Convex Optimization

Lecture 3: Simplex Method

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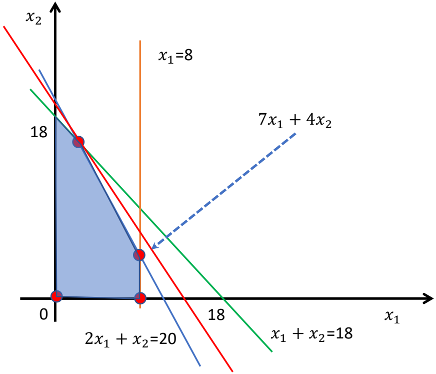
Outline

Linear Programming: the problem

$$
\text{Max. } 7x_1 + 4x_2 \\
\text{s. t.} \begin{cases} 2x_1 + x_2 \leq 20 \\ x_1 + x_2 \leq 18 \\ x_1 \leq 8 \\ x_1, x_2 \geq 0 \end{cases} \qquad (1)
$$



- In the previous lecture, we solve it by graph and linear transformations
- We are going to do linear transformations with tableau

Solve LP by Tableau (1)

- Introduce three slack variables s_1, s_2 and s_3
- We have

$$
\text{Max. } 7x_1 + 4x_2 \\
\text{s. t. } \begin{cases} 2x_1 + x_2 + s_1 & = 20 \\ x_1 + x_2 + \quad + s_2 & = 18 \\ x_1 + \qquad\quad + s_3 & = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{2}
$$

Solve LP by Tableau (2)

- We observe that increasing x_1 will maximize the target faster
- We learn that x_1 should be tuned according to the 3rd constraint
- So we plug $x_1 = 8 - s_3$ into the other constraints and the target
- We have

$$\text{Max. } 56 + 4x_2 - 7s_3$$
$$\text{s. t.} \begin{cases} x_2 + s_1 \quad - 2s_3 = 4 \\ x_2 + \quad + s_2 - s_3 = 10 \\ x_1 + \quad + s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

$$z - 7x_1 - 4x_2 = 0$$
$$\text{s. t.} \begin{cases} 2x_1 + x_2 + s_1 \quad = 20 \\ x_1 + x_2 + \quad + s_2 = 18 \\ x_1 + \quad + s_3 = 8 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

Solve LP by Tableau (3)

$$z - 7x_1 - 4x_2 = 0 \quad (0)$$

$$\text{s.t.} \begin{cases} 2x_1 + x_2 + s_1 = 20 & (1) \\ x_1 + x_2 + \quad + s_2 = 18 & (2) \\ x_1 + \quad + s_3 = 8 & (3) \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

- Plug $x_1 = 8 - s_3$ into (0)-(2)
- We have

$$\text{Max. } 56 + 4x_2 - 7s_3 \quad (0)$$

$$\text{s. t.} \begin{cases} x_2 + s_1 - 2s_3 = 4 & (1) \\ x_2 + \quad + s_2 - s_3 = 10 & (2) \\ x_1 + \quad + s_3 = 8 & (3) \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

$$\Downarrow$$

$$z - 4x_2 + 7s_3 = 56 \quad (0)$$

$$\text{s. t.} \begin{cases} x_2 + s_1 - 2s_3 = 4 & (1) \\ x_2 + \quad + s_2 - s_3 = 10 & (2) \\ x_1 + \quad + s_3 = 8 & (3) \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

- (0) = (0) +7*(3), (1)=(1)-2*(3), (2) = (2)-1*(3)

Solve LP by Tableau (4)

$$z - 7x_1 - 4x_2 = 0 \quad (0)$$

$$\text{s. t.} \begin{cases} 2x_1 + x_2 + s_1 = 20 & (1) \\ x_1 + x_2 + \quad + s_2 = 18 & (2) \\ x_1 + \quad + s_3 = 8 & (3) \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

	z	x_1	x_2	s_1	s_2	s_3	
	1	-7	-4	0	0	0	0
s_1	0	2	1	1	0	0	20
s_2	0	1	1	0	1	0	18
s_3	0	1	0	0	0	1	8

$\Downarrow$

Plug-in $\Downarrow$ $x_1 = 8 - s_3$

$$z - 4x_2 + 7s_3 = 56 \quad (0)$$

$$\text{s. t.} \begin{cases} x_2 + s_1 - 2s_3 = 4 & (1) \\ x_2 + \quad + s_2 - s_3 = 10 & (2) \\ x_1 + \quad + s_3 = 8 & (3) \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

	z	x_1	x_2	s_1	s_2	s_3	
	1	0	-4	0	0	7	56
s_1	0	0	1	1	0	-2	4
s_2	0	0	1	0	1	-1	10
x_1	0	1	0	0	0	1	8

- (0) = (0) +7*(3), (1)=(1)-2*(3), (2) = (2)-1*(3)

Solve LP by Tableau (5)

$$\text{Max. } z - 4x_2 + 7s_3 = 56 \quad (0)$$

$$\text{s.t.} \begin{cases} x_2 + s_1 \qquad - 2s_3 = 4 & (1) \\ x_2 + \quad + s_2 - s_3 = 10 & (2) \\ x_1 + \qquad + s_3 = 8 & (3) \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

	z	x_1	x_2	s_1	s_2	s_3	
	1	0	-4	0	0	7	56
s_1	0	0	1	1	0	-2	4
s_2	0	0	1	0	1	-1	10
x_1	0	1	0	0	0	1	8

$\Downarrow$

Plug-in $\Downarrow$ $x_2 = 4 - s_1 + 2s_3$

$$\text{Max. } z + 4s_1 - s_3 = 72 \quad (0)$$

$$\text{s.t.} \begin{cases} x_2 + s_1 - 2s_3 = 4 & (1) \\ -s_1 + s_2 + s_3 = 6 & (2) \\ x_1 + \qquad + s_3 = 8 & (3) \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

	z	x_1	x_2	s_1	s_2	s_3	
	1	0	0	4	0	-1	72
x_2	0	0	1	1	0	-2	4
s_2	0	0	0	-1	1	1	6
x_1	0	1	0	0	0	1	8

- (0) = (0) +4*(1), (2)=(2)-(1), (3)= (3)

Solve LP by Tableau (6)

$$\text{Max. } z + 4s_1 - s_3 = 72 \quad (0)$$

$$\text{s.t.} \begin{cases} x_2 + s_1 - 2s_3 = 4 & (1) \\ -s_1 + s_2 + s_3 = 6 & (2) \\ x_1 + \quad + s_3 = 8 & (3) \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

	z	x_1	x_2	s_1	s_2	s_3	
	1	0	0	4	0	-1	72
x_2	0	0	1	1	0	-2	4
s_2	0	0	0	-1	1	1	6
x_1	0	1	0	0	0	1	8

$\Downarrow$

Plug-in $\Downarrow$ $s_3 = 6 + s_1 - s_2$

$$\text{Max. } z + 3s_1 + s_2 = 78 \quad (0)$$

$$\text{s.t.} \begin{cases} x_2 - s_1 - 2s_2 = 16 & (1) \\ -s_1 + s_2 + s_3 = 6 & (2) \\ x_1 + s_1 - s_2 = 2 & (3) \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

	z	x_1	x_2	s_1	s_2	s_3	
	1	0	0	3	1	0	78
x_2	0	0	1	-1	2	0	16
s_3	0	0	0	-1	1	1	6
x_1	0	1	0	1	-1	0	2

- (0) = (0) + (2), (1)=(1)+2*(2), (3) = (3)-(2)

Solve LP by pure Tableau (1)

$$\text{Max.} \quad -5x_1 + 6x_2$$
$$\text{subject to} \begin{cases} x_1 + 2x_2 \leq 10 \\ 2x_1 - x_2 \leq 5 \\ x_1 - 4x_2 \leq 4 \\ x_1, x_2 \geq 0 \end{cases} \tag{3}$$

Introduce slack variables $\Downarrow$ s_1, s_2 and s_3

$$z + 5x_1 - 6x_2 = 0$$
$$\text{subject to} \begin{cases} x_1 + 2x_2 + s_1 = 10 \\ 2x_1 - x_2 + s_2 = 5 \\ x_1 - 4x_2 + s_3 = 4 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{4}$$

Solve LP by pure Tableau (2)

$$
z + 5x_1 - 6x_2 = 0
$$

$$
\text{subject to} \begin{cases} x_1 + 2x_2 + s_1 & = 10 \\ 2x_1 - x_2 \quad + s_2 & = 5 \\ x_1 - 4x_2 \qquad + s_3 & = 4 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{4}
$$

Construct $\Downarrow$ tableau for LP

		z	x_1	x_2	s_1	s_2	s_3	
0		1	5	-6	0	0	0	0
1	s_1	0	1	2	1	0	0	10
2	s_2	0	2	-1	0	1	0	5
3	s_3	0	1	-4	0	0	1	4

Solve LP by pure Tableau (3)

		z	x_1	x_2	s_1	s_2	s_3	
0		1	5	-6	0	0	0	0
1	s_1	0	1	2	1	0	0	10
2	s_2	0	2	-1	0	1	0	5
3	s_3	0	1	-4	0	0	1	4

swap x_2 in $\Downarrow$ s_1 out

		z	x_1	x_2	s_1	s_2	s_3	
0		1	8	0	3	0	0	30
1	x_2	0	1	2	1	0	0	10
2	s_2	0	2.5	0	0.5	1	0	10
3	s_3	0	3	0	2	0	1	24

- $0 = 0 + 3{*}1$, $2 = 2 + 0.5{*}1$, $3 = 3 + 2{*}1$

Solve LP by pure Tableau (4)

		z	x_1	x_2	s_1	s_2	s_3	
0		1	8	0	3	0	0	30
1	x_2	0	1	2	1	0	0	10
2	s_2	0	2.5	0	0.5	1	0	10
3	s_3	0	3	0	2	0	1	24

$$z + 5x_1 - 6x_2 = 0$$

$$\text{subject to} \begin{cases} x_1 + 2x_2 + s_1 & = 10 \\ 2x_1 - x_2 \quad + s_2 & = 5 \\ x_1 - 4x_2 \quad\quad + s_3 & = 4 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{4}$$

- We are lucky that no coefficient on 0 row is negative
- The maximum is 30
- We are almost done!!!

Solve LP by pure Tableau (5)

		z	x_1	x_2	s_1	s_2	s_3	
0		1	8	0	3	0	0	30
1	x_2	0	1	2	1	0	0	10
2	s_2	0	2.5	0	0.5	1	0	10
3	s_3	0	3	0	2	0	1	24

$$B = \left[\begin{array}{cccc} x_2 & s_2 & s_3 & b \\ \hline 2 & 0 & 0 & 10 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 24 \end{array}\right] \tag{5}$$

- The maximum is 30, when $x_1 = 0, x_2 = 5, s_1 = 0, s_2 = 10, s_3 = 24$

Solve LP by pure Tableau (6)

$$\text{Max. } 5x_1 + 4x_2$$
$$\text{s.t.} \begin{cases} 2x_1 - x_2 \leq 4 \\ x_1 + 2x_2 \leq 6 \\ 5x_1 + 3x_2 \leq 15 \\ x_1, x_2 \geq 0 \end{cases} \tag{6}$$

Introduce slack variables $\Downarrow$ s_1, s_2, and s_3

$$z - 5x_1 - 4x_2 = 0$$
$$\text{s.t.} \begin{cases} 2x_1 - x_2 + s_1 \qquad\qquad = 4 \\ x_1 + 2x_2 \quad + s_2 \qquad = 6 \\ 5x_1 + 3x_2 \qquad\quad + s_3 \quad = 15 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{7}$$

Solve LP by pure Tableau (7)

$$z - 5x_1 - 4x_2 = 0$$

$$\text{s.t.} \begin{cases} 2x_1 - x_2 + s_1 & = 4 \\ x_1 + 2x_2 \quad + s_2 & = 6 \\ 5x_1 + 3x_2 \quad\quad + s_3 & = 15 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{8}$$

$$\Downarrow$$

		z	x_1	x_2	s_1	s_2	s_3	
0		1	-5	-4	0	0	0	0
1	s_1	0	2	-1	1	0	0	4
2	s_2	0	1	2	0	1	0	6
3	s_3	0	5	3	0	0	1	15

Solve LP by pure Tableau (8)

		z	x_1	x_2	s_1	s_2	s_3	
0		1	-5	-4	0	0	0	0
1	s_1	0	2	-1	1	0	0	4
2	s_2	0	1	2	0	1	0	6
3	s_3	0	5	3	0	0	1	15

$\Downarrow$

		z	x_1	x_2	s_1	s_2	s_3	
0		1	0	$-\frac{13}{2}$	$\frac{5}{2}$	0	0	10
1	x_1	0	2	-1	1	0	0	4
2	s_2	0	0	$\frac{5}{2}$	$-\frac{1}{2}$	1	0	4
3	s_3	0	0	$\frac{11}{2}$	$-\frac{5}{2}$	0	1	5

Solve LP by pure Tableau (9)

		z	x_1	x_2	s_1	s_2	s_3	
0		1	0	$-\frac{13}{2}$	$\frac{5}{2}$	0	0	10
1	x_1	0	2	-1	1	0	0	4
2	s_2	0	0	$\frac{5}{2}$	$-\frac{1}{2}$	1	0	4
3	s_3	0	0	$\frac{11}{2}$	$-\frac{5}{2}$	0	1	5

$$\Downarrow$$

		z	x_1	x_2	s_1	s_2	s_3	
0		1	0	0	$-\frac{5}{11}$	0	$\frac{13}{11}$	$\frac{175}{11}$
1	x_1	0	2	0	$\frac{6}{11}$	0	$\frac{2}{11}$	$\frac{54}{11}$
2	s_2	0	0	0	$\frac{7}{11}$	1	$-\frac{5}{11}$	$\frac{19}{11}$
3	x_2	0	0	$\frac{11}{2}$	$-\frac{5}{2}$	0	1	5

Solve LP by pure Tableau (10)

		z	x_1	x_2	s_1	s_2	s_3	
0		1	0	0	$-\frac{5}{11}$	0	$\frac{13}{11}$	$\frac{175}{11}$
1	x_1	0	2	0	$\frac{6}{11}$	0	$\frac{2}{11}$	$\frac{54}{11}$
2	s_2	0	0	0	$\frac{7}{11}$	1	$-\frac{5}{11}$	$\frac{19}{11}$
3	x_2	0	0	$\frac{11}{2}$	$-\frac{5}{2}$	0	1	5

$$\Downarrow$$

		z	x_1	x_2	s_1	s_2	s_3	
0		1	0	0	0	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{1320}{77}$
1	x_1	0	2	0	0	$-\frac{6}{7}$	$\frac{4}{7}$	$\frac{264}{77}$
2	s_1	0	0	0	$\frac{7}{11}$	1	$-\frac{5}{11}$	$\frac{19}{11}$
3	x_2	0	0	$\frac{11}{2}$	0	$\frac{55}{14}$	$-\frac{11}{14}$	$\frac{165}{14}$

Solve LP by pure Tableau (10)

		z	x_1	x_2	s_1	s_2	s_3	
0		1	0	0	0	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{1320}{77}$
1	x_1	0	2	0	0	$-\frac{6}{7}$	$\frac{4}{7}$	$\frac{264}{77}$
2	s_1	0	0	0	$\frac{7}{11}$	1	$-\frac{5}{11}$	$\frac{19}{11}$
3	x_2	0	0	$\frac{11}{2}$	0	$\frac{55}{14}$	$-\frac{11}{14}$	$\frac{165}{14}$

$$B = \begin{array}{c} \begin{array}{cccc} x_1 & s_1 & x_2 & b \end{array} \\ \begin{bmatrix} 2 & 0 & 0 & \frac{264}{77} \\ 0 & \frac{7}{11} & 0 & \frac{19}{11} \\ 0 & 0 & \frac{11}{2} & \frac{165}{14} \end{bmatrix} \end{array} \tag{9}$$

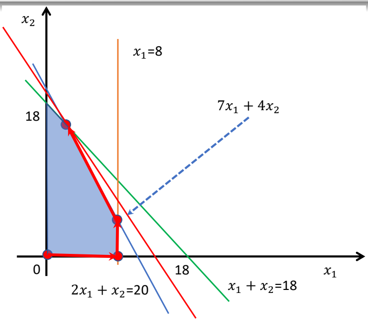
- The maximum is $\frac{1320}{77} \approx 17.1429$
- When $x_1 = \frac{132}{77} = 1.7143$, $x_2 = \frac{165}{77} = 2.1429$, $s_1 = \frac{19}{7} = 2.7143$

Things Attentions should be Paid

1. Keeping the coefficients of the basis variables in the objective to be zeros
2. If this is not the case, undertake a pre-processing step
3. When swap-in/swap-out happens, put the corresponding basis variable in place

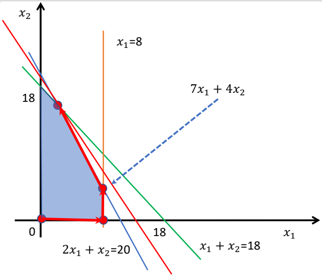
Outline

The Convergence of Simplex



- Solving $Ax = b$ results in a vertex on polyhedral
- Essentially, Simplex method jumps from one vertex to another
- Simplex method is non-degenerate (expected), there are limited num. of vertices
- As a result, it converges to an optimal solution

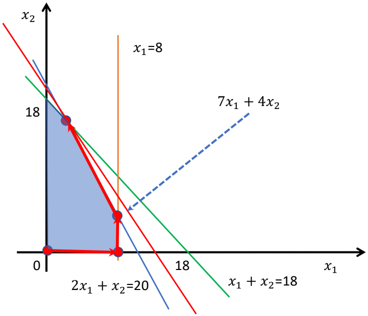
Complexity of Simplex method (1)



- Solving $Ax = b$ relies on matrix $B_{m\times m}$
- Given there are n variables
- We select m columns from n columns to work out a basic solution
- So the total number of basic solutions is

$$\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n\cdot(n-1)\cdots(n-m+1)}{m!} \tag{10}$$

Complexity of Simplex method (2)



$$\binom{n}{m} = \frac{n\cdot(n-1)\cdots(n-m+1)}{m!} \tag{11}$$

- Given $n = 50$, $m = 10$, the above number is 8.3813×10^{57}

Outline

Simplex by Matrix Operation (1)

- In the previous examples, we show you how to do Simplex by Tableau
- It is OK when we solve an LP problem manually
- Solving large-scale LP problem, we should do Simplex with matrix operations
- In the following, we will formulate the Simplex method in matrix operations

Simplex by Matrix Operation (2)

- The original problem

$$\text{Max. } 7x_1 + 4x_2$$
$$\text{s.t.} \begin{cases} 2x_1 + x_2 \leq 20 \\ x_1 + x_2 \leq 18 \\ x_1 \leq 8 \\ x_1, x_2 \geq 0 \end{cases}$$

- Introducing three slack variables x_3, x_4, and x_5

$$\text{Max. } 7x_1 + 4x_2$$
$$\text{s.t.} \begin{cases} 2x_1 + x_2 + x_3 \quad = 20 \\ x_1 + x_2 + \quad + x_4 \quad = 18 \\ x_1 + \quad\quad + x_5 = 8 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{cases}$$

- One constraint regularizes one plane
- Every two planes intersects into a line
- All the half-planes form a polyderal
- The optimal value locates on the vertex of the polyderal
- To find out the maximum, we only need to consider the vertices

Simplex by Matrix Operation (3)

$$\text{Max. } 7x_1 + 4x_2$$

$$\text{s.t.} \begin{cases} 2x_1 + x_2 + x_3 & = 20 \\ x_1 + x_2 + \quad + x_4 & = 18 \\ x_1 + \qquad + x_5 & = 8 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{cases}$$

- Given

$$X = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]^T$$

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 20 \\ 18 \\ 8 \end{bmatrix}$$

$$C = [7 \quad 4 \quad 0 \quad 0 \quad 0]$$

Simplex by Matrix Operation (4)

- Given

$$X = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]^T$$

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 20 \\ 18 \\ 8 \end{bmatrix}$$

$$C = [7 \quad 4 \quad 0 \quad 0 \quad 0]$$

- The original constraints are formulated as
- With target function $f = C * X$

$$AX = b \tag{1}$$

Simplex by Matrix Operation (5)

- The original constraints are formulated as
- With target function $f = C * X$

$$AX = b \tag{1}$$

- Further more, we divide A into two parts, B and N
- $X = [X_B \quad X_N]^T$, $X_B = [x_3 \quad x_4 \quad x_5]$, $X_N = [x_1 \quad x_2]$
- $C = [C_B \quad C_N]$, $C_B = [0 \quad 0 \quad 0]$, $C_N = [7 \quad 4]$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{m \times m} \qquad N = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}_{m \times p} \quad b = \begin{bmatrix} 20 \\ 18 \\ 8 \end{bmatrix}_{m \times 1}$$

- We have

$$[B_{m \times m} \quad N_{m \times p}] \begin{bmatrix} X_B^T \\ X_N^T \end{bmatrix} = b \tag{2}$$

Simplex by Matrix Operation (6)

$$[B_{m\times m} \quad N_{m\times p}]\begin{bmatrix} X_B^T \\ X_N^T \end{bmatrix} = b \tag{2}$$

$$\begin{aligned} B{\cdot}X_B^T + N{\cdot}X_N^T &= b \\ X_B^T + B^{-1}{\cdot}N{\cdot}X_N^T &= B^{-1}{\cdot}b \\ X_B^T &= B^{-1}{\cdot}b - B^{-1}{\cdot}N{\cdot}X_N^T \end{aligned}$$

$$\begin{aligned} f =& [C_B \quad C_N]{\cdot}[X_B \quad X_N]^T \\ f =& C_B{\cdot}X_B^T + C_N{\cdot}X_N^T \\ f =& C_B{\cdot}(B^{-1}{\cdot}b - B^{-1}{\cdot}N{\cdot}X_N^T) + C_N{\cdot}X_N^T \\ f =& C_B{\cdot}B^{-1}{\cdot}b - (C_B{\cdot}B^{-1}{\cdot}N - C_N){\cdot}X_N^T \end{aligned} \tag{3}$$

Simplex by Matrix Operation (7)

$$\begin{aligned} B{\cdot}X_B^T + N{\cdot}X_N^T &= b \\ X_B^T + B^{-1}{\cdot}N{\cdot}X_N^T &= B^{-1}{\cdot}b \\ X_B^T &= B^{-1}{\cdot}b - B^{-1}{\cdot}N{\cdot}X_N^T \end{aligned}$$

$$\begin{aligned} f =& [C_B \quad C_N]{\cdot}[X_B \quad X_N]^T \\ f =& C_B{\cdot}X_B^T + C_N{\cdot}X_N^T \\ f =& C_B{\cdot}(B^{-1}{\cdot}b - B^{-1}{\cdot}N{\cdot}X_N^T) + C_N{\cdot}X_N^T \\ f =& C_B{\cdot}B^{-1}{\cdot}b - (C_B{\cdot}B^{-1}{\cdot}N - C_N){\cdot}X_N^T \end{aligned} \tag{2}$$

- In above formula, $f = C_B{\cdot}B^{-1}{\cdot}b$, given $X_N = 0$
- $f = C_B{\cdot}B^{-1}{\cdot}b - (C_B{\cdot}B^{-1}{\cdot}N - C_N){\cdot}X_N^T$, given $X_N \neq 0$

Simplex by Matrix Operation (8)

$$\begin{aligned} f =& [C_B \quad C_N] \cdot [X_B \quad X_N]^T \\ f =& C_B \cdot X_B^T + C_N \cdot X_N^T \\ f =& C_B \cdot (B^{-1} \cdot b - B^{-1} \cdot N \cdot X_N^T) + C_N \cdot X_N^T \\ f =& C_B \cdot B^{-1} \cdot b - (C_B \cdot B^{-1} \cdot N - C_N) \cdot X_N^T \end{aligned} \tag{2}$$

- In above formula, $f = C_B \cdot B^{-1} \cdot b$, given $X_N = 0$
- $f = C_B \cdot B^{-1} \cdot b - (C_B \cdot B^{-1} \cdot N - C_N) \cdot X_N$, given $X_N \neq 0$
- $X_N = 0$, $f = C_B \cdot B^{-1} \cdot b$ is called the basic solution of the problem
- Given $(C_B \cdot B^{-1} \cdot N - C_N) \cdot X_N^T$ is negative, we can improve this basic solution

Simplex by Matrix Operation (9)

$$
\begin{aligned}
f &= [C_B \quad C_N] \cdot [X_B \quad X_N]^T \\
f &= C_B \cdot X_B^T + C_N \cdot X_N^T \\
f &= C_B \cdot (B^{-1} \cdot b - B^{-1} \cdot N \cdot X_N^T) + C_N \cdot X_N^T \\
f &= C_B \cdot B^{-1} \cdot b - (C_B \cdot B^{-1} \cdot N - C_N) \cdot X_N^T
\end{aligned} \tag{2}
$$

- $X_N = 0$, $f = C_B \cdot B^{-1} \cdot b$ is called the basic solution of the problem
- Given $(C_B \cdot B^{-1} \cdot N - C_N) \cdot X_N^T$ is negative, we can improve this basic solution

Simplex by Matrix Operation (10)

- $X = [X_B \quad X_N]^T$, $X_B = [x_3 \quad x_4 \quad x_5]$, $X_N = [x_1 \quad x_2]$
- $C = [C_B \quad C_N]$, $C_B = [0 \quad 0 \quad 0]$, $C_N = [7 \quad 4]$

$$B = \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{m\times m} \end{array} \qquad N = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}_{m\times p} \end{array} \quad b = \begin{bmatrix} 20 \\ 18 \\ 8 \end{bmatrix}_{m\times 1}$$

$$f = C_B{\cdot}B^{-1}{\cdot}b - (C_B{\cdot}B^{-1}{\cdot}N - C_N){\cdot}X_N^T \tag{2}$$

- The improvement of f relies on matrices **B**, **N** and X_N
- We can exchange columns between **B** and **N** to make f larger
- We set $f_0 = C_B{\cdot}B^{-1}{\cdot}b$

Simplex by Matrix Operation (11)

$$B = \begin{array}{c} \begin{matrix} x_3 & x_4 & x_5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{m\times m} \end{array} \qquad N = \begin{array}{c} \begin{matrix} x_1 & x_2 \end{matrix} \\ \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}_{m\times p} \end{array} \quad b = \begin{bmatrix} 20 \\ 18 \\ 8 \end{bmatrix}_{m\times 1}$$

$$\begin{aligned} f &= C_B \cdot B^{-1} \cdot b - (C_B \cdot B^{-1} \cdot N - C_N) \cdot X_N^T \\ f &= C_B \cdot B^{-1} \cdot b - \sum_{j \in R} (C_B \cdot B^{-1} \cdot p_j - c_j) \cdot x_j \end{aligned} \tag{2}$$

- **R** is the index set of non-basic variables, p_j is the vector in **N**
- We can exchange columns between **B** and **N** to make f larger
- When following inequation holds, we make f larger all the way

$$\sum_{j \in R} (C_B \cdot B^{-1} \cdot p_j - c_j) \cdot x_j < 0 \tag{3}$$

Simplex by Matrix Operation (12)

$$B = \begin{array}{c} \begin{matrix} x_3 & x_4 & x_5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{m\times m} \end{array} \qquad N = \begin{array}{c} \begin{matrix} x_1 & x_2 \end{matrix} \\ \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}_{m\times p} \end{array} \quad b = \begin{bmatrix} 20 \\ 18 \\ 8 \end{bmatrix}_{m\times 1}$$

$$f = C_B \cdot B^{-1} \cdot b - \sum_{j\in R} (C_B \cdot B^{-1} \cdot p_j - c_j) \cdot x_j \tag{2}$$

$$\sum_{j\in R} (C_B \cdot B^{-1} \cdot p_j - c_j) \cdot x_j < 0 \tag{3}$$

- We select one non-basic variable x_k from **R**, make it $x_k > 0$
- Keep the rest x_j in **R** to be 0
- Which x_k we should select??

$$k = \underset{j\in R}{\operatorname{argmin}} \{C_B \cdot B^{-1} \cdot p_j - c_j\} \tag{4}$$

Simplex by Matrix Operation (13)

- In our previous deduction, we have $X_B^T = B^{-1} \cdot b - B^{-1} \cdot N \cdot X_N^T$
- x_k is selected based on following equation

$$k = \underset{j \in R}{\text{argmin}} \{ C_B \cdot B^{-1} \cdot p_j - c_j \} \tag{3}$$

- Since the rest x_j in **R** are 0
- We have

$$X_B = B^{-1} \cdot b - B^{-1} \cdot p_k \cdot x_k \tag{4}$$

- Given $\overline{b} = B^{-1} \cdot b$, $y_k = B^{-1} \cdot p_k$
- We have

$$X_B = \overline{b} - y_k \cdot x_k \tag{5}$$

Simplex by Matrix Operation (14)

$$X_B^T = \overline{b} - y_k \cdot x_k \tag{5}$$

- Since X_B are required to be greater than 0

$$\overline{b} - y_k \cdot x_k \succeq 0 \tag{6}$$

- Namely, we have

$$\begin{bmatrix} \overline{b}_1 \\ . \\ \overline{b}_i \\ . \\ \overline{b}_m \end{bmatrix} - \begin{bmatrix} y_{1k} \\ . \\ y_{ik} \\ . \\ y_{mk} \end{bmatrix} \cdot x_k \succeq 0 \tag{7}$$

Simplex by Matrix Operation (15)

$$X_B^T = \overline{b} - y_k \cdot x_k \quad (5)$$

$$\begin{bmatrix} \overline{b}_1 \\ . \\ \overline{b}_i \\ . \\ \overline{b}_m \end{bmatrix} - \begin{bmatrix} y_{1k} \\ . \\ y_{ik} \\ . \\ y_{mk} \end{bmatrix} \cdot x_k \succeq 0 \quad (7)$$

$$\begin{bmatrix} \frac{\overline{b}_1}{y_{1k}} \\ . \\ \frac{\overline{b}_i}{y_{ik}} \\ . \\ \frac{\overline{b}_m}{y_{mk}} \end{bmatrix} \geq x_k \quad (8)$$

- We take the minimum of all values, namely

$$r = \underset{i}{\text{argmin}}\{\frac{\overline{b}_1}{y_{1k}}, \cdots, \frac{\overline{b}_i}{y_{ik}}, \cdots, \frac{\overline{b}_m}{y_{mk}}\} \quad (9)$$

- So $x_k = \frac{\overline{b}_r}{y_{rk}}$, $x_{B_r} = 0$
- We swap x_k into **B** and swap x_{B_r} out to **N**

Summarize on one round of Simplex

- Given $C_B = [0 \quad 0 \quad 0], C_N = [7 \quad 4]$

$$B = \begin{array}{c} \\ \end{array}\left[\begin{array}{ccc} x_3 & x_4 & x_5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]_{m\times m} \qquad N = \left[\begin{array}{cc} x_1 & x_2 \\ 2 & 1 \\ 1 & 1 \\ 1 & 0 \end{array}\right]_{m\times p} \quad b = \left[\begin{array}{c} 20 \\ 18 \\ 8 \end{array}\right]_{m\times 1}$$

1. Find out $k = \text{argmin}_{j\in R}\{C_B \cdot B^{-1} \cdot p_j - c_j\}$
2. $\overline{b} = B^{-1}b,\ y_k = B^{-1}p_k$
3. Find out $r = \text{argmin}_i\{\frac{\overline{b}_1}{y_{1k}}, \cdots, \frac{\overline{b}_i}{y_{ik}}, \cdots, \frac{\overline{b}_m}{y_{mk}}\}$
4. Swap-in x_k to **B**, swap-out x_{B_r} to **N**
5. Swap corresponding columns in C_B and C_N

Solve LP by Simplex in Matrix Operations: Example (1)

- Given $C_B = [0 \quad 0 \quad 0]$, $C_N = [7 \quad 4]$

$$B = \begin{array}{c} \begin{matrix} x_3 & x_4 & x_5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \quad B^{-1} = \begin{array}{c} \begin{matrix} x_3 & x_4 & x_5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} N = \begin{array}{c} \begin{matrix} x_1 & x_2 \end{matrix} \\ \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \quad b = \begin{bmatrix} 20 \\ 18 \\ 8 \end{bmatrix}$$

1. $k = \text{argmin}_{j \in R}\{C_B \cdot B^{-1} \cdot N_{x_1} - 7, C_B \cdot B^{-1} \cdot N_{x_2} - 4\}$
2. x_1 is selected, $\overline{b} = B^{-1} \cdot b = [20\ 18\ 8]^T$, $y_k = B^{-1} x_1 = [2\ 1\ 1]^T$
3. Find out $r = \text{argmin}_i\{\frac{20}{2}, \frac{18}{1}, \frac{8}{1}\}$
4. Swap-in x_1 to **B**, swap-out x_5 to **N**
5. $C_B = [0\ 0\ 7]$ and $C_N = [0\ 4]$

Solve LP by Simplex in Matrix Operations: Example (2)

- Given $C_B = [0 \quad 0 \quad 7], C_N = [0 \quad 4]$

$$B = \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_1 \end{array} \\ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \quad B^{-1} = \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_1 \end{array} \\ \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} N = \begin{array}{c} \begin{array}{cc} x_5 & x_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \quad b = \begin{bmatrix} 20 \\ 18 \\ 8 \end{bmatrix}$$

1. $k = \text{argmin}_{j \in R}\{[0\ 0\ 7]B^{-1}N_{x_5} - 0, [0\ 0\ 7]B^{-1}N_{x_2} - 4\}$
2. x_2 is selected, $\overline{b} = B^{-1} \cdot b = [4\ 10\ 8]^T$, $y_k = B^{-1}x_1 = [1\ 1\ 0]^T$
3. Find out $r = \text{argmin}_i\{\frac{4}{1}, \frac{10}{1}, \infty\}$
4. Swap-in x_2 to **B**, swap-out x_3 to **N**
5. $C_B = [4\ 0\ 7]$ and $C_N = [0\ 0]$

Solve LP by Simplex in Matrix Operations: Example (3)

- Given $C_B = [4 \quad 0 \quad 7], C_N = [0 \quad 0]$

$$B = \begin{array}{c} \begin{array}{ccc} x_2 & x_4 & x_1 \end{array} \\ \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \quad B^{-1} = \begin{array}{c} \begin{array}{ccc} x_2 & x_4 & x_1 \end{array} \\ \begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \quad N = \begin{array}{c} \begin{array}{cc} x_5 & x_3 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \end{array} \quad b = \begin{bmatrix} 20 \\ 18 \\ 8 \end{bmatrix}$$

1. $k = \text{argmin}_{j \in R}\{[4\ 0\ 7]B^{-1}N_{x_5} - 0, [4\ 0\ 7]B^{-1}N_{x_3} - 0\}$
2. x_5 is selected, $\overline{b} = B^{-1} \cdot b = [4\ 6\ 8]^T$, $y_k = B^{-1}x_5 = [-2\ 1\ 1]^T$
3. Find out $r = \text{argmin}_i\{-\frac{4}{2}, \frac{6}{1}, \frac{8}{1}\}$
4. Swap-in x_5 to **B**, swap-out x_4 to **N**
5. $C_B = [4\ 0\ 7]$ and $C_N = [0\ 0]$

Solve LP by Simplex in Matrix Operations: Example (4)

- Given $C_B = [4 \quad 0 \quad 7], C_N = [0 \quad 0]$

$$B = \begin{array}{ccc} x_2 & x_5 & x_1 \end{array} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad B^{-1} = \begin{array}{ccc} x_2 & x_5 & x_1 \end{array} \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad N = \begin{array}{cc} x_4 & x_3 \end{array} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 20 \\ 18 \\ 8 \end{bmatrix}$$

1. $k = \text{argmin}_{j \in R}\{[4\ 0\ 7]B^{-1}N_{x_4} - 0, [4\ 0\ 7]B^{-1}N_{x_3} - 0\} = \{1, 4\}$
2. **Converges**

Solve LP by Simplex in Matrix Operations: Example (5)

- Given $C_B = [4 \quad 0 \quad 7], C_N = [0 \quad 0]$

$$B = \begin{array}{c} \begin{matrix} x_2 & x_5 & x_1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \quad B^{-1} = \begin{array}{c} \begin{matrix} x_2 & x_1 & x_1 \end{matrix} \\ \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array} \quad N = \begin{array}{c} \begin{matrix} x_4 & x_3 \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \quad b = \begin{bmatrix} 20 \\ 18 \\ 8 \end{bmatrix}$$

1. $k = \text{argmin}_{j \in R}\{[3\ 1\ 0]N_{x_4} - 0, [3\ 1\ 0]N_{x_3} - 0\} = \{1, 4\}$
2. **Converges**

- In our previous deduction, we have $X_B^T = B^{-1} \cdot b - B^{-1} \cdot N \cdot X_N^T$
- So we have $X_B = [x_2, x_5, x_1]^T$, $x_N = [x_4, x_3] = [0\ 0]$

$$X_B = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 20 \\ 18 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 6 \\ 2 \end{bmatrix}. \tag{2}$$