ECS 178 Course Notes

A MATRIX REPRESENTATION FOR UNIFORM B-SPLINE CURVES

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Overview

The blending functions $N_{i,k}(t)$ for the uniform B-spline curve are just shifted versions of each other. This implies that we should be able to get a “nice” formula to represent these curves. In these notes, we develop a matrix formulation for the curve.

The normalized B-spline blending functions are defined recursively by

$$N_{i,1}(t) = \begin{cases} 1 & \text{if } u \in [t_i, t_{i+1}) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and if $k > 1$,

$$N_{i,k}(t) = \left(\frac{t - t_i}{t_{i+k-1} - t_i} \right) N_{i,k-1}(t) + \left(\frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} \right) N_{i+1,k-1}(t) \tag{2}$$

where $\{t_0, t_1, ..., t_{n+k}\}$ is a non-decreasing sequence of knots, and k is the order of the curve.

The Quadratic Blending Functions using a Uniform Knot Sequence

Assume that $\{t_0, t_1, t_2, ..., t_n\}$ is a uniform knot sequence, *i.e.,* $\{0, 1, 2, ..., n\}$. For the case

$k = 3$, we obtain

$$\begin{aligned}
N_{0,3}(t) &= \frac{t - t_0}{t_2 - t_0} N_{0,2}(t) + \frac{t_3 - t}{t_3 - t_1} N_{1,2}(t) \\
&= \frac{t}{2} N_{0,2}(t) + \frac{3 - t}{2} N_{1,2}(t) \\
&= \begin{cases} \frac{t^2}{2} & \text{if } 0 \le t < 1 \\ \frac{t}{2}(2 - t) + \frac{3-t}{2}(t - 1) & \text{if } 1 \le t < 2 \\ \frac{(3-t)^2}{2} & \text{if } 2 \le t < 3 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{t^2}{2} & \text{if } 0 \le t < 1 \\ \frac{-3+6t-2t^2}{2} & \text{if } 1 \le t < 2 \\ \frac{(3-t)^2}{2} & \text{if } 2 \le t < 3 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

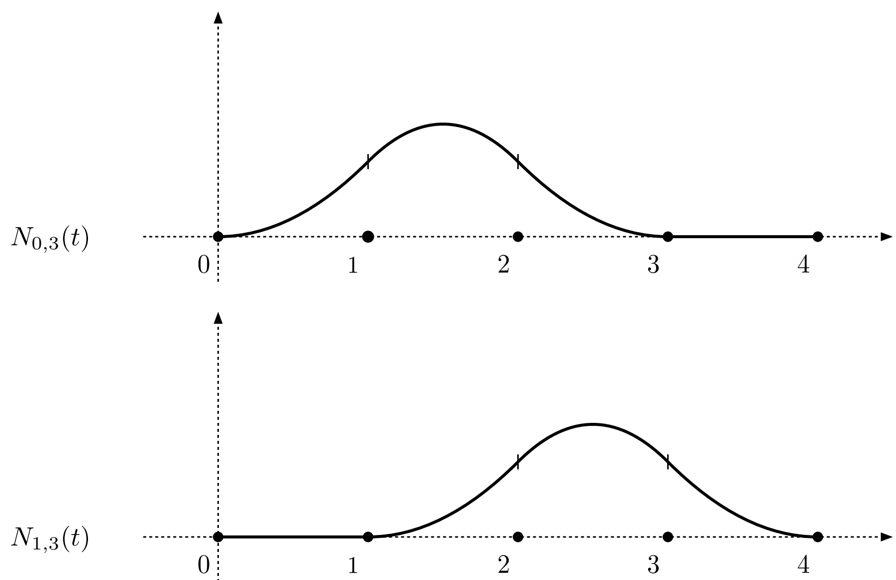
and by nearly identical calculations,

$$\begin{aligned}
N_{1,3}(t) &= \frac{t - t_1}{t_3 - t_1} N_{1,2}(t) + \frac{t_4 - t}{t_4 - t_2} N_{2,2}(t) \\
&= \frac{t - 1}{2} N_{1,2}(t) + \frac{4 - t}{2} N_{2,2}(t) \\
&= \begin{cases} \frac{(t-1)^2}{2} & \text{if } 1 \le t < 2 \\ \frac{-3+6(t-1)-2(t-1)^2}{2} & \text{if } 2 \le t < 3 \\ \frac{(3-(t-1))^2}{2} & \text{if } 3 \le t < 4 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and similarly

$$
\begin{aligned}
N_{2,3}(t) &= \frac{t-t_2}{t_4-t_2}N_{2,2}(t) + \frac{t_5-t}{t_5-t_3}N_{3,2}(t) \\
&= \frac{t-2}{2}N_{2,2}(t) + \frac{5-t}{2}N_{3,2}(t) \\
&= \begin{cases} \frac{(t-2)^2}{2} & \text{if } 2 \le t < 3 \\ \frac{-3+6(t-2)-2(t-2)^2}{2} & \text{if } 3 \le t < 4 \\ \frac{(3-(t-2))^2}{2} & \text{if } 4 \le t < 5 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

These curves are shown in the following figure. They are piecewise quadratic curves, each made up of three parabolic segments that are joined at the knot values



The nonzero portion of these two curves each span the interval between four consecutive knots – e.g., the nonzero portion of $N_{1,3}$ spans the interval $[1,4]$. $N_{1,3}$ is just a shifted version of $N_{0,3}$, and in general we can write

$$N_{i,3}(t) = N_{0,3}(t-i)$$

So, for example, consider the segment of the curve that corresponds to the interval $[2,3)$, then the uniform B-spline can be written as

$$\mathbf{P}(t) = \sum_{i=0}^{2} \mathbf{P}_i N_{i,3}(t)$$

and suppose we want to parameterize this curve segment between $[0,1]$. This can be written in matrix form as

$$\begin{aligned}
\mathbf{P}(t) &= \left[\begin{array}{ccc} N_{0,3}(t+2) & N_{0,3}(t+2) & N_{0,3}(t+2) \end{array}\right] \left[\begin{array}{c} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \end{array}\right] \\
&= \left[\begin{array}{ccc} \frac{1}{2}(1-t)^2 & \frac{1}{2}(1+2t+2t^2) & \frac{1}{2}t^2 \end{array}\right] \left[\begin{array}{c} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \end{array}\right] \\
&= \left[\begin{array}{ccc} 1 & t & t^2 \end{array}\right] \frac{1}{2} \left[\begin{array}{rrr} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{array}\right] \left[\begin{array}{c} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \end{array}\right]
\end{aligned}$$

or

$$\mathbf{P}(t) = \left[\begin{array}{ccc} 1 & t & t^2 \end{array}\right] M \left[\begin{array}{c} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \end{array}\right]$$

where

$$M = \frac{1}{2} \left[\begin{array}{rrr} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{array}\right]$$

This can be extrapolated to the kth segment of the curve, where

$$\mathbf{P}(t) = \left[\begin{array}{ccc} 1 & t & t^2 \end{array}\right] M \left[\begin{array}{c} \mathbf{P}_k \\ \mathbf{P}_{k+1} \\ \mathbf{P}_{k+2} \end{array}\right]$$

where

$$M = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

Matrix Equation for $k = 4$

It is not too difficult to conclude that the $N_{i,4}$ blending functions will be piecewise cubic functions. The support of $N_{i,4}$ will be the interval $[i, i+4]$ and each of the blending functions will be shifted versions of each other, allowing us to write

$$N_{i,4}(t) = N_{0,4}(t - i)$$

In this case, the kth segment can be written as

$$\mathbf{P}(t) \; = \; \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M \begin{bmatrix} \mathbf{P}_k \\ \mathbf{P}_{k+1} \\ \mathbf{P}_{k+2} \\ \mathbf{P}_{k+3} \end{bmatrix}$$

for $k = 0, 1, ..., n-3$, and $0 \le t \le 1$, and where

$$M \; = \; \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Summary

In the case of the uniform knot sequence, the blending functions are shifted versions of each other, and we can exploit this to write each segment of the curve in a matrix form.
