On the Recursive Teaching Dimension of VC Classes

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November 22, 2019

Overview

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Basic Definitions

Definition

A (x,y) concept class is a concept class, which every x instances of it has maximum y patterns.

Definition

$f(x, y) = max_{C \in (x,y)} TD_{min}(C)$

Theorem

If $VCD(C) = d$, Then $TS_{min}(C) = 2^{d+1}(d-2) + d + 4$.

Proof.

We'll prove by induction, base of induction is clear.
We'll define $C_b^Y = \{c \in C : C_{|Y} = b\}$, imagine $k = 2^d(d-1)+1$. we'll denote Y^*, b^*, smallest size C_b^Y which is not empty and $|Y| = |b| = k$. Without loss of generality imagine $Y^* = [k]$ and $b^* = 0$. If $VCD(C_{b^*}^{Y^*}) \leq d-1$, the by induction we'll drive $TS_{min}(C) \leq TS_{min}(C_{b^*}^{Y^*}) + k \leq 2^{d+1}(d-2) + d + 4$. By the sake of contradiction imagine $VCD(C_{b^*}^{Y^*}) = d$. consider a the size d set in $[n] \backslash Y^*$, which is shattered, then for all i between 1 and k $C_1^{X_i}$ must miss a pattern p_i in a, or else VC-Dimension will be $d+1$.

□

Exponential Upper Bound on RTD (cont.)

Proof.

number of possible patterns is 2^d, and $k = 2^d(d-1)+1$, so there exist a pattern p, which d of $p_i = p$. Replace those indexes in Y^* with a, since non of those indexes had pattern p, the number of concepts in new $C_{b^*}^{Y^*}$ is strictly less than the previous one, which is in contradiction with our assumption.

$\square$

0	$\underline{0}$	$\underline{0}$	$\underline{0}$	0	$\underline{0}$ 0 1 1	$\underline{0}$ 1 0 1
1					$\cancel{0}$	$\cancel{0}$
	1				$\cancel{0}$	$\cancel{1}$
		1			$\cancel{1}$	$\cancel{0}$
			1		$\cancel{1}$	$\cancel{1}$
				1	$\cancel{0}$	$\cancel{0}$

Figure 1: An illustration for the proof of Lemma 4, $\mathrm{TS}_{\min}(\mathcal{C}) \leq 6$ when $d = 2$. We prove by contradiction that the smallest nonempty set $\mathcal{C}_{\boldsymbol{b}^*}^{Y^*}$, after fixing five bits, has $\mathrm{VCD}(\mathcal{C}_{\boldsymbol{b}^*}^{Y^*}) = 1$, where $Y^* = \{1, 2, 3, 4, 5\}$ and $\boldsymbol{b}^* = \mathbf{0}$. In this example, we have $Z = \{6, 7\}$, $Y' = \{2, 3, 4, 6, 7\}$ and $\boldsymbol{b}' = \mathbf{0}$. Note that $\mathcal{C}_{\mathbf{0}}^{Y'}$ is indeed a nonempty proper subset of $\mathcal{C}_{\mathbf{0}}^{Y^*}$.

Lower Bound on Worst-Case RTD

Warmuth represented a concept classes C_W which, $RTD(C_W) = 3$, and $VCD(C_W) = 2$. Here we a concept class is represented which $RTD(C_O) = 5$, and $VCD(C_O) = 3$. note that succinct representation is a representation that for every concept we have all of the rotations of it in concept class, and C_O have 100 concepts.

Lower Bound on Worst-Case RTD (cont.)

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
$\underline{0}$	$\underline{0}$	0	$\underline{0}$	$\underline{0}$	1	$\underline{0}$	1	0	1	0	1
$\underline{0}$	$\underline{0}$	0	$\underline{0}$	$\underline{0}$	1	$\underline{1}$	1	0	1	0	1
$\underline{0}$	$\underline{0}$	0	$\underline{0}$	$\underline{1}$	1	$\underline{0}$	1	0	1	0	1
0	$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{0}$	1	$\underline{1}$	1	0	1	0	1
$\underline{0}$	$\underline{0}$	0	$\underline{1}$	$\underline{1}$	1	0	1	0	1	0	$\underline{1}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	1	0	1	$\underline{0}$	1	0	1	0	$\underline{1}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	1	$\underline{0}$	1	$\underline{1}$	1	0	1	0	1
0	$\underline{1}$	$\underline{0}$	1	0	$\underline{1}$	$\underline{1}$	1	0	1	$\underline{1}$	1
$\underline{0}$	1	$\underline{1}$	1	$\underline{0}$	1	$\underline{1}$	1	0	1	$\underline{1}$	1

Figure 4: The succinct representation of a concept class $\mathcal{C}_0$ with $\mathrm{RTD}(\mathcal{C}_0) = 5$ and $\mathrm{VCD}(\mathcal{C}_0) = 3$. The teaching set of each concept is marked with underline.

Quadratic Upper Bound on RTD

Lemma

For any x,y,z, that $y \leq 2^x - 1$, *and* $z \leq 2y + 1$ *following inequality holds:*

$$f(x+1, z) \leq f(x, y) + \lceil \frac{(y+1)(x-1)+1}{2y - z + 2} \rceil$$

Proof.

Imagine C is a $(x+1, z)$, just like previous section, we'll denote Y^*, b^*, smallest size C_b^Y which is not empty and $|Y| = |b| = k$. Without loss of generality imagine $Y^* = [k]$ and $b^* = 0$. Now, we only need to prove $C_{b^*}^{Y^*}$ is (x, y) class. Assume for the sake of contradiction that previous statement is false, it means that there exist a $|Z| \leq x$, which $|\{c_{|Z} : c \in C_{b^*}^{Y^*}\}| \geq y + 1$ (Generally assume that $Z \cap Y^* = \varnothing$, otherwise consider $Z \backslash Y^*$, instead of Z (Y^* has only one pattern which is zero so $Z \backslash Y^*$ cannot be empty)). □

Proof.

Now Define,

$$C_{b^*}^{Y^*,Z} = \{c_{|Z} : c \in C_{b^*}^{Y^*}\}.$$

Since C is $(x+1, z)$ class, the projection of C on the set $Z \cup \{w\}$ has no more than z patterns. Thus

$$|C_{b^*}^{Y^*,Z}| + |C_1^{\{w\},Z}| \leq z$$

we know $|C_{b^*}^{Y^*,Z}|$, so $|C_1^{\{w\},Z}| \leq z - y - 1$. Now find $\tilde{C}_{b^*}^{Y^*,Z} \subseteq C_{b^*}^{Y^*,Z}$ so that $|\tilde{C}_{b^*}^{Y^*,Z}| = y + 1$. $\forall W \in Y^* : |\tilde{C}_{b^*}^{Y^*,Z} \backslash C_1^{\{w\},Z}| \geq 2y + 2 - z \geq 1$. Thus,

$$\sum_{w \in Y^*} \geq k(2y-z+2) > (y+1)(x-1) = |\tilde{C}_{b^*}^{Y^*,Z}|(x-1) \geq |\tilde{C}_{b^*}^{Y^*,Z}|(|Z|-1).$$

□

Quadratic Upper Bound on RTD (cont.)

Proof.

It then follows from the Pigeonhole Principle that there exists $W \subseteq Y^*$ such that $|W| = |Z|$ and $\bigcap_{w \in W}(C_{b^*}^{Y^*,Z} \setminus C_1^{\{w\},Z}) \neq \varnothing$. Pick any string in it called s, now just like previous section $C_{0 \circ s}^{(Y^* \setminus W) \cup Z}$ is a non empty and proper subset of $C_{b^*}^{Y^*}$, which is contradiction. □

Corollary

$\forall 1 < \alpha < 2 : f(X, \lfloor \alpha^x \rfloor) \leq \frac{(x-1)^2}{4-2\alpha} + \frac{3-2\alpha}{4-2\alpha}.(x-1). = o(X^2)$

Lemma

For some constant c for every $x > cd$, $(\frac{ex}{d})^d \leq \alpha^x$.

Quadratic Upper Bound on RTD (cont.)

Theorem

$RTD(C) = O(VCD(C)^2)$.

Proof.

By Saur's lemma we know that every concept class with VC-Dimension d, is $(x, (\frac{ex}{d})^d)$, after using previous lemma we'll drive that $\forall x > cd$, every concept class with VC-Dimension d is $(x, \lfloor \alpha^x \rfloor)$ class. Using the corollary we'll drive $TD_{min}(C) \leq O(x^2) = O((cd)^2) = O(d^2)$. □

The End