Convex Optimization

Lecture 11: Portfolio Problem

Lecturer: *Dr*. Wan-Lei Zhao

Autumn Semester 2025

Contact: wlzhao@xmu.edu.cn

Outline

1. Portfolio Problem

2. Solve the Portfolio Problem

Opening Discussion (1)

- Given we have certain amount of capital to invest
- There are several options in the investment market
- How to design the investment strategy is a classic problem
- Intuitively, we want to maximize the returns/profit
- Famous saying: "**Capital is profit driven**"

Opening Discussion (2)

- On the one hand, we want to maximize the returns per unit investment
- On the other hand, we want to minimize the risk
- Given there exists a high-return and low-risk program
- A lot of capital will rush in to flatten the high-return, meanwhile increasing the risk
- For example, the real-estate market, and the futures market
- Actually, given there is no monopoly, all markets are subjected to above rules

Investment Problem (1)

- Given we have \$1000
- There are three non-dividend paying stocks, IBM (IBM), Walmart (WMT), and Southern Electric (SEHI)
- Namely, we buy the stocks at the end of one month, and sell them at the end of next month
- Suppose we bought the stock at price p, and sold it out at price s,
- The one-month profit we get from the stock is $r = \frac{s-p}{p}$
- Our expected return at the end of the month is \$50
- In the meantime, minimize the possible risk/losses

Table: Stock Prices at the 1st Day of Each Month

Month	IBM	WMT	SEHI
Nov-01	93.043	51.826	1.063
Dec-01	84.585	52.823	0.938
Jan-01	111.453	56.477	1.000
Feb-01	99.525	49.805	0.938
Mar-01	95.819	50.287	1.438
Apr-01	114.708	51.521	1.700
May-01	111.515	51.531	2.540
Jun-01	113.211	48.664	2.390
Jul-01	104.942	55.744	3.120
Aug-01	99.827	47.916	2.980
Sep-01	91.607	49.438	1.900
Oct-01	107.937	51.336	1.750
Nov-01	115.590	55.081	1.800

Observation and Analysis

- The price of one stock/product fluctuates
- We make profit when buying in low price and selling in higher price
- The prices of different products have impacts on each other
- It is impossible that the price of a product/collection of products monotonically increases

Raw Price data to Profit

Table: Prices at beg. of each Month

Month	IBM	WMT	SEHI
Nov-01	93.043	51.826	1.063
Dec-01	84.585	52.823	0.938
Jan-01	111.453	56.477	1.000
Feb-01	99.525	49.805	0.938
Mar-01	95.819	50.287	1.438
Apr-01	114.708	51.521	1.700
May-01	111.515	51.531	2.540
Jun-01	113.211	48.664	2.390
Jul-01	104.942	55.744	3.120
Aug-01	99.827	47.916	2.980
Sep-01	91.607	49.438	1.900
Oct-01	107.937	51.336	1.750
Nov-01	115.590	55.081	1.800

$\Rightarrow$

Table: Profit at the end of Each Month

Month	IBM	WMT	SEHI
Dec-01	-0.091	0.019	-0.118
Jan-01	0.318	0.069	0.066
Feb-01	-0.107	-0.118	-0.062
Mar-01	-0.037	0.010	0.533
Apr-01	0.197	0.025	0.182
May-01	-0.028	0.000	0.494
Jun-01	0.015	-0.056	-0.059
Jul-01	-0.073	0.145	0.305
Aug-01	-0.049	-0.140	-0.045
Sep-01	-0.082	0.032	-0.362
Oct-01	0.178	0.038	-0.079
Nov-01	0.071	0.073	0.029
Average	0.026	0.008	0.074

Observation and Analysis (continued)

- It is impossible that the price of a product/collection of products monotonically increases
- An intuitive way is to look at the history of profit
- Invest on a collection of products with the highest profit
- However, high profit always comes along with high risk
- So our target is
  1. To make profit while minimizing the risk
  2. To maximize the profit while keeping the risk at an acceptable level

Target: minimize the risk while making profit

Question: How to define the return/profit??

Where the risk comes from??

How to measure the risk??

Define the profit

- Given $R = [r_1, \cdots, r_i, \cdots, r_m]$ are the return rates from m stocks
- x_i is the amount of investment on S_i
- The total amount of return can be written as

$$P = \sum_i x_i \cdot r_i \tag{1}$$

- Notice that r_i[1] fluctuates across different months
- So the expected return is

$$\begin{aligned} \mathbf{E}P &= \mathbf{E}(\sum_{i=1}^{m} x_i \cdot r_i) \\ &= \sum_{i=1}^{m} x_i \cdot \mathbf{E}(r_i) \\ &= \sum_{i=1}^{m} x_i \cdot \bar{r}_i \end{aligned} \tag{2}$$

[1] Note that r_i is a random variable, while x_i is a constant.

Background Knowledge (1)

1. Given r is a random variable and C is a constant, $\mathbf{E}(cr)$=c$\mathbf{E}(r)$
2. Given n observations r_i, its mean is $\bar{r}$, the variance is

$$var(r_i) = \frac{\sum_i (r_i - \bar{r})^2}{n-1}$$

3. The covariance of two random variables is defined as

$$\begin{aligned} cov(r_i, r_j) &= \mathbf{E}[(r_i - \mathbf{E}r_i)(r_j - \mathbf{E}r_j)] \\ &= \mathbf{E}[r_i r_j] - \mathbf{E}r_i\mathbf{E}r_j \end{aligned}$$

4. Given there are n observations for random variable r_i and r_j, the covariance is

$$cov(r_i, r_j) = \frac{1}{n}\sum_{k=1}^{n}(r_{ik} - \mathbf{E}r_i)(r_{jk} - \mathbf{E}r_j) \tag{3}$$

Background Knowledge (2)

1. The covariance of two random variables is defined as

$$cov(r_i, r_j) = \mathbf{E}[(r_i - \mathbf{E}r_i)(r_j - \mathbf{E}r_j)]$$
$$= \mathbf{E}[r_i r_j] - \mathbf{E}r_i \mathbf{E}r_j$$

2. Given there are n observations for random variable r_i and r_j, the covariance is

$$cov(r_i, r_j) = \frac{1}{n-1} \sum_{k=1}^{n} (r_{ik} - \mathbf{E}r_i)(r_{jk} - \mathbf{E}r_j)$$

Define the risk (1)

Risk comes from uncertainty

- When r_i varies more, the more uncertain of the expected return
- As a result, the risk is defined as the Variance of the return

$$\mathbf{Var}(P) = \mathbf{Var}(\sum_i x_i \cdot r_i) \tag{4}$$

- Notice that r_i fluctuates across different months
- Given r_is are independent from each other, the var. on the return is

$$\begin{aligned}\mathbf{Var}(P) &= \mathbf{E}(P - \mathbf{E}(P))^2 \\ &= \mathbf{E}(\sum_i (x_i \cdot r_i - \mathbf{E}(x_i \cdot r_i)))^2 \\ &= \mathbf{E}(\sum_i x_i \cdot (r_i - \mathbf{E} r_i))^2 \end{aligned} \tag{5}$$

Define the risk (2)

- Given r_is are independent from each other
- The variation on the return is

$$
\begin{aligned}
\mathbf{Var}(P) &= \mathbf{E}(P - \mathbf{E}(P))^2 \\
&= \mathbf{E}(\sum_i (x_i \cdot r_i - \mathbf{E}(x_i \cdot r_i)))^2 \\
&= \mathbf{E}(\sum_i x_i \cdot (r_i - \mathbf{E} r_i))^2 \\
&= \sum_i x_i^2 \mathbf{E}((r_i - \mathbf{E} r_i)^2) \\
&= \sum_i x_i^2 \frac{1}{n-1} \sum_{k=1}^{n} (r_{ik} - \bar{r}_i)^2 \\
&= \sum_i x_i^2 Var(r_i)
\end{aligned} \tag{5}
$$

Define the risk (3)

- However, r_is are related to each other
- Given $R = [r_1, \cdots, r_m]^T$, $X = [x_1, \cdots, x_m]^T$
- The variation on the return is
$Var(P) = \mathbf{E}[(X^T R - \mathbf{E}[X^T R])(X^T R - \mathbf{E}[X^T R])^T]$

$$\begin{aligned} Var(P) =& \mathbf{E}[(X^T R - \mathbf{E}[X^T R])(X^T R - \mathbf{E}[X^T R])^T] \\ =& X^T \mathbf{E}[(R - \mathbf{E}R)(R - \mathbf{E}R)^T] X \\ =& X^T \cdot CoV(R) X \end{aligned} \tag{6}$$

Define the risk (4)

- However, r_is are related to each other
- The variation on the return is $Var(P) = X^T CoV(R)X$

$$CoV(R) = \mathbf{E}\begin{bmatrix} r_1 - \bar{r_1} \\ \cdot \\ \cdot \\ r_m - \bar{r_m} \end{bmatrix} [r_1 - \bar{r_1}, \cdots, r_m - \bar{r_m}] \tag{7}$$

$$= \mathbf{E}\begin{bmatrix} (r_1 - \bar{r_1})^2 & \cdots & (r_1 - \bar{r_1})(r_m - \bar{r_m}) \\ \cdots & \cdots & \cdots \\ (r_m - \bar{r_m})(r_1 - \bar{r_1}) & \cdots & (r_m - \bar{r_m})^2 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{E}(r_1 - \bar{r_1})^2 & \cdots & \mathbf{E}(r_1 - \bar{r_1})(r_m - \bar{r_m}) \\ \cdots & \cdots & \cdots \\ \mathbf{E}(r_m - \bar{r_m})(r_1 - \bar{r_1}) & \cdots & \mathbf{E}(r_m - \bar{r_m})^2 \end{bmatrix}$$

Define the risk (5)

$$
\begin{aligned}
\mathbf{Var}(P) =& \mathbf{E}(P - \mathbf{E}(P))^2 \\
=& \begin{bmatrix} \mathbf{E}(r_1 - \bar{r_1})^2 & \cdots & \mathbf{E}(r_1 - \bar{r_1})(r_m - \bar{r_m}) \\ \cdots & \cdots & \cdots \\ \mathbf{E}(r_m - \bar{r_m})(r_1 - \bar{r_1}) & \cdots & \mathbf{E}(r_m - \bar{r_m})^2 \end{bmatrix} \\
=& X^T \Sigma X
\end{aligned} \tag{6}
$$

$$
\Sigma = \frac{1}{n-1} \begin{bmatrix} \sum_{k=1}^{n} (r_{1k} - \bar{r_1})^2 & \cdots & \sum_{k=1}^{n} (r_{1k} - \bar{r_1})(r_{mk} - \bar{r_m}) \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^{n} (r_{mk} - \bar{r_m})(r_{1k} - \bar{r_1}) & \cdots & \sum_{k=1}^{n} (r_{mk} - \bar{r_m})^2 \end{bmatrix}
$$

Define the model: the profit and risk

- The profit is defined as

$$\mathbf{E}P = \mathbf{E}(\sum_i x_i \cdot r_i) = \sum_i x_i \cdot \mathbf{E}(r_i) \tag{2}$$

- The risk is defined as

$$\mathbf{Var}(P) = X^T \Sigma X \tag{4}$$

where Σ is the covariance matrix, and $X = [x_1, \cdots, x_m]^T$.

- Consider both the risk and the profit, we have

$$\begin{aligned} & Min.\ X^T \Sigma X \\ \text{s.t.} & \begin{cases} \sum_i x_i \cdot \mathbf{E}(r_i) \geq 50 \\ \sum_i x_i \leq 1000 \\ x_i \geq 0 \end{cases} \end{aligned} \tag{5}$$

The General Investment Model

- The left-side shows the model of the example
- The right-side shows the general investment model

$$
\begin{aligned}
&Min.\quad X^T\Sigma X\\
\text{s.t.}\ &\begin{cases} \sum_i x_i \cdot \mathbf{E}(r_i) \geq 50 \\ \sum_i x_i \leq 1000 \\ x_i \geq 0 \end{cases}
\end{aligned}
\qquad (5) \qquad
\begin{aligned}
&Min.\quad X^T\Sigma X\\
\text{s.t.}\ &\begin{cases} x^T\bar{r} \geq P_0 \\ x^T\mathbf{1} \leq M \\ x \succeq 0 \end{cases}
\end{aligned}
$$

- Σ in the general model is the covariance matrix
- It is symmetric and non-negative definite
- $\bar{r}$ is the average profit rate of one product in the considered period

Harry M. Markowitz proposed this investment model in his famous paper "The portfolio Selection" when he worked with RAND company in 1952. He was awarded Nobel Memorial Prize in Economic Sciences in 1990 for this contribution.

When he defended his Ph.D dissertation in University of Chicago, Milton Friedman argued his contribution was not economics.

Figure: Harry M. Markowitz (1927 - 2023)

Outline

Solve the Model

$$
\begin{aligned}
&Min. \quad [x_1 \ \ x_2 \ \ x_3] \cdot \Sigma \cdot [x_1 \ \ x_2 \ \ x_3]^T \\
&\text{s.t.} \quad \begin{cases} [x_1 \ \ x_2 \ \ x_3]\bar{r} \geq 50 \\ x_1 + x_2 + x_3 \leq 1000 \\ x_1, x_2, x_3 \geq 0 \end{cases}
\end{aligned} \tag{7}
$$

- In our case, we have $\bar{r} = [0.026 \ \ 0.008 \ \ 0.074]^T$
- The covariance matrix is

$$
\Sigma = \begin{bmatrix} 0.0186 & 0.0036 & 0.0013 \\ 0.0036 & 0.0064 & 0.0049 \\ 0.0013 & 0.0049 & 0.0686 \end{bmatrix} \tag{8}
$$

Solve it by Matlab

Standard QP:

$$\begin{aligned} \min \ & \frac{1}{2}x^t H_0 x + {c_0}^t x \\ \text{sub. to } & A^t x = r \\ & Bx \preccurlyeq b \\ & lb \preccurlyeq x \preccurlyeq ub \end{aligned}$$

```
C  = [0.0186 0.0036 0.0013;0.0036 0.0064 0.0049;0.0013 0.0049 0.0686];
br = [0.0260 0.0081 0.0737];
H  = 2*C;
C0 = zeros(1,3);
B  = [-br;ones(1,3)];
b  = [-50;1000];
lb = zeros(3,1);
ub = [1000,1000,1000]';
A  = []; r  = [];
[x, fval]=quadprog(H, C0, B, b, A, r, lb, ub)
```

- $x_1 = 496.8553$, $x_2 = 0$, $x_3 = 503.1447$, profit=50.0, risk=22608.0

Maximize the Profit

- Alternatively, we can think about maximizing the profit
- While keeping the potential loss at an acceptable level
- Given $X = [x_1, \cdots, x_m]$, $\Sigma = CoV(R)$
- This is the way most of the current investment products follow

$$
\begin{aligned}
&Min. \quad X^T \Sigma X \\
\text{s.t.} &\begin{cases} \sum_i x_i \cdot \mathbf{E}(r_i) \geq 50 \\ \sum_i x_i \leq 1000 \\ x_i \geq 0 \end{cases}
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&Max. \quad \sum_i x_i \cdot \mathbf{E}(r_i) \\
\text{s.t.} &\begin{cases} X^T \Sigma X \leq 30 \\ \sum_i x_i \leq 1000 \\ x_i \geq 0 \end{cases}
\end{aligned}
\tag{7}
$$

- Notice that this problem is no longer a quadratic programming problem

Combine both the Profit and Risk into Objective (1)

- Given $X = [x_1, \cdots, x_m]$, $\Sigma = CoV(R)$
- Alternatively, we can think about minizing the risk while maximizing the profit

$$\begin{aligned} & Min.\ \ X^T \Sigma X \\ \text{s.t.} & \begin{cases} \sum_i x_i \cdot \mathbf{E}(r_i) \geq 50 \\ \sum_i x_i \leq 1000 \\ x_i \geq 0 \end{cases} \end{aligned} \tag{5}$$

$$\begin{aligned} & Min.\ \ X^T \Sigma X - \sum_i x_i \cdot \mathbf{E}(r_i) \\ \text{s.t.} & \begin{cases} \sum_i x_i \leq 1000 \\ x_i \geq 0 \end{cases} \end{aligned} \tag{8}$$

- Notice that this problem becomes a quadratic programming problem again

Combine both the Profit and Risk into Objective (2)

- Alternatively, we can think about minizing the risk while maximizing the profit

$$
\begin{aligned}
&Min. \quad \mu X^T \Sigma X - \sum_i x_i \cdot \mathbf{E}(r_i) \\
&\text{s.t.} \begin{cases} \sum_i x_i \leq 1000 \\ x \succeq 0 \end{cases}
\end{aligned} \tag{8}
$$

$$\Downarrow$$

$$
\begin{aligned}
&Min. \quad \mu X^T \Sigma X - X^T \bar{r} \\
&\text{s.t.} \begin{cases} \sum_i x_i \leq 1000 \\ x \succeq 0 \end{cases}
\end{aligned} \tag{9}
$$

- μ is a parameter (constant) to indicate the weight of the risk in consideration
- Notice that this problem becomes a quadratic programming problem again

Solve the Problem with Different Objective

$$
\begin{aligned}
&Min.\ \ \mu X^T \Sigma X - X^T \bar{r} \\
&\text{s.t.} \begin{cases} \sum_i x_i \leq 1000 \\ x \succeq 0 \end{cases}
\end{aligned}
\tag{9}
$$

- Given $\bar{r} = [0.026 \ \ 0.008 \ \ 0.074]^T$, $\mu = 0.001$ and Σ is

$$
\Sigma = \begin{bmatrix} 0.0186 & 0.0036 & 0.0013 \\ 0.0036 & 0.0064 & 0.0049 \\ 0.0013 & 0.0049 & 0.0686 \end{bmatrix}
$$

Solve the Problem by Matlab

$$Min.\quad \mu X^T \Sigma X - X^T \bar{r}$$
$$\text{s.t.} \begin{cases} \sum_i x_i \leq 1000 \\ x \succeq 0 \end{cases} \tag{9}$$

```
mu = 0.001;
C  = [0.0186 0.0036 0.0013;0.0036 0.0064 0.0049;0.0013 0.0049 0.0686];
br = -1*[0.026 0.008 0.074];
H  = 2*mu*C;
c0 = br';
B  = [ones(1,3)];
b  = [1000];
lb = zeros(3,1);
ub = [1000,1000,1000]';
A  = []; r  = [];
[x, fval]=quadprog(H, c0, B, b, A, r, lb, ub)
```

- $x_1 = 496.8553$, $x_2 = 0$, $x_3 = 503.1447$, profit=50.01, risk=22608.0

Profit against Potential Risk

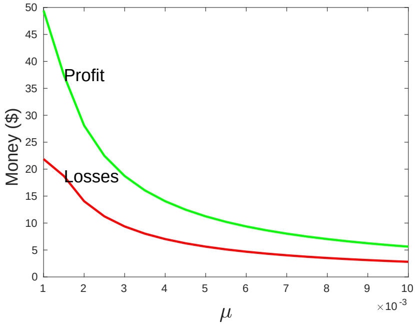

Figure: Profit against the acceptable risk weight $\mu = [0.001 \;\; 0.01]$.

- References

1. Markowitz, H.M. (March 1952). "Portfolio Selection". The Journal of Finance. 7 (1): 77–91.
2. Markowitz, H.M. (April 1952). "The Utility of Wealth". The Journal of Political Economy. LX (2): 151–158.
3. Shabbir Ahmed (Fall 2002) The Optimization Process: An example of portfolio optimization. Deterministic Optimization