Instructor: Shusen Pu

Student Name (print): Farooq Mahmud Group #\_\_\_\_\_10

Question:	1	2	3	4	Total
Points:	10	10	20	0	40
Bonus Points:	0	0	0	10	10
Score:					

- 1. (10 points) Find the derivatives of f(x) using basic rules of differentiation.
  - (a)  $f(x) = \sin x \cos x$

$$= [sin(x)(-sin(x))] + [cos(x)cos(x)]$$
 (product rule)  $= -sin^2(x) + cos^2(x)$ 

(b) 
$$f(x) = \tan x$$
 (hint:  $\tan x = \frac{\sin x}{\cos x}$ )

$$=rac{[cos(x)cos(x)]-[sin(x)(-sin(x))]}{cos^2(x)}$$
 (quotient rule)  $=rac{cos^2(x)+sin^2(x)}{cos^2(x)}$   $=rac{1}{cos^2(x)}$  (Pythagorean identity)  $=sec^2(x)$ 

(c) 
$$f(x) = \sin^3(x^2)$$
 (Note that  $\sin^3(x) = [\sin x]^3$ )

$$=(sin(x^2))^3$$
 (power rule)  $=(3sin^2(x^2))(sin(x^2))$  (chain rule)  $=(3sin^2(x^2))(2xcos(x^2))$   $=6xsin^2(x^2)cos(x^2)$ 

(d) 
$$f(x) = x\sin(x^3)$$

$$=x(sin(x^3))+(1(sin(x^3)))$$
 (product rule)  $=x(cos(x^3)3x^2))+sin(x^3)$  (chain rule)  $=3x^3cos(x^3)+sin(x^3)$ 

- 2. (10 points) Let  $f(x, y, z) = w_1 x^3 + w_2 x e^y + w_3 x y z^2$ , where  $w_1, w_2$  and  $w_3$  are weight constants. Find the following partial derivatives.
  - (a)  $\frac{\partial f}{\partial x}$

$$3x^2w_1 + w_2e^y + w_3yz^2$$

(b)  $\frac{\partial f}{\partial y}$ 

$$w^2xe^y+w_3xz^2$$

(c)  $\frac{\partial f}{\partial z}$ 

$$2zw_3xy$$

(d) Find the gradient of f(x, y, z), i.e.,  $\nabla f(x, y, z)$ 

$$3x^2w_1 + w_2e^y + w_3yz^2 + w^2xe^y + w_3xz^2 + 2zw_3xy$$

(e) What is the gradient of f(x, y, z) at (1, 2, 1)

$$3(1^{2})w_{1} + w_{2}e^{2} + w_{3}(2)(1^{2}) + w^{2}(1)e^{2} + w_{3}(1)(1^{2}) + 2(1)w_{3}(1)(2)$$

$$= 3w_{1} + w_{2}e^{2} + 2w_{3} + w^{2}e^{2} + w_{3} + 4w_{3}$$

3. (20 points) Suppose a  $3 \times 3$  matrix W has an eigenvalue decomposition as follows:

$$W = X \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} X^{-1} \tag{1}$$

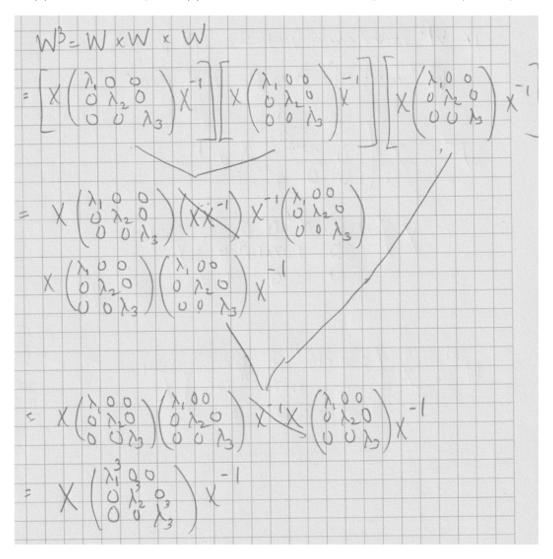
where X is the square  $n \times n$  matrix whose ith column is the eigenvector  $q_i$  of W,  $\lambda_1, \lambda_2$  and  $\lambda_3$  are eigenvalues of W arranged in descending order.

F.Y.I: matrix multiplication is defined in the following way:

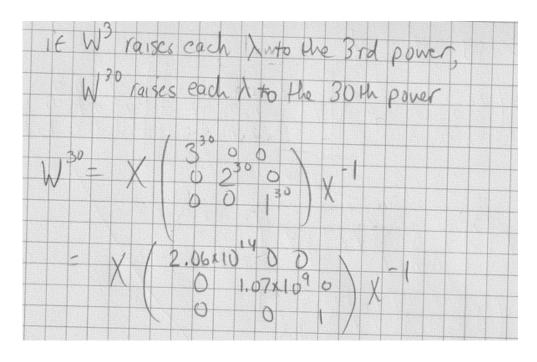
If  $A = (a_{ij})$  is an  $m \times n$  matrix and  $B = (b_{ij})$  is an  $n \times r$  matrix, then the product  $AB = C = (c_{ij})$  is the  $m \times r$  matrix whose entries are defined by

$$c_{ij} = \vec{\mathbf{a}}_i \mathbf{b}_j = \sum_{k=1}^n a_{ik} b_{kj}$$

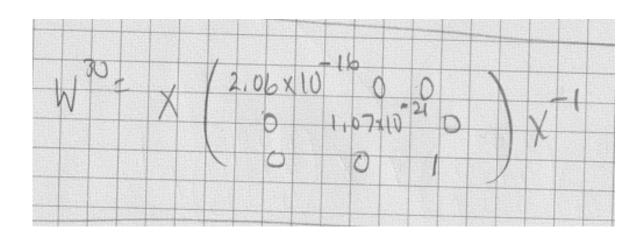
(a) Find  $W^3$  use equation (1) and the definition of matrix product. Show your steps.



(b) Let  $\lambda_1=3, \lambda_2=2$  and  $\lambda_3=1,$  find  $W^{30}$ 



(c) Let  $\lambda_1=0.3, \lambda_2=0.2$  and  $\lambda_3=0.1,$  find  $W^{30}$ 



(d) Note: this is an example of how the long-time dependence in RNN can lead to exploding or vanishing gradient.

- 4. (10 points (bonus)) Use the definition  $f'(x) = \lim_{h \to 0} \frac{f(x+h) f(x)}{h}$  to find f'(x) for the following functions.
  - a.  $f(x) = x^2$
  - b.  $f(x) = x^3$
  - c.  $f(x) = x^4$
  - d. Based on your answers to parts a-c, propose a formula for f'(x) if  $f(x) = x^n$ , where n is a positive integer.

$$(b) = \frac{f(x) = x^3}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \frac{h(3x^2 + 3xh + h^2)}{h} = 3x^2 + 3xh + h^2$$

$$\lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$$

$$f(x) = x^{4}$$

$$(c) = \frac{(x+h)^{4} - x^{4}}{h} = \frac{x^{4} + 4x^{3}h + 6x^{2}h^{2} + 4xh^{3} + h^{4} - x^{4}}{h} - \frac{4x^{3}h + 6x^{2}h^{2} + 4xh^{3} + h^{4}}{h}$$

$$= \frac{h(4x^{3} + 6x^{2}h + 4xh^{2} + h^{3})}{h}$$

$$= 4x^{3} + 6x^{2}h + 4xh^{2} + h^{3}$$

$$\lim_{h \to 0} (4x^{3} + 6x^{2}h + 4xh^{2} + h^{3}) = 4x^{3}$$

$$^{\scriptscriptstyle (\mathrm{d})} \ f(x) = nx^{n-1}$$