Supplementary Material on Solving SVM Duality

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1 Solve the dual problem of SVM model

Overall, W is a vector, it is a sum of m vectors:

$$W = \sum_{i=1}^{m} \alpha_i y^{(i)} x^{(i)} \tag{5}$$

Expand Eqn. 5, we have

$$W = \alpha_1 y^{(1)} x^{(1)} + \alpha_2 y^{(2)} x^{(2)} + \cdots + \alpha_i y^{(i)} x^{(i)} + \cdots + \alpha_m y^{(m)} x^{(m)}$$

So

$$\begin{aligned}
\|W\|^2 &= W^T \cdot W \\
&= (\alpha_1 y^{(1)} \cdot x^{(1)T} + \alpha_2 y^{(2)} \cdot x^{(2)T} + \cdots + \alpha_i y^{(i)} \cdot x^{(i)T} + \cdots + \alpha_m y^{(m)} \cdot x^{(m)T}) \\
&\quad \cdot (\alpha_1 y^{(1)} \cdot x^{(1)} + \alpha_2 y^{(2)} \cdot x^{(2)} + \cdots + \alpha_j y^{(j)} \cdot x^{(j)} + \cdots + \alpha_m y^{(m)} \cdot x^{(m)}) \\
&= \alpha_1 y^{(1)} \cdot x^{(1)T} \cdot (\alpha_1 y^{(1)} \cdot x^{(1)} + \alpha_2 y^{(2)} \cdot x^{(2)} + \cdots + \alpha_j y^{(j)} \cdot x^{(j)} + \cdots + \alpha_m y^{(m)} \cdot x^{(m)}) \\
&\quad + \alpha_2 y^{(2)} \cdot x^{(2)T} \cdot (\alpha_1 y^{(1)} \cdot x^{(1)} + \alpha_2 y^{(2)} \cdot x^{(2)} + \cdots + \alpha_j y^{(j)} \cdot x^{(j)} + \cdots + \alpha_m y^{(m)} \cdot x^{(m)}) \\
&\quad \cdots \\
&\quad + \\
&\quad \cdots \\
&\quad + \alpha_m y^{(m)} \cdot x^{(m)T} \cdot (\alpha_1 y^{(1)} \cdot x^{(1)} + \alpha_2 y^{(2)} \cdot x^{(2)} + \cdots + \alpha_j y^{(j)} \cdot x^{(j)} + \cdots + \alpha_m y^{(m)} \cdot x^{(m)}) \\
&= \sum_{i,j=1}^{m} y^i y^j \cdot \alpha_i \alpha_j \cdot x^{(i)T} \cdot x^{(j)} \\
&= \sum_{i,j=1}^{m} y^i y^j \cdot \alpha_i \alpha_j < x^{(i)}, x^{(j)} >
\end{aligned}$$

For

$$\begin{aligned}
&-\left[\sum_{i=1}^{m} \alpha_i y^{(i)} (W^T \cdot x^{(i)} + b) - \sum_{i=1}^{m} \alpha_i\right] \\
&\Rightarrow -\sum_{i=1}^{m} \alpha_i y^{(i)} W^T \cdot x^{(i)} - \sum_{i=1}^{m} \alpha_i y^{(i)} \cdot b + \sum_{i=1}^{m} \alpha_i \qquad \text{(according to Eqn. 6, } \sum_{i=1}^{m} \alpha_i y^{(i)} \cdot b = 0\text{)} \\
&\Rightarrow -\sum_{i=1}^{m} \alpha_i y^{(i)} \cdot (\alpha_1 y^{(1)} \cdot x^{(1)T} + \alpha_2 y^{(2)} \cdot x^{(2)T} + \cdots + \alpha_j y^{(j)} \cdot x^{(j)T} + \cdots) \cdot x^{(i)} \\
&\Rightarrow -\sum_{i,j=1}^{m} \alpha_i \alpha_j y^{(i)} y^{(j)} x^{(j)T} \cdot x^{(i)} + \sum_{i=1}^{m} \alpha_i
\end{aligned}$$

Summarize above:

$$\begin{aligned}
&\frac{1}{2}\|W\|^2 - \left[\sum_{i=1}^{m} \alpha_i y^{(i)} (W^T \cdot x^{(i)} + b) - \sum_{i=1}^{m} \alpha_i\right] \\
&= -\frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y^{(i)} y^{(j)} x^{(i)T} \cdot x^{(j)} + \sum_{i=1}^{m} \alpha_i \\
&= -\frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y^{(i)} y^{(j)} < x^{(i)}, x^{(j)} > + \sum_{i=1}^{m} \alpha_i
\end{aligned}$$