STA6856 (Summer, 2025) **Mid-term Examination**

Student Name (print): Farooq Mahmud

UWF Honor Code applies to this unit examination. By writing your name above, you agree with the following: *you have neither given nor received unauthorized aidfor this examination.*

This exam contains 6 pages (including this cover page) and 7 questions. The total number of possible points is 100 points.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear order will receive very little credit. If extra space is needed, use the back of the previous page. Indicate the location of the work clearly and be neat.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations may still receive partial credit.
- **Provide exact answers** unless otherwise instructed.
- **If question directed, simplify all answers as much as possible.** This means that you need to need to combine like terms, reduce fractions, rationalize denominators, etc.
- **Be sure to state units for applied problems.**
- **Clearly identify your answer for each problem.**
- **For Question 7, please provide R codes and the outputs.**

1. (15 points) Suppose $E(X) = 2$, $\text{Var}(X) = 9$, $E(Y) = 0$, $\text{Var}(Y) = 4$, and $\text{Corr}(X, Y) = 0.25$. Find

   (a) $\text{Var}(X + Y) = Var(x) + Var(y) + 2cov(x,y)$

   $Corr(x,y) = \dfrac{cov(x,y)}{\sqrt{var(x)}\sqrt{var(y)}}$

   $cov(x,y) = Corr(x,y)\sqrt{var(x)}\sqrt{var(y)}$
   $= (0.25)(\sqrt{9})(\sqrt{4}) = 1.5$

   $\boxed{var(x+y) = 9 + 4 + 2(1.5) = 16}$

   (b) $\text{Cov}(X, X + Y)$
   $= cov(x,x) + cov(x,y)$
   $cov(x,x) = var(x) = 9$
   $cov(x,y) = 1.5$

   $\boxed{cov(x, x+y) = 9 + 1.5 = 10.5}$

   (c) $\text{Corr}(X + Y, X - Y) = \dfrac{cov(x+y, x-y)}{\sqrt{var(x+y)}\sqrt{var(x-y)}}$

   $cov(x+y, x-y) = cov(x,x) - cov(x,y) + cov(y,x) - cov(y,y)$

   $cov(x,x) = var(x) = 9$
   $cov(x,y) = 1.5$
   $cov(y,x) = cov(x,y) = 1.5$
   $cov(y,y) = var(y) = 4$

   $cov(x+y, x-y)$
   $= 9 - 1.5 + 1.5 - 4 = 5$

   $var(x+y) = 16$
   $var(x-y) = var(x) + var(y) - 2cov(x,y)$
   $= 9 + 4 - 2(1.5) = 10$

   $\boxed{corr(x+y, x-y) = \dfrac{5}{\sqrt{16}\sqrt{10}} = \dfrac{5}{4\sqrt{10}}}$

2. (10 points) Consider the following model

$$Y_t = Y_{t-1} - 0.25Y_{t-2} + e_t - 0.5e_{t-1}$$

   (a) Characterize this model as models in the ARMA(p,q) family, that is identify p and q.

   $Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} = e_t + \theta_1 e_{t-1}$
   $= Y_t - Y_{t-1} + 0.25 Y_{t-2} = e_t - 0.5 e_{t-1}$

   AR goes to lag 2
   MA goes to lag 1 $\Rightarrow$ $\boxed{p = 2,\ q = 1 \Rightarrow ARMA(2,1)}$

   (b) Is this model causal? Is this a stationary model?

   $\phi B = 1 - B + 0.25B^2$
   $(4)0 = (1 - B + 0.25B^2)4$
   $0 = 4 - 4B + B^2$
   $0 = (B-2)^2$
   $B = 2, 2 \Rightarrow$ $\boxed{\text{causal because roots are} > 1 \text{ stationary for same reason}}$

3. (20 points) For each of the following, find the mean function and the autocovariance function, then state if it is a stationary process. Here $\{e_t\}$ is an i.i.d. $N(0,1)$ and $E(t) = t$

(a) $Y_t = t + e_3$ $E(Y_t) = E(t) + E(e_t) = t + 0 = t$

$E(Y_t) = t$

$\mu(Y_t) = t$ Mean function depends on t so not Stationary

$\gamma_Y(t,s) = Cov(Y_t, Y_s) = Cov(t+e_3, s+e_3)$

$Cov(t,s) = 0$

$Cov(t,e_3) = Cov(s,e_3) = 0$

$Cov(e_3,e_3) = Var(e_3) = 1$ $\boxed{\gamma_Y(t,s) = 1}$

not stationary since autocovariance does not depend only on lag

(b) $Y_t = e_t e_{t-2}$ $E(Y_t) = E(e_t)E(e_{t-2}) = 0$

$E(Y_t) = 0$ mean function

$\mu(Y_t) = 0$

$\gamma_Y(h) = Cov(Y_t, Y_{t-h}) = E(Y_t Y_{t-h}) - E(Y_t)E(Y_{t-h})$ $E(Y_t) = 0$

$\gamma_Y(h) = E(Y_t Y_{t-h}) = E(e_t e_{t-2} e_{t-h} e_{t-h-2})$

$t = t-h$ non zero when $h = 0$

$t-2 = t-h-2$ non zero when $h = 0$

$t = t-h-2$ non zero when $h = -2$

$t-2 = t-h$ non zero when $h = 2$

$\gamma_Y(h) = 1$ when $h = 0, \pm 2$ else 0

Since mean is 0 and autocovariance depends only on lag the process is stationary

4. (10 points) Identify the following as specific ARMA models. That is, what are p, q, and what are the values of the parameters (the ϕ's and θ's)

$$Y_t = 0.5Y_{t-1} - 0.5Y_{t-2} + e_t - 0.5e_{t-1} + 0.25e_{t-2}$$

$Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} = e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2}$

$Y_t - 0.5Y_{t-1} + 0.5Y_{t-2} = e_t - 0.5e_{t-1} + 0.25e_{t-2}$

AR goes to lag 2
MA goes to lag 2 $\Rightarrow$ $p = 2$, $q = 2$ $\Rightarrow$ ARMA(2,2)

$\phi_1 = 0.5$ $\theta_1 = -0.5$

$\phi_2 = -0.5$ $\theta_2 = 0.25$

5. (20 points) Consider the following AR process

$$Y_t = 1.1Y_{t-1} - 0.18Y_{t-2} + e_t, \qquad e_t \sim WN(0, \sigma^2)$$

(a) Show that this series is stationary

process is AR(2)

$\phi(B) = 1 - 1.1B + 0.18B^2$

$0 = 1 - 1.1B + 0.18B^2$

$B = \dfrac{1.1 \pm \sqrt{(1.1)^2 - 4(0.18)(1)}}{2(0.18)} = \dfrac{1.1 \pm \sqrt{1.21 - .72}}{0.36} = \dfrac{1.1 \pm .7}{.36}$

$B = 5,\ 1.11$

Since both roots >1 process is stationary

(b) Derive the autocovariance functions.

$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}$

$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2$

$\boxed{\gamma_0 = 1.1\gamma_1 - 0.18\gamma_2 + \sigma^2}$

$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1$

$\gamma_1 = 1.1\gamma_0 - 0.18\gamma_1$

$\gamma_1 + 0.18\gamma_1 = 1.1\gamma_0$

$1.18\gamma_1 = 1.1\gamma_0$

$\boxed{\gamma_1 = \frac{1.1}{1.18}\gamma_0 = \frac{1.1}{1.18} \times 7.7 = 7.2\sigma^2}$

$\gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0$

$\gamma_2 = 1.1\gamma_1 - 0.18\gamma_0$

$\gamma_2 = 1.1\left(\frac{1.1}{1.18}\right)\gamma_0 - 0.18\gamma_0$

$\gamma_2 = \frac{1.21}{1.18}\gamma_0 - 0.18\gamma_0$

$\gamma_2 = \left(\frac{1.21}{1.18} - 0.18\right)\gamma_0$

$0.18 = \frac{0.2124}{1.18}$

$\gamma_2 = \frac{1.21 - 0.2124}{1.18}\gamma_0$

$\boxed{\gamma_2 = \frac{0.9976}{1.18}\gamma_0 = \frac{.9976}{1.18}(7.7) = 6.5\sigma^2}$

$\gamma_0 = 1.1\left(\frac{1.1}{1.18}\gamma_0\right) - 0.18\left(\frac{0.9976}{1.18}\gamma_0\right) + \sigma^2$

$\gamma_0 = \left(\frac{1.21}{1.18} - \frac{0.18}{1.18}\right)\gamma_0 + \sigma^2$

$\gamma_0 - \frac{1.03}{1.18}\gamma_0 = \sigma^2$

$\gamma_0 - \left(\frac{1.03}{1.18}\right)\gamma_0 = \sigma^2$

$\gamma_0\left(1 - \frac{1.03}{1.18}\right) = \sigma^2$

$\gamma_0(1 - 0.87) = \sigma^2$

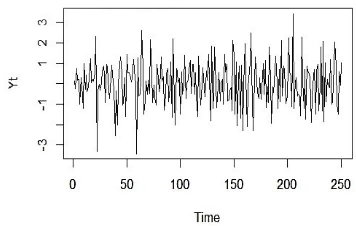
$\gamma_0(0.13) = \sigma^2$

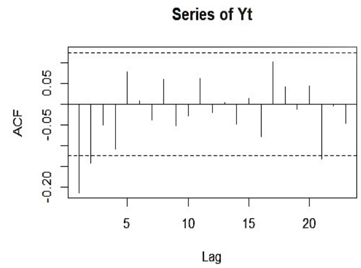
$\boxed{\gamma_0 = \frac{\sigma^2}{0.13} = 7.7\sigma^2}$

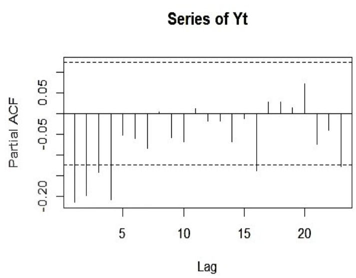
$\gamma_0 = 7.7\sigma^2$
$\gamma_1 = 7.2\sigma^2$
$\gamma_2 = 6.5\sigma^2$
$\gamma_k = 1.1\gamma_{k-1} - 0.18\gamma_{k-2}$

6. (10 points) Foe each case, identify an ARMA(p,q) model for the time series variable Y_t, given its ACF, PACF and EACF. Explain your reasoning for getting credits.

(a)







EACF

AR/MA	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	x	x	o	o	o	o	o	o	o	o	o	o	o	o
1	x	o	o	o	o	o	o	o	o	o	o	o	o	o
2	x	x	o	x	o	o	o	o	o	o	o	o	o	o
3	x	x	o	x	o	o	o	o	o	o	o	o	o	o
4	x	x	x	o	o	o	o	o	o	o	o	o	o	o
5	x	o	x	o	o	o	o	o	o	o	o	o	o	o
6	x	o	x	x	x	o	o	o	o	o	o	o	o	o
7	o	x	x	x	x	o	o	o	o	o	o	o	o	o

Mean is around zero, variance isn't changing which suggests a stationary process.

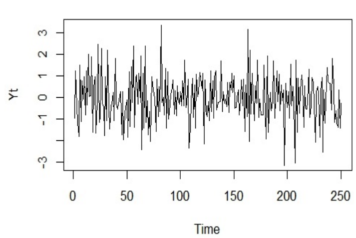
ACF tapers off instead of cutting off which suggests an AR components instead of pure MA.

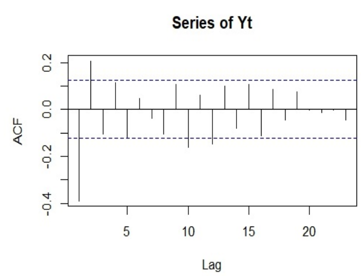
PACF also tapers off meaning it is likely not a pure AR process.

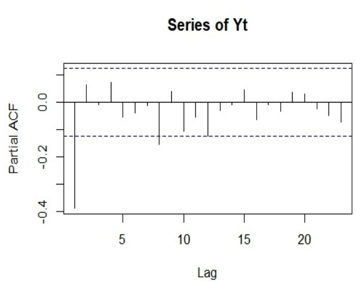
Given the above, the smallest clean triangle's vertex would be row 1, column 1. There are some X's but that could be noise.

Therefore a possible model would be ARMA(1, 1).

(b)







EACF

AR/MA	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	x	x	o	o	o	o	o	o	o	x	o	x	o	o
1	x	o	o	o	o	o	o	o	o	x	o	o	o	o
2	o	o	o	o	o	o	o	o	o	o	o	o	o	o
3	o	x	o	o	o	o	o	o	o	o	o	o	o	o
4	x	x	x	o	o	o	o	o	o	o	o	o	o	o
5	x	o	x	x	o	o	o	o	o	o	o	o	o	o
6	x	o	x	x	o	o	o	o	o	o	o	o	o	o
7	o	x	x	x	o	o	o	o	o	o	o	o	o	o

Mean is around zero, variance isn't changing which suggests a stationary process.

ACF tapers off instead of cutting off which suggests an AR components instead of pure MA.

PACF also tapers off meaning it is likely not a pure AR process.

Given the above, the smallest clean triangle's vertex would be row 1, column 1.

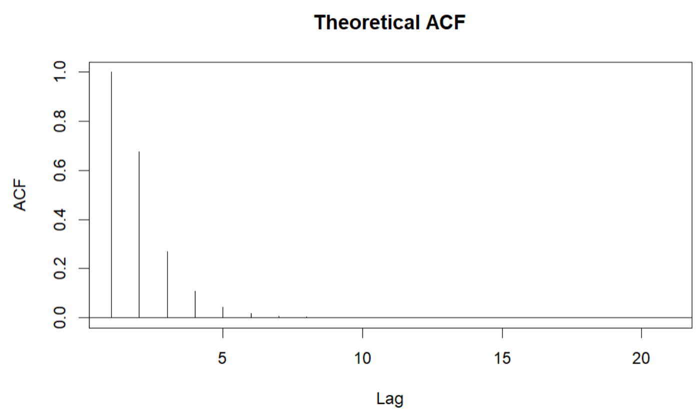
Therefore a possible model would be ARMA(1, 1).

7. (15 points) Simulate a mixed ARMA(1,1) model of length $n = 60$ with $\phi = 0.4$ and $\theta = 0.6$. Provide R codes and the outputs to achieve the following tasks.

   (a) Plot the theoretical autocorrelation function for this model. Plot sufficient lags until the correlations are negligible.

```r
phi <- c(0.4)
theta <- c(0.6)
n <- 70

theoretical_acf <- ARMAacf(ar = phi,
                           ma = theta,
                           lag.max = 20)
plot(
  theoretical_acf,
  type = "h",
  main = "Theoretical ACF",
  xlab = "Lag",
  ylab = "ACF"
)
abline(h = 0)
```

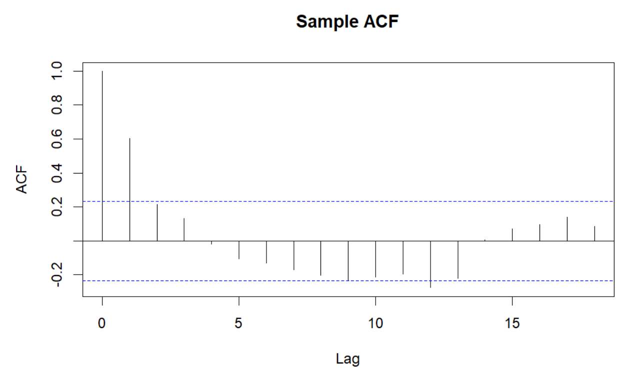


   (b) Plot the sample ACF for your simulated series. How well do the values and patterns match the theoretical ACF from part (a)?

```r
set.seed(123)

series <- arima.sim(
  n = n,
  model = list(ar = phi, ma = theta))

acf(series, main = "Sample ACF")
```



The sample ACF closely resembles the theoretical ACF in shape. The overall pattern of slow decay after significant values in lags 1 and 2 shows up in both plots.

   (c) Plot and interpret the sample EACF for this series. Does the EACF help you specify the correct orders for the model?

```r
library(TSA)
eacf(series)
```

```
AR/MA
  0 1 2 3 4 5 6 7 8 9 10 11 12 13
0 x o o o o o o o o o o  x  o  o
1 x o o o o o o o o o o  o  o  o
2 x o x o o o o o o o o  o  o  o
3 x x x o o o o o o o o  o  o  o
4 x x x o o o o o o o o  o  o  o
5 x x o x o o o o o o o  o  o  o
6 o x o x o o o o o o o  o  o  o
7 o x x x o o o o o o o  o  o  o
```

Yes, there is a clear triangle starting at (1,1) => ARMA(1,1). There are a few X's which is probably due to the small sample size.