Lensing in the Schwarzschild Spacetime Applicable to Neutron Stars

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ABSTRACT

I describe the effects of gravitational lensing on the image of the surface of a spherical object whose surface is well outside the Schwarszchild horizon.

1. INTRODUCTION

2. GEODESICS

The Schwarzchild line element in standard coordinates and natural units is

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1 - \frac{2}{r}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{2}{r}}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right); \tag{1}$$

t is the clock time of an observer at $r = \infty$, and $4\pi r^2$ is the area of the constant-r, constant-t hypersurfaces. The Schwarzschild radius (the coordinate of the event horizon) is $r_\mathrm{Schw} = 2$ in these units.

The metric has two killing vectors, ∂_t and ∂_ϕ, since it is independent of the corresponding coordinates. Write

$$\partial_t = \xi^\mu \partial_\mu, \qquad \partial_\phi = \zeta^\mu \partial_\mu; \tag{2}$$

then the only non-zero component of ξ^μ is $\xi^t = 1$, and the only non-zero component of ζ^μ is $\zeta^\phi = 1$. If p^μ is the momentum associated with a geodesic,

$$p^\mu = \frac{\mathrm{d}\chi^\mu(\lambda)}{\mathrm{d}\lambda}, \tag{3}$$

with λ either proper time (for time-like geodesics) or an affine parameter (null geodesics), and $\chi^\mu(\lambda)$ the coordinates of the points on the geodesic, then Nöther's Theorem gives

$$g_{\mu\nu}\xi^\mu p^\nu = -\left(1 - \frac{2}{\chi^r}\right)p^t = \mathrm{const} \equiv -e, \tag{4}$$

$$g_{\mu\nu}\zeta^\mu p^\nu = (\chi^r)^2 \sin^2(\chi^\theta)p^\phi = \mathrm{const} \equiv l \tag{5}$$

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along any geodesic. (e and l are the energy and angular momentum—per unit mass, if appropriate—of the particle traveling the geodesic measured by an observer at infinity.)

For a null geodesic, we also have the normalization condition

$$g_{\mu\nu}p^{\mu}p^{\nu} = 0. \tag{6}$$

Exploiting the rotational symmetry of the θ-ϕ subspace, we can, without loss of generality, work in the equatorial plane, considering only geodesics which have $\chi^{\theta}(\lambda) = \pi/2$. In this case, Equation 6 becomes

$$-\frac{e^2}{1 - \frac{2}{\chi^r}} + \frac{l^2}{(\chi^r)^2} + \frac{1}{1 - \frac{2}{\chi^r}}\left(\frac{d\chi^r}{d\lambda}\right)^2 = 0. \tag{7}$$

Rescaling the affine parameter, we obtain

$$\frac{d\chi^r(s)}{ds} = \pm\sqrt{1 - \frac{b^2\left(1 - \frac{2}{\chi^r(s)}\right)}{(\chi^r(s))^2}}, \tag{8}$$

with $b \equiv l/e$ called the "impact parameter"[1] of the trajectory. This equation and Equations 4 and 5 completely determine a trajectory given l and e.

3. INTERSECTION WITH THE SURFACE

For a given b, Equation 8 implies a point of closest approach to the mass at

$$\frac{d\chi^r(s_0)}{ds} = 0 = 1 - \frac{b^2\left(1 - \frac{2}{\chi^r(s_0)}\right)}{(\chi^r(s_0))^2}. \tag{9}$$

This gives a cubic equation for $\chi^r(s_0) \equiv r_0$, whose solution is not terribly illuminating. Note that the "point of closest approach" may be *inside* the star if the impact parameter is small enough; in this case the approach terminates at the stellar surface before achieving a close approach.

It is of interest to determine the impact parameter, b_{NS}, of the "grazing ray" whose point of closest approach, r_0, is equal to the radius of the neutron star, r_{NS}. Noting $r_{NS} \gg 2$ for physical neutron stars, we have

$$b_{NS}^2 = \frac{r_{NS}^2}{1 - \frac{2}{r_{NS}}} = r_{NS}^2\left(1 + \frac{2}{r_{NS}} + \frac{4}{r_{NS}^2} + \dots\right), \tag{10}$$

or

$$b_{NS} = r_{NS}\left(1 + \frac{1}{r_{NS}} + \frac{3}{2r_{NS}^2} + \dots\right). \tag{11}$$

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[1] b is so named because it gives the distance at infinity between the geodesic and the nearest parallel line running radially from the center of the star.

That the impact parameter of the grazing ray is *larger* than the neutron star radius is a result of gravitational lensing. It is interesting that the leading order addition to the impact parameter is one half the Schwarzschild radius.

We wish to determine the longitude of the intersection of a ray with impact parameter $b < b_{\mathrm{NS}}$ and the surface; this is χ^ϕ, the ϕ coordinate of the null geodesic, when the ray intersects the surface. To do this, it will be more convenient to parameterize the trajectory by χ^r. Using the trajectory derived in the last section (and remembering the scaling on the affine parameter s), we have

$$\frac{\mathrm{d}\chi^\phi}{\mathrm{d}\chi^r} = \frac{\mathrm{d}\chi^\phi/\mathrm{d}s}{\mathrm{d}\chi^r/\mathrm{d}s} = \frac{-b}{(\chi^r)^2\sqrt{1 - \frac{b^2\left(1-\frac{2}{\chi^r}\right)}{(\chi^r)^2}}}. \tag{12}$$

Thus, the longitude at "impact" is

$$\chi^\phi(r_{\mathrm{NS}}) = \int_{r_{\mathrm{NS}}}^\infty \mathrm{d}u\, \frac{b}{u^2\sqrt{1 - \frac{b^2\left(1-\frac{2}{u}\right)}{u^2}}}. \tag{13}$$

(The impact parameter of the grazing ray is $b_{\mathrm{NS}} = r_{\mathrm{NS}}(1 + 1/r_{\mathrm{NS}} + \ldots)$, so the argument of the square root approaches zero for the grazing ray at impact, but otherwise is guaranteed to be positive.)

It will be convenient to transform variables from u to $w = u/r_{\mathrm{NS}}$, whence

$$\chi^\phi(r_{\mathrm{NS}}) = \int_1^\infty \mathrm{d}w\, \frac{b}{r_{\mathrm{NS}}w^2\sqrt{1 - \frac{b^2\left(1-\frac{2}{r_{\mathrm{NS}}w}\right)}{r_{\mathrm{NS}}^2 w^2}}}. \tag{14}$$

It is also convenient to make the substitution

$$b = x b_{\mathrm{NS}} = x r_{\mathrm{NS}}\left(1 + \frac{1}{r_{\mathrm{NS}}} + \frac{3}{2r_{\mathrm{NS}}^2} + \ldots\right), \tag{15}$$

whence $-1 < x < 1$ for rays that impact the neutron star surface. This produces

$$\chi^\phi(r_{\mathrm{NS}}) = \int_1^\infty \mathrm{d}w\, \frac{x\left(1 + \frac{1}{r_{\mathrm{NS}}} + \ldots\right)}{w^2\sqrt{1 - \frac{x^2\left(1+\frac{2}{r_{\mathrm{NS}}}+\ldots\right)\left(1-\frac{2}{r_{\mathrm{NS}}w}\right)}{w^2}}}. \tag{16}$$

Expanding to leading order in $1/r_{\mathrm{NS}}$, we obtain

$$\chi^\phi(r_{\mathrm{NS}}) = \arcsin x + \frac{2\left(1 - \sqrt{1-x^2}\right)}{x r_{\mathrm{NS}}} + \ldots. \tag{17}$$

We see that the grazing ray $(x = 1)$ impacts the stellar surface at longitude

$$\phi_{\mathrm{NS}} = \frac{\pi}{2} + \frac{2}{r_{\mathrm{NS}}} + \ldots = \frac{\pi}{2} + \frac{r_{\mathrm{Schw}}}{r_{\mathrm{NS}}} + \ldots \tag{18}$$

It will be useful to have the momentum of the trajectory at the point of impact, when $\chi^r = r_{\text{NS}}$, $\chi^\theta = \pi/2$, and χ^ϕ given in Eq. (17). From Eq. (4), we see that

$$p^t = \frac{e}{1 - \frac{2}{r_{\text{NS}}}} \simeq e\left(1 + \frac{2}{r_{\text{NS}}} + \ldots\right);$$
(19)

from Eq. (5), and replacing the specific angular momentum l with the impact parameter $b = l/e$, we see that

$$p^\phi = \frac{be}{r_{\text{NS}}^2} = \frac{xe}{r_{\text{NS}}}\left(1 + \frac{1}{r_{\text{NS}}} + \ldots\right);$$
(20)

and from Eq. (7), we see that

$$p^r = e\sqrt{1 + \left(\frac{2 + 3r_{\text{NS}}}{r_{\text{NS}}^3} - 1\right)x^2} \simeq e\sqrt{1 - x^2}\left(1 + \frac{3x^2}{2\left(1 - x^2\right)r_{\text{NS}}^2} + \ldots\right).$$
(21)

For a ray that does not approach the surface in the equatorial plane, but rather strikes the surface at a colatitude θ_0 (but still with impact parameter $b = xb_{\text{NS}}$), the tangent vector ϕ component rotates into θ and ϕ components, and the tangent is

$$p^t \simeq e\left(1 + \frac{2}{r_{\text{NS}}} + \ldots\right)$$
(22)

$$p^r \simeq e\sqrt{1 - x^2}\left(1 + \ldots\right)$$
(23)

$$p^\theta \simeq \frac{xe\cos\theta_0}{r_{\text{NS}}}\left(1 + \frac{1}{r_{\text{NS}}} + \ldots\right)$$
(24)

$$p^\phi \simeq \frac{xe}{r_{\text{NS}}}\left(1 + \frac{1}{r_{\text{NS}}} + \ldots\right),$$
(25)

where we have kept only terms $\mathcal{O}\left(1/r_{\text{NS}}\right)$ beyond leading order.

4. RELATIVISTIC EFFECTS ON ENERGIES

For a neutron star that is observed to pulse with period P, observers at fixed position on the surface follow coordinate trajectories that have

$$\chi^\theta\left(\lambda\right) = \theta_0$$
(26)

$$\chi^r\left(\lambda\right) = r_{\text{NS}}$$
(27)

$$\chi^\phi\left(\lambda\right) = \frac{2\pi\chi^t\left(\lambda\right)}{P}$$
(28)

The last equation follows from the fact that P is the proper time difference between pulses arriving at infinity at fixed location on the celestial sphere; and thus the *coordinate* time difference between pulses is $\Delta t = P$. (Beware! This is not the *proper* time between pulses as measured by observers fixed to the neutron star surface; this latter quantity differs by a gravitational redshift, a time dilation due to rotational motion, and a redshift due to the radial acceleration of the rotational motion.)

The final coordinate trajectory for the fixed-position observer can be obtained from the normalization condition:

$$g_{\mu\nu} \frac{\mathrm{d}\chi^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}\chi^\nu}{\mathrm{d}\lambda} = -1, \tag{29}$$

which fixes

$$\frac{\mathrm{d}\chi^t}{\mathrm{d}\lambda} = \frac{1}{\sqrt{1 - \frac{2}{r_{\mathrm{NS}}} - v_{\mathrm{rot}}^2 \sin^2 \theta_0}} = 1 + \frac{1}{2} v_{\mathrm{rot}}^2 \sin^2 \theta_0 + \frac{1 + \frac{3}{2} v_{\mathrm{rot}}^2 \sin^2 \theta_0}{r_{\mathrm{NS}}} + \ldots, \tag{30}$$

where the last expansion is to first contributing order in $v_{\mathrm{rot}}^2 = (2\pi r_{\mathrm{NS}}/P)^2$ and $1/r_{\mathrm{NS}}$. In the expansion you can see the effects of special relativistic time dilation, the gravitational redshift, and the radial acceleration (the latter being proportional to $v_{\mathrm{rot}}^2/r_{\mathrm{NS}}$). Since this expression is constant, the coordinate time of the observer increases linearly (but not with coefficient equal to one) with proper time:

$$\Delta\chi^t = \Delta\lambda \frac{\mathrm{d}\chi^t}{\mathrm{d}\lambda}. \tag{31}$$

For a photon striking the surface of the neutron star with momentum $p^\mu (e, l)$, the energy measured by the observer fixed to the surface is

$$-e_{\mathrm{surf}} = g_{\mu\nu} \frac{\mathrm{d}\chi^\mu}{\mathrm{d}\lambda} p^\nu. \tag{32}$$

If we reverse the spatial components of the momentum, then for a photon *emitted* from the surface with energy e_{surf} in the frame fixed to the stellar surface (e.g. thermally from particles on the surface), the above equation relates the emitted energy to the energy as measured by observers at infinity (e).

Plugging in Eqs. (22), (23), (24), and (25), remembering to reverse the spatial components of the momentum, we obtain the relation between the energy observed at infinite distance (e) and the emitted energy at the surface (e_{surf}) for a ray with impact factor $b = x b_{\mathrm{NS}}$ hitting the surface of the neutron star at colatitude θ_0:

$$\frac{e}{e_{\mathrm{surf}}} = 1 - y v_{\mathrm{rot}} \sin \theta_0 + \left(y^2 - \frac{1}{2} \right) v_{\mathrm{rot}}^2 \sin^2 \theta_0$$
$$- \frac{1}{r_{\mathrm{NS}}} \left(1 - \left(y^2 - \frac{1}{2} \right) v_{\mathrm{rot}}^2 \sin^2 \theta_0 \right) + \ldots, \tag{33}$$

where $y = x \sin \theta_0$ is the "equatorial" component of the (dimensionless) impact factor. (Recall $x, y < 0$ for points west of the meridian and $x, y > 0$ for points to the east; $\sin \theta_0 > 0$ always because $0 < \theta_0 < \pi$.)

The velocity-independent term in Eq. (33) gives the gravitational redshift; the term linear in velocity gives the doppler shift from the rotational motion (blueshift—increased energy—west of the meridian, redshift—decreased energy—for points to the east); and the term quadratic in the velocity comes from relativistic beaming.

5. CALCULATING THE TIME DELAY

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1 - \frac{2}{r})dt^2 + \frac{1}{1 - \frac{2}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{34}$$

$$p^\mu = \frac{d\chi^\mu(\lambda)}{d\lambda} \tag{35}$$

In order to calculate the time delay, we need to calculate $\frac{d\chi^t(\lambda)}{d\lambda}$ and $\frac{d\chi^r(\lambda)}{d\lambda}$. Then, the time delay can be calculated as

$$\Delta t = \int_{r_{NS}}^\infty dr \left(\frac{d\chi^t}{d\chi^r} - \frac{d\chi^t}{d\chi^r}|_{b=0} \right) \tag{36}$$

Let's calculate $\frac{d\chi^t(\lambda)}{d\lambda}$ and $\frac{d\chi^r(\lambda)}{d\lambda}$ assuming $\theta = \pi/2$ and therefore $p^\theta = 0$. The we have in null geodesic:

$$-(1 - \frac{2}{r})(p^t)^2 + \frac{1}{1 - \frac{2}{r}}(p^r)^2 + r^2\sin^2\theta(p^\phi)^2 = 0 \tag{37}$$

Solving eq. 37 leads to $p^t = \frac{d\chi^t(\lambda)}{d\lambda} = \frac{e}{1 - \frac{2}{r}}$ and

$$\left(\frac{d\chi^r(\lambda)}{d\lambda} \right)^2 = \left(\frac{e^2}{1 - \frac{2}{r}} - \frac{l^2}{r^2} \right) \left(1 - \frac{2}{r} \right) \tag{38}$$

which, in terms of b becomes

$$\frac{d\chi^r(\lambda)}{d\lambda} = e\sqrt{\left(\frac{1}{1 - \frac{2}{r}} - \frac{b^2}{r^2} \right) \left(1 - \frac{2}{r} \right)} \tag{39}$$

Therefore, the integral 36 becomes:

$$\Delta t = \int_{r_{NS}}^\infty dr \left[\left(\frac{1}{1 - \frac{2}{r}} \right) - \frac{1}{\sqrt{\left(1 - \frac{2}{r} \right)^2 - \frac{b^2}{r^2} \left(1 - \frac{2}{r} \right)^3}} \right] \tag{40}$$

Following the substitution in Eq. 14, we then get

$$\Delta t = r_{NS} \int_1^\infty dw \left[\left(\frac{1}{1 - \frac{2}{wr_{NS}}} \right) - \frac{1}{\sqrt{\left(1 - \frac{2}{wr_{NS}} \right)^2 - \frac{b^2}{(wr_{NS})^2} \left(1 - \frac{2}{wr_{NS}} \right)^3}} \right] \tag{41}$$

It is also convenient to make the substitution

$$b = xb_{NS} = xr_{NS}\left(1 + \frac{1}{r_{NS}} + \frac{3}{2r_{NS}^2} + \dots \right), \tag{42}$$

which produces

$$\Delta t = r_{NS} \int_1^\infty dw \left[\left(\frac{1}{1 - \frac{2}{wr_{NS}}} \right) - \frac{1}{\sqrt{\left(1 - \frac{2}{wr_{NS}}\right)^2 - \frac{x^2 r_{NS}^2 \left(1 + \frac{1}{r_{NS}} + \frac{3}{2r_{NS}^2} + \ldots\right)^2}{(wr_{NS})^2} \left(1 - \frac{2}{wr_{NS}}\right)^3}} \right] \quad (43)$$

Substituting $1/r_{NS} = y$, and expanding around $y = 0$, the integral gives

$$\Delta t = r_{NS} \left(-1 + \sqrt{1 - x^2} \right)$$
$$\times \left(1 + \frac{1}{r_{NS}} \left(1 + \frac{2}{-1 + \sqrt{1 - x^2}} \log \left(\frac{x^2}{2 - 2\sqrt{1 - x^2}} \right) \right) + \mathcal{O} \left(\frac{1}{r_{NS}^2} \right) \right) \quad (44)$$

where $-1 < x < 1$.

6. CREATING MATRIX E FOR THE ENERGY FACTOR

To express the expression in 33 in matrix form using the spherical harmonic basis, we need to define matrices that correspond to the spherical harmonic functions and their associated coefficients.

Let's denote the spherical harmonic basis as $Y_{lm}(\theta, \phi)$. The spherical harmonic basis is expressed as:

$$\mathsf{Y_{lm}}(\theta, \phi) = \mathsf{P_{lm}}(\theta, \phi) \cdot e^{i \cdot m \cdot \phi} \quad (45)$$

where $P_{lm}(\theta, \phi)$ are the associated Legendre polynomials.

To express the given expression in matrix form using the spherical harmonic basis, we need to expand the function $F(y, v_{rot}, \theta_0, r_s)$ into a series of spherical harmonic functions and their coefficients. The general form of the spherical harmonic expansion is:

$$\mathsf{F(y, v_{rot}, \theta_0, r_s)} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathsf{C_{lm} Y_{lm}}(\theta, \phi) \quad (46)$$

Now, the spherical harmonic expansion involves integrating F against each spherical harmonic function Y_{lm}. The coefficients C_{lm} can be found using the following formula:

$$\mathsf{C_{lm}} = \int \mathsf{F(y, v_{rot}, \theta_0, r_s)} \cdot \mathsf{Y_{lm}^*}(\theta, \phi) \, d\Omega \quad (47)$$

Here's how the matrix elements are formed:

$$\mathsf{C_{lm}} = \int \mathsf{F(y, v_{rot}, \theta_0, r_s)} \cdot \mathsf{Y_{lm}^*}(\theta, \phi) \, d\Omega \quad (48)$$

For C_{00} (the $l = 0, m = 0$ component):

$$C_{00} = \int F(y, v_{\rm rot}, \theta_0, r_s) \cdot Y_{00}^*(\theta, \phi) \, d\Omega = \int F(y, v_{\rm rot}, \theta_0, r_s) \, d\Omega \qquad (49)$$

In the spherical harmonic basis, we can express the coefficients of the basis functions in matrix form:

$$\mathbf{C} = \begin{bmatrix} 1 & -v_{\rm rot} \sin \theta_0 & v_{\rm rot}^2 \sin^2 \theta_0 - \frac{1}{r_s} & 0 & \cdots \\ 0 & y\sqrt{2}v_{\rm rot} \sin \theta_0 & 0 & \cdots & \cdots \\ 0 & 0 & (y^2 - \frac{1}{2})\sqrt{6}v_{\rm rot}^2 \sin^2 \theta_0 & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \qquad (50)$$

Here, the rows of the matrix correspond to different values of "l" and the columns correspond to different values of "m". The diagonal elements of the matrix correspond to the coefficients of the spherical harmonic basis functions with the same "l" and "m" values, while off-diagonal elements are zero.

Spherical harmonics for $m = -1$, $m = 0$, and $m = 1$:

$$Y_{l,-1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} e^{-i\phi} \sin \theta$$

$$Y_{l,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{l,1}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} e^{i\phi} \sin \theta$$

Matrix elements for $m = -1$:

$$M_{-1} = -\sqrt{\frac{3}{8\pi}} v_{\rm rot} e^{-i\phi_0} \sin \theta_0$$

Matrix elements for $m = 0$:

$$M_0 = -\frac{\sqrt{6}}{2} v_{\rm rot}^2 e^{-2i\phi_0} \sin^2 \theta_0$$

Matrix elements for $m = 1$:

$$M_1 = -\sqrt{\frac{3}{8\pi}} v_{\rm rot}^2 e^{-2i\phi_0} \sin^2 \theta_0$$

$$\begin{bmatrix} 1 & -yv_{\rm rot} \sin \theta_0 & (y^2 - \frac{1}{2})v_{\rm rot}^2 \sin^2 \theta_0 - \frac{1}{r_s}(1 - (y^2 - \frac{1}{2})v_{\rm rot}^2 \sin^2 \theta_0) \\ 0 & -\sqrt{\frac{3}{8\pi}} v_{\rm rot} e^{-i\phi_0} \sin \theta_0 & -\frac{\sqrt{6}}{2} v_{\rm rot}^2 e^{-2i\phi_0} \sin^2 \theta_0 \\ 0 & 0 & -\sqrt{\frac{15}{8\pi}} v_{\rm rot}^2 e^{-2i\phi_0} \sin^2 \theta_0 \end{bmatrix} \qquad (51)$$

REFERENCES