Convex Optimization

Lecture 2: Linear Programming

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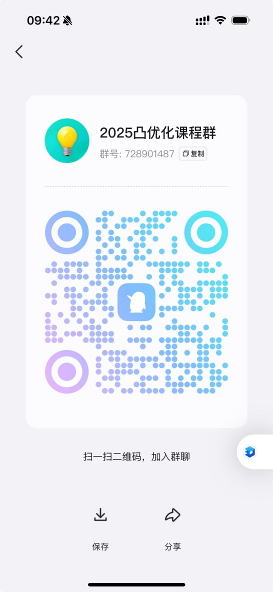
Outline

1. Linear Programming: the problem

2. Numerical Solution of LP

QQ Group



Linear Programming: the problem (1)

- Given following problem:
- An oil refinery produces two products: jet fuel and gasoline. The profit for the refinery is 0.10$ per barrel for jet fuel and 0.20$ per barrel for gasoline. The following conditions must be met.
  1. Only 10,000 barrels of crude oil are available for processing;
  2. Government contract requires at least 1,000 barrels of jet fuel;
  3. Private contract requires at least 2,000 barrels of gasoline;
  4. The delivery capacity of the truck fleet is 180,000 barrel-miles;
  5. The jet fuel is delivered to an airfield 10 miles from the refinery;
  6. The gasoline is transported 30 miles to the distributor.
- How to maximize the profit?
- x_1: quantity of jet fuel; x_2: quantity of gasoline
- Target: maximize $0.1 * x_1 + 0.2 * x_2$

Linear Programming: the problem (2)

- x_1: quantity of jet fuel; x_2: quantity of gasoline
- Target: maximize $0.1 * x_1 + 0.2 * x_2$
- Formularize the conditions:
  1. Only 10,000 barrels of crude oil are available for processing;
     - $x_1 + x_2 \leq 10000$
  2. Government contract requires at least 1,000 barrels of jet fuel;
     - $x_1 \geqslant 1000$
  3. Private contract requires at least 2,000 barrels of gasoline;
     - $x_2 \geqslant 2000$

Linear Programming: the problem (3)

- x_1: quantity of jet fuel; x_2: quantity of gasoline
- Target: maximize $0.1 * x_1 + 0.2 * x_2$
- Conditions:
  1. $x_1 + x_2 \leq 10000$
  2. $x_1 \geqslant 1000$
  3. $x_2 \geqslant 2000$
- Formularize the conditions:
  4. The delivery capacity of the truck fleet is 180,000 barrel-miles;
  5. The jet fuel is delivered to an airfield 10 miles from the refinery;
  6. The gasoline is transported 30 miles to the distributor;
  - $10 * x_1 + 30 * x_2 \leq 180000$

Linear Programming: the complete model

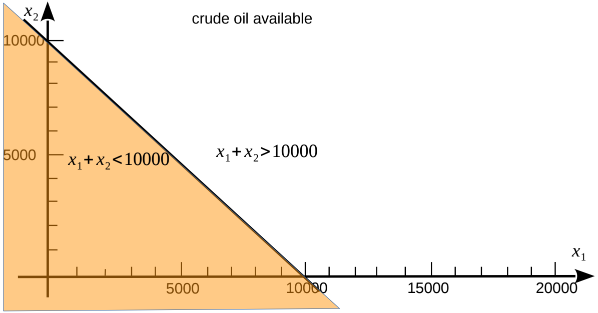
- x_1: quantity of jet fuel; x_2: quantity of gasoline
- Target: maximize $0.1 * x_1 + 0.2 * x_2$
- Conditions:
  1. $x_1 + x_2 \leq 10000$
  2. $x_1 \geqslant 1000$
  3. $x_2 \geqslant 2000$
  4. $10 * x_1 + 30 * x_2 \leq 180000$
- The formal linear programming form:

$$
\begin{aligned}
& \text{maximize } 0.1 * x_1 + 0.2 * x_2 \\
\text{subject to } & \begin{cases} x_1 + x_2 \leq 10000 \\ x_1 \geqslant 1000 \\ x_2 \geqslant 2000 \\ 10 * x_1 + 30 * x_2 \leq 180000 \end{cases}
\end{aligned}
\tag{1}
$$

Linear Programming: solve the problem with graph (1)

$$\text{maximize } 0.1 * x_1 + 0.2 * x_2$$

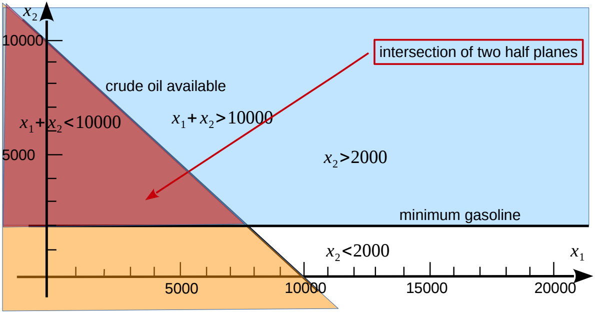
$$\text{subject to } \begin{cases} x_1 + x_2 \leq 10000 \\ x_1 \geqslant 1000 \\ x_2 \geqslant 2000 \\ 10 * x_1 + 30 * x_2 \leq 180000 \end{cases} \tag{1}$$



Linear Programming: solve the problem with graph (2)

$$\text{Max. } 0.1 * x_1 + 0.2 * x_2$$

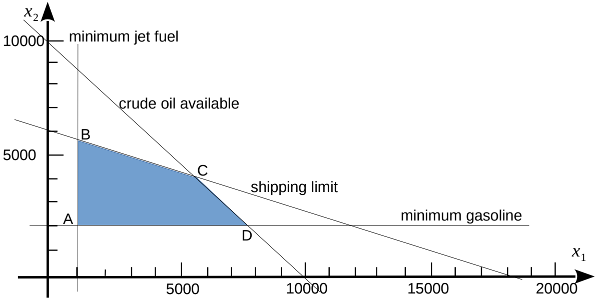
$$\text{subject to} \begin{cases} x_1 + x_2 \leq 10000 \\ x_1 \geqslant 1000 \\ x_2 \geqslant 2000 \\ 10 * x_1 + 30 * x_2 \leq 180000 \end{cases} \tag{1}$$



Linear Programming: solve the problem with graph (3)

$$\text{Max. } 0.1 * x_1 + 0.2 * x_2$$

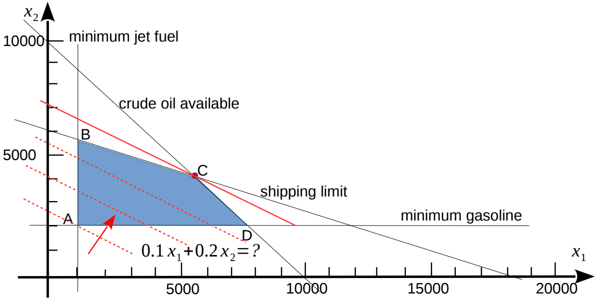
$$\text{subject to} \begin{cases} x_1 + x_2 \leq 10000 \\ x_1 \geqslant 1000 \\ x_2 \geqslant 2000 \\ 10 * x_1 + 30 * x_2 \leq 180000 \end{cases} \quad (1)$$



Linear Programming: solve the problem with graph (4)

$$\text{Max. } 0.1 * x_1 + 0.2 * x_2$$

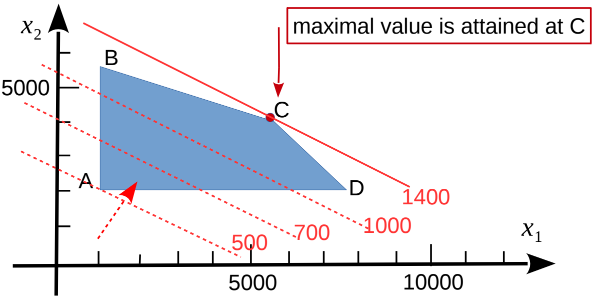
$$\text{subject to } \begin{cases} x_1 + x_2 \leq 10000 \\ x_1 \geqslant 1000 \\ x_2 \geqslant 2000 \\ 10 * x_1 + 30 * x_2 \leq 180000 \end{cases} \tag{1}$$



Linear Programming: solve the problem with graph (5)

$$\text{Max. } 0.1 * x_1 + 0.2 * x_2$$

$$\text{subject to} \begin{cases} x_1 + x_2 \leq 10000 \\ x_1 \geqslant 1000 \\ x_2 \geqslant 2000 \\ 10 * x_1 + 30 * x_2 \leq 180000 \end{cases} \tag{1}$$



Observation (1): problem is linear

$$\text{Max. } 0.1 * x_1 + 0.2 * x_2$$
$$\text{subject to} \begin{cases} x_1 + x_2 \leq 10000 \\ x_1 \geqslant 1000 \\ x_2 \geqslant 2000 \\ 10 * x_1 + 30 * x_2 \leq 180000 \end{cases} \quad (1)$$

- Two parts: target function and constraints
  1. All the constraints are linear inequations or equations (or both)
  2. The target function is linear too!
- This class of problems are called **linear programming**
- Otherwise, it is called **non-linear programming**

Observation (2): valid set forms a convex region (1)

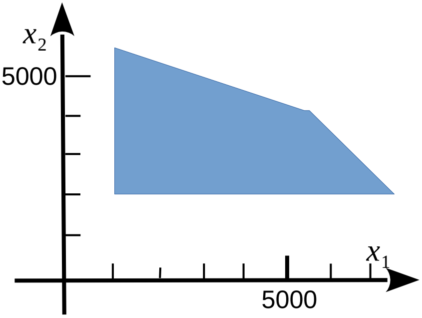
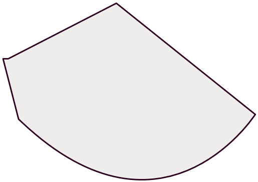
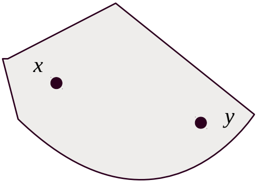


Figure: feasible set of Eqn. 1

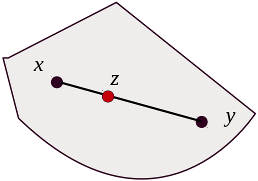
Convex set: the concept (2)



(a) Given a shape

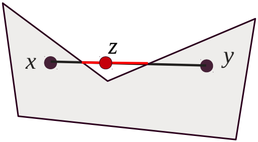


(b) Select x, y randomly



$z = \alpha \cdot x + (1 - \alpha) \cdot y$

(c) Linear combination

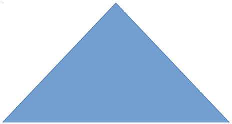


$z = \alpha \cdot x + (1 - \alpha) \cdot y$

(d) non-convex case

Convex set: verify your understanding

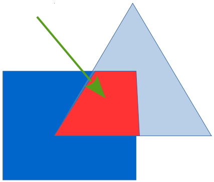
- Which is convex, which is not?



(a) Triangle

(b) Rectangle

(c) Pentagon

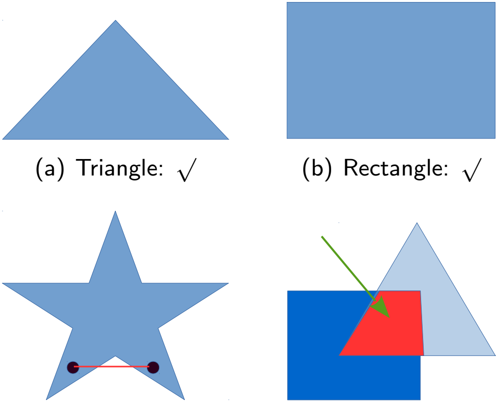


(d) Intersection

Figure: 2D shapes

Convex set: verify your understanding

- Which is convex, which is not?



(a) Triangle: $\surd$

(b) Rectangle: $\surd$

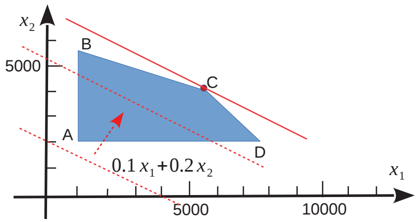
(c) Pentagon: $\times$

(d) Intersection: $\surd$

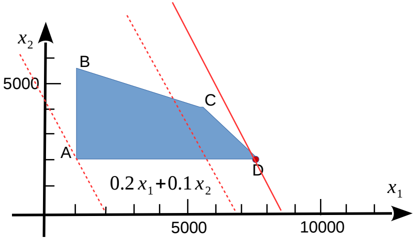
Figure: 2D shapes

Observation (3): extremal value is attained at vertices

- Let's fix the feasible set, and try different target functions



(a) target: $0.1x_1 + 0.2x_2$



(b) target: $0.2x_1 + 0.1x_2$

1. The optimal solution changes, but they are all located in vertices
2. It is provable that extremal value is ONLY attained at vertices
3. Solving the problem is to test different vertices for the best answers

Simplex: searching exremal value over vertices

- Solution can be found by searching over the vertices
- This is the idea of classic **Simplex** method

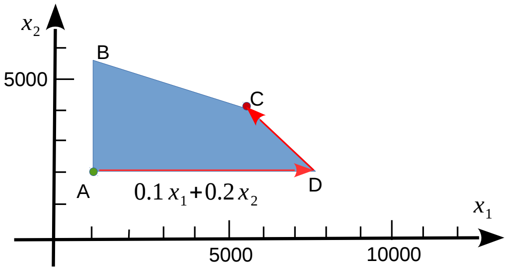
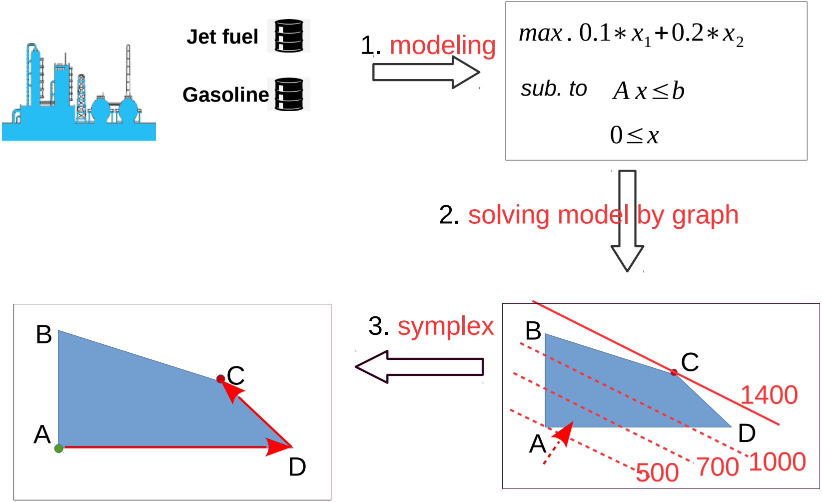


Figure: Nondegenerate jumping from one vertex to another for better solution

Review on the whole flow



Outline

Scientists behind Linear Programming

(a) L. Kantorovich (1912 - 1986)

(b) G. Dantzig (1914 - 2005)

(c) J. V. Neumann (1903 - 1957)

(d) N. Karmarkar (1956 -)

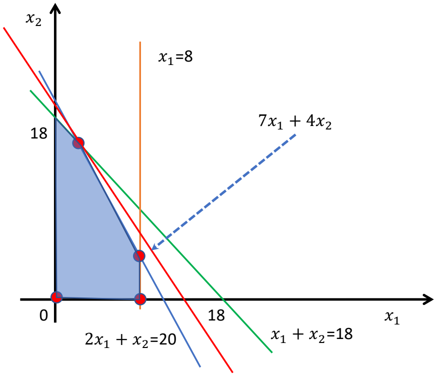
- In the year of 1939, L. Kantorovich proposed LP
- In the year of 1947, G. Danzig invented "Simplex" method
- In the year of 1944, John V. Neumann proposed game theory
- In 1984, Narendra Karmarkar proposed "inner-point" method of LP
- "Inner-point" guarantees to solve LP in polynomial time complexity

Solve LP by Linear Transformations (1)

- In our opening example, there are only two unknowns x_1 and x_2
- The problem can be easily solved by graph
- In practice, there could be hundreds of unknowns
- Hundreds of linear constraints
- It is unrealistic to solve it by graph

Solve LP by Linear Transformations (2)

$$\text{Max. } 7x_1 + 4x_2$$
$$\text{subject to } \begin{cases} 2x_1 + x_2 \leq 20 \\ x_1 + x_2 \leq 18 \\ x_1 \leq 8 \\ x_1, x_2 \geq 0 \end{cases} \qquad (1)$$



- First of all, it is not a linear equation problem
- We cannot solve it as the way of $Ax = b$
- We introduce three slack variables for three inequations s_1, s_2 and s_3
- To convert them into equations

Solve LP by Linear Transformations (3)

- We introduce three slack variables for three inequations s_1, s_2 and s_3

$$
\text{Max. } 7x_1 + 4x_2
$$
$$
\text{sub. to} \begin{cases} 2x_1 + x_2 + s_1 & = 20 \\ x_1 + x_2 + \quad +s_2 & = 18 \\ x_1 + \qquad\qquad +s_3 & = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{2}
$$

- We introduce three slack variables for three inequations s_1, s_2 and s_3
- Even for above problem, we cannot solve it by Ax=b
- There are five variables and three equations
- We are going to address it in another way

Solve LP by Linear Transformations (4)

$$
\begin{aligned}
&\text{Max. } 7x_1 + 4x_2 \\
\text{sub. to } &\begin{cases}
2x_1 + x_2 + s_1 \qquad = 20 \\
x_1 + x_2 + \quad + s_2 \quad = 18 \\
x_1 + \qquad\qquad + s_3 = 8 \\
x_1, x_2, s_1, s_2, s_3 \geq 0
\end{cases}
\end{aligned}
\tag{2}
$$

- Given initially $x_1 = 0$, $x_2 = 0$
- If we want to maximize $7x_1 + 4x_2$
- Increasing x_1 is more profitable as its coefficient 7 is greater

Solve LP by Linear Transformations (5)

- If we want to maximize $7x_1 + 4x_2$
- Increasing x_1 is more profitable as its coefficient 7 is greater
- How much we could increase x_1??

$$\text{Max. } 7x_1 + 4x_2$$
$$\text{sub. to} \begin{cases} 2x_1 + x_2 + s_1 & = 20 \\ x_1 + x_2 + \quad + s_2 & = 18 \\ x_1 + \qquad\qquad + s_3 & = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{2}$$

- x_1 is bounded by three constraints
- Let's look at the three constraints

Solve LP by Linear Transformations (6)

$$\text{Max. } 7x_1 + 4x_2$$
$$\text{sub. to} \begin{cases} 2x_1 + x_2 + s_1 & = 20 \\ x_1 + x_2 + \quad + s_2 & = 18 \\ x_1 + \qquad\qquad + s_3 & = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{2}$$

- x_1 is bounded by three constraints
- Let's look at the three constraints

1. For the 1st constraint, $2x_1$ should be no greater than 20, $x_1 \leq 10$
2. For the 2nd constraint, x_1 should be no greater than 18, $x_1 \leq 18$
3. For the 3rd constraint, x_1 should be no greater than 8, $x_1 \leq 8$

Solve LP by Linear Transformations (7)

$$\text{Max. } 7x_1 + 4x_2$$
$$\text{sub. to } \begin{cases} 2x_1 + x_2 + s_1 & = 20 \\ x_1 + x_2 + \quad +s_2 & = 18 \\ x_1 + \qquad\quad +s_3 & = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{2}$$

1. For the 1st constraint, $2x_1$ should be no greater than 20, $x_1 \leq 10$
2. For the 2nd constraint, x_1 should be no greater than 18, $x_1 \leq 18$
3. For the 3rd constraint, x_1 should be no greater than 8, $x_1 \leq 8$

- We can see x_1 should be increased to 8 at most

Solve LP by Linear Transformations (8)

- With the 3rd constraint, we have $x_1 = 8 - s_3$
- We plug this into the target and the other constraints

$$\text{Max.}\, 7(8 - s_3) + 4x_2$$

$$\text{sub. to} \begin{cases} 2(8 - s_3) + x_2 + s_1 \quad = 20 \\ (8 - s_3) + x_2 + \quad + s_2 = 18 \\ x_1 + \qquad\qquad + s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

$$\Downarrow$$

$$\text{Max.}\, 56 + 4x_2 - 7s_3$$

$$\text{sub. to} \begin{cases} \quad x_2 + s_1 \quad - 2s_3 = 4 \\ \quad x_2 + \quad + s_2 - s_3 = 10 \\ x_1 + \qquad\qquad + s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{3}$$

Solve LP by Linear Transformations (9)

- With the 3rd constraint, we have $x_1 = 8 - s_3$
- We plug this into the target and the other constraints

$$\text{Max. } 56 + 4x_2 - 7s_3$$
$$\text{sub. to} \begin{cases} \quad x_2 + s_1 \quad - 2s_3 = 4 \\ \quad x_2 + \quad s_2 - s_3 = 10 \\ x_1 + \qquad\qquad s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{3}$$

- We look at the target $56 + 4x_2 - 7s_3$
- Increasing s_3 is not helpful
- Increasing x_2 could make the target larger
- How much we can increase about x_2?

Solve LP by Linear Transformations (10)

- We look at the target $56 + 4x_2 - 7s_3$
- Increasing x_2 could make the target larger
- How much we can increase about x_2?
- We look at the three constraints

$$
\begin{aligned}
&\text{Max. } 56 + 4x_2 - 7s_3 \\
\text{sub. to } &\begin{cases}
\quad x_2 + s_1 \quad - 2s_3 = 4 \\
\quad x_2 + \quad + s_2 - s_3 = 10 \\
x_1 + \qquad\qquad s_3 = 8 \\
x_1, x_2, s_1, s_2, s_3 \geq 0
\end{cases}
\end{aligned}
\tag{3}
$$

1. For the 1st constraint, $x_2 \leq 4$
2. For the 2nd constraint, $x_2 \leq 10$
3. For the 3rd constraint, x_2 is not bounded

Solve LP by Linear Transformations (11)

1. For the 1st constraint, $x_2 \leq 4$
2. For the 2nd constraint, $x_2 \leq 10$
3. For the 3rd constraint, x_2 is not bounded
4. We should consider the 1st constraint
5. Plug $x_2 = 4 - s_1 + 2s_3$ into the target and the other constraints

$$\begin{aligned} &\text{Max.}\ 56 + 4x_2 - 7s_3 \\ \text{sub. to} &\begin{cases} x_2 + s_1 - 2s_3 = 4 \\ x_2 + \ \ + s_2 - s_3 = 10 \\ x_1 + \ \ s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \end{aligned} \tag{3}$$

Solve LP by Linear Transformations (12)

1. For the 1st constraint, $x_2 \leq 4$
2. For the 2nd constraint, $x_2 \leq 10$
3. For the 3rd constraint, x_2 is not bounded

- We should consider the 1st constraint
- Plug $x_2 = 4 - s_1 + 2s_3$ into the target and the other constraints

$$\text{Max.}\ 56 + 4x_2 - 7s_3$$
$$\text{sub. to} \begin{cases} \quad x_2 + s_1 \quad - 2s_3 = 4 \\ \quad x_2 + \quad + s_2 - s_3 = 10 \\ x_1 + \qquad\qquad s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{3}$$

Solve LP by Linear Transformations (13)

- We should consider the 1st constraint
- Plug $x_2 = 4 - s_1 + 2s_3$ into the target and the other constraints

$$\text{Max. } 56 + 4(4 - s_1 + 2s_3) - 7s_3$$
$$\text{sub. to} \begin{cases} x_2 + s_1 \quad - 2s_3 = 4 \\ (4 - s_1 + 2s_3) + s_2 - s_3 = 10 \\ x_1 + \quad s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

$$\Downarrow$$

$$\text{Max. } 72 - 4s_1 + s_3$$
$$\text{sub. to} \begin{cases} x_2 + s_1 \quad - 2s_3 = 4 \\ -s_1 + s_2 + s_3 = 6 \\ x_1 + \quad s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{4}$$

Solve LP by Linear Transformations (14)

- We should consider the 1st constraint
- Plug $x_2 = 4 - s_1 + 2s_3$ into the target and the other constraints

$$
\begin{aligned}
& \text{Max. } 72 - 4s_1 + s_3 \\
\text{sub. to } & \begin{cases}
x_2 + s_1 - 2s_3 = 4 \\
-s_1 + s_2 + s_3 = 6 \\
x_1 + s_3 = 8 \\
x_1, x_2, s_1, s_2, s_3 \geq 0
\end{cases}
\end{aligned}
\tag{4}
$$

- Now let's look at target $72 - 4s_1 + s_3$ again
- Increasing s_1 do nothing good for us
- Increasing s_3 will make the target larger

Solve LP by Linear Transformations (15)

- Now let's look at target $72 - 4s_1 + s_3$ again
- Increasing s_1 do nothing good for us
- Increasing s_3 will make the target larger

$$\text{Max.}\, 72 - 4s_1 + s_3$$
$$\text{sub. to} \begin{cases} x_2 + s_1 \quad - 2s_3 = 4 \\ -s_1 + s_2 + s_3 = 6 \\ x_1 + \quad s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{4}$$

1. For the 1st constraint, $-2s_3 \leq 4$ is not bounded
2. For the 2nd constraint, $s_3 \leq 6$
3. For the 3rd constraint, $s_3 \leq 8$

Solve LP by Linear Transformations (16)

1. For the 1st constraint, $-2s_3 \leq 4$ is not bounded
2. For the 2nd constraint, $s_3 \leq 6$
3. For the 3rd constraint, $s_3 \leq 8$

- Therefore, we should consider the 2nd constraint $s_3 = 6 + s_1 - s_2$

$$\text{Max. } 72 - 4s_1 + s_3$$
$$\text{sub. to} \begin{cases} x_2 + s_1 \quad - 2s_3 = 4 \\ -s_1 + s_2 + s_3 = 6 \\ x_1 + \qquad s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{4}$$

Solve LP by Linear Transformations (17)

- Therefore, we should consider the 2nd constraint $s_3 = 6 + s_1 - s_2$
- Plug $s_3 = 6 + s_1 - s_2$ into the target and the other constraints

$$\text{Max. } 72 - 4s_1 + (6 + s_1 - s_2)$$

$$\text{sub. to} \begin{cases} x_2 + s_1 - 2(6 + s_1 - s_2) = 4 \\ -s_1 + s_2 + s_3 = 6 \\ x_1 + (6 + s_1 - s_2) = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

$$\Downarrow$$

$$\text{Max. } 78 - 3s_1 - s_2$$

$$\text{sub. to} \begin{cases} x_2 - s_1 + 2s_2 = 16 \\ -s_1 + s_2 + s_3 = 6 \\ x_1 + s_1 - s_2 = 2 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{5}$$

Solve LP by Linear Transformations (18)

$$
\text{Max. } 78 - 3s_1 - s_2 \\
\text{sub. to} \begin{cases} x_2 - s_1 + 2s_2 = 16 \\ -s_1 + s_2 + s_3 = 6 \\ x_1 + s_1 - s_2 = 2 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{5}
$$

- We look at the target $78 - 3s_1 - s_2$ again
- Given $s_1 \geq 0$ and $s_2 \geq 0$
- There is no way to improve the target further

Solve LP by Linear Transformations (19)

$$\text{Max. } 78 - 3s_1 - s_2$$
$$\text{sub. to } \begin{cases} \quad x_2 - s_1 + 2s_2 \;= 16 \\ \quad -s_1 + s_2 + s_3 = 6 \\ x_1 + \quad s_1 - s_2 \quad = 2 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{5}$$

- The maximum is achieved when s_1 and s_2 is 0
- Plug $s_1 = 0$ and $s_2 = 0$ into the first two constraints
- We have $x_1 = 2$, $x_2 = 16$, and $s_3 = 6$

$$\text{Max. } 7x_1 + 4x_2$$
$$\text{sub. to } \begin{cases} 2x_1 + x_2 + s_1 \qquad = 20 \\ x_1 + x_2 + \quad + s_2 \;= 18 \\ x_1 + \qquad\qquad s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{2}$$