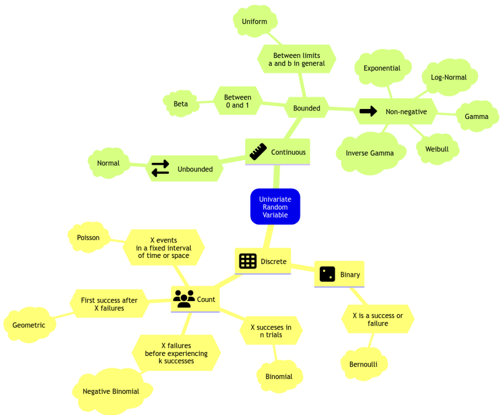
552 Quiz 1



Review of Frquentist Statistical Inference

- Use **Observed Data** (from random sample) to make **Inferences** about **Population Parameters**
  - e.g. μ, σ^2, median, etc.
  - Find **Point Estimates** and **Confidence Intervals** for these parameters.
- **Latent Population vs. Observable Variables**
  - **Latent Population** is the population that we are interested in, but we can't observe it directly.
  - **Observable Variables** are the variables that we can observe directly.
  - e.g. Online ad click data to estimate the total lifetime revenue.

Introduction to Bayesian Statistics

- Very flexible
  - Can handle: missing data, complex models, non-iid, etc.
- Valid inference for any (finite) amount of data
- The population parameters are treated as **random variables**
  - Easy to interpret uncertainty of the population parameters

$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$

- **Prior**: What we know about the parameter before we see the data (prior knowledge)
- **Likelihood**: How the data is generated given the parameter
- **Posterior**: What we know about the parameter after we see the data.
  - Good for *prediction, inference, and decision making*.

- **Recursive Updating**: As we get more data, we can update our prior to get a new posterior.

Generative Models

- A simplified mathematical model for some reality (For both Frequentist and Bayesian)
- **Generative** because it can make synthetic data
- Examples:
  1. We can incorporate **noise in measurements** (e.g., outputs coming from the model).
  2. They can be **overly simplified models with incomplete measurements** (e.g., rainy day model).
  3. They can even incorporate **unobservable latent variables** (e.g., hypothetical tennis rankings).

`Stan` and `rstan` Basics

- `Stan` is a probabilistic programming language for Bayesian inference

- `rstan` is an R interface to `Stan`

- List of common distributions in stan:

  - `bernoulli(theta)`
  - `bernoulli_logit(alpha)`
  - `binomial(n, theta)`
  - `beta_binomial(n, alpha, beta)`
  - `poisson(lambda)`
  - `neg_binomial(alpha, beta)`
  - `gamma_poisson(lambda, alpha)`

General Steps for Bayesian Modeling

1. Code the generative model in `Stan`
2. Specify observed values of data to estimate using `rstan`
3. Generate synthetic data from the model
4. Perform inference on the synthetic data
   - only data generated from the model is used for inference

Note: Generative model is all you need (and get).

Likelihood Vs Probability

Likelihood	Probability
how plausible a given distributional parameter is given some observed data	chance that some outcome of interest will happen for a particular random variable
$P(\theta \| X = x)$	$P(X = x \| \theta)$
bounded to 0 and 1	unbounded to 0 and 1

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- $P(A^c|B) = 1 - P(A|B)$
- IF A and B are independent, then $P(A|B) = P(A)$
  - So $P(B|A) = P(B)$

Bayes' Theorem

Let θ be a parameter of interest and Y be the observed data.

- **Prior**: $P(\theta)$
  - $P(\theta^c) = 1 - P(\theta)$
- **Likelihood** of the data given the parameter:
  - $\ell(\theta|Y) = P(Y|\theta)$
- **Posterior** (what we want to find): $P(\theta|Y)$

$$P(\theta|Y) = \frac{P(Y|\theta)P(\theta)}{P(Y)}$$

$$\text{posterior} = \frac{\text{prior} \times \text{likelihood}}{\text{normalization constant}}$$

$$\text{posterior} \propto \text{prior} \times \text{likelihood}$$

- Once we have the **posterior**, we have everything we need to make decisions.
  - MAP (Maximum A Posteriori) Estimation = mode of the posterior
  - Mean, median, quantiles of the posterior

Bayesian Inference

- **Properties**:
  1. Hidden variables of interest are random (prior distribution)
     - Benefits:
       - Can incorporate prior knowledge
       - Can handle small sample sizes + complex models
       - Can recursively update the prior
     - **Drawbacks**:
       - Forces us to use prior knowledge (which can be subjective)
       - computational complexity
  2. Use **posterior** (conditional distribution of hidden variables given observation) to capture uncertainty
- E.g. posterior: $P(A|B) = 0.3$,
  - There is a 30% chance of A if B is true.

Maximum a Posteriori Estimation (MAP) and Maximum Likelihood Estimation (MLE)

- MAP is a Bayesian approach to MLE

MLE	MAP
Finding value that maximizes **likelihood**	Finding value that maximizes **posterior**
Only uses observed data	Uses observed data and **prior**
$\hat{\theta}_{\text{MLE}} = \text{argmax}_\theta P(D \| \theta)$	$\hat{\theta}_{\text{MAP}} = \text{argmax}_\theta P(\theta \| D)$

The Bayesian Modelling

- **Big advantage**:
  - It formulates every problem in one common framework
- **Final goal**: Take samples from the posterior distribution
- Computer does most of the heavy lifting
  - All we need to do is good model design and critical analysis of the results
- **Characteristics:**

Big Idea

1. **Question**: Pose a scientific question
2. **Design**: Formulate variables and create a probabilistic model for them. **Prior knowledge** is included here!
3. **Inference**: Get **posterior** samples from the model
4. **Check**: If the samples are from your posterior
5. **Analyze**: Use the samples to answer the question

Pose a Scientific Question

1. **Inferential**: Using observed data Y to make inferences about the population/ latent variable θ

2. **Predictive**: Using observed data Y to make predictions about future data Y'

Beta-Binomial Model

- One of the most foundational Bayesian models
- Recall $Posterior \propto Likelihood \times Prior$
  - **Binomial**: The likelihood function
    - $Y|\pi \sim Binomial(n, \pi)$ where $\pi \in [0,1]$
  - Bayesian thinking: Y is a random variable (population parameters are no longer fixed)
  - **Beta**: Prior distribution of **parameter of interest** π
    - $\pi \sim Beta(a, b)$

Beta Distribution (The Prior)

$$\pi \sim Beta(a, b)$$

- **PDF**: $f(\pi) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\pi^{a-1}(1-\pi)^{b-1}$ for $0 \leq \pi \leq 1$
- **Mean**: $\frac{a}{a+b}$
- **Variance**: $\frac{ab}{(a+b)^2(a+b+1)}$
- **Mode**: $\frac{a-1}{a+b-2}$ when $a, b > 1$

Choosing the right Beta Prior

- One of the biggest challenges in Bayesian statistics
- Need to rely on subject matter prior knowledge
- e.g. Collect information from previous studies and plot a histogram of the data, then fit a beta distribution to it
- `bayesrule` package in R has a function `summarize_beta_binomial(a, b)` to summarize the beta distribution
- PDF of binomial distribution:
  - $f(y|\pi) = \binom{n}{y}\pi^y(1-\pi)^{n-y}$

Bayes Rules in Action

$$Posterior \propto Likelihood \times Prior$$

$$f(\pi|Y) \propto f(Y|\pi) \times f(\pi)$$

using the beta-binomial model:

$$f(\pi|Y) \propto \binom{n}{y}\pi^y(1-\pi)^{n-y} \times \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\pi^{a-1}(1-\pi)^{b-1}$$

Simplify (remove non-π terms):

$$f(\pi|Y) \propto \pi^{y+a-1}(1-\pi)^{n-y+b-1}$$

We recognize this as the **kernel of a beta distribution**:

$$f(\pi|Y) \propto Beta(a+y, b+n-y)$$

- **Kernel**: The part of the expression that depends on the variable of interest

Posterior Distribution

- **Posterior**: $f(\pi|Y) = Beta(a+y, b+n-y)$
  - **Mean**: $\frac{a+y}{a+b+n}$
  - **Variance**: $\frac{(a+y)(b+n-y)}{(a+b+n)^2(a+b+n+1)}$
  - **Mode**: $\frac{a+y-1}{a+b+n-2}$
    - Mode is the value of π that maximizes the posterior distribution/ peak (MAP/ Maximum A Posteriori)
- Can also use `summarize_beta_binomial(a, b, n, y)` to summarize the posterior distribution
  - `a` and `b` are the parameters of the prior beta distribution
  - `n` is the number of trials
  - `y` is the number of successes

Posterior Credible Interval

- **Credible Interval**: Range of plausible values for the parameter.
  - Width: measures variability of the posterior distribution
- Use function `qbeta` in R to calculate the quantiles of the beta distribution
- For a given a,b,n,y, the 95% credible interval is `qbeta(c(0.025, 0.975), shape1 = a + y, shape2 = b + n - y)`
- **95% CI means**:
  - There is a 95% posterior probability that the true value of π is between L and U

Designing the Model

- Bayesian model is a big joint probability distribution $P(Y, Y', \theta)$
  - Observations: Y
  - Latent variables: θ
  - Predictions: Y'
- Did inferential approach in Beta-Binomial model (previous section). Now another approach:

$$P(Y, Y', \theta) = P(Y, Y'|\theta) \times P(\theta)$$

1. Generate θ from the **prior** $P(\theta)$
2. Generate Y, Y' given θ from **likelihood** $P(Y, Y'|\theta)$

Markov Chain Monte Carlo (MCMC)

- **Goal**: Generate samples from the posterior distribution
- **Problem**: The posterior is often intractable (can't be solved analytically)
- **Solution**: Use MCMC to generate samples from the posterior
- **Monte Carlo Algorithm**
  - Need closed analytical form of the posterior $f(\theta|Y)$ (e.g. Beta-Binomial model or Gamma-Poisson model)
  - Build independent MC sample $\{\Theta_1, \Theta_2, \ldots, \Theta_n\}$ from $f(\Theta|Y)$ by:
    1. Drawing Θ_i from $f(\Theta|Y)$
    2. Go there
- Is a **random walk** in the space of θ
- Called a Markov Chain because the next state depends only on the current state $\theta^{(t)} \to \theta^{(t+1)}$
- **Disadvantages**:
  - Can be slow to converge (need burn-in period)
  - Samples are not independent
  - Gives approximate posterior not exact posterior
  - Posterior samples might get stuck in local modes (certain areas of the param space) => sample is not an adequate representation of the posterior

Metropolis-Hastings Algorithm

- Allows us to obtain an approximation of the posterior distribution $f(\Theta|Y)$ via MC $\{\Theta_1, \Theta_2, \ldots, \Theta_n\}$.
- Next Θ_{t+1} is selected by:
  1. Proposing a new value Θ' from a proposal distribution $q(\Theta'|\Theta_t)$ (e.g. Uniform, Normal, etc.)
  2. Decide whether to accept or reject Θ' based on acceptance probability α:

$$\alpha = min\left(1, \frac{f(\Theta')\ell(\Theta'|Y)}{f(\Theta_t)\ell(\Theta_t|Y)}\right)$$

- Then obtain the next via bernoulli trial with probability α for success $\Theta^{(t+1)} = \Theta'$

Sampling MCMC using rstan

- **Some considerations**:
  - **Warm-up**: Discard the first n samples to allow the chain to converge
  - **Thinning**: Only keep every nth sample to reduce autocorrelation
    - skip the first n samples and then keep every nth sample

$$\text{Num of approx posterior samples} = \frac{\text{iter} - \text{warmup}}{\text{thin}}$$

Example: Gamma Poisson Model

- **Prior**: $\lambda \sim Gamma(s, r)$
- **Likelihood**: $Y_i|\lambda \sim Poisson(\lambda)$

$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$

$$f(\lambda|Y) \propto \ell(\lambda|Y) \times f(\lambda)$$

- **Posterior**: $\lambda|Y \sim Gamma(s + \sum Y_i, r + n)$

```
data {
  int<lower=1> n; // number of rows in training data
  int<lower=0> count[n]; // array of observed counts (integer)
  real<lower=0> s; // prior shape Gamma parameter
  real<lower=0> r; // prior rate Gamma parameter
}
parameters {
  real<lower=0> lambda; // parameter of interest
}
model {
  lambda ~ gamma(s,r); // prior distribution of lambda
  count ~ poisson(lambda); // Poisson likelihood, can be complex formula too
}
```

```
bird_dictionary <- list(
  n = nrow(observed_evidence),
  count = as.integer(observed_evidence$count),
  s = 150,
  r = 40
)
posterior_lambda <- sampling(
  object = gamma_poisson_stan,
  data = bird_dictionary,
  chains = 1,
  iter = 10000,
  warmup = 1000,
  thin = 5,
  seed = 553,
  algorithm = "NUTS"
)
posterior_lambda <- as.data.frame(posterior_lambda)
```