The WDM Wavelet Basis

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Some notes deriving the forward and inverse transforms used in the code. See [1] or [2] for full details.

The underlying "mother wavelet" for the WDM basis is compact in the frequency domain, has parameters A, B, and d, and is defined by

$$\tilde{\varphi}(f \mid A, B, d) \propto \begin{cases} 1, & |f| < A \\ \cos\left(\frac{\pi}{2}\nu_d\left(\frac{|f|-A}{B}\right)\right), & A \leq |f| < A + B, \\ 0, & |f| \geq A + B \end{cases}$$

where ν_d is the normalized incomplete Beta function,

$$\nu_d(x) = \frac{\int_0^x y^{d-1}(1-y)^{d-1}\,\mathrm{d}y}{\int_0^1 y^{d-1}(1-y)^{d-1}\,\mathrm{d}y}.$$

For a bandwidth ΔF, A and B satisfy $2A + B = \Delta F$; d controls the sharpness of the decay to zero at frequencies between A and $A + B$.

Given a choice of the number of time bins N_t, and number of frequency bins N_f, with $N_t N_f = N$, the nm ($n = 0, ..., N_t - 1$, $m = 0, ..., N_f$) component of the wavelet basis in the Fourier domain is given by

$$\tilde{g}_{nm}(f) := \begin{cases} e^{-\pi i n f \Delta T}\tilde{\varphi}(f), & n = 0 \\ e^{-2\pi i n f \Delta T}\left(C_{nm}\tilde{\varphi}(f - m\Delta F) + C_{nm}^*\tilde{\varphi}(f + m\Delta F)\right), & n > 0 \end{cases}$$

with

$$C_{nm} = \begin{cases} 1, & n + m \text{ even} \\ i, & n + m \text{ odd} \end{cases},$$

and $\Delta T \Delta F = \frac{1}{2}$. Each basis element corresponds to a version of the mother wavelet time-shifted by $n\Delta T$ ($m > 0$) or $2n(\Delta T)$ ($m = 0$) and centered at frequencies $\pm m\Delta F$. For a uniformly sampled data segment of length N, over a time T, we have the definitions

$$\delta t = \frac{T}{N}$$
$$\delta f = \frac{1}{T}$$
$$\Delta T = N_f \delta t$$
$$\Delta F = \frac{1}{2\Delta T} = \left(\frac{N_t}{2}\right)\delta f,$$

so that the parameters N_t and N_f scale the frequency and time resolution of the wavelet basis.

The treatment of the zero-frequency ($m = 0$) and Nyquist frequency ($m = N_f$) bands is special —and I don't quite understand it. Currently the code does not handle the $m = 0$ wavelet band at all (enforces zeros in both directions).

The forward transform can be derived by writing

$$\begin{aligned} w_{nm} &= \sum_k x_k g_{nm}(k\delta t) \\ &= \sum_k \left(\frac{1}{N^2}\right)\left(\sum_l \tilde{x}_l e^{2\pi i \frac{kl}{N}}\right)\left(\sum_j \tilde{g}_{nm}(j\delta f) e^{2\pi i \frac{kj}{N}}\right). \end{aligned}$$

Performing the sum over k gives N times a delta function enforcing $j + l = N$; summing over j yields $j = N - l$, and we have

$$w_{nm} = \sum_l \tilde{x}_l \tilde{g}_{nm}((N - l)\delta f).$$

Inserting the defition of $\tilde{g}_{nm}$ in terms of the mother wavelet gives

$$w_{nm} = \sum_l e^{-2\pi i n((N-l)\delta f)\Delta T} \tilde{x}_l (C_{nm} \tilde{\varphi}((N - l)\delta f - m\Delta F) + C^*_{nm} \tilde{\varphi}((N - l)\delta f + m\Delta F)).$$

Simplifying the exponential, recalling that $\delta f \Delta T = \frac{1}{N_t}$ gives

$$w_{nm} = \sum_l e^{2\pi i \frac{nl}{N_t}} \tilde{x}_l (C_{nm} \tilde{\varphi}((N - l)\delta f - m\Delta F) + C^*_{nm} \tilde{\varphi}((N - l)\delta f + m\Delta F))$$

The arguments of the frequency domain mother wavelets simplify to

$$(N - l)\delta f - m\Delta F = \left(N - l - m\frac{N_t}{2}\right)\delta f \sim \left(l + m\frac{N_t}{2}\right)\delta f$$

and

$$(N - l)\delta f + m\Delta F = \left(N - l + m\frac{N_t}{2}\right)\delta f \sim \left(l - m\frac{N_t}{2}\right)\delta f,$$

where we have used the fact that, in a Fourier transform of the mother wavelet with frequency resolution δf, the coefficients at index i and $N-i$ are equal to express the final equivalence.

Thus we have

$$w_{nm} = \sum_l e^{2\pi i \frac{nl}{N_t}} \tilde{x}_l \left(C_{nm} \tilde{\varphi}_{l+m\frac{N_t}{2}} + C_{nm}^* \tilde{\varphi}_{l-m\frac{N_t}{2}} \right),$$

where $\tilde{\varphi}_k$ is the kth coefficient of the length-N Fourier transform of the mother wavelet. Conveniently, the exponential is periodic in l with period N_t, and due to the band-limited structure of the mother wavelet, $\tilde{\varphi}_k$ is non-zero only for $0 \leq k \leq \frac{N_t}{2}$ and $N - \frac{N_t}{2} < k < N$.

Since the transform is real, we are free to compute the transform of one of these terms and then double the real part, so we have

$$w_{nm} = 2\mathbb{R} C_{nm}^* y_{nm},$$

where

$$y_{nm} = \sum_l e^{2\pi i \frac{nl}{N_t}} \tilde{x}_l \tilde{\varphi}_{l-m\frac{N_t}{2}}.$$

The quantity y_{nm} can be computed for each m by inverse Fourier transforming the length N_t slice of $\tilde{x}$ centered at a zero frequency index at $l = m\frac{N_t}{2}$ extending from $(m-1)\frac{N_t}{2} < l \leq (m+1)\frac{N_t}{2}$ against the corresponding frequencies of the mother wavelet and storing the result into the n entries. There are N_f such length-N_t FFTs (again, ignoring the $m=0$ band, which must be treated differently).

The inverse transform is given similarly, starting from

$$\begin{aligned}
x_k &= \sum_{nm} w_{nm} g_{nm}(k\delta t) \\
&= \sum_{nm} w_{nm} \frac{1}{N} \sum_l \tilde{g}_{nm}(l\delta f) e^{2\pi i \frac{kl}{N}} \\
&= \sum_{nm} w_{nm} \frac{1}{N} \sum_l e^{2\pi i \frac{kl}{N}} e^{-2\pi i \frac{nl}{N_t}} \left(C_{nm} \tilde{\varphi}_{l-m\frac{N_t}{2}} + C_{nm}^* \tilde{\varphi}_{l+m\frac{N_t}{2}} \right) \\
&= \sum_l \frac{1}{N} e^{2\pi i \frac{kl}{N}} \sum_m \left(\tilde{\varphi}_{l-m\frac{N_t}{2}} \tilde{v}_{lm} + \tilde{\varphi}_{l+m\frac{N_t}{2}} \tilde{v}_{(N-l)m} \right),
\end{aligned}$$

where

$$\tilde{v}_{lm} = \sum_n e^{-2\pi i \frac{nl}{N_t}} w_{nm} C_{nm}$$

is the Fourier transform of w on C over the n index. Because w_{nm} is real, the C_{nm}^* term is the complex conjugate of the C_{nm} term, and therefore

$$\sum_n e^{-2\pi i \frac{nl}{N_t}} w_{nm} C^*_{nm} = \tilde{v}_{(N-l)m}.$$

Let

$$\tilde{x}_l = \sum_m \Big(\tilde{\varphi}_{l-m\frac{N_t}{2}} \tilde{v}_{lm} + \tilde{\varphi}_{l+m\frac{N_t}{2}} \tilde{v}_{(N-l)m} \Big);$$

$\tilde{x}$ can be computed by packing the results of m length-N_t Fourier transforms that produce $\tilde{v}$ into the appropriate locations of the l index. Then the time-domain signal is given by

$$x_k = \frac{1}{N} \sum_l e^{2\pi i \frac{kl}{N}} \tilde{x}_l,$$

a single length-N inverse Fourier transform.

Bibliography

[1] N. J. Cornish, Time-frequency analysis of gravitational wave data, Phys. Rev. D **102**, 124038 (2020).

[2] V. Necula, S. Klimenko, and G. Mitselmakher, *Transient analysis with fast Wilson-Daubechies time-frequency transform*, in *Journal of Physics Conference Series*, Vol. 363 (IOP, 2012), p. 12032.