Convex Optimization

Lecture 13: Integer Programming (IP)

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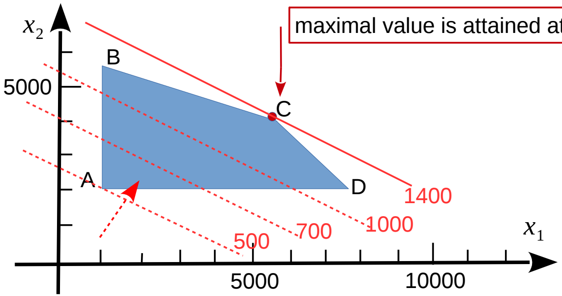
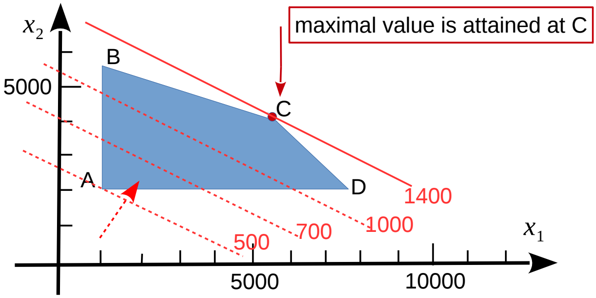
Outline

Opening Discussion (1)



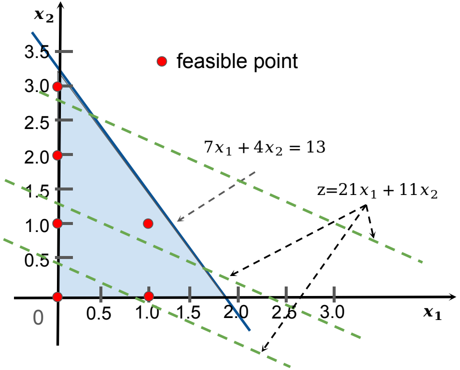
- For the Linear programming problems we handle
- The variables are all in R^+
- In practice, we may require the variables are all integers
- In the refinery problem, the crude oil and the refined oil are counted in barrels

Opening Discussion (2)



- An intuitive solution is round-off the real solution
- On the one hand, the solution may not be feasible
- On the other hand, simply try over these round-off solutions could be even more complicated

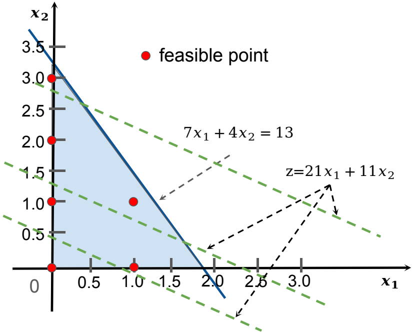
Example-1 (1)



$$\text{Max.} \quad 21x_1 + 11x_2$$

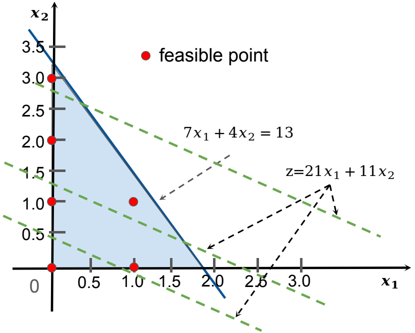
$$s.t. \begin{cases} 7x_1 + 4x_2 \leq 13 \\ x_1, \ x_2 \geq 0, x_1, x_2 \in I \end{cases} \tag{1}$$

Example-1 (2)



- The feasible region consists of the following set of points
- $S = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1)\}$
- Unlike LP, it is not convex set
- By trying all feasible points, we know the maximum is achieved when $x_1 = 0, x_2 = 3$

Example-1 (3)



- Feasible points $S = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1)\}$
- By trying all feasible points, we know the maximum is achieved when $x_1 = 0, x_2 = 3$
- In real cases, there could be millions of feasible points
- It is impossible to enumerate all feasible points

Example-2 (1)

$$
\begin{aligned}
&\text{Max.} \quad 4x_1 + x_2 \\
&s.t. \begin{cases} 2x_1 + x_2 \leq 5 \\ 2x_1 + 3x_2 = 5 \\ x_1,\ x_2 \geq 0, x_1, x_2 \in I \end{cases}
\end{aligned} \tag{2}
$$

- The optimal solution to the LP relaxation for this IP is $z = 10, x_1 = \frac{5}{2}, x_2 = 0$
- Rounding off this solution we get either $x_1 = 2, x_2 = 0$ or $x_1 = 3, x_2 = 0$
- Neither of them are feasible
- In general IP problems are much harder than LP

Outline

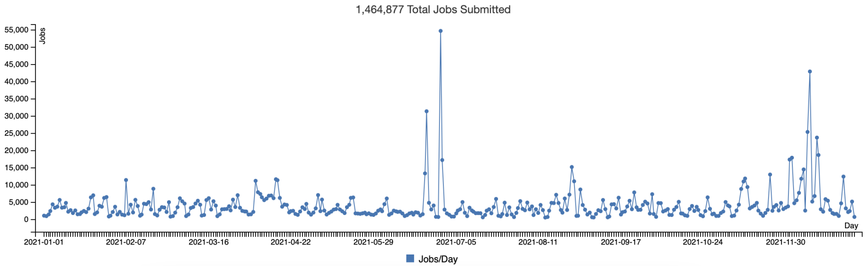


Figure: Statistics on 2021-01-01 - 2021-12-31 from NEOS[1].

- According to NEOS[2], *1,464,877* optimization problems are submitted
- Among them, there are *581,752* IP problems, takes up 40%

[1] https://neos-server.org/neos/report.html
[2] https://neos-server.org/neos/

Problem-1: Capital Budgeting (1)

- **Problem**: Suppose we wish to invest \$14,000. We have identified four investment opportunities. Investment-1 requires an investment of \$5,000 and has a present value (a time-discounted value) of \$8,000; investment-2 requires \$7,000 and has a value of \$11,000; investment-3 requires \$4,000 and has a value of \$6,000; and investment-4 requires \$3,000 and has a value of \$4,000. Into which investments should we place our money so as to maximize our total present value?
- Given variable x_1, x_2, x_3, and $x_4 \in \{0, 1\}$
- '0' indicates no investment on a product
- '1' indicates the investment on a product

Problem-1: Capital Budgeting (2)

1. Investment-1 requires \$5,000 and has a value of \$8,000
2. Investment-2 requires \$7,000 and has a value of \$11,000
3. Investment-3 requires \$4,000 and has a value of \$6,000
4. Investment-4 requires \$3,000 and has a value of \$4,000

- Given variable x_1, x_2, x_3, and $x_4 \in \{0, 1\}$
- The objective is to maximize $8x_1 + 11x_2 + 6x_3 + 4x_4$
- It is subjective to $5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$

Problem-1: Capital Budgeting (3)

- It is possible that we may add more constraints on this problem

1. We can only make two investments, that means $x_1 + x_2 + x_3 + x_4 \leq 2$
2. If investment-2 is made, investment-4 must be made $x_2 - x_4 \leq 0$
3. If investment-1 is made, investment-3 cannot be made $x_1 + x_3 \leq 1$

On-class exercise-1 (1)

- **Problem**: As the leader of an oil exploration drilling venture, you must determine the best selection of 5 out of 10 possible sites. Label the sites, $s_1, s_2, \cdots, s_{10}$. and the expected profits associated with each as $p_1, p_2, \cdots, p_{10}$. If site s_2 is explored, then site s_3 must also be explored. Furthermore, regional development restrictions are such that Exploring sites s_1 and s_7 will prevent you from exploring site s_8. Exploring sites s_3 or s_4 will prevent you from exploring site s_5.

Do it in 5 minutes ...

On-class exercise-1, the answer (2)

$$
\text{Max.} \quad \sum_{i=1}^{10} x_i p_i
$$

$$
\text{s. t.} \begin{cases} x_1 + x_2 + \cdots + x_{10} \leq 5 \\ x_2 - x_3 \leq 0 \\ x_1 + x_7 + x_8 \leq 2 \\ x_3 + x_5 \leq 1 \\ x_4 + x_5 \leq 1 \\ x_1, x_2, \cdots, x_{10} \in \{0, 1\} \end{cases} \tag{3}
$$

Problem-2: Multi-period Capital Budgeting (1)

- **Problem**: We wish to invest \$14,000, \$12,000, and \$15,000 in each month of the next quarter. We have identified four investment opportunities. Investment-1 requires an investment of \$5,000, \$8,000, and \$2,000 in month 1, 2, and 3, respectively, and has a present value (a time-discounted value) of \$8,000; Investment-2 requires \$7,000 in month 1 and \$10,000 in period 3, and has a value of \$11,000; Investment-3 requires \$4,000 in period 2 and \$6,000 in period 3, and has a value of \$6,000; and Investment-4 requires \$3,000, \$4,000, and \$5,000 and has a value of \$4,000.

Problem-2: Multi-period Capital Budgeting (2)

Product	Month-1	Month-2	Month-3	Value
Investment-1	5,000	8,000	2,000	8,000
Investment-2	7,000	-	10,000	11,000
Investment-3	-	4,000	6,000	6,000
Investment-4	3,000	4,000	5,000	4,000
Budget	14,000	12,000	15,000	

- Please note that one can only invest or not invest a product

Problem-2: Multi-period Capital Budgeting (3)

- The problem model can be built as follows.

$$
\begin{aligned}
&\text{Max.}\quad 8x_1 + 11x_2 + 6x_3 + 4x_4 \\
&\text{s. t.} \begin{cases}
5x_1 + 7x_2 + 3x_4 \leq 14 \\
8x_1 + 4x_3 + 4x_4 \leq 12 \\
2x_1 + 10x_2 + 6x_3 + 5x_4 \leq 15 \\
x_1, x_2, x_3, x_4 \in \{0, 1\}
\end{cases}
\end{aligned} \tag{4}
$$

Problem-3: Knapsack

- **Problem**: Given N items where each item has some weight and profit associated with it and also given a bag with capacity W, [i.e., the bag can hold at most W weight in it]. The task is to put the items into the bag such that the sum of profits associated with them is the maximum possible.

Item	Profit	Weight
I_1	p_1	w_1
I_2	p_2	w_2
$\dots$	$\dots$	$\dots$
I_n	p_n	w_n

$$\text{Max.} \quad \sum_{i=1}^{n} p_i x_i$$

$$\text{s. t.} \begin{cases} \sum_{i=1}^{n} w_i x_i \leq W \\ x_1, x_2, \cdots, x_n \in \{0, 1\} \end{cases} \tag{5}$$

Problem-4: Set Covering Problem (SCP)(1)

- **Problem**: Suppose you have a set of elements $\{1, 2, 3, 4, 5\}$ that need to be covered and a collection of subsets:

$$
\begin{aligned}
S_1 &= \{1, 2, 3\} \\
S_2 &= \{2, 4\} \\
S_3 &= \{3, 5\} \\
S_4 &= \{1, 4, 5\}
\end{aligned}
$$

- We want to select a minimum number of these subsets such that every element in the universal set is included in at least one selected subset.
- One subset can be either selected $x_i = 1$ or not selected $x_i = 0$
- For this simple example, the answer is $x_1 = 1, x_4 = 1,\ x_2 = x_3 = 0$
- But ... how to model it as an IP problem??

Problem-4: Set Covering Problem (SCP)(2)

- x_1: S_1 is selected 1, otherwise 0
- x_2: S_2 is selected 1, otherwise 0
- x_3: S_3 is selected 1, otherwise 0
- x_4: S_4 is selected 1, otherwise 0
- The objective is to minimize: $z = x_1 + x_2 + x_3 + x_4$
- So what are the constraints?

Problem-4: Set Covering Problem (SCP)(3)

- The objective is to minimize: $z = x_1 + x_2 + x_3 + x_4$
- So what are the constraints?
- One element should be covered by at least one subset
    - For element 1, $x_1 + x_4 \geq 1$
    - For element 2, $x_1 + x_2 \geq 1$
    - For element 3, $x_1 + x_3 \geq 1$
    - For element 4, $x_2 + x_4 \geq 1$

$$\begin{aligned} &\text{Min.} \quad x_1 + x_2 + x_3 + x_4 \\ &\text{s. t.} \begin{cases} x_1 + x_4 \geq 1 \\ x_1 + x_2 \geq 1 \\ x_1 + x_3 \geq 1 \\ x_2 + x_4 \geq 1 \\ x_1, x_2, x_3, x_4 \in \{0, 1\} \end{cases} \end{aligned} \tag{6}$$

Problem-5: Set Covering Problem (SCP)(4)

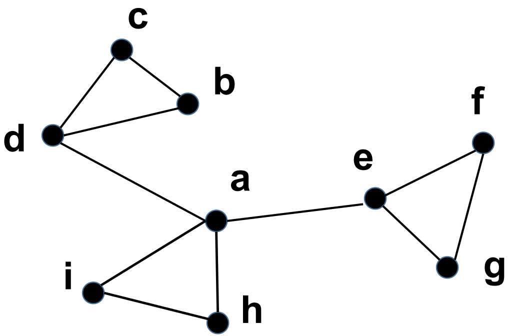


Figure: The distribution of *9* towns. Two cities between which the distance is less than *10* miles are connected by an edge.

- **Problem**: We are going to establish school in these towns. Students can go to school by bike if the distance is less than 10 miles. How many schools at least we should establish?

Problem-5: Set Covering Problem (SCP)(5)

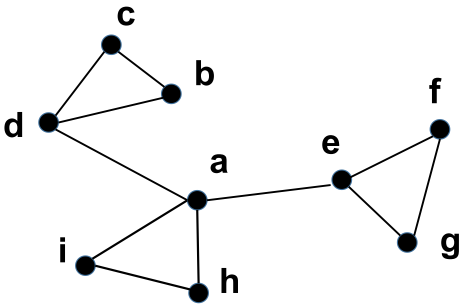
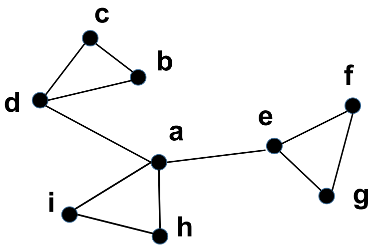


Figure: The distribution of *9* towns. Two cities between which the distance is less than *10* miles are connected by an edge.

- Given $x_{i=1\cdots 9} \in \{0, 1\}$ indicates whether establish school in town T_i
- The objective is to minimize $z = \sum_{i=1}^{9} x_i$

Problem-5: Set Covering Problem (SCP)(6)



1. For T_a, $x_1 + x_4 + x_5 + x_8 + x_9 \geq 1$
2. For T_b, $x_2 + x_3 + x_4 \geq 1$
3. For T_c, $x_2 + x_3 + x_4 \geq 1$
4. For T_d, $x_1 + x_2 + x_3 + x_4 \geq 1$
5. For T_e, $x_1 + x_5 + x_6 + x_7 \geq 1$
6. For T_f, $x_5 + x_6 + x_7 \geq 1$
7. For T_g, $x_5 + x_6 + x_7 \geq 1$

7. For T_h, $x_1 + x_8 + x_9 \geq 1$
8. For T_i, $x_1 + x_8 + x_9 \geq 1$

Problem-5: Set Covering Problem (SCP)(7)

$$\text{Min.} \quad \sum_{i=1}^{9} x_i$$

$$\text{s. t.} \begin{cases} x_1 + x_4 + x_5 + x_8 + x_9 \geq 1 \\ x_2 + x_3 + x_4 \geq 1 \\ x_1 + x_2 + x_3 + x_4 \geq 1 \\ x_1 + x_5 + x_6 + x_7 \geq 1 \\ x_5 + x_6 + x_7 \geq 1 \\ x_1 + x_8 + x_9 \geq 1 \\ x_{1\dots 9} \in \{0, 1\} \end{cases} \tag{7}$$

Outline

Motivation for Branch and Bounding (1)

- Let's consider the following problem

$$\text{Max. } 20x_1 + 10x_2 + 10x_3$$
$$\text{s.t.} \begin{cases} 2x_1 + 20x_2 + 4x_3 \leq 5 \\ 6x_1 + 20x_2 + 4x_3 = 20 \\ x_1, x_2, x_3 \in I^+ \end{cases} \tag{8}$$

$$\Downarrow$$

$$\text{Max. } z - 20x_1 - 10x_2 - 10x_3 = 0$$
$$\text{s.t.} \begin{cases} 2x_1 + 20x_2 + 4x_3 + s_1 = 5 \\ 6x_1 + 20x_2 + 4x_3 = 20 \\ x_1, x_2, x_3 \in I^+ \\ s_1 \geq 0 \end{cases} \tag{9}$$

Solve the LP relaxation

	z	x_1	x_2	x_3	s_1	C
	1	-20	-10	-10	0	0
s_1	0	2	20	4	1	5
x_3	0	6	20	4	0	20

$\Downarrow$

	z	x_1	x_2	x_3	s_1	C
	1	0	190	30	-10	50
x_1	0	2	20	4	1	5
x_3	0	0	-40	-8	3	5

$\Downarrow$

	z	x_1	x_2	x_3	s_1	C
	1	0	170/3	10/3	0	200/3
x_1	0	2	20/3	4/3	0	20/3
s_1	0	0	-40	-8	3	5

- We have $x_1 = 10/3, s_1 = 5/3, \ x_2 = x_3 = 0, \ \text{max-value} = 200/3$

Simple Round-off Does not Work

$$\text{Max. } 20x_1 + 10x_2 + 10x_3$$
$$\text{s.t.} \begin{cases} 2x_1 + 20x_2 + 4x_3 \leq 5 \\ 6x_1 + 20x_2 + 4x_3 = 20 \\ x_1, x_2, x_3 \in I^+ \end{cases} \tag{8}$$

- We have $x_1 = 10/3, s_1 = 5/3,\ x_2 = x_3 = 0,\ \text{max-value} = 200/3$
- Round-off the solution, we have $x_1 = 3, s_1 = 1,\ x_2 = x_3 = 0$
- Or $x_1 = 4, s_1 = 1,\ x_2 = x_3 = 0$
- Or $x_1 = 4, s_1 = 2,\ x_2 = x_3 = 0$
- Or $x_1 = 3, s_1 = 2,\ x_2 = x_3 = 0$
- **None of them are feasible**

Branch-bounding Method (1-1)

- Let's look at the 0-1 Knapsack problem given before

$$
\begin{aligned}
\text{Max.} \quad & 8x_1 + 11x_2 + 6x_3 + 4x_4 \\
\text{s. t.} & \begin{cases} 5x_1 + 7x_2 + 3x_4 \leq 14 \\ x_1, x_2, x_3, x_4 \in \{0, 1\} \end{cases}
\end{aligned} \tag{9}
$$

relaxed $\Downarrow$ LP problem

$$
\begin{aligned}
\text{Max.} \quad & 8x_1 + 11x_2 + 6x_3 + 4x_4 \\
\text{s. t.} & \begin{cases} 5x_1 + 7x_2 + 3x_4 \leq 14 \\ x_1 \leq 1 \\ x_2 \leq 1 \\ x_3 \leq 1 \\ x_4 \leq 1 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}
\end{aligned} \tag{10}
$$

Branch-bounding Method (1-2)

- Solve the following relaxed LP problem

$$
\begin{aligned}
\text{Max.} \quad & 8x_1 + 11x_2 + 6x_3 + 4x_4 \\
\text{s. t.} \quad & \begin{cases} 5x_1 + 7x_2 + 3x_4 \leq 14 \\ x_1 \leq 1 \\ x_2 \leq 1 \\ x_3 \leq 1 \\ x_4 \leq 1 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}
\end{aligned}
\tag{11}
$$

- Linear relaxation solution is $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0.667$ with maximal value **27.667**
- From this solution, one thing is certain that the optimal integer solution should be ≤ 27.667
- Since x_4 is not integer, we add integer constraint on the problem
  1. $x_4 = 0$
  2. $x_4 = 1$

Branch-bounding Method (1-3)

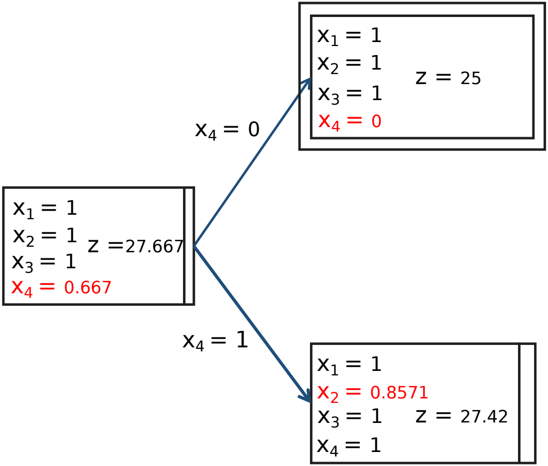
$x_4 = 0$

$$\text{Max.} \quad 8x_1 + 11x_2 + 6x_3$$
$$\text{s. t.} \begin{cases} 5x_1 + 7x_2 \leq 14 \\ x_1 \leq 1 \\ x_2 \leq 1 \\ x_3 \leq 1 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

$x_4 = 1$

$$\text{Max.} \quad 8x_1 + 11x_2 + 6x_3 + 4$$
$$\text{s. t.} \begin{cases} 5x_1 + 7x_2 \leq 11 \\ x_1 \leq 1 \\ x_2 \leq 1 \\ x_3 \leq 1 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Branch-bounding Method (1-4)



Branch-bounding Method (1-5)

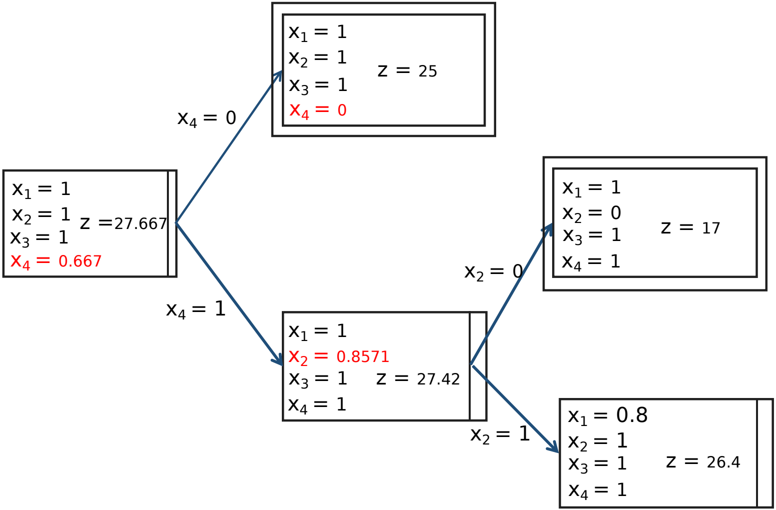
$x_4 = 1, x_2 = 0$

$$\text{Max.} \quad 8x_1 + 6x_3 + 4$$
$$\text{s. t.} \begin{cases} 5x_1 \leq 11 \\ x_1 \leq 1 \\ x_3 \leq 1 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

$x_4 = 1, x_2 = 1$

$$\text{Max.} \quad 8x_1 + 6x_3 + 15$$
$$\text{s. t.} \begin{cases} 5x_1 \leq 4 \\ x_1 \leq 1 \\ x_3 \leq 1 \\ x_1, x_3 \geq 0 \end{cases}$$

Branch-bounding Method (1-6)



Branch-bounding Method (1-7)

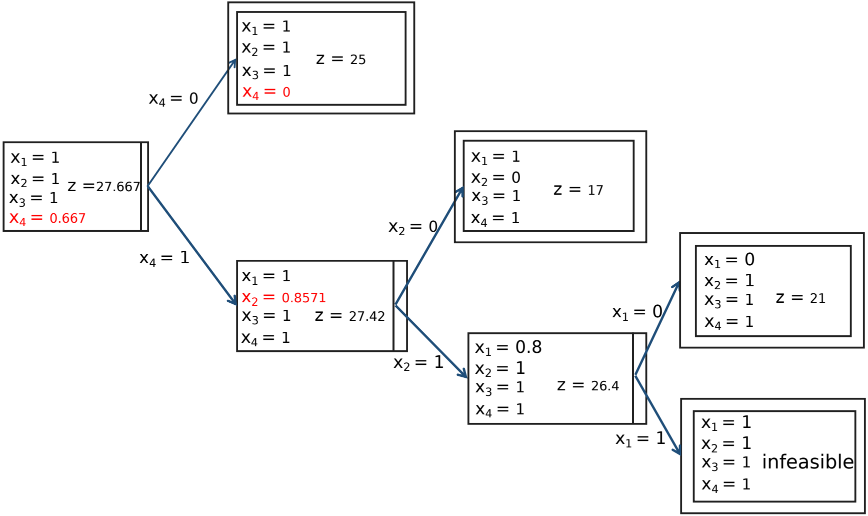
$x_4 = 1, x_2 = 1, x_1 = 0$

$$\text{Max.} \quad 6x_3 + 15$$
$$\text{s. t.} \begin{cases} x_3 \leq 1 \\ x_3 \geq 0 \end{cases}$$

$x_4 = 1, x_2 = 1, x_1 = 1$

$$\text{Max.} \quad 6x_3 + 23$$
$$\text{s. t.} \begin{cases} x_3 \leq 1 \\ x_3 \geq 0 \end{cases}$$

Branch-bounding Method (1-8)



Branch-bounding Method: exercise

$$
\begin{aligned}
&\text{Max. } x_1 + x_2 \\
\text{s. t.} &\begin{cases} 2x_1 + 5x_2 \leq 16 \\ 6x_1 + 5x_2 \leq 30 \\ x_1, x_2 \in I^+ \end{cases}
\end{aligned} \tag{12}
$$

- Please work it out by branch-bounding

Branch-bounding Method: the answer-1

	z	x_1	x_2	s_1	s_2	C
	1	-1	-1	0	0	0
s_1	0	2	5	1	0	16
s_2	0	6	5	0	1	30

$\Downarrow$

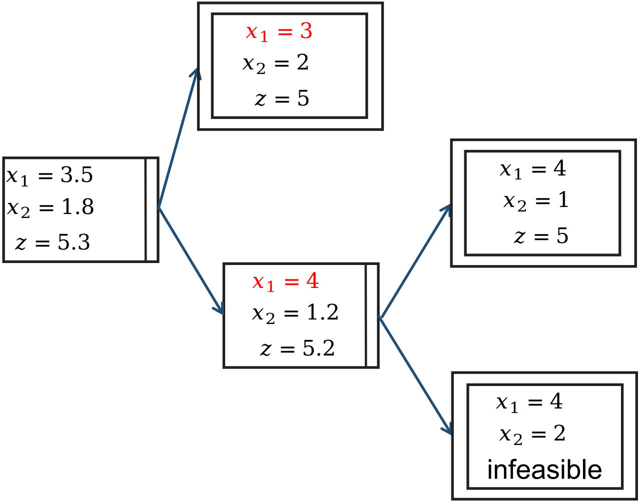
	z	x_1	x_2	s_1	s_2	C
	1	0	-1/6	0	1/6	5
s_1	0	0	10/3	1	-1/3	6
x_1	0	6	5	0	1	30

$\Downarrow$

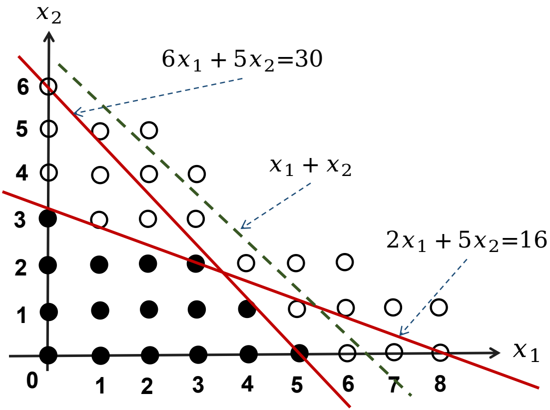
	z	x1	x2	s1	s2	C
	1	0	0	1/20	3/20	5.3
x_2	0	0	10/3	1	-1/3	6
x_1	0	6	0	-1.5	1.5	21

$x_1 = 3.5, x_2 = 1.8, z = 5.3$

Branch-bounding Method: the answer

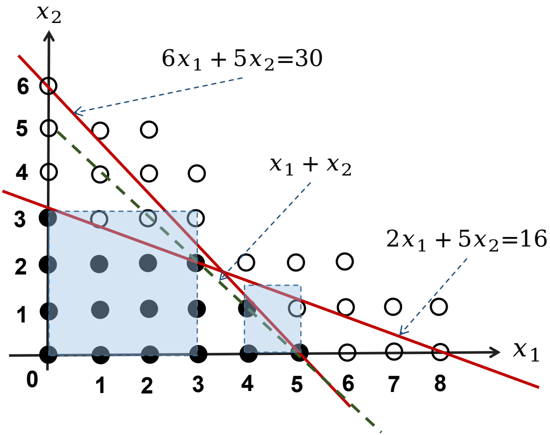


Branch-bounding Method: explained (1)



- As shown in the figure, there are many integer points to be enumerated

Branch-bounding Method: explained (2)



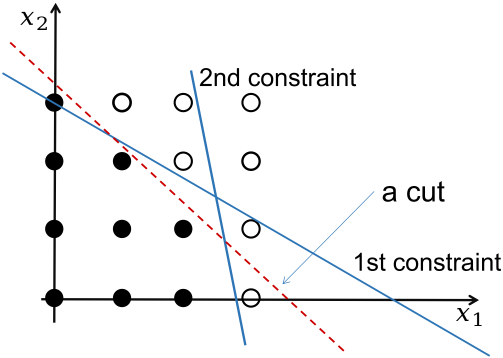
- The branch-bounding method cut the feasible set into two
- It does not ignore any feasible integer solution

Branch-bounding Method: Summary

- The general procedure is as follows
  1. Solve the LP relaxation
  2. Choose the most fractional variable to branch
  3. Solve the LP relaxation of the sub-problems
  4. If the sub-problem has either integer solution/no feasible solution, terminate on the branch
  5. Choose the best/depth-first branch and repeat Steps 2-4
- Disadvantage: could be slow
- Advantage: Suitable for integer programming (IP) and mixture integer programing (MIP)

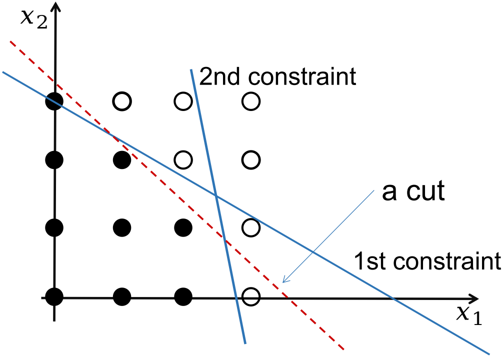
Outline

Motivation for Cutting Plane Method (1)



- Given the IP problem shown in the figure
- We try to cut out a fraction of the feasible set
- Such that the feasible set is smaller to explore
- Key issue: find out the cutting plane, namely the valid inequality

Motivation for Cutting Plane Method (2)



- Principle
  1. The cutting plane will cut out the current best fractional solution[3]
  2. It will keep all integer feasible set

[3] It becomes infeasible in other words.

$$
\begin{aligned}
&\text{Max. } x_1 + x_2 \\
\text{s. t.} &\begin{cases} 3x_1 + 2x_2 \leq 5 \\ \qquad\quad x_2 \leq 2 \\ x_1, x_2 \in I^+ \end{cases}
\end{aligned}
\tag{13}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\text{Max. } z - x_1 - x_2 = 0 \\
\text{s. t.} &\begin{cases} 3x_1 + 2x_2 + s_1 = 5 \\ \qquad\quad x_2 + s_2 = 2 \\ x_1, x_2, s_1, s_2 \in I^+ \end{cases}
\end{aligned}
\tag{14}
$$

	z	x_1	x_2	s_1	s_2	C
	1	-1	-1	0	0	0
s_1	0	3	2	1	0	5
s_2	0	0	1	0	1	2

$\Downarrow$

	z	x_1	x_2	s_1	s_2	C
	1	0	-1/3	1/3	0	5/3
x_1	0	3	2	1	0	5
s_2	0	0	1	0	1	2

$\Downarrow$

	z	x_1	x_2	s_1	s_2	C
	1	0	0	1/3	1/3	7/3
x_1	0	3	0	1	-2	1
x_2	0	0	1	0	1	2

- Solution for the relaxed LP problem, $x_1 = \frac{1}{3}$, $x_2 = 2$, $z = \frac{7}{3}$
- Unfortunately, x_1 is fractional

	z	x_1	x_2	s_1	s_2	C
	1	0	0	1/3	1/3	7/3
x_1	0	3	0	1	-2	1
x_2	0	0	1	0	1	2

$\Downarrow$

	z	x_1	x_2	s_1	s_2	C
	1	0	0	1/3	1/3	7/3
x_1	0	**1**	0	1/3	-2/3	1/3
x_2	0	0	**1**	0	1	2

- We normalize the coefficients of all basic variable to be **1**
- We are going to derive new constraint from equation corresponding to the fractional variable
- We have $x_1 + \frac{1}{3}s_1 - \frac{2}{3}s_2 = \frac{1}{3}$

	z	x_1	x_2	s_1	s_2	C
	1	0	0	1/3	1/3	7/3
x_1	0	**1**	0	1/3	-2/3	1/3
x_2	0	0	**1**	0	1	2

- Look at this constraint: $x_1 + \frac{1}{3}s_1 - \frac{2}{3}s_2 = \frac{1}{3}$
- We can manipulate on this
- We have $x_1 + \frac{1}{3}s_1 + (-1 + \frac{1}{3})s_2 = \frac{1}{3}$
- Keep all integer terms on the left hand, move all fractions on the right
- $x_1 - s_2 = \frac{1}{3} - \frac{1}{3}s_1 - \frac{1}{3}s_2$
- Since $x_1 - s_2$ is an integer, $\frac{1}{3} - \frac{1}{3}s_1 - \frac{1}{3}s_2$ must be integer
- The possible values for the right side can only be 0, -1, -2, $\cdots$
- So we have following inequation holds

$$\frac{1}{3} - \frac{1}{3}s_1 - \frac{1}{3}s_2 \leq 0 \tag{15}$$

- Since $x_1 - s_2$ is an integer, $\frac{1}{3} - \frac{1}{3}s_1 - \frac{1}{3}s_2$ must be integer
- The possible values for the right side can only be 0, -1, -2, $\cdots$
- So we have following inequation holds

$$\frac{1}{3} - \frac{1}{3}s_1 - \frac{1}{3}s_2 \leq 0 \tag{13}$$

- This inequality is satisfied by every feasible integer solution to our original problem
- However, current solution $x_1 = 1/3, x_2 = 2, s_1 = 0,\ s_2 = 0$ is not feasible
- As a result, we cut out the current optimal fractional solution from the our problem

	z	x_1	x_2	s_1	s_2	C
	1	0	0	1/3	1/3	7/3
x_1	0	**1**	0	1/3	-2/3	1/3
x_2	0	0	**1**	0	1	2

- We incorporate this new constraint to our problem

$$-\frac{1}{3}s_1 - \frac{1}{3}s_2 + s_3 = -\frac{1}{3} \tag{14}$$

	z	x_1	x_2	s_1	s_2	s_3	C
	1	0	0	1/3	1/3	0	7/3
x_1	0	**1**	0	1/3	-2/3	0	1/3
x_2	0	0	**1**	0	1	0	2
s_3	0	0	0	-1/3	-1/3	1	-1/3

- To this end, we are going to address another relaxed LP problem
- Notice that basic variable $s_3 = -1/3$, which is not feasible

Solve it by **Dual Simplex** Method (1)

		z	x_1	x_2	s_1	s_2	s_3	C
R_0		1	0	0	1/3	1/3	0	7/3
R_1	x_1	0	**1**	0	1/3	-2/3	0	1/3
R_2	x_2	0	0	**1**	0	1	0	2
R_3	s_3	0	0	0	-1/3	-1/3	1	-1/3

- We solve this problem by Dual Simplex
- Instead of making the objective value increase, we aim to tune down the objective value
- First of all, basic variable s_3 should be swapped out
- Divide coefficients of R_0 by the corresponding ones from R_3
- Select the non-basic variable corresponding to column that taking the minimum absolute value to swap with s_3

Solve it by **Dual Simplex** Method (2)

		z	x_1	x_2	s_1	s_2	s_3	C
R_0		1	0	0	1/3	1/3	0	7/3
R_1	x_1	0	**1**	0	1/3	-2/3	0	1/3
R_2	x_2	0	0	**1**	0	1	0	2
R_3	s_3	0	0	0	-1/3	-1/3	1	-1/3
$\left\lvert\frac{R_0}{R_3}\right\rvert$		∞	∞	∞	1	1	0	

$$\Downarrow$$

		z	x_1	x_2	s_1	s_2	s_3	C
R_0		1	0	0	0	0	0	2
R_1	x_1	0	**1**	0	0	-1	0	0
R_2	x_2	0	0	**1**	0	1	0	2
R_3	s_1	0	0	0	**1**	1	-3	1

- To this end, we have the optimal integer solution
- $x_1 = 0, x_2 = 2, s_1 = 1, s_2 = 0, s_3 = 0$

Cutting Plane Method: Example-2

$$
\text{Max. } 5x_1 + 6x_2 \\
\text{s. t.} \begin{cases} x_1 + x_2 \le 5 \\ 4x_1 + 7x_2 \le 28 \\ x_1, x_2 \ge 0 \quad \& \quad \textit{integer} \end{cases} \tag{15}
$$

Try to address it with cutting plane method ...

- Solve the relaxed LP problem, we have

	z	x_1	x_2	s_1	s_2	C
	1	0	0	11/3	1/3	83/3
x_1	0	1	0	7/3	-1/3	7/3
x_2	0	0	1	-4/3	1/3	8/3

1. Select one of the fractional variables and write equation based on the optimal table

$$
\begin{aligned}
x_1 + \frac{7}{3}s_1 - \frac{1}{3}s_2 &= \frac{7}{3} \\
x_1 + (2 + \frac{1}{3})s_1 + (-1 + \frac{2}{3})s_2 &= 2 + \frac{1}{3} \\
x_1 + 2s_1 - s_2 - 2 &= -\frac{1}{3}s_1 - \frac{2}{3}s_2 + \frac{1}{3}
\end{aligned}
$$

2. Work out the new constraint inequation (the cutting plane)

$$-\frac{1}{3}s_1 - \frac{2}{3}s_2 + \frac{1}{3} \leq 0$$

- Solve the relaxed LP problem, we have

	z	x_1	x_2	s_1	s_2	C
	1	0	0	11/3	1/3	83/3
x_1	0	1	0	7/3	-1/3	7/3
x_2	0	0	1	-4/3	1/3	8/3

3. Append the new constraint to the opitmal table

$$-\frac{1}{3}s_1 - \frac{2}{3}s_2 + \frac{1}{3} \leq 0$$
$$-\frac{1}{3}s_1 - \frac{2}{3}s_2 + s_3 = -\frac{1}{3}$$

	z	x_1	x_2	s_1	s_2	s_3	C
	1	0	0	11/3	1/3	0	83/3
x_1	0	1	0	7/3	-1/3	0	7/3
x_2	0	0	1	-4/3	1/3	0	8/3
s_3	0	0	0	-1/3	-2/3	1	-1/3

- Solve the relaxed LP problem, we have

	z	x_1	x_2	s_1	s_2	s_3	C
	1	0	0	11/3	1/3	0	83/3
x_1	0	1	0	7/3	-1/3	0	7/3
x_2	0	0	1	-4/3	1/3	0	8/3
s_3	0	0	0	-1/3	-2/3	1	-1/3

$\Downarrow$

	z	x_1	x_2	s_1	s_2	s_3	C
	1	0	0	7/2	0	1/2	55/2
x_1	0	1	0	5/2	0	-1/2	5/2
x_2	0	0	1	-3/2	0	1/2	5/2
s_2	0	0	0	1/2	1	-3/2	1/2

- We arrive the optima of the new relaxed LP problem
- Unfortunately, it is not an integer solution

	z	x_1	x_2	s_1	s_2	s_3	C
	1	0	0	7/2	0	-1/2	55/2
x_1	0	1	0	5/2	0	1/2	5/2
x_2	0	0	1	-3/2	0	1/2	5/2
s_2	0	0	0	1/2	1	-3/2	1/2

- Let's select the row of x_2 to derive the new constraint

$$\begin{aligned} x_2 + (-2 + \frac{1}{2})s_2 + \frac{1}{2}s_3 &= 2 + \frac{1}{2} \\ x_2 - 2s_2 - 2 &= -\frac{s_2}{2} - \frac{s_3}{2} + \frac{1}{2} \\ -\frac{s_2}{2} - \frac{s_3}{2} + \frac{1}{2} &\leq 0 \end{aligned}$$

- Append the new constraint, and solve the relaxed LP problem

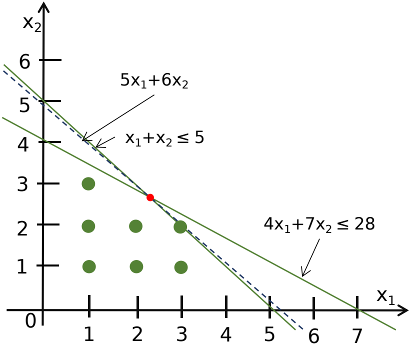
	z	x_1	x_2	s_1	s_2	s_3	s_4	C
	1	0	0	7/2	0	1/2	0	55/2
x_1	0	1	0	5/2	0	-1/2	0	5/2
x_2	0	0	1	-3/2	0	1/2	0	5/2
s_2	0	0	0	1/2	1	-3/2	0	1/2
s_4	0	0	0	-1/2	0	-1/2	1	-1/2

$$\Downarrow$$

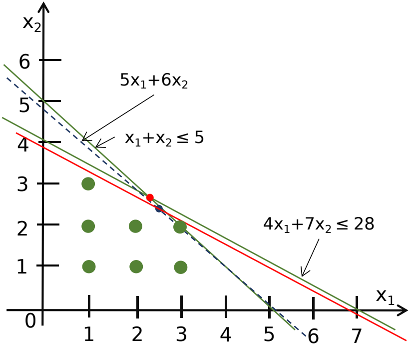
	z	x_1	x_2	s_1	s_2	s_3	s_4	C
	1	0	0	-3	0	0	1	27
x_1	0	1	0	3	0	0	0	3
x_2	0	0	1	-2	0	1	-1	2
s_2	0	0	0	2	1	0	-3	2
s_4	0	0	0	1	0	1	-2	1

- To this end, $x_1 = 3, x_2 = 2, s_2 = 1, s_3 = 1, \ z = 27$
- It is an integer solution

Interpretation in Geometry (1)

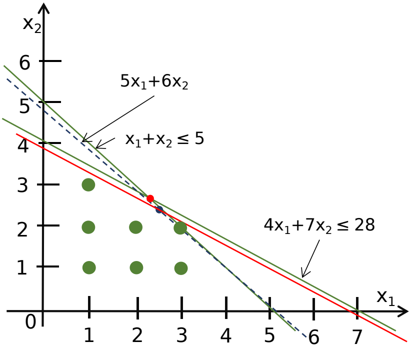


(a)

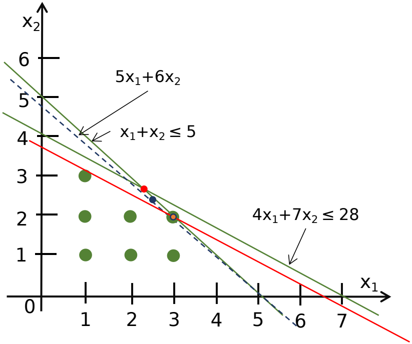


(b)

Interpretation in Geometry (2)



(b)



(c)

Gomory's cuts

- In general, let $\lfloor a \rfloor$ be defined as the largest integer less than or equal to a,
- For example, $\lfloor 3.9 \rfloor = 3$, $\lfloor 5 \rfloor = 3$, and $\lfloor -1.3 \rfloor = -2$
- Given we consider the constraint $x_k + \sum a_i x_i = b$, we write each $a_i = \lfloor a_i \rfloor + a'_i$
- $b = \lfloor b \rfloor + b'$, $0 < b' < 1$, we get

$$x_k + \sum \lfloor a_i \rfloor x_i - \lfloor b \rfloor = b' - \sum a'_i x_i \tag{16}$$

- The cut is

$$b' - \sum a'_i x_i \leq 0. \tag{17}$$

The man behind the Cutting-plane method

Ralph Edward Gomory is an American applied mathematician and executive. Gomory worked at IBM as a researcher and later as an executive. During that time, his research led to the creation of new areas of applied mathematics.

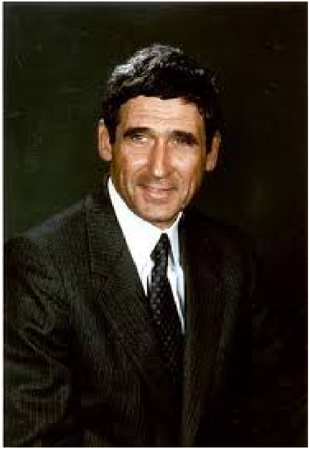


Figure: Ralph Gomory (1929 -)

Pitfalls of Cutting-plane Method

- The fractional cuts work with the assumption that all variables are integers
- Therefore, they do not allow fractional values for slack or excess variables
- Solution
  1. Multiply a factor on the constraint to change the coefficients to be integers
  2. Using mixed cuts, which restrict the integer assumption only to a subset of values

Available Softwares for LP and IP

1. GUROBI
2. IBM-CPLEX
3. FICO-Xpress
4. ZIB-SCIP, open source from Germany
5. Huaiwei OptVerse
6. Cardinal Operation-COPT

1. A tutorial on Integer Programming, Gerard Cornuejols, Michael A. Trick Matthew J. Saltzman, 1995.