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Lensing in the Schwarzschild Spacetime Applicable to Neutron Stars

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ABSTRACT

I describe the effects of gravitational lensing on the image of the surface of a spherical object whose surface is well outside the Schwarszchild horizon.

1. INTRODUCTION

2. GEODESICS

The Schwarzchild line element in standard coordinates and natural units is

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1 - \frac{2}{r}\right)\mathrm{d}t^2 + \frac{1}{1-\frac{2}{r}}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right); \tag{1}$$

t is the clock time of an observer at $r = \infty$, and $4\pi r^2$ is the area of the constant-r, constant-t hypersurfaces. The Schwarzschild radius (the coordinate of the event horizon) is $r_{\mathrm{Schw}} = 2$ in these units.

The metric has two killing vectors, ∂_t and ∂_ϕ, since it is independent of the corresponding coordinates. Write

$$\partial_t = \xi^\mu \partial_\mu, \qquad \partial_\phi = \zeta^\mu \partial_\mu; \tag{2}$$

then the only non-zero component of ξ^μ is $\xi^t = 1$, and the only non-zero component of ζ^μ is $\zeta^\phi = 1$. If p^μ is the momentum associated with a geodesic,

$$p^\mu = \frac{\mathrm{d}\chi^\mu(\lambda)}{\mathrm{d}\lambda}, \tag{3}$$

with λ either proper time (for time-like geodesics) or an affine parameter (null geodesics), and $\chi^\mu(\lambda)$ the coordinates of the points on the geodesic, then Nöther's Theorem gives

$$g_{\mu\nu}\xi^\mu p^\nu = -\left(1 - \frac{2}{\chi^r}\right)p^t = \mathrm{const} \equiv -e, \tag{4}$$

$$g_{\mu\nu}\zeta^\mu p^\nu = (\chi^r)^2 \sin^2(\chi^\theta) p^\phi = \mathrm{const} \equiv L \tag{5}$$

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along any geodesic. (e and L are the energy and angular momentum—per unit mass, if appropriate—of the particle traveling the geodesic measured by an observer at infinity.)

For a null geodesic, we also have the normalization condition

$$g_{\mu\nu}p^{\mu}p^{\nu} = 0. \tag{6}$$

Exploiting the rotational symmetry of the θ-ϕ subspace, we can, without loss of generality, work in the equatorial plane, considering only geodesics which have $\chi^{\theta}(\lambda) = \pi/2$. In this case, Equation 6 becomes

$$-\frac{e^2}{1-\frac{2}{\chi^r}} + \frac{l^2}{(\chi^r)^2} + \frac{1}{1-\frac{2}{\chi^r}}\left(\frac{\mathrm{d}\chi^r}{\mathrm{d}\lambda}\right)^2 = 0. \tag{7}$$

Rescaling the affine parameter, we obtain

$$\frac{\mathrm{d}\chi^r(s)}{\mathrm{d}s} = \pm\sqrt{1 - \frac{b^2\left(1-\frac{2}{\chi^r(s)}\right)}{(\chi^r(s))^2}}, \tag{8}$$

with $b \equiv L/e$ called the "impact parameter"[1] of the trajectory. This equation and Equations 4 and 5 completely determine a trajectory given L and e.

3. INTERSECTION WITH THE SURFACE

For a given b, Equation 8 implies a point of closest approach to the mass at

$$\frac{\mathrm{d}\chi^r(s_0)}{\mathrm{d}s} = 0 = 1 - \frac{b^2\left(1-\frac{2}{\chi^r(s_0)}\right)}{(\chi^r(s_0))^2}. \tag{9}$$

This gives a cubic equation for $\chi^r(s_0) \equiv r_0$, whose solution is not terribly illuminating. Note that the "point of closest approach" may be *inside* the star if the impact parameter is small enough; in this case the approach terminates at the stellar surface before achieving a close approach.

It is of interest to determine the impact parameter, b_{NS}, of the "grazing ray" whose point of closest approach, r_0, is equal to the radius of the neutron star, r_{NS}. Noting $r_{\mathrm{NS}} \gg 2$ for physical neutron stars, we have

$$b_{\mathrm{NS}}^2 = \frac{r_{\mathrm{NS}}^2}{1-\frac{2}{r_{\mathrm{NS}}}} = r_{\mathrm{NS}}^2\left(1 + \frac{2}{r_{\mathrm{NS}}} + \frac{4}{r_{\mathrm{NS}}^2} + \ldots\right), \tag{10}$$

or

$$b_{\mathrm{NS}} = r_{\mathrm{NS}}\left(1 + \frac{1}{r_{\mathrm{NS}}} + \frac{3}{2r_{\mathrm{NS}}^2} + \ldots\right). \tag{11}$$

[1] b is so named because it gives the distance at infinity between the geodesic and the nearest parallel line running radially from the center of the star.

That the impact parameter of the grazing ray is *larger* than the neutron star radius is a result of gravitational lensing. It is interesting that the leading order addition to the impact parameter is one half the Schwarzschild radius.

We wish to determine the longitude of the intersection of a ray with impact parameter $b < b_{\rm NS}$ and the surface; this is χ^ϕ, the ϕ coordinate of the null geodesic, when the ray intersects the surface. To do this, it will be more convenient to parameterize the trajectory by χ^r. Using the trajectory derived in the last section (and remembering the scaling on the affine parameter s), we have

$$\frac{\mathrm{d}\chi^\phi}{\mathrm{d}\chi^r} = \frac{\mathrm{d}\chi^\phi/\mathrm{d}s}{\mathrm{d}\chi^r/\mathrm{d}s} = \frac{-b}{(\chi^r)^2\sqrt{1-\frac{b^2\left(1-\frac{2}{\chi^r}\right)}{(\chi^r)^2}}}. \tag{12}$$

Thus, the longitude at "impact" is

$$\chi^\phi\left(r_{\rm NS}\right) = \int_{r_{\rm NS}}^{\infty} \mathrm{d}u\, \frac{b}{u^2\sqrt{1-\frac{b^2\left(1-\frac{2}{u}\right)}{u^2}}}. \tag{13}$$

(The impact parameter of the grazing ray is $b_{\rm NS} = r_{\rm NS}\left(1 + 1/r_{\rm NS} + \ldots\right)$, so the argument of the square root approaches zero for the grazing ray at impact, but otherwise is guaranteed to be positive.)

It will be convenient to transform variables from u to $w = u/r_{\rm NS}$, whence

$$\chi^\phi\left(r_{\rm NS}\right) = \int_{1}^{\infty} \mathrm{d}w\, \frac{b}{r_{\rm NS}w^2\sqrt{1-\frac{b^2\left(1-\frac{2}{r_{\rm NS}w}\right)}{r_{\rm NS}^2w^2}}}. \tag{14}$$

It is also convenient to make the substitution

$$b = xb_{\rm NS} = xr_{\rm NS}\left(1+\frac{1}{r_{\rm NS}}+\frac{3}{2r_{\rm NS}^2}+\ldots\right), \tag{15}$$

whence $-1 < x < 1$ for rays that impact the neutron star surface. This produces

$$\chi^\phi\left(r_{\rm NS}\right) = \int_{1}^{\infty} \mathrm{d}w\, \frac{x\left(1+\frac{1}{r_{\rm NS}}+\ldots\right)}{w^2\sqrt{1-\frac{x^2\left(1+\frac{2}{r_{\rm NS}}+\ldots\right)\left(1-\frac{2}{r_{\rm NS}w}\right)}{w^2}}}. \tag{16}$$

Expanding to leading order in $1/r_{\rm NS}$, we obtain

$$\chi^\phi\left(r_{\rm NS}\right) = \arcsin x + \frac{2\left(1-\sqrt{1-x^2}\right)}{xr_{\rm NS}} + \ldots. \tag{17}$$

We see that the grazing ray ($x = 1$) impacts the stellar surface at longitude

$$\phi_{\rm NS} = \frac{\pi}{2} + \frac{2}{r_{\rm NS}} + \ldots = \frac{\pi}{2} + \frac{r_{\rm Schw}}{r_{\rm NS}} + \ldots \tag{18}$$

It will be useful to have the momentum of the trajectory at the point of impact, when $\chi^r = r_{\rm NS}$, $\chi^\theta = \pi/2$, and χ^ϕ given in Eq. (17). From Eq. (4), we see that

$$p^t = \frac{e}{1-\frac{2}{r_{\rm NS}}} \simeq e\left(1+\frac{2}{r_{\rm NS}}+\dots\right); \tag{19}$$

from Eq. (5), and replacing the specific angular momentum l with the impact parameter $b = l/e$, we see that

$$p^\phi = \frac{be}{r_{\rm NS}^2} = \frac{xe}{r_{\rm NS}}\left(1+\frac{1}{r_{\rm NS}}+\dots\right); \tag{20}$$

and from Eq. (7), we see that

$$p^r = e\sqrt{1+\left(\frac{2+3r_{\rm NS}}{r_{\rm NS}^3}-1\right)x^2} \simeq e\sqrt{1-x^2}\left(1+\frac{3x^2}{2\left(1-x^2\right)r_{\rm NS}^2}+\dots\right). \tag{21}$$

So far we have worked in the equatorial plane, with $\theta \equiv \pi/2$; but because of the spherical symmetry of our spacetime, our results generalize. For a ray with two-dimensional cartesian impact parameter coordinates x and y; or, equivalently, with impact parameter magnitude $l = \sqrt{x^2+y^2}$ and angle with the positive x axis $\psi = \arctan y/x$, the impact point on the surface becomes

$$\chi^\theta\left(r_{\rm NS}\right) = \cos^{-1}\left(\sin\alpha\sin\psi\right) \tag{22}$$

$$\chi^\phi\left(r_{\rm NS}\right) = \tan^{-1}\left(\cos\psi\tan\alpha\right), \tag{23}$$

where the angle α (which corresponds to ϕ if the orbit is in the equatorial plane) is given in terms of the magnitude of the impact parameter vector l by

$$\alpha = \sin^{-1} l + \frac{2\left(1-\sqrt{1-l^2}\right)}{l r_{\rm NS}}. \tag{24}$$

[Work out the rotation of the p^ϕ momentum vector above.]

From the Equation 5, we can write

$$p^\phi = \frac{L}{(\chi^r)^2\sin^2(\chi^\theta)} = \frac{L}{(r_{\rm NS})^2\sin^2(\cos^{-1}\left(\sin\alpha\sin\psi\right))} \tag{25}$$

and after applying the normalization condition, we get

$$p^\theta = -\frac{\sqrt{\frac{l^2\left(\frac{e^2\left(l^2\left(3-2r_{\rm NS}^2\right)^2-4r_{\rm NS}^2\left(r_{\rm NS}^2-3\right)\right)}{(l^2-1)(r_{\rm NS}-2)}+\frac{4r_{\rm NS}}{\sin^2(\alpha)\sin^2(\psi)-1}\right)}{r_{\rm NS}^3}}}{2r_{\rm NS}} \tag{26}$$

or, if we expand in series, then we get

$$p^\theta = -\frac{\sqrt{e^2l^2}}{r_{\rm NS}} - \frac{\sqrt{e^2l^2}}{r_{\rm NS}^2} - \frac{\sqrt{e^2l^2}}{2r_{\rm NS}^3\left(e^2\left(\sin^2(\alpha)\sin^2(\psi)-1\right)\right)} + O\left(\left(\frac{1}{r_{\rm NS}}\right)^4\right) \tag{27}$$

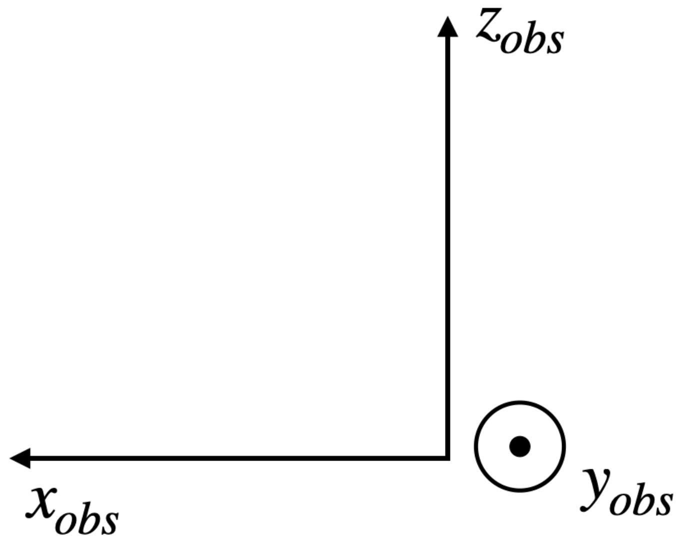

Figure 1. The coordinates in the observer frame.

4. RELATIVISTIC EFFECTS ON ENERGIES

For a neutron star that is observed to pulse with period P, observers at fixed position on the surface follow coordinate trajectories in the body frame that have

$$\chi^t(\lambda) \propto \lambda \tag{28}$$

$$\chi^\theta(\lambda) = \text{const} \tag{29}$$

$$\chi^r(\lambda) = r_{\text{NS}} \tag{30}$$

$$\chi^\phi(\lambda) = \frac{2\pi\chi^t(\lambda)}{P} \tag{31}$$

The last equation follows from the fact that P is the proper time difference between pulses arriving at infinity at fixed location on the celestial sphere; and thus the *coordinate* time difference between pulses is $\Delta t = P$. (Beware! This is not the *proper* time between pulses as measured by observers fixed to the neutron star surface; this latter quantity differs by a gravitational redshift, a time dilation due to rotational motion, and a redshift due to the radial acceleration of the rotational motion.)

To convert this world-line tangent vector to the *observer* coordinate system, let the inclination of the neutron star spin be ι and the obliquity (meassered ccw from "vertical" in the observer frame) be ω. Then the neutron star spin three-vector has cartesian components in the observer frame of

$$\vec{s} = (\cos\iota, -\sin\iota\sin\omega, \sin\iota\cos\omega)\,. \tag{32}$$

See the Figure 1. If the incoming ray strikes the surface at a point with colatitude θ_0

and longitude ϕ_0, then the vector pointing to the impact point has cartesian components in the observer frame of

$$\vec{r}_0 = (\cos\phi_0 \sin\theta_0, \sin\phi_0 \sin\theta_0, \cos\theta_0)\,. \tag{33}$$

The instantaneous world-line tangent of the observer fixed to the NS surface at the point of impact is given in cartesian coordinates by

$$\vec{\tau}_0 = \frac{2\pi r_{NS}}{P}\frac{\partial\chi^t}{\partial\lambda}\vec{s}\times\vec{r}_0. \tag{34}$$

The cross product is

$$\begin{aligned}\vec{s}\times\vec{r}_0 = (&-\sin\iota\sin\omega\cos\theta_0 - \sin\iota\cos\omega\sin\phi_0\sin\theta_0,\\ &\cos\iota\cos\theta_0 - \sin\iota\cos\omega\cos\phi_0\sin\theta_0,\\ &\cos\iota\sin\phi_0\sin\theta_0 + \sin\iota\sin\omega\cos\phi_0\sin\theta_0)\end{aligned}$$

Plugging in the cross product into the equation 34 and solving for $r-, \theta-, \phi-$ components of the vector, then, in the observer spherical coordinate system the components of the tangent vector are given by

$$\frac{\mathrm{d}\chi^r}{\mathrm{d}\lambda} = 0 \tag{35}$$

$$\frac{\mathrm{d}\chi^\theta}{\mathrm{d}\lambda} = \frac{\mathrm{d}\chi^t}{\mathrm{d}\lambda}\frac{2\pi}{P}\left(-\cos\iota\sin\phi_0 - \cos\phi_0\sin\iota\sin\omega\right) \tag{36}$$

$$\frac{\mathrm{d}\chi^\phi}{\mathrm{d}\lambda} = \frac{\mathrm{d}\chi^t}{\mathrm{d}\lambda}\frac{2\pi}{P}\left(-\cos\iota\cos\phi_0\cot\theta_0 + \sin\iota\left(\cos\omega + \cot\theta_0\sin\phi_0\sin\omega\right)\right) \tag{37}$$

The time-component of the tangent vector can be obtained from the normalization condition

$$g_{\mu\nu}\frac{\mathrm{d}\chi^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}\chi^\nu}{\mathrm{d}\lambda} = -1, \tag{38}$$

which gives to leading order in $1/r_{\mathrm{NS}}$ and $v_{\mathrm{rot}} \equiv 2\pi r_{\mathrm{NS}}/P$

$$\begin{aligned}&\frac{d\chi^t}{\mathrm{d}\lambda} = 1 + \frac{1}{r_{\mathrm{NS}}} - v_{\mathrm{rot}}^2\\ &\times\Big(-\sin^2(\iota)\left(\sin^2\theta_0\cos^2\omega + \sin^2\omega\left(\cos^2\theta_0\sin^2\phi_0 + \cos^2\phi_0\right) + 2\sin\theta_0\cos\theta_0\sin\omega\cos\omega\sin\phi_0\right)\\ &+2\sin\theta_0\sin\iota\cos\iota\cos\phi_0\left(\cos\theta_0\cos\omega - \sin\theta_0\sin\omega\sin\phi_0 - \left(\cos^2\iota\left(\cos^2\theta_0\cos^2\phi_0 + \sin^2\phi_0\right)\right)\right)\Big)\end{aligned} \tag{39}$$

[Edited up to here.]

For a photon striking the surface of the neutron star with momentum $p^\mu(e, l)$, the energy measured by the observer fixed to the surface is

$$-e_{\mathrm{surf}} = g_{\mu\nu}\frac{\mathrm{d}\chi^\mu}{\mathrm{d}\lambda}p^\nu. \tag{40}$$

If we reverse the spatial components of the momentum, then for a photon *emitted* from the surface with energy $e_{\rm surf}$ in the frame fixed to the stellar surface (e.g. thermally from particles on the surface), the above equation relates the emitted energy to the energy as measured by observers at infinity (e).

Plugging in Eqs. (??), (??), (??), and (??), remembering to reverse the spatial components of the momentum, we obtain the relation between the energy observed at infinite distance (e) and the emitted energy at the surface ($e_{\rm surf}$) for a ray with impact factor $b = xb_{\rm NS}$ hitting the surface of the neutron star at colatitude θ_0:

$$\frac{e}{e_{\rm surf}} = 1 - yv_{\rm rot}\sin\theta_0 + \left(y^2 - \frac{1}{2}\right) v_{\rm rot}^2 \sin^2\theta_0 - \frac{1}{r_{\rm NS}}\left(1 - \left(y^2 - \frac{1}{2}\right) v_{\rm rot}^2 \sin^2\theta_0\right) + \ldots, \quad (41)$$

where $y = x\sin\theta_0$ is the "equatorial" component of the (dimensionless) impact factor. (Recall $x, y < 0$ for points west of the meridian and $x, y > 0$ for points to the east; $\sin\theta_0 > 0$ always because $0 < \theta_0 < \pi$.)

The velocity-independent term in Eq. (41) gives the gravitational redshift; the term linear in velocity gives the doppler shift from the rotational motion (blueshift—increased energy—west of the meridian, redshift—decreased energy—for points to the east); and the term quadratic in the velocity comes from relativistic beaming.

5. CALCULATING THE TIME DELAY

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1-\frac{2}{r})dt^2 + \frac{1}{1-\frac{2}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (42)$$

$$p^\mu = \frac{d\chi^\mu(\lambda)}{d\lambda} \quad (43)$$

In order to calculate the time delay, we need to calculate $\frac{d\chi^t(\lambda)}{d\lambda}$ and $\frac{d\chi^r(\lambda)}{d\lambda}$. Then, the time delay can be calculated as

$$\Delta t = \int_{r_{NS}}^{\infty} dr \left(\frac{d\chi^t}{d\chi^r} - \frac{d\chi^t}{d\chi^r}|_{b=0}\right) \quad (44)$$

Let's calculate $\frac{d\chi^t(\lambda)}{d\lambda}$ and $\frac{d\chi^r(\lambda)}{d\lambda}$ assuming $\theta = \pi/2$ and therefore $p^\theta = 0$. The we have in null geodesic:

$$-(1-\frac{2}{r})(p^t)^2 + \frac{1}{1-\frac{2}{r}}(p^r)^2 + r^2\sin^2\theta(p^\phi)^2 = 0 \quad (45)$$

Solving eq. 45 leads to $p^t = \frac{d\chi^t(\lambda)}{d\lambda} = \frac{e}{1-\frac{2}{r}}$ and

$$\left(\frac{d\chi^r(\lambda)}{d\lambda}\right)^2 = \left(\frac{e^2}{1-\frac{2}{r}} - \frac{l^2}{r^2}\right)\left(1 - \frac{2}{r}\right) \quad (46)$$

which, in terms of b becomes

$$\frac{d\chi^r(\lambda)}{d\lambda} = e\sqrt{\left(\frac{1}{1-\frac{2}{r}} - \frac{b^2}{r^2}\right)\left(1-\frac{2}{r}\right)} \tag{47}$$

Therefore, the integral 44 becomes:

$$\Delta t = \int_{r_{NS}}^{\infty} dr \left[\left(\frac{1}{1-\frac{2}{r}}\right) - \frac{1}{\sqrt{\left(1-\frac{2}{r}\right)^2 - \frac{b^2}{r^2}\left(1-\frac{2}{r}\right)^3}}\right] \tag{48}$$

Following the substitution in Eq. 14, we then get

$$\Delta t = r_{NS}\int_{1}^{\infty} \mathrm{d}w \left[\left(\frac{1}{1-\frac{2}{wr_{NS}}}\right) - \frac{1}{\sqrt{\left(1-\frac{2}{wr_{NS}}\right)^2 - \frac{b^2}{(wr_{NS})^2}\left(1-\frac{2}{wr_{NS}}\right)^3}}\right] \tag{49}$$

It is also convenient to make the substitution

$$b = xb_{\mathrm{NS}} = xr_{\mathrm{NS}}\left(1+\frac{1}{r_{\mathrm{NS}}}+\frac{3}{2r_{\mathrm{NS}}^2}+\dots\right), \tag{50}$$

which produces

$$\Delta t = r_{NS}\int_{1}^{\infty} \mathrm{d}w \left[\left(\frac{1}{1-\frac{2}{wr_{NS}}}\right) - \frac{1}{\sqrt{\left(1-\frac{2}{wr_{NS}}\right)^2 - \frac{x^2 r_{\mathrm{NS}}^2\left(1+\frac{1}{r_{\mathrm{NS}}}+\frac{3}{2r_{\mathrm{NS}}^2}+\dots\right)^2}{(wr_{NS})^2}\left(1-\frac{2}{wr_{NS}}\right)^3}}\right] \tag{51}$$

Substituting $1/r_{NS} = y$, and expanding around $y = 0$, the integral gives

$$\begin{aligned}\Delta t = {} & r_{NS}\left(-1+\sqrt{1-x^2}\right)\\ & \times\left(1+\frac{1}{r_{NS}}\left(1+\frac{2}{-1+\sqrt{1-x^2}}\log\left(\frac{x^2}{2-2\sqrt{1-x^2}}\right)\right)+\mathcal{O}\left(\frac{1}{r_{NS}^2}\right)\right)\end{aligned} \tag{52}$$

where $-1 < x < 1$.

6. CREATING MATRIX E FOR THE ENERGY FACTOR

To express the expression in 41 in matrix form using the spherical harmonic basis, we need to define matrices that correspond to the spherical harmonic functions and their associated coefficients.

Let's denote the spherical harmonic basis as $Y_{lm}(\theta,\phi)$. The spherical harmonic basis is expressed as:

$$Y_{lm}(\theta,\phi) = P_{lm}(\theta,\phi)\cdot e^{i\cdot m\cdot\phi} \tag{53}$$

where $P_{lm}(\theta, \phi)$ are the associated Legendre polynomials.

To express the given expression in matrix form using the spherical harmonic basis, we need to expand the function $F(y, v_{\mathrm{rot}}, \theta_0, r_s)$ into a series of spherical harmonic functions and their coefficients. The general form of the spherical harmonic expansion is:

$$F(y, v_{rot}, \theta_0, r_s) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} Y_{lm}(\theta, \phi) \tag{54}$$

Now, the spherical harmonic expansion involves integrating F against each spherical harmonic function Y_{lm}. The coefficients C_{lm} can be found using the following formula:

$$C_{lm} = \int F(y, v_{rot}, \theta_0, r_s) \cdot Y_{lm}^*(\theta, \phi)\, d\Omega \tag{55}$$

Here's how the matrix elements are formed:

$$C_{lm} = \int F(y, v_{\mathrm{rot}}, \theta_0, r_s) \cdot Y_{lm}^*(\theta, \phi)\, d\Omega \tag{56}$$

For C_{00} (the $l = 0, m = 0$ component):

$$C_{00} = \int F(y, v_{\mathrm{rot}}, \theta_0, r_s) \cdot Y_{00}^*(\theta, \phi)\, d\Omega = \int F(y, v_{\mathrm{rot}}, \theta_0, r_s)\, d\Omega \tag{57}$$

In the spherical harmonic basis, we can express the coefficients of the basis functions in matrix form:

$$\mathbf{C} = \begin{bmatrix} 1 & -v_{\mathrm{rot}} \sin\theta_0 & v_{\mathrm{rot}}^2 \sin^2\theta_0 - \frac{1}{r_s} & 0 & \dots \\ 0 & y\sqrt{2} v_{\mathrm{rot}} \sin\theta_0 & 0 & \dots & \dots \\ 0 & 0 & (y^2 - \frac{1}{2})\sqrt{6} v_{\mathrm{rot}}^2 \sin^2\theta_0 & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{58}$$

Here, the rows of the matrix correspond to different values of "l" and the columns correspond to different values of "m". The diagonal elements of the matrix correspond to the coefficients of the spherical harmonic basis functions with the same "l" and "m" values, while off-diagonal elements are zero.

Spherical harmonics for $m = -1$, $m = 0$, and $m = 1$:

$$Y_{l,-1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} e^{-i\phi} \sin\theta$$

Symbol	**Description**	**Reference**
θ_0	colatitude on the surface of the star	
ϕ_0	longitude on the surface of the star	
L	the specific angular momentum	Equation 5
t	the clock time of an observer at $r = \infty$	–
r_{NS}	radius of a neutron star	
p^μ	the momentum associated with a geodesic	Equation 3
χ^μ	the coordinates of the points on the geodesic	–
λ	affine parameter	–
$g_{\mu\nu}$	components of the metric tensor	–
e	the specific energy	Equation 4
ω	the stellar obliquity	
ι	the stellar inclination	
ψ	the angular components of the impact parameter associated with the positive x-axis	

$$Y_{l,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{l,1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta$$

Matrix elements for $m = -1$:

$$M_{-1} = -\sqrt{\frac{3}{8\pi}}v_{\mathrm{rot}}e^{-i\phi_0}\sin\theta_0$$

Matrix elements for $m = 0$:

$$M_0 = -\frac{\sqrt{6}}{2}v_{\mathrm{rot}}^2 e^{-2i\phi_0}\sin^2\theta_0$$

Matrix elements for $m = 1$:

$$M_1 = -\sqrt{\frac{3}{8\pi}}v_{\mathrm{rot}}^2 e^{-2i\phi_0}\sin^2\theta_0$$

$$\begin{bmatrix} 1 & -yv_{\mathrm{rot}}\sin\theta_0 & (y^2-\frac{1}{2})v_{\mathrm{rot}}^2\sin^2\theta_0 - \frac{1}{r_s}(1-(y^2-\frac{1}{2})v_{\mathrm{rot}}^2\sin^2\theta_0) \\ 0 & -\sqrt{\frac{3}{8\pi}}v_{\mathrm{rot}}e^{-i\phi_0}\sin\theta_0 & -\frac{\sqrt{6}}{2}v_{\mathrm{rot}}^2e^{-2i\phi_0}\sin^2\theta_0 \\ 0 & 0 & -\sqrt{\frac{15}{8\pi}}v_{\mathrm{rot}}^2e^{-2i\phi_0}\sin^2\theta_0 \end{bmatrix} \tag{59}$$

7. NOTATIONS

REFERENCES