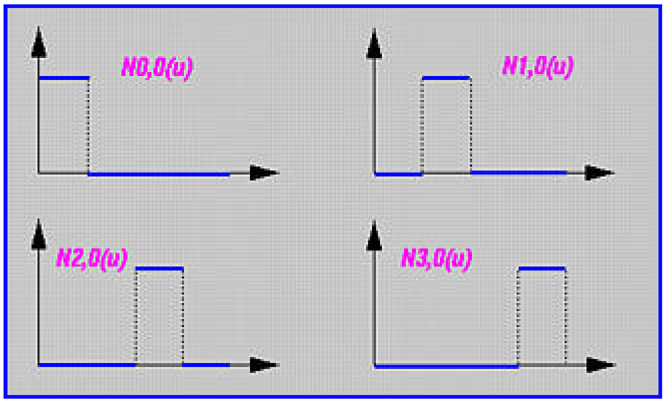
B-spline Basis Functions: Computation Examples

Two examples, one with all simple knots while the other with multiple knots, will be discussed in some detail on this page.

Simple Knots

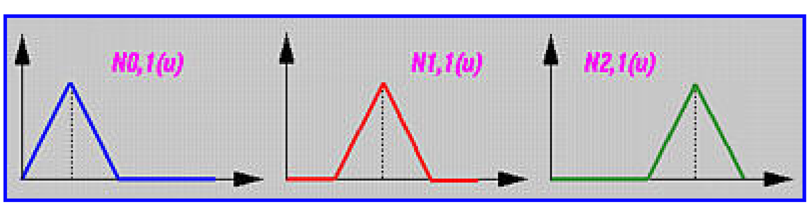
Suppose the knot vector is $U = \{ 0, 0.25, 0.5, 0.75, 1 \}$. Hence, $m = 4$ and $u_0 = 0$, $u_1 = 0.25$, $u_2 = 0.5$, $u_3 = 0.75$ and $u_4 = 1$. The basis functions of degree 0 are easy. They are $N_{0,0}(u)$, $N_{1,0}(u)$, $N_{2,0}(u)$ and $N_{3,0}(u)$ defined on knot span [0,0.25,), [0.25,0.5), [0.5,0.75) and [0.75,1), respectively, as shown below.



The following table gives the result of all $N_{i,1}(u)$'s:

Basis Function	***Range***	***Equation***
$N_{0,1}(u)$	[0, 0.25)	$4u$
	[0.25, 0.5)	$2(1 - 2u)$
$N_{1,1}(u)$	[0.25, 0.5)	$4u - 1$
	[0.5, 0.75)	$3 - 4u$
$N_{2,1}(u)$	[0.5, 0.75)	$2(2u - 1)$
	[0.75, 1)	$4(1 - u)$

The following shows the graphs of these basis functions. Since the internal knots 0.25, 0.5 and 0.75 are all simple (*i.e.*, $k = 1$) and $p = 1$, there are $p - k + 1 = 1$ non-zero basis function and three knots. Moreover, $N_{0,1}(u)$, $N_{1,1}(u)$ and $N_{2,1}(u)$ are C^0 continuous at knots 0.25, 0.5 and 0.75, respectively.

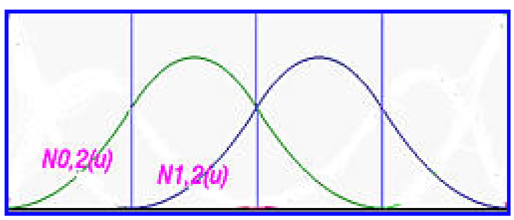


From $N_{i,1}(u)$'s, one can compute the basis functions of degree 2. Since $m = 4$, $p = 2$, and $m = n + p + 1$, we have

$n = 1$ and there are only two basis functions of degree 2: $N_{0,2}(u)$ and $N_{1,2}(u)$. The following table is the result:

Basis Function	***Range***	***Equation***
$N_{0,2}(u)$	$[0, 0.25)$	$8u^2$
	$[0.25, 0.5)$	$-1.5 + 12u - 16 u^2$
	$[0.5, 0.75)$	$4.5 - 12u + 8 u^2$
$N_{1,2}(u)$	$[0.25, 0.5)$	$0.5 - 4u + 8u^2$
	$[0.5, 0.75)$	$-1.5 + 8u - 8u^2$
	$[0.75, 1)$	$8(1 - u)^2$

The following figure shows the two basis functions. The three vertical blue lines indicate the positions of knots. Note that each basis function is a composite curve of three degree 2 curve segments. For example, $N_{0,2}(u)$ is the green curve, which is the union of three parabolas defined on [0,0.25), [0.25, 0.5) and [0.5,0.75). These three curve segments join together forming a smooth bell shape. Please verify that $N_{0,2}(u,)$ (*resp*., $N_{1,2}(u)$) is C^1 continuous at its knots 0.25 and 0.5 (*resp*., 0.5 and 0.75). As mentioned on the previous page, at the knots, this composite curve is of C^1 continuity.



Knots with Positive Multiplicity

If a knot vector contains knots with positive multiplicity, we will encounter the case of 0/0 as will be seen later. Therefore, we shall define 0/0 to be 0. Fortunately, this is only for hand calculation. For computer implementation, there is an efficient algorithm free of this problem. Furthermore, if u_i is a knot of multiplicity k (*i.e.*, $u_i = u_{i+1} = ... = u_{i+k-1}$), then knot spans $[u_i,u_{i+1})$, $[u_{i+1},u_{i+2})$, ..., $[u_{i+k-2},u_{i+k-1})$ do not exist, and, as a result, $N_{i,0}(u)$, $N_{i+1,0}(u)$, ..., $N_{i+k-1,0}(u)$ are all zero functions.

Consider a knot vector $U = \{ 0, 0, 0, 0.3, 0.5, 0.5, 0.6, 1, 1, 1 \}$. Thus, 0 and 1 are of multiplicity 3 (*i.e.*, 0(3) and 1(3)) and 0.5 is of multiplicity 2 (*i.e.*, 0.5(2)). As a result, $m = 9$ and the knot assignments are

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
0	0	0	0.3	0.5	0.5	0.6	1	1	1

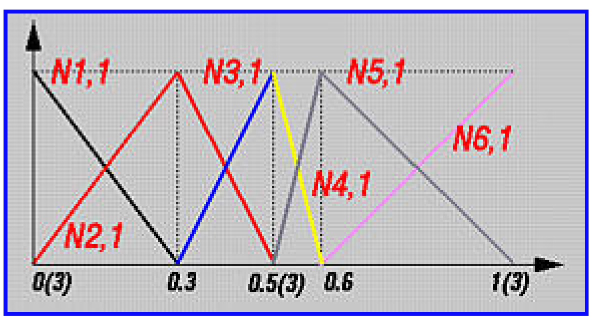
Let us compute $N_{i,0}(u)$'s. Note that since $m = 9$ and $p = 0$ (degree 0 basis functions), we have $n = m - p - 1 = 8$. As the table below shows, there are only four non-zero basis functions of degree 0: $N_{2,0}(u)$, $N_{3,0}(u)$, $N_{5,0}(u)$ and $N_{6,0}(u)$.

Basis Function	***Range***	***Equation***	***Comments***
$N_{0,0}(u)$	all u	0	since $[u_0, u_1)$ = [0,0) does not exist
$N_{1,0}(u)$	all u	0	since $[u_1, u_2)$ = [0,0) does not exist
$N_{2,0}(u)$	$[0, 0.3)$	1	
$N_{3,0}(u)$	$[0.3, 0.5)$	1	
$N_{4,0}(u)$	all u	0	since $[u_4, u_5)$ = [0.5,0.5) does not exist
$N_{5,0}(u)$	$[0.5, 0.6)$	1	
$N_{6,0}(u)$	$[0.6, 1)$	1	
$N_{7,0}(u)$	all u	0	since $[u_7, u_8)$ = [1,1) does not exist
$N_{8,0}(u)$	all u	0	since $[u_8, u_9)$ = [1,1) does not exist

Then, we proceed to basis functions of degree 1. Since p is 1, $n = m - p - 1 = 7$. The following table shows the result:

Basis Function	***Range***	***Equation***
$N_{0,1}(u)$	all u	0
$N_{1,1}(u)$	$[0, 0.3)$	$1 - (10/3)u$
$N_{2,1}(u)$	$[0, 0.3)$	$(10/3)u$
	$[0.3, 0.5)$	$2.5(1 - 2u)$
$N_{3,1}(u)$	$[0.3, 0.5)$	$5u - 1.5$
$N_{4,1}(u)$	$[0.5, 0.6)$	$6 - 10u$
$N_{5,1}(u)$	$[0.5, 0.6)$	$10u - 5$
	$[0.6, 1)$	$2.5(1 - u)$
$N_{6,1}(u)$	$[0.6, 1)$	$2.5u - 1.5$
$N_{7,1}(u)$	all u	0

The following figure shows the graphs of these basis functions.



Let us take a look at a particular computation, say $N_{1,1}(u)$. It is computed with the following expression:

$$N_{1,1}(u) = \frac{u - u_1}{u_2 - u_1} N_{1,0}(u) + \frac{u_3 - u}{u_3 - u_2} N_{2,0}(u)$$

Plugging $u_1 = u_2 = 0$ and $u_3 = 0.3$ into this equation yields the following:

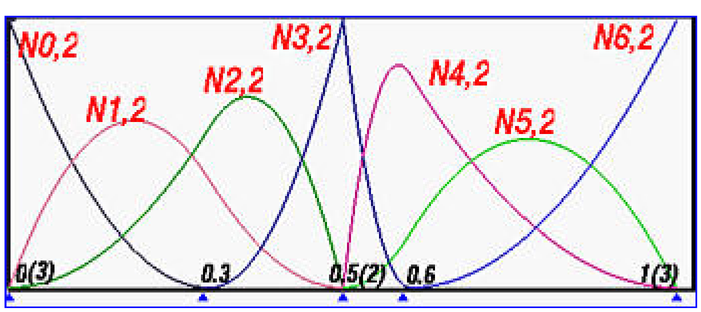
$$N_{1,1}(u) = \frac{u}{0} N_{1,0}(u) + \left(1 - \frac{10}{3}u\right) N_{2,0}(u)$$

Since $N_{1,0}(u)$ is zero everywhere, the first term becomes 0/0 and is defined to be zero. Therefore, only the second term has an impact on the result. Since $N_{2,0}(u)$ is 1 on [0,0.3), $N_{1,1}(u)$ is 1 - (10/3)u on [0,0.3).

Next, let us compute all $N_{i,2}(u)$'s. Since $p = 2$, we have $n = m - p - 1 = 6$. The following table contains all $N_{i,2}(u)$'s:

Basis Function	***Range***	***Equation***
$N_{0,2}(u)$	[0, 0.3)	$(1 - (10/3)u)^2$
$N_{1,2}(u)$	[0, 0.3)	$(20/3)(u - (8/3)u^2)$
	[0.3, 0.5)	$2.5(1 - 2u)^2$
$N_{2,2}(u)$	[0, 0.3)	$(20/3)u^2$
	[0.3, 0.5)	$-3.75 + 25u - 35u^2$
$N_{3,2}(u)$	[0.3, 0.5)	$(5u - 1.5)^2$
	[0.5, 0.6)	$(6 - 10u)^2$
$N_{4,2}(u)$	[0.5, 0.6)	$20(-2 + 7u - 6u^2)$
	[0.6, 1)	$5(1 - u)^2$
$N_{5,2}(u)$	[0.5, 0.6)	$20u^2 - 20u + 5$
	[0.6, 1)	$-11.25u^2 + 17.5u - 6.25$
$N_{6,2}(u)$	[0.6, 1)	$6.25u^2 - 7.5u + 2.25$

The following figure shows all basis functions of degree 2.



Let us pick a typical computation as an example, say $N_{3,2}(u)$. The expression for computing it is

$$\boxed{N_{3,2}(u) = \frac{u - u_3}{u_5 - u_3} N_{3,1}(u) + \frac{u_6 - u}{u_6 - u_4} N_{4,1}(u)}$$

Plugging in $u_3 = 0.3$, $u_4 = u_5 = 0.5$ and $u_6 = 0.6$ yields

$$\boxed{N_{3,2}(u) = (5u - 1.5) N_{3,1}(u) + (6 - 10u) N_{4,1}(u)}$$

Since $N_{3,1}(u)$ is non-zero on [0.3, 0.5) and is equal to $5u - 1.5$, $(5u - 1.5)^2$ is the non-zero part of $N_{3,2}(u)$ on [0.3, 0.5). Since $N_{4,1}(u)$ is non-zero on [0.5, 0.6) and is equal to $6 - 10u$, $(6 - 10u)^2$ is the non-zero part of $N_{3,2}(u)$ on [0.5, 0.6).

Let us investigate the continuity issues at knot 0.5(2). Since its multiplicity is 2 and the degree of these basis functions is 2, basis function $N_{3,2}(u)$ is C^0 continuous at 0.5(2). This is why $N_{3,2}(u)$ has a sharp angle at 0.5(2). For knots not at the two ends, say 0.3, C^1 continuity is maintained since all of them are simple knots.