Berkeley
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MATH 222B:

Partial Differential Equations II

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CONTENTS

CHAPTER 1

SOBOLEV SPACES

This chapter mostly develops the theory of Sobolev spaces,which turn out often to be the proper settings in which to apply ideas of functional analysis to glean information concerning PDE.

1.1 Distributions

We recall some basic knowledge of distributions here.

Definition 1.1.1 (Test function).
A test function is a smooth function with compact support, and we denote the space of such functions on an open domain $\Omega \subseteq \mathbb{R}^n$ by $C_c^\infty(\Omega)$, or, in the classical literature, $\mathcal{D}(\Omega)$. We endow the test functions with a countable family of semi-norms

$$\|\phi\|_\alpha := \sup_{x\in\mathbb{R}^n} |\partial^\alpha \phi(x)|$$

We say $\{u_n\}_n \subseteq C_c^\infty(\Omega)$ converges to $u \in C_c^\infty(\Omega)$ in the sense of test functions if

a. for every $\alpha \in \mathbb{N}^n$, we have uniform convergence $\partial^\alpha u_n \to \partial^\alpha u$ as $n \to \infty$,
b. there exists $K \subseteq \mathbb{R}^n$ compact such that $\operatorname{supp} u_n \subseteq K$ for all $n \in \mathbb{N}$.

Given a locally integrable function $u \in L^1_{\text{loc}}(\Omega)$, we can identify it with its action on test functions; define $\langle u, -\rangle : C_c^\infty(\Omega) \to \mathbb{C}$ via the pairing

$$\langle u, \phi\rangle := \int_{\mathbb{R}^n} u\phi \, dx.$$

Definition 1.1.2 (Distribution).
A distribution is a linear functional $T : C_c^\infty(\Omega) \to \mathrm{C}$ continuous with respect to convergence of test functions, denoting the space of distributions on Ω by $\mathcal{D}'(\Omega)$. We endow $\mathcal{D}'(\Omega)$ with the weak topology of pointwise convergence; $\{T_n\}_n \subset \mathcal{D}'(\Omega)$ converges to $T \in \mathcal{D}(\Omega)$ if $T_n(\phi) \to T(\phi)$ for all $\phi \in C_c^\infty(\Omega)$.

1.2 Sobolev Spaces

We start by introducing the Sobolev norms.

Definition 1.2.1 (Sobolev norm).
Let $u \in \mathcal{D}'(U), U$ is an open subset in $\mathbb{R}^d$. The k-th order L^p-based Sobolev norm of u is defined as

$$\|u\|_{W^{k,p}(U)} = \sum_{\alpha:|\alpha|\leq k} \|D^\alpha u\|_{L^p}$$

Here α is the multi-index, $D^\alpha u$ is distributional derivatives (or, derivatives in the weak sense).

For the Sobolev norm to be finite, we need $D^\alpha u \in L^p$. The definition of Sobolev spaces follow naturally.

Definition 1.2.2 (Sobolev spaces).
Sobolev spaces $W^{k,p}(U)$ is defined as

$$W^{k,P}(U) = \left\{u \in \mathcal{D}'(U) : \|u\|_{\omega^{k,p}}(U) < +\infty\right\}$$

and is called the L^P-Sobolev space of order K on U.

Let $C_c^\infty(U)$ be the space of compactly supported smooth functions. It is a subset of $W^{k,p}(U)$. We define its closure in $W^{k,p}(U)$ as

Definition 1.2.3 ($(W_0^{k,p}(U))$).
$W_0^{k,p}(U)$ is defined as the closure of $C_c^\infty(U)$ (with respect to $\|\cdot\|_{W^{k,p}}$)

$$W_0^{k,p}(U) = \overline{C_c^\infty(U)} \subseteq W^{k,p}(U)$$

This is the set of $u \in W^{k,p}(U)$ that vanishes (to appropriate orders) on ∂U.

For $p = 2$, we introduce the following notation:

$$H^k(U) = W^{k,2}(U), \quad H_0^k(U) = W_0^{k,2}(U)$$

Now we are ready to discuss some preliminary properties of Sobolev spaces.

Theorem 1.2.4 (Sobolev spaces are Banach spaces).
For any $k \in \mathbb{Z}_+, 1 \leq p \leq +\infty$,

$$\left(W^{k,p}(U), \|\cdot\|_{W^{k,p}}\right) \text{ is a Banach space.}$$
$$\left(W_0^{k,p}(U), \|\cdot\|_{W^{k,p}}\right) \text{ is a Banach space.}$$

Proof.
The proof of this is first to verify that $\|u\|_{W^{k,p}(U)}$ is a norm. Then, assert that for a Cauchy sequence $\{u_m\}$ in $W^{k,p}(U), \{u_m\}$ and $D^\alpha u_m$ have limit in L^p, denoted as u and u_α. Finally, verify that $D^\alpha u = u_\alpha$.

Then, it is straightforward to have:

Theorem 1.2.5.
For any $k \in \mathbb{Z}_+, 1 \le p \le +\infty$,

$$\left(H^k(U), \langle\cdot,\cdot\rangle\right) \text{ is a Hilbert space.}$$
$$\left(H_0^k(U), \langle\cdot,\cdot\rangle\right) \text{ is a Hilbert space.}$$

Theorem 1.2.6 (Fourier analytic characterization of $H^k(U)$)**.**
For $u \in H^k(U)$, we have

$$\|u\|_{H^k} \simeq \|\hat{u}\|_{L^2} + \left\||\xi|^k \hat{u}\right\|_{L^2} \simeq \left\|\left(1+|\xi|^2\right)^{\frac{k}{2}} \hat{u}\right\|_{L^2}.$$

Here we use the notation that $A \simeq B \Leftrightarrow \exists c_1, c_2 > 0, A \le c_1 B \& B \le c_2 A$.

Proof.

$$\begin{aligned}\|u\|_{H^k} = \|u\|_{W^{k,2}} = \sum_{\alpha:|\alpha|\le k} \|D^\alpha u\|_{L^2} = \sum_{\alpha:|\alpha|\le k} \|\xi^\alpha \tilde{u}\|_{L^2} \simeq \|\hat{u}\|_{L^2} + \left\||\xi|^k \hat{u}\right\|_{L^2} \\ \simeq \left\|\left(1+|\xi|^2\right)^{\frac{k}{2}} \hat{u}\right\|_{L^2}\end{aligned}$$

□

Now we can define the negative order Sobolev spaces:

Definition 1.2.7 (Negative order Sobolev spaces)**.**
For $k \in \mathbb{Z}_+, 1 < p < +\infty$, we define the negative order Sobolev norm:

$$\|u\|_{W^{-k,p}(U)} = \inf\left\{\sum_{\alpha:|\alpha|\le k} \|g_\alpha\|_{L^p} : u = \sum_{\alpha:|\alpha|\le k} D^\alpha g_\alpha\right\}$$

and the negative order Sobolev Spaces:

$$W^{-k,p}(U) = \left\{u \in \mathcal{D}'(U) : u = \sum_{\alpha:|\alpha|\le k} D^\alpha g_\alpha, g_\alpha \in L^p(U)\right\}$$

$W^{-k,p}(U)$ is a Banach space.

If $g \in L^p$, then $D_{x^1} g \in W^{-1,p}(U)$. If $u \in W^{k,p}(U)$, then $D_{x^j} g \in W^{k-1,p}(U)$. Acutually, the negative order Sobolev spaces serves as some kind of dual space:

Theorem 1.2.8.
For $k \in \mathbb{Z}_+, 1 < p < \infty, \frac{1}{p'} + \frac{1}{p} = 1$, we have

$$\left(W_0^{k,p}(U)\right)^* = W^{-k,p'}(U)$$

Proof.

- $W^{-k,p'}(U) \subset \left(W_0^{k,p}(U)\right)^*$

  For any $u = \sum_{\alpha:|\alpha|\le k} D^\alpha g_\alpha \in W^{-k,p'}(U), v \in W_0^{k,p}(U)$, we define

$$\langle u, v\rangle = \sum_{\alpha:|\alpha|\le k} \int D^\alpha g_\alpha v = \sum_{\alpha:|\alpha|\le k} \int (-1)^\alpha g_\alpha D^\alpha v \le C\|u\|_{W^{-k,p'}(U)} \cdot \|v\|_{W^{k,p}(U)},$$

  where in the last equation we can find $\{g_\alpha\}$ such that $\sum_{\alpha:|\alpha|\le k} \|g_\alpha\|_{L^{p'}} \le 2\|u\|_{W^{-k,p'}(U)}$. Therefore $\langle u, \cdot\rangle$ is a bounded linear operator in the dual space of $W_0^{k,p}(U)$.

- $W^{-k,p'}(U) \supset \left(W_0^{k,p}(U)\right)^*$

  In this part, we need Hahn-Banach Theorem:

  > **Theorem 1.2.9** (Hahn-Banach Theorem).
  > For a normed vector space X, M is a subspace of X. If f is bounded linear functional on M, s.t. $|f(u)| \le c\|u\|$, then there exists a bounded linear functional g on X such that
  > $$|g(u)| \le c\|u\| \text{ and } g|_M = f.$$

  Let f be a bounded linear functional on $W_0^{k,p}(U)$. Because for any $\{\alpha : |\alpha| \le k\}$, we have $g_\alpha \in L^p(U)$, we can regard $W_0^{k,p}$ as a subspace of $L^p(U)^{\oplus H}$ where $H = \#\{\alpha : |\alpha| \le k\}$. With Hahn-Banach thm, we have g, such that $g : L^p(U)^{\oplus H} \to \mathbb{R}$. Then we have

$$g(v) = \sum_{\alpha:|\alpha|\le k} \int g_\alpha D^\alpha v,$$

  for any $v \in W_0^{k,p}(U)$. Note that $g \in L^{p'}(U)^{\oplus H}$. Let $\tilde{g}_\alpha = (-1)^{|\alpha|} g_\alpha$ and

$$u = \sum_{\alpha:|\alpha|\le k} D^\alpha \tilde{g}_\alpha,$$

  hence $f(v) = g(v) = \langle u, v\rangle$.

□

1.3 A Digression on Functional Analysis

Let X, Y be Banach spaces. $P : X \to Y$. We are often concerned with the following two problems:

- Existence: For $f \in Y$, is there $u \in X$, s.t. $Pu = f$?
- Uniqueness: If $u, u' \in X$, $Pu = Pu'$, is $u = u'$? THis is equivalent to if $Pu = 0$, is $u = 0$?

P would be a linear differential operator, such as $-\Delta$ or $\Box$. We often prove the a-priori estimate for a PDE, i.e. if $u \in X, Pu = f$, then $\|u\|_X \le c \|f\|_Y$.

These two problems (existence and uniqueness) are related to each other by duality.

Theorem 1.3.1.
Let X, Y be Banach spaces, $P : X \to Y$ is bounded linear operator. Denote $P^* : Y^* \to X^*$ as the adjoint of P, i.e.

$$\langle v, Pu \rangle = \langle P^* v, u \rangle, \quad \forall u \in X, \forall v \in Y^*$$

Suppose $\exists C > 0$ s.t.

$$\|u\|_X \le C \|Pu\|_Y, \forall u \in X$$

then

- (Uniqueness for $Pu = f$) If $u \in X, Pu = 0$, then $u = 0$
- (Existence for $P^* v = g$) $\forall g \in X^*, \exists v \in Y^*$, s.t. $P^* v = g$ and $\|v\|_{Y^*} \le C \|g\|_{X^*}$. (C is the same constant as above.)

Proof.
The first part is trivial. For the second part, we need the Hahn-Banach theorem. Since $u \to Pu$ is injective, we can define a linear map $f(Pu) = \langle g, u \rangle$. Hence, there exists $v \in X^*$ such that

$$\langle v, Pu \rangle = f(Pu) = \langle g, u \rangle.$$

Because $\|f(Pu)\|_Y \le \|g\|_{X^*} \|u\|_X \le C \|g\|_{X^*} \|Pu\|_Y$, we also have $\|v\|_{Y^*} \le C \|g\|_{X^*}$. $\Box$

Now, how about existence theorem for $Pu = f$? Let's find an easy way out, and assume that X is reflexive. (i.e. $X \to (X^*)^* : u \mapsto (u' \mapsto \langle u^*, u' \rangle)$ is an isomorphism.)

Theorem 1.3.2.
Let X, Y be Banach spaces, $P : X \to Y$ is bounded linear operator. X is reflexive. Suppose

$$\|v\|_{Y^*} \le C \|P^* v\|_{X^*}$$

then

- (Uniqueness for $P^* v = g$) If $v \in Y^*, P^* v = 0$, then $v = 0$.
- (Existence for $Pu = f$) $\forall f \in Y, \exists u \in X$, s.t. $Pu = f$ and $\|u\|_X \le C \|f\|_Y$. (C is the same constant as above.)

The proof is similar as above. Note that all sobolev spaces $W_0^{k,p}(U), 1 < p < \infty$ are reflexive. This is left as an exercise. Also note that

$$(\text{Ran}\, P)^\perp = \ker P^* \qquad \ker P = {}^\perp (\text{Ran}\, P^*)$$

Here for $U \subseteq Y$, we define $U^\perp = \{v \in Y^* : \langle v, f \rangle = 0 \quad \forall f \in U\}$. And for $V \subseteq X^*$, we define ${}^\perp V = \{u \in \bar{X} : \langle g, u \rangle = 0 \quad \forall g \in V\}$. As a consequence, if $\ker P^* = 0$, then

$$(\text{Ran}\, P)^\perp = \{0\} \iff \overline{\text{Ran}\, P} = Y$$

In finite dimensions, $\overline{\mathrm{Ran}\, P} = \mathrm{Ran}\, P$. In infinite dimensions, $\overline{\mathrm{Ran}\, P} = \mathrm{Ran}\ P$ not necessarily holds.

Example 1.3.3.
Consider the map:

$$\begin{aligned} S : l(\mathbb{N}) &\longrightarrow l(\mathbb{N}) \\ (x_1, x_2, \dots, x_n, \dots) &\longmapsto (x_1, \frac{x_2}{2}, \dots, \frac{x_n}{n}, \dots). \end{aligned}$$

Hence

$$S(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots) = (1, \frac{1}{4}, \dots, \frac{1}{n^2}, 0, 0, \dots).$$

The limit is $(\frac{1}{n^2})_{n\in\mathbb{N}}$, which is not in the image of S. Therefore, $S(l(\mathbb{N}))$ is not a closed subset of $l(\mathbb{N})$.

Now we demonstrate that $\|v\|_{Y^*} \le C \left\|P^* v\right\|_{X^*}$ is not too much to ask.

Theorem 1.3.4.
X, Y are Banach spaces, $P : X \to Y$ is bounded linear operator. If $P(X) = Y$, then there exists some $C > 0$ such that

$$\|v\|_{Y^*} \le C \left\|P^* v\right\|_{X^*}$$

Proof.
We need the open mapping theorem.

Theorem 1.3.5 (Open mapping theorem for Banach spaces)**.**
If X and Y are Banach spaces and $A : X \to Y$ is a surjective continuous linear operator, then A is an open map.

Then we have $P(B(0,1))$ is open in Y and there exists $B(0,\delta) \subset P(B(0,1))$. Therefore,

$$\|v\|_{Y^*} = \sup \frac{v(x)}{\|x\|_Y} = \sup \frac{v(P(x))}{\|P(x)\|_Y} \le \sup \frac{v(P(x))}{\|(x)\|_X/\delta} \le \delta \|P^* v\|_{X^*}.$$

$\square$

We use the above results on an example.

Example 1.3.6.
Consider

$$\begin{cases} -u'' = f & \text{in} \quad (0,1) \\ \quad u = 0 & \text{at} \quad x = 0, 1 \end{cases}$$

We want to discuss its solvability in $H_0^1((0,1))$.

- (Uniqueness)
  Recall that $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2$. and $\left(H_0^1((0,1))\right)^* = H^{-1}(0,1)$. In this case $X = H_0^1((0,1)), Y = H^{-1}(0,1), Pu = -u''$. Therefore we want to prove $\|u\|_{H^1} \le c\|f\|_{H^{-1}}$ where $f = -u'', u \in H_0^1((0,1))$.
  We only need to consider $u \in C_c^\infty((0,1))$, according to the approximation theorem that we will introduce later. We have

$$\int -u'' u \, \mathrm{d}x = \int f u \, \mathrm{d}x$$

and $\int -u'' u \, \mathrm{d}x = \int (u')^2 \, \mathrm{d}x$. Note that there is no boundary term since $u \in C_c^\infty((0,1))$. For $\|u\|_{L^2}$, we have

$$u(x) = \int_0^x u'(x') \, \mathrm{d}x'$$

then $|u(x)| \leq \int_0^1 |u'(x')| \, \mathrm{d}x' \leq \|u'\|_{L^2}$. The second inequality is because of Cauchy-Schwartz inequality. Then $\int_0^1 |u(x)|^2 \, \mathrm{d}x \leq \sup_{(0,1)} |u|^2 \leq \|u'\|_{L^2}^2$. Therefore

$$\|u\|_{H^1}^2 \leq C\langle f, u\rangle \leq C\|f\|_{H^{-1}}\|u\|_{H^1}$$

Hence, with the first theorem, we have the uniqueness of this equation.

- (Existence)
  Let's compute P^*. $\langle P^* v, u\rangle = \langle v, Pu\rangle$ for all $v \in \left(H^{-1}\right)^*, u \in H_0^1$. Note that by reflexibility of H_0^1, we have $\left(H^{-1}\right)^* = H_0^1$. Then

$$\langle v, Pu\rangle = \int_0^1 v\left(-u''\right) \mathrm{d}x = \int_0^1 v' u' \mathrm{d}x = \int_0^1 -v'' u \, \mathrm{d}x$$

  Therefore $P^* v = -v'', Y^* = H_0^1((0,1)), X^* = H^{-1}((0,1))$. Then according to the second theorem, we have $\forall f \in H^{-1}, \exists u \in H_0^1$, s.t. $Pu = f$.

1.4 Approximation and Extension

The main tools of approximation theorems are convolution, mollifiers and smooth partition of unity.

Lemma 1.4.1 (Mollifiers).
Let φ be a smooth and compactly supported function. $\int \varphi \mathrm{d}x = 1$. Mollifiers are defined as

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi\left(\frac{x}{\varepsilon}\right)$$

Note that $\int \varphi_\varepsilon = 1$. For $u \in L^p\left(\mathbb{R}^d\right), 1 \leq p < \infty$, we have

$$\|\varphi_\varepsilon * u - u\|_{L^p} \to 0 \text{ as } \varepsilon \to 0$$

where

$$\varphi_\varepsilon * u = \int \varphi_\varepsilon(x-y)u(y)dy$$

Here $\varphi_\varepsilon * u$ is C^∞.

Proof.
We consider translation in L_p. We define

$$\tau_z f(x) = f(x-z)$$

In real analysis, we have

$$\lim_{z\to 0}\|\tau_z f - f\|_{L^p} = 0.$$

The point is that compact continuous functions are dense in L^p. Then we have

$$\begin{aligned}
\|\varphi_\varepsilon * u - u\|_{L^p} &= \|\int (u(x-y)-u(x))\varphi_\varepsilon(y)\,dy\|_{L^p} \\
&\leq \int \|u(\cdot - y) - u(\cdot)\|_{L^p}|\varphi_\varepsilon(y)|\,dy \\
&= \int \|u(\cdot - \varepsilon y) - u(\cdot)\|_{L^p}|\varphi(y)|\,dy \to 0.
\end{aligned}$$

Note that in the second line, we used the Minkowski's integral inequality:

Lemma 1.4.2 (Minkowski's integral inequality).
Suppose that (S_1,μ_1) and (S_2,μ_2) are two σ-finite measure spaces and $F: S_1\times S_2 \to \mathbf{R}$ is measurable. Then Minkowski's integral inequality:

$$\left[\int_{S_2}\left|\int_{S_1}F(x,y)\mu_1(\,\mathrm{d}x)\right|^p\mu_2(\,\mathrm{d}y)\right]^{\frac{1}{p}} \leq \int_{S_1}\left(\int_{S_2}|F(x,y)|^p\mu_2(\,\mathrm{d}y)\right)^{\frac{1}{p}}\mu_1(\,\mathrm{d}x).$$

□

Then, we have the following Lemma for smooth cutoff functions.

Lemma 1.4.3 (Smooth partition of unity).
Lemma 3.2 (Smooth partition of unity). $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open covering of U in $\mathbb{R}^d$. Then there exists a smooth partition of unity $\{\chi_\alpha\}$ on U subordinate to $\{U_\alpha\}_{\alpha\in\mathcal{A}}$, i.e.:

- $\sum_\alpha \chi_\alpha(x) = 1$ on U, and $\forall x \in U, \exists$ only finitely many nonzero $\chi_\alpha(x)$.
- $\operatorname{supp}\chi_\alpha \subseteq U_\alpha$.

- χ_α is smooth.

Proof.
Please check the notes for Math 214 Differential Manifold.

Approximation theorems basically says that: given $u \in W^{k,p}(U)$, we want to approximate it with something better, for example smooth and/or has a nice support property. We first state the approximation theorem in $\mathbb{R}^d$.

Theorem 1.4.4 (Approximation theorem in $\mathbb{R}^d$).
For integer $k \geq 0, 1 \leq p < \infty$, we have

- $C^\infty\left(\mathbb{R}^d\right)$ is dense in $W^{k,p}\left(\mathbb{R}^d\right)$;
- $C_c^\infty\left(\mathbb{R}^d\right)$ is dense in $W^{k,p}\left(\mathbb{R}^d\right)$

Proof.
For part 1, we just need the mollifiers. For part 2, we need cut off functions $\chi(\cdot/R)$ where $R \to \infty$. □

Now we turn to the approximation in open subset U. Let's look at a global approximation.

Theorem 1.4.5 (Global approximation theorem for open set U).
For integer $k \geq 0, 1 \leq p < \infty$, let U be any open subset of $\mathbb{R}^d$. Then $C^\infty(U)$ is dense in $W^{k,p}(U)$.

Proof.
Consider the following set

$$U_k = \{x \in U : dist(x, \partial U) > \frac{1}{k}\}, \quad V_k = U_{k+2}/\bar{U}_{k+1},$$

where $k = 1, 2, \ldots$. Then we have

$$U = \bigcup_{k=0}^{\infty} V_k.$$

Therefore, in terms of smooth partition of unity, we have χ_k which $\operatorname{supp} \chi_k \subset V_k$. For any $u \in W^{k,p}(U)$, we have $u_k = \chi_k u \in W^{k,p}(\mathbb{R}^n)$. Hence, there exists $v_k \in C^\infty(\mathbb{R}^n)$ such that

$$\|v_k - u_k\|_{L^p} \leq \epsilon_i \leq 2^{-i}\varepsilon, \quad \operatorname{supp} v_k \subset U_{k+3}/\bar{U}_k.$$

Consider $v = \sum_k v_k$, we have

$$\|v - u\|_{L^p} \leq \varepsilon.$$

□

What if we want to approximate u with smooth functions on $\bar{U}$? The answer is yes we can as long as $\bar{U}$ has a good enough boundary. Let us first define what is $C^\infty(\bar{U})$:

Definition 1.4.6 ($C^\infty(\bar{U})$).
$\left(C^\infty(\bar{U})\right).C^\infty(\bar{U})$ is defined as

$$C^\infty(\bar{U}) = \left\{u : U \to \mathbb{R} : u \text{ is the restriction to } U \text{ of a smooth function } \tilde{u} \in C^\infty(\tilde{U}), U \subset \tilde{U}\right\}$$

and recall the definition of boundary being C^k:

Definition 1.4.7 (C^k boundary).
∂U is of class C^k if $\forall x_0 \in \partial U, \exists \gamma = \gamma(x_0) > 0$, s.t. up to relabeling the variables,

$$B_r(x_0) \cap U = \left\{ x \in B_r(x_0) : x^d > \gamma\left(x^1, \cdots, x^{d-1}\right)\right\}$$

for some C^k function $\gamma = \gamma\left(x^1, \cdots, x^{d-1}\right)$ on $B_r(x_0) \cap \mathbb{R}^{d-1} \times \left\{x_0^d\right\}$.

Theorem 1.4.8 (Global approximation theorem for bounded open set U with $C^\infty(\bar{U})$).
For integer $k \geq 0, 1 \leq p < \infty$, let U be an open subset of $\mathbb{R}^d$ with boundary ∂U of class C^1, then $C^\infty(\bar{U})$ is dense in $W^{k,p}(U)$.

Proof.
Since U is bounded, ∂U is compact. Therefore, we can use finite balls $B(x, r)$ with $x \in \partial U$ to cover ∂U. Hence $\hat{U} = U \cup \left(\bigcup_{i=1}^k B(x_i, r_i)\right)$, we can use partition of unity again. For this problem, we just need to focus on $B(x_i, r_i)$ and $\chi_i u$.

WLOG, we can assume $x_i = 0$ and $B(x_i, r_i) \cap U = \{x_i^d > f(x_i^1, x_i^2, \ldots, x_i^{d-1})\}$. Here we will use a two-step approximation:

$$\|u - v\| \leq \|u - w_\eta\| + \|w_\eta - v\|$$

Here we let $w_\eta(x) = u(x + \eta e^d)$. The goal of this is to deal with the boundary points of U. What we do here is to make $V = B(0, \frac{r_i}{2}) \cap U$ move up and sit in the interior of $B(0, r_i) \cap U$, and then we have enough space for mollifying. With continuity of L_p translation, we have

$$\|u - w_\eta\|_{U \cap B(x_i, r_i)} \leq \frac{\varepsilon}{2}.$$

Then, we can let $v = \varphi_\delta * w_\eta$. Hence, $v \in C^\infty\overline{(U \cap B(x_i, r_i))}$. □

Now we discuss extension theorem. Extension theorem is a tool to deal with $u \in W^{k,p}(U)$, where U is a bounded domain.

Theorem 1.4.9 (Extension).
For $k \geq 0$ nonnegative integer, $1 \leq p < \infty, U$ is a bounded domain with C^k boundary. V is an open set such that $\bar{U} \subset V$. Then $\exists \mathcal{E} : W^{k,p}(U) \to W^{k,p}\left(\mathbb{R}^d\right)$, such that

- (Extension) $\mathcal{E}u|_U = u$,
- (Linear and bounded) $\mathcal{E}$ is linear and

$$\|\mathcal{E}u\|_{W^{k,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{k,p}(U)}$$

- (Support prescription) $\operatorname{supp} \mathcal{E}u \subset V$.

Proof.
Since we have $C^\infty(\bar{U})$ is dense in $W^{k,p}(U)(U$ bounded), we just consider $u \in C^\infty(\bar{U})$.

Step 1: Reduce to half ball case.

Similarly, we can construct $U_0, U_1, \cdot, U_k$ and $\chi_0, \chi_1, \ldots, \chi_k \in C^\infty$. Define $u_k = \chi_k u$. We have $u_0 \in W^{k,p}(\mathbb{R}^d)$. We just have to consider u_i. We have the following transform for $U \cap U_i$,

$$\begin{cases} y^i = x^i - x_k^i, i = 1, \ldots, d-1 \\ y^d = x^d - \gamma(x^1, \ldots, x^d) \end{cases}$$

This is a C^k map from $U \cap U_i$ to $\tilde{U} = \{y \in B_{\tilde{r}}(0), y^d \geq 0\}$. let $u_k(y) = u_k(x(y))$. We have

$$\|u_k(y)\|_{W^{k,p}(\tilde{U})} \leq C\|u_k(x)\|_{W^{k,p}(U_i \cap U)}.$$

Step 2: Extention in half-ball case. Given $U = B_r^+(0), W = B_{r/2}^+(0), \operatorname{supp} u \subset W$. We consider higher order reflection method.

$$\tilde{u} = \begin{cases} u, & x^d > 0 \\ \sum_{i=0}^K \alpha_j u\left(x^1, \cdots, x^{d-1}, -\beta_j x^d\right), & x^d < 0 \end{cases}$$

where $0 < \beta_j < 1$ but not yet determined. We need to match the normal derivatives on $\left\{x^d = 0\right\}$ up to order k. Note that

$$\partial_{x^d}^j \left(u\left(x^1, \cdots, x^{d-1}, -\beta_i x^d\right)\right) = (-1)^j \beta_i^j \left(\partial_{x^d}^j u\right)\left(x^1 \cdots, x^{d-1}, -\beta_j x^d\right)$$

Therefore we have

$$\begin{cases} u\left(x^1, \cdots, x^{d-1}, 0+\right) = \sum_{j=0}^k \alpha_j u\left(x^1, \cdots, x^{d-1}, 0+\right) \\ \partial_{x^d} u\left(x^1, \cdots, x^{d-1}, 0+\right) = \sum_{j=0}^k \alpha_j \left(-\beta_j\right)\left(\partial_{x_d} u\right)\left(x^1, \cdots, x^{d-1}, 0+\right) \\ \cdots \\ \partial_{x^d}^k u\left(x^1, \cdots, x^{d-1}, 0+\right) = \sum_{j=0}^k \alpha_j \left(-\beta_j\right)^k \left(\partial_{x_d}^k u\right)\left(x^1, \cdots, x^{d-1}, 0+\right) \end{cases}$$

which means

$$\begin{cases} 1 = \sum_{j=0}^k \alpha_j \\ 1 = \sum_{j=0}^k \alpha_j \left(-\beta_j\right) \\ \vdots \\ 1 = \sum_{j=0}^k \alpha_j \left(-\beta_j\right)^k \end{cases}$$

and as long as all β_j are distinct, then there exists such α_j to satisfy this equation system. (Recall the property of Vandermonde matrix.)

Finally, use an appropriate smooth cutoff χ_V (s.t. $\chi_V = 1$ on U and supp $\chi_V \subset V$) to define $\mathcal{E}u$, i.e. $\mathcal{E}u = \chi_v \tilde{u}$.

□

1.5 Trace Theorem

Let U be an open subset of $\mathbb{R}^d$ with ∂U being C^1 and $1 < p < \infty$. Recall that for any integer $k \geq 0, C^\infty(\bar{U})$ is dense in $W^{k,p}(U)$. In particular, $C^\infty(\bar{U})$ is dense in $W^{1,p}(U)$. Our aim is to discuss the restriction of $u \in W^{1,p}(U)$ to ∂U. Since the boundary is a measure 0 set, this is hard to specify directly (as L^p functions are only well-defined modulo null sets), so we will achieve this by appealing to the dense subset $C^\infty(\bar{U})$.

Definition 1.5.1 (Trace).
For $u \in C^1(\bar{U})$, we define the trace to be $\operatorname{tr}_{\partial U} u = u|_{\partial U}$.

We wish to extend this operation to all of $W^{1,p}(U)$. Note that $\operatorname{tr} \partial U$ is linear, so we can extend it if we know it is bounded.

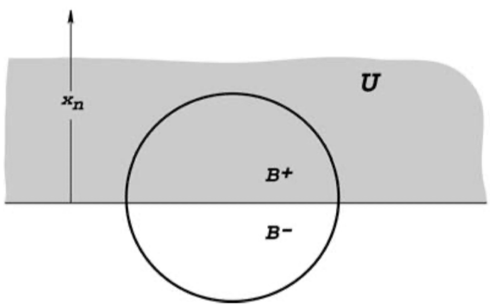
Theorem 1.5.2 (Trace theorem, non-sharp).
Let U be a bounded, open subsets of $\mathbb{R}^d$ with C^1 boundary ∂U, and let $1 < p < \infty$. Then for $u \in C^1(\bar{U})$, we have

$$\|\operatorname{tr}_{\partial U} u\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}.$$

- As a consequence, $\operatorname{tr}_{\partial U}$ is extended (uniquely) by continuity (and density of $C^1(\bar{U}) \subseteq W^{1,p}(U)$) to $\operatorname{tr}_{\partial U} : W^{1,p}(U) \to L^p(\partial U)$.
- Moreover, $u \in W_0^{1,p}(U) \Leftrightarrow \operatorname{tr}_{\partial U} u = 0$.

Proof.
Assume first $u \in C^1(\bar{U})$. Let us also initially suppose $x^0 \in \partial U$ and ∂U is flat near x^0, lying in the plane $\{x_n = 0\}$. Choose an open ball B as in the previous proof and let $\hat{B}$ denote the concentric ball with radius $r/2$.

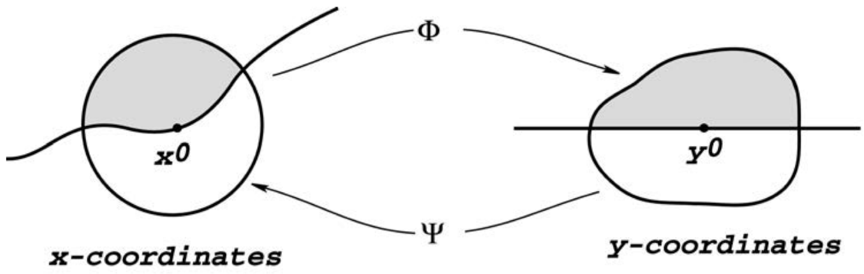
Select $\zeta \in C_c^\infty(B)$, with $\zeta \geq 0$ in $B, \zeta \equiv 1$ on $\hat{B}$. Denote by Γ that portion of ∂U within $\hat{B}$. Set $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} = \{x_n = 0\}$. Then



A half-ball at the boundary

$$\begin{aligned}
\int_\Gamma |u|^p dx' &\leq \int_{\{x_n=0\}} \zeta |u|^p dx' = -\int_{B^+} (\zeta|u|^p)_{x_n} dx \\
&= -\int_{B^+} |u|^p \zeta_{x_n} + p|u|^{p-1}(\operatorname{sgn} u) u_{x_n} \zeta dx \\
&\leq C \int_{B^+} |u|^p + |Du|^p dx
\end{aligned}$$

where we employed Young's inequality.



Straightening out the boundary

Remark.
The map $tr_{\partial u} : W^{1,p}(U) \to L^p(\partial U)$ is not surjective.

Now, we will understand the sharp trace theorem in a restricted setting.

The setting we have in mind is $p = 2$. The advantage here is that we may use the theory of the Fourier transform and Plancherel's theorem. We will also focus on the domain $U = \mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^d > 0\}$ with boundary $\{(x', 0) \in \mathbb{R}^d\} \cong \mathbb{R}^{d-1}$, where $x' := (x^1, \ldots, x^{d-1})$. Recall the Fourier transform characterization of the H^k norm:

$$\|u\|^2_{H^k} \simeq \left\| \left(1 + |\xi|^2\right)^{k/2} \widehat{u} \right\|^2_{L^2_\xi}, \quad k \geq 0 \text{ an integer.}$$

If we replace k with any $s \in \mathbb{R}$, we can talk about **fractional (L^2-based) Sobolev spaces**.

Definition 1.5.3 (Fractional Sobolev spaces).
The space of distributions equipped with the norm

$$\|u\|^2_{H^k} \simeq \left\| \left(1 + |\xi|^2\right)^{k/2} \widehat{u} \right\|^2_{L^2_\xi},$$

where k is any real number.

Theorem 1.5.4 (Sharp trace theorem).
For $u \in C^1\left(\overline{\mathbb{R}^d_+}\right) \cap H^1\left(\mathbb{R}^d_+\right)$,

$$\|\mathrm{tr}_{\partial U}\, u\|_{H^{1/2}\left(\mathbb{R}^{d-1}\right)} \leq C\|u\|_{H^1\left(\mathbb{R}^d_+\right)}.$$

Proof.
Proof. Take $u \in C^1\left(\overline{\mathbb{R}^d_+}\right) \cap H^1\left(\mathbb{R}^d_+\right)$. Using the extension procedure from last time, we can find a $\tilde{u} \in C^1\left(\mathbb{R}^d\right)$ such that

$$\|\tilde{u}\|_{H^1\left(\mathbb{R}^d\right)} \leq C\|u\|_{H^1\left(\mathbb{R}^d_+\right)}$$

Then

$$\begin{aligned} \mathrm{tr}_{\partial U}\, u\left(x'\right) &= u\left(x', 0\right) \\ &= \widetilde{u}\left(x', 0\right) \\ &= \int \mathcal{F}_{x^d} \widetilde{u}\left(x', \xi_d\right) \frac{1}{2\pi} d\xi_d \end{aligned}$$

On the other hand,

$$\mathcal{F}_{x',0} \operatorname{tr}_{\partial U} u\left(\xi'\right)=\int \mathcal{F} \widetilde{u}\left(\xi', \xi_d\right) \frac{1}{2 \pi} d \xi_d$$

For now, let us not assume $s=1/2$ so we can see where this choice comes from.

$$\begin{aligned}
\|\operatorname{tr} \partial U u\|_{H^s\left(\mathbb{R}^{d-1}\right)} & \sim\left\|\left(1+\left|\xi'\right|^2\right)^{s/2} \mathcal{F}_{x'} \operatorname{tr} u\left(\xi', 0\right)\right\|_{L_{\xi'}^2} \\
& =\left\|\left(1+\left|\xi'\right|^2\right)^{s/2} \int \mathcal{F} \widetilde{u}\left(\xi', \xi_d\right) \frac{1}{2 \pi} d \xi_d\right\|_{L_{\xi'}^2}
\end{aligned}$$

Writing $|\xi|^2=\left|\xi'\right|^2+\xi_d^2$,

$$=\left\|\int \frac{\left(1+|\xi'|^2\right)^{s/2}}{\left(1+|\xi'|^2+\xi_d^2\right)^{1/2}}\left(\left(1+\left|\xi'\right|^2+\xi_d^2\right)^{1/2} \mathcal{F} \widetilde{u}\right) \frac{1}{2 \pi} d \xi_d\right\|_{L_{\xi'}^2}$$

Applying Cauchy-Schwarz,

$$\begin{aligned}
& \leq\left\|\left(\int \frac{\left(1+|\xi'|^2\right)^s}{1+|\xi'|^2+\xi_d^2} d \xi_d\right)^{1/2}\left\|\left(1+|\xi|'|^2+\xi_d^2\right)^{1/2} \mathcal{F} \widetilde{u}\right\|_{L_{\xi_d}^2}\right\|_{L_{\xi'}^2} \\
& \leq \sup _{\xi' \in \mathbb{R}^{d-1}}\left(\int \frac{\left(1+|\xi'|^2\right)^s}{1+|\xi'|^2+\xi_d^2} d \xi_d\right)^{1/2} \underbrace{\left\|\left\|\left(1+|\xi|'|^2+\xi_d^2\right)^{1/2} \mathcal{F} \widetilde{u}\right\|_{L_{\xi_d}^2}\right\|_{L_{\xi}^2}}_{\|u\|_{H^1}}
\end{aligned}$$

For what s is this supremum $<+\infty$? This is $s \leq 1/2$. □

In fact, it turns out that the image of $\operatorname{tr}_{\partial U}$ is exactly $H^{\frac{1}{2}}$.

Theorem 1.5.5 (Extension from ∂U).
There exists a bounded linear map

$$\operatorname{ext}_{\partial U}: H^{1/2}\left(\mathbb{R}^{d-1}\right) \rightarrow H^1\left(\mathbb{R}_{+}^d\right)$$

such that $\operatorname{tr}_{\partial U} \circ \operatorname{ext}_{\partial U}=\mathrm{id}$.

Proof.
We will use the Poisson semigroup. Suppose we are given $g \in \mathcal{S}\left(\mathbb{R}^{d-1}\right)$, and let $\eta \in C_c^{\infty}(\mathbb{R})$ be such that $\eta=1$ for $|s|<1$ and $\eta=0$ for $|s|>2$. Define $u=\operatorname{ext}_{\partial U} g$ by

$$\mathcal{F}_{x'} u\left(\xi', x^d\right)=\eta\left(x^d\right) e^{-x^d|\xi'|} \widehat{g}(\xi)$$

This right term is the solution to the Laplace equation on the half-space with boundary data g. We need to show that

$$u \in H^1\left(\mathbb{R}_{+}^d\right) \Longleftrightarrow \text{(i)} u, \partial_1 u, \ldots, \partial_{d-1} u \in L^2$$

(ii) $\partial_d u \in L^2$. (i) implies:

$$\begin{aligned}\|u\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2 + \cdots + \|\partial_{d-1} u\|_{L^2}^2 &= \left\| \left(1+|\xi'|^2\right)^{1/2} \mathcal{F}_{x'} u\left(\xi', x^d\right) \right\|_{L'}^2 L_{x^d} \\ &= \| \left(1+|\xi'|^2\right)^{1/2} \eta\left(x^d\right) e^{-x^d|\xi'|} \widehat{g}\left(\xi'\right) \|_{L^2_{\xi'} L^2_{x^d}}^2\end{aligned}$$

We can integrate in any order, so integrate the x^d integral first.

$$= \| \underbrace{\left(1+|\xi'|^2\right)^{1/4} \left\|\eta\left(x^d\right) e^{-x^d|\xi'|}\right\|_{L^2_{x^d}}}_{\text{NTS this is unif. bdd. } \xi' \in \mathbb{R}^{d-1}} \left(1+|\xi'|^2\right)^{1/4} \widehat{g}\left(\xi'\right) \|_{L^2_{\xi'}}^2$$

We can use the bound

$$\left\|\eta\left(x^d\right) e^{-x^d|\xi'|}\right\|_{L^2_{x^d}}^2 \lesssim 1$$

and the substitution bound

$$\int \eta^2\left(x^d\right) e^{-2x^d|\xi'|} dx^d \lesssim \frac{1}{|\xi'|}$$

This gives

$$\left\|\eta\left(x^d\right) e^{-x^d|\xi'|}\right\|_{L^2_{x^d}} \lesssim \min\left\{1, \frac{1}{|\xi'|^{1/2}}\right\} \lesssim \left(1+|\xi'|^2\right)^{-1/4}$$

Hence,

$$\|u\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2 + \cdots + \|\partial_{d-1} u\|_{L^2}^2 \leq \|(1+|\xi'|^2)^{1/4} \hat{g}(\xi')\|_{L^2_{\xi'}}^2 = \|g\|_{H^{1/2}}^2.$$

(ii) implies:

$$\begin{aligned}\partial_{x^d} u &= \partial_{x^d}\left(\eta\left(x^d\right) v\right), \quad \mathcal{F}_{x'} v = e^{-x^d|\xi'|}\widehat{g}(\xi) \\ &= \eta'\left(x^d\right) v + \eta \partial_{x^d} v\end{aligned}$$

The norm of the first term is bounded by $\|v\|_{L^2\left(x^d \in \operatorname{supp} \eta\right)}$, and the norm of the second term is

$$\begin{aligned}\|\eta \partial_{x^d} v\|_{L^2_{x'} L^2_{\xi_d}} &= \left\|\eta \partial_{x^d}\left(e^{-x^d|\xi'|}\widehat{g}\left(\xi'\right)\right)\right\|_{L^2_{\xi'} L^2_{x^d}} \\ &= \| \, |\xi'| \underbrace{e^{-x^d|\xi'}| \, \widehat{g}\left(\xi'\right) \eta\left(x^d\right)}_{\mathcal{F}_{x'} u} \|_{L^2_{\xi'} L^2_{x^d}}\end{aligned}$$

We can use tricks in (i) again,

$$\|\eta \partial_{x^d} v\|_{L^2_{x'} L^2_{\xi_d}} \leq C\|g\|_{H^{1/2}}.$$

□

Remark.

- In fact, by the usual smooth partition of unity argument with boundary straightening, one can define $H^{1/2}(\partial U)$ for ∂U of class C^1 and prove the sharp trace theorem. The independence of this space from the smooth partition of unity and boundary straightening follows from interpolation theory (which you can find in the 1970 textbook of

Stein).

- For $p \neq 2, \operatorname{im}\left(\operatorname{tr}_{\partial U} W^{1,p}(U)\right) = B_p^{1-1/p,p}(\partial U)$. This is called the L^p- Besov space of order $1 - 1/p$ and summability index p. This is also covered in Stein's book.

1.6 Sobolev Inequalities

Sobolev-type inequalities are quantitative generations of fundamental theorem of calculus, which use derivatives to control function. Below we will prove them for smooth functions, and then according to density theorems, these inequalities hold in the corresponding Sobolev spaces.

Theorem 1.6.1 (Gaglierdo-Nirenberg-Sobolev inequality).
Let $d \geq 2$. For $u \in C_c^\infty\left(\mathbb{R}^d\right)$, we have

$$\|u\|_{L^{\frac{d}{d-1}}\left(\mathbb{R}^d\right)} \leq C_d \|Du\|_{L^1\left(\mathbb{R}^d\right)},$$

where C_d is a constant depending only on d.

Remark.
The exponent on the left hand side need not be remembered because it can be derived from scaling considerations (dimensional analysis). In particular, first observe that both sides are homogeneous: if $u \mapsto u_\lambda(x) = u(x/\lambda)$ for $\lambda > 0$, then

$$\begin{aligned} \|u_\lambda\|_{L^p} &= (\lambda^d \underbrace{\int \left|u\left(\frac{x}{\lambda}\right)\right|^p \frac{1}{\lambda^d} dx}_{=\int |u|^p dx'})^{1/p} \\ &= \lambda^{d/p} \|u\|_{L^p} \end{aligned}$$

On the other hand, $D\left(u_\lambda\right) = \frac{1}{\lambda}(Du)_\lambda$, so

$$\|D\left(u_\lambda\right)\|_{L^p} = \frac{1}{\lambda} \lambda^{d/p} \|Du\|_{L^p}$$

Now compare these:

$$\begin{aligned} \|u_\lambda\|_{L^p} \leq c \|Du_\lambda\|_{L^1} \quad \forall \lambda > 0 &\iff \lambda^{d/p} \|u\|_{L^p} \leq c \lambda^{-1+d} \|Du\|_{L^1} \quad \forall \lambda > 0 \\ &\iff \frac{d}{p} = d - 1 \\ &\iff p = \frac{d}{d-1}. \end{aligned}$$

All we are doing here is changing the unit of length and requiring that the inequality is invariant under our unit of length.

To prove this theorem, the key ingredient is another inequality. Denoting $\left(x^1, \ldots, \widehat{x}^j, \cdots, x^d\right) = \left(x^1, \ldots, x^{j-1}, x^{j+1}, \ldots, x^d\right)$, we have the following.

Lemma 1.6.2 (Loomis-Whitney inequality).
. Let $d \geq 2$. For $j = 1, \ldots, d$, suppose $f_j = f_j\left(x^1, \ldots, \widehat{x}^j, \ldots, x^d\right)$. Then

$$\left\|\prod_{j=1}^d f_j\right\|_{L^1\left(\mathbb{R}^d\right)} \leq \prod_{j=1}^d \|f_j\|_{L^{d-1}\left(\mathbb{R}^{d-1}\right)}$$

Proof.

Integrate in each variable and apply Hölder:

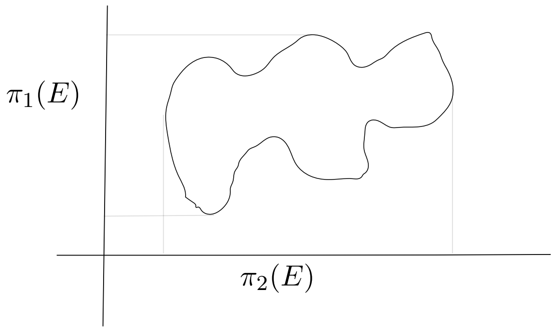
$$\int \left| \prod_{j=1}^{d} f_j \right| dx^1 = |f_1| \int |f_2| \cdots |f_d| \, dx^1$$
$$\leq |f_1| \, \|f_2\|_{L^{d-1}_{x^1}} \cdots \|f_d\|_{L^{d-1}_{x^1}}$$

This is a function of $\left(x^2, \dots, x^d\right)$. Now integrate with respect to the next variable:

$$\iint \left| \prod_{j=1}^{d} f_j \right| dx^1 dx^2 \leq \int |f_1| \, \|f_2\|_{L^{d-1}_{x^1}} \cdots \|f_d\|_{L^{d-1}_{x^1}} \, dx^2$$
$$= \|f_2\|_{L^{d-1}_{x^1}} \|f_1\|_{L^{d-1}_{x^2}} \|f_3\|_{L^{d-1}_{x^1,x^2}} \cdots \|f_d\|_{L^{d-1}_{x^1,x^2}}$$

Iterating this gives the inequality. □

Remark.
The Loomis-Whitney inequality answers the following geometric question. Suppose $E \subseteq \mathbb{R}^d$, and know the projections $\pi_j(E)$. Can we bound the measure of E by $|\pi_j(E)|$?



Yes!

$$|E| = \int \mathbf{1}_E dx$$
$$\leq \int \prod_{j=1}^{d} \mathbf{1}_{\pi_j(E)} \left(x^1, \dots, \widehat{x}^j, \dots, x^d\right) dx$$
$$\overset{\text{L-W}}{\leq} \prod_{j=1}^{d} |\pi_j(E)|^{\frac{1}{d-1}}.$$

Now let's prove the GNS inequality.

Proof (Proof of GNS inequality).
Observe that if we take a point $x \in \mathbb{R}^d$, then we can write

$$u(x) = \int_{-\infty}^{x^j} \partial_{x^j} u \left(x^1, \dots, x^{j-1}, y, x^{j+1}, \dots, x^d\right) dy,$$

using the fundamental theorem of calculus. Here, we use the compact support assumption to be sure this converges. This means that

$$|u(x)| \leq \int_{-\infty}^{x^j} \left|\partial_{x^j} u\left(x^1, \ldots, x^{j-1}, y, x^{j+1}, \ldots, x^d\right)\right| dy.$$

We can upper bound this by replacing x^j by ∞ and ∂_{x^j} by D :

$$|u(x)| \leq \underbrace{\int_{-\infty}^{\infty} \left|Du\left(x^1, \ldots, x^{j-1}, y, x^{j+1}, \ldots, x^d\right)\right| dy}_{\tilde{f}_j\left(x^1, \ldots, \widehat{x}^j, \ldots, x^d\right)}.$$

This means that we have

$$|u(x)|^d \leq \left(\prod_{j=1}^{d} \widetilde{f}_j\right)$$

which we can write as

$$|u(x)|^{\frac{d}{d-1}} \leq \left(\prod_{j=1}^{d} \widetilde{f}_j^{\frac{1}{d-1}}\right),$$

Using the Loomis-Whitney inequality and let $f_j = \tilde{f}_j^{\frac{1}{d-1}}$,

$$\begin{aligned}
\|u\|_{L^{\frac{d}{d-1}}}^{\frac{d}{d-1}} &= \int |u|^{\frac{d}{d-1}} dx \\
&\leq \int \prod_{j=1}^{d} f_j dx \\
&\leq \prod_{j=1}^{d} \|f_j\|_{L^{d-1}} \\
&= \prod_{j=1}^{d} \left(\int |f_j|^{d-1} dx^1 \cdots \widehat{dx^j} \cdots dx^d\right)^{\frac{1}{d-1}}
\end{aligned}$$

Observe that $|f_j|^{d-1} = \int_{-\infty}^{\infty} \left|Du\left(x^1, \ldots, x^j, \ldots, x^d\right)\right| dx^j$, so

$$\leq \|Du\|_{L^1}^{\frac{d}{d-1}}.$$

□

Remark.
GNS is the functional counterpart of the isoperimetric inequality. Given a function, we can make a layer cake decomposition in the y axis and apply the isoperimetric inequality to each part. This is useful for functions on manifolds where we have some geometric information.

1.6.1 Sobolev inequalities with $p < d$

Now we will upgrade this to the case where we have other L^p spaces on the right hand side.

Theorem 1.6.3 (Sobolev inequalities for L^p-based spaces).
Let $d \geq 2$, and assume that $1 < p < d$. For $u \in C_c^\infty\left(\mathbb{R}^d\right)$, we have

$$\|u\|_{L^q\left(\mathbb{R}^d\right)} \leq C\|Du\|_{L^p\left(\mathbb{R}^d\right)}$$

where $q = \frac{dp}{d-p}$.

What is q ? We do dimensional analysis to figure out the exponent. On the left hand side, we have $[x]^{d/q}$, and on the right hand side, we have $[x]^{-1+d/p}$. If we solve for q, we get $q = \frac{dp}{d-p}$. This also gives us the restriction that $p < d$.

Proof.
Take $v = |u|^{\widetilde{q}}$, where $\widetilde{q} = \frac{q}{d/(d-1)}$. Its derivative is $|Dv| = \tilde{q}|u|^{\tilde{q}-1}|Dv|$. This can be justified using approximation: approximate $|x|$ by $\left(\varepsilon^2 + x^2\right)^{1/2}$ v and let $\varepsilon \to 0$. Then

$$\int |u|^q dx = \int |v|^{\frac{d}{d-1}} dx$$

Using the GNS inequality,

$$\leq \left(\int |Dv| dx\right)^{\frac{d-1}{d}}$$

It is at this point that we need the above approximation. But it works, using the dominated convergence theorem.

$$= \left(\int |u|^{\tilde{q}-1}|Du| dx\right)^{\frac{d-1}{d}}$$

Using Hölder's inequality, we can put $|Du|$ into L^p, which puts $|u|^{\tilde{q}-1}$ in $L^{p'}$. By dimensional analysis, it must happen that

$$\leq (\int |u|^{(\tilde{q}-1)\frac{p}{p-1}} dx)^{\frac{(p-1)(d-1)}{pd}} \|Du\|_{L^p}^{\frac{d-1}{d}} = \|u\|_{L^q}^{q-\frac{d-1}{d}} \|Du\|_{L^p}^{\frac{d-1}{d}}.$$

This completes the proof. □

Now we will upgrade this to every element in the abstract Sobolev space and to situations where we have a function which is bounded on an abstract domain.

Theorem 1.6.4.
Let $d \geq 2$, and assume that $1 \leq p < d$.

- (i) For $u \in W^{1,p}\left(\mathbb{R}^d\right)$,
$$\|u\|_{L^q\left(\mathbb{R}^d\right)} \leq C\|Du\|_{L^p\left(\mathbb{R}^d\right)},$$
where $q = \frac{dp}{d-p}$.
- Let U be a bounded domain with C^1 boundary ∂U. Then for $u \in W^{1,p}(U)$,
$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{1,p}(U)}$$
where $q = \frac{dp}{d-p}$.

Proof.
For the first part, we can use density theorem of C_c^∞ in $W^{1,p}\left(\mathbb{R}^d\right)$. For the second theorem, we need to first use extension theorem then use density theorem. Note that G-N-S inequality only works for C_c^∞ functions. □

We have another further result for $W_0^{1,p}(U)$:

Theorem 1.6.5 (Estimates for $W_0^{1,p}(U)$).
For $1 \leq p < d$ (which makes $d \geq 2$), U is a bounded domain. For $u \in W_0^{1,p}(U)$, we have

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}, \quad \forall 1 \leq q \leq p^*$$

Note that the difference from the previous theorem is that the right hand side is $\|Du\|_{L^p(U)}$ instead of $\|u\|_{W^{1,p}(U)}$. The proof of this theorem requires the approximation of C_c^∞ directly, without extension. Particularly, we have

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

This is called **Poincare-Fredrich inequality**, one of the **Poincare-type inequalities**.

1.6.2 Sobolev Inequalities with $p > d$

Next, we investigate: What does $\|u\|_{W^{1,p}}$ tell us when $p \geq d$? This will be based on another way to relate u with its derivative, Du. Start with $u \in C^\infty\left(\mathbb{R}^d\right)$, and write down what we get by applying the fundamental theorem of calculus:

$$u(x) - u(y) = \int_0^1 \frac{d}{ds} u(x + s(y-x)) ds.$$

The key idea is to average to take advantage of the fact that we are in multiple dimensions. Take absolute values and average this in y : Fix $r > 0$, so

$$\frac{1}{|B_r(x)|} \int_{B_r} |u(x) - u(y)| dy \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} \int_0^1 \left| \frac{d}{ds} u(x + s(y-x)) \right| ds dy$$

By the chain rule, this derivative is $(y-x) \cdot Du(x + s(y-x))$.

$$\leq C \frac{1}{r^d} \int_{B_r(x)} \int_0^1 |x-y| |Du(x + s(y-x))| ds dy$$

Let $\rho\omega = y - x$, so that $\rho = |y-x|$.

$$= C \frac{1}{r^d} \int_0^r \int_{\mathbb{S}^{d-1}} \int_0^1 \rho |Du(x + s\rho\omega)| ds \rho^{d-1} d\omega d\rho$$

Make another change of variables, so we can make $x + s\rho\omega$ into an actual radius and then evaluate on of the integrals. We do $t = s\rho$

$$= C \frac{1}{r^d} \int_0^r \int_{\mathbb{S}^{d-1}} \int_0^1 \frac{t^d}{s^d} \frac{1}{s} |Du(x + t\omega)| ds d\omega dt$$

Simplify the s integral and upper bound $t \leq r$:

$$\leq C \int_0^r \int_{\mathbb{S}^{d-1}} |Du(x + t\omega)| d\omega dt$$
$$= C \int_{B_r(x)} \frac{|Du|}{|x-y|^{d-1}} dy$$

We can summarize this as a lemma:

Lemma 1.6.6.
Let $p > d$, let $d \geq 2$, and let $u \in C^\infty\left(\mathbb{R}^d\right)$. Then

$$\frac{1}{|B_r(x)|}\int_{B_r} |u(x) - u(y)| dy \leq C \int_{B_r(x)} \frac{|Du|}{|x-y|^{d-1}} dy$$

Theorem 1.6.7 (Sobolev ineq, $p > d$).
Let $p > d$ with $d \geq 2$, and take $u \in C^\infty\left(\mathbb{R}^d\right)$. Then

$$|u(x) - u(y)| \leq C|x-y|^\alpha \|Du\|_{L^p\left(\mathbb{R}^d\right)},$$

where $\alpha = 1 - \frac{d}{p}$.

Proof.
We will use the lemma. The idea is to introduce an auxiliary variable z and take the average over z on some domain U : $\frac{1}{|U|}\int_U |u(x)-u(y)|dz \leq \frac{1}{|U|}\int_U |u(x)-u(z)|dz + \frac{1}{|U|}\int_U |u(y) - u(y)|dz$ Since $\frac{|B_r(x)|}{|U|} \simeq 1$,

$$\begin{aligned}
&\lesssim \frac{|B_r(x)|}{|U|}\int_{B_r(x)} |u(x)-u(z)|dz + \frac{|B_r(y)|}{|U|}\int_{B_r(y)} |u(y)-u(z)|dz \\
&\lesssim \int_{B_r(x)} \frac{|Du|}{|x-z|^{d-1}} dz + \int_{B_r(y)} \frac{|Du|}{|y-z|^{d-1}} dz \\
&\lesssim \|Du\|_{L^p} \left\| \frac{1}{|x-z|^{d-1}} \right\|_{L^{p'}(B_r(x))} + \|Du\|_{L^p} \left\| \frac{1}{|y-z|^{d-1}} \right\|_{L^{p'}(B_r(y))}
\end{aligned}$$

Now we just need to evaluate

$$\int_{B_r(0)} \frac{1}{|z|^{(d-1)p'}} dz \simeq r^{\alpha p'}$$

$\square$

Remark.
Again, we can find the value of α by dimensional analysis: $1 = \alpha + (-1) + \frac{d}{p}$ gives $\alpha = 1 - \frac{d}{p}$.

We want to rephrase this as an inequality for $u \in W^{1,p}(U)$. To do this, we need a space that has a regularity property relating to the theorem above.

Definition 1.6.8 (Hölder seminorm).
Let $u \in C(I)$. The Hölder seminorm of order α is

$$[u]_{C^\alpha(U)} = \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

By a **seminorm**, we mean that $[\cdot]_{C^\alpha(U)}$ satisfies all the properties of a norm except the property that $[u]_{C^\alpha(U)} = 0 \implies u = 0$. Instead, this implies that u is constant. Here is how we make it into a norm. . .

Definition 1.6.9 (Hölder norm & Hölder space).
The Hölder norm of order α is

$$\|u\|_{C^\alpha(U)} = [u]_{C^\alpha(U)} + \|u\|_{L^\infty}.$$

The Hölder space of order α is

$$C^{0,\alpha}(U) = \{u \in C(U) : \|u\|_{C^\alpha} < \infty\}.$$

This definition could be generalized to define Holder space $C^{k,\alpha}$:

$$C^{k,\alpha}(U) = \left\{u \in C^k(U), \left\|D^\beta u\right\|_{C^{0,\alpha}} < \infty, \forall \beta, |\beta| = k\right\}$$

and

$$\|u\|_{C^{k,\alpha}(U)} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\alpha}(U)}.$$

Theorem 1.6.10 (Morrey's inequality).
Let $d \ge 2$, let $p > d$, and let U be a bounded domain in $\mathbb{R}^d$ with C^1 boundary ∂U [a]. If $u \in W^{1,p}(U)$, then $u \in C^\alpha(U)$ with $\alpha = 1 - \frac{d}{p}$. Moreover,

$$\|u\|_{C^{0,\alpha}(U)} \le C\|u\|_{W^{1,p}(U)}.$$

[a] This is sometimes called Morey's embedding

Proof.
By extension and density theorems, it suffices to consider $u \in C^\infty\left(\mathbb{R}^d\right)$ with supp $u \subseteq V$, where V is a bounded, open set with $V \supseteq \bar{U}$ (chosen independently of u). By the previous theorem,

$$[u]_{C^\alpha(V)} \le C\|u\|_{W^{1,p}}.$$

So all that remains is to bound $\|u\|_{L^\infty}$ in terms of $\|u\|_{W^{1,p}}$. For this purpose, we will again use the lemma to approximate u by its average. Let $x \in V$, then

$$\begin{aligned}
\left|u(x) - \frac{1}{|B_r(x)|}\int_{B_r(x)} u dz\right| &\le \left|\frac{1}{|B_r(x)|}\int_{B_r(x)} u(x) - u(z) dz\right| \\
&\le \frac{1}{|B_r(x)|}\int_{B_r(x)} |u(x) - u(z)| dz \\
&\le C\int_{B_r(x)} \frac{|Du(z)|}{|z-x|^{d-1}} dz \\
&\le Cr^\alpha \|Du\|_{L^p(B_r(x))}.
\end{aligned}$$

Take $r = 1$, then

$$\begin{aligned}
|u(x)| &\le C \underbrace{\left|\int_{B_r(x)} u dz\right| dz}_{\le \int_{B_1(x)} |u|\, dz \le C\|u\|_{L^p(B_1(0))}} + C\|Du\|_{L^p} \\
&\le C\left(\|u\|_{L^p} + \|Du\|_{L^p}\right).
\end{aligned}$$

$\square$

1.6.3 Sobolev Inequalities with $p = d$

What about when $p = d$ (and $d \geq 2$) ? In this case, the inequality $\|u\|_{L^\infty(U)} \leq \|u\|_{W^{1,d}(U)}$ fails.

Example 1.6.11.
Here is a counterexample to the above inequality when $p = d = 2$. Take $U = B_1(0) \subseteq \mathbb{R}^2$ and

$$u(x) = \log\log\left(10 + \frac{1}{|x|}\right).$$

A popular remedy for $p = d$ is to think about bounded mean oscillation:

Definition 1.6.12 (BMO seminorm).
Let $u \in L^1_{\text{loc}}(U)$. The BMO seminorm is

$$[u]_{\text{BMO}} = \sup_{B_r(x_0)\subseteq U} \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} \left| u(z) - \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u \right| dz.$$

Recall that we have Hardy-Littlewood theorem.

Theorem 1.6.13 (Hardy-Littlewood).
Let $u \in L^1_{\text{loc}}$, and define

$$\mathcal{M}u(x) = \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |u|.$$

(Note that $|\mathcal{M}u| \leq \|u\|_{L^\infty}$). For $1 < p \leq \infty$,

$$\|\mathcal{M}u\|_{L^p} \leq C\|u\|_{L^p}.$$

We have the following theorem:

Theorem 1.6.14.
Let $d \geq 2, U \subseteq \mathbb{R}^d$, and $u \in W^{1,d}(\mathbb{R}^d)$. Then $[u]_{\text{BMO}} < \infty$, and

$$[u]_{\text{BMO}} \leq C\|Du\|_{L^d}.$$

Proof.
Proof. Assume $u \in C^\infty(\mathbb{R}^d)$. We want to show that

$$[u]_{\text{BMO}} \leq C\|Du\|_{L^d}.$$

Fix $B_r(x)$. We want to show that

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} \left| u(z) - \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy \right| dz \leq C\|Du\|_{L^d}.$$

with some fixed constant C. We can rewrite the left hand side to get

$$\begin{aligned}\frac{1}{|B_r(x)|}\int_{B_r(x)} &\left| \frac{1}{|B_r(x)|} u(z)dy - \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy \right| dz \\ &\leq \frac{1}{|B_r(x)|^2}\int_{B_r(x)}\int_{B_r(x)} |u(z) - u(y)|dydz\end{aligned}$$

Since $B_r(x) \subseteq B_{2r}(y)$,

$$\leq \frac{1}{|B_r(x)|^2} \int_{B_r(x)} \int_{B_{2r}(y)} |u(z) - u(y)| dy dz$$

Using the lemma 1.6.6,

$$\leq \frac{1}{|B_r(x)|} \int_{B_r(x)} \underbrace{\int_{B_{2r}(y)} \frac{|Du(z)|}{|z-y|^{d-1}} dz}_{F(y)} \, dy$$

This is a convolution, so you might be tempted to use Young's inequality: $\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$, where $1 \leq p \leq q \leq r \leq \infty$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. However, this barely fails, since $\frac{1}{|z-x|^{d-1}} \notin L^q$. Instead, we use Thm 1.6.13:

Whenever you are faced with something that is hard to understand, it is a good idea to decompose the region into pieces where the function is mostly constant. The power function $|y|^\alpha$ has the property that if $2^{k-1} \leq |y|, |y'| \leq 2^k$, then $|y|^\alpha \simeq |y'|^\alpha$. For our problem, write $A_k = \left\{2^{k-1} \leq |z-y| \leq 2^k\right\}$, so

$$\begin{aligned}
\int_{B_{2r}(y)} \frac{|Du(z)|}{|z-y|^{d-1}} dz &\leq C \sum_{2^k \leq 2r + c} \int_{A_k} \frac{1}{(2^k)^{d-1}} |Du(z)| dz \\
&\leq C \sum_{2^k \leq 2cr} \frac{1}{(2^k)^{d-1}} \int_{B_{2^k}(y)} |Du(z)| dz \\
&\leq C \sum_{2^k \leq 2cr} 2^k \mathcal{M}(|Du|)(y).
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{|B_r(x)|} \int_{B_r(x)} \int_{B_{2r}(z)} \frac{|Du(y)|}{|z-y|^{d-1}} \text{ d}y \text{ d}z \\
\leq &C_2 \frac{1}{|B_r(x)|} \int_{B_r(x)} \text{d}z \sum_{2^k \leq 2r} 2^k M(|Du|)(z) \\
\leq &C_3 \frac{1}{|B_r(x)|} \int_{B_r(x)} r M(|Du|)(z) \text{d}z = C''' r \frac{1}{|B_r(x)|} \int_{B_r(x)} M(|Du|)(z) \text{d}z \\
\leq &C_3 \frac{1}{|B_r(x)|} r \|M(|Du|)\|_{L^d(B_r(x))} \|1\|_{L^{\frac{d}{d-1}}(B_r(x))} = C_4 \|M(|Du|)\|_{L^d(B_r(x))} \leq C \||Du|\|_{L^d}
\end{aligned}$$

where we use the Hardy-Littlewood inequality in the last step. □

1.7 Compactness

Definition 1.7.1 (Compact operator).
Let X, Y be normed spaces, and let $T : X \to Y$ be linear. We say that T is a compact operator if $T(B_X)$, the image of the unit ball in X, is compact in Y. Equivalently, we may require that for all bounded $\{x_n\} \subseteq X, \{Tx_n\}$ has a convergent subsequence.

Definition 1.7.2 (Compact embedding).
Consider an embedding $l : X \to Y$, where $X \subset Y$ and l is a bounded linear injective mapping. We say that X is compactly embedded in Y if and only if l is compact as a mapping, i.e. $\forall$ bounded sequence $\{x_n\} \subset X$, it has a convergent subsequence with respect to $\|\cdot\|_Y$.

Since we have $W^{1,p}(U) \subset L^p(U)$, it is natural to ask whether such embedding is compact. A natural starting point to discuss compactness is the Arzela-Ascoli theorem:

Theorem 1.7.3 (Arzela-Ascoli).
K is compact set, $F \subset C(K)$, if

- (Local boundedness) $\forall x \in K, \exists M_x > 0$, s.t. $\forall f \in K, |f(x)| \leq M_x$ holds.
- (Equicontinuity) $\forall \varepsilon > 0, \exists \delta > 0, \forall f \in F, |f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$.

Then F has a convergent subsequence. (uniformly)

Let us first discuss the compactness of the embedding of Holder spaces.

Theorem 1.7.4 (Compactness of $C^{0,\alpha}(U) \subseteq C^{0,\alpha'}(U)$).
U is a bounded open subset of $\mathbb{R}^d, 0 < \alpha' < \alpha < 1$. Then the embedding $C^{0,\alpha}(U) \subset C^{0,\alpha'}(U)$ is compact.

Proof.
Step 1: Using Thm 1.7.3, we know $C^{0,\alpha}(U) \subset C(U)$ is a compact embedding.
Step 2: according to step 1, if $\{u_n\} \subset C^{0,\alpha}(U)$, s.t. $\|u_n\|_{C^{0,\alpha}} \leq M, \exists$ subsequence u_{n_j} s.t. $\{u_{n_j}\} \subset C(U), u_{n_j} \to u_\infty$ in $C(U)$. We claim that in fact $\left\|u_{n_j} - u_\infty\right\|_{C^{0,\alpha'}} \to 0$ when $j \to \infty$. Since

$$\frac{|v(x) - v(y)|}{|x - y|^{\alpha'}} \leq \left(\frac{|v(x) - v(y)|}{|x - y|^{\alpha}}\right)^{\frac{\alpha'}{\alpha}} (|v(x)| + |v(y)|)^{1 - \frac{\alpha'}{\alpha}}$$

therefore $\forall v \in C^{0,\alpha'}(U)$, then

$$[v]_{C^{0,\alpha'}} \leq c\|v\|_{L^\infty}^{1-\frac{\alpha'}{\alpha}} [v]_{C^{0,\alpha}}^{\frac{\alpha'}{\alpha}}$$

$$\left[u_{n_j} - u_\infty\right]_{C^{0,\alpha'}} \leq c\left\|u_{n_j} - u_\infty\right\|_{L^\infty}^{1-\frac{\alpha'}{\alpha}} \left[u_{n_j} - u_\infty\right]_{C^{0,\alpha}}^{\frac{\alpha'}{\alpha}}$$

On the right hand side, the first term goes to 0 , the second term is bounded, and we are done. □

Our main theorem would be proving that for $1 \leq p < d$, the embedding $W^{1,p}(U) \subset L^q(U)$ is compact, where $1 \leq q < p^*$. We'll need the following lemma:

Lemma 1.7.5 (Mollifer).
Recall that if $v \in L^p\left(\mathbb{R}^d\right), \varphi \in C_c^\infty\left(\mathbb{R}^d\right), \int \varphi = 1, \varphi\varepsilon * v \to v(\varepsilon \to 0)$ in L^p. Now consider $v \in W^{k,p}\left(\mathbb{R}^d\right)$, and for φ assume that $\int \varphi = 1$ and $\int x^\alpha \varphi dx = 0$[a] for $1 \leq |\alpha| \leq k$. Then we have

$$\|\varphi_\varepsilon * v - v\|_{L^p} \leq C\varepsilon^k \left\|\partial^{(k)} v\right\|_{L^p}$$

[a]The conditions $\int x^\alpha \varphi, dx = 0$ are called **moment conditions**.

Proof.
We'll only prove this in the case of $k = 2$. First, write

$$\int \varphi_\varepsilon(y) v(x-y) dy - \underbrace{v(x)}_{=\int \varphi_\varepsilon(y) v(x) dy} = \int \varphi_\varepsilon(y)(v(x-y) - v(x)) dy.$$

Here, we should think of $|y| \lesssim \varepsilon$. To quantify the convergence of the v part, we Taylor expand in y. We will be using the integral form of the Taylor expansion with remainder. [2] Write

$$\begin{aligned}\int_0^1 \frac{d}{ds} v(x-sy) ds &= -\int \frac{d}{ds}(1-s)\frac{d}{ds} v(x-sy) dx \\ &= \frac{d}{ds} v(x-sy)\Big|_{s=0} + \int_0^1 (1-s)\frac{d^2}{ds^2} v(x-sy) ds\end{aligned}$$

The first term gves $y \cdot \nabla v(x)$, and the second term gives $y^i y^j \int_0^1 (1-s)\partial_i \partial_j v(x-sy) ds$. The contribution of the first term is 0 by the moment condition, and we are left with the remainder, which we can control. In all, we get

$$\left|\int \varphi_\varepsilon(y) v(x-y) dy - v(x)\right| \leq \int |\varphi_\varepsilon(y)| \, |y|^2 \int_0^1 \left|\partial^2 v(x-sy)\right| ds dy.$$

This tells us that

$$\begin{aligned}\|\cdot\|_{L^p} &\leq \left\|\partial^2 v\right\|_{L^p} \int |\varphi_\varepsilon(y)| \underbrace{|y|^2}_{\lesssim \varepsilon^2} dy \\ &\lesssim \varepsilon^2 \left\|\partial^2 v\right\|_{L^p}.\end{aligned}$$

□

Theorem 1.7.6 (Rellich-Kondrachov).
Let $d \geq 2$, and let U be a bounded domain in $\mathbb{R}^d$ with C^1 boundary ∂U. (Recall that if $1 \leq p < d$, we have the embedding $W^{1,p}(U) \to L^{p^*}(U)$, where $\frac{d}{p^*} = \frac{d}{p} - 1$.) Let $1 \leq p < d$, and let $1 \leq q < p^*$. Then the embedding $W^{1,p}(U) \to L^q(U)$ is compact.

Proof.
Step 1: Reduce to the compactness of $W^{1,p}(U) \to L^p(U)$. This is sufficient because of the following two cases:

- Case 1: $W^{1,p} \to L^q(U)$ with $1 \leq q \leq p$. In this case, if U is bounded, then Hölder gives $\|v\|_{L^q(U)} \leq |U|^{1/q - 1/p}\|v\|_{L^p}$, and we already have control in L^p.
- Case 2: $W^{1,p} \to L^q(U)$ with $p < q < p^*$. Again by Hölder, we have

$$\|v\|_{L^q} \leq \|v\|_{L^p}^\theta \|v\|_{L^{p^*}}^{1-\theta},$$

where $\frac{d}{q} = \frac{d}{p}\theta + \frac{d}{p^*}(1-\theta)$. The condition that $p < q < p^*$ tells us that $0 < \theta < 1$.

The L^p term goes ti 0 by compactness of $W^{1,p} \to C^p$, and the L^{p^*} term is bounded by the Sobolev inequality.

Step 2: Prove compactness of $W^{1,p}(U) \to L^p(U)$: Given $\{u_n\} \subseteq W^{1,p}(U)$ with $\|u_n\|_{W^{1,p}(U)} \leq M < \infty$, by extension, we can find a sequence of extensions $\widetilde{u}_n$ of u_n defined on $\mathbb{R}^d$ such that

$$\|\widetilde{u}_n\|_{W^{1,p}(\mathbb{R}^d)} \leq C\,\|u_n\|_{W^{1,p}(U)} \leq CM$$

and supp $\widetilde{u}_n \subseteq V$, where V is a bounded open set containing $\bar{U}$. It suffices to find a subsequence of $\widetilde{u}_n$ that converges in L^p. Introduce $\varphi \in C_c^\infty\left(\mathbb{R}^d\right)$ with $\int \varphi dx = 1$, and write

$$\tilde{u}_n = \underbrace{\varphi * \widetilde{u}_n}_{v_{n,\varepsilon}} + \underbrace{(\widetilde{u}_n - \varphi * \widetilde{u}_n)}_{e_{n,\varepsilon}}.$$

By the Lemma 1.7.5, we know that $\|(\widetilde{u}_n - \varphi_\varepsilon * \widetilde{u}_n)\|_{L^p} \leq C\varepsilon M$, independent of n. For $v_{n,\varepsilon} = \varphi_\varepsilon * \widetilde{u}_n$ for each fixed ε, we have

$$\|v_{n,\varepsilon}\|_{L^\infty} + \|\nabla v_{n,\varepsilon}\|_{L^\infty} \leq C_\varepsilon M$$

because of Holder inequality

$$\left|\int \varphi_\varepsilon(x-y)\widetilde{u}_n(y)\mathrm{d}y\right| \leq \|\widetilde{u}_n\|_{L^p}\,\|\varphi_\varepsilon\|_{L^{p'}} = C_\varepsilon\,\|\widetilde{u}_n\|_{L^p}\,,$$

$$\left|\int \varphi_\varepsilon(y)\partial_x\widetilde{u}_n(x-y)\mathrm{d}y\right| \leq \|\partial_x\widetilde{u}_n\|_{L^p}\,\|\varphi_\varepsilon\|_{L^{p'}} = C_\varepsilon\,\|\partial_x\widetilde{u}_n\|_{L^p}$$

and also $v_{n,\varepsilon}$ for fixed ε satisfy the condition of using Arzela-Ascoli theorem. Therefore any $l, \exists$ a subsequence $\left\{\widetilde{u}_{n_k^l}\right\}_{k=1}^\infty$ and a ε_l, such that

- $\left\|\widetilde{u}_{n_k^l} - \varphi_{\varepsilon_l} * \widetilde{u}_{n_k^l}\right\|_{L^p} < 2^{-l}$;
- $\left\|v_{n_{k'}^l,\varepsilon_l} - v_{n_{k''}^l,\varepsilon_l}\right\| < 2^{-l}$, for all $k', k'' \geq k$. (This follows from Arzela-Ascoli theorem.)

Then use a diagonal argument, we have constructed a sequence $\left\{\widetilde{u}_{n_l^l}\right\}$ such that it is a Cauchy sequence in L^p. □

A direct result of this theorem is that

Corollary 1.7.7 (Compactness of $W^{1,p}(U) \subseteq L^p(U)$).
The embedding of $W^{1,p}(U) \to L^p(U)$ is compact for any p for bounded U with C^1 boundary. The embedding of $W_0^{1,p}(U) \to L^p(U)$ is compact for any p for bounded U.

1.8 Poincaré Inequality

A Poincaré-type inequality refers to any inequality that controls u in terms of information on Du, along with some additional condition to fix the ambiguity.

Theorem 1.8.1 (Poincaré inequality).
Let $1 \leq p < \infty$, and let U be a bounded domain in $\mathbb{R}^d$ with C^1 boundary ∂U. For $u \in W^{1,p}(U)$ with $\int_U u dx = 0$,

$$\|u\|_{L^p} \leq C_U \|Du\|_{L^p}$$

Proof.
Proof. We argue by contradiction. For contradiction, assume that for each $n \geq 1$, there exists $u_n \in W^{1,p}(U)$ such that $\int u_n = 0$ and

$$\|u_n\|_{L^p} \geq n \|\nabla u_n\|_{L^p}$$

By normalization, we may assume that $\|u_n\|_{L^p} = 1$. Then it follows that

$$\|\nabla u_n\|_{L^p} \leq \frac{1}{n}$$

In particular, this means that $\|u_n\|_{W^{1,p}(U)} \leq 2$, and by Thm 1.7.6, there is a subsequence such that $u_n \to u_\infty$ in L^p. Moreover, $1 = \|u_n\|_{L^p} \to \|u_\infty\|_{L^p}$. Since $Du_n \to Du$ weakly in L^p, we must have $Du = 0$. That is, u is constant on U. But $0 = \int u_n \to \int u$, which tells us that $u = 0$ on U. However, this contradicts $\|u\|_{L^p} = 1$. □

Remark.
For $p = 1$ the proof requires a bit more effort than what we will say. And another popular form of the Poincaré inequality is

$$\left\| u - \frac{1}{|U|} u \right\|_{L^p} \leq C_U \|Du\|_{L^p}.$$

Here are some other examples of Poincaré-type inequalities:

Theorem 1.8.2 (Friedrich inequality).
Let $1 \leq p < \infty$, and let U be a bounded domain in $\mathbb{R}^d$ with C^1 boundary ∂U. For $u \in W^{1,p}(U)$ with $u|_{\partial U} = 0$,

$$\|u\|_{L^p} \leq C_U \|Du\|_{L^p}.$$

We can prove this in the same way using compactness. On the other hand, we can also prove this just from the Sobolev inequality for $W_0^{1,p}(U)$.

Theorem 1.8.3 (Hardy's inequality).

- (i) If $u \in W^{1,p}(U)$ and $u|_{\partial U} = 0$, then

$$\left\| \frac{1}{\text{dist}(\cdot, \partial U)} u \right\|_{L^p(U)} \leq C \|Du\|_{L^p(U)}.$$

- (ii) If $u \in W^{1,p}\left(\mathbb{R}^d\right)$ with $p < d$, then

$$\left\| \frac{1}{|x|} u \right\|_{L^p} \leq C \|Du\|_{L^p}.$$

Proof.
Switch to polar coordinates (r, ω). It suffices to show that this inequality holds with the radial derivative: For each fixed ω,

$$\int \frac{1}{r^2} u^2 r^{d-1} dt \leq C \int |\partial_r u|^2 r^{d-1} dr$$

and then we integrate over ω on both sides. The idea is to complete the square. We will subtract one side from the other and show it is ≥ 0. Without motivation, let's examine

$$\left(\partial_r u + \frac{\alpha}{r} u\right)^2 = (\partial_r u)^2 + \frac{2\alpha}{r} u \partial_r u + \frac{\alpha^2}{r^2} u^2.$$

The left hand side is ≥ 0. Now integrate both sides:

$$\begin{aligned}
0 &\leq \int \left(\partial_r u + \frac{\alpha}{r} u\right)^2 r^{d-1} dr \\
&= \int ((\partial_r u)^2 + \frac{2\alpha}{r} \underbrace{u \partial_r u}_{\frac{1}{2} \partial_r u^2} + \frac{\alpha^2}{r^2} u^2) r^{d-1} dr \\
&= \int (\partial_r u)^2 r^{d-1} dr + \alpha^2 \int \frac{1}{r^2} u^2 r^{d-1} dt + \alpha \int_0^\infty \partial_r u^2 r^{d-2} dr
\end{aligned}$$

We want to integrate by parts. Since $d > 0$, the boundary term will be 0 . In particular, $\int_0^\infty \partial_r u^2 r^{d-2} dr = u^2 r^{d-2}|_0^\infty - (d-2) \int_0^\infty u^2 r^{d-3} dr = -(d-2) \int_0^\infty u^2 r^{d-3} dr$. Hence, we have

$$= \int (\partial_r u)^2 r^{d-1} dr - \left((d-2)\alpha - \alpha^2\right) \int_0^\infty \frac{1}{r^2} u^2 r^{d-1} dr.$$

Really, what we need here is $(d-2) > 0$ because we want the coefficient of α in the above quadratic term to be positive. We can upper bound this by plugging in $\alpha = \frac{d-2}{2}$. We can also upper bound

$$\int_0^\infty \frac{1}{r^2} u^2 r^{d-1} dr \leq \left(\frac{2}{d-2}\right)^2 \int (\partial_r u)^2 r^{d-1} dr.$$

□

Remark.
Not only do we get the inequality, but we also get that

$$\left(\frac{d-2}{2}\right)^2 \int_0^\infty \frac{1}{r^2} u^2 dr = \int_0^\infty (\partial_r u)^2 r^{d-1} dr - \int_0^\infty \left(\partial_r u + \frac{d-2}{2r} u\right)^2 r^{d-1} dr.$$

This tells us that the extremizer is $r^{-(d-2)/2}$. However, this is not an element of H^1, so we can get near extremizers by including appropriate cutoffs.

CHAPTER 2

LINEAR ELLIPTIC PDES

This chapter investigates the solvability of uniformly elliptic, second- order partial differential equations, subject to prescribed boundary condi- tions. We will exploit two essentially distinct techniques, energy methods within Sobolev spaces and maximum principle methods.

2.1 Linear Elliptic PDEs

Elliptic PDEs are a generalization of the Laplace equation

$$-\Delta u = f.$$

Definition 2.1.1 (Symbol).
The symbol of a partial differential operator is what we get when we replace ∂_j with $i\xi_j$.

It turns out that an important property is that $-\Delta = -\sum_j \partial_j \partial_j$ has (principal) symbol $-\sum_j (i\xi_j)^2 = |\xi|^2$. What's important is that $|\xi|^2$ is nonzero and thus invertible for $\xi \neq 0$:

$$|\xi|^2 \widehat{u} = \widehat{f} \Longrightarrow \widehat{u} = \frac{1}{|\xi|^2} \widehat{f}.$$

This leads to the general definition of ellipticity of a partial differential operator. Suppose that P is a linear partial differential operator such that if $u = \left(u^I\right)_{I=1}^N : U \to \mathbb{R}^N$, then (Pu) takes values in $\mathbb{R}^N$ and

$$(Pu)^I = \underbrace{\sum_{\substack{J,\alpha \\ |\alpha|=K}} A^I_{J,\alpha_1,\ldots,\alpha_d} \partial^\alpha u^J}_{\text{principal part}} + (\text{ lower order terms }).$$

Here, K is called the **order** of P.

Definition 2.1.2 (Principal symbol).
The principal symbol of an operator is

$$\sigma_{\text{prin}}(P) = i^K \sum_{\alpha=K} A^I_{J,\alpha_1,\ldots,\alpha_d}(x) \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}.$$

Here, we allow the coefficients to be functions of x. We say that P is **elliptic** if $\sigma_{\text{prin}}(P)$ is larger than 0 for all $x \in U$ and $\xi \neq 0$. The case $N = 1$ is called the scalar case, where this looks like

$$Pu = \sum_{|\alpha|=K} a_\alpha(x) \partial^\alpha u$$

Then the principal symbol is

$$\sigma_{\text{prin}}(P) = i^K \sum_\alpha a_\alpha(x) \xi^\alpha$$

The first nontrivial example is when $K = 2$, so

$$Pu = -a^{i,j} \partial_i \partial_j u + b^i \partial_i u + cu.$$

In this case, ellipticity is equivalent to $a^{i,j} \xi_i \xi_j \neq 0$ for all $x \in U$ and $\xi \neq 0$. This is equivalent to $a = \left[a^{i,j}\right]$ being a positive definite matrix for all $x \in U$.

We will assume that a is a symmetric matrix and require the following property.

Definition 2.1.3 (Uniform ellipticity).
Uniform ellipticity is the property that there exists a uniform constant $\lambda > 0$ such that $a^{i,j} \xi_i \xi_j \geq \lambda$ for all $x \in U$ and $|\xi| = 1$.

This is equivalent to saying that all eigenvalues of the matrix $a(x)$ are bounded below by λ. Why do we care about elliptic PDEs?

- These arise naturally in optimization problems in math, physics, etc. In the latter part of the course, we will discuss these in the context of calculus of variations.
- They also often arise as a part of evolutionary problems.

Example 2.1.4 (Incompressible Euler equations).
Let $u : \mathbb{R}_t \times \mathbb{R}^3 \to \mathbb{R}^3$ represent the velocity of a fluid element at each point in time and space. This follows the equation

$$\begin{cases} \partial_t u + \cdot \nabla u + \nabla p = 0 \\ \nabla \cdot u = 0 \end{cases}$$

This is one of the most infamous PDEs because of how difficult it is to understand. How do we figure out p ? Take the divergence of the first equation to get that

$$-\Delta p = \nabla (u \cdot \nabla u).$$

This is the **pressure equation**.

2.2 Boundary Value Problems

Assume $d \geq 2$ and $N = 1$ (scalar case). Also assume uniform ellipticity of P and some "nice" regularity for the coefficients a, b, c. We will focus mostly on the case where U is a bounded domain in $\mathbb{R}^d$ with "nice" boundary.

When it comes to boundary value problems, you cannot prescribe both function values and values of the normal derivative at the boundary; this stems from the various uniqueness properties that arise for these PDEs. We will mostly focus on **Dirichlet boundary problems**,

$$\begin{cases} Pu = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

We will focus less on boundary problems such as **Neumann boundary problems**,

$$\begin{cases} Pu = f & \text{in } U \\ \frac{\partial}{\partial \nu} u = g & \text{on } \partial U \end{cases}$$

We will study solvability for $u \in H^1(U)$. We will first study the Dirichlet boundary value problem ($u|_{\partial u} = g$ is okay due to the trace theorem). We will later discuss the Neumann boundary value problem, which needs to be studied in H^2 because we need to use the trace theorem on the derivative.

The standard reduction is that it suffices to understand $g = 0$. This is because if we take any extension (with correct regularity) $\widetilde{g} : \bar{U} \to \mathbb{R}$ of g, then we can work with $v = u - \widetilde{g}$ and solve the problem

$$\begin{cases} Pv = f - P\widetilde{g} = \widetilde{f} & \text{in } U \\ v = 0 & \text{on } \partial U \end{cases}$$

Definition 2.2.1 (Divergence form).
P is in divergence form if

$$Pu = -\partial_j \left(a^{i,j} \partial_i u \right) + b^i \partial_i u + cu.$$

Note that if a is smooth, then

$$Pu = a^{i,j} \partial_i \partial_j u + \left(\partial_j a^{i,j} + b^i \right) \partial_i u + cu.$$

We will see, however, that there are definite advantages to considering the two different representations of P separately. The divergence form is most natural for energy methods, based upon integration by parts , and the nondivergence form is most appropriate for maximum principle techniques.

Our discussion of existence and uniqueness of the Dirichlet boundary value problem would be based on a-priori estimates.

Theorem 2.2.2 (a-priori estimate).
Suppose that $u \in H^1$ solves the Dirichlet boundary problem, and assume that $b, c \in L^\infty$ with $\|b\|_{L^\infty} + \|c\|_{L^\infty} \leq$ A. Then there exist constants $C > 0$ and $\gamma \geq 0$ such that

$$\|u\|_{H^1(U)} \leq C \|f\|_{H^{-1}} + \gamma \|u\|_{L^2(U)}$$

Proof.
The proof is essentially integration by parts. We can use approximation to justify the

integration by parts. Write

$$\int_U f u\, dx = \int_U Puudx = \int_U \left(\partial_i \left(a^{i,j}\partial_j u + b^i u\right) + cu\right) u\, dx$$
$$= \int -a^{i,j}\partial_j u \partial_i u - b^i u \partial_i u + cuu\, dx$$

Uniform ellipticity tells us that $\lambda|Du|^2 \leq a^{i,j}\partial_i u \partial_j u$; integrate this to take care of the first term. The second term can be dealt with using Cauchy-Schwarz, and the third term is $\gamma\|u\|_{L^2}^2$.

Putting this all together gives

$$\lambda\|Du\|_{L^2(U)}^2 \leq C\|f\|_{H^{-1}}\|u\|_{H^1} + \int_U |b||\partial u||u|dx + \int_U |c||u|^2 dx$$
$$\leq C\|f\|_{H^{-1}}\|u\|_{H^1} + A\underbrace{\int_U |\partial u||u|dx}_{\leq\|\partial u\|\|u\|_{L^2}} + A\underbrace{\int_U |u|^2 dx}_{\leq\gamma\|u\|_{L^2}^2}.$$

If we make γ large enough so that we have put an $\|u\|_{L^2}^2$ on the right hand side and abosrb the second term, we get

$$\|u\|_{H^1(U)}^2 \leq C\|f\|_{H^{-1}}\|u\|_{H^1} + \gamma\|u\|_{L^2}\|u\|_{H^1}.$$

□

Remark.
We can alter this argument to only require $b \in L^{d+}$ and $c \in L^{d/2+}$.

Recall that in order to prove existence statements with a priori estimates, we also needed to think about the dual problem for the adjoint P^*. (In finite dimensional linear algebra, $Ax = y$ has a solution x if and only if $r \in \operatorname{ran} A = {}^\perp(\ker A^*)$. For P as above, let's compute P^* with respect to $\langle u, v\rangle = \int uvdx$:

$$\int \partial_j uvdx = -\int u\partial_j vdx$$

So

$$P^* = -\partial_j\left(a^{j,k}\partial_k u\right) - \partial_j\left(b^j u\right) + cu,$$

where we are assuming everything is real-valued. Note that the energy estimate also applies to P^*.

2.2.1 Case 1: P and P^* Obey A-priori Estimates

In our discussion of Sobolev spaces, we introduce Thm 1.3.1 and Thm 1.3.2 from functional analysis.

In our previous proof, we assumed that X is reflexive to reduce (ii) to (i), but this assumption can be dropped. To see this argument, look for the "closed range theorem." The key idea is that $\overline{\operatorname{ran} P} = \perp^\perp(\ker P^*)$.

We want to apply this lemma to our $P, X = H_0^1$, and $Y = H^{-1}(U)$. In this setting, $X^* = H^{-1}(U) = Y$, and $Y^* = H_0^1(U) = X$.

In the energy estimate, we have an extra term $\gamma\|u\|_{L^2(U)}$ in the bound. For now, we will get rid of it by cheating. We will deal with it in full later. Here is when we have the energy estimate with $\gamma = 0$:

Lemma 2.2.3.
If $b = 0$ and $c = 0$, i.e. $Pu = -\partial_j \left(a^{j,k}\partial_j u\right)$, then the energy estimate holds with $\gamma = 0$.

Proof.
Proof. By density, $u \in C_0^\infty$.

$$\begin{aligned}
\int_U Puudx &= \int_U -\partial_j \left(a^{j,k}\partial_k u\right) udx \\
&= \int_U a^{j,k}\partial_j u \partial_k udx \\
&\geq \lambda \int |Du|^2 dx
\end{aligned}$$

Using Thm 1.8.2,

$$\geq C \int_U |u|^2 dx.$$

As in the proof of the energy estimate, we cancel a factor of $\|u\|_{H^1}$ on both sides of the inequality to get the result. □

Theorem 2.2.4.
For every $f \in H^{-1}(U)$, there exixsts a unique $u \in H_0^1(U)$ such that $-\partial_j \left(a^{j,k}\partial_j u\right) = f$ in U.

Remark.
- Since P^* has the same form with the same constants, this condition gives the energy estimate with $\gamma = 0$ for P^*, as well.
- For the proof of this, Evans' textbook uses the Lax-Milgram lemma, but our lemma is actually stronger.

2.2.2 Case 2: General P

To obtain stronger results for our general problem, we will develop tools which are specifically useful for this problem. In particular, we will discuss Fredholm theory.

Recall the notion of a compact operator $K : X \to Y$ from functional analysis: $K\left(\bar{B}_X\right)$ is compact, where $B_X = \{x \in X : \|x\| < 1\}$.

Lemma 2.2.5 (Solvability with compact operator).
- For $K : X \to Y$, K is compact if and only if K^* is compact.
- (Solvability of $(I + K)x = y$): Let $K : X \to Y$ be compact, and let $T = I + K$.
  - $ker(I + K)$ is finite dimensional;
  - There exists an $n_0 \geq 1$ such that $\ker(I + K)^n = \ker(I + K)^{n_0}$ for $n \geq n_0$;
  - $ran(I + K)$ is closed, so $rank(I + K) = {}^\perp(\ker(I + K^*))$;
  - $\dim\ker(I + K) = \dim\ker(I + K^*)$.

Proof (Pfsk).
For the proof when X is a Hilbert space, see the appendix of Evans' textbook. What is the idea? Here is how to think about compact operators: Notice that if A has $\dim \operatorname{ran} A < \infty$, then A is compact. Also notice that if $K_n \to K$ in the operator norm topology on $\mathcal{L}(X, Y)$, then K is compact. Combining these two facts tells us that the closure of the set of finite rank operators is a subset of the compact operators; in separable Hilbert spaces, this is

what all compact operators look like.

Remark.
The last part is the general equivalent of the fact that in finite dimensional linear algebra, the row rank of a matrix is equal to the column rank of a matrix. This statement is that index $(I+K)=0$, where the index of an operator is the difference of these two quantities. The index tends to be very stable under perturbation.

Why is this lemma relevant for us? Take any general

$$Pu = -\partial_j\left(a^{j,k}\partial_k u\right) + \partial_j(b_j u) + cu.$$

In general, the energy estimate gives

$$\|u\|_{H_0^1(U)} \le C\|Pu\|_{H^{-1}(U)} + \gamma\|u\|_{L^2(U)}$$

But if we consider instead $(P+\mu I)u = -\partial_j\left(a^{j,k}\partial_k u\right) + \partial_j(b^j u) + (c+\mu)u$ with $\mu \gg 1$, then we can remove γ on the right hand side. Indeed,

$$\int (P+\mu)udx = \underbrace{\int -\partial_j a^{i,k}\partial_k udx}_{\ge \lambda\int |Du|^2dx} + b, c \text{ terms } + \int \mu u^2 dx,$$

where the $\int \mu u^2 dx$ term is favorable if $\mu > 0$. By case 1 , for μ sufficiently positive, for all $f \in H^{-1}$, ther exists a unique $u \in H_0^1$ such that

$$(P+\mu I)u = f.$$

We then have a well-defined map $(P+\mu I)^{-1} : H^{-1}(U) \to H_0^1(U)$. Now go back to

$$(P+\mu)u - \mu u = Pu = f$$

Apply $(P+\mu)^{-1}$ to get

$$u - \mu(P+\mu)^{-1}u = (P+\mu)^{-1}f.$$

By Rellich-Kondrachov (recalling that U is bounded), the embedding $\iota : H_0^1(U) \to L^2$ is compact. From this, it follows that

$$(P+\mu)^{-1} : L^2(U) \to H^{-1}(U) \xrightarrow{(P+\mu)^{-1}} H^1(U) \to L^2(U)$$

is compact (since $A \circ K$ or $K \circ A$ is compact whenever A is bounded and linear and K is compact). Thus, $-\mu(P+\mu)^{-1} : L^2(U) \to L^2(U)$ is compact. Thus, our repackaging of the problem,

$$u - \mu(P+\mu)^{-1}u = (P+\mu)^{-1}f,$$

is of the form $(I+K)x = y$.

Theorem 2.2.6 (Fredholm alternative).
Let P be as before, and let U be a bounded domain with C^1 boundary.

(i) Exactly one of the following holds:

(a) (Solvability) For all $f \in H^{-1}(U)$, there exists a unique $u \in H_0^1(U)$ such that $Pu = f$, and there exists a $C > 0$ independent of u, f such that $\|u\|_{H^1(U)} \le C\|f\|_{H^{-1}(U)}$.

(b) (Existence of nonzero homogeneous solution) There exists a nonzero $u \in H_0^1(U)$ (or equivalently in $L^2(U)$) such that $Pu = 0$.

(ii) If (b) holds, then $\dim\ker P < \infty$ and $\dim\ker P^* < \infty$. Given $f \in H^{-1}(U)$, there exists a $u \in H_0^1(U)$ such that $Pu = f$ if and only if $\langle f, v \rangle = 0$ for all $v \in \ker P^*$.

Remark.

- While our initial approach didn't really care about boundedness, this approach essentially relies on this condition.
- Part (ii) is a statement about norms. This will be an exercise and follows from compactness.
- Here is a very nice consequence of this theorem. Take

$$\widetilde{P}u = -\partial_j \left(a^{j,k} \partial_k u \right) + b^j \partial_j u.$$

  There is a weak maximum principle which says that

$$\sup_{\bar{U}} |u| = \sup_{\partial U} |u|.$$

  This gives uniqueness in this Dirichlet problem. Then the Fredholm alternative gives us solvability from the uniqueness. We will properly discuss this later, when we go over maximum principles.

2.3 L^2-based Interior Regularity

Last time, we discussed solvability for elliptic PDEs. Now we will talk about the regularity of solutions to elliptic PDEs. Here is a prototypical example.

Example 2.3.1 (Poisson equation).
Consider the Poisson equation $-\Delta u = f$ in U, where $f \in H^k(U)$ or

$$f \in C^{k,\alpha} = \left\{ u \in C^k(U) : \partial^\alpha u \in C^{0,\alpha}(U) \forall |\alpha| = k \right\}.$$

The idea is that u should be more regular than f by order 2. Interior regularity says that for all $V \subseteq\subseteq U$ (notation meaning V is bounded and $\bar{V} \subseteq U$),

$$\|u\|_{H^{k+2}(V)} \leq C\|f\|_{H^k(V)} + C\|u\|_{L^2(U)}$$

Similarly,

$$\|u\|_{C^{k+2,\alpha}(V)} \leq C\|f\|_{C^{k,\alpha}(V)} + C\|u\|_{L^\infty(U)}.$$

In general, the constant C can depend on the domain V.

The first of these statements is referred to as **L^2-based regularity theory**, and the second is referred to as **Schauder theory**. We will think about L^2-based regularity theory for now and discuss Schauder theory later.

For L^2-based regularity theory, the key idea is integration by parts (the energy method)[1]. We will make a simplifying that $u \in H^{k+2}(V)$; this is not assuming everything because from this qualitative fact, we will derive a quantitative bound. This assumption allows us to commute the equation with derivatives. We have not said any assumptions about the boundary, which may seem like an issue with integration by parts, but this is why we are discussing interior regularity. We will solve this with a cutoff function. Let ζ be a nonnegative, smooth cutoff function which equals 1 in V and equals 0 near ∂U. Then (squaring ζ in anticipation of a nice L^2 trick),

$$\begin{aligned} \int_U f u \zeta^2 dx &= \int_U -\Delta u u \zeta^2 \\ &= \sum_{j=1}^d \int_U \underbrace{\partial_j u \partial_j \left(u\zeta^2\right)}_{\partial_j u \zeta^2 + 2u\zeta\partial_j\zeta} dx \end{aligned}$$

Note that we have no boundary term in the integration by parts thanks to ζ.

$$= \sum_{j=1}^d \int \left(\partial_j u\right)^2 \zeta^2 + 2\partial_j u u \zeta \partial_j \zeta dx$$

Rearrange this to get

$$\int_U |Du|^2 \zeta^2 dx \leq \left| \int_U f u \zeta^2 dx \right| + \underbrace{2\left| \int_U u\zeta Du \cdot D\zeta dx \right|}_{\leq 2\left(\int_U |Du|^2\zeta^2\right)^{1/2}\left(\int_U u^2|D\zeta|^2 dx\right)^{1/2}}$$

To control this right term, we use the AM-GM inequality $ab \leq \frac{a}{2} + \frac{b}{2}$. But we can weight this by $\sqrt{\varepsilon}$ on a and $\frac{1}{\sqrt{\varepsilon}}$ on b to get the inequality $ab \leq \varepsilon \frac{a^2}{2} + \frac{1}{\varepsilon}\frac{b^2}{2}$. This bounds

$$2\left(\int_U |Du|^2\zeta^2\right)^{1/2} \left(\int_U u^2 |D\zeta|^2 dx\right)^{1/2} \leq \varepsilon \int_U |Du|^2 \zeta^2 dx + \frac{1}{\varepsilon}\int_U u^2 |D\zeta|^2 dx.$$

[1] Fraydoun Rezakhanlou says that he is an analyst, a PDE-ist, and a probabilist. He is an analyst because he uses the Cauchy-Schwarz inequality, a probabilist because he uses Chebyshev's inequality, and a PDE-ist because he uses integration by parts.

Now set $\varepsilon = 1/2$ to absorb the first term to the right hand side. This gives

$$\begin{aligned}\frac{1}{2}\int_U |Du|^2\zeta^2 dx &\leq \left|\int_U fu\zeta^2\right| + 2\int_U u^2|D\zeta|^2 dx \\ &\leq \|f\|_{L^2(U)} + \|u\|_{L^2(U)},\end{aligned}$$

and we lower bound the left hand side by $\frac{1}{2}\int_V |Du|^2 dx$. For the actual result, we could have upgraded the $\|f\|_{L^2(U)}$ to $\|f\|_{H^1(U)}$ by using an additional cutoff argument. What about higher regularity? Suppose $k+2=2$. Then if $-\Delta u = f$, we get

$$-\Delta\partial_j u = \partial_j f,$$

where $\partial_j u \in H^1$, so we can do integration by parts. Now apply the case $k=1$ to get

$$\int_V |D\partial_j u|^2\, dx \leq \left|\int_U \partial_j f \partial_j u\zeta^2 dx\right| + \|\partial_j u\|_{L^2(U)}$$

Bound the first term by (using the same AM-GM trick)

$$\left|\int_U f\partial_j^2 u\zeta^2 dx\right| \leq \frac{1}{4\varepsilon}\int_U f^2\zeta^2 dx + \varepsilon\int_U |\partial_j u|^2\,\zeta^2 dx$$

Absorb the second term to the right hand side to get

$$\int_U |D\partial_j u|^2\,\zeta^2 dx \leq C\int_U f^2 dx + C\|Du\|_{L^2(U)}^2.$$

We want to change the last term into $\|u\|_{L^2(U)}$. Our tool to do this is the H^1 bound we just proved. But this needs us to have a domain in the interior of U. However, note that if we define $V \subseteq\subseteq W \subseteq\subseteq U$, we can replace this term on the the right hand side by $C\|Du\|_{L^2(W)}$. Then we use the H^1 bound $\|Du\|_{L^2(W)} \leq \|f\|_{L^2(U)} + \|u\|_{L^2(U)}$. In conclusion, we get

$$\|D\partial_j u\|_{L^2(V)} \leq C\|f\|_{L^2(U)} + C\|u\|_{L^2(U)}$$

for all j. Combined with the H^1 bound, this gives the H^2 bound

$$\|u\|_{H^2(V)} \leq C\|f\|_{L^2(U)} + C\|u\|_{L^2(U)}.$$

For the full L^2-regularity theorem, we have an elliptic operator

$$Pu = -\partial_j\left(a^{j,k}\partial_k u\right) + b^j\partial_j u + cu,$$

where $u : U \to \mathbb{R}$ and U is an open subset of $\mathbb{R}^d$. We also assume $a(x) \succ \lambda I$ for some $\lambda > 0$ for all $x \in U$. Also assume $a, b, c \in L^\infty(U)$ (although the natural assumption for $d \geq 3$ is actually $a \in L^\infty, b \in L^d, c \in L^{d/2}$). For the H^2 bound, we also make the assumption that $\partial a \in L^\infty(U)$; this comes from the fact that if we want to commute the derivative as in the argument above, we must be able to deal with the derivative of the coefficients $a^{i,j}$.

Theorem 2.3.2 (H^2 elliptic regularity).
Let $u \in H^1(U)$ be a weak solution to $Pu = f$ on U, and let $f \in L^2(U)$. Then for all $V \subseteq\subseteq U, u \in H^2(V)$, and

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right).$$

The proof of this theorem is the same as the previous argument but with some minor adjustments. The main step is integration by parts. Formally,

$$\begin{aligned}
\int_U -\partial_j \left(a^{j,k}\partial_k v\right) v\zeta^2 dx &= \int_U a^{j,k}\partial_k v \partial_j v\zeta^2 dx + \int_U a^{j,k}\partial_k v v\zeta\partial_j\zeta dx \\
&\geq \lambda \int_U |Du|^2\zeta^2 dx - \|a\|_{L^\infty} \cdot \qquad \underbrace{\int |Du|\zeta|v||D\zeta|dx}_{\leq \frac{\lambda}{2\|a\|_{L^\infty}}|Dv|^2\zeta^2 + \frac{1}{\lambda}\|a\|_L^\infty |v|^2|D\zeta|^2} \\
&\geq \frac{\lambda}{2}\int_U |Dv|^2\zeta^2 dx - \frac{\|a\|^2_{L^\infty}}{\lambda}\int_U |v|^2|D\zeta|^2 dx.
\end{aligned}$$

Since we do not know a priori that $u \in H^2(V)$, need to modify the proof idea to commute the equation with difference quotients instead of derivatives.

Definition 2.3.3 (Difference quotient)**.**
If $k \in \{1, \ldots, d\}$ and $h \in \mathbb{R}\backslash\{0\}$, the difference quotient is

$$D_k^h v(x) = \frac{v(x + he_k) - v(x)}{h}$$

This converges to $\partial_k v(x)$ as $h \to 0$.

Proof (Proof of Thm 2.3.2)**.**
Here we only proof the case $k = 0$.

Step 0: Note that for $u \in H^1(U)$,

$$Pu = f \text{ in } U \iff \langle Pu, \varphi\rangle = \langle f, \varphi\rangle \quad \forall \varphi \in C_c^\infty(U)$$

Here, $Pu \in H^{-1}(U), f \in L^2 \subseteq H^{-1}$.

$$\iff \langle Pu, \varphi\rangle = \langle f, \varphi\rangle \quad \forall \varphi \in H_0^1(U) \quad \left(= \left(H^{-1}(U)\right)^*\right)$$

When we did our a priori estimate last time, we used approximation of u by smooth functions. However, here, we want to show that we have extra regularity, so the equivalent of approximation is this step above.

$$\iff \int_U a^{j,k}\partial_j u\partial_k\varphi + b^j\partial_j u\varphi + cu\varphi dx = \int_U f\varphi dc \quad \forall\varphi \in H_0^1(U)$$

a
Step 1: Now commute the equation with D_j^h. Note that the Leibniz rule holds:

$$D_j^h(uv)(x) = D_j^h u(x)v(x) + u(x+h)D_j^h v(x).$$

This comes from

$$uv(x+h) - uv(x) = (u(x+h) - u(x))v(x) + \underbrace{u(x+h)}_{=:u^h(x)}(v(x+h) - v(x)).$$

Now

$$D_j^h f = D_j^h \left(-\partial_j a^{j,k}\partial_k u + b^j \partial_j u + cu\right)$$
$$= -\partial_\ell \left(a^h\right)^{j,k} \partial_k D_j^h u + \left(b^h\right)^j \partial_\ell D_j^h u + c^h D_j^h u - \partial_\ell \left(D^h a\right)^{\ell,k} \partial_k u + \left(D_j^h b\right)^\ell \partial_\ell u + D_j^h cu.$$

Rearrange this as

$$-\partial_\ell \left(\left(a^h\right)^{\ell,k} \partial_k D_j^h u\right) = \widetilde{f}_1^h$$

where f_1^h is everything else. Now

$$\left\langle -\partial_\ell \left(a^h\right)^{\ell,k} \partial_k D_j^h u, \varphi \right\rangle = \left\langle \widetilde{f}_1^h, \varphi \right\rangle \quad \forall \varphi \in H_0^1(U)$$

where the left hand side equals

$$\int \left(a^h\right)^{\ell,k} \partial_k \left(D_j^h\right) u \partial_\ell \varphi dx$$

Step 2: " $\varphi = \partial_j u \zeta^2$" : Choose $\varphi = D_j^h \zeta^2 \in H_0^1(U)$. By the integration by parts idea, we get

$$\frac{\lambda}{2} \int_U \left|DD_j^h u\right|^2 \zeta^2 dx \leq \cdots \widetilde{f}_1 D_j^h u$$

One treats the right hand side like before, treating $D_j^h u$ like $\partial_j u$. To make this precise, we need the following lemma

Lemma 2.3.4 (Diffrence quotient to derivative).
Let $V \subset\subset U$.

- If $u \in W^{1,p}$, $\left\|D_j^h u\right\|_{L^p(V)} \leq C \left\|\partial_j u\right\|_{L^p(U)}$ for $|h| \ll 1$.
- Assume $u \in L^p$. For $h \ll 1$, if $\|D_j^h u\|_{L^p(V)} \leq A$, then $\partial_j u \in L^p$, and $\|\partial_j u\|_{L^p(V)} \leq A$.

This finishes the proof. □

Remark.
The constant C is independent of u and f but dependent on λ, Λ, V, U. The basic ideas in the proof were:

- Integration by parts and ellipticity give us control over the highest order term.
- Commute the equation with ∂_j.

In the proof, we looked at the equation for $\partial_j u$, then applied ellipticity to control $\|\zeta D \partial_j u\|_{L^2}$, where ζ was a smooth curoff which equals 1 on B but is 0 near ∂U. In reality, however, to deduce that $u \in H^2(V)$, we have to work with the difference quotient $D_j(u) = \frac{u(x+he_j)-u(x)}{h}$.

Here is the higher regularity version of this theorem.

Theorem 2.3.5 (H^k elliptic interior regularity).
Assume the same hypotheses as before except

- $|D^\alpha a| \leq$ A for all $|\alpha| \leq k-1, |D^\alpha b| + |D^\alpha c| \leq$ A for all $|\alpha| \leq k-2$,
- $f \in H^{k-2}(U)$.

Then for all $V \subseteq\subseteq U, u \in H^k(V)$, and

$$\|u\|_{H^k(V)} \leq C\left(\|f\|_{H^{k-2}(U)} + \|u\|_{L^2(U)}\right).$$

Proof.

Here is a sketch. The proof follows the same idea, except we commute D^β for $|\beta| \leq k-1$. Then look at the equation for $D^\beta u$:

$$\begin{aligned} D^\beta f &= D^\beta(Pu) \\ &= D^\beta\left(-\partial_j\left(a^{j,k}\partial_k u\right) + b^j\partial_j u + cu\right) \\ &= -\partial_j\left(a^{j,k}\partial_k D^\beta u\right) + D^\beta\left(b^j\partial_j u\right) + D^\beta(cu) \end{aligned}$$

Multiply both sides by $\zeta^2 D^\beta u$. The first term on the right is

$$\sum a^{j,k}\partial_k D^\beta u \partial_j D^\beta u \zeta^2$$

This gives us control of $\left\|DD^\beta u\zeta\right\|_{L^2(U)}$. For the rest of the terms, you do not see more than $k-1$ derivatives of of u and $k-2$ derivatives of b and c after integration by parts. In reality, the details need to be carried out with difference quotients, using induction to take care of lower derivative terms. The full proof is in Evans' book. $\square$

2.4 L^2-based Boundary Regularity

Previously, we have been looking at regularity away from the boundary. You may also notice that we have not been putting conditions on boundary behavior of u (we only required, for example, $u \in H^1$ rather than $u \in H_0^1$).

Theorem 2.4.1 (L^2-based boundary regularity).
Assume the same hypotheses as in the H^2 interior regularity theorem, except:

- $u \in H_0^1(U)$ (i.e. $u|_{\partial U} = 0$ in the sense of traces).
- ∂U is C^2.

Then $u \in H^2(U)$, and

$$\|u\|_{H^2(U)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

Proof.
Assume for simplicity that $u \in H^2(U)$; we can take care of this by doing the argument with difference quotients instead of derivatives. We will omit the contribution of b and c because they do not contribute much, as we have seen. Start with the equation

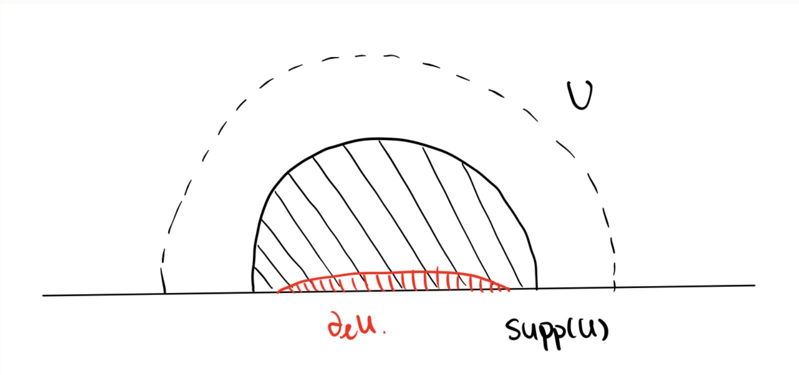
$$f = \partial_j\left(a^{j,k}\partial_k u\right) + \cdots .$$

We want to take a derivative to say

$$\partial_\ell f = -\partial_\ell\left(\partial_j\left(a^{j,k}\partial_k u\right)\right)$$

but we cannot necessarily take the derivative at the boundary. However, notice that if the boundary is flat (wlog $\left\{x^d = 0\right\}$), then all ∂_ℓ exist for $\ell = 1, \ldots, d-1$. The only problem is the normal derivative $\partial_{x^d} = -\nu$. In other words only $(d-1)$-many directions (tangential to ∂U) are admissible.

For the sake of simplicity, take the special case when $U = B_1(0) \cap \mathbb{R}^d_+$ and supp $u \subseteq B_{1/2}(0) \cap \mathbb{R}^d_+$.



In this case, $\partial_l f = -\partial_j\left(a^{j,k}\partial_k\partial_\ell u\right) - \partial_j\left(\partial_\ell a^{j,k}\partial_k u\right)$ for $\ell = 1, \ldots, d-1$. For these $d-1$ terms, we can use the cutoff ζ which equals 1 on $B_{1/2}(0)$ and is 0 near $\partial B_1(0)$ to get

$$\|\zeta D\partial_\ell u\|_{L^2} \leq C\left(\|\zeta f\|_{L^2} + \|u\|_{L^2}\right).$$

In the integration by parts, there is an additional boundary term from $B_{1/2}(0) \cap \left\{x^d = 0\right\}$. However, this contribution is zero because $u|_{\partial U} = 0$, which also implies $\partial_\ell u|_{\partial U} = 0$ for $\ell = 1, \ldots, d-1$.

In this special case, it now remains to control $\|\zeta \partial_{x^d}\partial_{x^d} u\|_{L^2}$. The key observation is that the equation allows us to express $D_\nu D_\nu u$ in terms of everything else. Recall that the original equation is

$$f = -\partial_j \left(a^{j,k}\partial_k u\right) + \cdots .$$

The condition that $a \succ \lambda I$ is equivalent to $a^{j,k}\xi_j\xi_k \geq \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^d$. If we take $\xi = e_d$, this tells us that $a^{d,d} \geq \lambda$. Now write the equation as

$$f = \underbrace{-\partial_d \left(a^{d,d}\partial_d u\right)}_{=a^{d,d}\partial_d^2 u - \left(\partial_d a^{d,d}\right)\partial_d u} - \sum_{\substack{j,k \\ j,k\neq d}} \partial_j \left(a^{j,k}\partial_k u\right).$$

We can divide the equation by $a^{d,d}$ to get

$$\partial_d^2 u = \frac{1}{-a_{d,d}}\left(a \cdot D_{\tan} u + (\partial a), b\right) Du + cu + f.$$

This lets us control $\partial_d^2 u$ by the other derivatives, completing the proof in this special case. In general, we reduce to this special case by first using a smooth partition of unity and boundary straightening. In particular, for every $x \in \partial U$, there exists a ball $B_r(x)$ such that, after relabeling of the coordinate axes, $U \cap B_r(x) = \left\{x^d > \gamma\left(x^1, \ldots, x^{d-1}\right)\right\}$ for some C^2 function γ. We then take a boundary straightening map y, defined by

$$\begin{cases} y^\ell = x^\ell & \ell = 1, \ldots, d-1 \\ y^d = x^d - \gamma\left(x^1, \ldots, x^d\right). & \end{cases}$$

By compactness, $U \subseteq \left(\bigcup_{k=1}^K U_k\right) \cup U_0$, where U_k are balls covering the boundary and U_0 contains the rest of the interior. Then there exists a smooth partition of unity $\{\chi_k\}_{k=0}^K$ subordinate to this cover, which gives

$$u = \chi_0 u + \sum_{k=1}^K \chi_k u$$

The first term is supported on the interior of U, so we can apply our interior regularity theorem to it. For each other $\chi_k u$, when we change $x \mapsto y = y(x)$, we are reduced to the half-ball case already covered (both in terms of geometry and support of u). Check that the ellipticity constant of the resulting equation is still $\simeq \lambda$ and that $\partial\widetilde{a}(y), \widetilde{b}(y)$ obey same bounds as before; this comes from writing the equation in terms of derivatives in y and checking that the change of variables formula $a^{j,k} = \frac{\partial x^j}{\partial y^{j'}}\widetilde{a}^{j',k'}\frac{\partial x^k}{\partial y^{k'}}$ preserves the $a \succ \lambda I$ condition. From the H^2 bound for $u_k(y)$, come back to $u\chi_k(x)$ (which needs the C^2 condition on ∂U). $\square$

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constant of the resulting equation is still $\simeq \lambda$ and that $\partial\widetilde{a}(y), \tilde{b}(y)$ obey same bounds as before; this comes from writing the equation in terms of derivatives in y and checking that the change of variables formula $a^{j,k} = \frac{\partial x^j}{\partial y^{j'}}\widetilde{a}^{j',k'}\frac{\partial x^k}{\partial y^{k'}}$ preserves the $a \succ \lambda I$ condition. From the H^2 bound for $u_k(y)$, come back to $u\chi_k(x)$ (which needs the C^2 condition on ∂U).

2.5 Schauder Theory

Schauder theory can be summarized as "Hölder-based elliptic regularity theory." Here are some of the main theorems.

Theorem 2.5.1 (Chaucer, interior regularity, divergence form).
Let U be an open subset of $\mathbb{R}^d$, and suppose that $Pu = f$, where $Pu = -\partial_j \left(a^{j,k}\partial_k u\right), a \succ \lambda I$, and $a \in C^{k-1,\alpha}(\bar{U})$, Assume that $u \in C^{k,\alpha}(\bar{U})$ (with $k \geq 1$ and $0 < \alpha < 1$) and $f \in C^{k-2,\alpha}(\bar{U})$ (if $k = 1$, we assume that $f = f^0 + \sum_{j=1}^{d} \partial_j f^j$ with $f^0, f^j \in C^{0,\alpha}(\bar{U})$). Then for all $V \subseteq\subseteq U$, there exists a constant $C = C_V$ such that

$$\|u\|_{C^{k,\alpha}(V)} \leq C \left(\|u\|_{L^\infty(U)} + \|f\|_{C^{k-2,\alpha}(U)}\right).$$

(If $k = 1$, we define $\|f\|_{C^{-1,\alpha}} := \left\|f^0\right\|_{C^{0,\alpha}} + \sum_{j=1}^{d} \left\|f^j\right\|_{C^{0,\alpha}.}$)

Remark.
We omit the $b^j + \partial_j u + cu$ parts because they can be easily added, and they are generally dealt with on a case-by-case basis to determine what regularity you need for b and c.

Theorem 2.5.2 (Schauder, interior regularity, non-divergence form).
Let U be an open subset of $\mathbb{R}^d$, and suppose that $Qu = f$, where $Qu = -a^{j,k}\partial_j\partial_k u, a \succ \lambda I$, and $a \in C^{k-2,\alpha}(\bar{U})$, Assume that $u \in C^{k,\alpha}(\bar{U})$ (with $k \geq 2$ and $0 < \alpha < 1$) and $f \in C^{k-2,\alpha}(\bar{U})$. Then for all $V \subseteq\subseteq U$, there exists a constant $C = C_V$ such that

$$\|u\|_{C^{k,\alpha}(V)} \leq C \left(\|u\|_{L^\infty(U)} + \|f\|_{C^{k-2,\alpha}(U)}\right).$$

Definition 2.5.3 ($C^{k,\alpha}$ boundary).
Definition 12.1. We say that U has $C^{k,\alpha}$ boundary if for all $x \in \partial U$, there exists an $r > 0$ such that (after possibly rearranging the axes)

$$U \cap B_r(x) = \left\{y \in B_r(x) : y^n > \gamma\left(y^1, \ldots, y^{d-1}\right), \gamma \in C^{k,\alpha}\right\}$$

Theorem 2.5.4 (Schauder, boundary regularity, divergence form).
Assume the same hypotheses in the interior divergence form theorem, and assume that ∂U is $C^{k,\alpha}$ and U is bounded. Take $Pu = f$ with the boundary condition $u|_{\partial U} = 0$. Then there exists a constant C such that

$$\|u\|_{C^{k,\alpha}(U)} \leq C \left(\|u\|_{L^\infty(U)} + \|f\|_{C^{k-2,\alpha}(U)}\right).$$

Theorem 2.5.5 (Schauder, boundary regularity, non-divergence form).
Assume the same hypotheses in the interior divergence form theorem, and assume that ∂U is $C^{k,\alpha}$ and U is bounded. Take $Pu = f$ with the boundary condition $u|_{\partial U} = 0$. Then there exists a constant C such that

$$\|u\|_{C^{k,\alpha}(U)} \leq C \left(\|u\|_{L^\infty(U)} + \|f\|_{C^{k-2,\alpha}(U)}\right).$$

Here are strategies to prove these theorems.

Interior:

Step 1. Prove the result in the constant coefficient case ($a^{j,k}$ constant).

Step 2. Prove the general case using the constant coefficient case by the method of freezing the coefficients: Elliptic regularity is local, so we can split the space into small balls and prove the statement on each ball. The regularity of $a^{j,k}$ allows us to approximate the general problem by constant coefficient problems.

Boundary:

Step 0. Locally straighten the boundary to reduce to the case of half balls.

Step 1+2. Use the same method as for interior regularity. Step 0 makes the relevant constant coefficient problems be the half-space case.

We will provide two proof for the constant coefficient case:

A. Littlewood-Paley theory proof.
B. Compactness + contradiction proof.

2.5.1 Littlewood-Paley Proof

Theorem 2.5.6 (Constant coefficient Schauder estimate).
Let $Pu = -\partial_j \left(a_0^{j,k} \partial_k u \right) = -a_0^{j,k} \partial_j \partial_k u$, where $a_0^{j,k}$ is constant on $\mathbb{R}^d$, and $a_0 \succ \lambda I$. Assume that $\left| a_0^{j,k} \right| \leq \Lambda$, where $\Lambda \geq \lambda > 0$. For $u \in C_c^{k,\alpha}\left(\mathbb{R}^d\right)$ and $f \in C^{k-2,\alpha}\left(\mathbb{R}^d\right)$ such that $Pu = f$,

$$\|u\|_{C^{k,\alpha}\left(\mathbb{R}^d\right)} \leq C \|f\|_{C^{k-2,\alpha}\left(\mathbb{R}^d\right)}.$$

Let us emphasize that we assume that u has compact support. We will focus on the case $k = 2$.

Definition 2.5.7 (Little-Paley projection).
Define

$$\chi_{\leq 0}(\xi) = \begin{cases} 1 & |\xi| \leq 1 \\ 0 & |\xi| > 1 \\ \geq 0 & \forall \xi \end{cases}$$

$$\chi_{\leq k}(\xi) = \chi_{\leq 0}\left(\xi / 2^k\right),$$

$$\chi_k(\xi) = \chi_{\leq k+1}(\xi) - \chi_{\leq k}(\xi) \qquad \left(\text{so } \operatorname{supp} \chi_k \subseteq \left\{ \xi : 2^k \leq |\xi| \leq 2^{k+1} \right\} \right).$$

The Littlewood-Paley projections are

$$P_k v = \mathcal{F}^{-1}\left(\chi_k(\xi) \widehat{v}\right), \quad P_{\leq k} v = \mathcal{F}^{-1}\left(\chi_{\leq k}(\xi) \widehat{v}\right).$$

Observe that for all $v \in \mathcal{S}'\left(\mathbb{R}^d\right)$,

$$v = P_{\leq k_0} v + \sum_{k > k_0} P_k v.$$

If v satisfies certain regularirt conditions in the same norm, $P_{\leq k_0} v \to 0$ as $k_0 \to -\infty$. Note that $|\xi| \simeq 2^k$ on $\operatorname{supp} \chi_k$.

Lemma 2.5.8 (Littlewood-Paley characterization of $C^{0,\alpha}(\mathbb{R}^d)$).
Let $v \in C^{0,\alpha}\left(\mathbb{R}^d\right)$. Then

$$[v]_{C^{0,\alpha}} = \sup_{\substack{x,y \\ x\neq y}} \frac{|v(x)-v(y)|}{|x-y|^\alpha} \simeq \sup_{k\in\mathbb{Z}} 2^{k\alpha} \|P_k v\|_{L^\infty}. .$$

Proof.

- Step 1: $\gtrsim$
  Both seminorms are invariant to scaling, so it suffices to consider $k=0$. So we just have to show that
  $$|P_0 v| \lesssim [v]_{C^{0,\alpha}}.$$
  Since $\int \overset{\vee}{\chi_0}(y)dy = 0$ iff $\chi_0(0)=0$,
  $$\begin{aligned} P_0 v = \int \overset{\vee}{\chi_0}(x-y)v(y)dy &= \int \overset{\vee}{\chi_0}(x-y)(v(y)-v(x))dy \\ &\leq \underbrace{\int \overset{\vee}{\chi_0}(x-y)|x-y|^\alpha dy}_{\text{fixed } \mathcal{S}(\mathbb{R}^d) \text{ function}}[v]_{C^{0,\alpha}}. \end{aligned}$$

- Step 2: $\lesssim$
  Whenever we work with Littlewood-Paley theory, we should think about what scale we are working on. Let $L=|x-y|$, and choose k_0 so that $L^{-1}\simeq 2^{k_0}$. Decompose
  $$v(x)-v(y) = P_{\leq k_0}v(k) - P_{\leq k_0}v(y) + \sum_{k>k_0} P_k v(x) - P_k v(y)$$
  We can bound the latter two terms as
  $$\begin{aligned} \left\|\sum_{k\geq k_0} P_{\leq k_0} v\right\|_{L^\infty} &\leq \sum_{k<k_0} \|P_k v\|_{L^\infty} \\ &\leq \sum_{k\geq k_0} 2^{-k\alpha}[v]_{C^{0,\alpha}} \\ &\simeq L^\alpha [v]_{C^{0,\alpha}}. \end{aligned}$$
  We can bound the first terms using the fundamental theorem of calculus:
  $$\begin{aligned} |P_{\leq k_0}v(x) - P_{\leq k_0}v(y)| &\leq \|\nabla P_{\leq k_0} v\|_{L^\infty L} \\ &\leq \sum_{k\leq k_0} \|\nabla P_k v\|_{L^\infty L} \\ &\lesssim L \sum_{k\leq k_0} 2^k 2^{-k\alpha}[v]_{C^{0,\alpha}} \\ &\simeq L L^{-(1-\alpha)}[v]_{C^{0,\alpha}} \end{aligned}$$

□

Now we can proof the theorem.

Proof (Proof of Thm 2.5.6).
We have $P(P_k u) = P_k f$, so after fourier transform

$$a^{j,l}\xi_j\xi_l\widehat{P_k u} = \widehat{P_k f}.$$

Since $\lambda|\xi|^2 \leq a_0^{j,\ell}\xi_j\xi_k$

$$\widehat{P_k u} = \frac{2^{2k}}{a^{j,\ell}\xi_j\xi_\ell}\widehat{P_k f}\widetilde{\chi}_k\frac{1}{2^{2k}} = \frac{1}{2^{2k}}\underbrace{\frac{2^{2k}}{a^{j,\ell}\xi_j\xi_\ell}}_{\eta_k(\xi)}\widetilde{\chi}_k\widehat{P_k f},$$

where $\widetilde{\chi}_k = 1$ on $\operatorname{supp}\chi_k$ and $\operatorname{supp}\widetilde{\chi}_k \subseteq \left\{|\xi| \simeq 2^k\right\}$. Then

$$P_k u = 2^{-2k}\eta_k^\vee * P_k f,$$

so

$$\|P_k u\|_{L^\infty} \leq C2^{-2k}\|P_k f\|_{L^\infty} \leq C2^{-2k-k\alpha}[f]_{C^{0,\alpha}},$$

which completes the proof. [a] □

[a] Why?

2.5.2 Compactness and Contradiction Proof

Proof (Proof of Thm 2.5.6).
Here are the steps:

1. Assume that the desired inequality fails. Then there exist $a_n^{j,k}, u_n, f_n$ such that (after normalization)
$$P_n u_n = f_n, \quad [u_n]_{C^{2,\alpha}} = 1, \quad [f_n]_{C^{0,\alpha}} \leq \frac{1}{n}.$$
After translation, we may also ensure that for some $\eta_n \in \mathbb{R}^d$,
$$\left|D^2 u_n(\eta_n) - D^2 u_n(0)\right| \geq c\,|\eta_n|^\alpha$$
Using scaling, we can assume that $|\eta_n| = 1$.

- Another massaging: Define $v_n(x) = u_n(x) - u_n(0) - xDu_n(0) - \frac{1}{2}x^2 D^2 u_n(0)$ to make $D^2 v_n(0) = 0$. Then
$$P_n v_n = \tilde{f}_n, \quad \tilde{f}_n \to 0, \left[D^2 v_n\right]_{C^{0,\alpha}} = 1, \quad \left|D^2 v_n(\eta_j)\right| \geq c.$$
- Take the limit: Let $a_n^{j,k} \to a_\infty^{j,k}, \widehat{f}_n \to 0, v_n \to v$, and $\eta_n \to \eta_\infty$. Then $P_\infty v = 0$ on $\mathbb{R}^d$, while
$$\left[D^2 v\right]_{C^{0,\alpha}} = 1, \quad D^2 v(\eta_\infty) \neq 0$$
But now use Liouville's theorem for P_∞ (using Liouville's theorem for the Laplace equation) to get that $D^2 v(\eta_\infty) = 0$, a contradiction.

□

2.6 Maximum Principles

Today, we will cover maximum principles. This material corresponds to section 6.5 in Evans' textbook. This is a theory for solutions to elliptic PDEs in terms of their pointwise values (inherently scalar). Here, it is very important that $u : U \to \mathbb{R}$ is real-valued.

For today's lecture, it is more convenient to consider operators in non-divergence form:

$$Pu = -a^{j,k}\partial_j\partial_k u + b^j\partial_j u + cu.$$

We assume the ellipticity condition, that $a \succ \lambda I$ for some $\lambda > 0$, and we assume that $a, b, c \in L^\infty$. (Often, we will start with $c = 0$.)

The theory of maximum principles should be thought of as a generalization of the theory of convex functions on $\mathbb{R}$.

2.6.1 The Weak Maximum Principle

Definition 2.6.1 ((Classical) subsolution).
We say that $u \in C^2(U)$ is a classical subsolution if $Pu \leq 0$.

Remark.
When $d = 1$ and $P = -a\partial_x^2$ with $a > 0, Pu \leq 0$ if and only if u is convex.

Theorem 2.6.2 (Weak maximum principle).
Let U be a connected, bounded, open subset of $\mathbb{R}^d$. Let $u \in C^2(U) \cap C(\bar{U})$ with $Pu \leq 0$. Assume for now that $c = 0$. Then

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

Proof.

- Step 1:
  Consider strict subsolutions $Pu < 0$. We will show that no interior maximum is possible. Suppose, for contradiction, that $x_0 \in U$ is a (local) maximum. Then $Du(x_0) = 0$, and the second derivative test tells us that $D^2u(x_0) \leq 0$. We have

$$0 > Pu(x_0) = -a^{j,k}\partial_j\partial_k u\Big|_{x_0} + b^j \underbrace{\partial_j u|_{x_0}}_{=Du=0} + \underbrace{c}_{=0} u = -a^{j,k}\partial_j\partial_k u\Big|_{x_0} = -tr(aD^2u)|_{x_0}.$$

  Note that we can write

$$a = \sum_{i=1}^n \lambda_i v_i v_i^T.$$

  Hence

$$-tr(aD^2u) = -\sum_{i=1}^n \lambda_i \operatorname{tr}(v_i v_i^T D^2 u) = -\sum_{i=1}^n \lambda_i v_i^T D^2 u v_i.$$

  Therefore, we can find one direction such that $v_i^T D^2 u v_i > 0$, contradiction.
- Step 2:
  Upgrade to all subsolutions u. Introduce the approximation

$$u_\varepsilon = u + \varepsilon v,$$

where v is a strict subsolution: $Pv < 0$ with $v \in C^2(U) \cap C(\bar{U})$. Then $u_\varepsilon \to u$ uniformly on $\bar{U}$, and

$$Pu_\varepsilon = Pu + \varepsilon Pv \leq \varepsilon Pv < 0.$$

How do we construct a strict subsolution v ? We want something that is convex. A good candidate is $v = e^{x^1}$ because

$$-a^{j,k}\partial_j\partial_k\left(e^{x^1}\right) = -a^{1,1}e^{x^1} < 0$$

We want to introduce a function which has a second order derivative much larger than a first order derivative. So instead consider $e^{\mu x^1}$, where μ is large. Then

$$\begin{aligned} -a^{j,k}\partial_j\partial_k\left(e^{\mu x^1}\right) &= -a^{1,1}e^{\mu x^1} \leq -\lambda\mu^2 e^{\mu x^1}, \\ \left|b^j\partial_j e^{\mu x^1}\right| &= \left|-b^j\mu e^{\mu x^1}\right| \leq \sup|b|\cdot\mu e^{\mu x^1} \end{aligned}$$

So if μ is large, $Pv < 0$.

□

Definition 2.6.3 ((Classical) supersolution).
We say that $u \in C^2(U)$ is a classical supersolution if $Pu \geq 0$.

Similarly, we have the following minimum principle.

Theorem 2.6.4 (Weak minimum principle).
Have the same hypotheses except assume that $Pu \geq 0$ and $c = 0$. Then

$$\min_{\bar{U}} u = \min_{\partial U} u$$

Remark.
u is a solution if and only if it is a subsolution and a super solution. So under the same hypotheses with $Pu = 0$, we get

$$\max_{\bar{U}}|u| = \max_{\partial U}|u|$$

Corollary 2.6.5 (Weak maximum principle, $c \geq 0$).
Suppose U is a bounded, open connected subset of $\mathbb{R}^d$ and $u \in C^2(U) \cap C(\bar{U})$. For $Pu \leq 0$.

$$\begin{aligned} Pu \leq 0 &\implies \max_{\bar{U}} u \leq \max_{\partial U} u^+, \\ Pu \geq 0 &\implies \min_{\bar{U}} u \leq \min_{\partial U} u^-, \end{aligned}$$

where

$$u^+ = \begin{cases} u & \text{if } u > 0 \\ 0 & \text{if } u \leq 0, \end{cases} \qquad u^+ = \begin{cases} 0 & \text{if } u \geq 0 \\ -u & \text{if } u < 0 \end{cases}$$

Proof.
Here is the max part: Let $V = \{x \in U : u(x) > 0\}$, and let $Qu = Pu - cu$. Q satisfies the hypotheses and has no zero order term: $u \leq -cu \leq 0$ in V. The weak maximum principle

for Q on V gives $\max_{\bar{V}} u \leq \max_{\partial V} u$. Note that the maximum of u on ∂V is the maximum of u on ∂U. So we get the claim. □

We also have the comparison principle.

Theorem 2.6.6 (Comparison principle).
Let U be an open, bounded, connected subset of $\mathbb{R}^d$. Let P be elliptic with $c \geq 0$. Suppose $u, v \in C^2(U) \cap C(\bar{U})$ with $Pu \leq 0$ in U and $Pv \geq 0$ in U. If $u \leq v$ on ∂U, then $u \leq v$ on U.

2.6.2 The Strong Maximum Principle

Theorem 2.6.7 (Strong maximum principle).
Let U be an open, bounded, connected subset of $\mathbb{R}^d$, and let $c = 0$. Let $u \in C^2(U) \cap C(\bar{U})$ be such that $Pu \leq 0$. If u has a maximum at $x_0 \in U$ (i.e.,$u(x) = \max_{\bar{U}} u$), then u is constant on U.

Think of the picture of convex functions. The only way to have a maximum in the interior is if the whole function is constant (the graph is a horizontal straight line). Before, we prove this theorem, we need the following lemma.

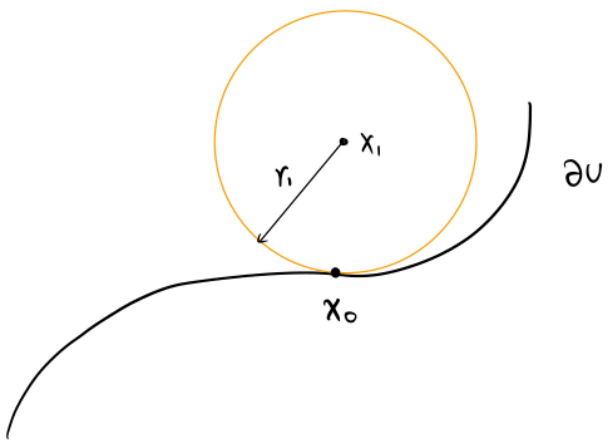
Theorem 2.6.8 (Hopf's lemma).
Let U be an open, bounded, connected subset of $\mathbb{R}^d$ and $Pu \leq 0$ in U. Suppose that $x_0 \in \partial U$ is such that

(1) there exists some $x_1 \in U$ and $r_1 > 0$ such that $B_{r_1}(x_1) \subset U$, and $\overline{B_{r_1}(x_1)} \cap \partial U = \{x_0\}$,
(2) $u(x_0) > u(x)$ in $\overline{B_{r_1}(x_1)}$,
(3) $u(x_0) > u(x)$ in $B_{r_1}(x_1)$.

Then the normal derivative $\frac{\partial u}{\partial \nu}|_{x_0} > 0$.

Proof.
Without loss of generality, take $x_1 = 0$. Consider $v = e^{-\mu|x|^2} - e^{-\mu r_1^2}$ so that $v(x) = 0$ on $\{|x| = r_1\}$. Then $Pv \leq 0$ on $B_{r_1} \backslash B_{r_1/2}$ for large μ (this is the same type of computation as before). Try to compare u to $w = -\varepsilon v + u\left(x_0\right)$, where

$$Pw = -\varepsilon Pv + Pu\left(x_0\right) = -\varepsilon Pv \geq 0.$$



Let $V = B_{r_1} \backslash B_{r_1/2}$, so $\partial V = \partial B_{r_1} \cup \partial B_{r_1/2}$. On the outer boundary ∂B_{r_1}, $w = u\left(x_0\right) \geq$

u. On the inner boundary $\partial B_{r_1/2}, w = -\varepsilon v + u\,(x_0)$. So for small enough ε, on the inner boundary, $u\,(x_0) > u(x) + \varepsilon v$. By the comparison principle, $w \geq u$ on $V = B_{r_1} \backslash B_{r_1/2}$. Since

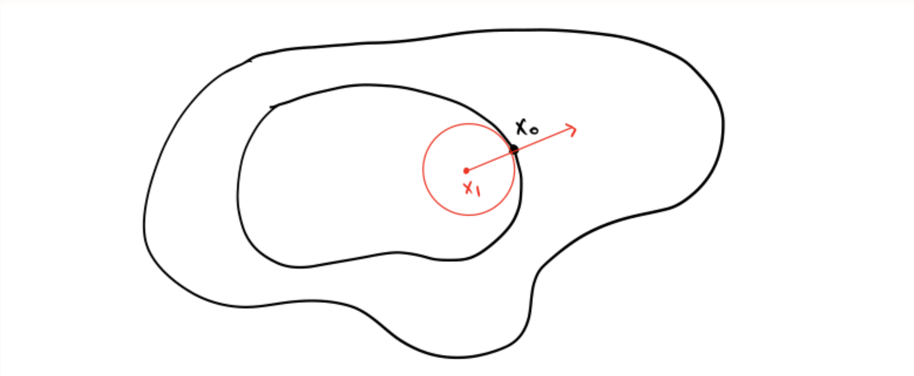
$$u(x) + \varepsilon v - u(x_0) < 0, \quad u(x_0) + \varepsilon v(x_0) - u(x_0) = 0,$$

we have

$$\left.\frac{\partial(u+\varepsilon v)}{\partial \nu}\right|_{x_0} \geq 0 \Rightarrow \left.\frac{\partial u}{\partial \nu}\right|_{x_0} \geq -\varepsilon \left.\frac{\partial v}{\partial \nu}\right|_{x_0} = 2\varepsilon r_1 e^{-\mu r_1^2} > 0.$$

□

Proof (Proof of Thm 2.6.7).
Let $V = \{x \in U : u(x) \leq M\}$, where $M = \sup_{\bar{U}} u$. Then for $x_0 \in U$, if $u\,(x_0) = M$, then $V \subsetneq U$. Assume for contradiction that $V \neq \varnothing$. Find a point x_1 closer to ∂V than ∂U and consider the biggest r_1 such that $B_{r_1}\,(x_1) \subseteq V$. Let $x_0 \in B_{r_1}\,(x_1) \cap \partial V$.



We may arrange, by taking x_1 close enough to ∂V, so that Hopf's lemma is applicable. This tells us that $\frac{\partial}{\partial \nu} u\big|_{x=x_0} \neq 0$. But this contradicts the facr that $u\,(x_0') = M$ implies $Du\big|_{x=x_0} = 0$ □

Similarly, we can prove the following strong maximum principle with $c \geq 0$:

Theorem 2.6.9 (Strong maximum principle with $c \geq 0$).
For U being open, bounded and connected subset of $\mathbb{R}^d$, and $u \in C^2(U) \cap C(\bar{U}), c \geq 0$, then we have

- If $Pu \leq 0$ and u has a nonnegative maximum at $x_0 \in U$, then $u = \text{const}$ in U.
- If $Pu \geq 0$ and u has a nonpositive minimum at $x_0 \in U$, then $u = \text{const}$ in U.

2.7 General Boundary Value Problems

In this lecture, we will assume that P is an elliptic operator in divergence form:

$$Pu = -\partial_j \left(a^{j,k} \partial_k u \right) + b^j \partial_j u + cu.$$

Let U be an open, bounded, connected subset of $\mathbb{R}^d$ with C^1 boundary ∂U. A general boundary value problem might be of the form

$$\begin{cases} Pu = 0 & \text{in } U \\ Bu|_{\partial_U} = g & (\text{ on } \partial U) \end{cases}$$

for some operator B. So far, we have focused on the Dirichlet boundary condition

$$\begin{cases} Pu = 0 & \text{in } U \\ u|_{\partial_U} = g & (\text{ on } \partial U) \end{cases}$$

By introducing an extension $\tilde{g}$ of g to U, we could set, without loss of generality, $g = 0$. With this reduction, the problem we have considered is

$$\begin{cases} Pu = 0 & \text{in } U \\ u|_{\partial_U} = 0 & (\text{ on } \partial U) \end{cases}$$

Our goal now is to generalize our elliptic theory to other boundary conditions. This will force us to consider what is a "regular" boundary value problem for PDEs. In order to solve a k-th order ODE, you need k pieces of data on the boundary. For the wave equation, which is a second order PDE, you impose boundary values and normal derivative values. Unlike ODEs, the wave equation, or Cauchy-Kovalevskaya, when we work with an elliptic PDE like $-\Delta u = f$, we do not prescribe the full u, $\frac{\partial}{\partial v} u$ on ∂U. How do we rigorously justify this high level discussion? We will see two approaches.

2.7.1 The Weak Formulation

We can prove a uniqueness theorem via the energy method.

Example 2.7.1 (Wave equation).
If $P = -\Delta$ and we are solving

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ u|_{\partial_U} = g & (\text{ on } \partial U) \end{cases}$$

then

$$0 = \int_U -\Delta u u dx = \int |Du|^2 dx.$$

Note the parallel between this basic consideration and our weak formulation of the Dirichlet problem: $u \in H^1$ solves the Dirichlet problem

$$\begin{cases} Pu = f & \text{in } U \\ u|_{\partial_U} = g & (\text{ on } \partial U) \end{cases}$$

if and only if $u \in H_0^1(U)$ and $-\Delta u = f$ in the sense of $\mathcal{D}'(U)$. This is equivalent to

$$\int_U a^{j,k} \partial_j u \partial_k \varphi + b^j \partial_j u \varphi + cu\varphi dx = \int_U f \varphi dx \quad \forall \varphi \in H_0^1(U).$$

We will try to generalize this weak formulation to other boundary conditions.

Example 2.7.2 (Neumann boundary condition).
Consider the Neumann boundary condition

$$\begin{cases} Pu = f & \text{in } U \\ \nu^j \partial_j u\big|_{\partial_U} = g & (\text{ on } \partial U) \end{cases}$$

We can rewrite this as

$$\begin{cases} Pu = f & \text{in } U \\ a^{j,k}\nu_k \partial_j u\big|_{\partial_U} = g & (\text{ on } \partial U) \end{cases}$$

In the case of the Laplace equation, this is the same. From the point of view of differential geometry, this is a more natural quantity to look at because ν_k is dh, where h is the boundary defining form. The natural Riemannian metric in this problem is a. By an extension procedure, we can write the problem as

$$\begin{cases} Pu = f & \text{in } U \\ a^{j,k}\nu_k \partial_j u\big|_{\partial_U} = 0 & (\text{ on } \partial U) \end{cases}$$

For simplicity, assume $b = c = 0$. Then we have the formal computation

$$\int_U f\varphi dx = \int_U -\partial_j \left(a^{j,k}\partial_j u\right)\varphi dx = \int_U a^{j,k}\partial_j u \partial_k \varphi dx - \int_{\partial U} \underbrace{\nu_j a^{j,k}\partial_k u}_{=0}\varphi dA.$$

This motivates the following definition:

Definition 2.7.3 (Neumann boundary problem).
We say that u satisfies the Neumann boundary problem if for all $\varphi \in H^1(U)$,

$$\int_U a^{j,k}\partial_j u \partial_k \varphi dx = \int_U f\varphi dx$$

Remark.
If $u \in C^1$ then this formulation should be equivalent to the classical one. Onece we formulate the problem like this, the L^2 theory is easy to generalize.

Theorem 2.7.4.

1. For any $\mu \in \mathbb{R}$, the map $u \mapsto Pu - \mu u$ associated to the Neumann boundary value problem

$$(\mathrm{NP}_\mu) \begin{cases} Pu - \mu u = f & \text{in } U \\ a^{j,k}\nu_j \partial_j u\big|_{\partial_U} = g & (\text{on } \partial U) \end{cases}$$

   is Fredholm with index μ from $H^1(U) \to \left(H^1(U)\right)^* \subseteq H^{-1}(U)$. That is, one of the following holds:

   (i) For all $f \in L^2(U)$, there exists a unique $u \in H^1$ which solves the Neumann boundary problem (NP_μ).

   (ii) There exists a solution $v \neq 0$ to (NP_μ) with $f = 0$. Furthermore, for $\mu \gg 1$, alternative (i) applies.

2. There exists a solution $v \neq 0$ to (NP_μ) with $f = 0$. Furthermore, for $\mu \gg 1$, alternative (i) applies.

Example 2.7.5.
ake $P = -\Delta$ and solve

$$\begin{cases} -\Delta u = 0 \\ u|_{\partial U} = 0 \end{cases}$$

This has a nontrivial solution $v = \text{const} \neq 0$. This leads to solvability for f orthogonal to the kernel of the adjoint. In this case, this is equivalent to $\int_{IJ} f dx = 0$.

We also have another definition for the Robin boundary problem.

Example 2.7.6 (Oblique Dirichlet boundary condition)**.**
Assume $b = c = 0$:

$$\begin{cases} Pu = f \text{ in } U \\ X^j \partial_j u = 0 \text{ on } \partial U \end{cases}$$

where X is transversal to ∂U, outward. Then $X = X^\perp + X^\top$, where $X^\perp$ is parallel to $a^{j,k} \nu_k \overrightarrow{e_k}$. Normalize to make $X^\perp = a^{j,k} \nu_j \overrightarrow{e_k}$. This tells us that

$$\int_U a^{j,k} \partial_j u \partial_k \varphi + \int_{\partial U} X^\top u \varphi dA = \int_U f \varphi dx$$

The second term is trickier to make sense of, since we need to make sense of the trace. As an exercise, check that $\int_{\mathbb{R}^{d-1}} \partial u v dx$ is well defined for $u, v \in H^{1/2}\left(\mathbb{R}^{d-1}\right)$. This is just barely well-defined, however, in the sense of the trace theorem needing $H^{1/2}$.

2.7.2 The "Microlocal" Formulation

The reference for this section is volume 1 of Taylor's PDE book, section 5.11. Look at the Laplace equation $-\Delta u = 0$ in the half space $\mathbb{R}^d_+$. Write z for the last variable and x for the remaining $d-1$ variables, so this is $-\partial_z^2 - \Delta_x u = 0$. Suppose we have boundary conditions $u|_{\partial U} = g$ and $\partial_z u|_{\partial U} = h$. We can view this as an evolution equation in the z variable and take the Fourier transform in x to get

$$\left(-\partial_z^2 + |\xi|^2\right) \widehat{u} = 0$$

with boundary conditions $\widehat{u}|_{z=0} = \hat{g}$ and $\partial_z \widehat{u}|_{z=0} = \hat{h}$. This gives

$$\widehat{u}(z, \xi) = a_+(\xi) e^{|\xi| z} + a_-(\xi) e^{-|\xi| z}.$$

However, the first term $e^{|\xi| z}$ is a problem because growth in Fourier space corresponds to a lack of regularity in physical space. So in order to have boundary regularity, we want $a_+(\xi) = 0$. This means that we are only left with half of the full freedom to choose $\widehat{g}$ and The claim is that the constant coefficient picture generalizes to the variable coefficient picture. The idea is that using the technique of "freezing the coefficients," we can formulate the notion of a "regular" elliptic boundary value problem, for which we have elliptic regularity and the Fredholm property, based on the constant coefficient computation.

Here, we assume that $a, b, c \in C^\infty(\bar{U})$ and that ∂U is C^∞.

Definition 2.7.7 ($H^{k-1/2}$).
For $k \geq 1$, define

$$H^{k-1/2}(\partial U) = \left\{ g = v|_{\partial U} : v \in H^k(U) \right\}$$

with the norm

$$\|g\|_{H^{k-1/2}(\partial U)} = \inf_{u:\, u|_{\partial U} = g} \|u\|_{H^k(U)}$$

Remark.
If we define fractional Sobolev spaces on manifolds, this will actually be the $k - 1/2$ Sobolev space on ∂U.

Now consider the boundary problem

$$\begin{cases} Pu = f \\ Bu|_{\partial U} = g. \end{cases} \text{in } U$$

Here, we assume that $P : C^\infty(U) \to C^\infty(U)$ and $B(\cdot)|_{\partial U} : C^\infty(U) \to C^\infty(\partial U)$. Given $x_0 \in \partial U$, there exists a boundary straightening map near x_0. In these variables, write

$$\begin{gathered} P = -\partial_z^2 + P_1(y, z, D_y)\, \partial_z + P_0\left(y, z, D_y, D_y^2\right) \\ B = b\partial_z + B_0(y, z, \partial_y) \end{gathered}$$

Say x_0 is mapped to 0 , and let P_{x_0} be the frozen constant coefficient operator

$$\begin{gathered} P_{x_0} = -\partial_z^2 + P_1(0, 0, D_y)\, \partial_z + P_0\left(0, 0, D_y, D_y^2\right) \\ B_{x_0} = b(0,0)\partial_z + B_0(0, 0, \partial_y) \end{gathered}$$

Definition 2.7.8 (Loputinski-Shapiro condition).
A boundary value problem is a **regular elliptic boundary value problem** if for all $x_0 \in \partial U$, for all $\xi \in \mathbb{R}^{d-1}$, and for all ζ, there exists a unique bounded solution to the ODE

$$P_{x_0} \widehat{u}(z, \xi) = 0, \quad B_{x_0} \widehat{u}(z, \xi) = \zeta.$$

This is called the Loputinski-Shapiro condition. This is like if we pretend we take the Fourier transform and replace ∂_y by $c\xi$. This condition gives an ODE in z.

Theorem 2.7.9.
For a regular elliptic boundary value problem, the map $H^{k+2}(U) \ni u \mapsto (Pu, Bu) \in H^k(U) \times H^{k-(\text{ order B })-1/2}(\partial U)$ is Fredholm, and we have elliptic (boundary) regularity

$$\|u\|_{H^{k+2}(U)} \lesssim \|f\|_{H^k(U)} + \|Bu\|_{H^{k-(\text{ order B } -1/2}(\partial U)} + \|u\|_{H^{k+1}(U)}$$

2.8 Unique Continuation

The original plan was for this lecture to cover one final topic for elliptic PDEs: unique continuation. Here is the main theorem.

Theorem 2.8.1 (Aronszajn).
Let $U \subseteq \mathbb{R}^d$ be open and connected, and consider the elliptic partial differential operator P with

$$Pu = -\partial_j \left(a^{j,k} \partial_k u \right) + b^j \partial_j u + cu,$$

where $a^{j,k} b^j, c \in C^\infty(U)$ with $a \succ \lambda I$ in U. Let $u \in H^1(U)$. If $Pu = 0$ in U and $u = 0$ in a nonempty open subset $Q \subseteq U$, then $u = 0$ in U.

For holomorphic functions, the way we prove this is to say that holomorphic functions are analytic and look at the domain of convergence of a Taylor series. The way we prove this for solutions to elliptic PDEs is via an a priori estimate.

Lemma 2.8.2 (Carleman estimate).
Let $v \in C_c^\infty \left(\mathbb{R}^d \right)$. and suppose that $\nabla \psi \neq 0$. Then

$$t^2 \left\| e^{t\psi} v \right\|_{L^2} + t \left\| e^{t\psi} \nabla v \right\|_{L^2} \leq C \left\| e^{t\psi} P v \right\|_{L^2}.$$

A good reference for this is the book Carlesman Estimates by Lerner [2019]. This is related to inverse problems and other non-well-posed problems in PDEs.

CHAPTER 3

LINEAR HYPERBOLIC PDES

In mathematics, a hyperbolic partial differential equation of order n is a partial differential equation (PDE) that, roughly speaking, has a well-posed initial value problem for the first $n-1$ derivatives. More precisely, the Cauchy problem can be locally solved for arbitrary initial data along any non-characteristic hypersurface. Many of the equations of mechanics are hyperbolic, and so the study of hyperbolic equations is of substantial contemporary interest.

3.1 Linear Hyperbolic PDEs

Instead of formally defining what a hyperbolic PDE is, which is difficult and not entirely productive. Instead, we will give a "working definition" of how people think of hyperbolic PDEs.

Definition 3.1.1 (Hyperbolic PDE).
A hyperbolic PDE is an evolutionary PDE with two characteristics:

- # order of time derivatives = # order of space derivatives
- (local) well-posedness of the initial value problem

$$\begin{aligned} &P\phi = 0 \\ &\left(\phi, \partial_t \phi, \ldots, \partial_t^{N-1}\phi\right)\Big|_{t=0} = (g_0, \ldots, g_{N-1}) \end{aligned}$$

  where N is the order of the time derivatives.

This second condition is really what people think of when they talk about hyperbolic PDEs.

Example 3.1.2.

- The wave equation $\left(-\partial_t^2 + \Delta\right)\phi = 0$.
- $\left(-\partial_t + x^j \partial_j\right)\phi = 0$.
- (Non-examples) The heat equation $(\partial_t - \Delta)\phi = 0$ and the Schrödinger equation $(\partial_t - i\Delta)\phi = 0$ are dispersive but not hyperbolic.
- The Laplace equation $\left(\partial_t^2 + \Delta\right)\phi = 0$ is not hyperbolic because it does not have local well-posedness of the initial value problem.
  Local well-posedness of the initial value problem is related to the energy estimate [a].

[a] Why this is true?

Example 3.1.3 (Linear constant coefficient system).
Let

$$\Phi = \begin{bmatrix} \Phi^{(1)} \\ \vdots \\ \Phi^{(n)} \end{bmatrix}$$

and suppose we have a system of linear, constant coefficient PDEs

$$B\partial_t \Phi = A^j \partial_{x^j} \Phi,$$

where A is an $n \times n$ matrix. Without loss of generality, assume we have

$$\partial_t \Phi = A^j \partial_{x^j} \Phi,$$

What guarantees uniqueness of a solution to the initial value problem? That is, what condition do we need on A to guarantee the validity of the energy estimate? Now we assume a more general PDE with

$$\partial_t \Phi + A^j \partial_{x^j} \Phi = F.$$

Then

$$\int_{\mathbb{R}^d} \underbrace{\Phi^{(k)} \partial_t \Phi^{(k)}}_{=\frac{1}{2}\partial_t \int \Phi^{(k)}\Phi^{(k)}} + \underbrace{\Phi^{(k)} \left(A^j\right)^{(k)}_{(\ell)} \partial_j \Phi^{(\ell)}}_{\frac{1}{2}\int (A^j)^{(k)}_{(\ell)} \Phi^{(k)} \partial_j \Phi^{(\ell)} - \frac{1}{2} (A^j)^{(k)}_{(\ell)} \partial_j \Phi^{(k)} \Phi^{(\ell)}} dx = \int \Phi^{(k)} F^{(k)} dx.$$

We get the following identity:

$$\frac{1}{2}\partial_t \int |\Phi|^2 dx + \frac{1}{2}\int \left(\left(A^j\right)^{(k)}_{(\ell)} - \left(A^j\right)^{(\ell)}_{(k)}\right) \Phi^{(k)} \partial_j \Phi^{(\ell)} dx = \int F \cdot \Phi dx$$

where the second term is 0 if A^j is symmetric. This tells us that if A^j is symmetric, then the energy estimate holds:

$$\int |\Phi|^2(t) dx = \int |\Phi|^2(0) dx + \int_0^t \int F \cdot \Phi dx\, dt$$

This gives uniqueness.

Theorem 3.1.4.
The linear, constant coefficient system

$$\partial_t \Phi = A^j \partial_{x^j} \Phi$$

is hyperbolic if and only if the A^j are symmetric. That is the initial value problem is well-posed in L^2, meaning for every $\Phi_0 \in L^2\left(\mathbb{R}^d\right)$, and $F \in L^1_t\left((-\infty, \infty); L^2_x\right)$, there exists a unique $\Phi \in C_t\left((-\infty,\infty); L^2_x\right)$ solving the system.

We use the notation $\phi \in C_t(I; X)$ to mean that the function $\phi : I \to X$ sending $t \mapsto \phi(t)$ is continuous, where $C_t(I; X)$ has the norm

$$\|\phi\|_{C_t(I;X)} := \sup_{t \in I} \|\phi(t, \cdot)\|_X = \|\phi\|_{L^\infty_t(X)} < \infty.$$

Example 3.1.5 (1st order formulation of $\Box\phi = f$).
Let the d'Alembertian be $\Box = -\partial_t^2 + \Delta$. Then

$$\Box\phi = f \iff \partial_t\phi = \psi, \partial_t\psi = \Delta\phi - f$$

We can write this system as

$$\partial_t \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} - \begin{bmatrix} 0 \\ f \end{bmatrix}$$

If we take the Fourier transform of the matrix, we get

$$\begin{bmatrix} 0 & 1 \\ -|\xi|^2 & 0 \end{bmatrix}$$

and if we diagonalize this, we get

$$\begin{bmatrix} +i|\xi| & 0 \\ 0 & -i|\xi| \end{bmatrix},$$

which is anti-Hermitian. This means that the energy estimate will hold in the diagonalized variables.[a]

[a] Why energy estimate will hold?

Goals for Studying Hyperbolic PDEs

Here are our goals for studying hyperbolic PDEs:

1. (Local) well-posedness of the initial value problem for variable-coefficient wave equations,

$$P\phi = \partial_\mu \left(g^{\mu,\nu}\partial_\nu\phi\right) + b^\mu\partial_\mu\phi + c\phi,$$

   where g is a **Lorentzian metric**, a non-degenerate symmetric $(d+1)\times(d+1)$ matrix with signature $(-,+,+\cdots,+)$ (meaning that the eigenvalues of g have signs $-,+,+\ldots,+$). This condition can also be stated as: for every (t,x), there exists an invertible matrix M such that $M^{-1}g(t,x)M = \mathrm{diag}(-1,+1,+1,\ldots,+1)$.
   Note that when $g = \mathrm{diag}(-1,+1,+1,\ldots,+1)$ and $b = c = 0, P = \Box$.
2. Long-time behavior of the solutions: If we look at this in general, it immediately becomes a research topic. [1] Instead, we will focus on long-time behavior of solutions to equations where P is a small variant of $\Box$.

[1] Scattering theory is devoted to studying these problems.

3.2 Grönwall's Inequality

Our treatment for the well-posedness of the initial value problem for variable coefficient wave equations will be closer to Ringström's book the Cauchy Problem in General Relativity than it will be to Evans' book. Our setting is

$$P\phi = \partial_\mu \left(g^{\mu,\nu}\partial_\nu \phi\right) + b^\mu \partial_\mu \phi + c\phi$$

We want to derive energy estimates for

$$\begin{cases} P\phi = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^d \\ (\phi, \partial_t \phi)|_{t=0} = (g, h) & \text{on } \{t = 0\} \times \mathbb{R}^d \end{cases}$$

We need the following preliminary tool, which was discussed in Math 222 A.

Lemma 3.2.1 (Grönwall's inequality).
Suppose that $E(t) \in C_t([0,T])$ and $r(t) \in L^1_t([0,T])$ with $E, r \geq 0$ satisfy the inequality

$$E(t) \leq E_0 + \int_0^t r\left(t'\right) E\left(t'\right) dt' \quad \forall 0 \leq t \leq T$$

Then

$$E(t) \leq E_0 \exp\left(\int_0^t r\left(t'\right) dt'\right) \quad \forall 0 \leq t \leq T$$

We give a proof that uses a bootstrap argument, i.e. continuous induction on time. First, here is a motivating computation: Take the inequality we are given, and plug in the answer into the right hand side. We get

$$E(t) \leq E_0 + E_0 \int_0^t r\left(t'\right) \exp(\underbrace{\int_0^{t'} r\left(t''\right) dt''}_{R(t')})dt$$

where R is just an antiderivative of r.

$$\begin{aligned} &= E_0 + E_0 \left(\exp\left(\int_0^t r\left(t'\right) dt'\right) - 1\right) \\ &= E_0 \exp\left(\int_0^t r\left(t'\right) dt'\right) \end{aligned}$$

This tells us that the solution is what we get if we try to find a fixed point when iterating the use of this bound.

Proof.
We are going to prove

$$E(t) \leq E_0(1+\delta) \exp(\int_0^t r(t')dt')$$

on $[0,T]$ for any $\delta > 0$. Fix δ, assume this inequality holds on $[0, T_0]$ by continuity (when $E_0 = 0$, we need more tricks). Now we plan to extend this interval.

Assume

$$E(t) \leq E_0(1+\delta) \exp\left(\int_0^t r\left(t'\right) dt'\right)$$

on $[0, T']$. If we plug this bound into the iteration, we get

$$\begin{aligned} E(t) &\leq E_0 + E_0(1+\delta) \int_0^t r\left(t'\right) \exp\left(\int_0^{t'} r\left(t''\right) dt''\right) dt' \\ &= E_0 + E_0(1+\delta)\left(\exp\left(\int_0^t r\left(t'\right) dt'\right) - 1\right) \\ &= E_0(1+\delta) \exp\left(\int_0^t r\left(t'\right) dt'\right) - \underbrace{\delta E_0}_{>0}. \end{aligned}$$

Hence we can extend this a little to $[0, T' + \varepsilon]$. It's obvious that this inequality holds on $[0, T]$.

Because for any $\delta > 0$ is true. This must be true for $\delta = 0$. $\square$

3.3 Variable Coefficient Wave Equations

Today, we are interested in a concrete goal. We will be studying **variable-coefficient wave equations**, PDEs of the form

$$P\phi = \partial_\mu \left(g^{\mu,\nu}\partial_\nu\phi\right) + b^\mu\partial_\mu\phi + c\phi,$$

where the key assumption is that g is a symmetric matrix with signature $(-,+,+\dots,+)$. The example we should keep in mind is $g = \mathrm{diag}(-1,1,1,\dots,1), b = 0, c = 0$; this makes $P = \Box$. We are solving the initial value problem

$$\begin{cases} P\phi = f & \text{in } (0,\infty)_t \times \mathbb{R}^d \\ (\phi, \partial_t\phi)|_{t=0} = (g,h) & \text{on } \{t=0\}\times\mathbb{R}^d \end{cases}$$

We further assume that $g^{\mu,\nu}, b^\mu, c$ are bounded with bounded derivatives of all orders. We also assume a restricted form of g (which we will later show is not much of a restriction): $g^{tt} = -1$ and $g^{t,x^j} = 0$. This means that if we write g as a matrix,

$$g = \begin{bmatrix} -1 & 0_{1\times d} \\ 0_{d\times 1} & \bar{g} \end{bmatrix},$$

where $\bar{g}$ is **uniformly elliptic** ($\bar{g} \succ \lambda I$). Our concrete goal is to prove the following theorem:

Theorem 3.3.1.
The initial value problem is well-posed in $H^k \times H^{k-1}$ for all $k \in \mathbb{Z}$. That is,

- (Existence) Given $(g,h) \in H^k \times H^{k-1}$ and $f \in L^1_t\left(H^{k-1}\right)$, there exists a solution ϕ to the initial value problem in the class $C_t\left(\mathcal{H}^k\right)$.
- (Uniqueness) The solution ϕ in $C_t\left(\mathcal{H}^k\right)$ to the initial value problem with (f,g,h) as in (i) is unique.
- (Continuous dependence)

$$\sup_t \|\phi\| + \sup_t \|\partial_t\phi\| \le C_k\left(\|(g,h)\|_{\mathcal{H}^k} + \|f\|_{L^1_t(H^{k-1})}\right).$$

Here, $\mathcal{H}^k = H^k \times H^{k-1}$, and by $\phi \in C_t\left(I;\mathcal{H}^k\right)$, we mean that $\phi \in C_t\left(I;H^k\right)$ and $\partial_t\phi \in C_t\left(I;H^{k-1}\right)$.

We will use the convention that $\mathbb{R}^{1+d} = \left\{\left(t = x^0, x^1, \dots, x^d\right)\right\}$. The Greek indices μ,ν will range from $0,1,\dots,d$, while the indices j,k,ℓ will range from $1,\dots,d$. We will also denote $g^{t,t} = g^{0,0}, g^{t,x^j} = g^{0,j}$.

Remark.
- The problem is time reversible. If we send $t \mapsto -t$, the equation is essentially unchanged.
- The reference for this topic is chapters 6-7 of Ringström's book.

3.3.1 Energy Inequality for P

The basic ingredient in this proof is an energy inequality for P. Suppose $P\phi = f$. The idea is to multiply the equation by $\partial_t\phi$ and "integrate by parts." Why should we multiply by $\partial_t\phi$ instead of ϕ ? This is a generalization of what we do in the classical wave equation, and we will be able to give a more insightful answer to this once we discuss calculus of variations for problems of this type. The key observation is this integration by parts idea, but in divergence form:

$$\begin{aligned}\partial_\mu\left(g^{\mu,\nu}\partial_\nu\phi\right)\partial_t\phi &= -\partial_t^2\phi\partial_t\phi+\partial_j\left(\bar{g}^{j,k}\partial_k\phi\right)\partial_t\phi\\ &= \partial_t\left(-\frac{1}{2}\left(\partial_t\phi\right)^2\right)+\partial_j\left(\bar{g}^{j,k}\partial_k\phi\partial_t\phi\right)-\bar{g}^{j,k}\partial_k\phi\partial_j\partial_t\phi\end{aligned}$$

Since g is symmetric, this last term can be written as $-\frac{1}{2}\bar{g}^{j,k}\partial_t\left(\partial_k\phi\partial_j\phi\right)$ by symmetrizing. Moving the ∂_t to the outside, we get

$$=\partial_t\left(-\frac{1}{2}\left(\partial_t\phi\right)^2\right)-\frac{1}{2}\partial_t\left(\bar{g}^{j,k}\partial_j\phi\partial_k\phi\right)+\partial_j\left(\bar{g}^{j,k}\partial_k\phi\partial_t\phi\right)+\frac{1}{2}\partial_t\bar{g}^{j,k}\partial_j\phi\partial_k\phi.$$

This form is nice because the terms that have the maximum number of derivatives are all in divergence form, while the terms that don't have the maximum number of derivatives are not in divergence form. Integrate this on $(t_0,t_1)\times\mathbb{R}^d=:R_{t_0}^{t_1}$ (assuming the boundary term vanishes):

$$\begin{aligned}&\iint_{R_{t_0}^{t_1}}\partial_\mu\left(g^{\mu,\nu}\partial\nu\phi\right)\partial_t\phi-\frac{1}{2}\iint_{R_{t_0}^{t_1}}\partial_t\bar{g}^{j,k}\partial_j\phi\partial_k\phi\\ &=-\int_{\Sigma_{t_1}}\frac{1}{2}\left(\left(\partial_t\phi\right)^2+\bar{g}^{j,k}\partial_j\phi\partial_k\phi\right)+\int_{\Sigma_{t_0}}\frac{1}{2}\left(\left(\partial_t\phi\right)^2+\bar{g}^{j,k}\partial_j\phi\partial_k\phi\right)z^{\text{‘}\prime}\\ &\quad+\underbrace{\lim_{R\to\infty}\int_{t_0}^{t_1}\int_{\partial B_R}\nu_j\left(\bar{g}^{j,k}\partial_k\phi\partial_t\phi\right)dAdt,}_{=0}\end{aligned}$$

where $\Sigma_t=\{t\}\times\mathbb{R}^d$. Denote $\vec{\phi}=(\phi,\partial_t\phi)$, so $(\phi,\partial_t\phi)\in\mathcal{H}^k$ if and only if $\vec{\phi}\in C_t\left(\mathcal{H}^k\right)$.

Lemma 3.3.2 (Energy estimate for variable-coefficient wave equations)**.**
For $\phi\in C_t\left(\mathcal{H}^1\right)$,

$$\sup_{t\in[0,T]}\|\vec{\phi}\|_{\mathcal{H}^k}\le C_T\left(\|\vec{\phi}(0)\|_{\mathcal{H}^1}+\int_0^T\|P\phi\|_{L^2}dt\right).$$

Proof.
We may assume without loss of generality that $\phi\in C^\infty\left(R_0^T\right)$ and $\phi(t,\cdot)$ has compact support for each $t\in[0,T]$. By the computation above, if

$$E[\phi](t)=\frac{1}{2}\int_{\Sigma_t}\left(\partial_t\phi\right)^2+\bar{g}^{j,k}\partial_j\phi\partial_k\phi dx$$

then

$$\mathbb{E}[\phi]\left(t_1\right)=\mathbb{E}[\phi](0)-\iint_{R_0^{t_1}}\partial_\mu\left(g^{\mu,\nu}\partial_\nu\phi\right)\partial_t\phi+\frac{1}{2}\iint_{R_0^{t_1}}\partial_t\bar{g}^{j,k}\partial_j\partial_k\phi$$

(Note that $\lim_{R\to\infty}\int_{\partial B_R}=0$ thanks to the support assumption). Now

$$\partial_\mu\left(g^{\mu,\nu}\partial_\nu\phi\right)=P\phi-b^\mu\partial_\mu\phi-c\phi$$

which tells us that

$$\mathbb{E}[\phi]\left(t_1\right)=E[\phi](0)-\iint_{R_0^t}P\phi\partial_t\phi dxdt+\iint_{R_0^t}\left(b^\mu\partial_\mu\phi\partial_t\phi+c\phi\partial_t\phi+\partial_t\bar{g}^{j,k}\partial_j\phi\partial_k\phi\right)dxdt$$

Call the error

$$\mathcal{E}_0^t=\iint_{R_0^t}\left|b^\mu\partial_\mu\phi\partial_t\phi+c\phi\partial_t\phi+\partial_t\bar{g}^{j,k}\partial_j\phi\partial_k\phi\right|dxdt$$

We get an inequality:

$$\sup_{t_1 \in [0,T]} E[\phi](t_1) \leq E[\phi](0) + \sup_{t \in [0,T} \left| \iint_{R_0^t} P\phi \partial_t \phi dx dt \right| + \mathcal{E}_0^T$$

Note that $E[\phi] \geq \frac{1}{2} \int (\partial_t \phi)^2 (t) dx + \frac{\lambda}{2} \int |D_t \phi|^2 (t) dx$. Using the fundamental theorem of calculus,

$$\int |\phi|^2(t) dx = \int_0^t \int \partial\phi\phi dx dt' + \int |\phi|^2(0) dx$$

Using Cauchy-Schwartz,

$$\leq 2 \int E(t')^{1/2} \left(\int |\phi|^2 (t') dx \right)^{1/2} dt' + \int |\phi|^2(0) dx$$

Skipping a few steps, we get[a]

$$\sup_{t \in [0,T]} \int |\phi|^2(t) dt \leq \int |\phi|^2(0) dx + C_T \sup_{t \in [0,T]} E(t)$$

Note $\|\vec{\phi}(t)\|_{\mathcal{H}^1}^2 \simeq \|\phi(t)\|_{L^2}^2 + \|\partial_j \phi(t)\|_{L^2}^2 + \|\partial_t \phi(t)\|_{L^2}^2$. The point here is that

$$\sup_{t \in [0,T]} \|\vec{\phi}\|_{\mathcal{H}^1}^2 \leq C_T \left(\|\vec{\phi}(0)\|_{\mathcal{H}^1}^2 + \sup_{t \in [0,T]} \left| \iint_{R_0^T} P\phi \partial_t \phi dx dt \right| + \sup_{t \in [0,T]} E(t) + \mathcal{E}_0^T \right).$$

If we use Cauchy-Schwarz, we get

$$\begin{aligned} \sup_{t \in [0,T]} \left| \iint_{R_0^t} P\phi \partial_t \phi dx dt \right| &\leq \int_0^T \|P\phi(t)\|_{L^2} \|\partial_t \phi\|_{L^2} dt \\ &\leq C \int_0^T \|P\phi(t)\|_{L^2} E[\phi]^{1/2} dt \\ &\leq \int_0^T \|P\phi(t)\|_{L^2} dt \sup_{[0,T]} E[\phi]^{1/2} \end{aligned}$$

We can use Cauchy-Schwarz to absorb the energy term to the left hand side, since $E[\phi] \leq C \int (\partial_t \phi)^2 + (D_x \phi)^2$. We get[b]

$$\sup_{t \in [0,t_1]} \|\vec{\phi}\|_{\mathcal{H}^1}^2 \leq C_T \left(\|\vec{\phi}(0)\|_{\mathcal{H}^1}^2 + \int_0^T \|P\phi\|_{L^2} dt + \int_0^{t_1} \|\phi(t)\|_{\mathcal{H}^1}^2 dt \right)$$

□

[a] How to get this?
[b] How to deal with $\mathcal{E}_0^T$?

3.3.2 Higher Order Regularity Estimates

We want to study something like $P : C_t(\mathcal{H}^k) \to L_t^1(H^{k-1})$. This means that we should look at the adjoint $P^* : C_t(H^{-(k-1)}) \to L_t^1(H^{-k})$. The dual problem here includes negative Sobolev spaces.

Lemma 3.3.3.

For any $k \in \mathbb{Z}$ and $\phi \in C_t\left(\mathcal{H}^{1+k}\right) \cap C^\infty_{t,x}$,

$$\sup_{t\in[0,T]} \|\vec{\phi}(t)\|_{\mathcal{H}^{1+k}} \le C_{T,k}\left(\|\vec{\phi}(0)\|_{\mathcal{H}^{1+k}} + \int_0^T \|P\phi\|_{H^k} dt\right).$$

The positive regularities will give us uniqueness for the initial value problem. The negative regularities will give us existence.

Proof.

For $k > 0$, we commute the equation with D^α for $|\alpha| \le k$. Then apply the previous lemma and Grönwall's inequality. (This technique is very similar to our previous proof of higher elliptic regularity bounds. However, we don't need to use a difference quotient.)

For $k < 0$, we work with $\Phi = (1-\Delta)^{-|k|}\phi$. (This means that we want to look at the solution to the elliptic problem $(1-\Delta)^{|k|}\Phi = \phi$ in $\mathbb{R}^d$. Another way to write this is $\widehat{\Phi} = \left(1-|\xi|^2\right)^{-|k|}\widehat{\phi}$.) We do this so that we don't have to deal with negative Sobolev spaces; we can study an operator that commutes well with P and use positive Sobolev spaces, instead. The key thing to notice is that $(1-\Delta)^{-\ell} : H^s \to H^{s+2\ell}$. We also use the following:

Lemma 3.3.4.

For any $s \in \mathbb{R}$, the H^s norm has the Fourier characterization

$$\begin{aligned}\|v\|_{H^s} &= \left\|\left(1+|\xi|^2\right)^{s/2}\widehat{v}\right\|^2_{L^2_\xi}\\ &= \left\|(1-\Delta)^{s/2}v\right\|^2_{L^2}.\end{aligned}$$

When $s \in 2\mathbb{Z}$, this agrees with our sense of derivatives. We want to compute

$$\begin{aligned}\|P\Phi\|^2_{H^{|k|}} &= \left\|\left(1+\|\ xi|^2\right)^{|k|/2}\widehat{P\Phi}\right\|^2_{L^2}\\ &= \left\langle\left(1+|\xi|^2\right)^{|k|/2}\widehat{P\Phi}, \left(1+|\xi|^2\right)^{|k|/2}\widehat{P\Phi}\right\rangle\\ &= \left\langle\left(1+|\xi|^2\right)^{|k|/2}\widehat{P\Phi}, \widehat{P\Phi}\right\rangle\\ &= \left\langle(1-\Delta)^{|k|}P\Phi, P\Phi\right\rangle.\end{aligned}$$

Now observe that [a]

$$\begin{aligned}(1-\Delta)^{|k|}P\Phi &= P\left((1-\Delta)^{|k|}\Phi\right) + \left[(1-\Delta)^{|k|}, P\right]\Phi\\ &= P\phi + \underbrace{\left[(1-\Delta)^{|k|}, P\right]}_{\text{order } 2|k|+2-1}\Phi\end{aligned}$$

This tells us that [b]

$$\begin{aligned}\|\vec{\Phi}(t)\|_{\mathcal{H}^{1+|k|}} &= \|\vec{\phi}(t)\|_{\mathcal{H}^{1+|k|-2|k|}}\\ &= \|\widehat{\phi}(t)\|_{\mathcal{H}^{1+k}}\end{aligned}$$

for $k < 0$. ☐

[a] Is this Lie bracket?
[b] Why we have this?

3.3.3 Local Well-Posedness

We have been looking at linear hyperbolic PDEs $P\phi = f$, where

$$P\phi = \partial_\mu \left(g^{\mu,\nu}\partial_\nu \phi\right) + b^\mu \partial_\mu \phi + c\phi.$$

We want to solve the initial value problem

$$\begin{cases} P\phi = f \\ (\phi, \partial_t \phi)|_{t=0} = (g, h). \end{cases}$$

To discuss existence and uniqueness, we made further assumptions on the coefficients:

- $g^{\mu,\nu}$ is a symmetric $(1+d)\times(1+d)$ matrix with signature $(-,+,+,\dots,+)$.
- $g^{0,j}(t,x) = 0$ and $g^{0,0}(t,x) = -1$.
- For $\xi \in \mathbb{R}^d$, $g^{j,k}\xi_j\xi_k \geq \lambda|\xi|^2$.
- $g^{\mu,\nu}, b, c$ are uniformly bounded, with uniformly bounded derivatives.

Example 3.3.5.
$b = c = 0$, and $g = \mathrm{diag}(-1, 1, 1, \dots, 1)$. Then $P = \Box$.

We take the convention that $x^0 = t$. We also use Greek indices $\mu, \nu \in \{0, 1, \dots, d\}$ and indices $j, k \in \{1, \dots, d\}$. Last time, we were proving the following theorem.

Theorem 3.3.6 (Local well-posedness of the initial value problem)**.**
Let $s \in \mathbb{Z}_+$. Given $(g,h) \in H^{s+1} \times H^s\left(\mathbb{R}^d\right)$ and $f \in L^1_t\left([0,t]; H^s\left(\mathbb{R}^d\right)\right)$, there exists a unique solution ϕ to the initial value problem with $\phi \in C_t\left([0,T], H^{s+1}\right)$ and $\partial\phi \in C_t\left((0,T); H^s\right)$. Moreover, the unique solution ϕ satisfies the estimate

$$\|\phi\|_{C_t([0,T];H^{s+1})} + \|\partial_t \phi\|_{C_t([0,T];H^s)} \lesssim_{g^{\mu,\nu},b,c,T,s} \|(g,h)\|_{H^{s+1}\times H^s} + \|f\|_{L^1_t([0,T];H^s)}$$

Remark.
Local well-posedness entails continuous dependence of ϕ on (f, g, h). Because of linearity, this a priori estimate implies continuous dependence (and in fact Lipschitz dependence).

We will give a proof if we have a priori estimate.

Proposition 3.3.7.
Let $s \in \mathbb{Z}$. Let $\phi \in C_t\left([0,T]; H^{s+1}\right)$ and $\partial_t\phi \in C_t\left([0,T]; H^s\right)$. Then

$$\|\phi\|_{C_t([0,T];H^{s+1})} + \|\partial_t\phi\|_{C_t((0,t):H^s)} \lesssim \|\left(\phi, \partial_t\phi\right)|_{t=0}\|_{H^{s+1}\times H^s} + \|P\phi\|_{L^1_t([0,T];H^s)}.$$

Proof.

- Step 1: $s \geq 0$
  We want to use the energy method. The natural strategy would be to commute $P\phi$ with D^α for $|\alpha| \leq s$ and apply the energy estimate (multiply by $\partial_t\phi$ and integrate by parts). Instead, we vary the multiplier:

  $$\langle P\phi, (1-\Delta)^s\partial_t\phi\rangle := \int P\phi(1-\Delta)^s\partial_t\phi dx$$

  - On the one hand, we know by duality that

  $$\int_0^T \langle P\phi, (1-\Delta)^s\partial_t\phi\rangle dt \lesssim \|P\phi\|_{L^1_t([0,T];H^s)}\|\partial_t\phi\|_{C_t([0,T];H^s)}.$$

This is basically integrating by parts s times and using Cauchy-Schwartz. We can also think of this a as the general bound

$$|\langle f,g\rangle| \lesssim \|f\|_{H^s}\|g\|_{H^{-s}}.$$

In general, if Q is an order r differential operator with that have uniformly bounded derivatives to all order, then (with some Fourier analysis), we can say that

$$\|Qg\|_{H^s} \lesssim \|g\|_{H^{r+s}}, \quad (s\in\mathbb{R}).$$

For negative s, we get the inequality by duality:

$$\begin{aligned}\|Qf\|_{H^s} &= \sup_{\|g\|_{H^s}=1} |\langle Qf,g\rangle| \\ &= \sup_{\|g\|_{H^s}=1} |\langle f,Q^*g\rangle| \\ &\lesssim \|f\|_{H^{s+r}}\|Q^*g\|_{H^{s-r}}.\end{aligned}$$

We also have the fact that

$$\|(1-\Delta^s)g\|_{L^2} \simeq \|g\|_{H^{2s}}, \langle (1-\Delta)^s g,g\rangle \simeq \|g\|^2_{H^s},$$

which we get by using the Fourier transform:

$$\langle (1-\Delta)^s g,g\rangle = \langle (1+|\xi|^2)^s\hat{g},\hat{g}\rangle = \|(1+|\xi|^2)^{s/2}\hat{g}\|^2_{L^2}.$$

- On the other hand, we have

$$P\phi = \underbrace{\partial_\mu(g^{\mu,\nu}\partial_\nu\phi)}_{-\partial_t^2\phi+\partial_j(g^{j,k}\partial_k\phi)} + b^\mu\partial_\mu\phi + c\phi.$$

Now, we can observe that

$$-\langle \partial_t^2\phi,(1-\Delta)^s\partial_t\phi\rangle = -\partial_t\langle\partial_t\phi,(1-\Delta)^s\partial_t\phi\rangle + \langle\partial_t\phi,(1-\Delta)^s\partial_t^2\phi\rangle.$$

Since $\langle\partial_t\phi,(1-\Delta)^s\partial_t^2\phi\rangle = \langle(1-\Delta)^s\partial_t\phi,\partial_t^2\phi\rangle$ we get

$$= -\frac{1}{2}\partial_t\langle\partial_t\phi,(1-\Delta)^s\partial_t\phi\rangle$$

For the other term, we have

$$\begin{aligned}\langle \partial_j(g^{j,k}\partial_k\phi),(1-\Delta)^s\partial_t\phi\rangle &= -\langle g^{j,k}\partial_k\phi,(1-\Delta)^s\partial_t\partial_j\phi\rangle \\ &= -\partial_t\langle g^{j,k}\partial_k\phi,(1-\Delta)^s\partial_j\phi\rangle \\ &\quad + \langle\partial_t g^{j,k}\partial_k\phi,(1-\Delta)^s\partial_j\phi\rangle \\ &\quad + \langle g^{j,k}\partial_k\partial_t\phi,(1-\Delta)^s\partial_j\phi\rangle.\end{aligned}$$

Write the last term as

$$-\langle\partial_t\phi,\partial_k(g^{j,k}(1-\Delta)^s\partial_j\phi)\rangle = -\langle\partial_t\phi,\partial_k([g^{j,k},(1-\Delta)^s\partial_j]\phi)\rangle - \underbrace{\langle\partial_t\phi,\partial_k\big((1-\Delta)^s(g^{j,k}\partial_j\phi)\big)\rangle}_{=\langle(1-\Delta)^s\partial_t\phi,\partial_k(g^{j,k}\partial_j\phi)\rangle}.$$

Overall, this equals

$$-\frac{1}{2}\partial_t\langle g^{j,k}\partial_k\phi,(1-\Delta)^s\partial_j\phi\rangle+\frac{1}{2}\langle\partial_t g^{j,k}\partial_k\phi,(1-\Delta)^s\partial_j\phi\rangle-\frac{1}{2}\langle\partial_t\phi,\partial_k([g^{j,k},(1-\Delta)^s]\partial_j\phi)\rangle.$$

This point of this messy calculation is as follows: for the terms with the highest number of derivatives, we want to put things into this total derivative form. The other terms will have at least 1 derivative that is not falling on ϕ. This is the purpose of using the commutator. What we get is that

$$\begin{aligned}&\langle P\phi,(1-\Delta)^s\partial_t\phi\rangle\\ &\quad=\underbrace{-\frac{1}{2}\Big(\langle\partial_t\phi,(1-\Delta)^s\partial_t\phi\rangle+\langle g^{j,k}\partial_k\phi,(1-\Delta)^s\partial_j\phi\rangle\Big)}_{\mathbb{E}_s[\phi](t)}\\ &\quad+\underbrace{O\left(\left\langle q_1\partial\phi,\partial^{2s}\partial\phi\right\rangle\right)+O\left(\left\langle q_2\partial\phi,\partial^{2s-1}\partial\phi\right\rangle\right)+\cdots+O\left(\left\langle q_{2s+1}\partial\phi,\partial\phi\right\rangle\right)}_{R_s}\end{aligned}$$

where $q_1=\partial g, q_2=\partial^2 g\partial bc$, etc.

So our energy argument says

$$\int_0^t\langle P\phi,(1-\Delta)^s\partial_t\phi\rangle dt'\geq\mathbb{E}_s[\phi](0)-\mathbb{E}_s[\phi](t)-C\int_0^t\|\phi\|_{H^{s+1}}^2+\|\partial_t\phi\|_{H^s}^2dt',$$

where we are just using the estimate for the remainder:

$$|R_s(t')|\lesssim(\|\phi\|_{H^{s+1}}+\|\partial_t\phi\|_{H^s})^2$$

Now we have

$$\mathbb{E}_s[\phi](t)\leq\mathbb{E}_s[\phi](0)+\|P\phi\|_{L_t^1([0,T];H^s)}\|\partial_t\phi\|_{C_t((0,T);H^s)}+\int_0^t\|\phi\|_{H^{s+1}}^2+\|\partial_t\phi\|_{H^s}^2\,dt'.$$

Note that $\mathbb{E}_s[\phi](t)\simeq\|\phi\|_{H^{s+1}}^2+\|\partial_t\phi\|_{H^s}^2$, so our proprties of H^s and the elliptic estimate for $\partial_j g^{j,k}\partial_k$ gives:

$$\mathbb{E}_s\Bigg[\phi(t)\leq\mathbb{E}_s[\phi](0)+\|P\phi\|_{L_t^1([0,T];H^s)}\|\partial_t\phi\|_{C_t((0,T);H^s)}+\int_0^t E_s\left[\phi\left(t'\right)\right]dt'$$

So Grönwall's inequality tells us that

$$\mathbb{E}_s[\Phi](t)\lesssim\mathbb{E}_s[\phi](0)+\|P\phi\|_{L_t^1([0,T];H^s)}\sup_{t\in[0,T]}E_s[\phi](t).$$

Now we can take the sup over $t\in[0,T]$ on the left hand side and use the AM-GM inequality with an epsilon to absorb the $\sup_{t\in[0,T]}E_s[\phi](t)$ on the right into the left hand side.

- Step 2: $s<0$
Let $\Phi=(1-\Delta)^{-|s|}\phi$. We have the equivalence:

$$\|\Phi\|_{H^{|s|+1}}\simeq\|\phi\|_{H^{-|s|+1}}=\|\phi\|_{H^{s+1}}.$$

Similarly,

$$\|\partial_t\Phi\|_{H^{|s|}}\simeq\|\partial_t\phi\|_{H^s}.$$

Now, we do the same argument with s replaced by $|s|$ and ϕ replaced by Φ. The only thing that is different is part 1 above. So we need to estimate

$$\begin{aligned} \left|\left\langle P\Phi, (1-\Delta)^{|s|}\partial_t\Phi\right\rangle\right| &= \left|\left\langle (1-\Delta)^{|s|}P\Phi, \partial_t\Phi\right\rangle\right| \\ &= \left| \left\langle P \underbrace{(1-\Delta)^{|s|}\Phi, \partial_t\Phi\right\rangle \right| + \left|\left\langle\left[(1-\Delta)^{|s|}, P\right]\Phi, \partial_t\Phi\right\rangle\right|}_{\phi} \end{aligned}$$

The right term has order $2|s|+2-1$. Using duality,

$$\lesssim \|P\phi\|_{H^s}\|\partial_t\Phi\|_{H^{|s|}} + \|\Phi\|_{H^{|s|+1}}\|\partial_t\Phi\|_{H^{|s|}}$$

This completes the proof.

□

Now we can quickly conclude the proof of existence and uniqueness theorem.

Proof (Thm 3.3.1).

Note that uniqueness and the a priori estimate follow from the proposition. It remains to prove existence. It remains to prove existence.

- Step 1: First, view this as trying to find the inverse of the operator $P : L_t^\infty\left([0,T],\mathcal{H}^{s+1}\right) \to L_t^1\left([0,T];H^s\right)$. We want to reduce to the case when the initial data $g,h=0$; we may achieve this using extension and modifying f.
- Step 2: By duality, $\phi \in L_t^\infty\left([0,T];H^{s+1}\right) = \left(L_t^1\left([0,T];H^{-s-1}\right)\right)^*$. We want

$$\begin{aligned} \int_0^T \langle f,\psi\rangle dt &= \int_0^T \langle P\phi,\psi\rangle dt \\ &= \int_0^T \langle \phi, P^*\psi\rangle\, dt. \end{aligned}$$

  Define $\ell : P^*\left(L_t^1\left([0,T];H^{-s}\right)\right) \to \mathbb{R}$ by $\ell\left(P^*\psi\right) = \int_0^T\langle f,\psi\rangle dt$. This is well-defined by our a-priori estimate:

$$\|\ell\| \le \|f\|_{L^1(H^s)}\|\psi\|_{L^\infty(H^{-s})} \le \|f\|_{L^1(H^s)}\|P^*\psi\|_{L^1(H^{-s-1})}.$$

  By Hahn-Banach, there exists an extension $\ell^* \in \left(L_t^1\left(H^{-s-1}\right)\right)^*$ which is an extension with the bound $\|\ell^*\| \lesssim \|f\|_{L^1(H^s)}$. Here, $\phi = \ell^* \in L_t^\infty\left(H^{s+1}\right)$.
- Step 3: Upgrade $\phi \in L_t^\infty\left(H^{s+1}\right)$ to $\phi \in C_t\left(H^{s+1}\right)$ with $\partial_t\phi \in C_t\left(H^s\right)$. The way to do this is to approximate by smooth objects and try to take the limit. The a priori estimate will stay intact through the limit.

□

CHAPTER 4

CALCULUS OF VARIATIONS AND NONLINEAR ELLIPTIC/HYPERBOLIC PDES

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