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A Derivation of the Likelihood Function for a Statistical H_0 Measurement

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ABSTRACT

We derive the likelihood function for a "statistical" measurement of the Hubble constant that combines a distance measured by a gravitational wave (GW) observation and a catalog of redshifts from possible host galaxies measured by electromagnetic (EM) observations.

1. LIKELIHOOD

We have two data sets: a GW set d_{GW}, corresponding to a single detected GW event, and an EM data set d_{EM}, corresponding to a survey of galaxies that overlaps on the sky with the GW event's inferred sky position. A likelihood function describes the generation of the GW data given GW parameters θ_{GW} such as masses, spins, etc:

$$\mathcal{L}\left(\theta_{\mathrm{GW}}\right) \equiv p\left(d_{\mathrm{GW}} \mid \theta_{\mathrm{GW}}\right). \tag{1}$$

Crucially, the luminosity distance to the event, d_L, and the position of the event on the sky Ω_{GW} are among the GW parameters. Similarly, we have an EM likelihood that describes the generation of the survey data given parameters for the galaxies in the survey:

$$\mathcal{L}\left(\theta_{\mathrm{EM}}\right) \equiv p\left(d_{\mathrm{EM}} \mid \theta_{\mathrm{EM}}\right). \tag{2}$$

Among the parameters for the EM survey are the redshifts of the galaxies $\{z_i \mid i = 1, \ldots, N_{\mathrm{gal}}\}$ and also the positions of the galaxies on the sky $\{\Omega_i \mid i = 1, \ldots, N_{\mathrm{gal}}\}$.

The joint likelihood for the two sets of observations is just the product of the individual likelihoods because the data generating processes in each instrument are

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independent of each other:

$$p\left(d_{\mathrm{GW}}, d_{\mathrm{EM}} \mid \theta_{\mathrm{GW}}, \theta_{\mathrm{EM}}\right) = p\left(d_{\mathrm{GW}} \mid \theta_{\mathrm{GW}}\right) p\left(d_{\mathrm{EM}} \mid \theta_{\mathrm{EM}}\right). \tag{3}$$

To measure the Hubble constant, we will not be concerned with the masses, spins, etc, of the GW source, nor with EM data besides the redshifts and sky positions, so let us impose some standard priors and integrate these variables out. This leaves us with a *marginal* likelihood for each observation expressed in terms of the luminosity distance, sky positions, and redshifts:

$$\begin{aligned} p\left(d_{\mathrm{GW}} \mid d_L, \Omega_{\mathrm{GW}}\right) p\left(d_{\mathrm{EM}} \mid \left\{z_i, \Omega_i \mid i = 1, \ldots, N_{\mathrm{gal}}\right\}\right) = \\ \int \mathrm{d}\bar{\theta}_{\mathrm{GW}}\, \mathrm{d}\bar{\theta}_{\mathrm{EM}}\, p\left(\bar{\theta}_{\mathrm{GW}}\right) p\left(\bar{\theta}_{\mathrm{EM}}\right) p\left(d_{\mathrm{GW}} \mid \theta_{\mathrm{GW}}\right) p\left(d_{\mathrm{EM}} \mid \theta_{\mathrm{EM}}\right), \end{aligned} \tag{4}$$

where the "bar" indicates the set of ignorable parameters and $p\left(\bar{\theta}_{\mathrm{GW}}\right)$ and $p\left(\bar{\theta}_{\mathrm{EM}}\right)$ are the priors we impose on these parameters.

Now let us impose the constraint that one of the galaxies in the survey must have hosted the GW event; suppose that we weight the choice of hosts by a set of weights $\{w_i \mid i = 1, \ldots, N_{\mathrm{gal}}\}$. This constraint will tie together both pairs Ω_{GW} and Ω_i and d_L and z_i; the relation between d_L and z_i depends on H_0 (and other cosmological parameters). We can use these relationships to marginalize over (i.e. eliminate) d_L and Ω_{GW} from the likelihood, obtaining for a single galaxy host, i

$$\begin{aligned} p\left(d_{\mathrm{GW}} \mid i, z_i, \Omega_i, H_0\right) \equiv \\ \int \mathrm{d}d_L\, \mathrm{d}\Omega_{\mathrm{GW}}\, p\left(d_{\mathrm{GW}} \mid d_L, \Omega_{\mathrm{GW}}\right) \delta\left(d_L - d_L\left(z_i, H_0\right)\right) \delta\left(\Omega_{\mathrm{GW}} - \Omega_i\right). \end{aligned} \tag{5}$$

The integrals are trivial, leaving

$$p\left(d_{\mathrm{GW}} \mid i, z_i, \Omega_i, H_0\right) = p\left(d_{\mathrm{GW}} \mid d_L\left(z_i, H_0\right), \Omega_i\right). \tag{6}$$

In the absence of other information, we do not know which galaxy hosted the GW event, so we should marginalize over the choice of galaxy, i, with the correct weights:

$$p\left(d_{\mathrm{GW}} \mid \left\{z_i, \Omega_i \mid i = 1, \ldots, N_{\mathrm{gal}}\right\}, H_0\right) = \sum_{i=1}^{N_{\mathrm{gal}}} w_i p\left(d_{\mathrm{GW}} \mid d_L\left(z_i, H_0\right), \Omega_i\right). \tag{7}$$

Note that this is just the GW part of the likelihood, and it depends on the true redshifts and sky locations of the host galaxies. The galaxy redshifts and sky positions are nuisance parameters in an inference of H_0 so we should marginalize over them with a suitable choice of prior, but first we must specify the EM likelihood since this also depends on those parameters. We assume that the EM data consists of independent measurements for each galaxy in the catalogue, with perfect sky location determination and imperfect redshift determination, i.e.,

$$p\left(d_{\mathrm{EM}} \mid \{z_i, \Omega_i \mid i = 1, \ldots, N_{\mathrm{gal}}\}\right) \quad = \quad \prod_{j=1}^{N_{\mathrm{gal}}} \delta(\Omega_{j,obs} \quad - \quad \Omega_j) p(z_{j,obs}|z_j, \Omega_j) \quad (8)$$

in which we have now replaced d_{EM} by individual estimated sky locations, $\Omega_{i,obs}$, and redshifts, $z_{i,obs}$, for each galaxy i. To marginalise the true redshift and sky location out of the likelihood we need to specify a prior on these parameters. A natural prior would be to assume that galaxies are uniformly distributed in comoving volume and on the sky, so

$$p\left(z_i, \Omega_i\right) \mathrm{d}z_i \, \mathrm{d}\Omega_i = \frac{1}{4\pi V_{\mathrm{max}}} \frac{\mathrm{d}V}{\mathrm{d}z_i} \, \mathrm{d}z_i \, \mathrm{d}\Omega_i \quad (9)$$

where V_{max} is some suitably large comoving volume that encompasses all galaxies in the survey. At sufficiently small redshifts, $p\left(z_i\right) \propto z_i^2$. Note that at *all* redshifts, $p\left(z_i\right)$ is independent of H_0.

The above prior is needed for each galaxy and the combined prior for all galaxies is therefore $\prod_j p(z_j, \Omega_j)$. Finally, we arrive at the marginal likelihood for H_0 given the joint observations:

$$\begin{aligned} p\left(d_{\mathrm{GW}}, d_{\mathrm{EM}} \mid H_0\right) &= \sum_{i=1}^{N_{\mathrm{gal}}} w_i \prod_j \int \mathrm{d}z_j \, \mathrm{d}\Omega_j \, p\left(z_j, \Omega_j\right) p\left(d_{\mathrm{GW}} \mid d_L\left(z_i, H_0\right), \Omega_i\right) \\ &\qquad \times \delta(\Omega_{j,obs} - \Omega_j) p(z_{j,obs}|z_j, \Omega_j) \\ &= \sum_{i=1}^{N_{\mathrm{gal}}} w_i \prod_j \int \mathrm{d}z_j \, p\left(z_j, \Omega_{j,obs}\right) p\left(d_{\mathrm{GW}} \mid d_L\left(z_i, H_0\right), \Omega_{i,obs}\right) \\ &\qquad \times p(z_{j,obs}|z_j, \Omega_{j,obs}). \end{aligned} \quad (10)$$

The GW data term depends only on the galaxy i appearing in the sum. Therefore, most of the integrals can be evaluated directly. We introduce the evidences

$$\mathcal{Z}_j = \int p(z_{j,obs}|z, \Omega_{j,obs}) p\left(z, \Omega_{j,obs}\right) dz \quad (11)$$

and then the previous expression reduces to

$$\begin{aligned} p\left(d_{\mathrm{GW}}, d_{\mathrm{EM}} \mid H_0\right) \propto \sum_{i=1}^{N_{\mathrm{gal}}} w_i \int \mathrm{d}z \, \frac{p\left(z, \Omega_{i,obs}\right) p(z_{i,obs}|z, \Omega_{i,obs})}{\mathcal{Z}_i} \\ \times p\left(d_{\mathrm{GW}} \mid d_L\left(z, H_0\right), \Omega_{i,obs}\right). \end{aligned} \quad (12)$$

The first term inside the integral is the posterior on the true redshift of galaxy i based on the EM data and so this can be interpreted as using the EM data as a prior on the possible redshifts of the host of the GW source.

1.1. *Selection Effects*

So far, we have been ignoring selection effects. Selection effects can be modeled by including a factor, $\beta\left(H_0\right)$ ensuring that the marginal likelihood integrates to one over all *detectable* data sets (Mandel et al. 2018). In other words, we write

$$p\left(d_{\mathrm{GW}}, d_{\mathrm{EM}} \mid H_0\right) = \frac{1}{\beta\left(H_0\right)} \sum_{i=1}^{N_{\mathrm{gal}}} w_i \int \mathrm{d}z_i\, \mathrm{d}\Omega_i\, p\left(z_i, \Omega_i\right) p\left(d_{\mathrm{GW}} \mid d_L\left(z_i, H_0\right), \Omega_i\right) \\ \times p\left(d_{\mathrm{EM}} \mid \left\{z_i, \Omega_i \mid i = 1, \ldots, N_{\mathrm{gal}}\right\}\right) \quad (13)$$

such that

$$\int \mathrm{d}d_{\mathrm{GW}}\, \mathrm{d}d_{\mathrm{EM}}\, P_{\mathrm{det}}\left(d_{\mathrm{GW}}\right) P_{\mathrm{det}}\left(d_{\mathrm{EM}}\right) p\left(d_{\mathrm{GW}}, d_{\mathrm{EM}} \mid H_0\right) = 1, \quad (14)$$

where $P_{\mathrm{det}}\left(d\right)$ refers to the probability that a particular data set d will result in a "detection" of a GW (typically, P_{det} is either 0 or 1, depending on the output of a pipeline applied to the data; it serves to truncate the integral to detectable data sets only). Thus,

$$\beta\left(H_0\right) = \int \mathrm{d}d_{\mathrm{GW}}\, \mathrm{d}d_{\mathrm{EM}}\, P_{\mathrm{det}}\left(d_{\mathrm{GW}}\right) P_{\mathrm{det}}\left(d_{\mathrm{EM}}\right) \\ \times \sum_{i=1}^{N_{\mathrm{gal}}} w_i \int \mathrm{d}z_i\, \mathrm{d}\Omega_i\, p\left(z_i, \Omega_i\right) p\left(d_{\mathrm{GW}} \mid d_L\left(z_i, H_0\right), \Omega_i\right) \\ \times p\left(d_{\mathrm{EM}} \mid \left\{z_i, \Omega_i \mid i = 1, \ldots, N_{\mathrm{gal}}\right\}\right). \quad (15)$$

Note that the issue of selection effects has nothing to do with how "detectable" the observed events are (by definition, in the absence of a stochastic detection pipeline, $P_{\mathrm{det}}\left(d_{\mathrm{GW,observed}}\right) \equiv 1$), but rather how the detectability of the *population* of GW events varies with H_0.

Current catalogs are complete (i.e. $P_{\mathrm{det}}\left(d_{\mathrm{EM}}\right) = 1$) for any GW detection assuming reasonable values of H_0. In this document we assume that the galaxy catalog is selected from a flux-limited survey, so (to sufficient approximation) it is complete to some $d_{L,\mathrm{max}}$; this corresponds to a $z_{\mathrm{max}}\left(d_{L,\mathrm{max}}, H_0\right)$ that depends on H_0. An alternative perspective is to assume that the catalogs are complete to some redshift z_{max} in which case $d_{L,\mathrm{max}}\left(z_{\mathrm{max}}, H_0\right)$ depends on H_0. Final expressions for the likelihood are identical between these two approaches, but intermediate expressions differ in some places, as I note below.

Performing the integral over d_{EM} (remember that the EM likelihood is normalised), we obtain

$$\beta\left(H_0\right) = \int \mathrm{d}d_{\mathrm{GW}}\, P_{\mathrm{det}}\left(d_{\mathrm{GW}}\right) \\ \times \sum_{i=1}^{N_{\mathrm{gal}}} w_i \int \mathrm{d}z_i\, \mathrm{d}\Omega_i\, p\left(z_i, \Omega_i\right) p\left(d_{\mathrm{GW}} \mid d_L\left(z_i, H_0\right), \Omega_i\right). \quad (16)$$

The integrals in each term of the sum are now all identical (only the EM data distinguishes the different galaxies) and the sum over weights is normalized, so we are left with

$$\beta\left(H_0\right) = \int \mathrm{d}d_{\mathrm{GW}}\, \mathrm{d}z\, \mathrm{d}\Omega\, P_{\mathrm{det}}\left(d_{\mathrm{GW}}\right) p\left(z, \Omega\right) p\left(d_{\mathrm{GW}} \mid d_L\left(z, H_0\right), \Omega\right). \tag{17}$$

We can change variables from z to d_L in the integral, and integrate up to the GW horizon, $d_{L,\mathrm{max,GW}}$, where $P_{\mathrm{det}}\left(d_{\mathrm{GW}}\right) \to 0$, obtaining

$$\beta\left(H_0\right) = \int \mathrm{d}d_{\mathrm{GW}}\, \mathrm{d}d_L\, \mathrm{d}\Omega\, P_{\mathrm{det}}\left(d_{\mathrm{GW}}\right) p\left(d_L, \Omega\right) p\left(d_{\mathrm{GW}} \mid d_L, \Omega\right) = \frac{V\left(d_{L,\mathrm{max,GW}}\right)}{V\left(d_{L,\mathrm{max}}\right)} \tag{18}$$

But this latter integral is independent of H_0 for Euclidean universes, and weakly dependent at larger distances under our assumption that the catalog is complete to some fixed $d_{L,\mathrm{max}}$, so $\beta\left(H_0\right) \simeq$ const; had we assumed instead a maximum redshift for completeness, then $d_{L,\mathrm{max}}$ depends on H_0 and $\beta\left(H_0\right) \propto H_0^3$ as in Fishbach et al. (2018).

Eventually, EM catalogs will not be complete for all GW events. If only the set of GW events for which the EM catalog is complete is used to estimate H_0 then $\beta\left(H_0\right)$ can be computed by returning to Eq. (15). Alternately, the full set of GW events can be used, and a fictitious catalog members can be added with weights corresponding to the incompleteness fraction at various redshifts and in different directions; the limit of this procedure yields the formulas in Fishbach et al. (2018) for incomplete catalogs.

2. PRACTICAL CONSIDERATIONS

Evaluating Eq. (12) requires the ability to compute the GW marginal likelihood at arbitrary d_L and Ω. Happily, a standard data product provided by LIGO (Singer et al. 2016) can be adapted to produce a function that approximates the marginal GW likelihood at fixed locations on the sky corresponding to HEALPix pixels (Górski et al. 2005). The method in Singer et al. (2016) fits the *posterior* over distance and sky position from a LIGO analysis using a per-pixel ansatz where

$$p\left(d_L, \Omega \mid d_{\mathrm{GW}}\right) = \frac{\rho_i}{A} \frac{N_i}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{1}{2}\left(\frac{d_L - \mu_i}{\sigma_i}\right)^2\right] d_L^2, \tag{19}$$

with Ω in HEALPix pixel i; N_i, ρ_i, μ_i, and σ_i per-pixel fitting parameters; and A the area of the pixel. Since the standard LIGO distance prior is proportional to d_L^2, this means that the GW marginal likelihood when Ω is in pixel i is (up to ignorable constants)

$$p\left(d_{\mathrm{GW}} \mid d_L, \Omega\right) \propto \frac{\rho_i}{A} \frac{N_i}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{1}{2}\left(\frac{d_L - \mu_i}{\sigma_i}\right)^2\right]. \tag{20}$$

(Note the missing factor of d_L^2 relative to Eq. (19).)

If we are in the linear regime, where

$$d_L = \frac{cz}{H_0} \tag{21}$$

to sufficient accuracy, the EM likelihood is Gaussian in the redshift,

$$p\left(d_{\mathrm{EM}} \mid z_i, \Omega_i\right) \propto \frac{1}{\sqrt{2\pi}\sigma_{z,i}} \exp\left[-\frac{1}{2}\left(\frac{z_i - z_{i,\mathrm{obs}}}{\sigma_{i,z}}\right)^2\right], \tag{22}$$

and we assume that the EM data determines the sky position of each galaxy precisely, then the marginalization integrals over redshift and sky position in Eq. (12) can be performed analytically, as we now show. The evidence integral, assuming a $p\left(z_i\right) \propto z_i^2$ prior becomes

$$\mathcal{Z}_i = \int \mathrm{d}z_i\, z_i^2 \frac{1}{\sqrt{2\pi}\sigma_{z,i}} \exp\left[-\frac{1}{2}\left(\frac{z_i - z_{i,\mathrm{obs}}}{\sigma_{i,z}}\right)^2\right] = \sigma_{i,z}^2 + z_{i,\mathrm{obs}}^2. \tag{23}$$

The result, when galaxy i lies inside pixel k_i, is

$$\begin{aligned} p\left(d_{\mathrm{GW}} \mid d_{\mathrm{EM}}, H_0\right) = \\ \frac{3H_0^3}{z_{\max}^3} \sum_{i=1}^{N_{\mathrm{gal}}} \frac{w_i \rho_{k_i} N_{k_i}}{A\sqrt{2\pi}\Sigma\mathcal{Z}_i} \exp\left[-\frac{\left(H_0\mu_{k_i} - cz_{i,\mathrm{obs}}\right)^2}{2\Sigma^2}\right] \\ \times \left(\frac{\sigma_{k_i}^2\sigma_{i,z}^2}{\Sigma^2} + \frac{\left(c\mu_{k_i}\sigma_{i,z}^2 + H_0\sigma_{k_i}^2 z_{i,\mathrm{obs}}\right)^2}{\Sigma^4}\right), \end{aligned} \tag{24}$$

with

$$\Sigma^2 = H_0^2\sigma_{k_i}^2 + c^2\sigma_{i,z}^2. \tag{25}$$

We are assuming that the galaxy catalog is complete to some $d_{L,\max}$, so the factor $H_0/z_{\max}$ is independent of H_0 and equal to $c/d_{\max}$:

$$\begin{aligned} p\left(d_{\mathrm{GW}} \mid d_{\mathrm{EM}}, H_0\right) = \\ \frac{3c^3}{d_{L,\max}^3} \sum_{i=1}^{N_{\mathrm{gal}}} \frac{w_i \rho_{k_i} N_{k_i}}{A\sqrt{2\pi}\Sigma\mathcal{Z}_i} \exp\left[-\frac{\left(H_0\mu_{k_i} - cz_{i,\mathrm{obs}}\right)^2}{2\Sigma^2}\right] \\ \times \left(\frac{\sigma_{k_i}^2\sigma_{i,z}^2}{\Sigma^2} + \frac{\left(c\mu_{k_i}\sigma_{i,z}^2 + H_0\sigma_{k_i}^2 z_{i,\mathrm{obs}}\right)^2}{\Sigma^4}\right). \end{aligned} \tag{26}$$

If, alternately, we had assumed that the catalog is complete to some $z_{\max}$ then $\beta\left(H_0\right) \propto H_0^3$ and this factor would cancel the H_0^3 in the numerator, again leading to a likelihood that lacks a pre-factor dependent on H_0. Both approaches agree on the likelihood *when the selection effects are dominated by the GW horizon.*

It is reasonable to truncate the sum above by ignoring galaxies for which ρ_{k_i} (the integrated per-pixel GW posterior probability) is sufficiently small.

2.1. *The Uninformative Limit*

In the limit where the individual redshift measurements are too uncertain to meaningfully constrain H_0, the measurement should be uninformative. In this limit

$$H_0^2 \sigma_{k_i}^2 \ll c^2 \sigma_{z,i}^2 \to \infty \tag{27}$$

in the relevant pixels. Since the $c^2\sigma_{i,z}^2$ term dominates Σ and also eliminates the exponential, the likelihood function for H_0 becomes constant and the measurement is uninformative about H_0.

2.2. *The Maximally Informative Limit*

Suppose there is only a single galaxy in the entire sky area (maybe our sky area is very small, or maybe we have an EM counterpart that eliminates all the galaxies in the catalog but one). Then the sum collapses to one term, and the pixel-dependent parts become ignorable constants, so

$$\begin{aligned} p\left(d_{\mathrm{GW}} \mid d_{\mathrm{EM}}, H_0\right) \propto \frac{1}{\sqrt{2\pi}\Sigma \mathcal{Z}_i} \exp\left[-\frac{\left(H_0 \mu_{k_i} - c z_{i,\mathrm{obs}}\right)^2}{2\Sigma^2}\right] \\ \times \left(\frac{\sigma_{k_i}^2 \sigma_{i,z}^2}{\Sigma^2} + \frac{\left(c\mu_{k_i}\sigma_{i,z}^2 + H_0 \sigma_{k_i}^2 z_{i,\mathrm{obs}}\right)^2}{\Sigma^4}\right). \end{aligned} \tag{28}$$

If we now imagine that $c\sigma_{i,z} \ll H_0 \sigma_{k_i}$ for all reasonable values of H_0 and that $\sigma_{i,z} \ll z_{i,\mathrm{obs}}$, then we do not distinguish any more between $z_{i,\mathrm{obs}}$ and z_i and the likelihood becomes

$$p\left(d_{\mathrm{GW}} \mid d_{\mathrm{EM}}, H_0\right) \propto \frac{1}{\sqrt{2\pi} H_0 \sigma_{k_i}} \exp\left[-\frac{\left(H_0 \mu_{k_i} - c z_i\right)^2}{2 H_0^2 \sigma_{k_i}^2}\right] \left(\frac{1}{H_0^2}\right). \tag{29}$$

If the GW measurement becomes infinitely precise as well, then $\sigma_{k_i} \to 0$, and the likelihood becomes

$$p\left(d_{\mathrm{GW}} \mid d_{\mathrm{EM}}, H_0\right) \propto \delta\left(H_0 \mu_{k_i} - c z_i\right) \left(\frac{1}{H_0^3}\right), \tag{30}$$

which selects $H_0 = c z_i / \mu_{k_i}$ as expected.

2.3. *The Many Galaxies Limit*

Suppose there are so many galaxies within the GW region on the sky that we can treat them as being continuously distributed with density $p(z) \propto z^2$. To make the expressions tractable, suppose the measurement uncertainty on redshift is small, $\sigma_{i,z} \to 0$ and that the GW source is localized to a single pixel k. Then the GW likelihood becomes

$$p\left(d_{\mathrm{GW}} \mid d_{\mathrm{EM}}, H_0\right) \propto \sum_i \frac{w_i}{\sqrt{2\pi} H_0^3 \sigma_k} \exp\left[-\frac{\left(H_0 \mu_k - c z_i\right)^2}{2 H_0^2 \sigma_k^2}\right]. \tag{31}$$

Note that the evidence term $\mathcal{Z}_i$ has cancelled the z_i^2 appearing from the prior we placed on the true redshift of each galaxy. There are so many galaxies that the sum over galaxies becomes an integral, leading to

$$p\left(d_{\mathrm{GW}} \mid d_{\mathrm{EM}}, H_0\right) \propto \int \mathrm{d}z \, \frac{p(z)}{\sqrt{2\pi} H_0^3 \sigma_k} \exp\left[-\frac{\left(H_0 \mu_k - cz\right)^2}{2 H_0^2 \sigma_k^2}\right] \propto \text{const.} \tag{32}$$

In other words, if the distribution of galaxies is uniform across the GW localization region then the measurement is uninformative.

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