Note 3

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In this note, we will review some results concerning continuous-time Markov chain and introduce two reduced models. In fact, we find that time t is independent of the transition probabilities, which indicates that considering embedded Markov chains is enough to observe gamma oscillations. Therefore, we propose two reduced models, in terms of the full model introduced in Note 1.

1 Continuous-Time Markov Chain

According to the introduction in Note 1, we know that our full model is a continuous-time Markov Chain. Thus, we have a Q-matrix, where each element is a parameter for its corresponding Poisson process. Since the time intervals of a Poisson process follow an exponential distribution, we list some properties of exponential distribution here.

Proposition 1. *Given two independent random variables $X_1 \sim Exp(\lambda_1), X_2 \sim Exp(\lambda_2)$, we have $\min\{X_1, X_2\} \sim Exp(\lambda_1 + \lambda_2)$.*

Corollary 1. *Given n independent random variables $X_1 \sim Exp(\lambda_1), X_2 \sim Exp(\lambda_2), ..., X_n \sim Exp(\lambda_n)$, we have $\min\{X_1, X_2, ..., X_n\} \sim Exp(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$.*

Proposition 2. *Given two independent random variables $X_1 \sim Exp(\lambda_1), X_2 \sim Exp(\lambda_2)$, we have for all $t \geq 0$,*

$$\mathbb{P}(\min\{X_1, X_2\} = X_1 | \min\{X_1, X_2\} = t) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Corollary 2. *Given n independent random variables $X_1 \sim Exp(\lambda_1), X_2 \sim Exp(\lambda_2), ..., X_n \sim Exp(\lambda_n)$, we have for all $t \geq 0$,*

$$\mathbb{P}(\min\{X_1, X_2, ..., X_n\} = X_i | \min\{X_1, X_2, ..., X_n\} = t) = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \cdots + \lambda_n}.$$

In fact, given a continuous-time Markov chain, we can imagine there are k independent exponential distributed clocks for each states and once a clock rings, the chain will move to its corresponding state. Thus, according to **Corollary 2**, the waiting time will not influence which clock rings first, since the conditional probabilities are fixed.

In other words, it's fine for us to consider embedded Markov chain, since for the time interval between two states, we can generate it later according to the states.

2 Reduced Models

In this section, we will introduce two reduced models. This first model is 2-state reduced model, which is similar to the full model but only has 2 states. We hope the 2-state model can help us find a good partition of neurons into *gate* and *base* neurons. The second model is statistical reduced model. In this model, we focus on the population of the E-neurons and I-neurons instead of caring each single neuron. Hence, we can use four quantities $(N_{GE}, N_{BE}, H^E, H^I)$ to characterize each state.

2.1 2-State Reduced Network Model

As for the elements in the 2-state Reduced model, they are similar to those in the full model, except the **Membrane Potential**.

- We assume that N_E excitatory neurons are labeled $1, 2, ..., N_E$ and N_I inhibitory neurons are labeled $N_E + 1, N_E + 2, ..., N_E + N_I$.
- The **membrane potential** of a neuron i, denoted V_i, only has two states, *gate* and *base*.
- Each neuron receives synaptic input from an **external source** in the form of Poisson kicks; these kicks are independent from neuron to neuron.

Since membrane potential only takes value in two states, the interactions in the reduced model is completely different from those in the full model.

- **External drive to neurons**: The action from external drive to neurons are inputs delivered in the form of impulsive kicks, arriving at random (Poissonian) times and the Poisson processes are independent from neuron to neuron. We assume there are two parameters $\lambda^E, \lambda^I > 0$, representing the rate of the Poisson kicks. When a kick arrives, if $V_i = base$, with probability P^B_{Ex}, V_i will change to *gate*. Otherwise, if $V_i = gate$, with probability P^G_{Ex}, the neuron will spike and V_i change to *base*.
- **Spikes of neurons**: When the membrane potential of a neuron i change from *gate* to *base*, the neuron spikes immediately.
- **Connections of neurons**: We assume the connectivity in our model is random and time-dependent, so that every time a neuron spikes, a random set of postsynaptic neurons is chosen anew. More precisely, for $Q, Q' \in \{E, I\}$, we let $P_{Q,Q'} \in [0, 1]$ be the probability that a neuron of type Q is postsynaptic when a neuron of type Q' spikes, and the set of postsynaptic neurons is determined by a coin flip with these probabilities following each spike.
- **Effects of kicks**: Firstly, we assume an kick received by an neuron takes effect at a random time after its arrival. This delay is given by an exponential random variable with mean τ^E for the excitatory kick and τ^I for the inhibitory kick. We let the H^E_i, H^I_i to denote the number of E-kicks and I-Kicks received by neuron i. Thus the state of neuron i at any moment in time can be described by the triplet (V_i, H^E_i, H^I_i).

  As for the effect of E-kicks, it depends on the membrane potential and the class of neuron i. With different membrane potentials and classes, the effect will cause the neuron to change its membrane potential with different probabilities. Thus, we have parameters $P^{GE}_E, P^{GI}_E, P^{BI}_E, P^{BE}_E$.

  As for the effect of I-kicks, it's a little different from the effect of E-kicks. Firstly, the I-kicks won't have any effect on a *base* neuron. As for a *gate* neuron, I-kicks will change its membrane potential with probabilities P^{GE}_I, P^{GI}_I.

This completes the description of the 2-state reduced model. We can observe the parameters are

$$\{N_I, N_E, \lambda^E, \lambda^I, \tau_{\mathcal{R}}, P_{Q,Q'}, \tau^E, \tau^I, P^{GE}_E, P^{GI}_E, P^{BI}_E, P^{BE}_E, P^{GE}_I, P^{GI}_I, P^G_{Ex}, P^I_{Ex}\}.$$

2.2 Coarse Grained Model

In coarse grained model, we only care the populations of E-neurons and I-neurons. Therefore, we don't have to record each neurons and can characterize the populations with fewer variables, which greatly decrease the total states. About the elements in the model:

- We assume there are N_E excitatory neurons and N_I inhibitory neurons.
- The **membrane potential** of a neuron only has two states, *gate* and *base*. We use $N_{GE}, N_{GI}, N_{BE}, N_{BI}$ to denote the numbers of *gate* E-neurons, *gate* I-neurons, *base* E-neurons and *base*-I neurons. In fact, we can observe that $N_{GE}+N_{BE} = N_E, N_{BI}+N_{GI} = N_I$.
- Each neuron receives synaptic input from an **external source** in the form of Poisson kicks; these kicks are independent from neuron to neuron.

Then, we are going to introduce the interactions between different elements in the model:

- **External drive to neurons**: The action from external drive to neurons are inputs delivered in the form of impulsive kicks, arriving at random (Poissonian) times and the Poisson processes are independent from neuron to neuron. We assume there are one parameter λ^E, λ^I, representing the rate of the Poisson kicks. When a kick arrives, if the state is $(N_{GE}, N_{GI}, H_E, H_I)$, we have four different cases:
  - With probability $\frac{N_{GE}}{N_E+N_I}$, this kick works on a *gate* E-neuron. Then, with probability $\frac{N_{GE}}{N_E+N_I} \cdot P^G_{Ex}$, $N_{GE} = N_{GE}-1, N_{BE} = N_{BE}+1, H_E = H_E+S_E$, where S_E is a constant we give. With probability $\frac{N_{GE}}{N_E+N_I} \cdot (1 - P^G_{Ex})$, the state will keep the same.
  - Similarly, with probability $\frac{N_{BE}}{N_E+N_I} \cdot P^B_{Ex}$,this kick works on a *base* E-neuron and this E-neuron's membrane potential switches to *gate*. $N_{GE} = N_{GE} + 1, N_{BE} = N_{BE} - 1$.
  - With probability $\frac{N_{BI}}{N_E+N_I} \cdot P^B_{Ex}$,this kick works on a *base* I-neuron and this I-neuron's membrane potential switches to *gate*. $N_{GI} = N_{GI} + 1, N_{BI} = N_{BI} - 1$.
  - With probability $\frac{N_{GI}}{N_E+N_I} \cdot P^B_{Ex}$,this kick works on a *gate* I-neuron and this I-neuron's membrane potential switches to *base*. $N_{GI} = N_{GI}-1, N_{BI} = N_{BI}+1, H_I = H_I+S_I$, where S_I is a constant we give.
- **Effect of kicks**: Firstly, we use H_E, H_I to denote the total pending E-kicks and I-kicks, and these pending spikes delay after an exponential distributed time with parameters $\tau^{EE}, \tau^{IE}, \tau^I$.Then we are going to introduce the effects of two different kicks:
  - **Effect of E-kicks**: As for the E-kicks, we assume with probability a_{EE}, it takes effect on an E-neuron and with probability a_{IE} on an I-neuron. Therefore,
    * With probability $a_{EE} \cdot \frac{N_{GE}}{N_E} \cdot P^{GE}_E$, the E-kick takes effect on a *gate* E-neuron and changes it's membrane potential. $N_{GE} = N_{GE} - 1, N_{BE} = N_{BE} + 1, H_E = H_E + S_E - 1$, where S_E is decided by the connectivity.
    * With probability $a_{IE} \cdot \frac{N_{GI}}{N_I} \cdot P^{GI}_E$, the E-kick takes effect on a *gate* I-neuron and changes it's membrane potential. $N_{GI} = N_{GI} - 1, N_{BI} = N_{BI} + 1, H_I = H_I + S_I, H_E = H_E - 1$, where S_I is decided by the connectivity.
    * With probability $a_{EE} \cdot \frac{N_{BE}}{N_E} \cdot P^{BE}_E$, the E-kick takes effect on a *base* E-neuron and changes it's membrane potential. $N_{BE} = N_{BE}-1, N_{GE} = N_{GE}+1, H_E = H_E-1$.
    * With probability $a_{IE} \cdot \frac{N_{BI}}{N_I} \cdot P^{BI}_E$, the E-kick takes effect on a *base* I-neuron and changes it's membrane potential. $N_{BI} = N_{BI}-1, N_{GI} = N_{GI}+1, H_E = H_E-1$.
    * As for the other probability, $H_E = H_E - 1$.
  - **Effect of I-kicks**: The effect of I-kicks is relatively simpler, since it has no effect on *base* neurons. Assume an I-kicks takes effect on an E-neuron with probability a_{EI} and a_{II} on an I-neuron.
    * With probability $a_{EI} \cdot \frac{N_{GE}}{N_E} \cdot P^{GE}_I$, the I-kick takes effect on a *gate* E-neuron and changes it's membrane potential. $N_{GE} = N_{GE}-1, N_{BE} = N_{BE}+1, H_I = H_I-1$.

* With probability $a_{II} \cdot \frac{N_{GI}}{N_I} \cdot P_I^{GI}$, the I-kick takes effect on a *gate* I-neuron and changes it's membrane potential. $N_{GI} = N_{GI} - 1, N_{BI} = N_{BI} + 1, H_I = H_I - 1$.
* As for the other probability, $H_I = H_I - 1$.

This finishes our statistical reduced model. We can observe that the parameters are:

$$\{N_E, N_I, \lambda^E, \lambda^I, S_E, S_I, P_{Ex}^G, P_{Ex}^B, a_{Q,Q'}, P_E^{GQ}, P_E^{BQ}, P_I^{GQ}, \tau^{EE}, \tau^{IE}, \tau^I\},$$

where $Q \in \{E, I\}$.

In conclusion, if we are going to consider the embedded Markov chain, for a given state, there are 12 possible transitions listed in Table 1 while the corresponding transition rate (of $\mathcal{Q}$-matrix) listed in Table 2. Note that quantities not mentioned will not change.

external input operates	pending E spikes operates	pending I spikes operates
$N_{GE} + 1$	$N_{GE} + 1 \quad H_E - 1$	$N_{GE} - 1 \quad H_I - 1$
$N_{GI} + 1$	$N_{GI} + 1 \quad H_E - 1$	$N_{GI} - 1 \quad H_I - 1$
$N_{GE} - 1 \quad H_E + S_E$	$N_{GE} - 1 \quad H_E - 1 + S_E$	$H_I - 1$
$N_{GI} - 1 \quad H_I + S_I$	$N_{GI} - 1 \quad H_E - 1 \quad H_I + S_I$	
	$H_E - 1$	

Table 1: possible transitions

external input operates	pending E spikes operates	pending I spikes operates
$P_{Ex}^{BE} N_{BE}/\lambda_E$	$P_E^{BE} a_{EE} \frac{N_{BE}}{N_E} H_E/\tau_{EE}$	$p_I^{GE} a_{EI} \frac{N_{GE}}{N_E} H_I/\tau_I$
$P_{Ex}^{BI} N_{BI}/\lambda_I$	$p_E^{BI} a_{IE} \frac{N_{BI}}{N_I} H_E/\tau_{IE}$	$p_I^{GI} a_{II} \frac{N_{GI}}{N_I} H_I/\tau_I$
$P_{Ex}^{GE} N_{GE}/\lambda_E$	$p_E^{GE} a_{EE} \frac{N_{GE}}{N_E} H_E/\tau_{EE}$	$(1 - p_I^{GE} a_{EI} \frac{N_{GE}}{N_E} - p_I^{GI} a_{II} \frac{N_{GI}}{N_I}) \cdot H_I/\tau_I$
$P_{Ex}^{GI} N_{GI}/\lambda_I$	$p_E^{GI} a_{IE} \frac{N_{GI}}{N_I} H_E/\tau_{IE}$	
	$(1 - P_E^{BE}) a_{EE} \frac{N_{BE}}{N_E} H_E/\tau_{EE}$ $+(1 - p_E^{BI}) a_{IE} \frac{N_{BI}}{N_I} H_E/\tau_{IE}$ $+(1 - p_E^{GE}) a_{EE} \frac{N_{GE}}{N_E} H_E/\tau_{EE}$ $+(1 - p_E^{GI}) a_{IE} \frac{N_{GI}}{N_I} H_E/\tau_{IE}$	

Table 2: transition rate