Contents

Chapter 1

Topological Vector Spaces

1. Exercise 1. Suppose X is a vector space. All sets mentioned below are understood to be subsets of X. Prove the following statements from the axioms as given in Section 1.4. (Some of these are tacitly used in the text.)

   (a) If $x \in X$ and $y \in X$ there is a unique $z \in X$ such that $x + z = y$.

   - Let $z = y + (-x)$. If $x, y \in X$, then $z \in X$. Therefore:

     $$x + z = x + (y + (-x)) = y$$

     Therefore, $\exists z$, st $x+z = y$. Now suppose there is $z_1, z_2 \in X$, such that $x+z_1 = y$ and $x+z_2 = y$. Therefore,

     $$x + z_1 = x + z_2 \implies x + z_1 + (-x) = x + z_2 + (-x) \ implies z_1 = z_2$$

   - **Incorret:** Suppose there is $z_1, z_2 \in X$, such that $x+z_1 = x+z_2 = y$. Let $\{x_i\}_i^n$ be a basis for X, such that $x = a_1x_1+a_2x_2+\cdots+a_nx_n$, $z_1 = b_1x_1+b_2x_2+\cdots+b_nx_n$, $z_2 = c_1x_1+c_2x_2+\cdots+c_nx_n$, and $y = d_1x_1 + d_2x_2 + \cdots + d_nx_n$, thus

     $$x + z_1 = (a_1 + b_1)x_1 + (a_2 + b_2)x_2 + \cdots + (a_n + b_n)x_n,$$
     $$x + z_2 = (a_1 + c_1)x_1 + (a_2 + c_2)x_2 + \cdots + (a_n + c_n)x_n.$$

     Since $x + z_1 = x + z_2$, then $(a_i + b_i) = (a_i + c_i)$ for all i. Therefore, $b_i = c_i$ $\forall i$ and thus $z_1 = z_2$

   (b) $0x = 0 = \alpha 0$ if $x \in X$ and α is a scalar

   - Let $0x = (0 + 0)x = 0x + 0x$. Get the additive inverse on both sides

     $$0 = 0x - 0x = (0x + 0x) - 0x = 0x + (0x - 0x) = 0x.$$

     Now, let $\alpha 0 = \alpha(x - x)$, for some $x \in X$, then

     $$\alpha 0 = \alpha x - \alpha x = 0.$$

   (c) $2A \subset A + A$; it may happen that $2A \neq A + A$

   - Since $2A = \{2a : a \in A\}$, and $A+A = \{a_1+a_2 : a_1 \in A, a_2 \in A\}$, then let $a \in A$, $a+a \in 2A$ and $a + a \in A + A$, therefore $2A \subset A + A$. Now, let $a_1, a_2 \in A$, s.t. $a_1 \neq a_2$, and $a_1 + a_2 \in A + A$, but $a_1 + a_2 \notin 2A$. Therefore, $2A \neq A + A$.

   (d) A is convex iff $(s + t)A = sA + tA$ for all positive scalar s and t

   - Let A convex, then $tA + (1 - t)A \subset A$ for all $0 \leq t \leq 1$. Let $r, s \in \mathbb{R}$, and $a_2, a_3 \in A$, st. $sa_2 + ra_3 = N$. Since A is convex, then, there is $a_2 = ta_1 + (1 - t)a_5$ and $a_3 = ta_1 + (1 - t)a_6$, then:

     $$(r + s)a_1 = t(r + s)a_2 + (1 - t)(r + s)a_3 = tra_2 + (1 - t)ra_3 + tsa_2 + (1 - t)sa_3 = ra_4 + sa_5.$$

     Now, let $(s + t)a_1 = sa_2 + ta_3$, $a_2 = \frac{s+t}{s}a_1 - \frac{t}{s}a_3$. Let $s = -1$, then $a_2 = (1 - t)a_1 + ta_3$ $\forall$ t. Therefore, A is convex.

   (e) Every union (and intersection) of balanced sets is balanced

- Let A, B be balanced sets, then $\forall\ |\alpha| \leq 1,\ \alpha A \subset A$ and $\alpha B \subset B$. Take $\alpha x \in \alpha(A \cup B) = \{\alpha x : x \in A \cup B\}$, st $x \in A \cup B$. If $x \in A$, then $\alpha x \in \alpha A \implies \alpha x \in A$, and if $x \in B$, $\alpha x \in \alpha B \implies \alpha x \in B$, because A and B are balanced. Therefore, $\alpha x \in A \cup B$.
  Now, let $\alpha x \in \alpha(A \cap B) = \{\alpha x : x \in A \cap B\}$. Since $|\alpha| \leq 1$, and $x \in A \cap B$, then $\alpha x \in \alpha A$, but also $\alpha x \in B$. Therefore, $\alpha(A \cap B) \in A \cap B$.

(f) Every intersection of convex sets is convex

- Let A, and B st $\forall |t| \leq 1\ \exists a_1, a_2 \in A$, st $ta_1 + (1-t)a_2 \in A$ and $\exists b_1, b_2 \in B$, st $tb_1 + (1-t)b_2 \in B$. Suppose $x \in A \cap B$, then $x \in A$, therefore $\exists t \in \mathbb{R}$ and $a_1, a_2 \in A$ st $x = ta_1 + (1-t)a_2$, and since $x \in B$, $\exists t' \in \mathbb{R}$ and $b_1, b_2 \in B$ st $x = t'b_1 + (1-t')b_2$. Suppose $t'' \in \mathbb{R}$ and $c_1,\ c_2 \in A \cap B$, then:

$$\begin{aligned}
&t''c_1 + (1-t'')c_2 \\
&= t''[ta_1 + (1-t)a_2] + (1-t'')[t'a_3 + (1-t')a_4] \\
&= t''[ta_1 + (1-t)a_2] + (1-t'')[ta_3 + (1-t)a_4] \\
&= t(t''a_1 + (1-t'')a_3) + (1-t)[t''a_2 + (1-t'')a_4].
\end{aligned}$$

  Then $t''c_1 + (1-t'')c_2 \in A$ and, by a similar argument, $t''c_1 + (1-t'')c_2 \in B$.

(g) If Γ is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of Γ is convex;

(h) If A and B are convex, so is $A + B$

(i) If A and B are balanced, so is $A + B$

(j) Show that parts (f), (g), (h) hold with subspaces in place of convex sets

2. The convex hull of a set A in a vector space X is the set of all convex combinations of members of A, that is, the set of all sums

$$t_1x_1 + \cdots + t_nx_n$$

in which $x_i \in A$, $t_i \geq 0$, $\sum_i t_i = 1$; n is arbitrary. Prove that the convex hull of A is convex and that it is the intersection of all convex sets that contain A.

- Let $a \in \tilde{A} = \{t_1x_1 + \cdots + t_nx_n : t_i \geq 0,\ \sum_i t_i = 1,\ x_i \in X\}$

$$\begin{aligned} t_1(tx_{11} + (1-t)x_{12}) + \cdots + t_n(tx_{n1} + (1-t)x_{n2}) = \\ t(t_1x_{11} + t_2x_{21} + \cdots + t_nx_{n1}) + (1-t)(t_1x_{12} + t_2x_{22} + \cdots + t_nx_{n2}). \end{aligned}$$

  $t_1x_{11} + t_2x_{21} + \cdots + t_nx_{n1}$ and $t_1x_{12} + t_2x_{22} + \cdots + t_nx_{n2}$ is in the convex hull of A, and, therefore, it is convex.
  Let B_1 convex and $b_{11}, b_{12} \in B_1$ and $A \subset B_1$, therefore

$$x = tb_{11} + (1-t)b_{12}$$

  Now let B_2 also convex, such that $b_{12} \in B_1 \cap B_2$, then there is t', st

$$\begin{aligned} x = tb_{11} + (1-t)(t'b_{21} + (1-t')b_{22}) &= tb_{11} + (1-t)t'b_{21} + (1-t)(1-t')b_{22} \\ &= tb_{11} + (1-t)t'b_{21} + (1-t)(1-t')t''b_{31} + (1-t)(1-t')(1-t'')b_32 \end{aligned}$$

  Subsequentially:

$$\begin{aligned} x = tb_{11} + (1-t)(t'b_{21} + (1-t')b_{22}) &= tb_{11} + (1-t)t'b_{21} + (1-t)(1-t')b_{22} \\ &= tb_{11} + (1-t)t'b_{21} + (1-t)(1-t')t''b_{31} + (1-t)(1-t')(1-t'')b_32 \end{aligned}$$

  Letting, the existence of B_n, then

$$t^{(1)} + (1-t^{(1)})t^{(2)} + (1-t^{(1)})(1-t^{(2)})t^{(3)} + \cdots + \sum_i^n \prod_j^{i-1} (1-t^{(j)})t^{(i)} = 1$$

  Therefore, $A \subset \bigcap_i B_i$ and that it is the convex hull of A

3. Let X be a topological vector space. All sets mentioned below are understood to be the subsets of X. Prove the following statements:

   (a) The convex hull of every open set is open

   - If A is open, then there is $x \in A$, such that $B_\varepsilon(x) \subset A$, $\forall \varepsilon > 0$. Let B be the convex hull of A, then there is $x_1, \cdots, x_n \in A$, and $t_i \in \mathbb{R}$, such that $t_1x_1 + \cdots + t_nx_n \in B$ and $\sum_i t_i = 1$. Take then the set

   $$t_1B_{\varepsilon_1}(x_1) + t_2B_{\varepsilon_2}(x_2) + \cdots + t_nB_{\varepsilon_n}(x_n) \qquad \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n > 0.$$

   Since A is open, then $B_{\varepsilon_i}(x_i) \subset A$. Since $B_\varepsilon(x_i) = \{x : |x - x_i| \le \varepsilon\}$, then $t_iB_\varepsilon(x_i) = \{t_ix : |x - x_i| \le \varepsilon\}$ and $B_\varepsilon(t_ix_i) = \{x : |x - t_ix_i| \le \varepsilon\}$. If we take $x \in B_\varepsilon(t_ix_i)$, then $x' \in B_\varepsilon(x_i)$, such that $x = t_ix'$ and, therefore, $t_ix' \in t_iB_\varepsilon(x_i)$. Therefore, $t_iB_\varepsilon(x_i) = B_\varepsilon(t_ix_i)$. Now, take $B_\varepsilon(x_i)$ and $B_\varepsilon(x_j)$, it is obvious that $B_\varepsilon(x_i) + B_\varepsilon(x_j) = B_\varepsilon(x_i + x_j)$. Finally, take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$, then

   $$x \in t_1B_\varepsilon(x_1) + t_2B_\varepsilon(x_2) + \cdots + t_nB_\varepsilon(x_n) = B_\varepsilon(t_1x_1 + t_2x_2 + \cdots + t_nx_n).$$

   Due to the openness of A, $x \in A$, and therefore the convex hull is open.

   (b) If X is locally convex then the convex hull of every bounded set is bounded (This is false without local convexity; see Section 1.47)

   - Since X is locally convex, then there's U a neighborhood of 0 such that $\exists\ V \subset U$ convex. Since V is convex, then conv $V = V$. Also we now that conv $tV = t$conv V. Take $A \subset tV$, then

   $$A \subset tV \implies \text{conv } A \subset \text{conv } (tV) = t\text{conv } (V) \subset tU$$

   (c) If A and B and bounded, so is $A + B$

   (d) If A and B and compact, so is $A + B$

   (e) If A is compact and B is closed, then $A + B$ is closed;

   (f) The sum of two closed sets may fail to be closed [The inclusion in (b) of Theorem 1.13 may therefore be strict.]

4.

5. Consider the definition of bounded set given in Section 1.6. Would the content of this definition be altered if it were required merely that to every neighborhood V of 0 corresponds *some* $t > 0$ such that $E \subset tV$?

    - Yes. Let E be a bounded set and V a neighborhood of 0. Then, let $s, t > 0$ and $t > s$, such that $E \subset sV$, but not $E \subset tV$.

6. Let X be a vector space of all complex functions on the unit interval $[0, 1]$, topologized by the family of seminorms

$$p_x(f) = |f(x)|, \quad x \in [0, 1].$$

This topology is called the **topology of pointwise convergence**. Justify this terminology

- A sequence converges pointwise if for every $x \in [0, 1]$, the sequence of complex numbers $(f_n(x))$ converges in $\mathbb{C}$. That is to say for x and $\varepsilon > 0$, there exists $N = N(x, \varepsilon)$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$. Here the seminorm $p_x(f_n(x) - f(x))$ converges pointwise to 0 as n increases.

7. Suppose

   (a) X and Y are topological vector spaces,

   (b) $\Lambda : X \to Y$ is linear,

   (c) N is a closed subspace of X,

   (d) $\pi : X \to X/N$ is the quotient map, and

   (e) $\Lambda x = 0 \; \forall \; x \in N$

   Prove that there is a unique $f : X/N \to Y$ which satisfies $\Lambda = f \circ \pi$, that is, $\Lambda x = f(\pi(x))$ for all $x \in X$. Prove that this f is linear and that Λ is continuous iff f is continuous. Also, Λ is open iff f is open.

   - Let $f(z) := \Lambda x$ and $z = \pi(x) = x + N$. Let $x_1, x_2 \in X$ and $x_1 - x_2 \in N$, then

   $$\Lambda x_1 - x_2 = 0 \implies \Lambda x_1 - \Lambda x_2 \implies \Lambda x_1 = \Lambda x_2.$$

     Additionally, $g(z) = g(\pi(x)) = \Lambda(x) = f(\pi(x)) = f(z)$, so $g = f$. So its unique.

   Hints:

   - $\pi : X \to X/N \implies \pi(x) = x + N = \{x + n : n \in N\}$
   - $\Lambda x = f(\pi(x))$

8. If N is a subspace of a vector space X, the *codimension* of N in X is, by definition, the dimension of the quotient space X/N.

   Suppose $0 < p < 1$ and prove that every subspace of finite codimension is dense in L^p.

   - Let N be a subspace of X with finite codimension X/N with basis $\{x_1, x_2, ..., x_N\}$.

   Notes:

   - $L^p = \{f : \int_0^1 |f(t)|^p dt < \infty\}$
   - Dense Set:
     - The smallest closed subset of X containing A is X itself.
     - For every $x \in X$, every neighborhood U of x intersects A; that is, $U \cap A \neq \emptyset$.

9. Let C be the vector space of all complex continuous functions on $[0,1]$. Define

$$d(f,g) = \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}dx.$$

Let (C,σ) be C with the topology induced by this metric. Let (C,τ) be the topological vector space defined by the seminorms

$$p_x(f) = |f(x)| \quad (0 \le x \le 1)$$

in accordance with Theorem 1.37.

(a) Prove that every τ-bounded set in C is also σ-bounded and that the identity map $id : (C,\tau) \to (C,\sigma)$ therefore carries bounded sets into bounded sets.

(b) Prove that $id : (C,\tau) \to (C,\sigma)$ is nevertheless not continuous, although it is sequentially continuous (by Lebesgue's dominated convergence theorem). Hence (C,τ) is not metrizable. (See Appendix A6, or Theorem 1.132.) Show also directly that (C,τ) has no countable local base.

(c)

10. Prove that the space $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in Section 1.44. Do the same for $C^\infty(\Omega)$ (Section 1.46.).

    - Let $\{K_n\}$ and $\{K'_m\}$ be compact sets as given by Section 1.44, with which seminorms are $p_n(f) = \sup\{|f(x)| : x \in K_n\}$, $p'_m(f) = \sup\{|f(x)| : x \in K'_m\}$. Suppose $U \subset C(\Omega)$ open, and $f_j \in U$, such that for each $\varepsilon > 0$ $\exists j > J \in \mathbb{N}$, such that $p_n(f_j - f) < \varepsilon$. Since $\bigcup_n K_n = \bigcup_m K'_m = \Omega$ and $K_n \subset K_{n+1}$ and $K'_m n \subset K'_{m+1}$, there exists $N \in \mathbb{N}$, such that $K'_m \subset K_n$. Therefore

    $$p'_m(f) = \sup_{x \in K'_m} |f(x)| \leq \sup_{x \in K_n} |f(x)| = p_n(f).$$

    Therefore, $p'_m(f_j - f) \leq p'_n(f_j - f) \leq \varepsilon$.

11. In the setting of Section 1.46, prove that $f \to D^\alpha f$ is a continuous mapping of $C^\infty(\Omega)$ into $C^\infty(\Omega)$ and also $\mathcal{D}_K$ into $\mathcal{D}_K$, for every multi-index α.

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Chapter 2