574 Quiz 1

Time Series

- Collection of observations made sequentially in time
- Data Types:
  - **Univariate**: Single observation at each time point (e.g. bike sale over time)
  - **Multivariate**: Multiple observations at each time point (e.g. bike sale + profit over time)
  - **Heirarchical**: Multiple time series, each with a hierarchical structure (e.g. bike sale + profit for each store over time)
- Common Tasks:
  - **Prediction/ Forecasting** (Supervised Learning)
    - Difficult since many factors
  - **Clustering/ Anomaly Detection** (Unsupervised Learning)

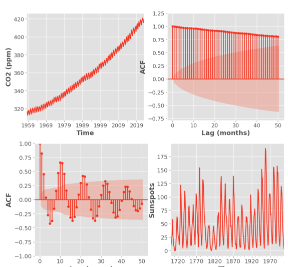
Features of Time Series

Temporal Dependence

- Observations close in time are often correlated
  - Can quanitify using **autocorrelation**
- **Autocorrelation**: Correlation of a time series with a lagged version of itself
  - **Lag**: Time difference between two observations
  - **ACF**: Autocorrelation function
    - Plots autocorrelation for different lags
  - **PACF**: Partial autocorrelation function
    - Plots correlation between two observations after removing the effect of other lags
  - e.g. `data[t (lag=1)] = data[t].shift(t)`

Correlogram

- Plot of ACF vs. lag
- Helps identify patterns in time series
- Use `statsmodels.graphics.tsaplots.plot_acf()`



- Shading indicates if correlation is significantly different from 0
  - $CI = \pm z_{\alpha/2} SE(r_k)$, $z_{\alpha/2} \approx 1.96$ for 95% CI
  - $SE(r_k) = \frac{1}{\sqrt{T}}$, where T is the number of observations - Or Bartlett's formula: $SE(r_k) = \sqrt{\frac{1+2\sum_{j=1}^{k-1} r_j^2}{T}}$
- CO2 plot has a trend so ACF for smaller lags tend to be higher
- **General Key Observations**:
  1. ACF almost always decays with lag
  2. If a series alternates (oscillates about mean), ACF will alternates too
  3. If a series has seasonal or cyclical fluctuations, the ACF will oscillate at the same frequency
  4. If there is a trend, ACF will decay slower (due to high correlation of the consecutive observations)
  5. Experience is required to interpret ACF

Time Series Patterns

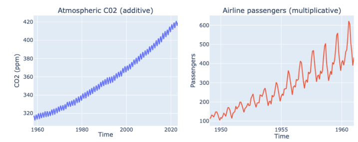
1. **Trend**: Long-term increase/ decrease
2. **Seasonality**: Regular pattern of up and down fluctuations (fixed interval)
   - typically over smaller time frame
3. **Cyclic**: Fluctuations not of fixed period (unknown and changing interval)
   - typically over larger time frame

White Noise

- Time series with: 0 mean, constant variance, no autocorrelation
- Further assumed that it is iid and gaussian: $N(0, \sigma^2)$
- **Why do we care?**
  - Cannot predict white noise
  - If residuals from time series for a forecast should resemble white noise
    - Implies that the model has captured all the information in the data

Time Series Decomposition

- When we decompose, we split the time series into 3 components:
  1. **Trend-cycle (T)**: Long-term increase/ decrease
  2. **Seasonal (S)**: same as seasonal above
  3. **Residual**: Random fluctuations



- **Additive Model**: $Y_t = T_t + S_t + R_t$
  - When the magnitude of the seasonal fluctuations **does not change** with the level of the time series
- **Multiplicative Model**: $Y_t = T_t \times S_t \times R_t$
  - When the magnitude of the seasonal fluctuations **does change** with the level of the time series

Estimating the Trend

1. **Curve Fitting**: Fit a polynomial of degree n to the time series

```
detrended = data - detrend(data, order=2) # order=2 for quadratic
```

2. **Moving Average**: Smooths out short-term fluctuations and highlights longer-term trends

```
# rolling is a pandas function
rolling_mean = df.rolling(window=5, center=True).mean()
# For even window, common practice to do:
window = 4
df.rolling(window).mean().rolling(2).mean().shift(-window//2)
```

   - `window`: Number of observations used for calculating the statistic
   - `center`: Set the labels at the center of the window
     - if odd, the label is at the center
     - if even, the label is at the right

Estimating Seasonality

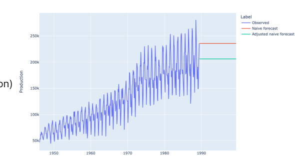
- Simple steps:
  1. Remove the trend from the data (the detrended data above)
  2. Estimate the seasonal component by averaging the detrended data over each season

Estimating the Residual

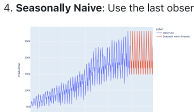
- The residual is the remainder after removing the trend and seasonal components
- If additive model: $R_t = Y_t - T_t - S_t$
- If multiplicative model: $R_t = \frac{Y_t}{T_t \times S_t}$
- Use `seasonal_decompose` from `statsmodels.tsa.seasonal` to do all of this

Forecasting

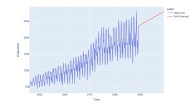
1. **Average**: Use average of all past observations
2. **Naive**: Use the last observation as the forecast
3. **Seasonally Adjusted Naive**: Same as Naive but with seasonally adjusted data (classical decomposition)



4. **Seasonally Naive**: Use the last observation from the same season (only one with seasonality)



5. **Drift**: Linearly extrapolate the trend (only one that is not a straight horizontal line)



Exponential Models

Simple Exponential Smoothing

- Forecast is a weighted average of all past observations
- Recursively defined: $\hat{y}_{t+1|t} = \alpha y_t + (1-\alpha)\hat{y}_{t|t-1}$
- α: **Smoothing parameter**
  - Close to 0: More weight to past observations
  - Close to 1: More weight to current observation (closer to Naive forecast)
- **Initial Forecast**:
  - $\hat{y}_{1|0} = y_1$
  - Heuristic: linear interpolation of the first few observations
  - Learn it by optimizing SSE
- Forecasts are flat

Holt's Method

- Extend SES to include a trend component

$$\hat{y}_{t+h|t} = \ell_t + hb_t$$

$$\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

$$b_t = \beta(\ell_t - \ell_{t-1}) + (1-\beta)b_{t-1}$$

- ℓ_t: Level
- b_t: Smoothness of the trend
  - Close to 0: Trend is more linear
  - Close to 1: Trend changes with each observation
- α: Smoothing parameter for level

Holt's Winter Method

- Extend Holt's method to include a seasonal component

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t-m+h_m}$$

$$b_t = \beta(\ell_t - \ell_{t-1}) + (1-\beta)b_{t-1}$$

- For Additive Seasonal:

$$\ell_t = \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

$$s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1-\gamma)s_{t-m}$$

- For Multiplicative Seasonal:

$$\ell_t = \alpha\frac{y_t}{s_{t-m}} + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

$$s_t = \gamma\frac{y_t}{\ell_{t-1} + b_{t-1}} + (1-\gamma)s_{t-m}$$

Trend component	Seasonal Component
None `(N)`	None `(N)`
Additive `(A)`	Additive `(A)`
Additive Damped `(Ad)`	Multiplicative `(M)`

- Simple Exponential Smoothing `(N,N)`
- Holt's Method `(A,N)`
- Holt's Winter Method `(A,A)`

ETS (Error, Trend, Seasonal) Models

- Components:
  - Error: `{A, M}`
  - Trend: `{N, A, Ad}`
  - Seasonal: `{N, A, M}`
- Can generate prediction intervals (confidence intervals):
  1. `model.get_prediction()` (analytical)
  2. `model.simulate()`

Selecting a Model

- **Metrics**, Commonly used:
  - AIC, BIC
  - SSE/ MSE/ RMSE
- **Residuals**:
  - Visual inspection (should be uncorrelated, zero mean, normally distributed)
  - Running diagnostic Portmanteau tests:
    - **Ljung-Box Test**: H_0: Residuals are uncorrelated (white noise)

- p-value < 0.05: Reject H_0 (bad)
- **Jarque-Bera Test**: H_0: Residuals are normally distributed
  - p-value < 0.05: Reject H_0 (bad)
- **Out-of-sample Forecasting**:
  - Split data into training and testing
  - Fit model on training data
  - Forecast on testing data
  - Compare forecast with actuals

ARIMA Models

- **ARIMA**: AutoRegressive Integrated Moving Average
- Commonly used for time series forecasting (other than exponential smoothing)
- Based on autocorrelation of data
- Do not model trend nor seasonality, so it is typically constrained to **stationary** data

Stationarity

- Statistical properties of a time series do not change over time
  - Mean, variance is constant
  - Is roughly horizontal (no strong trend)
  - Does not show predictable patterns (no seasonality)
- DOES not mean that the time series is constant, just that the way it changes is constant
- It is one way of modelling dependence structure
  - Can only be independent in one way but dependent in many ways

Strong vs Weak Stationarity

Property	Strong Stationarity	Weak Stationarity
Mean, Variance, Autocovariance	Constant	Constant
Higher order moments (skewness, kurtosis)	Constant	Not necessarily constant

- Weak stationarity is often sufficient for time series analysis

Checking for Stationarity

1. **Visual Inspection**: Plot the time series
   - Look for trends, seasonality, and variance (none of these should be present)
   - Make a correlogram plot (ACF plot should rapidly decay to 0)
2. **Summary Statistics**: Calculate mean, variance, and autocovariance
   - Mean and variance should be roughly constant over time
3. **Hypothesis Testing**: Use statistical tests
   - **Augmented Dickey-Fuller (ADF) test/ AdFuller**
     - Null hypothesis: Time series is non-stationary
     - small p: it is stationary (reject null)
     - Use `statsmodels.tsa.stattools.adfuller`
   - **Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test**
     - Null hypothesis: Time series is stationary
     - small p: it is non-stationary (reject null)

Making a Time Series Stationary

- **Stabilizing the variance using transformations**
  - Log or box-cox transformation

$$w_t = \begin{cases} \frac{y_t^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \ln(y_t) & \text{if } \lambda = 0 \end{cases}$$

- **Stabilize the mean using differencing**
  - First difference: $y' = y_t - y_{t-1}$
  - Second difference: $y'' = y' - y'_{t-1} = y_t - 2y_{t-1} + y_{t-2}$
  - Seasonal difference: $y' = y_t - y_{t-m}$, where m is the seasonal period

AR and MA Models

AR (AutoRegressive) Model	MA (Moving Average) Model
Regression of the time series on its own lagged values	Regression of the time series on past forecast errors
$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \epsilon_t$	$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \ldots + \theta_q \epsilon_{t-q}$
p: order of the AR model	q: order of the MA model
ϕ: AR coefficients	θ: MA coefficients
ϵ_t: white noise	ϵ_t: white noise
Long memory model: y_1 has a direct effect on y_t for all t	**Short memory model**: y_t is only affected by recent values of ϵ
Good for modeling time-series with dependency on past values	Good for modeling time-series with a lot of volatility and noise
Less sensitive to choice of lag or window size	More sensitive to choice of lag or window size

- Both values are between -1 and 1
- AR value of 1 means that the time series is a random walk

ARMA Model

- **ARMA**: AutoRegressive Moving Average
- Combines AR and MA models
- Key Idea: Parsimony
  - fit a simpler, mixed model with fewer parameters, than either a pure AR or a pure MA model

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \ldots + \theta_q \epsilon_{t-q}$$

- ϕ: AR coefficients, θ: MA coefficients
- Usually write it as `ARMA(p, q)`

ARIMA Model

- **ARIMA**: AutoRegressive Integrated Moving Average
- Combines ARMA with differencing
- `ARIMA(p, d, q)`
  - `p`: order of the AR model
  - `d`: degree of differencing
  - `q`: order of the MA model
- Use `statsmodels.tsa.arima.model.ARIMA`
- Hyperparameter tune using `pmdarima.auto_arima`

SARIMA

- **SARIMA**: Seasonal ARIMA
- `SARIMA(p, d, q)(P, D, Q, m)`
  - `p, d, q`: ARIMA parameters
  - `P, D, Q`: Seasonal ARIMA parameters
  - `m`: seasonal period
- Also have **SARIMAX** (with exogenous variables)
  - adds exogenous variables (other time series) to the model
  - Not the most effective model

Choosing Orders

- **ACF and PACF plots**
  - ACF: Autocorrelation Function
  - PACF: Partial Autocorrelation Function
  - Use these to determine the order of the AR and MA models

ACF Plot	PACF Plot
Measures correlation between an observation and its lagged values	same but removes intermediate correlations (kinda isolates the direct effect)
For **MA(q)**, cuts off after lag q	For **AR(p)**, cuts off after lag p
Else, tails off (exp or like damped sin)	Else, tails off (no clear pattern)

- See the cutoff when the peaks are lower than the shaded region

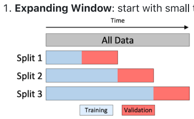
Time Series Forecasting in ML

Key Differences vs. Traditional ML

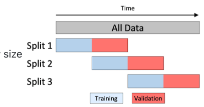
Traditional ML	Time Series ML
Data is IID	Data is ordered
CV is random	Use sliding window CV
Use feature engineering	Use lags, rolling windows, etc.
Predict new data	Predict future (specify horizon)

`sktime` Library

1. Load Data
2. Feature Engineering: shift/ lag
3. Train-Test Split
   1. Use `sktime.split.temporal_train_test_split` or `sklearn.model_selection.train_test_split` but with `shuffle=False`
4. CV with time series
   1. **Expanding Window**: start with small training set and increase it



   2. **Fixed/sliding Window**: use a fixed window size



5. Model Fitting
6. Forecasting

Forecasting Strategies

1. **One-step forecasting**: one step ahead
2. **Multi-step forecasting**: multiple steps ahead
   a. **Recursive strategy**: predict `t`, then it becomes part of the input for `t+1`
   b. **Direct strategy**: have a model for each step (model for `t+1`, another for `t+2`, etc)
   c. **Hybrid strategy**: is dumb and bad
   d. **Multi-output strategy**: 2 different series (e.g. temperature and humidity)

- Use `make_reduction`: `make_reduction(regressor, window_length=12, strategy="recursive")`

Feature Preprocessing and Engineering

Preprocessing

1. Coerce to stationary (via diff or transforms)
2. Smoothing (e.g. moving average)
3. Impute missing value (e.g. linear interpolation)
4. Removing outliers

Feature Engineering

1. Lagging features/ responses
2. Adding time stamps (e.g. day of week, month, etc) 3, Rolling Statistics (e.g. rolling mean, rolling std)

Multivariate Time Series

- Means time series with multiple variables (e.g. temperature and humidity)