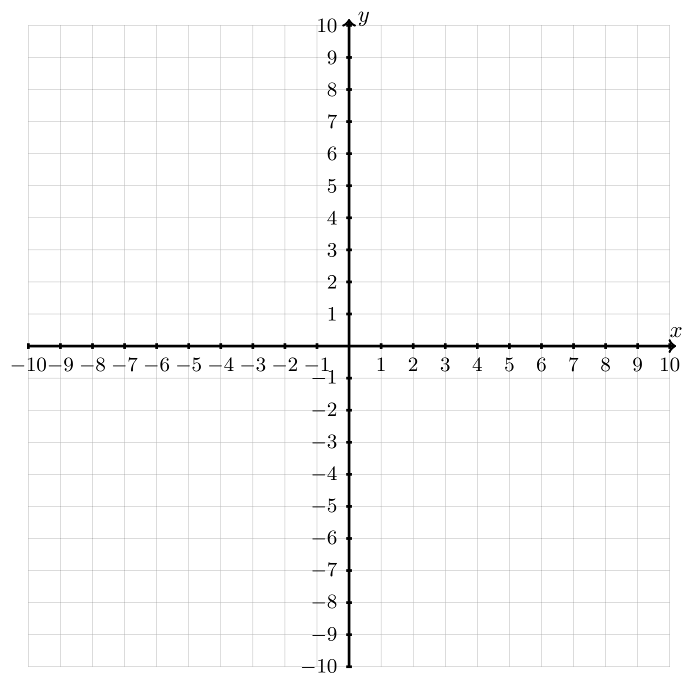
4. Use the graph of $y = 1/x$ and your list above to graph the function

$$y = \frac{-2}{x-3} + 1.$$









2.3 Symmetry

Intuitively, an object is *symmetric* if there is some line that divides the object into two (or more) pieces that are mirror images of one another. In this section, we utilize the methods in Section 2.2 to detect two types of symmetry mathematically.

2.3.1 Symmetry about the y-axis

The first type of symmetry we are interested in is symmetry about the y-axis in the plane. We begin with the intuitive understanding of this symmetry, and formalize a concrete method for detecting it algebraically.

Definition 2.3.1 We say a function f has **y-axis symmetry** if the y-axis divides the graph into two pieces that are mirror images of one another. ◊

For example, consider the graph of $y = x^2$. The graph intersects the y-axis at the point $(0, 0)$. If we cut the parabola in half at this point, then we get two pieces. We distinguish the two halves by coloring the left-hand side red and the right-hand side blue.

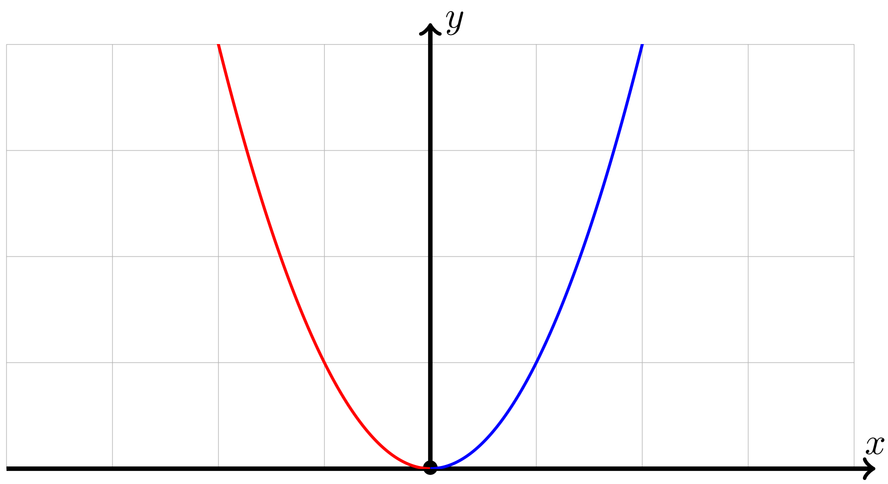


Figure 2.3.2 The graph of $y = x^2$ cut in half at $(0, 0)$.

To say these are mirror images of one another, means if we reflect the graph across the y-axis, then we obtain the same graph just with the colors switched.

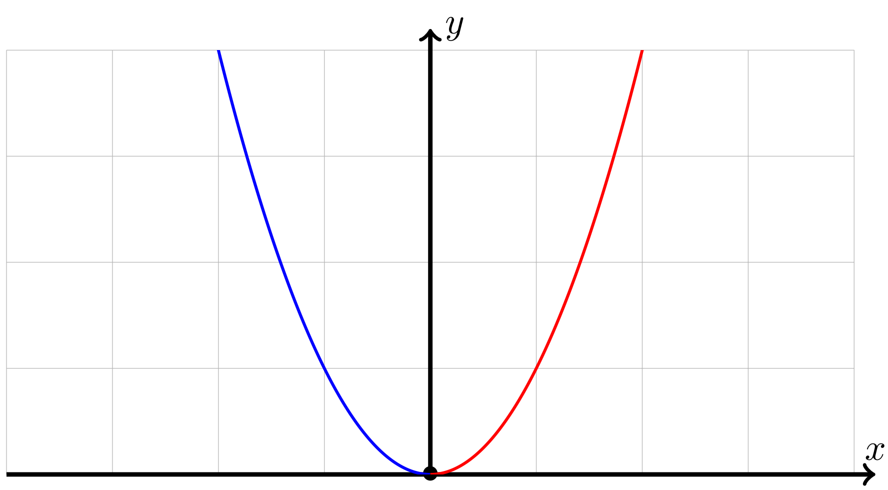


Figure 2.3.3 The reflection of $y = x^2$ across the y-axis.

Had we not colored the two pieces, we would be unable to tell which one is which. We say the graph is **invariant** under reflection across the y-axis to mean the original graph and its reflection across the y-axis are the same. This observation allows us to rephrase Definition 2.3.1 in terms of reflections.

Theorem 2.3.4 *The function f has y-axis symmetry if it is invariant under reflection across the y-axis.*

The characterization of y-axis symmetry above is purely geometric. Just like the intuitive understanding of symmetry, it can be difficult to verify whether two pictures are actually the same. Fortunately, the language of graph transformations provides us a dictionary between algebraic statements and geometric statements.

Definition 2.3.5 Even Function. A function $f\colon A \to \mathbb{R}$ is **even** if for every $x \in A$,

$$f(-x) = f(x)$$

◊

Theorem 2.3.6 *The function $f\colon A \to \mathbb{R}$ has y-axis symmetry if and only if f is even.*

The algebraic condition allows us to easily verify our intuition about images concretely.

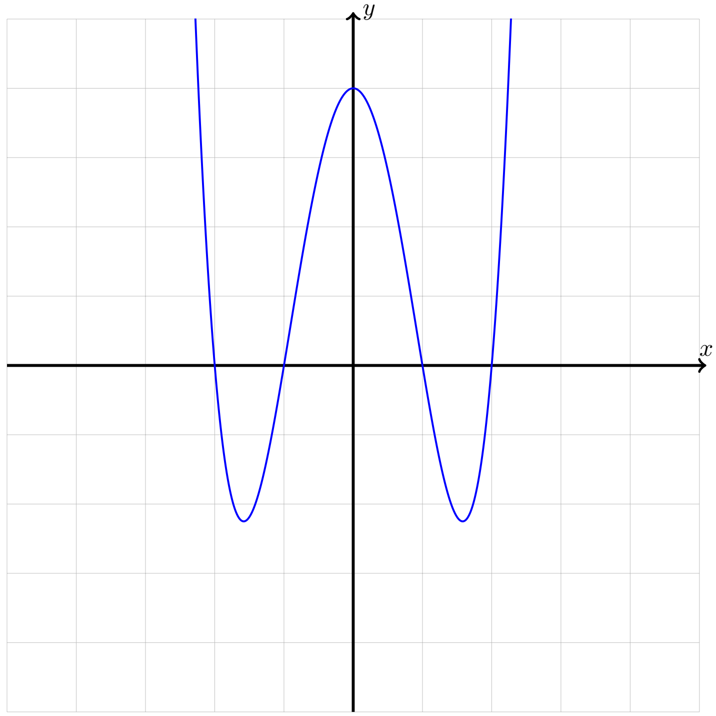


Figure 2.3.7 The graph of $y = x^4 - 5x^2 + 4$

Example 2.3.8 Verify the function $f(x) = x^4 - 5x^2 + 4$ has y-axis symmetry algebraically. $\square$

2.3.2 Origin Symmetry

The second type of symmetry we consider is slightly more subtle than y-axis symmetry.

Definition 2.3.9 We say a function f is **symmetric about the origin** if the graph of f is invariant under rotation by 180 degrees. $\Diamond$

To understand this type of symmetry, consider the graph of $y = x^3$ with the origin and another marked point.

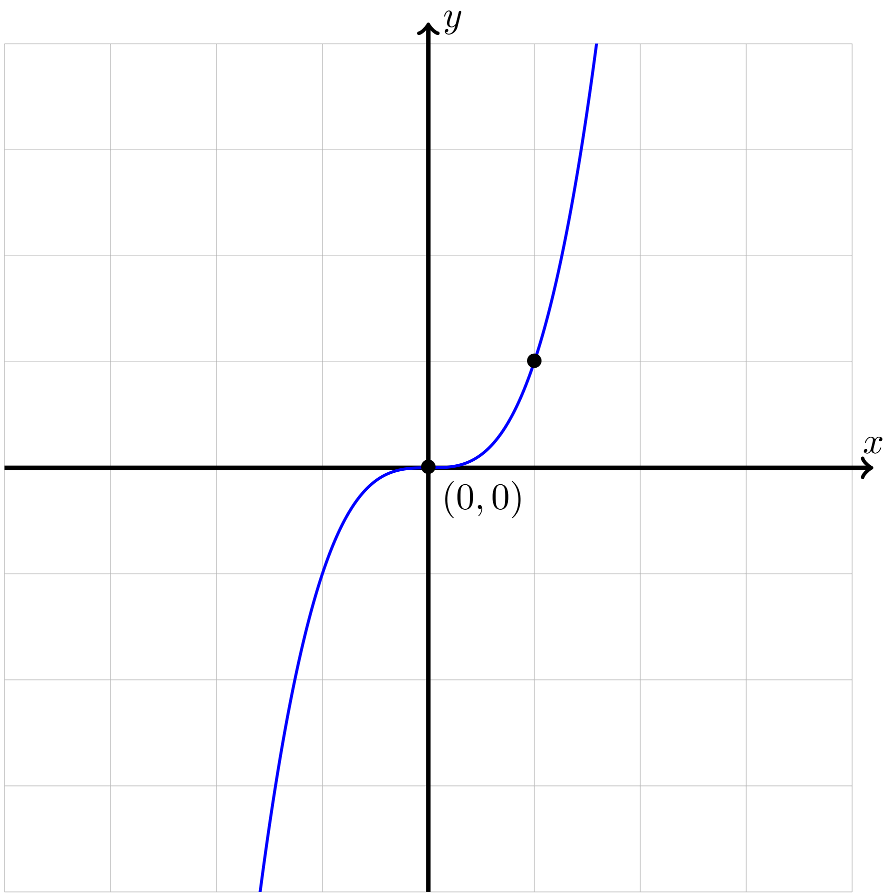


Figure 2.3.10 The graph of $y = x^3$

First, we rotate the graph by 90 degrees. We note that this rotation does not change the shape of the graph, so the distance of our marked point from the origin should not change.

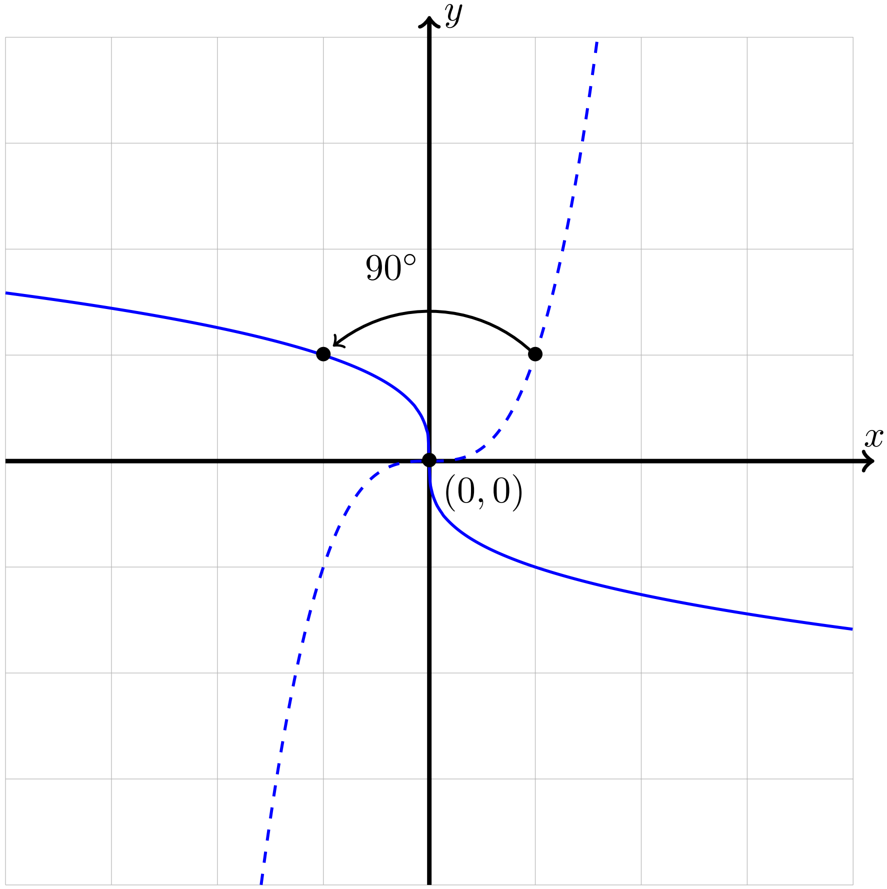


Figure 2.3.11 The graph of $y = x^3$ rotated 90 degrees counter-clockwise.

Rotating once more by 90 degrees, we still have not changed the distance of our marked point from the origin. The graph looks the same, except we have moved our marked point from the first quadrant into the third quadrant.

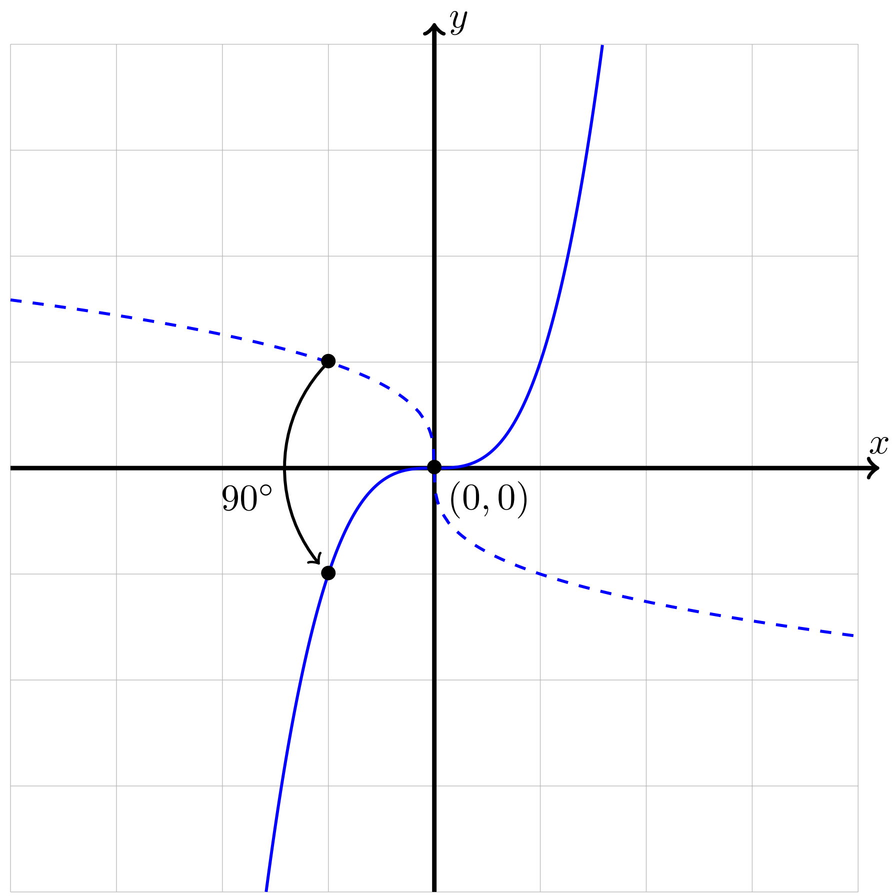


Figure 2.3.12 The graph of $y = x^3$ rotated 180 degrees counter-clockwise.

The path the marked point follows through the rotation traces out a piece of a circle. Plotting the entirety of this circle, we can see the starting and ending location of the marked point create a diameter.

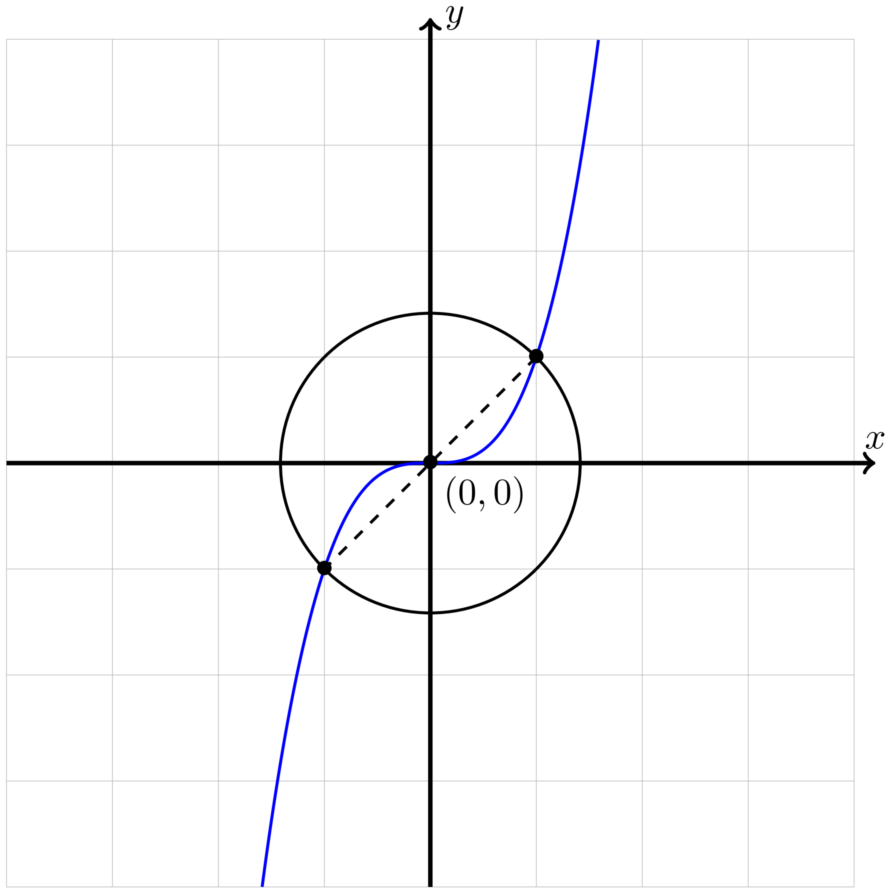


Figure 2.3.13 The graph of $y = x^3$ with a marked point and the rotation of that point by 180°

Assume the original point has coordinates (a, a^3) and its rotation by 180° has coordinates (b, b^3). Since the midpoint of the diameter is $(0, 0)$, we know that $0 = (a + b)/2$ and $0 = (a^3 + b^3)/2$. Solve the first equation for b in terms of a

$$\begin{aligned} 0 &= \frac{a+b}{2} && \text{Multiply both sides by 2} \\ 0 &= a + b && \text{Subtract } a \text{ from both sides} \\ b &= -a. \end{aligned}$$

The second coordinate is then

$$b^3 = (-a)^3 = (-1)^3 a^3 = -a^3.$$

Rewriting this in function notation, this tells us

$$f(-a) = (-a)^3 = -a^3 = -f(a).$$

This provides us with an algebraic condition for testing origin symmetry.

Definition 2.3.14 Odd Function. A function $f\colon A \to \mathbb{R}$ is **odd** if for every $x \in A$,

$$f(-x) = -f(x)$$

◊

Theorem 2.3.15 *The function $f\colon A \to \mathbb{R}$ has origin symmetry if and only if f is odd.*

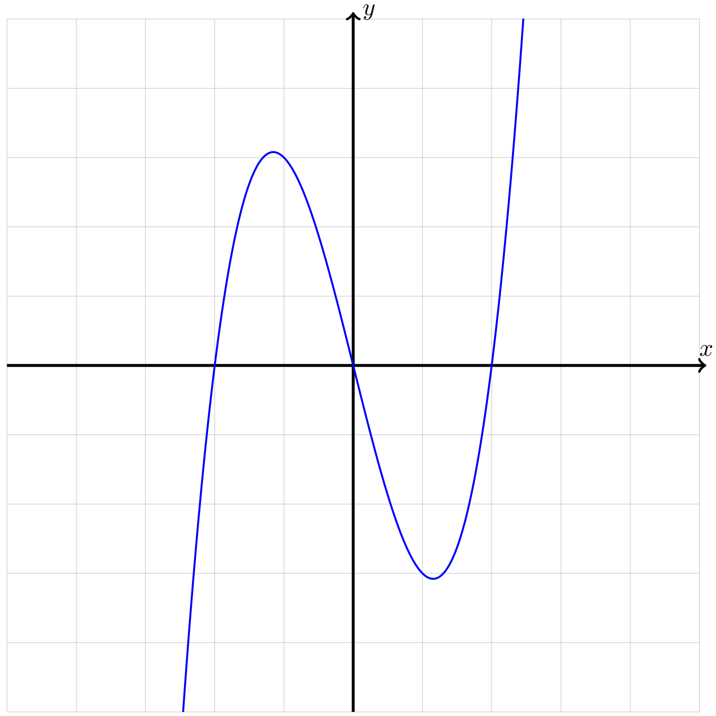


Figure 2.3.16 The graph of $y = x^3 - 4x$

Example 2.3.17 Verify the function $f(x) = x^3 - 4x$ has origin symmetry algebraically. $\square$

2.3.3 Worksheet: Symmetry

Objectives

- Identify if a function has y-axis symmetry.
- Identify if a function has origin symmetry.

Determine whether the following funtions have y-axis symmetry, origin symmetry, or neither.

1. $f(x) = |x| + 3$

2. $g(x) = x^5 + x^2$

3. $h(x) = x^4 + x^2 + 1$

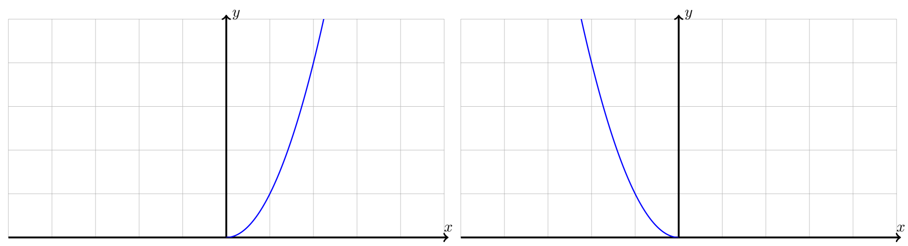
4. $k(x) = x^5 + x^3 + x$

2.4 Measures of Change

Functions are important tools in the application of mathematics because they allow us to formalize the relationship between two quantities. We are often interested in answering questions precisely about *how* the dependent variable responds to changes in the independent variable. While it is useful to be able to precisely quantify this relationship, it is often easier to understand these relationships visually. Our goal is to build a dictionary between the algebraic measures of change and the features of the graph.

2.4.1 Intervals of Increase and Decrease

The terms *increasing* and *decreasing* are useful qualitative descriptions of a function's behavior. A function is increasing if its y-values increase as its x-values increase and decreases if its y-values decrease as its x-values increase. This is easy to detect visually because either the height of the graph increases or decreases as you trace the graph from left to right.



(a) An increasing function **(b)** A decreasing function

Figure 2.4.1 Graphs of increasing and decreasing functions

Unfortunately, many interesting functions are not simply increasing or decreasing. Rather, most interesting functions have pieces that are increasing and pieces that are decreasing.

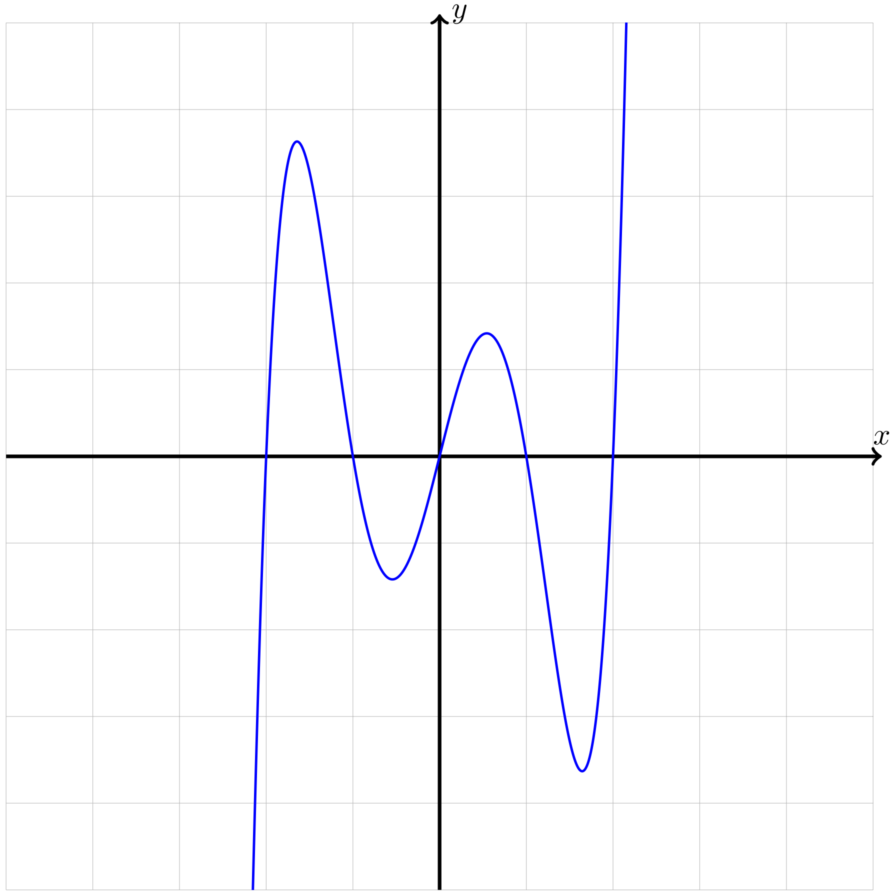


Figure 2.4.2 The graph of a function that is neither increasing nor decreasing

For this reason, we define increasing and decreasing on an interval to provide the flexibility to describe a range of interesting functions.

Definition 2.4.3 Assume $A \subseteq \mathbb{R}$ and $[a, b] \subseteq A$. We say the function $f: A \to \mathbb{R}$ is

- **increasing** on the interval $[a, b]$ if for every $a \leq x_1 < x_2 \leq b$,

$$f(x_1) < f(x_2).$$

- **decreasing** on the interval $[a, b]$ if for every $a \leq x_1 < x_2 \leq b$,

$$f(x_2) < f(x_1).$$

◊

Points where the graph changes between increasing and decreasing are often interesting features of the graph. These points are known collectively as **local extrema**.

Definition 2.4.4 Assume $A \subseteq \mathbb{R}$. The function $f: A \to \mathbb{R}$ has a **local maximum** at $(c, f(c))$ if there exist values $a, b \in A$ such that

1. f is increasing on the interval $[a, c]$, and
2. f is decreasing on the interval $[c, b]$.

We say $f(c)$ is a **local maximum value** ◊

Definition 2.4.5 Assume $A \subseteq \mathbb{R}$. The function $f\colon A \to \mathbb{R}$ has a **local minimum** at $(c, f(c))$ if there exist values $a, b \in A$ such that

1. f is decreasing on the interval $[a, c]$, and

2. f is increasing on the interval $[c, b]$.

We say $f(c)$ is a **local minimum value** $\Diamond$

Example 2.4.6 Use the following graph to find the intervals of increase and decrease. Identify all local extrema.

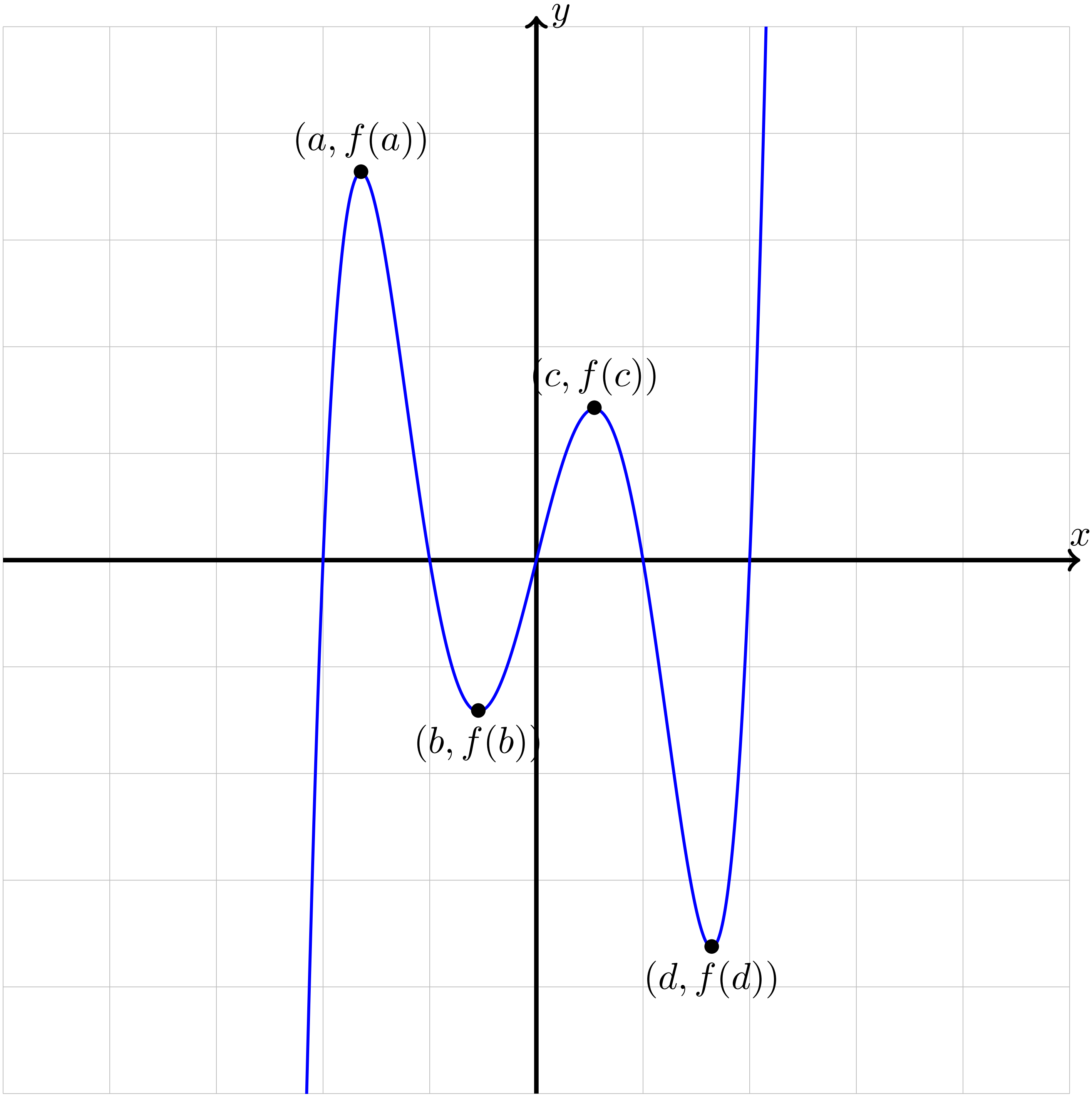


Figure 2.4.7 The graph of $y = f(x)$

□

2.4.2 Average Rate of Change

The *rate* at which $y = f(x)$ changes as x changes is a quantitative measure of change. Suppose we have the following data about a five hour road trip.

Table 2.4.8 Distance Driven as a Function of Time

Time	Miles Driven
12:00 PM	0
1:00 PM	25
2:00 PM	95
3:00 PM	165
4:00 PM	235
5:00 PM	270

While we cannot pinpoint exactly how fast the car was moving at any given time, we can use this data to give a rough estimate for the speed of the car over a period of time. If we wanted to estimate the speed of the car over the two hour period from 3:00 PM until 5:00 PM, we could take the distance traveled

$$270 - 165 = 105 \text{ miles}$$

and divide it by the number of hours traveled to obtain

$$\frac{105 \text{ miles}}{2 \text{ hours}} = 52.5\ \frac{\text{miles}}{\text{hours}}.$$

This represents the *average* speed of the car during this period of time. This idea generalizes to arbitrary functions naturally.

Definition 2.4.9 The Average Rate of Change. Assume f is a function and the domain of f contains the interval $[a, b]$. The **average rate of change** of f on the interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

◊

An object is thrown straight up in the air. The object reaches a maximum height of 1 meter after 1 second, and returns to the thrower's hand after 2 seconds. The function $h\colon [0,2] \to \mathbb{R}$ models the height of the object in meters as a function of x seconds.

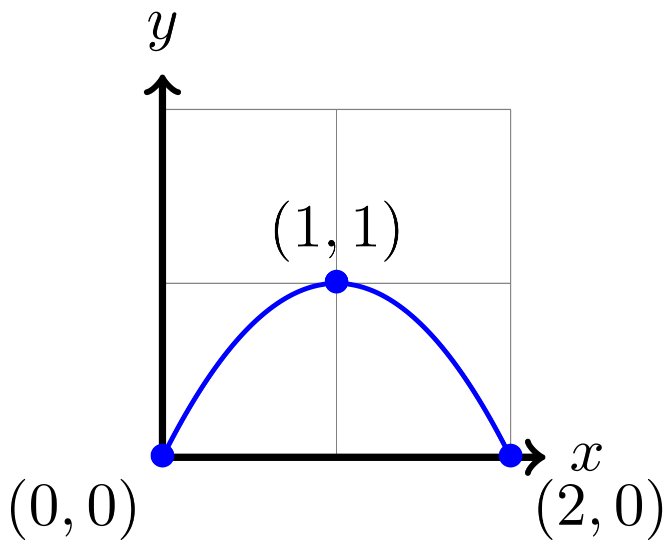


Figure 2.4.10 The graph of $h(x)$

Example 2.4.11

(a) What is the average rate of change for the function on the interval $[0,1]$? Interpret the meaning of this number.

(b) What is the average rate of change for the function on the interval $[0,1]$? Interpret the meaning of this number.

(c) What is the average rate of change for the function on the interval $[0,2]$?

□

2.4.3 Worksheet: Measures of Change

Objectives

- Identify intervals of increase and decrease of a function.
- Identify local extrema of a function.
- Find the average rate of change of a function on an interval.

Find the average rate of change for the given function over the given interval.

1. $f(x) = 3x + 2$; $[3, 5]$

2. $g(x) = \frac{1}{x}$; $\left[\frac{1}{4}, \frac{1}{2}\right]$

3. $h(x) = \sqrt{x}$; $[4, 9]$

4. Use the graph of the function f below to find the intervals of increase, decrease, and local extrema.

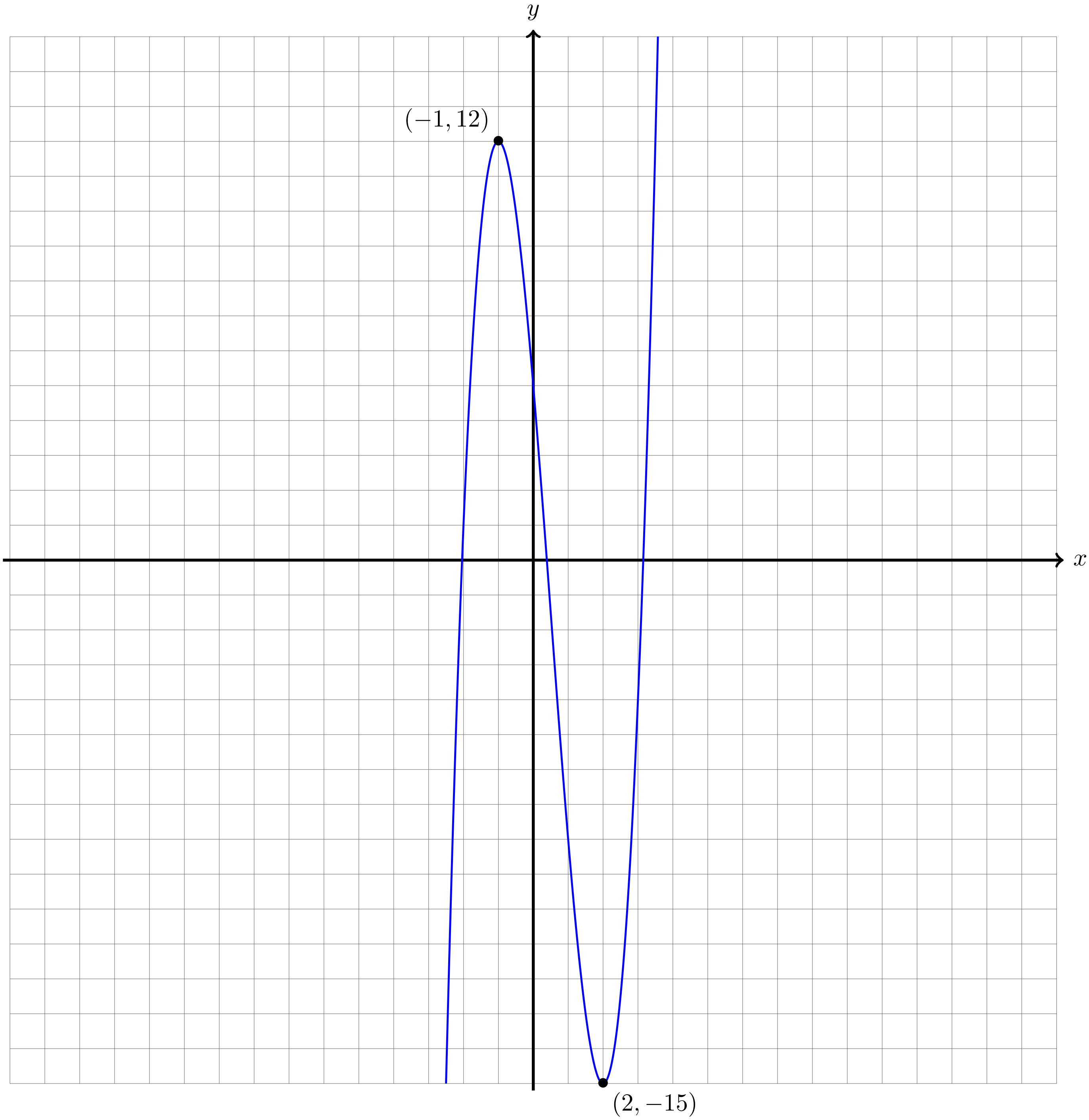


Figure 2.4.12 The graph of f

Chapter 3

Polynomial Functions

3.1 Polynomial Functions

The simplest algebraic functions are constructed using the operations of addition and multiplication with a variable. While simple to construct, it is often difficult or even impossible to answer basic questions about these functions in general. However, these functions are extraordinarily important in nearly every branch of mathematics. We discuss foundational results about general polynomials in this section, and specialize to some easy to classify cases in the later sections.

3.1.1 Roots of Polynomials

Definition 3.1.1 Polynomial Function. A **polynomial function** in the variable x is a function $f\colon \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is an integer, $a_0, a_1, \ldots, a_n$ are numbers and $a_n \neq 0$.

We say f is a polynomial of **degree** n and write $\deg f = n$. The numbers $a_0, a_1, \ldots, a_n$ are called the **coefficient** of $1, x, \ldots, x^n$, respectively. ◊

Polynomial functions are most interesting because they are relatively simple functions that allow us to model real world phenomena.

Example 3.1.2 Assume an object is thrown straight up in the air. If we let h_0 denote the distance from the thrower's hand to the ground when the object is released, and let v_0 denote the velocity of the object in meters/second as it leaves the thrower's hand, then the height of the object as a function of time can be expressed using the polynomial

$$h(t) = -4.9t^2 + v_0 t + h_0.$$

□

Question 3.1.3 How long will it take for the object to return to the hand of the thrower? □

To answer this question, we want the solutions to the equation

$$-4.9t^2 + v_0 t + h_0 = h_0.$$

If we subtract h_0 from both sides of the equation, then it is equivalent to find solutions to the polynomial equation

$$-4.9t^2 + v_0 t = 0.$$

Questions of this type arise frequently when discussing polynomials.

Definition 3.1.4 Root. A **root** of the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is a number, r, such that

$$0 = f(r) = a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0.$$

◊

Questions about roots of polynomials have been posed and studied for millenia. The most important result about roots of polynomials is the following.

Theorem 3.1.5 The Fundamental Theorem of Algebra. (Carl Friedrich Gauss) *Given a polynomial f of degree $n > 0$, there exist a constant c and n roots $r_1, r_2, \ldots, r_n$ such that*

$$f(x) = c(x - r_1)(x - r_2)\cdots(x - r_n).$$

*We call this expression the **factorization** of the polynomial f.*

Up to the constant c, the factorization in The Fundamental Theorem of Algebra is unique.

Example 3.1.6 The roots of the polynomial $f(x) = 2x^2 - 3x + 1$ are $r_1 = 1$ and $r_2 = 1/2$. Use this information to factor f.
$\square$

Another important observation is the roots in the list $r_1, r_2, \ldots, r_n$ may not necessarily be distinct.

Definition 3.1.7 Assume f is a polynomial of degree n and r is a root of f. The **multiplicity** of r, is the number of times of times r appears in the factorization of f. $\Diamond$

Example 3.1.8

(a) Write down the roots of the polynomial $f(x) = (x+1)^2(x-1)$.

(b) For each root, determine its multiplicity.

$\square$

Finally, we note the roots of a polynomial may not always be real numbers. For example, the polynomial $f(x) = x^2+1$ does not have any real roots because every real number, r, satisfies $r^2 \geq 0$ and thus

$$f(r) = r^2 + 1 \geq 0 + 1 = 1.$$

Definition 3.1.9 The Imaginary Unit The **imaginary unit**, i, is defined to be one of the roots of the polynomial $f(x) = x^2+1$. In particular, $i^2+1 = 0$ or, equivalently, $i^2 = -1$. It is common to write $i = \sqrt{-1}$. ♢

Since

$$(-i)^2 + 1 = (-1)^2 i^2 + 1 = i^2 + 1 = 0$$

the roots i and $-i$ provide the factorization

$$x^2 + 1 = (x - i)(x + i).$$

Numbers that involve i arise frequently in the study of roots of polynomials of degree 2.

Definition 3.1.10 Complex Number. A **complex number** has the form $a + bi$, where a and b are real numbers. The set of all complex numbers is

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}.$$

♢

Example 3.1.11 The number $3 + i\sqrt{2}$ is a complex number. □

It is important to note The Fundamental Theorem of Algebra tells us that every polynomial of degree n has precisely n roots (counting multiplicities), some of which may not be real numbers. However, it does ***not*** tell us *how* to find those roots.

Question 3.1.12 Is there a formula to find all of the roots of the polynomial $f(x) = a_n x^n + \cdots + a_0$ in terms of the coefficients? □

In general, the answer to this question is deeply unsatisfying. We will find formulas for roots of polynomials of degree 1 in Section 3.3 and of degree 2 in Section 3.7. While formulas exist for roots of polynomials of degree 3[1] and degree 4[2], they are beyond the scope of this text. However, there are no such general formulas for polynomials of degree 5 and higher. This means we will need to develop other techniques for finding roots of polynomials. One very useful tool for finding roots of general polynomials is the rational root theorem.

Theorem 3.1.13 The Rational Root Theorem. *Assume*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with $a_0, a_1, \ldots, a_n$ integers and $a_n \neq 0$. If p and $q \neq 0$ are integers and $r = p/q$ is a root of f, then p is a factor of a_0 and q is a factor of a_n.

[1] `en.wikipedia.org/wiki/Cubic_equation#General_cubic_formula`
[2] `en.wikipedia.org/wiki/Quartic_equation#The_general_case`

Example 3.1.14 Finding Roots of Polynomials.

(a) List all possible rational roots of the function $f(x) = 2x^4 - 7x^3 + 5x^2 - 7x + 3$.

(b) Determine which numbers from the list above are roots of f.

□

3.1.2 Worksheet: Polynomials

Objectives

- Find rational roots of polynomials

Use the The Rational Root Theorem to find all rational roots of the following polynomials.

1. $f(x) = x^3 - 3x^2 - 4x + 8$

2. $g(x) = 2x^4 - 5x^3 - 3x^2 + 6x - 3$

3. $h(x) = 4x^3 + 2x^2 - 9x - 3$

4. $k(x) = x^2 - 7x + 10$

3.2 Polynomial Division

Another method for finding roots that is often paired with The Rational Root Theorem is polynomial division. Besides finding roots, the method of polynomial division will be important in Section 5.2.

Recall that if we have an integer n and a smaller integer, d, then we can always find integers q and r with $r < d$ such that

$$n = qd + r.$$

Long division provides a convenient method for finding q and r.

Example 3.2.1 To divide 6935 by 30, we would perform the long division using the symbols

$$\begin{array}{r} 231 \\ 30\overline{)6935} \\ \underline{-60} \\ 93 \\ \underline{-90} \\ 35 \\ \underline{-30} \\ 5 \end{array}$$

to find $6935 = 30 \times 231 + 5$. □

For polynomials, the form is nearly identical, with numbers replaced by polynomials.

Theorem 3.2.2 Polynomial Division. *Assume f and d are polynomials with $\deg d < \deg f$. There exist polynomials q and r with $\deg r < \deg d$ such that*

$$f(x) = q(x)d(x) + r(x).$$

*In analogy with the integers, we call the polynomials d the **dividend**, q the **quotient**, and r the **remainder**, and we say that f divided by d is q with remainder r*

Recall for integers n and d, we say that d divides n (or d is a divisor of n, or d is a factor of n) if the remainder is zero n is divided by d. We use this same language for polynomials.

Definition 3.2.3 Assume f and d are polynomials. The polynomial d **divides** the polynomial f if the remainder when f is divided by d is the zero polynomial, $r(x) = 0$. ◊

To understand how to perform division with polynomials, it is helpful to understand the division algorithm with numbers written in base 10. The four steps in the long division above are shorthand for the following equations.

$$6935 = 30 \times 200 + 935 \tag{3.2.1}$$
$$935 = 30 \times 30 + 35 \tag{3.2.2}$$
$$35 = 30 \times 1 + 5 \tag{3.2.3}$$

We arrive at the final answer by repeated substitution for the remainders starting with (3.2.1).

$$\begin{aligned} 6935 &= 30 \times 200 + 935 && \text{Substitute the RHS of (3.2.2) for 935} \\ &= 30 \times 200 + 30 \times 30 + 35 && \text{Substitute the RHS of (3.2.3) for 35} \\ &= 30 \times 200 + 30 \times 30 + 30 \times 1 + 5 && \text{Factor 3 from the first three terms} \\ &= (200 + 30 + 1) \times 3 + 2 && \text{Simplify} \\ &= 231 \times 30 + 5. \end{aligned}$$

Long division with integers terminates because we reduce the *number of digits* in the remainder by at least one at each stage. Our goal will be similar for long division with polynomials, but we will reduce the *degree* of the remainder polynomial by at least one at every stage. The process terminates when the degree of the remainder is smaller than the degree of the divisor.

We will mimic the steps for division with integers to divide the polynomial $f(x) = 6x^3 + 9x^2 + 3x + 5$ by the polynomial $d(x) = 3x$. Since the degree of the divisor, $d(x) = 3x$, is 1, the process terminates when the degree of the remainder is zero — i.e. the remainder is simply a real number.

Example 3.2.4

(a) Use a multiple of $3x$ to eliminate the leading term, $6x^3$. Rewrite $f(x)$ in the form

$$6x^3 + 9x^2 + 3x + 5 = q_1(x) \times 3x + r_1(x).$$

(b) Repeat the same process with $r_1(x)$ and $3x$ to obtain an expression of the form

$$r_1(x) = q_2(x) \times 3x + r_2(x).$$

(c) Repeat the process one last time with $r_2(x)$ and $3x$ to obtain an expression of the form

$$r_2(x) = q_3(x) \times 3x + r_3(x).$$

(d) Substitute the expression for $r_1(x)$ from b into the expression

$$6x^3 + 9x^2 + 3x + 5 = q_1(x) \times 3x + r_1(x).$$

(e) Substitute the expression for $r_2(x)$ from c into the expression from d.

(f) Factor $3x$ from as many terms in e as possible to obtain an expression of the form

$$6x^3 + 9x^2 + 3x + 5 = q(x) \times 3x + r(x),$$

where $r(x)$ is a number.

(g) Rewrite this process using the same shorthand for long division with integers.

□

Remark 3.2.5 Note that if we evaluate each of the polynomials at $x = 10$, then we find

$$f(10) = 6000 + 900 + 30 + 5 = 6935$$

divided by $d(10) = 30$ is

$$q(10) = 200 + 30 + 1 = 231$$

with remainder $r(10) = 5$. Indeed, polynomial long division is a generalization of long division in base 10.

When paired with The Rational Root Theorem, polynomial long division is useful for factoring polynomials. The main tool is the following theorem.

Theorem 3.2.6 *The number r is a root of the polynomial f if and only if $(x - r)$ divides f.*

Example 3.2.7

(a) Divide $f(x)$ by the polynomial $x - 1/2$. Write f in the form

$$f(x) = \left(x - \frac{1}{2}\right) q_1(x).$$

(b) Divide $q_1(x)$ by the polynomial $x - 3$. Write f in the form

$$f(x) = \left(x - \frac{1}{2}\right)(x - 3)q_2(x).$$

(c) Factor the polynomial $q_2(x)$. Write f as the product of four degree one polynomials.

□

Worksheet: Polynomial Division

Objectives

- Use long division with polynomials to factor

1. Consider the polynomial $f(x) = x^3 - 3x^2 + 2x$

(a) Use the The Rational Root Theorem to list all possible rational roots of f.

(b) Use your list to find all the roots of f.

(c) Use the roots of f to find its factorization.

2. Combining Division and the Rational Root Theorem. Consider the polynomial $f(x) = x^3 - x^2 - 2x + 2$.

(a) Use the The Rational Root Theorem to list all possible rational roots of f.

(b) Use your list to find all *rational* roots of f.

(c) Using the rational roots, use polynomial division to factor f as the product of a degree 1 polynomial and a degree 2 polynomial.

(d) What can you say about the other roots of f?

3.3 Linear Functions

In this section, we study a special kind of polynomial known as a *linear function*. As we will see, the name linear comes from the fact that the graph of such a function is always a line.

3.3.1 Linear Functions and Slope

Definition 3.3.1 Linear Function. A function $f\colon \mathbb{R} \to \mathbb{R}$ is **linear** if the The Average Rate of Change of f is constant. That is, there exists a number m such that for all numbers a and b

$$m = \frac{f(b) - f(a)}{b - a}.$$

We call the number m the **slope** of f. ♢

In order to find the equation of a linear function, it is enough to know the slope, m, and the value of the function for a single x-value. The simplest point to find is the y-intercept: the x-coordinate is 0 and the y-coordinate is $b = f(0)$. If we write $y = f(x)$, then by Definition 3.3.1

$$m = \frac{f(x) - f(0)}{x - 0} = \frac{y - b}{x}.$$

We obtain the equation for the function by solving the equation for y in terms of x

$$\begin{aligned}
m &= \frac{y-b}{x} && \text{Multiply both sides by } x \\
mx &= y - b && \text{Add } b \text{ to both sides} \\
mx + b &= y && \text{Swap sides} \\
y &= mx + b && \text{Replace } y \text{ with } f(x) \\
f(x) &= mx + b
\end{aligned}$$

Theorem 3.3.2 The Slope-Intercept Form of a Line. *The* ***slope-intercept form of a linear function*** *is*

$$f(x) = mx + b \quad \textit{or} \quad y = mx + b,$$

where m is the slope or average rate of change of f.

The number b is the y-coordinate of the y-intercept $(0, b)$.

3.3.2 Graphing Linear Functions

Consider a linear function $y = mx + b$ with $m \neq 0$. Recall that the graph of an equation consists of all points (x, y) that are a solution to the equation. The graph of a linear equation is always a line. We can graph the equation by finding any two points on the line and drawing a line through them. In general, the two points that are simplest to find are the two intercepts.

We find the x-intercept by substituting 0 for y and solving the resulting equation for x

$$\begin{aligned}
0 &= mx + b && \text{Subtract } b \text{ from both sides} \\
mx &= -b && \text{Divide both sides by } m \\
x &= -\frac{b}{m}.
\end{aligned}$$

Theorem 3.3.3 *The x-intercept of the linear function $y = mx + b$ with $m \neq 0$ is the point*

$$\left(-\frac{b}{m}, 0\right).$$

Consider the linear function

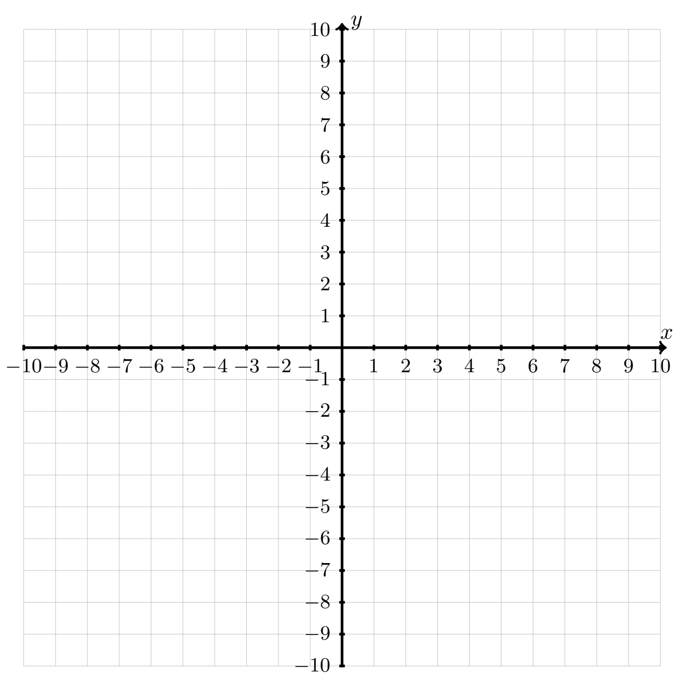
$$f(x) = -\frac{2}{3}x + 2$$

Example 3.3.4

(a) Find the x-intercept of f.

(b) Find the y-intercept of f.

(c) Graph the equation $y = f(x)$.



□

3.3.3 Worksheet: Linear Functions

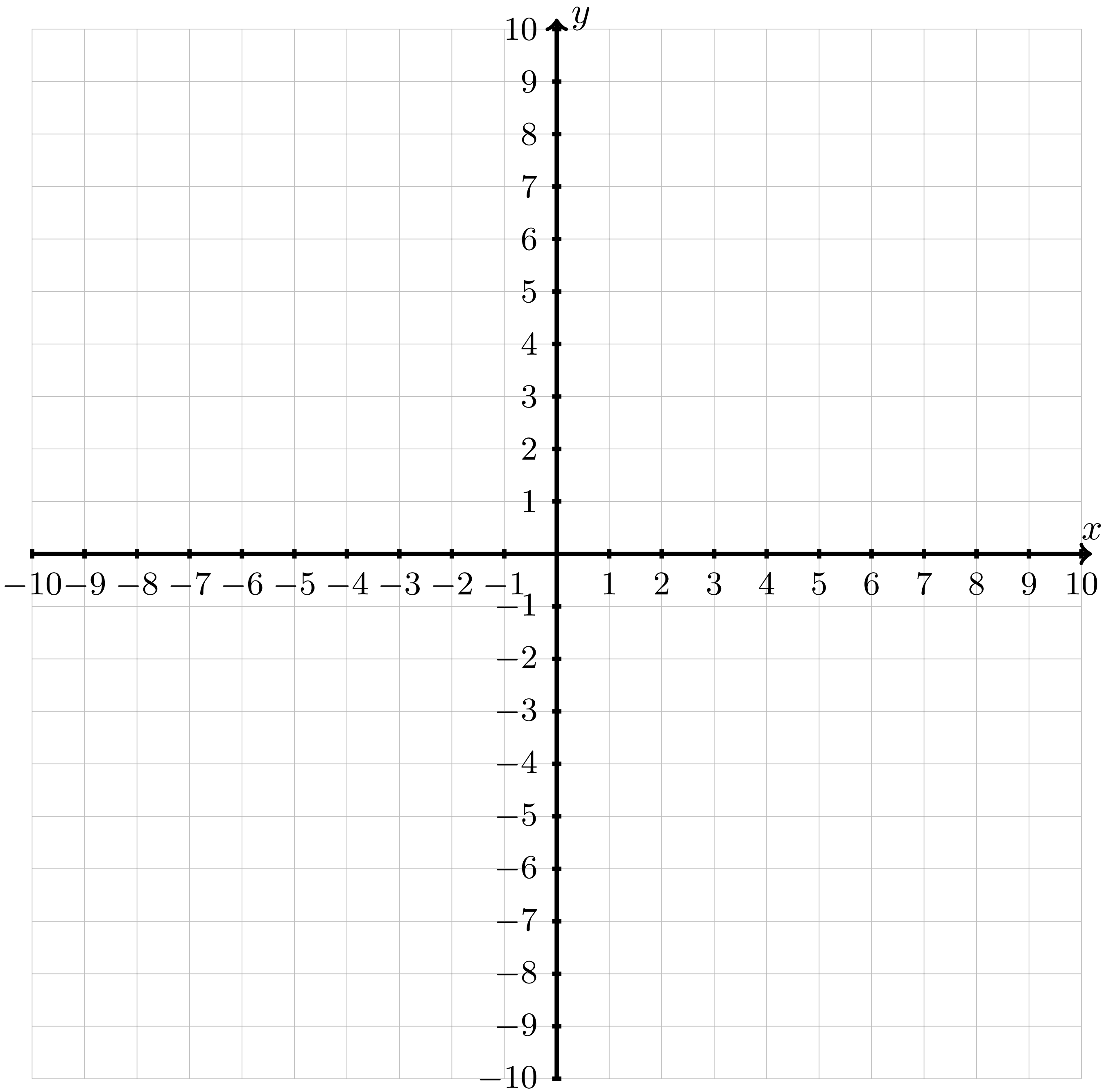
Objectives

- Recognize linear functions.
- Find x-intercept of a linear function.
- Find the y-intercept of a linear function.
- Graph a linear function.

1. Determine whether the equation $2x + 3yx = 3xy + 7y - 4$ represents a linear function of y in terms of x. If so, express the equation in the form $y = mx + b$. Otherwise, explain why the equation does not represent a linear function.

2. Determine whether the equation $2(x-2)^2 + y = 3x$ represents a linear function of y in terms of x. If so, express the equation in the form $y = mx + b$. Otherwise, explain why the equation does not represent a linear function.

3. Find the x- and y-intercepts of the linear equation $5x - 2y = 10$ and sketch the graph of the equation.



4. Match the following equations with their graphs.

(a) $y + \dfrac{6}{4}x - 2 = 0$

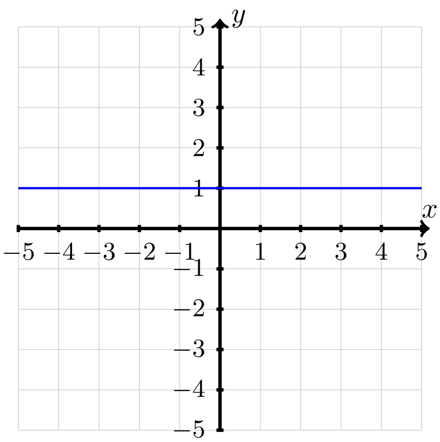
(b) $2y - x + 12 = 2x + 16$

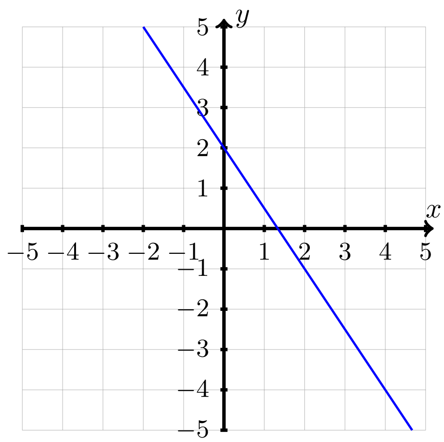
(c) $3y = 2x - 6$

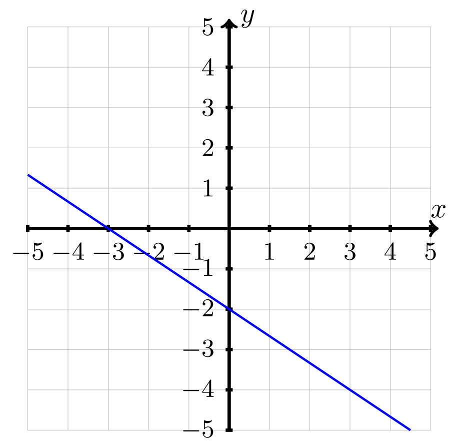
(d) $y + x - xy = \dfrac{1}{3}x - 2 - xy$

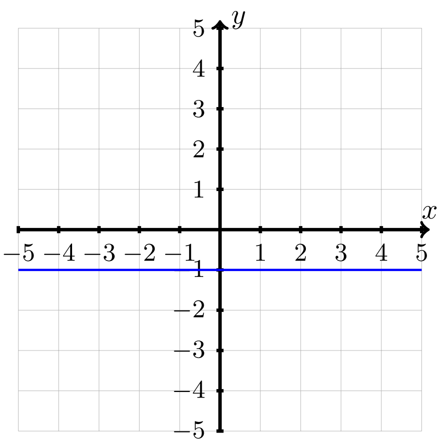
(e) $x + 2y = 1 + x + y$

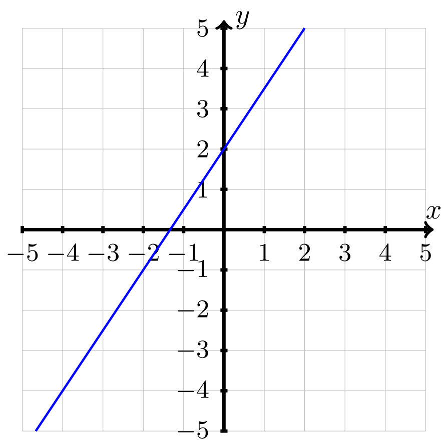
(f) $y + x = -1 + x$

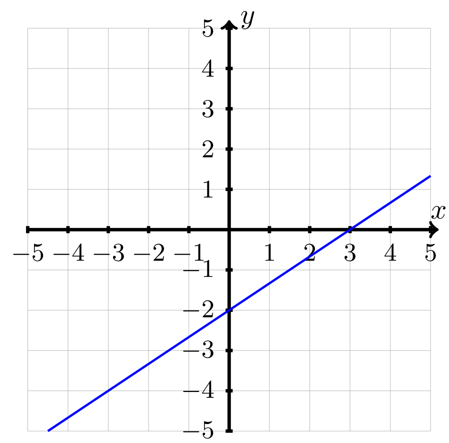












3.4 Representations of Lines

In the previous section, we encountered linear functions and saw that we can express these functions in the form $y = mx + b$. When we graphed a linear function, we used two points to determine the line that represents the graph. We can reverse this process to find a linear function. The first step is to determine the slope. In doing this, we will find two other representations for lines.

3.4.1 The Slope of a Line through two Points

Definition 3.4.1 The Slope of a Line. The **slope** of the line through the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1},$$

provided that $x_1 \neq x_2$. The slope measures the rate of change in y with respect to x, as depicted in Figure 3.4.2. ♢

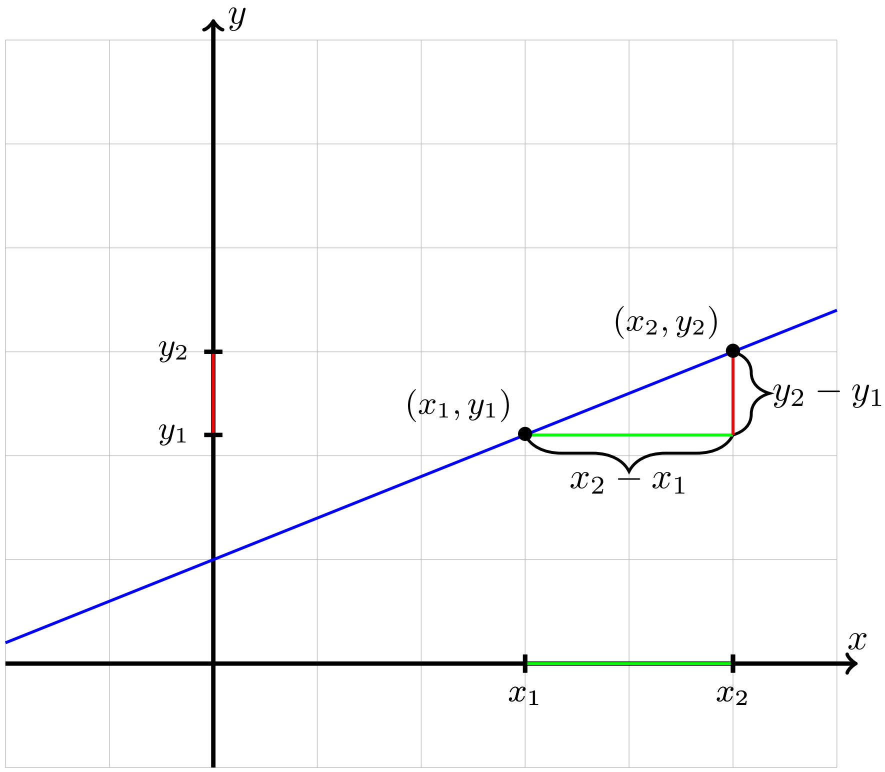


Figure 3.4.2 The slope of the line through the points (x_1, y_1) and (x_2, y_2).

Remark 3.4.3 In the definition of The Slope of a Line, we required that $x_1 \neq x_2$. While there is a line that passes through two distinct points with the same x-coordinate, that line is a vertical line. Since a vertical line is *not* a function, the slope of a vertical line is undefined.

Suppose we know the slope, m, and a point (x_0, y_0) that lines on the line. We can use the fact that (x_0, y_0) is a Solution to an Equation to determine the y-intercept. We substitute x_0 for x and y_0 for y in the The Slope-Intercept Form of a Line and solve for b:

$$\begin{aligned} y_0 &= mx_0 + b && \text{Subtract } mx_0 \text{ from both sides} \\ b &= y_0 - mx_0. \end{aligned}$$

Theorem 3.4.4 *The y-intercept of the line with slope m through the point (x_0, y_0) is*

$$b = y_0 - mx_0.$$

Example 3.4.5

(a) Find the slope of the line through the points $(3, 5)$ and $(6, 2)$.

(b) Find the equation of the line through the points $(3, 5)$ and $(6, 2)$.

□

3.4.2 The Point-Slope Form of a Line

Assume we know the slope, m, of a line and a point, (x_0, y_0), on the line. Suppose (x, y) is any other point on the line. Since the slope of the line is constant, we can write

$$\begin{aligned} m &= \frac{y - y_0}{x - x_0} && \text{Multiply both sides by } x - x_0 \\ m(x - x_0) &= \left(\frac{y - y_0}{x - x_0}\right)(x - x_0) && \text{Simplify} \\ m(x - x_0) &= y - y_0 && \text{Swap sides} \\ y - y_0 &= m(x - x_0) \end{aligned}$$

to find the next form of a line.

Theorem 3.4.6 The Point-Slope Form of a Line. *The* ***point-slope form of the line*** *through the point* (x_0, y_0) *with slope* m *is*

$$y - y_0 = m(x - x_0).$$

This form of a line is convenient to use when we know the slope of a line and a point on the line.

Worksheet

Example 3.4.7

(a) Find the The Point-Slope Form of a Line using the point $(1, 2)$ and the slope 3.

(b) Find an equation for the line that passes through the points $(3, 2)$ and $(4, 3)$.

□

3.4.3 The Standard Equation of a Line

So far, we have only been interested in working with lines with a well-defined slope. Sometimes, it is convenient to work with lines that may be vertical. The standard equation of a line allows us to do just that.

Definition 3.4.8 Standard Equation of a Line. The **standard equation of a line** in the variables x and y is

$$ax + by = c,$$

where a and b are not both zero. ◊

Warning 3.4.9 The standard equation of a line may not always represent y as a function of x! If $b = 0$, then the standard equation of a line reduces to

$$\begin{aligned} ax &= c && \text{Divide both sides by } a \\ x &= \frac{c}{a}, \end{aligned}$$

which is a vertical line.

Example 3.4.10 Place the linear equation

$$x + 3y - 2x + 7 = 5 + y - 4x$$

into standard form. □

3.4.4 Worksheet: Representations of Lines

Objectives

- Determine the slope of a line through two points.
- Represent a line using
  - Slope-Intercept Form,
  - Point-Slope Form, and
  - Standard Form.

1. Find the slope of the line through the points $(2, 11)$ and $(7, 1)$.

2. Write the equation of the line through the points $(2, 11)$ and $(7, 1)$ in Point-Slope Form.

3. Write the equation of the line through the points $(2, 11)$ and $(7, 1)$ in Slope-Intercept Form.

4. Write the equation of the line through the points $(2, 11)$ and $(7, 1)$ in Standard Form.

5. Find the slope of the line represented by the equation $\frac{1}{2}x - \frac{2}{3}y = 1$.

6. You and a friend used the same parking garage. You parked your car for 2 hours and it cost \$12. Your friend parked her car for 4 hours and it cost \$22.

(a) Let x represent the number of hours of parking and let y represent the cost. Assuming the relationship is linear, how much does parking cost per hour?

(b) Find a linear function that models the cost of parking.

(c) The garage charges a flat fee to enter the parking garage. How much is this fee?

3.5 Parallel and Perpendicular Lines

We have seen that a linear function is determined by a slope and a point. In this section, we extend the dictionary between algebra and geometry using two familiar geometric constructs: parallel and perpendicular lines.

3.5.1 Parallel Lines

Definition 3.5.1 Parallel Lines. Two lines in the Cartesian plane are said to be **parallel** if they do not intersect. ◊

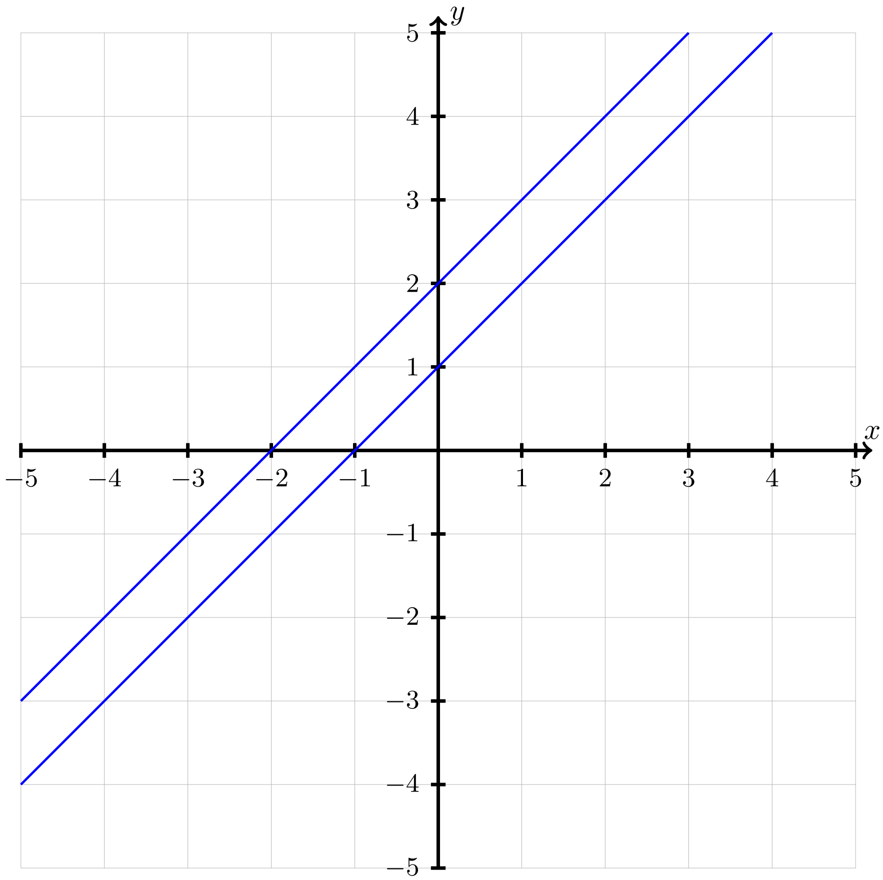


Figure 3.5.2 Parallel lines.

Theorem 3.5.3 *Two lines are parallel if and only if they have the same slope.*

Example 3.5.4

(a) Determine whether the lines

$$-x + 3y = -4 \quad \text{and} \quad -4x + 12y = 9$$

are parallel.

(b) Find the equation of the line parallel to $4x + 2y = 9$ that passes through the point $(-1, 5)$. Write your answer in slope-intercept form.

$\square$

3.5.2 Perpendicular Lines

Definition 3.5.5 Perpendicular Lines. Two lines are **perpendicular** if they intersect at a right angle. ◊

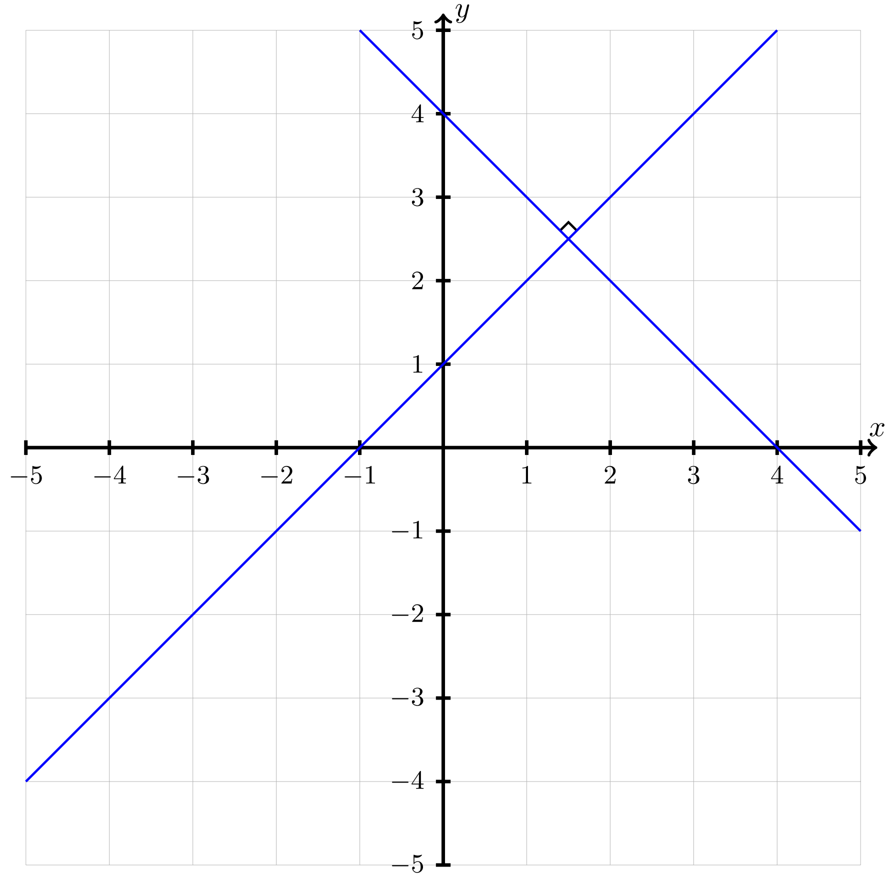


Figure 3.5.6 Perpendicular lines.

Theorem 3.5.7 *Two lines are perpendicular if and only if the product of their slopes is* -1.

Example 3.5.8

(a) Determine whether the lines

$$3x + 5y = 7 \quad \text{and} \quad -5x + 3y = 21$$

are perpendicular.

(b) Find the equation of a line perpendicular to $2x + 3y = 5$ that passes through the point $(3, 5)$. Write your answer in slope-intercept form.

$\square$

3.5.3 Worksheet: Parallel and Perpendicular Lines

Objectives

- Determine whether two lines are parallel, perpendicular, or neither.
- Find the equation of a line parallel to a given line.
- Find the equation of a line perpendicular to a given line.

1. Determine whether the given lines are parallel, perpendicular, or neither.

(a) $2x + 3y = 5$ and $2y - 3x - 1 = 0$.

(b) $5x + 7y = 11$ and $2x + 13y = 23$.

(c) $11x + 2y - 3 = 0$ and $11x + 2y - 23 = 0$.

2. Find the equation of the line parallel to $5x + 7y = 11$ that passes through the point $(2, 2)$.

3. Find the equation of the line perpendicular to $5x + 7y = 11$ that passes through the point $(2, 2)$.

3.6 Linear and Absolute Value Inequalities

We have focused primarily on functions of the form $y = mx + b$ or, more generally, on linear equations of the form $ax + by = c$. In this section, we consider linear inequalities in two variables.

3.6.1 Linear Inequalities

Definition 3.6.1 Linear Inequality. A **linear inequality** in the variables x and y is an expression that can be placed in one of the following forms

- $ax + by < c$,
- $ax + by \leq c$,
- $ax + by > c$, or
- $ax + by \geq c$.

◊

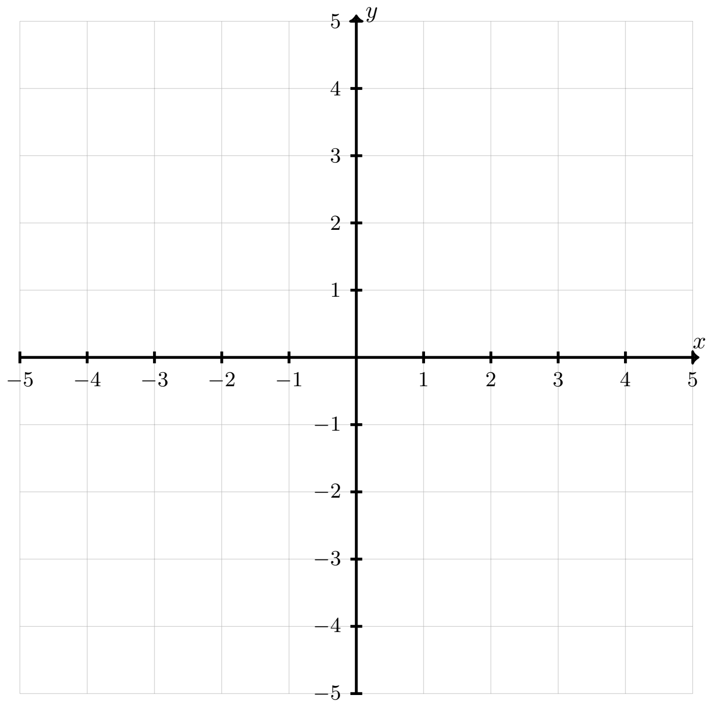
Definition 3.6.2 Solution to an Inequality. A **solution** to an inequality in the variables x and y is an ordered pair (a, b) that make the inequality true when

1. the first coordinate, a, is substituted into the inequality for x, and
2. the second coordinate, b, is substituted into the inequality for y.

◊

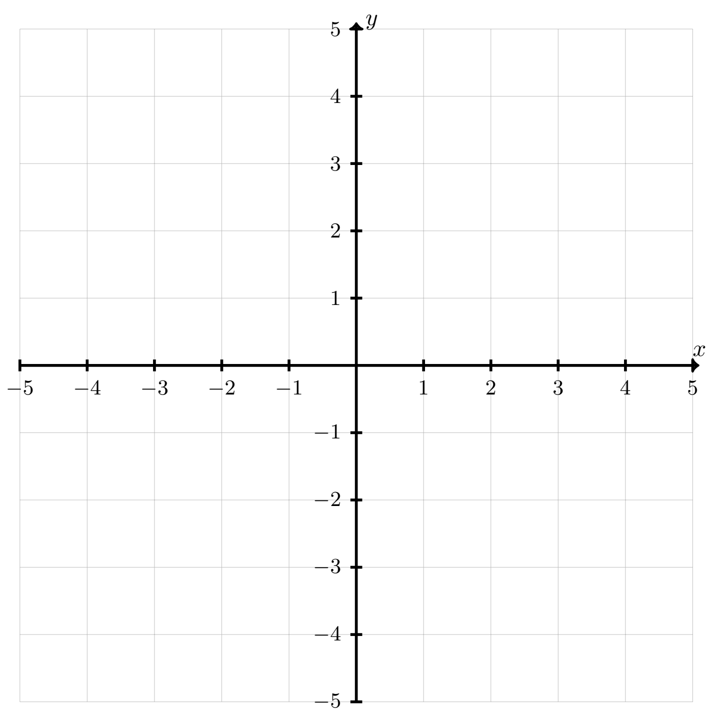
Definition 3.6.3 Graph of an Inequality. The **graph** of an inequality in the variables x and y consists of all the points in the plane that are solutions to the inequality. ◊

Example 3.6.4 Graph the linear inequality $2x + 3y < 5$



□

Example 3.6.5 Graph the linear inequality $2x + 3y \leq 5$



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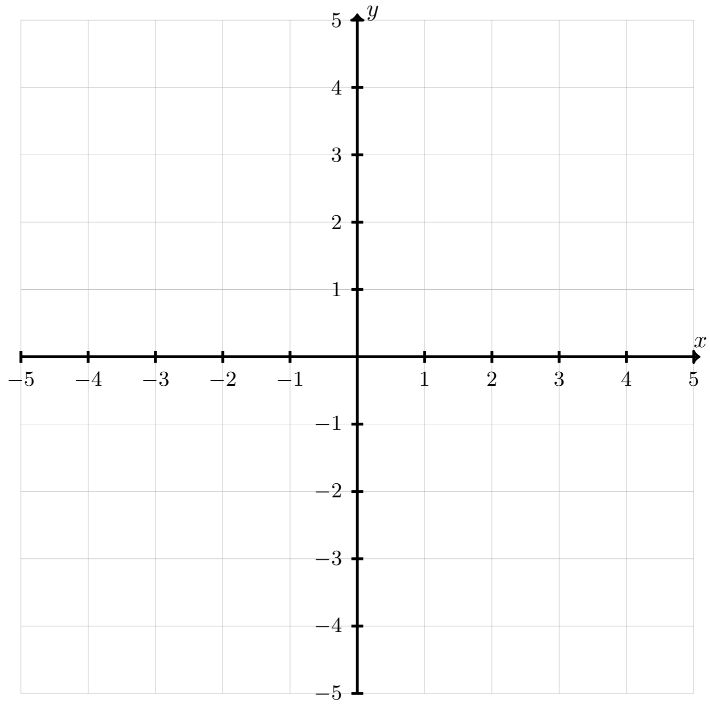
Example 3.6.6 Graph the linear inequality $2x + 3y > -5$



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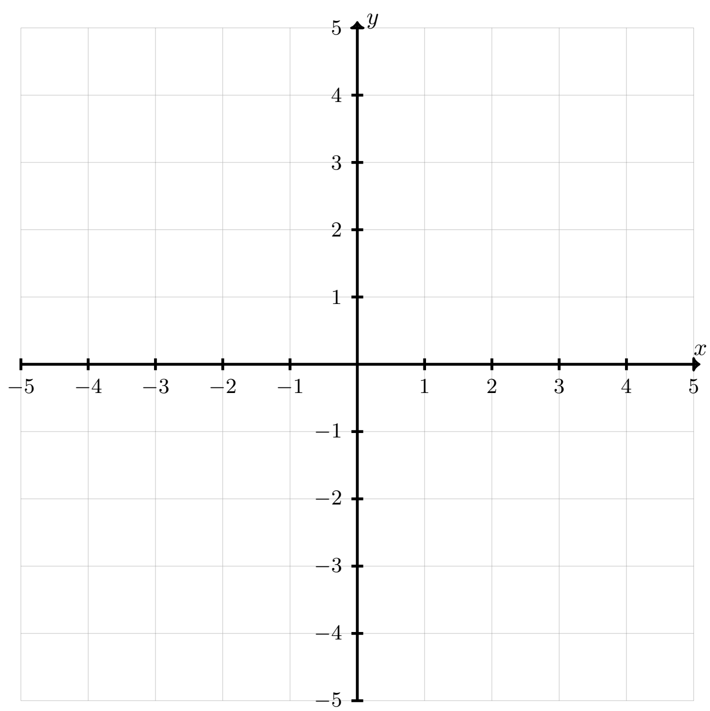
Example 3.6.7 Graph the linear inequality $2x + 3y \geq -5$



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3.6.2 Absolute Value and Inequality

Definition 3.6.8 Linear Absolute Value Inequality. A **linear absolute value inequality** in the variables x and y is an expression that can be placed in one of the following forms

- $|ax+by+c| < d$,
- $|ax+by+c| \leq d$,
- $|ax+by+c| > d$, or
- $|ax+by+c| \geq d$.

◊

Let d be any positive real number. Recall from Definition 1.4.1 that

$$|d-0| = |0-d| = |d| = d$$

measures the distance between d and the origin on the number line. By looking at the number line, we can see the distance between d and the origin is the same as the distance from $-d$ to the origin.

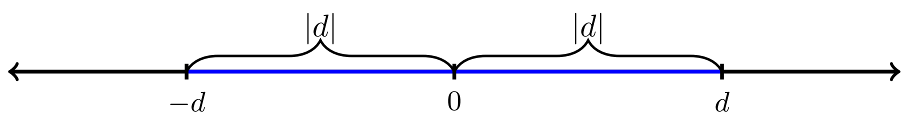


Figure 3.6.9 The distance between d and the origin, and between $-d$ and the origin.

We can confirm this algebraically using Definition 1.4.1:

$$|-d-0| = |-d| = \sqrt{(-d)^2} = \sqrt{d^2} = |d| = |d-0|$$

Hence for any positive number d, there are precisely two numbers that are d units away from the origin: d and $-d$. All other numbers are either more than d units away from the origin or less than d units away from the origin.

Now consider an equation of the form $|ax+by+c| = d$. The observation above tells us the solutions to this equation are the points (x, y) in the plane such that the number $ax+by+c$ is *exactly* d units away from the origin. As the only two numbers that are d units away from the origin are d and $-d$, either

$$ax+by+c = d \quad \text{or} \quad ax+by+c = -d$$

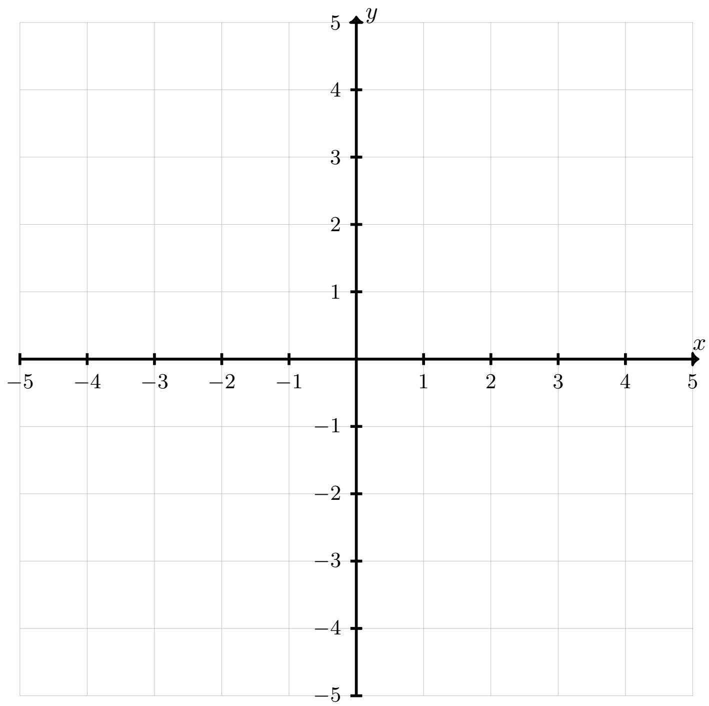
In standard form, we can recognize these as two parallel lines

$$ax+by = -c+d \quad \text{and} \quad ax+by = -c-d$$

Geometrically, this tells us the points between these lines make the number $ax+by+c$ *less* than d, while the points outside these lines make the number $ax+by+c$ *more* than d. This provides a pleasing geometric description of the solutions to the four types of linear absolute value inequalities:

- The solutions to $|ax+by+c| < d$ are the points that are between the lines $ax+by = -d-c$ and $ax+by = -c+d$,
- The solutions to $|ax+by+c| \leq d$ are the points that are either on or between the lines $-d-c = ax+by$ and $ax+by = -c+d$,
- The solutions to $|ax+by+c| > d$ are the points that are outside the lines $ax+by = -c-d$ and $ax+by = -c+d$.
- The solutions to $|ax+by+c| \geq d$ are the points that are either on or outside the lines $ax+by = -c-d$ and $ax+by = -c+d$.

Example 3.6.10 Graph the linear absolute value inequality $|2x + 3y| < 5$.



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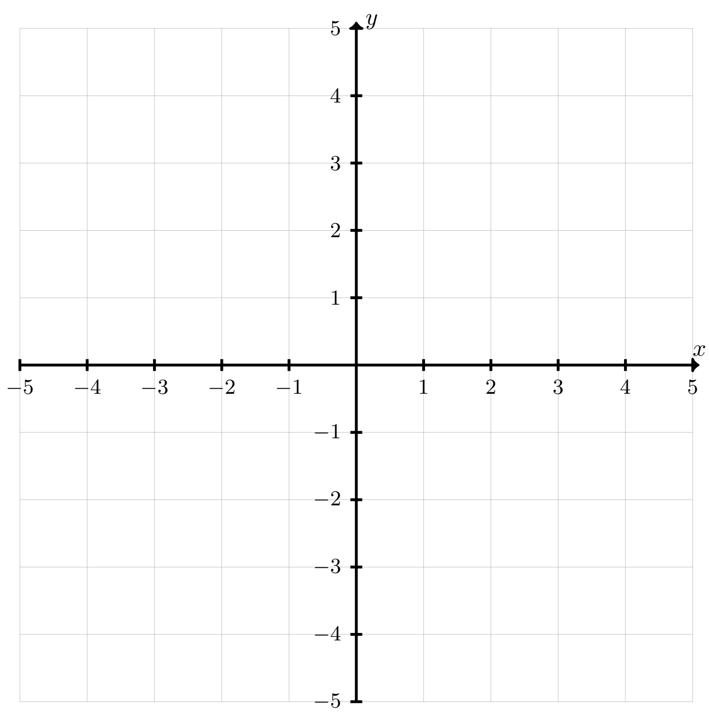
Example 3.6.11 Graph the linear absolute value inequality $|2x + 3y| \leq 5$.



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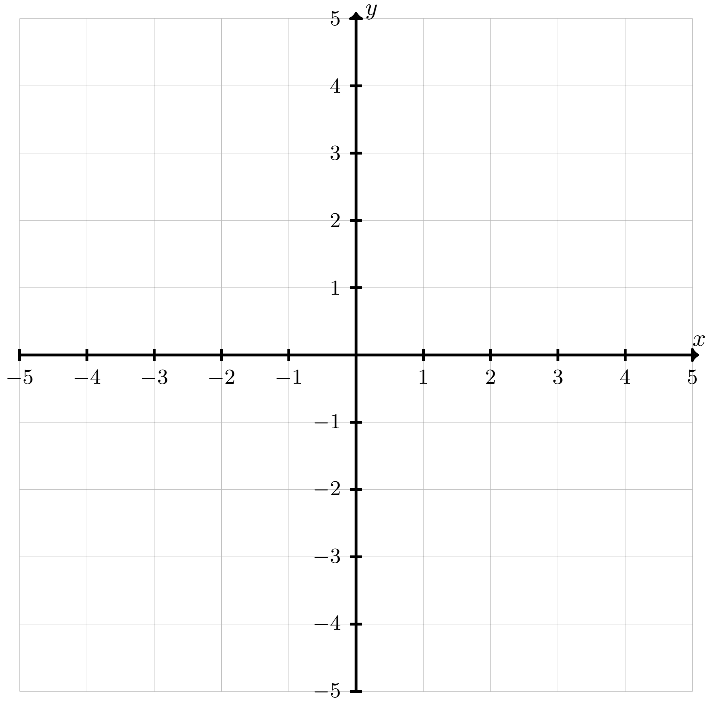
Example 3.6.12 Graph the linear absolute value inequality $|2x + 3y| > 5$.



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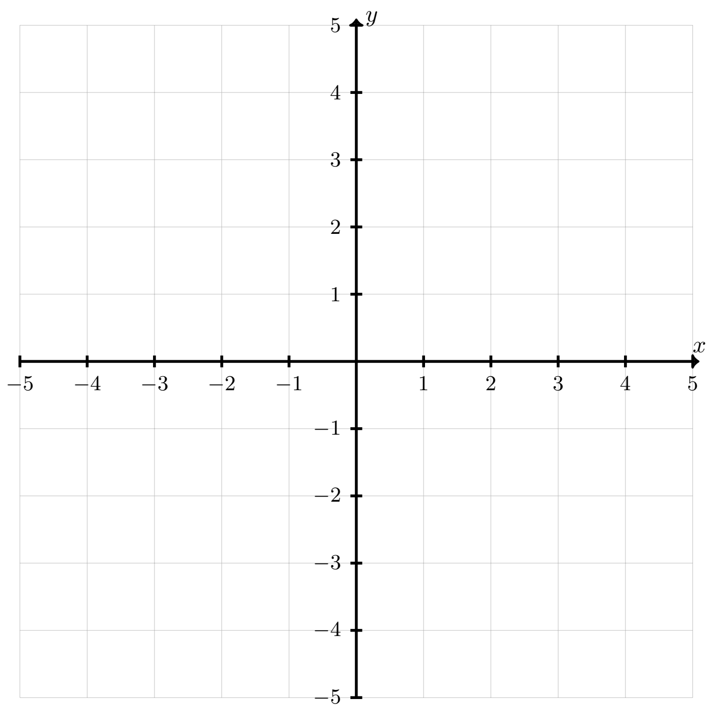
Example 3.6.13 Graph the linear absolute value inequality $|2x + 3y| \geq 5$.



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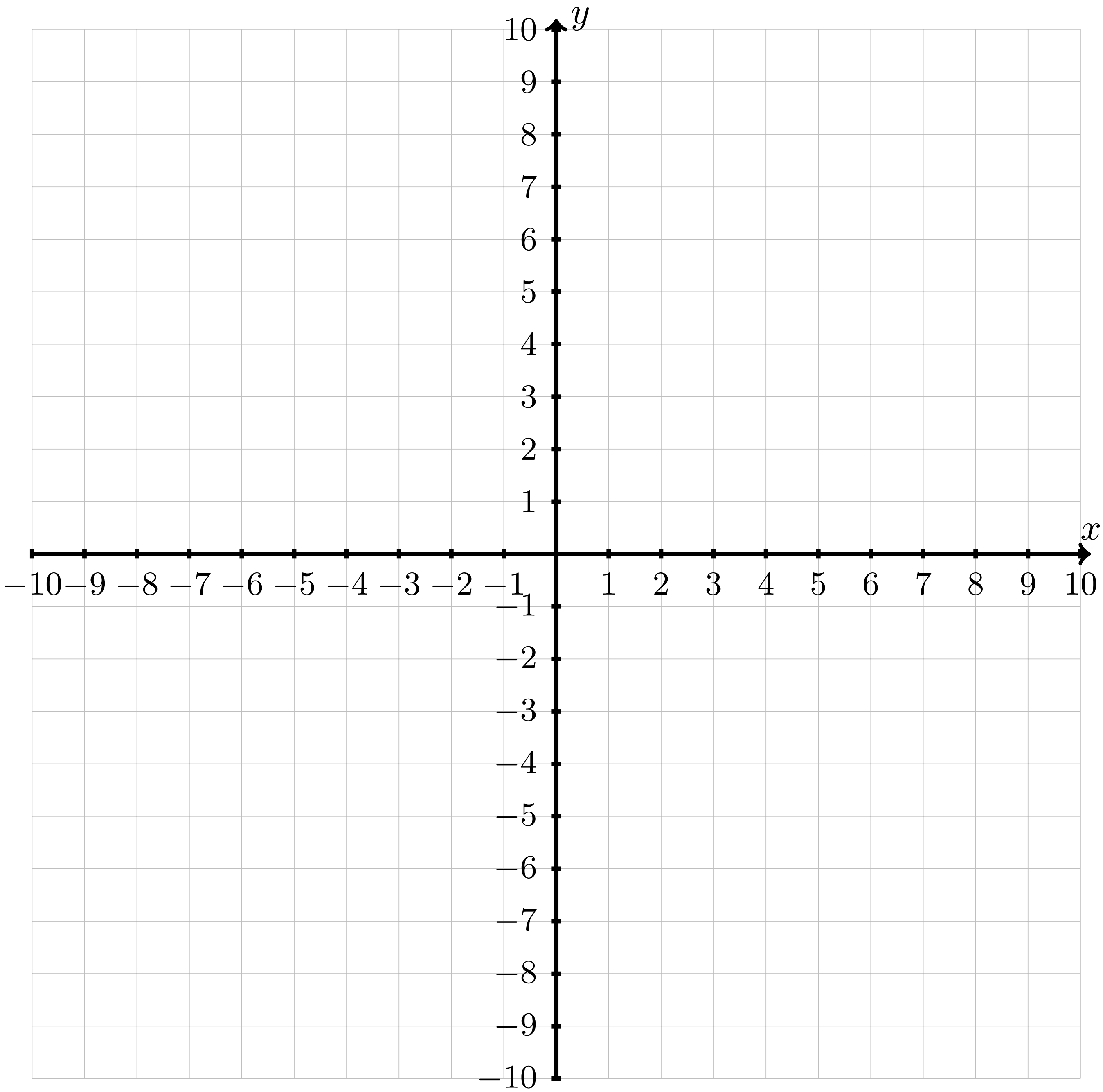
3.6.3 Worksheet: Linear and Absolute Value Inequalities

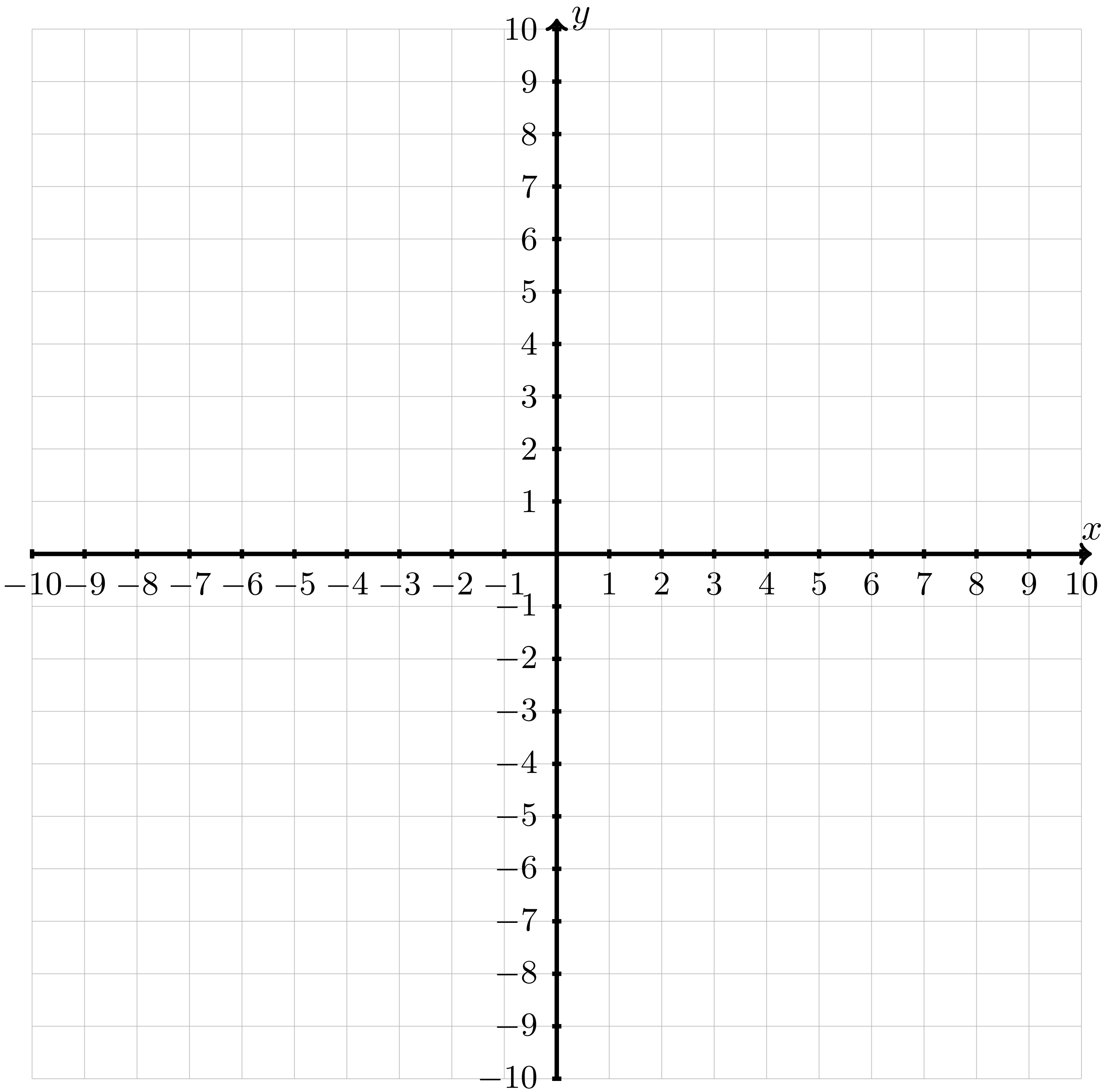
Objectives

- Graph linear inequalities.
- Graph linear absolute value inequalities.

1. Sketch graph of the linear inequality $5x - 2y < 10$.



2. Sketch graph of the linear inequality $5x - 2y \geq 10$.



3. Match the following equations with their graphs.

(a) $y + \dfrac{6}{4}x - 2 > 0$

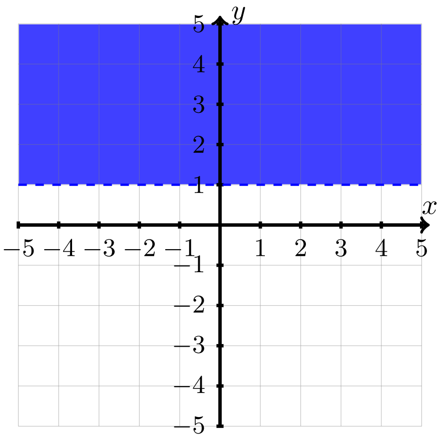
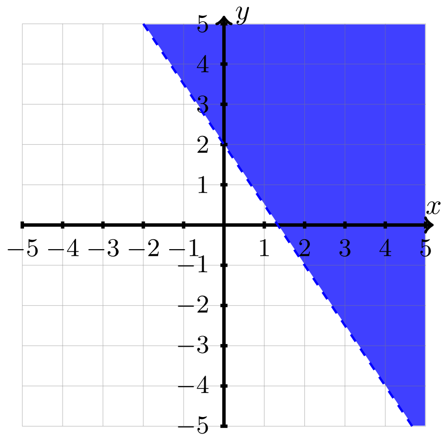
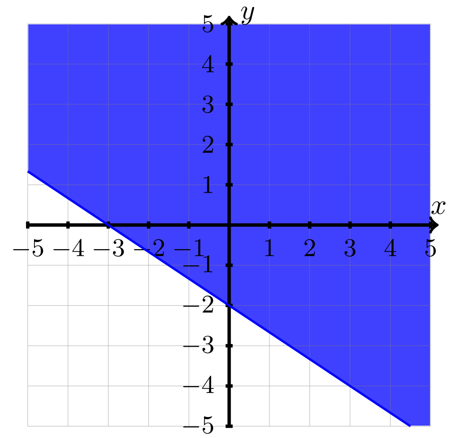
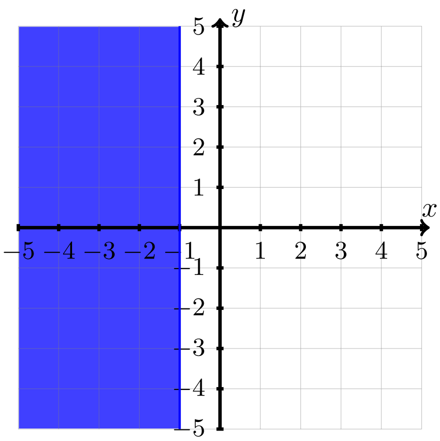
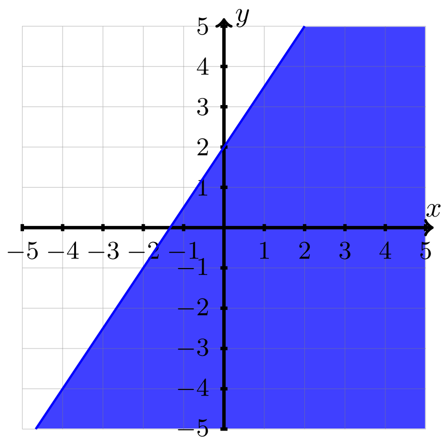
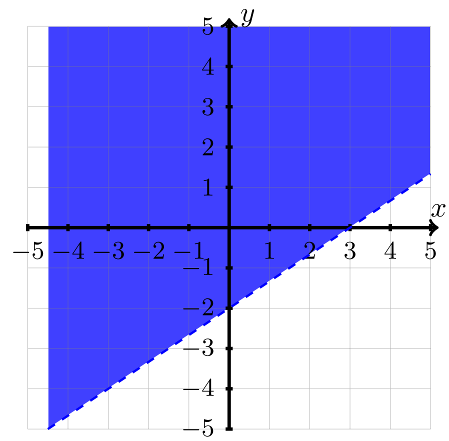
(b) $2y - x + 12 \leq 2x + 16$

(c) $3y > 2x - 6$

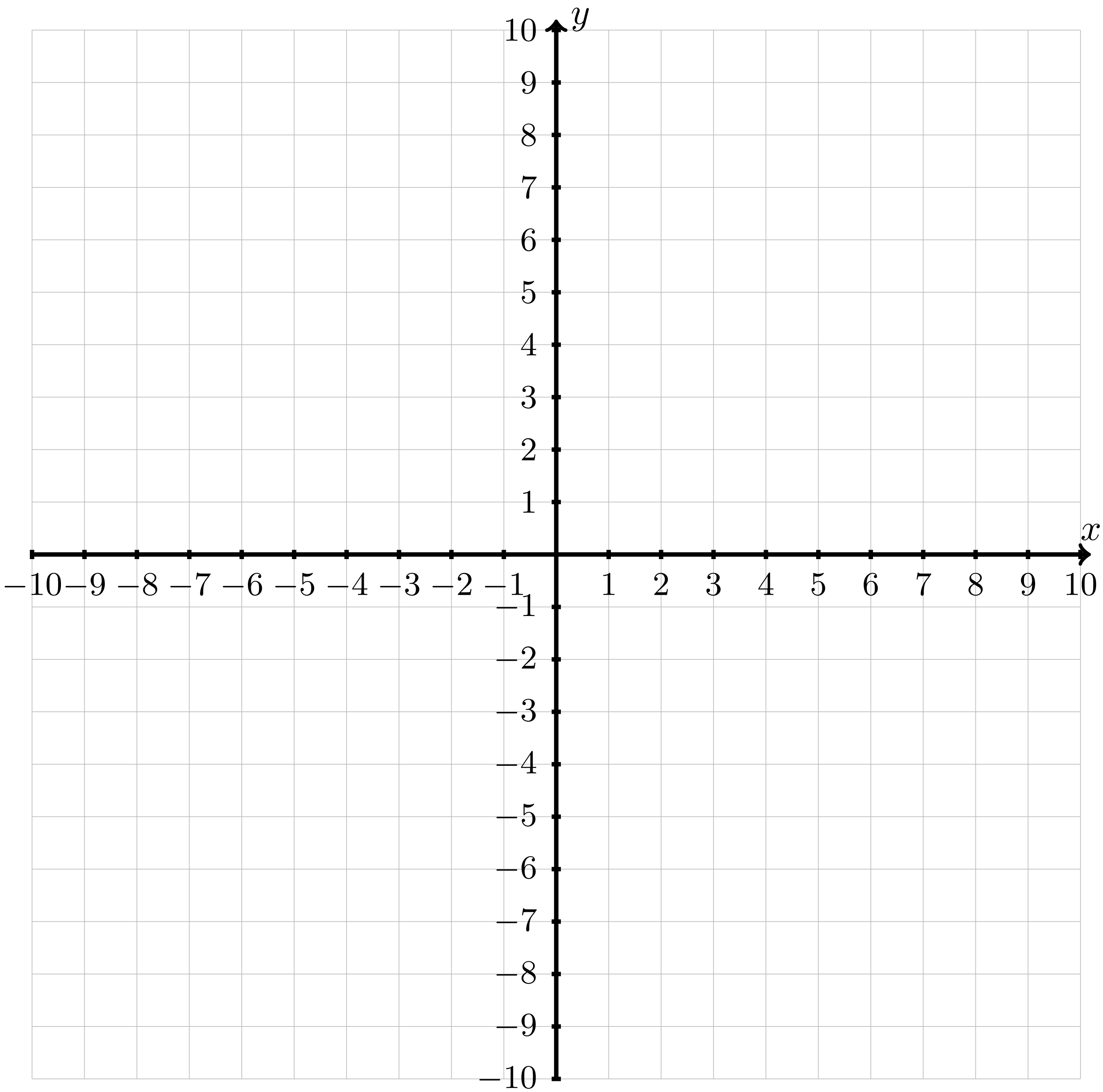
(d) $y + x - xy \geq \dfrac{1}{3}x - 2 - xy$

(e) $x + 2y > 1 + x + y$

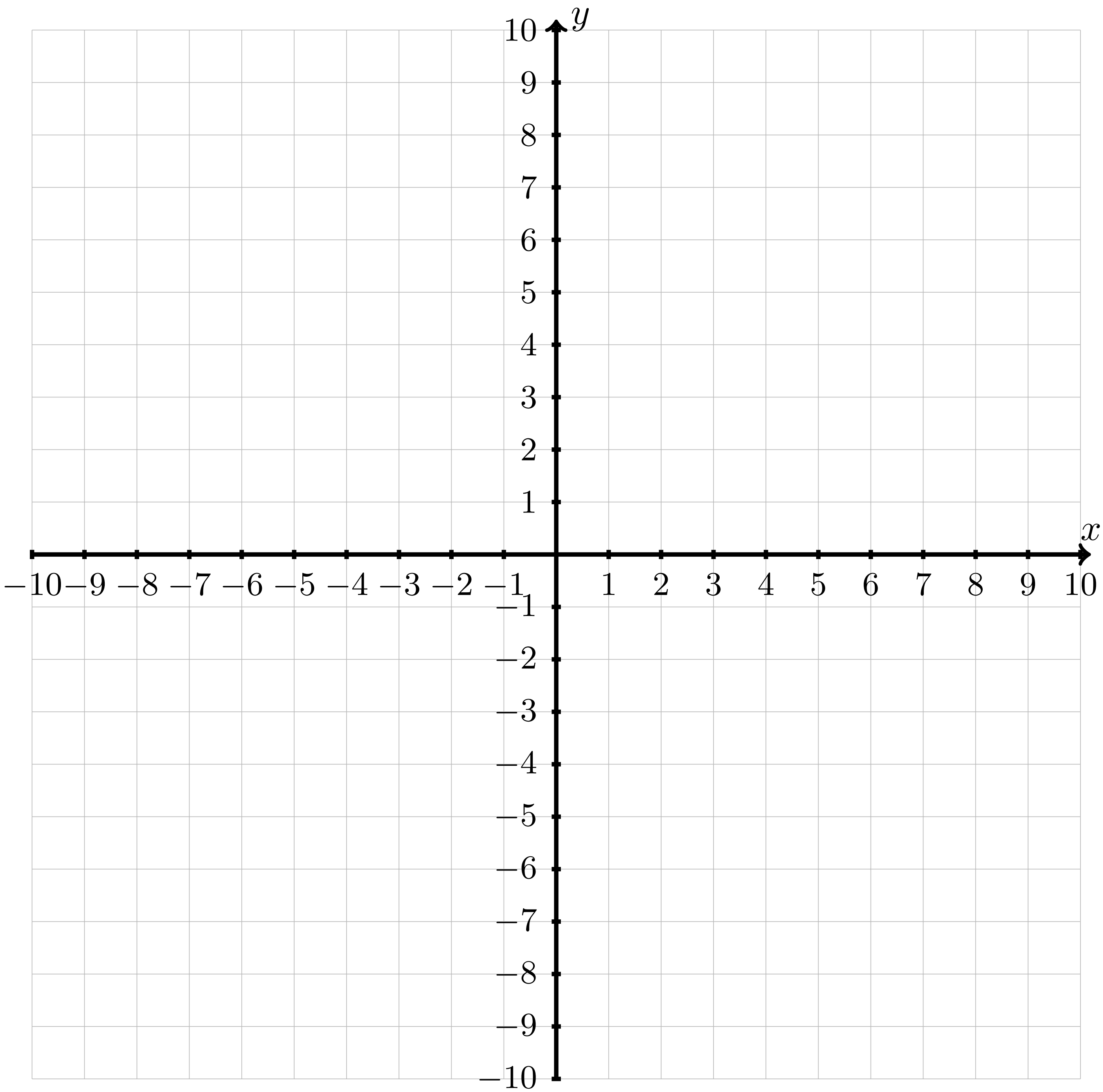
(f) $y + x \leq -1 + y$

4. Graph the linear absolute value inequality $|5x - 2y| \le 10$



5. Graph the linear absolute value inequality $|6x - 8y| > 12$.



3.7 Quadratic Functions

In this section, we study polynomial functions of degree 2. These polynomial functions are often called *quadratic functions*.

3.7.1 Vertex Form

Definition 3.7.1 Quadratic Function. A function $f\colon \mathbb{R} \to \mathbb{R}$ is **quadratic** if it has the form

$$f(x) = ax^2 + bx + c \quad \text{or} \quad y = ax^2 + bx + c,$$

where a, b, and c are numbers, and $a \neq 0$. ◊

Just as with lines, quadratic functions are easier to understand through visualizations. The simplest quadratic function to understand is $y = x^2$. We call the graph of $y = x^2$ a **parabola**. The point $(0, 0)$ is called the **vertex**.

As we have already seen in Section 2.3, the graph of this function is symmetric about the y-axis, which is the vertical line $x = 0$ through the vertex. While general quadratics are slightly more complicated, we can use the properties of $y = x^2$ to graph the class of quadratics in the following form.

Definition 3.7.2 The Vertex Form of a Quadratic. A quadratic function, $f(x)$, is in **vertex form** if there are numbers h and k such that

$$f(x) = a(x - h)^2 + k.$$

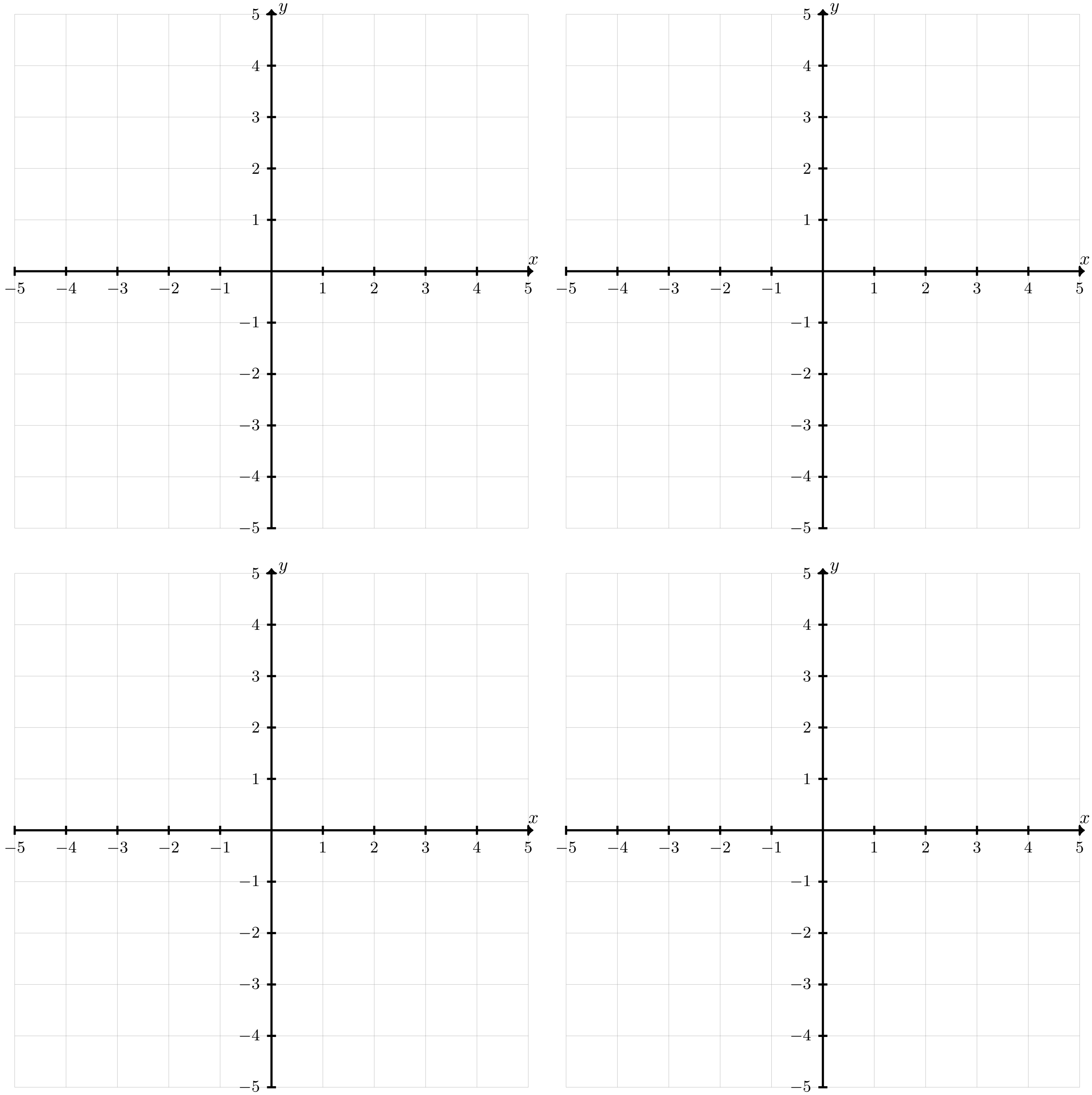
◊

Using the methods of Section 2.2, we can graph quadratics in this form by following the steps below.

Algorithm 3.7.3 Graphing Quadratic Functions in Vertex Form. *To graph of the quadratic function $y = a(x - h)^2 + k$*

1. *Translate the graph of $y = x^2$ horizontally by h units.*
   - *If $0 < h$, then the translation is to the right.*
   - *If $h < 0$, then the translation is to the left.*
2. *Scale the graph of $(x - h)^2$ by $|a|$.*
   - *If $1 < a$, then the graph is stretched.*
   - *If $a < 1$, then the graph is compressed.*
3. *If $a < 0$, reflect the graph of $|a|\,(x - h)^2$ over the x-axis.*
4. *Translate the graph of $a(x - h)^2$ vertically by k units.*
   - *If $k < 0$, then the translation is down.*
   - *If $0 < k$, then the translation is up.*

Example 3.7.4 Graph the function $f(x) = -2(x+1)^2 + 1$.



□

If we follow the vertex, we can see the first operation moves the point $(0,0)$ to $(h,0)$. Since the point $(h,0)$ lies on the x-axis, the second and third operation do not move the point $(h,0)$. The fourth operation moves the point $(h,0)$ to the point (h,k). Since these operations preserve the general shape of the graph, a quadratic function in the form

$$y = a(x-h)^2 + k$$

is a parabola with vertex at (h,k). Hence the reason for the name *vertex form*.

Similarly, if we follow the line of symmetry for $y = x^2$, then we can see the first operation moves the vertical line $x = 0$ to the vertical line $x = h$. The following three operations do not change the vertical line, so the graph of $y = a(x-h)^2 + k$ is symmetric about the vertical line $x = h$.

Finally, we observe the third operation may change the direction in which the parabola opens.

Definition 3.7.5 The graph of a parabola is

- **concave up** if the parabola opens upwards like $y = x^2$.
- **concave down** if the parabola opens downwards like $y = -x^2$.

◊

For quadratics in vertex form $f(x) = a(x-h)^2 + k$, we can see that f is concave up whenever $0 < a$ and concave down whenever $a < 0$.

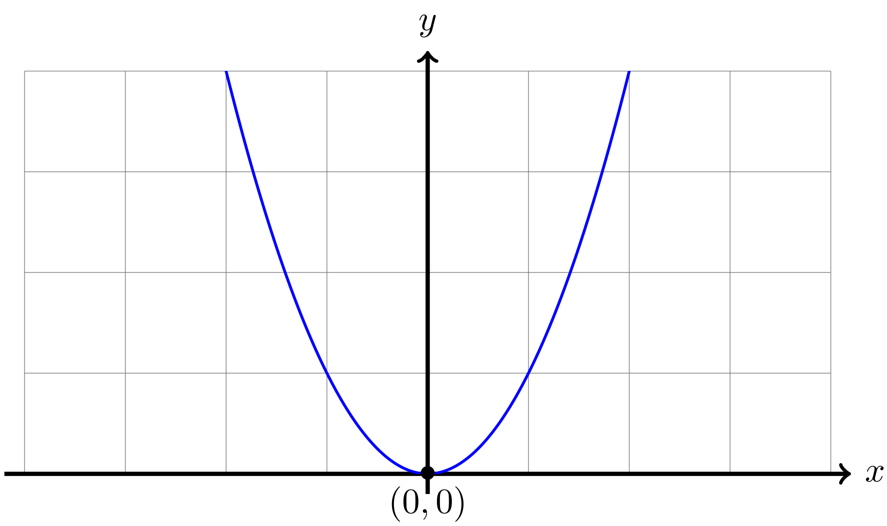


Figure 3.7.6 The graph of $y = x^2$

3.7.2 Completing the Square

Surprisingly, every quadratic function can be placed into vertex form. Paired with the observations above, this tells us the graph of *every* quadratic function is a parabola. Placing a general quadratic function into vertex form will require the method of completing the square.

Remember that when we multiply two binomials, we must distribute. We can remember how to distribute using the mnemonic FOIL: ***F***irst, ***O***uter, ***I***nner, ***L***ast:

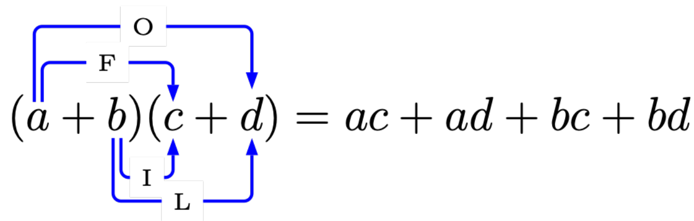


Figure 3.7.7 Multiplying binomials using the FOIL method.

If we use this method to multiply a binomial by itself, then we find the useful formula

$$\begin{aligned}(x+h)^2 &= (x+h)(x+h)\\ &= xx + xh + hx + hh\\ &= x^2 + hx + hx + h^2\\ &= x^2 + 2hx + h^2.\end{aligned}$$

We call a quadratic function of the form $f(x) = x^2 + 2hx + h^2$ a **perfect square**.

Given a quadratic equation $f(x) = ax^2 + bx + c$, we can use the formula

$$(x+h)^2 = x^2 + 2hx + h^2$$

to **complete the square** as follows.

Algorithm 3.7.8 Completing the Square. *To complete the square with the quadratic function $f(x) = ax^2 + bx + c$*

1. *Factor an a from the first two terms to get*

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c.$$

2. *Add and subtract $\left(\frac{b}{2a}\right)^2$ inside the parentheses to get*

$$f(x) = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c$$

3. *Factor the first three terms inside the parentheses as a perfect square*

$$f(x) = a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c$$

4. *Distribute the a through the sum*

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c.$$

The number $h = -\frac{b}{2a}$ is the x-coordinate of the vertex, and the number

$$\begin{aligned}k &= f(h)\\ &= f\left(-\frac{b}{2a}\right)\end{aligned}$$

$$
\begin{aligned}
&= a\left(-\frac{b}{2a}+\frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c \\
&= -a\left(\frac{b}{2a}\right)^2 + c \\
&= -a\left(\frac{b^2}{4a^2}\right) + c \\
&= -\frac{b^2}{4a} + \frac{4ac}{4a} \\
&= -\frac{b^2-4ac}{4a}.
\end{aligned}
$$

is the y-coordinate of the vertex.

Formula 3.7.9 The Vertex Formula. *Let $f(x) = ax^2+bx+c$ be a quadratic function. The x-coordinate of the vertex is*

$$h = -\frac{b}{2a}$$

and the y-coordinate of the vertex is

$$k = f(h) = f\left(-\frac{b}{2a}\right) = -\frac{b^2-4ac}{4a}.$$

As an ordered pair, the vertex of the graph of $f(x)$ occurs at

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right).$$

Remark 3.7.10 Using The Vertex Formula, the vertex form of the quadratic function $f(x) = ax^2 + bx + c$ is

$$
\begin{aligned}
a(x-h)^2 + k &= a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + f\left(-\frac{b}{2a}\right) \\
&= a\left(x + \frac{b}{2a}\right)^2 + f\left(-\frac{b}{2a}\right)
\end{aligned}
$$

Example 3.7.11

(a) Find the vertex form of the quadratic function $f(x) = x^2 + 4x + 5$.

(b) Find the vertex of the quadratic function $f(x) = 3x^2 - 4x + 5$.

$\square$

3.7.3 Applications to Circles

In Section 1.6 we only encountered circles in standard form. However, circles will not always be presented this way. We study how the The Vertex Formula can help us to identify the equation of a circle, even if it is not presented in standard form.

Consider the equation $x^2 - 4x + y^2 + 6x = -9$. We can view $f(x) = x^2 - 4x$ and $g(y) = y^2 + 6x$ as quadratic functions in the variables x and y, respectively. The first coordinate of the vertex of f is

$$h = -\frac{-4}{2} = \frac{4}{2} = 2$$

and the second coordinate of the vertex as

$$k = f(2) = 2^2 - 4(2) = 4 - 8 = -4.$$

This allows us to write

$$f(x) = x^2 - 4x = (x-2)^2 - 4.$$

Similarly, the first coordinate for the vertex of g is

$$h = -\frac{6}{2(1)} = -3,$$

the second coordinate of the vertex of $y^2 + 6x$ is

$$k = g(-3) = (-3)^2 + 6(-3) = 9 - 18 = -9,$$

and so

$$g(y) = (x - (-3))^2 - 9 = (x+3)^2 - 9.$$

These two observations allow us to rewrite the original equation

$$\begin{aligned}
x^2 - 4x + y^2 + 6x &= -9 && \text{Replace with vertex forms}\\
(x-2)^2 - 4 + (y+3)^2 - 9 &= -9 && \text{Add 9 to both sides}\\
(x-2)^2 - 4 + (y+3)^2 &= -9 + 9 && \text{Simplify}\\
(x-2)^2 - 4 + (y+3)^2 &= 0 && \text{Add 4 to both sides}\\
(x-2)^2 + (y+3)^2 &= 4
\end{aligned}$$

Therefore the equation $x^2 - 4x + y^2 + 6x = -9$ represents the circle of radius 2 centered at the point $(2, -3)$.

3.7.4 Worksheet: Quadratic Functions

Objectives

- Place a quadratic function in vertex form
- Place the equation of a circle in standard form

Quadratic Functions. Place each of the quadratic functions below in vertex form. Use graph transformations to sketch a graph of the function. Label the vertex of the parabola.

1. $f(x) = -8x^2 + 10x + 5$

2. $g(x) = 3x^2 - 5x - 4$

Degree 2 Equations in 2 Variables. Graph the following equations.

3. $x^2 - 4x + y^2 + 2y = 4$

4. $4x^2 - 4x + 4y^2 + 16y = 19.$

3.8 Roots of Quadratic Functions

In order to accurately graph a quadratic function $y = f(x) = ax^2 + bx + c$, it is necessary to identify the x-intercepts for the function. A point $(x, 0)$ in the Cartesian plane is an x-intercept for this function if

$$0 = ax^2 + bx + c.$$

This reduces the geometric problem of finding the x-intercepts of a quadratic to the algebraic problem finding solutions to equations.

3.8.1 The Quadratic Formula

Definition 3.8.1 An equation of the form $ax^2+bx+c = 0$ is called a **quadratic equation** ♢

To find the solutions to the quadratic equation $ax^2 + bx + c = 0$, we use The Vertex Form of a Quadratic to write

$$0 = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

and solve this equation for x

$$\begin{aligned}
a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} &= 0 && \text{Add } \frac{b^2-4ac}{4a} \text{ to both sides}\\
a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} && \text{Divide both sides by } a\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} && \text{Take the square root of both sides}\\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} && \text{Simplify}\\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} && \text{Simplify}\\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} && \text{Subtract } \frac{b}{2a} \text{ from both sides}\\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} && \text{Simplify}\\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Simplify}
\end{aligned}$$

It is important to note that

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are real numbers only when $b^2 - 4ac$ is a non-negative number. The number $b^2 - 4ac$ is an important invariant of a quadratic function.

Definition 3.8.2 The Discriminant. The **discriminant** of the quadratic function $f(x) = ax^2 + bx + c$ is the number

$$D = b^2 - 4ac$$

♢

The discriminant provides both algebraic and geometric information about quadratic functions. On the algebraic side, the discriminant tells us how many solutions there are to the quadratic equation $ax^2 + bx + c = 0$.

Formula 3.8.3 The Quadratic Formula. *The quadratic equation*

$$ax^2 + bx + c = 0$$

has either two, one, or zero real solutions, depending on The Discriminant, $D = b^2 - 4ac$*.*

1. *If* $D > 0$*, then the two solutions are*

$$x = \frac{-b + \sqrt{D}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{D}}{2a}.$$

2. *If* $D = 0$*, then the only solution is*

$$x = \frac{-b + \sqrt{D}}{2a} = \frac{-b}{2a}.$$

3. *If* $D < 0$*, then there are no real solutions.*

If we pair The Quadratic Formula with Theorem 3.2.6, then we have an algorithmic way to factor any quadratic function.

Theorem 3.8.4 *The quadratic function* $f(x) = ax^2 + bx + c$ *factors as*

$$f(x) = a\left(x - \frac{-b + \sqrt{D}}{2a}\right)\left(x - \frac{-b - \sqrt{D}}{2a}\right),$$

where $D = b^2 - 4ac$ *is the The Discriminant.*

Example 3.8.5

(a) Find the roots of the quadratic function $f(x) = 10x^2 - 11x + 3$.

(b) Use the roots of the quadratic function to factor f into the product of two linear polynomials.

$\square$

On the geometric side, the discriminant tells us how many x-intercepts the graph of $y = ax^2 + bx + c$ has.

Theorem 3.8.6 The x-intercepts of a Quadratic Function. *The quadratic function $y = ax^2 + bx + c$ has either two, one, or zero x-intercepts, depending on the The Discriminant, $D = b^2 - 4ac$.*

1. *If $D > 0$, then there are two x-intercepts,*

$$\left(\frac{-b+\sqrt{D}}{2a}, 0\right) \quad \textit{and} \quad \left(\frac{-b-\sqrt{D}}{2a}, 0\right).$$

2. *If $D = 0$, then there is one x-intercept*

$$\left(\frac{-b}{2a}, 0\right).$$

   Note this is the vertex.

3. *If $D < 0$, then there are no x-intercepts.*

Example 3.8.7

(a) Find the x-intercepts of the function $f(x) = 3x^2 + 2x + 4$.

(b) Find the x-intercepts of the function $f(x) = -3x^2 - 5x + 2$.

□

3.8.2 Graphing Quadratics

We can simplify the process of graphing a quadratic function by using the tools we have developed in this section.

Algorithm 3.8.8 Graphing a Quadratic Function.

1. *Use the The Vertex Formula to find the vertex, (h, k), of the parabola. Plot the vertex in the plane.*
2. *Factor or use the The Quadratic Formula to find the x-intercept(s). If there are x-intercepts, plot them in the plane.*
3. *Plot the y-intercept, $(0, f(0))$, in the plane.*
4. *Draw a parabola through the points in Steps (1) through (3).*
   - *The parabola will open upwards if $0 < a$.*
   - *The parabola will open downwards if $a < 0$.*

3.8.3 Worksheet: Roots of Quadratic Functions

Objectives

- Find the roots of a quadratic function

Sketch a graph of each of the quadratic functions below. Label the y-intercept, the vertex, and any x-intercepts.

1. $f(x) = x^2 - 5x + 6$

2. $g(x) = 2x^2 + 4x - 3$

3. $h(x) = -8x^2 + 10x + 5$

4. $k(x) = x^2 + x + 1$

5. $p(x) = x^2 - 3x - 4$

6. $q(x) = -2x^2 + 6x - 5$

3.9 Polynomial Inequalities

Similar to Section 3.6, we now turn to **polynomial inequalities**, which are relations of the form

- $p(x) < 0$,
- $p(x) \leq 0$,
- $p(x) > 0$, or
- $p(x) \geq 0$.

where $p\colon \mathbb{R} \to \mathbb{R}$ is a polynomial function.

3.9.1 Solutions to Inequalities

First we start with what it means to be a solution to an inequality.

Definition 3.9.1 Solution to a Polynomial Inequality. A **solution** to an inequality in the variable x is a number a that makes the expression true when a is substituted for x. ◊

We can interpret this definition graphically as saying

- The solutions to $p(x) < 0$ are the x-values where the graph of $y = p(x)$ is below the x-axis
- The solutions to $p(x) \leq 0$ are the x-values where the graph of $y = p(x)$ is on or below the x-axis
- The solutions to $p(x) > 0$ are the x-values where the graph of $y = p(x)$ is above the x-axis
- The solutions to $p(x) \geq$ are the x-values where the graph of $y = p(x)$ is on or above the x-axis

This tells us precisely how to solve a polynomial inequality using the roots.

3.9.2 Solving Polynomial Inequalities Algebraically

The first method is purely algebraic. It is useful when you are unable to sketch the graph of the polynomial.

Algorithm 3.9.2 Solving Polynomial Inequalities Algebraically. *To find the solutions to a polynomial inequality involving the polynomial $p(x)$*

1. *Find the* ***distinct*** *real roots of p and write them in order from smallest to largest*
$$r_1 < r_2 < r_3 < \ldots < r_n.$$
2. *Find numbers $x_1, x_2, x_3, \cdots, x_n, x_{n+1}$ such that*
$$x_1 < r_1 < x_2 < r_2 < x_3 < r_3 < \cdots < x_n < r_n < x_{n+1}$$
3. *Evaluate p at each x_i.*
    - *If the inequality is satisfied for x_1, then the inequality holds on the interval $(-\infty, r_1)$. Otherwise, the inequality does not hold on the interval $(-\infty, r_1)$.*
    - *If the inequality is satisfied for x_i where $2 \leq i \leq n$, then the inequality holds on the interval (r_{i-1}, r_i). Otherwise, the inequality does not hold on the interval (r_{i-1}, r_i).*
    - *If the inequality is satisfied for x_{n+1}, then the inequality holds on the interval (r_n, ∞). Otherwise, the inequality does not hold on the interval (r_n, ∞).*

4. *Write down each interval where the inequality holds. Include the endpoints if the inequality is not strict — i.e.* $p(x) \geq 0$ *or* $p(x) \leq 0$. *Otherwise, exclude the endpoints.*

Example 3.9.3 Find all solutions to the inequality

$$(x+4)(x+2)x(x-2)(x-5) \leq 0.$$

□

3.9.3 Solving Polynomial Inequalities Graphically

Solving polynomial inequalities graphically is significantly simpler. However, it has the disadvantage that it requires you to be able to sketch the graph of the polynomial function.

Example 3.9.4 Find all the values of x that satisfy the inequality

$$x^2 < 6 - x.$$

□

3.9.4 Worksheet: Polynomial Inequalities

Objectives

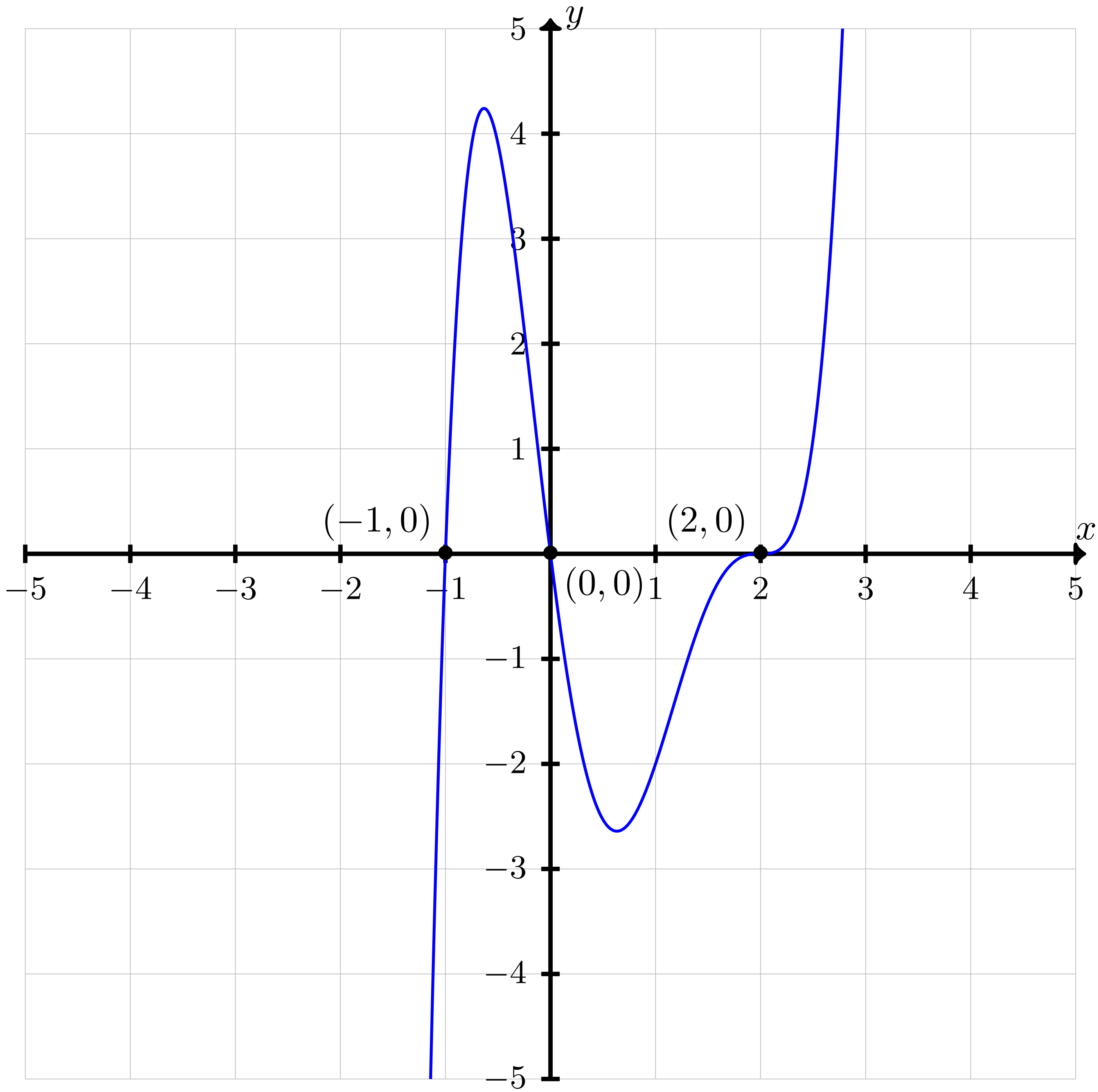
- Solve polynomial inequalities.

Solving Polynomial Inequalities Algebraically. Use Algorithm 3.9.2 to find all values of x that satisfy the following inequalities.

1. $2x + 3 < 7$

2. $x^3 - 6x^2 + 11x \geq 6$

Solving Polynomial Inequalities Graphically. Use the graph of the function below to answer the following questions



3. Find all values of x that satisfy $f(x) < 0$.

4. Find all values of x that satisfy $f(x) \leq 0$.

5. Find all values of x that satisfy $f(x) > 0$.

6. Find all values of x that satisfy $f(x) \geq 0$.

Chapter 4

Linear Systems

4.1 Two Dimensions

It is common to encounter situations involving multiple polynomials where we would like to find the points that are solutions to *all* of the polynomials at once. In the most general situation, we do not place any restrictions on the polynomials involved. This leads to a branch of mathematics called algebraic geometry[1]. The simplest case occurs when each of the polynomials involved is linear, and is known as linear algebra[2]. Both topics are well beyond the scope of this course, so we focus on finding solutions in two very special cases.

4.1.1 Systems of Linear Equations in Two Variables

First we begin with a rigorous definition of a linear system.

Definition 4.1.1 A **system of** m **linear equations in the variables** x **and** y is a collection of m equations

$$\begin{array}{ccccc} a_1x & + & b_1y & = & c_1 \\ a_2x & + & b_2y & = & c_2 \\ \vdots & & \vdots & & \vdots \\ a_mx & + & b_my & = & c_m \end{array}$$

A **solution** to the system is a point (a, b) that is a solution for *every* equation in the system. ◊

Solutions to system of equations have geometric meaning. Given a system of m linear equations in the variables x and y, each equation $a_ix + b_iy = c_i$ represents a line in standard form. A solution to the linear system is a point that lies on each line — we call this a point of **intersection**

[1] en.wikipedia.org/wiki/Algebraic_geometry
[2] en.wikipedia.org/wiki/Linear_algebra

Example 4.1.2

(a) Verify that $(1,2)$ is a solution to the system of two equations in x and y

$$\begin{aligned} -3x + y &= -1 \\ -x + y &= 1 \end{aligned}$$

(b) Interpret the solution, $(1,2)$, to the system of two equations in x and y

$$\begin{aligned} -3x + y &= -1 \\ -x + y &= 1 \end{aligned}$$

geometrically.

□

4.1.2 Dependent Systems

A slightly stranger way for a linear system in the variables x and y to have a solution is when the lines are the same. In this case, there are infinitely many solutions.

Definition 4.1.3 We say a system of linear equations is **dependent** if there are infinitely many solutions to the system. $\Diamond$

Example 4.1.4 Verify the system

$$
\begin{aligned}
x + 2y &= 4 \\
2x + 4y &= 8
\end{aligned}
$$

has infinitely many solutions. □

4.1.3 Consistency

In general, a system is not guaranteed to have *any* solutions. The simplest way this can happen is if two lines in the system are parallel. However, if there are more than two lines, then the system may fail to have any solutions even if no two lines are parallel.

Definition 4.1.5 We say a linear system is **consistent** if there is at least one solution. Otherwise, we say the linear system is **inconsistent**. $\diamondsuit$

4.1.4 Characterizing Solutions

It is a surprising fact of linear algebra that this essentially characterizes the behavior of *every* linear system — regardless of the number of variables and equations. That is to say, there are three possible behaviors for a linear system:

- The system is inconsistent (i.e. no solutions),
- The system has exactly one solution, or
- The system has infinitely many solutions.

Example 4.1.6 Verify geometrically that the system

$$\begin{aligned} -2x + 3y &= 6 \\ x + 2y &= 2 \\ 4x + y &= 4 \end{aligned}$$

does not have any solutions. $\square$

4.1.5 Finding Solutions Algebraically

The simplest way to find solutions to a linear system in two variables is to use a method called **substitution** The method relies on solving one equation in the system for a single variable of your choice and then substituting that into the other equations.

Example 4.1.7 Use substitution to solve the system

$$\begin{aligned} x + y &= 2 \\ -x + y &= 0 \end{aligned}$$

□

4.1.6 Worksheet: Systems of Linear Equations in Two Variables

Objectives

For each system below, sketch a graph of the lines and determine whether the system is consistent or inconsistent. If the system is consistent, find all solutions. Express the solutions as a set and plot them.

1.

$$\begin{aligned} 2x + 3y &= 8 \\ 4x - y &= -5 \end{aligned}$$

2.

$$\begin{aligned} 3x - 2y &= 7 \\ 2x + y &= 4 \end{aligned}$$

3.

$$\begin{aligned} 2x + y &= 5 \\ 4x + 2y &= 10 \end{aligned}$$

4.

$$\begin{aligned} 3x - 2y &= 7 \\ 6x - 4y &= 5 \end{aligned}$$

5.

$$\begin{aligned} -x + 2y &= 3 \\ 3x + y &= 10 \end{aligned}$$

6.

$$\begin{aligned} -3x + 5y &= 2 \\ 6x - 10y &= 4 \end{aligned}$$

4.2 Three Dimensions

When we increase the number of variables in a linear system, we increase the dimension of the ambient space. For linear equations with two variables, x and y, we work in the Cartesian plane. Linear equations with three variables, x, y, and z, require us to work in three dimensional space

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

This adds additional complexity to the geometric interpretation of solutions to linear equations and systems. In particular, a linear equation in three variables no longer represents a line, but rather a **plane** — a figure in space that looks like the Cartesian plane. Since this is *not* a course on linear algebra or multivariable calculus — two courses dedicated to the study of higher dimensional spaces — we will leave our discussion of the geometry at that, and focus on solving systems algebraically.

4.2.1 Systems of Linear Equations in Three Variables

The same terminology and intuition from linear systems in two variables apply to systems of linear equations in three variables. First, we start with an adaptation of solutions to equations in two variables to three variables.

Definition 4.2.1 Solution to an Equation in Three Variables. A **solution** to an equation in the variables x, y, and z is an ordered pair (a, b, c) that make the equation true when

1. the first coordinate, a, is substituted into the equation for x,
2. the second coordinate, b, is substituted into the equation for y, and
3. the third coordinate, c, is substituted into the equation for z.

$\Diamond$

Next, we generalize the notion of a linear equation from two variables to three.

Definition 4.2.2 Standard Form a Linear Equation in Three Variables. The **standard form a linear equation** in the variables x, y, and z is

$$ax + by + cz = d,$$

where a, b, and c are not all zero. $\Diamond$

Now, the notion of a linear system in three variables generalizes naturally.

Definition 4.2.3 A **system of m linear equations in the variables x, y, and z** is a collection of m equations

$$\begin{array}{ccccccc} a_1x & + & b_1y & + & c_1z & = & d_1 \\ a_2x & + & b_2y & + & c_2z & = & d_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_mx & + & b_my & + & c_3z & = & d_m \end{array}$$

A **solution** to the system is a point (a, b, c) that is a solution for *every* equation in the system. $\Diamond$

4.2.2 Solving Linear Systems with Three Equations

Fortunately, the method of substitution also generalizes naturally. To solve a system of equations in three variables, solve one equation of your choosing for one of the variables and substitute into the other equations. This reduces to a linear system in two equations, which we already know how to solve.

Example 4.2.4 Solve the system

$$\begin{aligned} 2x + z &= 5 \\ 2x + y + z &= 7 \\ x + z &= 4 \end{aligned}$$

□

Unlike linear systems with two equations, it is possible for to obtain a dependent system even when the equations are not the same.

Example 4.2.5 Solve the system

$$\begin{aligned} x + y + z &= 1 \\ x + y + 2z &= 2 \end{aligned}$$

□

4.2.3 Worksheet: Systems of Linear Equations in Three Variables

Objectives

- Sovle systems of three linear equations in three variables

Solve each system below. If the system is consistent, express the solutions as a set.

1.

$$\begin{aligned} 2x - y + z &= 5 \\ 3x + 2y - 4z &= -2 \\ x + 3y + 2z &= 8 \end{aligned}$$

2.

$$\begin{aligned} 4x - 2y + 3z &= 7 \\ 2x - y + \frac{3}{2}z &= 3 \\ \frac{1}{2}x - \frac{1}{2}y + \frac{3}{4}z &= 1 \end{aligned}$$

3.

$$\begin{aligned} x - 2y + z &= 3 \\ 2x + y - 3z &= 5 \\ 3x + 4y - 2z &= 1 \end{aligned}$$

4.

$$\begin{aligned} 3x - y + 2z &= 4 \\ 2x + 4y - z &= 7 \\ x - 2y + 3z &= 1 \end{aligned}$$

5.

$$
\begin{aligned}
2x + 3y - z &= 5 \\
4x + 6y - 2z &= 10 \\
6x + 9y - 3z &= 15
\end{aligned}
$$

6.

$$
\begin{aligned}
2x + y - z &= 4 \\
3x - 2y + 2z &= 1 \\
-x + 3y + 3z &= 5
\end{aligned}
$$

Chapter 5

Combining Functions

5.1 Algebraic Combinations of Functions

The simplest way to construct new functions from existing functions is to use the four arithmetic operations.

5.1.1 Sums and Differences of Functions

Definition 5.1.1 Sum of Functions. Assume $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ are functions. The **sum** of f and g is the function $f + g\colon A \cap B \to \mathbb{R}$ defined by

$$(f + g)(x) = f(x) + g(x).$$

$\Diamond$

Remark 5.1.2 Suppose $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ are functions. Since addition of real numbers is commutative

$$x + y = y + x,$$

the order of addition of functions is also commutative because

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x).$$

That is to say $f + g = g + f$ or, in words, the sum of f and g is the same as the sum of g and f.

Definition 5.1.3 Difference of Functions. Assume $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ are functions. The **difference** of f and g is the function $f - g\colon A \cap B \to \mathbb{R}$ defined by

$$(f - g)(x) = f(x) - g(x).$$

$\Diamond$

Example 5.1.4

(a) Consider the functions $f\colon [0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ and $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$. Find the domain of the sum of f and g, and an expression for $(f+g)(x)$.

(b) Consider the functions $f\colon [0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ and $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$. Find the domain of the difference of f and g, and an expression for $(f-g)(x)$.

$\square$

Warning 5.1.5 Unlike the sum of functions, the difference of functions is ***not commutative***.

Example 5.1.6 Consider the functions $f\colon [0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ and $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$. Find the domain of the difference of g and f, and an expression for $(g - f)(x)$. Conclude that $f - g \neq g - f$ as functions. □

5.1.2 Products and Ratios of Functions

Definition 5.1.7 Product of Functions. Assume $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ are functions. The **product** of f and g is the function $fg\colon A \cap B \to \mathbb{R}$ defined by

$$(fg)(x) = f(x)g(x).$$

$\Diamond$

Remark 5.1.8 Suppose $f\colon A \to \mathbb{R}$ and $g\colon A \to \mathbb{R}$ are functions that output real numbers. Since multiplication of real numbers is commutative

$$xy = yx,$$

the order of multiplication of functions is also commutative because

$$(fg)(x) = f(x)g(x) = g(x)f(x) = (gf)(x).$$

That is to say $fg = gf$ or, in words, the product of f and g is the same as the product of g and f.

Definition 5.1.9 Ratio of Functions. Assume $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ are functions, and for every $x \in B$, $g(x) \neq 0$. The **ratio** of f and g is the function $f/g\colon A \cap B \to \mathbb{R}$ defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

$\Diamond$

Example 5.1.10

(a) Consider the functions $f\colon [0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ and $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$. Find the domain of the product of f and g, and an expression for $(fg)(x)$.

(b) Consider the functions $f\colon [0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ and $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x$. Find the domain of the ratio of f and g, and an expression for $(f/g)(x)$.

□

Warning 5.1.11 Unlike the sum and product of functions, the ratio of functions is ***not commutative***.

Example 5.1.12 Consider the functions $f\colon [0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ and $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x$. Find the domain of the ratio of f and g, and an expression for $(f/g)(x)$. Conclude that $f/g \neq g/f$ as functions. $\square$

5.1.3 Worksheet: Algebraic Combinations of Functions

Objectives

- Combine functions algebraically,
- For a combination, find the implied
  - domain,
  - codomain, range
  - range

For each of the functions below, find the algebraic combination. Specify the domain, codomain, and range of the combination.

1. Find the sum of $f(x) = 2x + 5$ and $g(x) = -3x + 2$.

2. Find the difference of $p(x) = x^2 + 3$ and $q(x) = 2x - 1$.

3. Find the product of $a(x) = 4x$ and $b(x) = x + 1$.

4. Find the ratio of $u(x) = 2x^2 + 1$ and $v(x) = x - 3$.

5. Find an expression for the function $(m - 2n)(x)$, where $m(x) = 3x^2 + 2$ and $n(x) = 2x + 4$.

6. Find an expression for the function $st(x) + 4$, where $s(x) = x^2 - 1$ and $t(x) = 2x + 3$.

5.2 Rational Functions

A special case of ratios of functions arise when both the numerator and denominator are polynomials. These functions are of significant importance in many areas of mathematics, as well as its applications.

5.2.1 The Domain of a Rational Function

Definition 5.2.1 A **rational function** is a function of the form

$$f(x) = \left(\frac{p}{q}\right)(x) = \frac{p(x)}{q(x)},$$

where p and $q \neq 0$ are polynomial functions. $\Diamond$

Since polynomial functions always have domain $\mathbb{R}$ and codomain $\mathbb{R}$, the domain of the rational function $f = p/q$ is the set

$$D = \{x \in \mathbb{R} \mid q(x) \neq 0\}.$$

Example 5.2.2 Find the domain of the rational function

$$f(x) = \frac{x-2}{x^2-9}$$

□

5.2.2 Simplifying Rational Functions

Recall that we always write rational numbers in lowest terms. For example, instead of writing $2/4$, we cancel the common factor of 2 in the numerator and denominator to write $1/2$. The same is true for rational functions, although some extra care is required.

Example 5.2.3 Simplify the function $f\colon (-\infty, 3) \cup (3, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{x^2 - 9}{x - 3}$$

in lowest terms. What is the domain of this simplified function? $\square$

5.2.3 Sums and Differences of Rational Functions

Recall that in order to add or subtract rational numbers, we must find a common denominator. For example, to add 2/3 and 3/2, we use the common denominator $2 \times 3 = 6$

$$\begin{aligned}\frac{2}{3}+\frac{3}{2} &= \frac{2}{2}\frac{2}{3}+\frac{3}{3}\frac{3}{2}\\ &= \frac{4}{6}+\frac{9}{6}\\ &= \frac{4+9}{6}\\ &= \frac{13}{6}.\end{aligned}$$

The same is true for rational functions, provided we are careful with the denominator.

Consider the rational functions

$$f\colon \{x \in \mathbb{R} \mid x \neq 0,\, 1\} \to \mathbb{R} \quad \text{and} \quad g\colon \{x \in \mathbb{R} \mid x \neq 0,\, 2\} \to \mathbb{R}$$

defined by

$$f(x) = \frac{x+1}{x(x-1)} \quad \text{and} \quad g(x) = \frac{x-4}{x(x-2)}.$$

Example 5.2.4 Find the sum of f and g. What is the domain of this function? □

5.2.4 Products and Ratios of Rational Functions

Recall that when we multiply two rational numbers we multiply the numerators and denominators. For example,

$$\frac{1}{2}\left(\frac{3}{4}\right) = \frac{1(3)}{2(4)} = \frac{3}{8}.$$

When we divide one rational number by another, we multiply the one by the reciprocal of the other. For example

$$\frac{\frac{1}{2}}{\frac{3}{4}} = \frac{1}{2}\left(\frac{4}{3}\right) = \frac{4}{6} = \frac{2}{3}.$$

The same is true for products and ratios of rational functions.

Consider the rational functions

$$f\colon \{x \in \mathbb{R} \mid x \neq 0,\, 1\} \to \mathbb{R} \quad \text{and} \quad g\colon \{x \in \mathbb{R} \mid x \neq 0,\, 2\} \to \mathbb{R}$$

defined by

$$f(x) = \frac{x+1}{x(x-1)} \quad \text{and} \quad g(x) = \frac{x-4}{x(x-2)}.$$

Example 5.2.5

(a) Find the product of f and g.

(b) Find the ratio f and g

□

5.2.5 Rational Equations and Inequalities

Just as with polynomials, we may wish to solve equations or inequalities involving rational functions. Fortunately, with some care, rational equations and inequalities quickly reduce to polynomial equations or inequalities by simply multiplying both sides by the denominator.

Rational inequalities are defined nearly identically to polynomial inequalities.

Definition 5.2.6 Rational Inequality. Assume $p\colon \mathbb{R} \to \mathbb{R}$ and $q\colon \mathbb{R} \to \mathbb{R}$ are polynomial functions. A **rational inequality** is a relation of the form

- $\frac{p(x)}{q(x)} < 0$,
- $\frac{p(x)}{q(x)} \leq 0$,
- $\frac{p(x)}{q(x)} > 0$, or
- $\frac{p(x)}{q(x)} \geq 0$.

$\Diamond$

The solutions are also similarly defined.

Definition 5.2.7 Solution to a Rational Inequality. A **solution** to an inequality in the variable x is a number a that makes the expression true when a is substituted for x. $\Diamond$

Example 5.2.8 Solve the equation

$$\frac{x^2-1}{x^2-4} = 3$$

□

Example 5.2.9 Solve the rational inequality
$$\frac{x^2-1}{x^2-4} \leq 3$$
□

5.2.6 Worksheet: Rational Functions

Objectives

- Combine rational functions
- Solve rational inequalities

Solve the following rational inequalities.

1. $\dfrac{2x-1}{3} > 5$

2. $\dfrac{x^2-4}{x+2} \leq 0$

3. $\dfrac{3x-2}{x+1} < 4$

4. $\dfrac{2x+1}{x-3} \geq \dfrac{x+4}{2}$

5. $\dfrac{x-1}{x^2+3x+2} < 0$

6. $\dfrac{2x^2-5x+3}{x^2-x-6} \geq \dfrac{x+2}{x^2-4}$

5.3 Composition

When we study arithmetic in an elementary mathematics course, we focus primarily on operations that we can perform with numbers like addition and multiplication. While we have seen we can perform these same operations with functions, there is one operation that is particular to functions that we study here. Composition is important because it provides yet another way to create complicated functions from simpler functions.

Definition 5.3.1 Composition of Functions. Assume $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ are functions with $f(A) \subseteq B$. The **composition** of g with f is the function $g \circ f\colon A \to \mathbb{R}$ defined by

$$(g \circ f)(x) = g\left(f\left(x\right)\right).$$

$\Diamond$

Warning 5.3.2 It is important to ensure the range of the function f is contained within the domain of the function g. If not, the composition does not make sense.

Example 5.3.3 The function $g\colon [0, \infty) \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$ ***cannot*** be composed with the function $f\colon [0, \infty) \to (-\infty, 0]$ defined by $f(x) = -x$ because

$$(g \circ f)(x) = g(-x) = \sqrt{-x}$$

is ***not*** a real number when $x > 0$! $\square$

Example 5.3.4

(a) Consider the functions $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x+2$ and $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2+3$. Find the composition of g with f.

(b) Consider the functions $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x+2$ and $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2+3$. Find the composition of f with g. Verify that it is ***not*** the same as the composition of g with f.

$\square$

Example 5.3.5

(a) Let h be any non-zero real number and consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Find a simplified expression for the function

$$d(x) = \frac{f(x+h) - f(x)}{h}$$

(b) Let h be any non-zero real number and consider the function $f\colon [0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$. Find a simplified expression for the function

$$d(x) = \frac{f(x+h) - f(x)}{h}$$

□

Worksheet: Composition

Objectives

- Compose functions
- Find the implied domain, codomain, and range of a composition.

Compose each pair of functions. Find the implied domain, codomain, and range.

1. $f(x) = 2x + 3$ with $g(x) = 3x - 1$

2. $p(x) = x^2 + 1$ with $q(x) = 2x + 5$

3. $a(x) = x^3 - 2$ with $b(x) = 3x + 4$

4. $u(x) = 1/(x + 2)$ with $v(x) = 2x - 3$

5. $m(x) = |x - 2|$ with $n(x) = 3x - 1$

6. $s(x) = x^2 - 2$ with $t(x) = 4$

5.4 Composition Inverses

In keeping with the analogy with operations on numbers, recall the operations of addition and multiplication both have distinguished elements. The number 0 is called the **additive identity** because every real number x remains unchanged when 0 is added to it

$$x + 0 = x \quad \text{and} \quad 0 + x = x$$

Similarly, the number 1 is called the **multiplicative identity** because every real number x remains unchanged when it is multiplied by 1

$$x \times 1 = x \quad \text{and} \quad 1 \times x = x$$

5.4.1 Identities

We introduced the idea of an *identity* using two familiar operations to highlight that the ***identity element depends on the operation.*** However, the essential *behavior* of the identity element is the same: when we perform the operation using the identity, nothing changes. This tells us exactly how the identity element for the function composition operation should behave.

Definition 5.4.1 Identity Function. Assume A is a set. The **identity function** on A is the function $\mathrm{id}\colon A \to A$ defined by

$$\mathrm{id}_A(x) = x$$

◊

As we have already seen, the order of function composition matters. Complicating matters more, not all functions can be composed. For a function $f\colon A \to B$,

- the composition $g \circ f$ only makes sense when the domain of g contains the range of f, and
- the composition $f \circ g$ only makes sense when the range of g is contained in the domain of f.

For this reason, there is not one single identity function that satisfies the condition $f \circ \mathrm{id} = f$ and $\mathrm{id} \circ f = f$.

Instead, we need to consider the identity functions on the domain and codomain of f. The identity function on A satisfies $f \circ \mathrm{id}_A = f$ because

$$(f \circ \mathrm{id}_A)(x) = f\left(\mathrm{id}_A(x)\right) = f(x)$$

and the identity function on B satisfies $\mathrm{id}_B \circ f = f$ because

$$(\mathrm{id}_B \circ f)(x) = \mathrm{id}_B\left(f(x)\right) = f(x).$$

5.4.2 Composition Inverses

The identity elements for these operations are important because they provide a method for "undoing" the operation. This is easiest to understand by considering addition and multiplication first. For any real number c, the **additive inverse** is the real number $-c$ that satisfies

$$c + (-c) = 0 \quad \text{and} \quad -c + c = 0$$

If we add the number c to another number x, then we obtain the number $x + c$. We can "undo" this operation by performing the operation "add the additive inverse of c" -- which we normally just call "subtract c" -- in the sense that

$$(x + c) + (-c) = x + (c - c) = x + 0 = x. \tag{5.4.1}$$

Similarly, we note that the operation "add c" will "undo" the operation "subtract c" in the sense that

$$(x-c)+c = x+(-c+c) = x+0 = x. \tag{5.4.2}$$

The operations "add c" and "subtract c" define functions $\mathrm{add}_c\colon \mathbb{R}\to\mathbb{R}$ and $\mathrm{sub}_c\colon \mathbb{R}\to\mathbb{R}$ by

$$\mathrm{add}_c(x) = x+c \quad \text{and} \quad \mathrm{sub}_c(x) = x-c$$

We can view (5.4.1) and (5.4.2) in terms of composition:

$$\begin{aligned} \mathrm{sub}_c\circ\mathrm{add}_c(x) &= \mathrm{sub}_c(x+c) \\ &= (x+c)-c \\ &= x \end{aligned} \qquad\qquad \begin{aligned} \mathrm{add}_c\circ\mathrm{sub}_c(x) &= \mathrm{add}_c(x-c) \\ &= (x-c)+c \\ &= x. \end{aligned}$$

The story is similar for multiplication. For any non-zero real number c, the **multiplicative inverse** of c is the real number $1/c$ that satisfies

$$c\times\frac{1}{c} = 1 \quad \text{and} \quad \frac{1}{c}\times c = 1$$

Multiplication by c and division by c define functions $\mathrm{mul}_c, \mathrm{div}_c\colon \mathbb{R}\to\mathbb{R}$ by

$$\mathrm{mul}_c(x) = cx \quad \text{and} \quad \mathrm{div}_c(x) = \frac{x}{c}$$

Just as with addition, these two functions "undo" one another in the sense that

$$\begin{aligned} \mathrm{mul}_c\circ\mathrm{div}_c(x) &= \mathrm{mul}_c\left(\frac{x}{c}\right) \\ &= c\left(\frac{x}{c}\right) \\ &= x \end{aligned} \qquad\qquad \begin{aligned} \mathrm{div}_c\circ\mathrm{mul}_c(x) &= \mathrm{div}_c(cx) \\ &= \frac{cx}{c} \\ &= x. \end{aligned}$$

These are two canonical examples of functions that are *invertible*, and the ideas captured in inverting addition and multiplication motivate the more general definition.

Definition 5.4.2 Composition Inverse. Assume $f\colon A\to B$ is a function. The **composition inverse**, if it exists, is a function $f^{-1}\colon B\to A$ that satisfies

$$f^{-1}\circ f = \mathrm{id}_A \quad \text{and} \quad f\circ f^{-1} = \mathrm{id}_B$$

or, equivalently, for every $x\in A$, $f^{-1}\circ f(x) = x$ and for every $y\in B$, $f\circ f^{-1}(y) = y$.

If f has a composition inverse, then we say f is **invertible**. ◊

Warning 5.4.3 Mathematicians commonly use the word "inverse" or "invertible" without explicitly stating the operation. It is important to infer the meaning from the context. If the object being discussed is a function, then "inverse" means *composition* inverse, ***not*** multiplicative inverse. That is to say,

$$f^{-1}(x) \neq \frac{1}{f(x)} = f(x)^{-1}$$

In order to avoid confusion, we will avoid using the notation $f(x)^{-1}$ and write either $1/f(x)$ or

$$\frac{1}{f(x)}$$

whenever we refer to a fraction with the function $f(x)$ in the denominator.

5.4.3 Properties of Invertible Functions

For a function $f\colon A \to B$ with a composition inverse, the codomain of f and the domain of f^{-1} agree. For every $y \in B$, the element $x = f^{-1}(y) \in A$ satisfies

$$y = f \circ f^{-1}(y) = f(x)$$

by definition. This provides useful information about the range of an invertible function f.

Theorem 5.4.4 *If $f\colon A \to B$ is a function with a composition inverse, then the range of f is*

$$f(A) = \{f(x) \mid x \in A\} = B$$

The definition of a composition inverse is *symmetric* in the sense that the inverse of f is f^{-1} and the inverse of f^{-1} is f.

Theorem 5.4.5 *If $f\colon A \to B$ is a function with inverse $f^{-1}\colon B \to A$, then f^{-1} has a composition inverse and*

$$\left(f^{-1}\right)^{-1} = f$$

Example 5.4.6 In the language of Definition 5.4.2, the functions "add c" and "subtract c" are a pair of inverse functions

$$(\mathrm{add}_c)^{-1} = \mathrm{sub}_c \quad \text{and} \quad (\mathrm{sub}_c)^{-1} = \mathrm{add}_c$$

Similarly, the functions "multiply by c" and "divide by c" are a pair of inverse functions

$$(\mathrm{mul}_c)^{-1} = \mathrm{div}_c \quad \text{and} \quad (\mathrm{div}_c)^{-1} = \mathrm{mul}_c$$

□

Example 5.4.7 In general, a linear function $f(x) = mx + b$ is invertible if $m \neq 0$. To evaluate f at the real number x, we perform the following operations

1. Multiply x by m,
2. Add b to mx.

This allows us to decompose f as the function "add b" composed with the function "multiply by m"

$$f(x) = \mathrm{add}_b \circ \mathrm{mul}_m(x)$$

Since each function is invertible, we can invert f by performing the inverse operations in reverse order, like peeling back the layers of an onion.

Write $y = mx + b$ for the result of applying f to x. We would invert (or "undo") the operation of applying f to x by performing the operations

1. Subtract b from y
2. Divide $y - b$ by m (***Important:*** This is why we required $m \neq 0$!)

Note this is equivalent to solving the equation $y = mx + b$ for x in terms of y

In the language of Example 5.4.6, the inverse of f is the composition

$$\begin{aligned} f^{-1}(y) &= (\mathrm{mul}_m)^{-1} \circ (\mathrm{add}_b)^{-1}(y) \\ &= \mathrm{div}_m \circ \mathrm{sub}_b(y) \\ &= \mathrm{div}_m(y-b) \\ &= \frac{y-b}{m} \\ &= \frac{1}{m}y - \frac{b}{m}. \end{aligned}$$

Finally, we can check these are in fact inverses of one another by checking the two compositions explicitly:

$$\begin{aligned} f \circ f^{-1}(y) &= f\left(\frac{1}{m}y - \frac{b}{m}\right) \\ &= m\left(\frac{1}{m}y - \frac{b}{m}\right) + b \\ &= \frac{m}{m}y - \frac{mb}{m} + b \\ &= y - b + b \\ &= y \end{aligned}$$

$$\begin{aligned} f^{-1} \circ f(x) &= f^{-1}(mx+b) \\ &= \frac{1}{m}(mx+b) - \frac{b}{m} \\ &= \frac{m}{m}x + \frac{b}{m} - \frac{b}{m} \\ &= x. \end{aligned}$$

□

Our observations about compositions of invertible functions in Example 5.4.7 are true in general.

Formula 5.4.8 *Assume f and g are invertible functions. If the composition is well-defined, then $g \circ f$ is invertible and*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

As a final example, we illustrate why it is necessary to pay close attention to the domain of a function.

Example 5.4.9 Consider the quadratic function $f\colon \mathbb{R} \to [0, \infty)$ defined by $f(x) = x^2$. This function is ***not invertible*** .

To see why, we must observe that if an inverse function exists, then by Definition 5.4.2, it must have the form $f^{-1}\colon [0, \infty) \to \mathbb{R}$ and for every positive number y, we must assign a single number $x = f^{-1}(y)$ such that

$$x^2 = f(x) = f \circ f^{-1}(y) = y$$

However, there are always ***two*** such values: a positive number, x, and its negative, $-x$. This means we must *choose* one of these two values to be $f^{-1}(y)$. Unfortunately, both decisions fail to produce an inverse for f.

If we decide that $x = f^{-1}(y)$ is the positive number such that $x^2 = y$, then

$$\begin{aligned} f^{-1} \circ f(-x) &= f^{-1}\left((-x)^2\right) \\ &= f^{-1}(x^2) \\ &= f^{-1}(y) \\ &= x \neq -x. \end{aligned}$$

On the other hand, if we decide that $-x = f^{-1}(y)$ is the negative number such that $(-x)^2 = x^2 = y$, then

$$f^{-1} \circ f(x) = f^{-1}(x^2) = -x \neq x$$

Fortunately, these two problems suggest simple solutions.

In the first case, we simply exclude the negative numbers from the domain of f. This operation is called **restriction** and produces a new function $g\colon [0,\infty) \to [0,\infty)$ defined by $g(x) = x^2$. This function becomes invertible and its inverse is the **principle square root function** $g^{-1}\colon [0,\infty) \to [0,\infty)$ defined by $g^{-1}(y) = \sqrt{y}$. For every non-negative number x,

$$\begin{aligned} g^{-1} \circ g(x) &= g^{-1}(x^2) \\ &= \sqrt{x^2} \\ &= |x| \\ &= x, \end{aligned} \qquad \begin{aligned} g \circ g^{-1}(x) &= g(\sqrt{x}) \\ &= \sqrt{x}^2 \\ &= x. \end{aligned}$$

In the second case, we restrict the domain of f to the negative numbers. This produces a new function $h\colon (-\infty,0] \to [0,\infty)$ defined by $h(x) = x^2$. The function h is invertible and its inverse is the function $h^{-1}\colon [0,\infty) \to (-\infty,0]$ defined by $h^{-1}(y) = -\sqrt{y}$. For every non-negative real number x,

$$\begin{aligned} h^{-1} \circ h(-x) &= h^{-1}\left((-x)^2\right) \\ &= h^{-1}(x^2) \\ &= -\sqrt{x^2} \\ &= -\left|x\right| \\ &= -x \end{aligned} \qquad \begin{aligned} h \circ h^{-1}(x) &= h\left(-\sqrt{x}\right) \\ &= \left(-\sqrt{x}\right)^2 \\ &= (-1)^2\sqrt{x}^2 \\ &= \sqrt{x}^2 \\ &= x. \end{aligned}$$

$\square$

5.4.4 Worksheet: Composition Inverses

Objectives

- Determine whether an algebraic function has an inverse
- Find the inverse of an invertible function

For each function below, determine a subset of the domain where the function is invertible. Find the range of the function on that interval, then find the inverse.

1. $f(x) = 2x - 3.$

2. $g(x) = x^2 + 4.$

3. $h(x) = x^3 - 1.$

4. $p(x) = \dfrac{2x+1}{3}.$

5.5 Graphing Inverse Functions

5.5.1 A Little Symmetry

Assume $A, B \subseteq \mathbb{R}$ and $f\colon A \to B$ is a function with inverse $f^{-1}\colon B \to A$. In order to visualize the inverse of a function, it is helpful to recall from Definition 2.1.5 that the graph of f is the set of points

$$\{(x, f(x)) \mid x \in A\}$$

in the Cartesian plane.

Since we have been conditioned by training to think about the input to a function being named x and the output of a function being named y, we opt to write $f^{-1}(y)$ instead of $f^{-1}(x)$. This distinction is meaningless in practice, but prompts us to think about the input to f^{-1} as the output of f.

We made these observations more precise in Theorem 5.4.4, which asserts that every $y \in B$ can be written in the form $y = f(x)$ for *some* $x \in A$. This allows us to describe the graph of f^{-1} as the set of points

$$\begin{aligned}\left\{\left(y, f^{-1}(y)\right) \,\middle|\, y \in B\right\} &= \left\{\left(f(x), f^{-1}\left(f(x)\right)\right) \,\middle|\, x \in A\right\} \\ &= \left\{\left(f(x), f^{-1} \circ f(x)\right) \,\middle|\, x \in A\right\} \\ &= \left\{(f(x), x) \mid x \in A\right\}.\end{aligned}$$

Note that we started by describing the points on the graph of f^{-1} in terms of B and ended by describing the points on the graph of f^{-1} in terms of A. This description tells us that we can obtain the graph of f^{-1} from the graph of f by interchanging the x- and y-coordinates.

Using the The Midpoint Formula, we can exploit this relationship to describe symmetry between the graph of f and the graph of f^{-1} as follows. Each $a \in A$ corresponds to a point $(a, f(a))$ on the graph of f and a point $(f(a), a)$ on the graph of f^{-1}. The midpoint of the line that connects these two points has coordinates

$$x = \frac{a + f(a)}{2} \quad \text{and} \quad y = \frac{f(a) + a}{2}.$$

Since the x- and y-coordinates are the same, the midpoint lies on the line $y = x$. This says that if we were to plot $f$and f^{-1} together, the image would have symmetry about the line $y = x$. This allows us to conclude the following.

Theorem 5.5.1 *The graph of f^{-1} is obtained by reflecting the graph of f across the line $y = x$.*

5.5.2 The Horizontal Line Test

The importance of Theorem 5.5.1 is that it provides a geometric condition for testing whether a function is invertible.

Theorem 5.5.2 *Assume $A, B \subseteq \mathbb{R}$. A function $f\colon A \to B$ is invertible if and only if the graph*

$$\{(f(x), x) \mid x \in A\}$$

represents a function.

We can improve Theorem 5.5.2 significantly with a few key observations. Recall that we can test whether a graph represents a function using The Vertical Line Test. The graph of the vertical line $x = a$, where a is a constant, is the set of points

$$\{(a, t) \mid t \in \mathbb{R}\}.$$

Reflecting this graph across the line $y = x$ interchanges the two coordinates, so we can express the resulting graph as the set

$$\{(t, a) \mid t \in \mathbb{R}\}.$$

As the y-coordinate is fixed and the x-coordinate is allowed to vary, we can recognize this as the horizontal line $y = a$. This provides a geometric condition for invertibility that depends only on the graph of the original function f.

Theorem 5.5.3 The Horizontal Line Test. *Assume $A, B \subseteq \mathbb{R}$. The function $f\colon A \to B$ is invertible if and only if every horizontal line intersects the graph of f in* at most *one point.*

Example 5.5.4 Consider the graph of the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. When we observed that for every positive number y, there are two x-values that square to y, we were observing that every horizontal line intersects the parabola in two places.

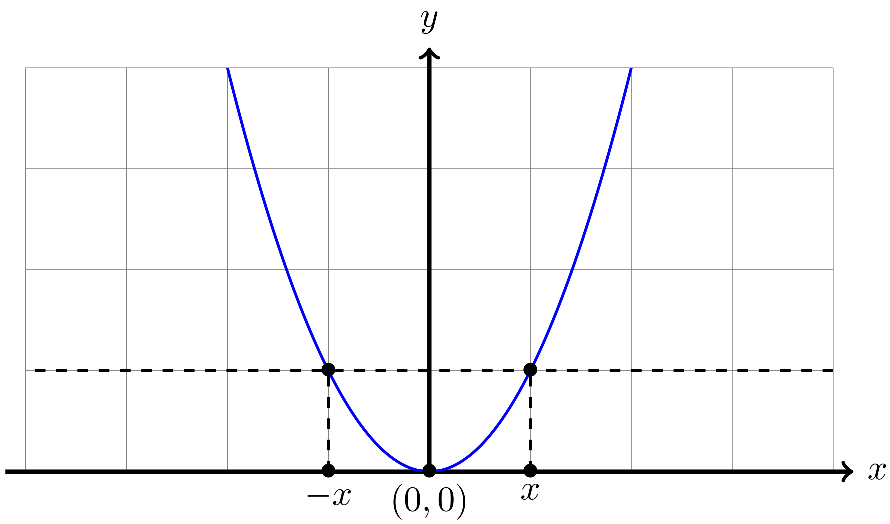
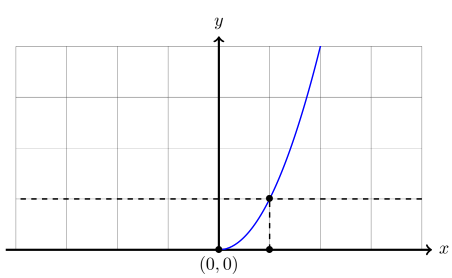
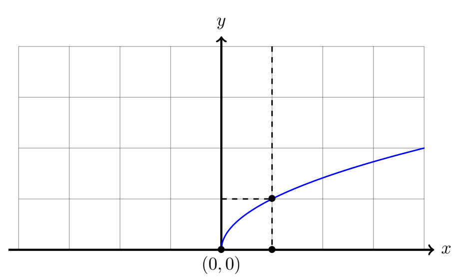


Figure 5.5.5 The two points of intersection between the parabola and a horizontal line

When we restrict the domain to the set $[0, \infty)$, we fix this problem by removing the left half of the parabola. This makes the function invertible with inverse the principal square root function.



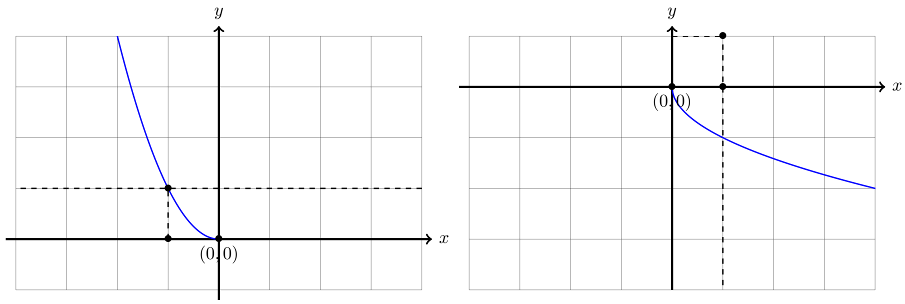
(a) The restriction of $f(x) = x^2$ to $[0, \infty)$



(b) The principal square root function

Figure 5.5.6 Removing the left-hand side of the parabola

On the other hand, we could also remove the right-hand side of the parabola. This creates another invertible function. The inverse of this function is the reflection of the principal square root across the x-axis.



(b) The inverse of the restriction.

(a) The restriction of $f(x) = x^2$ to $(-\infty, 0]$.

Figure 5.5.7 Removing the right-hand side of the parabola

□

5.5.3 Root Functions

A useful consequence of The Horizontal Line Test is that we may define functions with rational exponents.

Definition 5.5.8 Odd Root Function. Assume n is an odd, positive integer and $f\colon \mathbb{R} \to \mathbb{R}$ is the function defined by $f(x) = x^n$. The $n^{\mathbf{th}}$ **root function** $\sqrt[n]{}\colon \mathbb{R} \to \mathbb{R}$ given by $\sqrt[n]{x} = x^{1/n}$ is defined to be the inverse of f. ♢

Definition 5.5.9 Even Root Function. Assume n is an even, positive integer and $f\colon [0, \infty) \to \mathbb{R}$ is the function defined by $f(x) = x^n$. The $n^{\mathbf{th}}$ **root function** $\sqrt[n]{}\colon \mathbb{R} \to [0, \infty)$ given by $\sqrt[n]{x} = x^{1/n}$ is defined to be the inverse of f. ♢

5.5.4 Worksheet: Graphing Inverse Functions

Objectives

- Use the graph of a function to graph its inverse.

For each function below, graph the inverse.

1. $f(x) = 2x - 3.$

2. $g(x) = x^2 + 4.$

3. $h(x) = x^3 - 1.$

4. $p(x) = \dfrac{2x+1}{3}.$

5.6 Piecewise Defined Functions

A powerful method for constructing new functions from existing functions is to "glue" two or more functions together.

Definition 5.6.1 Piecewise Defined Function. Let n be a positive integer. Assume $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets, meaning whenever $i \neq j \in \{1, 2, \ldots, n\}$,

$$A_i \cap A_j = \emptyset,$$

and let $A = A_1 \cup A_2 \cup \cdots \cup A_n$. Suppose

$$f_1 \colon A_1 \to \mathbb{R},\ f_2 \colon A_2 \to \mathbb{R},\ \ldots,\ f_n \colon A_n \to \mathbb{R}$$

are functions. The function $f \colon A \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A_1, \\ f_2(x) & \text{if } x \in A_2, \\ & \vdots \\ f_n(x) & \text{if } x \in A_n \end{cases}$$

is called a **piecewise defined function**. ◊

Example 5.6.2 Write the absolute value function

$$|\cdot| : \mathbb{R} \to [0, \infty)$$

as a piecewise defined function. $\square$

Example 5.6.3

(a) Graph the piecewise defined function

$$f(x) = \begin{cases} x+2 & \text{if } x \leq 0 \\ x^2+3 & \text{if } x > 0 \end{cases}$$

(b) Find the inverse of the function f and graph it.

□

Worksheet: Piecewise Functions

- Evaluate piecewise functions,
- Graph piecewise functions

1. Use the piecewise function

$$f(x) = \begin{cases} -2x-4 & \text{if } x \in [-5,-3) \\ x^2+3t-1 & \text{if } x \in [-3,2] \\ 7 & \text{if } x \in (2,5] \end{cases}$$

to answer the following questions.

(a) What is the domain of f?

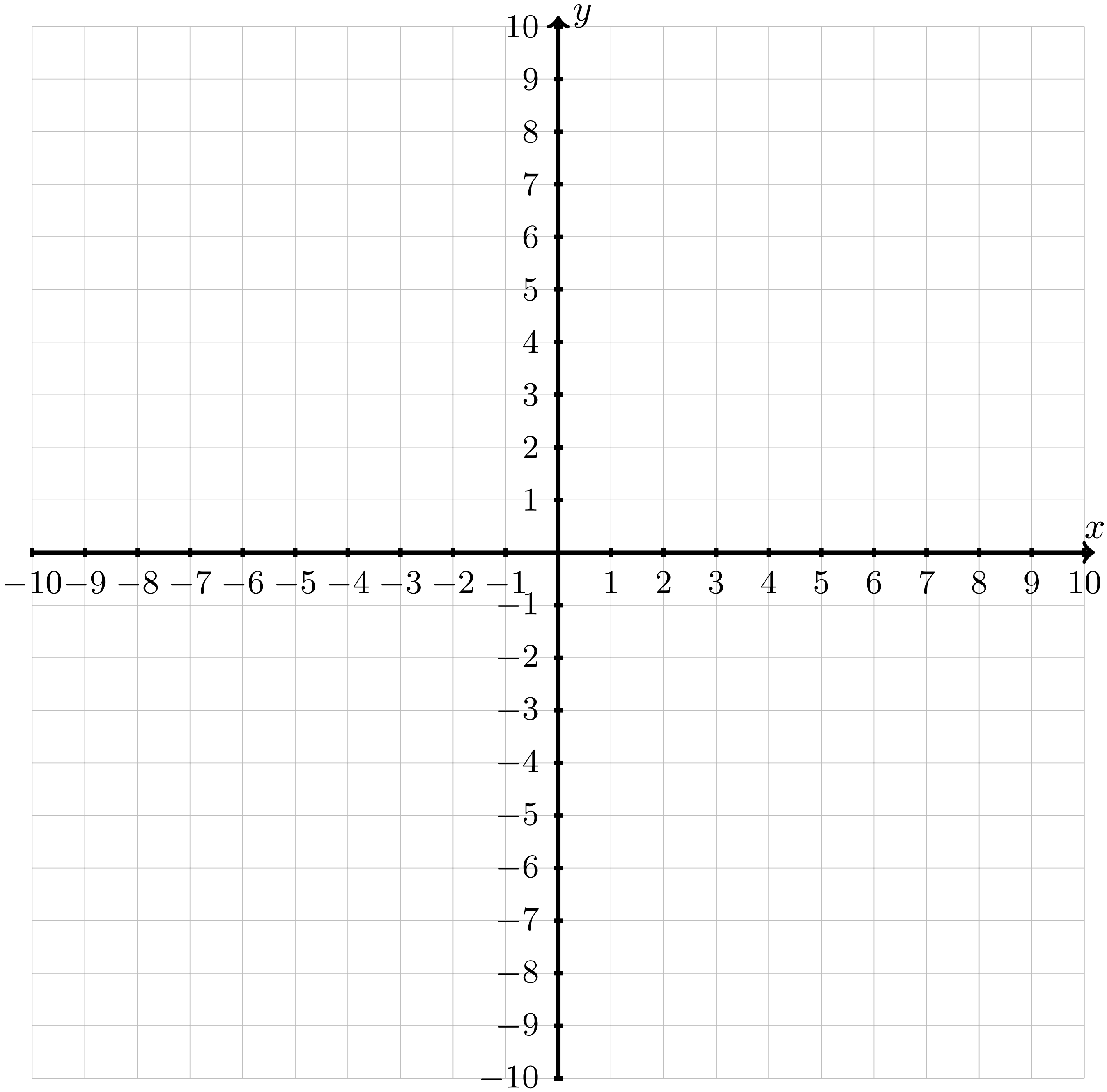
(b) What is the range of f?

(c) Evaluate the following

- $f(-5) =$
- $f(-3) =$
- $f(2) =$
- $f(5) =$

2. Graph the function

$$f(x) = \begin{cases} -2x - 4 & \text{if } x \in [-5, -3) \\ x^2 + 3t - 1 & \text{if } x \in [-3, 2] \\ 7 & \text{if } x \in (2, 5] \end{cases}$$



Chapter 6

Transcendental Functions

6.1 The Exponential Function

So far, all of the functions we have encountered are either polynomials or can be constructed from polynomial equations using algebraic operations. A **transcendental function** is any function that cannot be constructed in this manner. While it is certainly odd to discuss functions that cannot be constructed by algebraic means in a course on algebra, many calculus courses expect students to already be familiar with these functions. We focus solely on the algebraic properties of these functions.

6.1.1 Properties of the Exponential Function

We start with the definition of a constant. Much like the number π, this number *cannot* be constructed using algebraic methods. Indeed, the proper definition requires calculus, so we provide only a numerical approximation.

Definition 6.1.1 Euler's Number. **Euler's number** is the constant $e \approx 2.71828182846$. ◊

This constant is essential to the definition of the exponential function.

Definition 6.1.2 The Exponential Function. The **exponential function** $\exp\colon \mathbb{R} \to (0, \infty)$ is defined by $\exp(x) = e^x$. ◊

We can think about the value of the exponential function at a positive real number r as the result of multiplying r copies of the number e:

$$\exp(r) = e^r = \underbrace{eee\cdots e}_{r}.$$

This definition extends to all real numbers by defining

$$\exp(0) = e^r = 1 \quad \text{and} \quad \exp(-r) = e^{-r} = \frac{1}{e^r} = \frac{1}{\exp(r)}.$$

An interesting consequence of these definitions is that the exponential function transforms addition of the input into multiplication of the output and subtraction of the input into division of the output. For addition of the input, we take advantage of the fact that multiplication is associative to write

$$\begin{aligned}
\exp(r+s) &= e^{r+s} \\
&= \underbrace{eee\cdots e}_{r+s} \\
&= \left(\underbrace{eee\cdots e}_{r}\right)\left(\underbrace{eee\cdots e}_{s}\right) \\
&= e^r e^s \\
&= \exp(r)\exp(s).
\end{aligned}$$

For subtraction of the inverse, we note that $r - s = r + (-s)$, so

$$\begin{aligned}\exp(r - s) &= \exp(r + (-s))\\ &= \exp(r)\exp(-s)\\ &= \exp(r)\frac{1}{\exp(s)}\\ &= \frac{\exp(r)}{\exp(s)}.\end{aligned}$$

These two properties also provide a convenient observation about multiplication of the input. Recall that the product of two numbers is simply a notational convenience. Depending on your needs, the product rs can be regarded as either adding together r copies of the number s or s copies of the number r:

$$rs = \underbrace{s + s + s + \cdots + s}_{r} = \underbrace{r + r + r + \cdots + r}_{s}.$$

If we take first interpretation of rs, then we can apply the formula for the sum of inputs r times to get

$$\begin{aligned}\exp(rs) &= \exp\left(\underbrace{s + s + s + \cdots + s}_{r}\right)\\ &= \exp(s)\exp\left(\underbrace{s + s + s + \cdots + s}_{r-1}\right)\\ &= \exp(s)\exp(s)\exp\left(\underbrace{s + s + s + \cdots + s}_{r-2}\right)\\ &= \cdots\\ &= \underbrace{\exp(s)\exp(s)\exp(s)\cdots\exp(s)}_{r}\\ &= \exp(s)^r.\end{aligned}$$

Similarly, if we take the second interpretation of rs, then we can apply the formula for the sum of inputs s times to get

$$\begin{aligned}\exp(rs) &= \exp\left(\underbrace{r + r + r + \cdots + r}_{s}\right)\\ &= \exp(r)\exp\left(\underbrace{r + r + r + \cdots + r}_{s-1}\right)\\ &= \exp(r)\exp(r)\exp\left(\underbrace{r + r + r + \cdots + r}_{s-2}\right)\\ &= \cdots\\ &= \underbrace{\exp(r)\exp(r)\exp(r)\cdots\exp(r)}_{s}\\ &= \exp(r)^s.\end{aligned}$$

Theorem 6.1.3 Properties of Exponential Functions. *Assume x and x' are real numbers.*

1. $\exp(0) = a^0 = 1$.
2. $\exp(x + x') = \exp(x)\exp(x')$.
3. $\exp(x - x') = \dfrac{\exp(x)}{\exp(x')}$.
4. $\exp(xx') = \exp(x)^{x'} = \exp\left(x'\right)^{x}$.

6.1.2 The Graph of the Exponential Function

The graph of the exponential function is provided below.

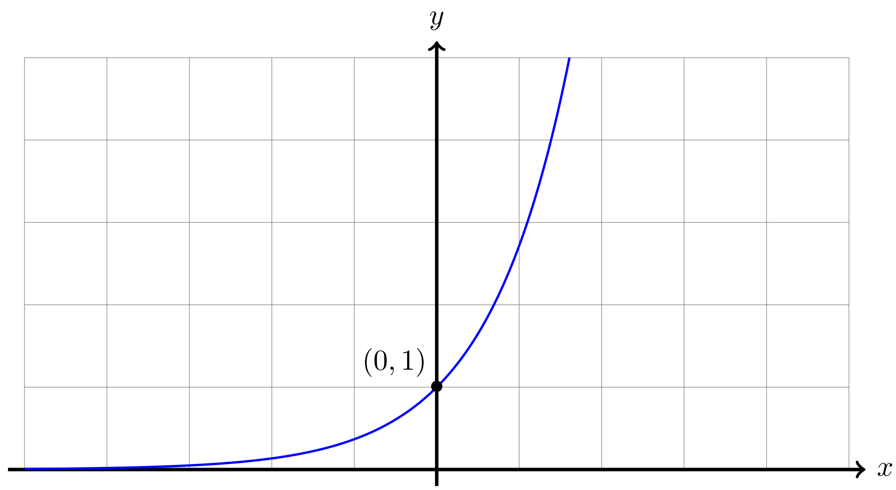


Figure 6.1.4 The graph of $y = \exp(x)$

6.1.3 Worksheet: The Exponential Function

Objectives

- Combine expressions involving the exponential
- Identify graphs involving the exponential

Use the Properties of Exponential Functions to rewrite each equation so there is only ***one*** exponential term.

1. $\exp(x^2) = \exp(-2x-1)$

2. $\exp(2x+1) = \exp(2)$

3. $\exp(-4x) = \exp(4)\exp\left(x^2\right)$

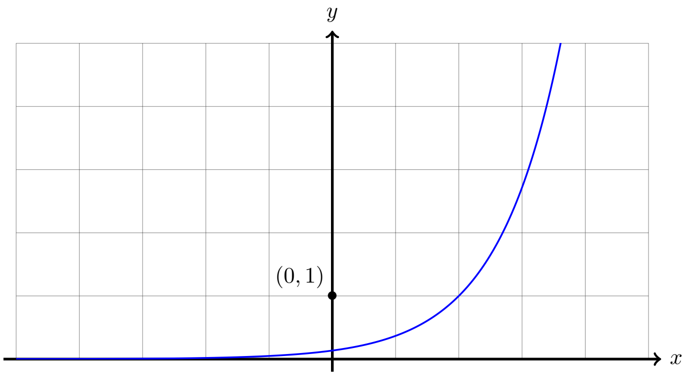
4. Match each function with its graph.

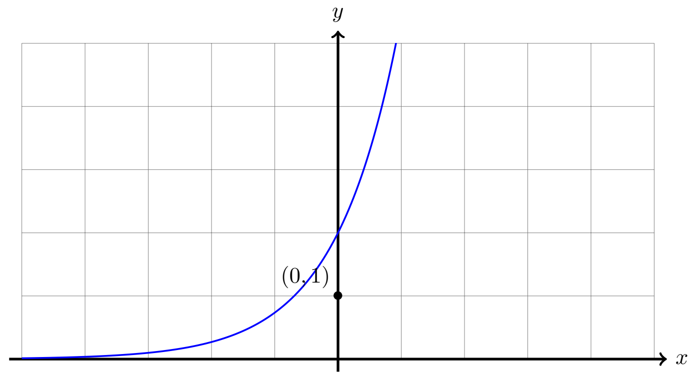
(a) $y = \exp(-x)$

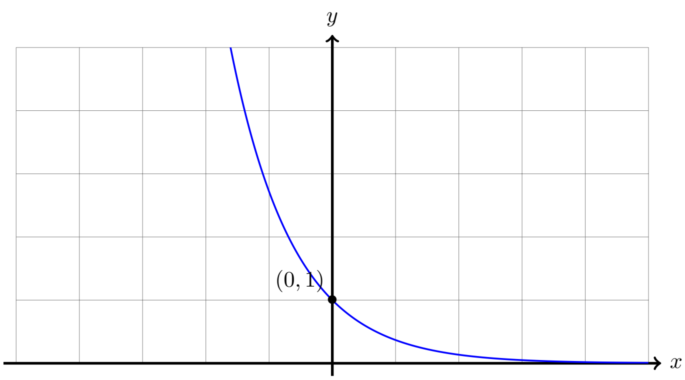
(b) $y = 2\exp(x)$

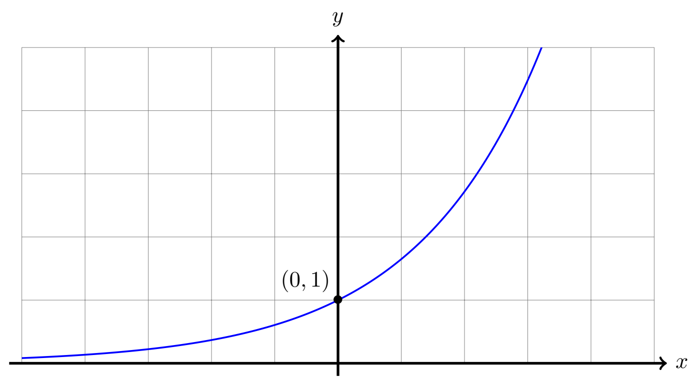
(c) $y = \exp\left(\dfrac{x}{2}\right)$

(d) $y = \exp(x-2)$









6.2 The Natural Logarithm

Looking at the graph of $\exp\colon \mathbb{R} \to (0,\infty)$, it appears to satisfy The Horizontal Line Test.

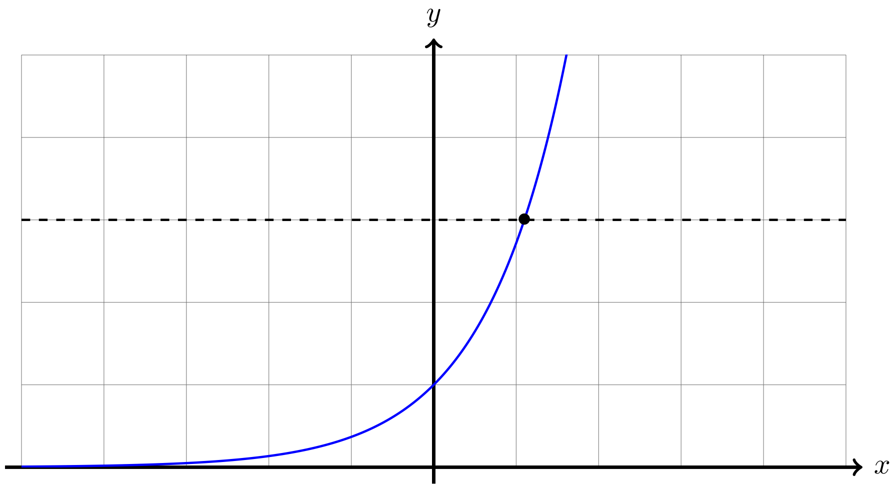


Figure 6.2.1 The horizontal line test for $\exp\colon \mathbb{R} \to (0,\infty)$

We can verify this algebraically as follows. Suppose the points $(x_1, \exp(x_1))$ and $(x_2, \exp(x_2))$ lie on the same horizontal line. These points must have the same y-coordinates, so

$$\begin{aligned} \exp(x_1) &= \exp(x_2) && \text{Divide both sides by } \exp(x_2) \\ \frac{\exp(x_1)}{\exp(x_2)} &= 1 && \text{Combine} \\ \exp(x_1 - x_2) &= 1 \end{aligned}$$

implies that $x_1 - x_2 = 0$ and so $x_1 = x_2$. That is to say, the two points are the *same* point and so any horizontal line intersects the graph in at most one point. This tells us the exponential function is invertible.

6.2.1 Properties of the Natural Logarithm

Definition 6.2.2 The Natural Logarithm. The **natural logarithm**, $\ln\colon (0,\infty) \to \mathbb{R}$, is the inverse of $\exp\colon \mathbb{R} \to (0,\infty)$. ◊

As the composition inverse of $\exp\colon \mathbb{R} \to (0,\infty)$, the logarithm inherits properties that are very similar to the Properties of Exponential Functions. To describe these properties algebraically it is helpful to phrase the relationship between the exponential and the logarithm as pairing each real number x with a positive real number y via

$$y = \exp(x) \quad \text{and} \quad x = \ln(y). \tag{6.2.1}$$

Starting from the property $\exp(0) = 1$, taking $x = 0$ and $y = 1$ in (6.2.1) allows us to write

$$1 = \exp(0) \quad \text{and} \quad 0 = \ln(1).$$

We should note this algebraic fact fits with our geometric understanding. The graph of $\exp\colon \mathbb{R} \to (0,\infty)$ passes through the point $(0,1)$. Since the graph of $\ln\colon (0,\infty) \to \mathbb{R}$ is the reflection of the graph of the exponential function across the line $y = x$, the graph of the natural logarithm passes through the point $(1,0)$.

If we introduce another real number, x', then we obtain a second positive real number, $y' = \exp(x')$. The function exp maps the sum of these numbers to the product of their pairs

$$\exp(x + x') = \exp(x)\exp(x') = yy',$$

so the right-hand side of (6.2.1) tells us that the logarithm pairs the product of positive real numbers with the sum of their pairs

$$x + x' = \log(yy').$$

Similarly, the function exp maps the difference of these numbers to the ratio of their pairs

$$\exp(x - x') = \frac{\exp(x)}{\exp(x')} = yy',$$

so the right-hand side of (6.2.1) tells us that the logarithm pairs the ratio of positive real numbers with the difference of their pairs

$$x - x' = \log\left(\frac{y}{y'}\right).$$

Finally, we make the following observation as a convenience. If y is any positive real number and r is any real number, then we think of y^r as the product of r copies of the number y

$$y^r = \underbrace{yyy\cdots y}_{r}.$$

If we transform the log of the product into the sum of the logs r times, then we obtain the expression

$$\begin{aligned}
\ln(y^r) &= \ln\left(\underbrace{yyy\cdots y}_{r}\right)\\
&= \ln(y) + \ln\left(\underbrace{yyy\cdots y}_{r-1}\right)\\
&= \ln(y) + \ln(y) + \ln\left(\underbrace{yyy\cdots y}_{r-2}\right)\\
&= \cdots\\
&= \underbrace{\ln(y) + \ln(y) + \cdots + \ln(y)}_{r}\\
&= r\ln(y).
\end{aligned}$$

We summarize these properties below.

Theorem 6.2.3 Properties of the Logarithm. *Assume y and y' are positive real numbers.*

1. $\ln(1) = 0$.
2. $\ln(yy') = \ln(y) + \ln(y')$.
3. $\ln\left(\frac{y}{y'}\right) = \ln(y) - \ln(y')$.
4. $\ln(y^r) = r\ln(y)$.

6.2.2 The Graph of the Natural Logarithm

As the composition inverse, the graph of $y = \ln(x)$ is obtained by reflecting the graph of $y = \exp(x)$ across the line $y = x$.

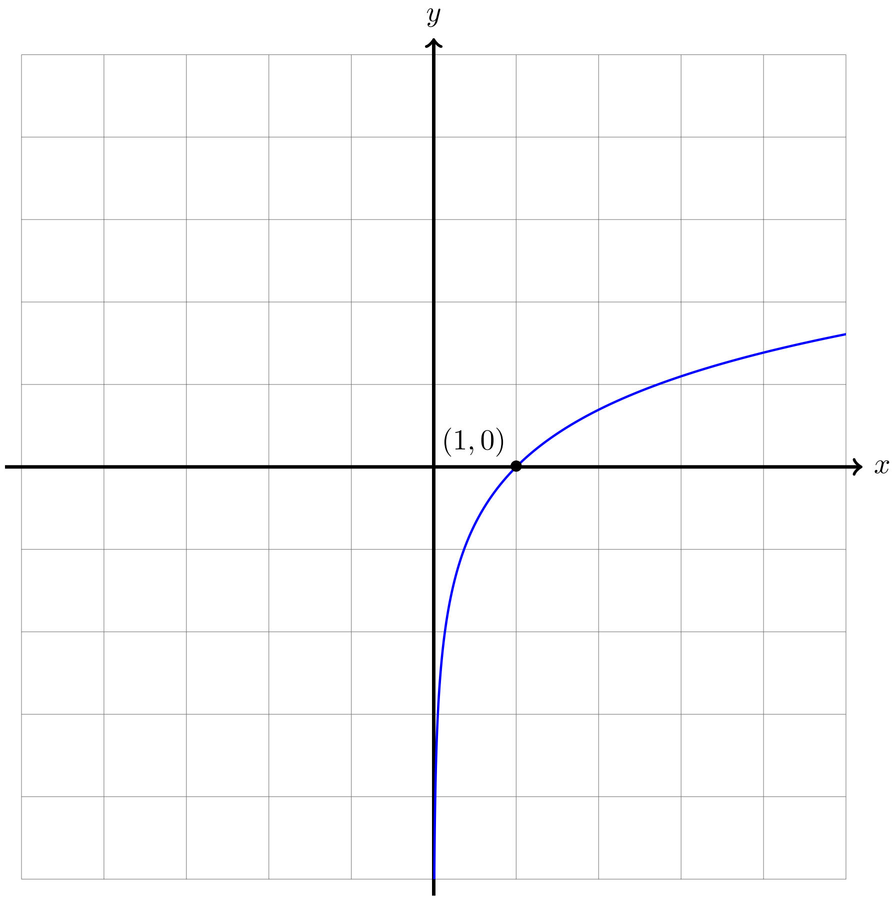


Figure 6.2.4 The graph of $y = \ln(x)$

6.2.3 Solving Equations with Transcendental Functions

Sometimes, equations arise that involve the exponential function or the natural logarithm. The general tactic for solving equations of this type is to combine the exponentials or logarithms using Properties of Exponential Functions and Properties of the Logarithm, then use the relation

$$\exp \circ \ln(x) = x \quad \text{or} \quad \ln \circ \exp(x) = x,$$

as appropriate, to obtain an equation without exponentials or logarithms.

Example 6.2.5

(a) Solve the equation

$$\exp\left(x^2\right)\exp(5) = \exp(6x)$$

for x.

(b) Solve the equation

$$2\ln\left(x-1\right) = \ln(1)$$

for x

□

6.2.4 Worksheet: The Natural Logarithm

Objectives

- Solve equations involving the exponential function.
- Solve equations involving the natural logarithm.

1. Solve the equation

$$\exp(2x + 1) = \exp(2)$$

for x.

2. Solve the equation

$$\exp(-4x) = \exp(4)\exp\left(x^2\right)$$

for x.

3. Solve the equation

$$\exp(x^2) = \exp(-2x-1)$$

for x

4. Solve the equation

$$\ln(x+3)+\ln(x-3)=\ln(16)$$

for x.

5. Solve the equation

$$\ln(x-5)-2\ln\left(\frac{1}{\sqrt{x}}\right)=\ln(6).$$

for x.

6. Solve the equation

$$-\ln\left(\frac{1}{5-x}\right)+2\ln\left(\sqrt{x+1}\right)=\ln\left(4\right)$$

for x.

6.3 Other Exponential Functions

The number e is chosen for the exponential function specifically because it has nice *analytic* properties, which are the domain of calculus. Since these properties are beyond the scope of an algebra course, we focus only on the algebraic similarities between exponential functions with different bases.

6.3.1 Changing the Base of the Exponential

When we defined *the* exponential function Section 6.1, there was nothing *algebraically* interesting about using the number e. We could just as well have chosen any other positive integer $a \neq 1$ — we exclude $a = 1$ simply because the constant function $1^x = 1$ is rather boring. Had we chosen a different positive integer, we would have started from the following definition.

Definition 6.3.1 Assume $a \neq 1$ is a fixed positive real number. The **exponential function with base** a is $\exp_a : \mathbb{R} \to (0, \infty)$ defined by $\exp_a(x) = a^x$. The number a is called the **base** of the exponential function. $\Diamond$

At this point, you might be trying to reconcile in your mind how there could be more than one. The answer, thankfully, is there really *is not* a second exponential function. Since the natural logarithm is defined for all positive real numbers, we can rewrite $a = \exp \circ \ln(a)$ so that with a little algebraic gymnastics,

$$\exp_a(x) = a^x = \left(e^{\ln(a)}\right)^x = e^{\ln(a)x} = \exp(\ln(a)x).$$

To understand the function $\exp_a$ in general, we need to consider two cases. When $a > 1$, $\ln(a) > 0$ and so $\exp_a$ is simply a Horizontal Scaling of exp, which we can see in Figure 6.3.2

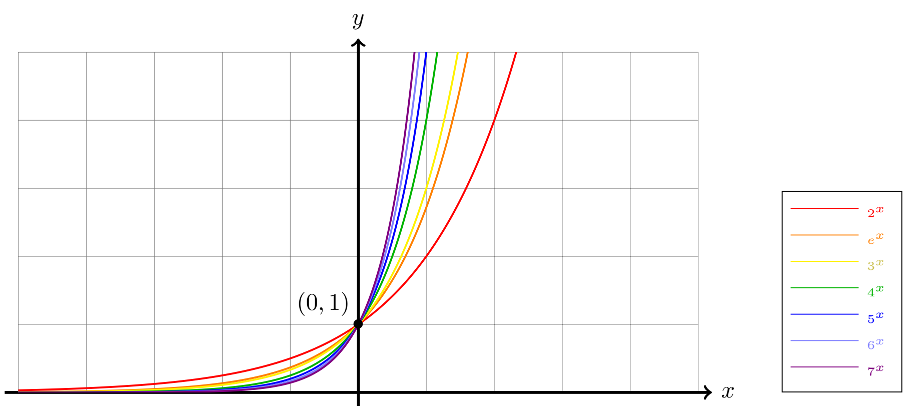


Figure 6.3.2 The graphs of the exponential functions with base 2, e, 3, 4, 5, 6, and 7

When $a < 1$, $\ln(a) < 0$ and so $\exp_a$ is obtained from exp by a Reflection and a Horizontal Scaling of exp.

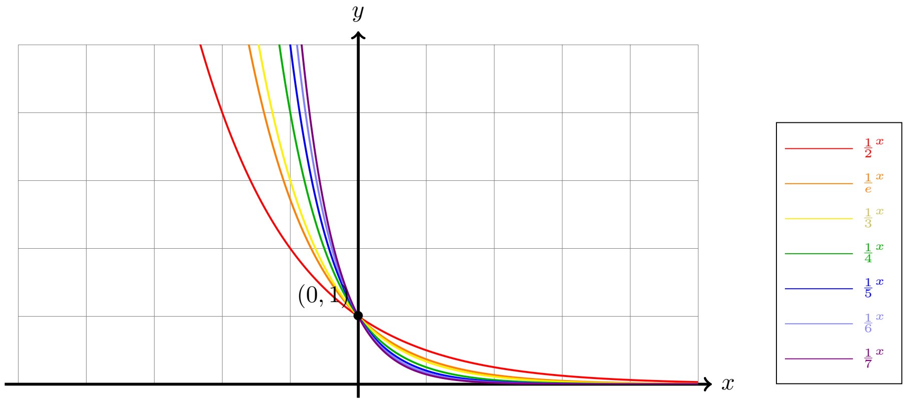


Figure 6.3.3 The graphs of the exponential functions with base $1/2$, $1/e$, $1/3$, $1/4$, $1/5$, $1/6$, and $1/7$

6.3.2 Properties of Exponential Functions

Unsurprisingly, all of the same properties enjoyed by exp are also enjoyed by $\exp_a$.

$$\begin{aligned}\exp_a(0) &= 1,\\ \exp_a(x+x') &= \exp_a(x)\exp_a(x'),\\ \exp_a(x-x') &= \frac{\exp_a(x)}{\exp_a(x')}, \text{ and}\\ \exp_a(xx') &= \exp_a(x)^{x'} = \exp_a(x')^{x}.\end{aligned}$$

We can use these properties to compare the effect of choosing a different base by measuring the *relative change*.

Definition 6.3.4 Let f be a function that is defined on the interval $[a,b]$. The **relative change** of f on the interval $[a,b]$ is

$$\frac{f(b)-f(a)}{f(a)}.$$

$\lozenge$

Since the exponential functions are defined for all real numbers, we can consider any interval of unit length by taking a number x as the left endpoint and $x+1$ as the right endpoint. Over this interval, the relative change is

$$\begin{aligned}\frac{\exp_a(x+1)-\exp_a(x)}{\exp_a(x)} &= \frac{\exp_a(x+1)-\exp_a(x)}{\exp_a(x)}\\ &= \frac{\exp_a(x)\exp_a(1)-\exp_a(x)}{\exp_a(x)}\\ &= \frac{\exp_a(x)(\exp_a(1)-1)}{\exp_a(x)}\\ &= \exp_a(1)-1\\ &= a-1.\end{aligned}$$

Definition 6.3.5 Assume $a \neq 1$ is a positive number. The number $r = a-1$ is called the **growth rate** of $\exp_a\colon \mathbb{R} \to (0,\infty)$ if $r > 1$ or the **decay rate** of $\exp_a\colon \mathbb{R} \to (0,\infty)$ if $r < 1$.

- When $r > 0$, we say the function $\exp_a\colon \mathbb{R} \to (0,\infty)$ models **exponential**

growth and we call the number $a = 1 + r$ the **growth factor**.

- When $r < 0$, we say the function $\exp_a : \mathbb{R} \to (0, \infty)$ models **exponential decay** and we call the number $a = 1 + r$ the **decay factor**.

$\Diamond$

6.3.3 Worksheet: Other Exponential Functions

Objectives

- Combine expressions involving exponentials
- Identify when an exponential function models growth
- Identify when an exponential function models decay

Use the properties of exponential functions to rewrite each equation so there is only ***one*** exponential term.

1. $\exp_7(2x+1) = \exp_7(2)$

2. $\exp_2(-4x) = \exp_2(4)\exp_2\left(x^2\right)$

3. $\exp_3(x^2) = \exp_3(-2x-1)$

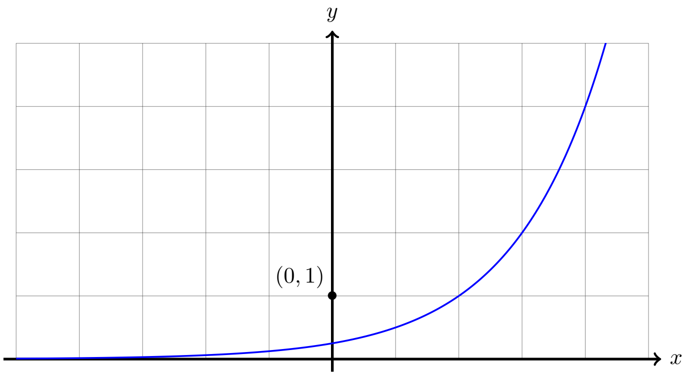
4. Match each function with its graph.

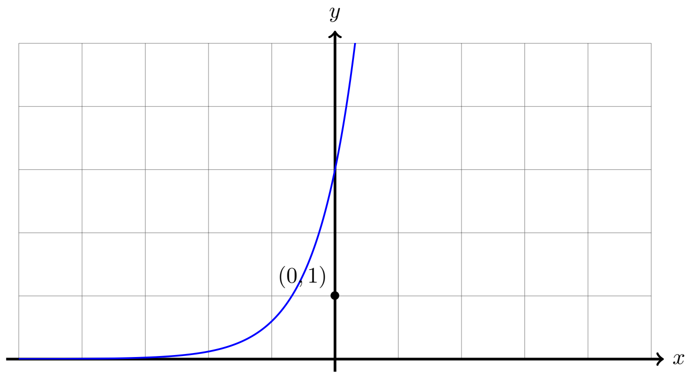
(a) $y = \exp_2(-x)$

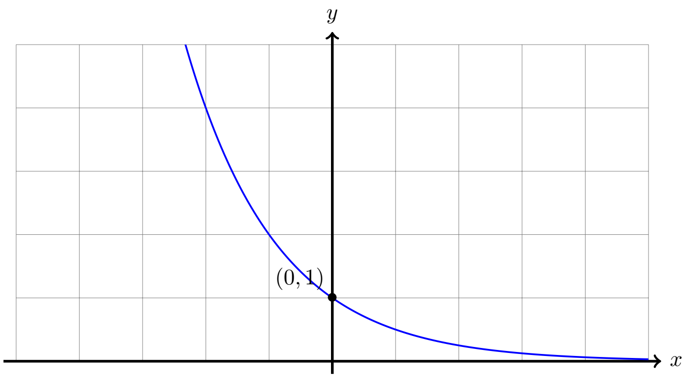
(b) $y = 3 \cdot \exp_5(x)$

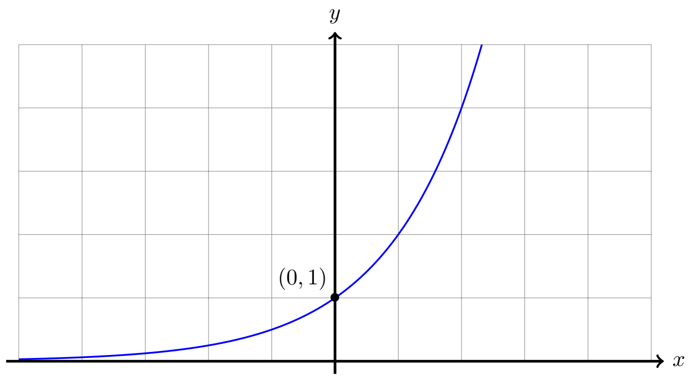
(c) $y = \exp_4\left(\dfrac{x}{2}\right)$

(d) $y = \exp_2(x - 2)$









6.4 Other Logarithmic Functions

Just as there is essentially only one exponential function, it should come as no surprise that there is essentially only one logarithm. We can regard the exponential function with base $a \neq 1$ as the composition

$$\exp_a(x) = \exp(\ln(a)x) = \exp \circ \mathrm{mul}_{\ln(a)}(x).$$

The condition $a \neq 1$ guarantees that $\ln(a) \neq 0$ and thus we may invert $\mathrm{mul}_{\ln(a)}$ using the division by $\ln(a)$ function

$$\mathrm{mul}_{\ln(a)}^{-1}(x) = \frac{x}{\ln(a)}.$$

By Formula 5.4.8, the inverse of the exponential with base a is the function defined by

$$\mathrm{mul}_{\ln(a)}^{-1} \circ \exp^{-1}(x) = \mathrm{mul}_{\frac{1}{\ln(a)}} \circ \ln(x) = \frac{\ln(x)}{\ln(a)}.$$

6.4.1 Changing the Base of the Logarithm

Definition 6.4.1 The Logarithm with Base a. Assume $a \neq 1$ is a positive real number. The **logarithm with base** a, is the function $\log_a\colon (0, \infty) \to \mathbb{R}$, defined by

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

In particular, for every real number x and every positive real number y,

$$\log_a \circ \exp_a(x) = \log_a(a^x) = x \quad \text{and} \quad \exp_a \circ \log_a(y) = a^{\log_a(y)} = y.$$

$\Diamond$

Our understanding of the natural logarithm and graph transformations allows us to very quickly visualize the logarithm with base a. Just as for exponentials, there are two cases: $a < 1$ and $1 < a$. When $a > 1$, $\ln(a) > 0$ and so the graph of

$$y = \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

is obtained as a vertical scaling of $\ln(x)$ by $1/\ln(a)$. We have plotted the logarithm for several bases in Figure 6.4.2.

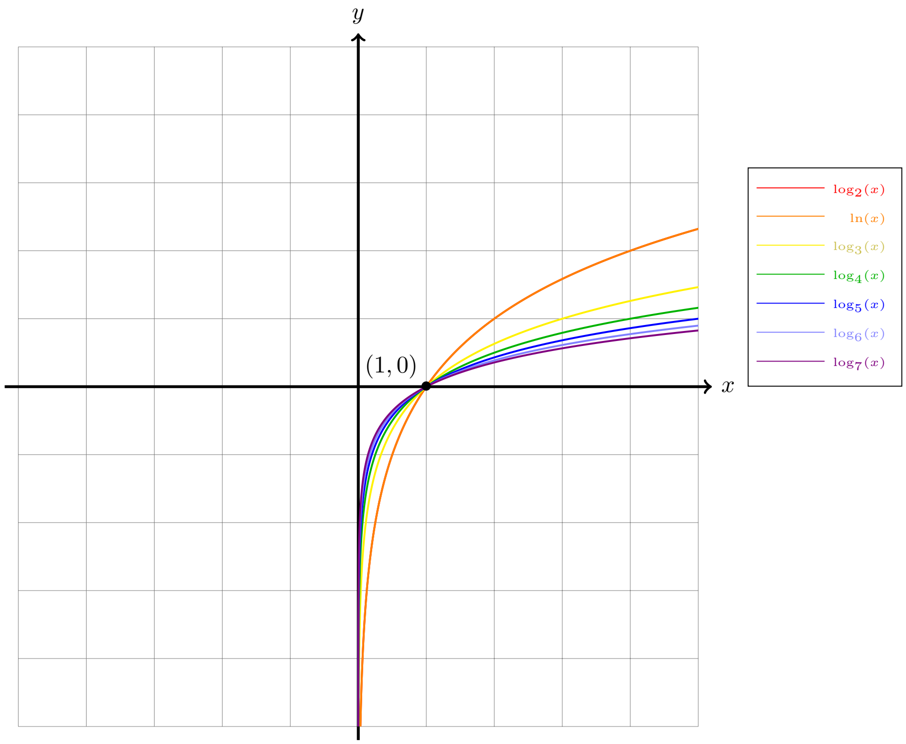


Figure 6.4.2 The graphs of the logarithms with base 2, e, 3, 4, 5, 6, and 7

When $a < 1$, $\ln(a) < 0$, and so the graph of

$$y = \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

is obtained by vertically scaling $y = \ln(x)$ by $1/\ln(a)$ and reflecting across the x-axis. We have plotted the logarithm for several bases in Figure 6.4.3.

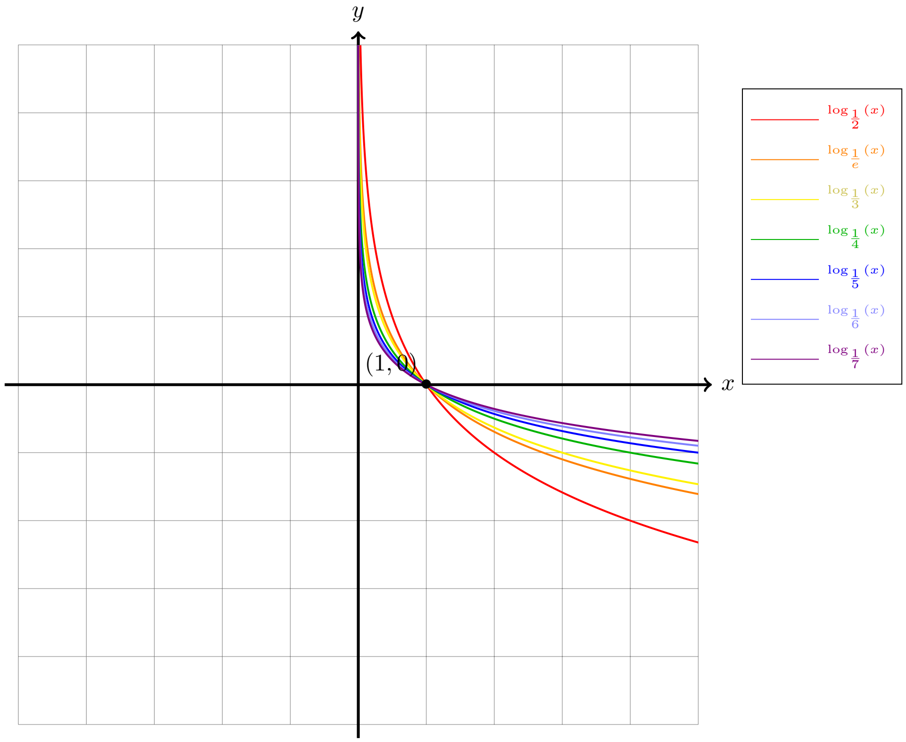


Figure 6.4.3 The graphs of the logarithms with base 1/2, 1/e, 1/3, 1/4, 1/5, 1/6, and 1/7

Explicitly defining the logarithm with base a in terms of the natural logarithm also allows us a simple way to convert between logarithms with different bases. Assume $a \neq 1$ and $b \neq 1$ are both positive numbers. Multiplying both sides of the equation

$$\log_b(y) = \frac{\ln(y)}{\ln(b)}$$

by the number $\ln(b)$ allows us to write

$$\ln(y) = \ln(b)\log_b(y).$$

We can substitute this for $\ln(y)$ in the definition of the logarithm base a to produce

$$\log_a(y) = \frac{\ln(y)}{\ln(a)} = \frac{\ln(b)\log_b(y)}{\ln(a)}.$$

It is often convenient to involve only two bases in this expression, so we rewrite

$$\frac{\ln(b)}{\ln(a)} = \frac{1}{\frac{\ln(a)}{\ln(b)}} = \frac{1}{\log_b(a)}.$$

Formula 6.4.4 Base Change for Logarithms. *Assume $a \neq 1$ and $b \neq 1$ are positive real numbers. For all positive real numbers y,*

$$\log_a(y) = \frac{\log_b(y)}{\log_b(a)}.$$

Remark 6.4.5 *The* logarithm — often written $\log(x)$ — means different things to different people, often depending on the context.

- Most mathematicians (and mathematical texts) will write $\log(x)$ to mean the logarithm with base e. This is because mathematicians are primarily interested in the functions and their behaviors, not numbers or numerical data.

- Most computer scientists will write $\log(x)$ to mean the logarithm with base 2. This is because computer scientists most frequently work with numbers and numerical data expressed in *base* 2 or *binary* — the number system used for computational devices. In base 2, numbers are represented as strings with digits that are either 0 or 1. For example, in base 2 the string 1011 stands for the number

$$\begin{aligned} 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 &= 1 \cdot 8 + 0 + 1 \cdot 2 + 1 \cdot 1 \\ &= 8 + 2 + 1 \\ &= 11 \end{aligned}$$

- Most scientists and engineers will write $\log(x)$ to mean the logarithm with base 10. This is because scientists and engineers most frequently work with numbers and numerical data expressed in the usual *base 10* or *decimal* number system. In contrast to base 2, the base 10 number system consists of strings of digits that are allowed to be one of 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. For example, in base 10 the string 1234 stands for

$$1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0 = 1000 + 200 + 30 + 4.$$

6.4.2 Worksheet: Other Logarithmic Functions

Objectives

- Solve equations involving exponential functions.
- Solve equations involving logarithms.

Solving Equations with Exponential Functions. Use the properties of exponentials and the appropriate logarithmic function to solve the following equations for x.

1. Solve the equation

$$\exp_7(2x+1) = 49$$

for x.

2. Solve the equation

$$\exp_2(-4x) = 16\exp_2\left(x^2\right)$$

for x.

3. Solve the equation

$$\exp_{\frac{1}{2}}(x^2) = \exp_{\frac{1}{2}}(-2x-1)$$

for x

Solving Equations with Logarithms. Use the properties of logarithms and the appropriate exponential function to solve the following equations for x.

4. Solve the equation

$$\log_4(x+3) + \log_4(x-3) = 2$$

for x.

5. Solve the equation

$$\log_6(x-5) - 2\log_6\left(\frac{1}{\sqrt{x}}\right) = 1.$$

for x.

6. Solve the equation

$$-\log_2\left(\frac{1}{5-x}\right) + 2\log_2\left(\sqrt{x+1}\right) = 2$$

for x.

6.5 Modeling with Exponential Functions

Exponential functions are of interest to scientists because they are useful in modeling natural phenomena. In this section, we explore some examples that demonstrate how exponential functions can be used for modeling.

6.5.1 Exponential Growth

Example 6.5.1 Population Growth. A biologist observes a population over a six year period. The biologist measures the size of the observed population at the beginning and end of each of the years. In an effort to understand how the population is changing, the biologist records the year-to-year change as a numeric value and as a percentage of the initial population for the year.

Table 6.5.2 Observed Population by Year

Year	Initial Population	Final Population	Change	Percent Change
1	64	96	+32	+50%
2	96	144	+48	+50%
3	144	216	+72	+50%
4	216	324	+108	+50%
5	324	486	+162	+50%
6	486	729	+243	+50%

Using the table, the biologist notices the following relations amongst the population sizes

$$\begin{aligned}
96 &= 64 + \frac{64}{2} = 64\left(1+\frac{1}{2}\right) = 64\left(\frac{3}{2}\right) = 64\exp_{\frac{3}{2}}(1)\\
144 &= 96\left(\frac{3}{2}\right) = 64\left(\frac{3}{2}\right)\left(\frac{3}{2}\right) = 64\left(\frac{3}{2}\right)^2 = 64\exp_{\frac{3}{2}}(2)\\
216 &= 144\left(\frac{3}{2}\right) = 64\left(\frac{3}{2}\right)^2\left(\frac{3}{2}\right) = 64\left(\frac{3}{2}\right)^3 = 64\exp_{\frac{3}{2}}(3)\\
324 &= 216\left(\frac{3}{2}\right) = 64\left(\frac{3}{2}\right)^3\left(\frac{3}{2}\right) = 64\left(\frac{3}{2}\right)^4 = 64\exp_{\frac{3}{2}}(4)\\
486 &= 324\left(\frac{3}{2}\right) = 64\left(\frac{3}{2}\right)^4\left(\frac{3}{2}\right) = 64\left(\frac{3}{2}\right)^5 = 64\exp_{\frac{3}{2}}(5)\\
729 &= 486\left(\frac{3}{2}\right) = 64\left(\frac{3}{2}\right)^5\left(\frac{3}{2}\right) = 64\left(\frac{3}{2}\right)^6 = 64\exp_{\frac{3}{2}}(6).
\end{aligned}$$

This leads the biologist to conclude that the size of the population can be measured with the exponential function

$$P(t) = 64\exp_{\frac{3}{2}}(t) = 64\left(\frac{3}{2}\right)^t,$$

where t is the amount of time in years that has elapsed. This function agrees with the observations for $t \in \{1,2,3,4,5,6\}$, and predicts the size of the population at the end of the seventh year will be

$$P(7) = 64\left(\frac{3}{2}\right)^7 = 1093.5 \approx 1094.$$

□

Example 6.5.3 Compound Interest. Your local bank offers a savings account with a 6% annual interest that is compounded monthly. This means the bank will add $6\%/12 = 0.5\%$ of your current balance to your account every month. In other words, your balance grows at a rate of 0.5% ***per month***.

The growth factor for the exponential function that models the amount of money in the account after t ***months*** is

$$a = 1 + 0.5\% = 1 + \frac{0.5}{100} = 1 + 0.005 = 1.005.$$

Assuming you deposit \$1,000 into a savings account, the balance of the account after t ***months*** is given by the function

$$B(t) = 1000 \exp_{1.005}(t) = 1000(1.005)^t.$$

Customarily, one measures investments as a function of the number of ***years*** that have elapsed, rather than months. Since there are 12 months in a year, we compress the time scale by a factor of 12 to obtain the function

$$A(t) = B(12t) = 1000(1.005)^{12t} = 1000\left(1.005^{12}\right)^t = 1000 \exp_{1.005^{12}}(t)$$

that measures the amount of money in the account after t **years**.

The table below shows the amount of money in the account for a five year period, with all figures rounded to two decimal places.

Table 6.5.4 Amount of Money in Savings Account

Year	Initial Amount (\$)	Final Amount (\$)	Change (\$)	Growth Rate (Annual)
1	1000	1061.68	+61.68	+6.17%
2	1061.68	1127.16	+65.48	+6.17%
3	1127.16	1196.68	+69.52	+6.17%
4	1196.68	1270.49	+73.81	+6.17%
5	1270.49	1348.85	+78.36	+6.17%

Note that the growth factor for this function is $a = (1.005)^{12}$, so the growth rate

$$a - 1 = (1.005)^{12} - 1 \approx 6.17\%$$

is slightly higher than the interest rate (6%). □

In general, we can model compound interest using the following formula.

Formula 6.5.5 Compound Interest Formula. *Assume P dollars are invested in an account at an annual interest rate of $r\%$ that is compounded n times each year. The amount of money in the account after t years is*

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

The growth rate for this function is

$$\left(1 + \frac{r}{n}\right) - 1,$$

which measures the increase in the balance over one year as a percentage of the previous balance.

For interest that compounds *continuously*, we have the following formula.

Formula 6.5.6 Continuously Compounding Interest Formula. *Assume P dollars are invested in an account at an annual interest rate of $r\%$ that is compounded* continuously. *The amount of money in the account after t years is*

$$A(t) = P\exp(rt) = Pe^{rt} = P(e^r)^t = P\exp_{e^r}(t).$$

The growth rate for this function is $e^r - 1$*, which measures the increase in the balance over one year as a percentage of the previous balance.*

Remark 6.5.7 We can always express compounding interest as continuously compounding interest using the logarithm.

Example 6.5.8 Converting to Continuously Compounding Interest. As we saw in Example 6.5.3, we can model the balance of a savings account with principal investment of $P = \$1000$ at an annual interest rate of $r = 6\%$ compounded monthly with the function

$$A(t) = 1000(1.005)^{12t}.$$

Since the growth factor of this function is $a = 1.005^{12}$, we have

$$\ln(a) = \ln\left(1.005^{12}\right) = 12\ln(1.005)$$

and thus we can rewrite with base e as

$$\begin{aligned} A(t) &= 1000\exp(\ln(a)t) \\ &= 1000\exp(12\ln(1.005)t) \\ &= 1000e^{12\ln(1.005)t}. \end{aligned}$$

□

6.5.2 Exponential Decay

Example 6.5.9 Half-Life. The half-life of the radioactive isotope technetium-99m is 6 hours [1]. Assume we start with a sample of 64 grams of technetium-99m. The table below lists the amount at the beginning and the end of each six hour period over the course of a day.

Table 6.5.10 Amount of technetium-99m over 24 hours

Period	Initial Amount (grams)	Final Amount (grams)	Change (grams)	Percent Change
1	64	32	-32	-50%
2	32	16	-16	-50%
3	16	8	-8	-50%
4	8	4	-4	-50%

We can model the amount of the sample remaining as a function of the number of 6 hour periods that have passed using the exponential function

$$f(t) = 64\left(\frac{1}{2}\right)^t = 64\exp_{\frac{1}{2}}(t).$$

However, it would be more convenient to model the amount of the sample as a function of the number of hours that have passed. To do this, we first recognize that 1 hour is $1/6^{\text{th}}$ of a period. We can stretch the time scale by a factor of 6 to obtain the function

$$A(t) = f\left(\frac{t}{6}\right) = 64\left(\frac{1}{2}\right)^{\frac{t}{6}} = 64\left(\frac{1}{\sqrt[6]{2}}\right)^t = 64\exp_{\frac{1}{\sqrt[6]{2}}}(t)$$

that measures the amount of the technetium-99m sample remaining after t hours have elapsed. Using this function, we have recorded the amount remaining after each hour for the first six hour period in the table below.

Table 6.5.11 Amount of technetium-99m over 24 hours

Hour	Amount Remaining (grams)	Percent Change
1	$A(1) \approx 57.02$	$\frac{A(1)-A(0)}{A(0)} \approx 0.11\%$
2	$A(2) \approx 50.80$	$\frac{A(2)-A(1)}{A(1)} \approx 0.11\%$
3	$A(3) \approx 45.25$	$\frac{A(3)-A(2)}{A(2)} \approx 0.11\%$
4	$A(4) \approx 40.32$	$\frac{A(4)-A(3)}{A(3)} \approx 0.11\%$
5	$A(5) \approx 35.92$	$\frac{A(5)-A(4)}{A(4)} \approx 0.11\%$
6	$A(6) = 32$	$\frac{A(6)-A(5)}{A(5)} \approx 0.11\%$

$\square$

Appendix A

List of Symbols

Symbol	Description	Page
$x \in S$	Membership operator	1
$A = B$	Equality of sets	4
$A \subseteq B$	Subset operator	4
$A \subset B$	Proper subset operator	4
$\emptyset$	The empty set	4
$\mathbb{N}$	The set of natural numbers	5
$\mathbb{Z}$	The set of integers	5
$\mathbb{Q}$	The set of rational numbers	5
$\mathbb{R}$	The set of real numbers.	5
$f\colon A \to B$	Function from A to B	39
$f(x)$	Value of the function f at x	39
$f(A)$	Range of f	39
$f = g$	Equality of functions	42
$\deg f$	Degree of a polynomial	85
i	The imaginary unit	90
$\mathbb{C}$	The set of complex numbers	90
$f + g$	Sum of f and g	174
$f - g$	Difference of f and g	174
fg	Product of f and g	177
f/g	Ratio of f and g	177
$g \circ f$	Composition of g with f	198
e	Euler's number	221
$\ln$	The natural logarithm	226
$\exp_a$	Exponential function with base a	232
$\log_a$	The logarithm with base a	237

Appendix B

Graphs of Basic Functions

B.0.1 Polynomials

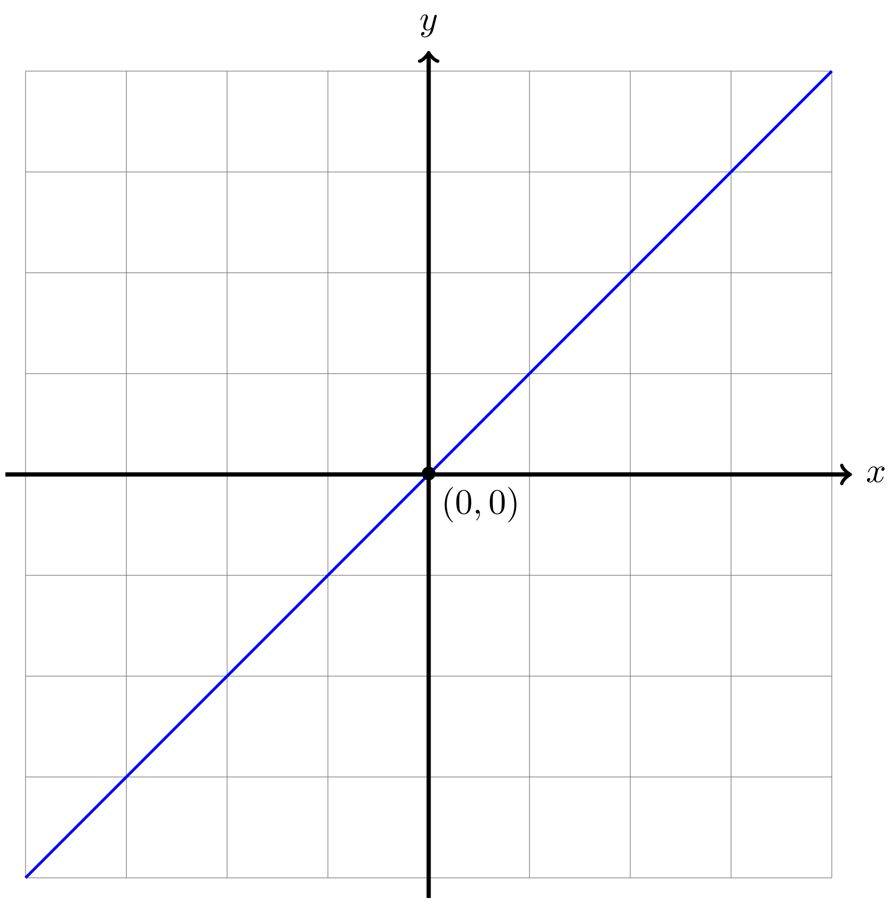


Figure B.0.1 The graph of $y = x$.

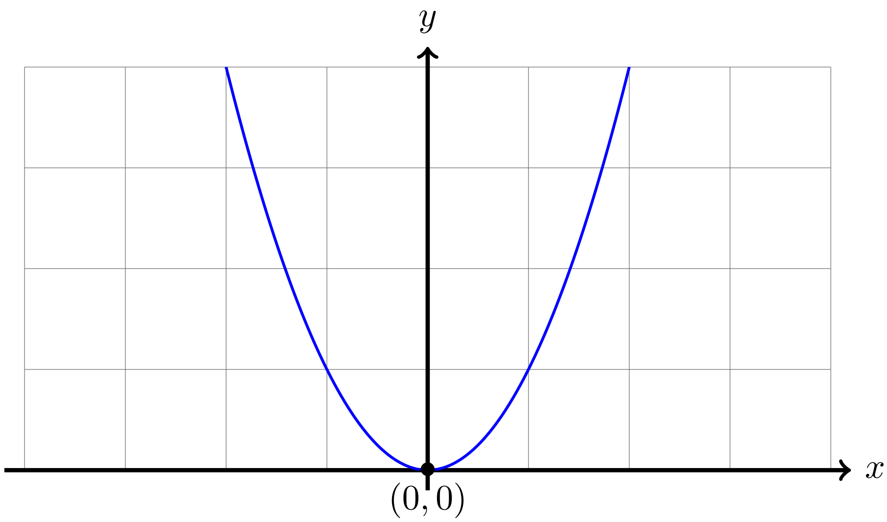


Figure B.0.2 The graph of $y = x^2$.

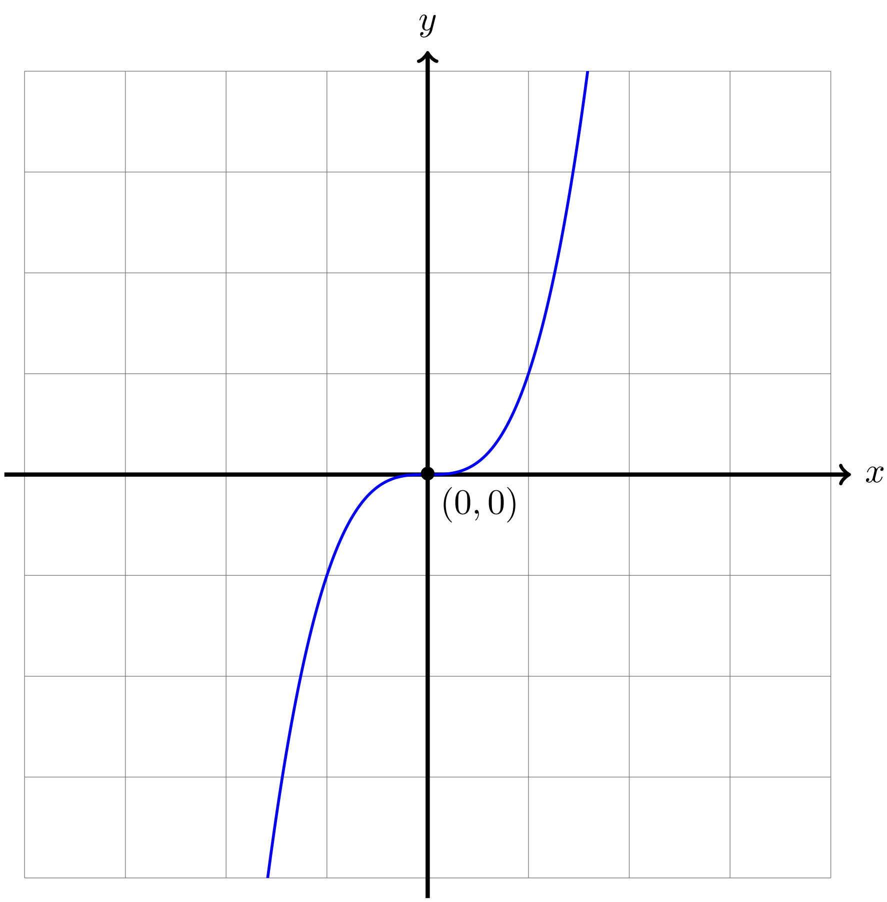


Figure B.0.3 The graph of $y = x^3$.

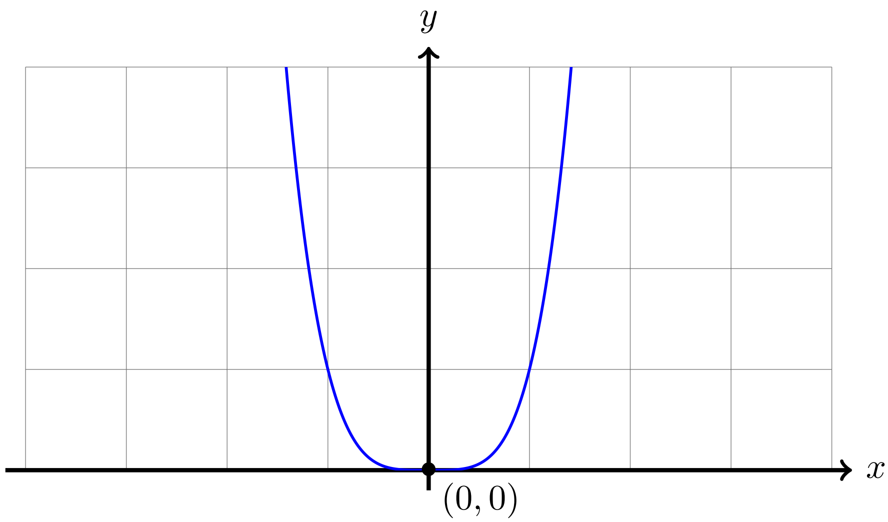


Figure B.0.4 The graph of $y = x^4$.

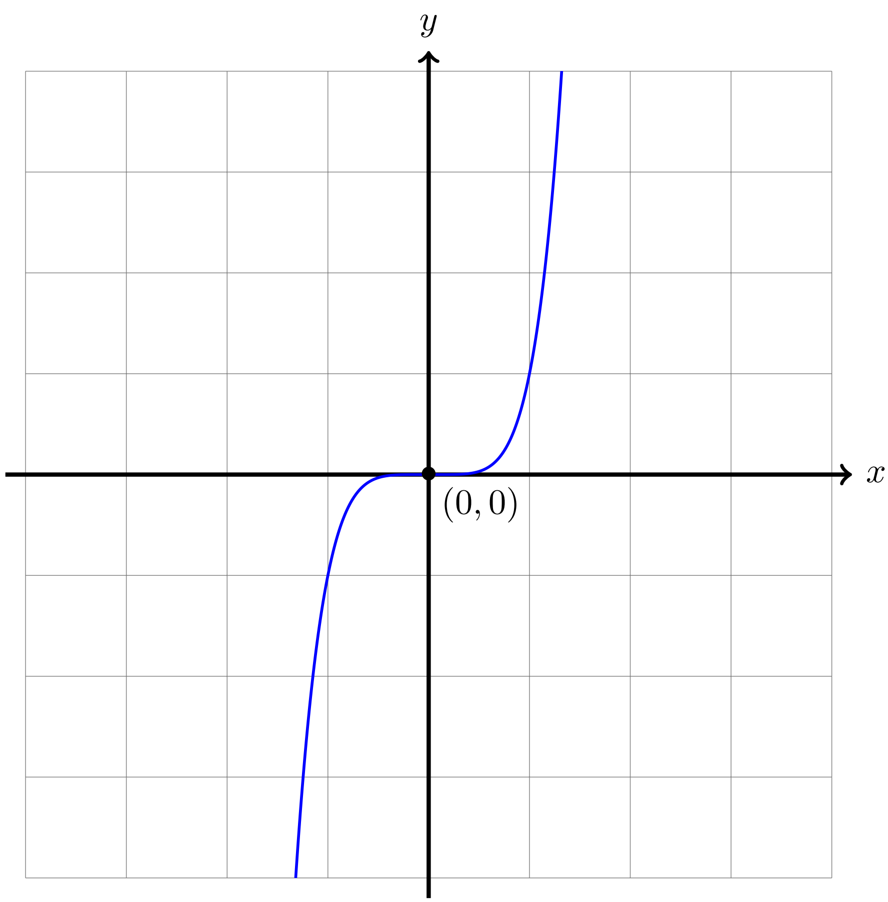


Figure B.0.5 The graph of $y = x^5$.

B.0.2 Roots

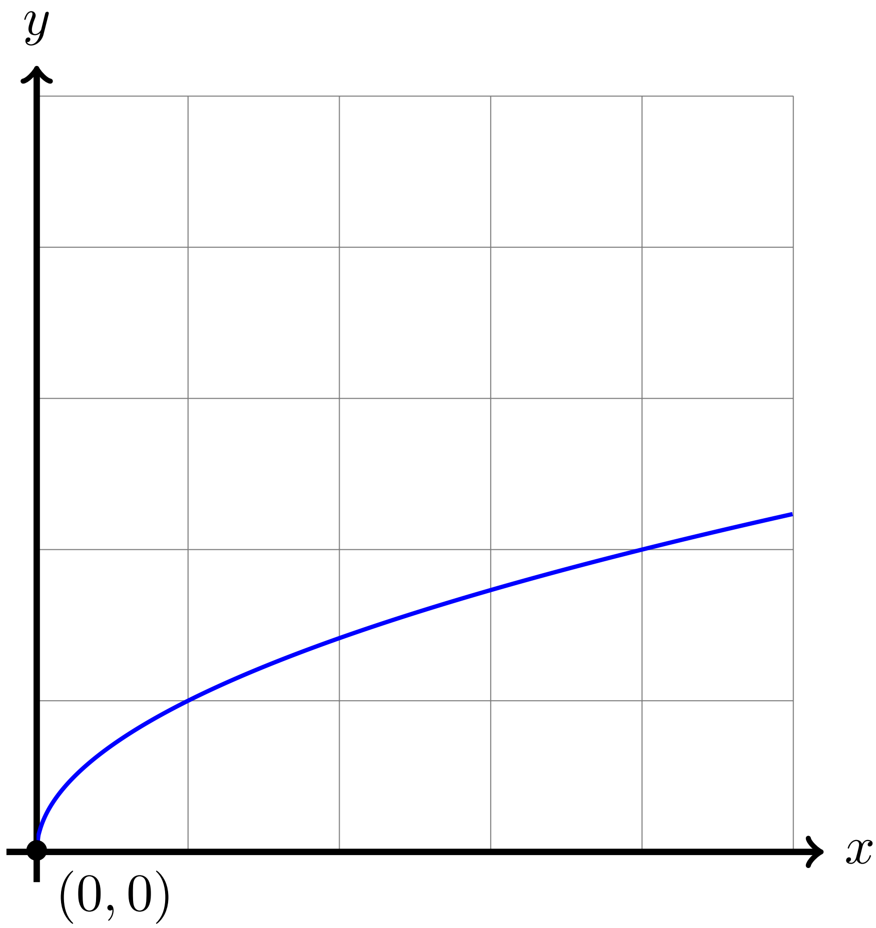


Figure B.0.6 The graph of $y = \sqrt{x}$.

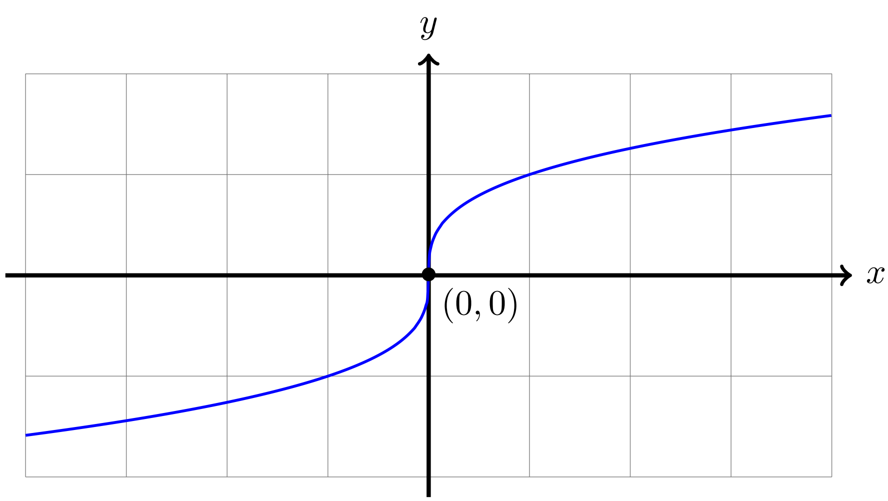


Figure B.0.7 The graph of $y = \sqrt[3]{x}$.

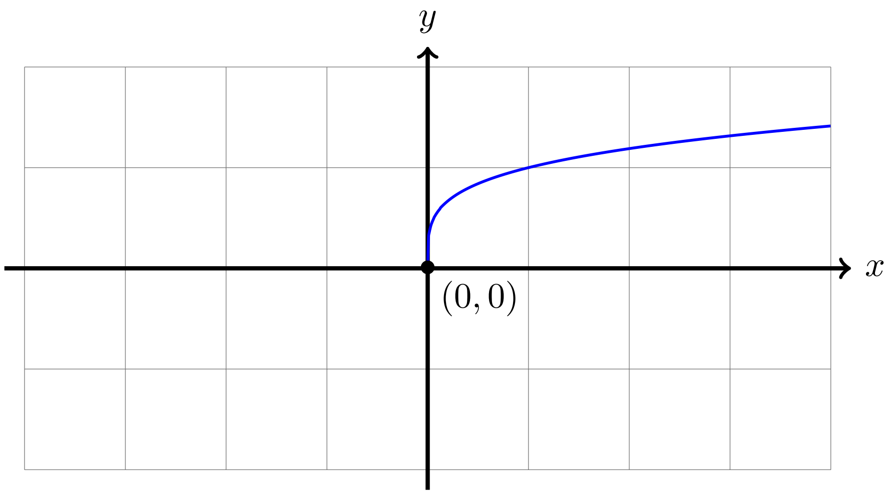


Figure B.0.8 The graph of $y = \sqrt[4]{x}$.

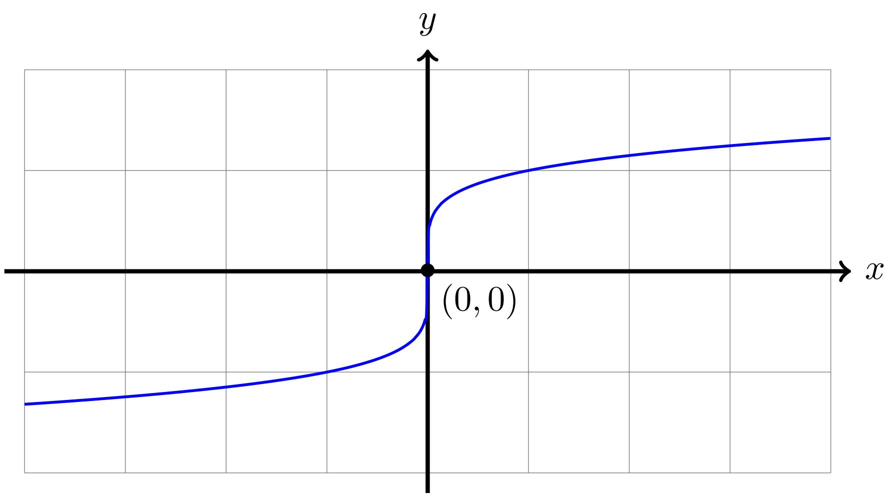


Figure B.0.9 The graph of $y = \sqrt[5]{x}$.

B.0.3 Rational Functions

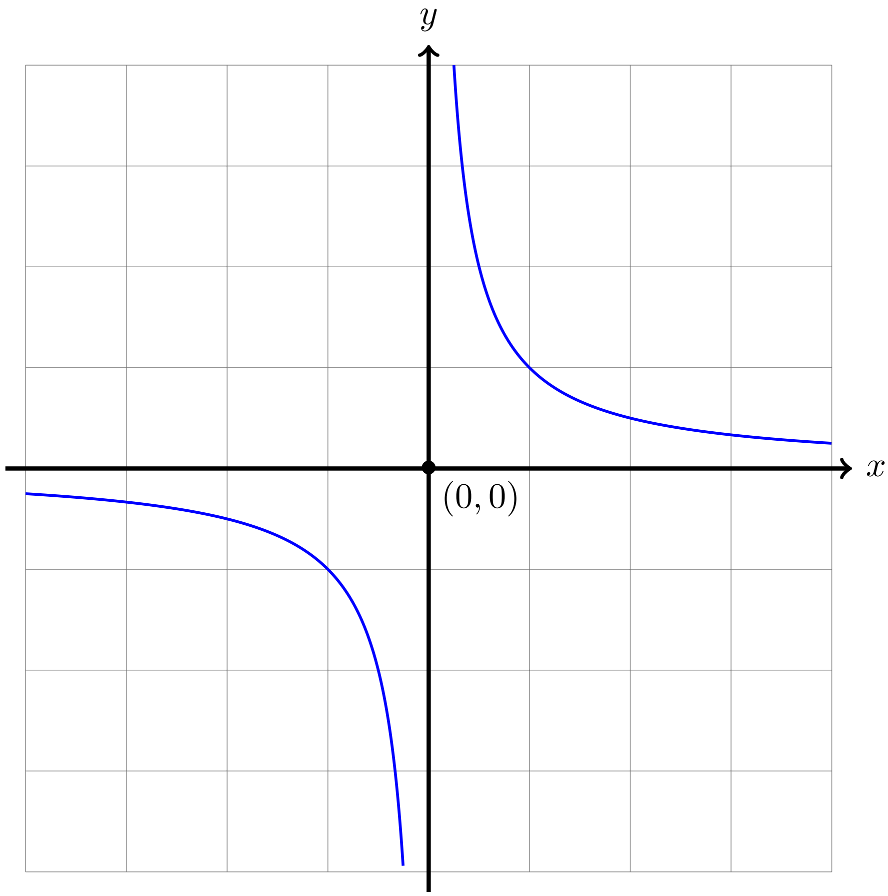


Figure B.0.10 The graph of $y = \frac{1}{x}$.

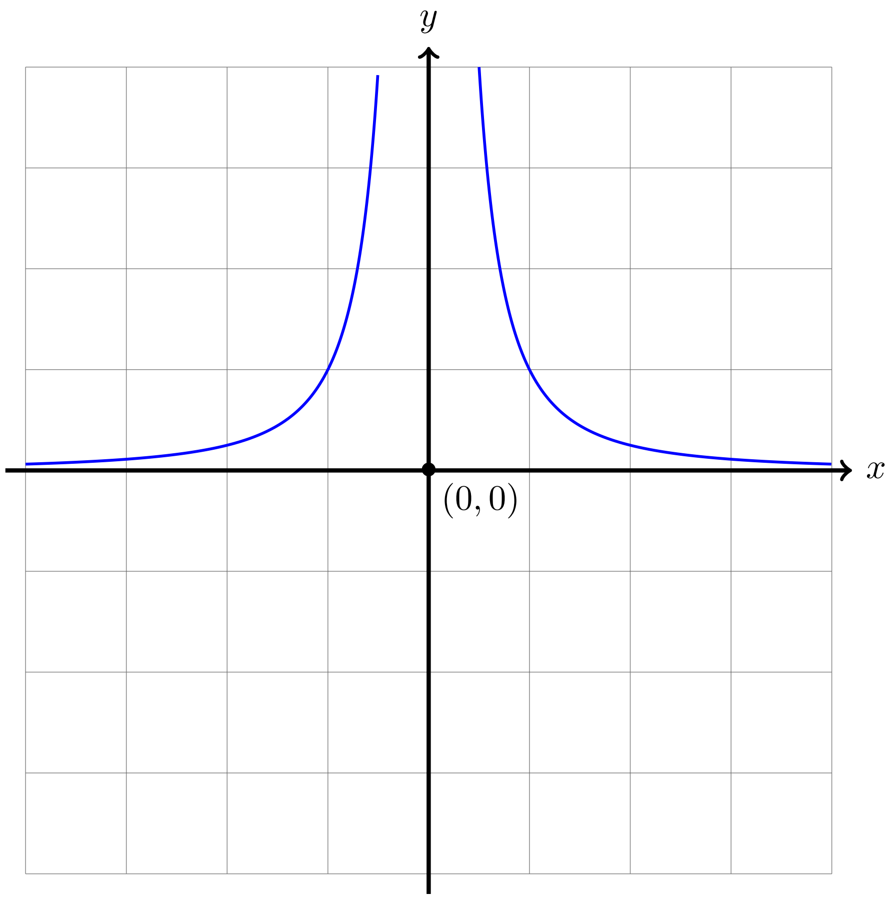


Figure B.0.11 The graph of $y = \frac{1}{x^2}$.

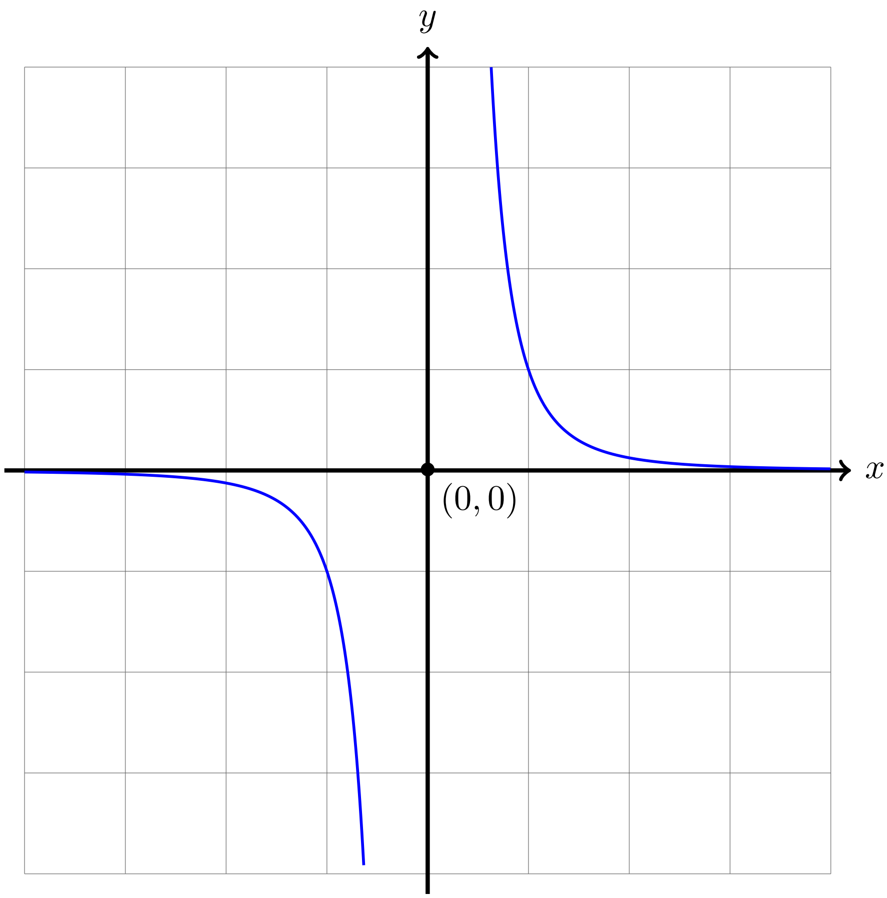


Figure B.0.12 The graph of $y = \frac{1}{x^3}$.

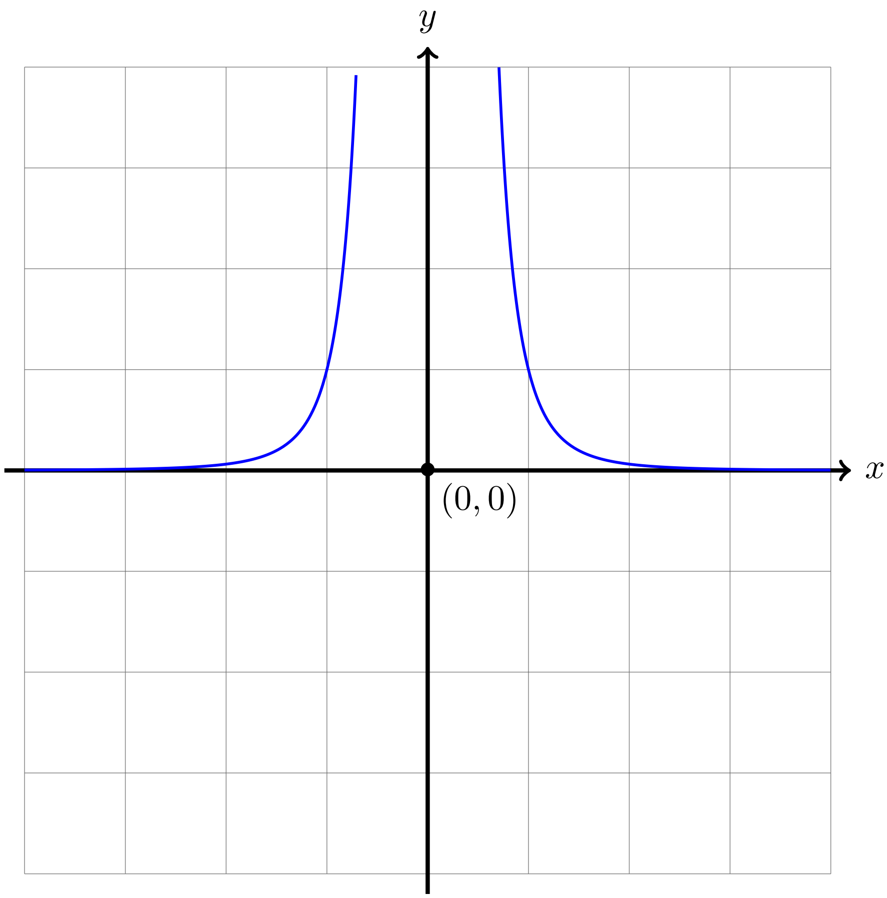


Figure B.0.13 The graph of $y = \frac{1}{x^4}$.

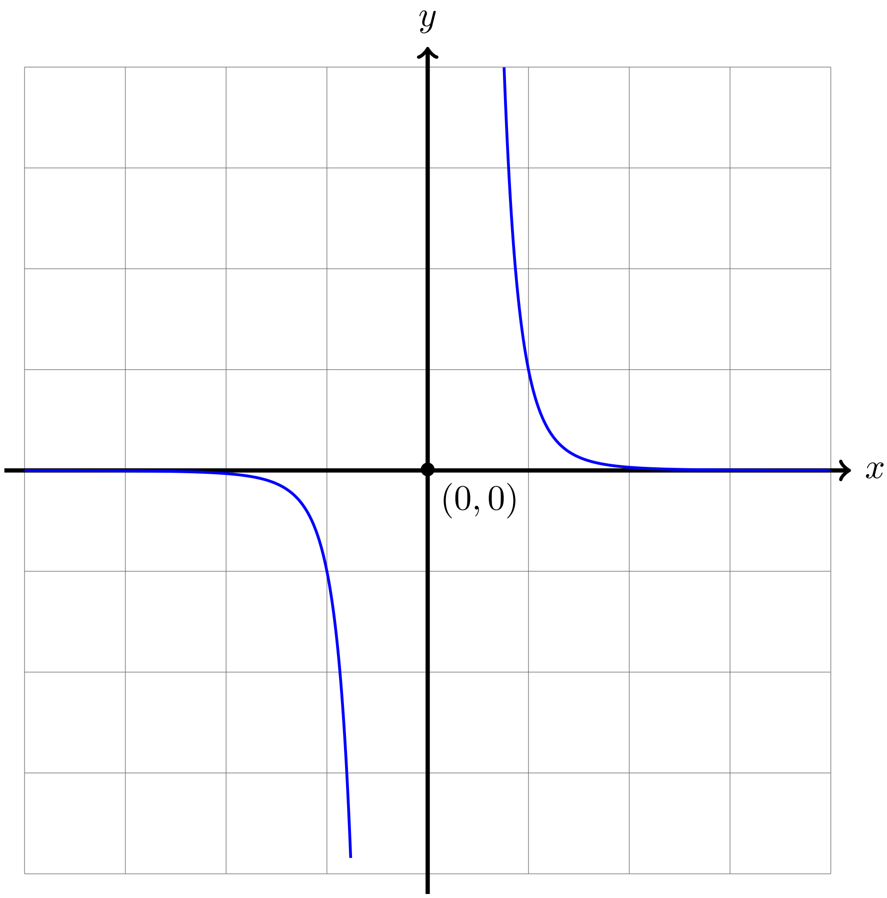


Figure B.0.14 The graph of $y = \frac{1}{x^5}$.

B.0.4 Miscellaneous Functions

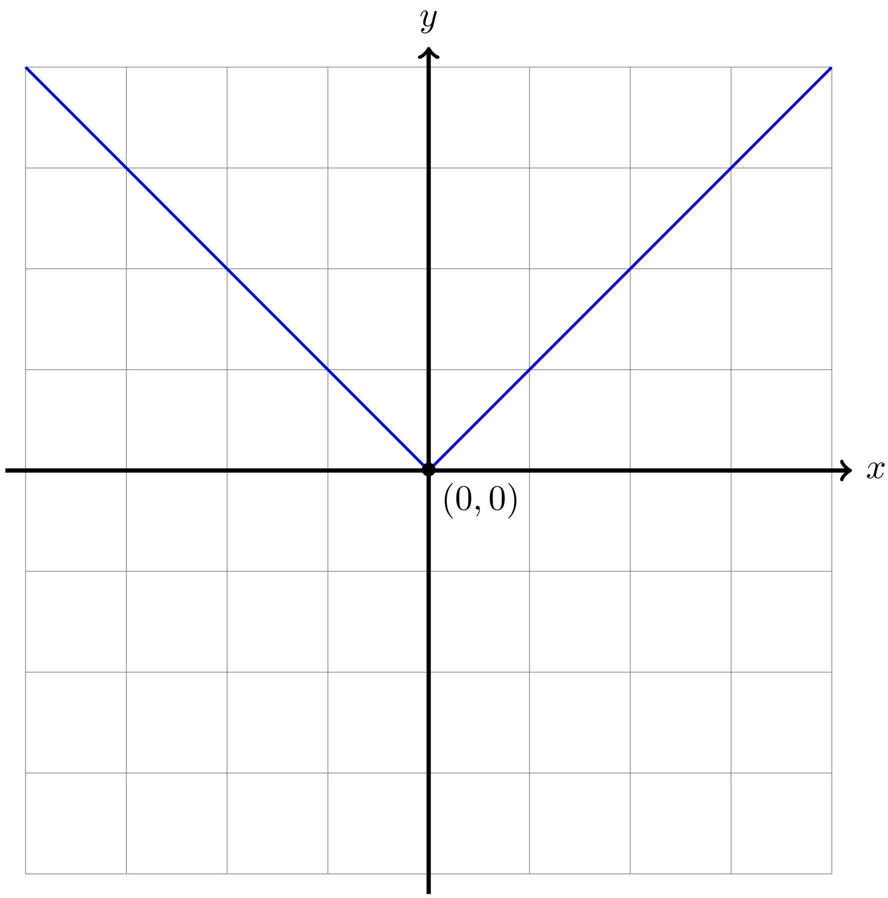


Figure B.0.15 The graph of $y = |x|$.

References

[1] Centers for Disease Control, *What is Radiation? Properties of Radioactive Isotopes*, https://www.cdc.gov/nceh/radiation/isotopes.html#halflife[1] (August 10, 2015) 2023-12-13

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Glossary

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Colophon

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