Convex Optimization

Lecture 6: Convex Function

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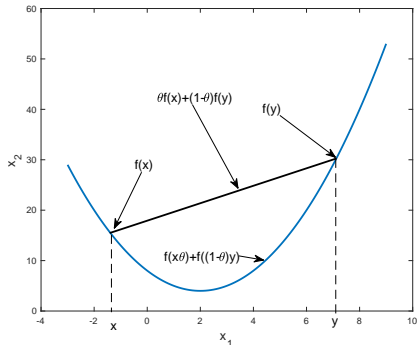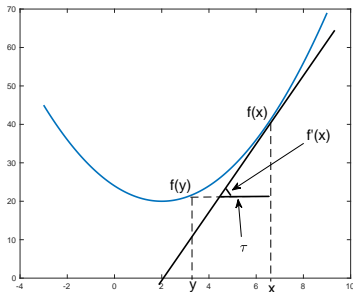
Outline

Convex Function: the definition-1 (1)

- **Definition 1:** Given function $f : R^n \rightarrow R$, it is convex *iff dom f* is convex and $\forall x, y \in dom\ f, \theta \in [0,\ 1]$, following inequation holds

$$f(\theta x + (1 - \theta)y) \le \theta f(x) + (1 - \theta)f(y) \tag{1}$$

Convex Function: the definition-1 (2)

- **Sufficient and Necessary Condition:** Given function $f : R^n \to R$ and it is differentiable , it is convex *iff*

  1 *dom f* is convex;

  2 $\forall x, y \in R^n, f(y) \geq f(x) + \bigtriangledown f^T(x)(y - x)$

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(x) - f(y)}{\tau} = f'(x)$$

$$f(x) - f(y) \leq f'(x)(x - y)$$

$$f(x) - f'(x)(x - y) \leq f(y)$$

$$f(x) + f'(x)(y - x) \leq f(y)$$

Figure: Example in 1D case.
Tangent: $y = f'(x_0)(x - x_0) + f(x_0)$

Convex Function: the definition-1 (3)

- **Sufficient and Necessary Condition:** Given function $f : R^n \to R$ and it is differentiable , it is convex *iff*
    1. *dom f* is convex;
    2. $\forall x, y \in R^n, f(y) \geq f(x) + \bigtriangledown f^T(x)(y - x)$

- **Proof:** $\Rightarrow$

- $\forall x, \ y \in domf, \ \forall t \in (0, 1]$, we have $x + t(y - x) \in domf$

$$f(x + t(y - x)) \leq (1 - t)f(x) + tf(y)$$
$$tf(y) \geq tf(x) + f(x + t(y - x)) - f(x)$$
$$f(y) \geq f(x) + \frac{f(x + t(y - x)) - f(x)}{t}$$
$$\xrightarrow{t \to 0} f(y) \geq f(x) + f'(x)(y - x)$$

— sufficient condition is proved –

Convex Function: the definition-1 (4)

- **Sufficient and Necessary Condition:** Given function $f : R^n \to R$ and it is differentiable , it is convex *iff*
  1. *dom f* is convex;
  2. $\forall x, y \in R^n, f(y) \geq f(x) + \bigtriangledown f^T(x)(y - x)$

- **Proof:** $\Leftarrow$

- $\forall x \neq y \in domf$, $\theta \in [0, 1)$, given $z = \theta x + (1 - \theta)y \in domf$

- Since *domf* is convex, we have

$$\boxed{1} \ f(x) \geq f(z) + f'(z)(x - z)$$
$$\boxed{2} \ f(y) \geq f(z) + f'(z)(y - z)$$

multiply θ on $\boxed{1}$ $\Downarrow$ and $(1 - \theta)$ on $\boxed{2}$

$$\theta f(x) \geq \theta f(z) + \theta(x - z) \cdot f'(z)$$
$$(1 - \theta)f(y) \geq (1 - \theta)f(z) + (1 - \theta)f'(z)(y - z)$$

Convex Function: the definition-1 (5)

- **Sufficient and Necessary Condition:** Given function $f: R^n \to R$ and it is differentiable , it is convex *iff*
  - ① *dom f* is convex;
  - ② $\forall x, y \in R^n, f(y) \geq f(x) + \bigtriangledown f^T(x)(y - x)$

- **Proof:** $\Leftarrow$

- $\forall x \neq y \in domf$, $\theta \in [0, 1)$, given $z = \theta x + (1 - \theta)y \in domf$

- Since *domf* is convex, we have

$$① \quad \theta f(x) \geq \theta f(z) + \theta(x - z) \cdot f'(z)$$

$$② \quad (1 - \theta)f(y) \geq (1 - \theta)f(z) + (1 - \theta)f'(z)(y - z)$$

$$①+② \Downarrow$$

$$\theta f(x) + (1 - \theta)f(y) \geq f(z) + (\theta(x - z) + (1 - \theta)(y - z)) \cdot f'(z)$$
$$\theta f(x) + (1 - \theta)f(y) \geq f(\theta x + (1 - \theta)y)$$

— necessary condition is proved —

Convex Function: the definition-2 & definition-3 (1)

- **Definition 2:** Given function $f : R^n \to R$, it is convex *iff dom f* is convex and $\forall v \in R^n$, $g(t) = f(x + tv)$ is convex, $dom g = \{t | x + tv \in dom f\}$

- **Definition 3:** $\forall x, y \in dom f$, function f, $f : R^n \to R$ and differentiable, it is convex *iff* $f(y) \geq f(x) + \bigtriangledown f^T(x)(y - x)$

- We are going to prove **Definition 3** for its multi-dimensional case

Convex Function: the definition-2 & definition-3 (2)

- **Definition 3:** $\forall x, y \in domf$, function f, $f : R^n \to R$ and differentiable, it is convex *iff* $f(y) \geq f(x) + \bigtriangledown f^T(x)(y - x)$

- **Proof:**$\Rightarrow$

- Given $f : R^n \to R$ is convex, to see whether $f(y) \geq f(x) + \bigtriangledown f^T(x)(y - x)$ holds

- Given $x, y \in domf$, according to **Definition 2**

$$g(t) = f(t \cdot y + (1 - t)x)$$
$$= f(x + t(y - x))$$

Take derivative $\Downarrow$ on $g(x)$

$$g'(t) = \bigtriangledown f^T(ty + (1 - t)x)(y - x)$$

According to $\Downarrow$ $f(y) \geq f(x) + \bigtriangledown f^T(x)(y - x)$

$$g(t_1) \geq g(t_2) + g'(t_2)(t_1 - t_2)$$

Convex Function: the definition-2 & definition-3 (3)

- Given $f : R^n \to R$ is convex, to see whether $f(y) \geq f(x) + \bigtriangledown f^T(x)(y - x)$ holds

- Given $x, y \in domf$, according to **Definition 2**
$$g'(t) = \bigtriangledown f^T(ty + (1 - t)x)(y - x)$$

    According to $\Downarrow f(y) \geq f(x) + \bigtriangledown f^T(x)(y - x)$

$$g(t_1) \geq g(t_2) + g'(t_2)(t_1 - t_2)$$

- Let $t_1 = 1, t_2 = 0$, plug-into above inequation

- $g(t) = f(t \cdot y + (1 - t)x)$
$$g(1) \geq g(0) + g'(0)$$
$$f(y) \geq f(x) + f'(x)(y - x)$$

    — sufficient condition is proved —

Convex Function: the definition-2 & definition-3 (4)

- **Definition 3:** $\forall x, y \in domf$, function f, $f : R^n \to R$ and differentiable, it is convex *iff* $f(y) \geq f(x) + \bigtriangledown f^T(x)(y - x)$

- **Proof:** $\Leftarrow$

- Given $f : R^n \to R$, given $f(y) \geq f(x) + \bigtriangledown f^T(x)(y - x)$, see whether f is convex

- $\forall x, y \in domf$, $ty + (1 - t)x \in domf$, $\widetilde{t}y + (1 - \widetilde{t})x$, we have

$$f(ty + (1 - t)x) \geq f(\widetilde{t}y + (1 - \widetilde{t})x) + \bigtriangledown f(\widetilde{t}y + (1 - \widetilde{t})x)(y - x)(t - \widetilde{t}) \quad (2)$$

$$g(t) = f(ty + (1 - t)x)$$
$$g'(\widetilde{t}) = \bigtriangledown f^T(\widetilde{t}y + (1 - \widetilde{t})x)(y - x)$$

- Plug above equations into Ineqn. 2, we have

$$g(t) \geq g(\widetilde{t}) + g'(t)(t - \widetilde{t}). \quad (3)$$

— necessary condition is proved –

Convex Function Definition: the second order condition

- Given the 2nd-derivative $f : R^n \to R$ exists, then f is convex *iff domf* is convex, and $\bigtriangledown^2 f(x) \succeq 0$, namely Hessian matrix is semi-definite.

$$H(x,y) = \left[\begin{array}{cc} \frac{\partial^2 I(x,y)}{\partial x^2} & \frac{\partial^2 I(x,y)}{\partial x \partial y} \\ \frac{\partial^2 I(x,y)}{\partial y \partial x} & \frac{\partial^2 I(x,y)}{\partial y^2} \end{array} \right] \succeq 0$$

Outline

Convex Functions: Quadratic Function (1)

- Given following quadratic function $f : R^n \to R$ exists $dom f = R^n$ is convex, if $P \in S^n_+$ then f is convex
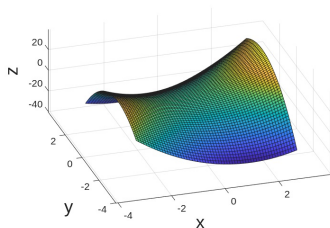

Figure: $f(x, y) = 4x^2 + y^2 + 6xy + 2x + y + 3$.

$$f(x) = \frac{1}{2}x^T P x + q^T x + r$$

(4)

where $P \in S^n_+, q \in R^n, r \in R$

Convex Functions: Quadratic Function (2)

- Given following quadratic function $f : R^n \to R$ exists $dom f = R^n$ is convex, if $P \in S_+^n$ then f is convex

Figure: $f(x,y) = 2x^2 - 3y^2 + 2xy + x + y + 2$.

$$f(x) = [x \ y] \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [1 \ 1] \begin{bmatrix} x \\ y \end{bmatrix} + 2 \tag{5}$$

$$\text{where } P \in S_+^n, q \in R^n, r \in R$$

Convex Functions: Affine Function and Exp. Function

- Given affine function $f(x) = Ax + b$, given $domf = R^n$, then f is convex
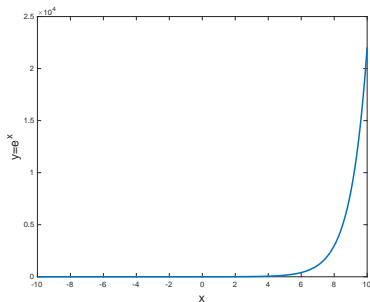
- Given function $f(x) = e^{ax}, x \in R$, then f is convex

Figure: $f(x) = e^x$.

Convex Functions: Affine Function and Exp. Function

- Given affine function $f(x) = Ax + b$, given $domf = R^n$, then f is convex

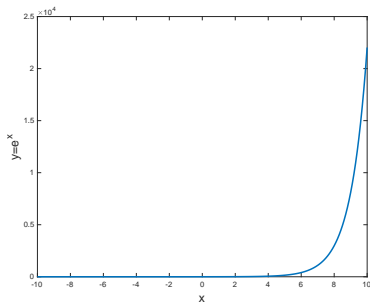- Given function $f(x) = e^{ax}, x \in R$, then f is convex

Figure: $f(x) = e^x$.

Convex Functions: Power Function

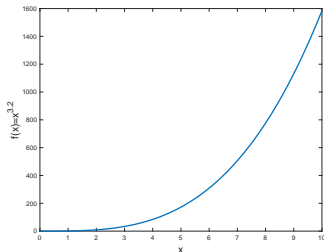- Given power function $f(x) = x^a$, $x \in R_{++}$, $f'(x) = ax^{a-1}$, $f''(x) = a(a-1)x^{a-2}$, we have

Figure: $f(x) = x^{3.2}$.

$$\bigtriangledown^2 f(x) = \left\{ \begin{array}{ll} \geq 0 & a \geq 1 \quad or \quad a \leq 0 \\ \leq 0 & [0, 1] \end{array} \right.$$

Convex Functions: Absolute Power Function

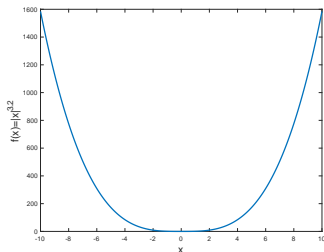- Given Absolute power function $f(x) = |x|^p$, $x \in R$, it is convex when $p \geq 1$

Figure: $f(x) = |x|^{3.2}$.

$$f'(x) = \left\{ \begin{array}{ll} px^{p-1} & x \geq 0 \\ -p(-x)^{p-1} & x < 0 \end{array} \right. \qquad f''(x) = \left\{ \begin{array}{ll} p(p-1)x^{p-2} & x \geq 0 \\ p(p-1)(-x)^{p-2} & x < 0 \end{array} \right.$$

Convex Functions: Logarithmic Function

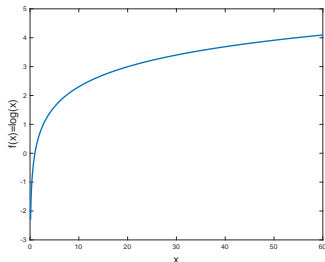- Logarithmic function $f(x) = lg(x)$, $x \in R_{++}$ is concave

Figure: $f(x) = log(x)$.

$$f'(x) = \frac{1}{x} \qquad\qquad f''(x) = -\frac{1}{x^2}$$

Convex Functions: Negative Entropy Function

- Logarithmic function $f(x) = x\log(x)$, $x \in R_{++}$ is concave
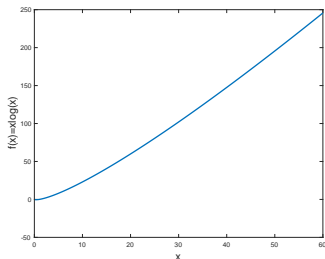- We have $f'(x) = \log x + 1$, $f''(x) = \frac{1}{x} > 0$

Figure: $f(x) = x\log(x)$.

Convex Functions: Norm in R^n Space (1)

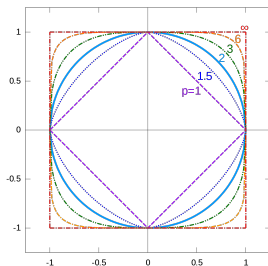- Norm $\|x\|_p$ is a $R^n \to R$ mapping, that satisfies following condition

1. $P(ax) = |a|P(x)$
2. $P(x + y) \leq P(x) + P(y)$
3. $P(x) = 0 \Leftrightarrow x = 0$

- The first and the third meet with the convexity definition
- Let's prove the 2nd
- **Proof:** $\forall x, y \in R^n, \theta \in [0, \ 1]$

$$P(\theta x + (1 - \theta)y) \leq P(\theta x) + P((1 - \theta)y)$$
$$\leq \theta P(x) + (1 - \theta)P(y)$$

- zero-Norm is non-convex!!!, namely $\|x\|_0$ is non-convex

Convex Functions: Norm in R^n Space (2)

- $L^p - norm$ is a typical example, given $p \in R_{++}, x \in R^n$

$$\| x \|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} \tag{6}$$

- 0-Norm is non-convex!!!, namely $\|x\|_0$ is non-convex
- $\| x \|_{\infty} = max\{|x_1|, |x_2|, \cdots, |x_n|\}$

Convex Functions: Max function in R^n Space

- Given $f(x) = max\{x_1, \cdots, x_n\}, x \in R^n$, it is convex
- **Proof:**
- $\forall x, \ y \in R^n, \forall \theta \in [0, 1]$,

$$\begin{aligned} f(\theta x + (1 - \theta)y) &= max\{\theta x_i + (1 - \theta)y_i, i = 1, \cdots, n\} \\ &\leq \theta max\{x_i\} + (1 - \theta)max\{y_i\} \\ &= \theta f(x) + (1 - \theta)f(y) \end{aligned}$$

Convex Functions: Logarithmic on Summation

- Given $f(x) = log(e^{x_1} + \cdots + e^{x_n}), x \in R^n$, it is convex
- This function is convex and satisfies the following function

$$max\{x_1, \cdots, x_n\} \leq f(x) \leq max\{x_1, \cdots, x_n\} + log(n)$$

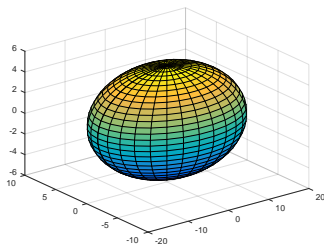# Convex Functions: Logarithmic on Determinant

- The following function is convex

Figure: 3D Ellipsoid.

$$f(X) = \begin{cases} \log \det(X) & X \succ 0 \\ +\infty & otherwise \end{cases}$$

- $\det(X)$ largely relates to the volume of an ellipsoid

Outline

1 Convex Function Definitions

2 Convex Functions

3 **Transformation Preserves Convexity**

4 Extending Function Maintaining its Convexity

Convex Transform: Non-neg. Weighted Summation

- Given functions $f_1, \cdots, f_i, \cdots, f_n$ are convex, $w_i \in R_+$
- Then $f = \sum_{i=1}^{m} w_i f_i$ is convex
- Given $f(x, y)$, $\forall y \in A$, f(x, y) is convex and $w(y) \geq 0$, we have

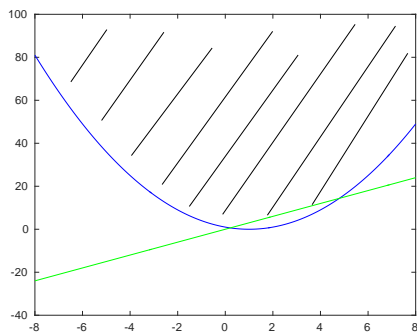$$g(x) = \int_{y \in A} w(y) f(x, y) dy \tag{7}$$

- $g(x)$ is convex

Convex Transform: Affine Transformation

- Given convex function $f : R^m \to R$, $A \in R^{m \times n}$, $b \in R^m$, we have
- $g(x) = f(Ax + b)$ is convex
- **Proof:**
- Given $x, y \in dom\, g, \forall \theta \in [0, 1]$,

$$\begin{aligned}
g(\theta x + (1 - \theta)y) &= f(\theta Ax + (1 - \theta)Ay + b) \\
&= f(\theta(Ax + b) + (1 - \theta)(Ay + b)) \\
&\leq \theta f(Ax + b) + (1 - \theta)f(Ay + b) \\
&= \theta g(x) + (1 - \theta)g(y)
\end{aligned}$$

Convex Transform: Max on Convex Functions (1)

- Given f_1, f_2 are convex, function $f(x) = max\{f_1(x), f_2(x)\}$ is convex
- First of all, $domf = domf_1 \cap domf_2$, it is convex

Convex Transform: Max on Convex Functions (2)

- Given f_1, f_2 are convex, function $f(x) = max\{f_1(x), f_2(x)\}$ is convex
- First of all, $domf = domf_1 \cap domf_2$, it is convex
- **Proof:**

$$
\begin{aligned}
f(\theta x + (1-\theta)y) &= max\{f_1(\theta x + (1-\theta)y), f_2(\theta x + (1-\theta)y)\} \\
&\leq max\{\theta f_1(x) + (1-\theta)f_1(y), \theta f_2(x) + (1-\theta)f_2(y)\} \\
&\leq max\{\theta f_1(x), \theta f_2(x)\} + max\{(1-\theta)f_1(y), (1-\theta)f_2(y)\} \\
&\leq \theta max\{f_1(x), f_2(x)\} + (1-\theta)max\{f_1(y), f_2(y)\}
\end{aligned}
$$

Convex Transform: Perspective Proj. (1)

- Given Euclidean function $f(x) = x^T x, x \in R^n$
- $t > 0$, its perspective projection is defined as

$$g(x, t) = t \cdot (\frac{x}{t})^T (\frac{x}{t}) = \frac{x^T x}{t}$$

- This function is convex w.r.t both x and t

Convex Transform: Perspective Proj. (2)

- Given function $f(x) = -log(x), dom\ f = R_{++}$
- Its perspective projection is defined as

$$g(x, t) = t \cdot (-log\frac{x}{t}) = t \cdot log\frac{t}{x}, dom\ g = R_{++}^2$$

- It is convex, **Proof:**

$$\frac{\partial g}{\partial x} = -\frac{t}{x} \qquad \frac{\partial g}{\partial t} = log\frac{t}{x} + 1$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{t}{x^2} \qquad \frac{\partial^2 g}{\partial x \partial t} = -\frac{1}{x}$$

$$\frac{\partial^2 g}{\partial t \partial x} = -\frac{1}{x} \qquad \frac{\partial^2 g}{\partial t^2} = \frac{1}{t}$$

Convex Transform: Perspective Proj. (3)

- Given $\forall u, v \in R_{++}^n$, define $g(u, v) = \sum_{i=1}^n u_i \cdot log \frac{u_i}{v_i}$
- It is called entropy, it is convex
- K-L divergence is defined as

$$D_{KL} = \sum_{i=1}^n (u_i \cdot log \frac{u_i}{v_i} - u_i + v_i)$$

- It is convex as well

Convex Transform: Conjugate Functions (1)

- Conjugate function is convex despite whether f is convex or not
- Given function $f : R^n \to R \quad f^* : R^n \to R$, Conjugate function is defined as
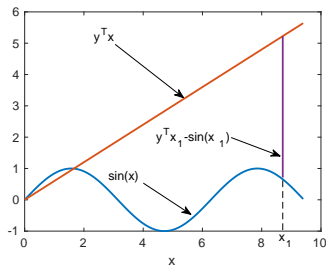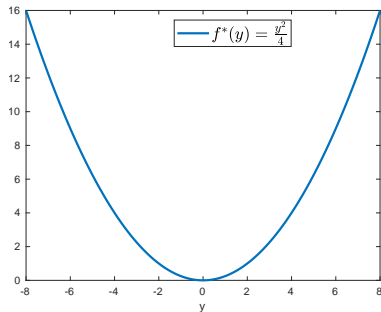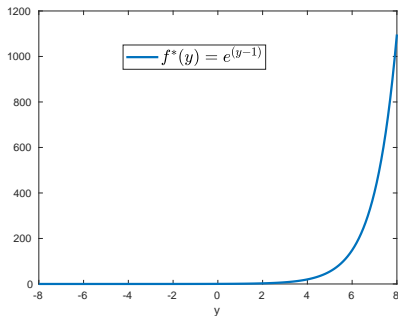
$$f^*(y) = sup_{x \in dom f}(y^T x - f(x)). \tag{8}$$

Figure: $f^*(y) = sup_{x \in dom f}(y^T x - f(x))$

Convex Transform: Conjugate Functions (2)

- Conjugate function is convex despite whether f is convex or not
- Given function $f : R^n \to R$ $f^* : R^n \to R$, Conjugate function is defined as
$$f^*(y) = sup_{x \in domf}(y^T x - f(x)). \qquad (9)$$

- Given $f(x) = x^2$, $f^*(y) = sup_{x \in domf}(y^T x - f(x))$
- In order to get the supreme of $f^*(y)$ w.r.t x
- Take partial derivative on $\frac{\partial f^*(y)}{\partial x} = y - 2x = 0$
- We have $x = \frac{y}{2}$, so $f^*(y) = \frac{y^2}{4}$

Convex Transform: Conjugate Functions (3)

(a) $f^*(y) = sup_{x \in dom f}(y^T x - x^2)$
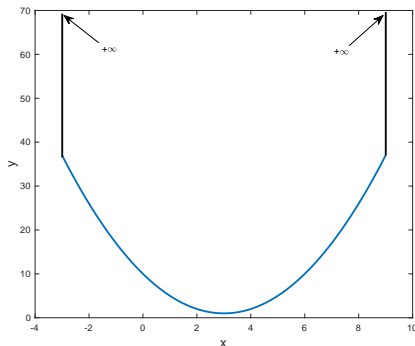
(b) $f^*(y) = sup_{x \in dom f}(y^T x - x ln(x))$

Outline

Function Extension Maintaining its Convexity (1)

- Given function $f : R^n \to R$, it is a convex function, $dom f = C \subseteq R^n$
- Now, we extend it to the whole R^n and maintaining its convexity

$$\widetilde{f(x)} = \begin{cases} f(x) & x \in dom f \\ +\infty & x \notin dom f \end{cases} \tag{10}$$

Function Extension Maintaining its Convexity (2)

- Given function $f : R^n \to R$, it is a concave function, $dom f = C \subseteq R^n$
- Now, we extend it to the whole R^n and maintaining its convexity

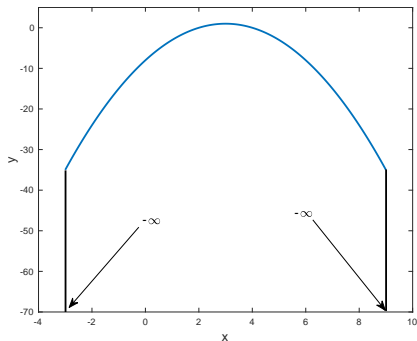$$\widetilde{f(x)} = \begin{cases} f(x) & x \in dom f \\ -\infty & x \notin dom f \end{cases} \tag{11}$$

Indicator Function

- Given convex set $C \in R^n$, and the indicator function defined on it

$$f_C = \begin{cases} \text{undefined} & x \notin C \\ 0 & x \in C \end{cases} \tag{12}$$

- It can be extended to the whole R^n while maintaining its convexity

$$f_C = \begin{cases} +\infty & x \notin C \\ 0 & x \in C \end{cases} \tag{13}$$