Riemann Surfaces

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Chapter 1

Introduction

The idea of a Riemann Surface is to employ the power of complex analysis on a general 2-D surface that satisfies a set of conditions. In this work, we, first, discuss analytic functions and the properties they fulfill. Then, we discuss the single-valued analytic functions and the emergent problem with multi-valued analytic functions and give some examples of multi-valued functions. In chapter 3, we turn the discussion to introduce a more rigorous definition of a Riemann Surface and give the uniformization theorem without proof. In chapter 4, we study some applications to shed the light on the usefulness of Riemann Surfaces.

Chapter 2

Analytic Functions

A function $f(z)$ is said to be analytic or holomorphic at a point, say z_0 if the following conditions are fulfilled:

1 - For $f(z) = u(x, y) + i\, v(x, y)$, u and v satisfy Cauchy-Riemann conditions which are: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. This is for every point z in the neighbourhood of z_0.

2 - $f(z)$ is differentiable at z_0 and at any other point in the neighbourhood of z_0, i.e., $\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$ exists $\forall$ points $\in$ the neighbourhood of z_0.

3 - $f(z)$ is represented by a convergent Taylor series expansion in the neighbourhood of z_0, i.e., $f(z) = \sum_{n\geq 0} a_n(z - z_0)^n$. As a result, $f(z)$ is infinitely differentiable on the whole neighbourhood of z_0.

2.1 Single-Valued Functions

For any function $f(z)$, we can use the parameterization $z = re^{i\,\theta}$ such that we can see the behavior of the function upon any number of periods via shifting the parameter θ to $\theta + 2\pi n$ with n periods. For example, the function $f(z) = \frac{1}{z}$ can be parameterized as $f(z) = \frac{1}{r}\, e^{-i\,\theta}$ and upon sending θ to $\theta + 2\pi n$, we send $\frac{1}{r}\, e^{-i\,\theta}$ to $\frac{1}{r}\, e^{-i\,\theta}\, e^{-2i\,\pi\, n}$. However, $e^{-2i\,\pi\, n} = 1$ and therefore, upon shifting the parameter θ with n periods, the function $f(z) = \frac{1}{z}$ does not change. We can generalize the statement to be: Any complex-valued function that does not change upon shifting the parameter θ with n periods, is called a *single-valued function*.

2.2 Multi-Valued Functions

However, if the function changed upon each period on n then, it is a *multi-valued function*. For example, $f(z) = \sqrt{z}$ which can be parameterized as $f(z) = \sqrt{r}\, e^{i\theta/2}$. Yet, upon shifting n periods, the function picks up $e^{i\pi n}$. This means for $n = 1$ (one cycle), the function picked a minus sign. But one cycle is essentially the same physical point! This is why they are called "mutli-valued" functions. After, two cycles, the function returns to itself. Fig-1 illustrates more on this definition.

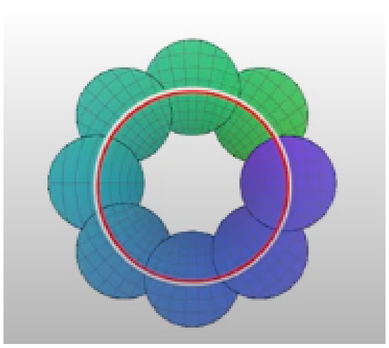

Fig-1-(a)

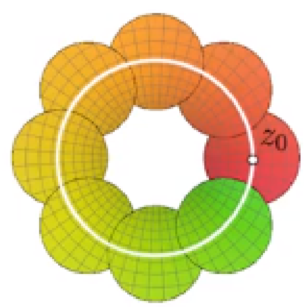

Fig-1-(b)

Fig-1-(a) is the original Taylor series of the function and Fig-1-(b) is what we have ended up with after one cycle. Clearly, they are not the same.

The set of all locally analytic functions, which are called multi-valued functions, is the set of functions defined on separate batches of the complex plane. Because multi-valued functions make a different response to the same batch of the complex plane comparing to single-valued ones, the usage of a single complex parameter z is no longer sufficient to identify such functions. So, we promote z by adding an additional label g to specify the set of all possible continuations reached by z, i.e., $z \to (z, g)$. g, in general, depends on z. Accordingly, a Riemann Surfaces is defined by all possible pairs of (z, g), i.e., $S(f) = (z, g) \in \mathbb{C} \times F : z \in \mathbb{C},\ g \in F_z$. Specifying g corresponds to specifying a particularly regular branch of a multi-valued function. Further, the set $S(f)$ can be thought of as the domain of the multi-valued function f because given (z, g), one can completely specify the value of the function f. Moreover, the projection $(z, g) \to z$ allows employing the power of complex analysis to be used on a Riemann surface. We can explore this more in the following examples.

2.2.1 Some Examples

Riemann Surface of $\sqrt{z}$

All possible values of $\sqrt{z}$ are enumerated by an integer n where $g(z) = e^{in\pi}\sqrt{z}$. Therefore, the Riemann surface of the $\sqrt{z}$ function is comprised of pairs, i.e., $S(\sqrt{z}) = (z, n),\ n \in \mathbf{Z}$. For every n, Riemann surface is a sheet. This sheet is a copy of a punctured complex plane slit along some line which allows to connect two adjacent sheets. Of course, there are only two sheets because $\sqrt{z}$ returns to its original value after two cycles. Fig-2 shows these sheets.

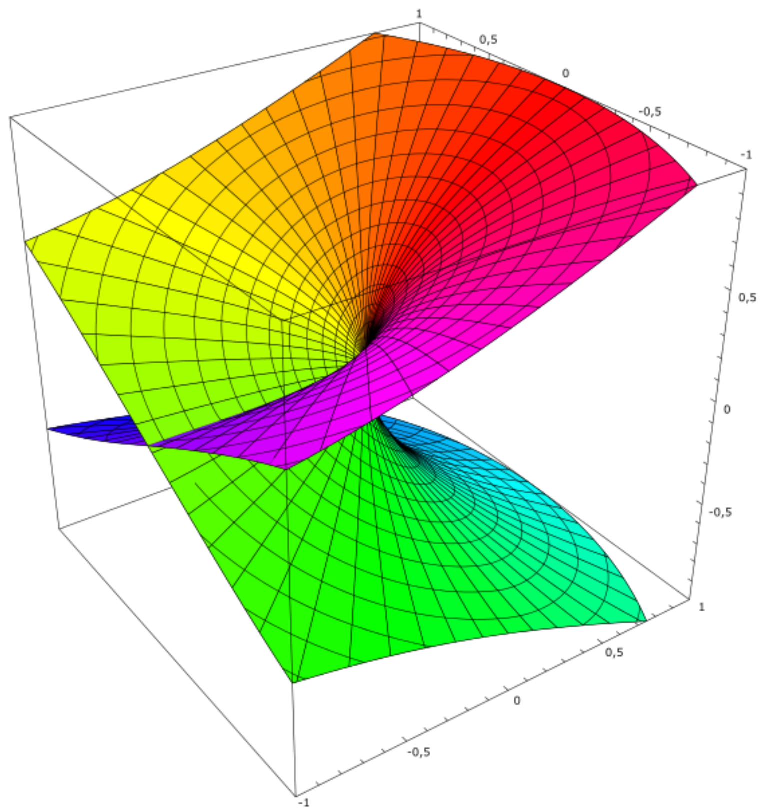

Fig-2

Riemann Surface of $\mathbf{z}^{1/4}$

All possible values of $z^{1/4}$ are enumerated by an integer n where $g(z) = e^{in\,\pi/2}\, z^{1/4}$. Therefore, the Riemann surface of the $z^{1/4}$ function is comprised of pairs, i.e., $S(z^{1/4}) = (z, n),\ n \in \mathbf{Z}$. For every n, Riemann surface is a sheet. This sheet is a copy of a punctured complex plane slit along some line which allows connecting two adjacent sheets. Of course, there are only four sheets because $z^{1/4}$ returns to its original value after four cycles. Fig-3 shows these sheets.

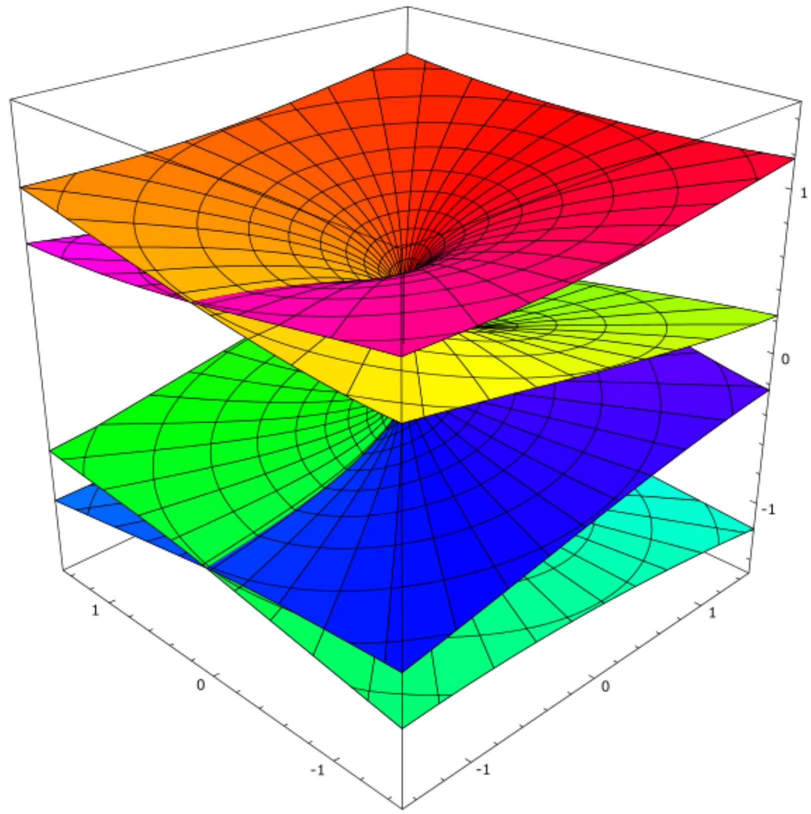

Fig-3

Riemann Surface of Log(z)

All possible values of $Log(z)$ are enumerated by an integer n where $g(z) = Log(z) + 2\pi\ in$. Therefore, the Riemann surface of the Log function is comprised of pairs, i.e., $S(Log) = (z, n)$, $n \in \mathbf{Z}$. For every n, Riemann surface is a sheet. This sheet is a copy of a punctured complex plane slit along some line which allows connecting two adjacent sheets. Of course, there are infinitely many sheets because the function never comes back to its original value. Fig-4 shows these sheets.

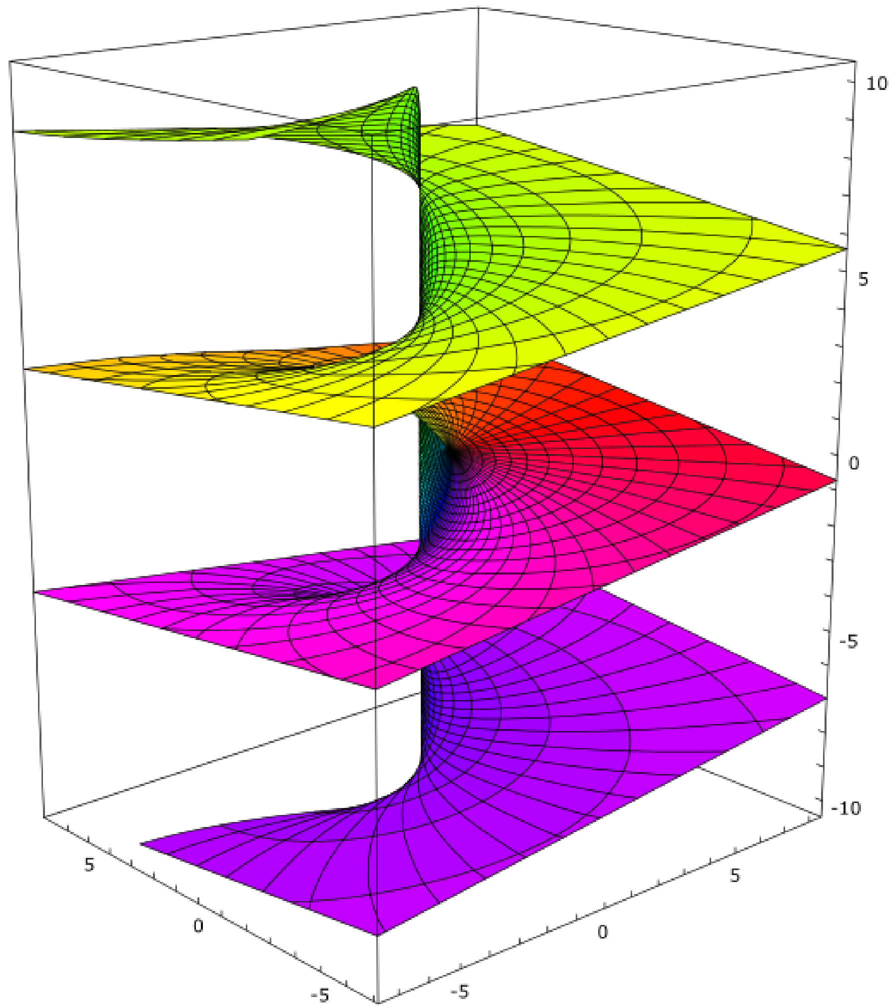

Fig-4

Chapter 3

The Idea of a Riemann Surface

In this chapter, we study a more general and rigorous definition of a Riemann Surface.

3.1 Intuitive Approach

Let us start with simple surfaces that can be visualized in $\mathbb{R}^3$, i.e., a sphere, cylinder, or tours. On a surface, X, given a point, say x_0, and given a small neighborhood, say D, of x_0 which looks like a disc. Topologically, such a small neighborhood looks like a disk in the complex plane $\mathbb{C}$. Also, given a mapping f from $D \mapsto \mathbb{C}$ which is defined $\forall$ points in D.

Now, if we want to define study holomorphic functions on such surface in the small neighborhood of x_0, it is instructive to use f as a holomorphic mapping at x_0. This will employ the power of complex analysis. Basically, this is the idea of a Riemann Surface, it is a surface where it is possible to use complex analysis. However, the question now is: How f could be holomorphic on D?

First, we identify D with an open set, say the unit disc $\{\Delta = z \in \mathbb{C} \quad | \quad |z| < 1\}$ by choosing a homeomorphism $\phi : D \mapsto \Delta$. Homeomorphism is topological isomorphism, i.e., $D \subset X \sim \Delta \subset \mathbb{C}$. This is illustrated by Fig-5.

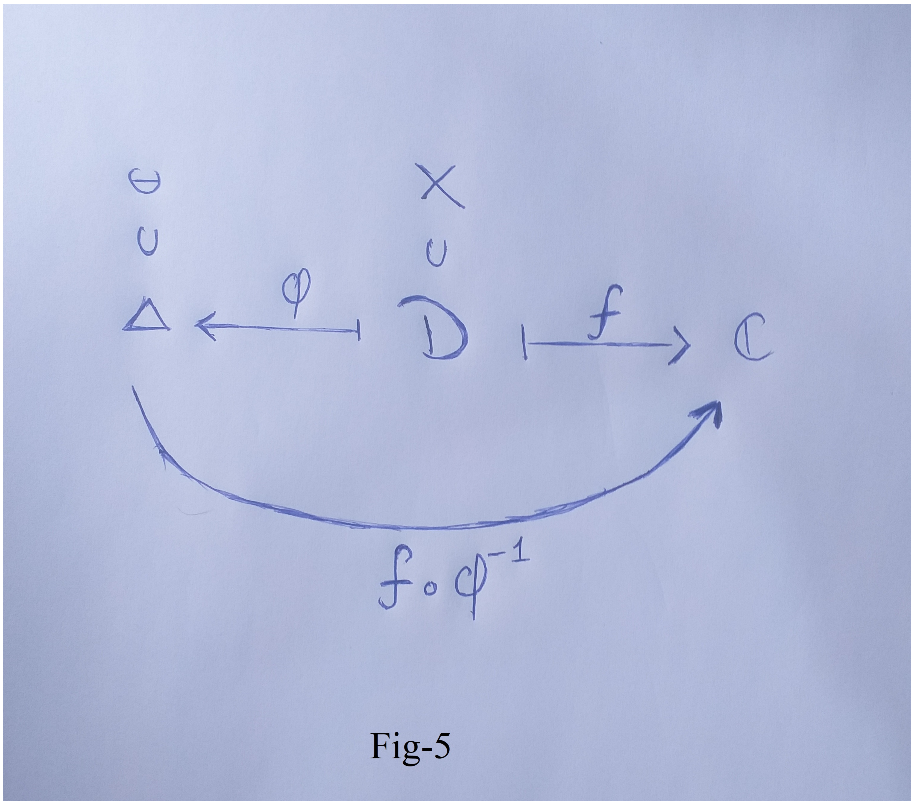

Fig-5

The figure shows the requirement that $f \circ \phi^{-1}$ is holomorphic at $\phi(x_0)$. Therefore, f is holomorphic on D, if and only if $f \circ \phi^{-1}$ is holomorphic on Δ. That is to define holomorphicity on an open subset D of the surface X, there must exist a holomorphic mapping $f \circ \phi^{-1}$ that takes Δ to the complex plane $\mathbb{C}$ such that, Δ is an open subset of the complex plane for which D can be mapped.

The identification of a disc-like neighbourhood $D \subset X$ and the real disc $\Delta \subset \mathbb{C}$ is a pair of data which can be written as (u, ϕ) while u is an open subset of the surface X and ϕ is a mapping to the real disc. Further, this identification is called *complex coordinate chart* or simply, *batch*.

3.1.1 preliminary Definition

Now, we can almost define what a Riemann surface is as follows: A Riemann surface is a surface X covered by a collection of charts $\{(u_\alpha, \phi_\alpha) \mid \alpha \in \mathbb{I}\}$ with $\mathbb{I}$ is an index in a set such that, for every element $\in \mathbb{I}$, there exists (u_α, ϕ_α). That is for every point $x_i \in D_\alpha$, there exists a chart (u_α, ϕ_α) that guarantees a proper definition of holomorphicity on the domain D_α. Note that, a collection of charts is called an *atlas*.

However, there is a possibility that a given point might occur on more than one chart! This would introduce ambiguity in the holomorphicity of f, i.e., the function could be holomorphic with respect to one chart and not holomorphic with respect to another chart. Here, we should modify the preliminary intuitive definition a bit to avoid such ambiguity.

3.2 A More General Approach

Because holomorphicity is an intrinsic property of f, it must be achieved using any chart. Therefore, the charts must be compatible in the sense that there must exist holomorphic mappings among the charts themselves. So, for any two intersecting charts, there exists a holomorphic mapping between these two intersecting charts.

More generally, a complex coordinate chart is a pair of (u, ϕ) while u is an open subset of a general surface X and v is an open subset of the complex plane $\mathbb{C}$ such that, $\phi : u \mapsto v$. This just means ϕ is a holomorphism of u onto the open subset v.

This is illustrated by Fig-6.

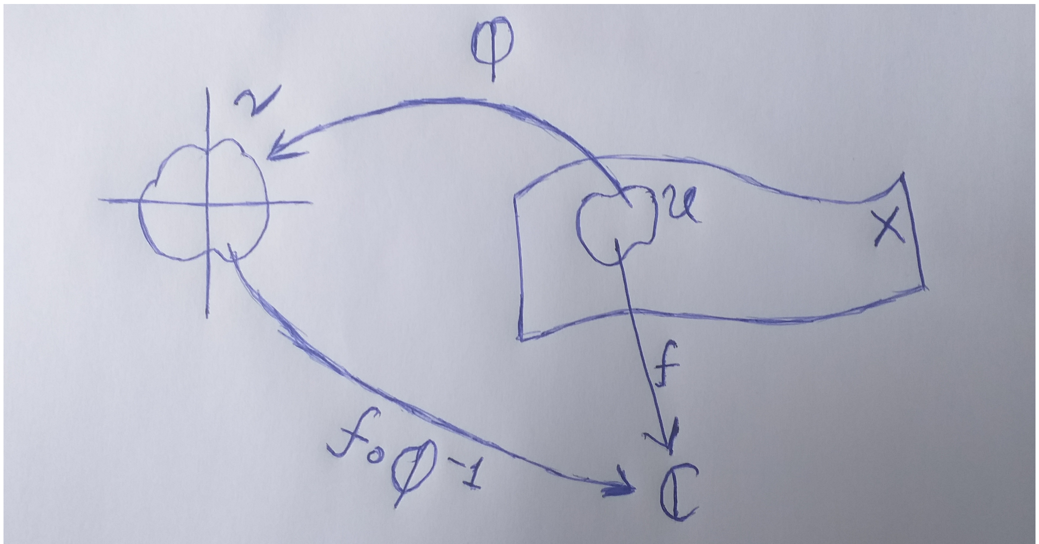

Fig-6

Again, f is holomorphic if an only if, $f \circ \phi^{-1}$ is holomorphic.

3.2.1 Concrete Definition

Now, we can give a concrete definition of what a Riemann surface structure is: A Riemann surface is a surface x covered by an atlas $\{(u_\alpha, \phi_\alpha) | \alpha \in \mathbb{I}\}$ with $\mathbb{I}$ is an index in a set, such that for every element $\in \mathbb{I}$, there exists (u_α, ϕ_α), and that atlas consists of a collection of compatible charts.

Chapter 4

Some Applications

Now, we use the previous knowledge to discuss some examples as applications to Riemann Surfaces.

4.1 The Complex Plane

Let a surface X be the $x - y$ plane in $\mathbb{R}^2$ and the atlas of X is $\{(u, \phi)\}$ where u is an open set ($u = \mathbb{R}^2$) and ϕ is a mapping ($\phi : u \mapsto \mathbb{C}, \quad \therefore \mathbb{R}^2 \mapsto \mathbb{C}$). Therefore, $(x, y) \in \mathbb{R}^2 \mapsto x + iy \in$ the Argand plane $\mathbb{C}$.

This natural identification is called the identity map. So, the complex palne $\mathbb{C}$ is a Riemann surface structure on $\mathbb{R}^2$ given by the standard identification of $\mathbb{R}^2$ with $\mathbb{C}$. Fig-7 illustrates more on this:

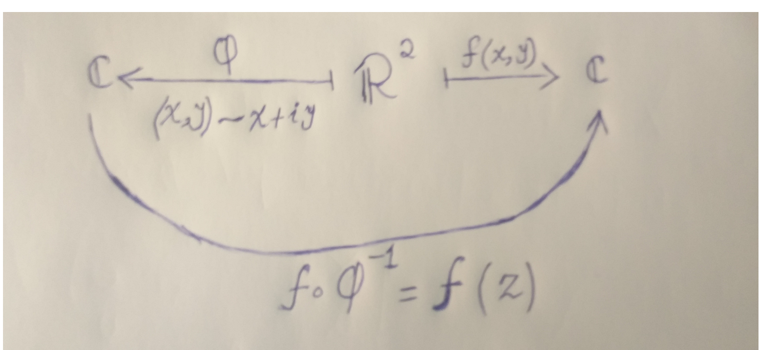

Fig-7

The question now is: Can we give to $\mathbb{R}^2$ some other Riemann surface structure that is different from the usual $\mathbb{C}$ structure?

The answer is yes. Let us consider $X = \mathbb{R}^2$. However, the atlas (u, ϕ) admits the mapping $\phi : u \mapsto \mathbb{C}$ and the image of ϕ exists on an open unit disc Δ where $\{\phi \in \Delta = z \in \mathbb{C} \ \mid \ |z| < 1\}$. Therefore,

$$(x, y) \in \mathbb{R}^2 \mapsto \frac{z}{1 + |z|} = \frac{x + iy}{1 + \sqrt{x^2 + y^2}}$$

. However, because ϕ^{-1} is no longer holomorphic due to its dependence on $\bar{z}$, the mapping $\mathbb{R}^2 \to \mathbb{C}$ is no longer holomorphic.

4.1.1 Uniformization Theorem in $\mathbb{R}^2$

Any simply-connected non-compact Riemann surface in $\mathbb{R}^2$ has to be isomorphic to either a unit disc or the complex plane.

4.2 Stereographic Projection

Let X be a unit sphere in $\mathbb{R}^3$ and the atlas of X is $S^2 = \{(u_\alpha, \phi_\alpha)\}$. The atlas consists of two charts: (u_1, ϕ_1) and (u_2, ϕ_2) where $u_1 = \{S^2 \setminus N \mapsto \mathbb{C}\}$ via ϕ_1 and $u_2 = \{S^2 \setminus S \mapsto \mathbb{C}\}$ via ϕ_2 with N as the north pole $(0, 0, 1)$ and and S as the south pole $(0, 0, -1)$ on the 2-sphere. This is called *Stereographic Projection* (see Fig-8).

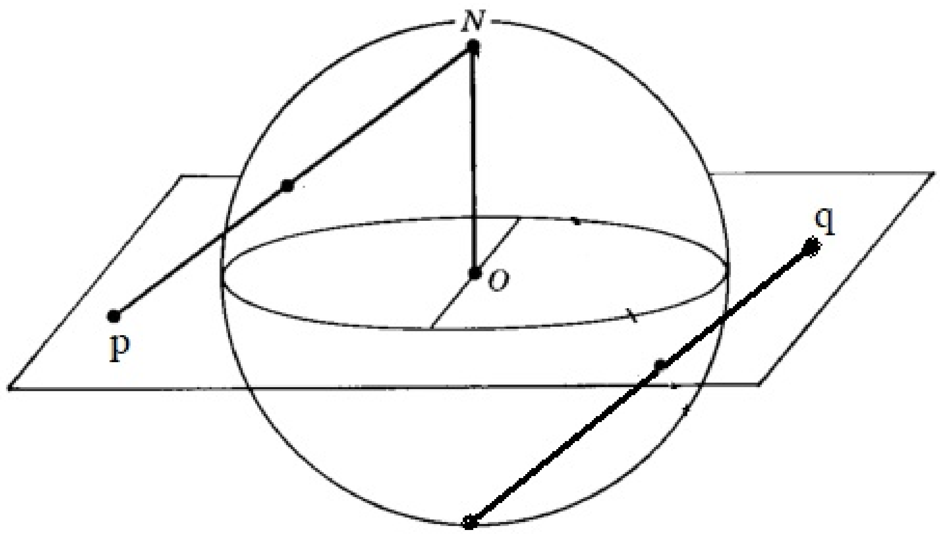

Fig-8

Clearly, $\phi_1(p)$ is the set of all points of the intersection between the north pole and p. Similarly, $\phi_2(q)$ is the set of all points of the intersection between the south pole and q.

If we seek, for example, an explicit realization of $\phi_1(p)$ and, one can show that:

$$x = \frac{x_1}{1-x_3} \;\; \& \;\; y = \frac{x_2}{1-x_3} \quad \text{where} \;\; p(x,y) \in \mathbb{C} \;\; \text{and} \;\; x_1^2 + x_2^2 + x_3^2 = 1 \in \mathbb{R}^3$$

.

Similarly, for $\phi_2(q)$:

$$\tilde{x} = \frac{x_1}{1+x_3} \;\; \& \;\; \tilde{y} = \frac{x_2}{1+x_3} \quad \text{where} \;\; q(\tilde{x},\tilde{y}) \in \mathbb{C}$$

.

On top of that, the compatibility condition holds where $u_1 \cup u_2 = S^2 \setminus \{N, S\}$ and the holomorphic mapping from $u_1 \mapsto u_2$ is $z \mapsto \frac{1}{z}$.

4.2.1 Uniformization Theorem in $\mathbb{R}^3$

For any simply-connected compact Riemann surface in $\mathbb{R}^3$, it is not possible to find another Riemann surface structure other than the Riemann Sphere S^2.

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