A Sufficient Condition for two Gravitational Waveforms to Furnish Equivalent Parameter Estimates

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Abstract—I derive a condition on the difference between two gravitational waveforms that is sufficient to guarantee that they will furnish equivalent parameter estimates in a Bayesian analysis. The condition is that the log-likelihood difference between the two waveforms at the same point in parameter space is small. I show that this condition is equivalent to the difference between the two waveforms being small compared to the signal-to-noise ratio of the waveforms; the condition does not require that either waveform be a good model for any signal in the data.

1. Introduction

TODO: dump some references here.

2. Bounding the Log Likelihood Difference

Consider the difference in log-likelihood between two waveforms g and h given data d (we will imagine that g and h are two different *models* for the waveform that is actually contained in d, but our argument here is fully general). We can write

$$\Delta \log \mathcal{L} := \log \mathcal{L}_h - \log \mathcal{L}_g = -\frac{1}{2}(< d-h|d-h > - < d-g \mid d-g >). \tag{1}$$

A sufficient condition that the two models are equivaent in a parameter analysis in some region in parameter space is that $\mid \Delta \log \mathcal{L}| \ll 1$ for parameter values in that region; we may refer to this as “pointwise equivalence” because the log-likelihood difference is evaluated at corresponding points in parameter space.

Applying the law of cosines, Equation 22, with $a = d - g$, $b = g - h$, and $c = d - h$, the log likelihood becomes

$$\Delta \log \mathcal{L} = -\frac{1}{2} < g-h|g-h > - < d-g|g-h > \tag{2}$$

Let us suppose that the two waveforms of interest, h and g, are close, so that we can write $g = h + \delta h$ with $|\delta h| \ll 1$. Then we see that the first term in Equation 2 is second order in $|\delta h|$; we will neglect it from here onward. Expanding the second term, and preserving only to first-order in $|\delta h|$, we have

$$|\Delta \log \mathcal{L}| \simeq | < d-h \mid \delta h > |. \tag{3}$$

In words: the change in the log likelihood is the projection of the residuals onto the wavefrom difference.

At this point, due to the presence of d in Equation 3 we cannot say much more in general (it is, in principle, possible for the projection of $d-h$ onto δh to be essentially arbitrarily large). But if we are willing to assume that the data contain a waveform H that is reasonably close to h (and g)

$$d = H + n, \tag{4}$$

plus noise n that is consistent with the spectral density used to define our inner product, then

$$\mid \Delta \log \mathcal{L} \mid \simeq \mid < n \mid \delta h > + < H \mid \delta h > - < h \mid \delta h > \mid. \tag{5}$$

(Note that it is not the case that $< H - h \mid H - h > \ll 1$, since the difference between the true waveform and a fitted waveform can be—in fact, must be—$\mathcal{O}(1)$ for waveforms h with good posterior support in a parameter estimation.)

The first term in the right hand side of Equation 5 is a random variable with zero mean and variance $< \delta h \mid \delta h > \ll 1$, and is therefore $\mathcal{O}(\delta h)$ in magnitude. We will see below that the other two terms are $\mathcal{O}(\rho \delta h)$ (see Equation 7); presuming that $\rho \gg 1$, we will ignore the first term. Applying the triangle inequality to the remainder of Equation 5, and using the Cauchy-Schwarz inequality, we have

$$\mid \Delta \log \mathcal{L} \mid \lesssim \left(\sqrt{< H|H >} + \sqrt{< h|h >} \right) \sqrt{< \delta h \mid \delta h >} \tag{6}$$

(we use $\lesssim$ to remind the reader that we are ignoring sub-leading-order contributions in $\rho \delta h$). From here we will assume that h is a sufficiently good model for H that the leading order behavior of $< H|H >$ is the same as $< h|h >$, both scaling as

$$\rho^2 := < h \mid h > = < H|H > + \mathcal{O}(\rho). \tag{7}$$

Under this assumption,

$$\Delta \log \mathcal{L} \lesssim 2\rho \sqrt{< \delta h \mid \delta h >} = 2\rho^2 \frac{\mid \delta h \mid}{\mid h \mid}. \tag{8}$$

We can relate $< \delta h \mid \delta h > = \mid \delta h \mid^2$ to the mismatch between g and h, which is defined by

$$\mathcal{M} := 1 - \frac{< g \mid h >}{\sqrt{< g|g >} \sqrt{< h|h >}}. \tag{9}$$

To linear order in δh this becomes

$$\mathcal{M} \simeq 2 \frac{< h| \, \delta h >}{< h|h >}. \tag{10}$$

Again applying Cauchy-Schwarz, we have

$$| \mathcal{M} | \lesssim 2 \frac{\sqrt{< \delta h \mid \delta h >}}{\sqrt{< h|h >}}. \tag{11}$$

This is the **wrong direction** to claim with certainty that

$$\Delta \log \mathcal{L} \lesssim \rho^2 \mathcal{M}, \tag{12}$$

but assuming there is no special orientational issues (i.e. that the projection of δh onto h is not particularly small compared to their magnitudes), we can say that

$$\Delta \log \mathcal{L} \sim \rho^2 \mathcal{M}; \tag{13}$$

but in any case, it is **bounded** by

$$\Delta \log \mathcal{L} \lesssim 2\rho^2 \frac{|\delta h|}{|h|}. \tag{14}$$

3. Definitions

3.1. Mismatch

The mismatch $\mathcal{M}$ between two waveforms g and h is given by

$$\mathcal{M} := 1 - \frac{< g|h >}{\sqrt{< g|g >}\sqrt{< h|h >}} = 1 - < \hat{g} \mid \hat{h} > \tag{15}$$

3.2. Log Likelihood

The log-likelihood for some waveform h given data d is

$$\log \mathcal{L} := -\frac{1}{2} < d - h|d - h > \tag{16}$$

3.3. Triangle Inequality

The triangle inequality applies to any vector space and states that for any vectors a and b

$$\| \ a + b \ \| \leq \|a\| + \|b\| \tag{17}$$

Equivalently, for vectors c and d

$$\|c\| - \|d\| \leq \|c - d\| \tag{18}$$

where $\|x\|$ is the norm of x. (this latter follows from the former by setting $a = c - d$ and $b = d$).

We can also square Equation 17 to obtain

$$\| \ a + b \ \|^2 \leq \|a\|^2 + 2 \ \|a\| \ \|b\| + \|b\|^2, \tag{19}$$

or

$$\|c\|^2 - \|d\|^2 \leq \|c - d\|^2 + 2 \ \|c - d\| \ \|d\|. \tag{20}$$

3.4. Cauchy-Schwarz Inequality

The Cauchy-Schwarz inequality for an inner product states that for all vectors a and b

$$| < a \mid b > | \leq \sqrt{< a|a >}\sqrt{< b|b >} \tag{21}$$

3.5. The Law of Cosines

For vectors a, b, and c with $c = a + b$ we have

$$< c|c > = < a|a > + < b|b > + 2 < a|b > \tag{22}$$